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Coarse topology of leaves of foliations

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Abstract

In this thesis, we investigate the question when a non-compact manifold can be quasi-isometric to a leaf in a foliation of a compact manifold. The point of departure is the result of Paul Schweitzer's that every non-compact manifold carries a Riemannian metric so that the resulting Riemannian manifold is not quasi-isometric to a leaf in a codimension one foliation of a compact manifold. We show that the coarse homology of these non-leaves is not finitely generated. This observation motivates the main question of this thesis: Does every leaf in a foliation of a compact manifold have finitely generated coarse homology?

The answer to this question is a double negative: Firstly, we show that there exists a large class of two-dimensional leaves in codimension one foliations that have non-finitely generated coarse homology. Moreover, we improve Schweitzer's construction by showing that every Riemannian metric can be deformed to a codimension one non-leaf without affecting the coarse homology. In particular, we find non-leaves with trivial coarse homology.

In order to answer these questions we develop computational tools for the coarse homology. Furthermore, we show that certain known criteria for manifolds to be a leaf are independent of one another and of the coarse homology.

Zusammenfassung

In dieser Arbeit beschäftigen wir uns mit der Frage, wann eine nicht-kompakte Mannigfaltigkeit quasi-isometrisch zu einem Blatt in einer Blätterung einer kompakten Mannigfaltigkeit sein kann. Ausgangspunkt der Arbeit ist ein Resultat von Paul Schweitzer, nach dem jede nicht-kompakte Mannigfaltigkeit eine Riemannsche Metrik trägt, sodass die resultierende Riemannsche Mannigfaltigkeit nicht quasi-isometrisch zu einem Blatt einer Kodimension 1 Blätterung einer kompakten Mannigfaltigkeit ist. Wir zeigen, dass die Grobhomologie dieser Nicht-Blätter nicht endlich erzeugt ist. Aus dieser Beobachtung motiviert sich die in dieser Arbeit untersuchte Frage, ob alle Blätter in kompakten Mannigfaltigkeiten endlich erzeugte Grobhomologie haben.

Wie sich herausstellt, ist sowohl dies als auch die Umkehrung im allgemeinen nicht wahr: Wir zeigen, dass es eine große Klasse zweidimensionaler Blätter in Kodimension 1 mit nicht endlich erzeugter Grobhomologie gibt. Ferner verbessern wir Schweitzers Konstruktion, indem wir zeigen, dass jede Riemannsche Metrik zu einem Kodimension 1 Nicht-Blatt deformiert werden kann, ohne die Grobhomologie dabei zu verändern. Insbesondere konstruieren wir Nicht-Blätter mit trivialer Grobhomologie.

Zur Behandlung dieser Fragestellungen entwickeln wir Berechnungsmethoden für die Grobhomologie und zeigen ferner, dass verschiedene bekannte Kriterien für Mannigfaltigkeiten Blatt zu sein voneinander und von der Grobhomologie unabhängig sind.

Contents

Chapter 1

Introduction

In this thesis we investigate the question when a non-compact manifold can occur as a leaf in a foliation of a compact manifold. If we equip the foliated manifold with a Riemannian metric, the induced metric on the leaves will, up to quasi-isometry, only depend on the foliation. It thus makes sense to ask when a non-compact Riemannian manifold can be quasi-isometric to a leaf in a foliation of a compact manifold. This will be our guiding question.

Given a Riemannian manifold L, it is in general very hard to determine whether there exists a foliation of a compact manifold such that L is quasi-isometric to one of the leaves. However, we can rule out certain Riemannian manifolds. The first examples of manifolds which are not quasi-isometric to leaves were given by Paul Schweitzer in the 1990s. Schweitzer proved that every non-compact surface carries a metric that cannot be bi-Lipschitz equivalent to a leaf in a foliation of a compact 3-manifold. Oliver Attie and Steven Hurder then found higher-dimensional examples of non-leaves, and in 2009 Schweitzer was able to generalize his previous results to any dimension. He showed that every non-compact manifold carries a metric such that the resulting Riemannian manifold cannot be diffeomorphically quasi-isometric to a leaf in a codimension one foliation of a compact manifold.

It is interesting to note that all of the above results are in need of some additional assumption about the quasi-isometry, such as it also being a diffeomorphism. Elaborating the counterexamples by Attie and Hurder, Abdelghani Zeghib was able to remove any additional regularity assumptions and produced Riemannian manifolds that cannot be coarsely quasi-isometric to simply connected leaves. It is still an open question whether every Riemannian manifold is coarsely quasi-isometric to a leaf, i.e. if we allow the maps to be discontinuous.

All of the aforementioned non-leaves are constructed through manifolds that violate certain conditions met by leaves in foliations of compact manifolds, and which is preserved under quasi-isometries. Since any non-compact leaf in a compact manifold has to accumulate against itself, these conditions measure in some way or another whether the manifold looks more or less the same around each point. Attie and Hurder introduce the *geometric entropy* of a metric space (X, d), which is defined via the number of quasi-isometry types of spaces of bounded diameter needed to cover increasingly large subsets of X. Schweitzer's criterion, *the bounded homology property* of a Riemannian manifold (M, g), is a condition on certain types of volumes of nullhomologous hypersurfaces in M. For an in-depth treatment, we refer the reader to Chapter 2 and the sources [Att-Hur] and [Schw2]. In particular Schweitzer's condition is specifically tailored to foliations and might be difficult to grasp. It is thus an interesting question whether established quasi-isometry invariants can detect whether a given Riemannian manifold is quasi-isometric to a leaf in a compact manifold. Schweitzer's and Zeghib's counterexamples are produced by deforming the metric on a given Riemannian manifold by inserting balloons of radius tending towards infinity. Computing the coarse homology of the resulting spaces, one notices that it is never finitely generated. What we want to find out is hence whether the coarse homology of a leaf in a compact manifold must always be finitely generated, and conversely, whether there exist non-leaves with finitely generated coarse homology. These are the main questions that we are going to investigate in this thesis. As we will describe in the following sections, both are to be answered in the negative.

Building on the work of Schweitzer we are able to give a non-leaf construction starting with any non-compact Riemannian manifold that does not affect the coarse homology. Moreover, Zeghib's construction of manifolds which are not even coarsely quasi-isometric to a simply connected leaf in a compact manifold can be improved to produce non-leaves with the coarse homology of $\mathbb{H}^n$. On the other hand, due to a connection between coarse homology and ends of manifolds, we can show that there exist leaves in foliations of compact manifolds that have non-finitely generated coarse homology.

The question which non-compact manifolds can be *homeomorphic* to leaves in compact manifolds has also spurred a lot of research. The first examples of topological non-leaves in codimension 1 were found by Étienne Ghys [Gh1] and by Takashi Inaba et. al. [I-N-T-T] in the 1980s. They produced manifolds as connected sums along trees in which the fundamental groups of the summands prevent them from accumulating against each other. In 2011, Souza and Schweitzer generalized these examples and gave criteria that prevent such connected sums along trees to be homeomorphic to leaves.

On the positive side, it is known that every surface can be topologically realized as a leaf in any compact 3-manifold, while there exist non-leaves in any higher dimension. For foliations of higher codimension very little is known. In particular, we do not know whether every manifold can be homeomorphic to a leaf in a codimension 2 foliation of a compact manifold.

Convention: Throughout this thesis, all foliated manifolds are compact and the foliations are of codimension one, unless explicitly stated otherwise. In particular, the statement *"(L, g) is not quasi-isometric to a leaf"* means *"There exists no compact manifold M and a codimension 1 foliation $\mathscr{F}$ of M such that (L, g) is quasi-isometric to a leaf of $\mathscr{F}$"*. Moreover, we take all manifolds to be connected and without boundary.

1.1 Coarse homology

Coarse homology is a theory of metric spaces which is designed to capture the homological large-scale properties of a space. Given a metric space X, we want to consider increasingly coarse versions of (X, d) by replacing all sets of diameter less than R by balls. If we let R go to infinity, we can think of having deleted all finite scale topology of X and thus only be left with the asymptotic topology. In practice, this is achieved by coarsening sequences $|\mathcal{U}_1| \to |\mathcal{U}_2| \to \ldots$, where each $|\mathcal{U}_i|$ is the nerve of increasingly coarse open coverings of X. Every $|\mathcal{U}_i|$ is quasi-isometric to X, but as i increases, the quasi-isometry constants will usually tend towards infinity. The coarse homology of (X, d) is then defined as the direct limit

$$HX_k(X, d) = \varinjlim_{i \to \infty} H_k^{lf}(|\mathcal{U}_i|),$$

where H_k^{lf} is locally finite homology, an adaptation of singular homology to non-compact spaces. We will always use $\mathbb{Z}$-coefficients for the homology theories.

While the idea of a coarsening sequence is geometrically intuitive, it is in general very hard to handle for concrete computations. Hence Chapter 3 develops tools that will allow us to compute the coarse homology of certain types of spaces. This will be important in later chapters, where we compute the coarse homology of known non-leaves and construct non-leaves with trivial coarse homology.

Since locally finite homology behaves somewhat differently from singular homology, our first results generalize theorems about singular homology to locally finite homology, which we didn't find elsewhere in the literature. We prove a locally finite Mayer-Vietoris sequence for families of coverings and generalize singular homology for Δ-complexes ([Hat]) to locally finite Δ-homology.

Proposition 1.1.1. *Let X be a finite dimensional Δ-complex. Then the locally finite Δ-homology X is naturally isomorphic to locally finite homology of X.*

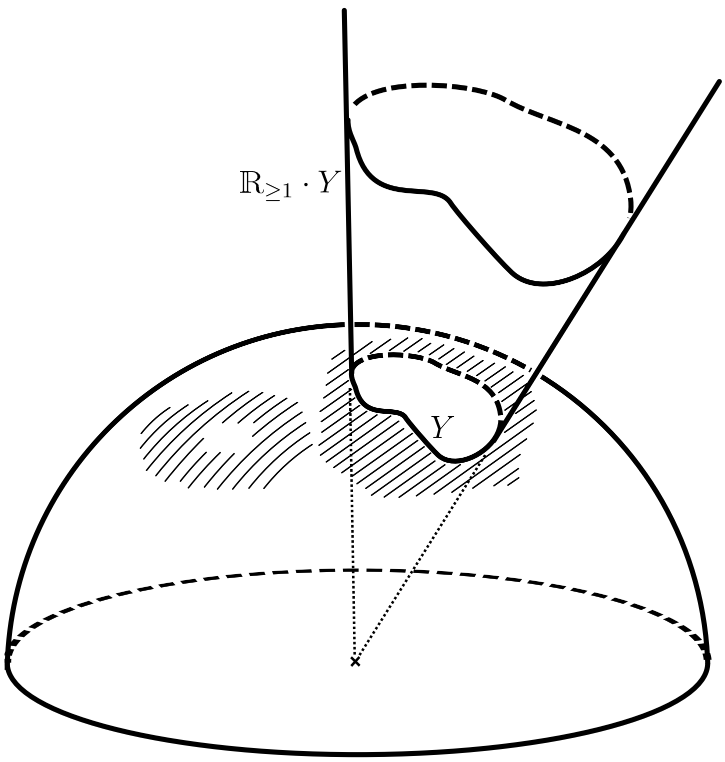

Figure 1.1: The truncated cone $\mathscr{T}(Y)$ over Y.

The following result allows us to compute the coarse homology of certain spaces, which we call *truncated cones*. These are manifolds which result from taking cones over subsets of the sphere. Truncated cones are convenient building blocks, which we will use in Chapter 6 to construct spaces with a desired coarse homology.

Proposition 1.1.2. *Let Y be a closed hypersurface of int(D^n_+). Then the coarse homology of the truncated cone $\mathscr{T}(Y)$ over Y is given by the following isomorphism*

$$HX_k\big(\mathscr{T}(Y)\big) \simeq H_{k-1}\big(Y\big).$$

1.2 Non-leaves with trivial coarse homology

Using the computational tools for coarse homology from Chapter 3, we are now able to compute the coarse homology of the non-leaves constructed by Schweitzer. We prove the following

Proposition 1.2.1. *Let (L, g) be an open Riemannian manifold, of dimension $n \geq 2$ and let g_S be the non-leaf metric described in [Schw2]. Then $HX_n(L, g_S)$*

is not finitely generated. In fact

$$\prod_i \mathbb{Z}/\bigoplus_i \mathbb{Z} \subset HX_n(L, g_S).$$

That is, Schweitzer's non-leaf construction blows up the top degree coarse homology by an infinite product of $\mathbb{Z}$s. This observation motivates the question whether every manifold quasi-isometric to a leaf in a foliation has to have finitely generated coarse homology. We then show that non-finitely generated coarse homology is not a necessary condition for a non-leaf:

Theorem 1.2.2. *On every non-compact manifold of bounded geometry (M, g), there exists a deformation of g to a bounded geometry metric g' by manipulating g on a sequence of balls in M such that (M, g') cannot be diffeomorphically quasi-isometric to a leaf of a codimension one $C^{2,0}$-foliation of a compact manifold. This deformation can be performed in such a way that $HX_*(M, g') = HX_*(M, g)$ and the growth type of (M, g) remains unchanged.*

While Schweitzer deforms the metric by gluing in spheres of increasing radius, we use building blocks modelled on certain trees. These trees are in a sense small enough to be ignored by the coarse homology, while adding enough complexity to make the resulting Riemannian manifold a non-leaf. One can use similar methods to produce manifolds which are not coarsely quasi-isometric to simply connected leaves. This improves the constructions by Zeghib [Zeg] and Attie-Hurder [Att-Hur].

Theorem 1.2.3. *In any dimension $n \geq 2$ there exist Riemannian manifolds (M, g) of bounded geometry such that (M, g) cannot be coarsely quasi-isometric to a simply connected leaf in neither a C^1-foliation of arbitrary codimension nor a $C^{1,0}$-foliation of codimension 1 of a compact manifold. Moreover, $HX_k(M, g)$ is trivial for all k but $k = \dim M$, where we have $HX_k(M, g) = \mathbb{Z}$.*

These manifolds are modelled on $\mathbb{H}^n$, where we have replaced a sequence of balls with geometrically more complex manifolds of which infinitely many cannot be quasi-isometric for fixed quasi-isometry constants. Attie's and Hurder's work then implies that such manifolds cannot be coarsely quasi-isometric to a leaf in a foliation of a compact manifold (see [Att-Hur] and Sections 2.3.1 and 4.3).

1.3 Leaves with non-finitely generated coarse homology

In Chapter 5, we conclude the discussion of the connection between the coarse homology of a Riemannian manifold and it being quasi-isometric to a leaf. We show that in every dimension, there exist leaves with non-finitely generated coarse homology. More precisely, we prove

Theorem 1.3.1. *In every dimension $n \geq 2$ there exist Riemannian manifolds (L, g) with $HX_1(L, g)$ containing an Abelian subgroup of infinite rank, such that (L, g) can be realized as a leaf in a foliation of a compact manifold of arbitrary codimension.*

This theorem is proved by exploiting the fact that in every proper geodesic space two distinct ends of the space yield a non-trivial element in the degree one coarse homology (cf. [Roe1]).

Proposition 1.3.2. *Let (X, d) be a proper connected length space with $k \in \mathbb{N} \cup \{\infty\}$ ends. Then $HX_1(X, d; \mathbb{Z})$ contains a subgroup isomorphic to $\oplus_{i=1}^{k-1}\mathbb{Z}$.*

For arbitrary metric spaces, we cannot expect to have any connection between topological objects such as ends and the coarse homology, which relies heavily on the given metric. For proper geodesic spaces, however, the set of ends is a quasi-isometry invariant and hence is preserved under coarsenings. Thus we can work with locally finite homology of the coarsenings and prove the analogous statement there. Because the locally finite 1-cycles on the coarsenings can be constructed very explicitly, it is then not hard to see that they in fact yield non-trivial elements in coarse homology.

Cantwell and Conlon have shown that any possible space of ends can be realized as the space of ends of a leaf of a codimension 1 foliation of a compact manifold [Cant-Co1], in particular there exist leaves with infinitely many ends, and by the above theorem their degree 1 coarse homology contains an Abelian subgroup of infinite rank. In particular, the coarse homology of such a leaf is non-finitely generated.

1.4 Independence of leaf criteria

The final chapter deals with the question whether leaf criteria such as Schweitzer's bounded homology property and Attie-Hurder's geometric entropy can be reduced to one another, to the Cheeger isoperimetric constant or the number of generators of the coarse homology. It turns out that there are no relations between these invariants if no further assumptions are made. As we are interested in leaves of foliations of compact manifolds, it is natural to restrict oneself to manifolds of bounded geometry. In this case the only dependence is given by the following proposition.

Proposition 1.4.1. *Let (M, g) be a Riemannian n-manifold of bounded geometry that does not satisfy the bounded homology property. Then the Cheeger constant of (M, g) vanishes.*

In particular, one cannot construct non-leaves of bounded geometry using Schweitzer's criteria without creating manifolds with vanishing Cheeger constant. On the other hand, the vanishing of the Cheeger constant does not imply that a manifold cannot be quasi-isometric to a leaf as is shown by the example of $\mathbb{R}^n$.

Other than the above, we have the following results.

Proposition 1.4.2. *There exist simply connected Riemannian manifolds satisfying the bounded homology property whose coarse homology is finitely generated as well as those whose coarse homology is non-finitely generated.*

This, together with Proposition 4.1.1 and Corollary 4.2.2 shows that the bounded homology property and the number of generators of the coarse homology are completely unrelated.

Moreover, in general there aren't any relations between the geometric entropy and the bounded homology property. That is, even though the counterexamples given by Attie-Hurder and by Schweitzer seemed very similar, their criteria do in fact measure different properties of a Riemannian manifold.

Chapter summary. Chapter 2 gives some more background on the realizability problem and provides basic definitions. Moreover, we present the leaf criteria of Schweitzer [Schw2] and Attie-Hurder [Att-Hur] as well as of Zeghib [Zeg] and their constructions of non-leaves. The following chapter is on coarse homology and besides the necessary definitions and previous results, presents computational tools that will be important in later chapters. In Chapter 4 we prove that there exist manifolds with trivial coarse homology that cannot be quasi-isometric to a leaf and Chapter 5 we prove that there exist leaves with non-finitely generated coarse homology. Hence we show that the property of a manifold to be quasi-isometric to a leaf is independent on the number of generators of the coarse homology. The final chapter concludes the discussion of the independence of leaf criteria, coarse homology and the Cheeger constant.

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I would also like to thank Jonathan Pfaff for being my friend and mathematical companion since the beginning of my studies.

Before, I never quite understood what partners had to do with doctoral theses and why people were thanking them. I would like to thank Malin for showing me.

Chapter 2

The realizability problem

If a non-compact Riemannian manifold (L, g) is homeomorphic or quasi-isometric to a leaf of a foliation of a compact manifold M, it will have to accumulate against itself in M. Hence we can expect some kind of periodicity in the topology or geometry of L. It is then interesting to ask in which ways this recursion can be measured. In the topological case, the earliest results by Ghys and by Inaba et al. ([Gh1], [I-N-T-T]) show that under certain circumstances, the fundamental group of L has to show some repetition. In the context of the question when a manifold can be quasi-isometric to a leaf, complexity can be measured in terms of how many different spaces of bounded diameter are needed to cover L [Att-Hur] or by bounds on the size of nullhomologous hypersurfaces [Schw2].

This chapter gives an account of previous results on the realizability problem and in particular explains the measures of recurrence outlined above. The first section recalls some basic definitions about foliations and coarse geometry. Section 2.2 gives an account of previous results on the topological realizability problem, that is the question of when a non-compact manifold can be homeomorphic to a leaf in a foliation of a compact manifold. The final and longest section of this chapter concerns the question when a Riemannian manifold can be quasi-isometric to a leaf of a foliation of a compact manifold. We give a detailed account of the construction of non-leaves by Attie-Hurder and by Zeghib, who developed a notion of entropy for metric spaces which has to be finite for leaves in compact manifolds. In addition, we present the bounded homology property developed by Schweitzer to show that every non-compact manifold carries a metric which makes it not quasi- isometric to a leaf. It turns out that the constructions by Schweitzer and Zeghib produce non-leaves whose coarse homology is not finitely generated. In Chapter 4 we will modify both constructions to show that there exist non-leaves with finitely generated coarse homology.

2.1 Basic definitions

A *codimension k foliation $\mathscr{F}$* of an n-manifold M is a decomposition of M into topologically immersed codimension k submanifolds L_α, called the *leaves* of $\mathscr{F}$, such that there exist an atlas $\{\varphi_i\colon U_i \longrightarrow B_\tau \times B_\pitchfork\}_{i\in I}$ of M which takes each component of the intersection $U_i \cap L_\alpha$ to a set of the form $B_\tau \times \{y\}$. By topologically immersed we mean that the inclusion $L_\alpha \hookrightarrow M$ is continuous but not necessarily open. In particular, the leaves need not carry the subspace topology.

More technically, a foliation can be described by foliated charts:

Definition 2.1.1 (foliated chart, plaque, transversal, foliated atlas)**.** A *foliated chart of class C^r* of a manifold M^n is a chart (U, φ) of class C^r such that $\varphi : U \to B_\tau \times B_\pitchfork$ is a diffeomorphism to the product of neighbourhoods of the origin $B_\tau \subset \mathbb{R}^{n-k}$ (the tangential or leafwise direction) and $B_\pitchfork \subset \mathbb{R}^k$ (the transversal direction).

The set $P_y := \varphi^{-1}(B_\tau \times \{y\}) \subset U$ for $y \in B_\pitchfork$ is called a *plaque* (through y) and $T_x := \varphi^{-1}(\{x\} \times B_\pitchfork) \subset U$ is called a *transversal* (through x) of the foliated chart φ.

A C^r-atlas of M consisting of foliated charts $\{U_\alpha, \varphi_\alpha\}$ is called a *C^r-foliated atlas* if the intersection $P_\alpha \cap P_\beta$ of plaques $P_\alpha \subset U_\alpha$ and $P_\beta \subset U_\beta$ is open both in P_α and in P_β.

A foliated atlas $\{\varphi_i, U_i\}_{i\in I}$ of a foliation $\mathscr{F}$ is of class $C^{r,k}, r > k \geq 0$ if the coordinate changes

$$\varphi_j \circ \varphi_i^{-1}(x_i, y_i) = (x_j(x_i, y_i), y_j(y_i))$$

with transversal coordinates y and leafwise coordinate x are of class C^k, but with x_j being of class C^r in x_i.

A foliation is then given by a foliated atlas and a leaf is given by the union of all plaques that intersect nontrivially. The topology on the leaves is induced by the topology on the plaques. In order to define the holonomy of a foliation, we need a somewhat more refined atlas.

Definition 2.1.2 (regular foliated atlas)**.** A foliated atlas $\mathcal{U} = \{U_\alpha, \varphi_\alpha\}$ is called *regular* if

i) each $\overline{U}_\alpha$ is a compact subset of a foliated chart (V_α, ψ_α) such that $\psi_\alpha|_{U_\alpha} = \varphi_\alpha$;

ii) the U_α form a locally finite cover of M;

iii) if U_α and U_β intersect, then each closed plaque of $\overline{U}_\alpha$ meets at most one plaque of $\overline{U}_\beta$.

It can be shown that any foliated atlas is equivalent to a regular atlas (Lemma 1.2.17, [Cand-Con]).

We now want to define the holonomy cocycles of a foliation. To this aim let $\{U_\alpha, \varphi_\alpha\}_{\alpha \in A}$ be a regular foliated atlas and let U_α and U_β be foliated charts that intersect nontrivially. For $p \in U_\alpha \cap U_\beta$ write

$$\begin{aligned} \varphi_\alpha(p) &= (x_\alpha(p), y_\alpha(p)) \in B^\alpha_\tau \times B^\alpha_\pitchfork \\ \varphi_\beta(p) &= (x_\beta(p), y_\beta(p)) \in B^\beta_\tau \times B^\beta_\pitchfork. \end{aligned}$$

Consider the coordinate change

$$\begin{aligned} g_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) &\to \varphi_\alpha(U_\alpha \cap U_\beta) \\ (x_\beta, y_\beta) &\mapsto \varphi_\alpha(\varphi_\beta^{-1}(x_\beta, y_\beta)) = (\psi_{\alpha\beta}(x_\alpha, y_\beta), \gamma_{\alpha\beta}(x_\beta, y_\beta)). \end{aligned}$$

Every plaque of P of U_α intersects at most one plaque of U_β (by property iii) of Definition 2.1.2). Hence for $p \in U_\alpha \cap U_\beta$ the transversal component of the coordinate change $\gamma_{\alpha\beta}(x_\beta(p), y_\beta(p)) = \gamma_{\alpha\beta}(y_\beta(p))$ is independent of the tangential coordinate $x_\alpha(p)$. The maps $\gamma_{\alpha\beta}$ are called the *holonomy cocycles* of the foliated atlas. They map the subset $y_\beta(U_\alpha \cap U_\beta)$ of $B^\beta_\pitchfork$ diffeomorphically to the subset $y_\alpha(U_\alpha \cap U_\beta)$ of $B^\alpha_\pitchfork$ and satisfy the usual cocycle conditions

i) $\gamma_{\alpha\beta} \circ \gamma_{\beta\delta} = \gamma_{\alpha\delta}$;

ii) $\gamma_{\alpha\alpha} = \mathrm{id}$;

iii) $\gamma_{\alpha\beta} = \gamma_{\beta\alpha}^{-1}$.

Observe that $y_\alpha : U_\alpha \to B^\alpha_\pitchfork$ induces a diffeomorphism $y_\alpha : T_\alpha \xrightarrow{\cong} B^\alpha_\pitchfork$ when restricted to any transversal of U_α. Up to restriction to the domain of $\gamma_{\alpha\beta}$ we get the following diagram of diffeomorphisms

$$\begin{tikzcd} B^\beta_\pitchfork \arrow[r, "\cong", "\gamma_{\alpha\beta}"'] & B^\alpha_\pitchfork \\ T_\beta \arrow[u, "y_\beta", "\cong"'] \arrow[r] & T_\alpha. \arrow[u, "y_\alpha", "\cong"'] \end{tikzcd}$$

This defines a diffeomorphism between open subsets of T_α and T_β which we will again call $\gamma_{\alpha\beta}$. It is clear that these maps satisfy the cocyle conditions whenever they are defined.

In what follows we will investigate whether a given Riemannian manifold is, in a stronger or weaker sense, "quasi-isometric" to a leaf in a compact manifold. In the most general sense a (λ, D, C) *quasi-isometry* between metric spaces (X, d_X) and (Y, d_Y) is just a map $f\colon X \longrightarrow Y$ with C-dense image (in particular not necessarily continuously or bijective) such that the following inequalities hold for all $x, x' \in X$:

$$\frac{1}{\lambda} d_X(x, x') - D \leq d_Y(f(x), f(x')) \leq \lambda d_X(x, x') + D.$$

We call λ the *dilation* and D the *distortion* of f. If we only require the above estimates to hold, but not C-denseness of the image, then we call f a *quasi-isometric embedding.*

One can show that if $f\colon X \longrightarrow Y$ is a quasi-isometry, then there exists a quasi-isometry $g\colon Y \longrightarrow X$ and a constant E such that $d_X(x, g \circ f(x)) < E$ and $d_Y(y, f \circ g(y)) < E$ for all $x \in X$ and $y \in Y$, that is $g \circ f$ and $f \circ g$ are uniformly close to the identity map on X and Y respectively. The map g is then called a *quasi-isometric inverse* of f.

We often require the map f to be a homeomorphism or even a diffeomorphism. In that case, C-denseness of $\mathrm{im}(f)$ is automatic and hence we call f a (λ, D)-*quasi-isometric homeomorphism* or *diffeomorphism* respectively. If we want to stress that a given quasi-isometry is not assumed to have any additional regularity, we call it a *coarse* quasi-isometry.

Recall that two non-decreasing functions $v, w\colon \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ are said to have the same growth type if there exists a constant $\lambda > 0$ such that

$$v(r) \leq \lambda w(\lambda r) + \lambda, \qquad w(r) \leq \lambda v(\lambda r + \lambda) + \lambda$$

for all $t \in \mathbb{R}_{\geq 0}$.

Definition 2.1.3 (growth type)**.** Let (M, g) be a Riemannian manifold, $x_0 \in M$. The *growth type* of (M, g) is growth type of the function $r \longmapsto \mathrm{vol}(B_r(x_0))$.

The growth type of (M, g) does not depend on the choice of base point x_0.

2.2 The topological realizability problem

The definition of a foliation is topological in nature, so it is natural to ask whether a given non-compact manifold can be homeomorphic to a leaf in a foliation of a compact manifold. This is what we will call the *topological realizability problem*, which was first posed by Sondow in [Son]. The restriction to foliations of compact manifolds is reasonable, as any manifold L is a leaf in the product foliation of $S^1 \times L$. Likewise, for compact L, the product foliation $S^1 \times L$ is an example of a foliation of a compact manifold in which L occurs as a leaf. Since the leaves do not necessarily carry the subspace topology – in fact non-compact leaves in compact manifolds never do – we may expect some non-compact manifolds to occur as leaves in foliations of compact manifolds. As an example, the Reeb foliation of the solid torus $S^1 \times D^2$ has leaves homeomorphic to $\mathbb{R}^2$.

If a non-compact L is a leaf in a foliation of a compact manifold M, then L has to accumulate somewhere in M and hence we expect L to show some type of recurrence. This recurrence or lack of it is then exploited to show that a given manifold cannot be homeomorphic to a leaf in a foliation of a compact manifold. The first such non-realizability results where obtained independently of each other by Ghys in [Gh1] and Inaba et al. in [I-N-T-T] in the 1980s.

Theorem 2.2.1 (Ghys, (1984), Inaba et al., (1985))**.** *In dimension greater than or equal to* 3*, there exist manifolds that cannot be realized as leaves in codimension one foliations of compact manifolds.*

Ghys examines an infinite connected sum W of compact manifolds with fundamental groups isomorphic to $\mathbb{Z}/p_k$, where p_k ranges over the prime numbers starting with 3 and shows that W cannot occur as a leaf in a codimension 1 $C^{1,0}$-foliation of a compact manifold.

Recall that the space of ends $\mathcal{E}(L)$ of a manifold L is a compact, totally disconnected, metrizable space (see also Section 5.1). Inaba et al. show that for any possible endspace E of a non-compact manifold and any $d \geq 3$, there exists a d-dimensional manifold L with endspace homeomorphic to E and such that L cannot be homeomorphic to a leaf of a C^2-foliation of codimension one of a compact manifold. To construct these non-leaves, they choose a tree T with endspace homeomorphic to E and take the connected sum of compact manifolds L_k with fundamental group $\mathbb{Z}/(2k+1)$ along T. The resulting manifold L has an endspace homeomorphic to E and cannot be homeomorphic to a leaf.

We can think of Ghys' construction as a special case of [I-N-T-T], where the connected sum is taken along the Cayley graph of $\mathbb{N}$. The above results were generalized by F. Souza and P. Schweitzer in [Sou-Schw] to so called *sum manifolds*, which are also connected sums of closed manifolds along a graph.

Theorem 2.2.2 (Theorem A, [Sou-Schw])**.** *Let W be a sum manifold patterned on an infinite tree such that the fundamental group of each summand is generated by torsion elements of odd order (or trivial) and infinitely many non-homeomorphic summands repeat finitely. Then W is not homeomorphic to a leaf of a C^0-foliation of codimension one of a compact manifold.*

The authors then define what it means for a manifold to be *non-periodic in homotopy (or homology)*:

Definition 2.2.3 (non-periodic in homotopy/homology)**.** A $(k-1)$-connected manifold M is *non-periodic in homotopy* in dimension $k \geq 2$ if its kth homotopy group $\pi_k(M)$ is isomorphic to the direct sum of cyclic groups of order p^n, where $p \geq 3$ is prime, and such that for an infinite number of prime powers p^n the number of summands of order p^n is finite but non-zero. A k-manifold is *non-periodic in homology* in dimension $k \geq 2$ if its kth homology group $H_k(M;\mathbb{Z})$ is isomorphic to a direct sum of cyclic groups satisfying the same property.

Theorem 2.2.2 implies the following more concise result.

Theorem 2.2.4 (Theorem B, [Sou-Schw], (2012))**.** *Sum manifolds patterned on a tree which are non-periodic in homotopy (or in homology) are not homeomorphic to any leaf of a C^0-foliation of codimension one of a compact manifold.*

In contrast to the results of Inaba et al., in codimension one foliations by surfaces any possible endspace can be realized as the endspace of a leaf in a foliation of a compact manifold. In particular, the endspace is no obstruction to a manifold occurring as a leaf:

Theorem 2.2.5 ([Cant-Co1])**.** *Let E be a compact, totally disconnected, metrizable space and let M be a 3-manifold with $H^1(M) \neq 0$. Then there exists a C^∞-foliation of M such that some leaf L has endspace homeomorphic to E.*

It is no coincidence that all topological non-leaves were at least 3-dimensional for J. Cantwell and L. Conlon also showed in [Cant-Co2] that every 2-manifold can be realized as a leaf on any 3-manifold.

Theorem 2.2.6 ([Cant-Co2], (1987))**.** *Given any open orientable surface Σ and a compact 3-manifold M or a non-orientable surface Σ and a compact non-orientable 3-manifold M, there exists a C^∞-foliation $\mathscr{F}$ of M such that Σ is diffeomorphic to a leaf of $\mathscr{F}$.*

While all the realizability results above trivially generalize to higher codimension by taking products, very little is known about non-realizability. In particular, it is still an open question whether there exist manifolds which cannot be diffeomorphic to a leaf in a codimension 2 (or higher) foliation on a compact manifold.

2.3 The geometric realizability problem

Let (M, g) now be a compact Riemannian manifold and let $\mathscr{F}$ be a foliation of arbitrary codimension of M. Then g induces a Riemannian metric on the leaves of $\mathscr{F}$ and if we choose a different metric g' on M, then the induced metrics on the leaves L will be quasi-isometric (in fact, even bi-Lipschitz) in the sense that there exists $\lambda \geq 0$ such that

$$\frac{1}{\lambda} g(X, X) \leq g'(X, X) \leq \lambda g(X, X)$$

for all $X \in T_pL$. For let $p \in M$. Then there exists a $\lambda_p > 0$ such that g_p and g'_p are bi-Lipschitz equivalent with Lipschitz constant λ_p. But λ_p varies continuously with p and by compactness of M attains its maximum λ. Then for any leaf L, the induced metrics $g|_L$ and $g'|_L$ will be $(\lambda, 0)$ quasi-isometric. Hence a leaf in a foliation of a compact manifold comes equipped with a natural quasi-isometry class of a metric. It thus makes sense to ask whether a given Riemannian manifold (L, g) is quasi-isometric to a leaf in a foliation of a compact manifold.

One should also note that the metric induced from M onto L will always be of *bounded geometry*, that is the injectivity radius of L is bounded away from zero and the sectional curvatures are bounded from above and below.

Various examples and constructions of manifolds which cannot be quasi-isometric to leaves have been found. Using a type of entropy, Attie and Hurder were able to construct simply connected 6-manifolds which cannot be homeomorphically quasi-isometric to a leaf in a compact manifold, while Schweitzer has shown that every non-compact manifold carries a metric so that it is not diffeomorphically quasi-isometric to a leaf in a compact manifold. That is, both non-realizability results do not rule out that the Riemannian manifolds that they constructed may be *coarsely* quasi-isometric to a leaf. A partial example of such a manifold is given by Zeghib, who adapted the methods of Attie and Hurder to find manifolds which cannot be coarsely quasi-isometric to *simply connected* leaves. The question whether every Riemannian manifold can be coarsely quasi-isometric to a leaf in a compact manifold is still open.

The abovementioned constructions will be presented in the following sections. In Chapter 4, we will further improve these non-leaf constructions to produce non-leaves with trivial coarse homology.

It may also be noteworthy that two natural invariants of Riemannian manifolds, the *Cheeger constant* (see Section 6.4) and the *growth type* (see Definition 2.1.3), defined as the growth class of the volume growth of balls, cannot serve as a criterion to decide whether a given non-compact manifold is quasi-isometric to a leaf in a compact manifold. This can be seen by considering the hyperbolic plane, which can be realized as a leaf in a foliated bundle over a surface of genus at least 2 (see Remark 2.3.13): It has both exponential growth and positive Cheeger constant, while Euclidean space, which can for example be realized as a leaf in the irrational foliation of the $n+1$-torus, has polynomial growth and Cheeger constant equal to zero.

2.3.1 Attie-Hurder, Zeghib: Geometric entropy

The techniques of Attie and Hurder, which were later generalized by Zeghib, exploit that the entropy of a foliation gives bounds on the complexity of its leaves. The latter is measured by the so called *geometric entropy*, which was defined in [Att-Hur]. Since the entropy of codimension 1 foliations and of C^1-foliations of arbitrary codimension is always finite [Eg], manifolds with infinite geometric entropy cannot be homeomorphically quasi-isometric to leaves.

Theorem 2.3.1 (Theorem 3, [Att-Hur]). *There exists a Riemannian manifold of bounded geometry that cannot be homeomorphically quasi-isometric to a leaf in neither a C^1-foliation nor a codimension 1 $C^{1,0}$-foliation of a compact manifold.*

Zeghib was able to partly improve this result to the construction of 2-dimensional manifolds which are not even *coarsely* quasi-isometric to a simply connected leaf in C^1-foliations or $C^{1,0}$-foliations of codimension 1 of a compact manifold. We note that his example is topologically just a 2-disk.

Theorem 2.3.2 ([Zeg]). *There exist 2-dimensional Riemannian manifolds that cannot be (coarsely) quasi-isometric to a simply connected leaf of a C^1-foliation of arbitrary codimension or to $C^{1,0}$-foliation of codimension 1 of a compact manifold.*

In this section we give an exposition of the results and techniques used to prove the above results as far as they are needed later in this thesis. Since [Att-Hur] uses a somewhat different notion of entropy than the sources they quote auxiliary results from, we give a more detailed account of the entropy of foliations hoping to make [Att-Hur] more accessible.

The geometric entropy of foliations gives a measure for the transverse dynamics of a foliation and was defined in [Gh-La-Wa]. We follow the exposition in [Eg].

Let $(M^n, \mathscr{F})$ be a codimension k foliated compact manifold with a foliated atlas $\mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ and let $B = \cup_\alpha B_\alpha \subset \mathbb{R}^{n-k}$, with B_α being the image of a transversal T_α in U_α under φ_α. Let $\{\gamma_1, \ldots, \gamma_l\}$ be the holonomy cocycles of $\{(U_\alpha, \varphi_\alpha)\}_\alpha$, which are local homeomorphisms of B (see Section 2.1). Set $\mathcal{H}_1 = \{\mathrm{id}_B\} \cup \{\gamma_1, \dots, \gamma_l\}$ and let $\mathcal{H}_n$ be the composition of at most n elements of $\mathcal{H}_1$. For each $n \geq 1$ we define a metric on B by

$$d_n^{\mathcal{H}_1}(x, y) = d_n(x, y) = \max_{\substack{f \in \mathcal{H}_n \\ x, y \in \mathrm{dom}(f)}} |f(x) - f(y)|_{\mathbb{R}^n}.$$

Thus d_n is a measure for the transverse dynamics of $\mathscr{F}$, measuring how far points on the transversal are spreading out under the holonomy of $\mathscr{F}$.

Definition 2.3.3 ((ε, n)-spanning set)**.** Let $0 < \varepsilon < 1$ and $n > 0$ and $K \subset B$. A subset $\{x_1, \dots, x_d\} \subset B$ is (ε, n)-*spanning* (for K) if

$$K \subset \bigcup_{i=1}^{d} B(x_i, \varepsilon; d_n).$$

Denote by $sp_n^{\mathcal{H}_1}(\varepsilon, K) = sp_n(\varepsilon, K)$ the minimal cardinality of an (ε, n)-spanning set for K. If $K = B$, we simply set $sp_n(\varepsilon, B) = sp_n(\varepsilon)$.

Remark 2.3.4. For $n \leq N$ we have $d_n \leq d_N$ and hence it follows that $B(x, \varepsilon, d_n) \supset B(x, \varepsilon, d_N)$. Consequently every (ε, N)-spanning set is also (ε, n)-spanning. Hence $sp_n(\varepsilon, K) \leq sp_N(\varepsilon, K)$ for all $n \leq N$ and all $K \subset B$. Clearly, $sp_n(\varepsilon', K) \geq sp_n(\varepsilon, K)$ for $\varepsilon' < \varepsilon$.

Definition 2.3.5 (geometric entropy of a foliation)**.** The *geometric entropy of* $\mathscr{F}$ is defined by

$$h(\mathscr{F}, \mathcal{U}) = h(\mathscr{F}) = \lim_{\varepsilon \to 0+} \limsup_{n \to \infty} \frac{1}{n} \log \left(sp_n^{\mathcal{H}_1}(\varepsilon, B) \right).$$

Even though $\mathcal{H}_1$ and consequently $d_n^{\mathcal{H}_1}$ and $sp_n^{\mathcal{H}_1}(\varepsilon, K)$ depend on the choice of the foliated atlas $\mathcal{U}$, Theorem 2.16 and Theorem 2.3 in [Eg] show that a suitable growth class of $sp_n^{\mathcal{H}_1}(\varepsilon, K)$ is independent of the choice of $\mathcal{U}$. It follows that $h(\mathscr{F}, \mathcal{U})$ is independent of $\mathcal{U}$ up to multiplication with a positive constant. In particular, the properties $h(\mathscr{F}) = 0, h(\mathscr{F}) = \infty$ and $0 < h(\mathscr{F}) < \infty$ depend only on the foliation $\mathscr{F}$.

It is not hard to see that there exists a constant $C(\varepsilon)$ such that $sp_n(\varepsilon) \leq C(\varepsilon)^{|\mathcal{H}_n|}$ which has the same growth type as e^{e^n}. For codimension 1 foliations and C^1-foliations of arbitrary codimension, we have the following stronger result.

Proposition 2.3.6 (Proposition 2.20, Lemma 3.1, [Eg])**.** *For C^1-foliations of arbitrary codimension and for codimension 1 foliations of class C^0, the growth of $n \mapsto sp_n(\varepsilon)$ is dominated by e^n for all $\varepsilon > 0$. In particular, for C^1-foliations and for codimension 1 foliations of class C^0, the geometric entropy is finite.*

Similarly to the entropy of *foliations*, Attie and Hurder define the entropy of *metric spaces* roughly by counting spanning sets of increasing diameter (see Definition 2.3.8). Their key observation is that the geometric entropy of a foliation bounds the geometric entropy of its simply connected leaves. Since the geometric entropy of C^1-foliations and of codimension 1 $C^{1,0}$-foliations is finite, any Riemannian manifold which is homeomorphically quasi-isometric to a leaf in such a foliation must have finite geometric entropy.

Definition 2.3.7 ((ε, R)-quasi-tiling)**.** Let (X, d) be a metric space. An (ε, R)-*quasi-tiling* of (X, d) is a finite collection of compact metric spaces $K_1, \dots K_s$ of diameter at most $R > 0$ and a countable collection of $(1+\varepsilon, \varepsilon)$-quasi-isometric topological embeddings $f_\alpha : K_{i_\alpha} \to X$ such that any set $K \subset X$ with diameter at most $R/4$ is covered by the image of some f_α.

In the above definition s is called the *cardinality* of the quasi-tiling. We define $H(X, d, \varepsilon, R) = H(X, \varepsilon, R)$ to be the minimal cardinality of an (ε, R)-quasi-tiling of X; if no (ε, R)-quasi-tilings exist, we set $H(X, \varepsilon, R) = \infty$. $H(X, \varepsilon, R) = s$ means roughly that X consists of s metrically and topologically distinct pieces of diameter $R/4$. As R increases the (ε, R)-quasi-tilings detect larger and larger topological and metrical features of (X, d).

Given $\varepsilon > 0$ the ε-*growth complexity function* of (X, d) is defined by

$$R \mapsto H(X, \varepsilon, R).$$

For $\varepsilon' > \varepsilon$ we find that $H(X, \varepsilon', R) \leq H(X, \varepsilon, R)$ as every (ε, R)-quasi-tiling is also an (ε', R)-quasi-tiling. For $R' > R$ no general assertions are possible.

Definition 2.3.8 (geometric entropy of a metric space)**.** Let (X, d) be a metric space. The *geometric entropy* of X is defined to be

$$h_g(X, d) = h_g(X) = \lim_{\varepsilon \to \infty} \limsup_{R \to \infty} \frac{\log \left(H(X, \varepsilon, R)\right)}{R}.$$

It comes as no surprise that the geometric entropy has some quasi-isometry invariance. The proof, however, is surprisingly tedious.

Lemma 2.3.9 (Proposition, [Att-Hur], p. 347)**.** *The geometric entropy of a metric space X depends up to multiplication with a positive constant only on the homeomorphic quasi-isometry class of the space. In particular the statements $h_g(X) = 0, 0 < h_g(X) < \infty$ and $h_g(X) = \infty$ depend only on the homeomorphic quasi-isometry class of X.*

Proof. It is not hard to see that if $f\colon X \longrightarrow X'$ is a (λ, D)-quasi-isometric homeomorphism and $\{K_1, \dots, K_d, f_\alpha\}$ an (ε, R)-quasi-tiling of X, then any subset $Y' \subset X'$ of diameter at most $\frac{1}{4}\left(\frac{R}{\lambda} - 2D\right)$ lies in some $(f \circ f_{\alpha_i})(K_i)$ and $f \circ f_{\alpha_i} \colon K_i \to X'$ is a $(\lambda(1+\varepsilon), \lambda\varepsilon + D)$-quasi-isometric topological embedding for every i. For sufficiently large R, we can rescale the metric on K_i by a factor of $\frac{1}{\lambda} - \frac{2D}{R}$ to get metric spaces K_i' of diameter at most $\frac{R}{\lambda} - 2D$ and quasi-isometric topological embeddings $f \circ f_{\alpha_i} \colon K_i' \longrightarrow X'$ with dilation $\left({}^1\!/\!_\lambda - {}^{2D}\!/\!_R\right)^{-1} \lambda(1+\varepsilon)$ and distortion $\left({}^1\!/\!_\lambda - {}^{2D}\!/\!_R\right)^{-1} (\lambda\varepsilon + D)$. Again, for $R \gg 0$, we have the estimate $\left({}^1\!/\!_\lambda - {}^{2D}\!/\!_R\right)^{-1} \leq 2\lambda$ and hence $\{K_1', \ldots, K_d', f \circ f_\alpha\}$ is an (ε', R')-quasi-tiling of X' with

$$\varepsilon' = \max\{2\lambda^2(1+\varepsilon) - 1, 2\lambda(\lambda\varepsilon + D)\}, \qquad R' = \frac{R}{\lambda} - 2D.$$

Hence $H(X', \varepsilon', R') \leq H(X, \varepsilon, R)$ for sufficiently large R. Then

$$\begin{aligned}
h_g(X) &= \lim_{\varepsilon \to \infty} \limsup_{R \to \infty} \frac{1}{R} \log(H(X, \varepsilon, R)) \\
&\geq \lim_{\varepsilon' \to \infty} \limsup_{R \to \infty} \frac{R'}{R} \frac{\log(H(X', \varepsilon', R'))}{R'} \\
&= \frac{1}{\lambda} \lim_{\varepsilon' \to \infty} \limsup_{R \to \infty} \frac{\log(H(X', \varepsilon', R'))}{R'} \\
&= \frac{1}{\lambda} h_g(X'). \qquad \square
\end{aligned}$$

The proof shows in particular, that $h_g(X)$ is insensitive to quasi-isometries with dilation equal to one. The geometric entropy changes if the metric is rescaled and by the lemma $h_g(X)$ is a quasi-isometry invariant only up to multiplication with a positive constant. Taking the limit $\varepsilon \to \infty$ as opposed to $\varepsilon \to 0$ is necessitated by the fact that a (λ, D)-quasi-isometry takes an (ε, R)-quasi-tiling to an (ε', R')-quasi-tiling with $\varepsilon' \geq D$ and hence we would not have the above quasi-isometry invariance up to a multiplicative constant.

We now want to relate the geometric entropy of a leaf $h_g(L)$ of a foliation $\mathscr{F}$ to the geometric entropy $h_g(\mathscr{F})$ of the foliation. As a preparation we recall the definition of geometric entropy of foliations used in [Att-Hur] and relate it to the Definition 2.3.5.

Let (M, g) be a codimension k foliated compact Riemannian manifold with a finite foliated atlas $\{(U_\alpha, \varphi_\alpha\}_{\alpha \in A}$ such that $B_\pitchfork = [-1, 1]^k$ (see Definition 2.1.1) for all α. Then $B = [-1, 1]^k$ and we call $T = \coprod_\alpha T_\alpha$, where $T_\alpha = \varphi_\alpha^{-1}\left(\{0\} \times [-1, 1]^k\right)$, *the transversal* of the foliation (or more precisely, the transversal of the foliated atlas). The holonomy cocycles of $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ induce local homeomorphisms of T, the set of which we again denote by $\mathcal{H}_1$. We define the following metric on the transversal T: Let $n > 0$. If p and p' lie in different transversals, set $D_n^{\mathcal{H}_1}(p,p') = \operatorname{diam}(M)$. If $p,p' \in T_\alpha$ set

$$D_n^{\mathcal{H}_1}(p,p') = \max_{|\gamma|\le n} d_M(h_\gamma(p), h_\gamma(p')),$$

where γ ranges over all leafwise paths that lie in a plaque chain of length at most N. We call a subset $\{p_1,\dots,p_i\} \subset T$ (ε,n)-spanning with respect $D_n^{\mathcal{H}_1}$ if the balls $B(x_i,\varepsilon; D_n^{\mathcal{H}_1})$ cover T.

Since the foliated atlas was chosen such that $B_\pitchfork = [-1,1]^k$ for each foliated chart and consequently $B = [-1,1]^k$, there exists a constant $D = D(\mathcal{U},g)$ such that for each α the embedding $B \hookrightarrow T_\alpha$ is a bi-Lipschitz map with Lipschitz constant $D(\mathcal{U},g)$.

We claim that the cardinalities of spanning sets of B with respect to $d_n^{\mathcal{H}_1}$ and those of T with respect to $D_n^{\mathcal{H}_1}$ are related as follows: For every $\varepsilon, n > 0$ there exist an $(D(\mathcal{U},g)\varepsilon, n)$-spanning set of T of cardinality at most $|\mathcal{U}| sp_n^{\mathcal{H}_1}(\varepsilon)$, where $|\mathcal{U}|$ is the number of charts in the foliation atlas $\mathcal{U}$ and $sp_n^{\mathcal{H}_1}(\varepsilon)$ is the minimal cardinality of an (ε,n)-spanning set of B. For let $\{x_1,\dots,x_s\}$ be an (ε,n)-spanning subset of B. Given $y \in T_\alpha \subset T$, there exists an x_i such that $d_n^{\mathcal{H}_1}(t_\alpha^{-1}(y), x_i) < \varepsilon$. But as the metrics $d_n^{\mathcal{H}_1}$ on B and $D_n^{\mathcal{H}_1}$ on T_α are bi-Lipschitz equivalent with Lipschitz constant D, we find that $D_n^{\mathcal{H}_1}(y, t_\alpha(x_i)) < D(\mathcal{U})\varepsilon$. Hence $\cup_\alpha\{t_\alpha(x_1),\dots,t_\alpha(x_s)\}$ is an $(D(\mathcal{U})\varepsilon, n)$-spanning subset of $\mathcal{T}$ and we have shown that

$$|\{\text{minimal } (D(\mathcal{U})\varepsilon, n)\text{-spanning set of } T\}| \le |\mathcal{U}| sp_n^{\mathcal{H}_1}(\varepsilon).$$

(In a similar fashion, every (ε,n)-spanning subset of T yields an $(D(\mathcal{U})\varepsilon,n)$-spanning subset of B.)

If we now, as in [Att-Hur], let $\mathcal{U}$ be a foliated atlas such that all plaques have diameter bounded by 1, we can rephrase their results relating the geometric entropy of a simply connected leaf with the geometric entropy of the foliation as follows.

Proposition 2.3.10 (Proposition, [Att-Hur], p. 348)**.** *Let $L \subset M$ be a simply connected leaf of a $C^{1,0}$-foliation $\mathscr{F}$. For each $\varepsilon, n > 0$ there exists an open covering $\{V_\beta\}_{\beta\in B}$ of M of cardinality $|\mathcal{U}|\, sp_n^{\mathcal{H}_1}(\varepsilon/D(\mathcal{U}))$ such that*

i) each V_β is a foliated product;

ii) for each leaf L' in the closure $\overline{L}$ of L in M the restriction $\{V_\beta \cap L'\}_\beta$ has Lebesgue number at least $n-3$;

iii) each component of the intersection $L \cap V_\beta$ has diameter bounded by $2n$.

Proof. Simply use the fact that there exists an $(D(\mathcal{U})\,\varepsilon, n)$-spanning subset of $\mathcal{T}$ of cardinality at most $|\mathcal{U}|\, sp_n^{\mathcal{H}_1}(\varepsilon/D(\mathcal{U}))$. The remainder of the proof is as in [Att-Hur]. □

Let K_β be a component of the intersection $L \cap V_\beta$. Since each V_β is a foliated product, we can write $V_\beta = K_\beta \times [-1,1]^k$ and the K_β together with the maps $K_\beta \longrightarrow K_\beta \times \{t\}$, for $t \in [-1,1]^k$ chosen such that $K_\beta \times \{t\} \subset L$ yields a $(1,2n)$-quasi-tiling of L and we get the following theorem.

Theorem 2.3.11 (Theorem 8, [Att-Hur]). *Let L be a simply connected leaf of a $C^{1,0}$-foliation $\mathscr{F}$ of a compact manifold. Then the cardinality of quasi-tilings of the leaf L and transversally spanning sets of the foliation $\mathscr{F}$ are related by $H(L,g,1,2n) \leq |\mathcal{U}|\, sp_n^{\mathcal{H}_1}(\varepsilon/D(\mathcal{U}))$ for all $\varepsilon, n > 0$.*

It follows in particular that

$$\begin{aligned}
h_g(L,g) &= \lim_{\delta\to\infty} \limsup_{n\to\infty} \frac{\log H(L,g,\delta,2n)}{2n} \\
&\leq \limsup_{n\to\infty} \frac{\log H(L,1,2n)}{2n} \\
&\leq \limsup_{n\to\infty} \frac{\log(|\mathcal{U}|\, sp_n^{\mathcal{H}_1}(1))}{2n} \\
&\leq \frac{1}{2} \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{\log sp_n^{\mathcal{H}_1}(\varepsilon)}{n} \\
&= \frac{1}{2} h_g(\mathscr{F},\mathcal{U}).
\end{aligned}$$

Now Lemma 2.3.9 implies that if (X,d) is homeomorphically quasi-isometric to a simply connected leaf of a $C^{1,0}$-foliation $\mathscr{F}$ of a compact manifold, then up to a multiplicative constant

$$h_g(X,d) \leq h_g(\mathscr{F},\mathcal{U}).$$

We summarize the above observation and Proposition 2.3.6 in the following proposition:

Proposition 2.3.12 (Corollary, [Att-Hur], p. 349). *Let (X,d) be homeomorphically quasi-isometric to a simply connected leaf of a C^1-foliation of arbitrary codimension or a codimension 1 foliation of class $C^{1,0}$ of a compact manifold. Then the geometric entropy $h_g(X,d)$ is finite.*

Attie-Hurder now prove their non-leaf result Theorem 2.3.1 by constructing simply connected Riemannian manifolds with infinite geometric entropy. By the above proposition, these can neither be leaves in C^1-foliations nor $C^{1,0}$-foliations of codimension 1 of a compact manifold.

The construction of Attie and Hurder is modelled on $\mathbb{H}^6$ and uses compact manifolds N_0, N_1, N_2 all of which are homotopy equivalent to $S^2 \times S^4$ but such that the first Pontryagin class $p_1(N_l) = l \in H^4(S^2 \times S^4; \mathbb{Z}) \simeq \mathbb{Z}, l = 0, 1, 2$. These are glued to $\mathbb{H}^6$ in such an irregular fashion that the geometric entropy of the resulting manifold Riemannian manifold L is infinite. It turns out though, that L is coarsely quasi-isometric to $\mathbb{H}^6$.

Since $\mathbb{H}^n$ has exponential growth, there exists a constant $c > 1$ such that, given $R > r > 0$, every ball in $\mathbb{H}^6$ of radius R contains at least $d(R, r) = \lfloor c^{R-r} \rfloor$ disjoint balls of radius r. Starting with any open ball $B(y; r) \subset \mathbb{H}^6$, choose $x_1, \ldots x_{d(r,1)} \in B(y; r) \subset \mathbb{H}^6$ be be such that the $B(x_i; 1)$ are pairwise disjoint and are contained in $B(y; r)$. Endow the N_l with metrics of injectivity radius at least 1 and remove a ball of radius 1 to perform the following connected sum operations: For each $1 \leq k < d(r, 1)$ we construct a building block $W^+(y, r, k)$ by gluing N_2 to $B(x_i; 1)$ for $1 \leq i \leq k$ and N_0 to $B(x_i; 1)$ for $k < i < d(r, 1)$. $W^-(y, r, k)$ is $W^+(y, r, k)$ with a copy of N_1 glued to $x_{d(r,1)}$ (cf. Figure 4.5 for a similar construction).

Note that collapsing each copy of N_0, N_1, N_2 to the respective x_i induces a coarse quasi-isometry from $W^\pm(y, r, k)$ to $B(y; r)$ with dilation 0 and distortion $\max_{l=0,1,2} \operatorname{diam}(N_l)$.

Given a function $\mathbf{j}\colon \{1, \ldots, d(r, 1) - 1\} \longrightarrow \{\pm\}$ we construct larger building blocks from the $W^\pm(y, r, k)$: Let $s \geq r$, then $d(r + s, r) = \lfloor c^{r+s-r} \rfloor \geq \lfloor c^{r-1} \rfloor = d(r, 1)$ and consequently $B(y; r + s) \subset \mathbb{H}^6$ contains at least $d(r, 1)$ many disjoint balls of radius r, say $B(y_k; r), k = 1, \ldots, d(r, 1)$. Now $N(y, r, s, \mathbf{j})$ is the ball $B(y, r + s)$ with $B(y_k, r)$ replaced by $W^{\mathbf{j}(k)}(y_k, r, k)$. There are as many choices of $N(y, r, s, \mathbf{j})$ as there are functions $\mathbf{j}$, that is $2^{d(r,1)-1}$. Note again that collapsing the N_l gives a *coarse* quasi-isometry from $N(y, r, s, \mathbf{j})$ to $B(y, r+s)$ with dilation 0 and distortion $\max \operatorname{diam}(N_l)$. In contrast, Attie and Hurder show that if there exists a *homeomorphic* quasi-isometry with dilation and distortion bounded by D between $N(y, r, s, \mathbf{j})$ and $N(y', r, s, \mathbf{l})$ and if additionally $s > 2D(2r + 1)$ holds, then the functions $textbfj$ and $\mathbf{l}$ are equal (see [Att-Hur], Proposition p. 350).

Recall that a *ray* in a Riemannian manifold is a path $\gamma\colon [0, \infty) \longrightarrow M$ such that $d(\gamma(0), \gamma(t)) = t$ for all $t \in [0, \infty)$. Rays always exist in complete noncompact Riemannian manifolds. Let γ be a ray in $\mathbb{H}^6$ and set $y_i = \gamma(i!)$. The non-leaf is now constructed by replacing the sequence of balls $B(y_i, n + \mu_n n)$ by the building blocks $N(y, n, \mu_n n, \mathbf{j}_n)$, where n runs through all natural numbers, μ_n through all numbers between 1 and n^2 and $\mathbf{j}_n$ through all functions $\mathbf{j}_n\colon \{1, \ldots, d(n, 1)\} \longrightarrow \{\pm\}$. Since the distance between the y_i grows faster than the radius of the $N(y, n, \mu_n n, \mathbf{j})$ it follows that the ε-growth complexity function of the resulting manifold L is superexponential for every $\varepsilon > 1$. (For a more detailed argument for a similar construction, see Lemma 4.3.2.) Hence its geometric entropy is infinite and since L is simply connected, by Proposition 2.3.12 it can neither be homeomorphically quasi-isometric to leaf of a

C^1-foliation nor to a leaf of a $C^{1,0}$-foliation of codimension 1.

As we have noted above, collapsing the N_i induces a (coarse) quasi-isometry from each $N(y, r, s, \mathbf{j})$ to $B(y; r+s) \subset \mathbb{H}^6$ with uniformly bounded dilation and distortion. Hence we find that the non-leaf L is still coarsely quasi-isometric to $\mathbb{H}^6$.

Remark 2.3.13. The non-leaves constructed in [Att-Hur] are coarsely quasi-isometric to $\mathbb{H}^6$. In particular, the coarse homology of the non-leaves of Attie-Hurder satisfy $HX_*(L, \mathbb{Z}) = HX_*(\mathbb{H}^6) = \{0\}$, unless $* = 6$ and thus have finitely generated coarse homology. In contrast, $\mathbb{H}^2$, $\mathbb{H}^3$ and more generally $\mathbb{H}^n$, for every n such that there exists a compact hyperbolic n-manifold whose fundamental group embeds into a right-angled Artin group, is isometric to a leaf in a $C^{\infty,0}$-foliated bundle.

Proof. The following argument was given by Ian Agol as an answer to a question I asked on mathoverflow.net [1]. I gratefully acknowledge his help.

For $n = 2$ simply note that the action of $\pi_1(\Sigma_g), g \geq 2$, on $\mathbb{H}^2$ by deck transformations extends smoothly to its boundary $\partial_\infty \mathbb{H}^2 = S^1$. The circle bundle given by the quotient of the diagonal action of $\pi_1(\Sigma_g)$ on $\widetilde{\Sigma_g} \times S^1 = \mathbb{H}^2 \times S^1$, is a foliated bundle over Σ_g with leaves the equivalence classes of $\mathbb{H}^2 \times \{z\}, z \in S^1$. It is not hard to see that there exists a smooth metric on $\mathbb{H}^2 \times S^1/\pi_1(\Sigma_g)$ which restricts to the pullback metric from Σ_g to each leaf. A point (p, z) is identified with (q, z) if there exists $\gamma \in \pi_1(\Sigma_2)$ such that $\gamma.p = q$ and $\gamma.z = z$. In order to get a leaf quasi-isometric to $\mathbb{H}^2$ we have to find $z \in S^1$ such that $\gamma.z \neq z$ for all γ or equivalently such that $\mathrm{stab}(z) = \{e\}$. But every deck transformation γ maps exactly one geodesic to itself and hence fixes two elements of the boundary. As $\pi_1(\Sigma_2)$ is countable and the number of geodesics in $\mathbb{H}^2$ is uncountable, there exists $z \in S^1 = \partial\mathbb{H}^2$ as desired.

Agol's argument now generalizes the above construction. Recall that a group G is called *left orderable* if there exists a total ordering $\preceq$ on G such that $a \preceq b$ implies $ga \preceq gb$ for all $g \in G$. It is a classical result that left orderable groups act faithfully on the real line by orientation preserving homeomorphisms and the action can be chosen so that there exists $x \in \mathbb{R}$ with $\mathrm{stab}(x) = \{e\}$ (see [Gh2], Theorem 6.8 and its proof; the points $v(\gamma_i) \in \mathbb{R}$ in the proof have trivial stabilizer). Now let M be a compact hyperbolic n-manifold whose fundamental group embeds into a right-angled Artin group. (By work of Agol, Haglund-Wise, et. al. 3-dimensional examples of such manifolds exist (see [Asch-Fr-Wi], p. 51). Since right-angled Artin groups are left-

[1] *Is $\mathbb{H}^n$ quasi-isometric to a leaf of a codimension 1 foliation of a compact manifold?* http://mathoverflow.net/questions/147026/is-mathbbhn-quasi-isometric-to-a-leaf-of-a-codimension-1-foliation-of-a-co/147082#147082

(and in fact bi-) orderable (see [Du-Kr]), $\pi_1(M)$ acts on the real line by orientation preserving homeomorphisms such that there exist elements with trivial stabilizers. This action extends to $S^1 = \mathbb{R} \cup \{\infty\}$ and we get a foliated circle bundle over M such that one leaf is topologically a copy of $\mathbb{H}^n = \overline{M}$. Note that since the action of $\pi_1(M)$ on S^1 is just by homeomorphisms, the foliation is of class $C^{\infty,0}$. We can now endow every leaf with the pullback metric coming from the hyperbolic metric on the base M. This metric is smooth on every leaf but just continuous in the transverse direction. In particular, with this metric the leaf which is a copy of $\mathbb{H}^n$ is in fact isometric to hyperbolic n-space.

It seems to be unclear, whether there exist smooth foliations of codimension 1 on compact manifolds which have a leaf quasi-isometric to hyperbolic n-space. But since every right-angled Artin group embeds into $\mathrm{Diff}(S^2)$ by a result of Kapovich [Kap], $\mathbb{H}^2, \mathbb{H}^3$ and each $\mathbb{H}^n$ such that there exists a compact hyperbolic n-manifold whose fundamental group embeds into a right-angled Artin group is isometric to a leaf in a smooth codimension 2 foliation of a compact manifold for every $n \geq 2$. □

Using different techniques, we will generalize these results in Chapter 4 to show that every open Riemannian manifold can be deformed to a manifold which is not diffeomorphically quasi-isometric to a leaf, while the coarse homology is unaffected by this deformation. Applying this deformation to one-ended cylinders, we find non-leaves with trivial coarse homology.

The construction of Zeghib on the other hand yields manifolds which are not even coarsely quasi-isometric to simply connected leaves in compact manifolds, but that also have non-finitely generated coarse homology.

Although not stated explicitly, Zeghib uses a coarser type of (ε, R)-quasi-tiling to show that the manifolds which he constructs cannot even be coarsely quasi-isometric to a leaf.

Definition 2.3.14 (coarse (ε, R)-quasi-tilings)**.** Let (X, d) be a metric space. A *coarse* (ε, R)*-quasi-tiling* of X is a finite collection of compact metric spaces $K_1, \dots K_s$ of diameter at most $R > 0$ and a countable collection of $(1+\varepsilon, \varepsilon)$-quasi-isometric embeddings $f_\alpha : K_{i_\alpha} \to X$ such that for any set $K \subset X$ with diameter at most $R/4$ some f_α is a $(1+\varepsilon, \varepsilon, \varepsilon)$-quasi-isometry between K_{i_α} and a neighbourhood of K.

If we compare the above definition with the Definition of an (ε, R)-quasi-tiling (see Definition 2.3.7), the main difference is that the requirement on the maps f_α to be topological embeddings has been relaxed to f_α to be a quasi-isometry. The condition that any subset of diameter bounded by $R/4$ has a neighbourhood K which is quasi-isometric to one of the $K_1, \dots, K_s$ is indeed the condition analogous to that in Definition 2.3.7 that any subset of diameter bounded by $R/4$ lies in the image of some f_α: Since the maps in coarse quasi-tilings are no longer homeomorphisms, their images might just be discrete sets. This necessitates the passing to a neighbourhood.

We define the *coarse ε-growth complexity function* $H^{cs}(X, \varepsilon, R)$ and the *coarse geometric entropy of a metric space* $h_g^{cs}(X)$ analogously to the homeomorphic case. Since every (ε, R)-quasi-tiling is in particular a coarse (ε, R)-quasi-tiling, is clear that $H^{cs}(X, \varepsilon, R) \leq H(X, \varepsilon, R)$ and hence $h_g^{cs}(X) \leq h_g(X)$. Moreover, we have a coarse quasi-isometry invariance for $h_g^{cs}(X)$ analogous to that of $h_g(X)$.

Lemma 2.3.15. *The coarse geometric entropy of a metric space X depends up to multiplication with a positive constant only on the coarse quasi-isometry class of the space. In particular the statements $h_g^{cs}(X) = 0, 0 < h_g^{cs}(X) < \infty$ and $h_g^{cs}(X) = \infty$ depend only on the coarse quasi-isometry class of X.*

Proof. Analogously to the proof of Lemma 2.3.9, we let $f\colon X \longrightarrow X'$ be a (λ, D, C)-quasi-isometry and $\{K_1, \ldots, K_d, f_\alpha\}$ be a coarse (ε, R)-quasi-tiling. Then again, $f \circ f_\alpha\colon K_{i_\alpha} \longrightarrow X'$ is a $(\lambda(1 + \varepsilon), \lambda\varepsilon + D)$-quasi-isometric embedding. It remains to show that every subset $K' \subset X'$ of diameter at most $\frac{1}{4}\left(\frac{R}{\lambda} - D\right)$ has a neighbourhood $U(K')$ such that $f_\alpha\colon K_{i_\alpha} \longrightarrow U(K')$ is a $(\lambda(1+\varepsilon), \lambda\varepsilon + D, C')$-quasi-isometry, with C' depending only on $\lambda, D, C, \varepsilon$ and R. First note that $\operatorname{diam}(f^{-1}(K')) < R/4$, since for $f(x), f(x') \in K'$ we find that
$$\frac{1}{\lambda} d_X(x, x') - D \leq d_{X'}(f(x), f(x')) < \frac{R}{4\lambda} - D$$
and thus $d_X(x, x') \leq R/4$ for every $x, x' \in f^{-1}(K')$. Since $\{K_1, \ldots, K_d, f_\alpha\}$ is a coarse (ε, R)-quasi-tiling, there exists a neighbourhood $U(K)$ of K and some α and i_α such that $f_\alpha\colon K_{i_\alpha} \longrightarrow U(K)$ is a $(1 + \varepsilon, \varepsilon, \varepsilon)$-quasi-isometry. Set $U(K') := B_C(f(U(K)))$. Since the image of f is C-dense in X', we find that $K' \subset U(K')$. Moreover, if $x' \in U(K')$, then there exists some $f(x) \in \operatorname{im} f|_{U(K)}$ with $d_{X'}(x', f(x)) \leq C$ and a $f_\alpha(k)$ with $d_X(x, f_\alpha(k)) \leq \varepsilon$. Hence
$$\begin{aligned} d_{X'}(x', (f \circ f_\alpha(k)) &\leq d_{X'}(x', f(x)) + d_{X'}(f(x), (f \circ f_\alpha)(k)) \\ &\leq C + \lambda d_X(x, f_\alpha(k)) \\ &\leq C + \lambda\varepsilon + D. \end{aligned}$$

Hence $f \circ f_\alpha\colon K_{i_\alpha} \longrightarrow U(K')$ is a $(\lambda(1+\varepsilon), \lambda\varepsilon + D, \lambda\varepsilon + C + D)$-quasi-isometry.

By rescaling the metric on the K_i as in the proof of Lemma 2.3.9, we find that $\{K_1, \ldots, K_d, f_\alpha\}$ is a coarse (ε', R')-quasi-tiling of X' with ε' tending to infinity if ε does and also $R' = R/\lambda - 2D$. Hence we get the same estimate $h_g^{cs}(X) \leq 1/\lambda h_g^{cs}(X')$ as in Lemma 2.3.9. □

Since the non-leaves constructed by Attie-Hurder are quasi-isometric to $\mathbb{H}^6$ by Remark 2.3.13, their coarse geometric entropy vanishes and the inequality in $h_g^{cs}(X) \leq h_g(X)$ is strict. The coarse geometric entropy is thus indeed a coarser invariant than the geometric entropy.

If a Riemannian manifold N is coarsely quasi-isometric to a simply connected leaf L, then by Lemma 2.3.9 and Lemma 2.3.15 there exists a constant $c > 0$ such that

$$h_g^{cs}(N) \leq c \cdot h_g^{cs}(L) \leq c \cdot h_g(L) < \infty,$$

where finiteness of $h_g(L)$ again follows from Proposition 2.3.12. In sum the following holds:

Proposition 2.3.16. *If a metric space (X, d) is coarsely quasi-isometric to a simply connected leaf of a C^1-foliation of arbitrary codimension or a codimension 1 foliation of class $C^{1,0}$ of a compact manifold, then the coarse geometric entropy $h_g^{cs}(X, d)$ is finite.*

Zeghib's construction of coarse non-leaves is similar in spirit to that of Attie and Hurder. The rôle of the N_l is played by 2-spheres $S(\rho)$ of radius $\rho = r, r^2, r^3$ with a disk of radius 1 removed around the south pole, where r runs through the natural numbers.

Consider a ball $B(y; r^4) \subset \mathbb{H}^2$ of radius r^4 and choose $y_i, i = 1, \ldots, d(r^4, r^3)$ such that the balls $B(y_i; r^3) \subset B(y, r^4)$ are pairwise disjoint. For each $1 \leq k < d(r^4, r^3)$ we define a building block $W^+(y, r^4, k)$ by replacing $B(x_i, 1)$ with $S(r)$ for $1 \leq i \leq k$ and $B(x_i; 1)$ with $S(r^2)$ for $k < i < d$. Moreover, we define $W^-(y, r^4, k)$, which is $W^+(y, r^4, k)$ with a copy of $S(r^3)$ replacing $B(x_d; 1)$ (cf. Figure 4.5 for a similar construction). Note that every $B(z; r^6) \subset \mathbb{H}^2$ contains $d(r^6, r^5)$ disjoint balls of radius r^5 with centers z_i, say. Again let $\mathbf{j}\colon \{1, \ldots, d(r^5, r^4) - 1\} \longrightarrow \{\pm\}$. Then $N(z, r^6, \mathbf{j})$ is the ball $B(z; r^6)$ with $W^{\mathbf{j}(i)}(z_i, r^4, i)$ replacing $B(z_i; r^4)$. Zeghib then shows that for fixed $\lambda > 1, D, C > 0$ and sufficiently large $r > 0$, two such building blocks $N(z, r^6, \mathbf{j})$ and $N(z', r^6, \mathbf{j'})$ are coarsely (λ, D, C)-quasi- isometric if and only if $\mathbf{j} = \mathbf{j'}$. Hence for fixed (λ, D, C) and large r there are $2^{d(r^6, r^5)-1}$ coarse quasi-isometry types of $N(z, r^6, \mathbf{j})$.

Again, the non-leaf (L, g) is constructed by replacing balls $B(z, r^6)$ by the $N(y, r^6, \mathbf{j})$, where r ranges over all natural numbers and $\mathbf{j}$ over all possible functions $\{1, \ldots, d(r^6, r^5)\} \longrightarrow \{\pm\}$. If we place the $N(z_i, r^6, \mathbf{j})$ increasingly far apart, a coarse (ε, R)-quasi-tiling of (L, g) with R sufficiently large will have to contain at least $2^{d(R^6, R^5)-1}$ distinct metric spaces to cover all $N(y, R^6, \mathbf{j})$, the coarse ε-growth complexity function then is exponential and the coarse geometric entropy of (L, g) is infinite (cf. 4.3.2 for a more detailed argument in a similar situation). Hence by Proposition 2.3.16 it cannot be coarsely quasi-isometric to a simply connected leaf in a compact manifold.

Since (L, g) results from gluing in balloons of increasing radius, similar arguments as in Proposition 4.1.1 show that the coarse homology of (L, g) is not finitely generated in top degree. In Section 4.3 we will adapt the above construction to yield non-leaves with the same coarse homology as hyperbolic space, which is finitely generated.

2.3.2 Schweitzer: Bounded homotopy and homology property

In [Schw2] Schweitzer proves that every Riemannian manifold of dimension at least 3 that is diffeomorphically quasi-isometric to a leaf of a codimension 1 foliation of a compact manifold has to satisfy the bounded homology property. Moreover, he proves in [Schw1] that every 2-manifold that is bi-Lipschitz equivalent to a leaf of a foliation of a 3-manifold has to satisfy the bounded homotopy property. By deforming any given Riemannian metric on a non-compact manifold so that the resulting Riemannian manifold violates the above criteria, Schweitzer shows that every non-compact manifold carries a metric such that it is not diffeomorphically quasi-isometric to a leaf in a codimension 1 foliation of a compact manifold.

Dimension 2: Bounded homotopy property

Definition 2.3.17 (bounded homotopy property)**.** A Riemannian manifold (L, g) has the *bounded homotopy property* if for each $k > 0$ there exists a $K > 0$ such that for every nullhomotopic loop in L of length less than k there exists a nullhomotopy via loops of length less than K.

Theorem 2.3.18 (Theorem 2, [Schw1])**.** *Every leaf of a codimension one C^1-foliation $\mathscr{F}$ of a compact 3-manifold satisfies the bounded homotopy property.*

One can show that the bounded homotopy property is invariant under bi-Lipschitz maps. Hence every surface which is bi-Lipschitz equivalent to a leaf in a compact 3-manifold must satisfy the bounded homotopy property. It is not hard to see that this is violated if we insert balloons of increasing diameter into a given surface. This construction will be discussed in some more detail at the end of the following section.

Higher dimension: Bounded homology property

The definition of the bounded homology property requires some preparations. The underlying idea is to generalize the bounded homotopy property – which is a statement about objects of codimension 1, namely nullhomotopic curves in surfaces – by considering nullhomologous hypersurfaces in manifolds of dimension at least 3. The requirement of loops in nullhomotopies to be uniformly small with respect to the original loop is translated into the condition that the manifolds bounding a nullhomologous hypersurface be uniformly small with respect to the size of the hypersurface. It is clear, though, that the notion of smallness needs to be specifically chosen since hypersurfaces of bounded diameter in the non-compact leaves of the Reeb foliation bound arbitrarily large subsets.

Definition 2.3.19 ($\mathrm{vol}_\beta(S)$, Definition 2.2 [Schw2])**.** Let S be a subset of a metric space and $\beta > 0$. The *β-volume* $\mathrm{vol}_\beta(S)$ of S is defined as the minimal number of balls of radius β needed to cover S. We have $\mathrm{vol}_\beta(S) \in \mathbb{N} \cup \{\infty\}$.

Definition 2.3.20 (Morse-β-Volume, Definition 2.3 [Schw2])**.** Let (C, g) be a compact Riemannian manifold with boundary and $f : C \to [0, \infty)$ a Morse function satisfying $f|_{\partial C} \equiv 0$. For $\beta > 0$, the *Morse-β-volume of C with respect to f* is defined to be the smallest natural number $\mathrm{MVol}_\beta(C, f)$ such that the β-volume of every level set of f is bounded by $\mathrm{MVol}_\beta(C, f)$, that is $\mathrm{vol}_\beta(f^{-1}(t)) \le \mathrm{MVol}_\beta(C, f)$ for all $t \ge 0$.

The Morse-β-volume of C is then defined to be the minimum of $\mathrm{MVol}_\beta(C, f)$ taken over all Morse functions vanishing on ∂C. In formulae, the Morse-β-volume is defined as

$$\mathrm{MVol}_\beta(C) = \min_{\substack{f|_{\partial C} \equiv 0 \\ f \ge 0 \text{ Morse}}} \max_{t \ge 0} \mathrm{vol}_\beta\left(f^{-1}(t)\right).$$

The Morse-β-volume and the β-volume are related by

$$\mathrm{vol}_\beta(\partial C) \le \mathrm{MVol}_\beta(C).$$

The inequality holds since we only consider Morse functions which vanish on ∂C. Moreover, the Morse-β-volume has the following monotonicity property: If $C' \subset C$, and $\partial C' \subset \partial C$, then $\mathrm{MVol}_\beta(C') \le \mathrm{MVol}_\beta(C)$. For the restriction to C' of any Morse function on C which vanishes on ∂C is a Morse function that vanishes on $\partial C'$. Hence

$$\begin{aligned}
\mathrm{MVol}_\beta(C) &= \min_{\substack{f|_{\partial C} \equiv 0 \\ f \ge 0 \text{ Morse}}} \max_{t \ge 0} \mathrm{vol}_\beta\left(f^{-1}(t)\right) \\
&\ge \min_{\substack{f|_{\partial C} \equiv 0 \\ f \ge 0 \text{ Morse}}} \max_{t \ge 0} \mathrm{vol}_\beta\left(f^{-1}(t) \cap C'\right) \\
&\ge \min_{\substack{\bar f|_{\partial C'} \equiv 0 \\ \bar f \ge 0 \text{ Morse}}} \max_{t \ge 0} \mathrm{vol}_\beta\left(\bar f^{-1}(t)\right) \\
&= \mathrm{MVol}_\beta(C').
\end{aligned}$$

However, if $\partial C'$ is not a subset of ∂C, then the second inequality does not hold anymore. In fact, for β sufficiently small, we have that $\mathrm{MVol}_\beta([-1,1]) = 2$, while $\mathrm{MVol}_\beta([-1, 1/2] \cup [1/2, 1]) = 4$.

The bounded homology property generalizes the bounded homotopy property in the following way: Nullhomotopic loops are replaced by nullhomologous $(n-1)$-chains and the requirement of being nullhomotopic via small loops is replaced by the requirement of being nullhomologous via small n-chains, hence the name *bounded homology property.* In terms of the above definitions, this is the requirement of subsets having uniformly small Morse-β-volume provided that the β-volume of their boundary is bounded by a constant. We define more precisely:

Definition 2.3.21 (bounded homology property)**.** A Riemannian manifold M has the *bounded homology property* if for all $k > 0$ and all sufficiently large $\beta > 0$, there exists a constant $K(\beta, k)$ such that the Morse-β-volume $\mathrm{MVol}_\beta(C)$ of all compact codimension 0 submanifolds C with smooth boundary is bounded by $K(\beta, k)$, provided they satisfy the following conditions:

i) $\mathrm{vol}_\beta(\partial C) \leq k$,

ii) C and ∂C are connected and simply connected,

iii) ∂C has a tubular neighbourhood V that contains

$$B_\beta(\partial C) = \{x \in M \mid \mathrm{dist}(x, \partial C) < \beta\}.$$

The bounded homology property can be viewed as a type of isoperimetric inequality involving the β-volume of the boundary and the Morse-β-volume of the interior. This, however, is in general not related to Cheeger's isoperimetric constant, which involves the Riemannian volumes of the boundary and the interior, as will be discussed in Section 6.4.

It is straightforward to show that the bounded homology property is invariant under quasi-isometric diffeomorphisms (see [Schw2]).

The main theorem of [Schw2] is that every manifold which is diffeomorphically quasi-isometric to a leaf satisfies the bounded homology property.

Theorem 2.3.22 ([Schw2])**.** *Every n-manifold, $n \geq 3$, that is diffeomorphically quasi-isometric to a leaf of a codimension one $C^{2,0}$-foliation of a compact manifold satisfies the bounded homology property.*

Every open Riemannian manifold (L, g) can be deformed by inserting balloons with radii tending to infinity. The Riemannian metric induced from the

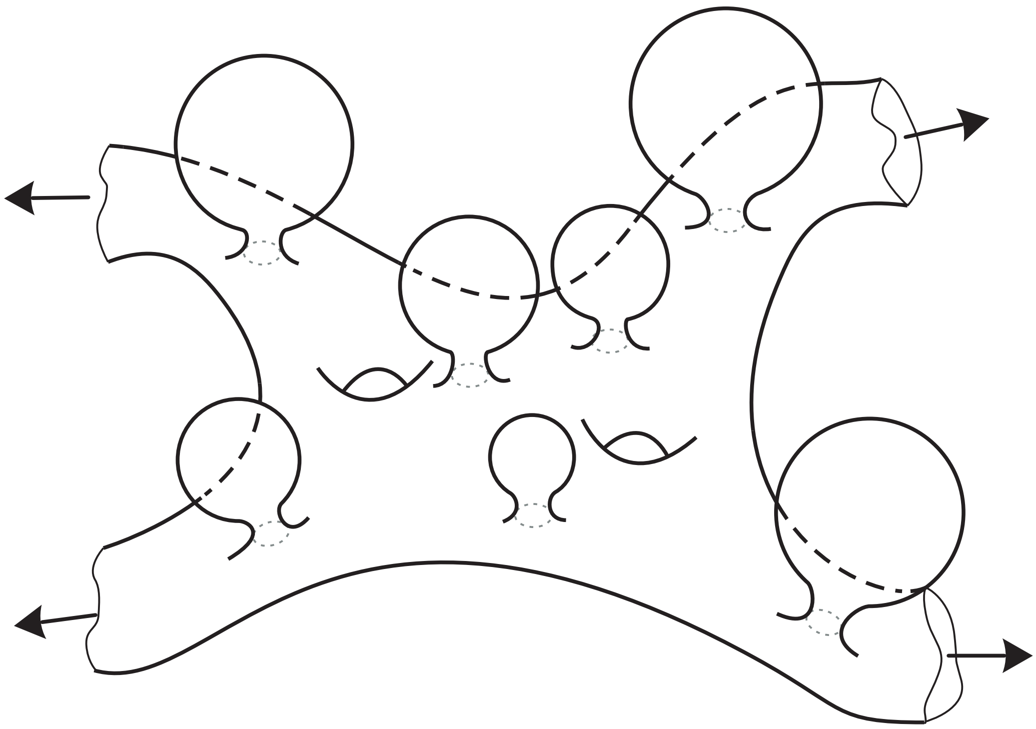

Figure 2.1: L with Schweitzer's non-leaf metric.[2]

balloons then does not satisfy the bounded homology property (respectively the bounded homotopy property in case L is a surface). More precisely, let (L, g) be a Riemannian manifold of bounded geometry. In particular, the injectivity radius of (L, g) is bounded from below by some $d > 0$ and hence all metric balls of radius d are also topologically balls. Now pick a sequence of points $x_i \in L$ with $d(x_0, x_i) = d_i$, where $d_i + 2d < d_{i+1}$ and replace $B_d(x_i) \subset L$ by an n-sphere $S^n(r_i) \setminus B_{d/2}(x_S)$ of radius $r_i \to \infty$ with a $d/2$-ball removed about the south pole. The resulting manifold is homeomorphic to L. On the inserted spheres, we equip L with the round metric from the sphere, which is smoothed around the gluing to yield a smooth Riemannian metric g_S.

The resulting Riemannian manifold (L, g_S) does not satisfy the bounded homology property, since $C_i := S^n(r_i) \setminus B_{d/2}(x_S), i \geq 1$, is a sequence of compact codimension 0 submanifolds with unbounded Morse-β-volume (see Section 4, [Schw2]), but $\mathrm{vol}_\beta(\partial C_i) = \mathrm{vol}_\beta(S^{n-1}(d))$ is constant.

In Chapter 4, we will show that L with the deformed metric has non-finitely generated coarse homology in top degree (see Proposition 4.1.1). We will then give a different construction of non-leaves which leaves the coarse homology of (L, g) unchanged (see Theorem 4.2.1).

[2] The picture is taken from Schweitzer's article [Schw2].

For the same reasons as for Schweitzer's construction, the non-leaves constructed by Zeghib in the end of Section 2.3.1 do neither satisfy the bounded homology property and hence Theorem 2.3.22 also implies that they cannot be diffeomorphically quasi-isometric to a leaf in a compact manifold. In general, though, neither criterion can be reduced to the other. This, together with various other independence results of obstructions for manifolds to be leaves will be proven in Chapter 6.

Chapter 3

Coarse and locally finite homology

Coarse geometry is the study of large-scale and asymptotic properties of metric spaces. As singular homology is a tool for topology, coarse homology is used to tackle coarse-geometric questions. The first definition of coarse homology was given by J. Roe in [Roe1] by combinatorial means. The approach via a coarsening sequence of a space, that is by successively deleting all finite scale topology, can be found in [Hig-Roe] and [Roe2]. There, coarse homology is defined to be the direct limit of a generalized homology theory applied to the coarsening sequence. We can thus view the coarse homology associated to the generalized homology theory as a coarsened version of it. Throughout this thesis, we understand coarse homology to be a coarsened version of locally finite homology, which is an adaptation of singular homology more suited to the study of non-compact spaces. Coarsening sequences can be quite cumbersome to handle thus making coarse homology hard to compute in general. It is important to note that coarse homology heavily depends on the metric a given topological space carries and thus is invariant under quasi-isometries but not under homeomorphisms.

This chapter provides some definitions and tools from coarse geometry that will be needed for the computations in the following sections. In particular, we give a somewhat more detailed exposition of locally finite homology. The first section introduces locally finite homology and presents generalizations of some computational tools from singular homology which we haven't found elsewhere in the literature. In Section 3.2, we define coarse homology as a coarsened version of locally finite homology. In particular, we define what a coarsening sequence of a metric space is and recall some results about coarse homology. Section 3.3 is concerned with cones over subspaces of S^n and we will show that they provide an easy way to produce spaces with prescribed coarse homology. These will be used in Chapter 6 to prove independence of coarse homology and certain foliation invariants.

Whenever possible, we will omit to mention the coefficient ring of the homology theories. All the results we prove hold for arbitrary coefficients, though.

3.1 Locally finite homology and locally finite Δ-homology

Locally finite homology, also known as Borel-Moore homology, can be seen as an adaptation of singular homology to non-compact spaces. In particular, every oriented non-compact manifold has a locally finite fundamental class, while for compact spaces it coincides with singular homology.

Among many equivalent definitions, we choose the one based on locally finite chains as it can be found in [Hu-Ra]. Through this approach, many properties of singular homology have natural generalizations to locally finite homology. In particular, we present a locally finite version of Δ-homology and prove that it is isomorphic to locally finite homology.

Definition 3.1.1 (locally finite homology)**.** The *locally finite homology* $H_*^{lf}(X)$ of a locally compact topological space X is the homology of the chain complex $(C_n^{lf}(X), \partial_n)_n$, where n-chains are formal, possibly infinite sums of n-simplices $\sum_\sigma a_\sigma \sigma$ such that each $x \in X$ has a neighbourhood that intersects only finitely many $\mathrm{im}(\sigma)$. The boundary map is the one induced by the boundary map on singular chains.

Equivalently, a formal sum of n-simplices $\sum r_\sigma \sigma$ is locally finite if each compact set $K \subset X$ intersects only finitely many $\mathrm{im}(\sigma)$. From the latter definition it is obvious that singular and locally finite homology agree for compact spaces.

The inclusion of finite chains $C_*^{sing}(X) \hookrightarrow C_*^{lf}(X)$ induces a homomorphism $i\colon H_*^{sing}(X) \longrightarrow H_*^{lf}(X)$. Note that this map need neither be injective nor surjective: Take for example X to be $S^1 \times \mathbb{R}$. Then $S^1 \times \{0\}$ defines a non-trivial singular cycle in $H_1(S^1 \times \mathbb{R})$, but $S^1 \times \mathbb{R}_{\geq 0}$ is a locally finite null-homology of it, that is $[S^1 \times \{0\}] = 0 \in H_1^{lf}(S^1 \times \mathbb{R})$. On the other hand, we can use Proposition 3.1.4 or the results of Chapter 5 to see that $[\{1\} \times \mathbb{R}]$ is a non-trivial locally finite cycle, which does not lie in the image of i, since it is not homologous to a finite chain by a locally finite 2-chain. This discrepancy between locally finite and singular homology is measured by the *singular homology at* ∞ or the *homology of the end* (see [Hu-Ra] and [Geo]).

Remark 3.1.2. Note that locally finite homology is functorial only under *proper* maps. Let $\alpha = \sum a_i \sigma_i$ be a locally finite chain on X, $f\colon X \to Y$

a proper map and $K \subset Y$ compact. If $\operatorname{im}(f \circ \sigma_i)$ intersects K, then $\operatorname{im}\sigma_i$ intersects $f^{-1}(K)$. But since $f^{-1}(K)$ is compact, only finitely many $f \circ \sigma_i$ intersect K and consequently, $f_*\alpha = \sum a_i(f \circ \sigma_i)$ is a locally finite chain on Y.

But if we let $X \subset Y$ be a subset with compact closure in Y, then the inclusion does in general not induce a map $C_*^{lf}(X) \to C_*^{lf}(Y)$. Take for example the interval $[0,1) \subset \mathbb{R}$. Then $[0,\frac{1}{2}]+[\frac{1}{2},\frac{3}{4}]+[\frac{3}{4},\frac{7}{8}]+\ldots$ is a locally finite chain on $[0,1)$ but not on $\mathbb{R}$ because any neighbourhood of 1 intersects infinitely many simplices.

If $A \subset X$ is closed (and hence the inclusion $A \hookrightarrow X$ proper), there exists a *long exact sequence of the pair* (X,A)

$$\ldots \longrightarrow H_n^{lf}(A) \longrightarrow H_n^{lf}(X) \longrightarrow H_n^{lf}(X,A) \longrightarrow H_{n-1}^{lf}(X) \longrightarrow \ldots$$

Moreover, the proof of the excision theorem for singular homology carries over verbatim to locally finite homology. This enables us to generalize the Mayer-Vietoris theorem to locally finite homology.

Corollary 3.1.3 (Mayer-Vietoris for families)**.** *Let X be a topological space and $\{A_i\}_{i\in\mathbb{N}}$ a locally finite covering of X by closed sets, $X = \cup_i \operatorname{int}(A_i)$ and $A_i \cap A_j \cap A_k = \emptyset$ for distinct i,j and k. Then there exists a long exact sequence*

$$\ldots \to \prod_{i<j} H_n^{lf}(A_i \cap A_j) \longrightarrow \prod_k H_n^{lf}(A_k) \longrightarrow H_n^{lf}(X) \xrightarrow{\partial} \prod_{i<j} H_{n-1}^{lf}(A_i \cap A_j) \to \ldots$$

Proof. Denote by $C_n^{lf}(X,\{A_i\})$ those n-chains on X whose simplices map to one of the A_i and let (c_i) or (c_{ij}) be an element of the product $\prod_k C_n^{lf}(A_k)$ or $\prod_{i<j} C_n^{lf}(A_i \cap A_j)$, respectively, while c_i and c_{ij} denote elements of the respective factors. Consider the following sequence of chain complexes

$$0 \longrightarrow \prod_{i<j} C_n^{lf}(A_i \cap A_j) \xrightarrow{\varphi} \prod_k C_n^{lf}(A_k) \xrightarrow{\psi} C_n^{lf}(X,\{A_i\}) \longrightarrow 0.$$

The map φ is induced by the inclusion

$$C_n^{lf}(A_i \cap A_j) \ni x \longmapsto (x,-x) \in C_n^{lf}(A_i) \oplus C_n^{lf}(A_j),$$

that is, an element (c_{ij}) in $\prod_{i<j} C_n^{lf}(A_i \cap A_j)$ is mapped to

$$(\ldots, \underbrace{\sum_{j_0>0} c_{0,j_0} - \sum_{i_0<0} c_{i_0,0}}_{\in C_n^{lf}(A_0)}, \ldots \underbrace{\sum_{j_k>k} c_{k,j_k} - \sum_{i_k<k} c_{i_k,k}}_{\in C_n^{lf}(A_k)}, \ldots)$$

and ψ is formal summation. Then $\psi(\ldots, c_{-1}, c_0, c_1, c_2, \ldots) := \sum_i c_i$ is indeed a locally finite chain since the covering $\{A_i\}$ and the chains c_i were locally finite and the inclusions proper.

By the excision theorem, the homology of $C_*^{lf}(X, \{A_i\})$ is isomorphic to $H_*^{lf}(X)$. The claim will thus follow from homological algebra, once we have shown that the above sequence is short exact.

Clearly, ψ is surjective and $\mathrm{im}(\varphi) \subset \ker(\psi)$. To see that φ is injective, note that for $\varphi((c_{ij}))$ to be zero, we need each sum $\sum_{j_k > k} c_{k,j_k} - \sum_{i_k < k} c_{i_k,k}$ to be zero. But for every summand $c_{k,j}$ we then have

$$c_{k,j} = -\sum_{\substack{j_k > k \\ j_k \neq j}} c_{k,j_k} + \sum_{i_k < k} c_{i_k,k},$$

implying that $c_{k,j}$ lies both in $A_k \cap A_j$ and $A_k \cap \left(\bigcup_{\substack{l \in \mathbb{Z} \\ l \neq j}} A_l \right)$. But since three distinct A_j intersect trivially, this can only be if $c_{k,j} = 0$. This proves that all $c_{k,j}$ are zero.

It remains to show that $\ker \psi \subset \mathrm{im}\, \varphi$. Let $\psi\left((c_i)\right) = \sum_i c_i = 0$. Then we can write each c_i as a sum of chains in $A_i \cap A_j$, i.e. $c_i = \sum_{j \neq i} c_{ij}$. For if there exists some chain σ in c_i such that $\sigma \not\subset A_i \cap A_j$ for all j, then σ will survive as a summand in $\psi((c_i))$. Now consider the c_{ij}. We have $c_{ji} = -c_{ij}$ since c_{ij} and c_{ji} are the only chains in $A_i \cap A_j$. Hence

$$(\ldots, c_{-1}, c_0, c_1, c_2, \ldots) = (\sum_{\substack{j_1 \neq 1 \\ j_1 \geq 1}} c_{1,j}, \sum_{\substack{j_2 \neq 2 \\ j_2 \geq 1}} c_{2,j_2}, \ldots) = \varphi(\sum_{j_1 > 1} c_{1,j_1}, \sum_{j_2 > 2} c_{2,j_2}, \ldots)$$

Hence $\ker \psi \subset \mathrm{im}\, \varphi$. $\square$

Recall the definition of a Δ-complex (see [Hat], p. 103), which is a a topological space X together with a family of maps $\sigma_\alpha \colon \Delta_\alpha^n \longrightarrow X$, where n depends on the index α such that the following holds:

- the restriction $\sigma_\alpha|_{\mathrm{int}(\Delta_\alpha^n)}$ is injective and each point $x \in X$ lies in the image of exactly one such restriction;
- the restriction of σ_α to each of the faces Δ^{n-1} of Δ^n is the characteristic map $\sigma_\beta \colon \Delta^{n-1} \longrightarrow X$ of some $(n-1)$-simplex;
- $A \subset X$ is open if and only if $\sigma_\alpha^{-1}(A) \subset \Delta^n$ is open for all α.

Δ-complexes are a combinatorially slightly less restrictive variant of simplicial complexes. Note though, that every Δ-complex can be subdivided to form a simplicial complex. Moreover, in a locally compact Δ-complex every vertex meets only finitely many simplices, i.e. the Δ-complex is locally finite.

In complete analogy to the definition of singular Δ-homology in Hatcher's book, we define the *locally finite Δ-homology* with R-coefficients of a locally compact Δ-complex X to be the homology of the chain complex with n-chains $C_n^{lf,\Delta}(X;R)$ being the product $\prod_\alpha R \cdot \sigma_\alpha^n$ over all characteristic maps $\sigma_\alpha^n \colon \Delta_\alpha^k \longrightarrow X$ of n-simplices in X. A locally finite Δ-chain of dimension n may then be viewed as an infinite formal sum $\sum_\alpha r_\alpha \sigma_\alpha^n$. The boundary from singular homology induces a well-defined map on locally finite Δ-chains since X was locally finite. We denote the locally finite Δ-homology of X by $H_n^{lf,\Delta}(X;R)$.

The term locally finite Δ-chain is justified by the following observation. Let x be a point in X and K a compact neighbourhood of x. Then K is contained in a finite subcomplex and thus K is a neighbourhood of x that intersects only finitely many simplices of any element in $C_n^{lf,\Delta}(X;R)$. Hence $C_*^{lf,\Delta}(X)$ may be viewed as a subcomplex of the locally finite chains $C_*^{lf}(X)$ and we denote the inclusion map by i_Δ.

We can now adapt the proof of the equality of singular homology and Δ-homology to prove the following

Proposition 3.1.4. *Let X be a finite dimensional Δ-complex. Then the inclusion i_Δ induces an isomorphism $H_*^{lf,\Delta}(X) \simeq H_*^{lf}(X)$ on homology.*

Proof. Since the 0-skeleton of a Δ-complex is totally disconnected, the Δ-inclusion induces an isomorphism between $H_0^{lf,\Delta}(X^0) \simeq \prod R \cdot \sigma_\alpha^0$ and $H_0^{lf}(X^0)$. The higher degree homology groups $H_k^{lf}(X^0)$ and $H_k^{lf,\Delta}(X^0)$ are all zero. Suppose now that the claim holds for the $(k-1)$-skeleton and consider the respective long exact sequences of the pair (X^k, X^{k-1}). Since the boundary map and the inclusion of X^{k-1} into X^k commute with the Δ-inclusion, we have the following commutative diagram

$$\begin{tikzcd}
H_{n+1}^{lf,\Delta}(X^k,X^{k-1}) \arrow[r, "\partial"] \arrow[d] & H_n^{lf,\Delta}(X^{k-1}) \arrow[r] \arrow[d, "\simeq"] & H_n^{lf,\Delta}(X^k) \arrow[r] \arrow[d] & H_n^{lf,\Delta}(X^k,X^{k-1}) \arrow[r, "\partial"] \arrow[d] & H_{n-1}^{lf,\Delta}(X^{k-1}) \arrow[d, "\simeq"] \\
H_{n+1}^{lf}(X^k,X^{k-1}) \arrow[r, "\partial"] & H_n^{lf}(X^{k-1}) \arrow[r] & H_n^{lf}(X^k) \arrow[r] & H_n^{lf}(X^k,X^{k-1}) \arrow[r, "\partial"] & H_{n-1}^{lf}(X^{k-1}).
\end{tikzcd}$$

The second and the fifth downward arrows are isomorphisms by the induction hypothesis. It remains to show that the Δ-inclusion induces isomorphisms on the relative homology groups of the pair (X^k, X^{k-1}) since the induction step will then follow from the five lemma.

To prove the claim, we will show that the Δ-inclusion fits into the following commutative diagram of isomorphisms:

$$\begin{tikzcd}
H_n^{lf,\Delta}(X^k, X^{k-1}) \arrow[r] & H_n^{lf}(X^k, X^{k-1}) \\
H_n^{lf,\Delta}(\coprod_{l\le k}\Delta_\alpha^l, \coprod_{l\le k}\partial\Delta_\alpha^l) \arrow[u, "\simeq"] \arrow[r, "\simeq"] & H_n^{lf}(\coprod_{l\le k}\Delta_\alpha^l, \coprod_{l\le k}\partial\Delta_\alpha^l), \arrow[u, "\simeq"']
\end{tikzcd}$$

where $\coprod_{l\le k}\Delta_\alpha^l$ denotes the set of simplices corresponding to the k-skeleton of X. The horizontal maps are the Δ-inclusions for the respective spaces and the vertical maps are induced by the characteristic maps of the simplices. In particular, the diagram is indeed commutative.

First observe that $C_n^{lf,\Delta}(X^k, X^{k-1}) := C_n^{lf,\Delta}(X^k)/C_n^{lf,\Delta}(X^{k-1})$ is non-trivial if and only if $n = k$ and hence

$$H_n^{lf,\Delta}(X^k, X^{k-1}) = \begin{cases} C_k^{lf,\Delta}(X^k, X^{k-1}) = \prod_\alpha R\cdot\sigma_\alpha^k & \text{if } n = k, \\ 0 & \text{if } n \neq k. \end{cases}$$

And the same holds for the locally finite Δ-homology of $\left(\coprod_{l\le k}\Delta_\alpha^l, \coprod_{l\le k}\partial\Delta_\alpha^l\right)$. It follows that the natural map $H_n^{lf,\Delta}(\coprod\Delta_\alpha^k, \coprod\partial\Delta_\alpha^k) \longrightarrow H_n^{lf,\Delta}(X^k, X^{k-1})$ taking $\mathrm{id}\colon \Delta_\alpha^k \longrightarrow \Delta_\alpha^k$ to σ_α^k is an isomorphism.

Moreover, the Δ-inclusion is an isomorphism between the two homology theories of the pair $(\coprod\Delta_\alpha^k, \coprod\partial\Delta_\alpha^k)$ because the identity of Δ_α^k viewed as an element of $C_k^{lf}(\coprod\Delta_\alpha^k, \coprod\partial\Delta_\alpha^k)$ is a cycle which generates the corresponding factor in $H_k^{lf}(\coprod\Delta_\alpha^k, \coprod\partial\Delta_\alpha^k)$.

It remains to show that the characteristic maps induce an isomorphism

$$\coprod\sigma_\alpha^k\colon H_n^{lf}(\coprod\Delta_\alpha^k, \coprod\partial\Delta_\alpha^k) \longrightarrow H_n^{lf}(X^k, X^{k-1}).$$

To this end, equip X with the metric induced from the Euclidean distance function on the simplices and let $C := \{x \in X_k \mid \mathrm{dist}(x, X_{k-1}) \le 1/8\}$ and $O := \{x \in X_k \mid \mathrm{dist}(x, X_{k-1}) < 1/8\}$. Then $X_k \supset C \supset O \supset X^{k-1}$ are closed respectively open neighbourhoods that deformation retract onto X^{k-1}. Furthermore, $\coprod\Delta_\alpha^k$ is homeomorphic to $X_k\setminus O$ via a compression of the characteristic maps σ_α and the restriction of this map to $\coprod\Delta_\alpha^{k-1}$ is a homeomorphism to $C\setminus O = \partial O$. The disjoint union of the characteristic maps then factorizes as in the following diagram of proper maps

$$\begin{tikzcd}
(\coprod\Delta_\alpha^k, \coprod\partial\Delta_\alpha^k) \arrow[r, "\coprod\sigma_\alpha^k"] \arrow[d] & (X^k, X^{k-1}) \\
(X^k\setminus O, C\setminus O) \arrow[r, hook] & (X^k, C). \arrow[u, "\text{def. retraction}"']
\end{tikzcd}$$

The left vertical map is the homeomorphism described above, which thus induces an isomorphism on locally finite homology. The lower horizontal map induces an isomorphism on homology by the excision theorem and since C is closed, the deformation retraction $(X^k, C) \longrightarrow (X^k, X^{k-1})$ is proper and thus also induces an isomorphism on homology. This proves that the characteristic maps induce an isomorphism on locally finite homology. This finishes the proof. □

Note that the usual proof of the induction step via the quotient X^k/X^{k-1} does not work for locally finite homology theories because X^k/X^{k-1} is in general not a locally finite Δ-complex.

3.2 Coarse homology

The fundamental idea of coarse homology is to adapt locally finite homology so that it captures the large scale features of a metric space, while ignoring everything on a bounded scale. To this end, we need to find a procedure that successively collapses all sets of increasingly large but finite diameter. This will be called a coarsening sequence of the metric space. We can then compute the locally finite homology of these coarsened versions and define coarse homology to be the limit over the coarsening sequence.

The following section makes the above notions precise by introducing coarsening sequences as a means to blur spaces. We define coarse homology and recall some methods to compute it.

While the definition of coarse homology is geometrically quite intuitive, it is notoriously hard to compute. Thus, in Section 3.3,we develop computational tools for the coarse homology of a certain class of spaces.

Recall that a *coarsening sequence* of a proper metric space X is a sequence of locally finite open coverings $\mathcal{U}_1, \mathcal{U}_2, \dots$ such that the diameter of the sets in $\mathcal{U}_i$ is bounded from above by a constant R_i and that the Lebesgue number of $\mathcal{U}_{i+1}$ is at least R_i. Moreover, R_i tends to infinity as i does. We denote by $|\mathcal{U}_i|$ the nerve of the covering $\mathcal{U}_i$, that is the simplicial complex with vertices (U) given by the sets $U \in \mathcal{U}_i$ and k-simplices $(U_0, \dots, U_k)$ spanned by $U_0, \dots, U_k \in \mathcal{U}_i$ with $U_0 \cap \ldots \cap U_k \neq \emptyset$. Note that each $U_j \in \mathcal{U}_i$ lies in some $V_l \in \mathcal{U}_{i+1}$ and the choice of such an assignment $U_j \mapsto V_l$ induces a proper map $|\mathcal{U}_i| \to |\mathcal{U}_{i+1}|$. In what follows, we will fix such an assignment and the induced maps will be called the *coarsening maps*. We will use term coarsening sequence both for sequence of locally finite coverings $\mathcal{U}_1, \mathcal{U}_2, \dots$ and for the sequence of their geometric realizations together with the coarsening maps $|\mathcal{U}_1| \to |\mathcal{U}_2| \to \ldots$ Since the coarsening maps are proper and the induced homomorphisms on locally finite homology give rise to a direct system

$$H_*^{lf}(|\mathcal{U}_1|) \longrightarrow H_*^{lf}(|\mathcal{U}_2|) \longrightarrow H_*^{lf}(|\mathcal{U}_3|) \longrightarrow \dots$$

Definition 3.2.1 (coarse homology)**.** The *coarse homology groups* of X are defined as

$$HX_*(X) = \varinjlim H_*^{lf}(|\mathcal{U}_i|),$$

One can show that up to natural isomorphism $HX_*(X)$ does not depend on the choice of the coarsening sequence.

Note that the coarsening maps $|\mathcal{U}_i| \longrightarrow |\mathcal{U}_{i+1}|$ are in general not the embedding of a subcomplex, but we can always find coarsening sequences where this is the case. To this aim let Y be a 1-dense subset of X, that is for any $x \in X$, there exists a $y \in Y$ such that $d_X(x, y) < 1$. Assume further that Y has no accumulation points. Then for any radius $i \geq 1$, the collection of open balls of radius i, $\mathcal{B}_i(Y) := \{B_i(y)\}_{y \in Y}$ forms a locally finite open covering of X with Lebesgue number at least $i - 1$. By letting the radii range over all natural numbers, we obtain a coarsening sequence $\{\mathcal{B}_i(Y)\}_{i \in \mathbb{N}}$. We set $|\mathcal{B}_i(Y)| = R_i(X; Y)$, but we will henceforth simply write $R_i(X)$ whenever the choice of a 1-dense subset Y as above is implicit. We have natural inclusions $B_i(y) \hookrightarrow B_{i+1}(y)$ and the induced coarsening maps hence are just the inclusion of a subcomplex $R_i(X) \hookrightarrow R_{i+1}(X)$. (The notation $R_i(X)$ is slightly abusive, since it usually denotes the ith Rips complex of X in which the simplices are spanned by *all* elements of X. In our notation $R(X; Y) = R_i(X)$ is the ith Rips complex of Y.)

The presence of a Mayer-Vietoris sequence for locally finite homology allows us to derive a Mayer-Vietoris sequence for coarse homology. In the setup of coarse homology, we must require the decomposition of a metric space $X = C \cup D$ to be *coarsely excisive*, that is for every $r > 0$ there exists an $R > 0$ such that

$$B_r(C) \cap B_r(D) \subset B_R(C \cap D).$$

Proposition 3.2.2 (Lemma 3.9, [Mit])**.** *Let X be a proper metric space and $X = C \cup D$ be a coarsely excisive decomposition. Then there exists a coarse Mayer-Vietoris sequence*

$$\dots \to HX_n(C \cap D) \to HX_n(C) \oplus HX_n(D) \to HX_n(X) \to HX_{n-1}(C \cap D) \to \dots$$

It is often quite hard and very inconvenient to compute the coarse homology via an explicit coarsening sequence. For certain spaces, though, Higson and Roe [Hig-Roe] showed that the coarse homology already equals the locally finite and hence no coarsening is necessary.

Under coarsening, all small scale topology of a metric space is deleted. Thus in order for locally finite and coarse homology to be isomorphic, we need to assume that all sets of bounded diameter are contractible in a controlled way.

Definition 3.2.3 (uniform contractibility)**.** A metric space X is called *uniformly contractible* if for every $r > 0$ there exists an $R > 0$ such that $B_r(x)$ is contractible in $B_R(x)$ for every $x \in X$.

The isomorphism between coarse and locally finite homology is proven for *metric simplicial complexes*, that is for simplicial complexes equipped with the path metric induced from the canonical metric on the simplices. One might think that the isomorphy holds true for uniformly contractible metric simplicial complexes. This however is not the case as has been shown in [Dra-Fe-We] and another bound on the complexity is needed.

Definition 3.2.4 (bounded coarse geometry)**.** A proper metric space has *bounded coarse geometry* if there exists some $\varepsilon > 0$ such that the ε-capacity of any ball of radius r (i.e. the maximal number of disjoint ε-balls in B_r) is bounded by some c_r.

Following the terminology of [Hig-Roe], we will simply call a 'metric simplicial complex of bounded coarse geometry' a *bounded geometry complex.* Note that every Δ-complex can be subdivided to form a simplicial complex. Hence the following results remain valid for bounded geometry Δ-complexes.

Proposition 3.2.5 (Proposition 3.8, [Hig-Roe])**.** *If (X, d) is a uniformly contractible bounded geometry complex, then $HX_*(X)$ and $H_*^{lf}(X)$ are isomorphic.*

3.3 Coarse homology of open and truncated cones

In later sections, we will want to construct spaces with given coarse homology groups. A convenient tool for this are the truncated cones which we introduce here. They are closely related to the open cones defined in [Hig-Roe]. Let Y be a closed subset of the sphere $S^n \subset \mathbb{R}^{n+1}$. The *open cone over* Y, denoted by $\mathcal{O}(Y)$ is defined to be

$$\mathcal{O}(Y) = \mathbb{R}_{\geq 0} \cdot Y = \{\lambda y \mid \lambda \geq 0, y \in Y\} \subset \mathbb{R}^{n+1}.$$

The following proposition relates the coarse homology of an open cone $\mathcal{O}(Y)$ to the locally finite homology of Y. It shows that the cone construction blows up the topological features of Y to make them discernible for coarse homology.

Proposition 3.3.1. *Let Y be a closed subset of S^n that has the structure of a finite Δ-complex. Then*

$$HX_k(\mathcal{O}(Y)) \simeq H_{k-1}(Y)$$

for all $k \geq 1$.

Proof. Since Y is a finite complex, prismatic subdivision of $\mathbb{R}_{\geq 0}Y$ makes $\mathcal{O}(Y)$ a uniformly contractible bounded geometry complex. Hence by Proposition 3.2.5

$$HX_*(\mathcal{O}(Y)) \simeq H_*^{lf}(\mathcal{O}(Y)).$$

It thus suffices to prove that $H_k^{lf}(\mathcal{O}(Y)) \simeq H_{k-1}^{lf}(Y)$ for $k \geq 1$.

We observe that $\mathcal{O}(Y)$ is homeomorphic to the quotient space $(Y\times[0,\infty))\,/\,(Y\times\{0\})$. Consider the long exact sequence of the pair $(Y\times[0,\infty), Y\times\{0\})$

$$\begin{aligned}\ldots \to H_k^{lf}\big(Y\times\{0\}\big) \xrightarrow{\iota_*} H_k^{lf}\big(Y\times[0,\infty)\big) \longrightarrow H_k^{lf}\big(Y\times[0,\infty), Y\times\{0\}\big) \xrightarrow{\partial_k} \\ \xrightarrow{\partial_k} H_{k-1}^{lf}\big(Y\times\{0\}\big) \longrightarrow H_{k-1}^{lf}\big(Y\times[0,\infty)\big) \to \ldots\end{aligned}$$

Contrary to singular homology, the embedding $Y\times\{0\} \hookrightarrow Y\times[0,\infty)$ induces the zero-map on locally finite homology: Let σ be a k-simplex on $Y\times\{0\}$. We can apply the prism operator from singular homology to the summands of $\sigma\times\mathrm{id}_{[0,\infty)} := \sum_i \sigma\times\mathrm{id}_{[i,i+1]}$ to form a locally finite $(k+1)$-chain on $Y\times[0,\infty)$, which we denote by $P^\infty(\sigma)$ or more suggestively as $\sigma\times\mathrm{id}_{[0,\infty)}$. (This step doesn't work for singular homology H_* since $P^\infty(\alpha)$ is not a finite chain.) The formula for the boundary of the prism operator generalizes to

$$\partial P^\infty(\sigma) = \sigma\times\{0\} - P^\infty(\partial\sigma)$$

or again more suggestively

$$\partial(\sigma\times\mathrm{id}_{[0,\infty)}) = \sigma\times\{0\} - \partial\sigma\times\mathrm{id}_{[0,\infty)}\,.$$

If $[\alpha] \in H_k^{lf}(Y\times\{0\})$, i.e. $\partial\alpha = 0$, then

$$\partial(\alpha\times\mathrm{id}_{[0,\infty)}) = (\partial\alpha)\times\mathrm{id}_{[0,\infty)} + \alpha\times\{0\} = \alpha\times\{0\},$$

that is, the image of $\alpha \in H_*^{lf}(Y\times\{0\})$ in $H_*^{lf}(Y\times[0,\infty))$ is the boundary of $\alpha\times\mathrm{id}_{[0,\infty)}$. This proves that $\iota_*\colon H_*^{lf}(Y\times\{0\}) \to H_*^{lf}(Y\times[0,\infty))$ is the zero map. The long exact sequence thus breaks down into short exact sequences

$$0 \to H_k^{lf}\big(Y\times[0,\infty)\big) \hookrightarrow H_k^{lf}\big(Y\times[0,\infty), Y\times\{0\}\big) \overset{\partial_k}{\twoheadrightarrow} H_{k-1}^{lf}\big(Y\times\{0\}\big) \to 0.$$

To see that the boundary map in the above sequence is injective and hence an isomorphism, it suffices to find a left-inverse. To do so, pick a representative α of a class in $H^{lf}_{k-1}(Y\times\{0\})$ and map it to $P^\infty(\alpha)$. Then we have seen above that $\partial P^\infty(\alpha)$ lies in $Y\times\{0\}$ and hence $P^\infty(\alpha)$ defines a class in $H^{lf}_k(Y\times[0,\infty), Y\times\{0\})$. This defines a map

$$P^\infty\colon H^{lf}_{k-1}\big(Y\times\{0\}\big)\to H^{lf}_k\big(Y\times[0,\infty), Y\times\{0\}\big)$$

and for all $\beta\in H^{lf}_k(Y\times[0,\infty), Y\times\{0\})$

$$\beta - P^\infty(\partial\beta) = \partial P^\infty(\beta).$$

But since ∂_k is nothing but the boundary map on locally finite chains, this proves that $P^\infty\circ\partial_k = \mathrm{id}_{H^{lf}_{k-1}(Y\times\{0\})}$. Thus the connecting homomorphism ∂_k is an isomorphism, that is $H^{lf}_k\big(Y\times[0,\infty), Y\times\{0\}\big)\simeq H^{lf}_{k-1}(Y)$ for all $k\geq 1$. But clearly $(Y\times[0,\infty), Y\times\{0\})$ is a good pair and consequently

$$H^{lf}_k\big(Y\times[0,\infty), Y\times\{0\}\big)\simeq H^{lf}_k\big(Y\times[0,\infty)/(Y\times\{0\}\big) = H^{lf}_k\big(\mathcal{O}(Y)\big)$$

for $k\geq 1$ by the analogous arguments as in the case of H^{sing}_*.

To summarize the argument, we have shown that there is the following sequence of isomorphisms

$$\begin{aligned} HX_k\big(\mathcal{O}(Y)\big) &\simeq H^{lf}_k\big(\mathcal{O}(Y)\big)\\ &\simeq H^{lf}_k\big(Y\times[0,\infty)/(Y\times\{0\}\big)\\ &\simeq H^{lf}_k\big(Y\times[0,\infty), Y\times\{0\}\big)\\ &\simeq H^{lf}_{k-1}(Y\times\{0\}\big)\end{aligned}$$

proving the claim. $\square$

The above proof shows moreover that $H^{lf}_k(Y\times[0,\infty)) = 0$ for all $k\geq 1$ since $H^{lf}_k\big(Y\times\{0\})\big)\hookrightarrow H^{lf}_k\big(Y\times[0,\infty), Y\times\{0\}\big)$ is the zero map.

When constructing manifolds with prescribed coarse homology groups, it will be convenient to glue in building blocks that have S^{n-1} as boundary. To this aim we modify the definition of open cones to make them more suitable for such constructions.

Definition 3.3.2 (Truncated cones)**.** Let Y be a closed hypersurface of the interior of the upper hemisphere $\mathrm{int}(D^n_+)\subset S^n$. Let Y_∂ be the component of $D^n_+\backslash Y$ that contains ∂D^n_+. The *truncated cone over* Y is then Y_∂ united with $\mathbb{R}_{\geq 1}\cdot Y = \{\lambda y \mid \lambda\geq 1, y\in Y\}$ (see Figure 1.1).

Since Y is a submanifold of S^n, smoothing $\mathscr{T}(Y)$ around Y, we find that the truncated cone over Y is quasi-isometric to a Riemannian manifold. When working with manifolds, we will often take $\mathscr{T}(Y)$ to be the smoothed version without further mention. Since $\mathscr{T}(Y)$ is quasi-isometric to $\mathcal{O}(Y)$, we immediately have the following corollary to Proposition 3.3.1.

Corollary 3.3.3. *Let Y be a closed hypersurface of int(D^n_+). Then*

$$HX_k\big(\mathscr{T}(Y)\big) \simeq H_{k-1}\big(Y\big).$$

Chapter 4

Non-leaves with trivial coarse homology

In his articles [Schw1] (1994) and [Schw2] (2011) Schweitzer gives necessary criteria for Riemannian manifolds to be quasi-isometric to leaves in codimension 1 foliations of compact manifolds. He uses these criteria to construct non-leaves by deforming a given metric on a non-compact manifold to a metric that does not satisfy his criterion. In this chapter we will show that the deformation performed by Schweitzer blows up the coarse homology so that the resulting non-leaves have non-finitely generated coarse homology. This led us to ask whether the bounded homology property could be reduced to a statement about the number of generators in the coarse homology, and hence whether the coarse homology could be used to show that a given Riemannian manifold cannot be quasi-isometric to a leaf in a compact manifold.

This, however, is not the case. We will present a different deformation to a non-leaf that does not affect the coarse homology groups of the original metric. This shows that non-leaves do not necessarily have a certain type of coarse homology groups. In particular there exist non-leaves with trivial coarse homology. In fact we will show in Section 6.1 that the number of generators of the coarse homology groups and the criteria of Schweitzer are completely unrelated. Moreover, we will show in Chapter 5 that there exists a large family of leaves with non-finitely generated coarse homology.

In Section 4.1, we compute the coarse homology groups of the non-leaves constructed by Schweitzer using tools from Chapter 3. Section 4.2 then presents the deformation of Riemannian manifolds to non-leaves without affecting the coarse homology. This shows that manifolds not satisfying the bounded homology property need not have non-finitely generated coarse homology. In Chapter 6, Section 6.1 we then show that also the opposite is possible: There exist manifolds with finitely and non-finitely generated coarse homology groups

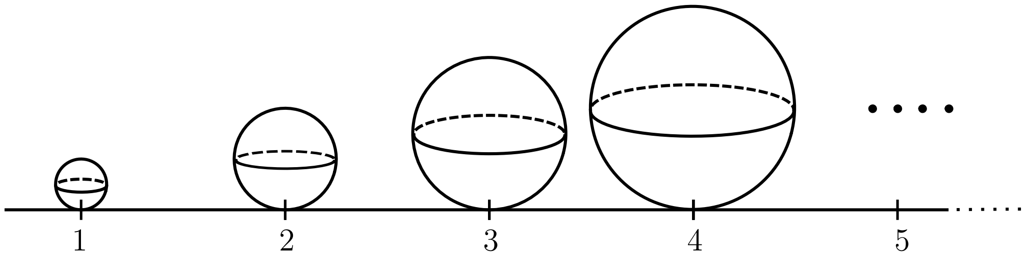

Figure 4.1: The balloon space B.

that do and that do not satisfy the bounded homology property.

4.1 Coarse homology of Schweitzer's counter-example

First recall how Schweitzer constructs non-leaves by deforming the metric on open manifolds by inserting balloons: Since (L, g) has bounded geometry, the injectivity radius is bounded from below by some $d > 0$. Hence every metric d-ball is topologically a ball. Now choose a sequence of real numbers d_i such that $d_i + 2d < d_{i+1}$ and let x_i be points in L such that $d(x_0, x_i) = d_i$. Choose moreover a sequence of real numbers r_i converging to infinity. And let $S^n(r_i) \backslash B_{d/2}(S)$ be the spheres of radius r_i, where a ball of radius $d/2$ has been removed around the south pole. Now replace every disk $B(x_i, d/2)$ by the "balloon" $S^n(r_i) \backslash B_{d/2}(S)$ and on the annulus $B_d(x_i) \backslash B_{d/2}(x_i)$ interpolated smoothly between the round metric of $S^n(r_i)$ and the original metric g on L. This defines the balloon metric g_S on L (see Figure 2.3.2). Note that topologically we have just replaced a ball by a ball and hence the new manifold is diffeomorphic to L.

The above construction is closely related to the following *balloon space* consisting of spheres with radii tending towards infinity attached to the real line, which is defined in [Ha-Ko-Roe-Sch]:

$$B = [0, \infty) \bigcup_{i \in \mathbb{N}} (\cup_i S^n(i)),$$

where $S^n(i)$ denotes the n-dimensional sphere of radius i (see Figure 4.1). They compute the coarse homology of B to be

$$HX_n(B) \simeq \big(\prod_i \mathbb{Z}\big) / \big(\bigoplus_i \mathbb{Z}\big).$$

In the following proposition we will show that an adapted balloon space B' coarsely embeds into the non-leaves (L, g_S) constructed by Schweitzer and show that this embedding is injective in coarse homology.

Proposition 4.1.1. *Let (L, g) be a complete connected open Riemannian manifold, of dimension $n \geq 2$ and let g_S be the deformation of g to the above described balloon metric performed along a ray in L. Then $HX_n(L, g_S)$ is not finitely generated. In fact*

$$HX_n(B) = \prod_i \mathbb{Z} / \bigoplus_i \mathbb{Z} \subset HX_n(L, g_S).$$

Proof. Let γ be a ray in (L, g) and let $x_i = \gamma(d_i), i = 0, 1, \ldots$ be the sequence of points in L where we have inserted balloons of radius r_i tending towards infinity. Let B' be the balloon space which is given by attaching an n-sphere $S^n(r_i)$ of radius i to $d_i \in [0, \infty)$ for every $i \in \mathbb{N}$. Note that (L, g_S) and the space L' which results from gluing the south poles of the $S^n(r_i)$ to the x_i are quasi-isometric if we equip L' with the path metric induced from the $S^n(r_i)$ and (L, g). Hence their coarse homology groups are equal and we can work with L' instead of (L, g_S). Then the embedding $\iota : B' \hookrightarrow L'$ mapping and $[0, \infty)$ to γ, and the $S^n(r_i) \subset B'$ to the corresponding spheres in L' is an isometry and thus a quasi-isometry onto its image. In particular, we can consider B' as a subspace of L' and ι to be an inclusion. In what follows, L is understood to be the 'original' Riemannian manifold (L, g).

Since L' carries the path metric, $L' = L \cup B'$ is a coarsely excisive decomposition with $L \cap B' = [0, \infty)$ and by Proposition 3.2.2 we have a Mayer-Vietoris sequence

$$\ldots \to HX_k([0, \infty)) \to HX_k(L) \oplus HX_k(B') \to HX_k(L') \to HX_{k-1}([0, \infty)) \to \ldots$$

But $[0, \infty)$ is a uniformly contractible bounded geometry complex and thus Proposition 3.2.5 implies that $HX_k([0, \infty)) \simeq H_k^{lf}([0, \infty))$. Using locally finite Δ-homology, it is easy to see that $H_k^{lf}([0, \infty)) = \{0\}$ for all $k \in \mathbb{N}$. Hence for all $k \in \mathbb{N}$ the above sequence breaks down to

$$0 \longrightarrow HX_k(L) \oplus HX_k(B') \longrightarrow HX_k(L') \longrightarrow 0$$

and thus

$$HX_k(L, g_S) \simeq HX_k(L') \simeq HX_k(L, g) \oplus HX_k(B').$$

As in [Ha-Ko-Roe-Sch], we can choose a coarsening sequence $\mathcal{U}_k$ for B' such that $|\mathcal{U}_k|$ is properly homotopic to B' with the first k spheres collapsed to their respective south soles. Hence again

$$HX_n(B') \simeq (\prod_i \mathbb{Z})/(\bigoplus_i \mathbb{Z}). \qquad \square$$

4.2 Non-leaves with trivial coarse homology

The following section shows that any Riemannian metric on a non-compact manifold can be deformed to a non-leaf without affecting the coarse homology. In particular, we construct non-leaves with trivial coarse homology. This, together with the results of Section 6.1 shows that contrary to what Proposition 4.1.1 might suggest, the bounded homology property and the number of generators of the coarse homology are independent from each other.

In [Schw2] Schweitzer shows that every metric g on a non-compact manifold M can be deformed to a new metric g_S that does not satisfy the bounded homology property and hence (M, g_S) cannot be a leaf in a codimension one foliation of a compact manifold. The construction can be performed in such a way that the growth types of g and g_S are the same. As we have seen in Proposition 4.1.1, the manifold (M, g_S) has non-finitely generated coarse homology groups.

The following theorem gives a modified construction of non-leaves such that both the coarse homology groups and the growth type of g remain unchanged.

Theorem 4.2.1. *On every non-compact Riemannian manifold of bounded geometry (M, g), $\dim M \geq 3$, there exists a deformation of g to a bounded geometry metric g' such that (M, g') cannot be diffeomorphically quasi-isometric to a leaf of a codimension one $C^{2,0}$-foliation of a compact manifold. This deformation can be performed in such a way that $HX_*(M, g') = HX_*(M, g)$ and that the growth type of M remains unchanged.*

The theorem is optimal in the sense that we cannot expect to find a non-leaf metric of bounded geometry with trivial coarse homology on every open manifold. This is because a Riemannian manifold of bounded geometry with at least 2 ends will always have non-trivial degree 1 coarse homology, as we will show in Proposition 5.0.5 in Chapter 5. However, there exist open manifolds with trivial coarse homology (which then necessarily have exactly one end), so if we apply the deformation to the one-ended cylinder $\partial D^n \times \mathbb{R}_{\geq 0} \cup D^n \times \{0\}$ we find the following corollary.

Corollary 4.2.2. *In every dimension $n \geq 3$, there exist Riemannian manifolds with trivial coarse homology which are not diffeomorphically quasi-isometric to a leaf of a codimension one $C^{2,0}$-foliation of a compact manifold.*

By Theorem 2.6 in [Schw2], every manifold that is diffeomorphically quasi-isometric to a leaf of a codimension one $C^{2,0}$ foliation must satisfy the bounded homology property. Thus Theorem 4.2.1 follows directly from the following lemma.

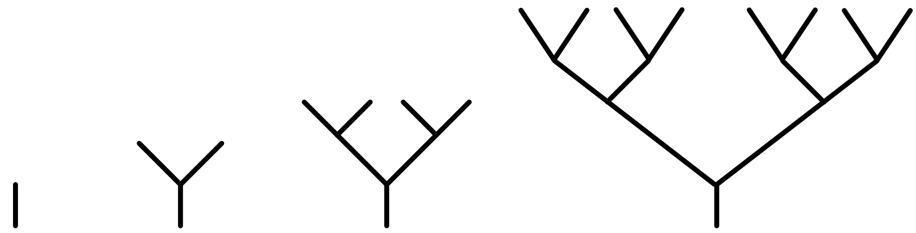

Figure 4.2: $\mathcal{T}_1, \ldots, \mathcal{T}_4$ (edge lengths are not true to scale).

Lemma 4.2.3. *On every non-compact manifold (M, g) of bounded geometry $\dim M \geq 3$, there exists a metric g' not satisfying the bounded homology property such that $HX_*(M, g') = HX_*(M, g)$. Moreover, we can choose g' such that it has the same growth type as g.*

Instead of gluing in balloons, our deformation of the metric uses building blocks that are modelled on perfect binary trees of height tending to infinity. We then exploit the fact that these model manifolds are coarsely quasi-isometric to the tree itself and hence do not add to the coarse homology, while still being complex enough to have unbounded Morse-β-volume and thus preventing the new metric from satisfying the bounded homology property. We recall that a *rooted tree* is a tree with a distinguished vertex, which we call the *root*. The *leaves* of a connected tree are the vertices with degree 1, while we do not consider the the root as a leaf. The *height* of a leaf is its distance from the root, where we let each edge have length equal to 1. By the *perfect binary tree of height* k we mean the rooted tree $\mathcal{T}_k$, where the root and the leaves have degree 1, while every other vertex has exactly 2 children (i.e. has degree 3) and every leaf has height k (see Figure 4.2). (Note that commonly the root in the perfect binary tree is also required to have 2 children.)

Proof of Lemma 4.2.3. Construction of the non-leaf metric: To construct a metric on M with the same coarse homology but not satisfying the bounded homology property, we let γ be a ray in (M, g) and set $x_i = \gamma(d_i), i = 0, 1, \ldots$ such that $d(x_{k-1}, x_k) > 3d$, where $1/2 > d > 0$ is a lower bound for the injectivity radius of M. Instead of doing gluing constructions with balloons as described in Section 4.1 of this chapter, we construct a thickened version of the perfect binary tree $\mathcal{T}_k$ of height $k \in \mathbb{N}$, where essentially we are replacing the edges of $\mathcal{T}_k$ by cylinders (one of them having length increasing with k) and connected them at the vertices using punctured spheres (see Figure 4.3).

Denote the root of a $\mathcal{T}_k$ by r, its leaves by $l_m, m = 1, \ldots 2^{k-1}$ and the other vertices by $v_i, i = 1, \ldots, \sum_{j=0}^{k-2} 2^{j-1}$. At each vertex v_i place a copy of the "T-piece" $T_i \approx S^n \backslash (B_1 \cup B_2 \cup B_3)$, which is an n-sphere with three disjoint balls of radius 1 removed. (The radius of the n-sphere is chosen large enough, so that

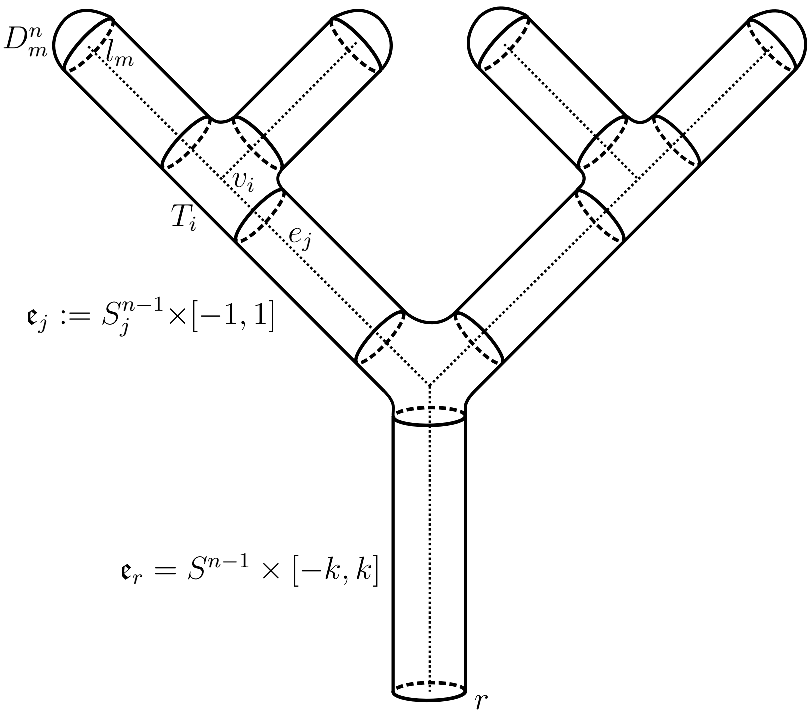

Figure 4.3: The manifold $\mathfrak{T}_3$ with dotted tree $\mathcal{T}_3$.

the sphere contains three disjoint balls of radius 1.) At each leaf l_m of $\mathcal{T}_k$ place a copy of the unit n-disk D^n. For every edge $e_j = [v_i, v_{i'}]$ glue one copy of the cylinder $\mathfrak{e}_j := S^{n-1} \times [-1,1]$ to the boundary components of the T-pieces T_i and $T_{i'}$. For an edge leading to a leaf $[v_i, l_m]$ glue one end of the cylinder to a boundary component of T_i and one to ∂D^n_m. For technical reasons, which we will explain later in this proof, for the unique edge $e_r = [r, v_i]$ starting from the root, we glue a boundary component of $\mathfrak{e}_r := S^{n-1}\times[-k,k]$ to a boundary component of T_i while leaving the other end open. All throughout this gluing process, we make sure that no boundary component of any T_i has two cylinders glued to it. The resulting manifold $\mathfrak{T}_k$ is then diffeomorphic to a closed disc.

For brevity, we will denote the different pieces $T_i, \mathfrak{e}_j, D^n_m$ as *building blocks*. Then $\mathfrak{T}_k$ carries the path metric induced from the building blocks, which we smoothen uniformly in small neighbourhoods where two building blocks were glued together. In what follows, $\mathfrak{T}_k$ will always be equipped with this metric. The uniformity of the smoothing and the fact that the $\mathfrak{T}_k$ are modelled on a finite number of building blocks, ensures that the curvature of all $\mathfrak{T}_k$ is uniformly bounded and that $\operatorname{inj}\mathfrak{T}_k > \iota > 0$ independently of k.

Then $\mathfrak{T}_k \approx B^n$ and its boundary $\partial\mathfrak{T}_k$ is isometric to the unit sphere S^{n-1}. Since we want to glue the boundary of $\mathfrak{T}_k$ to the boundary of $B_{d/2}(x_k)$, on $S^{n-1} \times [-k, 0]$ we interpolate the radii of the spheres $S^{n-1} \times \{t\}$ smoothly between 1 for $t = 0$ and $d/2$ for $t = -k$. Now for each $k \in \mathbb{N}$ we replace $B_{d/2}(x_k) \subset (M, g)$ by $\mathfrak{T}_k$. Since both $B_d(x_k)$ and $\mathfrak{T}_k$ are topologically balls, the new manifold is diffeomorphic to M, but the $\mathfrak{T}_k$ induce a new metric g'. We smoothen g' around $\partial B_{d/2}(x_k)$ to yield a smooth Riemannian metric which we again denote by g'. We claim that (M, g') does not have the bounded homology property.

Proof that (M, g') does not have the bounded homology property: In what follows, denote by $\mathfrak{T}'_k$ the building block $\mathfrak{T}_k$ without the lower part $S^{n-1} \times [-k, 0)$ of the cylinder starting from the root of $\mathcal{T}_k$. Thus $\partial\mathfrak{T}'_k = S^{n-1} \times \{0\}$. It suffices to show that the Morse-β-volume of the $\mathfrak{T}'_k$ is unbounded for every $\beta > 0$ as k goes to infinity. For then, the $\mathfrak{T}'_k$ form a sequence of closed codimension 0 submanifolds with $\mathrm{vol}_\beta(\partial\mathfrak{T}'_k) = \mathrm{vol}_\beta(S^{n-1})$, while there exists no constant $L > 0$ such that $\mathrm{MVol}_\beta(\mathfrak{T}'_k) \leq L$. It is easy to verify that the $\mathfrak{T}'_k$ satisfy the conditions i)-iii) in Definition 2.3.21: We have already seen that $\mathrm{vol}_\beta(\partial\mathfrak{T}'_k)$ is constant, and as $n \geq 3$ both $\mathfrak{T}'_k \approx D^n$ and $\partial\mathfrak{T}'_k \approx S^{n-1}$ are simply connected, thus conditions i) and ii) are fulfilled. To see that the boundary of $\mathfrak{T}'_k$ has a large tubular neighbourhood, i.e. satisfies condition iii), we use that the length of the cylinder at the root of $\mathcal{T}_k$ has length $2k$. For every $k > \beta$, this cylinder $S^{n-1} \times [-k, k]$ is a tubular neighbourhood of $\partial\mathfrak{T}'_k$ that contains the β-neighbourhood $B_\beta(\partial\mathfrak{T}_k)$.

To prove that $\mathrm{MVol}_\beta(\mathfrak{T}'_k)$ goes to infinity, we show that for every continuous (and in particular, for every Morse function) $f\colon \mathfrak{T}'_k \longrightarrow \mathbb{R}$ there exists an $x \in \mathbb{R}$ such that f takes the value x on at least $\lceil \frac{k}{2} \rceil$ building blocks $T_i, \mathfrak{e}_j, D^n_m \subset \mathfrak{T}'_k$. This is sufficient, because every ball of radius β contains at most finitely many building blocks, say c_β. For such x

$$\mathrm{vol}_\beta(f^{-1}(x)) \geq \frac{\lceil \frac{k}{2} \rceil}{c_\beta}$$

holds and hence

$$\mathrm{MVol}_\beta(\mathfrak{T}'_k) \geq \frac{\lceil \frac{k}{2} \rceil}{c_\beta}.$$

Note that this is somewhat more general than showing that the Morse-β-volume of $\mathfrak{T}'_k$ goes to infinity since we do not require our functions to vanish on $\partial\mathfrak{T}'_k$. This will enable us to do induction to the trees of lower height lying inside of $\mathcal{T}_k$.

We note that the above problem can be reduced to showing that for every continuous function $f\colon \mathcal{T}_k \longrightarrow \mathbb{R}$ there exists an $x \in \mathbb{R}$ such that f takes the value x on at least $\lceil \frac{k}{2} \rceil$ edges. We will give a direct proof instead.

In order to prove that for every continuous function $f\colon \mathfrak{T}'_k \longrightarrow \mathbb{R}$ there exists an $x \in \mathbb{R}$ such that $f^{-1}(x)$ contains at least $\lceil \frac{k}{2} \rceil$ building blocks, we

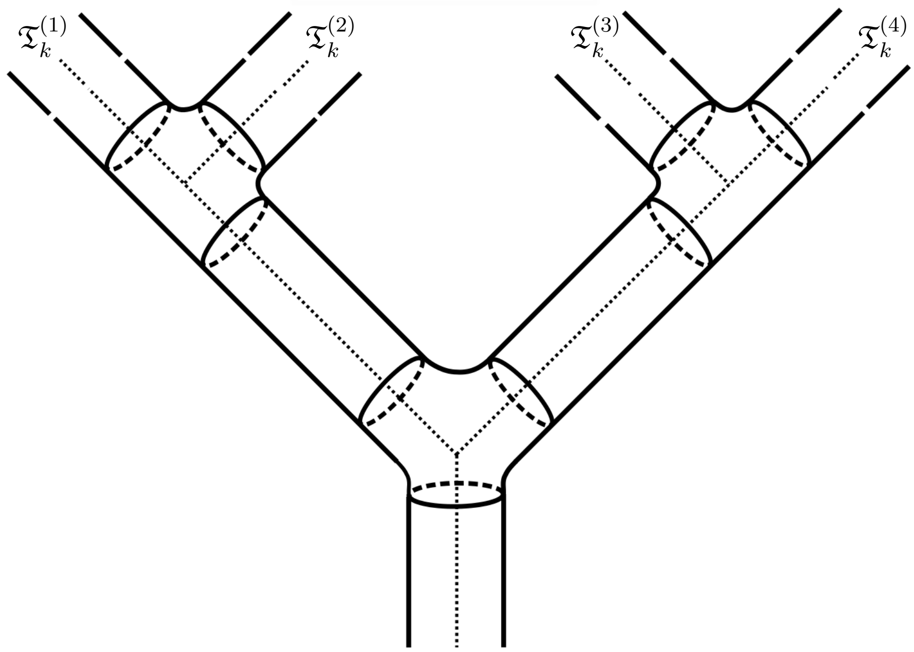

Figure 4.4: $\mathfrak{T}_{k+2}$ with four copies of $\mathfrak{T}_k$.

introduce

$$u(k) := \min_{f\colon \mathfrak{T}'_k \to \mathbb{R}} \max_{x \in \mathbb{R}} |\{T_i, \mathfrak{e}_j, D^n_m \subset \mathfrak{T}_k \mid x \in f(\mathfrak{e}_j)\}|.$$

That is, for every $f\colon \mathfrak{T}'_k \longrightarrow \mathbb{R}$ there exists an $x \in \mathbb{R}$ such that f takes the value x on at least $u(k)$ building blocks in $\mathfrak{T}'_k$. Since $\mathcal{T}_{k+1}$ contains $\mathcal{T}_k$, clearly $u(k+1) \geq u(k)$.

Lemma 4.2.4. *$u(k+2) > u(k)$.*

Proof. Let $f\colon \mathfrak{T}'_{k+2} \longrightarrow \mathbb{R}$. Note that $\mathfrak{T}'_{k+2}$ contains four copies $\mathfrak{T}_k^{'(1)}, \ldots, \mathfrak{T}_k^{'(4)}$ of $\mathfrak{T}'_k$ as shown in Figure 4.4 (up to scaling the length of the edge $\mathfrak{r}$ leading to the root down to 1). Let $t_{\max} \in \mathfrak{T}'_{k+2}$ be such that $f(t_{\max}) = \max f$ and $t_{\min}$ analogously. Then there exists a $\mathfrak{T}_k^{'(s)}$, say $\mathfrak{T}_k^{'(1)}$, such that the geodesic between $x_{\min}$ and $x_{\max}$ does not intersect $\mathfrak{T}_k^{'(1)}$. But for $f\mid_{\mathfrak{T}_k^{'(1)}}\colon \mathfrak{T}_k^{'(1)} \longrightarrow \mathbb{R}$ there exists some $x \in \mathbb{R}$ and $u(k)$ building blocks in $\mathfrak{T}_k^{'(1)}$ such that $f\mid_{\mathfrak{T}_k^{'(1)}}$ takes the value x on these blocks. But by construction $f(t_{\min}) \leq x \leq f(t_{\max})$ and since the geodesic between $t_{\min}$ and $t_{\max}$ does not intersect $\mathfrak{T}_k^{'(1)}$, there exists an additional edge in $\mathfrak{T}'_{k+2} \setminus \mathfrak{T}_k^{'(1)}$ such that f takes the value x on this edge. Hence $u(k+2) \geq u(k) + 1$. $\square$

The above lemma implies that $u(k) \geq \lceil \frac{k}{2} \rceil$. It remains to show that $HX_*(M, g') = HX_*(M, g)$.

Proof that $HX_*(M, g') = HX_*(M, g)$*:* Note that (M, g') is coarsely quasi-isometric to (M, g) with the trees $\mathcal{T}_k$ attached to x_k. Denote this metric space by M'. Then $HX_*(M, g') = HX_*(M')$. But M' has a coarsely excisive decomposition into (M, g) and the real line with the trees $\mathcal{T}_k$ attached to $r_k := d(x_0, x_k)$

$$\mathcal{T}' := [0, \infty) \bigcup_{\{r_1, r_2, \ldots\}} (\cup_k \mathcal{T}_k),$$

where $\mathcal{T}'$ carries the path metric induced from the $\mathcal{T}_k$ and $[0, \infty)$ and with $(M, g) \cap \mathcal{T}'$ being quasi-isometric to $[0, \infty)$. Then $\mathcal{T}'$ naturally has the structure of a metric simplicial complex by uniting the simplicial structure on the trees $\mathcal{T}_k$ and that on $[0, \infty)$. This space has bounded coarse geometry in the sense of Definition 3.2.4: Any ball of radius r in $\mathcal{T}'$ intersects at most $2^{\lceil r \rceil + 2}$-many edges. Since every subset of diameter 1 intersects at least one edge and since every edge is intersected by at most two disjoint sets of diameter 1, the 1-capacity of every ball of radius r in $\mathcal{T}'$ is bounded for all r. Hence $\mathcal{T}'$ is a bounded geometry complex. Moreover every ball in $\mathcal{T}'$ is contractible within itself, consequently $\mathcal{T}'$ is in particular uniformly contractible. Thus we can apply Proposition 3.2.5 to find that $HX_*(\mathcal{T}') \simeq H_*^{lf}(\mathcal{T}') \simeq H_*^{lf}([0, \infty)) = \{0\}$ since collapsing each $\mathcal{T}_k$ to its respective root is a proper homotopy equivalence from $\mathcal{T}'$ to $\mathbb{R}$.

This and the above coarsely excisive decomposition gives us a coarse Mayer-Vietoris sequence

$$\begin{aligned} \ldots \longrightarrow HX_k([0, \infty)) \longrightarrow HX_k(M, g) \oplus HX_k(\mathcal{T}') \longrightarrow HX_k(M') \longrightarrow \\ \longrightarrow HX_{k-1}([0, \infty)) \longrightarrow \ldots \end{aligned}$$

Again by Proposition 3.2.5 and Proposition 3.1.4 using locally finite Δ-homology, we see that $HX_k([0, \infty)) \simeq H_k^{lf}([0, \infty)) = \{0\}$ for all $k \in \mathbb{N}$. This and the above coarse Mayer-Vietoris sequence yield the desired isomorphism

$$HX_k(M') \simeq HX_k(M, g) \oplus HX(\mathcal{T}') \simeq HX_k(M, g).$$

Proof that g' can be chosen with the same growth type as g: Recall the definition of the growth type of (M, g) from Section 2.1. We claim that by placing the $\mathfrak{T}_k$ sufficiently far apart from each other, the growth type of (M, g) remains unchanged. It suffices to show that there exist $\lambda > 0$ such that $\mathrm{vol}_{g'}(B_r(x_0)) \leq \lambda \, \mathrm{vol}_g(B_r(x_0))$. To this aim, we set the distance $r_k = d(x_0, x_k)$ between the $\mathfrak{T}_k$ as follows: By bounded geometry $\mathrm{vol}(M, g)$ is infinite and hence the growth function $r \longmapsto \mathrm{vol}(B_r(x_0))$ is unbounded. Hence, there exist an

$r_1 > 0$ such that $\mathrm{vol}_g(B_{r_1}(x_0)) = \mathrm{vol}(\mathfrak{T}_1)$. Choose x_1 such that $d(x_0, x_1) = r_1$. Now r_i is chosen such that $\mathrm{vol}_g(B_{r_i}(x_0)) = \sum_{j=1}^{i} \mathrm{vol}(\mathfrak{T}_j)$. Then for $r < r_i$, we find that

$$\begin{aligned}\mathrm{vol}_{g'}(B_r(x_0)) &\leq \mathrm{vol}_g(B_r(x_0)) + \sum_{j=1}^{i-1} \mathrm{vol}(\mathfrak{T}_j)\\ &\leq \mathrm{vol}_g(B_r(x_0)) + \mathrm{vol}_g(B_{r_{i-1}}(x_0))\\ &\leq 2\,\mathrm{vol}_g(B_r(x_0)).\end{aligned}$$

This finishes the proof of the lemma. □

4.3 Coarse non-leaves with small coarse homology

Recall that Zeghib constructs manifolds which cannot be coarsely quasi-isometric to a simply connected leaf by gluing spheres of increasing radii in an irregular pattern to $\mathbb{H}^2$ (see [Zeg] and Section 2.3.1). As in the construction of Schweitzer, these balloons blow up the coarse homology in top degree and make it not finitely generated. The following theorem shows that non-finitely generated coarse homology is not a necessary condition for a Riemannian manifold to be a coarse non-leaf.

Theorem 4.3.1. *In any dimension greater than or equal to 2 there exist Riemannian manifolds (M, g) of bounded geometry which cannot be coarsely quasi-isometric to a simply connected leaf in neither a C^1-foliation of arbitrary codimension nor a $C^{1,0}$ foliation of codimension 1 of a compact manifold. Moreover, $HX_k(M, g)$ is trivial for all k but $k = \dim M$, where we have $HX_k(M, g) = \mathbb{Z}$.*

The construction of such manifolds is similar to that of Zeghib, which we presented in Section 2.3.1, and we make use of building blocks similar to those from the previous section. The idea is to construct manifolds with infinite coarse geometric entropy by gluing "slim" pieces, which are not seen by coarse homology, in a sufficiently irregular pattern to hyperbolic space. We would like to mention that in contrast to the constructions by Schweitzer and that in Section 4.2 this cannot be performed on every Riemannian manifold but relies on the exponential volume growth of hyperbolic space.

Construction of the building blocks $N(y, r^6, \boldsymbol{j})$: Recall the definition of the perfect binary tree from the previous section. In order to produce manifolds in which the volume of balls grows faster than that of hyperbolic n-space, we let

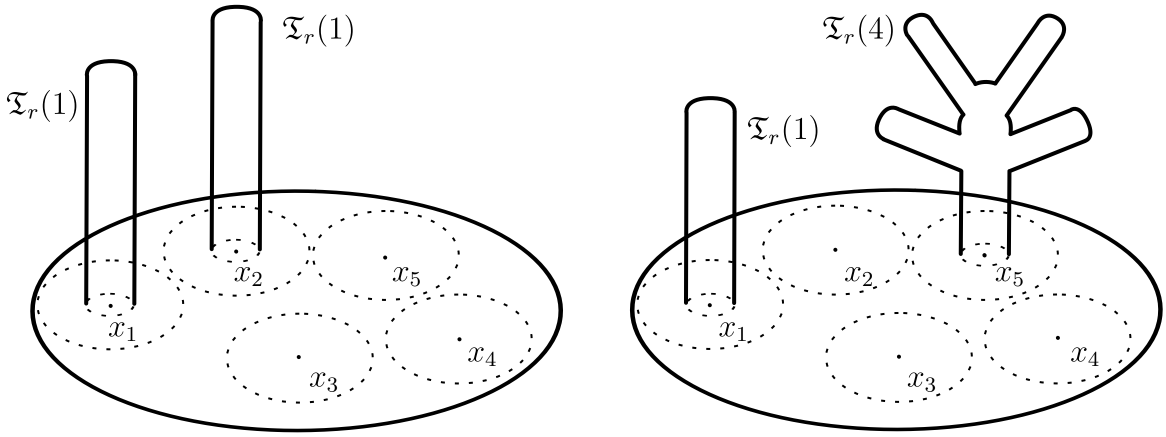

Figure 4.5: Schematic picture of $T^+(y, r^4, 2)$ (left) and $T^-(y, r^4, 1)$ (right) with dotted $B(y_i, r^3)$.

$\mathcal{T}_r(4^n)$ be the perfect 4^n-ary tree of height r. That is $\mathcal{T}_r(4^n)$ is the rooted tree, in which every vertex except for the root and the leaves has exactly 4^n children. The root has only one child (cf. Picture 4.2, where examples of perfect 2-ary trees are shown). As in the previous section, we let $\mathfrak{T}_r(4^n)$ be the manifold with boundary isometric to S^{n-1} that results from gluing cylinders to the edges of $\mathcal{T}_r(4^n)$. We let all edges of $\mathfrak{T}_r(4^n)$ have length 1 except for the one starting from the root, which we let have length equal to r. Moreover, we let $\mathfrak{T}_r(1)$ be the capped off cylinder

$$[0, 2r] \times S^{n-1} \cup_{\{2r\}\times S^{n-1}\sim\partial D^n} D^n.$$

(We could also view $\mathfrak{T}_r(1)$ to result from thickening the perfect 1-ary tree of height $2r$.)

Now that the basic manifolds are chosen, the construction of the non-leaf is analogous to that in [Att-Hur] and [Zeg]: We start with hyperbolic space $\mathbb{H}^n$ and note again, that there exists a constant $c > 1$ such that every ball of radius r^{k+1} contains at least $d(r) := d(r^6, r^5) = \lfloor c^{r^6-5} \rfloor \geq \lfloor c^r \rfloor$ pairwise disjoint balls of radius r^k for any $k, r > 1$.

For every integer $r > 0$ we construct $2^{d(r)-1}$ building blocks which will be glued to $\mathbb{H}^n$ to produce a non-leaf: Start with a ball $B(y; r^4) \subset \mathbb{H}^n$ and pick $d = d(r)$ points $y_1, \dots, y_d \in B(y, r^4)$ such that the $B(y_i, r^3) \subset B(y, r^4)$ are pairwise disjoint. For every $1 \leq k < d$ we let $T^+(y, r^4, k)$ be the manifold obtained from replacing the first k balls $B(y_1; 1), \dots, B(y_k; 1)$ with a copy of $\mathfrak{T}_r(1)$. $T^-(y, r^4, k)$ is $T^+(y, r^4, k)$ with a copy of $\mathfrak{T}_r(4^n)$ replacing $B(y_d, 1)$ (see Picture 4.5). Note that the $\mathfrak{T}_r(1)$ and $\mathfrak{T}_r(4^n)$ in $T^\pm(y, r^4, k)$ have distance at least $r^3 - 1 \geq r^2$ from each other.

Now let $z_1, \dots, z_{d(r)} \in B(z, r^6) \subset \mathbb{H}^n$ be such that the $B(z_i; r^5) \subset B(z, r^6)$ are pairwise disjoint and let $\mathbf{j}\colon \{1, \dots, d(r)-1\} \longrightarrow \{\pm\}$ be any function. The building block $N(z, r^6, \mathbf{j})$ then is $B(z, r^6)$ with $T^{\mathbf{j}(k)}(z_k, r^4, k)$ replacing $B(z_k, r^4)$ for $1 \leq k \leq d(r)-1$. Since there are $2^{d(r)-1}$ functions $\mathbf{j}$, we have constructed as many $N(z, r^6, \mathbf{j})$. Again note that the $T^{\mathbf{j}(k)}(z_k, r^4, k) \subset N(z, r^6, \mathbf{j})$ have distance at least $r^5 - r^4 \geq r^4$ from each other.

Construction of the non-leaf: Start with $\mathbb{H}^n$, let γ be a geodesic ray in $\mathbb{H}^n$ and pick points z_i on γ which are increasingly far apart, say $z_i := \gamma(i^{i+6})$. Now successively replace the balls $B(z_i; r^6)$ by the $N(z_i, r^6, \mathbf{j})$, where $\mathbf{j}$ runs through all functions $\{1, \dots, d(r)-1\} \longrightarrow \{\pm\}$ and r runs through all natural numbers $r \geq 2$. More precisely, we start by replacing the balls $B(z_i; 2^6), i = 1, \dots, 2^{d(2)-1}$ by the $N(z_i, 2^6, \mathbf{j})$, where $\mathbf{j}$ runs through all functions $\{1, \dots, d(2)-1\} \longrightarrow \{\pm\}$. Now suppose that for $r < R$ all $N(z_i, r^6, \mathbf{j})$ have already been used to replace the balls $B(z_i; r^6), i = 1, \dots, I$. Then in the next step the balls $B(z_i; R^6)$, $i = I, \dots, I + d(R) - 1$ are replaced by the $N(z_i, R^6, \mathbf{j})$, where, again, $\mathbf{j}$ runs through all functions $\mathbf{j}\colon \{1, \dots, d(R)-1\} \longrightarrow \{\pm\}$. Denote the resulting Riemannian manifold by (M, g).

We first show that (M, g) cannot be coarsely quasi-isometric to a simply connected leaf of a C^1-foliation of arbitrary codimension or a $C^{1,0}$-foliation of codimension 1. This will follow from the following lemma.

Lemma 4.3.2. *For fixed $\varepsilon > 0$ and sufficiently large $r > 0$, any coarse $(\varepsilon, 4r^6)$-quasi-tiling has cardinality at least $2^{d(r)-1}$, that is the coarse ε-growth complexity function satisfies the following bound from below: $H^{cs}(M, g, \varepsilon, 4r^6) \geq 2^{d(r)-1}$. In particular, the coarse geometric entropy of (M, g) is infinite.*

The proof relies on the fact that, while a quasi-isometry does not preserve volume, the volume of neighbourhoods can be controlled under quasi-isometries as follows.

Lemma 4.3.3 ([Zeg])**.** *Let $f\colon (M, g) \longrightarrow (N, h)$ be a (λ, D, C)-quasi-isometry between Riemannian manifolds of bounded geometry, let $A \subset M$ be any subset and let $\delta > 0$ be fixed. Denote by $B_\delta(A) \subset M$ the set of all points that are at most distance δ away from A. Then there exists a constant $K = K(\lambda, D, C, \delta) > 0$ independent of A such that*

$$\frac{1}{K} \leq \frac{\operatorname{vol}_g\left(B_\delta(A)\right)}{\operatorname{vol}_h\left(B_\delta(f(A)\right)} \leq K.$$

If the ratio of two quantities is bounded as above, we say that they are *proportional by a factor of K*. In that sense, the volume of a ball of radius r

in hyperbolic n-space is proportional to $\int_0^r \sinh^{n-1}(x)\,dx$. If we let z_1 be the child of the root in $\mathfrak{T}_r(1)$ the volume of $B(z_1; r) \subset \mathfrak{T}_r(1)$ is proportional to r, and for z_4 the child of the root in $\mathfrak{T}_r(4^n)$, the volume of $B(z_4, r) \subset \mathfrak{T}_r(4^n)$ is proportional to $4^{nr} + r$. In particular, given any $K > 0$, the volumes of $B(z_1, r), B(z_4, r)$ and $B(z, r) \subset \mathbb{H}^n$ are pairwise *not* proportional by a factor of K if r is chosen sufficiently large.

Proof of Lemma 4.3.2. Let $\varepsilon > 0$ be given and suppose that there exists a metric space K of diameter at most $4r^6$ and maps $f_\alpha, f_\beta \colon K_i \longrightarrow (M, g)$ which are $(1+\varepsilon, \varepsilon, \varepsilon)$-quasi-isometries to neighbourhoods of $N(z_i, r^6, \mathbf{j})$ and $N(z_j, r^6, \mathbf{j'})$, respectively. Denote these neighbourhoods by N and N' and let $f_\alpha^{-1}, f_\beta^{-1} \colon N, N' \longrightarrow K$ be the respective quasi-isometric inverses of f_α, f_β. Then $f_\beta \circ f_\alpha^{-1} \colon N \longrightarrow N'$ and $f_\alpha \circ f_\beta^{-1} \colon N' \longrightarrow N$ are (λ, C, D)-quasi-isometries with λ, C, D depending only on ε. By the construction of (M, g), the distance between the $N(z_i, r^6, \mathbf{j})$ increases exponentially in r, while the diameter of the $N(z_i, r^6, \mathbf{j})$ increases only polynomially. Since N, N' can have diameter at most $(1+\varepsilon)4r^6 + 2\varepsilon$, for sufficiently large r and $\varepsilon > 0$ fixed, the neighbourhood N contains only the building block $N(z_i, r^6, \mathbf{j})$, and likewise N' contains only $N(z_j, r^6, \mathbf{j'})$.

Now consider the restriction of $f_\beta \circ f_\alpha^{-1}$ to $N(z_i, r^6, \mathbf{j})$. Since the volumes of metric balls in hyperbolic space, $\mathfrak{T}_r(1)$ and $\mathfrak{T}_r(4)$ are pairwise not proportional, we will show - using the volume control given by Lemma 4.3.3 - that each copy of $\mathfrak{T}_r(4^n)$ in $N(z_i, r^6, \mathbf{j})$ has to be mapped closely to some $\mathfrak{T}_r(4^n)$ in $N(z_j, r^6, \mathbf{j'})$, and analogously for $\mathfrak{T}_r(1)$. From this it will follow that the $\mathfrak{T}_r(4^n)$s and the $\mathfrak{T}_r(1)$s appear in the same pattern in both $N(z_i, r^6, \mathbf{j})$ and $N(z_j, r^6, \mathbf{j'})$ and hence $\mathbf{j} = \mathbf{j'}$.

Consider some copy of $\mathfrak{T}_r(4^n)$ in $N(z_i, r^6, \mathbf{j})$ and set $A := B(z_4, r)$. Then there exists a constant $K_1 = K_1(\lambda, C, D)$ independent of r such that $\mathrm{vol}_g(B_1(A))$ is proportional to $4^{nr} + r$ by a factor of K_1. Moreover, since f is a (λ, C, D)-quasi-isometry, one easily computes that

$$f(A) \subset B\left(f(z_4), \lambda r + D\right)$$

and hence $\mathrm{vol}\left(B_1(f(A))\right) \leq \mathrm{vol}\left(B(f(z_4), \lambda r + D + 1\right)$. We now want to use this, to show that $f(z_4)$ has to be close to some copy of $\mathfrak{T}_r(4^n)$.

For this purpose consider the neighbourhood C_4 in N' of all points being at most distance $\lambda r + D + 1$ away from some copy of $\mathfrak{T}_r(4^n)$. Then for any $p \notin C_4$ the ball $B(p, \lambda r + D + 1)$ does not intersect any $\mathfrak{T}_r(4^n)$ and hence its volume is at most that of a ball with the same radius in hyperbolic space.

We claim that $f(z_4) \in C_4$. Suppose not. Then we have the following inequalities

$$\begin{aligned}
\frac{\mathrm{vol}_g\left(B_1(A)\right)}{\mathrm{vol}_g\left(B_1(f(A))\right)} &\geq \frac{\mathrm{vol}_g(B_1(A))}{\mathrm{vol}_g\left(B_{\lambda r+D+1}(f(z_4))\right)} \\
&\geq \frac{1}{K_1}\frac{4^{nr}+r}{\mathrm{vol}_{\mathbb{H}^n}(\lambda r+D+1)} \\
&\stackrel{r\to\infty}{\longrightarrow} \infty,
\end{aligned}$$

where $\mathrm{vol}_{\mathbb{H}^n}(R)$ denotes the volume of a ball of radius R in $\mathbb{H}^n$. Hence for sufficiently large r the volumes of $B_1(A)$ and $B_1(f(A))$ aren't proportional by the factor of $K(\lambda, C, D, 1)$ given by Lemma 4.3.3 if $f(z_4)$ does not lie in C_4. Hence $f(z_4) \in C_4$.

Since the $T^-(y_s, r^4, k)$ and hence the copies of $\mathfrak{T}_r(4^n)$ have distance at least r^5 from another, the neighbourhood C_4 is the disjoint union of neighbourhoods of the $\mathfrak{T}_r(4^n)$ and moreover, two distinct copies of $\mathfrak{T}_r(4^n)$ cannot be mapped into the neighbourhood of the same $\mathfrak{T}_r(4^n)$ in $N(z_j, r^6, \mathbf{j'})$ if r is large. Thus $N(z_j, r^6, \mathbf{j'})$ contains at least as many $\mathfrak{T}_r(4^n)$ as $N(z_i, r^6, \mathbf{j})$. Applying the same reasoning as above to $f_\beta \circ f_\beta^{-1}$, we find that they contain equally many $\mathfrak{T}_r(4^n)$. Hence $\mathbf{j}$ and $\mathbf{j'}$ map to $-$ (and hence to $+$) equally often.

It remains to be shown that $\mathbf{j}(k) = \mathbf{j'}(k)$. To this aim assume that $\mathbf{j}(k) = -$ and restrict $f_\alpha \circ f_\beta^{-1}$ to $T^-(y_s, r^4, k) \subset N(z_i, r^6, \mathbf{j})$. Let $z_4 \in \mathfrak{T}_r(4^n) \subset T^-(y_s, r^4, k)$ be chosen as above. We have already seen that $f(z_4)$ lies in a neighbourhood of some $\mathfrak{T}_r(4^n)$ which belongs to some $T^-(y'_s, r^4, m)$. We have to show that $m = k$, that is that $T^-(y'_s)$ contains as many copies of $\mathfrak{T}_r(1)$ as $T^-(y_s)$. But the volume of balls $B(z_1, r)$ in $\mathfrak{T}_r(1)$ is proportional to r, while volumes in $\mathbb{H}^n$ and $\mathfrak{T}_r(4^n)$ are exponential. Hence the same argument as above shows that for large r each $\mathfrak{T}_r(1) \subset T^-(y_i)$ is mapped closely to some $\mathfrak{T}_r(1) \subset T^-(y'_s)$ and hence $m = k$.

Hence, for fixed $\varepsilon > 0$ and sufficiently large r, any coarse $(\varepsilon, 4r^6)$-quasi-tiling contains at least as many distinct metric spaces K_i as there building blocks $N(z, r^6, \mathbf{j})$, that is $H^{cs}(M, g, \varepsilon, r^6) \geq 2^{d(r)-1}$.

It is now a plain computation to show that the coarse geometric entropy of (M, g) is infinite:

$$\begin{aligned}
h_g^{cs}(M,g) &= \lim_{\varepsilon\to\infty}\limsup_{r\to\infty}\frac{\log\left(H^{cs}(M,g,\varepsilon,4r^6)\right)}{4r^6} \\
&\geq \lim_{\varepsilon\to\infty}\limsup_{r\to\infty}\frac{\log 2^{d(r)-1}}{4r^6} \\
&= \limsup_{r\to\infty}\frac{(d(r)-1)\log 2}{4r^6} \\
&= \infty.
\end{aligned}$$

This finishes the proof of Lemma 4.3.2. □

Proof of Theorem 4.3.1. Since the coarse geometric entropy of (M,g) is infinite, Lemma 2.3.16 implies that (M,g) cannot be coarsely quasi-isometric to a simply connected leaf in neither a codimension 1 foliation of class $C^{1,0}$ nor a C^1-foliation of any codimension.

To see that (M,g) has the same coarse homology as $\mathbb{H}^n$, we note that (M,g) is quasi-isometric to $\mathbb{H}^n$ with the $\mathfrak{T}_r(1), \mathfrak{T}_r(4^n), r \in \mathbb{N}$, added at the same places as in (M,g). (That is, instead of *replacing* balls $B(y;1)$ by $\mathfrak{T}_r(1)$ and $\mathfrak{T}_r(4^n)$, we have just glued their boundaries to the boundaries of the balls.) Denote this space by $(\mathfrak{M}, d)$, where d is the path-metric induced from the metrics on $\mathbb{H}^n$ and on the $\mathfrak{T}_r(1), \mathfrak{T}_r(4^n)$. Denote the disjoint union of the $\mathfrak{T}_r(1), \mathfrak{T}_r(4^n)$ in $\mathfrak{M}$ by $\mathfrak{T}$ and note that $\mathfrak{T}$ is quasi-isometric to the corresponding disjoint union of trees $\mathcal{T}_r(1)$ and $\mathcal{T}_r(4^n)$, which we denote by $\mathcal{T}$. Let $\{y_\alpha\}_{\alpha\in A}$ be the set of all points where $B(y_\alpha;1)$ has been replaced by some $\mathfrak{T}_r(1)$ or $\mathfrak{T}_r(4^n)$ in (M,g). The balls $B(y_\alpha;1), \alpha \in A$ form a locally finite open covering of $\{y_\alpha\}_{\alpha\in A}$, which yields a coarsening sequence $R_j(\{y_{\alpha\in A}\})$. We extend this covering to a locally finite open covering of $\mathcal{T}$ by adding balls of radius 1 around each vertex in the $\mathcal{T}_r(1)$ and $\mathcal{T}_r(4^n)$. Denote the corresponding coarsening sequence by $R_j(\mathcal{T})$. Note that $\mathcal{T}_r(1)$ and $\mathcal{T}_r(4^n)$ properly deformation retract to their roots. We can extend these deformation retractions simplicially to $R_j(\mathcal{T})$ and thus get a proper deformation retraction of $R_j(\mathcal{T})$ to $R_j(\{y_\alpha\}_{\alpha\in A})$. Consequently $H_k^{lf}\left((R_j(\mathcal{T})\right) \simeq H_k^{lf}\left(R_j(\{y_\alpha\}_{\alpha\in A})\right)$ and we conclude that $HX_k(\mathcal{T}) \simeq HX_k(\{y_\alpha\}_{\alpha\in A})$.

Note that the decomposition $\mathfrak{M} = \mathbb{H}^n \cup \mathfrak{T}$ is coarsely excisive and $\mathbb{H}^n \cap \mathfrak{T}$ is quasi-isometric to $\{y_\alpha\}_{\alpha\in A}$ implying $HX_k(\mathbb{H}^n \cap \mathfrak{T}) \simeq HX_k(\{y_\alpha\}_{\alpha\in A})$. Then the map

$$HX_k(\mathbb{H}^n \cap \mathfrak{T}) \simeq HX_k(\{y_\alpha\}_\alpha) \longrightarrow HX_k(\mathbb{H}^n) \oplus HX_k(\{y_\alpha\}_\alpha) \simeq HX_k(\mathbb{H}^n) \oplus HX_k(\mathfrak{T})$$

in the coarse Mayer-Vietoris sequence is injective and the long exact sequence splits into short exact sequences

$$0 \to HX_k(\{y_\alpha\}_{\alpha\in A}) \xrightarrow{\varphi} HX_k(\mathbb{H}^n) \oplus HX_k(\{y_\alpha\}_{\alpha\in A}) \longrightarrow HX_k(\mathfrak{M}, d) \to 0.$$

Since $HX_k(M,g) \simeq HX_k(\mathfrak{M}, d)$, we find that

$$HX_k(M,g) \simeq \left(HX_k(\mathbb{H}^n) \oplus HX_k(\{y_\alpha\}_{\alpha\in A})\right)/\varphi(HX_k(\{y_\alpha\}_{\alpha\in A})) \simeq HX_k(\mathbb{H}^n).$$

This finishes the proof of Theorem 4.3.1, since $HX_k(\mathbb{H}^n)$ vanishes for all k but $k = n$. □

Chapter 5

Leaves with non-finitely generated coarse homology

In this chapter we show that there exist Riemannian manifolds which are isometric to a leaf in a foliation of a compact manifold whose coarse homology is non-finitely generated. This, together with the preceding chapther, where we showed that there exist non-leaves with trivial coarse homology, concludes the discussion whether the coarse homology can be used to decide whether a given Riemannian manifold can be quasi-isometric to a leaf.

Theorem 5.0.4. *In every dimension greater than or equal to* 2 *there exist Riemannian manifolds L with $HX_1(L)$ containing an Abelian subgroup of infinite rank, such that L is isometric to a leaf of a foliation of a compact manifold of any codimension.*

This will follow from the fact that there exist foliations of compact manifolds with leaves that have infinitely many ends. We will show that the degree 1 coarse homology of such leaves with any metric induced from the foliated manifold is non-finitely generated. This follows from the following proposition according to which k distinct ends in a proper length space span a free Abelian subgroup of rank $k-1$ in the degree 1 coarse homology. (Cf. Prop. 2.25, [Roe1] for the analogous statement for coarse cohomology.)

Proposition 5.0.5. *Let (X, d) be a proper connected length space with $k \in \mathbb{N} \cup \{\infty\}$ ends. Then $HX_1((X, d); \mathbb{Z})$ contains a subgroup isomorphic to $\oplus_{i=1}^{k-1}\mathbb{Z}$.*

Note that not all elements in the degree one coarse homology originate from the ends of the space. Take for example X to be the 1-dimensional balloon space from [Ha-Ko-Roe-Sch], then X has just one end, but the coarse homology $HX_1(X) = \prod_j \mathbb{Z}/\oplus_j \mathbb{Z}$ is not even finitely generated.

The proof of Proposition 5.0.5 will occupy the remainder of this chapter. Before, we show how the proposition implies Theorem 5.0.4:

Proof of Theorem 5.0.4. By a result of Cantwell and Conlon (Theorem A, [Cant-Co1]) every compact, totally disconnected, metrizable space E can be realized as the endspace of a 2-dimensional leaf Σ in a codimension one foliation of a compact 3-manifold M. In particular there exist surfaces with infinitely many ends which are diffeomorphic to a leaf in a foliation of a compact 3-manifold. (This can also be seen more elementary by turbulizing a linear foliation of T^3 by dense cylinders (Example 4.3.10, [Cand-Con].) Simply by taking products $M \times N^d \times N^c$ with compact manifolds and considering the leaf $\Sigma \times N^d$, we find that in every leaf-dimension at least 2 and every codimension at least 1, there exist leaves in foliations of compact manifolds with leaves that have infinitely many ends.

Let $(M, \mathscr{F})$ be such a foliation and L be a leaf of $\mathscr{F}$ with infinitely many ends. Then every metric g on M induces a complete metric $\iota^* g$ on L. In particular $(L, \iota^* g)$ is a proper length space with infinitely many ends. Hence we can apply Proposition 5.0.5 to see that $HX_1(L, \iota^* g)$ contains a free Abelian subgroup of infinite rank and hence cannot be finitely generated. $\square$

5.1 Ends of a topological space

Let X be a topological space. By an *end of* X we mean the equivalence class of a proper ray $r\colon [0, \infty) \longrightarrow X$, where two rays r, r' are equivalent if for each compact subset $K \subset X$ there exist a $t_K > 0$ such that $r([t_K, \infty))$ and $r'(t_K, \infty))$ lie in the same path component of $X \backslash K$. We denote by $\mathcal{E}(X)$ the set of ends and let $e(X)$ be the cardinality of $\mathcal{E}(X)$.

By a *length space* we mean a metric space, in which the distance between two points is given by the infimum of the lengths of rectifiable curves between them. We call a metric space *proper* if its closed and bounded subsets are compact. Note that by the Hopf-Rinow theorem (see [Br-Hae]) proper length spaces are *geodesic spaces*, that is any two points can be joined by a geodesic. We will henceforth assume our spaces to be proper geodesic spaces.

The number of ends is in general not a quasi-isometry invariant of metric spaces. Let for example $X = (\mathbb{Q} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\})$ with the metric induced from $\mathbb{R}^2$. Then X is a connected metric space and there exists a bijection from $\mathcal{E}(X)$ to $\mathbb{Q} \cup \mathbb{Q}$. But X is quasi-isometric to $\mathbb{R}^2$ which has only one end. For proper geodesic spaces, this situation does not occur.

Lemma 5.1.1 (8.29 Proposition, [Br-Hae])**.** *For proper geodesic spaces X and Y, every quasi-isometry $f\colon X \longrightarrow Y$ induces a bijection $\mathcal{E}(X) \longrightarrow \mathcal{E}(Y)$.*

It turns out that if f is just the isometric embedding of a subspace, the induced map on the end spaces can be chosen as the embedding of rays.

Lemma 5.1.2. *Let* (X, d) *be a proper geodesic space. Then any coarsening of* X *equipped with the metric induced by the Euclidean metric on the simplices is quasi-isometric to* (X, D).

Proof. Let $\mathcal{U}$ be a (locally finite) open covering of X with Lebesgue number $\lambda > 0$ and such that the diameter of the sets in $\mathcal{U}$ is bounded by some constant $r > 0$. For each $U \in \mathcal{U}$ choose $x_U \in U$. Then $\{x_U\}_{U \in \mathcal{U}} \subset X$ is r-dense and hence quasi-isometric to X. We claim moreover that the map that assigns to each x_U the vertex (U) in $|\mathcal{U}|$ is a quasi-isometry if we equip $|U|$ with the metric $d_{\mathcal{U}}$ induced by the Euclidean distance function on the simplices. But the image of the x_U is 1-dense in $|\mathcal{U}|$ and we claim that

$$\lambda \cdot d_{\mathcal{U}}\left((U), (V)\right) - 3\lambda \leq d_X(x_U, x_V) \leq r \cdot d_{\mathcal{U}}\left((U), (V)\right) + r.$$

For let $d_{\mathcal{U}}((U), (V)) = k$, then there exist a chain of sets $U_0 = U, U_1, \ldots, U_k = V$ in $\mathcal{U}$ such that $U_i \cap U_{i+1} \neq \emptyset$. But then

$$d_X(x_U, x_V) \leq r(k+1) = r \cdot d((U), (V)) + r.$$

For the other inequality, let $\gamma\colon [0,1] \longrightarrow X$ be a geodesic from x_U to x_V and let $t_0 = 0 < \ldots < t_s = 1$ be a subdivision of $[0,1]$ such that $d_X(\gamma(t_i), \gamma(t_{i+1})) = \lambda$ for $\lambda = 0, \ldots, s-2$ and $d_X(\gamma(t_{s-1}), \gamma(t_s)) \leq \lambda$. Since γ is a geodesic, we find that $s \leq \frac{d_X(x_U, x_V)}{\lambda} + 1$ and since $\operatorname{diam}(\gamma([t_i, t_{i+1}])) \leq \lambda$ for $i = 0, \ldots, s-1$, there exist $U_i \in \mathcal{U}$ with $\gamma([t_i, t_{i+1}]) \subset U_i$. Hence the chain of edges $(U, U_0), (U_0, U_1), \ldots, (U_{s-1}, U_s), (U_s, V)$ furnishes a path in $|\mathcal{U}|$ of length $s + 2$ from (U) to (V) and

$$d_{\mathcal{U}}((U), (V)) \leq s + 2 \leq \frac{d_X(x_U, x_V)}{\lambda} + 3.$$

We arrive at the inequality $\lambda \cdot d_{\mathcal{U}}((U), (V)) - 3\lambda \leq d(x_U, x_V)$. This proves the other inequality and hence $\{x_U\}_{U \in \mathcal{U}}$ is quasi-isometric to $|\mathcal{U}|$. By transitivity of the quasi-isometry relation, X is also quasi-isometric to $|\mathcal{U}|$. □

While true for any coarsening sequence, it is completely clear that the embedding $R_i(X) \longrightarrow R_{i+1}(X)$ is a quasi-isometry, since it is just a 1-dense isometric embedding. We summarize the above discussion in the following remark.

Remark 5.1.3. Let $\mathcal{U}_1, \mathcal{U}_2, \ldots$ be a coarsening sequence for X, then the above lemma implies that $e(X) = e(|\mathcal{U}_i|)$. That is, by coarsening a length space we do not create or loose ends. Moreover, for $|U_i| = R_i(X)$ the bijections $\mathcal{E}(R_i(X)) \longrightarrow \mathcal{E}(R_{i+1}(X))$ are induced by the inclusion of rays from $R_i(X)$ into $R_{i+1}(X)$.

5.2 Proof of Proposition 5.0.5

We will prove Proposition 5.0.5 as follows. First, we will show how two distinct ends of a space give rise to a locally finite 1-chain and when we are dealing with a locally finite simplicial complex S, these chains will generate a free Abelian subgroup on $e(S)-1$ generators in the degree 1 locally finite homology. If we let X be a proper geodesic space, Remark 5.1.3 implies that $e(X)=e(R_i(X))$ and hence all locally finite homology groups $H_1^{lf}(R_i(X))$ contain a free rank $e(X)-1$ Abelian subgroup. It will be easy to see, that these subgroups map to each other under the coarsening maps and hence $HX_1(X)$ also contains a free rank $e(X)-1$ Abelian subgroup. In particular, for infinitely many ends, $HX_1(X)$ cannot be finitely generated.

Lemma 5.2.1. *Let S be a locally finite simplicial complex with $k\in\mathbb{N}\cup\{\infty\}$ ends. Then $H_1^{lf}(S;\mathbb{Z})$ contains a subgroup isomorphic to $\oplus_{j=1}^{k-1}\mathbb{Z}$.*

Proof. Let $r_1, r_2, \ldots$ be the ends of S. Then $\sum_n r_j|_{[n,n+1]}$ can be viewed as a 1-chain, which we again denote by r_j. These chains are locally finite because the maps $r_j\colon [0,\infty)\longrightarrow$ are proper. (Let $K\subset S$ be compact. Then $r_j^{-1}(K)$ is compact and hence bounded. Thus only finitely many $r_j|_{[n,n+1]}$ intersect K.)

Without loss of generality, we may assume that all rays r_j emanate from the same point $x\in S$. Then $z_{j,j+1}:=-r_j+r_{j+1}$ are locally finite 1-cycles for all $j=1,\ldots,k-1$. We claim that the $z_{j,j+1}$ generate a free Abelian subgroup of rank $k-1$ in $H_1^{lf}(S)$.

Suppose first that S has finitely many ends. Then there exists a compact path-connected set $K\subset S$ such that for $t>t_K$ the $r_j|_{[t_K,\infty)}$ lie in distinct components of $S\setminus K$. Let $K'\supset K$ be such that K' properly deformation retracts onto K and such that $S=\operatorname{int}(K')\cup S\setminus K$. (This is possible, because S is a locally finite simplicial complex.) By adding compact sets, K can be chosen such that $S\setminus K$ has exactly k components. Now apply the locally finite Mayer-Vietoris sequence to $(K', \overline{S\backslash K})$.

$$\ldots\longrightarrow H_1^{lf}(S)\xrightarrow{\partial} H_0^{lf}(\overline{S\backslash K}\cap K')\xrightarrow{\varphi} H_0^{lf}(\overline{S\backslash K})\oplus H_0^{lf}(K')\longrightarrow\ldots$$

Then $\overline{S\backslash K}\cap K'$ is compact and has k components, corresponding to the r_j, $j=1,\ldots,k$, and clearly $H_0^{lf}(\overline{S\backslash K}\cap K')=H_0(\overline{S\backslash K}\cap K')=\oplus_{j=1}^{k}\mathbb{Z}$. Moreover, we can choose the rays r_j so that each factor in $H_0^{lf}(\overline{S\backslash K}\cap K')$ is generated by a point of the form $r_j(n_j)$ for some $n_j\in\mathbb{N}$.

The first component of φ is the zero map since $r_j(n_j)=\partial\big(\sum_{n\geq n_i} r_j|_{[n,n+1]}\big)$; the second component of φ is the map $(m_1,\ldots,m_k)\mapsto m_1+\ldots+m_k$. Hence $\operatorname{im}(\partial)=\ker(\varphi)=\oplus_{j=1}^{k-1}\mathbb{Z}\{r_{j+1}(n_{j+1})-r_j(n_j)\}$.

Recall that the boundary map of the Mayer-Vietoris sequence is defined by subdividing 1-chains on S into a sum of 1-chains on K' and on $S\backslash K$ and mapping to the boundary of either summand. Since $z_{j,j+1}$ can be decomposed as

$$\Big(-\sum_{n<n_j} r_j|_{[n,n+1]} + \sum_{n<n_{j+1}} r_{j+1}|_{[n,n+1]}\Big) + \Big(-\sum_{n\geq n_j} r_j|_{[n,n+1]} + \sum_{n\geq n_{j+1}} r_{j+1}|_{[n,n+1]}\Big),$$

where the first summand lies in K' and the second in $S \setminus K$, we find that

$$r_{j+1}(n_{j+1}) - r_j(n_j) = \partial z_{j,j+1}.$$

In particular, the $z_{j,j+1}$ map to a basis of $\ker\varphi$ and thus generate a free Abelian subgroup of rank $k-1$ in $H_1^{lf}(S)$. This proves the claim for $k < \infty$.

If $k = \infty$, we can choose $K_1 \subset K_2 \subset \dots$ such that $r_1|_{[t_{K_l},\infty)}, \dots, r_l|_{[t_{K_l},\infty)}$ lie in distinct components of $S \setminus K_l$. Then the case $k < \infty$ shows that $z_{j,j+1}, j = 1, \dots, l-1$, generate a free Abelian subgroup of rank $l-1$ in $H_1^{lf}(S)$ yielding an increasing sequence

$$\mathbb{Z}\{z_{1,2}\} \;\subset\; \mathbb{Z}\{z_{1,2}\}\oplus\mathbb{Z}\{z_{2,3}\} \;\subset \dots \subset\; \bigoplus_{j=1}^{l-1}\mathbb{Z}\{z_{j,j+1}\} \;\subset \dots \subset H_1^{lf}(S;\mathbb{Z}).$$

Thus $\oplus_{j=1}^{\infty}\mathbb{Z}\{z_{j,j+1}\} \subset H_1^{lf}(S;\mathbb{Z})$. □

The proof of Proposition 5.0.5 is now an easy consequence of the above facts.

Proof of Proposition 5.0.5. Recall that for a direct system of groups $G_1 \longrightarrow G_2 \longrightarrow \dots$, elements $\alpha_1, \dots, \alpha_n \in \varinjlim G_i$ generate a rank n Abelian subgroup in $\varinjlim G_i$ if for all sufficiently large i, the representatives $a_j \in G_i$ of the α_j generate a rank n Abelian subgroup in G_i.

For simplicity and geometric clearness, we take $\{R_i(X)\}_i$ as coarsenings of X and compute $HX_1(X)$ via $\varinjlim H_1^{lf}(R_i(X))$. By Remark 5.1.3 the $R_i(X)$ all have $k = e(X)$ ends and Lemma 5.2.1 then shows that the degree one locally finite homology groups $H_1^{lf}(R_i(X)), i \geq 1$, each contain a rank $k-1$ free Abelian subgroup. Moreover, we have seen that the coarsening maps $R_i(X) \hookrightarrow R_{i+1}(X)$ map the ends of $R_i(X)$ to the ends of $R_{i+1}(X)$ and hence the generators of the rank $k-1$ free Abelian subgroup in $H_1^{lf}(R_i(X))$ constructed in Lemma 5.2.1 to the generators of the respective subgroup in $H_1^{lf}(R_{i+1}(X))$. Thus the equivalence classes $[z_{j,j+1}] \in \varinjlim H_1^{lf}(R_i(X)) = HX_1(X)$ generate a free Abelian subgroup of rank $k-1$. □

Chapter 6

Independence of leaf criteria

This final chapter concludes the discussion about the independence of the criteria of Schweitzer and Attie-Hurder and the finite generation of the coarse homology. We show that all these three properties are independent of one another. Moreover, the balloon metrics constructed by Schweitzer suggest that every manifold not satisfying the bounded homology property has vanishing Cheeger constant. We confirm this expectation under the assumption of bounded geometry and give counterexamples with unbounded curvature.

6.1 Independence of bounded homology property and coarse homology

We have seen that there exist Riemannian manifolds that do not satisfy the bounded homology property and which are hence not diffeomorphically quasi-isometric to leaves, which have non-finitely generated coarse homology (Section 4.1) and those with trivial coarse homology (Section 4.2). The following section concludes the discussion of the independence of the bounded homology property and the number of generators of the coarse homology by proving that also the converse is true:

Proposition 6.1.1. *There exist simply connected Riemannian manifolds satisfying the bounded homology property whose coarse homology is trivial and those whose coarse homology is non-finitely generated.*

Proof. Consider the one-ended cylinder $M := (\partial D^n \times \mathbb{R}_{\geq 0}) \cup (D^n \times \{0\})$ equipped with the induced metric from $\mathbb{R}^{n+1}$. Then M is quasi-isometric to $\mathbb{R}_{\geq 0}$ which has trivial coarse homology. Moreover, it is easily seen that M satisfies the bounded homology property.

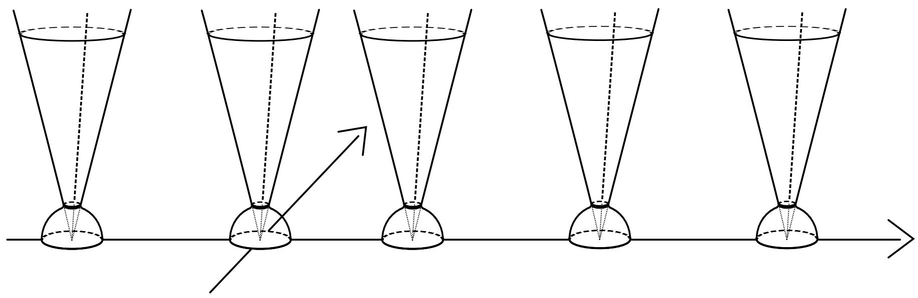

Figure 6.1: $\mathbb{R}^n$ with attached truncated cones $\mathscr{T}(B)$

In order to find a Riemannian manifold with non-finitely generated coarse homology which satisfies the bounded homology property, we start with $\mathbb{R}^n$ with the Euclidean metric. Let B be the boundary of a small disk around the north pole in $S^n(\frac{1}{4})$ and let $\mathscr{T}(B)$ be the truncated cone over B (see Definition 3.3.2). Replace the balls of radius $\frac{1}{4}$ around each $(k, \mathbf{0}) \in \mathbb{R}^n, k \in \mathbb{Z}$ by a copy of $\mathscr{T}(Y)$ and smoothly interpolate between the Euclidean metric and the metric on $\partial \mathscr{T}(B)$. Since $\mathscr{T}(B)$ is diffeomorphic to a disc with a point removed, the resulting manifold M is diffeomorphic to $\mathbb{R}^n \setminus \mathbb{Z}$ and we denote the induced smoothed Riemannian metric on M by g. For brevity, we denote by $\mathscr{T}_k$ the truncated cone $\mathscr{T}(Y)$ glued in at $(k, \mathbf{0})$.

We claim that $HX_n(M,g) \simeq \mathbb{Z} \oplus \bigoplus_k \mathbb{Z}\{\mathscr{T}_k\}$ and that (M,g) satisfies the bounded homology property.

To see that (M,g) satisfies the bounded homology property, let C be a compact, connected and simply connected codimension 0 submanifold of $\mathbb{R}^n \setminus \mathbb{Z} \times \{\mathbf{0}\}$ and let $\beta > 0$. We will show that there exists a constant $\widetilde{K}(\mathrm{vol}_\beta(\partial C), \beta)$ such that $\mathrm{MVol}_\beta(C) \leq \widetilde{K}$. To this aim consider the distance function $\mathrm{dist}(\cdot, \partial C) \colon C \to [0, \infty)$. Then for every $\varepsilon > 0$ there exists a Morse function $f \colon C \to [0,\infty)$ such that $f|_{\partial C} \equiv 0$ and $\| f - \mathrm{dist}(\cdot, \partial C)\| < \varepsilon$ and by the definition of the Morse-β-volume

$$\mathrm{MVol}_\beta(C) \leq \sup_{t \geq 0} \mathrm{vol}_\beta \left(f^{-1}(t) \right).$$

Let $\mathrm{vol}_\beta(\partial C) = k$ and $\partial C \subset B(x_1, \beta) \cup \ldots \cup B(x_k, \beta)$. Then by the triangle inequality

$$f^{-1}(t) \subset \cup_{i=1}^k B(x_i; \beta + t + \varepsilon).$$

holds for every $t \geq 0$. That is $\mathrm{vol}_{t+\beta+\varepsilon}(f^{-1}(t)) \leq \mathrm{vol}_\beta(\partial C)$ for all $t \geq 0$. But since (M,g) has bounded geometry, every ball of radius $(\beta + t + \varepsilon)$ in M can be covered by $K(\beta + t + \varepsilon, \beta)$ many balls of radius β, that is

$$\mathrm{vol}_\beta(f^{-1}(t)) \leq K(\beta + t + \varepsilon, \beta)\, \mathrm{vol}_\beta(\partial C).$$

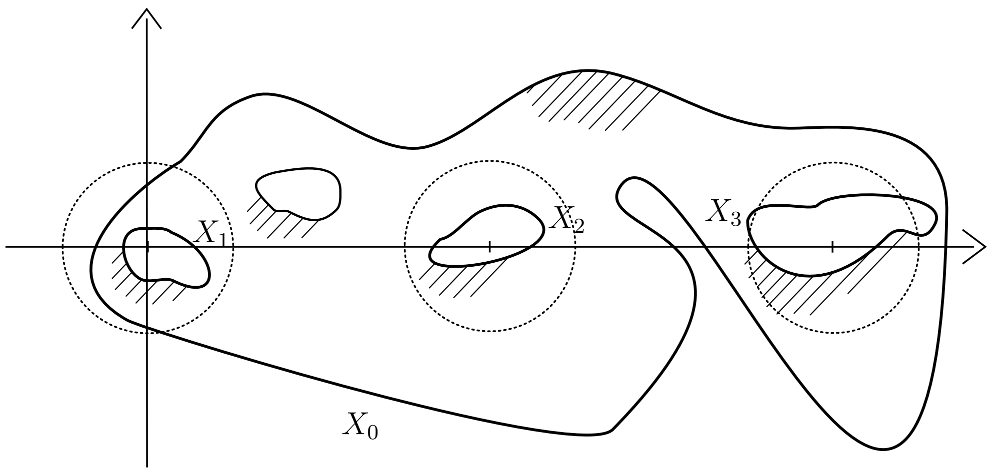

Figure 6.2: The relevant boundary components of C.

Note though, that $K(\beta + t + \varepsilon, \beta)$ goes to infinity as t does. It thus remains to show that we can control $\operatorname{diam}(C)$ by means of $\operatorname{vol}_\beta(\partial C)$. For suppose that there exists a constant $c(\operatorname{vol}_\beta(\partial C), \beta)$ such that $\operatorname{diam}(C) \leq c$, then

$$\operatorname{MVol}_\beta(C) \leq K(\beta + c + \varepsilon, \beta) \operatorname{vol}_\beta(\partial C).$$

That is, all C with fixed $\operatorname{vol}_\beta(\partial C)$ have uniformly bounded Morse-β-volume.

First note that $\operatorname{diam}(C)$ is assumed for $p, q \in \partial C$. For let $p \in C$ and $q \in \operatorname{int}(C)$. Then there exists a neighbourhood $U \subset C$ of q. Suppose that p lies in the flat part of M, i.e. $q \notin \mathscr{T}_i$ for every i. If q also lies in the flat part, then $d(p, p+(1+\varepsilon)(q-p)) > d(p,q)$ and $p+(1+\varepsilon)(q-p) \in C$ for sufficiently small $\varepsilon > 0$. If $q = (t, y) \in \mathscr{T}_i, t \in \mathbb{R}, y \in B$, then $d(p, (t+\varepsilon, y)) > d(p,q)$. A similar argument shows that if $p \in \mathscr{T}_k$ and $q \in \operatorname{int}C$, then $d(p,q)$ cannot be maximal.

If p, q lie in the same component X of ∂C, then we are done, since then $\operatorname{diam}(X) = \operatorname{diam}(C)$ and thus we need at least $\frac{\operatorname{diam}(C)}{2\beta}$ many balls of radius β to cover X, that is $\operatorname{diam}(C) \leq 2\beta \operatorname{vol}_\beta(X) \leq 2\beta \operatorname{vol}_\beta(\partial C)$.

Now suppose that $\operatorname{diam}(C)$ is assumed for p and q lying in distinct components of ∂C.

First note that there exists an "exterior component" X_0 of ∂C such that C lies in the bounded component of $\mathbb{R}^n \setminus X_0$. (We can prove the existence of X_0 by induction on the number of components of ∂C: First note that by Alexander-Lefshetz duality any component X_i of ∂C considered as a subset of $\mathbb{R}^n$ separates $\mathbb{R}^n$ into a bounded and an unbounded component. Now if ∂C has just one component, there is nothing to prove. Assume that such an exterior component always exists for C with n boundary components and let C' have $(n+1)$ boundary components. Let X be one of them. If C'

lies in the bounded component of $\mathbb{R}^n \setminus X$ we are done. Otherwise consider $C'' = C' \cup \{\text{the bounded component of } \mathbb{R}^n \setminus X\}$. Then C'' has n boundary components and there exists one, say Y, such that C'' and hence C' lies in the bounded component of $\mathbb{R}^n \setminus Y$.)

Whenever some $(k, \mathbf{0})$ lies in the bounded component of $\mathbb{R}^n \setminus X_0$, by compactness and connectedness of C, there exists exactly one corresponding "interior component" $X_i \neq X_0$ such that $(k, \mathbf{0})$ lies in the bounded component of $\mathbb{R}^n \setminus X_i$. Clearly, $\operatorname{diam}(C)$ is assumed for points in these components.

If X_i does not intersect any of the $\mathscr{T}_i$, then $\operatorname{diam}(C)$ is not assumed for one point lying in X_i. Thus suppose that $X_i \cap (\mathbb{R}_{\geq 1} \cdot B) \neq \emptyset$ and let h_i be the height of X_i in $\mathscr{T}_k$, i.e. h_i is the maximal real number $t \geq 1$ such that $t \cdot b \in X_i$ for some $b \in B$. Then there exists a constant $c > 0$ such that $h_i \leq c \operatorname{diam} X_i$. It is now clear that

$$\operatorname{diam}(C) \leq \operatorname{diam}(X_0) + 2(\max_i h_i + \frac{1}{8}\pi)$$

since for $p, q \in X_0$ we have $d(p,q) \leq \operatorname{diam}(X_0)$, for $p \in X_0, q \in X_j$ the inequality $d(p,q) \leq \operatorname{diam}(X_0) + (\max_i h_i + {}^1\!/\!_8\pi)$ holds and finally for $p \in X_j, q \in X_k$ we find that

$$d(p,q) \leq (\max_i h_i + {}^1\!/\!_8\pi) + \operatorname{diam}(X_0) + (\max_i h_i + {}^1\!/\!_8\pi).$$

Hence

$$\begin{aligned}
\operatorname{diam}(C) &\leq \operatorname{diam}(X_0) + 2\left(c \max_i \operatorname{diam}(X_i) + \frac{1}{8}\pi\right) \\
&\leq 2\beta \operatorname{vol}_\beta(X_0) + 2\left(2\beta c \max_i \operatorname{vol}_\beta(X_i) + \frac{1}{8}\pi\right) \\
&\leq 2\beta \operatorname{vol}_\beta(\partial C)(1 + 2c) + \frac{1}{2}\pi,
\end{aligned}$$

since we always have $\operatorname{diam}(Y) \leq 2\beta \operatorname{vol}_\beta(Y)$ for connected Y. Hence we can bound $\operatorname{diam}(C)$ by a constant depending only on β and $\operatorname{vol}_\beta(\partial C)$. This finishes the proof that M with the metric induced from the truncated cones satisfies the bounded homology property.

We now want to use the computational tools developed in Section 3.2 to show that

$$HX_n(M,g) \simeq \mathbb{Z} \times \prod_{k \in \mathbb{Z}} \mathbb{Z},$$

where the first factor is given by the image of the locally finite fundamental class of $\mathbb{R}^n$ and the factors indexed by the integers come from the cones $\mathscr{T}_k$.

Note first that (M, g) is quasi-isometric to $\mathbb{R}^n$ with open cones $\mathcal{O}(B)$ over B (see Section 3.3) attached to each $(k, \mathbf{0})$

$$X := \mathbb{R}^n \bigcup_{(k,0)\in\mathbb{Z}\times\{0\}} (\cup_i \mathcal{O}(B)),$$

that is $HX_*(M, g) \simeq HX_*(X)$. We again let $\mathcal{O}_k$ be the open cone attached to $(k, \mathbf{0})$. Since X is a uniformly contractible bounded geometry complex, Proposition 3.2.5 implies that $HX_*(M) \simeq H_*^{lf}(X)$. Now let A be a neighbourhood of $\cup_k \mathcal{O}_k \subset X$ that properly deformation retracts onto $\cup_k \mathcal{O}_k$ and B a neighbourhood of $\mathbb{R}^n \subset X$ that properly deformation retracts onto $\mathbb{R}^n$. Then $A \cap B$ is properly homotopic to $\cup_k \{(k, \mathbf{0})\}$. The locally finite Mayer-Vietoris sequence for A and B together with $H_n^{lf}(A) \simeq H_n^{lf}(\cup_k \mathcal{O}_k)$ and $H_n^{lf}(B) \simeq H_n^{lf}(\mathbb{R}^n)$ then takes the form

$$0 = H_n^{lf}(\cup_k\{(k,\mathbf{0})\}) \to H_n^{lf}(\cup_k \mathcal{O}_k) \times H_n^{lf}(\mathbb{R}^n) \to H_n^{lf}(X) \to H_{n-1}^{lf}(\cup_k\{(k,\mathbf{0})\}) = 0$$

and thus $H_n^{lf}(X) \simeq H_n^{lf}(\cup_k \mathcal{O}_k) \times H_n^{lf}(\mathbb{R}^n)$. Proposition 3.3.1 implies that $H_n^{lf}(\mathcal{O}_k) \simeq H_{n-1}(B) \simeq \mathbb{Z}$ and it follows that $H_n^{lf}(\cup_k \mathcal{O}_k) \simeq \prod_k \mathbb{Q}$. By giving $\mathbb{R}^n$ the structure of a Δ-complex and using Proposition 3.1.4, it is not difficult to see that $H_n^{lf}(\mathbb{R}^n) \simeq \mathbb{Z}$. We have now computed that $HX_n(M) \simeq \mathbb{Z} \times \prod_k \mathbb{Z}$ and thus the degree n locally finite homology of M is not finitely generated. □

Alternatively, we could have applied the coarse Mayer-Vietoris sequence from Proposition 3.2.2 to the decomposition $X = \mathbb{R}^n \cup (\cup_k \mathscr{T}_k)$. Then the intersection $\mathbb{R}^n \cap (\cup_k \mathscr{T}_k)$ is quasi-isometric to $\mathbb{R} \subset \mathbb{R}^n$ and we can argue as in Proposition 4.1.1.

Note also that M has infinitely many ends. By different argument than the above, we have shown in Chapter 5 that also $HX_1(M)$ is non-finitely generated.

6.2 Dimension 2: Bounded homotopy property and coarse homology

The corresponding statements of Chapter 5 and Section 6.1 also hold in dimension 2 for the bounded homotopy property. The arguments and constructions are somewhat simpler than in higher dimensions.

Proposition 6.2.1. *On $\mathbb{R}^2$ there exist metrics g_1, g_2 of bounded geometry both not satisfying the bounded homotopy property but such that $HX_*(\mathbb{R}^2, g_1)$ is trivial and $HX_*(\mathbb{R}^2, g_2)$ is not finitely generated.*

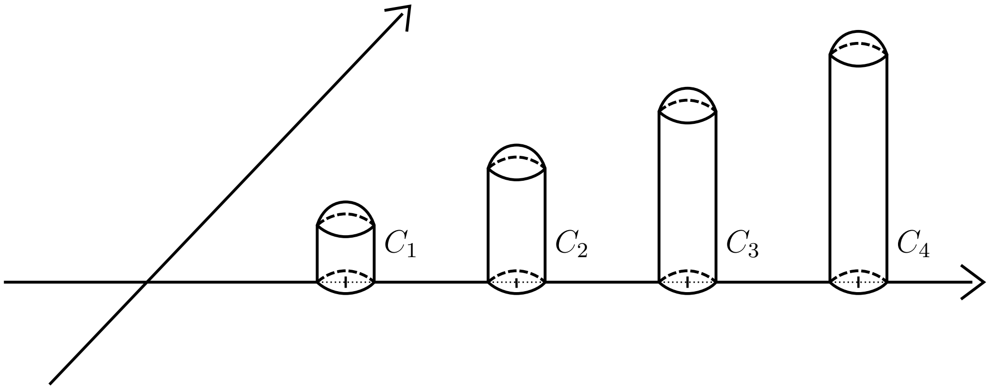

Figure 6.3: $\mathbb{R}^2$ with attached cylinders.

Proof. For an example with trivial coarse homology start with the one-sided cylinder $Z := (S^1 \times \mathbb{R}_{\geq 0}) \cup (D^2 \times \{0\})$ with the metric induced from $\mathbb{R}^3$ and replace the disks of radius $1/4$ around $(0, n) \in S^1 \times \mathbb{R}_{\geq 0}, n \in \mathbb{N}$, by capped off cylinders of height n, say $C_n := (S^1 \times [0, n]) \cup D^2$ (cf. the example of Section 5 in [Schw1]). We can smoothen the resulting space to be a Riemannian manifold (M, g_1) which is diffeomorphic to $\mathbb{R}^2$. A similar argument as in [Schw1] shows that (M, g_1) does not satisfy the bounded homotopy property.

We now determine the coarse homology groups of (M, g_1). Observe that (M, g_1) is quasi-isometric to $[0, \infty)$ with an interval $[0, n]$ attached to the points n for each $n \in \mathbb{N}$. Denote this space by M'. It is then not hard to see that M' is a uniformly contractible bounded geometry complex and hence Proposition 3.2.5 implies that

$$HX_*(M, g_1) \simeq HX_*(M') \simeq H_*^{lf}(M').$$

Moreover, M' is properly homotopic to $[0, \infty)$ and thus

$$HX_*(M, g_1) \simeq H_*^{lf}(M') \simeq H_*^{lf}([0, \infty)) \simeq \{0\}.$$

For an example with non-finitely generated coarse homology, just deform the Euclidean metric on $\mathbb{R}^2$ as described in [Schw1]. Then we have seen in Proposition 4.1.1 that the coarse homology of g contains $\prod_{j=1}^{\infty} \mathbb{Z} / \bigoplus_{j=1}^{\infty} \mathbb{Z}$. □

Proposition 6.2.2. *There exist surfaces satisfying the bounded homotopy property whose coarse homology is finitely generated and surfaces satisfying the bounded homotopy property whose coarse homology in degree* 2 *is non-finitely generated.*

Proof. For a surface with finitely generated coarse homology, simply take the Euclidean plane. $\mathbb{R}^2$ clearly satisfies the bounded homotopy property, since every loop $\gamma(t)$ is nullhomotopic via $\gamma_s(t) = (1-s)\gamma(t)$ and $l(\gamma_s) \leq l(\gamma)$. Its degree 2 coarse homology is isomorphic to $\mathbb{Z}$ in degree 2 and otherwise trivial.

For a surface with non-finitely generated coarse homology in degree 2, let M be as in Proposition 6.1.1, that M is $\mathbb{R}^2$ with truncated cones attached. Then we have already shown that $HX_2(M) \simeq \mathbb{Z} \times \prod_k \mathbb{Z}$. Moreover, M satisfies the bounded homotopy property, since every nullhomotopic loop in M is nullhomotopic via loops of lesser length. □

6.3 Independence of geometric entropy and bounded homology property

In Section 4.3 is was proven that there exist Riemannian manifolds having infinite coarse geometric entropy (and hence also infinite geometric entropy) with finitely generated coarse homology. These examples, though, do not satisfy the bounded homology property by the same argument as in the proof of Lemma 4.2.3.

In this section we want to show that the geometric entropy and the bounded homology property are independent conditions on a Riemannian manifold. This means that they both detect different properties and hence different ways in which a manifold cannot be diffeomorphically quasi-isometric to a leaf.

Proposition 6.3.1. *There exist Riemannian manifolds (M, g) of bounded geometry with $h_g^{cs}(M, g) = \infty$ that satisfy the bounded homology property and those that don't.*

Proof. A manifold with infinite coarse geometric entropy that does not satisfy the bounded homology property has been constructed in Section 4.3 by attaching spheres of increasing radius in an irregular pattern to $\mathbb{H}^n$. We can now adapt this example so that it satisfies the bounded homology property while retaining infinite coarse geometric entropy.

To this aim, we again start with $\mathbb{H}^n$, which satisfies the bounded homology property, and we modify the building blocks $\mathfrak{T}_r(4^n)$ from Section 4.3 that led to the violation of the bounded homology property: Instead of adding a cap D^n for each leaf of the tree $\mathcal{T}_r(4^n)$, we now glue an infinite cylinder $S^{n-1} \times \mathbb{R}_{\geq 0}$ to each vertex of $\mathcal{T}_r(4^n)$ leading to a leaf. Otherwise we leave $\mathfrak{T}_r(4^n)$ unchanged and call the new resulting manifolds with boundary $\mathfrak{E}_r(4^n)$. That is, we have added an end to $\mathfrak{T}_r(4^n)$ for each leaf of $\mathcal{T}_r(4^n)$. As an illustration, Figure 6.3 shows the analogous modification of $\mathfrak{T}_3$. As in Section 4.3 this yields building blocks which we again denote by $N(z, r^6, \mathbf{j})$ for every positive integer r and function $\mathbf{j}$. These blocks are again glued to $\mathbb{H}^n$ with increasing distance from each other,

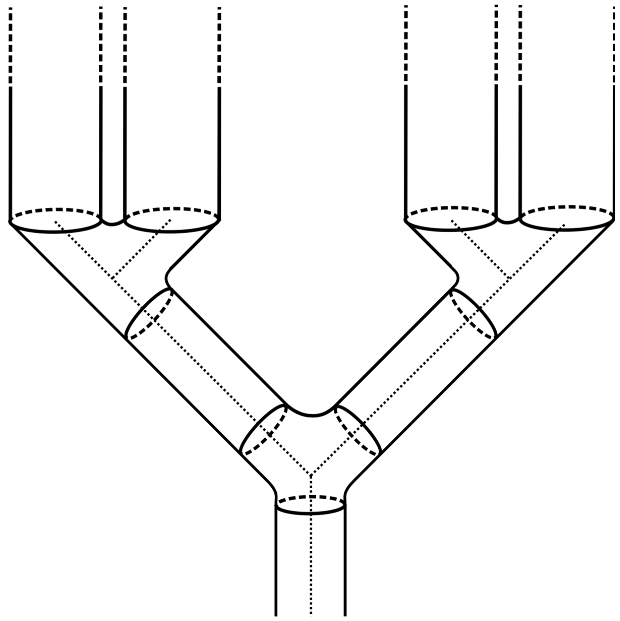

Figure 6.4: The manifold $\mathfrak{E}_3(2)$ with dotted tree $\mathcal{T}_3$.

as described in the proof of Theorem 4.3.1. Denote the resulting manifold by (M, g). The same arguments as in Section 4.3 show that $h_g^{cs}(M, g) = \infty$.

In order to see that (M, g) satisfies the bounded homology property, fix some $k, \beta > 0$ and let $C \subset (M, g)$ with $\mathrm{vol}_\beta(\partial C) \leq k$ be a compact codimension 0 submanifold. We have to show that there exists a constant $K(\beta, k)$ such that $\mathrm{MVol}_\beta(C) \leq K(\beta, k)$: If C does not intersect any of the $\mathfrak{E}_r(4^n)$, then by the fact that $\mathbb{H}^n$ satisfies the bounded homology property, there exists some $K'(\beta, k)$ such that $\mathrm{MVol}_\beta(C) \leq K'(\beta, k)$.

Thus assume that C lies in one of the $\mathfrak{E}_r(4^n)$. Since ∂C can be covered by at most k balls of radius β, there exists a constant $j = j(k, \beta)$ depending only on k and β such that C lies in a subtree $\mathfrak{E}_j(4^n)$ of height j of $\mathfrak{E}_r(4^n)$, which may have its root at some arbitrary vertex of $\mathfrak{E}_r(4^n)$. Let $h\colon \mathfrak{E}_j(4^n) \longrightarrow [0, \infty)$ be the height function. Then $\mathrm{vol}_\beta(h^{-1}(t)) \leq c(\beta) \cdot 4^{nj}$ for every $t \geq 0$. We restrict h to C but note that h does not necessarily vanish on ∂C. To amend this, we modify h on a collar neighbourhood $[0, \varepsilon) \times \partial C \subset C$ by setting

$$\overline{h}(x) = \begin{cases} h(x), & x \notin [0, \varepsilon) \times \partial C, \\ \frac{t}{\varepsilon} h(x), & x = (\varepsilon, y) \in [0, \varepsilon) \times \partial C. \end{cases}$$

Then $\overline{h}|_{\partial C} \equiv 0$ and for every $t \geq 0$ the level set $\overline{h}^{-1}(t)$ consists of elements of $h^{-1}(t)$ and possibly new elements from the collar neighbourhood $[0, \varepsilon) \times \partial C$, where we have altered h. We may choose this collar neighbourhood small enough so that it lies within the β-neighbourhood of ∂C. Now we choose

$x_1, \ldots, x_k$ such that $B_\beta(x_1), \ldots, B_\beta(x_k)$ cover ∂C. Then $B_\beta(\partial C)$ is covered by $B_{2\beta}(x_1), \ldots, B_{2\beta}(x_k)$. But since every ball of radius 2β can be covered by $d(\beta)$ balls of radius β, the β-volume of every level set is estimated by

$$\mathrm{vol}_\beta(\overline{h}^{-1}(t)) \leq \mathrm{vol}_\beta(h^{-1}(t)) + d(\beta)\,\mathrm{vol}_\beta(\partial C).$$

Now $\overline{h}$ may not be Morse but $\overline{h}$ is 1-close to a Morse function f vanishing on ∂C. By the same arguments as in the proof of Proposition 6.1.1 we find that there exists a constant $K(\beta) > 0$ such that for every $t \geq 0$

$$\begin{aligned} \mathrm{MVol}_\beta(C) &\leq \mathrm{vol}_\beta(f^{-1}(t)) \\ &\leq K(\beta)\,\mathrm{vol}_\beta(\overline{h}^{-1}(t)) \\ &\leq K(\beta)\left(c(\beta)\cdot 4^{nj(k,\beta)} + d(\beta)\,\mathrm{vol}_\beta(\partial C)\right) \\ &\leq K(\beta)\left(c(\beta)\cdot 4^{nj(k,\beta)} + d(\beta)k\right), \end{aligned}$$

where j was a function of k. The case where C intersects $\mathbb{H}^n$ and one or more of the $\mathfrak{E}_r(4^n)$ is handled by patching together suitable functions on $\mathbb{H}^n$ and the $\mathfrak{E}_r(4^n)$s. $\square$

The following proposition rounds off the discussion about the independence of geometric entropy and the bounded homology property.

Proposition 6.3.2. *There exist Riemannian manifolds of bounded geometry with finite geometric entropy that satisfy the bounded homology property and those that don't.*

A Riemannian manifold that satisfies the bounded homology property and also has finite geometric entropy (and hence also finite coarse geometric entropy) is given by Euclidean space of any dimension.

We apply the deformation of the metric described in Section 4.2 to Euclidean space to find a Riemannian manifold with vanishing geometric entropy that does not satisfy the bounded homology property. To be more precise, for every $k \geq 1$ we replace $B_1\left((k, \mathbf{0})\right)$ by $\mathfrak{T}_k$. It is part of Lemma 4.2.3 that $(\mathbb{R}^n, g')$ with the metric induced by the $\mathfrak{T}_k$ does not satisfy the bounded homology property.

It remains to show that $h_g(\mathbb{R}^n, g') = 0$. Let $(\varepsilon, 2R)$ be given. An $(\varepsilon, 2R)$-quasi-tiling is given by a Euclidean ball B_R of radius R, the distance balls $B_R\left((k, \mathbf{0})\right), k \in N$ in $(\mathbb{R}^n, g')$, a cylinder $S^{n-1} \times [0, R]$ and the distance balls $B_R(v_i)$, where v_i runs through all vertices of all $\mathfrak{T}_k$. Even though this list contains infinitely many spaces, we claim that up to isometry there are only are $2 + 4^{1/2R}$ spaces.

Note first that there are at most ${}^{R}\!/\!{}_{2}$ distinct $B_R((k,\mathbf{0})) \subset (\mathbb{R}^n, g')$ since for $k \geq {}^{R}\!/\!{}_{2}$ they are all isometric to an annulus with an open cylinder glued to the inner boundary,

$$B_R(0) \setminus B_1(0) \cup_{\partial B_1(0) \sim (S^{n-1}\times\{0\})} S^{n-1} \times [0, R].$$

Moreover, in a fixed tree $\mathfrak{T}_k$, $B_R(v_i)$ and $B_R(v_j)$ are isometric if v_i and v_j are at equal height, due to the symmetry of $\mathfrak{T}_k$. Thus in a fixed $\mathfrak{T}_k$ there are at most k non-isometric $B_R(v_i)$ in $\mathfrak{T}_k$. Let now k be arbitrary and $v_i \in \mathfrak{T}_k$ be at height $h(v_i)$. If $h(v_i) < k - R$, that is if v_i is more than R vertices away from the closest leaf, then $B_R(v_i)$ is contained in a subtree $\mathfrak{T}'_{2R}$ without caps at the leaves and due to the symmetry of $\mathfrak{T}'_{2R}$, there are at most $2R$ non-isometric balls $B_R(v_i)$ in $\mathfrak{T}'_{2R}$. If $h(v_i) \geq k - R$, then $B_R(v_i)$ is contained in a subtree isometric to $\mathfrak{T}_{2R}$ and again there are at most $2R$ non-isometric $B_R(v_i)$. Altogether, we have shown that there are at most $4R$ non-isometric $B_R(v_i)$. Thus the cardinality of our $(\varepsilon, 2R)$-quasi-tiling is bounded by $1 + {}^{R}\!/\!{}_{2} + 1 + 4R$.

The maps in the quasi-tiling will all be isometries, that is, the quasi-tilings we are going to construct are going to be $(\varepsilon, 2R)$-quasi-tilings for fixed R and arbitrary ε. We take the isometric embeddings $f_\alpha \colon B_R \longrightarrow B_R(\alpha), \alpha \in \mathbb{Z}^n$ whenever $B_R(\alpha)$ does not intersect any of the $B_1((k,\mathbf{0}))$, the natural embeddings f_k of $B_R((k,\mathbf{0}))$ and f_{v_i} of $B_R(v_i)$ and the for $k \geq {}^{R}\!/\!{}_{2}$ the embeddings

$$f_{k,l} \colon S^{n-1} \times [0, R] \longrightarrow S^{n-1} \times [l, l+R] \subset S^{n-1} \times [-k, k]$$

into the edge emanating from the root of $\mathfrak{T}_k$.

Let now $K \subset (\mathbb{R}^n, g')$ be a set of diameter at most ${}^{R}\!/\!{}_{2}$. If K does not intersect any of the $\mathfrak{T}_k$ or intersects both a $\mathfrak{T}_k$ and $\mathbb{R}^n \setminus \cup_{k\geq 1} B_1(k,\mathbf{0})$, then K clearly lies in the image of one of the f_α or of the f_k. If K lies in some $\mathfrak{T}_k$, we have two different cases: If K is completely contained in the edge emanating from the root of $\mathfrak{T}_k$, then K lies in the image of $f_{k,l}$ for some l. Otherwise, K lies in some $B_R(v_i)$ and hence in the image of f_{v_i}.

Hence, we have constructed an $(\varepsilon, 2R)$-quasi-tiling of cardinality at most $2 + 4{}^{1}\!/\!{}_{2}R$ and consequently

$$h_g(\mathbb{R}^n, g') \leq \lim_{\varepsilon\to\infty} \limsup_{R\to\infty} \frac{1}{R} \log(2 + 4{}^{1}\!/\!{}_{2}R) = 0.$$

6.4 Cheeger constant and bounded homology property

Recall that the *Cheeger isoperimetric constant* of a non-compact Riemannian n-manifold is defined as

$$h(M,g) = \inf_{\Omega\subset M} \frac{\mathrm{vol}_{n-1}(\partial\Omega)}{\mathrm{vol}_n(\Omega)},$$

where the infimum is taken over all compact codimension 0 submanifolds Ω with C^1-boundary.

The non-leaves constructed by Schweitzer ([Schw2] and Section 4.1) and those in Section 4.2 have vanishing Cheeger constant since the trees $\mathfrak{T}_k$ form a sequence of submanifolds with boundary isometric to a sphere of fixed radius, while the volume of $\mathfrak{T}_k$ tends to infinity as k does. This can be amended by scaling the metrics on the $\mathfrak{T}_k$ such that the volume of the $\mathfrak{T}_k$ is uniformly bounded while leaving the length of the edges unchanged. This can be done by shrinking the metric on the spherical part of the edges $S^{n-1} \times [-1, 1]$ and on the T-pieces $S^n \setminus (B_1 \cup B_2 \cup B_3)$ by a factor of $2^{-\operatorname{dist}(\partial \mathfrak{T}_k, \cdot)}$. The n-volume of the $\mathfrak{T}_k$ is then bounded and hence, if the deformation from Lemma 4.2.3 is applied to a manifold with positive Cheeger constant, the resulting manifold will have positive Cheeger constant, too. But as we shrink the spheres in the $\mathfrak{T}_k$, we loose the lower bound on the injectivity radius and the upper curvature bound and the resulting manifold won't be of bounded geometry anymore. This is no coincidence as the following proposition shows.

Proposition 6.4.1. *Let (M, g) be a Riemannian m-manifold of bounded geometry that does not satisfy the bounded homology property. Then the Cheeger constant of (M, g) vanishes.*

Proof. By assumption there exists $\beta, k > 0$ and a sequence of compact codimension 0 submanifolds C_n such that $\mathrm{vol}_\beta(\partial C_n) \leq k$ and $\mathrm{MVol}_\beta(C_n) \geq n$.

We first want to show that the m-dimensional Riemannian volume of C_n goes to infinity. Since $\mathrm{MVol}_\beta(C_n) \geq n$, in particular the Morse-β-volume of C_n with respect to $f_n := \operatorname{dist}(\partial C_n, \cdot)$ is at least n (up to approximation of continuous functions by Morse functions as in the proofs of Proposition 6.1.1 and Proposition 6.3.1). Thus there exists a sequence of $t_n \geq 0$ such that $\mathrm{vol}_\beta\left(f_n^{-1}(t_n)\right) \geq n$.

We claim that $t_n > \beta$ for all sufficiently large n. Since $\mathrm{vol}_\beta(\partial C_n)$ is bounded by k independently of n, for every $n \geq 1$ there exist $x_1^n, \ldots, x_k^n \in M$ such that $\partial C_n \subset \cup_{i=1}^k B(x_i, \beta)$. Since (M, g) has bounded geometry, every ball of radius 2β can be covered by $c(\beta)$ balls of radius β. Hence these x_i^n yield coverings for the level sets of f_n for every $t \leq \beta$ as follows:

$$f_n^{-1}(t) = \{x \in C_n \mid \operatorname{dist}(x, \partial C_n) = t\} \subset \bigcup_{i=1}^{k} B(x_i, 2\beta) \subset \bigcup_{i=1}^{c(\beta)k} B(y_i, \beta)$$

for appropriately chosen y_i. Hence $\mathrm{vol}_\beta(f_n^{-1}(t)) \leq c(\beta) \cdot k$ for all $t \leq \beta$.

One easily shows (Prop. 2.2, [Eg]) that $\mathrm{vol}_\beta(f_n^{-1}(t_n)) \geq n$ implies that there exist $z_1, \ldots z_n \in f_n^{-1}(t_n)$ such that the balls $B(z_i, \beta/2)$ are pairwise disjoint. Since $t_n \geq \beta$, this means that the $B(z_i, \beta/2)$ are completely contained in C_n and thus $\mathrm{vol}_m(C_n) \geq \sum_{i=1}^n \mathrm{vol}_m(B(z_i, \beta/2))$. Finally, by bounded geometry, the volume of $\beta/2$-balls in M is bounded away from 0, say $\mathrm{vol}_n(B(z, \beta/2)) \geq v(m, \beta) > 0$. Thus $\mathrm{vol}_m(C_n) \geq n \cdot v \to \infty$.

It remains to show that $h(M, g) = 0$. If $\mathrm{vol}_{m-1}(\partial C_n)$ is bounded from above, then $\lim_n {}^{\mathrm{vol}_{m-1}(\partial C_n)}/_{\mathrm{vol}_m(C_n)} = 0$ and we are done. If not, let $x_1^n, \ldots, x_k^n$ be as above and replace C_n by

$$C_n' = C_n \cup \bigcup_{i=1}^k B(x_i, \beta).$$

Then $\mathrm{vol}_n(C_n') \geq \mathrm{vol}_n(C_n)$ and $\partial C_n' \subset \cup_{i=1}^k B(x_i, \beta)$. Again by bounded geometry, the $(m-1)$-volume of distance spheres of radius β is bounded by a constant $V(m-1, \beta) < \infty$. It follows that $\mathrm{vol}_{m-1}(\partial C_n') \leq k \cdot V(m-1, \beta)$ and thus $\lim_n {}^{\mathrm{vol}_{m-1}(\partial C_n')}/_{\mathrm{vol}_m(C_n')} = 0$. $\square$

Clearly, the fact that a manifold satisfies the bounded homology property implies no restriction on the Cheeger constant since both $\mathbb{R}^m$ and $\mathbb{H}^m$ satisfy the bounded homology property but $h(\mathbb{R}^m) = 0$ and $h(\mathbb{H}^m) > 0$.

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Eidesstattliche Versicherung

(siehe Promotionsordnung vom 12.07.11, §8, Abs. 2 Pkt. 5.)

Hiermit erkläre ich an Eides statt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

(Name, Vorname)

München, den 21. Juli 2014

(Unterschrift Doktorand/in)