Non-zero Degree Maps between Manifolds and Groups Presentable by Products

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Abstract

We study non-zero degree maps between closed manifolds, in particular when the domain is a non-trivial direct product. In this context, we investigate the concept of "fundamental groups presentable by products".

On the one hand, we introduce an algebraic obstruction to domination by products for certain essential manifolds, termed "fundamental group not infinite-index presentable by products" (not IIPP). Using this condition, we extend previous non-domination results of Kotschick and Löh to rationally essential manifolds with fundamental groups presentable by products. On the other hand, we show that large classes of rationally inessential manifolds are dominated by products, by constructing simple maps of non-zero degree. Namely, we prove that certain products containing a sphere factor are realized as branched double covers of connected sums of sphere bundles.

In three dimensions, we apply the constructions of branched coverings to show that the fundamental class of every rationally inessential target is representable by products and we obtain a complete list of closed three-manifolds dominated by products. An ordering of three-manifolds, suggested by Wang, is then clarified and completed, following the geometrization picture of Thurston. Exploiting the obstruction "not IIPP", we furthermore determine which geometric closed four-manifolds are dominated by products and we extend Wang's ordering to non-hyperbolic aspherical four-dimensional Thurston geometries. To this end, we investigate non-zero degree maps between products in arbitrary dimensions, obtaining stable non-domination results.

Moreover, we show that every simply connected closed manifold in dimensions four and five is dominated by products. The proofs are constructive, relying on classification theorems and applying the branched coverings for rationally inessential manifolds.

Further applications are derived, in connection with the vanishing of Gromov's functorial semi-norms on homology and on the non-existence of degree -1 self-maps of products.

Zusammenfassung

Wir beschäftigen uns mit Abbildungen vom Grad ungleich Null zwischen geschlossenen Mannigfaltigkeiten, insbesondere wenn der Definitionsbereich der Abbildung ein nicht-triviales Produkt ist. In diesem Zussamenhang untersuchen wir den Begriff von "Fundamentalgruppen die durch Produkte präsentierbar sind".

Einerseits führen wir eine algebraische Obstruktion für die Dominierung durch Produkte für bestimmte rational essentielle Mannigfaltigkeiten ein, nämlich "Fundamentalgruppen die nicht präsentierbar sind durch Produkte von Untergruppen von unendlichem Index" ("nicht IIPP"). Unter dieser Bedingung erweitern wir frühere Nicht-Dominierungs-Ergebnisse von Kotschick und Löh auf rational essentielle Mannigfaltigkeiten deren Fundamentalgruppen präsentierbar durch Produkte sind. Andererseits zeigen wir, dass große Klassen von rational nicht essentiellen Mannigfaltigkeiten durch Produkte dominiert werden, indem wir einfache Abbildungen vom Grad ungleich Null konstruierien. Genauer beweisen wir, dass bestimmte Produkte mit Sphären verzweigte zweifache Überlagerungen von zusammenhängenden Summen von Sphären-Bündeln sind.

In Dimension drei wenden wir die Konstruktionen von verzweigten Überlagerungen an, um zu zeigen, dass die Fundamentalklasse jeder nicht rational essentiellen Mannigfaltigkeit präsentierbar durch Produkte ist und bekommen so eine vollständige Liste von geschlossenen drei-Mannigfaltigkeiten die durch Produkte dominiert werden. Ausgehend von der Thurston Geometrisierung vervollständigen wir dann eine Ordnung unter drei-Mannigfaltigkeiten, die auf Wang zurück geht. Wir benutzen die Obstruktion "nicht IIPP" um zu bestimmen welche geschlossenen geometrischen vier-Mannigfaltigkeiten von Produkten dominiert werden und erweitern Wangs Ordnung auf die nicht-hyperbolischen asphärischen vierdimensionalen Geometrien. Zu diesem Zweck untersuchen wir Abbildungen vom Grad ungleich Null zwischen Produkten in beliebigen Dimensionen und beweisen stabile Nicht-Dominierungs-Ergebnisse.

Weiterhin zeigen wir dass jede geschlossene einfach zusammenhängende Mannigfaltigkeit von Dimension vier oder fünf von Produkten dominiert wird. Die Beweise gehen von Klassifikations-Sätzen aus und sind konstruktiv.

Schließlich geben wir weitere Anwendungen auf das Verschwinden von funktoriellen semi-Normen im Sinne von Gromov auf der Homologie, und auf die Nicht-Existenz von Selbst-Abbildungen vom Grad -1 von Produkten.

Preface

A fundamental, long-standing topic in topology is the investigation of non-zero degree maps between manifolds. The existence of a map of non-zero degree defines a transitive relation on the homotopy types of closed oriented manifolds. Whenever such a map $M \longrightarrow N$ exists we say that M dominates N and denote this by $M \geq N$. Well-known obstructions to the existence of a dominant map can be derived from the size of the Betti numbers and the largeness of the fundamental groups. More sophisticated tools, such as functorial semi-norms on homology, were mainly developed by Gromov [34, 35] in his study of metric structures on Riemannian manifolds and of topological rigidity.

The field of non-zero degree maps consists of several topics. The most prominent one is to decide whether two given (classes of) manifolds are comparable under $\geq$. Important results have been obtained in the past, concerning maps to highly connected targets [20] and to manifolds admitting metrics of non-positive sectional curvature [34, 35, 13, 44]. Gromov suggested to think of the existence of dominant maps as defining a partial order on manifolds of the same dimension [13]. Naturally, the problem following the existence question is that of finding the simplest possible homotopy representatives for a dominant map. In this context, branched coverings constitute the most notable example, going back to Alexander's classical theorem on the realization of every piecewise linear oriented manifold as a branched cover of the sphere [2, 25], and passing through significant works of Edmonds in low dimensions [22, 21]. The investigation of the sets of mapping degrees is, finally, yet another essential topic, mainly because of its close connection to the study of numerical functorial invariants of manifolds as suggested by Milnor-Thurston [49].

The goal of this thesis is to improve our current understanding of the domination relation. We mainly focus on the existence question and on identifying simple homotopy representatives for dominant maps. In the process, we contribute to the topic of mapping degrees as well. A large part of this work is devoted to the study of non-zero degree maps with domain a non-trivial product of closed manifolds. This can naturally be viewed as specifying further Steenrod's question [23, Problem 25] on the realization of homology classes by manifolds under continuous maps (cf. Section 1.1). Now, we are interested to examine whether a fundamental class is realizable by the product of two (fundamental) classes of lower non-trivial degrees, which are quite often easier to understand.

For certain manifolds, called rationally essential, the study of domination by products is strongly motivated by Gromov's theory on functorial semi-norms on homology and on bounded cohomology, most notably the concept of simplicial volume. One of Gromov's predictions [35, Chapter $5G_+$] was that the fundamental classes of certain (rationally essential) manifolds, namely of irreducible locally symmetric spaces of non-compact type, might not be representable by products (of surfaces). Kotschick-Löh [44] verified that suggestion, by finding a purely algebraic obstruction to domination by products for every rationally essential manifold. Concretely, Kotschick-Löh proved that a rationally essential manifold whose fundamental group fulfills a property called "not presentable by products" cannot be dominated by products.

A natural motivating problem, stemming from Kotschick-Löh's work, is to examine to what extent the condition "fundamental groups presentable by products" suffices for domination by products for rationally essential manifolds. In this thesis, we extend the non-existence results of [44] to targets with fundamental groups presentable by products. We introduce a subclass of groups presentable by products, called groups "infinite-index presentable by products" or "IIPP". Using this notion, we prove that certain rationally essential manifolds with fundamental groups presentable by products, but not IIPP, cannot be dominated by products. We moreover show that large classes of aspherical circle bundles are dominated by products if and only if their fundamental groups are IIPP.

Another particular conclusion of our study is that many rationally essential manifolds are dominated by products if and only if they are finitely covered by products, which answers (for those manifolds) one of the motivating themes on finding simple representatives for dominant maps. At the other end, it raises the problem of determining which rationally inessential targets are dominated by products, and by what types of maps. These targets are far away from being aspherical or connected sums containing rationally essential summands. Important examples of inessential manifolds, which have been studied extensively in the literature, are the simply connected ones. In this thesis, we prove that certain rationally inessential manifolds are quotient spaces of products. More precisely, we show that manifolds of type $S^k \times N$ can be realized as branched double covers of connected sums of sphere bundles. Appealing furthermore to classification theorems, we prove that every simply connected manifold in dimensions four and five is dominated by a product of type $S^1 \times N$.

We shall moreover apply the results of this dissertation to understand the domination relation in low dimensions. Namely, we first determine which closed three-manifolds are dominated by products. Following Gromov's concept on studying the domination relation as an order, we then use Thurston's geometrization picture to clarify and complete an ordering of three-manifolds obtained by Wang [86]. The geometric form of the results in dimension three motivates the idea of investigating which higher dimensional geometric manifolds are dominated by products. We answer this question in dimension four, applying the obstructions "not presentable by products" and "not IIPP". In the sequel, we extend the three-dimensional ordering of Wang to non-hyperbolic aspherical geometric four-manifolds. Our results determine, moreover, in which cases the condition "fundamental group presentable by products" suffices for domination by products for low-dimensional rationally essential manifolds.

Some of the non-existence results are consequences of more general statements about maps between products in arbitrary dimensions. Although the proofs of those statements are for the most part elementary, the philosophy behind them has strong connections to the study of functorial semi-norms and to mapping degrees. On the one hand, we derive simple stable non-domination results which cannot be deduced using Gromov's semi-norms, for example that $M \not\geq N$ implies $M \times \cdots \times M \not\geq N \times \cdots \times N$ (same number of factors), whenever N is not dominated by products. On the other hand, we construct manifolds that do not admit self-maps of degree -1 or, more generally, self-maps of prime degree.

Short outline. In Chapter 1, we state our main results along with the basic terminology. The main body of this thesis consists of six chapters. In Chapter 2, we deal with the IIPP condition on the fundamental groups of rationally essential manifolds and in Chapter 3, we construct branched coverings for rationally inessential targets, where the domain is a non-trivial product. In the subsequent Chapters 4 and 5, we study domination by products and maps between geometric manifolds in low dimensions. Along the way, we obtain stable non-existence results for maps between products in arbitrary dimensions. In Chapter 6, we examine domination by products for manifolds with finite fundamental groups, especially in dimensions four and five. In the final Chapter 7, we deal briefly with self-mapping degrees of products and of simply connected rational homology spheres.

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Chapter 1

Background and main results

This chapter comprises the basic notions and the statements of the main results of this thesis. We begin with a brief introduction of the domination relation and of the concept of essentialness. The subsequent sections contain a concise summary of the main outcome of this work.

1.1 The domination relation and rational essentialness

In the early 1940s, Steenrod raised the question of whether every n-dimensional integral homology class can be realized as the image of the fundamental class of a closed oriented n-dimensional manifold under a continuous map [23, Problem 25]. About a decade later, Thom answered affirmatively Steenrod's question in degrees up to six and in any degree in homology with $\mathbb{Z}_2$ coefficients. He showed, however, that there exists a seven-dimensional integral homology class which is not realizable by a manifold. Since then, other non-realizability results have been obtained. Nevertheless, Thom proved that some multiple of each integral homology class is realizable in all degrees:

Theorem 1.1 (Thom [76])**.** *Suppose that X is a topological space and let $\alpha \in H_n(X;\mathbb{Z})$. Then there is a positive integer d and a closed oriented connected smooth n-dimensional manifold M together with a continuous map $f\colon M \longrightarrow X$ so that $H_n(f;\mathbb{Z})([M]) = d \cdot \alpha$, where $[M]$ denotes the fundamental class of M.*

In particular, every rational homology class in degree n is realizable by a closed oriented connected smooth n-dimensional manifold M.

In this thesis, we are interested in the realization of fundamental classes of closed oriented manifolds (especially by direct products of manifolds), and therefore we deal with the notion of *the degree of a continuous map.* Namely, suppose that $f\colon M \longrightarrow N$ is a continuous map between two closed oriented connected n-dimensional manifolds. The *degree* of f is defined to be the integer d so that $H_n(f;\mathbb{Z})([M]) = d \cdot [N]$, where $[M] \in H_n(M;\mathbb{Z})$ and $[N] \in H_n(N;\mathbb{Z})$

denote the fundamental classes of M and N respectively. Whenever d is not zero, we say that M *dominates* N (or that M *d-dominates* N), and write $M \geq N$ (or $M \geq_d N$). The degree of f is denoted by $\deg(f)$. Unless otherwise stated, we shall consider maps of non-zero degree between the homotopy types of closed oriented connected manifolds.

The domain of a non-zero degree map is usually a more complicated manifold than the target. For instance, whenever $M \geq N$, the fundamental group and the Betti numbers of M are at least as large[1] as the fundamental group and the Betti numbers of N respectively. These necessary domination conditions can be derived by the following simple properties:

Lemma 1.2. *Let M, N be two closed oriented connected n-dimensional manifolds and suppose that $f\colon M \longrightarrow N$ has non-zero degree. Then the following hold:*

(1) *The image $\mathrm{im}(\pi_1(f))$ of the induced homomorphism $\pi_1(f)\colon \pi_1(M) \longrightarrow \pi_1(N)$ is a finite index subgroup of $\pi_1(N)$.*

(2) *The induced homomorphisms $H_*(f;\mathbb{Q})\colon H_*(M;\mathbb{Q}) \longrightarrow H_*(N;\mathbb{Q})$ are surjective; equivalently $H^*(f;\mathbb{Q})\colon H^*(N;\mathbb{Q}) \longrightarrow H^*(M;\mathbb{Q})$ are injective.*

One of Gromov's suggestions was to investigate the domination relation as a tool to order manifolds and, especially, to understand the values of functorial semi-norms on homology [35, 34, 13]. For a space X, a semi-norm on $H_*(X;\mathbb{Q})$ is called functorial if it is not increasing under the induced homomorphisms $H_*(f)\colon H_*(X) \longrightarrow H_*(Y)$, for every continuous map $f\colon X \longrightarrow Y$. One of the most prominent functorial semi-norms, defined by Gromov, is the simplicial volume. In general, if $\alpha \in H_n(X;\mathbb{Q})$ then the *simplicial ℓ^1-semi-norm in degree n* is given by[2]

$$\|\alpha\|_1 := \inf_c \left\{ \sum_j |\lambda_j| \;\middle|\; c = \sum_j \lambda_j \sigma_j \in C_n(X;\mathbb{Q}) \text{ is a cycle representing } \alpha \right\}.$$

If M is a closed oriented n-dimensional manifold, then $\|M\| := \|[M]\|_1$ denotes *the simplicial volume of M*.

The functoriality of the simplicial ℓ^1-semi-norm implies that $\|M\| \geq |d| \cdot \|N\|$, when $M \geq_d N$. Thus, manifolds admitting self-maps of absolute degree higher than one have zero simplicial volume.

Example 1.3. Spheres and direct products with a sphere factor have vanishing simplicial volume. We remark here the obvious fact that spheres are minimal elements for the domination relation, being dominated by every other manifold M. Moreover, every integer can be realized as a degree of a map $f\colon M \longrightarrow S^n$ and this integer determines the homotopy type of f; see [51] for Hopf's theorem.

[1] A group G *is at least as large as* H if some finite index subgroup of G surjects onto a finite index subgroup of H.

[2] The original definition of the simplicial ℓ^1-semi-norm uses coefficients in $\mathbb{R}$ instead of $\mathbb{Q}$, yielding however the same semi-norm (by approximating boundaries with real coefficients by boundaries with rational coefficients).

Examples of closed oriented connected manifolds with non-vanishing simplicial volume are given by the negatively curved ones [34, 77]. Gromov proved that the simplicial ℓ^1-semi-norm is completely determined by the classifying space of the fundamental group:

Theorem 1.4 (Gromov [34]). *Let $c_M\colon M \longrightarrow B\pi_1(M)$ be the classifying map of the universal covering of a closed oriented connected manifold M. Then the induced homomorphism $H_*(c_M;\mathbb{Q})\colon H_*(M;\mathbb{Q}) \longrightarrow H_*(B\pi_1(M);\mathbb{Q})$ is an isometry with respect to the simplicial ℓ^1-semi-norm.*

In particular, the simplicial volume of an n-dimensional manifold M depends only on $H_n(c_M;\mathbb{Q})([M]) \in H_n(B\pi_1(M);\mathbb{Q})$. This gives rise to the following definition:

Definition 1.5 ([34]). A closed oriented connected n-dimensional manifold M is called *rationally essential* if $H_n(c_M;\mathbb{Q})([M]) \neq 0 \in H_n(B\pi_1(M);\mathbb{Q})$, where $c_M\colon M \longrightarrow B\pi_1(M)$ classifies the universal covering of M. Otherwise, M is called *rationally inessential.*

Clearly, every closed aspherical manifold (i.e. a closed manifold whose homotopy groups are trivial in degrees higher than one) is rationally essential. Nevertheless, the notion of essentialness expands widely the class of aspherical manifolds:

Example 1.6.

(1) A connected sum $M = M_1\#\cdots\#M_k$ of dimension higher than two is rationally essential if and only if it contains at least a rationally essential summand M_i, because $B\pi_1(M) = B\pi_1(M_1)\vee\cdots\vee B\pi_1(M_k)$. In particular, if there is an aspherical summand M_i, then M is rationally essential.

(2) Manifolds with non-vanishing simplicial volume are rationally essential by Gromov's Theorem 1.4.

1.2 An obstruction to domination by products

1.2.1 Kotschick-Löh's non-domination criterion

In the context of representing fundamental classes of manifolds by direct products, Gromov conjectured that there might exist "interesting" classes of (rationally essential) manifolds which do not fulfill that property, pointing out as potential candidates the fundamental classes of irreducible locally symmetric spaces of non-compact type; cf. [35, Chapter 5G$_+$].

Kotschick-Löh verified Gromov's suggestion, by finding a condition on the fundamental groups of rationally essential manifolds that are dominated by products:

Definition 1.7 ([44]). An infinite group Γ is called *presentable by products* if there is a homomorphism $\varphi\colon\Gamma_1\times\Gamma_2 \longrightarrow \Gamma$ onto a finite index subgroup of Γ so that the restriction of φ to each factor Γ_i has infinite image $\varphi(\Gamma_i)$.

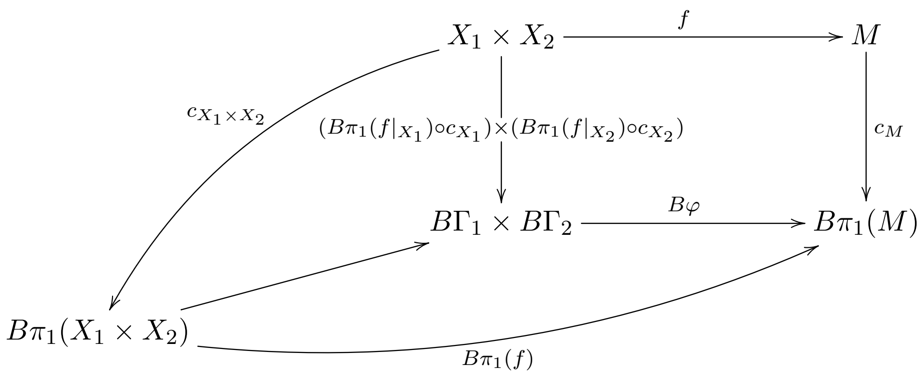

Figure 1.1: Domination by products on the level of classifying spaces.

The property of being (not) presentable by products is clearly preserved under passing to finite index subgroups.

Example 1.8.

(1) A *reducible group*, i.e. a group Γ that is a *virtual product* $\Gamma_1 \times \Gamma_2$ *of two infinite subgroups* $\Gamma_i \subset \Gamma$, is obviously presentable by products.

(2) Let Γ be a group which contains a finite index subgroup $\overline{\Gamma}$ with infinite center. Then Γ is presentable by products, via the multiplication homomorphism $C(\overline{\Gamma}) \times \overline{\Gamma} \longrightarrow \overline{\Gamma}$.

Actually, these two examples include every torsion-free group presentable by products; cf. Proposition 2.3.

Suppose that M is a rationally essential n-dimensional manifold and let $f\colon X_1 \times X_2 \longrightarrow M$ be a map of non-zero degree, where the X_i are closed oriented connected manifolds of positive dimensions. Consider the induced map $\pi_1(f)\colon \pi_1(X_1) \times \pi_1(X_2) \longrightarrow \pi_1(M)$ and set

$$\Gamma := \mathrm{im}(\pi_1(f)) \subset \pi_1(M) \;\text{ and }\; \Gamma_i := \mathrm{im}(\pi_1(f|_{X_i})) \subset \Gamma$$

for the image of $\pi_1(f)$ and the images under $\pi_1(f)$ of the restrictions of f to the two factors X_i respectively. The multiplication map $\varphi\colon \Gamma_1 \times \Gamma_2 \longrightarrow \Gamma$ is then a well-defined surjective homomorphism, because the Γ_i commute with each other (elementwise) and $\Gamma_1 \cup \Gamma_2$ generates Γ. Moreover, the outer commutative diagram in Figure 1.1 implies that $X_1 \times X_2$ is rationally essential as well [44].

Let $c_{X_i}\colon X_i \longrightarrow B\pi_1(X_i)$ be the classifying maps of the universal coverings of the X_i and $B\pi_1(f|_{X_i})\colon B\pi_1(X_i) \longrightarrow B\Gamma_i$ be the maps induced by $\pi_1(f|_{X_i})$ on the level of classifying spaces. Moreover, let $B\varphi\colon B\Gamma_1 \times B\Gamma_2 \longrightarrow B\Gamma$ be the map induced by φ between the classifying spaces;

here we apply the homotopy equivalence $B\Gamma_1 \times B\Gamma_2 \simeq B(\Gamma_1 \times \Gamma_2)$. We then have for $i = 1, 2$ the maps

$$B\pi_1(f|_{X_i}) \circ c_{X_i} \colon X_i \longrightarrow B\Gamma_i, \tag{1.1}$$

and the corresponding rational homology classes

$$\alpha_i := H_{\dim X_i}(B\pi_1(f|_{X_i}) \circ c_{X_i})([X_i]) \in H_{\dim X_i}(B\Gamma_i; \mathbb{Q}), \tag{1.2}$$

where $[X_i]$ denote the fundamental classes of the factors X_i.

According to this notation, the key observation of Kotschick-Löh, shown in the commutative rectangle of Figure 1.1, is that

$$0 \neq \deg(f) \cdot H_n(c_M)([M]) = H_n(B\varphi)(\alpha_1 \times \alpha_2).$$

This means that the α_i are not trivial and therefore the Γ_i are both infinite. In particular, Γ is presented by the product $\varphi \colon \Gamma_1 \times \Gamma_2 \longrightarrow \Gamma$. This proves the main result of [44]:

Theorem 1.9 (Kotschick-Löh [44])**.** *Let M be a rationally essential manifold. If $\pi_1(M)$ is not presentable by products, then $P \not\geq M$, for any non-trivial product P of closed oriented connected manifolds.*

Remark 1.10. A consequence of Theorem 1.9 is that Gromov's prediction was indeed correct. Namely, a locally symmetric space of non-compact type is dominated by a product if and only if it is virtually (isometric to) a product; cf. [44, Cor. 4.2]. (See also Theorem 2.24.)

The concept of "groups not presentable by products" is of independent interest, being related to other notions of geometric group theory, and it has been studied extensively in [45]. Below, we quote some basic examples of such groups:

Theorem 1.11 (Kotschick-Löh [44, 45])**.** *The following classes of groups are not presentable by products:*

(1) *Hyperbolic groups and non-trivial free products that are not virtually cyclic.*

(2) *Groups containing infinite acentral subgroups of infinite index.*

(3) *Fundamental groups of closed non-positively curved manifolds of rank one and dimension at least two.*

(4) *Mapping class groups of closed oriented surfaces of positive genus.*

One of the most notable examples of the above theorem is that of hyperbolic groups that are not virtually cyclic. We recall the proof of that case (as given in [44]) in Chapter 5. Moreover, we will deal with (virtually) free groups on at least two generators and with the concept of "acentrality", while studying fundamental groups of low-dimensional manifolds; see Sections 2.3.2 and 5.2.1 respectively.

1.2.2 The property IIPP

The non-domination criterion of Theorem 1.9 raises the following question:

Question 1.12. *When does the condition "fundamental group presentable by products" suffice for domination by products for rationally essential manifolds?*

Remark 1.13. Question 1.12 is quite delicate. First, it is not even known whether aspherical manifolds with reducible fundamental groups are virtual products of two manifolds. In higher dimensions, Lück [47] answered this problem affirmatively, relying, however, on strong assumptions (for instance, the hypothesis that a product group and its factors satisfy the Farrell-Jones conjecture). In addition, this question is closely related to significant open problems in topology, for example, to the Borel conjecture and on whether every finitely presentable Poincaré duality group is the fundamental group of a closed aspherical manifold [17].

It would be unnatural to expect that rationally essential manifolds with (irreducible) fundamental groups presentable by products are always dominated by products. Counterexamples to that already appear in Kotschick-Löh's work, where, however, rational essentialness is replaced by stronger assumptions, for example, by the non-vanishing of the simplicial volume; see [44, Section 6]. In addition, some of those rationally essential manifolds (but not aspherical; compare Remark 1.13) have fundamental groups which are direct products [44, Example 6.3].

Our goal is to extend the non-existence result of Theorem 1.9, using only properties of the fundamental group of the target. As we have seen, the proof of that theorem does not yield any information about the index of the factors of the product presenting $\pi_1(M)$. A priori, all the possibilities for the indices of the presenting factors in $\pi_1(M)$ can occur; cf. Example 2.4. In this thesis, we introduce the following class of groups extending the notion of groups not presentable by products:

Definition 1.14. An infinite group Γ is called *not infinite-index presentable by products* (not IIPP) if, for every homomorphism $\varphi\colon \Gamma_1 \times \Gamma_2 \longrightarrow \Gamma$ onto a finite index subgroup of Γ, at least one the images $\varphi(\Gamma_i)$ has finite index in Γ.

As before, the property of being (not) IIPP is preserved under subgroups of finite index. Clearly, if Γ is not presentable by products, then it is not IIPP as well, because for every homomorphism $\varphi\colon \Gamma_1 \times \Gamma_2 \longrightarrow \Gamma$ onto a finite index subgroup of Γ, one of the images $\varphi(\Gamma_i)$ is finite, and so the other image must have finite index in Γ. Actually, the class of groups not presentable by products is strictly contained in the class of not IIPP groups:

Example 1.15. The infinite cyclic group is obviously presentable by products, however, it is not IIPP, because all its non-trivial subgroups are of finite index.

Using Definition 1.14, we will extend the results of Kotschick-Löh to certain rationally essential manifolds with fundamental groups presentable by products but not IIPP. As we shall see (Lemma 2.7), if a finitely generated torsion-free group Γ is presented by a product of subgroups Γ_1 and Γ_2 so that $\Gamma_1 \cup \Gamma_2$ generates Γ, and Γ is not IIPP, then one of the Γ_i must be isomorphic to a free Abelian group of rank at most $\mathrm{rank} C(\Gamma)$. Topologically, this means that the condition "not IIPP" detects all the possible dimensions of the factors of a product that dominates (with a π_1-surjective map) a rationally essential manifold with torsion-free fundamental group (Proposition 2.9). For circle bundles whose fundamental groups have virtual center at most infinite cyclic we obtain the following:

Theorem 1.16 (Theorem 2.13)**.** *Let M be a circle bundle over a closed oriented connected aspherical manifold B so that $\pi_1(M)$ is not* IIPP *and it has virtual center at most $\mathbb{Z}$. Then $P \ngeq M$ for any non-trivial product P of closed oriented connected manifolds.*

As Example 1.15 illustrates, groups presentable by products need not be IIPP, and of course, they are not generally reducible. However, every IIPP group Γ with infinite cyclic center $C(\Gamma)$ is always reducible, whenever $\Gamma/C(\Gamma)$ is torsion-free and not presentable by products (cf. Proposition 2.17). This gives rise to the following topological characterization for the corresponding classes of circle bundles:

Theorem 1.17 (Theorem 2.27)**.** *Let $M \stackrel{\pi}{\longrightarrow} B$ be a circle bundle over a closed oriented connected aspherical manifold B whose fundamental group $\pi_1(B)$ is not presentable by products. Then the following are equivalent:*

(1) *$P \geq M$ for some non-trivial product P of closed oriented connected manifolds;*

(2) *M is finitely covered by a product $S^1 \times B'$, for some finite cover $B' \longrightarrow B$;*

(3) *$\pi_1(M)$ is reducible;*

(4) *$\pi_1(M)$ is* IIPP.

Since groups not IIPP enlarge the class of groups not presentable by products, we collect some main examples of groups not IIPP:

Theorem 1.18. *The following classes of (infinite) groups are not* IIPP*:*

(1) *Fundamental groups of circle bundles with non-trivial rational Euler class, over aspherical manifolds whose fundamental groups are not presentable by products.*

(2) *Fundamental groups of closed three-manifolds that are not virtually aspherical products.*

(3) *Irreducible fundamental groups of geometric solvable four-manifolds.*

(4) *Non-trivial free products.*

The proof of case (4) is now an obvious consequence of Kotschick-Löh's Theorem 1.11. The nature of the rest of the examples is geometric and, as item (1) suggests, it arises from the study of aspherical circle bundles. The proofs for the cases (1)-(3) will be derived along the way of our discussion; see the corresponding Sections 2.4, 2.3.2 and 5.2.2.

1.3 Branched coverings of inessential manifolds

Beyond answering Question 1.12 in many cases, our study on rationally essential targets identifies moreover simple homotopy representatives of dominant maps. Namely, Theorem 1.17 says that large classes of aspherical manifolds are dominated by products if and only if they are finitely covered by products. For rationally inessential targets, the algebraic obstructions on the fundamental groups are not applicable anymore. It is therefore natural to ask whether the fundamental classes of rationally inessential manifolds are representable by products.

Simply connected manifolds are prominent examples of inessential targets. Hence, another main question of this thesis is the following (see [44, Section 7.2]):

Question 1.19. *Is every simply connected closed manifold dominated by products?*

Of course, the above question can equivalently be reformulated for every closed (oriented) manifold with finite fundamental group.

In Chapter 3, we construct non-zero degree maps by products of type $S^k \times N$ for connected sums of sphere bundles. One of the main ingredients for those constructions will be the "pillowcase" map, i.e. a certain branched double covering $T^2 \longrightarrow S^2$, and its high-dimensional generalization; cf. Theorem 3.3. We first show that every connected sum of type $\#_{i=1}^{p}(S^n \times M_i)$, where M_i are arbitrary closed manifolds, admits a branched double covering by a product $S^k \times (\#_{i=1}^{p} S^{n-k} \times M_i)$.

Theorem 1.20 (Theorem 3.6)**.** *Let $\{M_i\}_{i=1}^{p}$ be a family of closed oriented connected m-dimensional manifolds. For every $n > k \geq 1$ there is a π_1-surjective branched double covering*

$$S^k \times (\#_{i=1}^{p} S^{n-k} \times M_i) \longrightarrow \#_{i=1}^{p}(S^n \times M_i).$$

The above statement is a generalization of a three-dimensional construction given in [46] and uses the fact that the connected summands of the target are direct products $S^n \times M_i$. This raises the problem of whether connected sums of (rationally inessential) twisted products are still dominated by products and, moreover, whether the type of the domain could be again $S^k \times N$. Using Steenrod's classification of sphere bundles, we answer both these questions affirmatively, when the connected summands are sphere bundles over the 2-sphere:

Theorem 1.21 (Theorem 3.8)**.** *For $n \geq 4$ and every $p \geq 0$ there is a branched double covering*

$$S^1 \times (\#_p S^{n-2} \times S^1) \longrightarrow \#_p(S^{n-2} \widetilde{\times} S^2),$$

where $S^{n-2}\widetilde{\times}S^2$ denotes the total space of the non-trivial S^{n-2}-bundle over S^2 with structure group $\mathrm{SO}(n-1)$*.*

In dimensions lower than four, the only simply connected closed manifolds are spheres, which are dominated by any other manifold, and therefore by products. Applying the constructions of Theorems 1.20 and 1.21, and relying on classification results, we answer Question 1.19 affirmatively in dimensions four and five:

Theorem 1.22 (Theorem 6.1)**.** *Every closed simply connected manifold in dimensions four and five is dominated by a non-trivial product.*

In dimension four, a non-constructive proof for the above result was previously obtained by Kotschick-Löh [44], applying a domination criterion of Duan-Wang [20] on the intersection form; see Section 6.2.3.

The investigation of the domination relation for simply connected closed manifolds raises the connection between the values of functorial semi-norms and the sets of (self-)mapping degrees. For instance, the generalized Hurewicz theorem by Serre (cf. Theorem 6.16) implies that every closed simply connected rational homology sphere is a minimal element with respect to the domination relation and the set of its self-mapping degrees is unbounded. Thus, in particular, every finite functorial semi-norm on a closed simply connected rational homology sphere vanishes. We will discuss these properties briefly in Section 7.3.

1.4 Low dimensions and maps between products

Domination by products for three-manifolds is very special, because the only products are those with a circle factor. The main topological result in dimension three, obtained in a joint work with Kotschick [46], is the following (cf. Theorem 4.5):

Theorem 1.23. *A closed oriented connected three-manifold M is dominated by a product if and only if*

(1) *either M is virtually a product $S^1 \times \Sigma$, for some closed aspherical surface Σ, or*

(2) *M is virtually a connected sum $\#_p(S^2 \times S^1)$, for some $p \geq 0$.*

During our discussion in dimension three, we will describe all rationally essential three-manifolds in terms of the Kneser-Milnor prime decomposition; see Theorem 4.7. Together with the characterizations of three-manifold groups (infinite-index) presentable by products (cf. Section 2.3.2), we can answer Question 1.12 in that dimension:

Corollary 1.24 (Corollary 4.10). *A rationally essential three-manifold M is dominated by a product if and only if $\pi_1(M)$ is a virtual product $\pi_1(\Sigma) \times \mathbb{Z}$ for some closed oriented aspherical surface Σ.*

As we shall see, the study of domination by products for rationally essential three-manifolds has many similarities to the study of domination by non-trivial circle bundles. Therefore, we will be able to determine which closed three-manifolds are dominated by non-trivial circle bundles as well; see Section 4.2.2.

In terms of Thurston geometries, our results say that a closed three-manifold is dominated by a product (resp. non-trivial circle bundle) if and only if either it carries one of the geometries $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{R}^3$ (resp. $\widetilde{SL_2}$ or Nil^3), or it is a connected sum of closed manifolds possessing one of the geometries $S^2 \times \mathbb{R}$ or S^3; cf. Theorem 4.14. Following the geometrization picture of Thurston, we then complete an ordering of three-manifolds obtained by Wang [86]; cf. Theorem 4.16.

The geometric form of the results in dimension three raises the problem of determining which higher dimensional geometric manifolds are dominated by products. To this end, we apply the conditions "not presentable by products" and "not IIPP" to the fundamental groups of aspherical geometric four-manifolds (cf. Theorems 5.4 and 5.5) to prove the following:

Theorem 1.25 (Theorem 5.29). *A closed oriented aspherical geometric four-manifold M is dominated by a non-trivial product if and only if it is finitely covered by a product. Equivalently, M carries one of the product geometries $\mathbb{X}^3 \times \mathbb{R}$ or it is a reducible quotient of the geometry $\mathbb{H}^2 \times \mathbb{H}^2$.*

In particular, we answer Question 1.12 for aspherical geometric four-manifolds:

Corollary 1.26 (Corollary 5.30). *A closed oriented aspherical geometric four-manifold M is dominated by a product if and only if*

(1) *either $\pi_1(M)$ is a virtual product $\pi_1(N) \times \mathbb{Z}$, for some closed oriented aspherical geometric three-manifold N, or*

(2) *$\pi_1(M)$ is a virtual product of two closed oriented hyperbolic surface groups.*

The study of four-manifolds will additionally yield an extension of the ordering of Wang in dimension three (Theorem 4.16) to the non-hyperbolic aspherical four-dimensional Thurston geometries; see Theorem 5.41. Some of our non-existence results will be obtained by applying the following more general statement on maps between products in arbitrary dimensions:

Theorem 1.27 (Theorem 5.31). *Let M, N be closed oriented connected n-dimensional manifolds such that N is not dominated by products and W be a closed oriented connected manifold of dimension m. Then $M \geq N$ if and only if $M \times W \geq N \times W$.*

The proof of the above statement is elementary, based on Thom's realization Theorem 1.1. However, it has interesting consequences, yielding non-domination results which cannot be obtained using well-known tools, such as Gromov's simplicial volume. In particular, it extends stable non-existence results of Kotschick-Löh for maps between products; cf. Corollary 5.32.

The concept of the above theorem has further applications to the investigation of the sets of self-mapping degrees of products. Namely, we give a simple method of constructing products that do not admit self-maps of degree -1. More generally, we construct products which do not admit self-maps of prime degree; see Section 7.2.

Chapter 2

Groups presentable by products

In this chapter we give an obstruction to domination by products for certain rationally essential manifolds with fundamental groups presentable by products. We pay particular attention to circle bundles over aspherical manifolds. First of all, these targets are aspherical (and hence rationally essential) and their fundamental groups are presentable by products having infinite center. Furthermore, irreducible fundamental groups of aspherical circle bundles constitute an important class of groups presentable by products, because every torsion-free group presentable by products either has virtually infinite center or it is reducible (or it satisfies both properties simultaneously). Here, we will mainly focus on circle bundles whose fundamental groups have infinite cyclic center.

We introduce a subclass of groups presentable by products, called "groups infinite-index presentable by products" (in short "IIPP"). We will see that the property "not IIPP" on the fundamental groups of certain rationally essential manifolds yields restrictions on the dimensions of the factors of a product that can dominate those manifolds. For large classes of circle bundles, we prove that the condition "not IIPP" is a complete obstruction to domination by products. Whenever the base is aspherical with fundamental group not presentable by products, we prove that a circle bundle is dominated by a product if and only if it is a virtual product. Our discussion yields non-trivial examples of finitely generated groups presentable by products, but not IIPP.

2.1 Groups infinite-index presentable by products (IIPP)

2.1.1 Preliminaries

Recall by Definition 1.7 that an infinite group Γ is presentable by products if there is a homomorphism $\varphi\colon \Gamma_1 \times \Gamma_2 \longrightarrow \Gamma$ onto a finite index subgroup of Γ so that both factors Γ_i have infinite image $\varphi(\Gamma_i) \subset \Gamma$. We begin this chapter by giving some elementary properties of groups presentable by products, mainly as introduced in [44, Section 3].

If a group Γ is presentable by a product through a homomorphism $\varphi\colon \Gamma_1 \times \Gamma_2 \longrightarrow \Gamma$, then the images $\varphi(\Gamma_i)$ commute with each other and $\varphi(\Gamma_1) \cup \varphi(\Gamma_2)$ generates $\mathrm{im}(\varphi)$. This means that, whenever a group Γ is presented by a product $\varphi\colon \Gamma_1 \times \Gamma_2 \longrightarrow \Gamma$, we can replace each Γ_i by its image $\varphi(\Gamma_i)$ and φ by the multiplication map. Therefore we may always assume that Γ can be presented by two commuting subgroups Γ_i through the multiplication map.

The following properties can be easily verified:

Lemma 2.1 ([44, Lemma 3.3]). *Suppose that Γ_1, Γ_2 are commuting subgroups of Γ so that $\Gamma_1 \cup \Gamma_2$ generates Γ. Then the multiplication map $\varphi\colon \Gamma_1 \times \Gamma_2 \longrightarrow \Gamma$ is a well-defined surjective homomorphism and the following statements hold:*

(1) *the intersection $\Gamma_1 \cap \Gamma_2$ is a subgroup of the center $C(\Gamma)$;*

(2) *the kernel of φ is isomorphic to the Abelian group $\Gamma_1 \cap \Gamma_2$.*

In particular, there exists a short exact sequence

$$1 \longrightarrow \Gamma_1 \cap \Gamma_2 \longrightarrow \Gamma_1 \times \Gamma_2 \stackrel{\varphi}{\longrightarrow} \Gamma \longrightarrow 1. \tag{2.1}$$

The isomorphism between $\Gamma_1 \cap \Gamma_2$ and the kernel of φ is given by the antidiagonal.

For groups with finitely generated center we moreover observe the following:

Lemma 2.2. *Let Γ be a finitely generated group with finitely generated center. Assume that Γ is presented by a product $\Gamma_1 \times \Gamma_2$ as in Lemma 2.1. Then each of the factors Γ_i is finitely generated.*

Proof. For $i, j \in \{1, 2\}$, $i \neq j$, there exist (two) short exact sequences

$$1 \longrightarrow \Gamma_i \longrightarrow \Gamma \longrightarrow \Gamma_j/(\Gamma_1 \cap \Gamma_2) \longrightarrow 1, \tag{2.2}$$

where $\Gamma \longrightarrow \Gamma_j/(\Gamma_1 \cap \Gamma_2)$ is obtained by composing the isomorphism $\Gamma \cong (\Gamma_1 \times \Gamma_2)/(\Gamma_1 \cap \Gamma_2)$ (cf. sequence (2.1)) with the homomorphism induced by the projection from $\Gamma_1 \times \Gamma_2$ to Γ_j (see also [45]).

Since Γ is finitely generated, the short exact sequence (2.2) implies that the quotient group $\Gamma_j/(\Gamma_1 \cap \Gamma_2)$ is also finitely generated. Moreover, the center $C(\Gamma)$ is finitely generated Abelian and thus the intersection $\Gamma_1 \cap \Gamma_2$ is also finitely generated Abelian by Lemma 2.1. This shows that Γ_j is also finitely generated. □

2.1.2 Definition of IIPP

Two basic examples of groups presentable by products are given by the reducible ones and by groups containing a finite index subgroup with infinite center; cf. Example 1.8. If follows by Lemma 2.1 that these two - not generally distinct - classes contain all torsion-free groups presentable by products:

Proposition 2.3 ([44, Prop. 3.2]). *Let Γ be a group whose every finite index subgroup has trivial center. Then the following are equivalent:*

(1) Γ *is reducible;*

(2) Γ *is presentable by products.*

The crucial difference between reducible groups and groups with infinite center is that a reducible group Γ can be presented (being a virtual product) by a product $\Gamma_1 \times \Gamma_2$ so that both subgroups Γ_i have infinite index in Γ, whereas groups with infinite center do not generally satisfy this property; a trivial example is given by the infinite cyclic group.

On the topological side, Theorem 1.9 states that whenever a rationally essential manifold M is dominated by a non-trivial product, its fundamental group must be presentable by products. However, (the proof of) that result does not provide any additional information on the index of the factors of a product presenting $\pi_1(M)$. The following example shows that all the possibilities can actually occur:

Example 2.4.

(1) Let M be a closed oriented manifold of positive dimension and infinite fundamental group. The identity map $\mathrm{id}_{M\times M}$ of the product $M \times M$ is obviously π_1-surjective of degree one and both subgroups $\mathrm{im}(\pi_1(\mathrm{id}_M)) = \pi_1(M)$ have infinite index in $\pi_1(M \times M)$.

(2) For $g \geq 1$, let $\Sigma_{g+1} = \Sigma_g \#(S^1_a \times S^1_b)$ be a closed oriented surface of genus $g+1$. Let the composition

$$\Sigma_g\#(S^1_a \times S^1_b) \stackrel{q}{\longrightarrow} \Sigma_g \vee (S^1_a \times S^1_b) \stackrel{\mathrm{id}\vee p}{\longrightarrow} \Sigma_g \vee S^1_b, \tag{2.3}$$

where q is the quotient map pinching to a point the essential circle defining the connected sum $\Sigma_g\#(S^1_a \times S^1_b)$, id is the identity map of Σ_g and the map p pinches to a point the meridian of the torus $S^1_a \times S^1_b$; cf. Figure 2.1. Denote by h the composite $(\mathrm{id}\vee p)\circ q$. Now, let the composition

$$\Sigma_{g+1} \times S^1_c \stackrel{h\times\mathrm{id}_c}{\longrightarrow} (\Sigma_g \vee S^1_b) \times S^1_c \stackrel{g}{\longrightarrow} \Sigma_g \times S^1_c.$$

The map h is given in (2.3) and id_c is the identity map of S^1_c. Finally, the map g restricts to the identity on Σ_g and S^1_c and sends the generator b of S^1_b to the generator c of S^1_c.

Let $f := g \circ (h \times \mathrm{id}_c)\colon \Sigma_{g+1} \times S^1_c \longrightarrow \Sigma_g \times S^1_c$. Then

$$H_3(f)([\Sigma_{g+1} \times S^1]) = [\Sigma_g \times S^1],$$

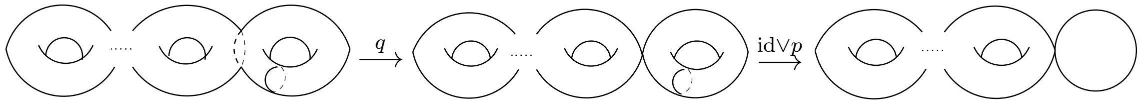

Figure 2.1: The map $(\mathrm{id} \vee p) \circ q\colon \Sigma_{g+1} \longrightarrow \Sigma_g \vee S^1_b$

i.e. $\deg(f) = 1$. By the definition of f, we obtain an index-one subgroup of $\pi_1(\Sigma_g \times S^1)$, namely $\mathrm{im}(\pi_1(f|_{\Sigma_{g+1}})) = \pi_1(\Sigma_g \times S^1)$, and the infinite-index subgroup $\mathrm{im}(\pi_1(f|_{S^1})) = \pi_1(S^1) \subset \pi_1(\Sigma_g \times S^1)$.

(3) Let two copies of a closed oriented surface of genus three,

$$\begin{aligned} \Sigma_3 &= (S^1_{g_1} \times S^1_{g_2})\#(S^1_{a_1} \times S^1_{a_2})\#(S^1_{b_1} \times S^1_{b_2}) \\ \Sigma_3' &= (S^1_{g_1'} \times S^1_{g_2'})\#(S^1_{a_1'} \times S^1_{a_2'})\#(S^1_{b_1'} \times S^1_{b_2'}). \end{aligned}$$

As in the previous example (cf. Figure 2.1), define $h\colon \Sigma_3 \longrightarrow (S^1_{g_1} \times S^1_{g_2}) \vee S^1_{a_2} \vee S^1_{b_2}$ as the composition

$$\Sigma_3 \stackrel{q}{\longrightarrow} (S^1_{g_1} \times S^1_{g_2}) \vee (S^1_{a_1} \times S^1_{a_2}) \vee (S^1_{b_1} \times S^1_{b_2}) \stackrel{\mathrm{id}\vee p\vee p}{\longrightarrow} (S^1_{g_1} \times S^1_{g_2}) \vee S^1_{a_2} \vee S^1_{b_2}$$

(see above for the notation). Now, let the composition

$$\Sigma_3 \times \Sigma_3' \stackrel{h\times h'}{\longrightarrow} ((S^1_{g_1} \times S^1_{g_2}) \vee S^1_{a_2} \vee S^1_{b_2}) \times ((S^1_{g_1'} \times S^1_{g_2'}) \vee S^1_{a_2'} \vee S^1_{b_2'}) \stackrel{g}{\longrightarrow} S^1_{g_1} \times S^1_{g_2} \times S^1_{g_1'} \times S^1_{g_2'},$$

where h, h' are defined above, and g restricts to the identity map on each $S^1_{g_j}$ and $S^1_{g_j'}$, and is given as follows on the rest of the circles:

$$a_2 \mapsto g_1', \quad b_2 \mapsto g_2', \quad a_2' \mapsto g_1, \quad b_2' \mapsto g_2.$$

We define $f\colon \Sigma_3 \times \Sigma_3' \longrightarrow T^4$ to be the composition $g \circ (h \times h')$. Again, f is a degree one map. However, both subgroups $\mathrm{im}(\pi_1(f|_{\Sigma_3}))$ and $\mathrm{im}(\pi_1(f|_{\Sigma_3'}))$ are now of index one in $\pi_1(T^4)$.

We note that this construction cannot be generalized when the target is not a product of two tori, $T^2 \times T^2$, because the generators of higher genus surfaces do not commute with each other. Actually, it will be transparent by the discussion in the upcoming section (cf. Lemma 2.7), that, if an n-dimensional aspherical manifold M admits a map $f\colon X_1 \times X_2 \longrightarrow M$ so that both subgroups $\mathrm{im}(\pi_1(f|_{X_i})) \subset \pi_1(M)$ are of finite index, then M is a virtual n-dimensional torus T^n.

In this chapter, we analyze groups presentable by products by adding a constraint on the index of the presenting factors. More precisely, we introduce the following class of groups presentable by products:

Definition 2.5. An infinite group Γ is called *infinite-index presentable by products* (IIPP) if there is a homomorphism $\varphi\colon \Gamma_1 \times \Gamma_2 \longrightarrow \Gamma$ onto a finite index subgroup of Γ so that for both factors Γ_i the images $\varphi(\Gamma_i) \subset \Gamma$ are of infinite index in Γ.

In the upcoming section, we will use this definition to show that large classes of rationally essential manifolds with fundamental groups presentable by products, but not IIPP, cannot be dominated by products. Furthermore, as we shall see in the last two sections, the condition "IIPP" is equivalent to "reducible" for the fundamental groups of certain aspherical circle bundles and this equivalence characterizes those bundles that are dominated by products.

Our interest in torsion-free groups stems from the problem of determining whether aspherical manifolds with reducible fundamental groups are always virtual products of two manifolds; cf. Remark 1.13.

We note that for non-torsion-free groups whose every finite index subgroup has finite center, the notions "presentable by products" and "IIPP" are equivalent:

Proposition 2.6. *If every subgroup of finite index in Γ has finite center, then Γ is presentable by products if and only if it is* IIPP.

Proof. It suffices to show that presentability by products implies IIPP. Suppose that Γ_1, Γ_2 are commuting infinite subgroups of Γ and that there is a short exact sequence

$$1 \longrightarrow \Gamma_1 \cap \Gamma_2 \longrightarrow \Gamma_1 \times \Gamma_2 \stackrel{\varphi}{\longrightarrow} \Gamma \longrightarrow 1,$$

where φ is the multiplication homomorphism and $\Gamma_1 \cap \Gamma_2$ lies in the center of Γ; cf. Lemma 2.1. Since $C(\Gamma)$ is finite, we deduce that $\Gamma_1 \cap \Gamma_2$ is also finite and therefore it has infinite index in both Γ_i. The proof now follows by the short exact sequence (2.2). □

2.2 Not IIPP as a non-domination criterion

In this section we extend Kotschick-Löh's non-domination results to rationally essential manifolds with fundamental groups presentable by products, but not IIPP. The strong feature of such torsion-free groups is that one of the presenting subgroups must be Abelian:

Lemma 2.7. *Let Γ be a finitely generated torsion-free group with finitely generated center. Suppose that there exist commuting subgroups $\Gamma_1, \Gamma_2 \subset \Gamma$ so that $\Gamma_1 \cup \Gamma_2$ generates Γ. If Γ is not* IIPP*, then one of the Γ_i is isomorphic to $\mathbb{Z}^k$ for some $k \leq \mathrm{rank} C(\Gamma)$.*

Proof. By Lemma 2.1, there is a short exact sequence

$$1 \longrightarrow \Gamma_1 \cap \Gamma_2 \longrightarrow \Gamma_1 \times \Gamma_2 \stackrel{\varphi}{\longrightarrow} \Gamma \longrightarrow 1,$$

where φ is the multiplication map and the intersection $\Gamma_1 \cap \Gamma_2$ is contained in the finitely generated center $C(\Gamma)$. Since Γ is torsion-free, we have that $\Gamma_1 \cap \Gamma_2$ is isomorphic to $\mathbb{Z}^k$ for some $k \leq \mathrm{rank} C(\Gamma)$. Moreover, each Γ_i is finitely generated by Lemma 2.2.

Because Γ is not IIPP, one of the Γ_i, say Γ_1, must have finite index in Γ. This means that Γ_2 is virtually $\Gamma_1 \cap \Gamma_2$, and so it is virtually Abelian. Moreover, the intersection $\Gamma_1 \cap \Gamma_2$ is central in Γ_2, which implies that $\Gamma_2/C(\Gamma_2)$ is finite (because Γ_2 is virtually $\Gamma_1 \cap \Gamma_2$). By Schur's theorem [63] (see Remark 2.8 below), we conclude that the commutator $[\Gamma_2, \Gamma_2]$ is also finite and so trivial, because Γ_2 is torsion-free. This shows that Γ_2 is Abelian itself and therefore isomorphic to $\mathbb{Z}^k$ since it is finitely generated. □

Remark 2.8. Schur's [63] theorem says that a group Γ has finite commutator subgroup $[\Gamma, \Gamma]$, if the quotient group $\Gamma/C(\Gamma)$ is finite. Whenever the center $C(\Gamma)$ is torsion-free, Schur's theorem can be proven using a modern language: For the (central) extension

$$1 \longrightarrow C(\Gamma) \longrightarrow \Gamma \longrightarrow \Gamma/C(\Gamma) \longrightarrow 1,$$

consider the following part of Stallings [71] five-term exact sequence:

$$H_2(\Gamma/C(\Gamma); \mathbb{Z}) \longrightarrow C(\Gamma) \longrightarrow \Gamma/[\Gamma, \Gamma].$$

If $\Gamma/C(\Gamma)$ is finite, then $H_2(\Gamma/C(\Gamma); \mathbb{Z})$ is also finite and so $H_2(\Gamma/C(\Gamma); \mathbb{Z}) \longrightarrow C(\Gamma)$ is the zero homomorphism, because $C(\Gamma)$ is torsion-free. Thus $C(\Gamma) \cap [\Gamma, \Gamma] = 1$, and the inequality

$$[[\Gamma, \Gamma] : C(\Gamma) \cap [\Gamma, \Gamma]] \leq [\Gamma : C(\Gamma)] < \infty$$

implies that $[\Gamma, \Gamma]$ is finite as well.

The above lemma yields the following dimension restrictions on the factors of a product that dominates a rationally essential manifold with torsion-free fundamental group:

Proposition 2.9. *Let M be a rationally essential manifold so that $\pi_1(M)$ is torsion-free and* $\mathrm{rank} C(\pi_1(M)) = r$. *If $\pi_1(M)$ is not* IIPP, *then there is no π_1-surjective non-zero degree map $X_1 \times X_2 \longrightarrow M$, whenever* $\min\{\dim X_1, \dim X_2\} > r$.

Proof. Suppose that there exist X_1, X_2 closed oriented connected manifolds of positive dimensions and a π_1-surjective non-zero degree map $f \colon X_1 \times X_2 \longrightarrow M$. Then there is a short exact sequence

$$1 \longrightarrow \Gamma_1 \cap \Gamma_2 \longrightarrow \Gamma_1 \times \Gamma_2 \stackrel{\varphi}{\longrightarrow} \pi_1(M) \longrightarrow 1, \qquad (2.4)$$

where φ is the multiplication map, $\Gamma_i := \pi_1(f|_{X_i})(\pi_1(X_i))$ and $\Gamma_1 \cap \Gamma_2 \subset C(\pi_1(M))$; see Section 1.2.1. In particular, $\Gamma_1 \cap \Gamma_2$ is isomorphic to $\mathbb{Z}^k$, for some $k \leq r = \mathrm{rank} C(\pi_1(M))$, because is torsion-free. We moreover observe that $k \geq 1$, otherwise $\pi_1(M)$ would be isomorphic to the product $\Gamma_1 \times \Gamma_2$ by (2.4) and so IIPP.

We now apply Lemma 2.7 to $\pi_1(M)$ to conclude that one of the Γ_i, say Γ_2, is isomorphic to $\mathbb{Z}^k$. This means that $B\Gamma_2 \simeq T^k$ and by the non-vanishing (cf. Section 1.2.1) of the homology class

$$\alpha_2 := H_{\dim X_2}(B\pi_1(f|_{X_2}) \circ c_{X_2})([X_2]) \in H_{\dim X_2}(T^k; \mathbb{Q}),$$

we deduce that $\dim X_2 \leq k$. This is possible only if $\min\{\dim X_1, \dim X_2\} \leq r$ and the proposition follows. $\square$

Example 2.10. Let M be a closed aspherical manifold. If $\pi_1(M)$ has infinite cyclic center and is not IIPP, then M can admit a degree one map by a product $X_1 \times X_2$ only if one of the X_i is a circle. (Recall that a map of degree one is π_1-surjective; cf. Lemma 1.2 (1).)

Manifolds whose fundamental groups have infinite cyclic center are of particular interest, because torsion-free virtually infinite cyclic groups are themselves infinite cyclic. This applies to the study of aspherical circle bundles. We begin with two general properties about finite coverings of circle bundles:

Lemma 2.11. *Let $M \xrightarrow{\pi} B$ be a circle bundle over a closed oriented connected manifold B.*

(1) *Every finite cover $\overline{M} \xrightarrow{p} M$ is a circle bundle over a finite cover $B' \xrightarrow{p'} B$. If moreover B is aspherical and $\pi_1(B)$ is not presentable by products, then $\pi_1(\overline{M})$ and $\pi_1(M)$ have infinite cyclic center.*

(2) ([9, Prop. 3]) *If the Euler class of M is torsion, then M is a virtually trivial circle bundle over a finite cover of B.*

Proof. (1) Since $\pi_1(p)(\pi_1(\overline{M}))$ has finite index in $\pi_1(M)$ and $\pi_1(\pi)(\pi_1(M)) = \pi_1(B)$, the image

$$H := \pi_1(\pi \circ p)(\pi_1(\overline{M}))$$

has finite index in $\pi_1(B)$. Let $B' \xrightarrow{p'} B$ be the finite covering corresponding to H. Then $\pi \circ p$ lifts to $\overline{M} \xrightarrow{\pi'} B'$, which is the desired circle bundle.

If B is aspherical, then the S^1 fiber is central in the fundamental group of M. If, in addition, $\pi_1(B)$ is not presentable by products, then it has trivial center (because it is torsion-free), and so the center of $\pi_1(M)$ is infinite cyclic. Now $\pi_1(B')$ has finite index in $\pi_1(B)$, and so it is not presentable by products as well and therefore the center of $\pi_1(\overline{M})$ is also infinite cyclic.

(2) Consider the abelianization $H_1(B) = \pi_1(B)/[\pi_1(B), \pi_1(B)]$. Since the Euler class of M is torsion, the Universal Coefficient Theorem implies that the torsion part of $H_1(B)$ is not trivial. Let now the composition

$$\pi_1(B) \longrightarrow H_1(B) \longrightarrow \mathrm{Tor} H_1(B),$$

where the first map is the quotient map and the second is the projection to the torsion of $H_1(B)$. If $B' \xrightarrow{p'} B$ is the finite covering corresponding to the kernel of the above composition, then the pullback bundle $(p')^*(M)$ is the desired product $S^1 \times B'$; see [9] for the details. □

Remark 2.12. Conversely to part (2) of the above lemma, let $\overline{M} = S^1 \times B' \xrightarrow{p} M$ be a finite cover, where $B' \xrightarrow{p'} B$ is a finite covering between the bases (the map p' is covered by p). The Euler class of $\overline{M}$ is trivial, that is $e_{\overline{M}} = H^2(p';\mathbb{Z})(e_M) = 0 \in H^2(B',\mathbb{Z})$, where $e_M \in H^2(B;\mathbb{Z})$ is the Euler class of M. By the fact that $H^2(p';\mathbb{Q})$ is injective (cf. Lemma 1.2 (2)), we conclude that e_M is torsion.

We can now obtain a complete non-domination result for aspherical circle bundles whose fundamental groups are not IIPP and whose virtual center is generated by multiples of the fiber:

Theorem 2.13. *Let M be a circle bundle over a closed oriented connected aspherical manifold B so that $\pi_1(M)$ is not* IIPP *and it has virtual center at most $\mathbb{Z}$. Then $P \not\geq M$ for any non-trivial product P of closed oriented connected manifolds.*

Proof. Since $\pi_1(M)$ is not IIPP, M is a non-trivial circle bundle and, moreover, its Euler class is not torsion by Lemma 2.11 (2). After passing to a finite cover, if necessary, suppose that there is a π_1-surjective non-zero degree map $f\colon P = X_1 \times X_2 \longrightarrow M$, where $\dim X_i > 0$ and $C(\pi_1(M)) = \mathbb{Z}$; cf. Lemma 2.11 (1). As before, there is a short exact sequence

$$1 \longrightarrow \Gamma_1 \cap \Gamma_2 \longrightarrow \Gamma_1 \times \Gamma_2 \xrightarrow{\varphi} \pi_1(M) \longrightarrow 1,$$

where $\Gamma_i := \mathrm{im}(\pi_1(f|_{X_i})) \subset \pi_1(M)$, φ is the multiplication map and $\Gamma_1 \cap \Gamma_2 \subset C(\pi_1(M)) = \mathbb{Z}$, see Section 1.2.1.

Lemma 2.7 implies that one of the Γ_i, say Γ_2, must be infinite cyclic, because $\pi_1(M)$ is not IIPP and torsion-free. Therefore, $B\Gamma_2 \simeq S^1$ and because the rational homology class

$$\alpha_2 := H_{\dim X_2}(B\pi_1(f|_{X_2}) \circ c_{X_2})([X_2]) \in H_{\dim X_2}(S^1;\mathbb{Q})$$

is not trivial we conclude that $\dim X_2 = 1$, i.e. $X_2 = S^1$. Now, we have a π_1-surjective dominant map $X_1 \times S^1 \longrightarrow M$, where $C(\pi_1(M)) = \mathbb{Z}$. The proof is completed by the next lemma, which generalizes [46, Lemma 1]. □

Lemma 2.14. *Let $M \xrightarrow{\pi} B$ be a non-trivial n-dimensional circle bundle over a closed oriented connected aspherical manifold B. Suppose that the Euler class of M is not torsion and that $\pi_1(M)$ has virtual center at most $\mathbb{Z}$. Then $X \times S^1 \not\geq M$ for any closed oriented connected manifold X.*

Proof. Since M is a non-trivial circle bundle whose integer Euler class $e_M \in H^2(B;\mathbb{Z})$ is not torsion, the rational Euler class of M is not trivial as well. The same property holds for every (fiber preserving) finite cover of M, by Lemma 2.11 and Remark 2.12.

By Poincaré duality, there exists a non-trivial class $\alpha \in H^{n-3}(B;\mathbb{Q})$ so that $e \cup \alpha$ is a non-zero multiple of the cohomology fundamental class ω_B of B. Since $H^{n-1}(B;\mathbb{Q}) = \mathbb{Q}$, the Gysin sequence

$$\cdots \longrightarrow H^{n-3}(B;\mathbb{Q}) \xrightarrow{\cup e} H^{n-1}(B;\mathbb{Q}) \xrightarrow{H^{n-1}(\pi)} H^{n-1}(M;\mathbb{Q}) \longrightarrow \cdots$$

implies that $\ker(H^{n-1}(\pi)) = \mathrm{im}(\cup e) = H^{n-1}(B;\mathbb{Q})$. Therefore $H^{n-1}(\pi) = 0$.

Suppose now that there exists a non-zero degree map $f\colon X \times S^1 \longrightarrow M$. After passing to a finite cover, if necessary, we may assume that f is π_1-surjective and that the center of $\pi_1(M)$ is infinite cyclic. The latter means that the circle fiber of M represents (up to multiples) the only central factor in $\pi_1(M)$. By the surjectivity of $\pi_1(f)$, we deduce that the composite $\pi \circ f$ kills the homotopy class of the S^1 factor of the product $X \times S^1$, because this factor is central in $\pi_1(X \times S^1)$. Since B is aspherical, we conclude that $\pi \circ f$ factors up to homotopy through the projection $p_1\colon X \times S^1 \longrightarrow X$. In particular, there is a continuous map $g\colon X \longrightarrow B$, so that $\pi \circ f = g \circ p_1$ up to homotopy. (We note that X is not necessarily aspherical. It is, however, rationally essential, because f has non-zero degree and M is aspherical.)

Let ω_X be the cohomology fundamental class of X. Since $H^{n-1}(p_1;\mathbb{Q})(\omega_X) = \omega_X \in H^{n-1}(X \times S^1;\mathbb{Q})$ and $H^{n-1}(\pi;\mathbb{Q})(\omega_B) = 0 \in H^{n-1}(M;\mathbb{Q})$, the homotopy equation $\pi \circ f = g \circ p_1$ implies that g must be of zero degree. Let now the pullback of M under g:

$$g^*M = \{(x,y) \in X \times M \mid g(x) = \pi(y)\} .$$

The map $f\colon X \times S^1 \longrightarrow M$ factors through g^*M as follows:

$$\begin{array}{ccccc} X \times S^1 & \longrightarrow & g^*M & \xrightarrow{\pi_2} & M \\ (x,t) & \mapsto & (x, f(x,t)) & \mapsto & f(x,t) . \end{array}$$

We have that the degree of the pullback map $\pi_2\colon g^*M \longrightarrow M$ is zero being equal to the degree of g. Thus f factors through a degree zero map, contradicting our assumption. This completes the proof. □

Remark 2.15. The requirement in the above lemma, that the center of $\pi_1(M)$ remains infinite cyclic in every finite cover, is essential and cannot be omitted. For example, let N be a virtually non-trivial circle bundle over a closed oriented hyperbolic surface Σ. Then, $N \times S^1$ is a product which has also the structure of a virtually non-trivial circle bundle over $\Sigma \times S^1$ and satisfies all the assumptions of Lemma 2.14, except that $\pi_1(N \times S^1)$ has a finite index subgroup whose center is $\mathbb{Z} \times \mathbb{Z}$, instead of $\mathbb{Z}$.

The manifold $N \times S^1$ is actually an example of a closed geometric four-manifold (with geometry modelled on $\widetilde{SL_2} \times \mathbb{R}$), with which we will deal in Chapter 5. A complete treatment of the fundamental groups of closed three-manifolds is given in the upcoming section.

Example 2.16. In Chapter 5, we will show that closed Nil^4-manifolds, i.e. nilpotent geometric closed four-manifolds that are not virtual products, fulfill all the conditions of Theorem 2.13 and therefore are never dominated by products; cf. Propositions 5.11 and 5.25. One can easily construct examples of such manifolds, as mapping tori of suitable self-homeomorphisms of T^3; see Remark 5.12.

2.3 Groups presentable by products but not IIPP

2.3.1 Groups with infinite cyclic center

As we have already mentioned, the class of IIPP groups is strictly contained in the class of groups presentable by products, because the infinite cyclic group is an example of a group presentable by products which is not IIPP. In this section, we give non-trivial examples of groups presentable by products but not IIPP. In particular, we determine all three-manifold IIPP groups.

We first show that certain torsion-free IIPP groups with infinite cyclic center are always reducible:

Proposition 2.17. *Let Γ be a group with infinite cyclic center so that $\Gamma/C(\Gamma)$ is torsion-free and not presentable by products. Then Γ is reducible if and only if it is* IIPP.

Proof. Clearly, the only non-trivial statement is that "IIPP" implies "reducible". We observe that Γ is torsion-free because it fits into the extension

$$1 \longrightarrow C(\Gamma) \longrightarrow \Gamma \stackrel{\pi}{\longrightarrow} \Gamma/C(\Gamma) \longrightarrow 1,$$

where $C(\Gamma) = \mathbb{Z}$ and $\Gamma/C(\Gamma)$ is torsion-free by assumption. (The map π is the quotient map.)

Suppose that Γ is IIPP. Then, after possibly passing to a finite index subgroup, there exist $\Gamma_1, \Gamma_2 \subset \Gamma$ commuting subgroups of infinite index and a short exact sequence

$$1 \longrightarrow \Gamma_1 \cap \Gamma_2 \longrightarrow \Gamma_1 \times \Gamma_2 \stackrel{\varphi}{\longrightarrow} \Gamma \longrightarrow 1, \qquad (2.5)$$

where φ is the multiplication map and $\Gamma_1 \cap \Gamma_2 \subset C(\Gamma)$; cf. Section 2.1.1.

Since $\Gamma/C(\Gamma)$ is not presentable by products and torsion-free, the composite homomorphism

$$\Gamma_1 \times \Gamma_2 \stackrel{\varphi}{\longrightarrow} \Gamma \stackrel{\pi}{\longrightarrow} \Gamma/C(\Gamma)$$

maps one of the Γ_i to the trivial element in $\Gamma/C(\Gamma)$. Without loss of generality, we may assume that $(\pi \circ \varphi)(\Gamma_2) = 1$, i.e. $\Gamma_2 \subset C(\Gamma)$. Since Γ_2 is not trivial, we deduce that it is isomorphic to $C(\Gamma) = \mathbb{Z}$.

We claim that the intersection $\Gamma_1 \cap \Gamma_2$ is trivial and therefore Γ is reducible, by the short exact sequence (2.5). Recall that $\Gamma_1 \cap \Gamma_2$ is a subgroup of $C(\Gamma) = \mathbb{Z}$. So, if $\Gamma_1 \cap \Gamma_2$ were not trivial, then it would be isomorphic to $\mathbb{Z}$ as well. Thus, $\Gamma_1 \cap \Gamma_2$ would have finite index in Γ_2, i.e. $[\Gamma\colon\Gamma_1] < \infty$, contradicting our assumption that both Γ_i have infinite index in Γ. Therefore, the intersection $\Gamma_1 \cap \Gamma_2$ is trivial and Γ is reducible being isomorphic to the direct product $\Gamma_1 \times \mathbb{Z}$. □

2.3.2 A low-dimensional example

We now deal with a low-dimensional example, namely with infinite fundamental groups of closed three-manifolds. Significant results on fibered three-manifolds and on their fundamental groups were obtained by Epstein and Stallings:

Theorem 2.18. *Let M be a compact three-manifold.*

(1) (Epstein [24]) *If $\pi_1(M)$ is isomorphic to a non-trivial product of two groups $\Gamma_1 \times \Gamma_2$ so that at least one of them is infinite, then either Γ_1 or Γ_2 is infinite cyclic.*

(2) (Stallings [70]) *If M is irreducible (i.e. every 2-sphere bounds an embedded 3-ball in M) and $\pi_1(M)$ fits into a short exact sequence*

$$1 \longrightarrow K \longrightarrow \pi_1(M) \longrightarrow \mathbb{Z} \longrightarrow 1,$$

where K is finitely generated and not isomorphic to $\mathbb{Z}_2$, then M fibers over the circle.

Putting together these results of Epstein and Stallings, we conclude that a closed oriented three-manifold with reducible fundamental group is a virtual product $\Sigma \times S^1$, for some closed oriented aspherical surface Σ.

Relying on the above factorization theorem of Epstein and on a result by Kotschick-Löh, that non-virtually cyclic free products are not presentable by products (Theorem 1.11), we now show that the existence of virtual center in three-manifold groups is equivalent to presentability by products:

Proposition 2.19 ([46, Theorem 8])**.** *For a closed three-manifold M with infinite fundamental group the following are equivalent:*

(1) *$\pi_1(M)$ is presentable by a product;*

(2) *$\pi_1(M)$ has virtually infinite center.*

Proof. Since (2) implies (1) for any group, we only need to show that the converse is also true for three-manifold groups.

By a result of Kotschick-Löh [45, Cor. 9.2] (cf. Theorem 1.11), the only non-trivial free product that is presentable by a product is $\mathbb{Z}_2 * \mathbb{Z}_2$, which is virtually $\mathbb{Z}$ and so it satisfies (2). Thus, we may assume that $\pi_1(M)$ is freely indecomposable, and M is prime; see Section 4.1 for a brief discussion about the prime decomposition of three-manifolds. If $\pi_1(M)$ is not virtually $\mathbb{Z}$, then M is irreducible and aspherical by the Sphere Theorem, cf. [50]. In particular, $\pi_1(M)$ is torsion-free. Now, Proposition 2.3 implies that $\pi_1(M)$ contains a finite index subgroup which either has infinite center, or it splits as a direct product of infinite groups. There is nothing to prove if the first alternative holds. For the latter one, Theorem 2.18 (1) says that $\pi_1(M)$ is a virtual product with an infinite cyclic direct factor, which is clearly central in the whole group. □

The two properties of the above proposition are moreover equivalent to M being Seifert fibered (with infinite fundamental group), by the Seifert fiber space conjecture, which was independently proven by Gabai [30] and Casson-Jungreis [14]; cf. Theorem 4.4. A closed three-manifold (with possibly finite fundamental group) is Seifert fibered if and only if it is virtually a circle bundle over a closed oriented surface, by the works of Seifert, Thurston and Scott; cf. Theorem 4.3. We shall recall these statements at the beginning of Chapter 4.

We therefore obtain the following consequence of Proposition 2.19:

Corollary 2.20. *Suppose that M is a closed three-manifold with infinite fundamental group. Then $\pi_1(M)$ is presentable by products if and only if M is a virtual circle bundle over a closed oriented surface.*

However, if the circle fiber of M is not (virtually) a direct factor, i.e. if M is not a (virtual) product of the circle with a closed surface, then $\pi_1(M)$ cannot be IIPP.

Proposition 2.21. *The fundamental group of a non-trivial circle bundle M over a closed oriented aspherical surface Σ is not* IIPP.

Proof. If Σ is hyperbolic, then $\pi_1(M)$ fits into a non-split central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(M) \longrightarrow \pi_1(\Sigma) \longrightarrow 1.$$

In particular, $\pi_1(M)$ fulfills the conditions of Proposition 2.17, because non-virtually cyclic hyperbolic groups are not presentable by products, by Theorem 1.11; see Section 5.2.1 for a proof. By Theorem 2.18, the fundamental group of a non-trivial circle bundle over a closed oriented surface is never reducible, and so Proposition 2.17 implies that $\pi_1(M)$ is not IIPP as well.

The remaining case is when Σ has genus one, i.e. when M is a non-trivial circle bundle over T^2 (and so is a Nil^3-manifold; cf. Chapter 4). In that case, $\pi_1(M)$ fits into a non-split central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(M) \longrightarrow \mathbb{Z}^2 \longrightarrow 1,$$

where $C(\pi_1(M)) = \mathbb{Z}$. Suppose that $\pi_1(M)$ is IIPP. Then we may assume that there exist non-trivial infinite-index commuting subgroups $\Gamma_1, \Gamma_2 \subset \pi_1(M)$ and a short exact sequence

$$1 \longrightarrow \Gamma_1 \cap \Gamma_2 \longrightarrow \Gamma_1 \times \Gamma_2 \stackrel{\varphi}{\longrightarrow} \pi_1(M) \longrightarrow 1,$$

where φ is the multiplication map and $\Gamma_1 \cap \Gamma_2 \subset C(\pi_1(M))$. (Note that both Γ_i are torsion-free, because $\pi_1(M)$ is torsion-free.)

We observe that the intersection $\Gamma_1 \cap \Gamma_2$ cannot be trivial, otherwise M would be a trivial circle bundle by Theorem 2.18. This means that $\Gamma_1 \cap \Gamma_2$ must be isomorphic to $C(\pi_1(M)) = \mathbb{Z}$. Moreover, since $[\pi_1(M) \colon \Gamma_i] = \infty$ and $\pi_1(M)$ has cohomological dimension[3] three, we conclude that each of the Γ_i is of cohomological dimension at most two [74].

Now, $\Gamma_1 \cap \Gamma_2$ is central in both Γ_i which means that the quotients $\Gamma_i/(\Gamma_1 \cap \Gamma_2)$ are finitely generated and virtually free groups F_{k_i}, by a result of Bieri [7, Cor. 8.7]. Passing to finite coverings, we may assume that these quotient groups are free and therefore the central extensions

$$1 \longrightarrow \Gamma_1 \cap \Gamma_2 \longrightarrow \Gamma_i \longrightarrow F_{k_i} \longrightarrow 1$$

split. We have finally reached the conclusion that $\pi_1(M)$ is virtually isomorphic to a direct product $\mathbb{Z} \times F_{k_1} \times F_{k_2}$, which contradicts Theorem 2.18. Alternatively, the latter conclusion is absurd, because if $\pi_1(M)$ were $\mathbb{Z} \times F_{k_1} \times F_{k_2}$, then the first Betti number of M would be at least three, which is impossible because M is a non-trivial circle bundle over T^2. □

Remark 2.22. We could handle the case of nilpotent groups in the above proposition in a different (and more elementary) way, using the fact that the cohomological dimension of a finitely generated torsion-free nilpotent group coincides to its Hirsch length. However, the dimensions here were suitable to appeal to Bieri's [7] result on central extensions. We will return to this interesting property about the Hirsch length of nilpotent groups in Section 5.2.2.

Because the fundamental group of $S^2 \times S^1$ is infinite cyclic and therefore not IIPP, we have now determined all fundamental groups of closed three-manifolds that are (not) IIPP:

Corollary 2.23. *Suppose that the fundamental group of a closed oriented three-manifold M is infinite. Then $\pi_1(M)$ is* IIPP *if and only if it is reducible. Equivalently, $\pi_1(M)$ is a virtual product $\pi_1(\Sigma) \times \mathbb{Z}$, where Σ is a closed oriented aspherical surface.*

The above corollary is case (2) of Theorem 1.18.

[3]The *cohomological dimension* of a group Γ is defined to be

$$\mathrm{cd}(\Gamma) := \sup\{n \mid H^n(\Gamma; A) \neq 0 \text{ for some } \Gamma\text{−module } A\},$$

if such integers n exist, otherwise we set $\mathrm{cd}(\Gamma) = \infty$.

2.4 Characterizations for circle bundles

As Remark 1.10 indicates, a main motivation behind Theorem 1.9 is to show that non-positively curved closed manifolds which are not virtual products cannot be dominated by products. Actually, the property "fundamental group presentable by products" suffices for domination by products for non-positively curved manifolds and is equivalent to reducibility:

Theorem 2.24 ([44, Theorem 4.1]). *Let M be a closed oriented connected Riemannian manifold of dimension $n \geq 2$, which admits a metric of non-positive sectional curvature. The following properties are equivalent:*

(1) *M is dominated by a non-trivial product of closed oriented connected manifolds;*

(2) *$\pi_1(M)$ is presentable by products;*

(3) *$\pi_1(M)$ is reducible;*

(4) *M is virtually diffeomorphic to a non-trivial product of closed oriented connected manifolds.*

Another consequence of the results of [44], which moreover contains examples of manifolds that do not admit any metric of non-positive sectional curvature, concerns fibrations whose fiber and base have fundamental groups not presentable by products:

Theorem 2.25 ([44, Theorem 5.1]). *Let $F \longrightarrow M \stackrel{\pi}{\longrightarrow} B$ be a fiber bundle whose fiber F and base B are closed oriented connected aspherical manifolds with fundamental groups not presentable by products. Then M is dominated by products if and only if it is a virtual product $F' \times B'$, where F' and B' are finite covers of F and B respectively.*

In particular, we have the following in dimension four:

Corollary 2.26 ([44, Cor. 5.3]). *Let M be a closed oriented four-manifold which is the total space of a surface bundle whose fiber F and base B are both hyperbolic surfaces. Then the following are equivalent:*

(1) *M is dominated by a non-trivial product of closed oriented connected manifolds;*

(2) *$\pi_1(M)$ is presentable by products;*

(3) *$\pi_1(M)$ is reducible;*

(4) *M is virtually diffeomorphic to a trivial surface bundle.*

We note that four-manifolds satisfying one (and therefore every) property in the above corollary constitute the class of closed reducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds; cf. Chapter 5. Moreover, we remark that examples of surface bundles over surfaces whose both fiber and base are hyperbolic, and that do not admit any metric of non-positive sectional curvature are given in [42].

Let us now replace the fiber F by S^1 in Theorem 2.25. Then, obviously $\pi_1(M)$ is presentable by products and it moreover fulfills the conditions of Proposition 2.17. As illustrated in the proof of Proposition 2.21 (for circle bundles over hyperbolic surfaces), the topological analogue of the algebraic statement of Proposition 2.17 corresponds to circle bundles over aspherical manifolds whose fundamental groups are not presentable by products. We can therefore prove that the conclusion of Theorem 2.25 still holds, if we replace F by S^1. However, domination by products is now equivalent to the conditions "$\pi_1(M)$ IIPP" and "$\pi_1(M)$ reducible"; compare the corresponding equivalences of Corollary 2.26.

Theorem 2.27. *Let $M \stackrel{\pi}{\longrightarrow} B$ be a circle bundle over a closed oriented connected aspherical manifold B whose fundamental group $\pi_1(B)$ is not presentable by products. Then the following are equivalent:*

(1) *$P \geq M$ for some non-trivial product P of closed oriented connected manifolds;*

(2) *M is finitely covered by a product $S^1 \times B'$, for some finite cover $B' \longrightarrow B$;*

(3) *$\pi_1(M)$ is reducible;*

(4) *$\pi_1(M)$ is* IIPP.

Proof. Since B is aspherical, M is also aspherical and its fundamental group fits into a short exact sequence

$$1 \longrightarrow \pi_1(S^1) \longrightarrow \pi_1(M) \stackrel{\pi_1(\pi)}{\longrightarrow} \pi_1(B) \longrightarrow 1, \tag{2.6}$$

where $\pi_1(S^1)$ is in the center of $\pi_1(M)$. Moreover, $\pi_1(B)$ has trivial center, because it is torsion-free and not presentable by products. Thus $C(\pi_1(M)) = \pi_1(S^1) = \mathbb{Z}$.

Suppose that there is a non-zero degree map $f\colon P = X_1 \times X_2 \longrightarrow M$. After passing to a finite cover, if necessary, we may assume that f is π_1-surjective. (The finite cover of M is a circle bundle with infinite cyclic center, by Lemma 2.11 (1).) As before, we have a short exact sequence

$$1 \longrightarrow \Gamma_1 \cap \Gamma_2 \longrightarrow \Gamma_1 \times \Gamma_2 \stackrel{\varphi}{\longrightarrow} \pi_1(M) \longrightarrow 1,$$

where $\Gamma_i := \mathrm{im}(\pi_1(f|_{X_i})) \subset \pi_1(M)$ and $\Gamma_1 \cap \Gamma_2 \subset C(\pi_1(M)) = \mathbb{Z}$. Moreover, we obtain two non-trivial rational homology classes

$$\alpha_i := H_{\dim X_i}(B\pi_1(f|_{X_i}) \circ c_{X_i})([X_i]) \neq 0 \in H_{\dim X_i}(B\Gamma_i; \mathbb{Q}),$$

see Sections 1.2.1 and 2.1.1 for the details.

As in the proof of Proposition 2.17, the composite homomorphism

$$\Gamma_1 \times \Gamma_2 \xrightarrow{\varphi} \pi_1(M) \xrightarrow{\pi_1(\pi)} \pi_1(B) \cong \pi_1(M)/\pi_1(S^1)$$

maps one of the Γ_i, say Γ_1, to the neutral element of $\pi_1(B)$, because $\pi_1(B)$ is not presentable by products and torsion-free. This means that Γ_1 is contained in $C(\pi_1(M)) = \pi_1(S^1) = \mathbb{Z}$ and is therefore isomorphic to $C(\pi_1(M)) = \mathbb{Z}$. In particular, $B\Gamma_1 \simeq B\mathbb{Z} = S^1$ and so the non-vanishing of $\alpha_1 \in H_{\dim X_1}(S^1;\mathbb{Q})$ implies that $\dim X_1 \leq 1$. Since $\dim X_1 > 0$, we have that $X_1 = S^1$, i.e. $S^1 \times X_2 \geq M$. It follows by Lemma 2.14 that M is a virtual product and, more precisely, that it is finitely covered by a product $S^1 \times B'$ for some finite cover $B' \longrightarrow B$. Thus (1) implies (2). The converse is trivially true and so (1) is equivalent to (2).

Next, we show that the properties (2) and (3) are equivalent. Obviously (2) implies (3). Assume now that $\pi_1(M)$ is reducible, i.e. there exists a finite cover $M' \longrightarrow M$ so that $\pi_1(M')$ is isomorphic to a direct product $\Delta_1 \times \Delta_2$, where Δ_i are infinite subgroups of $\pi_1(M')$. The cover M' is a circle bundle over a finite cover B' of B, where $\pi_1(B')$ is not presentable by products being a finite index subgroup of $\pi_1(B)$; cf. Lemma 2.11 (1). We therefore obtain a short exact sequence

$$1 \longrightarrow \pi_1(S^1) \longrightarrow \Delta_1 \times \Delta_2 \longrightarrow \pi_1(B') \longrightarrow 1, \tag{2.7}$$

where $\pi_1(S^1) = C(\pi_1(M')) \cong C(\Delta_1) \times C(\Delta_2)$. Since $\pi_1(B')$ is not presentable by products and torsion-free, one of the Δ_i, say Δ_1, maps trivially to $\pi_1(B') \cong \pi_1(M')/\pi_1(S^1)$ in (2.7). Thus $\Delta_1 \subset \pi_1(S^1)$ (and so Δ_1 is isomorphic to $\mathbb{Z}$) and Δ_2 surjects onto $\pi_1(B')$. Moreover, $\pi_1(S^1)$ maps trivially to Δ_2, otherwise Δ_2 would have finite index in $\pi_1(M')$, which is impossible, because $\pi_1(M') \cong \Delta_1 \times \Delta_2$ and both Δ_i are infinite. Therefore Δ_2 maps isomorphically onto $\pi_1(B')$. We have now proved that $\pi_1(M') \cong \pi_1(S^1 \times B')$ and so M' is homotopy equivalent to $S^1 \times B'$. Thus (3) implies (2).

Finally, $\pi_1(M)$ satisfies all the conditions of Proposition 2.17 and so (3) is equivalent to (4). □

Remark 2.28. The last step in the proof of the implication (3) ⇒ (2) could be simplified as follows: Having that Δ_1 maps trivially to $\pi_1(B') \cong \pi_1(M')/\pi_1(S^1)$ we can immediately conclude that $\pi_1(S^1)$ must be trivial in Δ_2, because the center of $\pi_1(M') \cong \Delta_1 \times \Delta_2$ is infinite cyclic, isomorphic to $\pi_1(S^1)$.

Actually, taking for granted that the the circle fiber of M' is the only central factor in $\pi_1(M')$, we can relax the condition "not presentable by products" for the fundamental group of the base B' to "irreducible"; see the proof of Lemma 5.19.

This discussion yields case (1) of Theorem 1.18:

Corollary 2.29. *Let M be a circle bundle with non-trivial rational Euler class over a closed* *oriented aspherical manifold B so that $\pi_1(B)$ is not presentable by products. Then $\pi_1(M)$ is not* IIPP.

Proof. Since $\pi_1(B)$ is not presentable by products, $\pi_1(M)$ is IIPP if and only if it is reducible, by the equivalence between (3) and (4) in Theorem 2.27 (or by Proposition 2.17). However, $\pi_1(M)$ is not reducible, otherwise M would be covered by $S^1 \times B'$, for some finite cover $B' \longrightarrow B$ (by the equivalence between (2) and (3) in Theorem 2.27), which is impossible because the Euler class of M is not torsion; cf. Remark 2.12. □

Example 2.30. As we shall see in Chapter 5, every closed four-manifold M carrying the geometry Sol_1^4 is virtually a circle bundle over a closed oriented Sol^3-manifold (which is a virtual mapping torus of T^2 with hyperbolic monodromy; cf. Table 4.1), and $\pi_1(M)$ is not IIPP; see Propositions 5.18 and 5.27 respectively. Since the fundamental groups of closed Sol^3-manifolds are not presentable by products (cf. Corollary 2.20 or Example 5.16), Theorem 2.27 implies that M is not dominated by products. Actually, a circle bundle over a closed oriented Sol^3-manifold is dominated by a product if and only if it possesses the geometry $Sol^3 \times \mathbb{R}$ (see also Theorem 5.1).

Chapter 3

Products as branched double covers

Beyond the existence question, another fundamental topic in the investigation of non-zero degree maps is that of identifying simple homotopy representatives of dominant maps between two (classes of) manifolds [22, 21, 55]. The discussion on rationally essential manifolds in the preceding chapter ended with a characterization for certain aspherical circle bundles, saying that those manifolds are dominated by products if and only if they are virtual products.

In this chapter, we will show that large classes of rationally inessential manifolds admit branched double coverings by non-trivial products $S^k \times N$. The inessential targets include

(i) connected sums of direct products of spheres with arbitrary manifolds, and

(ii) connected sums of sphere bundles over the 2-sphere.

The direct factor N in the domain $S^k \times N$ will also be a connected sum of a certain type, containing the same number of connected summands as the target.

3.1 The pillowcase

Definition 3.1. A *branched d-fold covering* between two closed smooth n-dimensional manifolds M and N is a smooth map $f\colon M \longrightarrow N$ with a critical set $B_f \subset N$, called the *branch locus* of f, such that the restriction $f|_{M\setminus f^{-1}(B_f)}\colon M \setminus f^{-1}(B_f) \longrightarrow N \setminus B_f$ is a d-fold covering in the usual sense and for each $x \in f^{-1}(B_f)$ there are local charts $U, V \longrightarrow \mathbb{C} \times \mathbb{R}^{n-2}_+$ about x, $f(x)$ on which f is given by $(z, v) \mapsto (z^m, v)$ for some positive integer m, called the *braching index* of f at x. The point x is called *singular* and its image $f(x)$ is called a *branch point*.

Example 3.2.

(1) Obviously, every covering map is a branched covering with empty branch locus.

(2) By a classical theorem of Alexander [2, 25], every piecewise linear oriented n-dimensional manifold M is a branched cover of S^n.

The starting point of the constructions given in the subsequent sections is the fact that the product of any two spheres of positive dimensions, k and $n-k$, can be realized as a branched double cover of the n-sphere. The existence of this map is inspired by the well-known pillowcase map from the 2-torus to the 2-sphere (see Example 3.4 below). Since this branched covering will be one of the basic ingredients for our proofs, and for the sake of completeness, we first construct the high-dimensional pillowcase.

Theorem 3.3. *For every $n > k \geq 1$, there is a branched double covering*

$$P\colon S^k \times S^{n-k} \longrightarrow S^n$$

with branch locus $B_P = S^{k-1} \times S^{n-k-1}$ and such that $P^{-1}(D^n) = S^k \times D^{n-k}$ for some n-ball $D^n \subset S^n$.

Proof. Let $k \geq 1$ and consider the k-sphere $S^k = \{(x_1, ..., x_{k+1}) \in \mathbb{R}^{k+1} \mid \sum_{i=1}^{k+1} x_i^2 = 1\}$. We define a continuous map on S^k by

$$\iota_k\colon S^k \longrightarrow S^k\colon (x_1, x_2, ..., x_{k+1}) \mapsto (-x_1, x_2, ..., x_{k+1}).$$

This map is an orientation reversing involution of S^k, which reflects the first coordinate and fixes the remaining k. For $n > k$, let $\iota_{n-k}\colon S^{n-k} \longrightarrow S^{n-k}$ be the corresponding orientation reversing involution of S^{n-k}. Then the product $\iota_k \times \iota_{n-k}$ is an orientation preserving involution of $S^k \times S^{n-k}$ given by

$$\begin{array}{ccc} S^k \times S^{n-k} & \longrightarrow & S^k \times S^{n-k} \\ (x_1, x_2, ..., x_{k+1}, y_1, y_2, ..., y_{n-k+1}) & \mapsto & (-x_1, x_2, ..., x_{k+1}, -y_1, y_2, ..., y_{n-k+1}). \end{array}$$

The involution $\iota_k \times \iota_{n-k}$ fixes the product of the two equators $S^{k-1} \subset S^k$ and $S^{n-k-1} \subset S^{n-k}$, corresponding to $x_1 = 0$ and $y_1 = 0$ respectively.

Now the quotient map for $\iota_k \times \iota_{n-k}$ is

$$\begin{array}{ccc} P\colon S^k \times S^{n-k} & \longrightarrow & S^n \\ (x_1, ..., x_{k+1}, y_1, ..., y_{n-k+1}) & \mapsto & \dfrac{(x_1 y_1, x_2, ..., x_{k+1}, y_2, ..., y_{n-k+1})}{\sqrt{x_1^2 y_1^2 + x_2^2 + \cdots + x_{k+1}^2 + y_2^2 + \cdots + y_{n-k+1}^2}}. \end{array}$$

The map P is a double covering outside the fixed point locus of $\iota_k \times \iota_{n-k}$, namely outside of the product $S^{k-1} \times S^{n-k-1}$ (for $x_1 = 0 = y_1$), which is the set of singular points of P. The image of this product under P is

$$P(S^{k-1} \times S^{n-k-1}) = \left\{ \frac{1}{\sqrt{2}}(0, x_2, ..., x_{k+1}, y_2, ..., y_{n-k+1}) \in S^n \;\middle|\; \sum_{i=2}^{k+1} x_i^2 = \sum_{i=2}^{n-k+1} y_i^2 = 1 \right\},$$

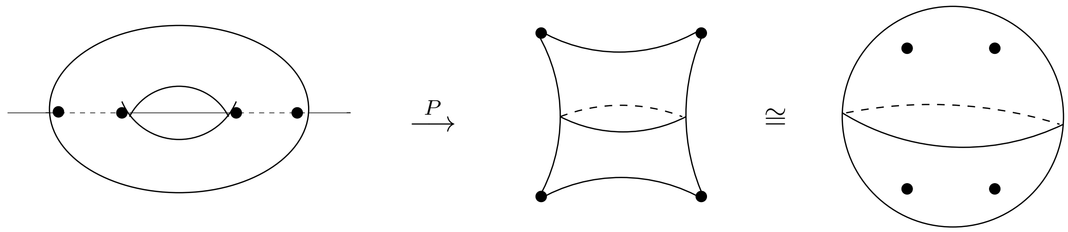

Figure 3.1: The usual pillowcase $P\colon T^2 \longrightarrow S^2$.

that is, the branch locus B_P of P is $S^{k-1} \times S^{n-k-1}$.

We now claim that there is an n-ball in the target S^n whose preimage under P is a product $S^k \times D^{n-k}$, i.e. it preserves the factor S^k of the domain of P. Let

$$\begin{aligned} D^n &= \left\{ \frac{(x_1 y_1, x_2, ..., x_{k+1}, y_2, ..., y_{n-k+1})}{\sqrt{x_1^2 y_1^2 + x_2^2 + \cdots x_{k+1}^2 + y_2^2 + \cdots y_{n-k+1}^2}} \in S^n \;\middle|\; \right. \\ & \left. -1 \leq x_1, ..., x_{k+1}, y_1, ..., y_{n-k} \leq 1,\ 0 \leq y_{n-k+1} \leq 1 \right\}. \end{aligned}$$

This n-ball contains the whole equator $S^{k-1} \subset S^k$ (corresponding to $x_1 = 0$) and the upper semi-sphere of the equator $S^{n-k-1} \subset S^{n-k}$ (corresponding to $y_1 = 0$). The preimage of D^n under P is

$$\begin{aligned} P^{-1}(D^n) &= \left\{ (x_1, x_2, ..., x_{k+1}, y_1, y_2, ..., y_{n-k+1}) \in S^k \times S^{n-k} \;\middle|\; \right. \\ & \sum_{i=1}^{k+1} x_i^2 = 1,\ -1 \leq x_i \leq 1, \\ & \left. \sum_{i=1}^{n-k+1} y_i^2 = 1,\ -1 \leq y_1, ..., y_{n-k} \leq 1,\ 0 \leq y_{n-k+1} \leq 1 \right\} \\ &= S^k \times D^{n-k}. \end{aligned}$$

This verifies our claim and finishes the proof. □

Example 3.4 (The usual pillowcase)**.** For $n = 2$ and $k = 1$ we obtain the usual pillowcase; see Figure 3.1. In that case, the branch locus consists of four points $\frac{1}{\sqrt{2}}(0, \pm 1, \pm 1)$. The 2-ball

$$D^2 = \left\{ \frac{(x_1 y_1, x_2, y_2)}{\sqrt{x_1^2 y_1^2 + x_2^2 + y_2^2}} \in S^2 \;\middle|\; -1 \leq x_1, x_2, y_1 \leq 1,\ 0 \leq y_2 \leq 1 \right\}$$

contains two of the branch points and its preimage under P is an annulus $S^1 \times I$ in T^2; see Figure 3.2.

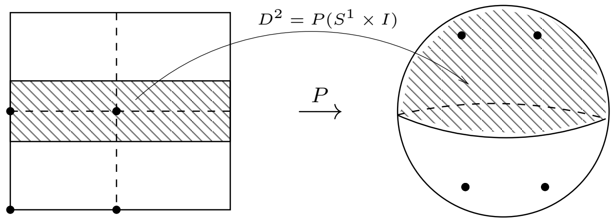

Figure 3.2: The branched double covering $P\colon T^2 \longrightarrow S^2$ is branched along four points (for $x_1 = y_1 = 0$), so that $P(S^1 \times I) = D^2$, where $S^1 \times I$ is an annulus in T^2 containing two singular points $((0, \pm 1, 0, 1))$.

Remark 3.5. The $\mathbb{Z}_2$-action given by $\iota_1 \times \iota_1$ on T^2 can be generalized for any closed oriented surface of genus $g \geq 0$. Namely, the rotation by π of $\Sigma_g \subset \mathbb{R}^3$ around the y-axis defines a $\mathbb{Z}_2$-action with $2g+2$ fixed points. The quotient map for this action is then a branched double covering $\Sigma_g \longrightarrow S^2$ with $2g+2$ branch points. In particular, S^2 is a branched double cover of itself with two branch points.

3.2 Connected sums of products of the sphere

We first deal with targets that are connected sums whose summands are direct products of the n-sphere ($n \geq 2$) with arbitrary closed oriented connected manifolds. The generalization of the pillowcase gives rise to the following construction:

Theorem 3.6. *Let $\{M_i\}_{i=1}^p$ be a family of closed oriented connected m-dimensional manifolds. For every $n > k \geq 1$ there is a π_1-surjective branched double covering*

$$S^k \times (\#_{i=1}^p S^{n-k} \times M_i) \longrightarrow \#_{i=1}^p (S^n \times M_i).$$

Proof. By Theorem 3.3, there is a branched double covering $P\colon S^k \times S^{n-k} \longrightarrow S^n$ and an n-ball $D^n \subset S^n$ so that $P^{-1}(D^n) = S^k \times D^{n-k}$. This proves the claim for the empty connected sum ($p = 0$).

For each $i \in \{1, ..., p\}$, we multiply P by the identity map on M_i to obtain p branched double coverings

$$P \times \mathrm{id}_{M_i}\colon S^k \times S^{n-k} \times M_i \longrightarrow S^n \times M_i. \tag{3.1}$$

Since S^n is simply connected for $n \geq 2$ and the identity map is π_1-surjective, each branched double covering $P \times \mathrm{id}_{M_i}$ is also π_1-surjective. In particular, this proves the statement for the case $p = 1$.

Let now D_i^m be an m-ball in M_i. Then the product $D^n \times D_i^m$ is an $(n+m)$-ball in $S^n \times M_i$ and its preimage under $P \times \mathrm{id}_{M_i}$ is

$$(P \times \mathrm{id}_{M_i})^{-1}(D^n \times D_i^m) = S^k \times D^{n-k} \times D_i^m.$$

We remove these $(n+m)$-balls $D^n \times D_i^m$ from each of the targets $S^n \times M_i$, and their preimages $S^k \times D^{n-k} \times D_i^m$ from the corresponding domains $S^k \times S^{n-k} \times M_i$ to obtain p branched double coverings

$$S^k \times ((S^{n-k} \times M_i) \setminus (D^{n-k} \times D_i^m)) \longrightarrow (S^n \times M_i) \setminus (D^n \times D_i^m). \tag{3.2}$$

For $p = 2$, we connected sum $S^n \times M_1$ with $S^n \times M_2$, by gluing over the two S^{n+m-1}-boundaries of the removed balls $D^n \times D_i^m$, and, simultaneously, we perform a fiber sum on the domain, by gluing the two $S^k \times S^{n-k+m-1}$-boundaries of the trivial S^k-bundles $S^k \times ((S^{n-k} \times M_i) \setminus (D^{n-k} \times D_i^m))$ so that the branching loci fit together. This gives a π_1-surjective branched double covering

$$S^k \times ((S^{n-k} \times M_1) \# (S^{n-k} \times M_2)) \longrightarrow (S^n \times M_1) \# (S^n \times M_2),$$

finishing the proof for $p = 2$.

For $p \geq 3$, we proceed as follows: We pick two of the $S^n \times M_i$ and from each of them we remove, as before, one $(n+m)$-ball $D^n \times D_i^m$ and its preimage $(P \times \mathrm{id}_{M_i})^{-1}(D^n \times D_i^m)$ from $S^k \times S^{n-k} \times M_i$ to obtain two branched double coverings as in (3.2). Let us use the indices $i = 1$ and p for those two branched coverings. From the remaining copies of $S^n \times M_i$, i.e. for $i \in \{2, ..., p-1\}$, we remove two disjoint $(n+m)$-balls $(D^n \times D_{i,1}^m) \amalg (D^n \times D_{i,2}^m)$, and their preimages under $P \times \mathrm{id}_{M_i}$, to obtain $p-2$ branched double coverings

$$S^k \times ((S^{n-k} \times M_i) \setminus ((D^{n-k} \times D_{i,1}^m) \amalg (D^{n-k} \times D_{i,2}^m))) \longrightarrow (S^n \times M_i) \setminus ((D^n \times D_{i,1}^m) \amalg (D^n \times D_{i,2}^m)). \tag{3.3}$$

We first glue the S^{n+m-1}-boundary of the manifold $(S^n \times M_1) \setminus (D^n \times D_1^m)$ to one of the two S^{n+m-1}-boundaries of the manifold

$$(S^n \times M_2) \setminus ((D^n \times D_{2,1}^m) \amalg (D^n \times D_{2,2}^m)).$$

We then obtain a connected sum

$$(S^n \times M_1) \# ((S^n \times M_2) \setminus (D^n \times D_{2,2}^m))$$

and on the remaining S^{n+m-1}-boundary we paste one of the two S^{n+m-1}-boundaries of the manifold

$$(S^n \times M_3) \setminus ((D^n \times D_{3,1}^m) \amalg (D^n \times D_{3,2}^m)).$$

By iterating the procedure, we finally obtain a connected sum

$$(S^n \times M_1) \# (S^n \times M_2) \# (S^n \times M_3) \# \cdots \# (S^n \times M_{p-2}) \# ((S^n \times M_{p-1}) \setminus (D^n \times D_{p-1,2}^m)). \tag{3.4}$$

On the remaining boundary of this connected sum, we glue the S^{n+m-1}-boundary of the manifold $(S^n \times M_p) \setminus ((D^n \times D^m_p)$.

Simultaneously to the above connected summing, we perform the corresponding fiber sums on the domains, as explained in the case $p = 2$. We then obtain the desired π_1-surjective branched double covering

$$S^k \times (\#_{i=1}^p S^{n-k} \times M_i) \longrightarrow \#_{i=1}^p (S^n \times M_i),$$

for every $p \geq 3$. This completes the proof. □

With this construction, we prove that large classes of inessential manifolds are indeed dominated by products, answering one of the motivating questions of this thesis affirmatively, for targets of type $\#_{i=1}^p (S^n \times M_i)$, where $n \geq 2$. In Chapter 6, we will apply Theorem 3.6 to study domination by products for simply connected manifolds.

We note that the branched double covering of Theorem 3.6 is a generalization of a three-dimensional construction obtained in [46] for connected sums of $S^2 \times S^1$ (see also [54] for the corresponding statement for connected sums of type $\#_p(S^2 \times M)$):

Example 3.7 ([46, Prop. 1])**.** If we set $n = 2$ and $M_i = S^1$ for all i in Theorem 3.6, then we obtain a branched double covering $\Sigma_p \times S^1 \longrightarrow \#_p(S^2 \times S^1)$. As we shall see in Chapter 4, every rationally inessential three-manifold is virtually a connected sum $\#_p(S^2 \times S^1)$, and therefore is dominated by products. Moreover, the genus of a closed surface Σ_g such that $\Sigma_g \times S^1 \geq \#_p(S^2 \times S^1)$ must be at least p (cf. Lemma 4.19) and so this construction gives a sharp value for the genus as well.

Furthermore, we remark that Theorem 3.6 identifies simple homotopy representatives for dominant maps by products to connected sums $\#_{i=1}^p (S^n \times M_i)$, where $n \geq 2$, and simultaneously yields precise mapping degrees. In addition, it shows that every such connected sum is a quotient space of a product $S^k \times (\#_{i=1}^p S^{n-k} \times M_i)$.

3.3 Connected sums of sphere bundles over the 2-sphere

In the construction of the preceding section, we considered targets that are connected sums of direct products. A natural question arising by that is whether the connected summands of the target could be replaced by twisted products.

At a first glance, this is a considerably difficult question for two main reasons. On the one hand, if we quit of the strong requirement of having summands that are direct products, then we are not able anymore to use product maps, as we did in the proof of Theorem 3.6, where we multiplied the generalized pillowcase map by the identity map on arbitrary manifolds. On the other hand, the comprehension of fiber bundles in general does not seem sufficiently enough to produce a precise argument for the existence of a branched covering by a non-trivial direct product.

We can, however, overcome these two constraints for certain fiber bundles whose both base and fiber are spheres. More precisely, Steenrod's [73] classification of sphere bundles over spheres (with linear structure group) gives the passage to the immediately next classes of connected summands, other than those considered in Section 3.2.

Theorem 3.8. *For $n \geq 4$ and every $p \geq 0$ there is a branched double covering*

$$S^1 \times (\#_p S^{n-2} \times S^1) \longrightarrow \#_p(S^{n-2} \widetilde{\times} S^2),$$

where $S^{n-2} \widetilde{\times} S^2$ denotes the total space of the non-trivial S^{n-2}-bundle over S^2 with structure group $\mathrm{SO}(n-1)$*.*

Proof. The interesting cases occur for $p \geq 1$, because Theorem 3.3 takes care for the empty connected sum ($p = 0$).

By the classification of Steenrod [73], oriented S^{n-2}-bundles over S^2 with structure group $\mathrm{SO}(n-1)$, where $n \geq 4$, are classified by $\pi_1(\mathrm{SO}(n-1)) = \mathbb{Z}_2$. This means that the total space of such bundles is either the product $S^{n-2} \times S^2$ or the twisted bundle $S^{n-2} \widetilde{\times} S^2$.

Let $\pi \colon S^{n-2} \widetilde{\times} S^2 \longrightarrow S^2$ denote the twisted bundle. We pull back π by the usual pillowcase map $P \colon T^2 \longrightarrow S^2$ (cf. Example 3.4) to obtain a branched double covering $P^* \colon P^*(S^{n-2} \widetilde{\times} S^2) \longrightarrow S^{n-2} \widetilde{\times} S^2$. Now $P^*(S^{n-2} \widetilde{\times} S^2)$ is the total space of an oriented S^{n-2}-bundle over T^2 with structure group $\mathrm{SO}(n-1)$. Again, there exist only two such bundles and since the degree of P is even, we deduce that $P^*(S^{n-2} \widetilde{\times} S^2)$ is the trivial bundle, namely the product $T^2 \times S^{n-2}$. Therefore, $T^2 \times S^{n-2}$ is a branched double cover of $S^2 \widetilde{\times} S^{n-2}$, proving the statement for $p = 1$. (Moreover, the branch locus of P^* consists of four copies of the S^{n-2} fiber of $S^2 \widetilde{\times} S^{n-2}$, given by the preimages under π of the four branch points of P.)

Next, we prove the claim for $p \geq 2$. Let the branched double covering $P^* \colon T^2 \times S^{n-2} \longrightarrow S^2 \widetilde{\times} S^{n-2}$ constructed above. We can think of $T^2 \times S^{n-2}$ as a trivial circle bundle over $S^1 \times S^{n-2}$. We thicken an S^1 fiber of this bundle to an annulus $S^1 \times I$ in T^2 so that $P(S^1 \times I) = D^2$, as in Figure 3.2. A fibered neighborhood of this S^1 fiber is a product $S^1 \times D^{n-1}$ in $T^2 \times S^{n-2}$ and $P^*(S^1 \times D^{n-1})$ is an n-disk D^n in $S^2 \widetilde{\times} S^{n-2}$. We now remove a fibered neighborhood $S^1 \times D^{n-1}$ from two copies of $T^2 \times S^{n-2}$ and perform a fiber sum by gluing the $S^1 \times S^{n-2}$-boundaries. Since $T^2 \times S^{n-2}$ is a trivial circle bundle, this fiber sum will produce another trivial circle bundle, namely a product $S^1 \times ((S^1 \times S^{n-2}) \# (S^1 \times S^{n-2}))$. At the same time, we connected sum two copies of $S^2 \widetilde{\times} S^{n-2}$ along D^n, so that the branch loci fit together. We have now obtained a branched double covering

$$S^1 \times ((S^1 \times S^{n-2}) \# (S^1 \times S^{n-2})) \longrightarrow (S^2 \widetilde{\times} S^{n-2}) \# (S^2 \widetilde{\times} S^{n-2}),$$

proving the claim for $p = 2$.

For $p > 2$ we iterate the above construction, by removing the appropriate number of n-balls from the connected summands of the target. □

The statement of Theorem 3.8 is the most general possible regarding oriented S^{n-2}-bundles over S^2. However, it presupposes the structure group of these bundles to be linear. This assumption is unnecessary in dimensions four and five, because the oriented diffeomorphism group of S^2 and S^3 is homotopy equivalent to SO(3) and SO(4) respectively, by the corresponding results of Smale [68] and Hatcher [37]. Resorting to that and to classification results, we will apply our constructions to study domination by products for simply connected manifolds in Chapter 6, answering Question 1.19 in dimensions four and five.

We finally note that Theorem 3.6 could be viewed as a special case of Theorem 3.8, when $M_i = S^n$ for all i. Actually, in order to have a connected sum $\#_p(S^{n-2}\widetilde{\times}S^2)$, as in the target of Theorem 3.8, it suffices only one of the summands to be the twisted S^{n-2}-bundle over S^2. This is a consequence of the next theorem of Wall:

Theorem 3.9 ([81])**.** *For $n \geq 4$, suppose that M is a simply connected closed non-spin n-dimensional manifold and $S^{n-2}\widetilde{\times}S^2$ is the total space of the non-trivial S^{n-2}-bundle over S^2 with structure group* $\mathrm{SO}(n-1)$*. Then $M\#(S^2 \times S^{n-2})$ is diffeomorphic to $M\#(S^2\widetilde{\times}S^{n-2})$.*

Chapter 4

Three-manifolds

The only closed three-manifolds that are products are those of type $S^1 \times \Sigma$, where Σ is a closed surface. In particular, the center of the fundamental group of those manifolds is infinite, which will play a decisive role for most of our non-domination results in this chapter. This constraint, together with the branched coverings constructed in Chapter 3, will yield a complete list of three-manifolds dominated by products. Furthermore, we will determine when the condition "fundamental group presentable by products" suffices to obtain domination by products for rationally essential three-manifolds (cf. Question 1.12).

The fact that the circle factor in the domain $S^1 \times \Sigma$ is central in the fundamental group raises the problem of which closed three-manifolds are dominated by (aspherical) non-trivial circle bundles, because the fiber of a circle bundle over an aspherical surface is always central in its fundamental group. In this chapter, we determine which three-manifolds are dominated by non-trivial circle bundles as well.

In the last section, we reformulate our results in terms of Thurston geometries. Using the geometrization theorem, we moreover complete an ordering of Wang [86] defined by the existence of non-zero degree maps between three-manifolds. This geometric interpretation will give rise to the discussion in the upcoming Chapter 5.

The material of this chapter is mostly contained in [46].

4.1 Prime decomposition and Seifert spaces

We begin this chapter with some basic facts on the prime decomposition of closed three-manifolds and on Seifert fibered spaces.

Recall that a closed n-dimensional manifold M is prime if $M = M_1 \# M_2$ implies that one of the M_i is homotopy equivalent to S^n. One of the earliest pioneer results about three-manifolds concerns their decomposition into prime pieces; cf. [50]:

Theorem 4.1 (Kneser-Milnor)**.** *Every closed oriented connected three-manifold M can be decomposed uniquely (up to order) as a connected sum $M_1 \# \cdots \# M_k$ of prime summands M_i, so that each M_i is either aspherical, is $S^2 \times S^1$ or has finite fundamental group.*

By Van Kampen's theorem, the fundamental group of a connected sum of three-manifolds, $M_1 \# M_2$, is isomorphic to the free product $\pi_1(M_1) * \pi_1(M_2)$. Conversely, if the fundamental group of a closed three-manifold is a free product, then this manifold is decomposed as a connected sum; cf. [72]:

Theorem 4.2 (Grushko)**.** *If M is a closed oriented connected three-manifold whose fundamental group is isomorphic to a free product $\Gamma_1 * \Gamma_2$, then M is homotopy equivalent to a connected sum $M_1 \# M_2$, where M_i are closed oriented three-manifolds so that $\pi_1(M_i) \cong \Gamma_i$, $i = 1, 2$.*

A class of three-manifolds which has attracted much of the interest of topologists is that of *Seifert fibered* spaces. The following characterization, due to Seifert, Thurston and Scott, will be used as our definition for Seifert three-manifolds (see also Section 2.3.2):

Theorem 4.3 ([77, 64, 65])**.** *A closed three-manifold M is Seifert fibered if and only if it is a virtual circle bundle over a closed oriented surface.*

For irreducible three-manifolds with infinite fundamental group, a purely algebraic characterization for Seifert spaces is given by the celebrated Seifert fiber space conjecture (cf. [64, pg. 484]), which was proven independently by Casson-Jungreis and by Gabai:

Theorem 4.4 ([14, 30])**.** *A closed oriented irreducible three-manifold M with infinite fundamental group is a Seifert fibered space if and only if $\pi_1(M)$ contains an infinite cyclic normal subgroup.*

4.2 Domination by circle bundles

In the main topological result of this chapter we determine which closed three-manifolds admit dominant maps by circle bundles. The following theorem contains two distinct statements, one for domination by products and one for domination by non-trivial circle bundles:

Theorem 4.5. *A closed oriented connected three-manifold M is dominated by a trivial (resp. non-trivial) circle bundle if and only if*

(1) *either M is virtually a trivial (resp. non-trivial) circle bundle over some closed aspherical surface, or*

(2) *M is virtually a connected sum $\#_p(S^2 \times S^1)$, for some $p \geq 0$.*

Obviously, a three-manifold which is finitely covered by a product (resp. non-trivial circle bundle) is dominated by that product (resp. non-trivial circle bundle). We will prove Theorem 4.5 in two steps, following the Kneser-Milnor prime decomposition.

First, we will show that, if a closed three-manifold which contains an aspherical summand in its prime decomposition is dominated by a product (resp. non-trivial circle bundle), then it must be prime and finitely covered by an aspherical product (resp. non-trivial circle bundle). If a closed three-manifold does not contain any aspherical summand in its prime decomposition, then we will see that it is finitely covered by a connected sum $\#_{p\geq 0}(S^2 \times S^1)$, where the case $p = 0$ corresponds to S^3. We will then finish the proof by showing that $\#_p(S^2 \times S^1)$ is indeed dominated by products (resp. non-trivial circle bundles).

The rest of this section is devoted to the proof of Theorem 4.5. Since we have two distinct statements to prove, we split our proof into two parts, according to whether the domain is a product or a non-trivial circle bundle.

4.2.1 Products

First, we prove Theorem 4.5 for three-manifolds dominated by trivial circle bundles, i.e. when the domain is a direct product $\Sigma \times S^1$ for some closed oriented connected surface Σ.

We split the proof into the next two cases, following the Kneser-Milnor prime decomposition for the target M:

Case I. There is an aspherical summand M_i in the prime decomposition of M.

Case II. Each summand M_i is either $S^2 \times S^1$ or has finite fundamental group.

Case I. Suppose that there is a non-zero degree map $f\colon \Sigma \times S^1 \longrightarrow M$. We may assume that f is π_1-surjective, after replacing M by a finite covering, if necessary. Since by assumption M contains an aspherical summand M_i in its prime decomposition, it is rationally essential (cf. Example 1.6) and therefore the genus of Σ must be positive. We compose f with the degree one map $M \longrightarrow M_i$ which collapses the connected summands other than M_i to obtain a map $\Sigma \times S^1 \longrightarrow M_i$ again of degree $\deg(f) \neq 0$. Thus we have a π_1-surjective dominant map $\Sigma \times S^1 \longrightarrow M_i$, where both the domain and the target are aspherical. Because this map has non-zero degree, it cannot factor through Σ, which means that $\mathrm{im}(\pi_1(f|_{S^1}))$ is not trivial. The infinite cyclic group generated by the S^1 factor is central in $\pi_1(\Sigma \times S^1)$ and so $\mathrm{im}(\pi_1(f|_{S^1}))$ is a non-trivial central subgroup in $\pi_1(M)$. This implies that $\pi_1(M)$ is freely indecomposable and so M is homotopy equivalent to M_i, i.e. M is prime and aspherical itself. Since $\pi_1(M)$ has infinite center, we conclude that M is Seifert fibered, by the proof of the Seifert fiber space conjecture; cf. Theorem 4.4. (If M is a Haken manifold, then the latter conclusion follows by an earlier result of Waldhausen [79].) In particular, M is a virtual circle bundle over a closed oriented aspherical surface (Theorem 4.3). Therefore, we have that f is a π_1-surjective map of non-zero degree, where the target M is a circle bundle over some closed oriented connected aspherical surface F. Since the domain is a product $S^1 \times \Sigma$, Lemma 2.14 implies that the Euler number of M must be zero as well. We have now proved that, if a closed oriented connected three-manifold with an aspherical summand in its prime decomposition is dominated by a product, then it must be prime and aspherical itself and finitely covered by a product of the circle with a closed oriented connected aspherical surface.

Case II. Suppose now that there is no aspherical summand in M's prime decomposition. Then M is a connected sum

$$M = \underbrace{(S^2 \times S^1)\# \cdots \#(S^2 \times S^1)}_{l} \# \underbrace{(C_{l+1})\# \cdots \#(C_k)}_{k-l} ,$$

where the case $k = 0$ corresponds to the 3-sphere S^3. The summands $S^2 \times S^1$ have infinite cyclic fundamental groups and the summands C_i have finite fundamental groups Q_i, $l+1 \leq i \leq k$. (After Perelman's proof of the Poincaré conjecture, we may write each C_i as a quotient S^3/Q_i.) Thus, the fundamental group of M is the free product

$$\pi_1(M) = F_l * Q_{l+1} * \cdots * Q_k ,$$

where F_l is a free group on l generators. We project this free product to the direct product of the Q_j to obtain the following exact sequence:

$$1 \longrightarrow \ker(\varphi) \longrightarrow \pi_1(M) = F_l * Q_{l+1} * \cdots * Q_k \stackrel{\varphi}{\longrightarrow} Q_{l+1} \times \cdots \times Q_k \longrightarrow 1 . \tag{4.1}$$

By the Kurosh Subgroup Theorem[4], $\ker(\varphi)$ is a free group F_p, and this group has finite index in $\pi_1(M)$, by the exact sequence (4.1). Thus M has a finite covering which is a connected sum of p copies of $S^2 \times S^1$, by the Kneser-Milnor prime decomposition and by Grushko's theorem; see Theorems 4.1 and 4.2 respectively. We have now proved that, if M has no aspherical summands in its prime decomposition, then it is finitely covered by a connected sum $\#_p(S^2 \times S^1)$, where p is the number of generators of the free group $\ker(\varphi)$ in the exact sequence (4.1).

In order to complete the proof of Theorem 4.5, we need to show that each connected sum $\#_p(S^2 \times S^1)$ is dominated by a product. This is already done in Example 3.7. Namely, setting $n = 2$ and $M_i = S^1$ for all i in Theorem 3.6, we obtain that $\#_p(S^2 \times S^1)$ admits a branched double covering by $S^1 \times \Sigma_p$, where Σ_p is a closed oriented connected surface of genus p.

Thus, we have proved that each target in statement (2) of Theorem 4.5 is indeed dominated by products.

This finishes the proof of Theorem 4.5 for the case of domination by products.

[4] **The Kurosh Subgroup Theorem.** Every subgroup of a free product $\Gamma_1 * \Gamma_2$ is a free product $F * (*_i \Delta_i)$, where F is a free group and each Δ_i is conjugate to a subgroup of one of the Γ_i.

4.2.2 Non-trivial circle bundles

We now prove Theorem 4.5 when the domain is a non-trivial circle bundle. Let $M_1\#\cdots\#M_k$ be the Kneser-Milnor prime decomposition of a closed oriented connected three-manifold M. We split again the proof into the Cases I and II, according to whether or not there is an aspherical summand M_i in the prime decomposition of M.

Case I. Suppose that there exists a π_1-surjective non-zero degree map $f\colon E\longrightarrow M$, where E is a non-trivial circle bundle over a closed oriented connected surface Σ and M contains an aspherical summand M_i in its prime decomposition. In particular, M is rationally essential (cf. Example 1.6) and therefore Σ (and E) must be aspherical. As in Case I for domination by products, we compose f with the degree one map $M\longrightarrow M_i$ which collapses the connected summands other than the aspherical M_i, to obtain a map $E\longrightarrow M_i$ of degree $\deg(f)\neq 0$. Since the Euler class of E is not trivial, its fundamental group $\pi_1(E)$ fits into a non-split central extension

$$1\longrightarrow\pi_1(S^1)\longrightarrow\pi_1(E)\longrightarrow\pi_1(\Sigma)\longrightarrow 1\ .$$

The composite map $E\longrightarrow M_i$ cannot factor through Σ, implying that $\pi_1(f)$ maps $\pi_1(S^1)$ non-trivially in $\pi_1(M)$. Moreover, $\pi_1(f)(\pi_1(S^1))$ is a central subgroup of $\pi_1(M)$, and so M is prime and therefore irreducible and aspherical itself. As in Case I in 4.2.1, we conclude that M is Seifert fibered (cf. Theorem 4.4). Therefore, we may assume that M is a circle bundle over some closed oriented connected aspherical surface F, by Theorem 4.3. We claim that M has non-trivial Euler class as well. In order to prove that, recall first that $\pi_1(f)$ maps the element of $\pi_1(E)$ that is represented by the circle fiber in E to a non-trivial element of the center of $\pi_1(M)$. This non-trivial element has infinite order in $\pi_1(M)$, because $\pi_1(M)$ is torsion-free, and (some multiple of it) is the circle fiber of M; see [41, p. 92/93] (for Seifert fibered spaces in general). The fiber in E has finite order in homology, because the Euler class of E is not zero. This implies that the circle fiber in M also has finite order in homology, being, up to a multiple, the image under $H_1(f)$ of the circle fiber in E. Thus the Euler class of M must be non-zero as well. We have now proved that, if a closed oriented connected three-manifold with an aspherical summand in its prime decomposition is dominated by a non-trivial circle bundle, then it must be prime and aspherical itself and finitely covered by a non-trivial circle bundle over a closed oriented connected aspherical surface.

Case II. We have already seen in Case II of Section 4.2.1 that, if M has no aspherical summands in its prime decomposition, then it is finitely covered by a connected sum $\#_p(S^2\times S^1)$, for some $p\geq 0$. We complete the proof of Theorem 4.5, for the case of domination by non-trivial circle bundles, by proving the following:

Proposition 4.6. *For every $p\geq 0$, the connected sum $\#_p(S^2\times S^1)$ admits a π_1-surjective branched double covering by a non-trivial circle bundle over a closed surface of genus p.*

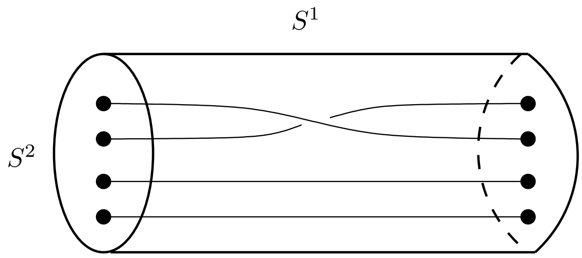

Figure 4.1: The mapping torus $M/\iota = S^2 \times S^1$ of the self-diffeomorphism of $S^2 = T^2/\iota$ induced by φ, and the branch locus of the branched double covering $P_M : M \longrightarrow M/\iota$.

Proof. The three-sphere S^3, corresponding to the case $p = 0$, is a non-trivial circle bundle over S^2, being the total space of the Hopf fibration. We pull back this circle bundle under a branched double covering $S^2 \longrightarrow S^2$ to obtain the desired branched double cover of S^3 (see Remark 3.5 for a construction of the branched double covering $S^2 \longrightarrow S^2$).

Let now $p = 1$. We will show that $S^2 \times S^1$ admits a π_1-surjective branched double covering by a non-trivial circle bundle over T^2. Define $M := M(\varphi)$ to be the mapping torus of the self-diffeomorphism of T^2 given by the matrix $\varphi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. (Note that M is also the total space of a non-trivial circle bundle over T^2 with Euler number one). Recall the (usual) pillowcase map from Section 3.1, which is a branched double covering $P : T^2 \longrightarrow S^2$. That branched covering was obtained as the quotient map for the involution $\iota = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ of T^2. This involution commutes with φ, inducing a fiber-preserving involution, also denoted by ι, of the mapping torus M. Therefore, the quotient M/ι is the mapping torus of the self-diffeomorphism of $T^2/\iota = S^2$ induced by φ. Since this diffeomorphism preserves the orientation, we conclude that $M/\iota = S^2 \times S^1$. The desired π_1-surjective branched double covering is given by the projection $P_M : M \longrightarrow M/\iota = S^2 \times S^1$ which is the quotient map for ι (see Figure 4.1 for a picture of the branch locus of P_M). This completes the proof for $p = 1$. (Clearly, P_M coincides with P on the T^2 fiber of M.)

Finally, we prove the claim for $p \geq 2$. We now think of M as a non-trivial circle bundle over T^2, with Euler number one. The S^1 fibers of M are contained in the T^2 fibers of the mapping torus projection $\pi : M \longrightarrow S^1$. We thicken one such S^1 fiber to an annulus $S^1 \times I$ contained in a torus fiber of π, so that $P(S^1 \times I)$ is a disk D^2 in S^2 containing exactly two branch points of P; cf. Example 3.4 and Figure 3.2. A fibered neighborhood of this S^1 fiber in M is a product of the annulus $S^1 \times I$ with an interval in S^1 (i.e. in the base of the mapping torus $M(\varphi)$), and the image under P_M of this fibered neighborhood is a three-ball D^3 in $S^2 \times S^1$. We now connected sum two copies of $S^2 \times S^1$ along this D^3, and, simultaneously, we perform a fiber sum of two copies of M by removing the fibered neighborhood $S^1 \times I \times I$ and gluing the boundary tori in a fiber-preserving way that matches up the branch loci. This gives a π_1-surjective branched double covering from a non-trivial circle bundle over Σ_2 (with Euler number 2) to $(S^2 \times S^1)\#(S^2 \times S^1)$, proving the claim for $p = 2$. The general case follows by iterating this construction. □

The proof of Theorem 4.5 for the case of domination by non-trivial circle bundles is now complete.

4.3 Rational essentialness and groups presentable by products

We now return to one of the main motivating themes of this thesis (Question 1.12), namely, to investigate when the condition "fundamental group presentable by products" suffices for domination by products for rationally essential three-manifolds.

The concept of rational essentialness for three-manifolds has an interesting interpretation in terms of the Kneser-Milnor prime decomposition, as one can derive along the lines of the proof of Theorem 4.5:

Theorem 4.7. *For a closed oriented connected three-manifold M the following conditions are equivalent:*

(1) M is rationally essential;

(2) M has an aspherical summand M_i in its prime decomposition;

(3) M is not finitely covered by a connected sum $\#_p(S^2 \times S^1)$.

Proof. A connected sum is rationally essential if and only if at least one of its summands is; cf. Example 1.6. Since $S^1 \times S^2$ and manifolds with finite fundamental group are not rationally essential, this proves the equivalence between (1) and (2).

It is obvious that (2) implies (3). The converse was proved in Case II in Section 4.2.1. □

Remark 4.8. The list in the statement of Theorem 4.7 is not exhaustive. In [46], we further show that the three equivalent properties of Theorem 4.7 are also equivalent to two more conditions from differential geometry:

(i) M is compactly enlargeable;

(ii) M does not admit a metric of positive scalar curvature.

For the proof of these two cases one needs to appeal to Perelman's proof of the geometrization conjecture, and to the classification of three-manifolds admitting metrics of positive scalar curvature, by Schoen-Yau and Gromov-Lawson. For more details, see [46] and the related references cited there. We will not refer to those properties again, because they are not directly connected to the main results of this thesis.

We can now reformulate our main Theorem 4.5 in terms of rational essentialness:

Theorem 4.9. *Let M be a closed oriented connected three-manifold.*

(1) *If M is rationally essential, then M is dominated by a trivial (resp. non-trivial) circle bundle over a closed surface if and only if it is finitely covered by a trivial (resp. non-trivial) circle bundle over some closed oriented aspherical surface.*

(2) *If M is rationally inessential, then it is dominated by both trivial and non-trivial circle bundles over some closed oriented surfaces.*

In Section 2.3.2, we determined which fundamental groups of closed three-manifolds are (infinite-index) presentable by products; cf. Proposition 2.19 and Corollary 2.23. We can therefore now answer Question 1.12 in dimension three.

Corollary 4.10. *A rationally essential three-manifold M is dominated by a product if and only if $\pi_1(M)$ is a virtual product $\pi_1(\Sigma) \times \mathbb{Z}$ for some closed oriented aspherical surface Σ.*

Thus, although the fundamental groups of (virtually) non-trivial circle bundles over closed aspherical surfaces are presentable by products, those three-manifolds are never dominated by products.

The above corollary reformulates on a purely algebraic level the topological characterization of Theorem 4.9 (1), regarding domination by products for rationally essential targets. At the other end, fundamental groups of rationally inessential three-manifolds are virtually free groups. Therefore, Theorem 4.9 (2) (concerning domination by products) can also be reformulated on a purely algebraic level:

Proposition 4.11. *A closed oriented three-manifold with virtually free fundamental group is dominated by a product.*

Similar algebraic characterizations can be obtained for three-manifolds dominated by non-trivial circle bundles as well:

Theorem 4.12. *A closed oriented connected three-manifold M is dominated by a non-trivial circle bundle over a surface if and only if*

(1) *either $\pi_1(M)$ has a finite index subgroup Γ which fits into a central extension*

$$1 \longrightarrow \mathbb{Z} \longrightarrow \Gamma \longrightarrow \pi_1(F) \longrightarrow 1$$

with non-zero Euler class for some aspherical surface F, or

(2) *$\pi_1(M)$ is virtually free.*

4.4 Thurston geometries

4.4.1 Geometric reformulation

Let $\mathbb{X}^n$ be a complete simply connected n-dimensional Riemannian manifold. We say that a closed manifold M is an $\mathbb{X}^n$*-manifold*, or that M *possesses the* $\mathbb{X}^n$ *geometry* in the sense of Thurston, if it is diffeomorphic to a quotient of $\mathbb{X}^n$ by a lattice Γ in the group of isometries of $\mathbb{X}^n$ (acting effectively). The group Γ denotes the fundamental group of M. We say that $\mathbb{X}^n$ and $\mathbb{Y}^n$ are the same geometries if there exists a diffeomorphism $\psi\colon \mathbb{X}^n \longrightarrow \mathbb{Y}^n$ and an isomorphism $\mathrm{Isom}(\mathbb{X}^n) \longrightarrow \mathrm{Isom}(\mathbb{Y}^n)$ mapping each $g \in \mathrm{Isom}(\mathbb{X}^n)$ to $\psi \circ g \circ \psi^{-1} \in \mathrm{Isom}(\mathbb{Y}^n)$.

The last thirty years three-manifold topology is guided by Thurston's pioneer geometrization program. Thurston [77] proved that there exist eight geometries, namely the geometries $\mathbb{H}^3$, Sol^3, $\widetilde{SL_2}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil^3, $\mathbb{R}^3$, $S^2 \times \mathbb{R}$ and S^3 (see also [64]).

Remark 4.13 (Rationally inessential geometric three-manifolds with infinite fundamental group). In the context of rational (in)essentialness, it is clear that there exist infinitely many geometric rationally essential three-manifolds and infinitely many geometric rationally inessential three-manifolds. On the one hand, every three-manifold possessing an aspherical geometry is rationally essential. On the other hand, all closed three-manifolds with finite fundamental group are geometric by Perelman's proof of the Poincaré conjecture and, of course, rationally inessential. However, there exist only two geometric rationally inessential three-manifolds with infinite fundamental group. The obvious one is $S^2 \times S^1$. The other one is $\mathbb{RP}^3 \# \mathbb{RP}^3$ which is double covered by $S^2 \times S^1$ (recall that $\pi_1(\mathbb{RP}^3 \# \mathbb{RP}^3) = \mathbb{Z}_2 * \mathbb{Z}_2$) and carries the geometry $S^2 \times \mathbb{R}$ as well. Note that $\mathbb{RP}^3 \# \mathbb{RP}^3$ is the only non-prime geometric three-manifold [64].

Table 4.1 gives characterizations (up to finite covers) of closed geometric three-manifolds. (For the hyperbolic geometry we refer to [77, 1]; for every other geometry to [77, 64, 65].) This description suggests that we can reformulate our main Theorem 4.5 in terms of Thurston geometries:

Theorem 4.14. *A closed oriented connected three-manifold M is dominated by a trivial (resp. non-trivial) circle bundle if and only if*

(1) *either M possesses one of the geometries $\mathbb{R}^3$ or $\mathbb{H}^2 \times \mathbb{R}$ (resp. Nil^3 or $\widetilde{SL_2}$), or*

(2) *M is a connected sum of manifolds possessing the geometries $S^2 \times \mathbb{R}$ or S^3.*

Proof. According to Theorem 4.5, a closed oriented connected three-manifold M is dominated by a trivial (resp. non-trivial) circle bundle if and only if

(i) either M is virtually a trivial (resp. non-trivial) circle bundle over some closed oriented aspherical surface F, or

(ii) M is virtually a connected sum $\#_p(S^2 \times S^1)$, for some $p \geq 0$.

Geometry $\mathbb{X}^3$	Virtual property
$\mathbb{H}^3$	Mapping torus of a hyperbolic surface with pseudo-Anosov monodromy
Sol^3	Mapping torus of T^2 with hyperbolic monodromy
$\widetilde{SL_2}$	Non-trivial circle bundle over a hyperbolic surface
Nil^3	Non-trivial circle bundle over T^2
$\mathbb{H}^2 \times \mathbb{R}$	Product of the circle with a hyperbolic surface
$\mathbb{R}^3$	The 3-torus
$S^2 \times \mathbb{R}$	The product $S^2 \times S^1$
S^3	The 3-sphere

Table 4.1: Virtual properties of closed geometric three-manifolds.

It therefore suffices to show that the items (1) and (2) of our statement are equivalent to the properties (i) and (ii) respectively.

The equivalence between (i) and (1) is given by the corresponding characterizations of the geometries $\mathbb{R}^3$, $\mathbb{H}^2 \times \mathbb{R}$, Nil^3 and $\widetilde{SL_2}$ in Table 4.1.

If now M is a connected sum of manifolds possessing the geometries $S^2 \times \mathbb{R}$ or S^3, then it is finitely covered by a connected sum $\#_p(S^2 \times S^1)$, as we have seen in Case II in Section 4.2.1. Conversely, if M is finitely covered by a connected sum $\#_p(S^2 \times S^1)$, then it has no aspherical summands in its prime decomposition. Appealing to Perelman's proof of the Poincaré conjecture for summands with finite fundamental groups, we see that M is a connected sum of manifolds carrying the geometries $S^2 \times \mathbb{R}$ or S^3. □

The geometric reformulation of our results has an additional interesting consequence in the case of rationally essential three-manifolds. Namely, we can weaken the standard characterization for Seifert spaces given in Theorem 4.3, by replacing covering maps by arbitrary dominant maps:

Corollary 4.15. *A rationally essential three-manifold M is Seifert fibered (equivalently carries one of the geometries $\widetilde{SL_2}$, Nil^3, $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{R}^3$) if and only it is dominated by a circle bundle over a closed oriented aspherical surface.*

4.4.2 Wang's ordering

In a lecture in 1978, Gromov's suggested to investigate the domination relation as defining an ordering of compact manifolds of the same dimension [13]. In dimension two, the relation $\geq$

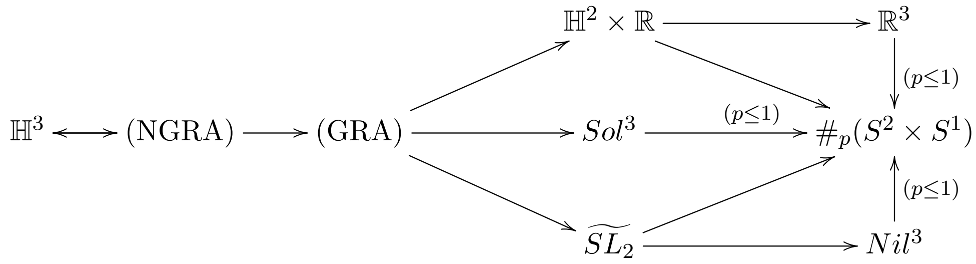

Figure 4.2: Ordering three-manifolds by maps of non-zero degree.

coincides with the order given by the genus, however, in higher dimensions is not generally an order. For example, S^3 and $\mathbb{RP}^3$ dominate each other, but obviously they are not homotopy equivalent.

Recall that, for a non-geometric closed aspherical three-manifold M, there is a finite family of splitting tori so that M can be cut into pieces, called JSJ pieces (after Jaco-Shalen and Johannson). If all the JSJ pieces are Seifert, then M is termed *non-trivial graph.* If there is a JSJ piece that is not Seifert, then this piece must be hyperbolic by Perelman's proof of Thurston's geometrization conjecture. In that case, M is called *non-graph.*

The discussion on rationally essential three-manifolds shows that there are no maps of non-zero degree between trivial and non-trivial circle bundles over aspherical surfaces. In particular, the three-dimensional version of Lemma 2.14 clarifies the proof of a claim made by Wang [86] about 20 years ago, that products cannot dominate non-trivial circle bundles over aspherical surfaces. In that paper, Wang determines the existence of dominant maps between all closed aspherical three-manifolds. According to that work and to our constructions for rationally inessential manifolds, we complete an ordering of three-manifolds defined by non-zero degree maps, following Thurston's geometrization picture:

Theorem 4.16 ([86, 46])**.** *Let the following classes of closed oriented three-manifolds:*

(i) *aspherical and geometric: possessing one of the geometries* $\mathbb{H}^3$, Sol^3, $\widetilde{SL_2}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil^3 *or* $\mathbb{R}^3$*;*

(ii) *aspherical and not geometric:* (GRA) *non-trivial graph or* (NGRA) *non-geometric irreducible non-graph;*

(iii) *rationally inessential: finitely covered by* $\#_p(S^2 \times S^1)$*, for some* $p \geq 0$.

If there exists an oriented path from a class X to another class Y in Figure 4.2, then any representative in the class Y is dominated by some representative of the class X. If there is no oriented path from the class X to the class Y, then no manifold in the class Y can be dominated by a manifold of the class X.

Some of the non-existence results in the above theorem can be easily deduced using other important tools, for example, Gromov's simplicial volume, Thurston's norm and the Seifert volume. For the existence part of Theorem 4.16, concerning maps between aspherical three-manifolds, we refer to Wang's paper [86]. In particular, aspherical three-manifolds containing a hyperbolic piece in their (possibly empty) JSJ decomposition are maximal with respect to the domination relation because of the following results:

Theorem 4.17.

(1) (Brooks [11].) Every closed three-manifold admits a branched double covering by a closed hyperbolic three-manifold which fibers over the circle.

(2) (Gordon-Litherland [33].) Let M be a closed three-manifold and a link L in M so that (M, L) admits a regular branched covering by a hyperbolic closed three-manifold. Then $M - L$ is hyperbolic.

Remark 4.18. Brooks [11] result is based on a construction of Sakuma [60], which states that every closed three-manifold admits a branched double covering by a surface bundle. Boileau-Wang [8] improved the degree of Brooks result, proving that every closed three-manifold admits a degree one map by a hyperbolic three-manifold that fibers over the circle. Gaifullin [31] showed that there exist closed hyperbolic three-manifolds that virtually dominate[5] every other closed three-manifold. Recently, Sun [75] proved that every closed hyperbolic three-manifold virtually 2-dominates every other closed three-manifold.

The only remaining case which is not contained in Wang's paper [86], or in [46], is that $S^2 \times S^1$ admits a non-zero degree map by a Sol^3-manifold. As we have mentioned above, every closed Sol^3-manifold is virtually a mapping torus of T^2 with hyperbolic monodromy. Therefore, a map for this remaining case can be obtained similarly to the construction of Proposition 4.6, for $p = 1$. We only need to replace the monodromy $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of the mapping torus of T^2 by a hyperbolic one, for example by $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

We furthermore note that the restriction $p \leq 1$ for the arrows from Sol^3, Nil^3 and $\mathbb{R}^3$ to $\#_p(S^2 \times S^1)$ in Figure 4.2 is required for the following reasons:

(i) The first Betti number of closed Sol^3-manifolds is one and so they cannot dominate $\#_p(S^2 \times S^1)$, if $p \geq 2$, by Lemma 1.2 (2).

(ii) Let M be a closed Nil^3-manifold. After passing to a finite covering, we may assume that M is a non-trivial circle bundle over T^2, and so $C(\pi_1(M)) = \mathbb{Z}$. Suppose that there

[5] A manifold M *virtually dominates* N if there is a finite cover $M' \xrightarrow{p} M$ and a map of non-zero degree $f\colon M' \longrightarrow N$. The *virtual degree* of the *virtual map* $v_f\colon M \dashrightarrow N$ is $\deg(v_f) := \frac{\deg(f)}{\deg(p)}$.

is a continuous map $f\colon M \longrightarrow \#_p(S^2 \times S^1)$, where $p \geq 2$, such that $\pi_1(f)(\pi_1(M))$ is a finite index subgroup of F_p, i.e. $\pi_1(f)(\pi_1(M)) = F_l$ for some $l \geq 2$. The homomorphism $\pi_1(f)\colon \pi_1(M) \longrightarrow F_l$ must factor through $\pi_1(T^2) = \mathbb{Z}^2$, because free groups on more than one generators do not have center, whereas $C(\pi_1(M)) = \mathbb{Z}$. However, $\mathbb{Z}^2$ cannot surject onto such a free group and so f must be of zero degree.

(iii) A flat three-manifold has fundamental group virtually $\mathbb{Z}^3$, which cannot surject onto a free group on more than one generators.

Finally, we observe that whenever a circle bundle dominates $\#_p(S^2 \times S^1)$, the genus of the base surface must be at least p:

Lemma 4.19. *Assume that M is a circle bundle over a surface of genus g so that M dominates a connected sum $\#_p(S^2 \times S^1)$. Then $g \geq p$.*

Proof. The interesting cases occur when $p \geq 2$. Suppose that $f\colon M \longrightarrow \#_p(S^2 \times S^1)$ is a map of non-zero degree. Then the base surface Σ_g is aspherical and $\pi_1(f)(\pi_1(M))$ is a free group on $l \geq p$ generators. The infinite cyclic group generated by the circle fiber of M is central in $\pi_1(M)$, and therefore is mapped trivially in F_l, which means that $\pi_1(f)$ factors through $\pi_1(\Sigma_g)$. Since the degree of f is not zero, we obtain an injective homomorphism (cf. Lemma 1.2 (2))

$$H^1(f)\colon H^1(F_l) \longrightarrow H^1(\Sigma_g).$$

(Note that both $H^1(F_l)$ and $H^1(\Sigma_g)$ are torsion-free.) The cup product of any two elements α_1, α_2 in $H^1(F_l)$ is trivial, because $H^2(F_l) = 0$. By the naturality of the cup product, we have that $H^1(f)(\alpha_1) \cup H^1(f)(\alpha_2)$ vanishes as well. This implies that $l \leq \frac{1}{2} \dim H^1(\Sigma_g) = g$, because otherwise the intersection form of $H^1(\Sigma_g)$ would be degenerate. (See Section 6.2.1 for a short introduction on the intersection form (of a four-manifold).) □

As we have already mentioned, the domination relation does not define a partial order in general, because the antisymmetric property is not always satisfied. Nevertheless, if we restrict to the class of aspherical three-manifolds and to degree one maps, then the domination relation indeed defines a partial order on those manifolds, because three-manifold groups are Hopfian [39] (i.e. every surjective endomorphism is an isomorphism), by Perelman's proof of the geometrization conjecture. For further details, we refer to the works of Wang [86, 87] and Rong [56].

Chapter 5

Geometric four-manifolds

In this chapter we investigate the domination relation in four dimensions. Following the geometric form of the results in dimension three, we determine which closed aspherical geometric four-manifolds are dominated by products, exploiting the obstructions "not presentable by products" and "not IIPP". In particular, we answer Question 1.12 for geometric four-manifolds with fundamental groups presentable by products.

Furthermore, we extend Wang's ordering defined in dimension three, to the non-hyperbolic aspherical four-dimensional Thurston geometries. Along the way, we investigate non-zero degree maps between products in arbitrary dimensions. This will yield certain other stable non-domination results with respect to direct products, which cannot be obtained using well-known tools, such as the simplicial volume.

5.1 Enumeration of the four-dimensional geometries

The classification of the four-dimensional geometries in the sense of Thurston (see Section 4.4 for the definition) is due to Filipkiewicz [26]. According to that, there exist eighteen geometries in dimension four with compact representatives. There is an additional geometry which, however, cannot be realized by any compact four-manifold. In this chapter, we deal only with the aspherical geometries, because the non-aspherical ones are not interesting for domination by products. Namely, the non-aspherical geometries are

- either products of a sphere with a non-compact factor ($\mathbb{H}^2 \times S^2$, $\mathbb{R}^2 \times S^2$, $S^3 \times \mathbb{R}$), or
- compact themselves ($S^2 \times S^2$, $\mathbb{CP}^2$, S^4),

and all of their representatives are dominated by products. Actually, the latter three geometries will be included (as trivial cases) in the discussion of the upcoming chapter, where we construct maps by products for every closed simply connected four-manifold.

Type	Geometry $\mathbb{X}^4$
Hyperbolic	$\mathbb{H}^4$, $\mathbb{H}^2(\mathbb{C})$
Product	$\mathbb{H}^3 \times \mathbb{R}$, $Sol^3 \times \mathbb{R}$, $\widetilde{SL_2} \times \mathbb{R}$, $Nil^3 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}^2$, $\mathbb{R}^4$, $\mathbb{H}^2 \times \mathbb{H}^2$
Solvable non-product	Nil^4, $Sol^4_{m \neq n}$, Sol^4_0 Sol^4_1

Table 5.1: The four-dimensional aspherical Thurston geometries with compact representatives.

Enumeration of the aspherical geometries

We begin by enumerating the aspherical geometries, following Wall's papers [84] and [85]. Our list is adapted to the main topic of the thesis, i.e. to domination by products, and it will be used as an organizing principle; see Table 5.1.

Hyperbolic geometries. There exist two aspherical irreducible symmetric geometries, namely the real and the complex hyperbolic, denoted by $\mathbb{H}^4$ and $\mathbb{H}^2(\mathbb{C})$ respectively.

Product geometries. Seven of the aspherical geometries are products of lower dimensional geometries: $\mathbb{H}^3 \times \mathbb{R}$, $Sol^3 \times \mathbb{R}$, $\widetilde{SL_2} \times \mathbb{R}$, $Nil^3 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}^2$, $\mathbb{R}^4$ and $\mathbb{H}^2 \times \mathbb{H}^2$.

Closed manifolds possessing a geometry of type $\mathbb{X}^3 \times \mathbb{R}$ satisfy the following property:

Theorem 5.1 ([40, Sections 8.5 and 9.2])**.** *Let $\mathbb{X}^3$ be a three-dimensional aspherical geometry. A closed four-manifold carrying the geometry $\mathbb{X}^3 \times \mathbb{R}$ is finitely covered by a product $N \times S^1$, where N is a closed oriented three-manifold carrying the $\mathbb{X}^3$ geometry.*

The geometry $\mathbb{H}^2 \times \mathbb{H}^2$ can be realized both by manifolds that are virtual products of two closed hyperbolic surfaces and by manifolds that are not even (virtual) surface bundles. These two types are known as the *reducible* and the *irreducible* $\mathbb{H}^2 \times \mathbb{H}^2$ geometry respectively; see [40, Section 9.5].

Solvable non-product geometries. Finally, there exist four aspherical non-product geometries of solvable type. Below, we describe their model Lie groups.

The nilpotent Lie group Nil^4 is defined as the semi-direct product $\mathbb{R}^3 \rtimes \mathbb{R}$, where $\mathbb{R}$ acts on $\mathbb{R}^3$ by

$$t \mapsto \begin{pmatrix} 0 & e^t & 0 \\ 0 & 0 & e^t \\ 0 & 0 & 0 \end{pmatrix}.$$

The model spaces for the three non-product solvable – but not nilpotent – geometries are defined as follows:

Let m and n be positive integers and $a > b > c$ reals such that $a + b + c = 0$ and e^a, e^b, e^c are roots of the equation $P_{m,n}(\lambda) = \lambda^3 - m\lambda^2 + n\lambda - 1 = 0$. If $m \neq n$, the Lie group $Sol^4_{m\neq n}$ is defined as $\mathbb{R}^3 \rtimes \mathbb{R}$, where $\mathbb{R}$ acts on $\mathbb{R}^3$ by

$$t \mapsto \begin{pmatrix} e^{at} & 0 & 0 \\ 0 & e^{bt} & 0 \\ 0 & 0 & e^{ct} \end{pmatrix}.$$

We remark that the case $m = n$ gives $b = 0$ and corresponds to the product geometry $Sol^3 \times \mathbb{R}$.

If we require two equal roots of the polynomial $P_{m,n}(\lambda)$, then we obtain the model space of the Sol^4_0 geometry, again defined as $\mathbb{R}^3 \rtimes \mathbb{R}$, where now the action of $\mathbb{R}$ on $\mathbb{R}^3$ is given by

$$t \mapsto \begin{pmatrix} e^{t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}.$$

The last solvable model space is an extension of $\mathbb{R}$ by the Heisenberg group

$$Nil^3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}.$$

Namely, the Lie group Sol^4_1 is defined as the semi-direct product $Nil^3 \rtimes \mathbb{R}$, where $\mathbb{R}$ acts on Nil^3 by

$$t \mapsto \begin{pmatrix} 1 & e^{-t}x & z \\ 0 & 1 & e^{t}y \\ 0 & 0 & 1 \end{pmatrix}.$$

Closed oriented four-manifolds possessing solvable non-product geometries are mapping tori:

Theorem 5.2 ([40, Sections 8.6 and 8.7])**.**

(1) *A closed Sol^4_0- or $Sol^4_{m\neq n}$-manifold is a mapping torus of a self-homeomorphism of T^3.*

(2) *A closed oriented Nil^4- or Sol^4_1-manifold is a mapping torus of a self-homeomorphism of a Nil^3-manifold.*

We note that non-orientable closed Nil^4- or Sol^4_1-manifolds are not mapping tori of Nil^3-manifolds [40, Theorem 8.9]. Further details about manifolds possessing a solvable non-product geometry, in particular concerning their fundamental groups, will be provided while examining each geometry individually.

A crucial property for our study is that the four-dimensional geometries are homotopically unique, by the following result of Wall:

Theorem 5.3 ([85, Theorem 10.1],[43, Prop. 1]). *If M and N are homotopy equivalent closed four-manifolds possessing geometries $\mathbb{X}^4$ and $\mathbb{Y}^4$ respectively, then $\mathbb{X}^4 = \mathbb{Y}^4$.*

In particular, a closed aspherical geometric four-manifold M is finitely covered by a closed $\mathbb{X}^4$-manifold if and only if it possesses the $\mathbb{X}^4$ geometry.

5.2 Fundamental groups of geometric four-manifolds

Unlike for three-manifolds, the fundamental group does not generally govern the topology of higher dimensional manifolds. However, as we have already seen, certain properties of the fundamental group play an essential role in the investigation of non-zero degree maps and, in particular, in the study of domination by products. Moreover, aspherical manifolds are characterized (at least) up to homotopy by their fundamental groups.

In the first result of this section we determine which closed aspherical geometric four-manifolds have fundamental groups (not) presentable by products:

Theorem 5.4. *The fundamental group of a closed aspherical geometric four-manifold M is presentable by products if and only if M possesses one of the geometries $\mathbb{X}^3 \times \mathbb{R}$, Nil^4, Sol_1^4 or the reducible $\mathbb{H}^2 \times \mathbb{H}^2$ geometry.*

However, the property "not presentable by products" will not suffice alone to determine which geometric four-manifolds are not dominated by products. In particular, we will not be able to decide whether closed four-manifolds with geometries modelled on Sol_1^4 or Nil^4 are (not) dominated by products. For these two geometries, we will first prove that the fundamental groups of their representatives are not IIPP, and then apply our results from Chapter 2.

The main algebraic characterization for the fundamental groups of aspherical geometric four-manifolds is the following:

Theorem 5.5. *The fundamental group of a closed aspherical geometric four-manifold M is reducible if and only if it is* IIPP*. Equivalently, M carries one of the product geometries $\mathbb{X}^3 \times \mathbb{R}$ or the reducible $\mathbb{H}^2 \times \mathbb{H}^2$ geometry.*

5.2.1 Groups presentable by products: The proof of Theorem 5.4

We first prove Theorem 5.4. We proceed by examining case by case all the aspherical geometries, following the enumeration of the previous section.

Hyperbolic geometries

As we have already seen in Theorem 1.11 (and used in the proof of Proposition 2.21), non-virtually cyclic hyperbolic groups are not presentable by products. The proof of this fact (given in [44]) is based on one of the basic features of hyperbolic groups, namely that the centralizers of their elements are small:

Proposition 5.6 ([10, Prop. 2.22 and Cor. 3.10])**.** *Let Γ be a hyperbolic group.*

(1) *If Γ is infinite, then it contains an element of infinite order.*

(2) *If $g \in \Gamma$ has infinite order, then the infinite cyclic group generated by g has finite index in the centralizer $C_\Gamma(g)$.*

Let now Γ be a hyperbolic group that is not virtually cyclic, and suppose that there exist commuting subgroups $\Gamma_1, \Gamma_2 \subset \Gamma$ and a short exact sequence

$$1 \longrightarrow \Gamma_1 \cap \Gamma_2 \longrightarrow \Gamma_1 \times \Gamma_2 \stackrel{\varphi}{\longrightarrow} \Gamma \longrightarrow 1,$$

where φ is the multiplication map and $\Gamma_1 \cap \Gamma_2 \subset C(\Gamma)$; cf. Section 2.1.1. We will show that one of the Γ_i must be finite and therefore Γ cannot by presented by products.

By Proposition 5.6 (2), every element of the center of Γ is of finite order, because Γ is assumed to be not virtually cyclic. However, Γ contains an element $g \notin C(\Gamma)$ of infinite order, by Proposition 5.6 (1); recall that Γ is infinite, because it is not virtually cyclic. This means that the center $C(\Gamma)$ (which is a torsion group) is finite as a subgroup of the virtually infinite cyclic group $C_\Gamma(g)$; cf. Proposition 5.6 (2). In particular, the intersection $\Gamma_1 \cap \Gamma_2$ is finite.

Now, we may assume that $g \in \Gamma_1$. Since Γ_i commute (elementwise) with each other, the group Γ_2 is virtually cyclic as a subgroup of the virtually infinite cyclic $C_\Gamma(g_1)$; cf. Proposition 5.6 (2). Thus, if Γ_2 were infinite, then there would exist an element $g' \in \Gamma_2$ of infinite order. But then $g' = g^k$ for some $k \in \mathbb{Z}$, otherwise Γ would contain $\mathbb{Z}^2$, contradicting the fact that Γ is hyperbolic; cf. Proposition 5.6 (2). We have now reached the absurd conclusion that the finite group $\Gamma_1 \cap \Gamma_2$ contains an element of infinite order. Thus Γ_2 must be finite. This proves the following:

Proposition 5.7 ([44, Prop. 3.6])**.** *Hyperbolic groups that are not virtually cyclic are not presentable by products.*

Closed four-manifolds with geometries modelled on $\mathbb{H}^4$ or $\mathbb{H}^2(\mathbb{C})$ are negatively curved and therefore they have hyperbolic fundamental groups. Moreover, these four-manifold groups are clearly not (virtually) infinite cyclic, and therefore not presentable by products.

Product geometries

An equivalent formulation of Theorem 5.1 is the following:

Corollary 5.8. *The fundamental group of a closed aspherical four-manifold M carrying a product geometry $\mathbb{X}^3 \times \mathbb{R}$ is a virtual product $\pi_1(N) \times \mathbb{Z}$, where N is a closed aspherical $\mathbb{X}^3$-manifold. In particular, $\pi_1(M)$ is presentable by products and $C(\pi_1(M))$ is virtually infinite.*

The geometry $\mathbb{H}^2 \times \mathbb{H}^2$ is an exceptional type among the product geometries. As we have already mentioned in the previous section, not every closed $\mathbb{H}^2 \times \mathbb{H}^2$-manifold has a finite cover which is a product of two closed hyperbolic surfaces, and this property distinguishes closed $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds into two classes, the reducible and the irreducible ones.

Since $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds admit metrics of non-positive sectional curvature, Theorem 2.24 implies that irreducible lattices in the group of isometries of $\mathbb{H}^2 \times \mathbb{H}^2$ are not presentable by products:

Proposition 5.9. *The fundamental group of a closed $\mathbb{H}^2 \times \mathbb{H}^2$-manifold M is presentable by products if and only if it is a virtual product of two closed hyperbolic surface groups. Equivalently, M carries the reducible $\mathbb{H}^2 \times \mathbb{H}^2$ geometry.*

An important feature of the fundamental groups of irreducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds is given by the next result of Margulis:

Theorem 5.10 ([48, Ch. IX, Theorem 6.14])**.** *Let Γ be an irreducible lattice in the group of isometries of $\mathbb{H}^2 \times \mathbb{H}^2$. Then Γ does not contain any non-trivial normal subgroup of infinite index.*

Obviously, Margulis's theorem yields an alternative proof of the fact that the fundamental groups of irreducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds are not presentable by products.

Solvable non-product geometries

We finally deal with the solvable non-product geometries, i.e. the geometries Nil^4, $Sol^4_{m\neq n}$, Sol^4_0 and Sol^4_1. As we shall see, only the fundamental groups of closed manifolds possessing one of the geometries Nil^4 or Sol^4_1 are presentable by products.

The geometry Nil^4. We first show every closed Nil^4-manifold is finitely covered by a non-trivial circle bundle over a closed oriented Nil^3-manifold.

Proposition 5.11. *A closed Nil^4-manifold M is a virtual circle bundle over a closed oriented Nil^3-manifold and the virtual center of $\pi_1(M)$ is at most $\mathbb{Z}$. In particular, $\pi_1(M)$ is presentable by products.*

Proof. Let M be a closed Nil^4-manifold. After possibly passing to a double cover, we may assume that M is oriented and so $\pi_1(M)$ fits into a short exact sequence

$$1 \longrightarrow \pi_1(N) \longrightarrow \pi_1(M) \longrightarrow \mathbb{Z} \longrightarrow 1,$$

where N is a closed oriented Nil^3-manifold and a generator $t \in \mathbb{Z}$ acts by conjugation on $\pi_1(N)$; cf. Theorem 5.2 (2). Passing to another finite cover, if necessary, we may assume that N is a non-trivial circle bundle over T^2 with fundamental group

$$\pi_1(N) = \langle x, y, z \mid [x, y] = z,\ xz = zx,\ yz = zy \rangle,$$

where $C(\pi_1(N)) = \langle z \rangle$; cf. [64].

Since M is a Nil^4-manifold, the automorphism of $\pi_1(N)/\langle z \rangle \cong \mathbb{Z}^2$, induced by the action of $t \in \mathbb{Z}$ on $\pi_1(N)$, is given (after possibly passing to another finite cover) by a matrix (conjugate to) $A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$, for some $k \neq 0$; cf. [40, Theorem 8.7]. The relation $x^m y^n = z^{mn} y^n x^m$ in $\pi_1(N)$ gives the following presentation of $\pi_1(M)$ (see also [29] and [78, pg. 522] for further details):

$$\pi_1(M) = \langle x, y, z, t \mid txt^{-1} = x,\ tyt^{-1} = x^k y z^l,\ tzt^{-1} = z^{\det A} = z, [x, y] = z,\ xz = zx,\ yz = zy \rangle,$$

where $C(\pi_1(M)) = \langle z \rangle$. Thus we have a short exact sequence

$$1 \longrightarrow \langle z \rangle \longrightarrow \pi_1(M) \longrightarrow Q \longrightarrow 1, \tag{5.1}$$

where $Q = \pi_1(M)/\langle z \rangle = \langle x, y, t \mid [t, y] = x^k,\ xt = tx,\ xy = yx \rangle$. In particular, the classifying space BQ is a non-trivial circle bundle over T^2 and thus a Nil^3-manifold. Now, the homotopy fiber sequence of the classifying spaces corresponding to (5.1) implies that M is homotopically a circle bundle over BQ.

Finally, the center of $\pi_1(M)$ remains infinite cyclic in finite covers, generated by multiples of z, because $k \neq 0$. $\square$

Since every nilpotent group has non-trivial center and the property of being "nilpotent" is closed under subgroups and quotient groups, the proof of the above proposition could be obtained using the fact that every nilpotent group of cohomological dimension three is either Abelian or isomorphic to Q (as in the above proof); see Lemma 5.22 and Remark 5.24.

Remark 5.12. Note that $\pi_1(M)$ is (virtually) an extension of $\mathbb{Z}^2 = \langle y, t \rangle$ by $\mathbb{Z}^2 = \langle z, x \rangle$, and so M is (virtually) a T^2-bundle over T^2, whose T^2-fiber contains the S^1-fiber of the circle bundle $S^1 \longrightarrow M \longrightarrow BQ$ of the above proposition. It is a result of Ue [78, Theorem B] that every closed Nil^4-manifold is a virtual T^2-bundle over T^2. We refer to a work of Fukuhara-Sakamoto [29] for a classification of T^2-bundles over T^2.

We furthermore observe that $\pi_1(M)$ is also (virtually) an extension of $\mathbb{Z} = \langle y \rangle$ by $\mathbb{Z}^3 = \langle z, x, t \rangle$, where the automorphism of $\mathbb{Z}^3$ is given by $\begin{pmatrix} 1 & -1 & -l \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{pmatrix}$ and it has infinite order. In particular, M is a (virtual) mapping torus of a self-homeomorphism of T^3; see also [40, Section 8.6].

The geometries $Sol^4_{m\neq n}$ and Sol^4_0. An interesting class of groups not presentable by products, given by Kotschick-Löh [45], is that of groups containing infinite acentral subgroups of infinite index (Theorem 1.11 (2)). In this paragraph, we will show that the fundamental groups of closed $Sol^4_{m\neq n}$-manifolds and Sol^4_0-manifolds fulfill that property.

Definition 5.13 ([45])**.** A subgroup A of a group Γ is called *acentral* if for every non-trivial element $g \in A$ the centralizer $C_\Gamma(g)$ is contained in A.

An extension of groups $1 \longrightarrow N \longrightarrow \Gamma \longrightarrow Q \longrightarrow 1$ is called *acentral extension* if the normal subgroup N is acentral.

Proposition 5.14 ([45, Prop. 3.2])**.** *If a group contains an infinite acentral subgroup of infinite index, then it is not presentable by products.*

This proposition has the following consequences on group extensions:

Corollary 5.15 ([45, Cor. 3.3 and 3.5])**.**

(1) *If the extension $1 \longrightarrow N \longrightarrow \Gamma \longrightarrow Q \longrightarrow 1$ is acentral, with N and Q infinite, then Γ is not presentable by products.*

(2) *Let N be a non-trivial Abelian group and Q an infinite group. If Γ is a semi-direct product $N \rtimes_\theta Q$, where the action of Q on N is free outside $0 \in N$, then the extension $0 \longrightarrow N \longrightarrow \Gamma \longrightarrow Q \longrightarrow 1$ is acentral and N is infinite. In particular, Γ is not presentable by products.*

Example 5.16 ([45, Section 3])**.** Let Γ be a lattice in Sol^3. Then Γ is virtually a semi-direct product $\mathbb{Z}^2 \rtimes_\theta \mathbb{Z}$, where the action θ is induced by a hyperbolic monodromy of T^2; cf. [64] (or Table 4.1). Thus, θ is free outside $0 \in \mathbb{Z}^2$ and Γ is not presentable by products, by Corollary 5.15 (2). This example can be generalized for semi-direct products $\mathbb{Z}^n \rtimes_\theta \mathbb{Z}$, if no non-trivial power of the matrix corresponding to the action θ has eigenvalue one.

The discussion in the above example applies to the fundamental groups of closed $Sol^4_{m\neq n}$-manifolds and Sol^4_0-manifolds as well:

Proposition 5.17. *The fundamental group of a closed four-manifold possessing one of the geometries $Sol^4_{m\neq n}$ or Sol^4_0 is not presentable by products.*

Proof. We will show that the fundamental groups of closed $Sol^4_{m\neq n}$- or Sol^4_0-manifolds contain acentral subgroups of infinite index.

By Theorem 5.2 (1), every manifold M with geometry modelled on $Sol^4_{m\neq n}$ or Sol^4_0 is a mapping torus of a self-homeomorphism of T^3 (see also [84, 85]) and its fundamental group is a semi-direct product $\mathbb{Z}^3 \rtimes_\theta \mathbb{Z}$, where the automorphism θ of $\mathbb{Z}^3$ is induced by the action by conjugation of a generator $t \in \mathbb{Z}$.

Now, if M is a $Sol^4_{m\neq n}$-manifold, then θ has three real distinct eigenvalues and none of them is equal to ± 1, because M is neither nilpotent nor carries the $Sol^3 \times \mathbb{R}$ geometry (which is the case $m = n$); cf. [84] and [40, pg. 164/165] (or Section 5.1).

If M is a Sol^4_0-manifold, then θ has two complex eigenvalues that are not roots of unity and a real eigenvalue not equal to ± 1; cf. [84] and [40, pg. 164/165] (or Section 5.1).

In both cases we derive that the centralizer $C_{\pi_1(M)}(g)$ of each element $g \in \mathbb{Z}^3 \setminus \{0\}$ is contained in $\mathbb{Z}^3$ (it is actually equal to $\mathbb{Z}^3$). This means that the infinite-index normal subgroup $\mathbb{Z}^3$ is acentral and so $\pi_1(M)$ is not presentable by products by Proposition 5.14 or Corollary 5.15. □

The geometry Sol^4_1. We finally show that closed Sol^4_1-manifolds are virtual circle bundles over closed oriented Sol^3-manifolds.

Proposition 5.18. *A closed Sol^4_1-manifold M is a virtual circle bundle over a mapping torus of T^2 with hyperbolic monodromy. In particular, $\pi_1(M)$ is presentable by products.*

Proof. Let M be a closed Sol^4_1-manifold. After passing to a double cover, we may assume that M is oriented, and so its fundamental group fits into a short exact sequence

$$1 \longrightarrow \pi_1(N) \longrightarrow \pi_1(M) \longrightarrow \mathbb{Z} \longrightarrow 1,$$

where N is a closed oriented Nil^3-manifold and a generator $t \in \mathbb{Z}$ acts by conjugation on $\pi_1(N)$; cf. Theorem 5.2 (2). If necessary, we pass to another finite cover of M and so we can assume that the fiber N is a non-trivial circle bundle over T^2 and that its fundamental group has presentation

$$\pi_1(N) = \langle x, y, z \mid [x, y] = z,\ xz = zx,\ yz = zy\rangle$$

with center $C(\pi_1(N)) = \langle z\rangle$; cf. [64].

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$ be the automorphism of $\pi_1(N)/\langle z\rangle \cong \mathbb{Z}^2$ induced by the action of $t \in \mathbb{Z}$ on $\pi_1(N)$. The eigenvalues λ_1, λ_2 of A satisfy $\det(A) = \lambda_1\lambda_2 = \pm 1$. Actually, $\det A = 1$, because M is oriented. Moreover, $\lambda_i \neq \pm 1$, because $\pi_1(M)$ is not nilpotent; see also [40, Theorem 8.7]. We conclude that A is a hyperbolic automorphism.

Now the relation $x^m y^n = z^{mn} y^n x^m$ in $\pi_1(N)$ implies that a presentation of the fundamental group of M is given by

$$\begin{aligned}\pi_1(M) = \ \langle x, y, z, t \mid \ & txt^{-1} = x^a y^c z^k,\ tyt^{-1} = x^b y^d z^l,\ tzt^{-1} = z^{\det A} = z, \\ & [x, y] = z,\ xz = zx,\ yz = zy\rangle,\ k, l \in \mathbb{Z},\end{aligned}$$

where the infinite cyclic group generated by z is central in $\pi_1(M)$. Thus we obtain a short exact sequence

$$1 \longrightarrow \langle z \rangle \longrightarrow \pi_1(M) \longrightarrow Q \longrightarrow 1, \tag{5.2}$$

where $Q = \pi_1(M)/\langle z \rangle = \langle x, y, t \mid txt^{-1} = x^a y^c,\ tyt^{-1} = x^b y^d,\ xy = yx \rangle$. Clearly, the group Q fits into an extension

$$1 \longrightarrow \mathbb{Z}^2 \longrightarrow Q \longrightarrow \mathbb{Z} \longrightarrow 1,$$

where $t \in \mathbb{Z}$ acts on $\mathbb{Z}^2$ by the hyperbolic automorphism A. We have now shown that Q is the fundamental group of a mapping torus of T^2 with hyperbolic monodromy, i.e. BQ is a closed oriented Sol^3-manifold. Therefore, M is homotopically a circle bundle over BQ, by the homotopy fiber sequence corresponding to the short exact sequence (5.2). □

The proof of Theorem 5.4 is now complete.

5.2.2 Groups not IIPP: The proof of Theorem 5.5

We now prove Theorem 5.5. Since reducible groups are IIPP, in order to complete the claim of Theorem 5.5 we need to show that the fundamental groups of manifolds possessing one of the geometries Nil^4 or Sol_1^4 are not IIPP.

Closed Nil^4-manifolds. As we have seen in Proposition 5.11, closed Nil^4-manifolds are virtual circle bundles over closed oriented Nil^3-manifolds and their fundamental groups have virtual center at most $\mathbb{Z}$. Using this description together with further properties of finitely generated torsion-free nilpotent groups, we will show that the fundamental groups of closed Nil^4-manifolds are not IIPP.

We begin by showing that closed Nil^4-manifolds have irreducible fundamental groups:

Lemma 5.19. *The fundamental group of a closed Nil^4-manifold is not a virtual product.*

Proof. Let M be a closed Nil^4-manifold. Suppose, for contrast, that $\pi_1(M)$ is reducible and so, after passing to a finite cover if required, isomorphic to a direct product $\Delta_1 \times \Delta_2$, where Δ_i are normal infinite subgroups of $\pi_1(M)$. The groups Δ_i are torsion-free, because $\pi_1(M)$ is torsion-free. By Proposition 5.11, we can moreover assume (after possibly passing to another finite cover) that $\Delta_1 \times \Delta_2$ fits into a short exact sequence

$$1 \longrightarrow \pi_1(S^1) \longrightarrow \pi_1(M) \cong \Delta_1 \times \Delta_2 \longrightarrow \pi_1(N) \longrightarrow 1,$$

where the center of $\pi_1(M)$ is isomorphic to $\pi_1(S^1)$ and N is a non-trivial circle bundle over T^2. In particular, $\pi_1(N)$ is not reducible by Epstein's factorization theorem and Stallings fibering criterion; cf. Theorem 2.18.

We observe that the center $\pi_1(S^1)$ of $\pi_1(M)$ maps non-trivially only to one of the Δ_i, otherwise it would have rank at least two, because $C(\pi_1(M)) \cong C(\Delta_1) \times C(\Delta_2)$. Let us assume that this infinite cyclic group maps trivially to Δ_2 and therefore $\pi_1(N)$ is isomorphic to the direct product $(\Delta_1/\mathbb{Z}) \times \Delta_2$. Since $\pi_1(N)$ is not reducible and Δ_i are not trivial, we deduce that Δ_1 is infinite cyclic. We have now reached the conclusion that M is (virtually) a product $S^1 \times N$, where N is a closed Nil^3-manifold. If that were true, then $S^1 \times N$ would carry the geometries $Nil^3 \times \mathbb{R}$ and Nil^4 simultaneously, which is not possible by Wall's Theorem 5.3. This proves that $\pi_1(M)$ is irreducible. □

In Chapter 2 (cf. Proposition 2.21), in order to show that closed Nil^3-manifolds have fundamental groups not IIPP, we resorted to Bieri's [7] results, because the cohomological dimensions of those groups (and of their subgroups) were suitable for that purpose. Passing now one dimension higher, we cannot appeal anymore to those results. However, we may use a property of the cohomological dimension of finitely generated torsion-free nilpotent groups, namely, that it is equal to its Hirsch length.

The Hirsch length generalizes the notion of the rank of free Abelian groups:

Definition 5.20 ([36])**.** Let Γ be a (virtually) polycyclic group with a series

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_n = 1,$$

so that the quotients Γ_i/Γ_{i+1} are cyclic. The sum of the ranks of these quotients is independent of the choice of the series of groups and is called the *Hirsch length* of Γ.

We denote the Hirsch length of Γ by $h(\Gamma)$.

Example 5.21. A finitely generated torsion-free nilpotent group Γ is polycyclic, admitting a central series

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_n = 1,$$

so that the quotients Γ_i/Γ_{i+1} are infinite cyclic for all $i = 0, 1, ..., n-1$. Therefore Γ has a well-defined Hirsch length equal to n.

Let us now recall some basic properties of nilpotent groups and of the Hirsch length. First of all, subgroups and quotient groups of nilpotent groups are nilpotent themselves. Moreover, if Γ is a torsion-free nilpotent group, then its center $C(\Gamma)$ has positive rank. More interesting is the fact that the quotient group $\Gamma/C(\Gamma)$ is again torsion-free. In order to prove the latter, let $g \in \Gamma$ such that $g^m \in C(\Gamma)$ for some m, i.e. the image of g under the projection $\Gamma \longrightarrow \Gamma/C(\Gamma)$ is a torsion element. We need to show that $g \in C(\Gamma)$. For any $\gamma \in \Gamma$, we define a recursive sequence by

$$x_1 := [g, \gamma], \ x_k := [g, x_{k-1}] \text{ for all } k \geq 2.$$

Since Γ is nilpotent, there exists $n \in \mathbb{N}$ such that $x_n = 1$, that is, $[g, x_{n-1}] = 1$. Now, the identity $[a^l, b] = a^{l-1}[a, b]a^{1-l}[a^{l-1}, b]$ for the commutator of any two elements, and our assumption that

$g^m \in C(\Gamma)$ yield

$$\begin{aligned}
1 = [g^m, x_{n-2}] &= g^{m-1}[g, x_{n-2}]g^{1-m}[g^{m-1}, x_{n-2}] \\
&= g^{m-1}x_{n-1}g^{1-m}[g^{m-1}, x_{n-2}] \\
&= g^{m-1}x_{n-1}g^{1-m}g^{m-2}[g, x_{n-2}]g^{2-m}[g^{m-2}, x_{n-2}] \\
&= g^{m-1}x_{n-1}g^{-1}x_{n-1}g^{2-m}[g^{m-2}, x_{n-2}] \\
&= g^m g^{-1}x_{n-1}g^{-1}x_{n-1}g^{2-m}[g^{m-2}, x_{n-2}] \\
&= \cdots \\
&= g^m \underbrace{g^{-1}x_{n-1}\cdots g^{-1}x_{n-1}}_{m} \\
&= g^m g^{-m} x_{n-1}^m \\
&= x_{n-1}^m.
\end{aligned}$$

Since Γ is torsion-free, we deduce that $x_{n-1} = 1$. Continuing this procedure we obtain that $x_k = 1$, for all $k \geq 1$. In particular, $[g, \gamma] = 1$ for all $\gamma \in \Gamma$, which means that $g \in C(\Gamma)$.

We furthermore observe that the Hirsch length is additive with respect to extensions of nilpotent groups. Indeed, let K be a normal subgroup of a finitely generated nilpotent group Γ. Then K and Γ/K are finitely generated nilpotent groups admitting series

$$K = K_0 \supset K_1 \supset \cdots \supset K_m = 1$$

and

$$\Gamma/K = \Gamma_0/K \supset \Gamma_1/K \supset \cdots \supset \Gamma_l/K = K/K = 1,$$

where the quotients K_i/K_{i+1} and Γ_j/Γ_{j+1} are all cyclic, but not necessarily infinite. (Note that the fact that K is finitely generated can be proven by induction on the nilpotency class of Γ.) Thus Γ admits a series

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_l = K = K_0 \supset K_1 \supset \cdots \supset K_m = 1,$$

which implies that $h(\Gamma) = h(K) + h(\Gamma/K)$. We remark that, if Γ is torsion-free and $K = C(\Gamma)$, then we can assume that all the quotients K_i/K_{i+1} and Γ_j/Γ_{j+1} are infinite cyclic, and so $m = h(K)$ and $l = h(\Gamma/K)$ in the above series for K and Γ/K respectively.

It is moreover clear that, if K has finite index in Γ, then $h(K) = h(\Gamma)$. The converse is also true: Let Γ be a (non-torsion) finitely generated nilpotent group and K be a subgroup so that $h(K) = h(\Gamma)$. Suppose that Γ admits a series

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_n = 1$$

where Γ/Γ_1 is infinite cyclic (otherwise replace Γ by Γ_1). Since the Hirsch length is additive for extensions of nilpotent groups, we obtain that $h(\Gamma) = h(\Gamma_1) + 1$, and so $h(K) = h(\Gamma_1) + 1$. The intersection $K \cap \Gamma_1$ is a normal subgroup of K (because Γ_1 is normal in Γ) and the quotient group $K/K \cap \Gamma_1$ is infinite cyclic (being isomorphic to a non-trivial subgroup of the infinite cyclic Γ/Γ_1). Therefore $h(K) = h(K \cap \Gamma_1) + 1$, and so $h(\Gamma_1) = h(K \cap \Gamma_1)$. Continuing the process, we obtain that

$$h(\Gamma_k) = h(K \cap \Gamma_k), \text{ for all } k = 0, 1, ..., n.$$

As far as $h(\Gamma_k) \neq 0$, each group $K \cap \Gamma_k$ is not trivial, and, by induction, is of finite index in Γ_k (note that $\mathbb{Z} \cong K \cap \Gamma_{k-1}/K \cap \Gamma_k \cong (K \cap \Gamma_{k-1})\Gamma_k/\Gamma_k \subset \Gamma_{k-1}/\Gamma_k \cong \mathbb{Z}$). In particular, $[\Gamma_1 : K \cap \Gamma_1] < \infty$ and the product $K\Gamma_1$ has finite index in Γ (again because $\mathbb{Z} \cong K/K \cap \Gamma_1 \cong K\Gamma_1/\Gamma_1 \subset \Gamma/\Gamma_1 \cong \mathbb{Z}$). This finally implies that

$$[\Gamma : K] = [\Gamma_1 : K \cap \Gamma_1] < \infty.$$

Using the above properties, we are able to determine all torsion-free nilpotent groups of Hirsch length three. We have used several times the fact that every closed Nil^3-manifold is a virtually non-trivial circle bundle over T^2 with non-zero Euler class n. Let us denote the corresponding nilpotent group by

$$G_n := \langle x, y, z \mid zy = yz, zx = xz, [x, y] = z^n \rangle.$$

Of course, for $n = 0$ the group G_0 is $\mathbb{Z}^3$. With this notation, we obtain the following:

Lemma 5.22. *Let Γ be a torsion-free nilpotent group of Hirsch length three. Then Γ is isomorphic to G_n, for some $n \geq 0$. In particular, Γ is the fundamental group of a circle bundle over T^2.*

Proof. First, we observe that Γ is finitely generated, because it is nilpotent of finite Hirsch length. Moreover, since Γ is torsion-free (and nilpotent), its center $C(\Gamma)$ is free Abelian of positive rank.

As we have seen above, the quotient group $Q := \Gamma/C(\Gamma)$ is again nilpotent and torsion-free. Since the Hirsch length is additive with respect to extensions of nilpotent groups, the short exact sequence

$$1 \longrightarrow C(\Gamma) \longrightarrow \Gamma \longrightarrow Q \longrightarrow 1, \tag{5.3}$$

yields that $0 \leq h(Q) \leq 2$, because $h(C(\Gamma)) \geq 1$.

If $h(Q) = 0$ or 1, then it is easy to see that Γ is free Abelian of rank three. Indeed, this is obvious if $h(Q) = 0$. If $h(Q) = 1$, then Q is infinite cyclic, and therefore the central extension (5.3) splits.

Suppose, finally, that $h(Q) = 2$. Since Q is torsion-free nilpotent, it has non-trivial center $C(Q)$. Therefore, it fits into a short exact sequence

$$1 \longrightarrow C(Q) \longrightarrow Q \longrightarrow Q/C(Q) \longrightarrow 1,$$

where the quotient $Q/C(Q)$ is again a torsion-free nilpotent group. By the additivity of the Hirsch length for the above exact sequence, we deduce that $h(Q/C(Q) \leq 1$. This finally implies that Q is free Abelian of rank two. Therefore, the central extension (5.3) takes the form

$$1 \longrightarrow \mathbb{Z} \longrightarrow \Gamma \longrightarrow \mathbb{Z}^2 \longrightarrow 1.$$

Choosing presentations $\mathbb{Z} = \langle z \rangle$ and $\mathbb{Z}^2 = \langle x, y \mid [x, y] = 1 \rangle$, we deduce that Γ is isomorphic to G_n for some $n \geq 0$. □

One of the most interesting properties of the Hirsch length of finitely generated torsion-free nilpotent groups is given by the following:

Theorem 5.23 (Gruenberg [36, §8.8] or Bieri [7])**.** *If Γ is a finitely generated torsion-free nilpotent group, then* $\mathrm{cd}(\Gamma) = h(\Gamma)$.

Remark 5.24. In the light of the above theorem, Lemma 5.22 determines all nilpotent groups of cohomological dimension three (note that groups of finite cohomological dimension are torsion-free). Moreover, it yields another proof of the fact that closed Nil^4-manifolds are virtual circle bundles over closed oriented Nil^3-manifolds; compare Proposition 5.11.

We finally show that Nil^4-manifold groups are not IIPP:

Proposition 5.25. *The fundamental group of a closed Nil^4-manifold M is not* IIPP.

Proof. We know that M is virtually a non-trivial circle bundle over a closed oriented Nil^3-manifold and that $\pi_1(M)$ is presentable by products (cf. Proposition 5.11). We now prove that $\pi_1(M)$ cannot be presented by a product of subgroups Γ_1 and Γ_2 so that both Γ_i have infinite index in $\pi_1(M)$. We proceed again by contradiction. After passing to suitable finite index subgroups, suppose that there exist two infinite-index commuting subgroups $\Gamma_i \subset \pi_1(M)$ and a short exact sequence

$$1 \longrightarrow \Gamma_1 \cap \Gamma_2 \longrightarrow \Gamma_1 \times \Gamma_2 \xrightarrow{\varphi} \pi_1(M) \longrightarrow 1, \tag{5.4}$$

where φ is the multiplication map and $\Gamma_1 \cap \Gamma_2 \subset C(\pi_1(M))$; cf. Section 2.1.1. We may also assume that $C(\pi_1(M)) = \mathbb{Z}$, by Proposition 5.11. Furthermore, we observe that Γ_i are finitely generated torsion-free and nilpotent.

Since $\pi_1(M)$ is not reducible (by Lemma 5.19), we conclude that the intersection $\Gamma_1 \cap \Gamma_2$ is not trivial and so it must be infinite cyclic, as a subgroup of $C(\pi_1(M)) = \mathbb{Z}$. The homotopy fiber sequence of the classifying spaces, corresponding to (5.4), implies that the product $B\Gamma_1 \times B\Gamma_2$ is a circle bundle over M, and so it is homotopy equivalent to a closed oriented five-manifold. In particular, $\Gamma_1 \times \Gamma_2$ has cohomological dimension $\mathrm{cd}(\Gamma_1 \times \Gamma_2) = 5$, that is $h(\Gamma_1 \times \Gamma_2) = 5$ by Theorem 5.23, because the product of two nilpotent groups is again nilpotent. Since both Γ_i have infinite index in $\pi_1(M)$, we deduce that $h(\Gamma_i) \leq 3$ (see the comments before Lemma 5.22). Therefore, one of the Γ_i must have Hirsch length equal to three and the other equal to two. Let us assume that $h(\Gamma_1) = 3$ and $h(\Gamma_2) = 2$.

Now we have that Γ_1 is torsion-free nilpotent of Hirsch length three. By Lemma 5.22, Γ_1 is isomorphic to G_n for some $n \geq 0$. Moreover, Γ_2 is isomorphic to $\mathbb{Z}^2$, because it is torsion-free nilpotent of Hirsch length two (see the proof of Lemma 5.22). We have now reached the conclusion that $B\Gamma_1 \times B\Gamma_2$ is homotopy equivalent to a product $T^2 \times BG_n$, and that the rank of the center of $\Gamma_1 \times \Gamma_2$ is at least three. This is however not possible, according to the next lemma, because $B\Gamma_1 \times B\Gamma_2$ is a circle bundle over a Nil^4-manifold M with $C(\pi_1(M)) = \mathbb{Z}$. □

Lemma 5.26. *Suppose that a group* Γ *with finitely generated center* $C(\Gamma)$ *fits into a central extension*

$$1 \longrightarrow \mathbb{Z}^k \longrightarrow \Gamma \stackrel{\pi}{\longrightarrow} Q \longrightarrow 1,$$

where Q *is torsion-free. Then* $\mathrm{rank}C(\Gamma) \leq \mathrm{rank}C(Q) + k$.

Proof. It follows by the fact that if $x \in C(\Gamma)$, then $\pi(x) \in C(Q)$. □

Closed Sol_1^4-manifolds. We finally deal with the Sol_1^4 geometry.

Proposition 5.27. *The fundamental group of a closed* Sol_1^4*-manifold is not* IIPP.

Proof. By Proposition 5.18, a closed Sol_1^4-manifold M is a virtual circle bundle over a closed oriented Sol^3-manifold N. In particular, its fundamental group $\pi_1(M)$ satisfies all the assumptions of Proposition 2.17 (recall that $\pi_1(N)$ is not presentable by products). Thus, $\pi_1(M)$ is IIPP if and only if it is a virtual product $\pi_1(N) \times \mathbb{Z}$. The latter is impossible by Wall's uniqueness Theorem 5.3. □

Remark 5.28. The discussion on the fundamental groups of closed manifolds which carry a solvable non-product geometry gives case (3) of Theorem 1.18.

This finishes the proof of Theorem 5.5.

5.3 Domination by products

We are now able to determine which geometric four-manifolds are (not) dominated by products. The algebraic pieces for the proof have been collected in the previous section. We combine them with the topological statements of Chapter 2 to prove the following:

Theorem 5.29. *A closed oriented aspherical geometric four-manifold M is dominated by a non-trivial product if and only if it is finitely covered by a product. Equivalently, M carries one of the product geometries $\mathbb{X}^3 \times \mathbb{R}$ or the reducible $\mathbb{H}^2 \times \mathbb{H}^2$ geometry.*

Proof. Clearly, if M is covered by a product, then it is dominated by that product. In particular, closed manifolds possessing one of the product geometries $\mathbb{X}^3 \times \mathbb{R}$ or the reducible $\mathbb{H}^2 \times \mathbb{H}^2$ geometry are dominated by products; cf. Theorem 5.1.

By the uniqueness of the four-dimensional geometries (cf. Theorem 5.3), it suffices to show that closed four-manifolds possessing either a hyperbolic geometry, the irreducible $\mathbb{H}^2 \times \mathbb{H}^2$ geometry, or a non-product solvable geometry cannot be dominated by products.

For the hyperbolic geometries $\mathbb{H}^4$ and $\mathbb{H}^2(\mathbb{C})$, the irreducible $\mathbb{H}^2 \times \mathbb{H}^2$ geometry, and the solvable geometries $Sol^4_{m \neq n}$ and Sol^4_0 our claim can be deduced by Theorem 1.9, because the fundamental groups of closed four-manifolds carrying one of those geometries are not presentable by products; see the corresponding Propositions 5.7, 5.9 and 5.17.

If now M is a closed Nil^4-manifold, then it is a virtual circle bundle over a closed oriented Nil^3-manifold and its fundamental group has virtual center at most $\mathbb{Z}$; cf. Proposition 5.11. Moreover, $\pi_1(M)$ is not IIPP, by Proposition 5.25, and so Theorem 2.13 implies that M cannot be dominated by products.

Finally, if M is a closed Sol^4_1-manifold, then it is a virtual circle bundle over a closed oriented Sol^3-manifold and $\pi_1(M)$ is not IIPP; cf. Propositions 5.18 and 5.27 respectively. Therefore, M is not dominated by products by Theorem 2.27, because closed Sol^3-manifolds have fundamental groups not presentable by products. (Equivalently, M is not dominated by products because of the equivalence between (1) and (2) in Theorem 2.27 and by Wall's Theorem 5.3.)

The proof is now complete. □

Combining the above theorem with the characterizations of groups (infinite-index) presentable by products in Section 5.2, we can now answer Question 1.12 for geometric aspherical four-manifolds:

Corollary 5.30. *A closed oriented aspherical geometric four-manifold M is dominated by a product if and only if*

(1) *either $\pi_1(M)$ is a virtual product $\pi_1(N) \times \mathbb{Z}$, for some closed aspherical geometric three-manifold N, or*

(2) *$\pi_1(M)$ is a virtual product of two closed hyperbolic surface groups.*

5.4 Ordering the four-dimensional geometries

In this section we order (in the sense of Wang; cf. Section 4.4.2) certain classes of four-manifolds, in particular, closed aspherical manifolds possessing a four-dimensional non-hyperbolic Thurston geometry. To this end, we begin by investigating maps of non-zero degree between products in arbitrary dimensions.

5.4.1 Non-zero degree maps between products and non-existence stability

A general consequence of Lemma 2.14 is that there do not exist (π_1-surjective) non-zero degree maps between trivial and non-trivial aspherical circle bundles, whenever the fundamental group of the target has infinite cyclic center. Lemma 2.14 was proved by showing that a π_1-surjective map between the total spaces is a bundle map covering a degree zero map (between the bases). In this section, we extend that idea to maps between direct products, since in that case we can always compose our dominant map with the corresponding inclusions (resp. projections) from (resp. to) each of the direct factors of the domain (resp. target).

Let, for example, T^n be the n-dimensional torus. Then, another consequence of Lemma 2.14 is that $M \times T^n \ngeq N \times T^n$, whenever N is aspherical, so that every finite index subgroup of $\pi_1(N)$ has trivial center, and $M \ngeq N$. Below, we generalize the latter observation, assuming that N is not dominated by products and replacing the torus factor by any manifold.

Theorem 5.31. *Let M, N be closed oriented connected n-dimensional manifolds such that N is not dominated by products and W be a closed oriented connected manifold of dimension m. Then $M \geq N$ if and only if $M \times W \geq N \times W$.*

Proof. Clearly, it suffices to prove that $M \times W \geq N \times W$ implies $M \geq N$, where W is any closed oriented connected manifold of (arbitrary) dimension m. Let $f : M \times W \longrightarrow N \times W$ be a map of non-zero degree, and suppose, for contrast, that $M \ngeq N$.

Let $[N] \otimes 1 \in H_n(N \times W)$, where $[N]$ denotes the fundamental class of N. Since the induced homomorphism $H_n(f;\mathbb{Q})\colon H_n(M \times W;\mathbb{Q}) \longrightarrow H_n(N \times W;\mathbb{Q})$ is surjective (cf. Lemma 1.2 (1)), there is a homology class $\beta_0 \in H_n(M \times W;\mathbb{Q})$ such that $H_n(f;\mathbb{Q})(\beta_0) = [N] \otimes 1$. Actually, we can assume that β_0 is an integral homology class so that $H_n(f;\mathbb{Z})(\beta_0) = \deg(f) \cdot ([N] \otimes 1)$.

By the Künneth theorem (with rational coefficients), the nth homology group of $M \times W$ is

$$H_n(M \times W) \cong (H_n(M) \otimes H_0(W)) \oplus [\oplus_{i=1}^{n-1}(H_{n-i}(M) \otimes H_i(W))] \oplus (H_0(M) \otimes H_n(W)). \quad (5.5)$$

Therefore,

$$\beta_0 = k \cdot ([M] \otimes 1) + \sum_{i=1}^{n-1} \lambda_i \cdot (x_{n-i}^M \otimes x_i^W) + \mu \cdot (1 \otimes x_n^W), \quad k, \lambda_i, \mu \in \mathbb{Z}, \quad (5.6)$$

where $[M]$ is the fundamental class of M and $x_{n-i}^M \in H_{n-i}(M)$, $x_i^W \in H_i(W)$, for $i \in \{1, ..., n\}$. (Actually, each summand $x_{n-i}^M \otimes x_i^W$ is a linear combination of elementary tensors, however this does not affect the following arguments of the proof.)

We claim that $\beta_0 \in H_n(W)$. Indeed, first of all we observe that $k=0$, otherwise M would dominate N through the composite map

$$M \overset{i_M}{\hookrightarrow} M \times W \overset{f}{\longrightarrow} N \times W \overset{p_N}{\longrightarrow} N,$$

where $i_M\colon M \hookrightarrow M \times W$ and $p_N\colon N \times W \longrightarrow N$ denote inclusion and projection respectively. Moreover, we may assume that all λ_i in (5.6) vanish: Suppose the contrary, i.e. that there exists a non-zero λ_{i_0}, for some $i_0 \in \{1,...,n-1\}$. Then, Thom's Theorem 1.1 implies that there exist positive integers d_1, d_2 and closed oriented connected smooth manifolds X and Y, of dimensions $n-i_0$ and i_0 respectively, together with continuous maps $g_1\colon X \longrightarrow M$ and $g_2\colon Y \longrightarrow N$, so that $H_{n-i_0}(g_1)([X]) = d_1 \cdot x^M_{n-i_0}$ and $H_{i_0}(g_2)([Y]) = d_2 \cdot x^W_{i_0}$. (The classes $[X]$ and $[Y]$ are the fundamental classes of X and Y respectively.) Therefore N is dominated by $X \times Y$ through the map

$$X \times Y \overset{g_1 \times g_2}{\longrightarrow} M \times W \overset{f}{\longrightarrow} N \times W \overset{p_N}{\longrightarrow} N.$$

However, this contradicts our hypothesis that N is not dominated by products. Thus $\lambda_i = 0$ for all $i \in \{1,...,n-1\}$, and so $\beta_0 \in H_n(W)$ as claimed.

Now, since $H_n(f;\mathbb{Q})$ is surjective, there exist an element $\beta_1 \in H_n(M \times W;\mathbb{Q})$, so that $H_n(f;\mathbb{Q})(\beta_1) = \beta_0$. As before, we may assume that β_1 is an integral homology class (after multiplying β_0 with $\deg(f)$). Likewise for β_0, the Künneth formula (5.5) gives

$$\beta_1 = k' \cdot ([M] \otimes 1) + \sum_{i=1}^{n-1} \lambda'_i \cdot (x^M_{n-i} \otimes x^W_i) + \mu' \cdot (1 \otimes x^W_n), \quad k', \lambda'_i, \mu' \in \mathbb{Z}; \tag{5.7}$$

see (5.6) for the notation.

We claim again that $\beta_1 \in H_n(W)$. Indeed, $k' = 0$, otherwise $M \geq N$ through the composite map

$$M \overset{i_M}{\hookrightarrow} M \times W \overset{f}{\longrightarrow} N \times W \overset{p_W}{\longrightarrow} W \overset{i_W}{\hookrightarrow} M \times W \overset{f}{\longrightarrow} N \times W \overset{p_N}{\longrightarrow} N.$$

(As usual, the maps i_M, i_W and p_W, p_N denote inclusions and projections respectively.) Also, $\lambda'_i = 0$ for all $i \in \{1,...,n-1\}$, otherwise Thom's Theorem 1.1 would imply that N is dominated by a non-trivial product, via the map

$$X \times Y \overset{g_1 \times g_2}{\longrightarrow} M \times W \overset{f}{\longrightarrow} N \times W \overset{p_W}{\longrightarrow} W \overset{i_W}{\hookrightarrow} M \times W \overset{f}{\longrightarrow} N \times W \overset{p_N}{\longrightarrow} N;$$

see above (in the proof that $\beta_0 \in H_n(W)$) for the details.

Now, we repeat the above procedure to show that there exists a $\beta_2 \in H_n(W)$ so that $H_n(f;\mathbb{Q})(\beta_2) = \beta_1$, a $\beta_3 \in H_n(W)$ so that $H_n(f;\mathbb{Q})(\beta_3) = \beta_2$, and so forth. However, $H_n(W)$ has finite rank, say r. (We may also assume that the β_i generate $H_n(W)$.) Thus, after repeating the above process at most r times, we have that the restriction of $H_n(f;\mathbb{Q})$ to $\langle \beta_0,...,\beta_r \rangle$, surjects onto $\langle\beta_0, ..., \beta_r\rangle \oplus H_n(N)$. This contradiction completes the proof. □

Another significant motivation for the investigation of maps between products stems by Gromov's simplicial volume. The functoriality of the simplicial ℓ^1-semi-norm provides a non-domination criterion. Namely, given two closed oriented n-dimensional manifolds M and N, then $M \not\geq N$, whenever $\|M\| < \|N\|$. However, it is not generally known whether the inequality $0 < \|M\| < \|N\|$ implies $\|M^k\| < \|N^k\|$, where $M^k = M \times \cdots \times M$ and $N^k = N \times \cdots \times N$ denote direct products with k-factors. This lack of multiplicativity[6] of the simplicial volume says, in particular, that we cannot decide when $M \not\geq N$ is stable under taking direct products, i.e. when it implies $M^k \not\geq N^k$.

However, a straightforward consequence of Theorem 5.31 is the following non-existence result, due to Kotschick-Löh (unpublished):

Corollary 5.32. *Suppose that M and N are closed oriented connected n-dimensional manifolds such that $M \not\geq N$ and N is not dominated by products. Then $M^k \not\geq N^k$.*

Furthermore, whenever N is not dominated by products, then no non-trivial product $N \times V$ is dominated by another direct product whose factors have lower dimensions than the dimension of N:

Corollary 5.33. *Let M, W and N be closed oriented connected manifolds of dimensions m, k and n respectively such that $m, k < n < m + k$. If N is not dominated by products, then $M \times W \geq N \times V$ for no $(m + k - n)$-dimensional closed oriented connected manifold V.*

Proof. Suppose, for contrast, that there exists a $(m + k - n)$-dimensional closed oriented connected manifold V and a map of non-zero degree $f\colon M \times W \longrightarrow N \times V$. As usual, let the projection $N \times V \xrightarrow{p_N} N$ and the composition

$$M \times W \xrightarrow{f} N \times V \xrightarrow{p_N} N. \tag{5.8}$$

Since $H_n(f;\mathbb{Q})$ is surjective and $H_n(M) = H_n(W) = 0$, the Künneth formula of $M \times W$ in degree n shows that $[N]$ must be represented by a non-trivial product of classes in $H_j(M) \otimes H_{n-j}(W)$, for some $0 < j < n$. Then Thom's realization theorem implies that N is dominated by products, contradicting our assumption. □

It is natural to examine to what extent the condition of Theorem 5.31, that N is not dominated by products, is necessary. The proof of Theorem 5.31 gives additional information related to the degree of the induced map: If a map $f\colon M \times W \longrightarrow N \times W$ of absolute degree one

[6] For any closed manifold M, the simplicial volume of the product $M \times N$ is bounded by

$$\|M\| \cdot \|N\| \leq \|M \times N\| \leq c \cdot \|M\| \cdot \|N\|,$$

where $c > 0$ is a constant which depends only on the dimension of $M \times N$; cf. [34, 6].

gives rise to a non-zero degree map $M \longrightarrow N$, then the latter map must be of absolute degree one as well (see also Chapter 7). In the following example, we will see that, if N is dominated by products, then the above conclusion is not generally true, giving therefore counterexamples to Theorem 5.31 (modulo the degree):

Example 5.34. Conner-Raymond [15] constructed aspherical manifolds which are not homotopy equivalent, but their products with the circle are homeomorphic. These examples are mapping tori of aspherical manifolds, whose monodromies are (distinct) powers of a periodic self-diffeomorphism of the fiber. (Clearly these manifolds are virtual products.)

In dimension three, let $M := M(\varphi)$ and $N := M(\varphi^k)$, $k \neq 1$, be two such mapping tori of a self-diffeomorphism φ (of prime period $p > k$) of an aspherical surface Σ_g, so that $\pi_1(M)$ is not isomorphic to $\pi_1(N)$, but $M \times S^1$ is homeomorphic to $N \times S^1$; we refer to [15, Sections 8, 9, 10 and 4] for the explicit constructions of M and N and the fact that $M \times S^1$ is homeomorphic to $N \times S^1$. Since the Betti numbers of M and N are equal, any dominant map $g\colon M \longrightarrow N$ is (homotopic to a) fiber preserving, by a result of Boileau-Wang [8, Cor. 2.2]. In particular, if $\deg(g) = \pm 1$, then the Hopfian property of surface groups [38] implies that the restriction of g to the fiber Σ_g must be a self-diffeomorphism, which is impossible because $\varphi \neq \varphi^k$. Therefore, we obtain examples of closed three-manifolds M and N (that are dominated products), such that M does not ± 1-dominate N, but there is a map $M \times S^1 \longrightarrow N \times S^1$ of degree ± 1.

Using the explicit presentation of the Seifert three-manifolds constructed in [15], one can derive the conclusion of the above example appealing to a result of Rong [57] for the existence of degree one maps between aspherical Seifert three-manifolds.

5.4.2 Four-manifolds covered by products

Targets that are virtual products with a circle factor. We now apply Theorem 5.31 to closed four-manifolds that are finitely covered by products of type $N \times S^1$. The main result of this paragraph extends the ordering of Theorem 4.16 in four dimensions:

Theorem 5.35. *Let X be one of the three classes* (i) – (iii) *of Theorem 4.16. We say that a closed four-manifold belongs to the class $X \times S^1$ if it is finitely covered by a product $N \times S^1$, where N is a closed three-manifold in the class X.*

If there exists an oriented path from the class X to the class Y in Figure 4.2, then any closed four-manifold in the class $Y \times S^1$ is dominated by a manifold of the class $X \times S^1$. If there is no oriented path from the class X to the class Y, then no manifold in the class $Y \times S^1$ can be dominated by a manifold of the class $X \times S^1$.

Proof of existence. The existence part follows easily by the corresponding existence results for maps between three-manifolds given in Theorem 4.16. Namely, let Z be a closed four-manifold in the class $Y \times S^1$ and suppose that there is an arrow from X to Y in Figure 4.2. By definition,

Z is finitely covered by a product $N \times S^1$ for some closed three-manifold N in the class Y. By Theorem 4.16, there is a closed three-manifold M in the class X and a map of non-zero degree $f: M \longrightarrow N$. Then $f \times \mathrm{id}_{S^1}: M \times S^1 \longrightarrow N \times S^1$ has degree $\deg(f)$ and the product $M \times S^1$ belongs to the class $X \times S^1$.

Proof of non-existence. We now prove the non-existence part of Theorem 5.35. Obviously, there is no four-manifold in the class $(\#_p S^2 \times S^1) \times S^1$ that can dominate a manifold of the other classes. Thus, the interesting cases appear when both the domain and the target are aspherical.

We first deal with targets whose three-manifold factor N in their finite cover $N \times S^1$ is not dominated by products:

Proposition 5.36. *Let W and Z be two closed oriented connected four-manifolds. Suppose that*

(1) *W is dominated by products, and*

(2) *Z is finitely covered by a product $N \times S^1$, where N is a closed oriented connected three-manifold which is not dominated by products.*

If $W \geq Z$, then there exists a closed oriented connected four-manifold $M \times S^1$ so that $M \times S^1 \geq W$ and $M \geq N$. In particular, M cannot be dominated by products.

Proof. Assume that $f: W \longrightarrow Z$ is a map of non-zero degree and $p: N \times S^1 \longrightarrow Z$ is a finite covering of Z, where N is a closed oriented three-manifold that is not dominated by products. Then the intersection

$$H := \mathrm{im}(\pi_1(p)) \cap \mathrm{im}(\pi_1(f))$$

is a finite index subgroup of $\mathrm{im}(\pi_1(f))$ and its preimage $G := \pi_1(f)^{-1}(H)$ is a finite index subgroup of $\pi_1(W)$. Let $p': \overline{W} \longrightarrow W$ be the finite covering of W corresponding to G and $\bar{f}: \overline{W} \longrightarrow N \times S^1$ be the lift of $f \circ p'$.

By assumption, there is a non-trivial product P and a dominant map $g: P \longrightarrow \overline{W}$. Thus, we obtain a non-zero degree map $\bar{f} \circ g: P \longrightarrow N \times S^1$. Now, since P is a four-manifold, there exist two possibilities: Either $P = M \times S^1$, for a closed oriented connected three-manifold M or $P = \Sigma_g \times \Sigma_h$, where Σ_g and Σ_h are closed oriented connected hyperbolic surfaces of genus g and h respectively. The latter possibility is excluded by Corollary 5.33, because N is not dominated by products. Thus $P = M \times S^1$, and so we obtain a non-zero degree map $M \times S^1 \longrightarrow N \times S^1$. Then $M \geq N$ by Theorem 5.31, again because N is not dominated by products. Clearly, M cannot be dominated by products. □

Corollary 5.37. *The non-existence part of Theorem 5.35 holds true for every aspherical target W in a class $Y \times S^1$, whenever Y is not one of the classes $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{R}^3$.*

Proof. By Theorem 4.5 or 4.14, the only closed aspherical three-manifolds that are dominated by products are those carrying one of the geometries $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{R}^3$. The result now follows by Proposition 5.36 and the non-existence part in dimension three given by Theorem 4.16. □

In terms of four-dimensional geometries of type $\mathbb{X}^3 \times \mathbb{R}$ we obtain the following straightforward consequence:

Corollary 5.38. *Suppose that W and Z are closed oriented aspherical four-manifolds carrying product geometries $\mathbb{X}^3 \times \mathbb{R}$ and $\mathbb{Y}^3 \times \mathbb{R}$ respectively. Assume that $\mathbb{Y}^3$ is not $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{R}^3$. If $W \geq Z$, then every closed $\mathbb{Y}^3$-manifold is dominated by a closed $\mathbb{X}^3$-manifold.*

In order to complete the proof of Theorem 5.35, we need to show that closed manifolds which belong to the classes $\mathbb{H}^2 \times \mathbb{R}^2$ or $\mathbb{R}^4$ are not dominated by closed manifolds of the classes $Sol^3 \times \mathbb{R}$, $\widetilde{SL_2} \times \mathbb{R}$ or $Nil^3 \times \mathbb{R}$.

Since the first Betti numbers of closed $Sol^3 \times \mathbb{R}$-manifolds are at most two, and of a closed $Nil^3 \times \mathbb{R}$-manifolds at most three, such manifolds cannot dominate closed manifolds possessing one of the geometries $\mathbb{H}^2 \times \mathbb{R}^2$ or $\mathbb{R}^4$, by Lemma 1.2 (2).

Finally, we deal with the $\widetilde{SL_2} \times \mathbb{R}$ geometry:

Lemma 5.39. *There is no closed oriented $\widetilde{SL_2} \times \mathbb{R}$-manifold that can dominate a closed oriented manifold possessing one of the geometries $\mathbb{H}^2 \times \mathbb{R}^2$ or $\mathbb{R}^4$.*

Proof. Every closed $\mathbb{R}^4$-manifold is finitely covered by T^4 and, therefore, is virtually dominated by every closed $\mathbb{H}^2 \times \mathbb{R}^2$-manifold. Thus, it suffices to show that T^4 cannot be dominated by a product $M \times S^1$, where M is a closed $\widetilde{SL_2}$-manifold. After passing to a finite cover, we can assume that M is a non-trivial circle bundle over a hyperbolic surface Σ (cf. Table 4.1).

Suppose now that $f\colon M \times S^1 \longrightarrow T^4$ is a continuous map. The product $M \times S^1$ carries the structure of a non-trivial circle bundle over $\Sigma \times S^1$, by multiplying by S^1 both the total space M and the base surface Σ of the circle bundle $M \longrightarrow \Sigma$. The S^1-fiber of the circle bundle $M \times S^1 \longrightarrow \Sigma \times S^1$ has finite order in $H_1(M \times S^1)$, being also the fiber of M. Therefore, its image under $H_1(f)$ has finite order in $H_1(T^4)$. Now, since $H_1(T^4)$ is isomorphic to $\pi_1(T^4) \cong \mathbb{Z}^4$, we deduce that $\pi_1(f)$ maps the fiber of the circle bundle $M \times S^1 \longrightarrow \Sigma \times S^1$ to the trivial element in $\pi_1(T^4)$. The latter implies that f factors through the base $\Sigma \times S^1$, because the total space $M \times S^1$, the base $\Sigma \times S^1$ and the target T^4 are all aspherical. This finally means that the degree of f must be zero, completing the proof. □

We have now finished the proof of Theorem 5.35.

Virtual products of two hyperbolic surfaces. We close this subsection by examining manifolds that are finitely covered by a product of two closed hyperbolic surfaces, i.e. closed reducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds.

On the one hand, it is clear that every closed four-manifold with geometry modelled on $\mathbb{H}^2 \times \mathbb{R}^2$ or $\mathbb{R}^4$ is dominated by a product of two hyperbolic surfaces (and therefore every target in the class $\#_p(S^2 \times S^1) \times S^1$ is dominated by such a product; see Example 3.7). However, as we have seen in the proof of Proposition 5.36 (see also Corollary 5.33), closed aspherical four-manifolds that are virtual products $N \times S^1$, where N does not belong to one of the classes $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{R}^3$, cannot be dominated by reducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds.

On the other hand, there is no manifold in the classes $X \times S^1$ which can dominate a product of two closed hyperbolic surfaces. This is obvious when $X = \#_p(S^2 \times S^1)$. If X is a class of aspherical three-manifolds, then the technique of factorizing dominant maps applies: The fundamental group of a product $M \times S^1$ has center at least infinite cyclic, whereas the center of the fundamental group of a product of two hyperbolic surfaces $\Sigma_g \times \Sigma_h$ is trivial. Therefore, every (π_1-surjective) map $f \colon M \times S^1 \longrightarrow \Sigma_g \times \Sigma_h$ kills the homotopy class of the S^1 factor of $M \times S^1$, and so it factors through an aspherical manifold of dimension at most three, because both $M \times S^1$ and $\Sigma_g \times \Sigma_h$ are aspherical. This means that $H_4(f)([M \times S^1]) = 0 \in H_4(\Sigma_g \times \Sigma_h)$, implying that the degree of f is zero.

Remark 5.40. Since Σ_g and Σ_h are hyperbolic, the fact that $M \times S^1 \ngeq \Sigma_g \times \Sigma_h$ is immediate, because $M \times S^1$ has vanishing simplicial volume (see Example 1.3), whereas the simplicial volume of $\Sigma_g \times \Sigma_h$ is positive (actually $\|\Sigma_g \times \Sigma_h\| = 24(g-1)(h-1)$; cf. [12]). However, we prefer to give the most elementary arguments, following simultaneously our methodology.

5.4.3 Ordering the non-hyperbolic geometries

We end our discussion by ordering (in the sense of Wang) all non-hyperbolic geometric aspherical four-manifolds:

Theorem 5.41. *Consider all closed oriented four-manifolds carrying a non-hyperbolic aspherical geometry. If there is an oriented path from a geometry $\mathbb{X}^4$ to another geometry $\mathbb{Y}^4$ in Figure 5.1, then any closed $\mathbb{Y}^4$-manifold is dominated by a closed $\mathbb{X}^4$-manifold. If there is no oriented path from $\mathbb{X}^4$ to $\mathbb{Y}^4$, then no closed $\mathbb{X}^4$-manifold dominates a closed $\mathbb{Y}^4$-manifold.*

In Figure 5.1, we distinguish the two types of the $\mathbb{H}^2 \times \mathbb{H}^2$ geometry. Namely, we denote the reducible geometry by $(\mathbb{H}^2 \times \mathbb{H}^2)_r$, and the irreducible one by $(\mathbb{H}^2 \times \mathbb{H}^2)_i$.

The proof for the right-hand side of the diagram in Figure 5.1, concerning maps between geometric aspherical four-manifolds that are virtual products, was obtained in the previous subsection.

We now deal with the remaining geometries and complete the proof of Theorem 5.41. The claim indicated in Figure 5.1 is that each of the geometries Nil^4, Sol_0^4, $Sol_{m\neq n}^4$, Sol_1^4 and $(\mathbb{H}^2 \times \mathbb{H}^2)_i$ is not comparable with any other (non-hyperbolic) geometry under the domination relation.

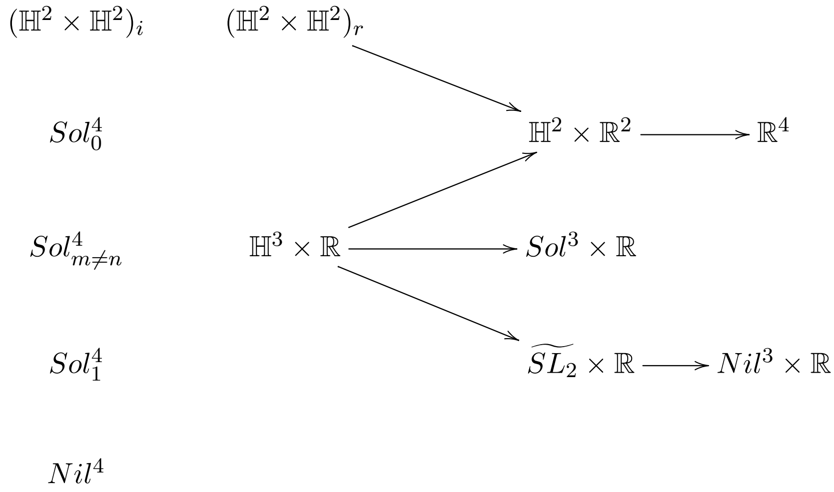

Figure 5.1: Ordering the non-hyperbolic aspherical Thurston geometries in dimension four.

As pointed out in Remark 5.40, some of the non-existence results can be obtained by applying well-known tools, such as the simplicial volume. However, we shall prove all the non-domination results in a rather uniform and elementary way, most of the time using only standard properties of the fundamental groups.

Non-product solvable geometries

We begin by showing that there are no maps of non-zero degree between any two closed manifolds possessing a different geometry among Nil^4, Sol_0^4, $Sol_{m\neq n}^4$ and Sol_1^4.

First, we show that there are no maps of non-zero degree between closed Nil^4-manifolds and Sol_1^4-manifolds. We need the following lemma, which is similar to Lemma 2.14:

Lemma 5.42. *For $i = 1, 2$ let $M_i \stackrel{p_i}{\longrightarrow} B_i$ be circle bundles over closed oriented aspherical manifolds B_i, so that the center of each $\pi_1(M_i)$ is virtually at most infinite cyclic. If $B_1 \not\geq B_2$, then $M_1 \not\geq M_2$.*

Proof. Suppose that $f\colon M_1 \longrightarrow M_2$ is a map of non-zero degree. After passing to a finite covering, if necessary, we may assume that f is π_1-surjective and that the center of each $\pi_1(M_i)$ remains infinite cyclic.

As in the proof of Lemma 2.14, we have that $\pi_1(p_2 \circ f)$ maps the infinite cyclic group generated by the circle fiber of M_1 trivially in $\pi_1(B_2)$. This implies that $p_2 \circ f$ factors through the bundle projection $p_1\colon M_1 \longrightarrow B_1$ (recall that B_2 is aspherical). In particular, there is a continuous map $g\colon B_1 \longrightarrow B_2$, so that $p_2 \circ f = g \circ p_1$. Finally, f factors through the pullback of M_2 under g, and so the degree of f is a multiple of $\deg(g)$. However, the degree of g is zero by our hypothesis that $B_1 \not\geq B_2$, and the lemma follows. □

Since closed Nil^4-manifolds and Sol_1^4-manifolds fulfill the assumptions of Lemma 5.42 (cf. Propositions 5.11 and 5.18 respectively), and since there are no maps of non-zero degree between closed Sol^3-manifolds and Nil^3-manifolds (cf. Chapter 4), we obtain the following:

Proposition 5.43. *Closed oriented Nil^4-manifolds are not comparable under $\geq$ with closed oriented Sol_1^4-manifolds.*

Next, we show that there are no dominant maps between closed Sol_0^4-manifolds and $Sol_{m\neq n}^4$-manifolds. As we have already mentioned, every closed manifold with geometry modelled on Sol_0^4 or $Sol_{m\neq n}^4$ is a mapping torus of T^3, so that the eigenvalues of the automorphism of $\mathbb{Z}^3$ induced by the monodromy of T^3 are not roots of unity.

In the following statement we show that every non-zero degree map between such mapping tori is π_1-injective:

Proposition 5.44. *Let M and N be closed manifolds that are finitely covered by mapping tori of self-homeomorphisms of T^n so that no eigenvalue of the induced automorphisms of $\mathbb{Z}^n$ is a root of unity. If $f\colon M \longrightarrow N$ is a non-zero degree map, then f is π_1-injective.*

Proof. Since we want to show that $f\colon M \longrightarrow N$ is π_1-injective, we may write $\pi_1(M) = \pi_1(T^n) \rtimes_{\theta_M} \langle t\rangle$, where $\pi_1(T^n) = \mathbb{Z}^n = \langle x_1, ..., x_n | \, [x_i, x_j] = 1\rangle$ and the automorphism $\theta_M\colon \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$ is induced by the action of $\langle t\rangle$ on $\mathbb{Z}^n$, given by

$$tx_it^{-1} = x_1^{k_{1i}} \cdots x_n^{k_{ni}}, \text{ for all } i = 1, ..., n.$$

(That is, the matrix of the automorphism θ_M is given by (k_{ij}), $i, j \in \{1, ..., n\}$.) We observe that $tx_it^{-1} \neq x_j$, for all $i, j \in \{1, ..., n\}$, because no eigenvalue of θ_M is a root of unity.

The image $f_*(\pi_1(M))$ of the induced homomorphism $f_*\colon \pi_1(M) \longrightarrow \pi_1(N)$ is a finite index subgroup of $\pi_1(N)$, generated by $f_*(x_1), ..., f_*(x_n), f_*(t)$. Also, the relations $[x_i, x_j] = 1$ and $tx_it^{-1} = x_1^{k_{1i}} \cdots x_n^{k_{ni}}$ in $\pi_1(M)$ give the corresponding relations $[f_*(x_i), f_*(x_j)] = 1$ and $f_*(t)f_*(x_i)f_*(t)^{-1} = f_*(x_1)^{k_{1i}} \cdots f_*(x_n)^{k_{ni}}$ in $f_*(\pi_1(M))$.

Since $\pi_1(N)$ (and therefore $f_*(\pi_1(M))$) is torsion-free and (virtually) a semi-direct product $\mathbb{Z}^n \rtimes \mathbb{Z}$, where the eigenvalues of the induced automorphism of $\mathbb{Z}^n$ are not roots of unity, we conclude that there no more relations between the generators $f_*(x_1), ..., f_*(x_n), f_*(t)$ and that $f_*(t)f_*(x_i)f_*(t)^{-1} \neq f_*(x_j)$, for all $i, j \in \{1, ..., n\}$. Therefore, $f_*(\pi_1(M))$ has a presentation

$$\begin{aligned} f_*(\pi_1(M)) &= \langle f_*(x_1), ..., f_*(x_n), f_*(t) | \, [f_*(x_i), f_*(x_j)] = 1, \\ &\qquad f_*(t)f_*(x_i)f_*(t)^{-1} = f_*(x_1)^{k_{1i}} \cdots f_*(x_n)^{k_{ni}}\rangle \\ &= \langle f_*(x_1), ..., f_*(x_n)\rangle \rtimes \langle f_*(t)\rangle. \end{aligned}$$

In particular, $f_*|_{\pi_1(T^n)}$ surjects onto $\pi_1(T^n) \cong \langle f_*(x_1), ..., f_*(x_n)\rangle \subset f_*(\pi_1(M))$. Since $\mathbb{Z}^n$ is Hopfian, we deduce that $f_*|_{\pi_1(T^n)}$ is injective.

Finally, we observe that $f_*(t^k) \notin \langle f_*(x_1), ..., f_*(x_n)\rangle$, for any non-zero integer k, otherwise the finite index subgroup $\langle f_*(x_1), ..., f_*(x_n)\rangle \rtimes \langle f_*(t^k)\rangle \subset \pi_1(N)$ would be isomorphic to $\mathbb{Z}^n$, which is impossible. This completes the proof. □

Since the four-dimensional geometries are homotopically unique (Theorem 5.3), we deduce the following:

Corollary 5.45. *Any two closed oriented manifolds M and N possessing the geometries $Sol^4_{m\neq n}$ and Sol^4_0 respectively are not comparable under $\geq$.*

Now, we show that closed Sol^4_1-manifolds are not comparable with closed manifolds possessing one of the geometries $Sol^4_{m\neq n}$ or Sol^4_0.

Proposition 5.46. *Closed oriented manifolds possessing the geometry Sol^4_1 are not dominated by closed oriented $Sol^4_{m\neq n}$- or Sol^4_0-manifolds. Conversely, closed oriented Sol^4_1-manifolds cannot dominate closed oriented manifolds with geometries modelled on $Sol^4_{m\neq n}$ or Sol^4_0.*

Proof. Let Z be a closed oriented Sol^4_1-manifold. By Theorem 5.2 (2), Z is a mapping torus of a self-homeomorphism of a closed Nil^3-manifold N. However, Z is not a mapping torus of a self-homeomorphism of T^3; cf. [40, Section 8.6].

Suppose that there is a non-zero degree map $f\colon W \longrightarrow Z$, where W is a closed oriented $Sol^4_{m\neq n}$- or Sol^4_0-manifold. By Theorem 5.2 (1), W is a mapping torus of a self-homeomorphism of T^3 and $\pi_1(W) = \mathbb{Z}^3 \rtimes_{\theta_W} \mathbb{Z} = \langle x_1, x_2, x_3\rangle \rtimes_{\theta_W} \langle t\rangle$, where θ_W is the automorphism of $\mathbb{Z}^3$ induced by the action by conjugation by t. Now $f_*(\pi_1(W))$ has finite index in $\pi_1(Z)$ and $\langle f_*(t)\rangle$ acts by conjugation (by $f_*(t)$) on $\langle f_*(x_1), f_*(x_2), f_*(x_3)\rangle$, that is $f_*(\pi_1(W))$ is a semi-direct product $\langle f_*(x_1), f_*(x_2), f_*(x_3)\rangle \rtimes \langle f_*(t)\rangle$ (recall also that our groups are torsion-free). However, the generators $f_*(x_1), f_*(x_2), f_*(x_3)$ commute with each other, contradicting the fact that $\pi_1(Z)$ cannot be (virtually) $\mathbb{Z}^3 \rtimes \mathbb{Z}$. Therefore $W \not\geq Z$.

For the converse, recall by Proposition 5.18 that a closed Sol^4_1-manifold Z is finitely covered by a non-trivial circle bundle over a closed oriented Sol^3-manifold and that the center of $\pi_1(Z)$ is virtually at most $\mathbb{Z}$. By Proposition 5.17, the fundamental group of every closed $Sol^4_{m\neq n}$- or Sol^4_0-manifold W is not presentable by products, because it contains an infinite-index acentral subgroup. In particular, every finite index subgroup of $\pi_1(W)$ has trivial center, because W is aspherical. Using the asphericity of our geometries and applying a standard factorization argument we derive that $Z \not\geq W$, because a dominant map $Z \longrightarrow W$ would factor through the base (Sol^3-manifold) of the domain. □

Finally, it has remained to show that there are no dominant maps between closed Nil^4-manifolds and closed manifolds possessing one of the geometries $Sol^4_{m\neq n}$ or Sol^4_0.

Proposition 5.47. *Closed oriented Nil^4-manifolds are not comparable under the domination relation with closed oriented manifolds carrying one of the geometries $Sol^4_{m\neq n}$ or Sol^4_0.*

Proof. It is an easy consequence of Lemma 1.2 (2) that a closed oriented Nil^4-manifold cannot be dominated by closed $Sol^4_{m\neq n}$- or Sol^4_0-manifolds, because the latter have first Betti number one, whereas closed Nil^4-manifolds have virtual first Betti number two (see Section 5.2.1 for the corresponding presentations of their fundamental groups).

Conversely, let M be a closed Nil^4-manifold. By Proposition 5.11, M is virtually a non-trivial circle bundle over a closed oriented Nil^3-manifold and $\pi_1(M)$ has (virtual) center at most $\mathbb{Z}$. Now, the fundamental group of every closed $Sol^4_{m\neq n}$- or Sol^4_0-manifold N is not presentable by products (Proposition 5.17), and in particular, it has trivial center. Therefore, every (π_1-surjective) map $M \stackrel{f}{\longrightarrow} N$ factors through the base Nil^3-manifold of the domain implying that $\deg(f) = 0$. Therefore $M \not\geq N$. □

Non-product solvable manifolds vs virtual products

Next, we show that there are no maps of non-zero degree between a closed manifold possessing one of the geometries Nil^4, Sol^4_0, $Sol^4_{m\neq n}$ or Sol^4_1 and a closed manifold carrying a product geometry $\mathbb{X}^3 \times \mathbb{R}$ or the reducible $\mathbb{H}^2 \times \mathbb{H}^2$ geometry.

It is part of Theorem 5.29 that closed four-manifolds possessing a non-product solvable geometry are not dominated by products. We therefore only need to show the converse. First, we deal with nilpotent domains.

Proposition 5.48. *A closed oriented Nil^4-manifold does not dominate any closed manifold possessing a geometry $\mathbb{X}^3 \times \mathbb{R}$ or the reducible $\mathbb{H}^2 \times \mathbb{H}^2$ geometry.*

Proof. Let W be a closed oriented Nil^4-manifold. The abelianization of $\pi_1(W)$ shows that W has virtual first Betti number at most two (cf. Proposition 5.11), and therefore it cannot dominate any closed manifold carrying one of the geometries $\mathbb{R}^4$, $\mathbb{H}^2 \times \mathbb{R}^2$, $(\mathbb{H}^2 \times \mathbb{H}^2)_r$, $\widetilde{SL_2} \times \mathbb{R}$, $Nil^3 \times \mathbb{R}$ or $\mathbb{H}^3 \times \mathbb{R}$, by Lemma 1.2 (2). (The proof for the $\mathbb{H}^3 \times \mathbb{R}$ geometry follows by the establishment of the celebrated Virtual Haken conjecture [1]; see also Table 4.1.)

We finally show that W does not dominate closed $Sol^3 \times \mathbb{R}$-manifolds. Suppose, for contrast, that there is a non-zero degree map $f\colon W \longrightarrow Z$, where Z is a closed four-manifold possessing the geometry $Sol^3 \times \mathbb{R}$. After passing to finite coverings, if necessary, we may assume that f is π_1-surjective, W is a non-trivial circle bundle over a closed oriented Nil^3-manifold M (cf. Proposition 5.11) and $Z = N \times S^1$, where N is a closed oriented Sol^3-manifold (cf. Theorem 5.1). If $p_1\colon Z \longrightarrow N$ denotes the projection to N, then $\pi_1(p_1 \circ f)\colon \pi_1(W) \longrightarrow \pi_1(N)$ kills the S^1-fiber of W, because the fundamental group of N has no center. Since our spaces are aspherical, we deduce that $p_1 \circ f$ factors through the bundle map $W \stackrel{p}{\longrightarrow} M$. However, $H^2(p;\mathbb{Q})$ is the zero homomorphism, because $W \stackrel{p}{\longrightarrow} M$ is a non-trivial circle bundle; see the proof of Lemma 2.14. This contradicts the fact that $H^2(p_1 \circ f;\mathbb{Q})$ is not trivial (cf. Lemma 1.2 (2)), and therefore $W \not\geq Z$. (Alternatively, the result follows by Lemma 5.42, because $M \not\geq N$, as we have seen in the previous chapter.) □

We remark that the argument we have used for the $Sol^3 \times \mathbb{R}$ geometry in the above proof could be applied as well for targets possessing the $\mathbb{H}^3 \times \mathbb{R}$ geometry.

Finally, the following result is a consequence of Lemma 1.2 (2), because the first Betti number of every closed Sol_0^4-, $Sol_{m\neq n}^4$- or Sol_1^4-manifold is one, by the corresponding presentation of their fundamental group (and therefore $b_2 = 0$; recall that the Euler characteristic of those manifolds is zero because they are virtual mapping tori).

Proposition 5.49. *A closed oriented manifold possessing one of the geometries Sol_0^4, $Sol_{m\neq n}^4$ or Sol_1^4 cannot dominate a closed manifold carrying a geometry $\mathbb{X}^3 \times \mathbb{R}$ or the reducible $\mathbb{H}^2 \times \mathbb{H}^2$ geometry.*

The irreducible $\mathbb{H}^2 \times \mathbb{H}^2$ geometry

We finally deal with irreducible closed $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds. We show that they cannot be compared under $\geq$ with any other closed manifold possessing a non-hyperbolic aspherical geometry.

Let M be a closed oriented irreducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifold. Suppose that $f\colon M \longrightarrow N$ is a map of non-zero degree, where N is a closed aspherical manifold not possessing the irreducible $\mathbb{H}^2 \times \mathbb{H}^2$ geometry. As usual, we can assume that f is a π_1-surjective map, after possibly passing to a finite cover. Then we obtain a short exact sequence

$$1 \longrightarrow \ker(\pi_1(f)) \longrightarrow \pi_1(M) \stackrel{\pi_1(f)}{\longrightarrow} \pi_1(N) \longrightarrow 1.$$

By Margulis's Theorem 5.10, the kernel $\ker(\pi_1(f))$ must be trivial, meaning that $\pi_1(f)$ is an isomorphism. Since M and N are aspherical, we deduce that M is homotopy equivalent to N. This contradicts our assumption that N is not an irreducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifold; cf. Theorem 5.3. Therefore $M \not\geq N$.

We now show that M cannot be dominated by any other non-hyperbolic geometric closed aspherical four-manifold N. Since M is not dominated by products, it suffices to show that M cannot be dominated by a closed manifold N possessing one of the geometries Sol_1^4, Nil^4, $Sol_{m\neq n}^4$ or Sol_0^4. For each of those geometries, $\pi_1(N)$ has a normal subgroup of infinite index, which is free Abelian of rank one (geometries Sol_1^4 and Nil^4) or three (geometries $Sol_{m\neq n}^4$ and Sol_0^4); see Section 5.2 for the details. If there were a (π_1-surjective) map of non-zero degree $f\colon N \longrightarrow M$, then by Margulis's theorem either f would factor through a lower dimensional aspherical manifold or $\pi_1(M)$ would be free Abelian of finite rank. This contradiction proves that $N \not\geq M$.

We have now shown the following:

Proposition 5.50. *Closed irreducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds are not comparable under $\geq$ with closed non-hyperbolic four-manifolds possessing a different aspherical geometry.*

This finishes the proof of Theorem 5.41.

Remark 5.51. Using (mostly) standard properties of the fundamental group or the obstruction "fundamental group not presentable by products", it is easy to see that closed hyperbolic four-manifolds (real and complex) are not dominated by geometric non-hyperbolic ones. Gaifullin [31] proved that there exist real hyperbolic closed four-manifolds that virtually dominate all closed four-manifolds. We finally note that complex hyperbolic four-manifolds cannot dominate real hyperbolic ones [13].

Chapter 6

Simply connected manifolds

A large part of this thesis is devoted to the study of domination by products for manifolds with infinite fundamental groups. In certain cases, we have shown that many of those manifolds are not dominated by products. At the other end of the spectrum, it is reasonable to ask whether every closed manifold with finite fundamental group is dominated by products [44]. These manifolds are significant examples of rationally inessential targets and are finitely covered by closed simply connected ones. Therefore, we can equivalently ask whether every closed simply connected manifold is dominated by products (Question 1.19). In dimensions two and three, the answer is obviously affirmative, because S^2 and S^3 respectively are the only simply connected closed manifolds.

In this chapter, we answer Question 1.19 affirmatively in dimensions four and five. More precisely, we apply the constructions of branched coverings from Chapter 3 to construct dominant maps by products to every closed simply connected manifold in those two dimensions. These maps will be obtained as compositions of a branched double covering followed by a certain simple map. For our constructions we will moreover rely on classification results of simply connected closed manifolds, by Wall and Freedman in dimension four, and by Smale and Barden in dimension five.

The content of this chapter is taken from [54].

6.1 Statement of the main result

In contrast to the obstructions on rationally essential manifolds, found in [44] and in this thesis, it could be possible that most of the rationally inessential manifolds are dominated by products. We have already constructed non-zero degree maps by products to many inessential targets in Chapter 3. In the present chapter, we focus our interest on domination by products for closed manifolds with finite fundamental groups. This problem was posed in [44] and it is (obviously) equivalent to the study of domination by products for closed simply connected manifolds.

The interesting cases occur in dimensions at least four, because S^2 and S^3 are the only simply connected closed manifolds in dimensions two and three. In dimension four, Kotschick-Löh [44] applied a domination criterion of Duan-Wang [20] on the intersection form (Theorem 6.7) to show that every closed simply connected four-manifold admits a non-zero degree map by a product; cf. Corollary 6.8. However, Kotschick-Löh's result does not come with an explicit construction of a dominant map.

In this chapter, we apply the results from Chapter 3 to construct dominant maps by products to every simply connected closed manifold in dimensions four and five:

Theorem 6.1. *Every closed simply connected manifold in dimensions four and five is dominated by a non-trivial product.*

The following two sections are devoted to the proof of Theorem 6.1, first in dimension four and then in dimension five.

6.2 Four-manifolds

In this section, we prove Theorem 6.1 in dimension four. Before that, we recall shortly the (stable diffeomorphism) classification of simply connected closed (smooth) four-manifolds by Wall [82] and Freedman [28]. At the end of this section, we discuss the issue of deforming dominant maps to simpler ones, especially to branched coverings, mainly as suggested by Edmonds [22, 21].

6.2.1 The intersection form and classification theorems

The *intersection form* of a closed oriented connected four-manifold M is the symmetric bilinear form

$$Q_M \colon H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

defined by $Q(\alpha,\beta) := \langle \alpha \cup \beta, [M] \rangle$, where $\alpha, \beta \in H^2(M;\mathbb{Z})$. By Poincaré duality, Q_M is unimodular (i.e. $\det Q_M = \pm 1$) and it can equivalently be defined on $H_2(M;\mathbb{Z})$.

Given a symmetric bilinear form Q on a finitely generated free Abelian group Γ, the *rank* of Q is defined to be the rank of Γ. Consider the diagonalization of Q over $\mathbb{R}$. If b_2^+ and b_2^- denote the number of $+1$s and -1s respectively on the diagonal, then the *signature* of Q is defined to be the difference $\mathrm{sign}(Q) := b_2^+ - b_2^-$. The *type* of Q is termed to be *even*, if $Q(g,g) \equiv 0 \pmod 2$ for all $g \in \Gamma$, otherwise it is called *odd*.

Hasse-Minkowski proved that any two unimodular symmetric bilinear forms which are indefinite (i.e. their rank is different from the absolute value of their signature) are isomorphic whenever they have the same rank, signature and type; see [52] for a proof.

Milnor-Whitehead proved first that the homotopy type of a closed simply connected four-manifold is determined completely by its intersection form. In the sequel, Wall obtained the following stable diffeomorphism classification:

Theorem 6.2 (Wall [82]). *If M and N are closed simply connected smooth four-manifolds with isomorphic intersection forms, then there is a $k \geq 0$ such that $M\#_k(S^2 \times S^2)$ and $N\#_k(S^2 \times S^2)$ are diffeomorphic.*

The intersection form of a closed simply connected four-manifold X is odd if and only if X is not spin, because of the Wu formula $\alpha \cup \alpha \equiv w_2 \cup \alpha \pmod 2$, for all $\alpha \in H^2(X;\mathbb{Z})$. (Here w_2 denotes the second Stiefel-Whitney class of X.) Moreover, the inclusion $\mathrm{SO}(3) \hookrightarrow \mathrm{Diff}^+(S^2)$ is a homotopy equivalence (cf. Smale [68]), which means that there exist only two oriented S^2-bundles over S^2, by the classification of sphere bundles (cf. Steenrod [73]). Therefore, if M and N have odd intersection forms, then the stable diffeomorphism of Theorem 6.2 can be obtained by adding connected summands of the twisted bundle $S^2 \widetilde{\times} S^2$, instead of $S^2 \times S^2$, by Theorem 3.9. The latter is equivalent to adding copies of $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, because the non-trivial S^2-bundle over S^2 is diffeomorphic to $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ (see for example [81, Lemma 1]). We further remark that Wall's first statement in [82] is that two closed simply connected four-manifolds are h-cobordant whenever they have isomorphic intersection forms.

Freedman classified completely up to homeomorphism simply connected closed smooth four-manifolds:

Theorem 6.3 (Freedman [28]). *For every even (resp. odd) unimodular symmetric bilinear form, there exists exactly one (resp. two), up to homeomorphism, simply connected closed four-manifold(s) realizing that form. In the odd case, at most one of the two homeomorphism types admits a smooth structure. In particular, two simply connected closed smooth four-manifolds are homeomorphic if and only if they have isomorphic intersection forms.*

The obstruction to the existence of a smooth structure lies in the fourth cohomology group with $\mathbb{Z}_2$ coefficients and is termed Kirby-Siebenmann invariant. Freedman [28] proved that every simply connected closed four-manifold is homotopy equivalent to one with vanishing Kirby-Siebenmann invariant.

6.2.2 The proof of Theorem 6.1 in dimension four

Recall that a collapsing map on a connected sum $P\#Q$ is a degree one map $P\#Q \longrightarrow P$ which pinches the gluing sphere of $P\#Q$ and then sends Q to a point. We have already used this map in the proof of Theorem 4.5 in Chapter 4.

If the connected sum is $P\#\overline{\mathbb{CP}^2}$, then instead of the whole summand $\overline{\mathbb{CP}^2}$, we may collapse the exceptional embedded sphere $\overline{\mathbb{CP}^1} \subset \overline{\mathbb{CP}^2}$ to obtain again a degree one map $P\#\overline{\mathbb{CP}^2} \longrightarrow P$. The collapsed $\overline{\mathbb{CP}^1}$ has self-intersection number -1 and is called -1-sphere. If P is smooth, then this is the usual blow-down operation. Similarly, for a connected sum $P\#\mathbb{CP}^2$ we obtain a degree one map $P\#\mathbb{CP}^2 \longrightarrow P$ by collapsing the embedded $+1$-sphere $\mathbb{CP}^1 \subset \mathbb{CP}^2$. In the smooth category, this operation is known as the antiblow-down of P.

We now prove Theorem 6.1 in four dimensions.

Theorem 6.4. *Every closed simply connected four-manifold M admits a degree two map by a product $S^1 \times (\#_k S^2 \times S^1)$, which is obtained by composing a branched double covering $S^1 \times (\#_k S^2 \times S^1) \longrightarrow \#_k \mathbb{CP}^2 \#_k \overline{\mathbb{CP}^2}$ with a collapsing map $\#_k \mathbb{CP}^2 \#_k \overline{\mathbb{CP}^2} \longrightarrow M$.*

Proof. As we mentioned above, the inclusion $\mathrm{SO}(3) \hookrightarrow \mathrm{Diff}^+(S^2)$ is a homotopy equivalence [68] and so there exist only two oriented S^2-bundles over S^2; cf. [73]. Moreover, the connected sum $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ is diffeomorphic to $S^2 \widetilde{\times} S^2$ (cf. [81]). Thus, setting $n = 4$ in Theorem 3.8, we see that for every k the product $S^1 \times (\#_k S^1 \times S^2)$ is a branched double covering of $\#_k \mathbb{CP}^2 \#_k \overline{\mathbb{CP}^2}$. It therefore suffices to show that for every closed simply connected four-manifold M there exists a k and a collapsing map $\#_k \mathbb{CP}^2 \#_k \overline{\mathbb{CP}^2} \longrightarrow M$.

First, if necessary, we perform connected sums of M with copies of $\mathbb{CP}^2$ or $\overline{\mathbb{CP}^2}$ (or both) to obtain a manifold $M \#_p \mathbb{CP}^2 \#_q \overline{\mathbb{CP}^2}$, whose intersection form is odd and indefinite (and therefore diagonal, by Hasse-Minkowski classification; cf. [52]).

If M is smooth, then Wall's stable diffeomorphism classification (Theorem 6.2) implies that $M \#_p \mathbb{CP}^2 \#_q \overline{\mathbb{CP}^2}$ is stably diffeomorphic to a connected sum $\#_l \mathbb{CP}^2 \#_m \overline{\mathbb{CP}^2}$. (The connected summing with $\overline{\mathbb{CP}^2}$, resp. $\mathbb{CP}^2$, is the blow-up, resp. antiblow-up, operation.) If M is not smooth, we may first assume that the homotopy type of M has trivial Kirby-Siebenmann invariant and then conclude that $M \#_p \mathbb{CP}^2 \#_q \overline{\mathbb{CP}^2}$ is homeomorphic to a connected sum $\#_l \mathbb{CP}^2 \#_m \overline{\mathbb{CP}^2}$, by Freedman's topological classification; cf. Theorem 6.3. In particular, $M \#_p \mathbb{CP}^2 \#_q \overline{\mathbb{CP}^2}$ inherits a smooth structure.

We now conclude that, in both cases, $M \#_p \mathbb{CP}^2 \#_q \overline{\mathbb{CP}^2}$ is (stably) diffeomorphic to a connected sum $\#_k \mathbb{CP}^2 \#_k \overline{\mathbb{CP}^2}$, after possibly connected summing with more copies of $\mathbb{CP}^2$ or $\overline{\mathbb{CP}^2}$.

Finally, a degree one collapsing map

$$\#_k \mathbb{CP}^2 \#_k \overline{\mathbb{CP}^2} \cong M \#_p \mathbb{CP}^2 \#_q \overline{\mathbb{CP}^2} \longrightarrow M$$

is obtained by collapsing the q embedded exceptional spheres $\overline{\mathbb{CP}^1} \subset \overline{\mathbb{CP}^2}$ and the p embedded spheres $\mathbb{CP}^1 \subset \mathbb{CP}^2$. If M is smooth, then the collapsing map is also smooth.

We now obtain the claimed composition

$$S^1 \times (\#_k S^2 \times S^1) \longrightarrow \#_k \mathbb{CP}^2 \#_k \overline{\mathbb{CP}^2} \longrightarrow M, \tag{6.1}$$

finishing the proof of Theorem 6.4 and therefore the proof of Theorem 6.1 in dimension four. □

Actually, the homotopy classification of simply connected four-manifolds suffices to obtain a map $\#_k \mathbb{CP}^2 \#_k \overline{\mathbb{CP}^2} \longrightarrow M$ between the homotopy types of manifolds, and therefore a map as in (6.1). In the above proof, however, we appealed to Wall's (stable) diffeomorphism classification in order to obtain a stronger result where all our maps are smooth (whenever M is smooth).

6.2.3 Deformation of maps to branched coverings

The construction of Theorem 6.4 raises once again the problem of deforming non-zero degree maps to simple ones, in particular, to branched coverings. The realization of manifolds as branched covers is a long-standing topic in topology. In Example 3.2, we mention one of the most classical results in this direction, namely Alexander's theorem [2, 25], which states that every piecewise linear oriented n-dimensional manifold is a branched cover of the n-sphere S^n.

In dimensions up to three, important results have been obtained in the past:

Theorem 6.5.

(1) (Edmonds [22]) *Every non-zero degree map between two closed surfaces is homotopic to the composition of a pinch map followed by a branched covering.*

(2) (Edmonds [21]) *Every π_1-surjective map of degree greater than two between two closed three-manifolds is homotopic to a branched covering.*

Remark 6.6. In addition, Edmonds [22] showed that a non-zero degree map between two closed surfaces, $f\colon \Sigma \longrightarrow F$, is homotopic to a branched covering if and only if either f is π_1-injective, or $|\deg(f)| > [\pi_1(F)\colon \pi_1(f)(\pi_1(\Sigma))]$. This implies that every non-zero degree map between two closed surfaces can be lifted to a (π_1-surjective) map which is homotopic either to a pinch map (absolute degree one) or to a branched covering.

We observe that there is an interesting analogy between Edmonds result in dimension two (Theorem 6.5 (1)) and the maps constructed in Theorem 6.4 for simply connected four-manifolds. In our constructions, however, the order between the pinch map and the branched covering is reversed compared to Edmonds theorem.

As we mentioned at the beginning of this chapter, Kotschick-Löh [44] obtained a positive, non-constructive answer to the question of domination by products for simply connected closed four-manifolds. That result is a consequence of the following domination criterion:

Theorem 6.7 (Duan-Wang [20])**.** *Let X and Y be closed oriented four-manifolds and suppose that Y is simply connected. A map $f\colon X \longrightarrow Y$ of non-zero degree d exists if and only if the intersection form of Y, multiplied by d, is embedded into the intersection form of X, where the embedding is given by $H^*(f)$.*

Corollary 6.8 (Kotschick-Löh [44, Prop. 7.1])**.** *Every closed simply connected four-manifold is dominated by the product of T^2 with a closed oriented connected surface of sufficiently large genus.*

Duan-Wang's theorem implies, for example, that every integer can be realized as the degree for a map from T^4 to $\#_3(S^2 \times S^2)$. However, no map $T^4 \longrightarrow \#_3(S^2 \times S^2)$ can be deformed to a branched covering, by the following result:

Theorem 6.9 (Pankka-Souto [55, Theorem 1.2])**.** *Suppose that N is a closed oriented connected manifold of dimension $n \geq 2$. If* $\mathrm{rank} H^k(N;\mathbb{Q}) = \mathrm{rank} H^k(T^n;\mathbb{Q})$ *for some* $1 \leq k < n$*, then every branched covering* $T^n \longrightarrow N$ *is a covering.*

Therefore, Edmonds deformation result in dimension three (Theorem 6.5 (2)) fails in four dimensions. Nevertheless, according to Theorem 6.4, it is natural to ask when a π_1-surjective non-zero degree map between two closed four-manifolds is homotopic to the composition of a branched covering with a pinch map.

6.3 Five-manifolds

In this section, we first recall briefly the classification of closed simply connected five-manifolds, by Smale [69] and Barden [4], and then, we give some existence results of dominant maps between closed simply connected five-manifolds. Using these results and the constructions of Chapter 3 we prove Theorem 6.1 for five-manifolds.

6.3.1 The classification of simply connected five-manifolds

Given two n-dimensional manifolds M and N with boundaries ∂M and ∂N respectively, we form a new manifold $M \cup_f N$, where f is an orientation reversing diffeomorphism of any $(n-1)$-dimensional submanifold of ∂N with one of ∂M.

Smale [69] classified simply connected spin closed five-manifolds and a few years later Barden [4] completed the classification including non-spin manifolds as well. The following constructions are given in [4]: Let $S^3 \times S^2$, $S^3 \widetilde{\times} S^2$ be the two S^3-bundles over S^2 and $A = S^2 \times D^3$, $B = S^2 \widetilde{\times} D^3$ be the two D^3-bundles over S^2 with boundaries $\partial A = S^2 \times S^2$ and $\partial B = S^2 \widetilde{\times} S^2 \cong \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ respectively. Note that, as in dimension four, we do not need to assume that the structure group of oriented S^3-bundles over S^2 is linear, because the inclusion $\mathrm{SO}(4) \hookrightarrow \mathrm{Diff}^+(S^3)$ is a homotopy equivalence, by the proof of the Smale conjecture [37].

A prime closed simply connected spin five-manifold is either $M_1 := S^5$ or $M_\infty := S^2 \times S^3$, if its integral homology groups have no torsion. If the second homology group is torsion, then

$$M_k := (A \#_\partial A) \cup_{\overline{f_k}} (\overline{A \#_\partial A}), 1 < k < \infty,$$

where $A \#_\partial A$ denotes the boundary connected sum of two copies of A and f_k is an orientation preserving self-diffeomorphism of $\partial(A \#_\partial A) = (S^2 \times S^2) \# (S^2 \times S^2)$, realizing the automorphism of $H_2(\partial(A\#_\partial A);\mathbb{Z}) = \mathbb{Z}^4$ given by

$$\begin{pmatrix} 1 & 0 & 0 & -k \\ 0 & 1 & 0 & 0 \\ 0 & k & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The second integral homology groups of the M_k are $H_2(M_k;\mathbb{Z}) = \mathbb{Z}_k \times \mathbb{Z}_k$, $1 < k < \infty$; see [4]. For details on the construction of the f_k we refer to [82].

A prime closed simply connected non-spin five-manifold with torsion-free integral homology is the non-trivial S^3-bundle over S^2, denoted by X_∞. Now, if a closed simply connected closed non-spin five-manifold has torsion second integral homology, then we have two cases:

(i) $X_{-1} := B \cup_{g_{-1}} \overline{B}$, where g_{-1} is an orientation preserving self-diffeomorphism of ∂B, realizing the automorphism of $H_2(\partial B;\mathbb{Z}) = \mathbb{Z}^2$ given by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(ii) $X_m := (B\#_\partial B) \cup_{\overline{g_m}} (\overline{B\#_\partial B})$, $1 \le m < \infty$, where $B\#_\partial B$ denotes the boundary connected sum of two copies of B and g_m is an orientation preserving self-diffeomorphism of $\partial(B\#_\partial B) = (S^2\widetilde{\times}S^2)\#(S^2\widetilde{\times}S^2)$, realizing the automorphism of $H_2(\partial(B\#_\partial B);\mathbb{Z}) = \mathbb{Z}^4$ given by

$$\begin{pmatrix} 1 & 2^{m-1} & -2^{m-1} & 0 \\ 2^{m-1} & 1 & 0 & 2^{m-1} \\ 2^{m-1} & 0 & 1 & 2^{m-1} \\ 0 & -2^{m-1} & 2^{m-1} & 1 \end{pmatrix}.$$

Except X_1, which is diffeomorphic to $X_{-1}\#X_{-1}$, all the other X_m are prime. Their second integral homology groups are $H_2(X_m;\mathbb{Z}) = \mathbb{Z}_{2^m} \times \mathbb{Z}_{2^m}$, $0 < m < \infty$ and $H_2(X_{-1};\mathbb{Z}) = \mathbb{Z}_2$; cf. [4]. We note that X_{-1} is the Wu manifold $SU(3)/SO(3)$; cf. [88]. Finally, we conventionally set $X_0 := S^5$, which is spin and has torsion-free homology.

According to the above data, we have the following classification of closed simply connected five-manifolds.

Theorem 6.10 (Barden [4]). *Every closed simply connected five-manifold M is diffeomorphic to a connected sum $M_{k_1}\#...\#M_{k_l}\#X_m$, where $-1 \le m \le \infty$, $l \ge 0$, $k_1 > 1$ and k_i divides k_{i+1} or $k_{i+1} = \infty$.*

A summand $X_{m\neq 0}$ exists if and only if M is not spin.

In particular, for a closed simply connected five-manifold with torsion-free homology one obtains the following classification result:

Theorem 6.11 (Smale [69], Barden [4])**.** *Every closed simply connected five-manifold M with torsion-free second homology group $H_2(M;\mathbb{Z}) = \mathbb{Z}^k$ is diffeomorphic*

(1) *either to a connected sum $\#_k(S^3 \times S^2)$, if M is spin, or*

(2) *to a connected sum $\#_{k-1}(S^3 \times S^2)\#(S^3 \widetilde{\times} S^2)$, if M is not spin.*

Remark 6.12. Recall that, by Wall's Theorem 3.9, the connected sum $\#_{k-1}(S^3 \times S^2)\#(S^3 \widetilde{\times} S^2)$ is diffeomorphic to $\#_k(S^3 \widetilde{\times} S^2)$ (without the additional assumption on structure groups).

6.3.2 The proof of Theorem 6.1 in dimension five

The classification results of Smale and Barden will be our guide to prove Theorem 6.1 for five-manifolds. First, we can prove the special case where the target has torsion-free second homology group:

Proposition 6.13. *Let M be a closed simply connected five-manifold with torsion-free second homology group $H_2(M;\mathbb{Z}) = \mathbb{Z}^k$. Then M admits a branched double covering by the product $S^1 \times (\#_k S^1 \times S^3)$.*

Proof. By Theorem 6.11 (and Remark 6.12) such M is diffeomorphic to a connected sum of copies of the trivial bundle $S^3 \times S^2$ or of the twisted product $S^2 \widetilde{\times} S^3$. Thus, Theorems 3.6 and 3.8 imply the proof, for $\#_k(S^3 \times S^2)$ and $\#_k(S^2 \widetilde{\times} S^3)$ respectively. □

We now proceed to complete the proof of Theorem 6.1 in dimension five. Observe that $S^2 \times S^3$ is a branched double covering of the other two prime closed simply connected five-manifolds with torsion-free homology, namely of $S^2 \widetilde{\times} S^3$ and S^5:

Example 6.14.

(1) A branched double covering $S^2 \times S^3 \longrightarrow S^2 \widetilde{\times} S^3$ is obtained by pulling back the S^3-bundle map $S^2 \widetilde{\times} S^3 \longrightarrow S^2$ by a branched double covering $S^2 \longrightarrow S^2$. (Recall that S^3-bundles over S^2 are classified by $\pi_1(\mathrm{SO}(4)) = \mathbb{Z}_2$.)

(2) There is a branched double covering $g\colon S^2 \times S^2 \longrightarrow \mathbb{CP}^2$ which is the quotient for the involution $(z, w) \mapsto (w, z)$ of $S^2 \times S^2$. Then, $S^2 \times S^3$ is a branched double covering of S^5, by pulling back the S^1-bundle map $S^5 \longrightarrow \mathbb{CP}^2$ by g; see [32, 19] for more details.

As we shall see below, every closed simply connected five-manifold M with torsion second integral homology group admits a dominant map by $S^2 \times S^3$. This map will be given as the composition of the above branched double covering $S^2 \times S^3 \longrightarrow S^5$ (Example 6.14 (2)) with a dominant map $S^5 \longrightarrow M$ whose degree depends on $H_2(M;\mathbb{Z})$. The map $S^5 \longrightarrow M$ will be obtained by applying the Hurewicz theorem modulo a Serre class of groups.

Definition 6.15. A *Serre class of Abelian groups* is a non-empty class $\mathcal{C}$ of Abelian groups such that for any exact sequence $A \longrightarrow B \longrightarrow C$, if $A, C \in \mathcal{C}$, then $B \in \mathcal{C}$. A Serre class $\mathcal{C}$ is called a *ring of Abelian groups* if it is closed under the tensor and torsion product operations. Moreover, $\mathcal{C}$ is said to be *acyclic* if any aspherical space X with $\pi_1(X) \in \mathcal{C}$ is $\mathcal{C}$-acyclic, i.e. $H_i(X;\mathbb{Z}) \in \mathcal{C}$, for all $i > 0$. We say that two groups A, B are *isomorphic* modulo $\mathcal{C}$ if there is a homomorphism between A and B whose kernel and cokernel belong to $\mathcal{C}$.

For instance, the class of all Abelian groups and the class of torsion Abelian groups are examples of Serre classes.

The Hurewicz theorem modulo a Serre class states the following:

Theorem 6.16 (Serre [67]). *Let X be a simply connected space and $\mathcal{C}$ be an acyclic ring of Abelian groups. Then the following are equivalent:*

(1) $\pi_i(X) \in \mathcal{C}$, *for all* $1 < i < n$,

(2) $H_i(X) \in \mathcal{C}$, *for all* $1 < i < n$.

Each of the above statements implies that the Hurewicz homomorphism $h\colon \pi_i(X) \longrightarrow H_i(X)$ is an isomorphism modulo $\mathcal{C}$ for all $i \leq n$.

As a consequence of this generalized version of the Hurewicz theorem, every closed simply connected n-dimensional manifold M, whose homology groups in degrees $\neq 0, n$ are all k-torsion, is minimal with respect to the domination relation:

Corollary 6.17 (Ruberman [59]). *Let M be a closed simply connected n-dimensional manifold whose homology groups $H_i(M;\mathbb{Z})$ in dimensions $0 < i < n$ are all k-torsion, for some integer k. Then the image of the Hurewicz homomorphism $\pi_n(M) \longrightarrow H_n(M)$ is given by $k^r\mathbb{Z}$, for some r. In particular, there is a map $S^n \longrightarrow M$ of degree k^r.*

We now conclude that every five-manifold M which is a connected sum of copies of M_k and X_m, where $1 \leq k < \infty$ and $-1 \leq m < \infty$, admits a dominant map by $S^2 \times S^3$ which is the composition of a branched double covering $S^2 \times S^3 \longrightarrow S^5$ (cf. Example 6.14 (2)) followed by a map $S^5 \longrightarrow M$. The degree of the latter map is determined by $H_2(M;\mathbb{Z})$, being a power of the least common multiple of the torsion second integral homology groups $H_2(M_k;\mathbb{Z})$ and $H_2(X_m;\mathbb{Z})$. As we have seen in Example 6.14 (1), the non-trivial S^3-bundle over S^2 admits a branched double covering by the product $S^2 \times S^3$. We now want to combine these maps together with our constructions from Chapter 3, to show that every closed simply connected five-manifold is dominated by products.

In general, it is not always possible to obtain a non-zero degree map $M_1\#M_2 \longrightarrow N_1\#N_2$ by connected summing any two non-zero degree maps $f_i\colon M_i \longrightarrow N_i$:

Example 6.18. Although there exists a map of non-zero degree (double covering) $p: S^5 \longrightarrow \mathbb{RP}^5$, every map $S^5 \# S^5 = S^5 \longrightarrow \mathbb{RP}^5 \# \mathbb{RP}^5$ has degree zero, because S^5 is simply connected, whereas $\mathbb{RP}^5 \# \mathbb{RP}^5$ has fundamental group $\mathbb{Z}_2 * \mathbb{Z}_2$; cf. Lemma 1.2 (1).

The obstruction in the above example is that the preimage of a 5-ball in $\mathbb{RP}^5$ is not a 5-ball in S^5. The π_1-surjectivity of the f_i is a sufficient condition to overcome this obstacle:

Lemma 6.19 ([72, 58])**.** *Let $f: M \longrightarrow N$ be a π_1-surjective non-zero degree map between two closed oriented connected manifolds of dimension $n \geq 3$. Then for any n-ball D^n in N there is a map g homotopic to f so that $g^{-1}(D^n)$ is an n-ball in M.*

Another (obvious) obstruction to connected summing dominant maps $f_i: M_i \longrightarrow N_i$ occurs whenever the degrees of f_1 and f_2 are not equal, because then we cannot paste the f_i along the gluing sphere (recall that maps between spheres are classified by their degrees).

Nevertheless, if $f_i: M_i \longrightarrow N_i$ are π_1-surjective maps of the same degree, then one can paste them together to obtain a new map $M_1 \# M_2 \longrightarrow N_1 \# N_2$ of the same non-zero degree:

Lemma 6.20 (Derbez-Sun-Wang [18])**.** *Let $n \geq 3$ and suppose that M_i, N_i are closed oriented connected n-dimensional manifolds, $i = 1, ..., k$. If there exist π_1-surjective maps $M_i \longrightarrow N_i$ of non-zero degree d, then there is a π_1-surjective map $\#_{i=1}^k M_i \longrightarrow \#_{i=1}^k N_i$ of degree d.*

Since our targets are simply connected, the π_1-surjectivity is automatically satisfied. Moreover, $S^2 \times S^3$ and S^5 admit self-maps of any degree. This means that every minimal summand (i.e. every connected sum of copies of M_k and X_m, where $1 \leq k < \infty$ and $-1 \leq m < \infty$) and every S^3-bundle over S^2 can be dominated by S^5 and by $S^2 \times S^3$ respectively by maps of the same non-zero degree. We therefore obtain the following statement which completes the proof of Theorem 6.1 for five-manifolds:

Theorem 6.21. *Let M be a closed simply connected five-manifold with* $\mathrm{rank} H_2(M;\mathbb{Z}) = k$*. Then M admits a non-zero degree map by the product $S^1 \times (\#_k S^1 \times S^3)$, which is given by the composition of a branched double covering $S^1 \times (\#_k S^1 \times S^3) \longrightarrow \#_k(S^2 \times S^3)$ with a map $\#_k(S^2 \times S^3) \longrightarrow M$ whose degree is determined by the torsion of $H_2(M;\mathbb{Z})$.*

Proof. By Theorem 6.10, a closed simply connected five-manifold M is diffeomorphic to a connected sum $M_{k_1} \# ... \# M_{k_l} \# X_m$. Clearly, the rank k of $H_2(M;\mathbb{Z})$ is equal to the number of M_∞ and X_∞, i.e. the number of S^3-bundles over S^2. Furthermore, we may assume that the torsion of $H_2(M;\mathbb{Z})$ is not trivial, otherwise we appeal to Proposition 6.13 to deduce that M admits a branched double covering by a product $S^1 \times (\#_k S^1 \times S^3)$.

By Corollary 6.17 (and by the comments following that) we have that the part of the connected sum $M_{k_1} \# ... \# M_{k_l} \# X_m$ which does not contain any summand M_∞ or X_∞ admits a dominant map by S^5. The degree of that map is a power of the least common multiple of the torsion second integral homology groups $H_2(M_{k_i};\mathbb{Z})$ and $H_2(X_m;\mathbb{Z})$, where $k_i, m \neq \infty$.

Now, Lemma 6.20 implies that M is dominated by a manifold diffeomorphic to $\#_k(S^2 \times S^3)$. (If $m = \infty$, we additionally use the fact that M_∞ is a branched double covering of X_∞, as we have seen in Example 6.14 (1).) The degree of $\#_k(S^2 \times S^3) \longrightarrow M$ is clearly determined (up to multiplication by two) by the torsion of $H_2(M;\mathbb{Z})$. Finally, Theorem 3.6 implies that $\#_k(S^2 \times S^3)$ admits a branched double covering by the product $S^1 \times (\#_k S^1 \times S^3)$, which completes the proof of our claim. □

Remark 6.22. In order to obtain the branched double covering $S^1 \times (\#_k S^1 \times S^3) \longrightarrow \#_k(S^2 \times S^3)$, we applied Theorem 3.6 to the S^2 factor of the direct product $S^2 \times S^3$. We could apply that theorem to the S^3 factor as well. In that case the branched double covering would be either $S^1 \times (\#_k S^2 \times S^2) \longrightarrow \#_k(S^2 \times S^3)$ or $S^2 \times (\#_k S^2 \times S^1) \longrightarrow \#_k(S^2 \times S^3)$. Similarly, applying Theorem 3.6 to both spheres, we obtain branched four-fold coverings $S^1 \times S^2 \times \Sigma_k \longrightarrow \#_k(S^2 \times S^3)$ and $T^2 \times (\#_k S^1 \times S^2) \longrightarrow \#_k(S^2 \times S^3)$.

6.3.3 Mapping degrees

In contrast to Theorem 6.4, the statement of Theorem 6.21 does not provide absolute control of the degree of the map $\#_k(S^2 \times S^3) \longrightarrow M$ following the branched covering $S^1 \times (\#_k S^1 \times S^3) \longrightarrow \#_k(S^2 \times S^3)$. This brings out the problem of studying the sets of (self-)mapping degrees between manifolds, i.e. the sets of integers that can be realized as degrees for continuous maps (see the introduction of the upcoming chapter for the formal definition).

In dimension five, not all the sets of mapping degrees between closed simply connected manifolds have been determined yet. Nevertheless, an aside result of our discussion is that the sets of degrees of maps between the summands M_k and X_m (for $k, m \neq \infty$) are all infinite. More precisely, we have seen that every multiple of a power of the (torsion) second homology of the target can be realized as a mapping degree between those manifolds. (Recall that S^5 admits self-maps of any degree and is minimal with respect to the domination relation.) This finally shows that every simply connected closed five-manifold has unbounded set of self-mapping degrees. Note that, in order to obtain this conclusion, we only need Serre's Theorem 6.16 and we do not use any differential geometric methods, such as formality [16]. Actually, Serre's theorem can be applied to show that every closed simply connected rational homology sphere is a minimal element with respect to the domination relation. We will extend briefly this discussion in the final Chapter 7.

6.4 Higher dimensions

6.4.1 Six-manifolds

In the light of the constructions of Theorems 3.6 and 3.8, we can further verify that the fundamental classes of certain simply connected closed manifolds in dimensions higher than five are representable by products.

As an example we deal with closed 2-connected six-manifolds. Wall [83] and Zhubr [89] classified closed smooth simply connected six-manifolds.

Theorem 6.23 (Wall [83])**.** *Let M be a closed smooth simply connected six-manifold. Then M is diffeomorphic to $N\#(S^3 \times S^3)\# \cdots \#(S^3 \times S^3)$, where $H_3(N)$ is finite.*

If now the target is 2-connected then $N = S^6$; cf. Smale [68]. In that case the topological and the diffeomorphism classification coincide (cf. Wall [83]) and so we obtain:

Corollary 6.24. *Every closed 2-connected six-manifold M admits a branched double covering by a product $S^1 \times (\#_k S^2 \times S^3)$, where $k = \frac{1}{2}\mathrm{rank} H_3(M;\mathbb{Z})$.*

Proof. By the above results of Wall [83] and Smale [68], every closed 2-connected six-manifold M is diffeomorphic to a connected sum $\#_k(S^3 \times S^3)$, for some non-negative integer $k = \frac{1}{2}\mathrm{rank} H_3(M;\mathbb{Z})$. The proof now follows by Theorem 3.6. □

As in Remark 6.22, we could again apply Theorem 3.6 to show that our M in the above corollary is dominated by other products as well. For example, M admits a branched four-fold covering by $S^2 \times S^2 \times \Sigma_k$ or a degree two map by $S^1 \times (\#_k T^5)$.

6.4.2 Final remarks

An interesting observation is that, for all the dominant maps in this chapter, the domain can be taken to be a product of the circle with a connected sum of copies of a torus. According to that, one could try to strengthen Question 1.19, by asking whether every closed simply connected manifold is dominated by a product of type $S^1 \times N$ or more generally by a product of type $S^m \times N$.

These questions are considerably stronger than Question 1.19, however, they are simultaneously less likely to be true, because they require the domain to be of a certain type $S^m \times N$. If one would attempt to find higher dimensional simply connected manifolds that are not dominated by products, at least of type $S^m \times N$, then closed simply connected manifolds admitting self-maps of absolute degree at most one (cf. [3, 16]) might be good candidates; see [54] or Remark 7.16. Manifolds with the latter property are termed inflexible. We refer shortly to (in)flexibility in the next chapter.

Chapter 7

Self-mapping degrees

The study of mapping degrees is yet another fundamental topic in topology. The set of degrees of maps between two closed oriented manifolds M and N is defined to be

$$D(M,N) = \{d \in \mathbb{Z} \mid \exists\ f\colon M \longrightarrow N,\ \deg(f) = d\}.$$

In particular, $D(M)$ denotes the set of self-mapping degrees of M and its investigation has a special importance. On the one hand, it has been a long-standing problem to characterize manifolds that (do not) admit orientation reversing self-maps, especially to study whether -1 belongs to $D(M)$. On the other hand, the existence of self-maps of absolute degree greater than one is a vanishing criterion for all (finite) functorial semi-norms.

The purpose of this final chapter is to give some results related to the sets of self-mapping degrees.

7.1 Motivating problems

In dimension three, the sets of (self-)mapping degrees have been studied by several people. The next result of Wang determines the finiteness of $D(M)$ for every closed three-manifold M. We state it following Thurston's geometrization picture and the notion of rational essentialness in dimension three (Theorem 4.7).

Theorem 7.1 (Wang [87]). *For a closed oriented connected three-manifold M the set $D(M)$ is infinite if and only if*

(1) *either M is aspherical and possesses one of the geometries Sol^3, Nil^3, $\mathbb{R}^3$ or $\mathbb{H}^2 \times \mathbb{R}$, or*

(2) *M is rationally inessential.*

In higher dimensions, only partial results are known about $D(M)$. For instance, as we have already mentioned in Section 6.3.3, the sets of self-mapping degrees of closed simply connected five-manifolds are not entirely determined, despite Barden's classification results since the 1960s.

In order to understand $D(M)$, we consider the following problems which will motivate the discussion in rest of this chapter:

(1) Characterize manifolds (not) admitting self-maps of degree -1. (Sasao [61, Problem 2].)

(2) Is there a manifold M and a prime number p so that $p\mathbb{Z} \cap D(M) = \{0\}$? (Sasao [61, Problem 3].)

(3) Does every finite functorial semi-norm on homology vanish on all simply connected manifolds? (Gromov [35, G_+ 5.35].)

Problems (1) and (2) were originally posed by Sasao for simply connected finite Poincaré complexes. Problem (3) and related questions have been investigated by Crowley-Löh [16].

7.2 Self-maps of products

Remaining in the general spirit of this thesis, the discussion in this section concerns self-maps of products with respect to Problems (1) and (2). We first deal with self-maps of prime degree. Then, following the same methodology, we study orientation reversing maps and, in particular, the concept of "chiral" products.

7.2.1 Prime degrees

For any two closed oriented manifolds M and N, it is trivial that

$$D(M) \cup D(N) \subset D(M) \cdot D(N) \subset D(M \times N). \tag{7.1}$$

In certain cases, we show that, if of the factors of $M \times N$ is not dominated by products, then for any self-map of $M \times N$ of prime degree p there is a self-map of M or N of absolute degree p:

Proposition 7.2. *Let M, N be two closed oriented connected manifolds of dimensions m and n respectively, so that M is not dominated by products. Suppose that*

(1) *either $M \not\geq N$ or $N \not\geq M$, if $m = n$, or*

(2) *$H^m(N;\mathbb{Q}) = 0$, if $m \neq n$.*

If the product $M \times N$ admits a self-map of prime degree p, then at least one of M or N admits a self-map of absolute degree p.

Obviously, the condition $H^m(N;\mathbb{Q}) = 0$ is automatically satisfied whenever $m > n$. For the proof below, we find it convenient to use the dual version (on cohomology) of Thom's realization Theorem 1.1.

Proof. Let p be a prime number and suppose that $f\colon M \times N \longrightarrow M \times N$ is a map of degree p. The Künneth formula (with rational coefficients) for $M \times N$ in degrees m and n respectively is

$$H^m(M \times N) \cong H^m(M) \oplus [\oplus_{i=0}^{m-1}(H^i(M) \otimes H^{m-i}(N))], \tag{7.2}$$

and

$$H^n(M \times N) \cong H^n(N) \oplus [\oplus_{i=1}^{n}(H^i(M) \otimes H^{n-i}(N))]. \tag{7.3}$$

We first assume that $m = n$. Denote by $\omega_M \in H^m(M)$ and $\omega_N \in H^n(N)$ the cohomology fundamental classes of M and N respectively. Since M is not dominated by products, (7.2) and Thom's realization Theorem 1.1 imply that

$$H^m(f;\mathbb{Q})(\omega_M) = k \cdot \omega_M + \mu \cdot \omega_N, \tag{7.4}$$

for some integers k and μ. (Recall that $H^*(f;\mathbb{Q})$ is injective and so at least one of k and μ is not zero.) Also, (7.3) implies that there exist integers λ, λ_i and ν so that

$$H^n(f;\mathbb{Q})(\omega_N) = \lambda \cdot \omega_N + \sum_{i=1}^{n-1} \lambda_i \cdot (x_M^i \otimes x_N^{n-i}) + \nu \cdot \omega_M, \tag{7.5}$$

where $x_M^i \in H^i(M)$ and $x_N^{n-i} \in H^{n-i}(N)$. (As in the proof of Theorem 5.31, we remark that each summand $x_M^i \otimes x_N^{n-i}$ is a linear combination of elementary tensors, which however does not change anything in our proof.) Since either $M \not\geq N$ or $N \not\geq M$, we deduce that either $\nu = 0$ in (7.5) or $\mu = 0$ in (7.4) respectively. In both cases the naturality of the cup product yields

$$H^{m+n}(f)(\omega_{M\times N}) = k\lambda \cdot \omega_{M\times N},$$

that is $p = k\lambda$. Because p is a prime number, we derive that, either $|k| = p$ or $|\lambda| = p$. In particular, $p \in D(M) \cup D(N)$ or $-p \in D(M) \cup D(N)$.

Assume now that $m \neq n$ and $H^m(N;\mathbb{Q}) = 0$. Then Thom's theorem and (7.2) imply that $H^m(f;\mathbb{Q})(\omega_M) = k \cdot \omega_M$, because M is not dominated by products. The proof follows as above. □

Remark 7.3. As we have mentioned, the condition $H^m(N;\mathbb{Q}) = 0$ is automatically satisfied whenever $m > n$. For $m \leq n$, the underline principle of Proposition 7.2 is that no multiple of a cohomology class $x_N^m \in H^m(N)$ is the image of ω_M under a continuous map (or that no multiple of ω_M is the image of ω_N under a continuous map if $m = n$). For instance, if a class $x_N^m \in H^m(N)$ is representable by a non-trivial product, then this class cannot be the image of ω_M unter a continuous map, because otherwise M would be dominated by products by Thom's theorem.

Moreover, whenever $m \leq n$, both items (1) and (2) of Proposition 7.2 could be replaced by the assumption that N cannot be dominated by a product $M \times X$, for any closed $(n-m)$-dimensional manifold X. Namely, if $M \times X \not\geq N$, then the coefficient λ_m of the product $\omega_M \otimes x_N^{n-m}$ in the following sum (cf. (7.3))

$$H^n(f)(\omega_N;\mathbb{Q}) = \lambda \cdot \omega_N + \sum_{i=1}^{m} \lambda_i \cdot (x_M^i \otimes x_N^{n-i})$$

vanishes (by Thom's theorem) and the conclusion of Proposition 7.2 follows. Note that for $m = n$, the hypothesis $M \times X \not\geq N$ becomes $M \not\geq N$, which is condition (1) of our proposition.

The proof of Proposition 7.2 says that, if $f\colon M \times N \longrightarrow M \times N$ is a map of non-zero degree d (not necessarily prime), then there exist $k \in D(M)$ and $\lambda \in D(N)$ so that $k\lambda = d$. Thus the right inclusion of (7.1) becomes equality under the hypothesis of Proposition 7.2:

Corollary 7.4. *Suppose that M, N are closed oriented connected manifolds of dimensions m and n respectively, so that M is not dominated by products. If either* (1) *or* (2) *in Proposition 7.2 holds, then $D(M) \cdot D(N) = D(M \times N)$.*

Concerning Problem (2), a consequence of Proposition 7.2 is the following:

Corollary 7.5. *Let M, N be closed oriented connected manifolds of dimensions m and n respectively and p a prime number. Suppose that M is not dominated by products and that either* (1) *or* (2) *in Proposition 7.2 holds. If $p\mathbb{Z} \cap D(M) = \{0\}$ and $p\mathbb{Z} \cap D(N) = \{0\}$, then $p\mathbb{Z} \cap D(M \times N) = \{0\}$*

Proof. Suppose that there exists a non-zero integer d so that $pd \in D(M \times N)$. Then, (the proof of) Proposition 7.2 implies that there exist $k \in D(M)$ and $\lambda \in D(N)$ so that $k\lambda = pd$. Therefore, the prime p divides $k\lambda$, which means that p divides at least one of k and λ. Thus, $p\mathbb{Z} \cap D(M) \neq \{0\}$ or $p\mathbb{Z} \cap D(N) \neq \{0\}$. □

7.2.2 Chiral products

Another problem in the context of investigating mapping degrees is that of characterizing manifolds that do not admit orientation reversing maps. This problem has an extensive interdisciplinary interest, because of its connection to themes in Biology and Chemistry, such as the study of molecular structures [27]. Following this relation, we use the following definition, cf. [62, 53, 27].

Definition 7.6. A closed oriented manifold M which does not admit an orientation reversing homotopy self-equivalence is termed *chiral*. More generally, if M does not admit a self-map of degree -1, then it is called *strongly chiral*.

With the above definition, Problem (1) asks to characterize strongly chiral manifolds. In this section, we obtain a characterization for certain strongly chiral products.

Several obstructions to the existence of self-maps of degree -1 have been obtained in the past. For manifolds of dimension $4n$, the intersection form yields such an obstruction. Namely, let M be a closed oriented $4n$-dimensional manifold and suppose that it admits a self-map $f\colon M \longrightarrow M$ of degree -1. Let A be a basis of $H^{2n}(M;\mathbb{Z})$ and denote by Q_M^A the matrix of the intersection form of M with respect to A. If B_A is the matrix of the induced homomorphism $H^{2n}(f)$, with respect to the basis A, then it is easy to see that $-Q_M^A = B_A^t Q_M^A B_A$ (compare Theorem 6.7). Therefore M must have zero signature. Standard examples of strongly chiral manifolds in every dimension divisible by four are given by the complex projective spaces $\mathbb{CP}^{2n}$.

Similarly, the linking form gives an obstruction to the existence of self-maps of degree -1 in dimensions $4n-1$; cf. [80]. For instance, there exist S^{2n+1}-bundles over S^{2n+2}, where $n \geq 1$, that are strongly chiral; cf. [53]. Those sphere bundles are rational homology spheres, which we denote by W_n.

Therefore, we have $\mathbb{CP}^{2n}$ and W_n as examples of closed simply connected strongly chiral manifolds in dimensions $4n$ and $4n+3$ respectively (where $n \geq 1$). In dimensions two, three, five and six, every simply connected closed manifold admits an orientation reversing self-diffeomorphism. For the non-obvious cases, the conclusions follow by Perelman's proof of the Poincaré conjecture in dimension three, and by the classification results of Barden and Zhubr in dimensions five and six respectively.

In the remaining dimensions, namely for $4n+1$ and $4n+2$ (where $n \geq 2$), Müllner [53] showed that strongly chiral simply connected closed manifolds exist. In almost all of those dimensions, a strongly chiral manifold in [53] is obtained by taking a direct product of a suitable W_n with some strongly chiral (simply connected) closed manifold that is not a rational homology sphere. The key idea for those constructions is the following:

Proposition 7.7 ([53, Section 3.2])**.** *Let M be a rational homology sphere of dimension m. Suppose that N is a closed oriented manifold of dimension n so that*

(1) *either N is not a rational homology sphere, if $m = n$, or*

(2) *$H^m(N;\mathbb{Q}) = 0$, if $m \neq n$.*

The product $M \times N$ is strongly chiral if and only if both M and N are strongly chiral.

The above result uses the fact that all rational cohomology groups of M in degrees $\neq 0, m$ are trivial, which implies that $H^m(M \times N)$ is isomorphic to $H^m(M) \oplus H^m(N)$. The corresponding philosophy of Proposition 7.2 is that M is not dominated by products, because, in that case, Thom's Theorem 1.1 implies that the (cohomology) fundamental class of M cannot be represented by a non-trivial product of (co)homology classes. Moreover, item (1) of Proposition 7.7 implies that $M \not\geq N$ (cf. Lemma 1.2 (2)) which is part of item (1) of Proposition 7.2.

Therefore, replacing in Proposition 7.2 the prime degree p by -1, we can obtain the characterization of Proposition 7.7 for other classes of products of manifolds:

Proposition 7.8. *Let M, N be two closed oriented connected manifolds of dimensions m and n respectively, so that M is not dominated by products. Suppose that*

(1) *either $M \not\geq N$ or $N \not\geq M$, if $m = n$, or*

(2) *$H^m(N;\mathbb{Q}) = 0$, if $m \neq n$.*

The product $M \times N$ is strongly chiral if and only if both M and N are strongly chiral.

The proof is essentially identical to that of Proposition 7.2 and is therefore omitted.

Remark 7.9. In order to make the statement of Proposition 7.8 exactly the corresponding to that of Proposition 7.7, we may modify the first assumption, by replacing

"$M \not\geq N$ or $N \not\geq M$, if $m = n$" by "N is dominated by products, if $m = n$".

Then it follows that $N \not\geq M$ (because M is not dominated by products) and the conclusion of Proposition 7.8 holds. Of course, this observation is included in Remark 7.3.

Since the non-trivial cases occur when the dimensions satisfy $m \leq n$, Remark 7.3 (1) suggests moreover the following, whenever one of the direct factors is a rational homology sphere:

Proposition 7.10. *Let M be a rational homology sphere of dimension m. Suppose that N is a closed oriented manifold of dimension $n \geq m$, which is not dominated by any product $M \times X$. Then the product $M \times N$ is strongly chiral if and only if both M and N are strongly chiral.*

Recall that closed hyperbolic manifolds have fundamental groups not presentable by products and so they are not dominated by products by Theorem 1.9 of Kotschick-Löh. A result of Belolipetsky-Lubotzky [5], that every finite group can be realized as the isometry group of (infinitely many) closed hyperbolic manifolds in every dimension greater than two, implies that there exist closed hyperbolic strongly chiral manifolds in every dimension at least three, by Mostow's rigidity theorem (recall that torsion-free hyperbolic groups are Hopfian [66]); see also [53]. We may therefore construct new strongly chiral products of manifolds:

Example 7.11.

(1) Let M, N be closed oriented strongly chiral hyperbolic manifolds of dimensions m and n respectively, such that $m > n$. Since $H^m(N) = 0$, we derive that $M \times N$ is strongly chiral, by Proposition 7.8.

(2) Let M be a closed hyperbolic strongly chiral four-manifold. Then Proposition 7.8 implies that $M \times \mathbb{CP}^2$ is strongly chiral, because $\mathbb{CP}^2$ is strongly chiral and $\mathbb{CP}^2 \not\geq M$.

7.3 Inflexibility

Mapping degrees is an important tool to understand numerical functorial homotopy invariants of manifolds and vice versa. Based on ideas of Milnor-Thurston [49] and Gromov [34, 35], by a numerical functorial invariant we mean the following:

Definition 7.12. A *numerical functorial homotopy invariant* of manifolds is a numerical value $I(M) \in [0, \infty]$, where M is a closed oriented connected manifold, which satisfies the following property (functoriality): If M d-dominates N, then $I(M) \geq |d| \cdot I(N)$. In particular, $I(M)$ is a homotopy invariant.

As we have already seen in Chapter 1, one of the most notable functorial invariants is the simplicial volume, being a special case of the simplicial ℓ^1-semi-norm on homology [34]. Löh-Crowley [16] generalized the notion of functorial semi-norms on homology, by attaching (associated) semi-norms in degree n homology on given functorial semi-norms on the fundamental classes of closed oriented n-dimensional manifolds.

The important correspondence between mapping degrees and functorial arithmetic invariants amounts to the following simple principle: If an n-dimensional manifold M admits a self-map of absolute degree greater than one, then every finite functorial numerical invariant of M vanishes. This correspondence gives rise to the following definition:

Definition 7.13 ([16])**.** A manifold M is called *inflexible* if $D(M)$ is finite, i.e. $D(M) \subset \{-1, 0, 1\}$, otherwise it is called *flexible.*

One of the conclusions in the preceding chapter (Section 6.3.3) is that every closed simply connected five-manifold is flexible. In particular, every finite functorial semi-norm on those five-manifolds vanishes. This is also true for every closed simply connected manifold in dimensions up to six, and it can be proven using differential geometric tools; see [16] and the related references.

The fact that every closed simply connected five-manifold has unbounded set of self-mapping degrees was an elementary consequence of the generalized Hurewicz Theorem 6.16. We can extend this observation to every simply connected rational homology sphere:

Proposition 7.14. *A closed simply connected n-dimensional manifold M is a rational homology sphere if and only if it is dominated by S^n.*

Proof. Clearly, every manifold that is dominated by a sphere is a rational homology sphere, because the induced homomorphisms in cohomology with rational coefficients are injective, by Lemma 1.2 (2).

Conversely, let M be a simply connected closed n-dimensional manifold and suppose that $H_k(M; \mathbb{Q}) = 0$ for all $1 < k < n$. Then the integral homology groups of M in degrees $1 < k < n$

are torsion or trivial, in fact, they are all finite Abelian groups. Then by the Hurewicz theorem modulo the Serre class $\mathcal{C}$ of finite Abelian groups (Theorem 6.16), we deduce that the Hurewicz homomorphism $h: \pi_n(M) \longrightarrow H_n(M)$ is an isomorphism mod $\mathcal{C}$ and so it has non-trivial image. In particular, $S^n \geq M$ (see also Corollary 6.17). □

Corollary 7.15. *Every closed simply connected rational homology sphere is a minimal element with respect to the domination relation and flexible.*

In particular, every finite numerical functorial invariant on closed simply connected rational homology spheres vanishes, giving a partially affirmative answer to Problem (3) in Section 7.1. It is not known any example of a non-vanishing functorial invariant on closed simply connected manifolds [16].

Remark 7.16. At the end of the preceding chapter (Section 6.4.2), we attempted to strengthen Question 1.19, following our results. Namely, we asked whether every simply connected closed manifold is dominated by a product of type $S^m \times N$. Obviously, every product $S^m \times N$ is flexible; actually $D(S^m \times N) = \mathbb{Z}$. It would be interesting to investigate a possible relation between the existence of a map of non-zero degree $X \longrightarrow Y$ and the sets of self-mapping degrees of X and Y. For instance, it seems natural to ask whether manifolds that are dominated by flexible manifolds are flexible themselves. The answer is completely known and affirmative in dimension three [87]. If this problem has also an affirmative answer for every closed simply connected target, then inflexible simply connected closed manifolds cannot be dominated by products $S^m \times N$. The existence of inflexible closed simply connected manifolds was first shown in [3].

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Eidesstattliche Versicherung

Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

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