Effective actions for F-theory compactifications and tensor theories

Federico Bonetti

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der Ludwig-Maximilians-Universität
München

vorgelegt von
Federico Bonetti
aus Mailand, Italien

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Erstgutachter: Prof. Dr. Dieter Lüst

Zweitgutachter: PD Dr. Ralph Blumenhagen

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Zusammenfassung

Diese Arbeit befasst sich mit dem Studium von vier und sechs dimensionalen Niederenergie, effektiven Wirkungen, die im Rahmen von F-Theorie Kompaktifizierungen entstehen. Motiviert durch den Versuch der Beschreibung eines Stapels M5-Branen im Limes großer Wellenlängen, bilden sechs dimensionale, super symmetrische Quantenfeldtheorien, welche selbst-duale Tensorfelder in ihrem Spektrum beinhalten, den zweiten Focus. Beide Themen sind Beispiele für intrinsisch nicht pertubative physikalische Systeme. Im Kontext von F-Theorie entspringt der nicht pertubative Charakter der geometrischen Formulierung dieser Klasse von String Vacua, in welcher die komplexifizierte String Kopplungskonstante im Raum variieren kann. Das infrarot Verhalten Infrarotverhalten von einem Stapel von mehreren M5-Branen andrerseits, wird durch die noch nicht vollkommen verstandenen, sogenannten „(2,0) Theorie“ beschrieben, für welche vermutlich kein schwach gekoppelter, pertubativer Sektor existiert. Es sei angemerkt, dass es in der Literatur keine Beschreibung dieser Theorien in Form von Lagrange-Dichten gibt.

Die überstehende Strategie, die hier angewendet wird, um dieser beiden Probleme Herr zu werden, ist eine analoge trans-dimensionale Behandlung dieser Systeme. Hierbei werden Informationen über d dimensionale Theorien aus dem Studium von $d-1$ dimensionalen Theorien gewonnen. Im Falle von F-Theorie Kompaktifizierungen gelingt dies durch die Dualität von M-Theorie zu F-Theorie. Durch die Wahl von elliptisch fibrierten Calabi-Yau Dreimannigfaltigkeiten als internen Raum ist es uns möglich sechs-dimensionale F-Theorie Vacua zu analysieren. Unsere neuartige F-Theorie Konstruktion, welche Spin(7) Holonomie Mannigfaltigkeit benutzt, bietet uns einen Zugang zu vier dimensionalen effektiven Theorien. Die sechs-dimensionalen (2,0) Theorien studieren wir indirekt durch die Analyse von fünf dimensionalen Theorien. Diese trans-dimensionale Herangehensweise ermöglicht uns eine Lagrange-Dichte in fünf Dimensionen zu konstruieren, die potentiell Aussagen über die sechs dimensionalen Wechselwirkungen der (2,0) Theorien zulässt. Diese Untersuchungen erweiterten unser Verständnis des Zusammenhangs zwischen fünf und sechs dimensionaler Physik, insbesondere fanden wir ein allgemeines Resultat für die ein „Loop“ Korrekturen der Chern-Simons Kopplungen in fünf Dimensionen.

Abstract

In this thesis we study the low-energy effective dynamics emerging from a class of F-theory compactifications in four and six dimensions. We also investigate six-dimensional supersymmetric quantum field theories with self-dual tensors, motivated by the problem of describing the long-wavelength regime of a stack of M5-branes in M-theory. These setups share interesting common features. They both constitute examples of intrinsically non-perturbative physics. On the one hand, in the context of F-theory the non-perturbative character is encoded in the geometric formulation of this class of string vacua, which allows the complexified string coupling to vary in space. On the other hand, the dynamics of a stack of multiple M5-branes flows in the infrared to a novel kind of superconformal field theories in six dimensions—commonly referred to as (2,0) theories—that are expected to possess no perturbative weakly coupled regime and have resisted a complete understanding so far. In particular, no Lagrangian description is known for these models. The strategy we employ to address these two problems is also analogous. A recurring Leitmotif of our work is a transdimensional treatment of the system under examination: in order to extract information about dynamics in d dimensions we consider a $(d-1)$-dimensional setup. As far as F-theory compactifications are concerned, this is a consequence of the duality between M-theory and F-theory, which constitutes our main tool in the derivation of the effective action of F-theory compactifications. We apply it to six-dimensional F-theory vacua, obtained by taking the internal space to be an elliptically fibered Calabi-Yau threefold, but we also employ it to explore a novel kind of F-theory constructions in four dimensions based on manifolds with Spin(7) holonomy. With reference to six-dimensional (2,0) theories, the transdimensional character of our approach relies in the idea of studying these theories in five dimensions. Indeed, we propose a Lagrangian that is formulated in five dimensions but has the potential to capture the six-dimensional interactions of (2,0) theories. This investigation leads us to explore in closer detail the relation between physics in five and in six dimensions. One of the outcomes of our exploration is a general result for one-loop corrections to Chern-Simons couplings in five dimensions.

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Contents

CHAPTER 1

Introduction

This first chapter is devoted to a presentation of the main ideas underlying our work. After motivating string theory from a theoretical perspective we highlight some general features of theories of strings as opposed to point particles. We also recall the importance of the effective field theory approach, we briefly comment on some features of supersymmetric theories, and we give a quick presentation of the setups analyzed in the following chapters. We conclude with the outline of the rest of the thesis.

1.1 The quest for a fundamental theory

The long sequence of successes of theoretical physics from its origins to the turn of the 21^{st} century represents an extraordinary intellectual achievement. Current physical theories provide a powerful framework that allows us to explain and predict a wide range of phenomena on length scales spanning over forty orders of magnitude, from the deep subatomic distances probed at the LHC ($\sim 10^{-19}$ m) to the size of the observable universe ($\sim 10^{27}$ m). Such a remarkable quantitative understanding relies on the possibility to identify the dominant aspects of the dynamics of a physical system at some energy or length scale, neglecting subdominant effects. By this token we can, for instance, study particle scattering processes ignoring gravitational forces, or analyze the large scale structure of the universe approximating its content by a uniform, isotropic fluid.

From this perspective, a description of the world based on several, coexisting models applying to different phenomena within different validity regimes is perfectly viable, as long as no experimental evidence is found against it. In the history of physics, however, an antithetic pattern seems to emerge. Indeed, a decisive element in the progress of this science has been the quest for a simple and unified description of seemingly different aspects of Nature. Galilei and Newton demolished the distinction between terrestrial and celestial mechanics, and Maxwell provided a unified framework for electric and magnetic phenomena, to mention some well-known examples from classical physics. Note that every step in the direction towards unification not only has proven aesthetically appealing, but has also been accompanied by a dramatic deepening of our understanding of the world.

During the 20$^{\text{th}}$ century yet another paradigm beside unification acquired primary importance in the development of theoretical physics: symmetry. Einstein's special relativity originated from the analysis of the symmetry transformations of Maxwell's equations, and was later extended to general relativity following the symmetry principle of general covariance. Conservations laws, one of the pillars of theoretical physics, were related to symmetries by Noether's theorem. In quantum mechanics, symmetries provide the quantum numbers that classify states in the system. They were instrumental, for instance, to unveil hidden patterns in the zoo of 'elementary' particles discovered in the 1960s. As a final example in this incomplete list, the role of symmetries and their spontaneous breaking in the context of quantum field theory cannot be overestimated.

The holy grail of theoretical physics, according to this paradigm, would be a 'theory of everything,' i.e. a unified and coherent framework that allows a consistent description of all known interactions. Ideally, such a theory would originate from a limited number of postulates combined with symmetry arguments. All known physical theories could then be recovered, at least in principle, taking appropriate limits of this underlying theory.

Modern-day theoretical physics is still far from the dream of a 'theory of everything' but embodies the principles of unification and symmetry to a great extent. It is remarkable that three out of the known fundamental interactions among the constituents of visible matter—the electromagnetic force and the weak and strong nuclear forces—can all be understood within the same framework of quantum field theories based on gauge symmetry. Even further, the electromagnetic and weak interactions have already been unified in the Standard Model of particle physics. Its experimental successes are compelling and even the most elusive of its ingredients, the Higgs boson, is likely to have been recently detected at the LHC [1]. Unfortunately, the theoretical tools that have been so successful in describing interactions at the subatomic scale are not directly applicable to the gravitational force. Einstein's general relativity, however, provides a beautiful theory of gravitation at the classical level, in which dynamics is geometrized in an elegant fashion. Besides its aesthetic appeal, general relativity has passed all direct tests at solar system scales [2, 3] and constitutes one of the main pillars of the concordance model of cosmology (or ΛCDM model), which provides a coherent framework for the history of the Universe from Big Bang nucleosynthesis up to the present day.

This optimistic account should not lead to the conclusion that our understanding of the fundamental principles of Nature is almost complete. Indeed, even though the current framework based on the Standard Model (supplemented by neutrino masses) and the concordance model of cosmology has not been falsified experimentally so far, it is unsatisfactory in many respects.

To begin with, there is evidence that some crucial physics is being missed by known theories. In fact, cosmological data indicate that known particles account for only about 5% of the total energy content of the Universe. The remaining part consists of so-called dark matter (27%) and dark energy (68%) [2], whose properties can be effectively parametrized to explain observations on macroscopic scales, but whose microscopic interpretation is obscure. Many particles have been proposed as dark matter candidates, but no conclusive answer has been found yet. Even more dramatic is the situation for dark energy. It is usually identified with the effect of a non-vanishing cosmological constant in Einstein's equations and interpreted microscopically as the energy associated to the vacuum. The theoretical prediction for this quantity in the Standard Model, however, differs from the observed value by 120 orders of magnitude. Better results can be obtained, for instance, considering supersymmetric extensions of the Standard Model, but the cosmological constant problem remains a pressing open question in physics, see e.g. [4].

A second reason to be dissatisfied with the present framework is related to the free parameters that enter known models. These constants cannot be predicted theoretically and have to be fixed by means of experimental or observational input. On the one hand, we need quite a large number of such free parameters: they are about 20 in the Standard Model (the precise number depending on the details of neutrino mass mechanism) and six in the concordance model of cosmology. On the other hand, some parameters require a high amount of fine-tuning that, although not inconsistent, appears to be extremely unnatural. The hierarchy problem and the strong CP problem in the Standard Model, along with the flatness problem in the hot Big Bang scenario, might be seen as examples of this issue.[1]

Finally, from a purely formal point of view the theoretical frameworks of quantum field theory and general relativity suffer from serious limitations that prevent us from viewing them as fundamental theories.

According to the modern effective field theory point of view, quantum field theories are most appropriately considered as effective descriptions valid only up to some definite energy scale. Beyond that scale new physics becomes relevant that is not captured by the theory. Under some circumstances, the formal properties of a quantum field theory allow us to extrapolate it to arbitrary high energies while retaining consistency and predictive power. This happens, for instance, for renormalizable asymptotically free gauge theories. In this case, the theory can be formulated without any reference to a possible UV completion. It is noteworthy that the electromagnetic, weak, and strong interactions can be described precisely in terms of renormalizable gauge theories.[2]

Any attempt to describe gravity along the same lines, though, is doomed to fail, due to the non-renormalizability of the perturbative expansion resulting from the Einstein-Hilbert action. Strictly speaking, this is not enough to rule out the possibility that gravity is described by a quantum field theory at the fundamental level, as the theory may sit at a non-perturbative non-trivial UV fixed point. In what follows, however, we will interpret the lack of renormalizability as a signal that the theory is only an effective description that has to be modified at high energies. The natural expectation for the scale at which quantum gravity effects are important is given by the Planck mass $M_{\rm Pl} \simeq 10^{19}\,$GeV.

[1]The hierarchy problem can be addressed, for instance, by means of low-energy supersymmetry, see e.g. [5]. The strong CP problem can be cured by introducing an axion according to the Peccei-Quinn mechanism, as reviewed for instance in [6]. The flatness problem is solved by inflation, see e.g. [7].

[2]Note, however, that the $U(1)$ hypercharge factor of the Standard Model gauge group is not asymptotically free and can therefore suffer from pathologies in the UV (Landau pole). This problem can be cured within the framework of quantum field theory. For instance, in GUT models the hypercharge $U(1)$ factor is embedded in a simple gauge group that leads to an asymptotically free gauge theory.

We are thus left with the problem of finding a suitable candidate for a UV completion for energies $E \gtrsim M_{\rm Pl}$.

Besides the difficulties of finding a microscopic theory of quantum gravity, general relativity seems to determine its own limitations already at the (semi)classical level. Singularity theorems [8] show that systems with well-behaved initial conditions—representing for example a realistic astrophysical object—can undergo spontaneous gravitational collapse until a singularity in spacetime is formed, at which the notions of geometry and general relativity break down. In the majority of cases (actually, in any case, if one accepts the cosmic censorship conjecture [9]) the singularity produced by gravitational collapse of physically sensible initial conditions is hidden behind the event horizon of a black hole. From the classical point of view this is enough to guarantee that no inconsistency arises, as the singularity in the interior of the black hole is causally disconnected to any external observer. If semiclassical effects such as Hawking radiation are taken into account, however, severe difficulties emerge, as the so-called information paradox. Discussion about this point is still ongoing and has been recently renewed with the introduction of the firewall proposal suggested in [10].

In view of the observations made above, it would be desirable to have a new framework that is able to overcome the difficulties of known theories and, ideally, provide the long-sought 'theory of everything.' In the next section we introduce the best known candidate of such a theory: string theory. It can be seen as the epitome of the implementation of unification and symmetry in theoretical physics. In fact, in string theory all different particles and interactions originate from a single kind of object, the string. Furthermore, the dynamics of strings is so highly constrained by symmetry and consistency that—as it will be clarified in section 3—there is essentially a unique string theory. It is the opinion of the author that these elements are enough to justify a thorough study of the subject. Such an investigation has already provided powerful insights about formal aspects of theoretical physics and intriguing connections to mathematics. It is worth pursuing the analysis further since string theory might have the potential to provide a solution to some of the main open problems in physics.

1.2 From particles to strings

Let us start with some general observations about string theory. The underlying fundamental idea is remarkably simple: replace point particles with extended, one-dimensional objects. More precisely, we are interested in a relativistic quantum theory of interacting strings. Such a theory automatically comes with a fundamental scale of mass, or equivalently, length. It is set by the string tension, or energy per unit length. For historical reasons this scale is usually encoded in the so-called Regge slope α', with dimensions of length squared. It is related to the string tension T by

$$T = \frac{1}{2\pi\alpha'} \ . \tag{1.1}$$

Eventually, we would like to interpret string theory as a theory of quantum gravity. This suggests a relation of the form $(\alpha')^{-1/2} \simeq M_{\rm Pl}$. In some scenarios, however, this naive expectation is not met, and the string scale $(\alpha')^{-1/2}$ can be as low as the TeV scale, opening up the possibility of string detection in colliders, see e.g. [11, 12].

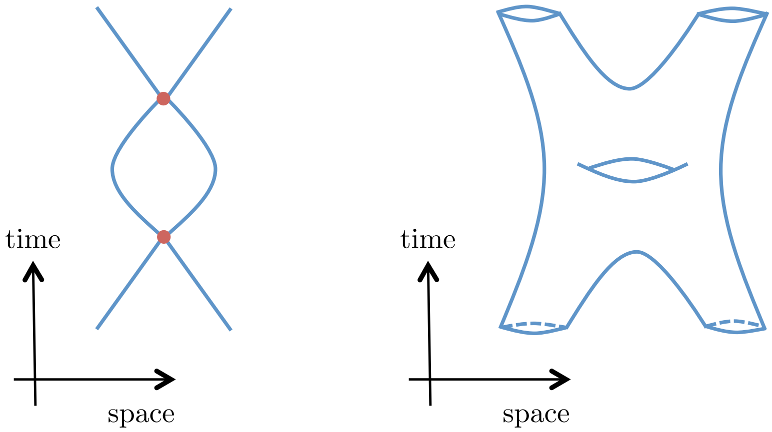


Figure 1.1: Pictorial comparison between the world-lines swept by interacting particles and the world-sheets swept by interacting strings. In the particle case interactions take place at a definite event in spacetime, but there is no Lorentz-invariant way of defining joining and splitting of relativistic strings.

Considering extended objects rather than point particles presents some advantages. First of all, a single fundamental string can account for a variety of particles. This can be intuitively seen by noting that the motion of a string can be decomposed into two parts: a translational displacement of its center of mass, and a tower of discrete oscillatory modes around the center of mass. If a propagating string is probed with energies $E \ll (\alpha')^{-1/2}$, its extended nature will not be detected. One will rather observe a point particle with mass and quantum numbers determined by the oscillatory motion of the string.

Secondly, replacing particles with strings can improve the UV properties of the theory, as can be argued heuristically by comparing interactions between point particles and strings (see figure 1.1). In a local theory of point particles, interactions take place at definite spacetime events. Effectively, this implies that spacetime is probed to arbitrarily high resolution when virtual states are considered in the path integral that defines the quantized theory. This can be considered to be the origin of the familiar UV divergences of quantum field theory. When particles are replaced by strings with typical length scale $(\alpha')^{1/2}$ interactions are effectively smeared out. Stated differently, spacetime cannot be probed to distances smaller than $(\alpha')^{1/2}$ due to the extended nature of the string. One can thus hope that interacting theories of strings free of UV divergences can be formulated. In the context of superstring perturbation theory, consistency and finiteness of the terms in the expansion have been proven up to two loops [13] and partial results such as [14] support the conjecture that the same holds for higher-loop terms.[3] Further observations can be found in the review [16].

An extended one-dimensional object can have two distinct topologies, since it can be homeomorphic to a circle or to an interval. In simpler terms, we can have closed strings, without endpoints, or open strings, with two endpoints. It is a generic feature of string theory that one of the oscillation modes

[3]Note, however, that it has been proven that the perturbative expansion of bosonic string theory is not Borel resummable [15], and the same is believed to be true for superstrings. This means that perturbative data are not sufficient to define the theory completely and that suitable non-perturbative input, related e.g. to instantons, is needed.

of a closed string is associated to a massless, spin-two particle, while for an open string one finds a massless, spin-one particle. These are interpreted as the graviton and as a gauge boson, respectively.[4] Gravitational and gauge interactions are thus beautifully unified thanks to the two different topologies of the string. This picture can provide further insight. For instance, the universal character of gravity can be seen from the fact that open strings can always join to form closed strings. As a result, closed strings, and therefore gravity, must be always included in a sensible string theory. Moreover, it turns out that the space of states of a closed string is, roughly speaking, the tensor product of two copies of the space of states of an open string. This observation leads to a new look on gravity, which can be seen as the 'square' of a Yang-Mills interaction. Remarkably, this heuristic picture has found concrete realizations in explicit computations of open and closed string amplitudes starting from the work of Kawai, Lewellen, and Tye [17].

Finally, the quantum theory of a relativistic string is much more constrained than its point particle counterpart, to the extent that symmetries determine the dynamics of strings completely. As we will see, perturbative string theory has no free parameters (except for the fundamental scale α') and mild consistency conditions—as the absence of quantum anomalies and stability of the vacuum—are powerful enough to single out five fundamental (super)string theories. When non-perturbative arguments are taken into account, however, these five theories are seen to be limits of a single, unifying theory, the so-called M-theory. This leads to the picture of the 'M-theory star' depicted in figure 1.2. This extreme degree of uniqueness should be contrasted to the situation in quantum field theory, in which no argument can be used to single out *a priori* some specific model from the set of anomaly-free, renormalizable, asymptotically free theories.

1.3 The effective action paradigm

Any candidate 'theory of everything' providing a unification of all known interactions is likely to contain new degrees of freedom at high energies subject to some intricate microscopic dynamics. This is the case of string theory, understood in a broad sense to include M-theory. In order to understand the low-energy behavior of these theories, however, most of the detailed structure underlying their microscopic fundamental degrees of freedom is probably unnecessary. We can indeed appeal to the Wilsonian approach to the formulation of effective field theories and argue that, if we are only interested in the dynamics of long-wavelength excitations around the vacuum, we should integrate out all states associated to short-distance physics. The resulting effective theory is formulated in terms of light degrees of freedom and the information about high-energy dynamics is encoded inside the couplings among light modes. The validity of the effective action is clearly limited to processes at energies much lower than the typical scale of the heavy degrees of freedom that have been integrated out. Its predictive power is therefore restricted and if we want to probe dynamics beyond that scale we have to resort to the original theory, which is referred to as the UV-completion of the effective theory.

[4] The situation is different in heterotic superstring theory, where both gravity and gauge interactions originate from closed strings. String dualities, however, relate the heterotic string to different string theories in which the association closed-gravity, open-gauge holds.

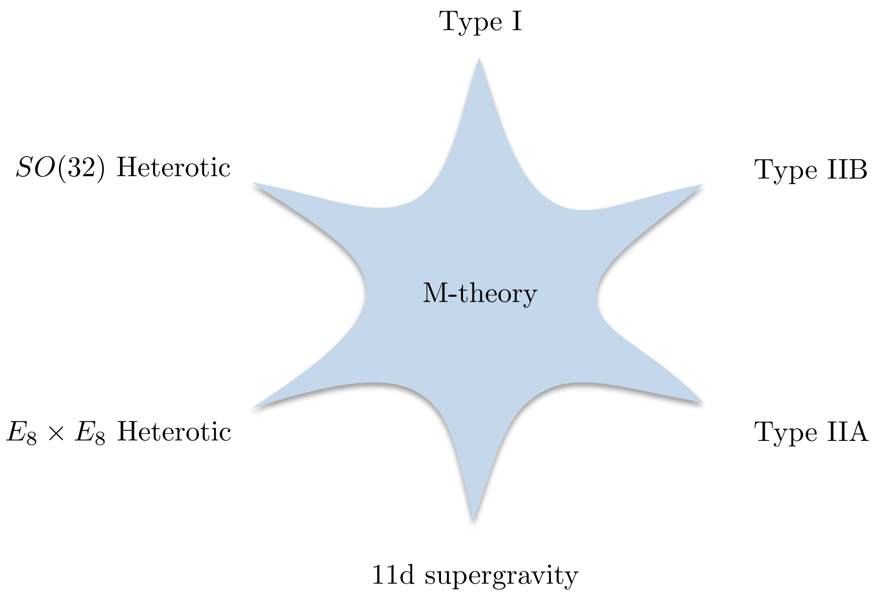


Figure 1.2: An impressionistic sketch of the moduli space of M-theory, showing that the five known consistent superstring theories and eleven-dimensional supergravity can be considered as special 'corners' in the 'M-theory star.'

The effective field theory program is best understood in those situations in which the original microscopic theory in the UV is a consistent, renormalizable quantum field theory. If we let the theory follow its renormalization group flow to the deep IR we find an effective theory for its long-wavelength dynamics. This is a quantum field theory whose properties and degrees of freedom might differ substantially from the ones of the UV theory. For example renormalizability can be lost. Probably the most familiar example of this phenomenon is furnished by the Standard Model of electroweak interactions. Because of spontaneous symmetry breaking the W and Z bosons get massive and are integrated out at sufficiently low energies. The resulting effective action is a non-renormalizable four-Fermi theory. In some situations even the relation between UV and IR degrees of freedom is highly non-trivial. Arguably the prototypical example is given by confining gauge theories, such as QCD: if in the UV we have an asymptotically-free theory of quarks and gluons, in the IR the correct variables for the description of the dynamics are mesons and baryons.

We can also reverse the logic of the last example and conclude that, given a low-energy effective theory, its UV-completion can be formulated in terms of completely different entities and might not be a quantum field theory. This is indeed the case in string theory, which is not a theory of relativistic particles but contains extended objects.

On general grounds the existence of a UV-completion serves as a proof of principle of the full quantum consistency of the effective action under examination. In this respect, effective theories that are known to admit a UV-completion are singled out from the set of all effective theories that are apparently consistent at low energies. From this perspective string theory, which is expected to be a fully consistent UV-complete theory valid at all energy scales, is especially fruitful in at least two different ways. On the one hand, we can use string theory as a guiding principle in the search for a consistent effective theory for some new physics. Instead of exploring the full set of apparently consistent effective theories we can exploit string theory to single out a preferred subset for us, hoping that this exploration will lead to phenomenologically viable models. On the other hand, we can also consider the problem from a more formal point of view and regard string theory as a tool to learn more about what conditions a low-energy effective field theory must meet in order to be a fully consistent theory coupled to quantum gravity. This approach has led to the introduction of the concept of 'swampland' [18]: this is the set of low-energy theories that pass all known consistency checks at low energies (for instance, absence of anomalies) but are nonetheless inconsistent if coupled to quantum gravity. A study of the swampland might lead to the discovery of novel kinds of low-energy consistency conditions that would restrict the set of allowed effective theories and would teach us something about matter-coupled quantum gravity.

Let us point out that in some situations the relevance of the existence of a UV-completion goes beyond formal considerations and can directly affect the physics. For example, in inflationary models of cosmology the so-called eta-problem is extremely sensitive to some details of the UV-completion [19, 20, 21]. In these cases string theory can be even more useful, since it actually furnishes a UV-completion from which—at least in principle—low-energy data can be reliably computed.

1.4 The power of supersymmetry

In quantum mechanics "everything not forbidden is compulsory," in the words of Murray Gell-Mann [22]. From this perspective the role of symmetries is instrumental since they are usually the underlying motivation for some physical process to be forbidden or some quantum correction to be absent. The prototypical example is furnished by the connection established by Noether's theorem between global continuous symmetries and conserved charges in Lagrangian field theory. As another example we can consider the photon in QED: gauge symmetry ensures that it remains exactly massless even when all quantum corrections are taken into account.[5]

Supersymmetry is one of the most powerful symmetries that can be enjoyed by a quantum field theory and it emerges naturally from string theory. Its crucial feature is to take bosonic states into fermionic states and *vice versa*. Since it will often play a very important role in our following considerations, we would like to draw the attention of the reader to a few general facts concerning supersymmetry. Some textbooks on the subject are for instance [23, 24, 25, 26].

From a purely theoretical perspective supersymmetry is arguably the most appealing extension of the familiar Poincaré symmetry of relativistic quantum field theory. A series of theorems, culminating in the celebrated Coleman-Mandula theorem [27], have classified all possible symmetries of a non-trivial relativistic field theory. Under mild assumptions about the spectrum of the model and its

[5]Gauge symmetries are actually best understood as redundancies of our description of the theory, rather than as actual local symmetries enjoyed by the system under examination. For the sake of exposition, however, we will adopt the common parlance and consider global/rigid and local/gauge symmetries in parallel.

S-matrix, it has been proven that the most general symmetry algebra must be the direct sum of the Poincaré algebra and of an internal algebra, i.e. an algebra acting on the internal quantum numbers of fields (for instance, flavor). Thus in particular there cannot be conserved charges carrying Lorentz indices different from the Poincaré generators.[6]

Supersymmetry represents the natural way to avoid these theorems and uncover new possible symmetries of interacting field theories. The crucial element that distinguishes supersymmetry algebras from ordinary symmetry algebras is the inclusion of both bosonic and fermionic generators. Correspondingly, the structure of a supersymmetry algebra is encoded by commutators and well as by anticommutators, according to the schematic pattern

$$[B, B] = B \ , \quad [B, F] = F \ , \quad \{F, F\} = B \ , \tag{1.2}$$

where B and F stand for boson and fermion, $[\cdot, \cdot]$ is a commutator, and $\{\cdot, \cdot\}$ is an anticommutator. A series of results, such as the Haag-Łopuszański-Sohnius theorem [29] or the classification performed in [30], have explored the features of supersymmetry algebras that are relevant in quantum field theory. The fermionic generators of these superalgebras are called supercharges and turn out to carry spinor indices, in accord with the correspondence between spin and statistic. From the theorist point of view it is useful to consider not only supersymmetry in four dimensions, but also its generalizations to lower and higher dimensions. These emerge naturally in string theory.

Since supercharges transform non-trivially under the Lorentz group, they connect states with different spins and in particular mix bosonic and fermionic states. Indeed, in the representations of the supersymmetry algebra on particle states we find an equal number of bosonic and fermionic degrees of freedom, forming a so-called supermultiplet. If supersymmetry were an unbroken symmetry of Nature, all members of the same supermultiplets would have the same mass in a Minkowski vacuum. This would imply that every known particle of the Standard Model should be accompanied by a particle of opposite statistic and same mass. Since this is clearly false, supersymmetry must be broken in order to be compatible with experimental data. The phenomenology of supersymmetric theories is a vast and rich subject that lies beyond the scope of our work. Let us mention that low-energy supersymmetry can be used to solve the hierarchy problem of the Standard Model, improves gauge coupling unification, and can provide dark matter candidates, just to name a few examples. The interested reader is referred for instance to the review [5].

For the purposes of our work we will be mainly focused on unbroken supersymmetry, with the exception of chapter 8 in which we will consider setups that break four-dimensional supersymmetry in

[6]Non-trivial interactions are an essential ingredient for these theorems. In free field theories it is not hard to construct counterexamples. For instance, in the free theory of a real scalar field ϕ of mass m in d dimensions one can build the tensorial current

$$X_{\mu\nu} = \phi \partial_\mu \partial_\nu \phi - \partial_\mu \phi \partial_\nu \phi \ .$$

It is conserved on-shell, since $\partial^\mu X_{\mu\nu} = 0$ for $\partial^\mu \partial_\mu \phi = m^2 \phi$. Therefore it gives rise to the conserved charge

$$X_\mu = \int d^{d-1}x \, X_{0\mu} \ ,$$

which transforms as a vector under the Lorentz group. By the same token one can actually construct conserved charges $X_{\mu_1 \dots \mu_n}$ for any $n \geq 1$. Let us mention that an $O(N)$ generalization of this construction in three dimensions proves to be relevant in the study of the CFT dual to higher-spin theories in AdS_4 [28].

a specific way. As a result, we can exploit a series of remarkable properties implied by supersymmetry. For example, we can specify only the bosonic part of the spectrum, since the fermionic part is fixed by supersymmetry: this is what we will systematically do in the next chapters. Furthermore, supersymmetry imposes powerful restrictions on the couplings of the theory. Thus, if we specify the purely bosonic part of the effective action all fermionic terms are implied and need not be spelled out explicitly. Let us also point out that, on general grounds, supersymmetry offers protection against various kinds of perturbative and non-perturbative quantum corrections and improves considerably the UV properties of the theory. An example is furnished by a supersymmetric extension of four-dimensional gauge theories known as maximally supersymmetric Yang-Mills, which is a non-trivial interacting field theory free of UV divergences.

Supersymmetry has the peculiar feature that, if it is promoted from a global symmetry to a gauge symmetry, it necessarily requires the inclusion of gravity and yields so-called supergravity theories. This can already be seen by the fundamental anticommutator of the supersymmetry algebra, which in crude approximation takes the form

$$\{Q, Q\} \sim P + \dots \ , \tag{1.3}$$

where Q are the supercharges, P are the components of momentum, i.e. the generators of translations, and we are neglecting possible other generators commonly referred to as central charges of the supersymmetry algebra. If supersymmetry is made local, the same has to hold for translations. But, intuitively speaking, a gauge theory of translations amounts to a theory with diffeomorphism invariance, and thus gravity. This argument can be made precise, in the sense that it is possible to construct supergravity theories by a suitable gauging of global supersymmetry algebras, see e.g. [26]. Supergravity provides a beautiful framework for the study of low-energy effective actions propagating particles of all spin from zero to two and thus potentially accommodating all known interactions in a unified fashion. The massless fields of spin 3/2 are referred to as gravitini and always belong to the same supermultiplet as the graviton in interacting theories.

As a final remark, we would like to mention some general results about the connection between massless particles of spin 1, 2, 3/2 and symmetries. In theories with a Lagrangian description it is customary to postulate a symmetry principle such as gauge invariance, general covariance, or local supersymmetry and deduce the properties of massless particles and their interactions. It is interesting to recall that this logic can also be reversed to a certain extent. More precisely, a theorem by Weinberg [31] states that if a massless particle of spin one has non-vanishing couplings at zero momentum, then it necessarily couples to a conserved current. The theorem does not rely on perturbation theory: its only assumptions are exact masslessness of the spin one particle, Lorentz invariance, and the pole structure of the S-matrix. In the same paper Weinberg proves also the spin two version of the theorem: if a massless spin two particle has couplings at zero momentum, then it couples universally to all particles with a strength proportional (in the non-relativistic limit) to their inertial mass.

It is amusing that a similar result holds for spin-3/2 particles. As proven in [32], if a massless spin-3/2 particle interacts at zero momentum, then it is coupled to a supersymmetry current and consistency of the theory also requires the presence of massless spin-two particles with interactions at zero momentum. Loosely speaking, interacting massless gravitini require supersymmetry and gravity.

1.5 Non-perturbative effective actions in string and M-theory

This work is devoted to the study of low-energy effective actions arising in string theory and M-theory is various contexts. In particular, we will be focusing our attention on two setups: F-theory compactifications on manifolds with special holonomy and conformal theories in six-dimensions with sixteen supercharges, commonly referred to as (2,0) theories.

Despite what the name might suggest, F-theory is not a fundamental theory but rather a reformulation of a class of superstring vacua. This reformulation makes extensive use of the language of topology and geometry to encode as much physical information as possible. More precisely, alongside with the ten spacetime dimensions predicted by superstring theory in F-theory one considers two additional auxiliary dimensions. The resulting twelve-dimensional space is a powerful mathematical tool to describe in a compact and elegant fashion interesting string dynamics. In particular, the formalism of F-theory is able to describe setups in which the parameter that governs the strength of string interactions is not merely a constant, but varies in spacetime. This remarkable feature makes F-theory intrinsically non-perturbative.

The interest in the study of F-theory compactifications can be motivated from different points of view. To begin with, four-dimensional vacua constructed using F-theory possess various phenomenologically appealing features. In fact, F-theory is able to combine the virtues of different corners of the 'M-theory star' (see figure 1.2) thanks to its non-perturbative nature. On the one hand, it is able to reproduce some of the features of the gauge and charged matter sector that can be constructed in heterotic string theories, which are particularly promising for the building of models of grand unification (i.e. unification of electroweak and strong interactions). On the other hand, F-theory inherits from Type IIB superstring theory some useful theoretical tools that improve our control on the gravitational sector of the setup and on so-called moduli. The latter are scalar fields that are not charged under the gauge group of the visible sector and that have to be made sufficiently massive in order not to interfere with bounds on fifth force experiments and with the concordance model of cosmology.

From a more formal point of view, F-theory compactifications provide a rich class of string theoretic constructions of consistent effective field theories. This makes F-theory an excellent playground for the analysis of general questions regarding the consistency of low-energy actions and the possible existence of a swampland of apparently consistent theories that are actually inconsistent, as mentioned in section 1.3. This analysis is most easily performed for F-theory compactifications to dimensions higher than four. For instance, six-dimensional F-theory vacua provide a good balance between computational feasibility and non-trivial structure. One can thus hope that six dimensions might teach us valuable lessons that we will be eventually able to apply to the search for a realistic model of new physics in four dimensions.

The approach to F-theory followed in this work is more closely related to this second, formal point of view. In particular, we will consider F-theory in six dimensions and we will see how this more controlled setup allows us to have a better theoretical control over the compactification and to gain valuable insights that can be also generalized to four-dimensional vacua. In a similar spirit, our exploration of compactifications of four-dimensional F-theory vacua on so-called Spin(7) manifolds is motivated by the curiosity to test to which extent the established theoretical framework of F-theory can be pushed to probe new directions in the space of string vacua. As a consequence, we do not claim that the four-dimensional effective theories we will be considering have any immediate phenomenological value. Nonetheless the investigation of new F-theory constructions has the potential to provide new tools that might be useful in the formulation of realistic theories.

The other main objects of interest of this work, six-dimensional (2,0) theories, share with F-theory an inherently non-perturbative character. In fact they constitute an example of strongly coupled quantum field theories that are expected to admit no perturbative expansion. The existence of these theories is inferred from the study of suitable string theory and M-theory setups, in which they emerge as the infrared conformal fixed points obtained by renormalization group flow and decoupling of gravitational interactions. The study of (2,0) theories constitutes a theoretical challenge. They are believed to be a novel kind of field theories possessing a peculiar sort of 'gauge invariance' that is far from being understood. No direct Lagrangian description for these theories is known, and there are arguments that suggest that it might not even exist. Nonetheless it would be extremely beneficial to have a better comprehension of six-dimensional (2,0) theories since they are directly related to so-called S-duality in four-dimensional gauge theories. An S-duality is a map connecting two seemingly distinct quantum field theories in such a way that the weak-coupling regime of one theory is mapped to the strongly-coupled regime of the other, and *vice versa*. By means of an S-duality transformation we can thus explore strongly coupled quantum field theories by analyzing the weakly coupled dual description. S-duality is best understood for theories with a too high degree of symmetry to be realistic, but it is conceivable that this kind of duality might be usefully exploited to learn more about the universality class of phenomenologically relevant quantum field theories such a QCD.

Given the non-perturbative nature of F-theory and (2,0) theories, how can we access their dynamics? Since we cannot rely on perturbative techniques we have to resort to indirect approaches. It is amusing that both for F-theory and for (2,0) theory a possible approach involves what might be defined a transdimensional treatment. Let us comment further on this point.

Suppose we want to study an F-theory setup in d dimensions, for instance $d = 4$. Our goal is the formulation of a low-energy effective theory that incorporates features such as Poincaré invariance and non-Abelian gauge interactions. Unfortunately, there is no direct way to access the effective action in this regime. What can be done instead is to appeal to a duality between F-theory and M-theory to get information about a deformed setup. More precisely, d-dimensional Poincaré invariance is broken by choosing a special direction in space and compactifying it on a circle of small radius. This yields a $(d-1)$-dimensional theory. Concurrently, non-Abelian gauge symmetry is broken to its maximal Abelian subgroup. The duality between F-theory and M-theory then dictates a well-defined prescription to recover the sought-for d-dimensional 'unbroken phase' from this $(d-1)$-dimensional 'broken phase.'

In this work we advocate a similar transdimensional approach to the study of (2,0) theories in six dimensions. As we will argue in chapter 11 if a six-dimensional (2,0) theory is compactified on a circle it yields a five-dimensional theory that is more tractable in several respects. In particular, a Lagrangian description in five dimensions appears to be more feasible than in six dimensions. We are indeed able to write down a supersymmetric Lagrangian that encodes the expected spectrum of a (2,0) theory and incorporates non-Abelian gauge interactions. Our approach is inspired by the considerations of [33, 34] and is similar to other strategies formulated in the literature [35, 36, 37, 38]. Clearly, once a five-dimensional description of the 'broken phase' is achieved one has to find a suitable recipe to extract information about the six-dimensional 'unbroken phase.' In the context of the F-theory setups we will analyze in our work this step is under reasonable control, but for (2,0) theories is necessarily much more conjectural given our limited understanding of the dynamics of these elusive theories. Nonetheless we will argue that some robust features of (2,0) theories that do not depend on the fine details of the interactions might be captured by our five-dimensional approach. The prominent example of such a feature is given by anomalies, which play a pivotal role throughout our work.

A quantum anomaly occurs when a symmetry of a classical theory is lost upon quantization. There are several kinds of anomalies, but we will be only concerned with so-called perturbative anomalies. The interested reader is referred to e.g. [39, 40] for a review on the subject. Perturbative anomalies are based on a subtle interplay between IR degrees of freedom and UV divergences. More precisely, they are computed through the examination of UV divergent one-loop diagrams, but they only depend on the massless spectrum of the theory. Let us stress that anomalies are a quantum effect that can be reliably computed perturbatively even in non-renormalizable theories and that they provide a window on matter-coupled quantum gravity. A close inspection of anomalies in six-dimensional F-theory setups will be the key for the correct understanding of some features of the duality between F-theory and M-theory. Recall that, according to this duality, a d-dimensional F-theory setup is studied in a $(d-1)$-dimensional 'broken phase.' In the case of six dimensions we are thus led to consider five-dimensional theories. We will find an intriguing correspondence between six-dimensional anomalies and a specific kind of one-loop corrections in five-dimensions. More precisely, these corrections involve topological Chern-Simons couplings and can be though of as a parity anomaly. The same logic supports our claim about the possibility of studying (2,0) theories using a five-dimensional action: loop corrections in five dimensions can encode anomalies in six dimensions.

1.6 Outline of the thesis

The remaining part of this thesis is articulated in three parts. In part I we present an overview of some preliminary material that will be useful in the rest of this work. In particular, we start in chapter 2 with a brief review of some basic aspects of string theory in general and Type II superstring in particular and we then move in chapter 3 to a quick discussion of three ingredients that will prove crucial in our following considerations: T-duality, S-duality, M-theory and its connections to Type II superstring theory. Chapter 4 is devoted to a presentation of the idea of compactification focussing on the cases in which the internal space is a circle or a Ricci-flat manifold with special holonomy. In the last two chapters of part I we address more specific topics. In chapter 5 we introduce F-theory with particular emphasis on its duality with M-theory. Finally, chapter 6 contains a brief overview about six-dimensional superconformal (2,0) theories.

In part II we perform a detailed analysis of two F-theory compactification setups. The subject of chapter 7 is the study of the low-energy effective action of F-theory compactified to six dimensions on an elliptically fibered Calabi-Yau threefold. A match between the data of the six-dimensional effective action and the geometry of the threefold is presented and the importance of quantum effects generated by massive Kaluza-Klein states is stressed. In chapter 8 we consider instead four-dimensional F-theory setups obtained by compactification on a suitable class of manifolds with Spin(7) holonomy, or Spin(7) manifolds for short. Our constructions represents a first concrete implementation of F-theory on Spin(7) manifolds and exhibits interesting features. In particular, for the manifolds we consider the resulting four-dimensional theory is formulated on a spacetime with codimension-one boundaries. We argue that, while the bulk dynamics is supersymmetric in four dimensions, the dynamics of the boundaries only respects half of this amount of supersymmetry. For those F-theory configurations that admit a weakly coupled Type IIB interpretation we identify explicitly the objects that are localized on the boundaries and we check the amount of supersymmetry that they preserve.

Part III is devoted to a study of six-dimensional self-dual tensors from a five-dimensional perspective. In chapter 9 we present some general results about one-loop corrections to gauge and mixed gauge-gravitational Chern-Simons terms in five dimensions. More precisely, we extend known results in the literature by computing the contribution of massive gravitini and of massive tensors to the quantum corrections to the aforementioned couplings. These results are applied to the study of six-dimensional tensors in chapter 10, in which we also discuss the possibility of exploring the space of five-dimensional supergravity theories using one-loop corrected Chern-Simons terms. Finally, chapter 11 is dedicated to a proposal for a five-dimensional Lagrangian for six-dimensional (2,0) theories. In particular we construct a five-dimensional action that contains an infinite number of massive fields, in such a way to account for all the degrees of freedom of a six-dimensional (2,0) theory on a circle. We argue that the couplings in our five-dimensional action are natural and have the potential to capture robust features of (2,0) theories, such as anomalies.

We collect some conclusive remarks in chapter 12. A series of appendices presents our notation and conventions together with some useful technical results.

PART I

Preliminary material

CHAPTER 2

Basic notions about Type II superstring theory

This chapter is devoted to a brief introduction to some fundamental aspects of Type II superstring theory. This is such a rich and vast field that we do not aim at a complete overview. We would rather like to guide the reader through a quick survey of the logical steps connecting the first principles upon which string theory is formulated to the aspects that will be relevant for our discussion in the next chapters. The interested reader is referred to textbooks such as [41, 42, 43, 44, 45, 46] for a more in-depth introduction and for a discussion of all the fascinating features of string theory that we are not able to cover here.

2.1 World-sheet perspective and massless spectra

The natural starting point for our introduction to string theory is a lightning review of the fundamental aspects of its world-sheet formulation. After some preliminary remarks about bosonic string theory, we move on to discuss Type II superstring theories with particular emphasis on their massless spectrum.

2.1.1 World-sheet action for the bosonic string

A covariant description of the string motion is furnished by the embedding of its world-sheet Σ in spacetime. The latter is usually referred to as target space and for our present discussion it is taken to be d-dimensional Minkowski spacetime, with d left arbitrary. In a flat coordinate system x^μ, $\mu = 0, \ldots, d-1$, the world-sheet embedding is described by a set of functions $X^\mu(\tau, \sigma)$, where τ, σ are the time and space coordinate on Σ, respectively. We also use the notation σ^α, $\alpha = 0, 1$, with $\sigma^0 = \tau$ and $\sigma^1 = \sigma$.

We would like to use an action principle to determine the dynamics of $X^\mu(\tau, \sigma)$. A suitable action is the Polyakov action

$$S_{\rm P} = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{-h}\,h^{\alpha\beta}\,\eta_{\mu\nu}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\ , \tag{2.1}$$

where α' is the Regge slope already introduced in (1.1), $\eta_{\mu\nu}$ is Minkowski metric with mostly plus signature, $h_{\alpha\beta}$ is a Lorentzian metric on the world-sheet Σ and $h = \det h_{\alpha\beta}$. Note that $h_{\alpha\beta}$ enters the action only algebraically. At the classical level it can be thus removed by means of its equation of motion. If this is done, the Polyakov action (2.1) becomes proportional to the area of the world-sheet computed with the pull-back of the Minkowski metric. This is a natural generalization of the action for a point particle, which is proportional to the proper time of its world-line. The introduction of the auxiliary metric $h_{\alpha\beta}$ renders (2.1) quadratic in X^μ and is instrumental to the formulation of more general string theories, as we will see shortly.

The Polyakov action (2.1) is manifestly invariant under world-sheet diffeomorphisms. Crucially it enjoys another local symmetry: it in invariant under Weyl rescalings of the world-sheet metric,

$$h_{\alpha\beta} \to e^{2\omega}\,h_{\alpha\beta}\ , \tag{2.2}$$

where ω is any locally defined function on the world-sheet. Even though (2.1) is straightforwardly generalized to higher-dimensional membranes, Weyl invariance is specific to two-dimensional extended objects. This is one of the reasons why strings are singled out among extended objects.

A two-dimensional metric has three independent components. This is the same number of gauge parameters of the theory: two parameters from world-sheet diffeomorphisms and one from Weyl rescalings. As a result, any world-sheet metric $h_{\alpha\beta}$ can be locally gauge-fixed to the flat metric $\eta_{\alpha\beta}$. Since we are considering the free propagation of a string, the world-sheet Σ is a cylinder for a closed string and a strip for an open string. In these simple cases the gauge-fixing of $h_{\alpha\beta}$ to $\eta_{\alpha\beta}$ can be performed globally. The Polyakov action (2.1) then yields

$$S_{\rm m} = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\,\partial_\alpha X^\mu\partial^\alpha X_\mu = \frac{1}{\pi\alpha'}\int_\Sigma d^2\sigma\,\partial_+ X^\mu\partial_- X_\mu\ , \tag{2.3}$$

where world-sheet indices are contracted with $\eta^{\alpha\beta}$, and spacetime indices with $\eta_{\mu\nu}$. In the second equality we have introduced coordinates $\sigma^\pm = \sigma^0 \pm \sigma^1$ on Σ. The subscript 'm' stands for 'matter', as we can regard the embedding functions X^μ as scalar fields on the world-sheet carrying an additional 'flavor' index μ. The corresponding 'flavor' global symmetry is nothing but Poincaré symmetry in the target space.

The gauge-fixed action (2.3) is still invariant under suitable combinations of diffeomorphisms and Weyl transformations. More precisely, if a diffeomorphism $\sigma^\alpha \to \sigma'^\alpha(\sigma)$ is such that

$$\eta'_{\alpha\beta}(\sigma') = e^{2\Lambda(\sigma)}\,\eta_{\alpha\beta}(\sigma)\ , \tag{2.4}$$

we can act with a compensating Weyl transformation to reabsorb the prefactor and restore the gauge condition $h_{\alpha\beta} = \eta_{\alpha\beta}$. Diffeomorphisms satisfying (2.4) are called conformal transformations. At the infinitesimal level, they are generated by conformal Killing vector fields ξ^α, satisfying

$$\partial_\alpha\xi_\beta + \partial_\beta\xi_\alpha = \partial_\gamma\xi^\gamma\,\eta_{\alpha\beta}\ . \tag{2.5}$$

This equation takes a particularly simple form if we use coordinates $\sigma^{\pm} = \sigma^0 \pm \sigma^1$ on the world-sheet, since we find

$$\partial_- \xi^+ = 0 \ , \qquad \partial_+ \xi^- = 0 \ . \tag{2.6}$$

We can immediately see that in two dimensions the space of solutions to (2.5) is infinite-dimensional: the action (2.3) thus admits an infinite number of conserved charges. This remarkable property allows one to use the powerful methods of conformal field theory (CFT). Since we will not make use of CFT techniques in what follows, we refer the reader to e.g. [47, 48] for introductions to the subject and its applications to string theory.

Our starting point has been the geometric picture of the world-sheet embedding in spacetime. From a more abstract point of view, a string theory is defined by the CFT living on its world-sheet. The string theory based on (2.3) is called bosonic string theory. It turns out that this theory is unable to describe spacetime fermions and furthermore its perturbative vacuum suffers from tachyonic instabilities. These difficulties can be overcome by introducing additional matter fields on the world-sheet.

2.1.2 World-sheet action for Type II superstrings

In Type II superstring theories in the Ramond-Neveu-Schwarz formulation one supplements the world-sheet scalars X^μ with a pair of opposite-chirality Majorana-Weyl world-sheet spinors ψ_+^μ, ψ_-^μ. The resulting world-sheet theory is supersymmetric.

More precisely, recall that in the Polyakov action (2.1) the scalars X^μ are coupled to two-dimensional gravity in a diffeomorphism and Weyl invariant way. In a similar fashion, one can couple X^μ, ψ_+^μ, ψ_-^μ to two-dimensional superconformal gravity: a generalization of (2.1) can be found that is invariant under diffeomorphisms, local world-sheet Lorentz transformations, local supersymmetry transformations, together with Weyl transformations and their supersymmetric partners, super-Weyl transformations. This huge amount of local symmetries can be reduced by a suitable gauge-fixing procedure, as in the previous bosonic discussion. The outcome is a generalization of (2.3) that in coordinates $\sigma^{\pm} = \sigma^0 \pm \sigma^1$ reads

$$S_{\rm sm} = \frac{1}{2\pi} \int_\Sigma d^2\sigma \left\{ \frac{2}{\alpha'} \partial_+ X^\mu \, \partial_- X_\mu + i \, \psi_+^\mu \, \partial_- \psi_{+\mu} + i \, \psi_-^\mu \, \partial_+ \psi_{-\mu} \right\} \ . \tag{2.7}$$

The label 'sm' in (2.7) stands for supersymmetric matter. Indeed, (2.7) is invariant under supersymmetry transformations generated by the analog of conformal Killing vectors, the so-called conformal Killing spinors. Explicitly, the transformations read

$$\delta X^\mu = \sqrt{\frac{\alpha'}{2}} \, i(\epsilon^+ \psi_+^\mu + \epsilon^- \psi_-^\mu) \ , \qquad \delta\psi_+^\mu = -\sqrt{\frac{2}{\alpha'}} \, \epsilon^+ \partial_+ X^\mu \ , \qquad \delta\psi_-^\mu = -\sqrt{\frac{2}{\alpha'}} \, \epsilon^- \partial_- X^\mu \ , \tag{2.8}$$

where the real anticommuting parameters $\epsilon^\pm$ satisfy the fermionic analog of (2.6), which is

$$\partial_- \epsilon^+ = 0 \ , \qquad \partial_+ \epsilon^- = 0 \ . \tag{2.9}$$

In total analogy to the purely bosonic case, we see that the two-dimensional conformal Killing spinor equation (2.9) admits infinite solutions. As a result (2.7) is invariant under an infinite-dimensional superconformal algebra, and is therefore a superconformal field theory (sCFT).

One easily verifies that the generic solutions to the classical equations of motion associated to (2.7) have the form

$$X^\mu(\tau,\sigma) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-) \ , \quad \psi^\mu_+(\tau,\sigma) = \psi^\mu_+(\sigma^+) \ , \quad \psi^\mu_-(\tau,\sigma) = \psi^\mu_-(\sigma^-) \ . \tag{2.10}$$

This is an important point: the dynamics of the string can be analyzed in terms of left-moving and right-moving modes that have no local interactions. Indeed, the left-moving and right-moving sectors are related only via considerations about the global topology of the world-sheet, to be discussed below. This observation is the starting point for the construction of heterotic superstrings: the left-moving sector of the bosonic string is combined with the right-moving sector of a superstring. We refrain from a more detailed account of heterotic string theories and we refer the reader to the textbooks [41, 42, 43, 44, 45, 46].

2.1.3 Constraints and Weyl anomaly

The matter action (2.3) and its supersymmetric extension (2.7) describe two-dimensional free-field theories that can be straightforwardly quantized. In deriving (2.3) or (2.7), however, one performs a gauge-fixing and this requires special care in the quantization procedure.

First of all, note that the bosonic gauge-fixed action (2.3) does not contain the information encoded in the $h_{\alpha\beta}$ equation of motion of the original Polyakov action (2.1), which imposes the vanishing of the world-sheet energy-momentum tensor $T_{\alpha\beta}$. Similarly, in the supersymmetric extension of the Polyakov action one finds a two-dimensional gravitino, whose equation of motion imposes the vanishing of the supercurrent $(T_F)_\alpha$, which is a two-dimensional vector-spinor. At the classical level $T_{\alpha\beta} = 0$ and $(T_F)_\alpha = 0$ can be imposed by hand on the space of solutions. In the quantum theory we rather have to impose

$$\langle\psi_1| T_{\alpha\beta} |\psi_2\rangle = 0 \ , \qquad \langle\psi_1| (T_F)_\alpha |\psi_2\rangle = 0 \ , \qquad \text{for any } \psi_1, \psi_2 \in \mathcal{H}_{\text{phys}} \ , \tag{2.11}$$

where $\mathcal{H}_{\text{phys}}$ denotes the physical Hilbert space of the theory. There are various equivalent ways to construct $\mathcal{H}_{\text{phys}}$ and thus determine the spectrum of the physical oscillations of the string. We are not going to discuss this procedure, but we will rather state the results in what follows. The interested reader is referred to the textbooks for a thorough derivation. Before that, however, a crucial issue has to be addressed.

As we have seen in the previous section, Weyl invariance of the Polyakov action (2.1) is fundamental to have enough gauge parameters to gauge-fix all local degrees of freedom of the world-sheet metric $h_{\alpha\beta}$. Nonetheless, nothing prevents this classical symmetry to suffer from quantum anomalies. Indeed, this is to be expected: Weyl transformations are closely related to scale transformations, and the latter are commonly violated in QFT due to a non-trivial RG flow of the couplings. In the supersymmetric case a similar discussion applies. The Weyl anomaly can be analyzed with different techniques, but all give the same results. For the bosonic string, if the target space is Minkowski the anomaly vanishes if and only if $d = 26$; for the superstring, if and only if $d = 10$. By means of self-consistency, string theory is thus able to predict the dimensionality of spacetime.

2.1.4 Closed string sectors

For a closed string the world-sheet is a cylinder and therefore we require the scalars X^μ to be periodic along the spatial direction on the world-sheet,

$$X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma) \ , \qquad \sigma \sim \sigma + 2\pi \ . \tag{2.12}$$

Since any physical observable is quadratic in fermions, for ψ_+^μ, ψ_-^μ we can either choose periodic or antiperiodic boundary conditions, which are referred to as Ramond and Neveu-Schwarz, respectively:

$$\begin{aligned} &\text{Ramond:} && \psi^\mu(\tau, \sigma + 2\pi) = +\psi^\mu(\tau, \sigma) \ , \\ &\text{Neveu-Schwarz:} && \psi^\mu(\tau, \sigma + 2\pi) = -\psi^\mu(\tau, \sigma) \ , \end{aligned} \tag{2.13}$$

where ψ^μ stands for ψ_+^μ or ψ_-^μ. If we want to preserve spacetime Lorentz invariance we have to choose the same periodicity for all values of μ, but we are still free to choose different periodicity for left- and right-movers. As a result, the superstring contains four sectors, which can be denoted (R,R), (R,NS), (NS,R), (NS,NS).

As noted after (2.10), left- and the right-moving classical solutions decouple. One can show that a similar decoupling is valid at the level of the quantum constraints (2.11). The physical states of the closed superstring are thus formed by combining the quantum states of left-moving and right-moving oscillators, which can be analyzed separately. This is reflected in the mass formula for closed string excitations. It is given by the sum of the contributions of left-moving and right-moving modes, which have to be equal:

$$m^2 = m_L^2 + m_R^2 \ , \qquad m_L^2 = m_R^2 \ . \tag{2.14}$$

The latter equation expresses the level-matching condition and is the only constraint relating left-movers and right-movers. It originates from the requirement of invariance under σ-translations on the world-sheet, which is motivated by the fact that no point should be preferred on a closed string.

Let us examine in more details the quantum states in the left-moving sector; similar considerations apply to the right-movers. One can show that both for Ramond and Neveu-Schwarz periodicities a well-defined $\mathbb{Z}_2$-grading on the space of states is given by the world-sheet left-moving fermion number operator $\mathscr{F}_L$. This grading induces a refinement of the R and NS sectors into subsectors with definite eigenvalues for $(-1)^{\mathscr{F}_L}$, which we denote R_+, R_-, NS_+, NS_+. In table 2.1 we collect the lightest states for each of these subsectors. The entries of the table denote the representations of the corresponding states with respect to the relevant little group in the critical dimension $d = 10$: this is $SO(8)$ for massless states and $SO(9)$ for all other mass levels. We use the standard notation for distinguishing the three eight-dimensional representations of $SO(8)$: $\mathbf{8}_v$ is the fundamental, or vector, representation, while $\mathbf{8}_c$ and $\mathbf{8}_s$ denote the two opposite-chirality Majorana-Weyl spinor representations. Note the presence of a scalar with negative mass squared in the NS_- sector, the tachyon. Let us also point out that all states in the NS_+ and NS_- sectors belong to representations with integer weights and are associated to spacetime bosonic statistic, while all states in the R_+ and R_- sectors transform in representations with half-integer weights and correspond to spacetime fermionic statistic.

$\alpha' m_L^2$	NS_+	NS_-	R_+	R_-
-1		$\mathbf{1}$		
0	$\mathbf{8}_v$		$\mathbf{8}_s$	$\mathbf{8}_c$
1		$\mathbf{36}$		
2	$\mathbf{44}+\mathbf{84}$		$\mathbf{128}$	$\mathbf{128}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 2.1: Lightest states in the left-moving sector of a closed superstring.

Type IIA	$(\mathrm{NS}_+,\mathrm{NS}_+)$	$\mathbf{8}_v \times \mathbf{8}_v$	$=$	$\mathbf{1}+\mathbf{28}+\mathbf{35}_v$	Φ, B_2, $g_{\mu\nu}$
	$(\mathrm{NS}_+,\mathrm{R}_-)$	$\mathbf{8}_v \times \mathbf{8}_c$	$=$	$\mathbf{8}_c+\mathbf{56}_c$	λ, Ψ_μ
	$(\mathrm{R}_+,\mathrm{NS}_+)$	$\mathbf{8}_s \times \mathbf{8}_v$	$=$	$\mathbf{8}_s+\mathbf{56}_s$	$\tilde{\lambda}$, $\tilde{\Psi}_\mu$
	$(\mathrm{R}_+,\mathrm{R}_-)$	$\mathbf{8}_s \times \mathbf{8}_c$	$=$	$\mathbf{8}_v+\mathbf{56}_v$	C_1, C_3
Type IIB	$(\mathrm{NS}_+,\mathrm{NS}_+)$	$\mathbf{8}_v \times \mathbf{8}_v$	$=$	$\mathbf{1}+\mathbf{28}+\mathbf{35}_v$	Φ, B_2, $g_{\mu\nu}$
	$(\mathrm{NS}_+,\mathrm{R}_+)$	$\mathbf{8}_v \times \mathbf{8}_s$	$=$	$\mathbf{8}_c+\mathbf{56}_c$	λ, Ψ_μ
	$(\mathrm{R}_+,\mathrm{NS}_+)$	$\mathbf{8}_s \times \mathbf{8}_v$	$=$	$\mathbf{8}_c+\mathbf{56}_c$	λ, Ψ_μ
	$(\mathrm{R}_+,\mathrm{R}_+)$	$\mathbf{8}_s \times \mathbf{8}_s$	$=$	$\mathbf{1}+\mathbf{28}+\mathbf{35}_s$	C_0, C_2, C_4

Table 2.2: Sectors, massless representations, and associated ten-dimensional fields in Type IIA and Type IIB superstring theories.

2.1.5 GSO projection and spacetime supersymmetry

Thanks to the $\mathbb{Z}_2$-grading induced by $\mathscr{F}_L$ and $\mathscr{F}_R$ on the left-moving and right-moving sectors, respectively, one can construct tachyon-free superstring theories. More precisely, it is possible to consistently restrict the theory to four out of 16 possible refined sectors $(\mathrm{R}_\pm,\mathrm{R}_\pm)$, $(\mathrm{R}_\pm,\mathrm{NS}_\pm)$, $(\mathrm{NS}_\pm,\mathrm{R}_\pm)$, $(\mathrm{NS}_\pm,\mathrm{NS}_\pm)$. This procedure is known as GSO projection and can be shown to be compatible with non-trivial string interactions. Furthermore, a GSO projection is also needed to ensure modular invariance of the one-loop string partition function. There exists two inequivalent GSO projections that yield a tachyon-free spectrum. They correspond to Type IIA and Type IIB superstring theories. For each of them table 2.2 summarizes the sectors that survive the GSO projection and the associated massless states, labelled by their $SO(8)$ representations.

As we can see, the (NS,NS) sector is the same for both theories. The corresponding massless states are interpreted as fluctuations of ten-dimensional fields, as follows. The singlet $\mathbf{1}$ is associated to a real scalar field, the dilaton Φ; the representation $\mathbf{28}$ is the antisymmetric rank-two representation and corresponds to the degrees of freedom of a two-form, the Kalb-Ramond field $B_{\mu\nu}$; the notation $\mathbf{35}_v$ denotes the symmetric traceless rank-two representation and is associated with the graviton, i.e. the fluctuation of the metric $g_{\mu\nu}$. Let us stress that the identification of the representations $\mathbf{35}_v$ and $\mathbf{28}$ with the metric and the Kalb-Ramond two-form is consistent with the local symmetries enjoyed by these two fields. For the metric, the local symmetry is furnished by diffeomorphisms, which ensure that only the transverse traceless polarizations of the graviton are physical. For the Kalb-Ramond field the local symmetry is a generalization of gauge symmetry of a $U(1)$ vector. In differential form notation it reads

$$B_2' = B_2 + d\Lambda_1 \ , \tag{2.15}$$

where Λ_1 is an arbitrary one-form. The gauge invariant field strength is $H_3 = dB_2$. Similarly to what happens in Maxwell theory, this local symmetry renders timelike and longitudinal polarizations of the Kalb-Ramond field unphysical.

The (R,R) sectors of both Type II theories contain massless states that are associated to p-form potentials C_p that enjoy an Abelian gauge symmetry completely analogous to (2.15). More precisely, we have

$$\text{Type IIA:} \qquad C_1, C_3 \ , \qquad\qquad \text{Type IIB:} \qquad C_0, C_2, C_4 \ . \tag{2.16}$$

The four-form C_4 is associated to the representation $\mathbf{35}_s$. This implies that its five-form field strength satisfies a self-duality constraint in ten dimensions, which will be given in (2.26).

Finally, the (NS,R) and (R,NS) massless states comprise the degrees of freedom of fermionic fields. They are Majorana-Weyl spin-3/2 fermions, the gravitini Ψ_μ, $\tilde{\Psi}_\mu$ associated to the representations $\mathbf{56}_c$, $\mathbf{56}_s$, and two Majorana-Weyl spin-1/2 fermions, the dilatini λ, $\tilde{\lambda}$ corresponding to $\mathbf{8}_c$, $\mathbf{8}_s$. In Type IIA the two spin-3/2 fermions have opposite chiralities, while in Type IIB their chiralities are the same. Recall from section 1.4 that if a massless spin-3/2 fermion has zero-momentum couplings it interacts with a conserved supercurrent. Therefore, the presence of spin-3/2 massless fermions hints towards the presence of spacetime supersymmetry. Indeed, one can show that for both Type II theories the number of bosonic and fermionic degrees of freedom are equal at all mass levels. Further evidence for spacetime supersymmetry comes from the vanishing of the one-loop partition function, resulting from a cancellation between bosonic and fermionic degrees of freedom, and from the study of the low-energy effective action for massless modes, discussed in section 2.2.

2.1.6 Open strings and D-branes

For a freely propagating open string the world-sheet Σ has the topology of a strip and suitable boundary conditions have to be imposed to ensure the vanishing of the surface terms in the variation of (2.7). Since the equation of motion for X^μ is second-order, two kinds of boundary conditions are possible,

$$\begin{aligned} &\text{Neumann:} \qquad && \partial_\sigma X^\mu|_{\sigma=\sigma^*} = 0 \ , \\ &\text{Dirichlet:} \qquad && X^\mu|_{\sigma=\sigma^*} = \text{const} \ , \end{aligned} \tag{2.17}$$

where σ^* stands for the σ-coordinate of one of the string endpoints. For both choices, at the endpoints right-moving modes are reflected into left-moving modes, and vice versa.

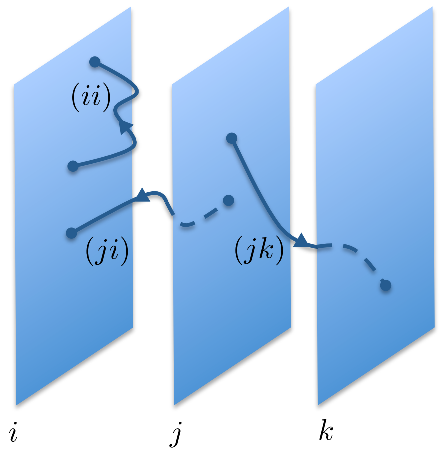


Figure 2.1: Schematic representation of a collection of parallel Dp-branes displaced along their common transverse directions. Each brane is labelled by a Chan-Paton index, while strings are labelled by an ordered pair of indices. String with labels (ij) with $i \neq j$ yield extra massless states when branes i and j come on top of each other.

Note that the conditions (2.17) can be imposed independently at each endpoint and along different directions in spacetime. If for an endpoint we choose Neumann boundary conditions for $p+1$ directions and Dirichlet boundary conditions for the orthogonal $d-p-1$ directions, that endpoint is bound to lie on a $(p+1)$-dimensional subspace of the target space. Such a subspace is referred to as a Dp-brane. Note that we can also set $p = d-1$ in such a way that the D-brane becomes spacetime-filling: this just corresponds to Neumann boundary conditions. In summary, the two endpoints of an open string always end on (possibly different) D-branes.

For a closed string and for spacetime-filling D-branes Poincaré symmetry is unbroken and the momentum and angular momentum of the string are thus conserved. In presence of a Dp-brane with $p < d-1$, however, boundary conditions break Poincaré symmetry and conservation laws are violated. The fact that Dp-branes can absorb and release momentum suggests that they should be considered as dynamical objects of the theory. This is indeed the correct viewpoint, as we will see in section 2.2.

The world-sheet fermions ψ^μ_+, ψ^μ_- always satisfy a boundary condition of the form

$$\psi^\mu_+|_{\sigma=\sigma^*} = \pm\psi^\mu_-|_{\sigma=\sigma^*} , \qquad (2.18)$$

where *a priori* the sign on the right hand side can be chosen independently at each endpoint and along each spacetime direction μ. Note also that any condition of the form (2.18) preserves only half of the world-sheet supersymmetry transformations (2.8), and that different sign choices can correspond to different preserved supercharges. For any given choice of Neumann or Dirichlet boundary conditions for X^μ, however, the signs in (2.18) can be chosen in such a way that the same half of world-sheet supersymmetry is preserved by the whole system of boundary conditions. Actually, this choice can be made in two inequivalent ways, corresponding to the Ramond and the Neveu-Schwarz sectors of the open string.

We refrain from a detailed discussion of open strings and rather illustrate their main features. The spectrum of the open string contains states from both the Neveu-Schwarz and the Ramond sectors. The details depend on the specific choice of boundary conditions, but the qualitative features are similar to those of table 2.1, so that in particular tachyonic states can emerge in the NS sector. Once again, however, a well-defined $\mathbb{Z}_2$ grading based on the world-sheet fermion number can be defined. Thus there is the hope that the GSO projection will be able to eliminate the tachyon and leave a massless spectrum invariant under spacetime supersymmetry. This expectation is indeed confirmed. It can be shown that as a consequence of the GSO projections that define Type IIA and Type IIB stable supersymmetric Dp-branes are found for p even in Type IIA and for p odd in Type IIB. By supersymmetric brane we mean an object that preserves half of the spacetime supersymmetries.

Let us consider in more detail the illustrative example of an open string with both ends attached to the same Dp-brane in the corresponding Type II theory. At the massless level, string states are interpreted as the quantum fluctuations of $(p{+}1)$-dimensional fields confined along the world-volume of the Dp-brane. The fact that D-branes carry localized degrees of freedom is an essential feature of string theory. More precisely, on the world-volume of a Dp-brane we find a $U(1)$ gauge field A_a $(a = 0, \ldots, p)$, a collection of scalars φ^I, $(I = 1, \ldots, 9 - p)$ and their fermionic superpartners. The scalars can be interpreted as Goldstone bosons associated to the breaking of ten-dimensional translations induced by the Dp-brane. Note that for any p the $p - 1$ transverse polarizations of A_a combine with the $9 - p$ scalars to give the eight bosonic degrees of freedom in the massless $\mathbf{8}_v$ of the NS sector of the open string, see table 2.1.

It is also interesting to consider the case of N parallel Dp-branes, displaced arbitrarily along the transversal $8 - p$ directions orthogonal to their world-volumes, see figure 2.1. It can be shown that they all preserve the same half of spacetime supersymmetry. If we assign a label $i = 1, \ldots, N$ to each Dp-brane, open string sectors are labeled by an ordered pair (ij), denoting a string starting on brane i and ending on brane j. These labels are called Chan-Paton factors. The case $i = j$ has just been discussed and yields a $U(1)$ gauge field. If $i \neq j$, the mass spectrum is shifted by an amount which is interpreted as the energy required to stretch a string between brane i and brane j. Thus, these sectors are not associated to massless vectors, but rather to massive ones. Nonetheless, we can consider the limit in which all Dp-branes become coincident and get a total of N^2 massless vectors. This suggests an enhancement of the gauge group living on their world-volume from $U(1)^N$ to $U(N)$. This expectation is confirmed by the study of open string scattering amplitudes. We have thus encountered another fundamental feature of string theory: a stack of coincident D-branes carries a non-Abelian gauge theory on its world-volume.

2.1.7 String coupling to background fields

The Polyakov action (2.1) describes the propagation of a string in flat spacetime with metric $\eta_{\mu\nu}$. The closed string spectrum, however, contains a massless spin-two excitation that is identified with the graviton. This suggests to consider the propagation of a string on a curved spacetime with metric $g_{\mu\nu}$, which can be interpreted as a coherent superposition of graviton states. By the same token, one is also led to include a non-trivial background for the Kalb-Ramond two-form and the dilaton. The resulting generalization of (2.1) takes the form of a non-linear sigma model on the world-sheet. Focusing on bosonic NSNS fields only, its action reads

$$S_{\text{nl}\sigma} = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{-h}\left[(h^{\alpha\beta}g_{\mu\nu} + \epsilon^{\alpha\beta}B_{\mu\nu})\,\partial_\alpha X^\mu\,\partial_\beta X^\nu + \alpha'\,\Phi\,R\right]\ , \tag{2.19}$$

where $\epsilon^{\alpha\beta}$ denotes the world-sheet Levi-Civita tensor and R is the world-sheet Ricci scalar. Let us point out that in the Ramond-Neveu-Schwarz formulation of the superstring, which we have followed in our review, there is no straightforward way to couple the world-sheet theory to background RR forms. In alternative formulations, however, this can be achieved in such a way that all massless bosonic fields of the closed superstring spectrum can appear as backgrounds for the string propagation.

As stressed in section 2.1.1, Weyl invariance of the quantum world-sheet action is necessary to ensure that the world-sheet metric does not contribute any physical degree of freedom to the propagation of the string. The non-linear sigma model action (2.19) is not Weyl invariant at the classical level, because of the dilaton term. At the quantum level, however, this classical non-invariance can be cancelled by anomalous variations of the first two terms involving $g_{\mu\nu}$ and $B_{\mu\nu}$. If the background fields are slowly varying in spacetime with respect to the string length scale $(\alpha')^{1/2}$, the conditions for quantum Weyl invariance can be written as differential constraints on $g_{\mu\nu}$, $B_{\mu\nu}$, Φ in the form of a derivative expansion, with higher-derivative terms suppressed by higher powers of α'. These constraints are referred to as (super)string equations of motion for the background fields. They can be alternatively derived from a spacetime effective action for massless modes, as discussed below in section 2.2.

A trivial example of a string background that solves the superstring equations of motion exactly in α' is furnished by flat space in ten dimensions,

$$g_{\mu\nu} = \eta_{\mu\nu}\ , \qquad B_{\mu\nu} = 0\ , \qquad \Phi = \Phi_0\ , \tag{2.20}$$

with constant Φ_0. In this case the first two terms in the non-linear sigma model action (2.19) reproduce the Polyakov action (2.1), while the dilaton term becomes the integral of a total world-sheet derivative. This is most conveniently treated by analytic continuation to Euclidean signature on the world-sheet. We then get the term

$$S_\chi = \frac{1}{4\pi}\Phi_0\int_\Sigma d^2\sigma\sqrt{h}\,R = \Phi_0\,\chi\ , \tag{2.21}$$

where in the second step we have used the fact that the integral reproduces the Euler number of the world-sheet, which is a topological invariant. For example, the Euler number of a Riemann surface is given by $\chi = 2 - 2g$, where g is the genus of the surface, i.e. the number of 'handles.' The topological term S_χ in (2.21) has no effect on the string spectrum but it plays a crucial role in the study of string interactions. Indeed, in the sum over topologically non-trivial world-sheets, each contribution comes weighted by a factor $e^{-S_\chi} = e^{-\Phi_0\,\chi}$. This implies that if $e^{\Phi_0} \ll 1$ string amplitudes can be computed as a perturbative expansion organized by the Euler number. We are thus led to identify e^{Φ_0} with the string coupling constant. This conclusion is confirmed by the study of the effective actions for massless modes, discussed below. As promised, string theory has no dimensionless tunable parameter: the strength of string interactions is not a property of the theory itself, but rather of the background or vacuum under consideration.

2.2 Low-energy dynamics of massless modes

Consider a closed string scattering process in Type IIA or Type IIB with typical energy scale $E \ll (\alpha')^{-1/2}$. Since the mass of the first excited level in the string spectrum is $\alpha' m^2 = 4$ only the massless states summarized in table 2.2 will play a relevant role in the string dynamics. Effectively, we are thus looking at strings in the so-called point particle limit $\alpha' \to 0$. As a result, the information about the interactions among these light states can be encoded in an effective action, according to the general paradigm of section 1.3. In what follows, we record the effective actions for ten-dimensional fields and for fields living on D-branes, without any derivation. The interested reader is referred to the textbooks for more details.

2.2.1 Effective actions

The low-energy limit of Type IIA and Type IIB superstring theories at leading order in α' yields Type IIA and Type IIB supergravities, which are the two inequivalent theories with maximal local supersymmetry in ten dimensions (32 supercharges). The corresponding actions up to two derivatives are uniquely determined up to field redefinitions. In what follows, we will give only the bosonic part of the effective actions, since fermionic terms are fixed by supersymmetry.

The effective action for the massless fields in the NSNS sector is the same for both Type IIA and Type IIB. In differential form notation, it is given by

$$S_{\text{NSNS}} = \frac{1}{2\kappa_{10}^2} \int_{\mathcal{M}_{10}} e^{-2\Phi} \left[R * 1 - \frac{1}{2} H_3 \wedge * H_3 + 4\, d\Phi \wedge * d\Phi \right] \,, \tag{2.22}$$

where $\mathcal{M}_{10}$ denotes ten-dimensional spacetime, R is the Ricci scalar of the spacetime metric $g_{\mu\nu}$, and $H_3 = dB_2$ is the field strength of the Kalb-Ramond field. The constant κ_{10} is not fixed at the moment and will be related to physical observables below. The effective action (2.22) is said to be written in the string frame, because the metric $g_{\mu\nu}$ is the metric that enters the world-sheet non-linear sigma model (2.19). In section 3.2 we will encounter a formulation of Type IIB in a different frame, the Einstein frame. Note also that the dilaton prefactor $e^{-2\Phi}$ signals that this effective action is obtained at tree-level in closed string perturbation theory. Indeed, the relevant closed string topology is a sphere, with Euler number $\chi = 2$.

The actions for the RR sectors of Type IIA and Type IIB are different, but are both given by the sum of kinetic terms and Chern-Simons terms. More precisely,

$$S_{\text{RR}}^{\text{IIA}} = \frac{1}{2\kappa_{10}^2} \int_{\mathcal{M}_{10}} \left[-\frac{1}{2} F_2 \wedge * F_2 - \frac{1}{2} F_4 \wedge * F_4 - \frac{1}{2} B_2 \wedge dC_3 \wedge dC_3 \right] \,, \tag{2.23}$$

$$S_{\text{RR}}^{\text{IIB}} = \frac{1}{2\kappa_{10}^2} \int_{\mathcal{M}_{10}} \left[-\frac{1}{2} F_1 \wedge * F_1 - \frac{1}{2} F_3 \wedge * F_3 - \frac{1}{4} F_5 \wedge * F_5 - \frac{1}{2} C_4 \wedge H_3 \wedge F_3 \right] \,, \tag{2.24}$$

where the field strengths are given by

$$\begin{aligned} &F_2 = dC_1 \ , \qquad F_4 = dC_3 - dB_2 \wedge C_1 \ , \\ &F_1 = dC_0 \ , \qquad F_3 = dC_2 - C_0\, dB_2 \ , \qquad F_5 = dC_4 - \frac{1}{2} C_2 \wedge dB_2 + \frac{1}{2} B_2 \wedge dC_2 \ , \end{aligned} \tag{2.25}$$

where B_2 is the Kalb-Ramond two-form. The specific form of these field strengths is dictated by invariance under the full set of RR gauge transformation—which we do not write down explicitly—and implies that all of them, except F_2 and F_1, satisfy non-trivial Bianchi identities, $dF_{p+1} \neq 0$. Let us remark that the RR actions also correspond to tree-level closed string amplitudes. One could extract an overall factor $e^{-2\Phi}$ by a suitable dilaton-dependent redefinition of the RR potentials. This is not convenient, however, since it would render the RR gauge transformation and Bianchi identities dilaton dependent.

The Type IIB action for RR fields (2.24) offers the opportunity to introduce the concept of pseudoaction, which will play a crucial role in chapters 7 and 10. Recall that a suitable self-duality condition has to be imposed on the field strength of C_4 in order to have the correct number of degrees of freedom. More precisely, the self-duality constraint reads

$$* F_5 = F_5 \ . \tag{2.26}$$

If this relation holds, however, the kinetic term $F_5 \wedge *F_5$ vanishes identically. Indeed, there is no simple covariant action that can yield the first-order differential constraint (2.26) upon variation of the four-form potential C_4. Taking the exterior derivative of (2.26) yields

$$d * F_5 = H_3 \wedge F_3 \ , \tag{2.27}$$

which precisely corresponds to the equation of motion that is derived from (2.24) if it is varied with respect to C_4 ignoring any constraint. In summary, the action (2.24) is a pseudoaction in the sense that its equations of motion are compatible with the self-duality constraint (2.26), which however has to be implemented by hand after taking variations of (2.24).

The RR actions (2.23) and (2.24) are written in terms of the form potentials of lowest degree. One can, however, encode the degrees of freedom of a $(p+1)$-form C_{p+1} into its magnetic dual C_{7-p}, defined by a relation of the form $F_{p+2} = \pm * F_{8-p}$, where the F's are the gauge invariant field strengths. The full RR form content of Type IIA is indeed given by C_1, C_3, C_5, C_7, C_9, where (C_1, C_7), (C_3, C_5) are dual pairs and C_9 carries no propagating degrees of freedom. In a similar fashion, in Type IIB we have C_0, C_2, C_4, C_6, C_8, C_{10}, with the dual pairs (C_0, C_8), (C_2, C_6), C_4 self-dual, and C_{10} non dynamical. Let us mention that there exists a so-called democratic formulation of Type II supergravities [49] in which the effective action contains also the higher degree RR forms and which is useful in the discussion of couplings to Dp-branes and in the study of string flux compactifications.

Let us now discuss the effective action for the massless fields living on the world-volume $\mathcal{W}_{p+1}$ of a Dp-brane. Recall that a Dp-brane respects half of the supersymmetry of the corresponding Type II bulk action. Therefore, the action for the fermionic degrees of freedom living on $\mathcal{W}_{p+1}$ is completely determined by the action for the bosonic fields. The latter are conveniently described in terms of a $U(1)$ gauge field A_a ($a = 0, \dots, p$) and the embedding functions $X^\mu(\xi)$ ($\mu = 0, \dots, 9$), where ξ^a are coordinates on $\mathcal{W}_{p+1}$. Intuitively speaking, out of the 10 directions in X^μ, the $9-p$ transversal directions provide the degrees of freedom of the scalars living on the brane, while the other $p+1$ directions are pure gauge. The Dp-brane effective action is given by the sum of the Dirac-Born-Infeld action, describing the coupling of the world-volume degrees of freedom to the bulk NSNS fields, and of the Chern-Simons action, which describes the coupling to the bulk RR forms.

The Dirac-Born-Infeld action reads

$$S_{\text{DBI}} = -T_p \int_{\mathcal{W}_{p+1}} d^{p+1}\xi \, e^{-\Phi} \sqrt{-\det\left(g_{ab} + B_{ab} + 2\pi\alpha' F_{ab}\right)} \;, \tag{2.28}$$

where $F_{ab} = 2\partial_{[a}A_{b]}$, Φ is the dilaton restricted to the brane, g_{ab} and B_{ab} are the pull-back of the metric and Kalb-Ramond field,

$$g_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} g_{\mu\nu} \;, \qquad B_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} B_{\mu\nu} \;. \tag{2.29}$$

The parameter T_p is related to the tension of the brane and will be discussed below in more detail. The dilaton prefactor $e^{-\Phi}$ is once again a signal that the action encodes string amplitudes at tree-level: for an open string the relevant topology is a disk, with Euler number $\chi = 1$.

The action (2.28) can be seen as a non-linear generalization of the Maxwell action, to which it reduces to lowest order in the field strength F_{ab},

$$S_{\text{DBI}} = -T_p \int_{\mathcal{W}_{p+1}} d^{p+1}\xi \, e^{-\Phi} \sqrt{-\det g_{ab}} \left\{ 1 + \frac{1}{4}(2\pi\alpha')^2 \, F^{ab}F_{ab} + \dots \right\} \;, \tag{2.30}$$

where the dots represent higher order terms and contributions from the B-field and the $9-p$ massless scalars. For a stack of D-branes with non-Abelian gauge group, the full non-linear analog of (2.28) is not completely known, but the lowest order action is given by the natural non-Abelian generalization of (2.30), obtained with the replacements

$$F_{ab} = 2\partial_{[a}A_{b]} \;\to\; F_{ab} = 2\partial_{[a}A_{b]} + \tfrac{1}{2}[A_a, A_b] \;, \qquad F^{ab}F_{ab} \;\to\; \operatorname{tr}(F^{ab}F_{ab}) \;, \tag{2.31}$$

where tr denotes the trace in the fundamental representation of $U(N)$. Taking into account that a stack of Dp-branes preserves 16 out of the 32 supercharges of the corresponding bulk theory, we then find maximally supersymmetric $U(N)$ gauge theory living on $\mathcal{W}_{p+1}$.

The full form of the Chern-Simons action for a Dp-brane is quite involved and will not be displayed. We rather consider only the case of a stack consisting of a single Dp-brane and focus on the simple coupling

$$S_{\text{CS}} \supset \mu_p \int_{\mathcal{W}_{p+1}} C_{p+1} \;, \tag{2.32}$$

which is the straightforward generalization of the coupling of a charged particle to a vector, which can be written covariantly as $q\int_\gamma A$, where γ is the world-line of the particle and q its charge. The physical interpretation is the following: a Dp-brane has a minimal electric coupling to the RR bulk form C_{p+1}; equivalently, it is magnetically charged under the dual form C_{7-p}. This makes sense thanks to the consistency of the even/odd grading of RR form degrees and D-brane dimensionalities in Type IIA and Type IIB.

2.2.2 The string coupling constant

Let us make some remarks on the coupling constants that enter the effective actions we have considered. To begin with, note that all the bulk and brane effective actions and the functional form of the RR field strengths (2.25) are left invariant by a simultaneous redefinition of the dilaton Φ, the RR fields C_p, F_p, the parameters κ_{10}, T_p, and μ_p,

$$\begin{aligned} &\Phi' = \Phi + a \ , && C_p' = C_p\, e^{-a} \ , && F_p' = F_p\, e^{-a} \ , \\ &\kappa_{10}' = \kappa_{10}\, e^{-a} \ , && T_p' = T_p\, e^{a} \ , && \mu_p' = \mu_p\, e^{a} \ , \end{aligned} \tag{2.33}$$

where a is an arbitrary constant. This ambiguity, however, does not affect measurable couplings. Indeed, the physical gravitational coupling, Dp-brane tension, and Dp-brane electric coupling to C_{p+1} are given respectively by

$$\kappa = \kappa_{10}\, e^{\langle\Phi\rangle} \ , \qquad \tau_p = T_p\, e^{-\langle\Phi\rangle} \ , \qquad e_p = \sqrt{2}\,\mu_p\,\kappa_{10} \ , \tag{2.34}$$

where $\langle\Phi\rangle$ denotes the VEV of the dilaton. These quantities can be seen as the interaction vertices that can be read off from the effective actions of the previous section after imposing canonical normalization on the kinetic terms for metric and RR field fluctuations.

By comparing string amplitudes and tree-level effective field theory computations one can show that the couplings (2.34) are not independent, but rather satisfy

$$2\kappa^2\tau_p^2 = e_p^2 = 2\pi\,\ell_s^{2(3-p)} \ , \qquad \ell_s = 2\pi\sqrt{\alpha'} \ , \tag{2.35}$$

where the normalization of ℓ_s is chosen in order to absorb all the dependence on p in the unit of measurement. The first equality in (2.35) can also be written in the form

$$T_p = |\mu_p| \tag{2.36}$$

and shows that the forces between two parallel Dp-branes due to the exchange of NSNS and RR fields have the same magnitude. In fact, they cancel exactly, leaving no net force between the branes. This is common for BPS objects in theories with extended supersymmetry.

Because of the relations (2.35) there is only one independent physical coupling. Moreover, (2.34) shows it is not a tunable parameter of the theory, but a property of the vacuum. As promised, there are no free dimensionless parameters is string theory. The ambiguity related to the field redefinitions (2.33) can be removed by choosing a conventional normalization. A standard choice is

$$\frac{1}{2\kappa_{10}^2} = 2\pi\,\ell_s^{-8} \ , \qquad T_p = \mu_p = 2\pi\,\ell_s^{-(p+1)} \ , \tag{2.37}$$

because in this way the VEV of the dilaton is related in a simple fashion to the ratio between the fundamental string tension and the D1-brane tension, which is usually taken as the definition of the string coupling constant g_s:

$$e^{\langle\Phi\rangle} = \frac{T_{\text{string}}}{\tau_1} = g_s \ . \tag{2.38}$$

The low-energy effective actions discussed in this section can be checked against string perturbation theory only if g_s is small. Thanks to the protection ensured by supersymmetry, however, their validity can be extrapolated to low-energy but strongly coupled regimes. This observation will be crucial in introducing F-theory in chapter 5.

Let us close this section with a remark about another kind of extended object in Type II string theory, the NS5-brane. The non-linear sigma model action (2.19) contains the coupling

$$S_{\text{nl}\sigma} \supset \frac{1}{2\pi\alpha'} \int_\Sigma B_2 \ , \tag{2.39}$$

which can be seen as an electric coupling of the string to the NSNS two-form. We thus expect the existence of a magnetic dual to the fundamental string, i.e. an extended objects with five spatial directions coupling to the dual potential given locally by $dB_6 = *dB_2$. Such an object is called NS5-brane and can be constructed as a solitonic solutions of the source-free equations of motion derived from the action (2.22). More precisely, the NS5-brane is a field configuration that admits a finite total energy per unit area in the six longitudinal spacetime directions, i.e. a finite tension,

$$\tau_{\text{NS5}} = \frac{2\pi \ell_s^{-6}}{g_s^2} \ . \tag{2.40}$$

The string coupling dependence $1/g_s^2$ differs from the $1/g_s$ factor for D-branes. This is consistent with the interpretation of the NS5-brane as a closed string soliton. It is topologically stable against decay thanks to a non-vanishing magnetic flux on the sphere S^3 in the four-dimensional transverse space,

$$\ell_s^{-2} \int_{S_3} H_3 = N \ , \tag{2.41}$$

where we have recored the expression for the general case of a stack of N coincident NS5-branes. Finally, one can show that NS5-branes are BPS objects preserving half of the 32 supercharges of the Type II bulk. In Type IIB, the 16 preserved supercharges have opposite chirality and the resulting world-volume supersymmetry algebra is $(1,1)$. Fluctuations around the NS5-brane solution can then be described in terms of an effective action for $(1,1)$ vector multiplets in six dimensions. In Type IIA we find instead the chiral world-volume supersymmetry algebra $(2,0)$. Correspondingly, massless fluctuations are described in terms of $(2,0)$ tensor multiplets. They will be described in detail in chapter 11.

2.3 Orientifold projections

In this section we present some foundational material on orientifolds in Type II theories that will be useful in chapter 8. The crucial ingredient of an orientifold projection is the world-sheet operator Ω_p which implements a parity transformation on world-sheet coordinates. For a closed string with $\sigma \sim \sigma + 2\pi$ one has

$$\Omega_\text{p}\, X^\mu(\tau,\sigma)\, \Omega_\text{p}^{-1} = X^\mu(\tau, 2\pi - \sigma) \ . \tag{2.42}$$

A similar relation holds for the world-sheet fermions $\psi_\pm^\mu$ in such a way to have compatibility with world-sheet supersymmetry (2.8). Since world-sheet parity reverses left-moving and right-moving oscillations it is a symmetry of Type IIB string theory, but not of Type IIA string theory, since in the latter different GSO projections are performed on the left-movers and on the right-movers.

We can, however, generalize the action of the world-sheet parity $\Omega_{\rm p}$ by combining it with a geometric action on spacetime coordinates. Let us consider the simple case in which the geometric action is a reflection $R_{(n)}$ of n spatial directions, say x^9, x^8, ..., x^{10-n}:

$$\begin{aligned} [\Omega_{\rm p}\, R_{(n)}]\, X^\mu(\tau,\sigma)\, [\Omega_{\rm p}\, R_{(n)}]^{-1} &= +X^\mu(\tau, 2\pi-\sigma) \qquad \text{for } \mu = 0,1,\ldots,9-n\ , \\ [\Omega_{\rm p}\, R_{(n)}]\, X^\mu(\tau,\sigma)\, [\Omega_{\rm p}\, R_{(n)}]^{-1} &= -X^\mu(\tau, 2\pi-\sigma) \qquad \text{for } \mu = 10-n,\ldots,9\ . \end{aligned} \tag{2.43}$$

For $n = 2k$, $R_{(n)}$ can be seen as the composition of k rotations with angle π in k different two-planes. This is a Pin-even transformation, i.e. $R_{(n)}$ acts on ten-dimensional spinors preserving their chirality. For $n = 2k+1$, in contrast, $R_{(n)}$ is the composition of k π-rotations and a reflection, and this amounts to a Pin-odd transformation that flips chirality. Recall also that a rotation of an angle π squares to minus the identity on spacetime fermions. In summary, we have

$$\begin{aligned} R_{(2k)}\ :& \qquad \text{Pin-even}\ , \qquad R^2_{(2k)} = (-1)^{k(F_L+F_R)}\ , \\ R_{(2k+1)}\ :& \qquad \text{Pin-odd}\ , \qquad R^2_{(2k+1)} = (-1)^{k(F_L+F_R)}\ , \end{aligned} \tag{2.44}$$

where $F_{L,R}$ denote the spacetime left- and right-moving fermion numbers, which are defined mod 2. For instance $(-1)^{F_L}$ is $+1$ on the (NS,R) sector and -1 on the (R,NS) sector. Please note that $F_{L,R}$ should not be confused with the world-sheet fermions numbers $\mathscr{F}_{L,R}$ discussed in section 2.1.4. Let us remind the reader that the GSO projection in the Ramond sector amounts to selecting a definite chirality for the ground state, which can be $\mathbf{8}_c$ or $\mathbf{8}_s$, as can be seen from table (2.1). Taking into account (2.44) we thus find the following symmetry operators for Type IIA and Type IIB:

$$\begin{aligned} \text{Type IIA:} & \qquad \mathcal{O}_{(2k+1)} = \Omega_{\rm p}\, R_{(2k+1)}\, (-1)^{kF_L}\ , \\ \text{Type IIB:} & \qquad \mathcal{O}_{(2k)} = \Omega_{\rm p}\, R_{(2k)}\, (-1)^{kF_L}\ , \end{aligned} \tag{2.45}$$

in which the factors of $(-1)^{kF_L}$ are inserted to ensure $\mathcal{O}^2_{(n)} = \mathbb{I}$ for all n. As a result, we can restrict the theory to the subspace invariant under the action of $\mathcal{O}_{(n)}$.

In the resulting theories the spacetime points related by the geometric action $R_{(n)}$ are identified. The geometry of the quotient space, in the simple non-compact example (2.43), is given by $\mathbb{R}^{1,9-n} \times (\mathbb{R}^n/\mathbb{Z}_2)$; the origin in of $\mathbb{R}^n/\mathbb{Z}_2$ corresponds to the $(10-n)$-dimensional subspace of $\mathbb{R}^{1,9}$ fixed under $R_{(n)}$. Moreover, after the $\mathcal{O}_{(n)}$ projection we have theories of non-oriented strings, in which non-orientable world-sheet topologies have to be included in the path integral that defines the string partition function. The effect of these new geometries on the low-energy physics of the system can be attributed to a new kind of extended object, the O-planes. More precisely, for (2.43) an Op-plane with $p = 9-n$ is located on top of the fixed space of $R_{(n)}$.

We refrain from giving the full form of the effective action for an Op-plane. Let us mention, however, that it couples to the NSNS sector via a non-vanishing tension $T_{{\rm O}p}$ and to the RR form C_{p+1} via a non-vanishing charge $\mu_{{\rm O}p}$. The values of $T_{{\rm O}p}$ and $\mu_{{\rm O}p}$ depend upon the specific action of $\Omega_{\rm p}$ on string states. All Op-planes that we consider have negative tension and negative charge, given by

$$T_{{\rm O}p} = -2^{p-4}\, T_p\ , \qquad \mu_{{\rm O}p} = -2^{p-4}\, \mu_p\ , \tag{2.46}$$

$\Omega_{\rm p} = +1$	$\Phi,\ g_{\mu\nu},\ C_1,\ C_2$
$\Omega_{\rm p} = -1$	$B_{\mu\nu},\ C_0,\ C_3,\ C_4$

Table 2.3: Intrinsic parities of NSNS and RR bulk fields under the action of the world-sheet parity operator $\Omega_{\rm p}$.

where T_p, μ_p are the tension and charge of a Dp-brane.

The previous discussion can be generalized from the reflection $R_{(n)}$ in Minkowski spacetime to a geometric action σ acting on a more general background $\mathcal{M}$ with $\sigma^2 = \mathrm{id}_{\mathcal{M}}$. For example, we will encounter orientifolds of Calabi-Yau compactifications in section 5.4. The qualitative features of the setup are the same as in the simplified case (2.43). If the geometric action admits a $(p+1)$-dimensional subspace of ten-dimensional spacetime, an Op-plane is located there, with tensions and charges given by (2.46).

In setups with compact directions the presence of RR charged objects can lead to inconsistencies, since, intuitively speaking, on a compact space the flux lines generated by a source can only end on sources with opposite charge and cannot 'escape to infinity.' In many situations is then necessary to add D-branes to the construction to counterbalance the negative tension of the O-planes and cancel all RR tadpoles. Note that the D-brane configuration included in the setup has to be compatible with the geometric action σ in order to take the orientifold projection consistently. On way to achieve this is to put a stack of $N = 2^{p-4}$ Dp-branes on top of an Op-plane, for $p = 5, \ldots, 9$. In this case the spectrum of open strings starting and ending on the stack of Dp-branes is modified by the orientifold projection. For O-planes with tension and charges given by (2.46) the gauge group is reduced from $U(N)$ to $SO(N)$. The prototypical example of this kind of constructions is furnished by the orientifold projection of Type IIB with respect to $\mathcal{O}_{(0)} = \Omega_{\rm p}$. To balance the negative charge and tension of the corresponding spacetime-filling O9-plane a stack of 32 D9-branes is introduced. The resulting theory is a theory of closed and open unoriented strings with $SO(32)$ gauge group known as Type I string theory. It has minimal supersymmetry in ten dimensions, corresponding to 16 supercharges.

From the point of view of the low-energy effective action, the orientifold projection induces an intrinsic $\Omega_{\rm p}$-parity on all bulk fields. This parity is determined by equation (2.42) and can be combined with parities induced by the geometric action σ to determine the fields that have total positive parity and then survive the orientifold projection. The $\Omega_{\rm p}$-parity of NSNS and RR bosons are collected in table 2.3. On top of this, the contribution of localized sources such as O-planes and D-branes has to be added to compute the total effective action. This program has been carried out for Calabi-Yau compactifications in [50, 51, 52, 53].

CHAPTER 3

T-duality, S-duality, M-theory

As recalled in the introduction in section 1.2, all known string theories and their compactifications to lower dimensions are related by an intricate web of dualities, some of which are strong/weak dualities and thus shed light on the dynamics of strings beyond perturbation theory. In what follows we refrain from an account of the vast and fascinating subject of string dualities, and rather focus on those elements that will be instrumental for our discussion of F-theory in chapter 5. Let us mention, however, that the resulting picture sees all consistent, perturbative ten-dimensional string theories unified by a new eleven-dimensional theory, called M-theory. More precisely, the five known superstring theories can be though of as special limits in the moduli space of M-theory, see figure 1.2. For our purposes we need to review in some detail important aspects of T-duality and S-duality in the contest of Type II string theory. We also discuss briefly the low-energy limit of M-theory and elucidate its relation to Type IIA and Type IIB.

3.1 T-duality of Type II superstring theories

T-duality is a perturbative duality relating string compactifications on spaces that admit continuous isometries. One of the simplest examples of this situation is furnished by ten-dimensional Type IIA or Type IIB compactified on a circle of radius R. The geometry of the circle is most conveniently described in terms of the quotient $\mathbb{R}/(2\pi R\mathbb{Z})$: explicitly, one of the coordinates, say x^9, is periodically identified,

$$x^9 \sim x^9 + 2\pi R \; . \tag{3.1}$$

Crucially, this identification allows for a generalization of the closed string bosonic periodicity (2.12) which reads

$$X^9(\tau, \sigma + 2\pi) = X^9(\tau, \sigma) + 2\pi w^9 R \; , \tag{3.2}$$

where the integer w^9 is referred to as winding number, since it counts the number of times that the closed string winds around the compact direction x^9. Note also that the momentum of the string along the x^9 direction must be quantized,

$$p^9 = \frac{n^9}{R} \ , \tag{3.3}$$

in such a way that the spacetime translation operator $e^{ip^9x^9}$ respects the circle identification (3.1). The integer n^9 is the Kaluza-Klein level of the circle compactification. In summary, closed string sectors are labelled by two integers (n^9, w^9). Fermionic periodicity conditions (2.13) are unaffected and still yield a Ramond and a Neveu-Schwarz sector for each value of (n^9, w^9).

We are now in a position to state T-duality in this simple example: Type IIA compactified on a circle of radius R is equivalent to Type IIB compactified on a circle of radius

$$\tilde{R} = \frac{\alpha'}{R} \ . \tag{3.4}$$

More precisely, let $\tilde{X}^9(\tau,\sigma)$ denote the coordinate in the dual Type IIB setup with radius $\tilde{R}$, so that we can introduce the dual winding numbers $\tilde{w}^9$ and momentum units $\tilde{n}^9$ by

$$\tilde{X}^9(\tau,\sigma+2\pi) = \tilde{X}^9(\tau,\sigma) + 2\pi\tilde{w}^9\tilde{R} \ , \qquad \tilde{p}^9 = \frac{\tilde{n}^9}{\tilde{R}} \ . \tag{3.5}$$

Then T-duality amounts to the statement the dual coordinate $\tilde{X}^9$ and its fermionic partners $\tilde{\psi}^9_\pm$ are given in terms of X^9, $\psi^9_\pm$ by

$$\tilde{X}^9_L(\tau,\sigma) = X^9_L(\sigma_+) - X^9_R(\sigma_-) \ , \qquad \tilde{\psi}^9_+(\sigma_+) = \psi^9_+(\sigma_+) \ , \qquad \tilde{\psi}^9_-(\sigma_-) = -\psi^9_-(\sigma_-) \ , \tag{3.6}$$

where we have made use of the decomposition (2.10) into left- and right-moving parts. We can thus see that T-duality is a reflection $x^9 \to -x^9$ acting on the right-moving sector only. One can show that this transformation is such that the physics encoded in the world-sheet CFT for the fields $\tilde{X}^9$, $\tilde{\psi}^9_\pm$ is the same as the physics of the CFT of X^9, $\psi^9_\pm$. This makes T-duality an exact perturbative symmetry of Type II.

In order to make the spacetime implications of (3.6) more manifest, let us consider how the four sectors of Type IIA are mapped to Type IIB: one can show that

$$\begin{array}{lcl} (\mathrm{NS}_+, \mathrm{NS}_+; n^9, w^9) & \to & (\mathrm{NS}_+, \mathrm{NS}_+; \tilde{n}^9, \tilde{w}^9) \ , \\ (\mathrm{NS}_+, \mathrm{R}_-; n^9, w^9) & \to & (\mathrm{NS}_+, \mathrm{R}_+; \tilde{n}^9, \tilde{w}^9) \ , \\ (\mathrm{R}_+, \mathrm{NS}_+; n^9, w^9) & \to & (\mathrm{R}_+, \mathrm{NS}_+; \tilde{n}^9, \tilde{w}^9) \ , \\ (\mathrm{R}_+, \mathrm{R}_-; n^9, w^9) & \to & (\mathrm{R}_+, \mathrm{R}_+; \tilde{n}^9, \tilde{w}^9) \ , \end{array} \tag{3.7}$$

where we have used the notation of section 2.1.4 and where

$$(\tilde{n}^9, \tilde{w}^9) = (w^9, n^9) \ . \tag{3.8}$$

The change from Type IIA to Type IIB is due to the change in chirality of the right-moving Ramond ground state. As in section 2.3, this is caused by the fact that $x^9 \to -x^9$ is a Pin-odd transformation.

Equation (3.8) states that Kaluza-Klein modes and winding modes are exchanged. This can also be checked from inspection of the mass formula for closed string states, which reads schematically

$$m^2 = \left(\frac{n^9}{R}\right)^2 + \left(\frac{w^9\,R}{\alpha'}\right)^2 + \text{osc} \ , \tag{3.9}$$

where the first term is the Kaluza-Klein mass, the second is the mass due to winding, and osc denotes the contribution from the oscillation modes of the string around its center of mass. As we can see, the combined replacements (3.4) and (3.8) leave the Kaluza-Klein and winding contributions to m^2 invariant.

For our discussion of the relation between M-theory and Type IIB string theory in section 3.4 it will be useful to know the explicit form of the T-duality map on massless bosonic fields. This map is furnished by so-called Buscher rules, see e.g. [54, 55, 56, 57] for a review. Let us consider them in a somewhat non-standard notation that is inspired by the interpretation of T-duality as a dimensional oxidation ambiguity from nine to ten dimensions [58] and that is best suited for our purposes. Let y be a compact dimensionless coordinate of period 1 along the T-duality circle. We adopt the following parametrization of the Type IIA string frame metric, dilaton, NSNS two-form, and RR p-forms,

$$\begin{aligned} &ds^2_{\rm IIA} = ds^2_9 + L^2(dy + \ell_s^{-1}\,V)^2 \ , \qquad \Phi_{\rm IIA} = \varphi + \frac{1}{2}\log\frac{L}{\ell_s} \ , \qquad B_2^{\rm IIA} = B_2 + B_1\wedge \ell_s\,dy \ , \\ &C_p^{\rm IIA} = C_p + C_{p-1}\wedge \ell_s\,dy \ , \quad p = 1,3 \ , \end{aligned} \tag{3.10}$$

where V is a vector. In a similar fashion, the massless bosonic fields of Type IIB are parametrized as

$$\begin{aligned} &ds^2_{\rm IIB} = d\tilde s^2_9 + \tilde L^2(dy + \ell_s^{-1}\,\tilde V)^2 \ , \qquad \Phi_{\rm IIB} = \tilde\varphi + \frac{1}{2}\log\frac{\tilde L}{\ell_s} \ , \qquad B_2^{\rm IIB} = \tilde B_2 + \tilde B_1\wedge \ell_s\,dy \ , \\ &C_p^{\rm IIB} = \tilde C_p + \tilde C_{p-1}\wedge \ell_s\,dy \ , \quad p = 0,2,4 \ , \end{aligned} \tag{3.11}$$

with the understanding $\tilde C_{-1} \equiv 0$. The Buscher rules can then be written in the following form,

$$\begin{aligned} &d\tilde s^2_9 = ds^2_9 \ , \qquad \tilde L = \frac{\ell_s^2}{L} \ , \qquad \tilde V = -B_1 \ , \qquad \tilde\varphi = \varphi \ , \\ &\tilde B_1 = -V \ , \qquad \tilde B_2 = B_2 - B_1\wedge A \ , \\ &\tilde C_0 = C_0 \ , \qquad \tilde C_1 = C_1 - C_0\,V \ , \qquad \tilde C_2 = C_2 + B_1\wedge(C_1 - C_0\,V) \ , \\ &\tilde C_3 = C_3 - \frac{1}{2}C_2\wedge V - \frac{1}{2}B_2\wedge(C_1 - C_0\,V) \ . \end{aligned} \tag{3.12}$$

They can be easily inverted to obtain the map from Type IIB to Type IIA. Note that the period of the compact coordinate y is unaffected, and the information about the length of the T-duality circumference is encoded in the metric functions L, $\tilde L$. They are indeed related in such a way that their vacuum expectation values $2\pi R$, $2\pi\tilde R$ satisfy (3.4). Let us point out that no rule was given for $\tilde C_4$ as it is not independent, because of the self-duality constraint (2.26). From the vacuum expectation value of the relation $\tilde\varphi = \varphi$ we can immediately read off the transformation rule for the string coupling constant under T-duality,

$$g_s^{\rm IIB} = g_s^{\rm IIA}\,\frac{\ell_s}{2\pi R} \ . \tag{3.13}$$

T-duality is a perturbative duality in the sense that taking the limit $g_s \to 0$ on one side corresponds to taking the same limit on the dual side. The change in the string coupling constant ensures that the nine-dimensional gravitational coupling is the same in both dual descriptions.

So far we have discussed the action of T-duality on closed strings. As far as open strings are concerned, we can see that the left/right asymmetric reflection (3.6) interchanges Neumann and Dirichlet boundary conditions (2.17). As a result, a Dp-brane extending along the x^9 direction is turned into a D$(p-1)$-brane localized at a point along the T-dual coordinate $\tilde{x}^9$, and vice versa. Note that this is consistent with the action on RR forms in (3.12). The same result holds for a stack of N coincident Dp-branes. In this case the information about the relative location of the dual D$(p-1)$-branes along the $\tilde{x}^9$ direction is encoded, in the original Dp-brane picture, in a Wilson line. The latter is a topologically non-trivial constant VEV of the component A_9 of the non-Abelian gauge field living on the brane. More precisely, the i-th brane on the Dp-brane stack ($i = 1, \ldots, N$) is mapped to a D$(p-1)$-brane located at

$$\tilde{x}^9_i = -2\pi\alpha'(A_9)_{ii} \ , \qquad A_9 = -\frac{1}{2\pi R}\mathrm{diag}(\theta_1, \ldots, \theta_N) \ , \tag{3.14}$$

where no sum over i is performed, the real parameters θ_i are defined modulo 2π, and an arbitrary additive constant has been set to zero. It is also possible to consider more complicated setups with magnetized branes and branes at angles, but we will not need to develop such generalizations.

3.2 S-duality of Type IIB superstring theory

Type IIB string theory is invariant under a $\mathbb{Z}_2$ transformation that inverts the string coupling constant and thus maps a weakly coupled vacuum to a strongly coupled vacuum. This duality is called S-duality and is actually a subgroup of a larger invariance under the action of the group $SL(2,\mathbb{Z})$. A first hint towards this duality is furnished by a judicious reformulation of the Type IIB effective action, given by the sum of (2.22) and (2.24). Let us perform the metric redefinition

$$g^{(\mathrm{E})}_{\mu\nu} = e^{-\Phi/2}\, g_{\mu\nu} \tag{3.15}$$

and introduce the complex combinations

$$\tau = C_0 + ie^{-\Phi} \ , \qquad G_3 = F_3 - ie^{-\Phi}H_3 = dC_2 - \tau\, dB_2 \ . \tag{3.16}$$

The complex scalar τ is referred to as the axio-dilaton. With this notation the Type IIB effective action takes the form

$$S_{\mathrm{IIB}} = \frac{1}{2\kappa_{10}^2}\int_{\mathcal{M}_{10}} \left[R * 1 - \frac{1}{2}\frac{d\tau \wedge * d\overline{\tau}}{(\mathrm{Im}\tau)^2} - \frac{1}{2}\frac{G_3 \wedge * \overline{G}_3}{\mathrm{Im}\tau} - \frac{1}{4}F_5 \wedge * F_5 - \frac{i}{4\,\mathrm{Im}\tau} C_4 \wedge G_3 \wedge \overline{G}_3 \right] \ , \tag{3.17}$$

in which the Ricci scalar and the Hodge star are computed with the new metric $g^{(\mathrm{E})}_{\mu\nu}$, but we drop the superscript (E). This form of the action is referred to as action in the Einstein frame because the new metric has a canonical Einstein-Hilbert term. Note, however, that the prefactor contains the parameter κ_{10} and not the ten-dimensional physical gravitational constant κ given in (2.34), because the full dilaton field, background and fluctuations, is reabsorbed in the metric redefinition (3.15). Let us also point out that the self-duality constraint (2.26) is always understood. It takes the same form when written in string frame or Einstein frame metric.

The form (3.17) is useful as it makes more transparent the invariance of the Type IIB classical action under the action of $SL(2,\mathbb{R})$. More precisely, let us consider an $SL(2,\mathbb{R})$ matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} , \qquad ad - bc = 1 \ . \tag{3.18}$$

It acts non-trivially on the axio-dilaton τ and on the two-forms C_2, B_2 as

$$\tau' = \frac{a\,\tau + b}{c\,\tau + d} \ , \qquad \begin{pmatrix} C_2' \\ B_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \ , \tag{3.19}$$

while the Einstein frame metric and the RR four-form are inert.

Semiclassical arguments suggest that only the discrete subgroup $SL(2,\mathbb{Z})$ of the classical invariance group $SL(2,\mathbb{R})$ can be realized in the quantum theory. For example, a rotation of the two-form potentials with real coefficients as in (3.19) can be in conflict with semi-classical quantization of fluxes. This can be seen as follows. The non-linear sigma model action (2.19) contains the coupling

$$S_{\rm F1} \supset \frac{1}{2\pi\alpha'} \int_\Sigma B_2 \ , \tag{3.20}$$

where F1 stands for fundamental string and Σ denotes the world-sheet. In order for the path integral weight e^{iS} to be invariant under large gauge transformations one has to impose the flux quantization condition

$$\frac{1}{4\pi^2\alpha'} \int_{X_3} dB_2 = \ell_s^{-2} \int_{X_3} dB_2 \in \mathbb{Z} \ , \tag{3.21}$$

where X_3 is an arbitrary three-cycle in spacetime. By a similar token, the D1-brane action contains the coupling

$$S_{\rm D1} \supset \mu_1 \int_{\mathcal{W}_2} C_2 \ , \tag{3.22}$$

and therefore we have to require

$$\frac{\mu_1}{2\pi} \int_{X_3} dC_2 = \ell_s^{-2} \int_{X_3} dC_2 \in \mathbb{Z} \ , \tag{3.23}$$

where we have adopted the conventions (2.37) to express μ_1 in terms of ℓ_s. As we can see, if we use (2.37) and measure lengths in units of ℓ_s both B_2 and C_2 are integrally quantized. This property is preserved by the action of $SL(2,\mathbb{Z})$ but not by the action of $SL(2,\mathbb{R})$. The same conclusion is confirmed by the study of D-instantons, that break the classical shift symmetry $C_0 \to C_0 + b$ from $b \in \mathbb{R}$ to $b \in \mathbb{Z}$.

The group $SL(2,\mathbb{Z})$ is generated by the transformations

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \ , \qquad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \tag{3.24}$$

which act on τ as

$$\tau' = \tau + 1 \ , \qquad \tau' = -\frac{1}{\tau} \ , \tag{3.25}$$

respectively. The first transformation amounts to a shift in C_0 and does not alter the string coupling. The second transformation, however, acts non-trivially on the dilaton. Indeed, in a simple background with $C_0 = 0$ the second equation in (3.25) implies

$$g_s' = \frac{1}{g_s} \ . \tag{3.26}$$

Clearly, this is a weak/strong duality and cannot be checked within perturbation theory. Nonetheless, the full group $SL(2,\mathbb{Z})$, including the generator S, is believed to be an exact symmetry group of Type IIB superstring theory.

Evidence in favor of this claim comes, for instance, from the study of the tension of a fundamental string and a D1-brane in the Einstein frame. As noted above, the Einstein metric $g^{(E)}_{\mu\nu}$ depends on both the VEV and the fluctuations of the dilaton. When the latter are neglected, we have schematically

$$(\text{length})^{(\text{E})} = g_s^{-1/4}\,(\text{length}) \ , \tag{3.27}$$

where the length on the right hand side is computed with the original string frame metric. Using (3.27) together with (2.37) we then find by dimensional analysis

$$T^{(\text{E})}_{\text{F1}} = 2\pi\,\ell_s^{-2}\,g_s^{1/2} \ , \qquad \tau^{(\text{E})}_{\text{D1}} = 2\pi\,\ell_s^{-2}\,g_s^{-1/2} \ , \tag{3.28}$$

which are indeed exchanged under (3.26). This result is significant since both the fundamental string and the D1-brane are BPS objects of the theory, so that their tension is related to their charge and is therefore protected against quantum corrections. We can then safely extrapolate the weak-coupling expressions for their tensions to strong coupling. A similar analysis can be carried out for all extended BPS objects of Type IIB. For instance, a D5-brane is S-dual to an NS5-brane; their Einstein frame tensions are consistent,

$$\tau^{(\text{E})}_{\text{D5}} = 2\pi\,\ell_s^{-6}\,g_s^{1/2} \ , \qquad \tau^{(\text{E})}_{\text{NS5}} = 2\pi\,\ell_s^{-6}\,g_s^{-1/2} \ , \tag{3.29}$$

where we recalled the string frame NS5-brane tension (2.40). The D3-brane is expected to be self-dual, since it couples to C_4 which is a singlet under $SL(2,\mathbb{Z})$. Indeed, its Einstein frame tension is independent of g_s,

$$\tau^{(\text{E})}_{\text{D3}} = 2\pi\,\ell_s^{-4} \ . \tag{3.30}$$

The behavior of D7-branes under $SL(2,\mathbb{Z})$ will be discussed in detail in section 5.1.2.

In the weak coupling limit D-branes do not participate to the dynamics as they acquire infinite tension. For finite or large coupling, however, we can consider dynamical BPS objects that can be thought of as bound states of F1 and D1 strings. They are called (p,q) strings and couple electrically to B_2 with charge p and to C_2 with charge q. Thus a $(1,0)$ string is a fundamental string, and a $(0,1)$ string is a D1-brane. The expressions for the tensions of an F1 and a D1 in the Einstein frame (3.28) can be generalized to an $SL(2,\mathbb{Z})$ invariant expression

$$\tau^{(\text{E})}_{(p,q)} = \frac{|p+\tau q|}{\tau_2^{1/2}} 2\pi\ell_s^2 \ . \tag{3.31}$$

More precisely, this is invariant under a combined $SL(2,\mathbb{Z})$ action on τ as in (3.19) and on the charges (p,q). The transformation of the latter is given by

$$\begin{pmatrix} q' & p' \end{pmatrix} = \begin{pmatrix} q & p \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} , \tag{3.32}$$

and is such that the bilinear $p\int B_2 + q\int C_2$ is invariant. Using (3.31) one can show that a (p,q)-string is stable against decay provided that p and q are relatively prime.

Since a fundamental string can end on a D1-brane one expects to find (p,q) strings configurations with several prongs. They are referred to as string junctions. Charge conservation demands that $\sum p = \sum q = 0$, where the sum extends over all prongs of the string junction. We will comment briefly on the role of string junctions in F-theory in section 5.2.3.

Suppose we start from a weakly coupled setup with fundamental strings and ordinary D-branes. We can perform an $SL(2,\mathbb{Z})$ transformation (3.32) that turns the fundamental string into a (p,q) string. What happens to D-branes under this transformation? By electric-magnetic duality, D5-branes are mixed with NS5-branes into (p,q)-five-branes. On the contrary, since D3-branes are self-dual under $SL(2,\mathbb{Z})$ they are unaffected. A D7-brane couples magnetically to $C_0 = \mathrm{Re}\,\tau$, which transforms non-trivially under $SL(2,\mathbb{Z})$, see (3.19). We then expect to find (p,q) seven-branes, but their analysis is more complicated and relies on the study of their backreaction on the geometry, which will be addressed in section 5.1. We also expect that a (p,q) string can end on a (p,q) five-brane or seven-brane with the same p, q, but on any D3-brane.

3.3 M-theory and Type IIA superstring theory

Inspection of (2.34) reveals that the physical tension τ_p of a Dp-brane is inversely proportional to the string coupling constant. As a result, if we start from the perturbative regime $g_s \ll 1$ and we increase the coupling, Dp-brane states become lighter and lighter and their dynamics starts to intertwine with that of fundamental strings. As we have seen in the previous section, in Type IIB S-duality gives us a way to describe the dynamics of D-branes at strong coupling. The situation is different in Type IIA: its strongly coupled dynamics is captured by a new eleven-dimensional theory, called M-theory.

The emergence of an eleventh dimension can be seen from the spectrum of D0-branes of Type IIA. Being the D-branes with the lowest dimensionality in Type IIA, they are the lightest ones. Working in the string frame and using (2.37), the D0-brane tension—or rather mass—is given by

$$\tau_{\mathrm{D0}} = \frac{2\pi}{\ell_s\, g_s} . \tag{3.33}$$

Compatibly with their BPS nature, D0-branes can form bound states whose energy is just the sum of the energy of the constituents, since gravitational and RR interactions balance against each other. A bound state of n D0-branes has therefore mass $n\,\tau_{\mathrm{D0}}$. In summary, we have a tower of equally spaced massive states that become light as the string coupling constant increases. As we will review in section 4.2, this can be interpreted as a Kaluza-Klein spectrum of a circle compactification and thus constitutes a first hint of a hidden direction of spacetime. Below we will review further evidence in support of M-theory.

A full formulation of M-theory at the quantum level is not known. Arguably, the best proposal for a microscopic description of its fundamental degrees of freedom is based on the matrix theory that describes a stack of D0-branes in a suitable kinematic limit [59, 60, 61]. Whatever the correct description of M-theory at the fundamental level may be, its low-energy dynamics must be captured by the unique eleven-dimensional supergravity theory. Its field content consists of the metric $\hat{g}_{\hat{\mu}\hat{\nu}}$, a three-form potential $\hat{C}_3$, and a Majorana gravitino $\hat{\Psi}_{\hat{\mu}}$. A hat is used to denote eleven-dimensional quantities and spacetime indices. The bosonic part of the action of eleven-dimensional supergravity reads

$$S_{\rm M} = \frac{1}{2\kappa_{11}^2} \int_{\mathcal{M}_{11}} \left[\hat{R} \,\hat{*}\, 1 - \frac{1}{2} \hat{G}_4 \wedge \hat{*} \hat{G}_4 - \frac{1}{6} \hat{C}_3 \wedge \hat{G}_4 \wedge \hat{G}_4 \right] \ , \tag{3.34}$$

where $\hat{G}_4 = d\hat{C}_3$ is the field strength of the three-form potential and κ_{11} is the eleven-dimensional gravitational constant. It is useful to introduce the associated fundamental length scale by the relation

$$\frac{1}{2\kappa_{11}^2} = \frac{2\pi}{\ell_{\rm M}^9} \ , \tag{3.35}$$

where the normalization is chosen for later convenience.

The interpretation of D0-brane bound states in terms of Kaluza-Klein modes suggests that M-theory compactified on a small circle should reproduce weakly coupled Type IIA string theory. Indeed, the eleven-dimensional low-energy effective action (3.34) reproduces the Type IIA supergravity action upon dimensional reduction on a circle. Let us review in some detail the relation between eleven-dimensional and ten-dimensional quantities. As our starting point we take the following Ansatz for the eleven-dimensional metric and three-form potential,

$$\begin{aligned} d\hat{s}^2 &= \lambda^{\frac{2}{3}}\, e^{-\frac{2}{3}\Phi}\, ds^2 + \mu^2\, \lambda^{-\frac{4}{3}}\, e^{\frac{4}{3}\Phi} (\ell_{\rm M}\, dx + \mu^{-1}\, \lambda\, C_1)^2 \ , \qquad x \sim x+1 \ , \\ \hat{C}_3 &= \lambda\, C_3 + \mu\, B_2 \wedge \ell_{\rm M}\, dx \ , \end{aligned} \tag{3.36}$$

where the adimensional coordinate x parametrizes the circle direction and $\lambda, \mu > 0$ are dimensionless constants. The scalar Φ, the ten-dimensional metric ds^2, the forms C_1, C_3, B_2 do not depend on x and are identified with the Type IIA dilaton, string frame metric, RR forms and NSNS two-form, respectively. The form of the Ansatz (3.36) is engineered is such a way to exactly reproduce the sum of the NSNS effective action (2.22) and the RR effective action (2.23). In particular, the parameters λ, μ enter the dimensionally reduced action only in the overall prefactor, which reads

$$\frac{1}{2\kappa_{10}^2} = \frac{2\pi}{\ell_{\rm M}^8}\, \lambda^2\, \mu \ . \tag{3.37}$$

In order to reproduce the correct normalization for the dilaton, in such a way that (2.38) holds, this prefactor must be matched with (2.37). As a result, we obtain a first relation between ℓ_s, $\ell_{\rm M}$ and the parameters λ, μ,

$$\ell_s = \ell_{\rm M}\, \lambda^{-\frac{1}{4}} \mu^{-\frac{1}{8}} \ . \tag{3.38}$$

Another relation can be extracted by comparing the Kaluza-Klein mass and the D0-tension. The former can be read off from the vacuum expectation value of the eleven-dimensional metric in (3.36),

$$\langle d\hat{s}^2 \rangle = \lambda^{\frac{2}{3}} g_s^{-\frac{2}{3}} \left\{ \langle ds^2 \rangle + (\lambda^{-1} \mu\, g_s\, \ell_{\rm M})^2 dx^2 \right\} \ , \tag{3.39}$$

where we made use of $e^{\langle \Phi \rangle} = g_s$. From (3.39) one can conclude that the Kaluza-Klein masses measured in the string frame metric are an integer multiple of

$$m_{\rm KK} = \frac{2\pi}{\lambda^{-1} \mu\, g_s\, \ell_{\rm M}} \ . \tag{3.40}$$

Kaluza-Klein masses will de discussed in more detail in section 4.2. Comparison with (3.33) yields $\ell_s = \ell_{\rm M}\, \lambda^{-1} \mu$ and therefore consistency with (3.38) imposes

$$\mu = \lambda^{\frac{2}{3}} \ . \tag{3.41}$$

We are thus left with one free parameter only, identified with the ratio between the string length and eleven-dimensional fundamental length,

$$\ell_s = \ell_{\rm M}\, \lambda^{-\frac{1}{3}} \ . \tag{3.42}$$

Note, however, that this parameter drops out from the expression of the circumference $L_{\rm M}$ of the circle measured by the background eleven-dimensional metric (3.39),

$$L_{\rm M} = \ell_{\rm M}\, g_s^{\frac{2}{3}} \ . \tag{3.43}$$

This relation shows that the string coupling constant is unambiguously determined by the compactification geometry. As a final comment on the match between eleven-dimensional and ten-dimensional bulk effective actions let us point out that the Ansatz (3.36) takes a transparent form if we trade the parameter λ for the ratio $\ell_s/\ell_{\rm M}$. In fact, we obtain

$$\begin{aligned} \ell_{\rm M}^{-2}\, d\hat{s}^2 &= e^{-\frac{2}{3}\Phi}\, \ell_s^{-2}\, ds^2 + e^{\frac{4}{3}\Phi} (dx + \ell_s^{-1}\, C_1)^2 \ , \quad x \sim x + 1 \ , \\ \ell_{\rm M}^{-3}\, \hat{C}_3 &= \ell_s^{-3}\, C_3 + \ell_s^{-2}\, B_2 \wedge dx \ . \end{aligned} \tag{3.44}$$

This shows that eleven-dimensional and ten-dimensional quantities are matched naturally if the former are measured in units of $\ell_{\rm M}$ and the latter in units of ℓ_s. By the same token, if the eleven-dimensional flux $\hat{G}_4$ is quantized in units of $\ell_{\rm M}$—or rather half-integrally quantized [62]—then Type IIA fluxes are quantized in units of ℓ_s.

The eleven-dimensional supergravity action (3.34) admits half-BPS solutions that describe a membrane and a five-brane, see for instance the review [63]. The membrane acts as an electric source for the three-form potential $\hat{C}_3$, while the five-brane is a soliton with non-trivial magnetic flux $\hat{G}_4$. In view of the interpretation of eleven-dimensional supergravity as the low-energy limit of M-theory, we are led to conclude that two kind of extended objects exist in M-theory, called M2-brane and M5-brane. Our current understanding of M-theory is not sufficient to determine if M2-branes of M5-branes are better understood as fundamental objects or as emergent excitations of some different microscopic constituent. Nonetheless, we can elucidate the role played by M-theory branes in the duality with Type IIA string theory.

We have already argued that Type IIA D0-branes should be identified with excited Kaluza-Klein modes of the eleven-dimensional graviton. The fundamental string F1 and the D2-brane are instead uplifted to an M2-brane. More precisely, an M2-brane wrapping the circle yields an F1, while an unwrapped M2-brane corresponds to a D2-brane. This picture is confirmed by the analysis of the tension of these objects. The complete action for the embedding of a supermembrane in eleven-dimensional spacetime is known [64] but for our purposes it suffices to consider the schematic action

$$S_{\mathrm{M2}} = -T_{\mathrm{M2}} \int_{\mathcal{W}_3} d^3\xi \sqrt{-\hat{\gamma}_3} \ , \tag{3.45}$$

where $\hat{\gamma}$ denotes the pullback of the eleven-dimensional metric to the world-volume $\mathcal{W}_3$ of the M2-brane, parametrized by coordinates ξ, and T_{M2} is the M2-brane tension. If the action (3.45) is reduced with one or no legs along the circle according to the Ansatz (3.44), it should reproduce the action for the embedding of an F1 or D2-brane in ten dimensions. Indeed, one finds

$$\begin{aligned} S_{\mathrm{M2}}^{\mathrm{wrapped}} &= -T_{\mathrm{M2}}\, \ell_{\mathrm{M}}^3\, \ell_s^{-2} \int_{\Sigma_2} d^2\xi \sqrt{-\gamma_2} \ , \\ S_{\mathrm{M2}}^{\mathrm{unwrapped}} &= -T_{\mathrm{M2}}\, \ell_{\mathrm{M}}^3\, \ell_s^{-3} \int_{\mathcal{W}_3} d^3\xi \, e^{-\Phi} \sqrt{-\gamma_3} \ , \end{aligned} \tag{3.46}$$

where γ_2, γ_3 denote the appropriate pull-back of the string frame metric. We can see that the Nambu-Goto action for F1 and the DBI action for D2-brane are correctly reproduced. The prefactors should be matched with the string and brane tensions read off from (2.34), (2.37),

$$T_{\mathrm{F1}} = 2\pi \ell_s^{-2} \overset{!}{=} T_{M2}\, \ell_{\mathrm{M}}^3\, \ell_s^{-2} \ , \qquad \tau_{\mathrm{D2}} = 2\pi \ell_s^{-3} g_s^{-1} \overset{!}{=} T_{M2}\, \ell_{\mathrm{M}}^3\, \ell_s^{-3}\, e^{-\langle \Phi \rangle} \ , \tag{3.47}$$

and indeed both these conditions can be simultaneously met if

$$T_{\mathrm{M2}} = 2\pi \ell_{\mathrm{M}}^{-3} \ . \tag{3.48}$$

Note that this holds irrespectively of the value of the ratio ℓ_s/ℓ_{M}, or equivalently of the parameter λ. In a completely analogous way the NS5-brane and the D4-brane are uplifted to the M5-brane. To crude approximation, the embedding of a super five-brane is described by the action

$$S_{\mathrm{M5}} = -T_{\mathrm{M5}} \int_{\mathcal{W}_6} d^6\xi \sqrt{-\hat{\gamma}_6} \ , \tag{3.49}$$

and its schematic dimensional reduction yields

$$\begin{aligned} S_{\mathrm{M5}}^{\mathrm{wrapped}} &= -T_{\mathrm{M5}}\, \ell_{\mathrm{M}}^6\, \ell_s^{-5} \int_{\mathcal{W}_5} d^5\xi \, e^{-\Phi} \sqrt{-\gamma_5} \ , \\ S_{\mathrm{M5}}^{\mathrm{unwrapped}} &= -T_{\mathrm{M5}}\, \ell_{\mathrm{M}}^6\, \ell_s^{-6} \int_{\mathcal{W}_6} d^6\xi \, e^{-2\Phi} \sqrt{-\gamma_6} \ . \end{aligned} \tag{3.50}$$

These results must be compatible with the D4-brane and NS5-brane tensions given in (2.34), (2.37), (2.40). This is indeed the case, since both conditions

$$\tau_{\mathrm{D4}} = 2\pi\, \ell_s^{-5}\, g_s^{-1} \overset{!}{=} T_{\mathrm{M5}}\, \ell_{\mathrm{M}}^6\, \ell_s^{-5}\, e^{-\langle \Phi \rangle} \ , \qquad \tau_{\mathrm{NS5}} = 2\pi\, \ell_s^{-6}\, g_s^{-2} \overset{!}{=} T_{\mathrm{M5}}\, \ell_{\mathrm{M}}^6\, \ell_s^{-6}\, e^{-2\langle \Phi \rangle} \tag{3.51}$$

hold provided that the M5-brane tension is furnished by

$$T_{\rm M5} = 2\pi\,\ell_{\rm M}^{-6}\ . \tag{3.52}$$

The M-theory uplift of the Type IIA D6-brane does not involve M2-branes or M5-branes. This can be expected by noting that a D6-brane couples magnetically to C_1, which is identified with the Kaluza-Klein vector of the circle compactification. In fact the D6-brane is uplifted to a Kaluza-Klein monopole, i.e. a regular solitonic solution of vacuum Einstein's equations in eleven dimensions in which the circle that connects M-theory and Type IIA is non-trivially fibered over ten-dimensional spacetime. More precisely, the Kaluza-Klein monopole metric takes the form

$$ds^2_{\text{KK-mon}} = \eta_{ij} dx^i_{\|}\, dx^j_{\|} + ds^2_{\text{Taub-NUT}}\ , \tag{3.53}$$

where $x^i_{\|}$, $i = 0,\dots,6$ are coordinates along the flat, seven-dimensional world-volume of the D6-brane and the four transverse directions are described by the Taub-NUT metric $ds^2_{\text{Taub-NUT}}$. The latter is a four-dimensional $U(1)$ fibration over $\mathbb{R}^3$ given by

$$ds^2_{\text{Taub-NUT}} = V(\vec{x})\, d\vec{x}\cdot d\vec{x} + V(\vec{x})^{-1}(dx_{\rm M} + \vec{A}\cdot\vec{x})^2\ . \tag{3.54}$$

In this expression $\vec{x}$ are flat coordinates in $\mathbb{R}^3$ with standard inner product and $x_{\rm M}$ is the compact coordinate that parametrizes the M-theory circle. A metric of the form (3.54) solves Einstein's vacuum equations provided that the scalar function $V(\vec{x})$ and the $U(1)$ vector potential $\vec{A} = \vec{A}(\vec{x})$ are chosen in such a way that

$$\vec{\nabla}\times\vec{A} = -\vec{\nabla}V\ , \tag{3.55}$$

so that V is harmonic on $\mathbb{R}^3$. A single D6-brane is lifted to a single-center Taub-NUT metric,

$$V(\vec{x}) = 1 + \frac{R}{2|\vec{x}|}\ , \tag{3.56}$$

but it is also possible to lift a stack of parallel D6-branes arbitrarily displaced along the transverse direction by means of a multi-center Taub-NUT metric,

$$V(\vec{x}) = 1 + \frac{R}{2}\sum_a \frac{1}{|\vec{x}-\vec{x}_{(a)}|}\ , \tag{3.57}$$

with the index a labeling branes. For the choices (3.56) or (3.57) for V, the vector potential $\vec{A}$ describes one or more magnetic monopoles and it cannot globally defined on $\mathbb{R}^3$. This signals the non-triviality of the $U(1)$ fibration. Let us point out that the length scale R in the previous equations is related to the radius of the M-theory circle. In fact, regularity of the metric imposes that $x_{\rm M}$ has period $2\pi R$ if V is given by (3.56) or (3.57). Note, however, that the actual radius of the M-theory circle measured by the Taub-NUT metric is $V^{-1/2}R$: it approaches the asymptotic value R as $|\vec{x}|\to\infty$ and it goes to zero at the locations $\vec{x}_{(a)}$ of the branes, see figure 3.1.

An interesting configuration is furnished by a periodic array of centers along some fixed direction in $\mathbb{R}^3$. Let us write $\vec{x} = (x_1, x_2, x_3)$ and consider the x_3 direction for definiteness. An infinite array of D6-branes along x_3 can be equivalently interpreted as a single D6-brane localized at a point along a

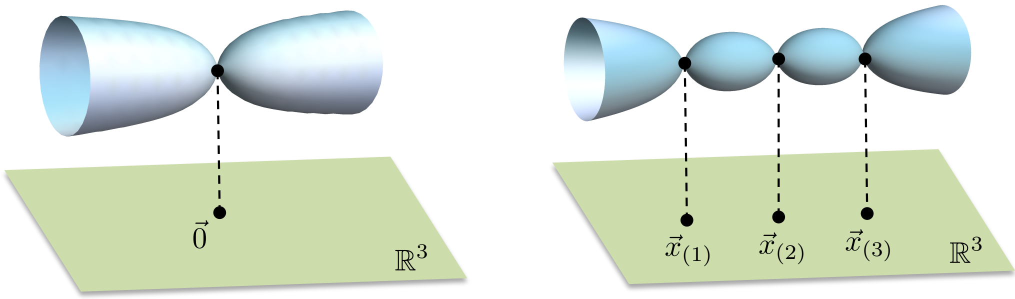


Figure 3.1: Schematic representation of a single-center and of a multi-center Taub-NUT geometry. The radius of the S^1 fiber varying over the base space $\mathbb{R}^3$ asymptotes to a fixed value for $|\vec{x}| \to \infty$ and goes to zero at the location of the centers of the Taub-NUT space. Note the presence of non-trivial two-cycles in the case of a multi-center geometry.

circle extending in the x_3 direction. This picture will be useful in clarifying the M-theory interpretation of Type IIB D7-branes in section 5.3.

Let us close this section with a remark about the free parameter λ. In order to fix its value and therefore determine the string length in terms of the fundamental length of M-theory we need a prescription for the relation between lengths in ten dimensions and in eleven dimensions. In the present context of M-theory on a circle a natural prescription consists in the requirement that distances measured by the background eleven-dimensional metric coincide with distances measured by the background string frame metric. Equivalently, the prefactor in (3.39) should be equal to one. This gives $\lambda = g_s$, so that

$$\ell_s = \ell_{\rm M}\, g_s^{-\frac{1}{3}} \ . \tag{3.58}$$

In the next section we will find it convenient to adopt a different prescription to relate ℓ_s and $\ell_{\rm M}$.

3.4 M-theory and Type IIB superstring theory

We know from the previous section that M-theory on a circle gives ten-dimensional Type IIA string theory. If we compactify on a further circle we obtain a nine-dimensional theory that can be equivalently though of as Type IIB on the dual circle, by means of T-duality, see section 3.1. We thus get that M-theory on a torus is dual to Type IIB on a circle. This duality furnishes a geometric interpretation of the $SL(2,\mathbb{Z})$ invariance of Type IIB string theory in terms of the invariance of M-theory under large diffeomorphisms of the compactification torus. Let us explore in more detail how this works.

An appropriate Ansatz for dimensional reduction of M-theory on a two-torus is

$$\begin{aligned} d\hat{s}^2 &= ds^2_{9({\rm M})} + \frac{A\,\ell_{\rm M}^2}{{\rm Im}\,\tau_{\rm M}} \left| dx + \ell_{\rm M}^{-1} V_{(x)} + \tau_{\rm M}\,(dy + \ell_{\rm M}^{-1} V_{(y)}) \right|^2 \ , \\ \hat{C}_3 &= C_3^{({\rm M})} + C_2^{(x)} \wedge \ell_{\rm M}\, dx + C_2^{(y)} \wedge \ell_{\rm M}\, dy + C_1^{(xy)} \wedge \ell_{\rm M}\, dx \wedge \ell_{\rm M}\, dy \ , \end{aligned} \tag{3.59}$$

where x, y are dimensionless coordinates with period 1, $\tau_{\rm M}$ is the complex structure parameter of the torus, A is its area in units of $\ell_{\rm M}$. We have decomposed the M-theory three-form in its components with zero, one, or two legs along the torus and we have also included the Kaluza-Klein vectors $V_{(x)}$, $V_{(y)}$ associated to the x- and y-cycles of the torus. Let us consider the coordinate transformation

$$\begin{pmatrix} y' & x' \end{pmatrix} = \begin{pmatrix} y & x \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} , \qquad a, b, c, d \in \mathbb{Z} \ , \qquad ad - bc = 1 \ , \tag{3.60}$$

in which we have adopted a standard matrix notation. The integrality of coefficients is required to ensure compatibility with the integral periods of y, x and the unimodular condition guarantees that the volume form $dy \wedge dx$ is preserved. If the $SL(2,\mathbb{Z})$ matrix determined by the coefficients a, b, c, d is non-trivial, this transformation describes a large diffeomorphism of the torus, since it cannot be continuously deformed to the identity map. The reduction Ansatz (3.59) can be made invariant under (3.60) provided that the nine-dimensional fields on the right hand side transform as

$$\tau_{\rm M}' = \frac{a\,\tau_{\rm M} + b}{c\,\tau_{\rm M} + d} \ , \quad \begin{pmatrix} V_{(y)}' & V_{(x)}' \end{pmatrix} = \begin{pmatrix} V_{(y)} & V_{(x)} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} , \quad \begin{pmatrix} C_2^{(y)\prime} \\ C_2^{(x)\prime} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C_2^{(y)} \\ C_2^{(x)} \end{pmatrix} . \tag{3.61}$$

Note that $C_1^{(xy)}$ does not transform thanks to the unimodular condition.

On the other side of the duality we have Type IIB compactified on a circle. Recall from section 3.2 that the S-duality properties of Type IIB are most conveniently studied in terms of the Einstein frame metric (3.15). Therefore, we compactify Type IIB on a circle with Ansatz

$$\begin{aligned}
& ds^2_{\rm E} = ds^2_{9({\rm B})} + L^2_{({\rm B})}\,(dx_{\rm B} + \ell_s^{-1}\,V_{({\rm B})})^2 \ , \\
& B_2 = B_2^{({\rm B})} + B_1^{({\rm B})} \wedge \ell_s\,dx_{\rm B} \ , \\
& C_2 = C_2^{({\rm B})} + C_1^{({\rm B})} \wedge \ell_s\,dx_{\rm B} \ , \qquad C_4 = C_4^{({\rm B})} + C_3^{({\rm B})} \wedge \ell_s\,dx_{\rm B} \ ,
\end{aligned} \tag{3.62}$$

where $ds^2_{\rm E}$ is the Einstein frame metric in ten dimensions. In these expressions $x_{\rm B}$ is a dimensionless coordinate with period 1 that parametrizes the circle upon which Type IIB is compactified and $V_{({\rm B})}$ denotes the corresponding Kaluza-Klein vector. Let us stress that the components $C_4^{({\rm B})}$ and $C_3^{({\rm B})}$ of the RR four-form are not independent because of the self-duality constraint (2.26). In what follows, we will focus on $C_3^{({\rm B})}$ only. Finally, the Type IIB dilaton $\Phi_{\rm IIB}$ and axion C_0 behave trivially under circle compactification.

The x-cycle in the M-theory Ansatz (3.59) is identified with the circle that connects M-theory to Type IIA. Correspondingly, the y-cycle is identified with the circle upon which Type IIA is further compactified and T-dualized to Type IIB, so that we have $x_{\rm B} = y$. Combining the results of the previous section with the transformations of Type IIA fields into Type IIB fields under T-duality (3.12) we can establish the precise map between the quantities that appear on the right hand sides of (3.59) and those that enter the right hand sides of (3.62). First of all, we obtain

$$C_0 = {\rm Re}\,\tau_{\rm M} \ , \qquad e^{\Phi_{\rm IIB}} = \frac{1}{{\rm Im}\,\tau_{\rm M}} \ , \tag{3.63}$$

which shows that the complex structure parameter of the M-theory torus is identified with the Type IIB axio-dilaton defined in (3.16). Correspondingly, the $SL(2,\mathbb{Z})$ transformation (3.19) is identified with the geometric action (3.61) on $\tau_{\rm M}$. Secondly, the Type IIB Einstein frame metric components are determined by

$$ds^2_{9({\rm B})} = \frac{\ell_s^2\, A^{\frac12}}{\ell_{\rm M}^2} ds^2_{9({\rm M})} \ , \qquad L_{({\rm B})} = \ell_s\, A^{-\frac34} \ , \qquad \ell_s^{-1}\, V_{({\rm B})} = \ell_{\rm M}^{-1}\, C_1^{(xy)} \ . \tag{3.64}$$

These expression are $SL(2,\mathbb{Z})$ invariant provided that the string length ℓ_s is related to the M-theory length $\ell_{\rm M}$ by means of an $SL(2,\mathbb{Z})$ invariant prescription. The relation (3.58) used in the previous section is not suitable in the present context, since it would imply

$$\ell_s \stackrel{?}{=} \ell_{\rm M}\, \langle A\rangle^{-\frac14}\, ({\rm Im}\langle\tau\rangle)^{\frac14} \ , \tag{3.65}$$

where as usual $\langle ...\rangle$ denotes the vacuum expectation value. Equation (3.64) suggests a natural alternative,

$$\ell_s = \ell_{\rm M}\, \langle A\rangle^{-\frac14} \ . \tag{3.66}$$

This condition ensures that distances measured by the background value of the Type IIB Einstein frame metric coincide with distances measured by the background metric in eleven dimensions. Finally, let us record the match between Type IIB forms and M-theory forms,

$$\begin{aligned} &\ell_s^{-1} \begin{pmatrix} C_1^{({\rm B})} \\ B_1^{({\rm B})} \end{pmatrix} = \ell_{\rm M}^{-1} \begin{pmatrix} V_{(x)} \\ -V_{(y)} \end{pmatrix} \ , \qquad \ell_s^{-2} \begin{pmatrix} C_2^{({\rm B})} \\ B_2^{({\rm B})} \end{pmatrix} = \ell_{\rm M}^{-2} \begin{pmatrix} C_2^{(y)} \\ C_2^{(x)} \end{pmatrix} + \ell_{\rm M}^{-2} \begin{pmatrix} V_{(x)} \\ -V_{(y)} \end{pmatrix} \wedge C_1^{(xy)} \ , \\ &\ell_s^{-3}\, C_3^{({\rm B})} = \ell_{\rm M}^{-3}\, C_3^{({\rm M})} - \frac12\, \ell_{\rm M}^{-3} \begin{pmatrix} V_{(y)} & V_{(x)} \end{pmatrix} \begin{pmatrix} C_2^{(y)} \\ C_2^{(x)} \end{pmatrix} \ . \end{aligned} \tag{3.67}$$

The same matrix notation used before has been adopted here in order to make the connection between S-duality transformations in Type IIB (3.19) and large diffeomorphisms in M-theory (3.61) transparent. Note, in fact, that the column vector $(V_{(x)}\ \ -V_{(y)})^{\rm T}$ transforms in the same way as $(C_2^{(y)}\ C_2^{(x)})^{\rm T}$, as can be easily checked from (3.61).

3.5 Low-energy dynamics of M-theory branes

No complete microscopic description of the M2-brane or the M5-brane is available at the quantum level. Nonetheless, one can address the problem of the determination of the low-energy dynamics of the massless fields localized along their world-volumes. In the case of D-branes in string theory, massless fields on the branes can be read off from the open string spectrum. For M2-branes and M5-branes we have to rely on a different, indirect approach: massless fields on their world-volumes are identified as Goldstone bosons of spontaneously broken large diffeomorphisms and gauge transformations of the membrane and five-brane supergravity solutions [65]. For a single M2-brane we find eight scalar fields, corresponding to the eight transverse directions to the brane, together with their fermionic partners in a multiplet of three-dimensional $\mathcal{N}=8$ supersymmetry (16 supercharges). For a single M5-brane, the five scalars associated with the transverse directions are accompanied by a self-dual two-form and fermionic partners into a tensor multiplet of (2,0) supersymmetry (16 supercharges).

For a stack consisting of one brane only the effective action is uninteresting, since it describes the free dynamics of the scalars that encode the position of the center of mass of the brane and their superpartners. If several branes are on top of each other, however, non-trivial interactions take place and the resulting theory is believed to be strongly coupled, since we can regard M2-branes and M5-branes as the strong-coupling limit of D2-branes and D4-branes is Type IIA, respectively. Information about the degrees of freedom of these interacting theories can be extracted indirectly, for instance by analyzing the thermal properties of brane solutions in supergravity, the cross section for absorption of a graviton, or anomaly cancellation in the case of the M5-brane. See e.g. [66] for a review. These computations show that the number of degrees of freedom of a stack of N coincident branes grows like $N^{3/2}$ for M2-branes and N^3 for M5-branes. For ordinary D-branes the scaling is N^2 and is readily explained in terms of the dimension of the $U(N)$ gauge group on their world-volume.

It is interesting to consider the limit in which gravity is decoupled from the stack of M2-branes or M5-branes. One then gets a three- or six-dimensional quantum field theory that flows in the IR to a non-trivial superconformal fixed point with 16 supersymmetries. Effective actions are available that capture the low-energy dynamics of a stack of M2-branes. It can be described by the ABJM model [67], a matter-coupled three-dimensional Chern-Simons theory. The matter fields include the scalars that parametrize the directions orthogonal to the world-volume of the brane, while the gauge vectors are non-dynamical. The theory has one free coupling constant, the Chern-Simons level k, and is weakly coupled for large k. From an M-theory perspective, k is not a parameter of the world-volume theory of the M2-branes, but of the eleven-dimensional background in which they live. Indeed, the theory at level k describes a stack of M2-branes on top of an orbifold singularity $\mathbb{C}^4/\mathbb{Z}_k$ in the eight-dimensional transverse space. Therefore, the case of M2-branes in flat spacetime corresponds to the strongly coupled regime $k = 1$. Let us also recall that in the case of a stack consisting of two M2-branes the ABJM model has additional symmetries and is equivalent to the Bagger-Lambert-Gustavsson theory [68, 69, 70, 71, 72]. The latter is based on a non-standard gauge symmetry formulated in terms of three-algebras rather than Lie algebras.

The superconformal theory living on the world-volume of a stack of M5-branes remains largely mysterious. It will be considered in more detail in chapter 6 as it constitutes one of the main motivations for the study of tensor theories in six and five dimensions.

CHAPTER 4

Compactifications and effective actions

In this chapter we review some useful notions about compactification of higher-dimensional supergravity theories and the resulting lower-dimensional effective action. We focus in particular on the cases of compactification on a circle and on Ricci-flat manifolds with special holonomy. The latter are motivated by supersymmetry considerations. After some general results, we present an overview of the classes of Ricci-flat manifolds we will be studying in more detail in the following chapters: Calabi-Yau manifolds and manifolds with Spin(7) holonomy.

4.1 The compactification paradigm

As we learn from general relativity, in any theory that captures gravitational degrees of freedom the geometry of spacetime is not fixed *a priori* but is rather determined dynamically. This allows for the possibility that a D-dimensional theory admits vacua whose low-energy physics is effectively d-dimensional, with $d < D$. One of the simplest ways to achieve this is to consider a spontaneous compactification scenario: the full D-dimensional theory has a solution that describes the product of a non-compact d-dimensional spacetime and a compact k-dimensional space, with $D = d + k$. We will refer to the former as external space and the latter as internal space. In such a scenario the dynamics of field fluctuations around the vacuum comes associated with an energy scale given by the inverse of the typical length scale of the internal space ℓ_{int}. Physical processes with energies much lower than ℓ_{int}^{-1} cannot excite fluctuations in the internal space and are therefore not able to probe all D dimensions of spacetime, but rather only the d non-compact directions.

This observation is particularly relevant in the context of string theory and M-theory, since compactification appears as a natural way to extract lower-dimensional physics from ten or eleven dimensions. On the one hand, this is clearly necessary in order to discuss phenomenological implications of string theory. On the other hand, it can also provide powerful insights on more formal aspects of quantum field theory. For instance, Type IIB compactifications to six dimensions shed some light on the nature of interacting superconformal $(2,0)$ theories.

In the upcoming sections we examine in detail two compactification scenarios that play a crucial role in the following chapters: circle compactifications and compactifications on Ricci-flat manifolds with special holonomy. Let us stress that we address the compactification problem from a purely field-theoretical perspective: effects due to the extended nature of strings or membranes will be neglected and our starting point consists of the low-energy supergravity actions discussed in sections 2.2, 3.3. This approximation is self-consistent provided that a hierarchy of scales exists

$$E \ll \ell_{\rm int}^{-1} \ll \ell_s^{-1} \qquad \text{or} \qquad E \ll \ell_{\rm int}^{-1} \ll \ell_{\rm M}^{-1} \tag{4.1}$$

among the typical energy scale E under consideration, the characteristic size of the internal space $\ell_{\rm int}$, and the string length ℓ_s or the M-theory fundamental length $\ell_{\rm M}$.

4.2 Circle compactification

The simplest example of compactification is obtained by choosing a one-dimensional internal space with the topology of a circle. This setup was originally considered by Klein [73] in relation to Kaluza's proposal [74] of unifying gravity and electromagnetism in four dimensions by means of pure gravity in five dimensions. For our discussion it is convenient to keep the dimensionality of the external space arbitrary. We then start with a theory in $D = d+1$ dimensions and we consider it on a background $\mathcal{M}_{d+1} = \mathcal{M}_d \times S^1$ where $\mathcal{M}_d$ is d-dimensional Minkowski spacetime. The background metric reads

$$\langle \hat g_{\hat\mu\hat\nu}(\hat x)\rangle d\hat x^{\hat\mu} d\hat x^{\hat\nu} = \eta_{\mu\nu}\, dx^\mu dx^\nu + R^2\, dy^2 \ , \qquad y \sim y + 2\pi \ , \tag{4.2}$$

where $\hat x^{\hat\mu}$, $\hat\mu = 0,\dots,d$, are $(d{+}1)$-dimensional coordinates that split into the d-dimensional coordinates x^μ, $\mu = 0,\dots d-1$, and the coordinate y parametrizing the circle. The constant R has dimensions of length: $2\pi R$ is the circumference of the compactification circle measured by the $(d+1)$-dimensional metric.

4.2.1 Mode expansions and dimensional reduction of gravity

In the original $(d+1)$-dimensional theory fields depend on both the external coordinates x^μ and the internal coordinate y. If we consider bosonic fields, consistency with the background (4.2) requires periodicity in y with period 2π.[1] As a result, $(d+1)$-dimensional fields can be written in a Fourier expansion with x-dependent coefficients. This is most clearly exemplified by the simple case of a real scalar $\hat\phi$ in $d+1$ dimensions,

$$\hat\phi(x,y) = \sum_{n\in\mathbb{Z}} \phi^{(n)}(x)\, e^{iny} \ . \tag{4.3}$$

[1] For fermions antiperiodicity is a viable option. More generally, the periodicity conditions can be twisted by the action of a generator of a global symmetry in the original $(d+1)$-dimensional theory, yielding generalized circle reductions à la Scherk-Schwarz [75, 76, 77]. Our discussion is restricted to compactifications without any twist.

The Fourier coefficients are interpreted as d-dimensional scalar fields and are referred to as Kaluza-Klein modes of $\hat{\phi}$. They satisfy the reality condition $\bar{\phi}^{(n)} \equiv (\phi^{(n)})^* = \phi^{(-n)}$. If we suppose that the dynamics of $\hat{\phi}$ in $d+1$ dimensions is governed by the massless Klein-Gordon equation in the background (4.2)

$$\langle \hat{g}^{\hat{\mu}\hat{\nu}} \rangle \partial_{\hat{\mu}} \partial_{\hat{\nu}} \hat{\phi} = 0 \ , \tag{4.4}$$

the d-dimensional dynamics of Kaluza-Klein modes is diagonal in the level n and is described by

$$\eta^{\mu\nu} \partial_\mu \partial_\nu \phi^{(n)} - \frac{n^2}{R^2} \phi^{(n)} = 0 \ . \tag{4.5}$$

We can then see that the zeromode $\phi^{(0)}$ is a free massless scalar in d dimensions, while excited modes are massive with mass $m_n = n/R$. Note that the functions $\{e^{iny}\}_{n \in \mathbb{Z}}$ constitute a complete set of eigenfunctions of the internal Laplacian,

$$g^{yy} \partial_y \partial_y \, e^{iny} = -\frac{n^2}{R^2} e^{iny} \ , \tag{4.6}$$

and that the mass term in (4.5) precisely corresponds to the eigenvalue of this operator. This is a general feature of compactifications which is valid also for more complicated internal spaces: higher-dimensional fields are expanded onto eigenfunctions of suitable differential operators in the internal space and modes associated to non-vanishing eigenvalues are massive, with a mass inversely proportional to the typical length scale of the internal geometry. This observation singles out zeromodes as they are the only fields that can participate directly to the low-energy dynamics in any compactification scenario.

The most general $(d+1)$-dimensional metric on a space with topology $\mathcal{M}_{d+1} = \mathcal{M}_d \times S^1$ can be written in the form

$$\hat{g}_{\hat{\mu}\hat{\nu}} \, d\hat{x}^{\hat{\mu}} d\hat{x}^{\hat{\nu}} = g_{\mu\nu} \, dx^\mu dx^\nu + r^2 (dy + A_\mu dx^\mu)^2 \ , \qquad y \sim y + 2\pi \ , \tag{4.7}$$

provided that the metric functions $g_{\mu\nu}$, r, A_μ are allowed to depend both on x and on y. In view of the discussion of the previous paragraph, however, we are led to consider the restricted case in which $g_{\mu\nu}$, r, A_μ are functions of x only, since this corresponds to retaining zeromodes and discarding excited modes. Under this assumption the relation (4.7) is known as Kaluza-Klein Ansatz. The d-dimensional fields $g_{\mu\nu}$, r, A_μ are interpreted as the d-dimensional metric, a scalar usually called dilaton, and a vector commonly referred to as the Kaluza-Klein vector. Note that the background metric (4.2) can be recovered from (4.7) by setting

$$\langle g_{\mu\nu} \rangle = \eta_{\mu\nu} \ , \qquad \langle r \rangle = R \ , \qquad \langle A_\mu \rangle = 0 \ . \tag{4.8}$$

Let us suppose for simplicity that the dynamics of gravity in $d+1$ dimensions is governed by the standard Einstein-Hilbert action

$$S_{\text{EH}} = \frac{1}{2\kappa_{d+1}^2} \int_{\mathcal{M}_{d+1}} \hat{R} \, \hat{*} 1 \ , \tag{4.9}$$

where $\hat{R}$ is the Ricci scalar built from $\hat{g}_{\hat{\mu}\hat{\nu}}$. If we plug the expression (4.7) for $\hat{g}_{\hat{\mu}\hat{\nu}}$ into the Einstein-Hilbert action (4.9) we obtain the action for the d-dimensional fields $g_{\mu\nu}$, r, A_μ,

$$S_{\text{EH}} = \frac{1}{2\kappa_d^2} \int_{\mathcal{M}_d} \left[r \, R * 1 - \frac{1}{2} r^3 F \wedge * F \right] \ . \tag{4.10}$$

In this expression R and $*$ denote the Ricci scalar and the Hodge star operator associated to the metric $g_{\mu\nu}$, $\kappa_d^2 = \kappa_{d+1}^2/(2\pi)$, and $F = dA$. Note that all terms involving gradients of r organize into a total derivative that has been discarded. Nonetheless a kinetic term for r is generated as soon as the first term in (4.10) is brought into standard Einstein-Hilbert form by means of the Weyl rescaling

$$g_{\mu\nu}^{\text{new}} = r^{\frac{2}{d-2}} g_{\mu\nu} \ . \tag{4.11}$$

Indeed, in terms of the new metric (but dropping the superscript 'new') we obtain

$$S_{\text{EH}} = \frac{1}{2\kappa_d^2} \int_{\mathcal{M}_d} \left[R * 1 - \frac{d-1}{d-2}\, d\log r \wedge *d\log r - \frac{1}{2} r^{\frac{2(d-1)}{d-2}} F \wedge *F \right] \ . \tag{4.12}$$

The reduction of the Einstein-Hilbert term we have just considered is the prototype of the circle reductions discussed in greater detail in chapters 7 and 8 in the context of F-theory compactifications.

The reduced action (4.12) is manifestly invariant under the expected gauge transformation of the Kaluza-Klein vector, $\delta A = d\lambda$. This transformation can be given a geometrical interpretation in terms of $(d+1)$-dimensional diffeomorphisms. In fact, the functional form of the Kaluza-Klein Ansatz (4.7) and the range of the periodic coordinate y are invariant under diffeomorphisms of the form

$$x'^{\mu} = x^{\mu} \ , \qquad y' = y - \lambda(x) \ , \tag{4.13}$$

with arbitrary function λ. These diffeomorphisms act trivially on $g_{\mu\nu}$ and r and precisely reproduce the expected finite $U(1)$ gauge transformation on the Kaluza-Klein vector,

$$g'_{\mu\nu} = g_{\mu\nu} \ , \qquad r' = r \ , \qquad A' = A + d\lambda \ . \tag{4.14}$$

Inspection of (4.13) reveals that excited modes transform non-trivially under this $U(1)$ gauge transformation.

4.2.2 Circle compactification of p-forms

It is instructive to consider the Kaluza-Klein expansion of an Abelian p-form in $d+1$ dimensions. The study of this system provides also preliminary material for the discussion of self-dual p-forms in chapter 10.

Let $\hat{C}_p$ be a $(d+1)$-dimensional Abelian p-form with free action

$$S_{\text{form}} = -\frac{1}{2} \int_{\mathcal{M}_{d+1}} \hat{F}_{p+1} \wedge \hat{*} \hat{F}_{p+1} \ , \qquad \hat{F}_{p+1} = d\hat{C}_p \ , \tag{4.15}$$

which is invariant under the $(d+1)$-dimensional gauge symmetry

$$\delta \hat{C}_p = d\hat{\Lambda}_{p-1} \ , \tag{4.16}$$

with arbitrary $(p-1)$-form parameter $\hat{\Lambda}_{p-1}$. Let us Fourier expand both the p-form and the gauge parameter as

$$\hat{C}_p = \sum_{n\in\mathbb{Z}} \left[C_p^{(n)} + C_{p-1}^{(n)} \wedge (dy + A) \right] e^{iny} \ , \qquad \hat{\Lambda}_{p-1} = \sum_{n\in\mathbb{Z}} \left[\Lambda_{p-1}^{(n)} + \Lambda_{p-2}^{(n)} \wedge (dy + A) \right] e^{iny} \ . \tag{4.17}$$

In these expressions all fields on the right hand side of the equations depend on x only. Note also the appearance of the combination $dy + A$, which is invariant under the combined transformations (4.13) and (4.14). Using $dy + A$ to expand $(d+1)$-dimensional fields ensures simple $U(1)$ transformation laws, for instance

$$C_p^{(n)\prime} = e^{in\lambda}\, C_p^{(n)} \ , \qquad C_{p-1}^{(n)\prime} = e^{in\lambda}\, C_{p-1}^{(n)} \ , \tag{4.18}$$

and similarly for modes in the expansion of the gauge parameter $\hat{\Lambda}_{p-1}$. This leads us to the introduction of a $U(1)$ covariant derivative $\mathcal{D}$ which acts on modes at level n as

$$\mathcal{D} = d - inA \ . \tag{4.19}$$

The p-form action (4.15) written in terms of Kaluza-Klein modes takes the form

$$\begin{aligned} S_{\text{form}} = & \int_{\mathcal{M}_d} 2\pi r \left[-\frac{1}{2} F_{p+1}^{(0)} \wedge *F_{p+1}^{(0)} - \frac{1}{2} r^{-2} F_p^{(0)} \wedge *F_p^{(0)} \right] \\ & + \sum_{n=1}^{\infty} \int_{\mathcal{M}_d} 2\pi r \left[-\bar{F}_{p+1}^{(n)} \wedge *F_{p+1}^{(n)} - r^{-2} \bar{F}_p^{(n)} \wedge *F_p^{(n)} \right] \ , \end{aligned} \tag{4.20}$$

where we have not performed the Weyl rescaling (4.11) and we have introduced the d-dimensional field strengths

$$F_{p+1}^{(n)} = \mathcal{D}C_p^{(n)} + (-)^{p-1}\, C_{p-1}^{(n)} \wedge F \ , \qquad F_p^{(n)} = \mathcal{D}C_{p-1}^{(n)} + (-)^p\, in\, C_p^{(n)} \ , \tag{4.21}$$

with $n \in \mathbb{Z}$. They are invariant under the d-dimensional gauge transformations induced by (4.16)

$$\delta C_p^{(n)} = \mathcal{D}\Lambda_{p-1}^{(n)} + (-)^{p-2}\, \Lambda_{p-2}^{(n)} \wedge F \ , \qquad \delta C_{p-1}^{(n)} = \mathcal{D}\Lambda_{p-2}^{(n)} + (-)^{p-1} in\, \Lambda_{p-1}^{(n)} \ , \tag{4.22}$$

which are again valid for any $n \in \mathbb{Z}$. Let us clarify the physical implications of (4.20) and (4.22).

The zeromodes $C_p^{(0)}$, $C_{p-1}^{(0)}$ describe a massless p-form and a massless $(p-1)$-form, respectively. Note that, while $C_{p-1}^{(0)}$ has standard gauge transformations and Bianchi identity, the gauge transformation of $C_p^{(0)}$ implies a non-standard Bianchi identity

$$dF_{p+1}^{(0)} = (-)^{p-1} F_p^{(0)} \wedge F \ . \tag{4.23}$$

This mechanism, for instance, is the source of the non-standard Bianchi identity for the Type IIA four-form field strength (2.25) if it is interpreted as the circle reduction of the four-form field strength of eleven-dimensional supergravity. It will also play an important role in the discussion of self-dual p-forms on a circle in chapter 10.

Let us now turn to excited modes. For $n \neq 0$ the $(p-1)$-form $C_{p-1}^{(n)}$ enjoys a shift symmetry with parameter $\Lambda_{p-1}^{(n)}$ that makes it possible to completely gauge it away. The gauge-fixed action contains the field $C_p^{(n)}$ only, has no residual local symmetries at level n, and its relevant terms read

$$S_{\text{form}} \supset \sum_{n=1}^{\infty} \int_{\mathcal{M}_d} 2\pi r \left[-d\bar{C}_p^{(n)} \wedge *dC_p^{(n)} - \frac{n^2}{r^2} \bar{C}_p^{(n)} \wedge *C_p^{(n)} \right] \ , \tag{4.24}$$

From this expression we see that the gauge-fixed forms $C_p^{(n)}$ describe massive fields with $m_n = n\langle r\rangle^{-1} = n/R$. Note that what we have described is a p-form generalization of the Stückelberg mechanism, which corresponds to the case $p = 1$: $C_0^{(n)}$ is the Stückelberg scalar with shift symmetry gauged by the vector $C_1^{(n)}$.

4.2.3 Some remarks

Let us conclude this section with some additional remarks about excited modes in circle compactifications. In the previous example of the Abelian p-form on a circle the original action in $d+1$ dimensions describes a free field and as a result in d dimensions Kaluza-Klein modes associated to different levels are decoupled. This makes it easy to retain them all in our discussion. As soon as interactions in $d+1$ dimensions are considered, however, non-trivial mixing among Kaluza-Klein levels occurs in the lower-dimensional theory.

This would be the case, for instance, if we allowed for a non-trivial y dependence of $g_{\mu\nu}$, r, A_μ in the Kaluza-Klein Ansatz, given the non-linear nature of gravity. Some preliminary information about the associated massive spectrum can be extracted from the study of linearized Einstein's equations on the background (4.2). This analysis confirms the known fact that the massless states associated to the y-independent part of $g_{\mu\nu}$, r, A_μ are accompanied by a tower of massive spin-two[2] particles of mass $m_n = n/R$, see for instance [78] for a discussion in $d = 10$.

By neglecting any y-dependence in the Kaluza-Klein Ansatz (4.7) we have effectively frozen to zero all these massive modes. In chapters 7 and 8 we will consider circle reductions with both gravity and matter fields and we will similarly switch off all excited modes. Can this procedure be justified? This question can be addressed from two points of view.

According to the effective action paradigm, at low-energies the dynamics of the system is governed by zeromodes only. Excited modes should then be integrated out and their presence affects the effective action only indirectly, via corrections to the couplings among zeromodes. For the vast majority of couplings all effects due to massive modes are suppressed by the Kaluza-Klein scale R^{-1} and are therefore negligible for energies $E \ll R^{-1}$. Instead of properly integrating out massive modes we can then simply set them to zero in this approximation. In chapter 9 we will study a notable exception to this prescription: topological couplings among zeromodes in five-dimensions can receive loop corrections from excited Kaluza-Klein modes that are independent of the scale R^{-1}. Some implications of this peculiar effect will be explored in chapter 10.

From a purely classical perspective, the truncation of the full theory to zeromodes only is consistent provided that any solution of the d-dimensional equations of motion for zeromodes can be uplifted to a solution of the full $(d+1)$-dimensional theory. From this perspective, an Ansatz like (4.7) is interpreted as a recipe to construct $(d+1)$-dimensional solutions starting from d-dimensional solutions. In the case of circle compactification, the truncation to zeromodes can be shown to be always consistent in this sense [79, 80]. This can be understood recalling that all excited modes are charged under the $U(1)$ symmetry associated to the Kaluza-Klein vector. As a result, in the equation of motion for a mode at level n all terms must have charge n and this forbids any term that is a function of zeromodes only. This implies that if all excited modes are set to zero, all their equations of motion are satisfied automatically.

[2]The spin of a massive particle in d dimensions is defined as the maximum absolute value of the eigenvalues of the Cartan generators of the massive little group $SO(d-1)$. Massive Kaluza-Klein modes of the graviton fall into the symmetric traceless representation of $SO(d-1)$.

4.3 More general compactification on Ricci-flat manifolds

In the typical compactification scenario in string theory or M-theory one is interested in the reduction of a D-dimensional theory on a k-dimensional compact space X_k down to d dimensions. It is customary to require that the external spacetime be a maximally symmetric space, i.e. Minkowski, de Sitter, or anti-de Sitter. Under this assumption, the most general background value of the D-dimensional metric that is compatible with all symmetries of external spacetime takes the form of a warped product,

$$\langle \hat{g}_{\hat{\mu}\hat{\nu}}(\hat{x})\rangle d\hat{x}^{\hat{\mu}} d\hat{x}^{\hat{\nu}} = e^{2w(y)}\, g_{\mu\nu}(x)\, dx^\mu dx^\nu + g_{mn}(y)\, dy^m dy^n \ . \tag{4.25}$$

In this expression $\hat{x}^{\hat{\mu}}$, $\hat{\mu} = 0, \dots D-1$ are coordinates in D dimensions, while x^μ, $\mu = 0, \dots d-1$ are coordinates in the external spacetime and y^m, $m = 1, \dots, k$ are coordinates in the internal space. Moreover, $g_{\mu\nu}$ is the Minkowski, de Sitter, or anti-de Sitter metric, and g_{mn} is the metric of the internal space. Finally, w is a scalar function referred to as the warp factor.

In what follows we restrict our analysis to the simplest situation in which the warp factor is constant (and can be set to zero with no loss of generality), external spacetime is Minkowski, and the internal metric is Ricci-flat,

$$\langle \hat{g}_{\hat{\mu}\hat{\nu}}(\hat{x})\rangle d\hat{x}^{\hat{\mu}} d\hat{x}^{\hat{\nu}} = \eta_{\mu\nu}\, dx^\mu dx^\nu + g_{mn}(y)\, dy^m dy^n \ , \qquad R_{mn} = 0 \ . \tag{4.26}$$

This metric is a solution to D-dimensional vacuum Einstein's equation. Note that if we consider the Type II or M-theory effective actions at two-derivative level (2.22), (2.23), (2.24), (3.34) and we set all matter fields to zero in the vacuum, the $D = 10$ or $D = 11$ equations of motion reduce precisely to vacuum Einstein's equation. Let us stress, however, that we are ignoring several complications related to higher-derivative corrections [81, 82] and the possibility of quantum anomalies that impose a half-integral quantization of fluxes, which prohibits to just set them to zero in the background [62]. Some of these crucial issues will be addressed in section 5.5 in the context of F-theory vacua, but will be neglected for the time being.

According to the lesson learned in the study of circle compactification, D-dimensional fields have to be expanded in eigenfunctions of some suitable operator in the internal space. Zeromodes of these operators will then yield massless fields in d dimensions. In order to identify the relevant operators we do not need to consider the full non-linear theory and we can rather analyze small perturbations around the background (4.26),

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \langle \hat{g}_{\hat{\mu}\hat{\nu}}\rangle + \hat{h}_{\hat{\mu}\hat{\nu}} \ . \tag{4.27}$$

One finds that metric fluctuations should be expanded in eigenfunctions of the operators

$$\Delta_0 Y = \nabla^p \nabla_p Y \ , \quad (\Delta_1 Y)_m = \nabla^p \nabla_p Y_m \ , \quad (\Delta_2 Y)_{mn} = \nabla^p \nabla_p Y_{mn} + 2 R_m{}^p{}_n{}^q\, Y_{pq} \ . \tag{4.28}$$

In these expression and henceforth indices and raised and lowered with the background internal metric g_{mn} in (4.26), which is also used to build the covariant derivative ∇_m. The operator Δ_0 is the connection Laplacian acting on scalar functions, while Δ_1 is the connection Laplacian acting on covectors.

The latter actually coincides up to a sign with the Laplace-de Rham operator acting on one-forms in a Ricci-flat manifold, by virtue of the identity

$$(\Delta_{\mathrm{dR}} Y)_m = (d\, d^\dagger Y + d^\dagger dY)_m = -\nabla^p \nabla_p Y_m + R_{mp} Y^p \ . \tag{4.29}$$

Finally, Δ_2 acts on symmetric tensors and is known as the Lichnerowicz operator. Note also that we restrict the action of Δ_1 and Δ_2 to functions Y_m, Y_{mn} that obey the transversality and tracelessness requirements

$$\nabla^m Y_m = 0 \ , \qquad \nabla^m Y_{mn} = 0 \ , \qquad g^{mn} Y_{mn} = 0 \ . \tag{4.30}$$

It is possible to prove that on a compact space Δ_0, Δ_1, and Δ_2 admit a finite number of zeromodes and an infinite tower of excited modes with strictly negative eigenvalues. As a result we find a spectrum with the same qualitative features as the Kaluza-Klein spectrum of a circle compactification: a finite number of massless fields is accompanied by an infinite tower of massive fields. In what follows we focus on zeromodes only. It is interesting to note that, in analogy to the case of a p-form on a circle in section 4.2, the masses of excited modes are generated via a Stückelberg-like mechanism.

Let us start our analysis of zeromodes by recalling that a zeromode of the scalar Laplacian Δ_0 on a compact manifold is constant. As a result, there is only one independent zeromode, which will be denoted Y^0. Secondly, the identification between $-\Delta_1$ and Δ_{dR} allows us to exploit Hodge theory of harmonic forms and conclude that zeromodes of Δ_1 are labelled by the first Betti number b_1 of the internal manifold: we thus have the zeromodes $Y^\beta_m(y)$ with $\beta = 1, \dots b_1$. In order to specify further the zeromodes of the Lichnerowicz operator one needs additional information about the compact Ricci-flat manifold. In the following sections we will explore in more detail zeromodes of Δ_2 on Calabi-Yau manifolds and Spin(7) manifolds, and in both cases they will be related to appropriate harmonic forms. For the time being, it suffices to recall that Δ_2 admits a finite number N_2 of zeromodes, that will be denoted $Y^\gamma_{mn}(y)$, with $\gamma = 1, \dots, N_2$.

The expansion of the components of the metric fluctuation with respect to zeromodes of Δ_0, Δ_1, Δ_2 reads

$$\hat{h}_{\mu\nu} = \gamma^0_{\mu\nu} Y^0 \ , \qquad \hat{h}_{\mu m} = \sum_{\beta=1}^{b_1} A^\beta_\mu Y^\beta_m \ , \qquad \hat{h}_{mn} = \sum_{\gamma=1}^{N_2} M^\gamma Y^\gamma_{mn} + \Phi^0 Y^0 g_{mn} \ , \tag{4.31}$$

where we have introduced a d-dimensional symmetric tensor $\gamma^0_{\mu\nu}$ identified with the graviton, a collection of vectors A^β_μ and a collection of scalars M^γ, Φ^0. Note that, thanks to (4.30), the scalar fluctuations M^γ leave the volume of the internal space invariant. One can show that the same holds for excited modes. As a result, the variation of the internal volume is entirely encoded in Φ^0. It is possible to check that plugging the metric perturbations (4.31) into the linearized Einstein's equation in D dimensions the expected massless equations of motion for $\gamma^0_{\mu\nu}$, A^β_μ and M^γ, Φ^0 are recovered. Let us just mention that one finds a mixing between the graviton $\gamma^0_{\mu\nu}$ and the scalar Φ^0: it is the manifestation at linearized level of the Weyl rescaling discussed below.

It is known that for compact Ricci-flat manifolds the first Betti number b_1 receives contributions from torus factors only, i.e. it is non-vanishing only if the space can be written as a product $Z_{k-\ell} \times$

T^ℓ, with $\ell > 1$. Equivalently, the only Killing vectors on a compact Ricci-flat manifold are those associated to such torus factors [46]. In chapters 7 and 8 we will consider internal Ricci-flat manifolds whose definition forbids torus factors. As a result, no massless vector appears in the spectrum of the dimensionally reduced theory.

The scalars Φ^0 and M^γ are commonly referred to as metric moduli. Their masslessness reflects the fact that they correspond to deformations of the internal manifold that respect Ricci-flatness and thus they parametrize a vacuum degeneracy. In this respect they are analogous to Goldstone bosons. Note that our discussion has considered only infinitesimal deformations. In many cases of interest, including Calabi-Yau and Spin(7) manifolds introduced below, it is possible to prove that these deformations are unobstructed and can be promoted to finite transformations. We therefore have a well-defined notion of moduli space. It is a manifold with dimension N_2+1 and can be parametrized by coordinates $X^{\mathcal{M}}$, $\mathcal{M} = 1, \dots, N_2+1$. From this perspective the background internal metric g_{mn} in (4.26) is though of as some point in moduli space, and the scalar fluctuations Φ^0 and M^γ are best understood as orthogonal vectors in the tangent space to moduli space at that point.

The dynamics of moduli can be extracted from the dynamics of pure gravity in the higher-dimensional theory. In fact, we will now discuss how the dimensional reduction of the D-dimensional Einstein-Hilbert action

$$S = \frac{1}{2}\int_{\mathcal{M}_D} \hat{R}\,\hat{*}\,1 \tag{4.32}$$

yields a non-linear sigma model for the moduli $X^{\mathcal{M}}$ coupled to d-dimensional gravity. For notational simplicity we have set the D-dimensional gravitational constant to one, $\kappa_D = 1$. The starting point of the dimensional reduction is the metric Ansatz

$$\hat{g}_{\hat{\mu}\hat{\nu}}(\hat{x})d\hat{x}^\mu d\hat{x}^{\hat{\nu}} = g_{\mu\nu}(x)dx^\mu dx^\nu + g_{mn}(y;X(x))dy^m dy^n\ , \tag{4.33}$$

where the notation $g_{mn}(y;X(x))$ indicates that the internal metric depends parametrically on the coordinates $X^{\mathcal{M}}$ on moduli space, which are in turn allowed to vary with the external coordinates x^μ. Note that we have included no d-dimensional vector in (4.33). This is because we are interested in Ricci-flat manifolds with $b_1 = 0$ for which the expansion of the internal metric yields no massless vector, as we have seen above.

The dimensional reduction of (4.32) is performed in detail in appendix A.3. Let us stress that the external metric has to be Weyl rescaled according to

$$g^{\text{old}}_{\mu\nu} = \mathcal{V}^{-\frac{2}{d-2}} g^{\text{new}}_{\mu\nu} \tag{4.34}$$

in order to achieve canonical normalization for the d-dimensional Einstein-Hilbert term. In terms of the new external metric, but dropping the superscript ‘new,’ one finds

$$S = \int_{\mathcal{M}_d}\left[\frac{1}{2}\,R*1 - \frac{d-1}{2(d-2)}d\log\mathcal{V}\wedge *d\log\mathcal{V} - \frac{1}{2}\mathcal{G}_{\mathcal{M}\mathcal{N}}\,dX^{\mathcal{M}}\wedge *dX^{\mathcal{N}}\right]\ , \tag{4.35}$$

where $\mathcal{V}$ denotes the volume of the internal space and sigma-model metric $\mathcal{G}_{\mathcal{M}\mathcal{N}}$ is given by an integral over the internal space,

$$\mathcal{G}_{\mathcal{M}\mathcal{N}} = \frac{1}{4\mathcal{V}}\int_{\mathcal{M}_k} d^k y\,\sqrt{g}\left[g^{mp}g^{nq}\frac{\partial g_{mn}}{\partial X^{\mathcal{M}}}\frac{\partial g_{pq}}{\partial X^{\mathcal{N}}} - g^{mn}\frac{\partial g_{mn}}{\partial X^{\mathcal{M}}}g^{pq}\frac{\partial g_{pq}}{\partial X^{\mathcal{M}}}\right]\ . \tag{4.36}$$

The partial derivatives $\partial g_{mn}/\partial X^{\mathcal{M}}$ encode the variations of the internal metric components with respect to the moduli, thought of as parameters. Equivalently, under an infinitesimal variation $\delta X^{\mathcal{M}}$ of the moduli the metric varies according to

$$\delta g_{mn} = \frac{\partial g_{mn}}{\partial X^{\mathcal{M}}} \delta X^{\mathcal{M}} \ . \tag{4.37}$$

When a specific class of internal manifolds is considered there is usually a natural choice for the coordinates $X^{\mathcal{M}}$ in moduli space. Furthermore the geometry of the internal manifold allows one to derive relations of the form (4.37), from which one can read off the quantities $\partial g_{mn}/\partial X^{\mathcal{M}}$ and compute $\mathcal{G}_{\mathcal{M}\mathcal{N}}$ using (4.36). We will see how this works explicitly for the case of the moduli space of Calabi-Yau manifolds and Spin(7) manifolds.

So far we have discussed the compactification of D-dimensional gravity. Another essential ingredient for our discussion in chapters 7 and 8 is furnished by the reduction of p-form fields. In this case the appropriate differential operator in the internal manifold is the Laplace-deRham operator. This can be seen by noting that the linearized equation of motion for a p-form $\hat{C}_p$ supplemented by the Lorenz-like gauge condition $d * \hat{C}_p = 0$ reads simply $\hat{\Delta}_{\rm dR} \hat{C}_p = 0$. In the factorized background metric (4.26) the Laplace-deRham operator of the total space splits into the sum of the corresponding operators for external spacetime and for the internal manifold. As a result, zeromodes are obtained by expanding onto a basis of harmonic forms of the compact space. Therefore we can make use of an Ansatz of the schematic form

$$\hat{C}_p(\hat{x}) = \sum_{r+s=p} \sum_{I=1}^{b_s} C_r^I(x) \wedge \Psi_s^I(y; X(x)) \ , \tag{4.38}$$

where C_r^I are r-forms in external spacetime, Ψ_s^I is a basis of harmonic s-form in the internal space, with b_s being the s-th Betti number. We have stressed that it is possible for the harmonic forms Ψ_s to depend parametrically on the moduli $X^{\mathcal{M}}$, just like the internal metric does in (4.33).

We would like to stress a crucial difference between (4.33) or (4.38) and the Kaluza-Klein Ansatz on a circle (4.7). Contrary to (4.7), in fact, (4.33) and (4.38) do not generically define a consistent truncation, i.e. it is not true that any solution to the d-dimensional equations of motion for $g_{\mu\nu}$, $X^{\mathcal{M}}$, C_r^I can be lifted to a solution of the D-dimensional equations of motion via (4.33) and (4.38). As a result, our approximation of retaining zeromodes only is strictly speaking inconsistent at the level of the classical equations of motion. We can still make sense of expressions like (4.33) and (4.38) within the framework of the low-energy effective action paradigm. From this perspective, these relations are convenient tools in the derivation of the effective action for massless fields in d-dimensions.

As mentioned above, in order to flesh out the content of formal expressions like (4.36) more information is necessary about the structure of the moduli space of the theory. For our purposes, we are interested in some features of the moduli spaces of Calabi-Yau manifolds and of Spin(7) manifolds. Before discussing them, however, let us briefly motivate the emergence of these special geometries and clarify the origin of their relevance for string and M-theory compactifications.

4.4 Supersymmetry and special holonomy

As noted above, the background metric (4.26) solves vacuum Einstein's equation in D dimensions and is therefore a viable vacuum if we consider a two-derivative action and we switch off all matter fields. In most cases of interest in string theory and M-theory the higher-dimensional action is supersymmetric and one in interested in backgrounds that not only do solve the equations of motion, but also preserve a fraction of the supersymmetry.

For definiteness let us consider supersymmetric compactifications of eleven-dimensional supergravity, whose two-derivative bosonic action was given in (3.34). We now need the expression for the supersymmetry variations to lowest order in the gravitino: they read [45]

$$\delta\hat{e}^{\hat{a}}{}_{\hat{\mu}} = \bar{\hat{\epsilon}}\,\hat{\Gamma}^{\hat{a}}\,\hat{\psi}_{\hat{\mu}}\ , \qquad \delta\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}} = -3\bar{\hat{\epsilon}}\,\hat{\Gamma}_{[\hat{\mu}\hat{\nu}}\,\hat{\psi}_{\hat{\rho}]}\ ,$$
$$\delta\hat{\psi}_{\hat{\mu}} = \hat{\nabla}_{\hat{\mu}}\hat{\epsilon} + \frac{1}{12}\left(\frac{1}{4!}\hat{G}_{\hat{\nu}_1\hat{\nu}_2\hat{\nu}_3\hat{\nu}_4}\,\hat{\Gamma}_{\hat{\mu}}\,\hat{\Gamma}^{\hat{\nu}_1\hat{\nu}_2\hat{\nu}_3\hat{\nu}_4} - \frac{1}{2}\hat{G}_{\hat{\mu}\hat{\nu}_1\hat{\nu}_2\hat{\nu}_3}\,\hat{\Gamma}^{\hat{\nu}_1\hat{\nu}_2\hat{\nu}_3}\right)\hat{\epsilon}\ . \tag{4.39}$$

In these expressions $\hat{\mu},\hat{\nu},\ldots$ are curved eleven-dimensional indices, $\hat{a}$ is a flat eleven-dimensional index, and $\hat{e}^{\hat{a}}{}_{\hat{\mu}}$ denotes the elfbein. The Majorana spinor $\hat{\epsilon}$ is the anticommuting supersymmetry parameter and the Majorana vector-spinor $\hat{\psi}_{\hat{\mu}}$ is the gravitino. The symbol $\hat{\nabla}_{\hat{\mu}}$ represents the Levi-Civita connection acting on spinors. The quantity $\hat{\Gamma}_{\hat{\mu}_1\ldots\hat{\mu}_p}$ denotes the antisymmetrized product of p gamma matrices in eleven dimensions. Finally, recall that $\hat{G}_4 = d\hat{C}_3$.

As noted above, our discussion is restricted to vacua in which all matter fields are set to zero. In the context of eleven-dimensional supergravity we then have

$$\langle\hat{C}_3\rangle = 0\ , \qquad \langle\hat{\psi}_{\hat{\mu}}\rangle = 0\ . \tag{4.40}$$

As a result, the supersymmetry variations of the elfbein $\hat{e}^{\hat{a}}{}_{\hat{\mu}}$ and of the three-form $\hat{C}_3$ are automatically zero if evaluated in the vacuum and the gravitino variation takes the simple form $\delta\hat{\psi}_{\hat{\mu}} = \hat{\nabla}_{\hat{\mu}}\hat{\epsilon}$. This implies that a fraction of supersymmetry is preserved if and only if a non-trivial solution to the Killing spinor equation

$$\hat{\nabla}_{\hat{\mu}}\hat{\epsilon} = 0 \tag{4.41}$$

can be found. Since this is a linear equation, its solutions constitute a linear space whose dimension determines the number of supercharges preserved by the compactification.

In order to search for solutions to the Killing spinor equation (4.41) on the background (4.26) we need to decompose the eleven-dimensional spinor $\hat{\epsilon}$ into a d-dimensional spinor and a k-dimensional spinor. The details of this decomposition depend on the specific dimensionalities involved. However, these complications can be avoided if we ignore temporarily the Majorana condition on $\hat{\epsilon}$ and we treat it as a Dirac spinor. We then search for solutions to (4.41) of the factorized form

$$\hat{\epsilon}(x,y) = \epsilon(x)\otimes\eta(y)\ , \tag{4.42}$$

where $\epsilon(x)$ is a Grassmann-odd Dirac spinor in external spacetime and $\eta(y)$ is a Grassmann-even Dirac spinor in internal spacetime. The external component of the Killing spinor equation in Minkowski spacetime is simply solved by any constant ϵ, while the internal component gives the Killing spinor equation on the internal manifold,

$$\nabla_m \eta = 0 \ , \tag{4.43}$$

where now ∇_m denotes the spinor covariant derivative built with the internal metric g_{mn} in (4.26). This equation implies the integrability condition

$$0 = [\nabla_n, \nabla_m]\eta = \frac{1}{4} R_{mnpq} \Gamma^{pq} \eta \ , \tag{4.44}$$

where Γ^{pq} is the antisymmetric product of two internal gamma matrices, which in turn implies

$$R_{mn} = 0 \ , \tag{4.45}$$

provided that $\eta \neq 0$ and making use of the Euclidean signature of the internal space. Ricci-flatness of the internal space was already demanded on grounds of the D-dimensional equations of motion, but we now see that it stems automatically from the requirement of unbroken supersymmetry in an unwarped background of the form (4.26).

In order for the decomposition (4.42) to give a well-defined spinor in eleven dimensions we have to demand that η be globally defined on the internal manifold. Since it is covariantly constant (4.43), it must be in particular nowhere vanishing. This generically imposes topological conditions on the internal manifold, such as a reduction of the structure group to a proper subgroup of $SO(k)$. This observation remains valid if the Killing spinor equation (4.43) is generalized, allowing for deformations of the Levi-Civita connection acting on the spinor η. Pursuing further these ideas would lead us to the discussion of G-structures in the context of string and M-theory compactifications, which however lies beyond the scope of this work. We refer the reader to e.g. [83] for an introduction to the subject.

On top of topological considerations, the Killing spinor equation (4.43) restricts the holonomy group of the Levi-Civita connection on the internal manifold. Since it plays a crucial role in our discussion, it is useful to recall a few facts about the notion of holonomy. Let E be a vector bundle on a Riemannian manifold. We are currently interested in the case in which E is an appropriate spinor bundle, such as the bundle of Dirac or Weyl spinors. Let p be a point on the Riemannian manifold and let γ be a piecewise smooth loop based at p. The Levi-Civita connection induces a well-defined notion of parallel transport in the vector bundle E. We can then take an element η in the fiber E_p and parallel transport it along γ all the way back to p. The outcome of this procedure will generically be a different $\eta' \in E_p$, related to η by a linear transformation $S \in GL(E_p)$. The holonomy group based at p, denoted Hol_p, is then defined as the subgroup of $GL(E_p)$ consisting of all transformations S that can be obtained in this way. For a spinor bundle in a k-dimensional Riemannian manifold Hol_p is generically contained in a (possibly reducible) spinor representation of $SO(k)$. It can be shown that the holonomy groups Hol_p, $\mathrm{Hol}_{p'}$ based at points p, p' are isomorphic on a connected manifold. Therefore in what follows we will simply refer to the holonomy group as Hol. Let us point out that Hol is a Lie group and that its identity component Hol^0, known as the restricted holonomy group, is generated by considering parallel transport along contractible loops only. The Lie algebra of Hol^0 is related to the curvature two-form of the connection by the Ambrose-Singer theorem [84]: intuitively speaking, curvature encodes the rotation induced by parallel transport along an infinitesimal loop.

For simplicity we will restrict to simply connected manifolds, for which it is possible to show that $\mathrm{Hol} = \mathrm{Hol}^0$. Let us now illustrate the relation between the Killing spinor equation and holonomy using some examples that will be relevant to the F-theory compactifications discussed in chapters 7 and 8.

Let us consider the case in which the dimension of the internal manifold is $k = 6$. Minimal spinors of $SO(6) \cong SU(4)$ can be represented as Weyl spinors in the complex irreducible representations $\mathbf{4}$ and $\overline{\mathbf{4}}$ of $SU(4)$, with complex conjugation interchanging chiralities. Let η be a globally defined spinor in the $\mathbf{4}$ representation satisfying the Killing spinor equation (4.43). Its complex conjugate η^* then furnishes a covariantly constant spinor in the $\overline{\mathbf{4}}$ representation. Clearly, both η and η^* are invariant under the action of the holonomy group since they behave trivially under parallel transport. If the holonomy group Hol of the manifold were all of $SU(4)$, it would act irreducibly on η and η^*, and the only possible invariant spinors would be $\eta = \eta^* = 0$. In order to have non-vanishing Killing spinors Hol must be a proper subgroup of $SU(4)$, $\mathrm{Hol} \subsetneq SU(4)$, such that in the decomposition of the representations $\mathbf{4}$ and $\overline{\mathbf{4}}$ under $SU(4) \to \mathrm{Hol}$ a singlet of Hol is found. It is possible to show that the existence of a pair of non-vanishing Killing spinors η, η^* forces $\mathrm{Hol} \subseteq SU(3)$. Indeed, we have

$$\begin{aligned} SU(4) &\to SU(3) \\ \mathbf{4} &\to \mathbf{3} + \mathbf{1} \ , \end{aligned} \tag{4.46}$$

and similarly for $\overline{\mathbf{4}}$. By the same token, if we require the existence of two pairs of non-vanishing, linearly independent Killing spinors η_1, η_1^* and η_2, η_2^* the holonomy group is further reduced to $\mathrm{Hol} \subseteq SU(2)$, consistently with the fact that

$$\begin{aligned} SU(4) &\to SU(2) \\ \mathbf{4} &\to \mathbf{2} + \mathbf{1} + \mathbf{1} \ . \end{aligned} \tag{4.47}$$

It turns out that the existence of three pairs of independent non-vanishing Killing spinors forces the holonomy group to be trivial, so that the space is a flat six-torus that actually admits four pairs of such spinors, in accordance with the trivial decomposition

$$\begin{aligned} SU(4) &\to 1 \\ \mathbf{4} &\to \mathbf{1} + \mathbf{1} + \mathbf{1} + \mathbf{1} \ . \end{aligned} \tag{4.48}$$

The internal spinors η, η^* can be used to build an eleven-dimensional Majorana spinor according to

$$\hat{\epsilon} = \sum_{i=1}^{\mathcal{N}} (\epsilon_i \otimes \eta_i + \epsilon_i^* \otimes \eta_i^*) \ , \qquad \mathcal{N} = 0, 1, 2, 4 \quad \text{for} \quad \mathrm{Hol} = SU(4), SU(3), SU(2), 1 \ , \tag{4.49}$$

where ϵ_i are constant Dirac spinors of $SO(1,4)$. This shows that for holonomies $SU(4), SU(3), SU(2), 1$ we have $\mathcal{N} = 0, 1, 2, 4$ supersymmetry in five dimensions, respectively, corresponding to $0, 8, 16, 32$ real supercharges.

One can analyze in a similar fashion the amount of supersymmetry preserved by compactification on 8-dimensional manifolds depending on their holonomy. We refrain from a complete account and we just discuss the two cases that will be relevant for F-theory compactifications. Recall that the minimal spinors of $SO(8)$ are Majorana-Weyl spinors. If we impose the existence of one non-vanishing Majorana-Weyl Killing spinor of positive chirality η_+ the holonomy group must satisfy Hol $\subseteq$ Spin(7). Indeed, it is possible to show that the two inequivalent Majorana-Weyl representations $\mathbf{8}_c$ and $\mathbf{8}_s$ of $SO(8)$ decompose as

$$\begin{aligned} SO(8) &\to \text{Spin}(7) \\ \mathbf{8}_c &\to \mathbf{7}+\mathbf{1} \ , \\ \mathbf{8}_s &\to \mathbf{8} \ , \end{aligned} \tag{4.50}$$

yielding only one singlet. (Chirality assignments are convention dependent.) This implies that if the internal manifold has Spin(7) holonomy the eleven-dimensional Majorana supersymmetry parameter can be written as

$$\hat{\epsilon} = \epsilon \otimes \eta_+ \ , \tag{4.51}$$

where ϵ is a Majorana spinor of $SO(1,2)$. The resulting three-dimensional theory has minimal $\mathcal{N}=1$ supersymmetry, corresponding to two real supercharges.

Next, we consider the case in which we have two non-vanishing Majorana-Weyl Killing spinors of positive chirality η_{1+}, η_{2+}, which can also be combined into a single complex Weyl spinor $\eta_+ = \eta_{1+} + i\,\eta_{2+}$. In this case the holonomy group must satisfy Hol $\subseteq SU(4)$ and the relevant group-theoretical decomposition is

$$\begin{aligned} SO(8) &\to SU(4) \\ \mathbf{8}_c &\to \mathbf{6}+\mathbf{1}+\mathbf{1} \ , \\ \mathbf{8}_s &\to \mathbf{8} \ . \end{aligned} \tag{4.52}$$

In this case the have

$$\hat{\epsilon} = \epsilon_1 \otimes \eta_{1+} + \epsilon_2 \otimes \eta_{2+} \ , \tag{4.53}$$

where $\epsilon_{1,2}$ are once again Majorana spinors of $SO(1,2)$, so that three-dimensional theory has $\mathcal{N}=2$ supersymmetry, or four real supercharges.

We have encountered a few examples of manifolds with special holonomy. A complete classification of these spaces is due to Berger [85]. More precisely, his classification applies to simply connected Riemannian manifolds that are irreducible and not locally symmetric spaces. On the one hand, irreducibility refers here to the requirement that the holonomy group act irreducibly on the tangent bundle. On the other hand, a local symmetric space is a Riemannian manifold in which for every point p is it possible to define an isometry in a neighborhood of p that fixes p and reverses geodesics through p. It can be proven that a Riemannian manifold is locally symmetric if and only if its Riemann tensor is covariantly constant, and that any locally symmetric space is locally isomorphic to a coset space G/H. These space are not relevant for the present discussion. Taking into account these preliminary remarks, Berger's classification is shown in table 4.1.

Since it is instrumental for our discussion, let us state explicitly the definition of Calabi-Yau manifold that we will be using henceforth: a Calabi-Yau n-fold is a compact Kähler manifold with

holonomy	dimension	class	remarks
$SO(n)$	n	Riemannian manifold	
$U(n)$	$2n$	Kähler manifold	
$SU(n)$	$2n$	Calabi-Yau manifold	Ricci-flat, Kähler
$Sp(n) \times Sp(1)$	$4n$	Quaternionic Kähler manifold	non-flat Einstein, non-Kähler
$Sp(n)$	$4n$	Hyper-Kähler manifold	Ricci-flat, Kähler
G_2	7	G_2 manifold	Ricci-flat
Spin(7)	8	Spin(7)-manifold	Ricci-flat

Table 4.1: Berger's classification of the holonomy groups of simply connected, irreducible, non locally symmetric Riemannian manifolds.

strict $SU(n)$ holonomy (and not a proper subgroup of $SU(n)$). For example, with this definition the complex torus T^{2n} with $n > 1$ is not a Calabi-Yau manifold. By the same token, by Spin(7) manifold we will always mean a manifold of strict Spin(7) holonomy. The next sections are devoted to a lightning account of some properties of Calabi-Yau threefolds and fourfolds and Spin(7) manifolds that are relevant in the context of string theory and M-theory compactifications.

4.5 A brief overview of Calabi-Yau manifolds

This section is devoted to a brief account about the main geometrical and topological properties of Calabi-Yau manifolds, together with a short summary of the distinctive features of their moduli spaces. We would like to warn the reader that we will not attempt to make our review self-contained and that some degree of familiarity with Kähler manifolds and (co)homology groups will be assumed. Books and reviews such as [86, 87, 88, 89] are good references for background material as well as for the properties of Calabi-Yau manifolds we are about to address.

4.5.1 Geometry and topology of Calabi-Yau manifolds

Consider a Kähler n-fold X, i.e. a complex manifold with $\dim_{\mathbb{C}} X = n$ endowed with a Hermitian metric whose Kähler form J is closed. The Kähler condition ensures that the Levi-Civita connection is compatible with the decomposition of the tangent bundle into holomorphic and antiholomorphic parts, $TX = TX^{1,0} \oplus TX^{0,1}$. As a result, the restricted holonomy group of X is contained in $U(n) = SU(n) \times U(1)$. The $U(1)$ factor is associated to the trace of the curvature two-form,

$$\mathcal{R}^k{}_\ell = R^k{}_{\ell i \bar{\jmath}}\, dz^i \wedge d\bar{z}^{\bar{\jmath}} \ , \qquad \operatorname{tr} \mathcal{R} = R^k{}_{k i \bar{\jmath}}\, dz^i \wedge d\bar{z}^{\bar{\jmath}} \ , \tag{4.54}$$

where z^i, $i = 1, \ldots, n$ are local complex coordinates on X_n and $R^k{}_{\ell i \bar{\jmath}}$ are the components of the Riemann tensor. On a Kähler manifold they enjoy the symmetry property $R_{i\bar{\jmath}k\bar{\ell}} = R_{k\bar{\jmath}i\bar{\ell}}$, so that the components of $\operatorname{tr} \mathcal{R}$ are proportional to the components of the Ricci tensor $R_{i\bar{\jmath}} = R^k{}_{ik\bar{\jmath}}$. One can then conclude that the restricted holonomy group of a Kähler manifold is contained in $SU(n)$ if and only if the Kähler metric is Ricci-flat.

The trace of the curvature two-form also enters the expression of the first Chern class of X,

$$c_1(X) \equiv c_1(TX^{1,0}) = \frac{i}{2\pi}\mathrm{tr}\,\mathcal{R}\ . \tag{4.55}$$

Therefore Ricci-flatness of the Kähler metric immediately implies $c_1(X_n) = 0$. The converse for compact Kähler n-fold was conjectured by Calabi and proved by Yau. More precisely, Yau's theorem states that if X_n is a compact Kähler n-fold with Kähler form J and $c_1(X_n) = 0$, then there exists a unique Ricci-flat metric in the same Kähler class as J. In what follows, we will always assume that X_n is endowed with this unique Ricci-flat representative of its Kähler class. As noted above, we will use the terminology Calabi-Yau n-fold to denote a compact Kähler n-fold whose restricted holonomy is exactly $SU(n)$ and not a proper subgroup. In what follows we will elucidate the consequences of this restriction.

Recall that the canonical line bundle K_X of a Kähler n-fold X is the n-th exterior power of the holomorphic cotangent bundle, or equivalently the bundle of $(n,0)$-forms. It can be shown that $c_1(K_X) = -c_1(X)$, so that the first Chern class of X vanishes if and only if its canonical line bundle is trivial. The latter condition is in turn equivalent to the existence of a global nowhere vanishing section of K_X, i.e. a globally defined, nowhere vanishing $(n,0)$-form, commonly denoted Ω. If X is endowed with its Ricci-flat metric, Ω transforms as a singlet under the restricted holonomy group $SU(n)$, and is therefore covariantly constant. By virtue of the Kähler condition and Ricci-flatness this implies that Ω is harmonic and holomorphic,

$$\Delta_{\mathrm{dR}}\Omega = 0\ , \qquad \overline{\partial}\Omega = 0\ , \tag{4.56}$$

where $\Delta_{\mathrm{dR}} = d\,d^\dagger + d^\dagger d$ is the Laplace-de Rham operator and $\overline{\partial} = \frac{\partial}{\partial \bar{z}^{\bar{\imath}}} d\bar{z}^{\bar{\imath}}$. Two holomorphic $(n,0)$-forms can only differ by multiplication by a scalar holomorphic function, which must be constant on a compact manifold. Therefore Ω is unique up to normalization. The (n,n)-form $\Omega \wedge \overline{\Omega}$ is proportional to the volume form of the Calabi-Yau n-fold. More precisely, one can show that[3]

$$\frac{1}{n!}J^n = i^n(-)^{\frac{n(n-1)}{2}}\frac{\Omega\wedge\overline{\Omega}}{\|\Omega\|^2}\ , \tag{4.57}$$

where

$$\|\Omega\|^2 = \frac{1}{n!}\Omega_{i_1\dots i_n}\overline{\Omega}^{i_1\dots i_n}\ . \tag{4.58}$$

Let us also recall the identity

$$\Omega_{i_1\dots i_r j_1\dots j_{n-r}}\,\overline{\Omega}^{i_1\dots i_r k_1\dots k_{n-r}} = r!(n-r)!\,\|\Omega\|^2\,\delta^{[k_1}_{[j_1}\dots\delta^{k_{n-r}]}_{j_{n-r}]}\ , \tag{4.59}$$

which is valid for $0 \le r \le n$.

The Kähler form J and the holomorphic $(n,0)$-form Ω can be constructed as spinor bilinears using the covariantly constant spinors introduced in section 4.4. More precisely, for a Calabi-Yau threefold

[3] An algebraic identity like (4.57) is most easily derived in a convention-independent fashion by exploiting the fact both the left hand side and the right hand side are covariantly constant so that if the relation holds at some specific point p, it holds everywhere. We are then free to choose complex coordinates in a neighborhood of p in such a way that

$$ds^2|_p = A\,\delta_{i\bar{\jmath}}\,dz^i d\bar{z}^{\bar{\jmath}}\ , \qquad J|_p = iA\,\delta_{i\bar{\jmath}}\,dz^i\wedge d\bar{z}^{\bar{\jmath}}\ , \qquad \Omega|_p = B\,dz^1\wedge\dots\wedge dz^n\ ,$$

with A, B constants that are eventually reabsorbed in $\|\Omega\|^2$.

we have a pair of conjugate Weyl spinors of opposite chiralities η, η^*. Supposing η has positive chirality for definiteness, we write $\eta_+ = \eta$ and $\eta_- = \eta^*$. It is then possible to check that the spinor bilinears

$$J = -i\,\eta_+^\dagger\,\gamma_{i\bar{\jmath}}\,\eta_+\,dz^i \wedge d\bar{z}^{\bar{\jmath}}\ , \qquad \Omega = \frac{1}{3!}\,\eta_-^\dagger\,\gamma_{ijk}\,\eta_+\,dz^i \wedge dz^j \wedge dz^k \tag{4.60}$$

satisfy all the expected properties of the Kähler form and of the holomorphic (3,0)-form. In particular, they are covariantly constant since the spinors $\eta_\pm$ are, and algebraic relations such as $J^2 = -\mathbb{I}$ or (4.57) can be derived by means of Fierz rearrangements. A similar construction holds for a Calabi-Yau fourfold, which admits a complex Weyl spinor of positive chirality η_+,

$$J = -i\,\eta_+^\dagger\,\gamma_{i\bar{\jmath}}\,\eta_+\,dz^i \wedge d\bar{z}^{\bar{\jmath}}\ , \qquad \Omega = \frac{1}{4!}\,\eta_+^\dagger\,\gamma_{ijk\ell}\,\eta_+\,dz^i \wedge dz^j \wedge dz^k \wedge dz^\ell\ . \tag{4.61}$$

Let us stress that the fact that η_+ is complex is crucial in order to build the Kähler form J: if η_+ were a Majorana-Weyl spinor the bilinear $\eta_+^\dagger\gamma_{mn}\eta_+ = \eta_+^\top C\gamma_{\mu\nu}\eta_+$ would vanish by virtue of the antisymmetry of $C\gamma_{\mu\nu}$ in its spinor indices (which we do not write explicitly) and the commuting character of η_+.

Let us now discuss some topological properties of Calabi-Yau n-folds. Recall that the Hodge numbers $h^{p,q}$ of a complex manifold X are defined as the complex dimension of the Dolbeault cohomology group $H^{p,q}_{\bar\partial}(X,\mathbb{C})$ and that on any compact Kähler n-fold they satisfy

$$h^{p,q} = h^{q,p}\ , \quad h^{p,q} = h^{n-q,n-p}\ , \quad h^{p,p} > 0\ , \tag{4.62}$$

where the first relation is a consequence of complex conjugation, the second is derived using Hodge duality, and the third observation stems from the fact that J^p is a closed but not exact (p,p)-form. We will also always assume that X is connected, so that $h^{0,0} = 1$. As we have seen a Calabi-Yau n-fold is endowed with a nowhere vanishing holomorphic $(n,0)$-form Ω which is unique up to normalization. As a result we have immediately

$$h^{n,0} = h^{0,n} = 1\ , \tag{4.63}$$

but also

$$h^{p,0} = h^{0,n-p}\ , \tag{4.64}$$

since the map $\alpha_{i_1\dots i_p} \mapsto \beta_{\bar{\jmath}_1\dots\bar{\jmath}_{n-p}} = \overline{\Omega}_{\bar{\jmath}_1\dots\bar{\jmath}_{n-p}}{}^{i_1\dots i_p}\alpha_{i_1\dots i_p}$ is a one-to-one map from $(p,0)$-forms to $(0,n-p)$-forms that preserves the harmonicity property. The equations we have written so far apply to any compact Kähler n-fold with restricted holonomy contained in $SU(n)$. As anticipated above, we insist that the holonomy group be exactly $SU(n)$ for a Calabi-Yau n-fold. As a consequence

$$h^{p,0} = 0 \qquad \text{for} \qquad p \neq 0, n\ . \tag{4.65}$$

This can be seen as follows. On the one hand, the restricted holonomy group $SU(n)$ acts on any $(p,0)$-form in the representation $\bigwedge^p \square$, where $\square$ denotes the fundamental representation of $SU(n)$. On the other hand, on a Ricci-flat Kähler manifold a $(p,0)$-form α is harmonic if and only if $\nabla^m\nabla_m\alpha_{i_1\dots i_p} = 0$, which in turn is equivalent to $\nabla_m\alpha_{i_1\dots i_p} = 0$ if the manifold is compact. The $(p,0)$-form α, being parallel, should then transform as a singlet of $SU(n)$, but the representation $\bigwedge^p \square$ contains no such singlet unless $p = 0, n$. Let us stress that the fact that $h^{1,0} = h^{0,1} = 0$ implies that the first Betti number of X vanishes, $b_1 = 0$. As a result, X cannot have any continuous isometries, since on a compact Ricci-flat manifold any solution to the Killing equation determines a harmonic one-form.

The observations made above imply that the Hodge diamond of a Calabi-Yau threefold is given by

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{1,2} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} , \tag{4.66}$$

and contains only two independent Hodge numbers $h^{1,1}$ and $h^{1,2}$. The Euler characteristic is given in terms of these Hodge numbers by

$$\chi = 2(h^{1,1} - h^{1,2}) \ . \tag{4.67}$$

In a similar fashion the Hodge number of a Calabi-Yau fourfold reads

$$\begin{array}{ccccccccc} & & & & h^{0,0} & & & & \\ & & & h^{1,0} & & h^{0,1} & & & \\ & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\ & h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} & \\ h^{4,0} & & h^{3,1} & & h^{2,2} & & h^{1,3} & & h^{0,4} \\ & h^{4,1} & & h^{3,2} & & h^{2,3} & & h^{1,3} & \\ & & h^{4,2} & & h^{3,3} & & h^{2,4} & & \\ & & & h^{4,3} & & h^{3,4} & & & \\ & & & & h^{4,4} & & & & \end{array} = \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & h^{1,1} & & 0 & & \\ & 0 & & h^{1,2} & & h^{1,2} & & 0 & \\ 1 & & h^{1,3} & & h^{2,2} & & h^{1,3} & & 1 \\ & 0 & & h^{1,2} & & h^{1,2} & & 0 & \\ & & 0 & & h^{1,1} & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array} , \tag{4.68}$$

but there are only three independent Hodge numbers, because of the relation[4]

$$h^{2,2} = 2(22 + 2h^{1,1} - h^{1,2} + 2h^{1,3}) \ . \tag{4.69}$$

The Euler characteristic of a Calabi-Yau fourfold in terms of the independent Hodge numbers $h^{1,1}$, $h^{1,2}$, and $h^{1,3}$ reads

$$\chi = 6(8 + h^{1,1} - h^{1,2} + h^{1,3}) \ . \tag{4.70}$$

Let us close this section by noting that Hodge diamonds are symmetric under reflection with respect to their vertical and horizontal axes by means of (4.62). According to mirror symmetry [91] there exists an additional reflection symmetry with respect to the main diagonal of the Hodge diamond. More precisely, given any Calabi-Yau n-fold X, there exists a mirror Calabi-Yau $\tilde{X}$ whose Hodge diamond is the same as that of X up to reflection with respect to the main diagonal. For Calabi-Yau threefolds mirror symmetry exchanges $h^{1,1}$ and $h^{1,2}$; for Calabi-Yau fourfolds it exchanges $h^{1,1}$ and $h^{1,3}$.

[4] This constraint can be derived by expressing the arithmetic genera $\chi_q = \sum_{p=0}^{n} (-)^p h^{p,q}$ in terms of the Chern classes of X using the Hirzebruch-Riemann-Roch theorem [90]. Recalling $c_1 = 0$ for a Calabi-Yau fourfold, one gets

$$\chi_0 = \frac{1}{720} \int_X -c_4 + c_2^2 \ , \qquad \chi_1 = \frac{1}{180} \int_X -31c_4 + 3c_2^2 \ , \qquad \chi_2 = \frac{1}{120} \int_X 79c_4 + 3c_2^2 \ .$$

The first two equalities allow us to express $\int_X c_4$ and $\int_X c_2^2$ in terms of $h^{1,1}$, $h^{1,2}$, $h^{1,3}$. Plugging these quantities back into the equation for χ_2 yields the desired expression for $h^{2,2}$.

4.5.2 Moduli space of Calabi-Yau manifolds

In what follows we discuss some general features of the moduli space of Calabi-Yau n-folds with $n = 3, 4$. The cases $n = 1, 2$ are special and require a separate treatment. Some of the results quoted below are valid also for $n > 4$, but the physical relevance of Calabi-Yau manifolds beyond complex dimension four is not clear.

The tangent space to the moduli space of a Calabi-Yau n-fold X can be thought of as the space of infinitesimal deformations of the metric on X that respect the Ricci-flatness condition. Recall from section 4.3 that such deformations are determined by zeromodes Y_{mn} of the Lichnerowicz operator Δ_2 defined in (4.28). The transversality condition $\nabla^m Y_{mn}$ imposed in (4.30) ensures in the present context that Y_{mn} encodes a deformation that is not generated by a diffeomorphism. The tracelessness requirement $g^{mn} Y_{mn}$ that we assumed in (4.30) will be henceforth relaxed, since in the discussion of the moduli space of Calabi-Yau manifolds it is not convenient to single out the overall volume fluctuation from volume-preserving deformations.

On a Kähler manifold the Lichnerowicz operator does not mix holomorphic and antiholomorphic components, which can be therefore analyzed separately. Let Y_{mn} be a symmetric tensor. The components with mixed indices $Y_{i\bar{\jmath}}$ can be used to define a $(1,1)$-form

$$\lambda_Y = Y_{i\bar{\jmath}}\, dz^i \wedge d\bar{z}^{\bar{\jmath}} \ , \tag{4.71}$$

while the components with two antiholomorphic indices can be used to build a $(0,1)$-form with values in the holomorphic tangent bundle $TX^{1,0}$,

$$\alpha^i_Y = g^{i\bar{\jmath}}\, Y_{\bar{\jmath}\bar{k}}\, d\bar{z}^{\bar{k}} \ . \tag{4.72}$$

The components Y_{ij} determine the components of the complex conjugate form $\bar{\alpha}^{\bar{\imath}}_Y$. A direct computation shows then that

$$\begin{aligned} (\Delta_2 Y)_{i\bar{\jmath}} = 0 \qquad &\Leftrightarrow \qquad \Delta_{\mathrm{dR}} \lambda_Y = 0 \ , \\ (\Delta_2 Y)_{\bar{\imath}\bar{\jmath}} = 0 \qquad &\Leftrightarrow \qquad \Delta_{\bar{\partial}} \alpha^i_Y = 0 \ , \end{aligned}$$

where $\Delta_{\mathrm{dR}} = d\, d^\dagger + d^\dagger d$ is the Laplace-de Rham operator and $\Delta_{\bar{\partial}} = \bar{\partial}\,\bar{\partial}^\dagger + \bar{\partial}^\dagger \bar{\partial}$ is the Laplace-Dolbeault operator acting on $(0,p)$-forms with values in $TX^{1,0}$. Using the holomorphic $(n,0)$-form Ω we can encode all the information of the components of the vector-valued $(0,1)$-form α^i_Y in a conventional $(n-1,1)$-form β_Y defined by

$$\beta_{Y\, i_1 \ldots i_{n-1} \bar{\jmath}} = \Omega_{i_1 \ldots i_{n-1} k}\, \alpha^k_{Y\, \bar{\jmath}} \ . \tag{4.73}$$

One can check that $\Delta_{\bar{\partial}} \alpha^i_Y = 0$ if and only if $\Delta_{\mathrm{dR}} \beta_Y = 0$. In summary, on a Calabi-Yau n-fold metric variations with mixed components are in one-to-one correspondence with harmonic $(1,1)$-forms, while metric variations with two antiholomorphic indices are in one-to-one correspondence with harmonic $(n-1,1)$-forms. The former are referred to as Kähler deformations, since they can be interpreted as arising from a variation of the Kähler form with complex structure kept fixed. The latter are complex structure deformations, since they are induced by a variation of the complex structure of the Calabi-Yau n-fold, as it will be discussed below in more detail.

It has been proven [92, 93] that the infinitesimal metric deformations discussed above are unobstructed and can be integrated to yield finite deformations. This ensures the existence of an actual moduli space for Calabi-Yau manifolds. The decoupling of mixed and purely antiholomorphic deformations $Y_{i\bar{\jmath}}$, $Y_{\bar{\imath}\bar{\jmath}}$ signals that the moduli space of a Calabi-Yau n-fold with $n > 2$ can be written locally as a product

$$\mathcal{M} = \mathcal{M}_{\text{Kähler}} \times \mathcal{M}_{\text{cstr}} \ , \tag{4.74}$$

where the factor $\mathcal{M}_{\text{Kähler}}$ is referred to as Kähler moduli space, and $\mathcal{M}_{\text{cstr}}$ denotes the complex structure moduli space. These are discussed in turn in what follows.

Suppose X is endowed with a fixed complex structure. Since $c_1(X) = 0$, Yau's theorem ensures that X admits exactly one Ricci-flat metric in each Kähler class. As a result, the Kähler moduli space can be identified with the space of inequivalent Kähler classes that can be considered on X for a given complex structure. A Kähler class is an element of $J \in H^{1,1}(X, \mathbb{R})$ that satisfies suitable positivity requirements. More precisely, we have to demand that the volume of all holomorphic non-trivial $2k$-cycles $\mathcal{C}_k$ inside X, $k = 0, \dots, n$, be positive,

$$\int_{\mathcal{C}_{2k}} J^k > 0 \ . \tag{4.75}$$

Note that if J satisfies (4.75), so does λJ for any $\lambda > 0$, so that the Kähler moduli space has a natural cone structure. At the boundaries of the cone some cycles collapse to zero volume and the Calabi-Yau manifold can develop singularities.

Since we are considering Calabi-Yau n-folds with $n > 2$, $h^{2,0} = 0$. This ensures that any harmonic two-form on X is automatically of $(1,1)$-type. The Kähler class J can thus be expanded onto a basis of the integral cohomology $H^2(X, \mathbb{Z})$ with real coefficients. More precisely, let ω_Λ with $\Lambda = 1, \dots, b_2 = 1, \dots, h^{1,1}$ be a fixed basis in $H^2(X, \mathbb{Z})$. We write

$$J = v^\Lambda \, \omega_\Lambda \ , \tag{4.76}$$

and we refer to v^Λ as (real) Kähler moduli of the Calabi-Yau manifold. The harmonic representative of the class J is a $(1,1)$-form whose components are given by the components of the Hermitian metric, $J = i \, g_{i\bar{\jmath}} \, dz^i \wedge d\bar{z}^{\bar{\jmath}}$. From (4.76) we can then conclude that the metric variation induced by a variation of the Kähler moduli reads simply

$$\delta g_{i\bar{\jmath}} = -i \, \omega_{\Lambda \, i\bar{\jmath}} \, \delta v^\Lambda \qquad \text{or} \qquad \frac{\partial g_{i\bar{\jmath}}}{\partial v^\Lambda} = -i \, \omega_{\Lambda \, i\bar{\jmath}} \ . \tag{4.77}$$

The Kähler moduli space is equipped with a natural metric determined by a positive-definite pairing between two real $(1,1)$ forms. More explicitly,

$$G^{\text{Kähler}}_{\Lambda\Sigma} = \frac{1}{2\mathcal{V}} \int_X d^{2n}y \sqrt{g} \, g^{i\bar{\ell}} \, g^{k\bar{\jmath}} \, \frac{\partial g_{i\bar{\jmath}}}{\partial v^\Lambda} \frac{\partial g_{k\bar{\ell}}}{\partial v^\Sigma} = \frac{1}{2\mathcal{V}} \int_X \omega_\Lambda \wedge * \omega_\Sigma \ , \tag{4.78}$$

where

$$\mathcal{V} = \frac{1}{n!} \int_X J^n = \int d^{2n}y \, \sqrt{g} \tag{4.79}$$

is the volume of the Calabi-Yau n-fold.[5] It is convenient to define the intersection numbers

$$\mathcal{V}_{\Lambda_1\dots\Lambda_n} = \int_X \omega_{\Lambda_1} \wedge \dots \wedge \omega_{\Lambda_n} \ , \tag{4.80}$$

which are integral because $\omega_\Lambda \in H^2(X,\mathbb{Z})$. The volume of the n-fold can thus be written as

$$\mathcal{V} = \frac{1}{n!}\,\mathcal{V}_{\Lambda_1\dots\Lambda_n}\, v^{\Lambda_1}\dots v^{\Lambda_n} \ . \tag{4.81}$$

The derivatives of the volume with respect to the real Kähler moduli v^Λ can be expressed in terms of the harmonic forms ω_Λ as

$$\begin{aligned} \frac{\partial}{\partial v^\Lambda}\mathcal{V} &= \frac{1}{(n-1)!}\int_X \omega_\Lambda \wedge J^{n-1} = -i(g^{i\bar{\jmath}}\omega_{\Lambda\, i\bar{\jmath}})\mathcal{V} \ , \\ \frac{\partial}{\partial v^\Lambda}\frac{\partial}{\partial v^\Sigma}\mathcal{V} &= \frac{1}{(n-2)!}\int_X \omega_\Lambda\wedge\omega_\Sigma\wedge J^{n-2} = -(g^{i\bar{\jmath}}\omega_{\Lambda\, i\bar{\jmath}})(g^{k\bar{\ell}}\omega_{\Sigma\, k\bar{\ell}})\mathcal{V} - \int_X \omega_\Lambda \wedge *\omega_\Sigma \ . \end{aligned} \tag{4.82}$$

Note that in deriving these expressions we have exploited the fact that, for any harmonic $(1,1)$-form ω on a compact Ricci-flat Kähler manifold, $g^{i\bar{\jmath}}\omega_{i\bar{\jmath}}$ is covariantly constant. Thanks to the identities (4.82) we can write the metric (4.78) as the second derivative of a scalar function,

$$G^{\text{Kähler}}_{\Lambda\Sigma} = -\frac{1}{2}\frac{\partial}{\partial v^\Lambda}\frac{\partial}{\partial v^\Sigma}\log\mathcal{V} \ . \tag{4.83}$$

So far we have discussed pure geometry. In string theory and M-theory compactifications the real Kähler moduli v^Λ usually combine with further real scalars into complexified Kähler moduli. The resulting moduli space is itself a Kähler manifold. Furthermore, care has to be taken in comparing the abstract metric $G^{\text{Kähler}}_{\Lambda\Sigma}$ and the physical metric $\mathcal{G}_{\Lambda\Sigma}$ that enters the non-linear sigma-model after Weyl rescaling, given in (4.36) for general dimensional reduction on a Ricci-flat manifold. In many circumstances the coordinates v^Λ have to be replaced by new coordinates in Kähler moduli space to make the supersymmetry properties of the lower-dimensional action manifest. Since the details of the computation depend on the compactification scenario, we refrain from a general discussion. In chapter 7 we will study M-theory compactified on a Calabi-Yau threefold, while chapter 8 contains some remarks on M-theory on a Calabi-Yau fourfold.

Let us now discuss the complex structure moduli space. We already know that the tangent space to the complex structure moduli space is spanned by deformations associated to harmonic $(n-1,1)$-forms. A convenient basis for such forms is introduced as follows. Let Z^κ, $\kappa = 1,\dots,h^{n-1,1}$ be local complex coordinates on the complex structure moduli space (we anticipate that this space is naturally a complex manifold). Under an infinitesimal change of the complex structure parametrized by δZ^κ the complex coordinates z^i of the Calabi-Yau n-fold vary according to

$$\delta z^i = M^i_\kappa(z,\bar{z})\,\delta Z^\kappa \ , \tag{4.84}$$

where the non-holomorphic functions M^i_κ are only locally defined. As a result, the holomorphic differential dz^i and the holomorphic tangent vector $\partial_i \equiv \frac{\partial}{\partial z^i}$ undergo the variations

$$\delta z^i = \delta Z^\kappa\,\partial_j M^i_\kappa\,dz^j + \delta Z^\kappa\,\bar{\partial}_{\bar{\jmath}} M^i_\kappa\,d\bar{z}^{\bar{\jmath}} \ , \qquad \delta\partial_i = -\delta Z^\kappa\,\partial_i M^j_\kappa\,\partial_j - \delta\bar{Z}^{\bar{\kappa}}\,\partial_i \bar{M}^{\bar{\jmath}}_{\bar{\kappa}}\,\bar{\partial}_{\bar{\jmath}} \ . \tag{4.85}$$

[5] We adopt conventions such that $dz^1\wedge d\bar{z}^{\bar{1}}\wedge\dots\wedge dz^n\wedge d\bar{z}^{\bar{n}} = (-i)^n dy^1\wedge\dots\wedge dy^{2n} \equiv (-i)^n d^{2n}y$. As a result $J_{i\bar{\jmath}} = i\,g_{i\bar{\jmath}}$ implies the identity $J^n = n!\sqrt{g}\,d^{2n}y$.

As a consequence of the first equation we infer that a (p,q)-form is generically deformed into the sum of a $(p-1,q+1)$-form, a (p,q)-form, and a $(p+1,q-1)$-form.

At any point in complex structure moduli space we have

$$g(\bar\partial_{\bar\imath},\bar\partial_{\bar\jmath}) = g_{\bar\imath\bar\jmath} = 0 \ , \qquad \Omega(\partial_{i_1},\ldots,\partial_{i_{n-1}},\bar\partial_{\bar\jmath}) = \Omega_{i_1\ldots i_{n-1}\bar\jmath} = 0 \ . \tag{4.86}$$

Under an infinitesimal change δZ^κ of the complex structure moduli the variations of g and Ω must balance the variation of the (anti)holomorphic tangent vector ∂_i, $\bar\partial_{\bar\imath}$, in such a way that (4.86) still holds after the deformation. One thus finds

$$\begin{aligned}(\delta g)(\bar\partial_{\bar\imath},\bar\partial_{\bar\jmath}) = \delta g_{\bar\imath\bar\jmath} = 2\delta Z^\kappa\, g_{(\bar\imath|\ell}\,\bar\partial_{\bar\jmath)}M^\ell_\kappa \ ,\\ (\delta\Omega)(\partial_{i_1},\ldots,\partial_{i_{n-1}},\bar\partial_{\bar\jmath}) = \delta\Omega_{i_1\ldots i_{n-1}\bar\jmath} = \delta Z^\kappa\,\Omega_{i_1\ldots i_{n-1}\ell}\,\bar\partial_{\bar\jmath}M^\ell_\kappa \ .\end{aligned} \tag{4.87}$$

Crucially, symmetrization in indices $\bar\imath$, $\bar\jmath$ on the right hand side of the first equation can be dropped. In fact, $g_{[\bar\imath|\ell}\,\bar\partial_{\bar\jmath]}M^\ell_\kappa$ would be the components of a closed but not exact $(0,2)$-form, which must vanish since we are considering Calabi-Yau n-fold with $n>2$ and $h^{0,2}=0$. We thus have $\delta Z^\kappa\,\bar\partial_{\bar\jmath}M^i_\kappa = \frac{1}{2}\,g^{i\bar\ell}\,\delta g_{\bar\ell\bar\imath}$, and since $\delta g_{\bar\imath\bar\jmath}$ is a zeromode of the Lichnerowicz operator we infer from the discussion at the beginning of this section that the $(n-1,1)$-forms

$$\chi_{\kappa\, i_1\ldots i_{n-1}\bar\jmath} = \Omega_{i_1\ldots i_{n-1}\ell}\,\bar\partial_{\bar\jmath}M^\ell_\kappa \tag{4.88}$$

are harmonic. In fact, the set $\{\chi_\kappa\}$ provides the anticipated basis of $H^{n-1,1}(X,\mathbb{C})$.

As a consequence of the remark below (4.85) the holomorphic $(n,0)$-form Ω varies under a change in complex structure in the sum of a $(n,0)$-form a $(n-1,1)$-form, $\delta\Omega = \delta\Omega|_{n,0} + \delta\Omega|_{n-1,1}$. Equation (4.5.2) tells us that $\delta\Omega|_{n-1,1} = \delta Z^\kappa\chi_\kappa$, which is a harmonic and hence closed form. Since the complex structure variation δ commutes with the exterior differential d, we have $d\delta\Omega=0$. As a result $d\delta\Omega|_{n,0} = \bar\partial\delta\Omega|_{n,0} = 0$, i.e. $\delta\Omega|_{n,0}$ is holomorphic. It must therefore be a multiple of Ω. Furthermore, one can check that $\delta\bar Z^{\bar\kappa}$ does not enter the variation of Ω. In summary, we can write

$$\frac{\partial}{\partial Z^\kappa}\Omega = k_\kappa\,\Omega + \chi_\kappa \ , \qquad \frac{\partial}{\partial \bar Z^{\bar\kappa}}\Omega = 0 \ , \tag{4.89}$$

where k_κ is a scalar that does not depend on the Calabi-Yau coordinates but generically depends on the complex structure moduli Z^κ.

The metric variation encoded in (4.5.2) can be equivalently formulated in terms of the harmonic forms χ_κ as

$$\delta g_{\bar\imath\bar\jmath} = \frac{2}{(n-1)!\|\Omega\|^2}\chi_{\kappa\,\ell_1\ldots\ell_{n-1}\bar\imath}\,\overline{\Omega}^{\ell_1\ldots\ell_{n-1}}{}_{\bar\jmath}\,\delta Z^\kappa \ . \tag{4.90}$$

Even though it is not manifest, the right hand side is symmetric in $\bar\imath,\bar\jmath$ by virtue of the absence of harmonic $(0,2)$-forms, as outlined above. The complex structure moduli space is naturally equipped with the metric

$$G^{\rm cstr}_{\kappa_1\bar\kappa_2} = \frac{1}{2\mathcal{V}}\int_X d^{2n}y\sqrt{g}\,g^{k\bar\imath}\,g^{\ell\bar\jmath}\,\frac{\partial g_{\bar\imath\bar\jmath}}{\partial Z^{\kappa_1}}\,\frac{\partial g_{k\ell}}{\partial \bar Z^{\bar\kappa_2}} = \frac{2}{(n-1)!\mathcal{V}\|\Omega\|^2}\int_X \chi_{\kappa_1}\wedge *\bar\chi_{\bar\kappa_2} \ . \tag{4.91}$$

This expression can be further manipulated recalling (4.57) and making use of the identity

$$*\,\chi_\kappa = (-)^{\frac{n(n+1)}{2}} i^{n-2}\,\chi_\kappa\ , \tag{4.92}$$

which can be derived using the fact that χ_κ is a primitive $(n-1,1)$-form, see for instance [46]. We are thus left with

$$G^{\rm cstr}_{\kappa_1\bar\kappa_2} = -\frac{2}{(n-1)!}\frac{\displaystyle\int_X \chi_{\kappa_1}\wedge\bar\chi_{\bar\kappa_2}}{\displaystyle\int_X \Omega\wedge\overline{\Omega}}\ , \tag{4.93}$$

This form of the metric is manifestly independent of the volume of the Calabi-Yau n-fold and of any Kähler modulus. Furthermore, it is useful to prove that the complex structure moduli space is itself a Kähler manifold. Indeed, one can easily check that (4.89) implies

$$\frac{\partial}{\partial Z^\kappa}\int_X \Omega\wedge\overline{\Omega} = k_\kappa \int_X \Omega\wedge\overline{\Omega}\ ,\quad \frac{\partial}{\partial Z^{\kappa_1}}\frac{\partial}{\partial \bar Z^{\bar\kappa_2}}\int_X \Omega\wedge\overline{\Omega} = k_{\kappa_1}\bar k_{\bar\kappa_2}\int_X \Omega\wedge\overline{\Omega} + \int_X \chi_{\kappa_1}\wedge\bar\chi_{\bar\kappa_2}\ . \tag{4.94}$$

As a result, we have

$$G^{\rm cstr}_{\kappa_1\bar\kappa_2} = \frac{\partial}{\partial Z^{\kappa_1}}\frac{\partial}{\partial \bar Z^{\bar\kappa_2}}\mathcal{K}^{\rm cstr}\ , \tag{4.95}$$

where the Kähler potential $\mathcal{K}^{\rm cstr}$ is given by

$$\mathcal{K}^{\rm cstr} = -\frac{2}{(n-1)!}\log\left[i^n(-)^{\frac{n(n-1)}{2}}\int_X \Omega\wedge\overline{\Omega}\right]\ . \tag{4.96}$$

Note that the argument of the logarithm is a positive real number, because (4.57) shows that it equals $\mathcal{V}\|\Omega\|^2$.

The geometry of Calabi-Yau moduli space exhibits even richer structures than those we have reviewed. For instance, both the complexified Kähler moduli space and the complex structure moduli space in Calabi-Yau threefold compactification of Type II superstring constitute examples of so-called special geometries. One of the key properties of these spaces is the fact that their Kähler potential can be written in terms of a holomorphic prepotential. Since we will not need to develop this interesting subject we refer the reader to e.g. [94, 95, 96, 97] for an account. We also refrain from a review of the global structure of the complex structure moduli space, which is essential for explicit computations of the Kähler potential $\mathcal{K}^{\rm cstr}$, see for instance [98].

4.6 A brief overview of Spin(7) manifolds

4.6.1 Geometry and topology of Spin(7) manifolds

Recall that by Spin(7) manifold we mean an eight-dimensional Riemannian manifold with restricted holonomy group given by Spin(7), and not a proper subgroup thereof. The Spin(7) group is defined as the universal cover of the rotation group $SO(7)$, but it can be equivalently characterized as follows.

Let y^m, $m = 1, \dots, 8$ be Cartesian coordinates in $\mathbb{R}^8$, and let $dy^{mnpq} = dy^m \wedge dy^n \wedge dy^p \wedge dy^q$. Next, define the self-dual four-form Φ_0 by setting

$$\begin{aligned} -\Phi_0 = {} & dy^{1234} + dy^{1256} + dy^{1278} + dy^{1357} - dy^{1368} - dy^{1458} - dy^{1467} \\ & + dy^{5678} + dy^{3478} + dy^{3456} + dy^{2468} - dy^{2457} - dy^{2367} - dy^{2358} \ . \end{aligned} \tag{4.97}$$

The right hand side is the mathematicians' standard form of the Spin(7) structure in $\mathbb{R}^8$. The minus sign is introduced to agree with the physics literature, see e.g. [99]. The subgroup of $SO(8)$ that preserves Φ_0 is isomorphic to Spin(7). This is related to the fact that the structure group of an eight-dimensional Riemannian manifold X can be reduced from $SO(8)$ to Spin(7) if and only if there exists a self-dual four-form Φ such that, for any $p \in X$, there exists an isomorphism from T_pX onto $\mathbb{R}^8$ such that $\Phi|_p$ is mapped to Φ_0. In particular Φ is globally defined and nowhere vanishing. If it is additionally covariantly constant with respect to the Levi-Civita connection, the restricted holonomy group of X is contained in Spin(7). The converse is also true: if the restricted holonomy is contained in Spin(7), there exists a four-form Φ with the expected properties. The four-form Φ is commonly referred to as Cayley calibration.

Let us mention a minor point: we do not fix the normalization of the Cayley form, but we fix its 'sign.' This can be made precise as follows. An orthonormal frame e^a_m induces at any $p \in X$ a map $e_p : T_pX \to \mathbb{R}^8$ given by $v^m \mapsto e^a_m v^m$. We then fix an orientation on X and we demand the existence of a positively oriented frame e^a_m such that $\Phi|_p = \lambda\, e^*_p \Phi_0$ at any $p \in X$ for some $\lambda > 0$. If we choose the opposite 'sign' small modifications have to be performed in some of the equations recorded below. For instance, the second term on the right hand side of (4.100) would change sign.

An alternative characterization of a manifold with restricted holonomy contained in Spin(7) is furnished by the existence of a Majorana-Weyl covariantly constant spinor of positive chirality η_+, as anticipated in section 4.4. The Cayley calibration can be constructed as a real Majorana bilinear as

$$\Phi = \frac{1}{4!}\, \eta_+^{\mathsf{T}} C\, \gamma_{mnpq}\, \eta_+\, dy^m \wedge dy^n \wedge dy^p \wedge dy^q \ . \tag{4.98}$$

Actually this bilinear, together with the norm of the spinor $\eta_+^{\mathsf{T}} C\, \eta_+$, is the only bilinear that is not trivially zero by virtue of the chirality and Majorana flip property of the commuting spinor η_+. In particular, as already noted below (4.61), it is not possible to build any two-form bilinear with a single Majorana-Weyl spinor. This is consistent with the fact that manifolds with restricted holonomy contained in Spin(7) do not generically admit any complex structure.

An immediate consequence of self-duality of Φ is the relation

$$\Phi \wedge \Phi = \|\Phi\|^2 * 1 \ , \qquad \|\Phi\|^2 = \frac{1}{4!} \Phi_{mnpq} \Phi^{mnpq} \ . \tag{4.99}$$

A Fierz rearrangement can be used to prove the more involved algebraic identity

$$\Phi^{m_1 m_2 m_3 p} \Phi_{n_1 n_2 n_3 p} = \frac{3}{7} \|\Phi\|^2 \delta^{[m_1}_{[n_1} \delta^{m_2}_{n_2} \delta^{m_3]}_{n_3]} - \frac{9}{\sqrt{14}} \|\Phi\| \delta^{[m_1}_{[n_1} \Phi_{n_2 n_3]}{}^{m_2 m_3]} \ . \tag{4.100}$$

Alternatively, (4.100) can be derived considering a fixed but arbitrary point p and choosing coordinates in such a way that $\Phi|_p = \lambda\, \Phi_0$ ($\lambda > 0$) and $ds^2|_p = \delta_{mn} dy^m dy^n$.

The Cayley calibration induces a split of the cohomology groups of X according to the reduction of the structure group from $SO(8)$ to Spin(7). One can show that

$$\begin{aligned}
&H^0(X,\mathbb{R}) = \mathbb{R} \ , \\
&H^1(X,\mathbb{R}) = 0 \ , \\
&H^2(X,\mathbb{R}) = H^2_{\mathbf{21}}(X,\mathbb{R}) \ , \\
&H^3(X,\mathbb{R}) = H^3_{\mathbf{48}}(X,\mathbb{R}) \ , \\
&H^4(X,\mathbb{R}) = H^4_{\mathbf{1}\mathrm{S}}(X,\mathbb{R}) + H^4_{\mathbf{27}\mathrm{S}}(X,\mathbb{R}) + H^4_{\mathbf{35}\mathrm{A}}(X,\mathbb{R}) \ ,
\end{aligned} \tag{4.101}$$

and similar relations hold for the cohomology group $H^p(X,\mathbb{R})$ with $p = 5,\ldots,8$ by virtue of Hodge duality. The boldface subscripts denote the relevant Spin(7) representation, while S and A denote self-dual and antiself-dual four-forms, respectively. Note that the Spin(7) singlet cohomology group $H^4_{\mathbf{1}\mathrm{S}}(X,\mathbb{R})$ is precisely generated by the Cayley calibration. Let us also remark that $H^1(X,\mathbb{R}) = 0$ is a consequence of strict Spin(7) holonomy: as noted in section 4.3, the first Betti number of a Ricci-flat manifolds receives contributions from torus factors only, and those are not allowed for a manifold with strict Spin(7) holonomy. This also means that Spin(7) manifold cannot have any continuous isometry. Let us close this section by recalling that a Spin(7) manifold has only three independent Betti numbers, by virtue of the constraint

$$b_2 - b_3 - b_{4\mathrm{S}} + 2b_{4\mathrm{A}} + 25 = 0 \ , \tag{4.102}$$

where $b_p = \dim_{\mathbb{R}} H^p(X,\mathbb{R})$, $b_{4\mathrm{S}} = \dim_{\mathbb{R}} H^4_{\mathrm{S}}(X,\mathbb{R})$, and $b_{4\mathrm{A}} = \dim_{\mathbb{R}} H^4_{\mathrm{A}}(X,\mathbb{R})$.

4.6.2 Moduli space of Spin(7) manifolds

Global properties of the moduli space of Spin(7) manifolds are under considerably less control than the corresponding properties for Calabi-Yau manifolds. Nonetheless, for the purpose of discussing the effective action of M-theory compactified on a Spin(7) manifolds some local considerations on infinitesimal deformations of the metric and Cayley calibration will suffice.

Let us begin by a simple class of metric deformations, given by an overall infinitesimal rescaling of the metric, $\delta g_{mn} = g_{mn} Y$. It is easily checked that this deformation satisfies the Lichnerowicz equation if and only if Y is a zeromode of the scalar Laplacian. As already noted in section 4.3, it is therefore a constant on the internal manifold, and can be related to the variation of the volume. A simple computation shows

$$\delta g_{mn} = \frac{1}{4} g_{mn} \delta \log \mathcal{V} \qquad \text{or} \qquad \frac{\partial g_{mn}}{\partial \mathcal{V}} = \frac{1}{4\mathcal{V}} g_{mn} \ , \tag{4.103}$$

where in the second equation we interpret the volume $\mathcal{V}$ as one of the coordinates of the Spin(7) moduli space. To identify the remaining coordinates we need to consider metric deformations $\delta g_{mn} = Y_{mn}$ with Y_{mn} transverse and traceless. It can be shown that such solutions of the Lichnerowicz equation are in one-to-one correspondence with harmonic antiself-dual forms. We can then write

$$\delta g_{mn} = \frac{7}{6\|\Phi\|^2} \, \xi_{A\, m p_1 p_2 p_3} \Phi_n{}^{p_1 p_2 p_3} \, \delta\varphi^A \qquad \text{or} \qquad \frac{\partial g_{mn}}{\partial \varphi^A} = \frac{7}{6\|\Phi\|^2} \, \xi_{A\, m p_1 p_2 p_3} \Phi_n{}^{p_1 p_2 p_3} \ , \tag{4.104}$$

where ξ_A is a basis of $H^4_{\rm A}(X,\mathbb{R})$ and φ^A are local real coordinates on the Spin(7) moduli space orthogonal to $\mathcal{V}$. Although it is not manifest, the right hand side of (4.104) is symmetric and traceless in indices m, n as a result of the fact that χ_A transforms in the **35** representation of Spin(7). We refer the reader to [100, 99, 101] for a proof of this claim and for a thorough discussion of various useful identities involving the Cayley calibration. The same antiself-dual four-forms ξ_A enter the variation of the Cayley calibration, which reads

$$\delta\Phi = K_{\mathcal{V}}\Phi\,\delta\mathcal{V} + (K_A\Phi + \xi_A)\,\delta\varphi^A\ , \tag{4.105}$$

where $K_{\mathcal{V}}$, K_A are constant on the Spin(7) manifold but generically depend on the moduli $\mathcal{V}$ and φ^A. Note that the numerical factor in (4.104) is indeed chosen to ensure mutual compatibility among (4.104), (4.105), and the identity (4.100).

The tracelessness of the metric variation in (4.104) ensures the absence of off-diagonal $d\mathcal{V}d\varphi^A$ terms in the metric of the Spin(7) moduli space. Let us focus on the metric components associated to the moduli φ^A. We have

$$G^{\rm Spin(7)}_{AB} = \frac{1}{4\mathcal{V}}\int_X d^8y\sqrt{g}g^{mp}g^{nq}\frac{\partial g_{mn}}{\partial\varphi^A}\frac{\partial g_{pq}}{\partial\varphi^B} = -\frac{7}{48}\frac{\displaystyle\int_X \xi_A\wedge\xi_B}{\displaystyle\int_X \Phi\wedge\Phi}\ , \tag{4.106}$$

where in the second step we have made use of (4.104), (4.99), (4.100), together with the antiself-duality of ξ_B and the identity

$$\xi_{A\,p_1p_2[mn}\Phi^{p_1p_2}{}_{rs]} = 0\ , \tag{4.107}$$

which is proven e.g. in [99]. This concludes our general analysis of the local structure of Spin(7) moduli space. In chapter 8 we discuss the full reduction of M-theory on a Spin(7) manifold, including the $d\mathcal{V}d\mathcal{V}$ component of the moduli space metric, which is not discussed here as it depends on the specific compactification scenario.

CHAPTER 5

An introduction to F-theory from the M-theory perspective

F-theory constitutes a geometric formulation of a class of Type IIB string theory vacua in which the $SL(2,\mathbb{Z})$ symmetry of the theory is interpreted as the modular group of an auxiliary two-torus varying over spacetime. The aim of this chapter is to clarify this statement and to elucidate the connection between F-theory, the S-duality properties of Type IIB discussed in section 3.2, and the reduction of M-theory on a two-torus introduced in section 3.4. Such an analysis will constitute the basis for the duality between F-theory and M-theory and will provide a justification to the prescription to compute the effective action of an F-theory compactification by means of the dual M-theory setup.

As we will see, the duality between F-theory and M-theory takes a simpler form when restricted to the gravity and moduli sector of the compactification, while it is more subtle in the gauge and matter sector. For instance, properties of charged matter will not be directly visible in the effective actions we discuss in part II, and indirect arguments are need to extract them from the M-theory setup. For this reason our exposition will only cover general aspects of the gauge and matter sectors of F-theory. In particular we will not develop the technology necessary to engineer specific non-Abelian gauge groups and therefore we will not be able to do justice to the vast and interesting subject of F-theory model-building. By the same token, many intriguing features of F-theory will not be covered, such as the duality with heterotic string theory, the geometry of singular genus-one fibrations, the development of local GUT-modes and their connection to the global properties of the compactification, and the rich physics of $U(1)$ gauge fields in F-theory, to name a few. The reader is referred for example to the lecture notes [102, 103] and the reviews [104, 105] for an account of the aspects of F-theory compactifications that we do not address in what follows.

5.1 Seven-brane backreaction

Most conservatively, F-theory can be thought of as a geometrized framework to incorporate seven-branes in a Type IIB compactification in a fully consistent, backreacted, and non-perturbative way. Achieving this goal is desirable from several points of view. The rich interplay between bulk and brane physics in Type II string theory is the key to many interesting directions for string phenomenology, such as intersecting D6-brane models in Calabi-Yau orientifolds of Type IIA or magnetized D7/D3-brane models in Calabi-Yau orientifolds of Type IIB, see e.g. for review [106]. In these compactification scenarios D-branes are usually treated in a suitable probe approximation, neglecting their backreaction in the large volume limit. As we will argue below this requires special care for D7-branes, and more generically seven-branes. The framework of F-theory exploits in a clever fashion the $SL(2,\mathbb{Z})$ duality of Type IIB in order to address this problem in an elegant and powerful way. At the same time, F-theory allows us to go beyond the scope of perturbative Type II superstring theory. As a result, for example, exceptional gauge groups can be realized on the world-volume of seven-brane stacks, paving the way to interesting GUT model building scenarios.

5.1.1 Problems with codimension-two branes

Localized objects with $d_\perp \geq 3$ transverse spatial direction, such as Dp-branes with $p < 7$ in Type II string theory, behave in a way that is qualitatively the same as point charges in ordinary three-dimensional electrostatics. In the transverse directions to their world-volume these lower-dimensional branes act as point-like sources for the bulk fields they couple to. Their effects are negligible far away from the brane: if they are electrically charged with respect to a $(p+1)$-form, the associated field strength decays according to a power law $r^{-(d_\perp -1)}$ with the distance r from the source; the dilaton approaches an asymptotic value that can be tuned to be small; the metric in a neighborhood of infinity is simply the standard flat Euclidean metric.

The situation for codimension-two objects, like D7-branes and more generally seven-branes in Type IIB, is qualitatively different. This can be anticipated from the fact that the Green's function of the Laplace operator in two dimensions is proportional to $\log r$. Recall that a D7-brane couples electrically to C_8 and magnetically to C_0. As a result, it acts as a source term in the equation of motion for C_8, or equivalently in the Bianchi identity for C_0. Schematically we have

$$d * F_9 = dF_1 = \delta_{\rm D7} \ , \tag{5.1}$$

where F_1, F_9 are the field strengths associated to C_0, C_8 respectively and $\delta_{\rm D7} = \delta(x)\,\delta(y)\,dx \wedge dy$ is a two-form with legs along the transverse space to the D7-brane stack with coordinates x, y that has a δ-function-like support on the world-volume of the D7-brane stack, which is located at the origin of the (x, y)-plane.[1]

The presence of a source term in the Bianchi identity for F_1 implies that $F_1 = dC_0$ can only hold

[1] Objects such as $\delta_{\rm D7}$ can be though of as a generalization of distributions to p-forms and are most properly understood in the framework of the theory of currents, see e.g. [107].

locally or alternatively that C_0 cannot be single-valued. This is readily seen integrating (5.1) over a small disk in the transverse space to the D7-brane centered at the origin,

$$\oint_\gamma C_0 = 1 \ , \tag{5.2}$$

where γ is the boundary of the disk with counterclockwise orientation. We thus learn that the axion C_0 undergoes the monodromy

$$C_0 \to C_0 + 1 \tag{5.3}$$

as we encircle once the location of a D7-brane. This implies that the D7-brane induces a non-trivial profile for the axio-dilaton $\tau = C_0 + ie^{-\Phi}$ in the transverse space.

The latter is most conveniently analyzed introducing the complex coordinate $u = x + iy$. In fact, it is possible to show that supersymmetry arguments and the Type IIB bulk equations of motion impose that τ is a holomorphic function of u. The monodromy (5.3) suggests therefore the following behavior of the axio-dilaton near the location of the brane,

$$\tau \overset{u\to 0}{\sim} \frac{1}{2\pi i} \log \frac{u}{\lambda} \ , \tag{5.4}$$

where λ is a complex constant. This equation implies

$$e^{-\Phi} \overset{u\to 0}{\sim} -\frac{1}{2\pi} \log \frac{|u|}{|\lambda|} \ , \tag{5.5}$$

from which we see that the string coupling constant approaches zero near the location of a D7-brane. In the region $|u| \ll |\lambda|$ string perturbation theory is therefore valid. It is clear, though, that this expression has to be modified for $|u| \gtrsim |\lambda|$, as it would yield negative values for $e^{-\Phi}$.

Extreme care is required in constructing complete solutions of the Type IIB bulk equations of motion that reproduce the desired behavior (5.4) near the location of a D7-brane. First of all, since τ is multi-valued in the u-plane it is convenient to describe the u-dependence of the axio-dilaton using the Klein invariant j-function $j(\tau)$. It is worth to recall a few properties of this special function. This is a meromorphic function defined on the upper half plane that provides a bijection of the fundamental domain of τ onto the Riemann sphere. The cusp $\tau = i\infty$ is mapped to the point at infinity $j = \infty$; more precisely, j admits a Laurent expansion in $q = e^{2\pi i\tau}$ of the form

$$j(q) = \frac{1}{q} + 744 + \mathcal{O}(q) \ , \tag{5.6}$$

which exhibits a simple pole at the cusp $q = 0$. Besides $\tau = i\infty$ the fundamental domain has two other special points, which are fixed under the action of some element of the modular group $PSL(2, \mathbb{Z})$: the point $\tau = i$ is fixed under the action of S, while $\tau = e^{2\pi i/3} \equiv \rho$ is fixed under ST. The action of generators T and S of the modular group on τ was given in (3.25). In a neighborhood of the special points $\tau = i$, ρ we have

$$\begin{aligned} j(\tau) &\overset{z\to i}{\sim} 1728 + k_1(\tau - i)^2 + \mathcal{O}((\tau - i)^3) \ , \\ j(\tau) &\overset{z\to \rho}{\sim} 0 + k_2(\tau - \rho)^3 + \mathcal{O}((\tau - \rho)^4) \ . \end{aligned} \tag{5.7}$$

The inverse map $w \mapsto \tau = j^{-1}(w)$ develops non-trivial monodromies around $w = 1728$ and $w = 0$,

$$
\begin{aligned}
j^{-1}(w) &= \tau \to -\frac{1}{\tau} && \text{around} \quad w = 1728 \ , \\
j^{-1}(w) &= \tau \to -1 - \frac{1}{\tau} && \text{around} \quad w = 0 \ .
\end{aligned} \tag{5.8}
$$

After these mathematical preliminaries we can go back to the problem of finding a solution matching the asymptotic behavior (5.4). The simplest possibility is

$$
j(\tau(u)) = \frac{\lambda}{u} \ . \tag{5.9}
$$

Indeed, as $u \to 0$ the right hand side develops a simple pole, implying $\tau(u) \to i\infty$. The Laurent expansion (5.6) then reproduces (5.4). As we increase $|u|$ we move away from the region of small coupling, until we reach the point $u = 1728/\lambda$. There $\tau = i$ (thus $g_s = 1$) and there is a non-trivial monodromy $\tau \to -1/\tau$. This special point breaks rotational invariance in the u-plane because of the phase of the complex number λ. As we increase $|u|$ further, up to a neighborhood of $u = \infty$, rotational invariance is approximately restored and $\tau \to \rho$, with a non-trivial monodromy $\tau \to -1 - 1/\tau$ as we encircle the point at infinity $u = \infty$. We have gone through a detailed analysis of this example in order to highlight the subtleties that are typically encountered in studying the τ-profile generated by a configuration of D7-branes at finite and large distances from the branes. In particular note that in this example the asymptotic value of the string coupling constant at large distances from the brane is fixed to the non-perturbative value $g_s = 2/\sqrt{3} \approx 1.15$. Obviously this effect cannot be neglected appealing to the usual large volume argument.

Additional complications arise as soon as gravity is considered. Our problem is effectively $(2+1)$-dimensional. In three dimensions the Riemann tensor is proportional to the Ricci tensor at every point in spacetime. As a result, Einstein's equation dictates that spacetime is everywhere locally flat, except at the precise location of the sources of the energy-momentum tensor. Localized sources, however, induce a deficit angle in the flat geometry that surrounds them: for instance, a point particle of mass m induces a deficit angle $\delta = m\kappa^2$, where κ is the effective gravitational constant in three dimensions. The case of D7-brane is somewhat peculiar. The Einstein-frame DBI action of the D7-brane contains the coupling

$$
S_{\text{DBI}} \supset \frac{2\pi}{\ell_s^8} \int_{\mathcal{W}_8} d^8\xi \, e^{\Phi} \sqrt{-g} \ , \tag{5.10}
$$

where ξ are coordinates of the D7-brane world-volume $\mathcal{W}_8$ and g is the pullback of the Einstein metric of the Type IIB bulk. Note the positive power of the dilaton prefactor. Since $g_s \to 0$ as we approach the location of the brane, the DBI action vanishes and the D7-brane behaves effectively as a zero-tension object, leading to no deficit angle in the limit $u \to 0$. The τ-profile generated by the brane, however, does carry energy-momentum and generates a deficit angle in the asymptotic region far away from the brane. Its effects can be dramatic: for example, a deficit angle $\delta = 4\pi$ turns the transverse space from a plane to a compact sphere. The precise value of this angle depends on the brane configurations and on the monodromy that are imposed on the functions that determine the metric. For instance, in the case of a single D7-brane the classic analysis in [108, 109] predicts $\delta = \pi/6$, but this setup was revisited in [110] yielding $\delta = 2\pi/3$. We do not need to discuss this interesting problem in detail. It has been mentioned to exemplify how subtle and difficult can be an explicit construction of Type IIB solution with backreacted D7-branes in eight dimensions. As soon as we consider lower-dimensional vacua the problem becomes soon intractable. Luckily, F-theory provides an alternative route to the description of backreacted setup with D7-branes. Indeed, it also captures naturally seven-branes beyond perturbative Type IIB.

5.1.2 Seven-brane monodromies

The only feature of (p, q)-seven-branes that we need to consider in detail is the monodromy they induce in the τ-profile as we move along a small loop around their location. To begin with, we can rewrite (5.3) in terms of the $SL(2, \mathbb{Z})$ monodromy matrix

$$M_{\mathrm{D7}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} . \tag{5.11}$$

The monodromy of a more general (p, q)-brane is then computed recalling that a (p, q)-string can be obtained from a fundamental string using a transformation of the form (3.32),

$$\begin{pmatrix} q & p \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} r & s \\ q & p \end{pmatrix} , \tag{5.12}$$

where the integers r, s are such that $pr - qs = 1$ but otherwise arbitrary. If follows that the monodromy matrix associated to a (p, q)-seven-brane reads

$$M_{(p,q)} = \begin{pmatrix} r & s \\ q & p \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r & s \\ q & p \end{pmatrix} = \begin{pmatrix} 1 + pq & p^2 \\ -q^2 & 1 - pq \end{pmatrix} , \tag{5.13}$$

and indeed only depends on p, q. This monodromy matrix acts on the charges $\tilde{q}$, $\tilde{p}$ of a $(\tilde{p}, \tilde{q})$-string as

$$\begin{pmatrix} \tilde{q} & \tilde{p} \end{pmatrix} \to \begin{pmatrix} \tilde{q}' & \tilde{p}' \end{pmatrix} = \begin{pmatrix} \tilde{q} & \tilde{p} \end{pmatrix} M_{(p,q)} . \tag{5.14}$$

Stated differently, if a $(\tilde{p}, \tilde{q})$-string is carried once around the location of a (p, q)-seven-brane, it reemerges as a $(\tilde{p}', \tilde{q}')$-string. If $\tilde{p}' = \tilde{p}$, $\tilde{q}' = \tilde{q}$ the string and the seven-brane are said to be mutually local. As a sanity check note that a (p, q)-string is mutually local with a (p, q)-seven-brane, so that is can actually end on such a brane. The monodromy matrix (5.14) of seven-branes is a distinctive feature that can be used to detect their presence: if the τ-profile is known, an analysis of its monodromies around some point reveals which kind of seven-brane configurations are located at the that point.

For our following considerations we need also to consider the monodromy matrix associated to an O7-plane. More precisely, the O7-plane sits at the fixed locus of the orientifold projection $\sigma_{\mathrm{hol}}\, \Omega_{\mathrm{p}} (-)^{F_L}$, where $\sigma_{\mathrm{hol}} : u \mapsto -u$ reflects two real directions and thus requires the introduction of the $(-)^{F_L}$ factor, see (2.45). The monodromy matrix reads

$$M_{\mathrm{O7}} = -M_{\mathrm{D7}}^{-4} = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix} . \tag{5.15}$$

This can be justified as follows. Recall from (2.46) that an O7-plane carries eight units of D7-brane tension and charge in the ‘upstairs’ picture, i.e. in the space which is acted upon by the involution σ_{hol}.

We would thus need four D7-branes and their images to cancel the tension and charge of the O7-plane. This explains the power -4 in $M_{\rm D7}^{-4}$, since monodromies take place in the quotient space and thus count the number of brane/image brane pairs. The factor -1 in front of $M_{\rm D7}^{-4}$ comes from the intrinsic parities of Type IIB fields under the action of $\Omega_{\rm p}(-)^{F_L}$. This is most easily seen looking at the monodromy of a system of one O7-plane and four pairs of D7-branes and images,

$$M_{\rm O7/D7} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{5.16}$$

This monodromy acts trivially on τ, as it should be since the combined O7/D7 system is neutral with respect to C_0. Nonetheless, (3.19) shows that both B_2 and C_2 change sign under the action of $M_{\rm O7/D7}$. This matches with the results of table (2.3) about $\Omega_{\rm p}$-parities and the fact that $(-)^{F_L}(B_2, C_2) = (B_2, -C_2)$. Finally, let us mention that the deficit angle close to an O7/D7 system is $\delta = \pi$, consistently with the $\mathbb{Z}_2$ action of $\sigma_{\rm hol}$ on the transverse space.

5.2 Elliptic fibrations and seven-branes

In section 3.4 we have established the nine-dimensional duality between M-theory compactified on a torus and Type IIB string theory compactified on a circle. One of the main outcomes of this duality is the reinterpretation of the $SL(2,\mathbb{Z})$ S-duality group of Type IIB as large diffeomorphism of a torus, or equivalently modular transformations of its complex structure parameter. In essence, the program of F-theory is to generalize this situation to lower-dimensional setups in which some of the nine dimensions are compactified. The duality between M-theory on a torus and Type IIB on a circle then becomes the duality between M-theory and F-theory, which we analyze in greater detail in section 5.3.

For the time being we can simply make the observation that the reinterpretation of the $SL(2,\mathbb{Z})$ symmetry of Type IIB in terms of the modular parameter of a torus can be extremely convenient to address the problem of Type IIB setups with seven-branes. Let us make this remark more precise. Suppose we are interested in a Type IIB setup with $2n$ real dimensions compactified on a Kähler n-fold B_n and $10-2n$ non-compact directions spanning Minkowski spacetime. We also allow for spacetime-filling seven-branes wrapping divisors in B_n, i.e. holomorphic cycles of complex codimension one. Our task would be to solve the equations of motion for the metric and the axio-dilaton with a non-trivial dependence on the coordinates of B_n and allowing for a multi-valued τ in order to account for the monodromies induced by the seven-brane backreaction. Note that we do not expect B_n to be Ricci-flat, since its curvature is counterbalanced by the non-trivial axio-dilaton profile.

An alternative strategy is to consider an auxiliary two-torus T^2 fibered over B_n, in such a way that at any point p on B_n the value of the axio-dilaton is identified with the complex structure parameter of the torus fiber at p. In this way we have an intrinsic description of the axio-dilaton profile and the problem is translated into the determination of the geometry of the total space X_{n+1} of the torus fibration over B_n. As we will see in section 5.3 supersymmetry demands that X_{n+1} be a Calabi-Yau $(n+1)$-fold and the fiber depend holomorphically on the space B_n, referred to as the base. We thus have to study Calabi-Yau spaces that admit a holomorphic genus-one fibration. In what follows we will also make the assumption that the genus-one fibration comes with a global holomorphic section. The fiber is thus a torus with a marked zero-point, and has therefore the structure of an elliptic curve. In summary, we are left with the study of elliptically fibered Calabi-Yau $(n+1)$-folds. Even though this might seem as intractable as our original task, a lot of information about the geometry of X_{n+1} can be extracted using powerful techniques from algebraic geometry. This gives a handle on the dynamics of seven-branes and automatically takes into account their backreaction. We can make the previous discussion more concrete. To this end we need to recall a few facts about elliptic curves and elliptic fibrations.

5.2.1 Weierstrass form of an elliptic curve

An elliptic curve can be realized as the vanishing locus of a homogeneous polynomial in the weighted projective space $\mathbb{P}_{2,3,1}$. The latter is defined as $\mathbb{C}^3 \setminus \{(0,0,0)\}$ modded out by the $\mathbb{C}^*$-action

$$(x,y,z) \sim (\lambda^2 x, \lambda^3 y, \lambda z) \ , \qquad \lambda \in \mathbb{C}^* \ , \tag{5.17}$$

where x, y, z are coordinates in $\mathbb{C}^3$. The equivalence class of (x,y,z) will be denoted $[x:y:z]$ and determines a point in $\mathbb{P}_{2,3,1}$. The coordinates x, y, z are referred to as homogeneous coordinates on $\mathbb{P}_{2,3,1}$. The equation that defines the elliptic curve reads

$$\mathbb{E}: \quad y^2 = x^3 + f\, x\, z^4 + g\, z^6 \ , \tag{5.18}$$

and is known as Weierstrass form. The quantities f and g are complex parameters. Note that (5.18) is compatible with (5.17) as all monomials have total weight 6 under the $\mathbb{C}^*$-action. In a patch of $\mathbb{P}_{2,3,1}$ where $z \neq 0$ we can make use of the $\mathbb{C}^*$-action to set $z = 1$. The remaining coordinates x, y are then referred to as affine coordinates in the patch $z \neq 0$. The original equation (5.18) gives an equation in x, y that describes a two-sheeted covering of the complex x-plane branched over the roots of the cubic polynomial in x that enters the right hand side of (5.18). This space (including the points at infinity) has the topology of a torus, with two non-trivial one-cycles, see figure 5.1.

It is possible to endow the elliptic curve $\mathbb{E}$ with an Abelian group structure giving a well-defined prescription for determining the sum $P + Q$ of two points P, Q on $\mathbb{E}$. The point $[x:y:z] = [1:1:0]$ in $\mathbb{P}_{2,3,1}$, which by (5.18) lies on $\mathbb{E}$ for any value of f, g, can be shown to be the neutral element of this Abelian group action. Intuitively speaking an elliptic curve is thus a torus with a marked, special point.

The complex structure parameter τ of the elliptic curve $\mathbb{E}$ defined by (5.18) is encoded in the parameters f, g. More precisely, let us define the discriminant

$$\Delta = 27\, g^2 + 4\, f^3 \ . \tag{5.19}$$

This quantity is engineered in such a way as to vanish whenever two roots of the cubic polynomial in x on the right-hand side of (5.18) in the patch $z \neq 0$ coincide. In fact, one has

$$x^3 + f\, x + g = \prod_{i=1}^{3} (x - x_i) \ , \qquad \Delta = -(x_1 - x_2)^2 (x_2 - x_3)^2 (x_3 - x_1)^2 \ . \tag{5.20}$$

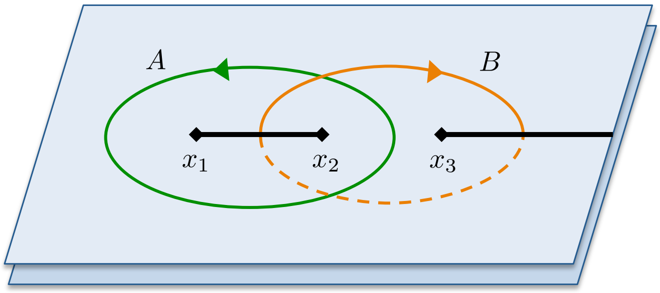


Figure 5.1: Schematic representation of the two-sheeted covering of the complex x-plane with the three roots x_1, x_2, and x_3 of the cubic polynomial on the right hand side of (5.18). Black solid lines denote branch cuts: one connects x_1 and x_2 while the other connects x_3 to the point at infinity. We also depict a basis for the independent, non-trivial one-cycles of the torus described by (5.18). A dashed line and a solid line are used to distinguish the two sheets of the covering of the x-plane.

It is then possible to prove that the complex structure parameter of the elliptic curve is given by

$$j(\tau) = \frac{4(24f)^3}{\Delta} , \tag{5.21}$$

where the j-invariant function has the same normalization as in (5.6).

Let us stress that the Weierstrass form is not the unique way of presenting an elliptic curve, even though any alternative representation is birationally equivalent to the Weierstrass form. Roughly speaking, birational equivalence is isomorphism up to lower-dimensional algebraic subsets. In what follows we will only make use of the Weierstrass form and will not consider different representations. Let us mention, however, that they can be extremely useful for the study of $U(1)$ symmetries in F-theory, see e.g. [111, 112, 113, 114, 115, 116, 117].

5.2.2 Weierstrass form of an elliptic fibration

An elliptic fibration X_{n+1} over a Kähler base B_n can be described by the same Weierstrass equation (5.18) provided we promote the constants f, g to objects depending of the base space B_n. More precisely, one can first construct an ambient space $\mathcal{A}_{n+2}$ by fibering $\mathbb{P}_{2,3,1}$ appropriately over B_n. The Weierstrass equation (5.18) then cuts out a hypersurface in $\mathcal{A}_{n+2}$ that yields the desired elliptic fibration X_{n+1}. In order to substantiate this program we would need to develop some technical tools from algebraic geometry. Since they will not be needed in our discussion of F-theory effective actions, we rather proceed with a heuristic account on elliptic fibrations.

We have seen that the patch $z \neq 0$ in $\mathbb{P}_{2,3,1}$ can be covered with two affine coordinates x, y. By the same token an appropriate patch of the base space B_n can be parametrized by n affine coordinates $u_1, \dots, u_n$. In this local picture, the Weierstrass equation that defines X_{n+1} has the form

$$X_{n+1} : \quad y^2 = x^3 + f(u_1, \dots, u_n)\, x + g(u_1, \dots, u_n) , \tag{5.22}$$

f and g are polynomials in $u_1, \ldots, u_n$. Let us stress that it can be proven that the marked point $[1:1:0]$ on the elliptic curve $\mathbb{E}$ is promoted in the present context to a global holomorphic section of the total fibration X_{n+1}. Intuitively speaking, as the torus fiber is varied on the base B_n its marked point swaps a copy of B_n inside X_{n+1}.

Since f and g now vary over the base B_n, the same happens to the complex structure parameter τ, which is still given by (5.19) and (5.21). A special locus on the base B_n is determined by the vanishing of the discriminant,

$$\Delta = 0 \ . \tag{5.23}$$

These equation generically determines a hypersurface inside B_n, possibly made of more than one irreducible component. Over this locus a one-cycle on the torus pinches. This can be seen from figure 5.1: if two of the roots x_1, x_2, x_3 of the cubic in the Weierstrass equation coincide, one of the non-trivial cycles of the torus is shrunk to a point. The pinching of the torus signals the presence of spacetime-filling seven-branes at the locus $\Delta = 0$. This can be argued as follows. Let A, B be a basis of the one-cycles of the torus fiber. One-cycles are equipped with a skew-symmetric, bilinear intersection pairing that satisfies

$$A \cdot A = 0 \ , \quad B \cdot B = 0 \ , \quad A \cdot B = 1 \ , \quad B \cdot A = -1 \ . \tag{5.24}$$

Suppose at some point along the $\Delta = 0$ locus on the base B_n the cycle

$$\alpha = pA + qB \tag{5.25}$$

is pinched, with p, q integer and coprime. According to the Picard-Lefschetz theorem, as we move around the point where α collapses an arbitrary cycle $\beta = nA + mB$ with $n, m \in \mathbb{Z}$ undergoes a monodromy

$$\beta \to \beta - (\beta \cdot \alpha)\alpha \ . \tag{5.26}$$

It is easily checked using (5.24) that this is equivalent to

$$\begin{pmatrix} m & n \end{pmatrix} \to \begin{pmatrix} m & n \end{pmatrix} \begin{pmatrix} 1 - pq & p^2 \\ -q^2 & 1 + pq \end{pmatrix} \ , \tag{5.27}$$

but the matrix on the right hand side coincides precisely with the monodromy matrix $M_{(p,q)}$ of a (p, q)-seven-brane given in (5.14). In summary, at the vanishing locus of the discriminant of the elliptic fibration the torus fiber pinches and a spacetime-filling seven-brane is located.

Suppose Δ has a simple zero at a point p on B_n and that f is non-vanishing at p. From (5.21) we see that the j-invariant function develops a simple pole. This is reminiscent of (5.9) considered in the previous section. If we had only one seven-brane in our setup we could safely argue using (5.6) that $\tau \to i\infty$ and $g_s \to 0$ near the location of the brane, which would then be identified with a D7-brane. Since generically we have several seven-branes (as it is also required by tadpole cancellation, addressed below) it is not possible to choose globally an $SL(2, \mathbb{Z})$ frame in which all of them look like D7-branes. In this case the solution to $j(\tau) = \infty$ is no longer $\tau = i\infty$ but rather one of its $SL(2, \mathbb{Z})$ images. Incidentally, note that if we write $\tau = \tau_1 + i\tau_2$, the $SL(2, \mathbb{Z})$ action (3.19) implies

$$\tau_2' = \frac{\tau_2}{(c\,\tau_1 + d)^2 + c^2\,\tau_2^2} \ . \tag{5.28}$$

If $c = 0$ the $SL(2,\mathbb{Z})$ image of $\tau = i\infty$ still has $\tau_2 = \infty$ and thus $g_s = 0$. But the subgroup of $SL(2,\mathbb{Z})$ with $c = 0$ is generated by T,

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : c = 0 \right\} = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \, , n \in \mathbb{Z} \right\} \, , \tag{5.29}$$

and encodes the perturbative monodromies of D7-branes around each other. As soon as we act with the generator S and we introduce non mutually local branes we have $c \neq 0$ and the image of $\tau = i\infty$ has $\tau_2 = 0$, corresponding to $g_s = \infty$. Thus the string coupling constant defined with respect to a reference D7-brane diverges near the location of a seven-brane that is not mutually local to that reference D7-brane.

Let conclude this section with a discussion of the Calabi-Yau condition for the total space X_{n+1}, which will be physically motivated in section 5.3. To enforce this requirement we need to express the first Chern class of the total space X_{n+1} in terms of topological data of the base space and of the fibration. It was proven by Kodaira that[2]

$$12\, c_1(X_{n+1}) = \pi^* \left(12 c_1(B_n) - \mathrm{PD}([\Delta]) \right) \, , \tag{5.30}$$

where π^* denotes the pullback induced by the projection map $\pi : X_{n+1} \to B_n$, $c_1(B_n)$ is the first Chern class of the base, and $\mathrm{PD}[\Delta] \in H^2(B_n, \mathbb{Z})$ is the Poincaré dual to the divisor class of the vanishing locus of the discriminant.

Let us give a brief and heuristic account on the terminology used in the previous sentence. Without entering mathematical details, we can intuitively think of a divisor D in a complex variety Z as a submanifold of complex codimension one that is locally described by an equation of the form $h = 0$, with h holomorphic. It is possible to construct formal integral linear combinations of divisors, such as $n_1 D_1 + n_2 D_2$, $n_1, n_2 \in \mathbb{Z}$. If F is a globally defined meromorphic function on Z, it can be locally written as $F = F_0/F_\infty$, where F_0, F_∞ are holomorphic and have no common factors. We can then define the so-called principal divisor associated to F as

$$(F) = \{F_0 = 0\} - \{F_\infty = 0\} \, . \tag{5.31}$$

Two divisors D, D' are then called linearly equivalent if $D - D' = (F)$ for some globally defined meromorphic function F on Z. The equivalence class of D with respect to linear equivalence is denoted $[D]$. The Poincaré dual $\mathrm{PD}([D])$ of $[D]$ is a cohomology class of two-forms defined by the property

$$\int_Z \mathrm{PD}([D]) \wedge \alpha = \int_D \alpha \, , \tag{5.32}$$

where α is (the cohomology class of) an arbitrary $2(\dim_{\mathbb{C}} Z - 1)$-form. The concept of Poincaré duality can be extended to higher-codimensions: if a subvariety is defined locally by $f^1 = \cdots = f^p = 0$, its Poincaré dual is the class of the $2p$-form that satisfies the analog of (5.32) where now α is (the cohomology class of) an arbitrary $2(\dim_{\mathbb{C}} Z - p)$-form.

[2] Strictly speaking this equation holds for $n = 1$, i.e. for elliptically fibered K3 surfaces. For $n > 1$ one has to take into account degenerations of the fibration that occur at higher codimension. Nonetheless, these additional contributions do not affect our argument [118, 119, 120].

After this detour, we can come back to (5.30) and enforce that X_{n+1} be a Calabi-Yau $(n+1)$-fold,

$$12\, c_1(B_n) = \mathrm{PD}([\Delta]) \ . \tag{5.33}$$

In many explicit constructions of elliptically fibered Calabi-Yau manifolds this equation has a simple interpretation: the polynomials f, g entering (5.22) have to have a definite degree in the affine coordinates $u_1, \ldots, u_n$ on B_n. More precisely, (5.33) fixes x, y, z, f, g in the Weierstrass equation to be sections of appropriate line bundles over B_n, but we will not develop this terminology any further. Note that for non-trivial fibrations the right hand side of (5.33) does not vanish so that B_n is not Ricci-flat. This confirms the physical intuition that the curvature of the base space has to balance the non-trivial axio-dilaton profile described by the elliptic fibration.

5.2.3 Non-Abelian gauge groups and matter from singularities

As stressed above, at the vanishing locus of the discriminant the elliptic fiber degenerates, since one of the one-cycles of the torus collapses. Given a point $p \in B_n$ such that $\Delta(p) = 0$, depending on the vanishing orders of Δ, f, and g the total space may or may not be singular at p. The presence of a singularity of the total space does not imply a breakdown of the setup we are considering: F-theory, just like perturbative Type IIB, probes a geometry different from that of point-particles and can be well-defined even on some classes of singular spaces. For many purposes, however, and most notably in the framework of F-theory/M-theory duality discussed below, it is desirable to replace the singular space X_{n+1} with a smooth space $\tilde{X}_{n+1}$ in a well-defined fashion, in such a way that the singular space is recovered as a suitable limit of $\tilde{X}_{n+1}$. Furthermore we would like the smooth space to respect the Calabi-Yau condition. Algebraic geometry offers various tools to address this task, but in what follows we will only consider resolution of singular Calabi-Yau spaces. Other options, such as deformation, are also useful in F-theory, see e.g. [121, 122] for recent progress.

Roughly speaking, a resolution of a singular Calabi-Yau space X_{n+1} is a smooth Calabi-Yau space $\tilde{X}_{n+1}$ together with a map $\varphi : \tilde{X}_{n+1} \to X_{n+1}$, called the blow-down map. The preimage under the blow-down map of the singular loci of X_{n+1} are cycles in $\tilde{X}_{n+1}$ such that, when they are collapsed to zero volume, the smooth space $\tilde{X}_{n+1}$ reproduces the singular space X_{n+1}. In other words, the singular Calabi-Yau is recovered as a limit point in the Kähler moduli space of the smooth Calabi-Yau sitting on the boundary of the Kähler cone.

The possible singularities that can occur at codimension one in the elliptically fibered Calabi-Yau X_{n+1} described by the Weierstrass model (5.22) have been classified by Kodaira [123, 124]. The type of singularity is determined by the vanishing orders of Δ, f, g. For all these singularities a resolution $\varphi : \tilde{X}_{n+1} \to X_{n+1}$ exists, such that the preimages of the singular loci under the blow-down map φ are divisors in $\tilde{X}_{n+1}$. They are commonly referred to as exceptional divisors of the resolved space. It is possible to provide a very pictorial description of exceptional divisors. Let p be a point on the base space B_n such that in the original fibration X_{n+1} the fiber over p is singular. After resolution, the fiber over p turns out to be a reducible variety consisting of a collection of $\mathbb{P}^1$'s intersecting at various points, see figure 5.2. One of these $\mathbb{P}^1$'s is identified with the original fiber, though of as a pinched

Figure 5.2: Pictorial representation of a smooth elliptic fiber and of a degenerate elliptic fiber. In the latter situation we have depicted the resolved geometry, characterized by a collection of $\mathbb{P}^1$'s with a specific intersection pattern. The case depicted in the figure corresponds to the affine Dynkin diagram of a singularity of the A series.

torus. The other $\mathbb{P}^1$'s are resolution cycles that are collapsed to zero volume when we recover X_{n+1} from the smooth space $\tilde{X}_{n+1}$. Exceptional divisors are then realized by fibering these resolution $\mathbb{P}^1$'s over the locus on the base where the fibration is singular.

The intersection pattern of the resolution $\mathbb{P}^1$'s in the fiber translates into the intersection pattern of exceptional divisors. Remarkably, for the singularities in Kodaira's classification this pattern reproduces exactly the affine Dynkin diagram of Lie algebras from the A, D, E series. Since we know that singularities of the elliptic fibration encode the location of seven-branes, we are naturally led to conclude that the Dynkin diagram determined by exceptional divisors should be identified with the Dynkin diagram of the gauge group G living on the seven-brane. In particular exceptional divisors are in one-to-one correspondence with the generators of the Cartan subalgebra of G, so that we can denote them as

$$D_i \ , \qquad i = 1, \dots, \operatorname{rank} G \ , \tag{5.34}$$

where $\operatorname{rank} G$ denotes the rank of the gauge group G. The duality between F-theory and M-theory will clarify the physical origin of these Cartan degrees of freedom as well as of the degrees of freedom associated to the roots of G.

In perturbative Type IIB with D7-branes and O7-branes it is only possible to engineer the gauge groups $U(N)$, $SO(N)$, or $Sp(N)$. Kodaira's classifications shows that, thanks to its non-perturbative character, F-theory allows for more general groups, and in particular for the exceptional groups E_6, E_7, and E_8. These are particularly interesting for GUT phenomenology and have been extensively exploited in heterotic string theory setups. F-theory is thus able to combine favorable features of heterotic string theory for particle physics phenomenology with other properties inherited from Type IIB and M-theory, such as a better control over moduli stabilization. The latter perspective on F-theory is emphasized e.g. in the review [102].

Let us mention that it is possible to understand the emergence of exceptional gauge groups in F-theory from BPS networks of string junctions [125, 126, 127]. The latter were briefly introduced in section 3.2. In particular the fact that string junctions are multi-pronged objects—as opposed to fundamental strings, which only have two ends—is instrumental for the realization of a larger class of gauge symmetries than those of perturbative Type IIB. String junctions play also a crucial role in the systematics of deformations of singularity in the recent works [121, 122].

So far we have been discussing only singularities that can occur at codimension-one loci. If we consider higher codimensions richer singularity structures are possible. For instance, at the intersection of the location of two seven-brane stacks the elliptic fibration undergoes an additional degeneration that corresponds to a codimension-two enhancement of the gauge groups associated to the stacks. The degrees of freedom responsible for this enhancement are interpreted as charged matter under the gauge groups living on the world-volume of the seven-branes. This is in complete analogy to the picture of charged massless fermions emerging at the intersection of D7-branes in perturbative Type IIB. Let us also remark that codimension-two singularities can sometimes trigger non-trivial monodromies on the world-volume of a seven-brane stack. As a result, the associated gauge theory is modified and non-simply laced gauge groups of the B and C series can be engineered, in addition to the A, D, E groups found in Kodaira's classification.

Finally, additional singularities are found at the codimension-three loci associated to the intersection of three seven-branes stacks. These are not present in F-theory compactification to six dimensions (the base B_2 is complex two-dimensional) but play a fundamental role in F-theory compactifications to four dimensions, where codimension-three loci on the base B_3 are just points. Indeed, these points of triple intersection are associated to the trilinear Yukawa couplings that enter the four-dimensional effective action. Once again, this constitutes a non-perturbative generalization of the corresponding situation in perturbative Type IIB with intersecting D7-brane stacks, see e.g. [106] for a pedagogical account.

5.3 Duality between F-theory and M-theory

In section 3.4 we have studied in detail the nine-dimensional duality between M-theory on $\mathbb{R}^{1,8} \times T^2$ and Type IIB on $\mathbb{R}^{1,8} \times S^1$. The duality between M-theory and F-theory emerges as we replace the direct product $\mathbb{R}^{1,8} \times T^2$ on the M-theory side with a non-trivial fibration of the torus over a base manifold. More precisely, we consider M-theory on the space

$$ds^2_{\rm M} = \eta_{\mu\nu} dx^\mu dx^\nu + ds^2_{B_n} + \frac{A\,\ell^2_{\rm M}}{{\rm Im}\,\tau_{\rm M}} \left|dx + \tau_{\rm M}\, dy\right|^2 \ , \tag{5.35}$$

where $\mu, \nu = 0, \dots, d-1$ with $d = 9 - 2n$, $ds^2_{B_n}$ denotes the metric on the base space, x, y are adimensional, period-one coordinates on the torus fiber, and the complex structure parameter of the torus $\tau_{\rm M}$ is now allowed to vary over B_n. We can therefore regard (5.35) as a generalization of (3.59), where for simplicity we have suppressed the Kaluza-Klein vectors $V_{(x)}$, $V_{(y)}$. It is not hard, though, to follow them in the duality from M-theory to F-theory applying the results of section 3.2 to the present setup. Let us remind the reader that the x-cycle in (5.35) is interpreted as the M-theory circle connecting M-theory to Type IIA, while the y-cycle is the T-duality circle that connects Type IIA to Type IIB.

The validity of the extension of the duality between M-theory on a torus and Type IIB on a circle to the case of non-trivial torus fibrations can be argued appealing to the so-called adiabatic argument [128]: if the torus fiber varies adiabatically over the base space, the system under examination looks locally like an open patch of the simple product space $\mathbb{R}^{1,8} \times T^2$. We can thus apply the duality fiberwise.

The identification (3.63) between the complex structure parameter of the torus on the M-theory side and the axio-dilaton of Type IIB remains valid. The duality thus predicts that a non-trivial torus fibration corresponds to a non-trivial axio-dilaton profile on the Type IIB side. Of course this is not unexpected, and it has actually been the observation that motivated the introduction of elliptic fibrations in the section 5.2. The crucial difference between the M-theory and the Type IIB/F-theory perspective is that in Type IIB/F-theory the torus is merely a mathematical artifact to describe the axio-dilaton profile, while in M-theory it is part of physical eleven-dimensional spacetime. As a result the amount of supersymmetry preserved by the setup under examination can be determined according to the paradigm outlined in section 4.4. In particular, we already know that if we consider an unwarped compactification to Minkowski spacetime (4.26) the compactification space has to be Ricci-flat. This justifies the imposition of the Calabi-Yau condition (5.33) on the elliptic fibration. In section 5.5 we will actually see that in compactifications to four dimensions a warp factor has to be included, but we will argue that the Calabi-Yau condition is preserved in a suitable sense.

The area of the torus has a very different status compared to its complex structure parameter. In an elliptically fibered Calabi-Yau manifold it must be constant over the base space, even for non-trivial fibrations. This follows from the fact that the fiber can be realized as a holomorphic complex one-dimensional submanifold in the Calabi-Yau, so that its area is given by $\int J$, which cannot vary over the base by virtue of $dJ = 0$. We known from (3.64), however, that the area A of the torus is mapped to the circumference $L_{(\mathrm{B})}$ of the circle on the Type IIB side of the duality. Since A does not vary over the base, we infer that the on the Type IIB side the circle fibration is trivial.[3] This observation is crucial. Using (3.64) and (3.66) it implies that the metric on the Type IIB/F-theory side of the duality reads

$$ds^2_{\mathrm{F}} = \eta_{\mu\nu} dx^\mu dx^\nu + \ell_s^2 A^{-3/2} dy^2 + ds^2_{B_n} \ , \tag{5.36}$$

where the prefactor of dy^2 is a constant. We can thus take the limit $A \to 0$ and decompactify one direction on the Type IIB/F-theory side of the duality, in such a way that Lorentz invariance is restored. This procedure is commonly referred to as the F-theory limit.

This program presents a clear difficulty. Since we do not know yet how to quantize M-theory, we have to rely on its approximation by means of eleven-dimensional supergravity coupled to membranes and five-branes. This approximation, however, makes only sense if M-theory is compactified on a smooth space and if all volumes are large compared to the eleven-dimensional Planck length ℓ_{M}. Since A is precisely the volume of the torus fiber measured in units of ℓ_{M}, it is clear that taking the F-theory limit forces us to go beyond the regime of validity of the supergravity approximation. By the same token we have seen in section 5.2.3 that interesting F-theory setups with non-Abelian gauge groups require singular elliptic fibrations X_{n+1}. In summary, if we want to achieve Lorentz invariance and non-Abelian gauge symmetries on the F-theory side of the duality we must consider M-theory on a singular space with some small or vanishing volumes.

[3] In general this does not hold for the intermediate Type IIA step connecting M-theory and Type IIB.

In order to overcome this difficulty we have to refine our understanding of M-theory/F-theory duality. Suppose we start with M-theory on the resolved elliptic fibration $\tilde{X}_{n+1}$ where the volumes of the torus fiber and of the resolution $\mathbb{P}^1$'s are large in units of ℓ_{M}. We can then reliably compute the effective action of the resulting d-dimensional theory ($d = 9 - 2n$) using the effective action (3.34), supplemented by suitable higher-derivative corrections addressed below. The light degrees of freedom of the d-dimensional theory thus originate from the Kaluza-Klein zeromodes in the expansion of the eleven-dimensional metric and three-form. In particular the expansion of the three-form along the two-forms $\omega_i = \mathrm{PD}([D_i])$ Poincaré dual to the classes of the exceptional divisors yields a collection of $\mathrm{rank}\, G$ massless $U(1)$ vectors,

$$\hat{C}_3 \supset A^i \wedge \omega_i \ . \tag{5.37}$$

These are interpreted as the Cartan vectors of the non-Abelian gauge group on the F-theory side.

The d-dimensional M-theory compactification on $\tilde{X}_{n+1}$ also features massive BPS states originating from M2-branes wrapping two-cycles of the geometry. More precisely, we can wrap an M2-brane along the torus fiber or along the resolution $\mathbb{P}^1$'s, yielding massive particle states in d dimensions. Those are automatically integrated out in the low-energy effective action in d dimensions obtained from Kaluza-Klein reduction of eleven-dimensional supergravity. Let us follow these states through the duality to the F-theory side.

An M2-brane wrapping the torus fiber becomes a winding string in Type IIA, which in turn after T-duality becomes a string state carrying non-vanishing Kaluza-Klein momentum along the y-cycle. As the torus fiber shirks on the M-theory side of the duality, the y-cycle grows large on the F-theory side, and excited Kaluza-Klein states become light. We thus see that M2-branes wrapping the fiber encode the degrees of freedom of the massive Kaluza-Klein states in the circle reduction of the $(d+1)$-dimensional F-theory effective action down to d dimensions.

To clarify the role of M2-branes wrapping the resolution $\mathbb{P}^1$'s we proceed as follows. A stack of D7-branes in Type IIB that fills the x^μ directions and wraps the y-cycle becomes, upon T-duality along y, a collection of D6-branes extended along x^μ and located at points along on the y-circle, see section 3.1. Such a D6-brane configuration is uplifted to M-theory to a multi-center Taub-NUT geometry, as reviewed in section 3.3. This geometry possesses two-cycles obtained by fibering the M-theory circle between the location of two centers of the Taub-NUT space, see figure 3.1. These two-cycles are identified with the resolution $\mathbb{P}^1$'s of the F-theory geometry. As a result, M2-branes states wrapping these $\mathbb{P}^1$'s are interpreted as Type IIA strings stretching between parallel D6-branes. As we recover the singular fibration from the resolved space these D6-branes become coincident and gauge symmetry is enhanced to a non-Abelian gauge group. Strings stretching between different branes provide the degrees of freedom associated to the roots of the group, or equivalently to its 'W-bosons.' Even though we have used the language of perturbative Type II and D-branes, this conclusion holds for more general seven-branes configurations in F-theory. In summary, we have identified the M-theory origin of all gauge bosons living on the world-volume of a seven-brane stack: the Cartan $U(1)$s come from the M-theory three-form $\hat{C}_3$ expanded along exceptional divisors (5.37), while W-bosons come from M2-branes wrapping the resolution $\mathbb{P}^1$'s. In the resolved elliptic fibrations the latter have finite volume and W-bosons are massive: the associated gauge theory is pushed to its Coulomb branch and

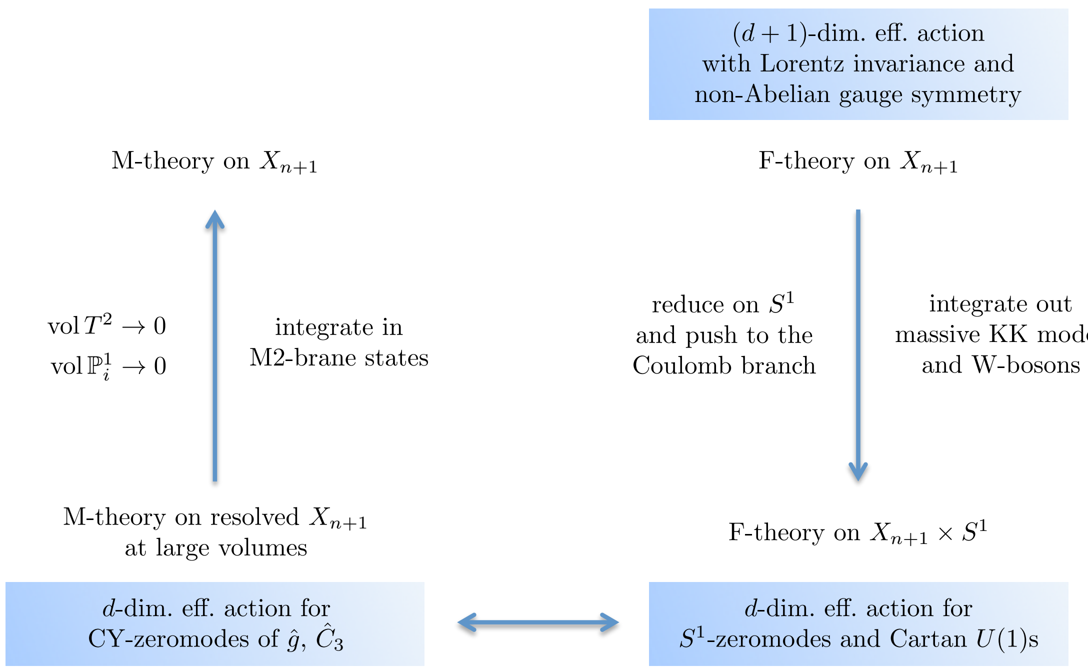


Figure 5.3: Schematic representation of M-theory/F-theory duality and of the resulting prescription for the computation of the $(d+1)$-dimensional effective action of F-theory on an elliptically fibered Calabi-Yau $(n+1)$-fold X_{n+1}, where $d = 9 - 2n$. On the M-theory side, T^2 denotes the fiber, while $\mathbb{P}^1_i$ denote the resolution $\mathbb{P}^1$'s.

the gauge group G is spontaneously broken to $U(1)^{\operatorname{rank} G}$. If we go back to the singular fibration, full non-Abelian symmetry is restored.

Similar considerations apply to the codimension-two singularity enhancements in the elliptic fibration associated to charged matter. In order to have a smooth space on the M-theory side a suitable resolution procedure has to be performed, which introduces additional resolution two-cycles in the geometry. Charged matter states originate from wrapped M2-branes that become massless in the limit in which these resolution two-cycles are shrunk to zero size. In the Coulomb phase of the d-dimensional gauge theory, however, these charged matter states are not directly accessible as they acquire a mass from spontaneous gauge symmetry breaking and are automatically integrated out from the M-theory effective action.

We are finally in a position to use the duality between M-theory and F-theory to determine a prescription for the computation of the F-theory effective action in $d+1$ dimensions. Figure 5.3 gives a schematic overview of the setup under examination.

The sought-for F-theory effective action is a Lorentz-invariant theory with a non-Abelian gauge group and some amount of supersymmetry, depending of the dimension $d+1$. This can be parametrized in terms of some characteristic data, which can either be discrete (e.g. anomaly coefficients) or continuous (e.g. coupling functions). If we compactify this theory on a circle and we push it to the Coulomb branch we obtain a setup with both massless states, given by Kaluza-Klein zeromodes and Cartan vectors, and massive states, given by all excited Kaluza-Klein states and/or states that become massive upon gauge symmetry breaking.

In the vast majority of cases the effects of these massive states on the low-energy d-dimensional dynamics of massless states are suppressed by the inverse of the compactification radius and are thus negligible. Instead of properly integrating out these states we can simply truncate them away. Some d-dimensional couplings, however, are sensitive to the presence of massive states via quantum corrections that are independent of the radius of the circle. We will analyze in detail an example of such a coupling in chapter 9. In this case more care is needed in order to obtain the correct effective action for d-dimensional massless states.

Performing the circle compactification on the F-theory side corresponds to inverting the blow-down process that connects M-theory on the resolved fibration $\tilde{X}_{n+1}$ to M-theory on the singular fibration X_{n+1} with vanishing fiber volume. As noted above, the d-dimensional effective action of M-theory on $\tilde{X}_{n+1}$ can computed in the supergravity approximation and the process of resolution automatically integrates out all massive M2-branes states at the classical level. We thus have to compare the quantum circle reduction of the sought-for F-theory effective action to the classical Calabi-Yau reduction of eleven-dimensional supergravity. This comparison gives us the necessary information to fix all the characteristic data that enter the parametrization of the F-theory effective action.

In chapter 7 this program will be carried out in detail in the case of F-theory compactified on an elliptically fibered Calabi-Yau threefold. A suitable generalization of this prescription will also be the starting point of our discussion of four-dimensional compactifications of F-theory on Spin(7) manifolds.

As a final remark, let us stress that the relations (3.67) between the p-forms resulting from expansion of the M-theory three-form and the p-forms of Type IIB on a circle can still be applied to the present context. We are therefore able to follow all light bosonic degrees of freedom through the duality between M-theory and F-theory. A similar analysis could be performed for fermions, but we will not need to consider them explicitly and we can rely on supersymmetry for determining all fermionic terms of the relevant effective actions in terms of their bosonic terms. The situation is more subtle in the context of F-theory compactified on a Spin(7) manifold: a discussion of this topic is postponed to chapter 8.

5.4 Sen's weak-coupling limit

The power of F-theory resides in its ability to encode the physics of the gauge fields and charged matter associated to seven-branes into the geometry of the elliptically fibered Calabi-Yau $(n+1)$-fold. Note in particular that from the perspective of the M-theory/F-theory duality seven-branes can be seen as solitonic excitations of the bulk fields of M-theory. This is ultimately related to the fact that Type IIA D6-branes are lifted to pure geometry in M-theory, as recalled in section 3.3.

There exists, however, a suitable limit in the complex structure moduli space of the elliptically fibered Calabi-Yau $(n+1)$-fold in which the F-theory setup can be described with the language of perturbative Type IIB superstring theory. This limit is due to Sen [129] and stems from the observation that the most general f, g entering the Weierstrass equation (5.22) can be conveniently parametrized as

$$f = C\,\eta - 3\,h^2 \;, \qquad g = h(C\,\eta - 2\,h^2) + C^2\,\chi \;, \tag{5.38}$$

where η, h, χ are locally given by polynomials in the affine coordinates $u_1, \ldots, u_n$ on the base B_n of the fibration, while C is a complex constant. The parametrization (5.38) is engineered in such a way that the series expansion of the discriminant Δ given in (5.19) as $C \to 0$ has no $\mathcal{O}(C^0)$ and $\mathcal{O}(C^1)$ terms,

$$\Delta = -9\,h^2(\eta^2 + 12\,\chi\,h)\,C^2 + \mathcal{O}(C^3) \;. \tag{5.39}$$

The Klein j-invariant of the elliptic fibration (5.21) has a Laurent expansion in C that starts with a second order pole,

$$j(\tau) = \frac{12 \cdot 24^3\,h^4}{\eta^2 + 12\,\chi\,h}\,\frac{1}{C^2} + \mathcal{O}\left(\frac{1}{C}\right) \;. \tag{5.40}$$

This expression shows that, away from the special loci $h = 0$ and $\eta^2 + 12\,\chi\,h = 0$, the string coupling can be made arbitrarily small by taking the limit $C \to 0$. In order to have control over the setup we have to show that the special loci can be described in terms of objects of perturbative Type IIB.

From (5.39) we see that the dominant term in the discriminant expansion has a factorized form. It thus described two distinct codimension-one loci on the base B_n. In order to identify the nature of the objects sitting at these loci one has to study the monodromies around them, as outlined schematically in section 5.1.2. We do not perform here this analysis, and rather state the result. The monodromy around the component $h = 0$ is encoded in the matrix (5.15), signaling the presence of an O7-plane. The locus $\eta^2 + 12\,\chi\,h = 0$ can be thought of in terms of its components $\eta = \pm\sqrt{-12\,\chi\,h}$; each of them can be shown to have monodromy given in (5.11) corresponding to a D7-brane. In summary,

$$\mathrm{O7}: \quad h = 0 \;, \qquad\qquad \mathrm{D7}: \quad \eta^2 + 12\,\chi\,h = 0 \;. \tag{5.41}$$

The presence of an O7-plane suggests that the base of the fibration B_n should be though of as the quotient of a suitable space X_n under the action of an involution whose fixed points lie at $h = 0$. This expectation is indeed confirmed: the space X_n can be described introducing an additional variable ξ together with an additional equation

$$X_n: \quad \xi^2 = h \;. \tag{5.42}$$

Actually, the Calabi-Yau condition (5.33) for the elliptic fibration X_{n+1} implies that X_n is a Calabi-Yau n-fold. From (5.42) we see that X_n is a double cover of the base space B_n branched over the locus $h = 0$. The geometric involution acting on X_n is

$$\sigma_{\mathrm{h}} : \xi \mapsto -\xi \;, \tag{5.43}$$

and yields the full orientifold action

$$\mathcal{O}_1 = \Omega_{\mathrm{p}}\,\sigma_{\mathrm{h}}\,(-1)^{F_L} \;, \tag{5.44}$$

where Ω_{p} is the world-sheet parity operator and $(-1)^{F_L}$ is the spacetime left-moving fermion number. The need for the inclusion of the latter was motivated in section 2.3.

5.5 Remarks on G_4-flux in four-dimensional F-theory setups

In F-theory compactifications to six dimensions with minimal supersymmetry both the bulk physics and the localized gauge and matter sectors can be understood purely in terms of the geometry of the underlying elliptically fibered threefold. The situation is qualitatively different for four-dimensional F-theory compactifications: the geometric data of the fourfold have to be supplemented by the inclusion of a suitable G_4-flux, i.e. a background value for the field strength of the M-theory three-form. A thorough analysis of the problem lies beyond the scope of this work, so we refer the reader to the review [102] and references therein for an introduction.

Our discussion of compactification of eleven-dimensional supergravity on an unwarped product background of the form (4.26) is valid only at the two-derivative level and is modified by the introduction of higher-curvature corrections. Some of them are known [81, 82] and can have a sizable effect also in the large volume limit, i.e. in the limit in which all length scales of the internal manifold are large compared to the eleven-dimensional fundamental length $\ell_{\rm M}$. When such effects are taken into account the correct Ansatz for the dimensional reduction must include a non-trivial warp factor, like in (4.25). A convenient parametrization for the problem at hand is

$$\langle \hat{g}_{\hat{\mu}\hat{\nu}}(\hat{x})\rangle d\hat{x}^{\hat{\mu}} d\hat{x}^{\hat{\nu}} = e^{-A(y)}\, g_{\mu\nu}(x)\, dx^\mu dx^\nu + e^{A(y)/2}\, g_{mn}(y)\, dy^m dy^n \ . \tag{5.45}$$

Collecting a prefactor $e^{A(y)/2}$ in the internal part of the metric is useful because the requirement of $\mathcal{N}=2$ supersymmetry in the three external dimensions (four real supercharges) implies that g_{mn} is still a Ricci-flat Calabi-Yau metric even for non-trivial warp factor $A(y)$. Moreover, supersymmetry requirements and three-dimensional Poincaré symmetry demand that the background G_4-flux be of the form

$$\hat{G}_4 = -\frac{1}{3!}\epsilon_{\mu\nu\rho}\partial_m e^{-3A/2}\, dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dy^m + G_4 \ , \tag{5.46}$$

where G_4 has all four legs along the internal space. This internal flux has to satisfy the celebrated tadpole cancellation condition

$$\frac{1}{2\ell_{\rm M}^6}\int_{X_4} G_4 \wedge G_4 + N_{\rm M2} = \frac{\chi}{24} \ , \tag{5.47}$$

where χ is the Euler characteristic of the Calabi-Yau fourfold X_4 and $N_{\rm M2}$ is the number of spacetime-filling M2-branes included in the compactification setup. (Since their world-volume fills external spacetime they are compatible with three-dimensional Poincaré symmetry and there is no *a priori* argument to exclude their presence.) As a consequence of (5.47) if the fourfold has $\chi > 0$ we have to turn on a non-vanishing G_4-flux and/or to include some spacetime-filling M2-branes in order to have a consistent vacuum. If $\chi < 0$ no supersymmetric vacua can be found, since the left hand side to (5.47) can be shown to be non-negative. Finally, if $\chi = 0$ it is consistent to turn off the flux and to introduce no spacetime-filling M2-branes. This special case has been recently revisited in [130], were it was found to have enhanced off-shell supersymmetry. The terminology ‘tadpole cancellation condition’ comes from the observation that (5.47) constitutes the M-theory/F-theory generalization of the Type IIB D3-brane tadpole cancellation condition reviewed e.g. in [131], which in turn can actually be interpreted as a condition for the absence of tadpoles in string diagrams.

The previous argument for the necessity of a non-trivial G_4-flux along some four-cycle of the Calabi-Yau fourfold X_4 is based on the physics of bulk fields and holds in M-theory irrespectively of the duality with F-theory. In the latter context, however, a non-zero flux is also needed in order to allow for massless fermions in chiral representations of the gauge group living on the world-volume of seven-branes. This can already be seen in perturbative Type IIB. The generic intersection of two spacetime-filling D7-brane stacks wrapping divisors on the base B_3 is six-dimensional and carries charged chiral fermions. In order for these six-dimensional chiral spinors to reduce to four-dimensional chiral spinors it is necessary to introduce a non-vanishing magnetic flux along the world-volume of the D7-branes. The same argument generalizes to seven-brane stacks in F-theory. The flux along the world-volume of seven-branes is uplifted to suitable components of the M-theory G_4-flux. The latter are related to chiral matter via the index formula [132, 133, 134, 135, 136, 137, 138, 139]

$$\chi(\mathbf{R}) = \frac{1}{\ell_{\mathrm{M}}^3} \int_{S_{\mathbf{R}}} G_4 \ . \tag{5.48}$$

Let us comment this equation. Suppose we have a codimension-two locus on the base B_3 of the elliptic fibration, i.e. a curve $\mathcal{C}_{\mathbf{R}} \subset B_3$, where two seven-branes stacks intersect yielding a singularity enhancement that is associated to the representation $\mathbf{R}$ of the gauge group of the F-theory setup. The quantity on the left hand side of (5.48) is the chiral index of the representation $\mathbf{R}$, defined as the net number of chiral massless four-dimensional fermions transforming in the representation $\mathbf{R}$. The resolution of the codimension-two singularity along $\mathcal{C}_{\mathbf{R}}$ introduces new resolution $\mathbb{P}^1$'s in the geometry of the smooth fourfold $\tilde{X}_4$. By fibering these $\mathbb{P}^1$'s over $\mathcal{C}_{\mathbf{R}}$ we obtain a surface $S_{\mathbf{R}} \subset \tilde{X}_4$, which is commonly referred to as matter surface and which is the four-cycle over which the G_4-flux is integrated in the right hand side of (5.48).

As pointed out in section 5.3, charged matter is massive on the Coulomb branch of the three-dimensional gauge theory on the M-theory side of the duality. It is therefore not possible to access chiral fermions directly at the level of the effective action derived by means of F-theory/M-theory duality. Nonetheless the identification between massive M2-brane states in M-theory and massive Kaluza-Klein modes and W-bosons in F-theory allows one to make contact to (5.48) in the context of F-theory/M-theory duality by taking into account quantum corrections of three-dimensional Chern-Simons terms induced at one-loop by massive Kaluza-Klein modes and W-bosons, see e.g. [140]. This three-dimensional mechanism is the direct analog of the five-dimensional mechanism that we will study in greater detail in chapter 9.

CHAPTER 6

The puzzle of (2,0) theories

This chapter is devoted to a brief overview about some known results in the study of six-dimensional theories with (2,0) supersymmetry. We first recall the string theory and M-theory arguments in favor of the existence of non-trivial interacting field theories with this amount of supersymmetry. Next, we consider some general features that can be inferred without any detailed knowledge of the structure of interactions. Finally, we review in particular the connection between (2,0) theories and five-dimensional maximally supersymmetric Yang-Mills. This is connected to the proposal for a five-dimensional description of (2,0) theories formulated in chapter 11.

6.1 Non-trivial infrared dynamics from string theory and M-theory

One of the most interesting insights provided by the so-called second superstring revolution has been the discovery of a novel kind of interacting quantum field theory in six dimensions. Such theories emerge as a non-trivial infrared fixed point in the RG flow of the dynamics of Type IIB compactified on a singular K3 surface [141] or of a stack of coincident M5-branes in M-theory [142, 143]. Let us briefly review the evidence in favor of the existence of these new theories.

Compactification of Type IIB superstring theory on a smooth K3 surface yields a six-dimensional low-energy effective theory with (2,0) supersymmetry, i.e. with 16 real supercharges organized into two complex Weyl spinors of the same chirality. It is useful to recall that in supergravity theories with (2,0) supersymmetry there exist only two kinds of supermultiplets with spins lower than or equal to two:

- supergravity multiplet, consisting of the graviton, five self-dual tensors, and two positive-chirality complex Weyl gravitini;

- tensor multiplet, consisting of one antiself-dual tensor, five real scalars, and two negative-chirality complex Weyl spinors, referred to as tensorini.

In particular (2,0) supersymmetry forbids the presence of massless vectors in the low-energy spectrum of the theory. In the case of Type IIB on a smooth K3 surface, the resulting six-dimensional theory consists of the supergravity multiplet together with 21 tensor multiplets. Indeed, a smooth K3 surface has three self-dual harmonic two-forms and 19 antiself-dual harmonic two-forms. Expansion of the self-dual four-form C_4 of Type IIB onto harmonic two-forms of K3 yields therefore three self-dual tensors and 19 antiself-dual tensors. Furthermore, the Type IIB two-forms B_2 and C_2 provide additional two self-dual tensors and two antiself-dual tensors. In total we thus find the five self-dual tensors of the gravity multiplet and the 21 antiself-dual tensors of the tensor multiplets. Let us mention that this is the precise number of tensor multiplets needed for anomaly cancellation in six-dimensional (2,0) theories, as we will see in section 10.2.1.

At special loci in its moduli space a K3 surface can develop orbifold singularities. Those have a specific structure and can be classified in terms of ADE Dynkin diagrams. Intuitively speaking, these singularities can be resolved by introducing a collection of resolution $\mathbb{P}^1$'s whose intersection pattern contains the same information as the corresponding ADE Dynkin diagram. We have encountered a very similar situation in chapter 5 when we discussed singularities of the elliptic fibration in the context of F-theory. There we argued that M2-brane states wrapping the resolution $\mathbb{P}^1$'s can provide particle-like states in external spacetime that become light as the $\mathbb{P}^1$'s are shrunk to zero size. In the case of Type IIB on a singular K3, one finds that D3-branes can wrap the resolution $\mathbb{P}^1$'s, yielding states in the six-dimensional theory that become light as the $\mathbb{P}^1$'s collapse to a point. Crucially, however, these are now string states in six-dimensions, rather than particle states.

These strings are commonly referred to as non-critical strings. Indeed they do not coincide with the fundamental Type IIB superstring and their tension—proportional to the area of the resolution $\mathbb{P}^1$'s—can be made arbitrarily small in units of the fundamental string tension. Furthermore, their world-volume does not carry a propagating graviton, i.e. a massless spin-two state. Another crucial property of these non-critical strings is the fact that they coincide with their own magnetic duals. This property is inherited by the same property of D3-branes in Type IIB. As a result, these strings couple naturally to (anti)self-dual tensors in six-dimensions, compatibly with (2,0) supersymmetry. Another consequence of self-duality of these strings is the fact that their interactions are governed by an order-one coupling, so that these objects are intrinsically non-perturbative.

As noted above, by tuning the volume of resolution $\mathbb{P}^1$'s it is possible to make the non-critical strings arbitrarily light. Their influence on the background spacetime can therefore be neglected and one expects that the dynamics of the system in the deep infrared can be studied decoupling the gravitational degrees of freedom. We are thus left with an interacting six-dimensional quantum field theory with (2,0) rigid supersymmetry. Since K3 singularities can be labelled by ADE Dynkin diagrams, the same holds for (2,0) theories: we thus have theories of type A_n, D_n, E_6, E_7, E_8. Loosely speaking we can then have 'gauge groups' $SU(n+1)$, $SO(2n)$, E_6, E_7, E_8, but clearly this is not a precise statement since there are no massless vectors in the spectrum and therefore we are dealing with a novel kind of theories.

Let us also stress that, since the theory does not have any mass scale, it is scale invariant and, by virtue of general arguments found e.g. in [144], it is expected to be conformally invariant. It is interesting to note that in Nahm's classification of rigid superconformal algebras [30], the six-dimensional (2,0) superconformal algebra is special, in the sense that it is the superconformal algebra with the highest possible dimension of spacetime.

An alternative realization of non-trivial (2,0) theories is furnished by the study of coincident M5-branes in M-theory. First of all, note that an M2-brane can end on an M5-brane. One can check that such a configuration is compatible with all charges possessed by these extended objects. This can also be expected by noting that, upon a suitable circle reduction, an M2-brane stretching between two M5-branes becomes a fundamental string of Type IIA stretching between two D4-branes, which is undoubtedly an allowed configuration. The world-volume theory of a single M5-brane has (2,0) supersymmetry. As a result, the only matter multiplet that can live on an M5-brane is a tensor multiplet, with one antiself-dual tensor. The boundary of an M2-brane ending on a M5-brane defines a string on the world-volume of the latter. It is natural to think that this string couples to the antiself-dual tensor on the world-volume of the M5-brane. Let us also point out that the five real scalars in the tensor multiplet have a clear interpretation in terms of the M5-branes: these scalars encode the fluctuations of the brane in the five spatial directions transverse to its world-volume.

Let us now consider a collection of N parallel M5-branes separated in the transverse five directions. The light degrees of freedom of the system are those of the eleven-dimensional supergravity multiplet and of the tensor multiplets on the world-volumes of the M5-brane. We can imagine to flow to the deep infrared and decouple gravity: we are thus left with a non-interacting theory of tensor multiplets of (2,0) supersymmetry. If we now put the N M5-branes on top of each other the states coming from M2-branes stretching between them become massless. By analogy to the D-brane picture one is led to conclude that these additional massless states are responsible for some sort of 'gauge symmetry enhancement' that yields a non-trivial interacting theory. Indeed, if we reduce on a circle we get a stack of N D4-branes in Type IIB with gauge group $U(N) = SU(N) \times U(1)$. The $U(1)$ factor is associated to the center-of-mass motion of the D-brane system and is thus uninteresting. We are led to conclude that a stack of N M5-branes gives rise, upon decoupling gravity and the center-of-mass degrees of freedom, to an interacting (2,0) theory of type A_{N-1}.

6.2 Some general features of (2,0) theories

Some properties of interacting (2,0) theories can be derived from general field theory arguments without any detailed knowledge of the interactions of the system. Let us briefly review some of these considerations, largely following [144] and [145].

As explained above, (2,0) theories emerge in the deep infrared dynamics of some Type IIB and M-theory setups, so that they are non-trivial IR fixed points of the RG flow. As a result, they cannot possess any dimensionful parameter. They are also expected, however, to be isolated fixed points, so that they do not have any dimensionless parameter either. This can be contrasted, for instance, with four-dimensional $\mathcal{N} = 4$ super Yang-Mills, which is a superconformal theory with vanishing beta-function whose coupling can be tuned at will.

Supersymmetry forces the moduli space of vacua of a (2,0) theory to be locally flat. Away from possible singularities in moduli space we have a free theory of r tensor multiplets. In the M5-brane picture they can be thought of as the 'Cartan' tensor multiplets living on the world-volume of M5-branes which are separated in the transverse directions (modulo the center-of-mass degree of freedom). More precisely the moduli space is

$$\mathcal{M} = \frac{\mathbb{R}^{5r}}{\mathcal{W}} , \tag{6.1}$$

where $\mathcal{W}$ is a discrete group. It is identified with the Weyl group of a Lie group G whose Lie algebra $\mathfrak{g}$ is determined by the ADE type of the given (2,0) theory.[1] By abuse of terminology we will refer to G as the gauge group of the (2,0) theory.

Let B^α, $\alpha = 1, \ldots, r$ be the antiself-dual tensors in the free theory at a generic point in moduli space. They have naturally mass dimension two, so that their integral $\int_{\mathcal{C}_2} B^\alpha$ on a two-cycle $\mathcal{C}_2$ is dimensionless. In order for the 'Wilson surface' operator $\exp\left(i \int_{\mathcal{C}_2} B^\alpha\right)$ to be invariant under large gauge transformations of B^α, the field strengths $H^\alpha = dB^\alpha$ must satisfy the quantization condition

$$\int_{\mathcal{C}_3} H^\alpha \in 2\pi\,\mathbb{Z} , \tag{6.2}$$

where $\mathcal{C}_3$ is an arbitrary three-cycle in six-dimensional spacetime. The possible H-fluxes define therefore an r-dimensional lattice Γ known as the charge lattice of the (2,0) theory. It has been shown that it has to be a self-dual lattice [147]. Moreover, it is identified with the weight lattice of the gauge group G of the theory.

No six-dimensional Lagrangian description is known for the interacting theories at the singular points in moduli space, and it is not even known if a Lagrangian can exist at all. This is due to the fact that, since the theory is superconformal, the coupling does not run and is therefore always fixed at some non-perturbative value. Nonetheless, it is possible to define the partition function of a (2,0) theory without reference to any action, as shown in [148]. The analysis there also reveals that, in order to define a (2,0) theory, a choice has to be made of suitable discrete topological data of six-dimensional spacetime, which can be intuitively thought of as the tensor analog of the spin structure required for defining fermions.

Different strategies can be applied to circumvent the absence of an action formulation for (2,0) theories. For instance, (2,0) theories of type A_n admit a formulation in terms of matrix models [149, 150, 151] that describes part of their dynamics. Gauge/gravity duality can also be used to infer some general properties of the local operators of the full interacting quantum theory, see e.g. [152]. Anomalies provide another robust window on some aspects of (2,0) that do not depend on the details

[1] Note that there can be different groups G with the same algebra $\mathfrak{g}$. A discussion of this subtle point is beyond our scope, and we refer the reader to [146, 145] for further explanations.

of the interactions. In particular the analysis of [153] reproduces the N^3 scaling of degrees of freedom of a (2,0) theory, already mentioned in section 3.5. The same scaling behavior has been deduced with different approaches in [154, 155, 156]. Anomalies are also related to proposals about certain topological couplings that the full interacting theory should possess [157, 158].

Finally, let us mention that, even if we do not have full control over the dynamics of interacting (2,0) theories, they have been successfully exploited to generate a rich class of four-dimensional $\mathcal{N} = 2$ gauge theories. Arguably the most known example is furnished by the theories defined in [159] by compactifying the A_1 (2,0) theory of a punctured Riemann surface. This construction prompted many further developments, most notably the AGT correspondence of [160].

6.3 (2,0) theories and five-dimensional super Yang-Mills

Let us consider the circle reduction of a (2,0) theory at a generic point in its moduli space, where we have the non-interacting tensors B^α, $\alpha = 1, \ldots, r$. As we will discuss in greater detail in section 10.1, a (anti)self-dual tensor on a circle yields one massless vector and a Kaluza-Klein tower of massive tensors. At sufficiently low energies these massive states can be neglected and we are left with a collection of vectors A^α with gauge group $U(1)^r$. Suppose we move to a singular point in the moduli space (6.1) of the (2,0) theory. It is possible to argue that, from a five-dimensional perspective, additional vectors become massless, which enhance $U(1)^r$ to a non-Abelian gauge group. If we start from a (2,0) theory of type A_{N-1}, so that $r = N - 1$, the discussion of the previous section indicates that this group is $SU(N)$. In summary, we obtain maximally supersymmetric Yang-Mills theory (MSYM) in five dimensions with gauge group $SU(N)$.

The gauge coupling of MSYM is given by [144]

$$g_{\rm YM}^2 = 4\pi^2 R \ , \tag{6.3}$$

where R is the radius of the compactification circle. This is consistent with the fact that $g_{\rm YM}$ has mass dimension $-1/2$ in five dimensions and that there are no dimensionful nor dimensionless parameters in the (2,0) theory. In (6.3) we have adopted the normalization of e.g. [161] and [34], which is natural in the discussion of Kaluza-Klein momentum and instanton number developed below.

From the Type IIB and M-theory analysis of section 6.1 we know that (2,0) theories possess six-dimensional string excitations. These are BPS states and become tensionless at the point of the moduli space (6.1) where the theory is non-trivial. Let us follow these string states in the circle reduction of the (2,0) theory.

If a six-dimensional string wraps the compactification circle it yields a particle state in five dimensions. We will always consider strings with winding number one along the circle. In the simplest possible case the string lies in its ground state and carries no Kaluza-Klein momentum along the circle. We identify such a state with a W-boson of five-dimensional MSYM. If instead the string has some Kaluza-Klein momentum,

$$m = \frac{n}{R} \ , \qquad n \neq 0 \ , \tag{6.4}$$

the associated particle state in five dimensions is a soliton of MSYM, which is conveniently thought of as the uplift of a four-dimensional instanton. These solitons can therefore carry a non-vanishing instanton number

$$n = \int_{\Sigma_4} p_1(E) = -\frac{1}{2}\frac{1}{(2\pi)^2}\int_{\Sigma_4} \mathrm{tr}(F \wedge F) \ , \tag{6.5}$$

where Σ_4 is a spatial slice of five-dimensional spacetime, E denotes the non-Abelian gauge bundle, $p_1(E)$ is its first Pontryagin class, F is the non-Abelian field strength, and tr is the trace in the fundamental representation of $SU(N)$. The integers n in (6.4) and (6.5) are actually identified: excited Kaluza-Klein modes of a wrapped string are the same as Yang-Mills instantons with non-vanishing instanton number [162]. In [34] these correspondence has been extended including states that carry electric charge under the five-dimensional gauge group.

If a six-dimensional string does not wrap the compactification circle it yields a string in five dimensions. These states can be realized in MSYM as an uplift of four-dimensional 't Hooft-Polyakov monopoles. As shown in [34] one can find BPS string-like states that have a non-vanishing instanton number and magnetic charge: they can be identified with the modes of a six-dimensional string that does not wrap the compactification circle, but has some Kaluza-Klein momentum along that direction.

Given the presence of BPS strings in six-dimensional (2,0) theories, it is not obvious *a priori* if these theories are better understood as quantum field theories satisfying the usual locality axioms, or rather as theories of strings, i.e. extended, non-local objects. The connection to five-dimensional MSYM suggests that the former interpretation is the correct one [144]. In fact, we have seen that a wrapped string with no Kaluza-Klein momentum is interpreted as a W-boson, which is an elementary field of MSYM. Wrapped strings with Kaluza-Klein momentum, or unwrapped strings, are identified with solitonic excitations of MSYM, which are intuitively speaking made out of W-bosons, and thus do not represent new independent elementary objects. If we were to regard the six-dimensional string as an elementary object we would overcount elementary degrees of freedom of MSYM.

It is important to recall that five-dimensional MSYM theory is power-counting non-renormalizable. Thus it is not clear if its classical Lagrangian suffices to describe physics at high energies, or if it has to be supplemented by additional information related to UV degrees of freedom. Note also that (2,0) theories are expected to be finite quantum field theories: they would then provide a possible UV completion for five-dimensional MSYM. In [33] it has been conjectured that this relation can be reversed: five-dimensional MSYM theory in the infinite-coupling limit (with all non-perturbative effects taken into account) is equivalent to the (2,0) theory, without the need of any new degree of freedom. The fact that Kaluza-Klein excited modes are already present in the form of five-dimensional solitons is consistent with this picture. If this conjecture is true it has deep implications on the structure of perturbative divergences of five-dimensional MSYM [33], which has been proven to suffer from UV divergences starting at six loops [163].

Other pieces of evidence in favor of a strong connection between (2,0) theories and five-dimensional MSYM comes from the computation of suitable conformal indices of the latter theory on manifolds such as S^5 or $\mathbb{CP}^2 \times S^1$, see for instance [164, 165, 166] and references therein. These computations show that a path integral based on the classical action for the massless fields of the Yang-Mills theory is able to reproduce the N^3 scaling of the degrees of freedom of a six-dimensional (2,0) theory. As a final remark we would like to draw the attention of the reader to [167], where the proposal of the equivalence between (2,0) theories and MSYM is contrasted with deconstructing techniques, and [168], where the relation between (2,0) theories and MSYM involves the emergence of a timelike, as opposed to spacelike, direction.

PART II

F-theory effective actions

CHAPTER 7

Effective action for six-dimensional F-theory compactifications

This chapter is devoted to the analysis of the six-dimensional effective action of F-theory compactifications on elliptically fibered Calabi-Yau threefolds. We will thus substantiate the duality between F-theory and M-theory described in section 5.3 by implementing explicitly the prescription for the extraction of the desired six-dimensional action from the dynamics of M-theory on the resolved Calabi-Yau threefold. In particular we will match the anomaly coefficients in six dimensions with topological data of the elliptic fibration. A careful analysis of the five-dimensional Chern-Simons terms on both sides of the duality reveals the importance of quantum corrections induced by massive Kaluza-Klein states.

7.1 F-theory and the space of six-dimensional (1,0) supergravities

The study of effective theories arising in string compactifications is clearly of crucial importance both from a conceptual as well as phenomenological point of view. It is now believed that there is a vast landscape of four-dimensional effective theories with minimal or no supersymmetry arising in string theory, but it is an open problem to systematically characterize these theories [169, 131, 102]. A systematic study becomes more tractable in compactifications to higher dimensions and with more supersymmetry. Highly supersymmetric compactifications have a more constrained effective theory, and arise from restricted classes of candidate string constructions. In the maximally supersymmetric case the theory and compactification geometry are in fact almost unique.

An intermediate scenario is provided by six-dimensional $(1,0)$ supergravity theories [170]. While there are strong constraints both from supersymmetry and anomalies in this dimension, the moduli space of these theories still permits a rich structure and is not fixed by the symmetries. In particular the effective action can feature various non-Abelian gauge groups and non-trivial matter representations. The $(1,0)$ multiplets in the spectrum are the gravity multiplet, a number of tensor and vector multiplets, as well as hypermultiplets. The latter can be either neutral, i.e. transform as singlets under the gauge group, or be charged, i.e. transform in non-trivial representations. We will refer to the second class of hypermultiplets as matter hypermultiplets.

In the last years a systematic study of six-dimensional $(1,0)$ supergravity theories has been undertaken to study the consistency conditions imposed by quantum gravity [170]. In six dimensions there are gravitational, gauge, and mixed gauge-gravitational anomalies. These impose constraints on the number of multiplets, and link the matter spectrum to the anomaly coefficients; see e.g. [171, 172, 173]. A fruitful starting point has been to ask for a realization of these supergravity theories as a compactification of F-theory on Calabi-Yau threefolds [174, 175, 176, 177, 178, 179, 180, 181, 182, 183]. Indeed, F-theory constructions cover a large part of the space of six-dimensional (1,0) theories that can be obtained from string theory. Furthermore the elliptic fibration structure of the Calabi-Yau threefolds entering the compactification makes it possible to undertake a classification of vacua based on the classifications of the possible Kähler two-folds that can be chosen as a basis of the fibration [184]. Topological transitions among the various bases translate into extremal tensionless string transitions from the point of view of the low-energy effective theory in six dimensions [185, 186].

These considerations constitute one of the main motivations for a detailed analysis of the duality between F-theory and M-theory in six dimensions. In particular this framework provides an excellent playground to get a better understanding of the correspondence between massive M2-brane states on the M-theory side and massive Kaluza-Klein and W-bosons in the circle compactification of the F-theory effective action. Indeed, we will be able to match the classical Chern-Simons terms of M-theory on the resolved Calabi-Yau threefold with the quantum-corrected Chern-Simons terms of the circle compactification. Our analysis applies to the case in which the gauge group is semi-simple, with no Abelian factor. In [187] it has been extended it to include $U(1)$ gauge bosons. The study of the low-energy F-theory effective action performed there confirms some results previously found in [182, 183] about the relation between geometry and anomalies in the presence of $U(1)$ factors. It also reveals interesting patterns in the quantum corrections to Chern-Simons levels induced by Kaluza-Klein modes and W-bosons.

Another motivation for the study of six-dimensional F-theory models comes from the observation that they can provide useful insights that can be successfully adapted to more complicated four-dimensional F-theory setups. For example, the relevance of quantum corrections to Chern-Simons couplings in the context of F-theory/M-theory duality can already be appreciated in six dimensions. The analysis of anomaly cancellation and Green-Schwarz mechanism for F-theory compactifications on Calabi-Yau fourfolds performed in [140] can be seen as an interesting generalization to four dimensions.

7.2 Elliptically fibered Calabi-Yau threefolds

The general features of an elliptically fibered Calabi-Yau space presented in Weierstrass form have already been described in section 5.2.2. In order to derive the effective action of a (1,0) F-theory setup in six dimensions, however, we need more information about the topology and geometry of elliptically fibered Calabi-Yau threefolds.

Let $\pi : X_3 \to B_2$ be a possibly singular elliptically fibered Calabi-Yau threefold described by the Weierstrass equation (5.22). Recall from section 5.2.2 that the vanishing locus of the discriminant Δ defined in (5.19) correspond to degenerations of the elliptic fiber. These may or may not correspond to singularities of the total space X_3. We are thus led to represent the divisor class $[\Delta]$ as

$$[\Delta] = \sum_A \nu_A [S_A] + [\Delta'] \ , \tag{7.1}$$

where $[S_A]$ are the classes of the irreducible, effective divisors S_A on which the Calabi-Yau threefold develops a singularity, while $[\Delta']$ is the residual class associated to singularities of the fibration which leave the total space smooth. Singularities of the Calabi-Yau threefold along S_A correspond to stacks of seven-branes on S_A which admit a non-Abelian gauge theory on their world-volume. As already mentioned in section 5.2.3, possible gauge groups can be classified looking at the possible singularities which occur in X_3 [175, 188, 189, 190]. The constants ν_A are related to group-theoretical invariants. The divisor Δ' is wrapped by a single seven-brane with no massless gauge bosons on its world-volume. Furthermore, if $[\Delta']$ and some of the $[S_A]$'s have non-vanishing intersection, singularity enhancements take place, which give rise to charged matter in the Type IIB picture. As explained in section 5.3 it is useful to resolve the singularities of X_3 to obtain a smooth Calabi-Yau threefold $\tilde{X}_3$. The canonical way of doing that, both if the singularity locus is a point and if it is a smooth curve, is discussed in [175, 188, 191]. For our purposes we do not need to perform the resolution explicitly, but we rather need only some general patterns of the topology of the resulting smooth space $\tilde{X}_3$.

Let us collect some results about divisors and intersection numbers of an elliptically fibered Calabi-Yau threefold. Recall that strict $SU(3)$ holonomy is always understood in our terminology. It is simpler to start with the case of a smooth threefold X_3. On such a space there is a natural set of divisors which span $H_4(X_3, \mathbb{R})$. Firstly, one has the section of the fibration which is homologous to the base B_2. Secondly, there is the set of vertical divisors D_α which are obtained as $D_\alpha = \pi^{-1}(D^{\mathrm{b}}_\alpha)$, where D^{b}_α is a divisor of B_2 and π is the projection to the base $\pi : X_3 \to B_2$. For these smooth elliptic fibrations one has $h^{1,1}(B_2) = h^{1,1}(X_3) - 1$ such divisors. Let ω_0, ω_α be the two-form cohomology classes Poincaré dual to the divisor classes $[B_2]$, $[D_\alpha]$. It is useful to record some facts concerning intersections of divisors for smooth elliptic fibrations. Due to the fibration structure one has

$$D_\alpha \cap D_\beta \cap D_\gamma = 0 \ . \tag{7.2}$$

We also introduce the matrix $\eta_{\alpha\beta}$ by defining

$$\eta_{\alpha\beta} = D^{\mathrm{b}}_\alpha \cap D^{\mathrm{b}}_\beta = B_2 \cap D_\alpha \cap D_\beta \ . \tag{7.3}$$

Note that $\eta_{\alpha\beta}$ is a non-degenerate symmetric matrix of signature $(+,-,\ldots,-)$ with $h^{1,1}(B_2)-1$ minus signs. Finally, let us recall the cohomological identity[1]

$$\omega_0 \wedge \omega_0 + c_1(B_2) \wedge \omega_0 = 0 \; . \tag{7.4}$$

We also introduce the vector K^α by expanding the first Chern class of the base B_2 onto a basis two-forms dual to vertical divisors as

$$-\, c_1(B_2) = K^\alpha \omega_\alpha \; . \tag{7.5}$$

Some basic formulas for the base B_2 of X_3 will be useful later. The Euler number $\chi(B_2)$ and the integral of $c_1^2(B_2)$ can be generally evaluated as

$$\chi(B_2) = \int_{B_2} c_2(B_2) = 2 + h^{1,1}(B_2) \; , \qquad \int_{B_2} c_1^2(B_2) = K^\alpha K^\beta \eta_{\alpha\beta} = 10 - h^{1,1}(B_2) \; , \tag{7.6}$$

where we have used (7.5) and the fact that $h^{1,0}(B_2) = h^{2,0}(B_2) = 0$ for a base of a Calabi-Yau manifold.

Let us now take into account a singular Calabi-Yau threefold X_3 and its resolution $\tilde{X}_3$. For the sake of simplicity, we will restrict ourselves to the case of a single seven-brane stack, thus omitting the sum over index A in (7.1). We thus have a simple gauge group G and we can write $[\Delta] = \nu[S] + [\Delta']$. Let D_i be the exceptional divisors of the resolved threefold $\tilde{X}_3$. They were introduced in general terms in section 5.2.3, where it was argued that the index i runs from 1 to $\operatorname{rank} G$. Recall also that the cohomology class Poincaré dual to $[D_i]$ is denoted ω_i. Furthermore, let us expand the divisor S wrapped by the stack of branes in a basis two-forms dual to vertical divisors as

$$\mathrm{PD}([S]) = C^\alpha \omega_\alpha \; . \tag{7.7}$$

Note that, after resolution, this is replaced by

$$\mathrm{PD}([\hat{S}]) = C^\alpha \omega_\alpha + a^i \omega_i \; , \tag{7.8}$$

where a^i are the Dynkin numbers characterizing the Dynkin diagram of G.[2] Exceptional divisors enjoy the following properties

$$\begin{aligned} B_2 \cap D_i &= 0 \; , \\ D_\alpha \cap D_i \cap D_j &= -C_{ij}\; B_2 \cap D_\alpha \cap S \; , \\ D_\alpha \cap D_\beta \cap D_i &= 0 \; , \end{aligned} \tag{7.9}$$

where C_{ij} is the Cartan matrix of the group G.

We are now in a position to summarize all intersection numbers on the resolved Calabi-Yau threefold $\tilde{X}_3$. We have found a cohomology basis $\{\omega_0, \omega_\alpha, \omega_i\}$ which can be denoted collectively as $\{\omega_\Lambda\}$. As in section 4.5.2 intersection numbers are defined as

$$\mathcal{V}_{\Lambda\Sigma\Theta} = \int_{\tilde{X}_3} \omega_\Lambda \wedge \omega_\Sigma \wedge \omega_\Theta \; . \tag{7.10}$$

[1] We will be slightly sloppy with the notation in the following, since we do not explicitly indicate that certain quantities, e.g. the first Chern class $c_1(B_2)$, have to be pulled back from B_2 to the Calabi-Yau threefold.

[2] Note that after singularity resolution also (7.5) is modified by the addition of non-trivial ω_i terms. Nonetheless, these terms do not affect the following discussion on intersection numbers, thanks to identities (7.9)

Identities and properties listed above imply that intersection numbers must satisfy

$$\begin{aligned} \mathcal{V}_{000} &= \eta_{\alpha\beta} K^\alpha K^\beta \ , & \mathcal{V}_{0i\Lambda} &= 0 \ , \\ \mathcal{V}_{00\alpha} &= \eta_{\alpha\beta} K^\beta \ , & \mathcal{V}_{\alpha ij} &= -C_{ij} \eta_{\alpha\beta} C^\beta \ , \\ \mathcal{V}_{0\alpha\beta} &= \eta_{\alpha\beta} \ , & \mathcal{V}_{\alpha\beta i} &= 0 \ , \\ \mathcal{V}_{\alpha\beta\gamma} &= 0 \ , \end{aligned} \tag{7.11}$$

where $\Lambda = 0, \alpha, j$. As far as $\mathcal{V}_{ijk}$ is concerned, in general it is non-vanishing, but otherwise unconstrained by our discussion so far. These intersection numbers arise from intersecting exceptional divisors. In fact, as we will discuss below, they will be linked to group-theoretical factors depending on the charged matter content of the gauge theory.

7.3 Generalities on six-dimensional (1,0) supergravity

In this section we review some basic facts about the spectrum and the dynamics of a generic six-dimensional supergravity model with $(1,0)$ supersymmetry, corresponding to 8 real supercharges.

7.3.1 Field content

Massless states in six dimensions are classified by representations of the little group $SO(4) \cong SU(2) \times SU(2)$ and are therefore labelled by a couple of integer or half-integer spins, (j_L, j_R). Four different kinds of supersymmetric multiplets can be constructed, restricting to spin less or equal to two [173]. We list them following the chirality conventions which are more common in the six-dimensional supergravity literature, see e.g. [192]:

- gravity multiplet: $(1,1) \oplus 2(1,\frac{1}{2}) \oplus (1,0)$, i.e. the graviton, one Weyl[3] left-handed gravitino, one self-dual two-form;
- vector multiplet: $(\frac{1}{2},\frac{1}{2}) \oplus 2(\frac{1}{2},0)$, i.e. one vector and one Weyl left-handed gaugino;
- tensor multiplet: $(0,1) \oplus 2(0,\frac{1}{2}) \oplus (0,0)$, i.e. one antiself-dual two-form, one Weyl right-handed tensorino, one real scalar;
- hypermultiplet: $2(0,\frac{1}{2}) \oplus 4(0,0)$, i.e. one Weyl right-handed hyperino and two complex scalars.

A general model features one gravity multiplet, n_V vector multiplets, n_H hypermultiplets, n_T tensor multiplets. The (anti)self-duality condition is incompatible with a naive Lagrangian formulation, because the usual kinetic term for two-forms vanishes identically once it is taken into account. This is a six-dimensional analog of the problem one faces in writing the Ramond-Ramond effective action for

[3] An equivalent formulation makes use of a $SU(2)$ doublet of Weyl left-handed gravitini ($SU(2)$ is the automorphism group of the supersymmetry algebra), supplemented by a symplectic Majorana condition. Similar remarks apply to all other fermions. This explains why this model is sometimes referred to as $\mathcal{N} = 2$ in the literature.

C_4 in Type IIB supergravity, see section 2.2. In six dimensions in the special case $n_T = 1$ the antiself-dual two-form from the gravity multiplet and the self-dual two-form from the tensor multiplet can be combined into a two-form without any self-duality property, and the standard Lagrangian formulation applies. Nonetheless, a set of consistent, supersymmetric, two-derivative, classical equations of motion is known for arbitrary n_T [192]. We can still derive them from variation of a suitable pseudoaction, imposing the self-duality condition after computation of functional derivatives, as usual.

We will always restrict ourselves to the bosonic content of the model, and adopt notations described below. First of all, we denote all six-dimensional two-forms collectively as $\hat{B}^\alpha$, where $\alpha = 1, ... n_T + 1$.[4] The scalars coming from the n_T tensor multiplets parameterize the quotient

$$SO(1, n_T)/SO(n_T) \ . \tag{7.12}$$

It is customary to describe this coset scalar manifold by means of a vielbein formalism. We refer the reader to e.g. [192] for a detailed account. For our present discussion we need only to recall that a constant $SO(1, n_T)$ metric $\Omega_{\alpha\beta}$ is introduced, along with a set of $n_T + 1$ scalar fields j^α. The metric $\Omega_{\alpha\beta}$ has mostly minus Lorentzian signature $(1, n_T)$, and the scalars j^α are subject to the constraint

$$\Omega_{\alpha\beta} j^\alpha j^\beta = 1 \ . \tag{7.13}$$

Moreover, the scalar manifold is endowed with another non-constant, positive definite metric $g_{\alpha\beta}$, which is given in terms of $\Omega_{\alpha\beta}, j^\alpha$ by

$$g_{\alpha\beta} = 2 j_\alpha j_\beta - \Omega_{\alpha\beta} \ , \tag{7.14}$$

where $j_\alpha = \Omega_{\alpha\beta} j^\beta$. This metric is needed to write down the (anti)-self-duality condition for $\hat{B}^\alpha$ in a $SO(1, n_T)$ covariant way, as we will see in equation (7.42).

As far as vectors are concerned, in this section we consider a supergravity model with semi-simple gauge group $G = \prod_i G_i$. For each simple factor G_i let $\mathfrak{g}_i$ be the corresponding Lie algebra. We denote the $\mathfrak{g}_i$-valued gauge one-form by $\hat{A}_i$, and matrix multiplication will always be understood. Moreover, we use anti-Hermitian generators so that the expression for the non-Abelian field strength two-form reads

$$\hat{F}_i = d\hat{A}_i + \hat{A}_i \wedge \hat{A}_i = d\hat{A}_i + \tfrac{1}{2}[\hat{A}_i, \hat{A}_i] \ , \tag{7.15}$$

where here and in what follows no sum over i is understood. The field strength transforms covariantly under the gauge transformation

$$\delta\hat{A}_i = d\hat{\lambda}_i + [\hat{A}_i, \hat{\lambda}_i] \ , \tag{7.16}$$

where the gauge parameter $\hat{\lambda}_i$ is a $\mathfrak{g}_i$-valued zero-form. Let us recall the definition of the Chern-Simons three-form

$$\hat{\omega}_i^{\rm CS} = {\rm tr}\left(\hat{A}_i \wedge d\hat{A}_i + \tfrac{2}{3}\hat{A}_i \wedge \hat{A}_i \wedge \hat{A}_i\right) \ , \tag{7.17}$$

[4] Later on we will identify $n_T + 1 = h^{1,1}(B_2)$ in the duality to M-theory. This provides the match of the indices of the present section with the ones of section 7.2.

where the trace is taken in a suitable representation of $\mathfrak{g}_i$. More details about our normalization for gauge traces will be given below when we review anomalies in six dimensions. It is also useful to recall two key properties of the Chern-Simons three-form,

$$\delta\hat{\omega}_i^{\rm CS} = {\rm tr}\, d\hat{\lambda}_i \wedge d\hat{A}_i \ , \qquad d\hat{\omega}_i^{\rm CS} = {\rm tr}\, \hat{F}_i \wedge \hat{F}_i \ . \tag{7.18}$$

Next, let us make some remarks about the hyper sector. Each hypermultiplet contains four real scalars and therefore we use the notation q^U $(U = 1, ..., 4n_H)$. These scalar fields can be considered as real coordinates for a quaternionic manifold, whose metric we write as h_{UV}. The geometric structures of quaternionic manifolds have been studied intensively, see e.g. [193, 194]. Since our main focus will be on the tensor and vector multiplet structure, we will refrain from giving a detailed account of these results here. However, in the following we will need to consider some aspects of charged hypermultiplets. The only piece of information relevant to our discussion is the six-dimensional covariant derivative, which reads schematically

$$\hat{\mathcal{D}}q^U = dq^U + \hat{A}^I (T_I^{\mathbf{R}} q)^U \ , \tag{7.19}$$

where the index I runs over all generators of the gauge group G, and $T_I^{\mathbf{R}}$ are the group generators acting on the scalars q^U in the representation $\mathbf{R}$. Several examples of gauged six-dimensional $(1,0)$ supergravities are known. We refer the reader to [195, 196, 197, 198] and references therein for a detailed account on the subject.

Finally, gravitational degrees of freedom are described by means of the vielbein formalism. The analogue of the one-form gauge connection $\hat{A}$ is provided by the $\mathfrak{so}(1,5)$-valued spin connection one-form $\hat{\omega}$, determined by the vielbein through the usual torsionless condition

$$d\hat{e} + \hat{\omega} \wedge \hat{e} = 0 \ , \tag{7.20}$$

where matrix multiplication is understood. If $\hat{\ell}$ is a $\mathfrak{so}(1,5)$-valued zero-form which we interpret as infinitesimal parameter of a local Lorentz transformation, we have

$$\delta\hat{\omega} = d\hat{\ell} + [\hat{\omega}, \hat{\ell}] \ . \tag{7.21}$$

The correct covariant field strength is the curvature two-form $\hat{\mathcal{R}}$, which is constructed out of the spin connection according to

$$\hat{\mathcal{R}} = d\hat{\omega} + \hat{\omega} \wedge \hat{\omega} \ , \tag{7.22}$$

and is related to the components of the six-dimensional Riemann tensor $\hat{R}^{\hat{\lambda}}{}_{\hat{\tau}\hat{\mu}\hat{\nu}}$ by

$$\hat{\mathcal{R}}^{\hat{a}}{}_{\hat{b}} = \tfrac{1}{2} \hat{e}^{\hat{a}}_{\hat{\lambda}} \hat{e}^{\hat{\tau}}_{\hat{b}} \hat{R}^{\hat{\lambda}}{}_{\hat{\tau}\hat{\mu}\hat{\nu}}\, d\hat{x}^{\hat{\mu}} \wedge d\hat{x}^{\hat{\nu}} \ , \qquad \hat{a}, \hat{b}, = 0, ..., 5 \ . \tag{7.23}$$

We also define a gravitational Chern-Simons three-form

$$\hat{\omega}^{\rm CS}_{\rm grav} = {\rm tr}\, \left(\hat{\omega} \wedge d\hat{\omega} + \tfrac{2}{3} \hat{\omega} \wedge \hat{\omega} \wedge \hat{\omega} \right) \ , \tag{7.24}$$

which satisfies

$$\delta\hat{\omega}^{\rm CS}_{\rm grav} = {\rm tr}\, d\hat{\ell} \wedge d\hat{\omega} \ , \qquad d\hat{\omega}^{\rm CS}_{\rm grav} = {\rm tr}\, \hat{\mathcal{R}} \wedge \hat{\mathcal{R}} \ . \tag{7.25}$$

Note that the right hand side of the last equation is proportional to a characteristic class build from the curvature two-form. In general, the proportionality constant is fixed by the requirement that suitable integrals of such classes take integer values. This standard normalization is achieved by inserting a factor of $(2\pi)^{-1}$ for each occurrence of the curvature two-form $\hat{\mathcal{R}}$ specified by (7.23). In order to improve readability, we will never write down these factors of $(2\pi)^{-1}$ in the following. Similar remarks apply to the five-dimensional curvature two-form introduced in section 7.4.1.

7.3.2 Anomaly cancellation

As we have seen above, the spectrum of a general six-dimensional (1,0) supergravity model contains chiral fermions and (anti)self-dual two-forms. As a result, gauge, gravitational, and mixed anomalies may appear once one-loop effects are taken into account. Nonetheless, a generalization of the ten-dimensional Green-Schwarz mechanism [171], due to Sagnotti [172, 177], can be implemented to generate consistent, anomaly-free theories. Let us review it in the case at hand of a semi-simple gauge group $G = \prod_i G_i$ with no Abelian factor, using the notation of [170].

At the heart of the Green-Schwarz-Sagnotti mechanism lies the observation that tree-level exchange of quanta of the tensor fields $\hat{B}^\alpha$ can counterbalance one-loop anomalous diagrams. For this to be possible, the total anomaly polynomial must be of the form

$$\hat{I}_8 = \tfrac{1}{2}\Omega_{\alpha\beta}\hat{X}_4^\alpha \wedge \hat{X}_4^\beta \, , \tag{7.26}$$

where we introduced the four-forms

$$\hat{X}_4^\alpha = \tfrac{1}{2}a^\alpha \mathrm{tr}\, \hat{\mathcal{R}} \wedge \hat{\mathcal{R}} + \sum_i 2b_i^\alpha\, \lambda_i^{-1} \mathrm{tr}_f \hat{F}_i \wedge \hat{F}_i \, . \tag{7.27}$$

In these expressions a^α, b_i^α are known as anomaly coefficients and transform as vectors in the space $\mathbb{R}^{1,T}$ with symmetric inner product $\Omega_{\alpha\beta}$. More precisely, a^α, b_i^α can be shown to be vectors in a lattice, commonly referred to as the anomaly lattice of the theory [180]. Furthermore, tr_f denotes the trace in the fundamental representation, and λ_i are normalization constants depending on the type of each simple group factor, as described in [178, 179, 180, 181].

If condition (7.26) is met, the theory can be made anomaly-free by introducing the generalized Green-Schwarz term

$$\hat{S}^{\mathrm{GS}} = -\frac{1}{2}\int_{\mathcal{M}_6} \Omega_{\alpha\beta}\hat{B}^\alpha \wedge \hat{X}_4^\beta \, . \tag{7.28}$$

where $\mathcal{M}_6$ denotes six-dimensional spacetime. By computation of the anomaly polynomial $\hat{I}_8$ in terms of the chiral matter content [199] and comparison with the factorized form (7.26), the following necessary conditions for anomaly cancellation are found:

$$n_H - n_V = 273 - 29 n_T \tag{7.29}$$
$$0 = B^i_{\rm adj} - \sum_{\mathbf{R}} x^i_{\mathbf{R}} B^i_{\mathbf{R}} \tag{7.30}$$
$$\Omega_{\alpha\beta} a^\alpha a^\beta = 9 - n_T \tag{7.31}$$
$$-\Omega_{\alpha\beta} a^\alpha b^\beta_i = \frac{1}{6}\lambda_i \Big(\sum_{\mathbf{R}} x^i_{\mathbf{R}} A^i_{\mathbf{R}} - A^i_{\rm adj} \Big) \tag{7.32}$$
$$\Omega_{\alpha\beta} b^\alpha_i b^\beta_i = \frac{1}{3}\lambda_i^2 \Big(\sum_{\mathbf{R}} x^i_{\mathbf{R}} C^i_{\mathbf{R}} - C^i_{\rm adj} \Big) \qquad \text{(no sum over } i) \tag{7.33}$$
$$\Omega_{\alpha\beta} b^\alpha_i b^\beta_j = \lambda_i \lambda_j \sum_{\mathbf{RS}} x^{ij}_{\mathbf{RS}} A^i_{\mathbf{R}} A^j_{\mathbf{S}} \qquad (i \neq j). \tag{7.34}$$

Recall that n_H, n_V, n_T are the numbers of hyper-, vector and tensor multiplets in the model. The constants $A^i_{\mathbf{R}}, B^i_{\mathbf{R}}, C^i_{\mathbf{R}}$ are group theory coefficients defined through

$$\mathrm{tr}_{\mathbf{R}} \hat{F}_i^2 = A^i_{\mathbf{R}} \mathrm{tr}_f \hat{F}_i^2 \tag{7.35}$$
$$\mathrm{tr}_{\mathbf{R}} \hat{F}_i^4 = B^i_{\mathbf{R}} \mathrm{tr}_f \hat{F}_i^4 + C^i_{\mathbf{R}} (\mathrm{tr}_f \hat{F}_i^2)^2 \,. \tag{7.36}$$

Finally, $x^i_{\mathbf{R}}$, $x^{ij}_{\mathbf{RS}}$ denote the number of matter fields that transform in the irreducible representation $\mathbf{R}$ of gauge group factor G_i, and $(\mathbf{R}, \mathbf{S})$ of $G_i \times G_j$, respectively. Note that for groups such as $SU(2)$ and $SU(3)$, which lack a fourth order invariant, $B^i_{\mathbf{R}} = 0$ and there is no condition 7.30. In order to simplify the notation, in the rest of this chapter we absorb the group-theoretical prefactor λ_i into the definition of the trace,

$$\mathrm{tr} = \lambda_i^{-1}\, \mathrm{tr}_f \;. \tag{7.37}$$

In equations (7.17) and (7.18) the symbol tr should be interpreted in this fashion.

The Green-Schwarz term (7.28) contributes to the anomaly polynomial because the tensor fields transform inhomogeneously under gauge transformations and local Lorentz transformations. More precisely, we have

$$\delta \hat{B}^\alpha = d\hat{\Lambda}^\alpha - \tfrac{1}{2} a^\alpha\, \mathrm{tr}\, \hat{\ell}\, d\hat{\omega} - 2 \sum_i b^\alpha_i\, \mathrm{tr}\, \hat{\lambda}_i\, d\hat{A}_i \;. \tag{7.38}$$

In this equation $\hat{\Lambda}^\alpha$ is a collection of one-forms which are the parameters of the usual Abelian gauge invariance of two-form potentials. The correct, gauge-invariant field strength three-form for $\hat{B}^\alpha$ turns out to be

$$\hat{G}^\alpha = d\hat{B}^\alpha + \tfrac{1}{2} a^\alpha \hat{\omega}^{\rm CS}_{\rm grav} + 2 \sum_i b^\alpha_i\, \hat{\omega}^{\rm CS}_i \;, \tag{7.39}$$

and satisfies a non-standard Bianchi identity,

$$d\hat{G}^\alpha = \hat{X}^\alpha_4 \;. \tag{7.40}$$

7.3.3 Effective action

For the sake of notational simplicity in the rest of this chapter we will consider the simpler case in which the gauge group G consists of a single simple factor. It is straightforward to reintroduce several simple factors labelled by the index i.

The bosonic terms of the pseudoaction for six-dimensional $(1,0)$ supergravity with simple gauge group G is given by

$$\hat{S}^{(6)} = \int_{\mathcal{M}_6} + \tfrac{1}{2}\hat{R}\hat{*}1 - h_{UV}\hat{\mathcal{D}}q^U \wedge *\hat{\mathcal{D}}q^V - \tfrac{1}{4}g_{\alpha\beta}\hat{G}^\alpha \wedge \hat{*}\hat{G}^\beta - \tfrac{1}{2}g_{\alpha\beta}dj^\alpha \wedge \hat{*}dj^\beta$$
$$- 2\Omega_{\alpha\beta}j^\alpha b^\beta \mathrm{tr}\,\hat{F} \wedge \hat{*}\hat{F} - \tfrac{1}{2}\Omega_{\alpha\beta}\hat{B}^\alpha \wedge \hat{X}_4^\beta - \hat{V}\hat{*}1 \ . \qquad (7.41)$$

The non-constant, positive-definite metric $g_{\alpha\beta}$ has been introduced in (7.14). In the second line, $\hat{V}$ is a potential generated by gauging the hypermultiplet scalars q^U. Its explicit form can be found e.g. in [198], but will not be crucial for our discussion. Let us stress that we have included some higher-derivative terms connected with the Green-Schwarz mechanism described in the previous subsection. In particular, note that some higher-derivative terms are implicitly contained in the definition of the gauge and local Lorentz invariant field strength $\hat{G}^\alpha$ (7.39). Let us remind the reader that (7.41) has to be supplemented by the self-duality constraint of for the tensors $\hat{B}^\alpha$. It is written in terms of the three-form field strengths as

$$g_{\alpha\beta}\hat{*}\hat{G}^\alpha = \Omega_{\alpha\beta}\hat{G}^\beta \ , \qquad (7.42)$$

where $g_{\alpha\beta}$ is the positive-definite, non-constant metric introduced in (7.14).

The classical action (7.41) might fail to be gauge and local Lorentz invariant because of the Green-Schwarz term. Indeed,

$$\delta\hat{S}^{(6)} = \tfrac{1}{2}\int_{\mathcal{M}_6} \Omega_{\alpha\beta}\left(\tfrac{1}{2}a^\alpha \mathrm{tr}\,\hat{\ell}d\hat{\omega} + 2b^\alpha \mathrm{tr}\,\hat{\lambda}d\hat{A}\right) \wedge \hat{X}_4^\beta \ , \qquad (7.43)$$

which in general is not just a surface contribution. Of course this is precisely the reason why the Green-Schwarz mechanism can work. Nonetheless, let us point out that there is a special case where the action is already classically gauge invariant. It is enforced by the conditions

$$\Omega_{\alpha\beta}a^\alpha a^\beta = 0 \ , \qquad \Omega_{\alpha\beta}a^\alpha b^\beta = 0 \ , \qquad \Omega_{\alpha\beta}b^\alpha b^\beta = 0 \ . \qquad (7.44)$$

These conditions on a^α, b^α can be related to the spectrum of fields, in particular the charge matter content, through the anomaly cancellation conditions (7.29)-(7.34) As we argue in section 7.6, the match between the F-theory set-up and the M-theory compactification is simpler in this special case.

7.4 Circle compactification from six to five dimensions

In this section we discuss the circle reduction of a general six-dimensional $(1,0)$ supergravity theory. This gives us the first opportunity to discuss the circle reduction of a six-dimensional pseudoaction for two-forms. In this chapter we retain only massless Kaluza-Klein modes. This topic will be revisited in chapter 10, where both zeromodes and excited modes will be kept.

The reduction from six to five dimensions yields a non-Abelian gauge theory that is further pushed to the Coulomb branch according to the paradigm explained in section 5.3. The five-dimensional action is brought into canonical $\mathcal{N}=2$ form in subsection 7.4.3. We point out an intriguing generalization of the $\mathcal{N}=2$ formalism which captures the full reduced action. Finally, in subsection 7.4.4 we perform the reduction of a specific higher-order curvature correction that carries crucial information about gravitational six-dimensional anomalies.

7.4.1 Kaluza-Klein reduction on the circle at two-derivative level

Let us now study the supergravity model outlined above on a background with one compact spatial dimension, i.e. with topology $\mathbb{R}^5 \times S^1$. As anticipated above, we restrict ourselves to zeromodes only. Some general features of circle reductions have been already discussed in section 4.2.

Ansätze for the reduction

The metric Ansatz reads

$$d\hat{s}^2_{(6)} = \tilde{g}_{\mu\nu} dx^\mu dx^\nu + r^2 Dy^2 \ , \qquad Dy = dy - A^0 \ , \qquad y \sim y + 2\pi \tag{7.45}$$

where $A^0 = A^0_\mu dx^\mu$ is the Kaluza-Klein vector and all five-dimensional field are independent of the coordinate y along S^1. A twiddle on the five-dimensional metric is used to stress that a Weyl rescaling will be necessary to cast the five-dimensional Einstein-Hilbert term into canonical form, as already anticipated in section 4.2 for a general circle compactification. Note that (7.45) is a rewriting of (4.7) in a slightly different notation which is better suited for the problem at hand. With the current notation the $U(1)$ symmetry coming from circle diffeomorphisms $y \to y + \chi$ acts on the Kaluza-Klein vector as $A^0 \to A^0 + d\chi$, in such a way that the one-form Dy is invariant. The field strength of A^0 reads

$$F^0 = dA^0 \ . \tag{7.46}$$

It is useful to write down the Kaluza-Klein Ansatz for the metric in the vielbein formalism, too. Up to local Lorentz transformations, we can take

$$\hat{e}^a = \tilde{e}^a_\mu dx^\mu \ , \qquad \hat{e}^5 = r\, Dy \ , \tag{7.47}$$

where Dy is given in (7.45), and $\tilde{e}^a_\mu, a = 0, \dots, 4$ is the five-dimensional vielbein (independent of y) before Weyl rescaling.

Let us now turn to the one-forms and two-forms, and take into account zeromodes only. As already pointed out for general circle compactifications in section 4.2, in order to get lower-dimensional massless fields that are uncharged under the aforementioned $U(1)$ symmetry we have to expand all fields on Dy defined in (7.45). To begin with, we set

$$\hat{A} = A + \zeta\, Dy \ , \tag{7.48}$$

where ζ is a $\mathfrak{g}$-valued five-dimensional zero-form. The gravitational analogue of this relation consists of the expression for the spin connection components, which can be computed from (7.47):

$$\hat{\omega}_{ab} = \tilde{\omega}_{ab} + \tilde{\mathfrak{a}}^{(0)}_{ab} Dy \ , \qquad \hat{\omega}_{a5} = \tilde{\mathfrak{b}}^{(1)}_a + \tilde{\mathfrak{c}}^{(0)}_a Dy \ , \tag{7.49}$$

where $\tilde{\omega}_{ab}$ is the five-dimensional spin connection determined by $\tilde{e}^a_\mu$. The zero-forms $\tilde{\mathfrak{a}}^{(0)}_{ab}, \tilde{\mathfrak{c}}^{(0)}_a$, and the one-form $\tilde{\mathfrak{b}}^{(1)}_a$ are given by

$$\tilde{\mathfrak{a}}^{(0)}_{ab} = \tfrac{1}{2} r^2 \tilde{e}^\mu_a \tilde{e}^\nu_b F^0_{\mu\nu} \ , \qquad \tilde{\mathfrak{b}}^{(1)}_a = \tfrac{1}{2} r \tilde{e}^\lambda_a F^0_{\lambda\mu}\, dx^\mu \ , \qquad \tilde{\mathfrak{c}}^{(0)}_a = -\tilde{e}^\lambda_a \tilde{\nabla}_\lambda r \ , \tag{7.50}$$

where $\tilde{\nabla}_\lambda$ is the five-dimensional Levi-Civita connection before Weyl rescaling.

We are now in a position to write down the Kaluza-Klein Ansatz for the two-forms $\hat{B}^\alpha$. Care has to be taken because the six-dimensional transformation rule (7.38) entangles the degrees of freedom encoded in $\hat{B}^\alpha$ with those of vectors and gravity. Thus, we set

$$\hat{B}^\alpha = B^\alpha - \Big[A^\alpha - \tfrac{1}{2} a^\alpha \, \mathrm{tr}\,(\tilde{\mathfrak{a}}^{(0)} \tilde{\omega}) - 2 b^\alpha \, \mathrm{tr}\,(\zeta A) \Big] \wedge Dy \ . \tag{7.51}$$

In this way A^α, B^α have the simplest possible gauge transformations,

$$\delta A^\alpha = d\mu^\alpha \tag{7.52}$$

$$\delta B^\alpha = d\Lambda^\alpha + \mu^\alpha F^0 - \tfrac{1}{2} a^\alpha \, \mathrm{tr}\,(\ell d\tilde{\omega}) - 2 b^\alpha \, \mathrm{tr}\,(\lambda dA) \ , \tag{7.53}$$

where the infinitesimal parameters are a $\mathfrak{g}$-valued five-dimensional zero-form λ, a $\mathfrak{so}(1,4)$-valued five-dimensional zero-form ℓ, five-dimensional zero-, one-forms $\mu^\alpha, \Lambda^\alpha$. The first relation implies that A^α has a standard, Abelian field strength

$$F^\alpha = dA^\alpha \ . \tag{7.54}$$

However, the naive field strength dB^α is not gauge invariant, and must be improved by setting

$$G^\alpha = dB^\alpha - A^\alpha \wedge F^0 + \tfrac{1}{2} a^\alpha \tilde{\omega}^{\mathrm{CS}}_{\mathrm{grav}} + 2 b^\alpha \omega^{\mathrm{CS}} \ , \tag{7.55}$$

where

$$\tilde{\omega}^{\mathrm{CS}}_{\mathrm{grav}} = \mathrm{tr}\, \left(\tilde{\omega} \wedge d\tilde{\omega} + \tfrac{2}{3} \tilde{\omega} \wedge \tilde{\omega} \wedge \tilde{\omega} \right) \ , \tag{7.56}$$

$$\omega^{\mathrm{CS}} = \mathrm{tr}\, \left(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A \right) \ . \tag{7.57}$$

The corresponding non-standard Bianchi identity reads

$$dG^\alpha = -F^\alpha \wedge F^0 + \tfrac{1}{2} a^\alpha \mathrm{tr}\, \tilde{\mathcal{R}} \wedge \tilde{\mathcal{R}} + 2 b^\alpha \mathrm{tr}\, F \wedge F \ . \tag{7.58}$$

Dimensional reduction of the two-derivative Lagrangian

In the rest of this subsection we will only focus on the two-derivative part of the total pseudoaction (7.41). As a consequence, we drop higher curvature terms from the six-dimensional pseudoaction, and we also neglect gravitational contribution to the gauge transformation of B^α and to the field strength G^α. A discussion of the higher curvature corrections is postponed to subsection 7.4.4.

Let us stress that, even if we start with a pseudoaction in six dimensions, the resulting five-dimensional action is a proper action: it captures the dynamics of zeromodes without any need for auxiliary conditions to be imposed at the level of the equations of motion. This is possible because the six-dimensional two-forms $\hat{B}^\alpha$ dimensionally reduce to two-forms B^α and vectors A^α as seen in (7.51). At the same time, we also have to dimensionally reduce the self-duality constraint (7.42). Explicitly we find

$$r\, g_{\alpha\beta}\, \tilde{*}G^\beta = -\Omega_{\alpha\beta}\mathcal{F}^\beta \;, \tag{7.59}$$

where we have introduced the shorthand notation

$$\mathcal{F}^\alpha = F^\alpha - 4b^\alpha \mathrm{tr}\,(\zeta F) + 2b^\alpha \mathrm{tr}\,(\zeta\zeta)F^0 \;. \tag{7.60}$$

The key point is that the five-dimensional duality condition (7.59) now relates two-forms and vectors. Since it does not involve a self-duality, it can be imposed on the level of the action itself. Hence, in computing the five-dimensional action we proceed in the two steps:

1. We perform a straightforward reduction of the pseudoaction (7.41) ignoring momentarily the self-duality constraint. The resulting five-dimensional pseudoaction, denoted $S^{(5)\mathrm{F}}_{\mathrm{pseudo}}$, should be written in a form such that B^α only appears through its field strength G^α. Moreover, G^α can be treated as an independent variable which enters the action only algebraically.
2. The five-dimensional pseudoaction $S^{(5)\mathrm{F}}_{\mathrm{pseudo}}$ is modified by adding a term $\Delta S^{(5)\mathrm{F}}$ with the following properties. If $\Delta S^{(5)\mathrm{F}}$ is regarded as a functional of A^α, B^α it is a total derivative. If it is regraded as a functional of A^α, G^α it is engineered in such a way that variation of $S^{(5)\mathrm{F}}_{\mathrm{pseudo}} + \Delta S^{(5)\mathrm{F}}$ with respect to G^α reproduces both the self-duality constraint and the Bianchi identity for G^α.

If we succeed in performing these two steps we can integrate out the five-dimensional three-forms G^α by means of their classical equation of motion and thus obtains a proper action $S^{(5)\mathrm{F}}$ for the vectors A^α. In the remaining part of this subsection we provide a detailed derivation of $S^{(5)\mathrm{F}}$ at two-derivative level. Crucially, however, the two steps described above can be performed even if we reintroduce the gravitational part of the generalized Green-Schwarz term, and all gravitational contributions to G^α, as discussed in section 7.4.4.

To begin with, let us record the expression for the functional $S^{(5)\mathrm{F}}_{\mathrm{pseudo}}$ obtained by naive dimensional reduction of the two-derivative part of (7.41) according to the metric, vector, and two-form Ansätze (7.45), (7.48), (7.51). One finds

$$\begin{aligned}
S^{(5)\mathrm{F}}_{\mathrm{pseudo}} = \int_{\mathcal{M}_5} & + \tfrac{1}{2} r\tilde{R}\,\tilde{*}1 - \tfrac{1}{4} r^3 F^0 \wedge \tilde{*}F^0 - \tfrac{1}{2} r g_{\alpha\beta} dj^\alpha \wedge \tilde{*}dj^\beta - r h_{UV}\mathcal{D}q^U \wedge \tilde{*}\mathcal{D}q^V \\
& - 2r\Omega_{\alpha\beta} j^\alpha b^\beta \mathrm{tr}\,(F - \zeta F^0) \wedge \tilde{*}(F - \zeta F^0) - 2r^{-1}\Omega_{\alpha\beta} j^\alpha b^\beta \mathrm{tr}\, D\zeta \wedge \tilde{*}D\zeta \\
& - \tfrac{1}{4} r g_{\alpha\beta} G^\alpha \wedge \tilde{*}G^\beta - \tfrac{1}{4} r^{-1} g_{\alpha\beta}\mathcal{F}^\alpha \wedge \tilde{*}\mathcal{F}^\beta \\
& - \tfrac{1}{2}\Omega_{\alpha\beta} G^\alpha \wedge (\mathcal{F}^\beta - F^\beta) + \Omega_{\alpha\beta} b^\alpha A^\beta \wedge \mathrm{tr}\, F \wedge F \\
& - 2\Omega_{\alpha\beta} b^\alpha b^\beta \omega^{\mathrm{CS}} \wedge \big(2\mathrm{tr}\,\zeta F - \mathrm{tr}\,\zeta\zeta F^0\big) \\
& - 2\Omega_{\alpha\beta} b^\alpha b^\beta \mathrm{tr}\,\zeta A \wedge \big(\mathrm{tr}\, F \wedge F - 2\mathrm{tr}\,\zeta F \wedge F^0 + \mathrm{tr}\,\zeta\zeta F^0 \wedge F^0\big) \\
& - \big[r\hat{V} + r^{-1} h_{UV}\zeta^I\zeta^J (T^{\mathbf{R}}_I q)^U (T^{\mathbf{R}}_J q)^V\big]\tilde{*}1 \;.
\end{aligned} \tag{7.61}$$

In this expression, $D\zeta = d\zeta + [A,\zeta]$ is the gauge covariant derivative for the adjoint scalars ζ, while $\mathcal{D}q^U = dq^U + A^I(T_I^{\mathbf{R}}q)^U$ are the five-dimensional gauge covariant derivatives for the scalars q^U in the hypermultiplets. The shorthand notation $\mathcal{F}^\alpha$ has been introduced in (7.60). Note that, by virtue of some integrations by parts, (7.61) has been cast in a form in which B^α only enters $S^{(5)\mathrm{F}}_{\mathrm{pseudo}}$ via G^α.

Next, we have to identify the suitable term $\Delta S^{(5)\mathrm{F}}$ for the implementation of the second step of the program outlined above. The correct form of $\Delta S^{(5)\mathrm{F}}$ reads

$$
\begin{aligned}
\Delta S^{(5)\mathrm{F}} &= \int_{\mathcal{M}_5} -\tfrac{1}{2}\Omega_{\alpha\beta} dB^\alpha \wedge F^\beta \\
&= \int_{\mathcal{M}_5} -\tfrac{1}{2}\Omega_{\alpha\beta} G^\alpha \wedge F^\beta + \tfrac{1}{2}\Omega_{\alpha\beta}(-A^\alpha F^0 + 2b^\alpha \omega^{\mathrm{CS}}) \wedge F^\beta \ .
\end{aligned} \tag{7.62}
$$

From the first line it is apparent that $\Delta S^{(5)\mathrm{F}}$ is a total derivative if it is regarded as a functional of A^α, B^α. Let us also stress that the Chern-Simons terms originating from the difference between dB^α and G^α in the second line of (7.62) are essential for our following discussion.

If we now consider $S^{(5)\mathrm{F}}_{\mathrm{pseudo}} + \Delta S^{(5)\mathrm{F}}$ as a functional of G^α and A^α, the equations of motion ensure both the self-duality condition (7.59) and the non-standard Bianchi identity (7.58). Moreover, G^α enters $S^{(5)\mathrm{F}}_{\mathrm{pseudo}} + \Delta S^{(5)\mathrm{F}}$ only quadratically, and is therefore readily integrated out. We thus obtain the desired five-dimensional proper action,

$$
\begin{aligned}
S^{(5)\mathrm{F}} = \int_{\mathcal{M}_5} & + \tfrac{1}{2} r\tilde{R}\,\tilde{*}1 - \tfrac{1}{4} r^3 F^0 \wedge \tilde{*} F^0 - \tfrac{1}{2} r g_{\alpha\beta} dj^\alpha \wedge \tilde{*} dj^\beta - r h_{UV} \mathcal{D}q^U \wedge \tilde{*}\mathcal{D}q^V \\
& - 2r\Omega_{\alpha\beta} j^\alpha b^\beta \mathrm{tr}\,(F - \zeta F^0) \wedge \tilde{*}(F - \zeta F^0) - 2r^{-1}\Omega_{\alpha\beta} j^\alpha b^\beta \mathrm{tr}\, D\zeta \wedge \tilde{*} D\zeta \\
& - \tfrac{1}{2} r^{-1} g_{\alpha\beta} \mathcal{F}^\alpha \wedge \tilde{*}\mathcal{F}^\beta - \tfrac{1}{2}\Omega_{\alpha\beta} A^0 \wedge F^\alpha \wedge F^\beta + 2\Omega_{\alpha\beta} b^\alpha A^\beta \wedge \mathrm{tr}\, F \wedge F \\
& - 2\Omega_{\alpha\beta} b^\alpha b^\beta \omega^{\mathrm{CS}} \wedge \big(2\mathrm{tr}\,\zeta F - \mathrm{tr}\,\zeta\zeta F^0\big) \\
& - 2\Omega_{\alpha\beta} b^\alpha b^\beta \mathrm{tr}\,\zeta A \wedge \big(\mathrm{tr}\, F \wedge F - 2\mathrm{tr}\,\zeta F \wedge F^0 + \mathrm{tr}\,\zeta\zeta F^0 \wedge F^0\big) \\
& - \big[r\hat{V} + r^{-1} h_{UV} \zeta^I \zeta^J (T_I^{\mathbf{R}} q)^U (T_J^{\mathbf{R}} q)^V\big]\tilde{*}1 \ .
\end{aligned} \tag{7.63}
$$

It is worth pointing out that $-\frac{1}{4} r g_{\alpha\beta} G^\alpha \wedge \tilde{*} G^\beta - \frac{1}{4} r^{-1} g_{\alpha\beta} \mathcal{F}^\alpha \wedge \tilde{*}\mathcal{F}^\beta$ vanishes identically after elimination of G^α, and that the kinetic term for vectors $-\frac{1}{2} r^{-1} g_{\alpha\beta}\mathcal{F}^\alpha \wedge \tilde{*}\mathcal{F}^\beta$ comes from the Chern-Simons term $-\frac{1}{2}\Omega_{\alpha\beta} G^\alpha \wedge \mathcal{F}^\beta$. Moreover, the term $+2\Omega_{\alpha\beta} b^\alpha A^\alpha \wedge \mathrm{tr}\, F \wedge F$ has a different prefactor because two different contributions must be taken into account: one was already present in (7.61), the other one is found in $\Delta S^{(5)\mathrm{F}}$.

The last step of the dimensional reduction consists of the Weyl rescaling

$$
\tilde{g}_{\mu\nu} = r^{-2/3} g_{\mu\nu} \ , \tag{7.64}
$$

which is simply a special case of (4.11) written in a slightly different notation. The final form of the action is therefore

$$\begin{aligned} S^{(5)\mathrm{F}} = \int_{\mathcal{M}_5} & + \tfrac{1}{2}R *1 - \tfrac{2}{3}r^{-2}dr \wedge *dr - \tfrac{1}{2}g_{\alpha\beta}dj^\alpha \wedge *dj^\beta \\ & - 2r^{-2}\Omega_{\alpha\beta}j^\alpha b^\beta \mathrm{tr}\, D\zeta \wedge *D\zeta - h_{UV}\mathcal{D}q^U \wedge *\mathcal{D}q^V \\ & - \tfrac{1}{4}r^{8/3}F^0 \wedge *F^0 - \tfrac{1}{2}r^{-4/3}g_{\alpha\beta}\mathcal{F}^\alpha \wedge *\mathcal{F}^\beta \\ & - 2r^{2/3}\Omega_{\alpha\beta}j^\alpha b^\beta \mathrm{tr}\,(F - \zeta F^0) \wedge *(F - \zeta F^0) \\ & - \tfrac{1}{2}\Omega_{\alpha\beta}A^0 \wedge F^\alpha \wedge F^\beta + 2\Omega_{\alpha\beta}b^\alpha A^\beta \wedge \mathrm{tr}\, F \wedge F \\ & - 2\Omega_{\alpha\beta}b^\alpha b^\beta \omega^{\mathrm{CS}} \wedge \big(2\mathrm{tr}\,\zeta F - \mathrm{tr}\,\zeta\zeta F^0\big) \\ & - 2\Omega_{\alpha\beta}b^\alpha b^\beta \mathrm{tr}\,\zeta A \wedge \big(\mathrm{tr}\, F \wedge F - 2\mathrm{tr}\,\zeta F \wedge F^0 + \mathrm{tr}\,\zeta\zeta F^0 \wedge F^0\big) \\ & - \big[r^{-1}\hat{V} + r^{-8/3}h_{UV}\zeta^I\zeta^J(T_I^{\mathbf{R}}q)^U(T_J^{\mathbf{R}}q)^V\big] * 1\,. \end{aligned} \tag{7.65}$$

7.4.2 Moving to the Coulomb branch

In the following sections, we will explore the dynamics of F-theory in six dimensions by means of the duality with M-theory on a Calabi-Yau threefold, as introduced in section 5.3. In this framework, we can access directly only the Coulomb branch of our non-Abelian gauge sector. The full gauge group G is spontaneously broken down to $U(1)^{\mathrm{rank}(G)}$, which is spanned by the Cartan generators T_i, $i = 1, ..., \mathrm{rank}(G)$. We take them to be normalized in such a way that

$$\mathrm{tr}\,(T_iT_j) = C_{ij} \tag{7.66}$$

where C_{ij} is the Cartan matrix of G.

The spontaneous break down of gauge symmetry is triggered by non-vanishing VEVs of some adjoint scalars ζ in the vector multiplets. In particular, inspection of the terms

$$-2r^{2/3}\Omega_{\alpha\beta}j^\alpha b^\beta \mathrm{tr}\, F \wedge *F - 2r^{-2}\Omega_{\alpha\beta}j^\alpha b^\beta \mathrm{tr}\, D\zeta \wedge *D\zeta \tag{7.67}$$

in the non-Abelian five-dimensional action (7.65) shows that the usual Higgs mechanism originates a mass term for the vectors lying outside of the Cartan subalgebra. As usual, we refer to these massive vectors as W-bosons. Their scalar partners acquire a mass, as well. From an effective field theory perspective, we are thus left only with the massless fields A^i, ζ^i associated to the Cartan subalgebra of the full gauge algebra. As a result, replacements such as

$$\begin{gathered} \mathrm{tr}\,(F \wedge *F) \to C_{ij}F^i \wedge *F^j\,, \qquad \mathrm{tr}\,(D\zeta \wedge *D\zeta) \to C_{ij}d\zeta^i \wedge *d\zeta^j \\ \omega^{\mathrm{CS}} \to C_{ij}A^i \wedge F^j \end{gathered} \tag{7.68}$$

have to be made in (7.65) to get the relevant five-dimensional action.

In a similar fashion, charged hypermultiplets acquire a mass through the five-dimensional scalar potential

$$V = r^{-1}\hat{V} + r^{-8/3}h_{UV}\zeta^I\zeta^J(T_I^{\mathbf{R}}q)^U(T_J^{\mathbf{R}}q)^V \tag{7.69}$$

given in the last line of (7.65). Note that the second term originates directly from dimensional reduction of the six-dimensional kinetic term $h_{UV}\hat{\mathcal{D}}q^U \wedge \hat{*}\hat{\mathcal{D}}q^V$. It is quadratic in the scalars of the charged hypermultiplets and is the source for their masses once gauge symmetry is spontaneously broken. Following the effective field theory paradigm, one should integrate out the massive hypermultiplets and only keep neutral hypermultiplets in the five-dimensional action in the Coulomb branch. We use lower-case indices $u, v = 1, ..., 4n_H^{\text{neutral}}$ to enumerate them. Hence, we have the replacement rule

$$h_{UV}\mathcal{D}q^U \wedge *\mathcal{D}q^V \to h_{uv}dq^u \wedge *dq^v \ , \tag{7.70}$$

where h_{uv} is a quantum corrected hypermultiplet metric. Determining h_{uv} after integrating out the massive states is in general a complicated task, but we will later give the M-theory expression for h_{uv} where certain corrections have been taken into account implicitly via the geometry. In accord with supersymmetry we also drop the scalar potential from the effective action for the massless modes.

The explicit form of the five-dimensional action pushed to the Coulomb branch according to the prescriptions above reads

$$\begin{aligned} S^{(5)\text{F}} = \int_{\mathcal{M}_5} & + \tfrac{1}{2}R *1 - \tfrac{2}{3}r^{-2}dr \wedge *dr - \tfrac{1}{2}g_{\alpha\beta}dj^\alpha \wedge *dj^\beta \\ & - 2r^{-2}\Omega_{\alpha\beta}j^\alpha b^\beta C_{ij}d\zeta^i \wedge *d\zeta^j - h_{uv}dq^u \wedge *dq^v \\ & - \tfrac{1}{4}r^{8/3}F^0 \wedge *F^0 - \tfrac{1}{2}r^{-4/3}g_{\alpha\beta}\mathcal{F}^\alpha \wedge *\mathcal{F}^\beta \\ & - 2r^{2/3}\Omega_{\alpha\beta}C_{ij}j^\alpha b^\beta(F^i - \zeta^i F^0) \wedge *(F^j - \zeta^j F^0) \\ & - \tfrac{1}{2}\Omega_{\alpha\beta}A^0 \wedge F^\alpha \wedge F^\beta + 2\Omega_{\alpha\beta}C_{ij}b^\alpha A^\beta \wedge F^i \wedge F^j \\ & - 2(\Omega_{\alpha\beta}b^\alpha b^\beta)(C_{kl}\zeta^k\zeta^l)C_{ij}\zeta^i A^j \wedge F^0 \wedge F^0 \\ & + 2(\Omega_{\alpha\beta}b^\alpha b^\beta)(C_{ij}C_{kl}\zeta^k\zeta^l + 2C_{ik}C_{jl}\zeta^k\zeta^l)A^i \wedge F^j \wedge F^0 \\ & - 6(\Omega_{\alpha\beta}b^\alpha b^\beta)C_{(ij}C_{k)l}\zeta^l A^i \wedge F^j \wedge F^k \ . \end{aligned} \tag{7.71}$$

In order to implement the F-theory lift discussed in section 7.6, however, it is essential to recast this result in a more transparent form. The aim of the following section is precisely the reformulation of the five-dimensional action in terms of new variables, in such a way to exploit the underlying supersymmetric structure. Hence, we begin our analysis with a concise review of five-dimensional $\mathcal{N} = 2$ supergravity.

7.4.3 The five-dimensional effective action and its canonical form

Let us briefly recall the field content of five-dimensional $\mathcal{N} = 2$ (8 real supercharges) supersymmetry multiplets [200]:

- gravity multiplet: the graviton, one vector (referred to as 'graviphoton'), one Dirac[5] gravitino;
- vector multiplet: one vector, one scalar, one Dirac gaugino;

[5] It is customary to replace one Dirac fermion by a $SU(2)$ doublet of Dirac fermions satisfying a symplectic Majorana condition. This explains the notation $\mathcal{N} = 2$.

- hypermultiplet: 2 complex scalars, one Dirac hyperino.

Let the spectrum consist of the gravity multiplet, $n_V^{(5)}$ vector multiplets, $n_H^{(5)}$ hypermultiplets, and let us focus on the bosonic sector. We are not going to study gauged supergravity models, and therefore the framework outlined in [201] is general enough for our purposes.[6] As usual, each hypermultiplet contributes four real scalars to the spectrum, and we will use notation q^u with $u = 1, ..., 4n_H^{(5)}$. The hypersector is entirely specified once a quaternionic structure with metric h_{uv} is given. Since the graviphoton and the vectors from the vector multiplets are naturally entangled by the dynamics of the theory, let us denote them collectively as $A^{\mathcal{I}}$ where $\mathcal{I} = 0, ..., n_V^{(5)}$. The scalars coming from the vector multiplets parameterize a $n_V^{(5)}$ manifold which is most conveniently described in terms of so-called very special coordinates $M^{\mathcal{I}}$. These are $n_V^{(5)} + 1$ real coordinates which describe an auxiliary $(n_V^{(5)} + 1)$-dimensional manifold in which the actual scalar manifold is embedded as an hypersurface, as explained below.

The dynamics of gravity-vector sector at two-derivative level is entirely specified once the cubic potential

$$\mathcal{N} = \tfrac{1}{3!} C_{\mathcal{I}\mathcal{J}\mathcal{K}} M^{\mathcal{I}} M^{\mathcal{J}} M^{\mathcal{K}} \tag{7.72}$$

is given in terms of very special coordinates and of a constant symmetric tensor $C_{\mathcal{I}\mathcal{J}\mathcal{K}}$. First of all, the scalar manifold is identified with the hypersurface described by the so-called very special geometry constraint

$$\mathcal{N} = 1\ . \tag{7.73}$$

Second of all, the gauge coupling function and the metric on the scalar manifold coincide and are constructed out of second derivatives of the cubic potential,

$$G_{\mathcal{I}\mathcal{J}} = \left[-\tfrac{1}{2}\partial_{M^{\mathcal{I}}}\partial_{M^{\mathcal{J}}} \log \mathcal{N}\right]_{\mathcal{N}=1} = \left[-\tfrac{1}{2}\mathcal{N}_{\mathcal{I}\mathcal{J}} + \tfrac{1}{2}\mathcal{N}_{\mathcal{I}}\mathcal{N}_{\mathcal{J}}\right]_{\mathcal{N}=1}\ . \tag{7.74}$$

In this expression, and in the following, downstairs indices $\mathcal{I}, \mathcal{J}$ denote partial derivative with respect to coordinates $N^{\mathcal{I}}, M^{\mathcal{J}}$. Finally, the constant tensor $C_{\mathcal{I}\mathcal{J}\mathcal{K}}$ itself appears in the action as Chern-Simons coupling. Indeed, the action is given by

$$\begin{aligned} S^{(5)\text{can}} = \int_{\mathcal{M}_5} & + \tfrac{1}{2} R * 1 - \tfrac{1}{2} G_{\mathcal{I}\mathcal{J}} dM^{\mathcal{I}} \wedge * dM^{\mathcal{J}} - h_{uv} dq^u \wedge * dq^v \\ & - \tfrac{1}{2} G_{\mathcal{I}\mathcal{J}} F^{\mathcal{I}} \wedge * F^{\mathcal{J}} - \tfrac{1}{12} C_{\mathcal{I}\mathcal{J}\mathcal{K}} A^{\mathcal{I}} \wedge F^{\mathcal{J}} \wedge F^{\mathcal{K}}\ . \end{aligned} \tag{7.75}$$

Let us now discuss the relation between the spectrum of a six-dimensional supergravity model and the spectrum of its Kaluza-Klein reduction on a circle. Suppose the numbers of six-dimensional tensor, vector and hypermultiplets are n_T, n_V, n_H respectively. To begin with, we note that the bosonic part

[6] In order to compare formulae below with the reference, the reader should be aware that we have changed notation, should recall our conventions on Riemann tensor contractions (cf. appendix A), and should also note that

$$C^{\text{there}}_{\mathcal{I}\mathcal{J}\mathcal{K}} = \frac{\sqrt{6}}{8} C^{\text{here}}_{\mathcal{I}\mathcal{J}\mathcal{K}}\ .$$

of a hypermultiplet behaves trivially under dimensional reduction on S^1. Hence, we can conclude that the number $n_H^{(5)}$ of five-dimensional hypermultiplets is given simply by

$$n_H^{(5)} = n_H^{\text{neutral}} \,, \tag{7.76}$$

where the label 'neutral' has been added to remind the reader that charged six-dimensional hypermultiplets are integrated out and do not appear in the five-dimensional effective theory.

As far as five-dimensional vectors are concerned, they are generated by three different mechanisms. First of all, one vector A^0 is introduced by the off-diagonal component of the Kaluza-Klein Ansatz for the six-dimensional metric. Second of all, $n_T + 1$ vectors A^α come from the (anti)-self-dual two-forms in six-dimensions. Finally, reduction of six-dimensional vectors gives us n_V additional A^i. We thus have a total of $1 + (n_T + 1) + n_V$ vectors, which we denote collectively as $A^{\mathcal{I}} = (A^0, A^\alpha, A^i)$. They fit into

$$n_V^{(5)} = n_V + n_T + 1 \tag{7.77}$$

five-dimensional vector multiplets, because one linear combination of $\{A^0, A^\alpha\}$ has to be identified with the graviphoton and sits in the gravity multiplet.[7] The corresponding scalar degrees of freedom are provided by j^α, ζ^i, r for a total of $(n_T + 1) + n_V + 1$ variables. However, they are subject to one constraint, which in six-dimensional language is given by (7.13). This counting is consistent with the existence of very special coordinates $M^{\mathcal{I}} = (M^0, M^\alpha, M^i)$ satisfying (7.73).

In the remaining part of this section we discuss in which way, and to which extent, the results of the dimensional reduction performed in 7.4.1 can be expressed in canonical form (7.75). The first step towards this direction is provided by the correct identification of the very special coordinates $M^{\mathcal{I}}$ on the vector multiplet scalar manifold. It turns out that these new coordinates are defined in terms of the old coordinates (r, j^α, ζ^i) by

$$\begin{aligned} M^0 &= r^{-4/3} \,, \\ M^\alpha &= r^{2/3}\left(j^\alpha + 2b^\alpha r^{-2} C_{ij}\zeta^i\zeta^j\right) \,, \\ M^i &= r^{-4/3}\zeta^i \,. \end{aligned} \tag{7.78}$$

Next, let us define

$$\mathcal{N}^{\text{F}} = \Omega_{\alpha\beta} M^0 M^\alpha M^\beta - 4\Omega_{\alpha\beta} b^\alpha C_{ij} M^\beta M^i M^j + 4\Omega_{\alpha\beta} b^\alpha b^\beta C_{ij} C_{kl} \frac{M^i M^j M^k M^l}{M^0} \,. \tag{7.79}$$

Expressions (7.78) and (7.79) are engineered in such a way that

$$\mathcal{N}^{\text{F}} = \Omega_{\alpha\beta} j^\alpha j^\beta = 1 \tag{7.80}$$

holds identically. In particular, note that this identity depends on the non-trivial interplay of the non-linear b^α-shifted redefinition of the coordinates M^α (7.78) and the fact that there is a non-polynomial term in the definition (7.79) of $\mathcal{N}^{\text{F}}$, including an inverse power of M^0. This non-polynomial term in

[7] We include A^α because we cannot exclude a contribution from the six-dimensional antiself-dual two-form in the gravity multiplet.

$\mathcal{N}$ is a significant deviation from the canonical case, in which $\mathcal{N}$ is a cubic polynomial, and will be discussed further in the following. However, note that $\mathcal{N}^{\mathrm{F}}$ is still a homogeneous function of degree three in the coordinates $M^{\mathcal{I}}$.

Once the new coordinates $M^{\mathcal{I}}$ are introduced, the five-dimensional effective action takes the form

$$S^{(5)\mathrm{F}} = \int_{\mathcal{M}_5} + \tfrac{1}{2}R \ast 1 - h_{uv} dq^u \wedge \ast dq^v - \tfrac{1}{2} G_{\mathcal{I}\mathcal{J}} dM^{\mathcal{I}} \wedge \ast dM^{\mathcal{J}} \\ - \tfrac{1}{2} G_{\mathcal{I}\mathcal{J}} F^{\mathcal{I}} \wedge \ast F^{\mathcal{J}} - \tfrac{1}{12} X_{\mathcal{I}\mathcal{J}\mathcal{K}} A^{\mathcal{I}} \wedge F^{\mathcal{J}} \wedge F^{\mathcal{K}} \,. \tag{7.81}$$

where the metric $G_{\mathcal{I}\mathcal{J}}$ and the coefficients $X_{\mathcal{I}\mathcal{J}\mathcal{K}} = X_{\mathcal{I}(\mathcal{J}\mathcal{K})}$ are functions of the scalar fields $M^{\mathcal{I}}$. Note that the gauge coupling function and the metric in the kinetic term for scalars $M^{\mathcal{I}}$ coincide, as expected for a five-dimensional $\mathcal{N} = 2$ theory. Moreover, both $G_{\mathcal{I}\mathcal{J}}$ and $X_{\mathcal{I}\mathcal{J}\mathcal{K}}$ are completely determined by the function $\mathcal{N}^{\mathrm{F}}$ introduced above, as explained in the following.

As far as the metric $G_{\mathcal{I}\mathcal{J}}$ is concerned, it is given precisely by (7.74). It is interesting to point out that the non-polynomial term in the definition of $\mathcal{N}^{\mathrm{F}}$ is crucial for (7.74) to hold for the Kaluza-Klein reduced action.

The Chern-Simons term in (7.81),

$$S^{(5)\mathrm{F}}_{\mathrm{CS}} = -\tfrac{1}{12} \int_{\mathcal{M}_5} X_{\mathcal{I}\mathcal{J}\mathcal{K}} A^{\mathcal{I}} \wedge F^{\mathcal{J}} \wedge F^{\mathcal{K}} \,, \tag{7.82}$$

deserves more discussion. Its variation under an Abelian gauge transformation $\delta A^{\mathcal{I}} = d\lambda^{\mathcal{I}}$ can be written as a boundary term, plus

$$\delta S^{(5)\mathrm{F}}_{\mathrm{CS}} = -\tfrac{1}{12} \int_{\mathcal{M}_5} \lambda^{\mathcal{I}} dX_{\mathcal{I}\mathcal{J}\mathcal{K}} \wedge F^{\mathcal{J}} \wedge F^{\mathcal{K}} \,. \tag{7.83}$$

For each value of indices $\mathcal{I}, \mathcal{J}, \mathcal{K}$, two possibilities may occur:

1. $X_{\mathcal{I}\mathcal{J}\mathcal{K}}$ is constant: the corresponding contribution to the Chern-Simons term is gauge invariant in five dimensions;
2. $X_{\mathcal{I}\mathcal{J}\mathcal{K}}$ depends non-trivially on the scalars $M^{\mathcal{I}}$: the corresponding contribution to the Chern-Simons term breaks five-dimensional gauge invariance explicitly.

Usually, only the first case is encountered in supergravity models. As a consequence, only the totally symmetric part of $X_{\mathcal{I}\mathcal{J}\mathcal{K}}$ effectively enters the action, because we are allowed to integrate by parts and permute indices on the vector and the field strengths in (7.82). This symmetry argument breaks down if some components of $X_{\mathcal{I}\mathcal{J}\mathcal{K}}$ are non-constant. In fact, the first slot of this tensor plays a distinguished role: exactly those gauge symmetries are broken, whose gauge vector has index $\mathcal{I}$ such that not all components $\{X_{\mathcal{I}\mathcal{J}\mathcal{K}}\}_{\mathcal{J},\mathcal{K}}$ are constant, as can be see from (7.83).

As already mentioned, all data needed to construct (7.82) can be extracted from the function $\mathcal{N}^{\mathrm{F}}$ introduced above. To this end, it is useful to note that $\mathcal{N}^{\mathrm{F}}$ naturally splits in a polynomial part $\mathcal{N}^{\mathrm{F}}_{\mathrm{p}}$

and a non-polynomial part $\mathcal{N}^{\rm F}_{\rm np}$,

$$\begin{aligned}\mathcal{N}^{\rm F}_{\rm p} &= \Omega_{\alpha\beta} M^0 M^\alpha M^\beta - 4\Omega_{\alpha\beta} b^\alpha C_{ij} M^\beta M^i M^j \\ \mathcal{N}^{\rm F}_{\rm np} &= 4\Omega_{\alpha\beta} b^\alpha b^\beta C_{ij} C_{kl} \frac{M^i M^j M^k M^l}{M^0} \ .\end{aligned} \tag{7.84}$$

On the one hand, since $\mathcal{N}^{\rm F}_{\rm p}$ is a homogeneous polynomial of degree three, its third derivatives with respect to coordinates $M^{\mathcal{I}}$ are constants. In fact, they turn out to be simply related to the coefficients of the gauge invariant part of (7.82). On the other hand, third derivatives of $\mathcal{N}^{\rm F}_{\rm np}$ are non-constant, and indeed they are proportional to the coefficient functions appearing in the gauge-anomalous contributions to (7.82). More precisely, we have

$$S^{(5)\rm F}_{\rm CS} = -\tfrac{1}{12}\int_{\mathcal{M}_5} (\mathcal{N}^{\rm F}_{\rm p})_{\mathcal{I}\mathcal{J}\mathcal{K}} A^{\mathcal{I}} \wedge F^{\mathcal{J}} \wedge F^{\mathcal{K}} - \tfrac{1}{16}\int_{\mathcal{M}_5} (\mathcal{N}^{\rm F}_{\rm np})_{i\mathcal{J}\mathcal{K}} A^i \wedge F^{\mathcal{J}} \wedge F^{\mathcal{K}} \ . \tag{7.85}$$

Two remarks are due at this point. Firstly, observe that the first term fits into the canonical form discussed above, since for a cubic polynomial as (7.72) one has precisely $\mathcal{N}_{\mathcal{I}\mathcal{J}\mathcal{K}} = C_{\mathcal{I}\mathcal{J}\mathcal{K}}$. Secondly, note that in the second term the first index never takes values $0, \alpha$. This means that the $U(1)$ gauge symmetries associated to vectors A^0, A^α are unbroken, while those associated to vectors A^i are broken.

It may be considered questionable, if not inconsistent, to construct a five-dimensional effective action which fails to be gauge invariant. However, this should not come as a surprise. Our starting point in six dimensions (7.41) is not gauge invariant as well, because of the introduction of the Green-Schwarz terms. As discussed in section 7.3, these terms are needed in order to implement the anomaly cancellation mechanism: they introduce tree-level gauge violations which counterbalance one-loop anomalous diagrams generated by the chiral matter content of the theory. As a result, the sum of the tree-level and one-loop contributions to the six-dimensional effective action is gauge invariant, while the two summands are not invariant separately. This suggests that a gauge invariant five-dimensional effective action could be obtained supplementing the computation of this section with the reduction of the one-loop six-dimensional effective action. However, we do not need to address this ambitious task, since we will show that all relevant data about the effective action of F-theory in six dimensions can already be extracted from the reduction of the tree-level action only.

It is worth mentioning a crucial distinction between anomalous terms in six and five dimensions. It is well known that five-dimensional theories do not develop quantum anomalies. Indeed, possible non-gauge invariant terms can always be cancelled by adding suitable local counter-terms to the tree level action, in such a way that the full effective action at one-loop is gauge-invariant. This kind of anomalies is referred to as 'irrelevant'. The aforementioned counterterms in five-dimensional take the form $\int A \wedge *J$, where A is one of the vectors whose gauge invariance is anomalous, and J is a gauge invariant five-dimensional current, such that $*J \propto F \wedge F$. It is precisely the gauge invariance of this current which makes the anomaly irrelevant. If we were to implement a similar mechanism to treat six-dimensional anomalies, we would have $*J \propto A \wedge F \wedge F$, which is manifestly non gauge invariant.

From this point of view, the non-gauge invariant Chern-Simons term which appears in (7.85) has the same form as the counterterms discussed above. More precisely, the corresponding gauge invariant current reads

$$* J_i = -\tfrac{1}{16}(\mathcal{N}^{\rm F}_{\rm np})_{i\mathcal{J}\mathcal{K}} F^{\mathcal{I}} \wedge F^{\mathcal{K}} \ . \qquad (7.86)$$

Note that all scalar fields in $(\mathcal{N}^{\rm F}_{\rm np})_{i\mathcal{J}\mathcal{K}}$ are neutral under the gauge group $U(1)^{{\rm rank}(G)}$ after spontaneous symmetry breaking to the Coulomb branch.

In summary, we are able to cast the Kaluza-Klein reduced action in canonical form, even though some subtle points have to be stressed:

- $\mathcal{N}$ has to be promoted from a cubic polynomial to a homogeneous function $\mathcal{N}^{\rm F}$ of degree three; the very special geometry constraint $\mathcal{N}^{\rm F} = 1$ and the metric $G_{\mathcal{I}\mathcal{J}}$ are formulated in terms of this non-polynomial $\mathcal{N}^{\rm F}$;
- the Chern-Simons term coming from Kaluza-Klein reduction and the Chern-Simons term obtained through the canonical prescription $C_{\mathcal{I}\mathcal{J}\mathcal{K}} = (\mathcal{N}^{\rm F})_{\mathcal{I}\mathcal{J}\mathcal{K}}$ share the same gauge-invariant part, and differ only for non gauge-invariant terms; these can be interpreted as local counterterms which make five-dimensional anomalies irrelevant.

Since counterterms are completely specified by the classical data of the model, all information about the effective five-dimensional action is encoded in the polynomial part of $\mathcal{N}^{\rm F}$ and the corresponding gauge-invariant Chern-Simons terms.

7.4.4 Higher order curvature corrections

As we have seen in subsection 7.3, anomaly cancellation requires the introduction of a higher curvature term in the six-dimensional action,

$$\hat{S}^{(6)}_{\mathcal{R}^2} = -\tfrac{1}{4}\int_{\mathcal{M}_6} \Omega_{\alpha\beta} a^\alpha \hat{B}^\beta \wedge {\rm tr}\,\hat{\mathcal{R}} \wedge \hat{\mathcal{R}} \ . \qquad (7.87)$$

Furthermore, local Lorentz transformations act non-trivially on the two-forms $\hat{B}^\alpha$, in such a way that the corresponding field strength $\hat{G}^\alpha$ receives a gravitational contribution, as can be seen from (7.39) specialized to the simpler case of a single simple factor in the gauge group. Even if we are not going to perform the dimensional reduction of the complete, higher-derivative action, we can make general remarks about some interesting feature of the resulting five-dimensional action.

First of all, as stated in subsection 7.4.1, inclusion of gravitational contributions does not interfere with the possibility to get rid of five-dimensional two-forms B^α in favor of vectors A^α. Indeed, gravitational terms modify the action in such a way that F^β in

$$\Delta S^{(5){\rm F}} = -\tfrac{1}{2}\int_{\mathcal{M}_5} \Omega_{\alpha\beta} dB^\alpha \wedge F^\beta \qquad (7.88)$$

is replaced by a more complicated expression, which is nonetheless exact. $\Delta S^{(5){\rm F}}$ is still a total derivative, and the elimination of B^α can proceed along the same line as in the two-derivative case.

Secondly, it can be verified that all possible non-gauge invariant terms in the final five-dimensional action are proportional to

$$\Omega_{\alpha\beta} a^\alpha a^\beta \qquad \text{or} \qquad \Omega_{\alpha\beta} a^\alpha b^\beta \qquad \text{or} \qquad \Omega_{\alpha\beta} b^\alpha b^\beta \ . \tag{7.89}$$

This observation will be relevant for the discussion of F-theory lift, in section 7.6.

Finally, let us present one particular higher-curvature contribution to the five-dimensional action, which will play a prominent role in the matching with M-theory on a Calabi-Yau threefold. It is the $A\mathcal{R}\mathcal{R}$ term coming from dimensional reduction of the $\hat{B}\hat{\mathcal{R}}\hat{\mathcal{R}}$ six-dimensional term written above. In order to extract this term from the total five-dimensional action, we can effectively set A^0 to zero and treat r as a constant. Note that the Weyl rescaling (7.64) has no effect on the leading, moduli-independent terms we are interested in. We thus obtain simply

$$\hat{\mathcal{R}}_{ab} = \mathcal{R}_{ab} + \dots \ , \qquad \hat{\mathcal{R}}_{a5} = 0 + \dots \ , \tag{7.90}$$

where $a, b, = 0, \dots, 4$ are five-dimensional flat spacetime indices, and '5' refers to the compact direction. As a consequence, we have

$$\operatorname{tr} \hat{\mathcal{R}} \wedge \hat{\mathcal{R}} = \operatorname{tr} \mathcal{R} \wedge \mathcal{R} + \dots \ . \tag{7.91}$$

A first contribution to the term we are looking for is then given by

$$\tfrac{1}{4} \int_{\mathcal{M}_5} \Omega_{\alpha\beta} a^\alpha A^\beta \wedge \operatorname{tr} \mathcal{R} \wedge \mathcal{R} \ , \tag{7.92}$$

in which the change of sign comes from the Ansatz (7.51). Note however that an addition contribution arises when $\Delta S^{(5)\mathrm{F}}$ is added in order to eliminate tensors from the five-dimensional action, as can be seen recalling the definition of G^α (7.55):

$$-\tfrac{1}{2} \int_{\mathcal{M}_5} \Omega_{\alpha\beta} dB^\alpha \wedge F^\beta \supset +\tfrac{1}{4} \int_{\mathcal{M}_5} \Omega_{\alpha\beta} a^\alpha \omega_{\mathrm{CS}}^{\mathrm{grav}} \wedge F^\beta = \tfrac{1}{4} \int_{\mathcal{M}_5} \Omega_{\alpha\beta} a^\alpha A^\beta \wedge \operatorname{tr} \mathcal{R} \wedge \mathcal{R} \ . \tag{7.93}$$

In summary, we find the five-dimensional higher curvature term

$$S^{(5)\mathrm{F}}_{A\mathcal{R}\mathcal{R}} = \tfrac{1}{2} \int_{\mathcal{M}_5} \Omega_{\alpha\beta} a^\alpha A^\beta \wedge \operatorname{tr} \mathcal{R} \wedge \mathcal{R} \ . \tag{7.94}$$

We conclude this subsection describing the effect of higher curvature terms on the canonical form of five-dimensional supergravity. As done in [202], superconformal techniques can be used to construct the five-dimensional supersymmetric completion of the $A\mathcal{R}\mathcal{R}$ term. In this formalism, the supersymmetry algebra closes off-shell, at the expense of introducing auxiliary fields in the gravity multiplet, vector multiplets, and hypermultiplets. The scalar manifold associated to vector multiplets is still described by constrained coordinates $M^{\mathcal{I}}$. However, the constraint is no longer

$$\tfrac{1}{3!} C_{\mathcal{I}\mathcal{J}\mathcal{K}} M^{\mathcal{I}} M^{\mathcal{J}} M^{\mathcal{K}} = 1 \ , \tag{7.95}$$

but gets corrected by terms proportional to the constants $c_{2\mathcal{I}}$ appearing in front of $A^{\mathcal{I}} \wedge \operatorname{tr} \mathcal{R} \wedge \mathcal{R}$ in the higher derivative Lagrangian [203]. More precisely,

$$\tfrac{1}{3!} C_{\mathcal{I}\mathcal{J}\mathcal{K}} M^{\mathcal{I}} M^{\mathcal{J}} M^{\mathcal{K}} = 1 - \tfrac{1}{72} c_{2\mathcal{I}} \left(D M^{\mathcal{I}} + v^{\mu\nu} F^{\mathcal{I}}_{\mu\nu} \right) \ , \tag{7.96}$$

where $D, v_{\mu\nu}$ are the auxiliary bosonic fields in the gravity multiplet. It is possible to integrate them out iteratively in a small $c_{2\mathcal{I}}$ expansion; the result reads schematically $CM^3 = 1 + cF^2$.

7.5 M-theory on an elliptically fibered Calabi-Yau threefold

In order to implement the duality between M-theory and F-theory we need to compute the effective action of M-theory on an elliptically fibered Calabi-Yau threefold. To begin with, we present the reduction on a general Calabi-Yau threefold at two-derivative level. Next, we specialize to the case of an elliptic fibration using the results collected before in section 7.2. Finally, we perform a partial dimensional reduction of a suitable higher-derivative correction in order to extract the terms that will be relevant for the matching with the F-theory side of duality.

7.5.1 Two-derivative effective action of M-theory on a Calabi-Yau threefold

This section is devoted to a review of the two-derivative five-dimensional effective action resulting from compactification of eleven-dimensional supergravity on a smooth Calabi-Yau threefold X_3, see for instance [204]. We will then specialize these results to the case of a resolved elliptic fibration in section 7.5.2.

Outcome of the dimensional reduction

Our starting point is the two-derivative action (3.34) of eleven-dimensional supergravity, which we record here again for ease of reference,

$$\hat{S}^{(11)} = \int_{\mathcal{M}_{11}} \tfrac{1}{2}\hat{R}\hat{*}1 - \tfrac{1}{4}\hat{G}_4 \wedge \hat{*}\hat{G}_4 - \tfrac{1}{12}\hat{C}_3 \wedge \hat{G}_4 \wedge \hat{G}_4 \ . \tag{7.97}$$

The eleven-dimensional gravitational constant has been suppressed for convenience. Let us also fix our notation for the metric and the three-form Ansätze. The former reads

$$ds^2_{11} = \tilde{g}_{\mu\nu}(x)dx^\mu dx^\nu + 2g_{\bar{\imath}j}(z)d\bar{z}^{\bar{\imath}}dz^j \tag{7.98}$$

where z^i $(i = 1, 2, 3)$ are complex coordinates on the threefold and the twiddle on the external metric is again a reminder of the necessity to perform a Weyl rescaling after reduction. As explained in detail in section 4.5.2, deformations of the internal metric $g_{i\bar{\jmath}}$ give rise to Kähler moduli and complex structure moduli. Let us remind the reader that the former are obtained by expanding the Kähler form of the threefold onto a basis $\{\omega_\Lambda\}$ of $H^2(X_3, \mathbb{Z})$, with $\Lambda = 1, \dots, h^{1,1}(X_3)$,

$$J = v^\Lambda \omega_\Lambda \ . \tag{7.99}$$

The complex structure moduli Z^κ are instead associated to $H^{1,2}(X_3)$, so that $\kappa = 1, \dots, h^{1,2}(X_3)$.

The Ansatz for the three-form reads

$$\hat{C}_3 = \xi^K \alpha_K - \tilde{\xi}_K \beta^K + A^\Lambda \wedge \omega_\Lambda + C_3 \ . \tag{7.100}$$

In this equation α_K, β^K are elements of an integral basis of the middle cohomology $H^3(X_3)$ of the threefold, $K = 1, \dots, h^{1,2}(X_3) + 1$. Some of their properties will be discussed below. The fields ξ^K, $\tilde{\xi}_K$

are scalars in the external five-dimensional spacetime, while the fields A^Λ are external five-dimensional vectors. Finally, C_3 is a five-dimensional three-form that will be later dualized into a real scalar Φ.

To begin with it is useful to discuss how the five-dimensional fields in (7.98), (7.100) fit into five-dimensional $\mathcal{N}=2$ supersymmetry multiplets. The gravity multiplet consists of $\tilde{g}_{\mu\nu}$ and of one (linear combination) of the A^Λ vectors. The remaining vectors fit into

$$n_V^{(5)} = h^{1,1}(X_3) - 1 \tag{7.101}$$

vector multiplets, along with the Kähler moduli v^Λ. It seems like there is a mismatch of degrees of freedom, since we have $h^{1,1}(X_3)$ scalars. This seeming difficulty is overcome by the observation that volume $\mathcal{V}$ of the threefold sits in a hypermultiplet. Recall from section 4.5.2 that $\mathcal{V}$ can be written as

$$\mathcal{V} = \frac{1}{3!}\int_{X_3} J\wedge J\wedge J = \frac{1}{3!}\mathcal{V}_{\Lambda\Sigma\Theta}v^\Lambda v^\Sigma v^\Theta \ , \tag{7.102}$$

where $\mathcal{V}_{\Lambda\Sigma\Theta}$ are the intersection numbers

$$\mathcal{V}_{\Lambda\Sigma\Theta} = \int_{X_3}\omega_\Lambda\wedge\omega_\Sigma\wedge\omega_\Theta \ . \tag{7.103}$$

We thus see that one of the degrees of freedom carried by the scalars v^Λ has to be subtracted from the counting of scalars in vector multiplets, in accord with (7.101).

To discuss hypermultiplets we need to recall the decomposition of the third cohomology into complex cohomologies,

$$H^3(X_3) = \left[H^{1,2}(X_3)\oplus H^{2,1}(X_3)\right]\oplus\left[H^{0,3}(X_3)\oplus H^{3,0}(X_3)\right] \ . \tag{7.104}$$

Real scalars $\xi^K, \tilde{\xi}_K$ provide $h^{1,2}(X_3)+1$ complex degrees of freedom: $h^{1,2}(X_3)$ of these correspond to the $H^{1,2}(X_3)\oplus H^{2,1}(X_3)$ component and combine with the complex structure moduli Z^κ to give $h^{1,2}(X_3)$ hypermultiplets; the remaining complex degree of freedom lives in $H^{0,3}(X_3)\oplus H^{3,0}(X_3)$ and combines with $\mathcal{V},\Phi$ in the so-called universal hypermultiplet. In conclusion, we have found

$$n_H^{(5)} = h^{1,2}(X_3) + 1 \tag{7.105}$$

hypermultiplets, which will be collectively denoted by q^u.

We can finally record the result of the dimensional reduction of (7.97) according to the Ansätze (7.98), (7.100). Since the overall volume sits in the universal hypermultiplet it is natural to define scalar fields

$$L^\Lambda = \mathcal{V}^{-\frac{1}{3}}v^\Lambda \ , \tag{7.106}$$

which are the real scalars in the vector multiplets. They only parameterize $h^{1,1}(X_3)-1$ degrees of freedom, since due to their definition they are subject to the constraint

$$\tfrac{1}{3!}\mathcal{V}_{\Lambda\Sigma\Theta}L^\Lambda L^\Sigma L^\Theta = 1 \ . \tag{7.107}$$

We are naturally led to interpret L^Λ as five-dimensional very special coordinates, in term of which the cubic potential reads

$$\mathcal{N} = \tfrac{1}{3!}\mathcal{V}_{\Lambda\Sigma\Theta}L^\Lambda L^\Sigma L^\Theta \ . \tag{7.108}$$

The reduced bosonic action is then given by

$$S^{(5)\mathrm{M}} = \int_{\mathcal{M}_5} + \tfrac{1}{2} R * 1 - \tfrac{1}{2} G_{\Lambda\Sigma} dL^\Lambda \wedge * dL^\Sigma - h_{uv} dq^u \wedge * dq^v \\ - \tfrac{1}{2} G_{\Lambda\Sigma} F^\Lambda \wedge * F^\Sigma - \tfrac{1}{12} \mathcal{V}_{\Lambda\Sigma\Theta} A^\Lambda \wedge F^\Sigma \wedge F^\Theta \ , \tag{7.109}$$

where, as expected from section 7.4.3,

$$G_{\Lambda\Sigma} = \left[-\tfrac{1}{2} \partial_{L^\Lambda} \partial_{L^\Sigma} \log \mathcal{N} \right]_{\mathcal{N}=1} \ , \tag{7.110}$$

and where the hypermultiplet moduli metric reads

$$h_{uv} dq^u \wedge * dq^v = + dD \wedge * dD + g_{\kappa_1 \bar{\kappa}_2} dZ^{\kappa_1} \wedge * d\bar{Z}^{\bar{\kappa}_2} \\ + \tfrac{1}{4} e^{4D} \left[d\Phi + (\xi^K d\tilde{\xi}_K - \tilde{\xi}_K d\xi^K) \right]^2 \\ - \tfrac{1}{2} e^{2D} (d\tilde{\xi}_K - \mathcal{M}_{KM} d\xi^M) (\mathrm{Im}\mathcal{M})^{-1\,KL} (d\tilde{\xi}_L - \mathcal{M}_{LN} d\xi^N) \ . \tag{7.111}$$

In this expression

$$D = -\tfrac{1}{2} \log \mathcal{V} \ , \tag{7.112}$$

and we have made use of a complex matrix $\mathcal{M}$ that encodes the relevant data about the middle cohomology of the threefold, to which we now turn.

Intermezzo: a few facts on the middle cohomology

As promised, let us take a short detour and discuss briefly some features of the middle cohomology of a Calabi-Yau threefold. The elements $\{\alpha_K, \beta^K\}$ $(K = 1, \ldots, h^{1,2}(X_3) + 1)$ are chosen in such a way as to form an integral symplectic basis of the middle cohomology $H^3(X_3)$ of the threefold. The only non-vanishing independent wedge product among basis elements is

$$\int_{X_3} \alpha_K \wedge \beta^L = \delta_L^K \ . \tag{7.113}$$

In order to discuss the metric on the moduli space of neutral hypermultiplets, we need to introduce matrices ${A_K}^L, B_{KL}, C^{KL}$, such that

$$* \alpha_K = {A_K}^L \alpha_L + B_{KL} \beta^L \ , \qquad * \beta^K = C^{KL} \alpha_L - {A_L}^K \beta^L \ , \tag{7.114}$$

where $*$ represents the Hodge star in X_3. These matrices can be conveniently expressed in terms of a symmetric, complex matrix $\mathcal{M}$,

$$\begin{aligned} {A_K}^L &= (\mathrm{Re}\mathcal{M})_{KH} (\mathrm{Im}\mathcal{M})^{-1\,HL} \ , \\ B_{KL} &= -(\mathrm{Im}\mathcal{M})_{KL} - (\mathrm{Re}\mathcal{M})_{KH} (\mathrm{Im}\mathcal{M})^{-1\,HM} (\mathrm{Re}\mathcal{M})_{ML} \ , \\ C^{KL} &= (\mathrm{Im}\mathcal{M})^{-1\,KL} \ . \end{aligned} \tag{7.115}$$

Of course, this matrix $\mathcal{M}$ is the same that enters the hypermultiplet moduli metric (7.111).

Derivation of the reduced action

We now discuss in detail the derivation of (7.109). The reader who is not interested in this computation can safely skip the rest of this section.

We do not need to discuss at length the reduction of the Einstein-Hilbert term, which has been discussed for general Ricci-flat manifolds in section 4.3 and appendix A.3. We only need to specialize those results to the case of a Calabi-Yau threefold, making use of the facts about Calabi-Yau moduli spaces collected in section 4.5.2. We also know already that after reduction we have to perform the Weyl rescaling

$$\tilde{g}_{\mu\nu} = \mathcal{V}^{-2/3} g_{\mu\nu} \ , \tag{7.116}$$

which is the specialization of (4.34) to $d = 5$, written in a slightly different notation. The result of the dimensional reduction of the Einstein-Hilbert term followed by the Weyl rescaling is

$$\int_{\mathcal{M}_{11}} \tfrac{1}{2}\hat{R}\hat{*}1 = \int_{\mathcal{M}_5} \tfrac{1}{2}R * 1 - \tfrac{1}{2}H_{\Lambda\Sigma}(v) dv^\Lambda \wedge * dv^\Sigma - g_{\kappa_1\bar{\kappa}_2} dZ^{\kappa_1} \wedge * d\bar{Z}^{\bar{\kappa}_2} \ . \tag{7.117}$$

The metric $g_{\kappa_1\bar{\kappa}_2}$ on the space of the complex structure moduli Z^κ was derived in section 4.5.2 and is recorded here again for ease of reference,

$$g_{\kappa_1\bar{\kappa}_2}(Z,\bar{Z}) = \partial_{Z^{\kappa_1}}\partial_{\bar{Z}^{\bar{\kappa}_2}}\mathcal{K}_{\text{cs}}(Z,\bar{Z}) \ , \qquad \mathcal{K}_{\text{cs}}(Z,\bar{Z}) = \log\left[i\int_{X_3} \Omega\wedge\overline{\Omega}\right] \ . \tag{7.118}$$

The metric on Kähler moduli space requires more care. The quantity $H_{\Lambda\Sigma}(v)$ in (7.117) is defined by

$$H_{\Lambda\Sigma}(v) = -G_{\Lambda\Sigma}(v) - \mathcal{V}^{-1}\mathcal{V}_{\Lambda\Sigma} \ , \tag{7.119}$$

where $G_{\Lambda\Sigma}(v)$ is the natural metric on the Kähler moduli space of the threefold, derived in section 4.5.2,

$$G_{\Lambda\Sigma}(v) = -\tfrac{1}{2}\partial_{v^\Lambda}\partial_{v^\Sigma}\log\mathcal{V}(v) = -\tfrac{1}{2}\mathcal{V}(v)^{-1}\mathcal{V}_{\Lambda\Sigma\Theta}v^\Theta + \tfrac{1}{8}\mathcal{V}(v)^{-2}\mathcal{V}_{\Lambda\Omega\Theta}\mathcal{V}_{\Sigma\Psi\Xi}v^\Omega v^\Theta v^\Psi v^\Xi \ . \tag{7.120}$$

As anticipated above, the natural variables in the Kähler moduli space are L^Λ defined in (7.106). Trading the scalars v^Λ for L^Λ we find

$$-\tfrac{1}{2}H_{\Lambda\Sigma}(v)dv^\Lambda\wedge * dv^\Sigma = -\tfrac{1}{2}G_{\Lambda\Sigma}(L)dL^\Lambda\wedge * dL^\Sigma - dD\wedge * dD \ , \tag{7.121}$$

where D was defined in (7.112). The symbol $G_{\Lambda\Sigma}(L)$ denotes the metric obtained by replacing v^Λ by L^Λ everywhere in (7.120). It is easily checked that it coincides precisely with (7.110).

Next, let us consider the reduction of the other terms in the eleven-dimensional action (7.97). As far as the three-form kinetic term is concerned, a straightforward computation shows that

$$\begin{aligned}\int_{\mathcal{M}_{11}} -\tfrac{1}{4}\hat{G}_4\wedge\hat{*}\hat{G}_4 = \int_{\mathcal{M}_5} & + \tfrac{1}{4}(d\tilde{\xi}_K - \mathcal{M}_{KM}d\xi^M)(\text{Im}\mathcal{M})^{-1\,KL}\wedge\tilde{*}(d\tilde{\xi}_L - \mathcal{M}_{LN}d\xi^N) \\ & - \tfrac{1}{2}\mathcal{V}G_{\Lambda\Sigma}(v)F^\Lambda\wedge\tilde{*}F^\Sigma - \tfrac{1}{4}\mathcal{V}G_4\wedge\tilde{*}G_4 \ ,\end{aligned} \tag{7.122}$$

where $F^\Lambda = dA^\Lambda$, $G_4 = dC_3$, and the matrix $\mathcal{M}$ is defined implicitly by the relations (7.115). For the Chern-Simons term, we find

$$\int_{\mathcal{M}_{11}} -\tfrac{1}{12}\hat{C}_3 \wedge \hat{G}_4 \wedge \hat{G}_4 = \int_{\mathcal{M}_5} -\tfrac{1}{12}\mathcal{V}_{\Lambda\Sigma\Theta}A^\Lambda \wedge F^\Sigma \wedge F^\Theta + \tfrac{1}{4}(\xi^K d\tilde{\xi}_K - \tilde{\xi}_K d\xi^K) \wedge G_4 \ . \tag{7.123}$$

As anticipated above, we can dualize the three-form C_3 into a real scalar Φ. To this end we add to the five-dimensional action the term

$$\Delta S^{(5)\mathrm{M}} = \int_{\mathcal{M}_5} \tfrac{1}{4} d\Phi \wedge G_4 \tag{7.124}$$

which implements Bianchi identity $dG_4 = 0$ if we consider G_4 rather than C_3 as independent variable. After elimination of G_4 via its equation of motion, we get

$$\begin{aligned} S^{(5)\mathrm{M}}_{\text{non-grav}} = \int_{\mathcal{M}_5} & + \tfrac{1}{4}(d\tilde{\xi}_K - \mathcal{M}_{KM}d\xi^M)(\mathrm{Im}\mathcal{M})^{-1\,KL} \wedge \tilde{*}(d\tilde{\xi}_L - \mathcal{M}_{LN}d\xi^N) \\ & - \tfrac{1}{2}\mathcal{V}G_{\Lambda\Sigma}(v)F^\Lambda \wedge \tilde{*}F^\Sigma - \tfrac{1}{12}\mathcal{V}_{\Lambda\Sigma\Theta}A^\Lambda \wedge F^\Sigma \wedge F^\Theta \\ & - \tfrac{1}{16\mathcal{V}}\Big[\xi^K d\tilde{\xi}_K - \tilde{\xi}_K d\xi^K + d\Phi\Big] \wedge \tilde{*}\Big[\xi^K d\tilde{\xi}_K - \tilde{\xi}_K d\xi^K + d\Phi\Big] \ . \end{aligned} \tag{7.125}$$

Let us stress that we still have to take into account the Weyl rescaling of the metric $\tilde{g}_{\mu\nu}$. It is interesting to note that it is crucial to get the equality between the inverse gauge coupling function and the metric of the moduli space of scalars L^Λ, since

$$-\tfrac{1}{2}\mathcal{V}G_{\Lambda\Sigma}(v)F^\Lambda \wedge \tilde{*}F^\Sigma = -\tfrac{1}{2}\mathcal{V}^{\frac{2}{3}}G_{\Lambda\Sigma}(v)F^\Lambda \wedge *F^\Sigma = -\tfrac{1}{2}G_{\Lambda\Sigma}(L)F^\Lambda \wedge *F^\Sigma \ , \tag{7.126}$$

where $G_{\Lambda\Sigma}(L)$ is the same as in (7.110). This concludes our derivation. Indeed, it is straightforward to check that the hypermultiplet moduli metric (7.111) collects the kinetic terms of all scalars different from the L^Λ's after Weyl rescaling.

7.5.2 The case of an elliptic fibration

Let us now specify this result to the elliptically fibered geometry introduced in subsection 7.2. We first split the index Λ into $(0, \alpha, i)$ and write

$$L^\Lambda = (R, L^\alpha, \xi^i) \ , \qquad A^\Lambda = (A^0, A^\alpha, A^i) \ . \tag{7.127}$$

Combining this notation with the intersection numbers (7.11) of an elliptic fibration we get

$$\begin{aligned} \mathcal{N} =& \tfrac{1}{2}\eta_{\alpha\beta}RL^\alpha L^\beta + \tfrac{1}{2}\eta_{\alpha\beta}K^\alpha R^2 L^\beta + \tfrac{1}{6}\eta_{\alpha\beta}K^\alpha K^\beta R^3 \\ & - \tfrac{1}{2}\eta_{\alpha\beta}C^\alpha C_{ij}L^\beta\xi^i\xi^j + \tfrac{1}{6}\mathcal{V}_{ijk}\xi^i\xi^j\xi^k \ . \end{aligned} \tag{7.128}$$

As we will discuss in section 7.6 couplings of the form R^2L^α in (7.128) are not compatible with the uplift from five to six dimensions. However, there is as simple field redefinition which allows us to get rid of these R^2L^α terms. More precisely, one introduces the shifted fields[8]

$$\check{L}^\alpha = L^\alpha + \tfrac{1}{2}K^\alpha R \ , \qquad \check{A}^\alpha = A^\alpha + \tfrac{1}{2}K^\alpha A^0 \ , \tag{7.129}$$

[8] This field redefinition is also crucial in the analogous treatment of F-theory on Calabi-Yau fourfolds as discussed in [205, 206].

where the shift of the vectors is required by supersymmetry. Clearly, the new $\check{L}^\alpha$ and new vectors can be obtained by expanding J and C_3 in a new basis of two-forms

$$\check{\omega}_0 = \omega_0 - \tfrac{1}{2} K^\alpha \omega_\alpha \ , \qquad \check{\omega}_\alpha = \omega_\alpha \ , \qquad \check{\omega}_i = \omega_i \ . \tag{7.130}$$

In fact, this new basis is better suited to identify the vectors $\check{A}^\alpha$ as dualizable into five-dimensional tensors. The cubic potential in the new coordinates given by

$$\begin{aligned}\mathcal{N}^{\mathrm{M}} =& \tfrac{1}{2}\eta_{\alpha\beta} R \check{L}^\alpha \check{L}^\beta + \tfrac{1}{24}\eta_{\alpha\beta} K^\alpha K^\beta R^3 \\ & - \tfrac{1}{2}\eta_{\alpha\beta} C^\alpha C_{ij} \check{L}^\beta \xi^i \xi^j + \tfrac{1}{4}\eta_{\alpha\beta} C^\alpha C_{ij} K^\beta R \xi^i \xi^j + \tfrac{1}{6}\mathcal{V}_{ijk}\xi^i\xi^j\xi^k \ .\end{aligned} \tag{7.131}$$

Using this expression of $\mathcal{N}$ the Chern-Simons term takes the form

$$\begin{aligned}S^{(5)\mathrm{M}}_{\mathrm{CS}} = \int_{\mathcal{M}_5} & - \tfrac{1}{4}\eta_{\alpha\beta} A^0 \wedge \check{F}^\alpha \wedge \check{F}^\beta + \tfrac{1}{4}\eta_{\alpha\beta} C^\alpha C_{ij} \check{A}^\alpha \wedge F^i \wedge F^j \\ & - \tfrac{1}{48}\eta_{\alpha\beta} K^\alpha K^\beta A^0 \wedge F^0 \wedge F^0 - \tfrac{1}{8}\eta_{\alpha\beta} C^\alpha C_{ij} K^\beta A^0 \wedge F^i \wedge F^j \\ & - \tfrac{1}{12}\mathcal{V}_{ijk} A^i \wedge F^j \wedge F^k \ ,\end{aligned} \tag{7.132}$$

where $\check{F}^\alpha$ is the usual field strength of the vectors $\check{A}^\alpha$ introduced in (7.129).

7.5.3 Higher order curvature corrections

Several higher-derivative corrections to the 11d M-theory action (7.97) are known [81, 82]. In the following, we will focus on the mixed gauge-gravitational topological correction[9]

$$\hat{S}^{(11)}_{C\mathcal{R}^4} = \tfrac{1}{96} \int_{\mathcal{M}_{11}} \hat{C}_3 \wedge \Big[\mathrm{tr}\, \hat{\mathcal{R}}^4 - \tfrac{1}{4} (\mathrm{tr}\, \hat{\mathcal{R}}^2)^2 \Big] \ . \tag{7.133}$$

Rather than performing a complete dimensional reduction of (7.133), we will extract the relevant terms and we will systematically neglect all contributions which involve gradients of the Kähler and complex structure moduli. This means that we can effectively neglect fluctuations and compute curvature invariants on the background, which is the product space $\mathcal{M}_{11} = \mathcal{M}_5 \times X_3$. As a result, we have simply [10]

$$\hat{\mathcal{R}} = \mathcal{R} + \mathcal{R}_{X_3} \ , \tag{7.134}$$

where $\mathcal{R}_{X_3}$ is the curvature two-form on the Calabi-Yau threefold, and $\mathcal{R}$ is the five-dimensional curvature two-form. A straightforward computation gives then

$$(\mathrm{tr}\, \hat{\mathcal{R}}^2)^2 = 2 \mathrm{tr}\, \mathcal{R}^2 \wedge \mathrm{tr}\, \mathcal{R}^2_{X_3} + \dots \ , \qquad \mathrm{tr}\, \hat{\mathcal{R}}^4 = 0 + \dots ; \ , \tag{7.135}$$

where the dots are a reminder of the moduli-dependent, neglected terms. It is useful to recall the definition of the first Pontryagin class of the Calabi-Yau threefold X_3,

$$p_1(X_3) = -\tfrac{1}{2} \mathrm{tr}\, \mathcal{R}^2_{X_3} \ , \tag{7.136}$$

[9] As discussed in section 7.3, factors of $(2\pi)^{-1}$ are understood in $\hat{\mathcal{R}}$. Moreover, the relative normalization of this higher-derivative term and the two-derivative action (7.97) depends on the value of the eleven-dimensional gravitational constant. It is suppressed everywhere, adopting a convention which is best suited to make contact with the six-dimensional Green-Schwarz term, in which the six-dimensional gravitational constant has been equally suppressed.

[10] Just like in the reduction from six to five dimensions, performing the Weyl rescaling on the five-dimensional metric does not affect the moduli-independent terms in the expression of the curvature two-form.

and its relation with the second Chern class,

$$p_1(X_3) = -2c_2(X_3) \ . \tag{7.137}$$

Combining these equations with the three-form expansion (7.100), we can deduce that the 11d correction (7.133) yields, among other terms, the following five-dimensional correction [207]

$$S^{(5)\mathrm{M}}_{A\mathcal{R}\mathcal{R}} = \tfrac{1}{48}\, \check{c}_\Lambda \int_{\mathcal{M}_5} \check{A}^\Lambda \wedge \mathrm{tr}\, \mathcal{R}^2 \ , \tag{7.138}$$

where we have defined

$$\check{c}_\Lambda = \int_{X_3} \check{\omega}_\Lambda \wedge c_2(X_3) \ . \tag{7.139}$$

To make further progress it is crucial to specialize to the case of an elliptically fibered Calabi-Yau threefold X_3. Let us discuss a smooth fibration first. The second Chern class of the total space can then be expressed in term of Chern classes on the base space B_2, by means of [208]

$$c_2(X_3) = c_2(B_2) + 11c_1^2(B_2) + 12\omega_0 \wedge c_1(B_2) \ . \tag{7.140}$$

Making use of (7.4) we get

$$\int_{X_3} \omega_0 \wedge c_2(X_3) = \int_{X_3} \omega_0 \wedge [c_2(B_2) - c_1^2(B_2)] = \int_{B_2} c_2(B_2) - c_1^2(B_2) \ . \tag{7.141}$$

This equation can be evaluated further by using the explicit expressions of the integrals of c_2 and c_1^2 on B_2 given in (7.6) as

$$\int_{X_3} \omega_0 \wedge c_2(X_3) = 2h^{1,1}(B_2) - 8 \ . \tag{7.142}$$

Furthermore, we can also evaluate the second Chern class on the basis ω_α as

$$\int_{X_3} \omega_\alpha \wedge c_2(X_3) = \int_{X_3} \omega_\alpha \wedge [c_2(B_2) + 11c_1^2(B_2) + 12\omega_0 \wedge c_1(B_2)] \ . \tag{7.143}$$

Since the first two terms have all their indices on the base, only the last term provides a non-vanishing contribution. Using $c_1(B_2) = -K^\alpha \omega_\alpha$, as introduced in subsection 7.2, we compute

$$\check{c}_\alpha = \int_{X_3} \check{\omega}_\alpha \wedge c_2(X_3) = -12\eta_{\alpha\beta} K^\beta \ , \tag{7.144}$$

where we have used $\check{\omega}_\alpha = \omega_\alpha$. In order to obtain $\check{c}_0$ from (7.142), (7.143) we have to recall the definition (7.130) of $\check{\omega}_0$, and find

$$\check{c}_0 = 52 - 4\, h^{1,1}(B_2) \ . \tag{7.145}$$

So far we have worked on a smooth elliptic fibration. We now include the effects of singularities and their resolution. Clearly, the presence of resolved singularities induces new couplings

$$\check{c}_i = \int_{\tilde{X}_3} \check{\omega}_i \wedge c_2(X_3) \ . \tag{7.146}$$

One expects that this expression evaluated for a given gauge group has a group theoretic interpretation. Giving its precise form is beyond the scope of this work. However, let us note that also the other couplings $\check{c}_0$ and $\check{c}_\alpha$ could be corrected by the inclusion of blow-up divisors. Indeed, a general shift of $c_2(\tilde{X}_3)$ with the blow-up divisors induces

$$\int_{\tilde{X}_3} \omega_0 \wedge \Delta c_2(\tilde{X}_3) = 0 \ , \qquad \int_{\tilde{X}_3} \omega_\alpha \wedge \Delta c_2(\tilde{X}_3) = \mathcal{C}^{ij} \int_{\tilde{X}_3} \omega_\alpha \wedge \omega_i \wedge \omega_j \ , \tag{7.147}$$

where we have used the vanishing of the intersections (7.11) with only one ω_i and two ω_α, and $\omega_i \wedge \omega_0 = 0$. Note that a shift in $\check{c}_0$ could still be induced due to the basis change (7.130) inducing a term proportional to $\check{c}_\alpha$. We claim that also $\check{c}_\alpha$ is uncorrected, and thus $\check{c}_0$ and $\check{c}_\alpha$ remain unchanged. Despite that we do not have a general proof, we have checked for many examples that (7.144) and (7.145) are still true:

$$\check{c}_\alpha = -12\eta_{\alpha\beta}K^\beta \ , \qquad \check{c}_0 = 52 - 4\, h^{1,1}(B_2) \ . \tag{7.148}$$

As we will show later, the fact that $\check{c}_\alpha$ is not changed is consistent with the F-theory lift. The fact that $\check{c}_0$ does not change in this case follows from (7.147).

7.6 F-theory lift and one-loop corrections

In this section we compare the result of the circle reduction of the general six-dimensional $(1,0)$ supergravity theory with the M-theory reduction on an elliptically fibered Calabi-Yau threefold. We identify terms which appear at classical level on both sides and can be immediately matched as discussed in subsection 7.6.1. We also comment on the matching of certain higher derivative terms. It is crucial insight that both reductions contain additional terms which have no immediate analogue in the dual reduction. We suggest in subsection 7.6.2 that these terms arise at the quantum level and encode the same information about the underlying fully quantized theory. In particular, we argue that certain intersections on the M-theory side correspond in the reduction from six to five dimensions on a circle to one-loop corrections with charged matter fermions and Kaluza-Klein modes of all six-dimensional chiral fields running in the loop. In conclusion this allows us to extract all data from M-theory required to specify the six-dimensional action including the complete information about six-dimensional anomalies. In chapter 9 the loop computation will be explicitly performed and will corroborate our present analysis.

7.6.1 Classical action in the F-theory lift

In order to extract information about F-theory in six dimensions, we have to compare the five-dimensional action coming from Kaluza-Klein reduction from six dimensions with the five-dimensional action of M-theory on an elliptically fibered Calabi-Yau threefold. Our strategy will be similar to the treatment of F-theory on Calabi-Yau fourfolds presented in [206].

As a first step, we present the match of the number of multiplets in five dimensions in order to give the number of six-dimensional multiplets in terms of the topological data of the F-theory compactification manifold X_3. This was already implicit in our choice of indices in sections 7.4 and 7.5. More precisely, for the α-index we find that the number of six-dimensional tensors is given by

$$n_T + 1 = h^{1,1}(B_2) \ , \tag{7.149}$$

where we recall that there are n_T six-dimensional tensor multiplets and one tensor in the gravity multiplet. In the F-theory reduction the tensors arise from the reduction of the Type IIB RR four-form into a base of $H^2(B_2)$. Since A^i parameterize the Coulomb branch of the circle-reduced six-dimensional gauge theory, one finds

$$\text{rank}(G) = h^{1,1}(\tilde{X}_3) - h^{1,1}(B_2) - 1 \ , \tag{7.150}$$

which counts the number of independent blow-up divisors induced to resolve the singular elliptic fibration to obtain $\tilde{X}_3$. Note that for ADE gauge groups G the number of six-dimensional vector multiplets is then given by

$$n_V = (c_G + 1)\text{rank}(G) \ , \tag{7.151}$$

where c_G is the dual Coxeter number of G. In F-theory these vectors arise from the seven-brane gauge potentials. Finally, one can match the number of hypermultiplets, simply by noting that a six-dimensional hypermultiplet becomes a five-dimensional hypermultiplet in the circle reduction. This leads to the following number of neutral six-dimensional multiplets

$$n_H^{\text{neutral}} = h^{2,1}(\tilde{X}_3) + 1 \ . \tag{7.152}$$

In F-theory on X_3 these neutral hypermultiplets contain the complex deformations of the seven-branes and their Wilson line moduli.[11] The universal hypermultiplet in the F-theory reduction contains as one complex scalar the volume of the base together with the scalar of the Type IIB RR four-form expanded in the volume form of B_2. The remaining two real scalar degrees of freedom in the universal hypermultiplet arise in the expansion of the Type IIB RR and NSNS two-forms into the universal two-form mode present for any B_2. The proof of the match (7.149)-(7.152) follows from the match of the effective theories presented in the following.

In order to systematically approach the match of the effective action, we would first like to identify the terms which are classical on both sides. This is not hard for the circle reduction from six to five dimensions. More complicated is the distinction of the various terms in the M-theory potential. We will address the two sides in turn.

In the reduction of the six-dimensional action on a circle performed in section 7.4 we found that there is a potential $\mathcal{N}^{\text{F}}$ given in (7.79) which encodes the kinetic terms of the gauge coupling functions and the Chern-Simons terms in the five-dimensional reduced action. It is crucial to recall the natural decomposition of $\mathcal{N}^{\text{F}}$ in (7.79) into a polynomial and a non-polynomial part:

$$\begin{aligned} \mathcal{N}^{\text{F}}_{\text{p}} &= \Omega_{\alpha\beta} M^0 M^\alpha M^\beta - 4\Omega_{\alpha\beta} b^\alpha C_{ij} M^\beta M^i M^j \ , \\ \mathcal{N}^{\text{F}}_{\text{np}} &= 4\Omega_{\alpha\beta} b^\alpha b^\beta C_{ij} C_{kl} \frac{M^i M^j M^k M^l}{M^0} \ . \end{aligned} \tag{7.153}$$

[11] See ref. [209], for a detailed matching with the orientifold picture with D7-branes.

The terms in $\mathcal{N}^{\rm F}_{\rm p}$ are cubic and hence encode a standard $\mathcal{N}=2$ five-dimensional action. In contrast $\mathcal{N}^{\rm F}_{\rm np}$ is only homogeneous of degree three, but non-polynomial. As argued in section 7.4.3 it can be interpreted as a counterterm of the five-dimensional one-loop effective action. Its six-dimensional origin is related to the classical lack of gauge invariance of the six-dimensional action. In fact, it vanishes precisely when

$$\Omega_{\alpha\beta} b^\alpha b^\beta = 0 \ . \tag{7.154}$$

This corresponds to the case where the six-dimensional action is gauge invariant as inferred from (7.44), and is consistent with the absence of six-dimensional anomalies, see section 7.3.2.

Let us now turn to the M-theory reduction. Here the identification of the classical terms is more subtle. We have worked on the resolved space with finite size elliptic fiber. Recall from section 5.3 that the F-theory limit corresponds to both shrinking the blow-up divisors as well as the size of the elliptic fiber. One expects that this selects classical terms in the potential $\mathcal{N}^{\rm M}$ of equation (7.131). It turns out to be useful to introduce an ϵ-scaling to distinguish various terms in $\mathcal{N}^{\rm M}$. For the volumes v^0, v^α, v^i appearing in the Kähler form $J = v^\Lambda \omega_\Lambda$, we make the formal replacements

$$v^0 \mapsto \epsilon v^0 \ , \qquad v^\alpha \mapsto \epsilon^{-1/2} v^\alpha \ , \qquad v^i \mapsto \epsilon^{1/4} v^i \ . \tag{7.155}$$

Note that these scalings satisfy some important consistency checks. Firstly, the size of the elliptic fiber v^0 and the blow-up fibers v^i vanish for $\epsilon \to 0$. Secondly, the total volume $\mathcal{V}$ of X_3 is finite, which is required by the fact that $\mathcal{V}$ sits in a five-dimensional hypermultiplet. Translated into the variables R, L^α, ξ^i one finds the replacements

$$R \mapsto \epsilon R \ , \qquad L^\alpha \mapsto \epsilon^{-1/2} L^\alpha \ , \qquad \xi^i \mapsto \epsilon^{1/4} \xi^i \ . \tag{7.156}$$

Since the redefined scalars $\check{L}^\alpha$ contain L^α linearly, they obey the same rescaling as L^α. In the limit $\epsilon \to 0$ two terms in (7.131) survive which we collect in $\mathcal{N}^{\rm M}_{\rm class}$. We thus divide the terms in (7.131) into

$$\begin{aligned} \mathcal{N}^{\rm M}_{\rm class} &= \tfrac{1}{2}\eta_{\alpha\beta} R \check{L}^\alpha \check{L}^\beta - \tfrac{1}{2}\eta_{\alpha\beta} C^\alpha C_{ij} \check{L}^\beta \xi^i \xi^j \ , \\ \mathcal{N}^{\rm M}_{\rm loop} &= \tfrac{1}{24}\eta_{\alpha\beta} K^\alpha K^\beta R^3 + \tfrac{1}{4}\eta_{\alpha\beta} C^\alpha C_{ij} K^\beta R \xi^i \xi^j + \tfrac{1}{6}\mathcal{V}_{ijk}\xi^i\xi^j\xi^k \ . \end{aligned} \tag{7.157}$$

It is now straightforward to match $\mathcal{N}^{\rm M}_{\rm class}$ with $\mathcal{N}^{\rm F}_{\rm p}$ given in (7.153). Note that the second term $\mathcal{N}^{\rm M}_{\rm loop}$ in (7.157) will be later reinterpreted as a loop correction, which gives another justification of the split induced by the F-theory limit (7.156).

Let us first start by matching the fields on the F-theory side and the M-theory side. In order to do that we have to fix the normalization of the fields, which cannot be uniquely extracted by comparing (7.153) and (7.157). Supersymmetry relates the normalization of the real scalars and vectors in the vector multiplets. Hence, given a fixed normalization of the vectors the complete match of the scalar components can be inferred. On the one hand, in the circle compactification from six dimension the vectors are normalized by the Green-Schwarz term (7.28), and the fixed definition of the anomaly coefficients b^α, a^α. On the other hand, in M-theory the normalization of the vectors is fixed by a choice of integral basis in the expansion (7.100) of $\hat{C}_3$. Appropriately rescaling the six-dimensional vectors to also adopt to an integral basis, one can infer the map

$$M^0 = 2R \ , \qquad M^\alpha = \tfrac{1}{2}\check{L}^\alpha \ , \qquad M^i = \tfrac{1}{2}\xi^i \ , \tag{7.158}$$

while the constants are identified as

$$\Omega_{\alpha\beta} = \eta_{\alpha\beta} \ , \qquad b^\alpha = C^\alpha \ . \tag{7.159}$$

Note that our result are consistent with the findings of [177, 178, 179, 180, 181].

So far we have only discussed the vector and gravity sectors of the M-theory to F-theory matching. Clearly, both the 6d/5d reduction as well as the M-theory reduction contain a hypermultiplet sector. As discussed in section 7.4.2, we found that in the dimensional reduction from six to five dimensions the charged hypermultiplets are massive in the Coulomb branch. Therefore, they are not visible in the effective action of the massless modes of M-theory. We will include them in the study of loop corrections in the next subsections. However, the neutral hypermultiplets are massless and their moduli space could be matched straightforwardly also leading to (7.152).

Let us close this subsection by also comparing the classical parts of the higher curvature terms dimensionally reduced in sections 7.4.4 and 7.5.3. We have focussed on the terms involving the five-dimensional vectors and two five-dimensional curvature forms $\mathcal{R}$. In (7.92) and (7.138) we found that such couplings are given by

$$S^{(5)\mathrm{F}}_{A\mathcal{R}\mathcal{R}} = -\tfrac{1}{2}\,\Omega_{\alpha\beta}a^\beta \int_{\mathcal{M}_5} A^\alpha \wedge \mathrm{tr}\,\mathcal{R}^2 \ , \qquad S^{(5)\mathrm{M}}_{A\mathcal{R}\mathcal{R}} = \tfrac{1}{48}\,\check{c}_\Lambda \int_{\mathcal{M}_5} \check{A}^\Lambda \wedge \mathrm{tr}\,\mathcal{R}^2 \ . \tag{7.160}$$

Recall that the coefficients $\check{c}_\Lambda$ have been determined in (7.148), and (7.146). Since in the circle reduction only the A^α appears, one suspects that, similar to the F-theory limit discussed above, that these are the only classical terms in the reduction. Using $\check{c}_\alpha = -12\eta_{\alpha\beta}K^\beta$, as given in (7.148), we can apply the identification (7.159) to infer

$$a^\alpha = K^\alpha \ . \tag{7.161}$$

Note that this is precisely the identification dictated by anomaly cancellation conditions as found in [177, 178, 179, 180, 181]. On the M-theory side we also found the non-vanishing couplings involving $\check{c}_i, \check{c}_0$. Similar to the split found for $\mathcal{N}^{\mathrm{M}}$ we believe that these couplings are induced by one-loop corrections on the F-theory side. The remainder of this chapter is devoted to the discussion of such one-loop quantum corrections.

7.6.2 Completing the duality using one-loop corrections

As we have seen in the previous subsection, only some terms of the five-dimensional cubic potential $\mathcal{N}^{\mathrm{M}}$ of M-theory compactified on a Calabi-Yau threefold admit a straightforward dual in the potential $\mathcal{N}^{\mathrm{F}}$ arising from circle compactification of six-dimensional supergravity. In this subsection, we will provide a framework for the interpretation of the remaining terms in $\mathcal{N}^{\mathrm{M}}$, which we record here again for the ease of the reader,

$$\mathcal{N}^{\mathrm{M}}_{\mathrm{loop}} = \tfrac{1}{24}\eta_{\alpha\beta}K^\alpha K^\beta R^3 + \tfrac{1}{4}\eta_{\alpha\beta}C^\alpha C_{ij}K^\beta R\xi^i\xi^j + \tfrac{1}{6}\mathcal{V}_{ijk}\xi^i\xi^j\xi^k \ . \tag{7.162}$$

Recall that five-dimensional $\mathcal{N}=2$ supersymmetry ensures that exactly the same amount of information is contained in the cubic potential $\mathcal{N}$ and in the Chern-Simons couplings of vectors. The following discussion is conveniently formulated in terms of the latter. As already anticipated, we relate these couplings to one-loop effects in the reduction from six to five dimensions.

In order to clarify the precise meaning of this statement, let us analyze in more detail the origin of Chern-Simons couplings in the effective five-dimensional theory arising from six-dimensional supergravity on a circle. A possible source of this kind of interactions is of course provided by dimensional reduction of the Green-Schwarz term in the classical six-dimensional action. These interactions are precisely the ones which we have considered in the previous subsection. However, additional contributions arise, which are understood in the framework of effective quantum field theory. In fact, from a quantum perspective, the five-dimensional effective action resulting from compactification on a circle of six-dimensional supergravity encodes all information about the low-energy dynamics, including interactions induced by massive fields which have to be integrated out when we restrict our attention to the lightest states of the theory.

In the case under examination, we identify two different families of massive fields which can alter five-dimensional effective couplings:

- *Kaluza-Klein modes.* As we know from section 4.2, all six-dimensional fields can be schematically expanded into Kaluza-Klein modes as
$$\hat{\varphi}(x,y) = \sum_{n\in\mathbb{Z}} \varphi^{(n)}(x) e^{iny} \; . \tag{7.163}$$
The modes $\varphi^{(n)}$ with non-zero n appear in the five-dimensional theory as massive fields, with mass inversely proportional to the radius r of the compactification circle, $m^{(n)} \sim |n|/r$.[12] Zeromodes only are sufficient to fix all data needed to specify the six-dimensional model we are compactifying, and this is why we have systematically neglected excited modes so far. Nonetheless, Kaluza-Klein modes can run in five-dimensional loop diagrams.
- *Fields which are given a mass by gauge symmetry breaking.* Recall that F-/M-theory duality can be applied in a geometric regime only if the five-dimensional gauge symmetry is spontaneously broken down to the Coulomb phase and the compactification threefold is resolved. This amounts to giving non-vanishing VEVs to some scalars in the vector multiplets. As described in subsection 7.4.2, these VEVs provide mass terms for the W-bosons and the scalars in charged hypermultiplets. Supersymmetry implies that their fermionic partners, gaugini and hyperini, get massive as well. We claim that these fields can run in five-dimensional loops in such a way as to induce effective Chern-Simons couplings.

In chapter 9 we will demonstrate that five-dimensional massive spin-1/2 and spin-3/2 fermions, as well as five-dimensional massive tensors, generate a shift of gauge and mixed gauge-gravitational Chern-Simons terms. In order for this mechanism to work, these massive fields have to be minimally coupled electrically to a massless five-dimensional vector. In the next subsection we will identify such couplings and thus justify our claim about the one-loop origin of the terms in (7.162). A precise match of one-loop Chern-Simons terms will be presented in section 10.2.

[12] This holds before possible Weyl rescalings are taken into account.

7.6.3 Identifying relevant couplings involving massive fields

We start discussing fermionic Kaluza-Klein modes. Let $\hat{\psi}_{(\pm)}$ denote a general six-dimensional spinor of given chirality. It is an 8-component spinor with complex entries, but the number of degrees of freedom is halved by restriction to definite chirality. This counting agrees with the number of degrees of freedom of the (off-shell) five-dimensional reduced spinor ψ, which can be represented as a 4-component vector with complex entries.

We can be more explicit. A representation of six-dimensional gamma matrices $\hat{\Gamma}_{\hat{a}}$, $\{\hat{\Gamma}_{\hat{a}}, \hat{\Gamma}_{\hat{b}}\} = 2\hat{\eta}_{\hat{a}\hat{b}}$, $\hat{a}, \hat{b} = 0, 1, ..., 5$ can be found, such that

$$\hat{\Gamma}_a = \sigma_1 \otimes \Gamma_a \; , \qquad \hat{\Gamma}_5 = \sigma_2 \otimes \mathbb{I}_4 \; . \tag{7.164}$$

In these equations, σ_i are the usual Pauli matrices, while Γ_a, $\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}$, $a, b = 0, 1, ..., 4$ are five-dimensional gamma matrices, satisfying

$$i\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4 = \mathbb{I}_4 \; . \tag{7.165}$$

As a result, the six-dimensional chirality matrix is simply given by

$$\hat{\Gamma} = \hat{\Gamma}_0\hat{\Gamma}_1\hat{\Gamma}_2\hat{\Gamma}_3\hat{\Gamma}_4\hat{\Gamma}_5 = \sigma_3 \otimes \mathbb{I}_4 \; . \tag{7.166}$$

We can thus write $\hat{\psi}_{(\pm)}$ in the factorized form

$$\hat{\psi}_{(\pm)} = \iota_{(\pm)} \otimes \psi \; , \tag{7.167}$$

where $\iota_{(\pm)}$ is a unit vector in $\mathbb{C}^2$, such that $\sigma_3\iota_{(\pm)} = \pm\iota_{(\pm)}$, and ψ is a five-dimensional spinor.

Using these conventions, dimensional reduction of the six-dimensional standard kinetic term for $\hat{\psi}_{(\pm)}$ yields[13]

$$\int d^6\hat{x} \; \bar{\hat{\psi}}_{(\pm)}\hat{\Gamma}^{\hat{\mu}}\hat{\partial}_{\hat{\mu}}\hat{\psi}_{(\pm)} \supset 2\pi \sum_{n\in\mathbb{Z}} \int d^5x \; r\big\{\bar{\psi}^{(n)}\Gamma^{\mu}\partial_{\mu}\psi^{(n)} \mp \tfrac{n}{r}\bar{\psi}^{(n)}\psi^{(n)} + inA^0_{\mu}\bar{\psi}^{(n)}\Gamma^{\mu}\psi^{(n)}\big\} \; . \tag{7.168}$$

On the left hand side, a hat denotes six-dimensional gamma matrices, indices, and coordinates. The modes $\psi^{(n)}$ of the fermion ψ are defined as in (7.163). On the right hand side, we find a result consistent with the general features of Kaluza-Klein models on a circle. In fact, the n-th excited Kaluza-Klein mode has a mass proportional to n and is electrically charged with respect to the vector A^0. The charge is proportional to n as well. Note that additional non-minimal Pauli-like couplings of the form $F^0_{\mu\nu}\,\bar{\psi}^{(n)}\Gamma^{\mu\nu}\psi^{(n)}$ are generated in the reduction. They are not relevant for our current purposes since, as we will show in chapter 9, the desired Chern-Simons one-loop correction is only sensitive to minimal couplings.

A very similar computation shows that dimensional reduction of the Rarita-Schwinger term for six-dimensional gravitini yields, among other terms, a mass term for the excited Kaluza-Klein modes $\psi_{\mu}^{(n)}$

[13] In order to keep the argument simple, we work in a flat background and we do not Weyl rescale the five-dimensional metric.

and a minimal coupling to the Kaluza-Klein vector A^0 with charge proportional to the Kaluza-Klein level n.

We can now turn to fermions in the vector multiplets. Let $\hat{\lambda}$ be a six-dimensional spinor in the adjoint representation of the simple gauge group G. Its gauge-covariant derivative is given by

$$\hat{D}\hat{\lambda} = d\hat{\lambda} + [\hat{A}, \hat{\lambda}] \ , \tag{7.169}$$

where $\hat{A}$ are the non-Abelian six-dimensional vectors introduced in section 7.3. In order to keep the discussion as simple as possible, we restrict our attention to Kaluza-Klein zeromodes only in this paragraph. As a consequence, dimensional reduction of the six-dimensional kinetic term for $\hat{\lambda}$ is of the form

$$\int d^6\hat{x}\mathrm{tr}\left(\hat{\bar{\lambda}}\hat{\Gamma}^{\hat{\mu}}\hat{D}_{\hat{\mu}}\hat{\lambda}\right) = 2\pi \int d^5x\, r\big\{\mathrm{tr}\left(\bar{\lambda}\Gamma^{\mu}D_{\mu}\lambda\right) + \tfrac{i}{r}\mathrm{tr}\left(\bar{\lambda}[\zeta,\lambda]\right)\big\} \ . \tag{7.170}$$

On the right hand side, $D\lambda = d\lambda + [A, \lambda]$ is the five-dimensional gauge-covariant derivative, while ζ is the adjoint scalar introduced in the Ansatz (7.48). Note that the sign of the last term is determined by the requirement of left-handedness for the gaugini, and that no A_0-coupling emerges for the Kaluza-Klein zeromodes precisely thanks to the shift of five-dimensional vectors described by (7.48). When the gauge symmetry is spontaneously broken to the Coulomb branch, the scalars ζ acquire a non-vanishing VEV orthogonal to the Cartan subalgebra. Furthermore, commutators $[A, \lambda], [\zeta, \lambda]$ vanish for the components of λ lying in this subalgebra. However, they are non-trivial for the components orthogonal to it. These components receive a mass from the second term in (7.170), while the first term in the same equation provides electric coupling to the Abelian vectors A^i associated to the generators of the Cartan subalgebra. We can thus see that Higgsed gaugini have the correct coupling to generate the effective Chern-Simons interaction under examination.

A similar argument can be used to conclude that charged hyperini can run in the loop and furnish a non-vanishing contribution. More precisely, dimensional reduction of their kinetic term gives

$$\int d^6\hat{x}\mathrm{tr}\left[h_{UV}\hat{\bar{\psi}}^U\hat{\Gamma}^{\hat{\mu}}(\hat{\mathcal{D}}_{\hat{\mu}}\hat{\psi})^V\right] = 2\pi\int d^5x\, r\big\{h_{UV}\bar{\psi}^U\Gamma^{\mu}(\mathcal{D}_{\hat{\mu}}\psi)^V - \tfrac{i}{r}h_{UV}\bar{\psi}^U\zeta^I(T_I^{\mathbf{R}}\psi)^V\big\} \ .$$

In this expression, the six-dimensional covariant derivative of the hyperino is defined as

$$(\hat{\mathcal{D}}_{\hat{\mu}}\hat{\psi})^U = \hat{\nabla}_{\hat{\mu}}\hat{\psi}^U + \hat{A}^I_{\hat{\mu}}(T_I^{\mathbf{R}}\hat{\psi})^U \ , \tag{7.171}$$

and an analogous expression is understood for the five-dimensional covariant derivative on the right hand side. Note that the sign of the last term has changed with respect to the gaugino reduction, because hyperini are right-handed. Upon spontaneous gauge symmetry breaking to the Coulomb branch, this term provides a mass for charged hyperini, while neutral hyperini are unaffected and remain in the massless five-dimensional spectrum.

The reader might wonder whether there are massive fermions which are electrically coupled to vectors A^α. Our analysis suggests that this is not the case. A thorough explanation would require dimensional reduction of the full six-dimensional pseudoaction, including fermionic terms. Such a pseudoaction can be found e.g. in [210]. However, it is crucial to recall that five-dimensional vectors

A^α are obtained by dimensional reduction of six-dimensional two-forms $\hat{B}^\alpha$. Such two-forms enter the six-dimensional action in a qualitatively different way as six-dimensional vectors. Geometrically, they are not connection forms, and cannot be used to build six-dimensional covariant derivatives. Therefore, the reduced five-dimensional action lacks electric couplings of vectors A^α to fermions. Nonetheless, different couplings are possible, which can be referred to as magnetic. They read schematically $m_\alpha \bar{\psi}\Gamma^{\mu\nu}F^\alpha_{\mu\nu}\psi$ where ψ stands for a five-dimensional fermion. Even though these interactions may play a role in the full one-loop five-dimensional effective action, in the absence of electric vertices they are not able to generate contributions to the Chern-Simons couplings.

It is interesting to point out the connection between this argument and the shift of vectors performed in (7.129). As explained in section 7.5.1, this shift is crucial to identify properly five-dimensional vectors coming from six-dimensional two-forms. As we can see by comparing (7.128) and (7.131), the field redefinition (7.129) is such that in the cubic potential $\mathcal{N}^{\mathrm{M}}$ the term $R^2 L^\alpha$ gets replaced by the term $R\xi^i\xi^j$. As argued in the previous paragraph, it would be impossible to generate the former term using five-dimensional fermion loops, while in the following we will show how the latter term can emerge from such Feynman diagrams.

After these general remarks about massive fermions in the five-dimensional theory, let us discuss in more detail each term in (7.162). The first term corresponds to a Chern-Simons coupling of the form $A^0 \wedge F^0 \wedge F^0$. As we argued above, Kaluza-Klein modes are the fields which are electrically charged under A^0. We therefore claim that this five-dimensional interaction is generated by diagrams in which Kaluza-Klein excited modes coming from reduction of all chiral six-dimensional fields can run in the loops. This claim will be verified in section 10.2.

The next term in (7.162) corresponds to a Chern-Simons vertex of the form $A^0 \wedge F^i \wedge F^j$. In order to reproduce this effective coupling using five-dimensional one-loop diagrams, we need fermions which are electrically coupled both to the Kaluza-Klein vector A^0 and to the Abelian vectors A^i in the Coulomb branch. Our discussion above singles out Kaluza-Klein modes of Higgsed gaugini and charged hyperini as natural candidates to run in the loop.

Finally, we focus our attention on the last term in (7.162), which gives rise to a Chern-Simons term $A^i \wedge F^j \wedge F^k$. We identify the source of this coupling in the Higgsed gaugini and the massive charged hyperini. The one-loop effect due to these fermions has been computed [211] for a five-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory decoupled from gravity. The full result for the purely gauge part of the five-dimensional cubic potential $\mathcal{N}$, including quantum corrections, reads

$$\mathcal{N}^{\mathrm{gauge}} = \tfrac{1}{2} m_0 C_{ij}\xi^i\xi^j + \tfrac{1}{6} c_{\mathrm{class}} d_{ijk}\xi^i\xi^j\xi^k + \tfrac{1}{12}\Big(\sum_{\mathbf{R}} |\mathbf{R}\cdot\xi|^3 - \sum_f \sum_{\mathbf{w}\in\mathbf{W}_f} |\mathbf{w}\cdot\xi + m_f|^3 \Big) . \qquad (7.172)$$

In this equation ξ is a vector whose component are the scalar fields ξ^i associated to vectors A^i. In $\xi\cdot\mathbf{R}$ it is contracted with a root of the simple gauge group G, while in $\xi\cdot\mathbf{w}$ it contracts with a weight of a the representation in which the charged fermions transform. The first sum in (7.172) runs over all the roots of G, and arises from integrating out the Higgsed gaugini, i.e. the fermionic partners of massive W-bosons. The second sum in (7.172) runs over all massive charged fermions f and all weights in $\mathbf{W}_f$, i.e. all elements of the set of weights of the representation in which the fermion f transforms. m_f is the classical mass of the fermion f. Finally, the group theoretical invariants C_{ij} and d_{ijk} are given by

$$C_{ij} = \mathrm{tr}\, T_i T_j \ , \qquad d_{ijk} = \tfrac{1}{2} \mathrm{tr}\, T_i (T_j T_k + T_k T_j) \ . \tag{7.173}$$

To apply the formula (7.172) to our circle compactification from six to five dimensions we recall the classical expression (7.153) for $\mathcal{N}^{\mathrm{F}}$. This leads to the identification

$$m_0 = -8 M^\alpha b^\beta \Omega_{\alpha\beta} \ , \qquad c_{\mathrm{class}} = 0 \ , \tag{7.174}$$

where we have used the fact that upon decoupling gravity the M^α are simply parameters. Following the discussion of section 7.6.1 this matches the classical M-theory result. A careful comparison of the loop terms in (7.172) and the intersection numbers $\mathcal{V}_{ijk}$ of the resolved Calabi-Yau threefold $\tilde{X}_3$ would require the introduction of new technical tools and lies out of the main line of development of this section. However, let us stress that the reader can find a detailed discussion of this point in [137], appendix A: as explained there, the match can be performed successfully in many examples of Calabi-Yau threefolds with $SU(N)$ singularities. The classical mass m_f is zero in this case.

In summary, we are confident that all terms in the M-theory expression (7.162) arise from one-loop quantum corrections in the 6d/5d dual picture. Moreover, this analysis can be extended to some higher-derivative couplings which appear naturally in the M-theory reduction on a Calabi-Yau threefold, but seem to be absent in the reduction of six-dimensional supergravity on a circle. In section 7.5.3 we have seen that M-theory higher-curvature correction induce a term (7.138) which has a non-vanishing contribution involving the Kaluza-Klein vector A^0. It is proportional to the shifted component $\check{c}_0$ of the second Chern class of the Calabi-Yau threefold $c_2(X_3)$ and reads schematically

$$A^0 \wedge \mathrm{tr}\, \mathcal{R} \wedge \mathcal{R} \ , \tag{7.175}$$

and corresponds to an amplitude with one Kaluza-Klein vector A^0 and two five-dimensional gravitons. It is impossible to extract such a coupling from the higher-curvature Green-Schwarz term (7.28) in the six-dimensional pseudoaction. Hence, we are led to the conclusion that on the F-theory side this interaction emerges as quantum effect, in a similar fashion as the $A^0 \wedge F^0 \wedge F^0$ coupling analyzed above. This claim will be substantiated in section 10.2, where a perfect match between one-loop effects and classical threefold geometry will be presented.

CHAPTER 8

F-theory on Spin(7) manifolds

In this chapter we explore the possibility to compactify F-theory down to four dimensions on an eight-dimensional Riemannian manifolds with Spin(7) holonomy. As explained in section 8.1 this idea goes back to the early days of F-theory but has resisted a complete understanding so far. In order to make some progress we are going to consider a special class of Spin(7) manifolds that are related to Calabi-Yau elliptically fibered fourfolds. This will allows us to get some insight on F-theory on these spaces. In particular we will be able to identify the Type IIB weak-coupling limit of a class of Spin(7) geometries. They exhibit an intriguing interplay between supersymmetry breaking and Lorentz symmetry breaking in four dimensions.

8.1 A long-standing challenge and a proposed duality

Over the last decades four-dimensional supersymmetric effective theories arising in string compactifications have been studied intensively. Minimally supersymmetric theories are considered as providing interesting physics beyond the Standard Model. Therefore it has been a crucial long-standing task to embed supersymmetric extensions of the Standard Model or Grand Unified Theories into string theory as reviewed, for example, in [212, 131, 213, 103, 105]. As we have seen in section 4.4, the established approach is to consider compactifications of string theory on manifolds with special holonomy, such that some of the underlying ten/eleven-dimensional supersymmetries are preserved in four dimensions and allow a supersymmetric effective theory to be determined. Precisely these supersymmetry-preserving geometries are also mathematically best studied and many powerful tools have been developed exploiting the interplay of geometry and low-energy physics. It is therefore natural to ask whether one can find a rich set of string compactifications with non-supersymmetric four-dimensional effective theories, and possibly interesting phenomenological properties, while still allowing the virtues of the remarkable mathematical tools developed for special holonomy manifolds to be used.

In section 5.3 we have reviewed the standard M-theory/F-theory duality, in which the four-dimensional theory is minimally supersymmetric (four real supercharges). This is achieved by choosing the torus-fibered geometry relevant to F-theory to be a Calabi-Yau fourfold, i.e. to have $SU(4)$ holonomy. On eight-dimensional manifolds, however, the classification by Berger reported in table 4.1 shows that $SU(4)$ is not the maximal possible special holonomy group within the local Lorentz group $SO(8)$. This maximal special holonomy group is instead given by Spin(7). For these geometries one therefore is led to ask:

(1) Is there a controlled construction of Spin(7) manifolds that can serve as backgrounds for F-theory?

(2) What are the characteristics of the four-dimensional effective theories arising from F-theory compactifications on such Spin(7) manifolds?

(3) What is the weak coupling Type IIB string interpretation of these theories?

In this chapter we will address these questions. This problem is particularly interesting because reduction of M-theory on a Spin(7) manifold yields a three-dimensional theory with half of the supersymmetries with respect to a Calabi-Yau compactification. One might therefore hope that in the F-theory limit a four-dimensional theory emerges with no supersymmetry. It should be noted these considerations were already mentioned in the original paper by Vafa [174], in connection with the proposals of Witten [214, 215]. However, this link has not been concretized. In particular, it is hard to characterize the most general Spin(7) geometry that allows for a four-dimensional theory to emerge in the appropriate F-theory limit.

In fact, before entering any analysis of the effective action, we have to answer the question of whether or not there are suitable Spin(7) manifolds that can be used for F-theory. In particular, it will be crucial to single out geometries that have an appropriate torus fibration structure to identify the F-theory compactification as a Type IIB string background. In building these manifolds we will be motivated by the constructions described by Joyce [216]. These constructions begin by considering a Calabi-Yau fourfold which is then quotiented in such a way that a Spin(7) manifold is generated. We will then proceed with an analysis of F-theory on this class of Spin(7) manifolds. Clearly, one expects that there exist many more examples of Spin(7) geometries that are not based on any Calabi-Yau fourfold. Definite statements about these more general cases turn out to be hard to extract, nevertheless various results of our analysis may well extend beyond the context that we consider. Importantly, these constructions based upon Calabi-Yau quotients give us control over the setup and allow our intuition about Calabi-Yau fourfold compactifications of F-theory to be used. Other explicit constructions of Spin(7) geometries appeared in [217, 218].

In section 5.3 we have outlined the program based on M-theory/F-theory duality to compute the effective action for F-theory on an elliptically fibered Calabi-Yau manifold. In the previous chapter this program has been successfully carried out for a Calabi-Yau threefold. The analogous discussion of the effective action for a Calabi-Yau fourfold has been studied in [206]. In this chapter we aim at a generalization of this program to the class of Spin(7) geometries mentioned above.

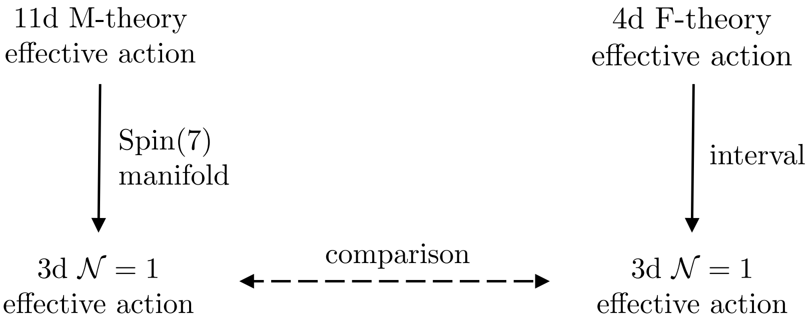


Figure 8.1: Summary of the effective actions considered in this chapter. The left column corresponds to the M-theory side of the duality (8.1), while the right column corresponds to the F-theory side. The comparison between the three-dimensional $\mathcal{N} = 1$ theories is performed in the case in which the Spin(7) manifold arises as an antiholomorphic quotient of an elliptically fibered Calabi-Yau fourfold. We consider a fibration structure that yields a simple non-Abelian gauge group. The match of three-dimensional actions is carried out in the Coulomb branch at the level of zero modes.

Our starting point must be the compactification of M-theory on a smooth Spin(7) manifold. This yields a three-dimensional effective theory with minimal supersymmetry, i.e. two supercharges [219]. In section 8.3.1 we describe the three-dimensional effective action of M-theory on a general Spin(7) manifold with probe fluxes extending and applying earlier works [220, 221, 222, 223, 224, 225] and determine the couplings in terms of the geometric data of the Spin(7) geometry.

According to the standard paradigm, the duality between M-theory and F-theory is based on a circle uplift of the M-theory effective action. To take the F-theory limit for the Spin(7) geometries under consideration we need a different prescription. More precisely, we propose the following duality:

$$\textit{M-theory on Spin(7) manifold} \;\cong\; \textit{F-theory on} \left\{ \begin{array}{c} \textit{Spin(7) manifold} \\ \textit{(with vanishing fiber)} \end{array} \times \textit{Interval} \right\} . \tag{8.1}$$

In the context of this work we can only make this claim in cases where the Spin(7) manifold is constructed from a Calabi-Yau fourfold as described in the previous paragraph. The extension of this to more general geometries remains an interesting open problem.

The essential new feature of the proposed duality is the replacement of the circle with an interval. This can be argued from different perspectives. First of all, the interval is crucial as the boundary conditions that are imposed project out half of the zero mode degrees of freedom that would arise in the circle reduction of a four-dimensional fermion. This observation will make it possible to obtain minimal supersymmetry in three dimensions by reduction of a four-dimensional theory. In contrast, it is impossible oxidize a minimally supersymmetric theory in three dimensions on a circle. Second of all, the appearance of an interval is also natural from the construction of Spin(7) manifolds that we have mentioned above for which the quotient of the fourfold may be associated with the quotient of the circle that gives rise to the interval. It is crucial in (8.1) that the core features of the theory in four dimensions and the boundary conditions for the interval are fixed by the Spin(7) geometry.

The presence of an interval presents some drawbacks, as well. In particular, we will find some ambiguities in the uplift of the effective action from three dimensions to four dimensions. These cannot be resolved within the framework of supergravity, and other arguments have to be invoked. Uplift ambiguities are essential, since they encode the fate of supersymmetry in the infinite interval limit. To cast light on this subtle issue we will examine in greater detail a special class of Spin(7) quotients that admit a clear Type IIB weak coupling limit. For these setups we will be able to establish the following points:

- the topology of four-dimensional spacetime is $\mathbb{R}^{1,2} \times I$, where the size of the interval I is macroscopic and goes to infinity in the F-theory limit;
- localized objects sit at the boundaries of the interval, while wrapping suitable submanifolds in the internal space;
- bulk physics is invariant under four real supercharges, while localized sources preserve only two real supercharges;
- in the strict limit of infinite interval the endpoints of the interval effectively disappear from the setup, four-dimensional Lorentz invariance is restored, and supersymmetry under four supercharges is achieved.

It is still an open problem to determine if these features are common for all compactifications of F-theory on Spin(7) manifolds that are obtained as quotients of Calabi-Yau fourfolds. It is therefore not possible to rule out the existence of Spin(7) manifolds that are able to yield a Lorentz invariant but non-supersymmetric four-dimensional theory.

8.2 Geometries with Spin(7) holonomy for F-theory

To set the stage for the discussions that follow we describe the construction of Spin(7) manifolds as antiholomorphic quotients of Calabi-Yau fourfolds. This construction is applied to elliptically fibered Calabi-Yau fourfolds in subsection 8.2.2. We discuss the fiber structures which arise and comment on seven-brane configurations that can appear.

8.2.1 Constructing Spin(7) manifolds from Calabi-Yau fourfolds

The key features of the topology and geometry of Spin(7) manifolds have been reviewed in section 4.6.1. We will now recall the basic ideas underlying the construction of Spin(7) manifolds from Calabi-Yau fourfolds performed in the work of Joyce [216]. Let Y_4 be a Calabi-Yau fourfold and let $\sigma : Y_4 \to Y_4$ be a antiholomorphic and isometric involution. In other terms, σ is required to satisfy

$$\sigma^2 = \mathbb{1}\,, \qquad \begin{cases} \text{isometric} & \sigma^*(g) = g \ , \\ \text{antiholomorphic} & \sigma^*(I) = -I \ , \end{cases} \tag{8.2}$$

where g and I are the metric and complex structure on Y_4, respectively. These conditions translate to the Kähler form J and on the holomorphic four-form Ω as

$$\sigma^* J = -J \ , \qquad\qquad \sigma^*\Omega = e^{2i\theta}\bar{\Omega} \ , \tag{8.3}$$

where θ is some constant phase factor. It is then shown in [216] that the quotient space

$$Z_8 = \frac{Y_4}{\sigma} \tag{8.4}$$

can be equipped with a natural Spin(7) structure. In particular, the Cayley calibration Φ of Z_8 can be expressed in terms of the Kähler form J and the holomorphic four-form Ω of Y_4 as

$$\Phi = \frac{1}{\mathcal{V}^2}\left[\frac{1}{||\Omega||}\mathrm{Re}(e^{-i\theta}\Omega) + \frac{1}{8}J\wedge J\right] \ . \tag{8.5}$$

In this equation $\mathcal{V}$ and $\|\Omega\|$ denote the volume of the Calabi-Yau fourfold and the norm of the holomorphic four-form, which we record again for ease of reference,

$$\mathcal{V} = \frac{1}{4!}\int_{Y_4} J^4 \ , \qquad\qquad \|\Omega\|^2 = \frac{1}{4!}\,\Omega_{i_1 i_2 i_3 i_4}\,\overline{\Omega}^{i_1 i_2 i_3 i_4} \ . \tag{8.6}$$

The derivation of the precise prefactors in front of $\mathrm{Re}(e^{-i\theta}\Omega)$ and $J\wedge J$ will be presented in section 8.3.2. The four-form Φ is invariant under the involution σ and an associated Spin(7) manifold may then be constructed by quotienting Y_4 by σ and resolving the singularities in a Spin(7) compatible way [216]. In this way Y_4 represents the double cover of Z_8 which relates the volumes as $\mathcal{V} = 2\hat{\mathcal{V}}$.

In preparation for the application to F-theory let us comment further on the geometries involved. We note that when considering F-theory on a Calabi-Yau space Y_4^s, the space can be chosen to be singular. The singularities arise, for example, when the four-dimensional theory has to have a non-Abelian gauge group. These non-Abelian singularities can be resolved in a way that is compatible with the Calabi-Yau condition to yield a manifold Y_4. We denote the antiholomorphic involution on the singular space Y_4^s by σ^s and on the resolved space by σ. The respective quotient spaces are denoted by $Z_8^s = Y_4^s/\sigma^s$ and $Z_8 = Y_4/\sigma$. The Spin(7) resolution of Z_8 will be denoted by $\hat{Z}_8$. By analogy with the standard M-theory/F-theory duality we thus expect that the duality (8.1) relates F-theory compactified on Z_8^s with M-theory compactified on $\hat{Z}_8$. It should be stressed that finding a resolution of Z_8 admitting a Spin(7) structure is a hard task and involves constructing local real Spin(7) ALE geometries that can be used to resolve possible orbifold singularities [216]. The Betti numbers of the resolved space can be computed as described in [216]. A stringy computation of the Betti numbers on the quotient geometry Z_8 can be found in [226]. In this work we will not be concerned with this real resolution $\hat{Z}_8$, and mostly work with Z_8 neglecting possible singularities. We will refer to the Spin(7) manifold Z_8 constructed in this way as a quotient torus fibration. Our goal is, however, to formulate the results in a general Spin(7) language such that they can be equally applied to the resolved geometries $\hat{Z}_8$. We summarize the relevant geometries in figure 8.2.

The construction that is carried out in [216] assumes certain additional properties of the orbifold singularities that are required for the Spin(7) ALE resolutions which are considered there to be applied.

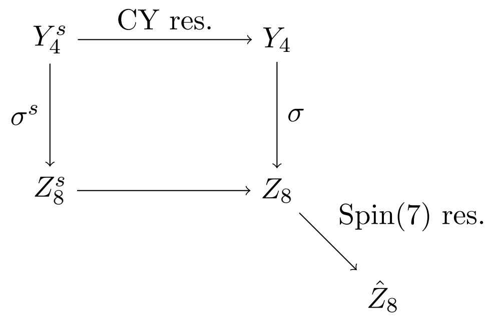


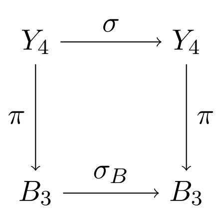
Figure 8.2: Construction of Spin(7) manifolds by using Calabi-Yau fourfolds with antiholomorphic involutions.

One such condition is that the singularities introduced by quotienting with respect to σ must be isolated points in Z_8 which lie at points that are already holomorphic orbifold singularities of Y_4. However it is anticipated that these resolution methods are by no means the only possibility. Therefore, in what follows, we will not limit ourselves to considering only the sorts of singularities which are required in [216], but will bear in mind these additional constraints. The analysis of the more general resolutions that would then be required but the physics associated with their structure will not be discussed in this work.

8.2.2 Spin(7) manifolds from elliptically fibered Calabi-Yau fourfolds

In order that the Spin(7) manifold Z_8 can be used as a background of F-theory we require that the Calabi-Yau fourfold Y_4 is an elliptic fibration with Kähler base B_3. The general remarks of section 5.2.2 on Calabi-Yau with elliptic fibration structure apply to the present case of Y_4. In particular, we suppose that Y_4 is presented in Weierstrass form (5.22). We refer the reader to (5.19) for the expression of the discriminant locus encoding the positions of seven-branes and to (5.21) for the complex structure parameter of the elliptic fiber.

The involutive symmetry σ on the elliptic fibration is demanded to have a definite action on B_3, i.e. σ should be compatible with the fibration and induce a well-defined action on the base that we denote σ_B. Diagrammatically we have



where $\pi : Y_3 \to B_3$ is the canonical projection onto the base of the fibration. The preimage of a point p on B_3 under π, i.e. the fiber over p, will be denoted $\mathcal{C}_p$.

Let us denote by $\hat{L}_\sigma$ the fixed-point space of σ in Y_4. Its projection to B_3 is denoted by $L_\sigma^B = \pi(\hat{L}_\sigma)$ and can equally be obtained as the fixed-point space of σ_B. On general grounds the space L_σ^B can be composed of several components that can be one- or three-dimensional. To see this we can perform the following local analysis. Let U be a given local patch U on B_3 containing a fixed point p of σ_B. We introduce local complex coordinates (z_1, z_2, z_3) in such a way that that the coordinates of p are $(0, 0, 0)$. A first possibility for the action of σ_B, referred to as case (a) in what follows, is given by

$$(a) \qquad (z_1, z_2, z_3) \to (\bar{z}_1, \bar{z}_2, \bar{z}_3) \ , \qquad \Rightarrow \qquad L_\sigma^B(U) \text{ is three-dimensional.} \tag{8.7}$$

This is the simplest case, since the geometry of the base B_3 around the fixed locus is smooth. A possible alternative that we refer to as case (b) is when L_σ^B is one-dimensional. In this situation B_3 cannot be smooth and instead is replaced by an orbifold with singularities associated with a discrete group G that contains $\mathbb{Z}_2$. For simplicity we will focus here on the case where $G = \mathbb{Z}_2$ but the extension to more general orbifold singularities may be easily performed. A patch U of B_3 near such a singularity takes locally the form $\mathbb{C}^3/\mathbb{Z}_2$ and may be described locally by the complex coordinates (z_1, z_2, z_3) identified by $\rho_U : (z_1, z_2, z_3) \to (-z_1, -z_2, z_3)$. The action of σ_B on these coordinates is given by

$$(b) \qquad (z_1, z_2, z_3) \to (\bar{z}_2, -\bar{z}_1, \bar{z}_3) \ , \qquad \Rightarrow \qquad L_\sigma^B(U) \text{ is one-dimensional,} \tag{8.8}$$

which is an involution on the patch U as σ_B squares to the identification ρ_U. Let us point out two special occurrences of case (b). Firstly, one could start with a non-singular threefold admitting a global $\mathbb{Z}_2$ and quotient by this symmetry to find the base B_3. In fact, this sort of situation naturally arises in toroidal orbifolds. Secondly, one may consider the case that B_3 is described as a hypersurface or complete intersection in a higher-dimensional ambient space exhibiting orbifold singularities as a result of scaling identifications. This allows σ_B to act as an involution on B_3 if it is induced by a symmetry of the ambient space that squares to the identity upon using the scalings. Both types of constructions appear in [216]. Finally, we would like to furnish an explicit example of freely acting antiholomorphic involution σ_B,

$$(c) \qquad (z_1, z_2, z_3) \to \left(\frac{\bar{z}_2}{\bar{z}_3}, -\frac{\bar{z}_1}{\bar{z}_3}, -\frac{1}{\bar{z}_3}\right) \ , \qquad \Rightarrow \qquad L_\sigma^B(U) \text{ is empty ,} \tag{8.9}$$

even though we will not analyze this case (c) in detail in the upcoming sections.

The fixed space $\hat{L}_\sigma$ of Y_4 can have components that are either two- or four-dimensional, or σ can be freely acting. To investigate the action of σ on Y_4 further we must analyze several cases which are distinguished by the location of the point p on B_3:

(1) $p \notin L_\sigma^B$: For each point p on B_3 that is not a fixed point of σ_B the corresponding elliptic curve $\mathcal{C}_p$ is mapped onto another elliptic curve $\mathcal{C}_{\sigma_B(p)}$ over the image point $\sigma_B(p)$. However, since σ is antiholomorphic the orientations of $\mathcal{C}_{\sigma_B(p)}$ and $\sigma(\mathcal{C}_p)$ will differ. In this case σ will be freely acting on all points of Y_4 that project to p or $\sigma_B(p)$, see figure 8.3.

(2) $p \in L_\sigma^B$ and $\Delta(p) \neq 0$: If a point p on B_3 is a fixed point of σ_B the elliptic curve over this point will be mapped to itself. In particular, this implies that if p is not on a seven-brane that

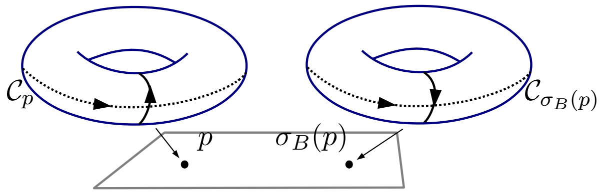


Figure 8.3: Generic torus fibers exchanged by the antiholomorphic involution.

a smooth two-torus is mapped onto itself. Recall that the fixed point set of an antiholomorphic involution on a smooth complex two-torus either consists of up to two real lines or is empty.

(2.1) If the torus is fixed point free this implies that each point on Y_4 that projects to p is actually not fixed by σ and hence does not give rise to a singularity of Z_8. This means that σ will be freely acting on all points of Y_4 that project to p. If L_σ^B is one-dimensional then the additional singularities associated with the σ^2 identification can be resolved using standard tools in algebraic geometry. Interestingly, if σ is fixed point free on the torus but not on the base then the quotient fiber at such p is a Klein bottle, see figure 8.4.

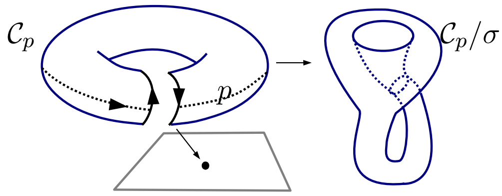


Figure 8.4: Fiber modded by antiholomorphic involution to Klein bottle fibers.

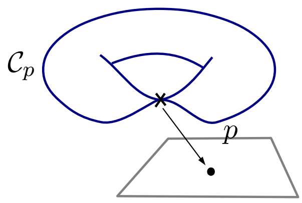


Figure 8.5: Nodal fiber at fixed point p. Involution fixes pinch-point.

(2.2) If the torus has a fixed line on it then the dimension of $\hat{L}_\sigma$ may be up to one greater than the dimension of L_σ^B, depending on the dimension of the subspace of L_σ^B over which the fixed space on the torus is a line. Since $\hat{L}_\sigma$ must then have even dimensions greater than one, it must have dimension of either 2 or 4. The quotient of the elliptic curve by σ then gives rise to a cylinder.[1]

(3) $p \in L_\sigma^B$ and $\Delta(p) = 0$: The most interesting case is if a point p on B_3 is both a fixed point of σ_B and lies on a seven-brane. In this case $\mathcal{C}_p$ is actually a singular curve. There are various possibilities for such singular curves and a systematic study should investigate all possible antiholomorphic involutions and their fixed points. Here, let us only consider the simplest case

[1] We note that in certain cases an antiholomorphic involution of a smooth torus with a one-dimensional fixed space can also yield a Möbius band.

where $\mathcal{C}_p$ is a nodal curve (I_1 type), as schematically depicted in figure 8.5. In this case there can exist an involution σ that has one fixed point exactly at the node of the elliptic curve. One can think of this nodal point as arising by shrinking the real one-dimensional fixed point set of an antiholomorphic involution on a smooth elliptic curve. In this case the dimension of $\hat{L}_\sigma$ may be an even integer less than the dimension of L^B_σ, so it can be either zero or two.

From this we see that if the action of σ on Y_4 is to be fixed point free then it can have only points for which situations (1) or (2.1) apply. Alternatively if we restrict the fixed space to consist only of isolated fixed points, which is imposed in [216], then we find that situation (3) must apply in which the torus is pinched at these points. In addition to this if we also wish to consider fixed points which are already holomorphic orbifold singularities of Y_4, as is also imposed in [216], then we find that L^B_σ must be one-dimensional. An example of a space which has singularities of this sort is shown in appendix C in section C.2.

Let us now analyze the action of the antiholomorphic involution σ on the elliptic fiber. To this end, we consider the case in which the elliptic fiber is presented in Weierstrass form (5.18) without specializing to the patch $z \neq 0$. We can then let the antiholomorphic involution σ act antilinearly on the projective coordinates of $\mathbb{P}^2_{2,3,1}$. Any σ action of this type may then be brought into the form

$$\sigma : \quad (x, y, z) \to (\bar{x}, \bar{y}, \bar{z}) \tag{8.10}$$

by an appropriate coordinate redefinition. Comparison between (5.18) and (8.10) reveals that, in order for the antiholomorphic involution to be well-defined on the Calabi-Yau fourfold Y_4, the sections f and g have to satisfy

$$f_{\sigma_B(p)} = \overline{f_p} \ , \qquad g_{\sigma_B(p)} = \overline{g_p} \ , \tag{8.11}$$

for every p on the base B_3. Recalling (5.21), we conclude that for any point p on the base B_3

$$j(\tau_{\sigma_B(p)}) = \overline{j(\tau_p)} = j(-\overline{\tau_p}) \ . \tag{8.12}$$

In the last step we have made use of the fact that all coefficients entering the Laurent series (5.6) are integers and therefore real. In summary, we can infer that

$$\tau_{\sigma_B(p)} = -\overline{\tau_p} \qquad \text{up to } SL(2, \mathbb{Z}) \text{ transformations.} \tag{8.13}$$

Note that this condition is perfectly compatible with a non-trivial holomorphic dependence of the modular parameter on the base coordinates. In particular, it can be satisfied for τ profiles with non-trivial monodromies associated to the presence of seven-branes. Only in the special case in which τ is constant over the base, as in the weak coupling limit away from orientifold planes, (8.13) enforces a reality condition on τ, which has to be purely imaginary.

8.3 M-theory on Spin(7) spaces and Calabi-Yau quotients

Having discussed the geometry of the Spin(7) holonomy manifolds that we wish to consider, we will now describe the effective theories which arise in the reduction of M-theory on these spaces. In subsection

8.3.1 we will begin this analysis by considering the reduction on general Spin(7) manifolds. Then in subsection 8.3.2 we will analyze how this may be related to the quotient of the effective theories that arise from compactification on Calabi-Yau fourfolds. In subsection 8.3.3 we will then restrict to the case where these Calabi-Yau manifolds are elliptically fibered and study the redefinitions that must be made in order to move into a frame that can be lifted to the four-dimensional F-theory dual.

8.3.1 Effective action of M-theory on Spin(7) manifolds

The compactification of M-theory on a Spin(7) manifold $\hat{Z}_8$ yields a three-dimensional effective theory with minimal $\mathcal{N} = 1$ supersymmetry. The action, to quadratic order in the fermions, for a general three-dimensional theory with $\mathcal{N} = 1$ supersymmetry can always be written in the form [227, 228]

$$\begin{aligned} S^{(3)}_{\mathcal{N}=1} = \int d^3x\, e \Big[& \frac{1}{2}R - \frac{1}{4}\Theta_{IJ}\epsilon^{\mu\nu\rho}A^I_\mu(\partial_\nu A^J_\rho + \frac{1}{3}f_{KL}{}^J A^K_\nu A^L_\rho) - \frac{1}{2}g_{\Lambda\Sigma}\mathcal{D}_\mu\phi^\Lambda\mathcal{D}^\mu\phi^\Sigma - V(\phi) \\ & - \frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_r - \frac{1}{2}g_{\Sigma\Lambda}\bar{\chi}^\Sigma\gamma^\mu\mathcal{D}_\mu\chi^\Lambda + \frac{1}{2}g_{\Sigma\Lambda}\bar{\chi}^\Sigma\gamma^\mu\gamma^\nu\psi_\mu\mathcal{D}_\nu\phi^\Lambda \\ & - \frac{1}{2}F\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu + \partial_\Lambda F\bar{\psi}_\mu\gamma^\mu\chi^\Lambda + \frac{1}{2}(g_{\Sigma\Lambda}F - 2D_\Sigma\partial_\Lambda F + 2X^I_\Sigma X^J_\Lambda\Theta_{IJ})\bar{\chi}^\Sigma\chi^\Lambda \Big] , \end{aligned} \tag{8.14}$$

with covariant derivatives and scalar potential given by

$$\mathcal{D}_\mu\phi^\Lambda = \partial_\mu\phi^\Lambda + \Theta_{IJ}X^{I\Lambda}A^I_\mu , \qquad V(\phi) = 2g^{\Lambda\Sigma}\partial_\Lambda F\partial_\Sigma F - 4F^2 . \tag{8.15}$$

Here $X^{I\Lambda}$ is the Killing vector of the target space symmetry that is gauged via (8.15). The action (8.14) contains the ϕ^Λ-dependent metric $g_{\Lambda\Sigma}(\phi)$ that is non-degenerate and positive definite. The coefficient Θ_{IJ} of the Chern-Simons term is symmetric in I, J, and constant which ensures the gauge invariance of the action. This represents the embedding tensor for the three-dimensional gauged supergravity theory. The real function $F(\phi)$ depends on the scalars ϕ^Λ and is required to satisfy $\Theta_{IJ}X^{I\Lambda}\partial_\Lambda F = 0$ for gauge invariance.

For smooth Spin(7) geometries $\hat{Z}_8$ the $\mathcal{N} = 1$ vacua where studied in [219, 220, 229, 101]. The three-dimensional effective theory can be derived by reducing the action for eleven-dimensional supergravity, which we record here again,

$$S^{(11)} = \int \tfrac{1}{2}R * 1 - \tfrac{1}{4}G_4 \wedge *G_4 - \tfrac{1}{12}C_3 \wedge G_4 \wedge G_4 . \tag{8.16}$$

This reduction is discussed in [220, 221, 223, 225]. In the full reduction one must also take into account the higher derivative terms along with the Spin(7) analog of the Calabi-Yau fourfold tadpole cancellation condition (5.47). Since we do not consider spacetime-filling M2-branes, this constraint reads

$$\frac{\chi(\hat{Z}_8)}{24} = \frac{1}{2}\int_{\hat{Z}_8} G_4 \wedge G_4 . \tag{8.17}$$

We will describe this reduction in the following and reconsider some aspects of the derivation presented in [225]. We stress that this reduction is actually a warped compactification, and we will neglect this backreaction in the following leading order analysis.

We carry out the reduction by decomposing the metric and three-form of eleven-dimensional supergravity as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + g_{mn}dy^m dy^n \,, \qquad\qquad C_3 = A^I \wedge \omega_I \,, \tag{8.18}$$

where g_{mn} is the metric on $\hat{Z}_8$ and ω_I form a basis for $H^2(\hat{Z}_8, \mathbb{R})$ with $I = 1, \dots, b^2(\hat{Z}_8)$. We will restrict to the case of $b^3(\hat{Z}_8) = 0$ for simplicity. The three-dimensional theory will then admit $U(1)$ gauge symmetries associated with the vectors A^I.

In performing the Kaluza-Klein reduction one has to allow the metric of the internal geometry $\hat{Z}_8$ to vary without leaving the class of Spin(7) geometries. The analysis of such deformations was performed in section 4.6.2. Let us remind the reader that zeromodes of the internal metric are in one-to-one correspondence with the set of antiself-dual four-forms ξ_A, $A = 1, \dots, b^4_{\mathrm{A}}(\hat{Z}_8)$, along with one additional zero mode that corresponds to a rescaling of the overall volume. This implies that there will be $b^4_A(\hat{Z}_8)+1$ real scalar fields φ^A and $\hat{\mathcal{V}}$ parameterizing the deformations of the Spin(7) structure. We refer the reader to (4.103) and (4.104) for the expressions of the variations of the Spin(7) metric in terms of scalars φ^A, $\hat{\mathcal{V}}$. The analogous expression for the Cayley calibration was given in (4.105) but is repeated here for convenience,

$$\delta\Phi = K_{\hat{\mathcal{V}}}\Phi \; \delta\hat{\mathcal{V}} + (K_A\Phi + \xi_A) \; \delta\varphi^A \; . \tag{8.19}$$

Upon performing the dimensional reduction, followed by a Weyl rescaling of the three-dimensional metric to move into the Einstein frame, the bosonic part of the effective action is given by

$$S^{(3)}_{\hat{Z}_8} = \int \tfrac{1}{2}R{*}1 - \tfrac{1}{2}h_{IJ}F^I\wedge{*}F^J - \tfrac{1}{4}\Theta_{IJ}A^I\wedge F^J - \tfrac{1}{2}g_{\hat{\mathcal{V}}\hat{\mathcal{V}}}d\hat{\mathcal{V}}\wedge{*}d\hat{\mathcal{V}} - \tfrac{1}{2}g_{AB}d\varphi^A\wedge{*}d\varphi^B - V(\varphi){*}1 \; , \tag{8.20}$$

where

$$g_{\hat{\mathcal{V}}\hat{\mathcal{V}}} = \tfrac{9}{8}\hat{\mathcal{V}}^{-2} \,, \qquad g_{AB} = -\frac{7}{2}\frac{\int_{\hat{Z}_8}\xi_A\wedge\xi_B}{\int_{\hat{Z}_8}\Phi\wedge\Phi} \,, \qquad h_{IJ} = \frac{1}{2\hat{\mathcal{V}}}\int_{\hat{Z}_8}\omega_I\wedge{*}\omega_J \,, \tag{8.21}$$

and the scalar potential $V(\varphi)$ is of the form (8.15). We do not discuss the details of the dimensional reduction since they are conceptually similar to the Calabi-Yau threefold reduction of the previous chapter. In particular, the reduction of the Einstein-Hilbert term can be performed by combining the general results of section 4.3 with the observations about the Spin(7) moduli metric collected in section 4.6.2

Let us point out that the action (8.21) is less general then (8.14). Firstly, we have only included Abelian vectors. More importantly, we did not dualize all dynamical vector degrees of freedom into scalar degrees of freedom as it is always possible in three dimensions. Therefore the kinetic terms of the vectors with φ^A-dependent metric h_{IJ} still appears in (8.20). Dualizing all vector degrees of freedom yields new scalars ζ_I with metric h^{IJ}, the inverse of h_{IJ}. The presence of a Chern-Simons term in (8.20) implies that the ζ_I are in general gauged with covariant derivative

$$\mathcal{D}\zeta_I = d\zeta_I + \Theta_{IJ}A^J \; . \tag{8.22}$$

Hence, the action (8.20) allows us to determine all couplings in (8.14): $\phi^\Lambda = (\hat{\mathcal{V}}, \varphi^A, \zeta_I)$, $g_{\Lambda\Sigma} = (\frac{9}{8\hat{\mathcal{V}}^2}, g_{AB}, h^{IJ})$, and $X^I_J = \delta^I_J$, $X^{IA} = 0$.

So far we have not discussed the scalar potential V and the Chern-Simons coupling Θ_{IJ}. In fact, in a compactification without fluxes both vanish identically. They are, however, induced if one allows for a non-trivial flux background of the field strength dC_3. Let us denote the background flux on $\hat{Z}_8$ by G_4. A direct reduction of eleven-dimensional supergravity then implies that a flux-induced Chern-Simons term takes the form

$$\Theta_{IJ} = \int_{\hat{Z}_8} G_4 \wedge \omega_I \wedge \omega_J \ . \tag{8.23}$$

More involved is the derivation of the flux-induced scalar potential from a real function F. After dimensional reduction of the full action including the higher curvature term, one uses the tadpole cancellation condition (8.17) to show that the scalar potential takes the form

$$V = \frac{1}{4\hat{\mathcal{V}}^3}\Big(\int_{\hat{Z}_8} G_4 \wedge *G_4 - \int_{\hat{Z}_8} G_4 \wedge G_4\Big) = -\frac{1}{2\hat{\mathcal{V}}^3}\int_{\hat{Z}_8} G_4^A \wedge G_4^A \ , \tag{8.24}$$

where G_4^A is the antiself-dual part of the background flux G_4. To generally derive F let us first note that it was argued in [225] that F should be proportional to $\int_{\hat{Z}_8} G_4 \wedge \Phi$. The factor in front of this flux integral can, however, be field-dependent. In fact the correct form of F is given by

$$F = \frac{\sqrt{7}}{4\sqrt{2}||\Phi||\hat{\mathcal{V}}^2}\int_{\hat{Z}_8} G_4 \wedge \Phi \, . \tag{8.25}$$

In this expression we have made use of the quantity

$$\|\Phi\| = \frac{1}{4!}\Phi_{mnpq}\Phi^{mnpq} \ , \tag{8.26}$$

which has been already introduced in section 4.6.1. The derivatives of F satisfy

$$\frac{\partial F}{\partial \varphi^A} = \frac{\sqrt{7}}{4\sqrt{2}||\Phi||\hat{\mathcal{V}}^2}\int_{\hat{Z}_8} G_4 \wedge \xi_A \ , \qquad \frac{\partial F}{\partial \hat{\mathcal{V}}} = -\frac{3}{2\hat{\mathcal{V}}}F \ . \tag{8.27}$$

Note that the derivation of these results does not depend on the precise form of moduli-dependent coefficients $K_{\hat{\mathcal{V}}}$ and K_A entering the variation of the Cayley calibration (4.105) as these cancel when taking the derivative.[2] Inserting (8.27), (8.25) and the inverse metrics $g^{AB}, g^{\hat{\mathcal{V}}\hat{\mathcal{V}}}$ obtained from (8.21) into the general form of the $\mathcal{N} = 1$ scalar potential (8.15) one readily shows match with (8.24).

We conclude this section by performing a rearrangement of the Spin(7) moduli that will be useful in the comparison to the Calabi-Yau reduction of section 8.3.2. To begin with, we divide the Spin(7) moduli φ^A into two subsets, $\varphi^A = (\varphi^{\mathcal{K}}, \varphi^{\tilde{I}-})$. This notation is chosen to make contact to section 8.3.2.

[2] One can also show that given a general Cayley calibration Φ, which varies as (4.105), it is possible to define an alternatively normalized self-dual four-form $\hat{\Phi}$ which is also a singlet of Spin(7) and satisfies

$$\hat{\Phi} = \frac{1}{||\Phi||\hat{\mathcal{V}}^2}\Phi \, , \qquad \hat{K}_{\hat{\mathcal{V}}} = -\tfrac{3}{2}\hat{\mathcal{V}}^{-1} \ , \qquad \hat{K}_A = 0 \ . \tag{8.28}$$

This corresponds to the normalization for Φ chosen in (8.5).

Note that this partition of the Spin(7) moduli is supposed to be such that the associated antiself-dual four-forms satisfy the orthogonality condition

$$\int_{\hat{Z}_8} \xi_{\mathcal{K}} \wedge \xi_{\tilde{I}_-} = 0 \ . \tag{8.29}$$

Next we extend the range of the index $\tilde{I}_-$ by defining a new index I_- that includes one additional entry and define $\phi^{I_-} = (\hat{\phi}, \hat{\phi}\varphi^{\tilde{I}_-})$. This definition is such that that the variation of Φ in (4.105) is now given by

$$\delta\Phi = K_{\hat{\mathcal{V}}}\Phi \ \delta\hat{\mathcal{V}} + (\mathbb{K}_{I_-}\Phi + \eta_{I_-})\delta\phi^{I_-} + (K_{\mathcal{K}}\Phi + \xi_{\mathcal{K}})\delta\varphi^{\mathcal{K}} \ , \tag{8.30}$$

where

$$\mathbb{K}_{I_-} = \Big(- \frac{\varphi^{\tilde{J}_-} K_{\tilde{J}_-}}{\hat{\phi}}, \frac{K_{\tilde{I}_-}}{\hat{\phi}} \Big) , \qquad \eta_{I_-} = \Big(- \frac{\varphi^{\tilde{J}_-} \xi_{\tilde{J}_-}}{\hat{\phi}}, \frac{\xi_{\tilde{I}_-}}{\hat{\phi}} \Big) \ . \tag{8.31}$$

These definitions then imply the constraints

$$\phi^{I_-}\mathbb{K}_{I_-} = 0 \ , \qquad \phi^{I_-}\eta_{I_-} = 0 \ , \tag{8.32}$$

which means that the action (8.20) develops a new local symmetry under under which

$$\phi^{I_-} \to \lambda\phi^{I_-} \ , \qquad \Phi \to \lambda\Phi \ . \tag{8.33}$$

As anticipated above, this constrained formulation will be helpful in section 8.3.2. It might also be useful, however, in finding generalizations of the F-theory construction to Spin(7) manifolds that are not obtained as Calabi-Yau quotients.

8.3.2 Effective action of M-theory on Spin(7) manifolds from Calabi-Yau quotients

In the following we would like to introduce Spin(7) geometries whose effective theories can be uplifted to four dimensions via the M-theory to F-theory limit. It is an outstanding question to characterize such geometries generally. In order to approach this problem we therefore restrict our analysis to Spin(7) geometries arising from elliptically fibered Calabi-Yau fourfolds as introduced in Section 8.2.2. Our aim is to first show, that the three-dimensional $\mathcal{N} = 2$ theories arising in Calabi-Yau fourfold compactifications of M-theory are truncated to $\mathcal{N} = 1$ when performing the antiholomorphic quotient Y_4/σ, with an involution σ as in (8.3). We note that the following steps bear many similarities to the construction of four-dimensional Type IIA Calabi-Yau orientifold actions [51]. However, here we are truncating three-dimensional $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$ supersymmetry.[3] Truncations of $\mathcal{N} = 2$ Chern-Simons theories to $\mathcal{N} = 1$ induced by an antiholomorphic involution have been also considered in [231].

Let us first recall the general form of a three-dimensional $\mathcal{N} = 2$ action. The bosonic part of this can always be brought to the form

$$S^{(3)}_{\mathcal{N}=2} = \int \tfrac{1}{2}R * 1 - \tfrac{1}{4}\Theta_{IJ}A^I \wedge (dA^J + \tfrac{2}{3}f_{KL}{}^J A^K \wedge A^L) - g_{\mathcal{A}\bar{\mathcal{B}}}\mathcal{D}M^{\mathcal{A}} \wedge *\mathcal{D}\bar{M}^{\mathcal{B}} - \tilde{V} * 1 \ , \tag{8.34}$$

[3] A systematic study of spontaneous $\mathcal{N} = 2$ to $\mathcal{N} = 1$ breaking in three dimensions can be found in [230].

where $g_{\mathcal{A}\bar{\mathcal{B}}} = \partial_{\mathcal{A}}\partial_{\bar{\mathcal{B}}}K$ is a Kähler metric and $\tilde{V}(M, \bar{M})$ is the scalar potential. This scalar potential is generally of the form

$$\tilde{V} = e^K\big(K^{\mathcal{A}\bar{\mathcal{B}}}D_{\mathcal{A}}W\overline{D_{\mathcal{B}}W} - 4|W|^2\big) + \big(K^{\mathcal{A}\bar{\mathcal{B}}}\partial_{\mathcal{A}}\mathcal{T}\overline{\partial_{\mathcal{B}}\mathcal{T}} - \mathcal{T}^2\big)\ , \tag{8.35}$$

where $W(M)$ is a holomorphic superpotential and $\mathcal{T}$ is a real potential. One may also note that in the $\mathcal{N} = 2$ case the presence of a non-vanishing $\mathcal{T}$ is linked to the gaugings $\mathcal{D}M^{\mathcal{A}}$.

The three-dimensional $\mathcal{N} = 2$ effective action for a Calabi-Yau fourfold compactification of eleven-dimensional supergravity was derived in [232, 233]. For the case $b^3(Y_4) = 0$ it takes a particularly simple form. From our general discussion of the moduli space of Calabi-Yau n-folds with $n > 2$ in section 4.5.2 we know that fluctuations of the internal metric yield both complex structure moduli and Kähler moduli. The former are denoted $z^{\mathcal{K}}$ in this chapter and are labelled by $\mathcal{K} = 1, \dots, h^{3,1}(Y_4)$. Real Kähler deformations v^I are indexed by $I = 1, \dots, h^{1,1}(Y_3)$ and arise, as usual, from the expansion of the Kähler form, $J = v^I\omega_I$.

The expansion of the M-theory three-form $C_3 = A^I \wedge \omega_I$ yields $h^{1,1}(Y_4)$ three-dimensional vectors A^I. The vectors A^I together with v^I form the bosonic components of three-dimensional $\mathcal{N} = 2$ vector multiplets. In contrast to the five-dimensional reduction of the previous chapter, the vectors A^I and dualized into scalars ζ_I that provide the degrees of freedom necessary for the complexification of the real Kähler moduli v^I. More precisely, the natural coordinates in the complexified Kähler moduli space are

$$T_I = \tfrac{1}{3!}\int_{Y_4}\omega_I \wedge J^3 + i\zeta_I\ . \tag{8.36}$$

They can be interpreted as the classical action of a Euclidean M5-brane wrapping the divisor class $[D_I]$ Poincaré dual to ω_I. After dualization the kinetic terms of the three-dimensional $\mathcal{N} = 2$ supergravity theory are encoded by a Kähler potential

$$K(z, T) = -\log\int_{Y_4}\Omega \wedge \bar{\Omega} - 3\log\mathcal{V}\,, \tag{8.37}$$

which is evaluated as a function of the $h^{3,1}(Y_4) + h^{1,1}(Y_4)$ complex coordinates $z^{\mathcal{K}}$ and T_I.

In the presence of background fluxes G_4 a non-trivial Chern-Simons term with Θ_{IJ} exactly as in (8.23) is induced. As above in (8.22) this also implies the presence of gaugings $\mathcal{D}T_I = dT_I + i\Theta_{IJ}A^J$. Furthermore, a scalar potential arises from the functions

$$\mathcal{T} = \frac{1}{4\mathcal{V}^2}\int_{Y_4} G_4 \wedge J^2\ , \qquad\qquad W = \int_{Y_4} G_4 \wedge \Omega\ , \tag{8.38}$$

where $\mathcal{T}$ is in accord with the gauged shift symmetries.

In order to implement the $\mathcal{N} = 1$ truncation we first note that the relevant forms have to transform under σ^* as

$$\sigma^* J = -J\ , \qquad\qquad \sigma^*(C\Omega) = \overline{C\Omega}\ , \qquad\qquad \sigma^* C_3 = C_3\ , \tag{8.39}$$

where the first two conditions already appeared in (8.3) when inserting the definition

$$C = e^{-i\theta} e^{K/2} \ , \tag{8.40}$$

with K as defined in (8.37). To perform the reduction one thus has to split the cohomology of Y_4 into parity-even and parity-odd eigenspaces as

$$H^n(Y_4, \mathbb{R}) = H^n_+(Y_4, \mathbb{R}) \oplus H^n_-(Y_4, \mathbb{R}) \ . \tag{8.41}$$

The surviving vectors in the expansion of C_3 only arise from elements of $H^2_+(Y_4)$, while the surviving Kähler structure scalars arise from elements of $H^2_-(Y_4)$. Thus, one has

$$C_3 = A^{I_+} \wedge \omega_{I_+} \ , \qquad I_+ = 1, \ldots, h^{1,1}_+(Y_4) \ , \qquad J = v^{I_-} \omega_{I_-} \ , \qquad I_- = 1, \ldots, h^{1,1}_-(Y_4) \ . \tag{8.42}$$

Applying this to the dual complex scalars T_I introduced in (8.36) one finds the split

$$T_I = (T_{I_+}, T_{I_-}) = (-i\text{Im}T_{I_+}, \text{Re}T_{I_-}) \ , \qquad \text{Im}T_{I_-} = \text{Re}T_{I_+} = 0 \ . \tag{8.43}$$

In other words, out of the $h^{1,1}(Y_4)$ complex coordinates T_I only $h^{1,1}(Y_4)$ real coordinates survive in the quotient theory. Similarly, the $h^{3,1}(Y_4)$ complex fields $z^{\mathcal{K}}$ encoding complex structure deformations are reduced to $h^{3,1}(Y_4)$ real complex structure deformations $\varphi^{\mathcal{K}}$. This can be inferred by considering all complex structure deformations of Ω preserving the condition (8.39). One can chose local coordinates such that $\varphi^{\mathcal{K}} = \text{Re}\, z^{\mathcal{K}}$. In summary, the involution truncates the $\mathcal{N}=2$ Kähler manifold spanned by T_I and $z^{\mathcal{K}}$ to a real Lagrangian submanifold $\mathcal{L}_\sigma$ parametrized by ζ_{I_+}, $\text{Re}T_{I_-}$ and $\varphi^{\mathcal{K}}$.

To compare these degrees of freedom which survive the quotient with those described in the Spin(7) reduction of subsection 8.3.1 it is necessary to redefine the fields. The vectors A^{I_+} and the volume $\mathcal{V}$ are simply identified with the vectors A^I and the volume $\mathcal{V}$ in (8.20), while the $b^4_A(Z_8)$ scalar fields φ^A in (8.20) parametrize the independent degrees of freedom of the constrained fields

$$\phi^{\hat{A}} = (\varphi^{\mathcal{K}}, \phi^{I_-}) \ , \qquad \text{where} \qquad \hat{A} = 1, \ldots, 1 + b^4_A(Z_8) \, , \qquad \phi^{I_-} = \mathcal{V}^{-\frac{1}{4}} v^{I_-} \ . \tag{8.44}$$

They satisfy the constraint

$$N \equiv \tfrac{1}{4!} \mathcal{K}_{I_- J_- K_- L_-} \phi^{I_-} \phi^{J_-} \phi^{K_-} \phi^{L_-} = 1 \, , \tag{8.45}$$

as a result of the definition (8.44). This condition can be viewed as a gauge fixing of the additional symmetry introduced in (8.33). In terms of these fields the bosonic part of the effective theory describing the projected Calabi-Yau reduction is given by

$$\begin{aligned} S^{(3)}_{Y_4/\sigma} = \int & \tfrac{1}{2} R * 1 - \tfrac{1}{2} h_{I_+ J_+} F^{I_+} \wedge * F^{J_+} - \tfrac{1}{4} \Theta_{I_+ J_+} A^{I_+} \wedge dA^{J_+} - \tfrac{1}{2} g_{\mathcal{V}\mathcal{V}} d\mathcal{V} \wedge * d\mathcal{V} \\ & - \tfrac{1}{2} \tilde{g}_{I_- J_-} d\phi^{I_-} \wedge * d\phi^{J_-} - \tfrac{1}{2} \tilde{g}_{\mathcal{K}\mathcal{I}} d\varphi^{\mathcal{K}} \wedge * d\varphi^{\mathcal{I}} - V * 1 \ , \end{aligned} \tag{8.46}$$

where the scalar metrics may be written as

$$\begin{aligned} g_{\mathcal{V}\mathcal{V}} &= \tfrac{9}{8} \mathcal{V}^{-2} \, , & h_{I_+ J_+} &= \frac{1}{2\mathcal{V}} \int_{Y_4} \omega_{I_+} \wedge * \omega_{J_+} \, , \\ \tilde{g}_{I_- J_-} &= -4\mathcal{V}^3 \int_{Y_4} \eta_{I_-} \wedge \eta_{J_-} \, , & \tilde{g}_{\mathcal{K}\mathcal{L}} &= -4\mathcal{V}^3 \int_{Y_4} \xi_{\mathcal{K}} \wedge \xi_{\mathcal{L}} \, , \end{aligned} \tag{8.47}$$

and where

$$\eta_{I_-} = \tfrac{1}{4}\mathcal{V}^{-\frac{3}{2}} P_{I_-}{}^{J_-}\omega_{J_-} \wedge J_\phi \;, \qquad P_{I_-}{}^{J_-} = \delta_{I_-}{}^{J_-} - \frac{1}{4!}\mathcal{K}_{I_-K_-L_-M_-}\phi^{K_-}\phi^{L_-}\phi^{M_-}\phi^{J_-} \;,$$
$$\xi_{\mathcal{K}} = \mathrm{Re}(C\chi_{\mathcal{K}}) \;, \qquad \mathcal{K}_{I_-J_-K_-L_-} = \int_{Y_4} \omega_{I_-} \wedge \omega_{J_-} \wedge \omega_{K_-} \wedge \omega_{L_-} \;. \tag{8.48}$$

We have used the definition $J_\phi = \phi^{I_-}\omega_{I_-}$. Note that the constraint (8.45) is responsible for the projection matrices $P_{I_-}{}^{J_-}$ that appear in the definition of the scalar metric. The Chern-Simons terms in (8.46) are induced by G_4 fluxes as in (8.23) and read

$$\Theta_{I_+J_+} = \frac{1}{2}\int_{Y_4} \omega_{I_+} \wedge \omega_{J_+} \wedge G_4 \;. \tag{8.49}$$

By considering the potential of the truncated theory and matching this with (8.15) we see that

$$F = e^{K/2}\mathrm{Re}W + \tfrac{1}{2}\mathcal{T} = \int_{Y_4} G_4 \wedge \big(\mathrm{Re}(C\Omega) + \tfrac{1}{8}\mathcal{V}^{-2} J\wedge J\big)\,. \tag{8.50}$$

By comparing this with (8.25) we may then read off $\Phi = \big(\mathrm{Re}(C\Omega) + \tfrac{1}{8}\mathcal{V}^{-2}J \wedge J\big)$ up to a choice of normalization. This is the expression for Φ that we already quoted in (8.5). In the remainder of this subsection we discuss the structure of the resulting Spin(7) field space in more detail.

To investigate the metric on the Spin(7) field space we need to determine its variations with respect to the coordinates introduced in (8.44). This again requires the constraint (8.45) to be consistently implemented. One way to achieve this is to first express Φ in terms of $\mathcal{V}$ and N before taking derivatives and later impose (8.45). Concretely, one has

$$\Phi = \frac{1}{\mathcal{V}^{3/2}}\left(\frac{\mathrm{Re}(e^{-i\theta}\Omega)}{\big(\int_{Y_4}\Omega\wedge\bar{\Omega}\big)^{1/2}} + \frac{1}{8}\frac{J_\phi \wedge J_\phi}{N^{1/2}}\right) \;. \tag{8.51}$$

Then taking the variations of this with respect to $\mathcal{V}$, ϕ^{I_-}, and $\varphi^{\mathcal{K}}$ we find

$$\delta\Phi|_{N=1} = -\tfrac{3}{2}\mathcal{V}^{-1}\,\Phi\,\delta\mathcal{V} + \eta_{I_-}\;\delta\phi^{I_-} + \xi_{\mathcal{K}}\;\delta\varphi^{\mathcal{K}}\,, \tag{8.52}$$

and in addition find that the normalization of Φ is such that

$$\int_{\hat{Z}_8} \Phi \wedge \Phi = \tfrac{7}{16}\mathcal{V}^{-3} \;. \tag{8.53}$$

Then by comparing the variation (8.52) with (8.19) we may identify the forms $\xi_{\mathcal{K}}$ and η_{I_-} with the Spin(7) forms ξ_A. More precisely, note that the constraint (8.45) implies $\phi^{I_-}\,\eta_{I_-} = 0$. We thus identify the coordinates ϕ^{I_-} and forms η_{I_-} with the quantities constructed after (8.29). Moreover, we find that the projected Y_4 moduli metric (8.47) matches the Spin(7) moduli metric (8.21). As expected from the general Spin(7) analysis, η_{I_-} and $\xi_{\mathcal{K}}$ also form a basis for the complete set of antiself-dual four-forms of Y_4 which are invariant under σ.[4]

[4] In fact the set formed by η_{I_-} and $\xi_{\mathcal{K}}$ is complete but also degenerate as a result of the projection matrix $P_{I_-}{}^{J_-}$ which appears in the definition of η_{I_-}.

8.3.3 Effective action of M-theory on Spin(7) quotients of elliptically fibered Calabi-Yau fourfolds

In order to derive the four-dimensional effective action of F-theory on a Spin(7) manifold, we must now restrict our M-theory reduction of section 8.3.2 to be based on elliptically fibered Calabi-Yau fourfolds. In doing this we will denote the base of the elliptically fibered Calabi-Yau Y_4 by B_3. Recall that the Calabi-Yau condition for an elliptic fibration is (5.33), which can be written more precisely as

$$12c_1(B_3) = \mathrm{PD}_{B_3}([\Delta]) \ , \tag{8.54}$$

where we have stressed that Poincaré duality is performed with respect to the base B_3. We note that both $c_1(B_3)$ and $\mathrm{PD}_{B_3}([\Delta])$ have to transform with a negative sign under the antiholomorphic and isometric involution σ. This requirement also ensures that Δ has a finite volume, i.e. $\int_\Delta J \wedge J$ does not vanish.

The two-form associated to the zero section of the elliptic fibration is denoted by ω_0. In this work we will be only dealing with Calabi-Yau fourfold geometries with holomorphic zero sections. Note that ω_0 must transform with a negative sign under σ^*. In fact, as we discussed in section 8.2.2 the homology class of the torus fiber is negative under σ, since σ reverses the orientation of the two-torus. This property can also be seen by noting that the base intersects the fiber exactly once. As we will discuss later, this allows us to perform the uplift by sending the coefficient ϕ^0 in the expansion of J to zero.

As the involution σ also descends to the base to σ_B, the cohomologies of B_3 may be decomposed under the action of σ_B as $H^p(B_3) = H^p_+(B_3) \oplus H^p_-(B_3)$. This means that one can write

$$(\omega_\alpha) = (\omega_{\alpha_+}, \omega_{\alpha_-}) \ , \qquad \alpha_\pm = 1, \dots, h^{1,1}_\pm(B_3) \ , \tag{8.55}$$

where $\omega_{\alpha_\pm}$ are obtained by pulling back elements of $H^2_\pm(B_3)$ to $H^2_\pm(Y_4)$.

We will also allow for resolved singularities of the elliptic fibration of Y_4 that correspond to simple non-Abelian gauge groups G in the dual F-theory compactification on Y_4. The location of these non-Abelian singularities defines a divisor S in B_3. In the simple analysis that follows we will assume that there is only one stack of seven-branes on B_3 that describe a non-Abelian gauge group and so S has only one connected component. This significant simplification by no means represents the most general setup which we will not address here. As a result the actions that follow will not represent the most general possibilities.

The Poincaré dual two form $\mathrm{PD}_{B_3}([S])$ lifted to Y_4, admits the expansion $b_S^{\alpha-}\omega_{\alpha-}$ defining constant coefficients $b_S^{\alpha-}$. As noted above, $\mathrm{PD}_{B_3}([\Delta])$ and hence $\mathrm{PD}_{B_3}([S])$ have negative parity under σ so only the $\omega_{\alpha-}$ appear in the expansion. The non-Abelian singularities are resolved by introducing new two-forms ω_i, $i = 1, \dots, \mathrm{rank}(G)$. Assuming the absence of Abelian gauge factors one has $\mathrm{rank}(G) = h^{1,1}(Y_4) - h^{1,1}(B_3) - 1$. Let us note that all $\mathrm{rank}(G)$ forms ω_i are in fact negative under σ^*. To infer this we stress that each exceptional divisor is a $\mathbb{P}^1$-fibration over the seven-brane locus in the base B_3. Within B_3 the seven-brane divisor S and its volume form are positive under σ by

Poincaré duality.[5] Since the antiholomorphic σ reverses the sign of the volume form of the $\mathbb{P}^1$-fiber, we conclude that the exceptional divisors and their Poincaré dual two-forms ω_i are negative under σ. In summary, we find that the two-forms representing $H^2(Y_4)$ are split according to

$$(\omega_{I_+}) = (\omega_{\alpha_+}) , \qquad (\omega_{I_-}) = (\omega_0, \omega_{\alpha_-}, \omega_i) . \tag{8.56}$$

This implies that the truncated spectrum of the three-dimensional $\mathcal{N} = 1$ theory is given by $h^{1,1}_+(B_3)$ vectors A^{α_+}, and $h^{1,1}(Y_4) - h^{1,1}_+(B_3) + h^{3,1}(Y_4)$ scalars $v^{I_-} = (v^0, v^{\alpha_-}, v^i)$ and $\varphi^{\mathcal{K}}$.

One can now systematically study all intersection numbers that are not forbidden by the σ-parity. The intersection numbers of the fourfold will be denoted

$$\mathcal{K}_{IJKL} = \int_{Y_4} \omega_I \wedge \omega_J \wedge \omega_K \wedge \omega_L . \tag{8.57}$$

Since the volume form on Y_4 is positive under σ^* some of them vanish automatically,

$$\mathcal{K}_{I_+ J_+ K_+ L_-} = 0 , \qquad \mathcal{K}_{I_+ J_- K_- L_-} = 0 . \tag{8.58}$$

Combined with the intersection structure on elliptic fibrations one thus finds that for the potential $\hat{K} = K|_{\mathcal{L}_\sigma}$ the relevant non-vanishing intersections are

$$\begin{aligned} \mathcal{K}_{0\alpha_-\beta_-\gamma_-} &\equiv \kappa_{\alpha_-\beta_-\gamma_-} , & \mathcal{K}_{0\alpha_-\beta_+\gamma_+} &\equiv \kappa_{\alpha_-\beta_+\gamma_+} , \\ \mathcal{K}_{ij\alpha_-\beta_-} &= -C_{ij} b_S^{\gamma_-} \kappa_{\gamma_-\alpha_-\beta_-} , & \mathcal{K}_{ij\alpha_+\beta_+} &= -C_{ij} b_S^{\gamma_-} \kappa_{\gamma_-\alpha_+\beta_+} , \end{aligned} \tag{8.59}$$

where $\kappa_{\alpha_-\beta_-\gamma_-}$ and $\kappa_{\alpha_-\beta_+\gamma_+}$ are the triple intersections on B_3. The matrix C_{ij} is the Cartan matrix of the non-Abelian gauge group G. Let us stress that there are numerous other intersection numbers that are in general non-zero on Y_4/σ. In particular, intersection numbers involving $(\omega_0)^n$, $n > 0$ will play a crucial role when matching the F-theory and M-theory reduction at the one-loop level [137, 140, 114].[6] Crucially, this requires a redefinition of the coordinates

$$\hat{\phi}^{\alpha_-} = \phi^{\alpha_-} + \frac{1}{2} K^{\alpha_-} \phi^0, \tag{8.60}$$

where the coefficients K^{α_-} enter the expansion of $c_1(B_3)$ onto the basis $\{\omega_{\alpha_-}\}$,

$$c_1(B_3) = -K^{\alpha_-} \omega_{\alpha_-} . \tag{8.61}$$

Clearly no coefficients K^{α_+} are found because $c_1(B_2)$ is negative under the involution. The shift (8.60) is analogous to the one in (7.129) in the previous chapter and to the shift found in [205].

The splitting of the v^{I_-} coordinates then induces a splitting of the constrained Spin(7) moduli ϕ_{I_-} defined in (8.44). After performing the redefinition (8.60) we may then move into a set of redefined coordinates that are appropriate for performing the F-theory lift. Firstly, ϕ^0 is mapped the length of the interval and we set

$$\frac{1}{r^2} = \phi^0 \mathcal{V}^{-\frac{3}{4}} , \tag{8.62}$$

[5] Recall that formally $\sigma(B_3) = -B_3$, since σ reverses the orientation of B_3.

[6] They can be reduced by repeatedly using $(\omega^0)^2 = -c_1(B_3) \wedge \omega_0$, see (7.4).

where r is the circumference of the circle in $S^1/\mathbb{Z}_2$. Hence, ϕ^0 captures degrees of freedom of the four-dimensional metric. The $\hat{\phi}^{\alpha_-}$ become four-dimensional scalars, while the ϕ^i are the scalar part of four-dimensional vectors with index along the interval $\phi^i_{\mathrm{b}} = A^i_3$. It is convenient to set

$$\phi^{\alpha_-}_{\mathrm{b}} = (\phi^0)^{\frac{1}{3}}\hat{\phi}^{\alpha_-} - \tfrac{1}{2}(\phi^0)^{-\frac{2}{3}} b^\alpha C_{ij}\phi^i\phi^j \ , \qquad \mathcal{V}_{\mathrm{b}} = (\phi^0)^{\frac{1}{2}}\mathcal{V}^{\frac{9}{8}} \ , \qquad \phi^i_{\mathrm{b}} = (\phi^0)^{-1}\phi^i \ . \tag{8.63}$$

These redefinitions can be motivated by the fact that, when taking the F-theory limit with large r, the constraint (8.45) only depends on $\phi^{\alpha_-}_{\mathrm{b}}$, while r and ϕ^i_{b} are unconstrained. In addition, following [206] the vectors A^{α_+} will become four-dimensional scalars with a real shift symmetry. We will consider the lift more explicitly in section 8.4.2.

Let us finally also consider the flux-induced Chern-Simons couplings $\Theta_{I_+J_+}$ and potential F, given in (8.49) and (8.50). From the split (8.56) we infer that the Chern-Simons coupling $\Theta_{\alpha_+\beta_+}$ only involves vectors that become four-dimensional scalars and therefore, by the considerations of [205], have to be absent

$$\Theta_{\alpha_+\beta_+} = 0 \ . \tag{8.64}$$

The real potential F can be expressed in terms of $\Theta_{I_-I_J}$ as

$$F = \int_{Y_4} G_4 \wedge \mathrm{Re}(C\Omega) + \tfrac{1}{8}\mathcal{V}^{-1}\Theta_{I_-J_-}\phi^{I_-}\phi^{J_-} \ . \tag{8.65}$$

Again using (8.56) and following [205] one has to additionally impose

$$\Theta_{00} = 0 \ , \quad \Theta_{0\alpha_-} = 0 \ , \quad \Theta_{0i} = 0 \ , \quad \Theta_{\alpha_-\beta_-} = 0 \ , \quad \Theta_{i\beta_-} = 0 \ . \tag{8.66}$$

This choice of fluxes allows that a four-dimensional theory might exist, no fluxes are included in reduction from four to three dimensions, and the gauge group G is unbroken in four dimensions.[7] The resulting potential F will contain a term that is classical on the F-theory side and a one-loop contribution as we will discuss at the end of the next section.

8.4 F-theory on Spin(7) manifolds

In the previous section we studied M-theory on Spin(7) manifolds and later focused on examples constructed as quotients of elliptically fibered Calabi-Yau fourfolds by an antiholomorphic involution. As a next step we discuss in subsection 8.4.1 the dual interval reduction of a four-dimensional theory. Concretely, we will identify the boundary conditions on various four-dimensional fields on an interval that have to be imposed in order to make a duality of the form (8.1) possible. Aspects of the non-supersymmetric four-dimensional effective theories are discussed in subsection 8.4.2. We particularly focus on the couplings of the uncharged scalar fields that are real both in three and four dimensions and satisfy Neumann boundary conditions at the ends of the interval.

[7] These conditions will be modified in the presence of $U(1)$ gauge factors [137, 140, 114].

8.4.1 Dimensional reduction of the four-dimensional theory on an interval

One of the crucial ingredients of the new kind of M-theory/F-theory duality claimed in (8.1) is the use of an interval in the dimensional reduction from four to three dimensions on the F-theory side of the duality. In this subsection we discuss some general features of dimensional reduction on an interval and consider candidate four-dimensional parent actions.

Due to the presence of an interval $I = S^1/\mathbb{Z}_2$ in (8.1) the uplift of a three-dimensional theory on $\mathcal{M}_3$ to a four-dimensional theory on $\mathcal{M}_4 = \mathcal{M}_3 \times I$ is further complicated, since boundary conditions have to be given for each field. These have to be appropriately specified in order that the duality suggested in (8.1) holds. In the following we will discuss vectors, fermions, and scalars in turn.

Let us first consider a four-dimensional Abelian vector A_m. Since its components satisfy a second-order equation of motion we can choose Dirichlet or Neumann conditions. This choice, however, has to be such that each component of the field strength F_{mn} has a definite parity under the $\mathbb{Z}_2$ action. In particular, inspection of the the mixed component

$$F_{\mu 3} = \partial_\mu A_3 - \partial_3 A_\mu \tag{8.67}$$

reveals that if A_μ satisfies Dirichlet boundary conditions A_3 has to satisfy Neumann boundary conditions, and vice versa. This gives the two choices

$$\begin{array}{lllll} \text{(A)} & D: \quad A_\mu \,|_{\partial\mathcal{M}_4} = 0 & \text{and} & N: \quad \partial_3\, A_3 \,|_{\partial\mathcal{M}_4} = 0 \,, \\ \text{(B)} & D: \quad A_3 \,|_{\partial\mathcal{M}_4} = 0 & \text{and} & N: \quad \partial_3\, A_\mu \,|_{\partial\mathcal{M}_4} = 0 \,, \end{array} \tag{8.68}$$

that may be made without over constraining the equation of motion. When carrying out the interval reduction the Dirichlet boundary conditions will remove the would-be zero mode of the corresponding four-dimensional field. So fields with Dirichlet boundary conditions will not be seen in the three-dimensional effective theory. This implies that reduction of A_m can yield either one massless scalar or one massless vector in the three-dimensional effective action, but not both. This fact can be extended to non-Abelian gauge fields for a four-dimensional gauge group G. To do this let us denote the generators of the algebra of G by $(T_i, T_{\mathcal{I}})$, with T_i labeling the Cartan generators. Then for each vector $A^i_m, A^{\mathcal{I}}_m$ one can choose different boundary conditions.

To conform with the theory arising in the Spin(7) reduction it turns out that one needs to chose option (A) in (8.68) for the Cartan vectors to keep three-dimensional scalars $\phi^i_{\mathrm{b}} = A^i_3$ and option (B) for the non-Cartan vectors in order to keep three-dimensional vectors $A^{\mathcal{I}}_\mu$.[8] In this case one notes that the non-Cartan three-dimensional vectors $A^{\mathcal{I}}_\mu$ acquire a mass term for which the mass is determined by the vacuum expectation value of the three-dimensional massless scalars ϕ^i_{b}. This mass term arises in the effective theory from the reduction of the gauge kinetic term. This analysis is consistent with the fact that the three-dimensional theory arising in the reduction described in section 8.3.3 is a Wilsonian effective action with no non-Cartan vectors and only the scalars ϕ^i, $i = 1, \dots, \mathrm{rank}(G)$. Let

[8] These boundary conditions imply that the gauge coupling constant should be effectively assigned odd parity under the $\mathbb{Z}_2$ action.

us stress, however, that we are still able to extract the classical couplings using the Spin(7) reduction by uplifting the couplings of the scalars $\phi^i_{\rm b}$. The Lorentz transformations and gauge transformations of the four-dimensional vector mix all components of $A^i_m, A^{\mathcal{I}}_m$ and thus allow to recover the couplings of the four-dimensional vectors from the couplings of $\phi^i_{\rm b}$, for a large interval on which these symmetries are restored.

Let us next consider a four-dimensional fermion given by a Majorana spinor χ. Since its equations of motion are first-order, we can only impose a Dirichlet boundary condition of the form

$$\tfrac{1}{2}(1 \pm \gamma^3)\chi\,|_{\partial\mathcal{M}_4} = 0 \tag{8.69}$$

without over constraining the dynamics. The sign is related to the intrinsic parity of the spinor under the $\mathbb{Z}_2$ action on the interval. For both choices, reduction of χ furnishes a massless Majorana spinor in the three-dimensional effective action. This implies that when focusing on zero modes, the degrees of freedom of the fermions are halved. However, there is no ambiguity when uplifting a fermion from three to four dimensions. Four-dimensional Lorentz invariance implies that the three-dimensional dynamics of the spinor encodes its four-dimensional couplings. A similar argument applies to the gravitino.

The comparison can, however, be more involved if the four-dimensional fermion is charged under the gauge group G. In an interval reduction the Coulomb branch scalars can give dimensionally reduced fermions a mass proportional to $\phi^i_{\rm b}$ if the coupling to $\phi^i_{\rm b}$ is non-vanishing. This implies that these fermions are not part of the low-energy effective theory and have to be integrated out. As with the vectors we find that the Cartan fermions remain dynamical in the three-dimensional low-energy effective theory. These then comprise the three-dimensional, $\mathcal{N} = 1$ supersymmetric partners of $\phi^i_{\rm b}$ moduli.

Finally, we turn to the reduction of a four-dimensional scalar field ϕ with standard two-derivative action yielding a second-order equation of motion. As a result, we can impose Dirichlet or Neumann boundary conditions

$$\phi\,|_{\partial\mathcal{M}_4} = 0 \qquad \text{or} \qquad \partial_3\,\phi\,|_{\partial\mathcal{M}_4} = 0 \tag{8.70}$$

without over constraining the equation of motion. As a result the degree of freedom of a four-dimensional scalar might be entirely lost (for Dirichlet b.c.) or kept (for Neumann b.c.) when considering only the zero mode in the three-dimensional effective theory. This is in contrast to the vectors and fermions discussed above. In other words, one can add an arbitrary number of Dirichlet scalars to a candidate four-dimensional action without changing the three-dimensional effective theory on a small interval.

These features of interval reductions lead us to first specify a minimal four-dimensional Lorentz invariant Ansatz for the four-dimensional action containing only those couplings that can be uniquely fixed by comparison with the three-dimensional $\mathcal{N} = 1$ zero mode action. This non-supersymmetric theory is given to quadratic order in the fermions by

$$\begin{aligned} S^{(4)}_{\rm min} = \int d^4x\, e \Big[&- \tfrac{1}{2}R - \tfrac{1}{2}\,\mathcal{G}_{\mathcal{AB}}\,\partial_m\varphi^{\mathcal{A}}\partial^m\varphi^{\mathcal{B}} - \tfrac{1}{4}f\,\mathrm{Tr}(F_{mn}F^{mn}) - V^{(4)} \\ &- \tfrac{1}{2}\bar\psi_m\gamma^{mnr}D_n\psi_r - \tfrac{1}{2}\mathcal{G}_{\mathcal{AB}}\bar\chi^{\mathcal{A}}\gamma^m D_m\chi^{\mathcal{B}} - \tfrac{1}{2}f\,\mathrm{Tr}(\bar\lambda\gamma^m D_m\lambda) + \tfrac{1}{4}f\,\bar\psi_m\gamma^{rs}\gamma^m\mathrm{Tr}(\lambda F_{rs}) \\ &+ \tfrac{1}{2\sqrt{2}}\mathcal{G}_{\mathcal{AB}}\bar\psi_m\gamma^n\gamma^m\chi^{\mathcal{A}}D_n\varphi^{\mathcal{B}} + \tfrac{1}{2}A^1\bar\psi_m\gamma^{mn}\psi_n + \tfrac{1}{\sqrt{2}}A^2_{\mathcal{A}}\bar\psi_m\gamma^m\chi^{\mathcal{A}} \\ &- \tfrac{1}{2}A^3_{\mathcal{AB}}\bar\chi^{\mathcal{A}}\chi^{\mathcal{B}} + \tfrac{1}{4\sqrt{2}}A^4_{\mathcal{A}}\,\mathrm{Tr}(F_{mn}\bar\lambda)\gamma^{mn}\chi^{\mathcal{A}} - \tfrac{1}{2}\mathcal{G}^{\mathcal{AB}}A^4_{\mathcal{A}}\,A^2_{\mathcal{B}}\mathrm{Tr}(\bar\lambda\lambda)\Big]\,, \end{aligned} \tag{8.71}$$

where the covariant derivatives of the Majorana fermions are given by

$$\begin{aligned} D_m\psi_n = \partial_m\psi_n + \tfrac{1}{4}\omega_{mrs}\gamma^{rs}\psi_n\,, \qquad & D_m\lambda = \partial_m\lambda + \tfrac{1}{4}\omega_{mrs}\gamma^{rs}\lambda + [A_m,\lambda]\,, \\ D_m\chi^{\mathcal{A}} = \partial_m\chi^{\mathcal{A}} + \tfrac{1}{4}&\omega_{mrs}\gamma^{rs}\chi^{\mathcal{A}} + D_m\phi^{\mathcal{B}}\Gamma_{\mathcal{BC}}{}^{\mathcal{A}}\chi^{\mathcal{C}}\,. \end{aligned} \tag{8.72}$$

In this action $G_{\mathcal{AB}}$ is a real metric for the scalar target space and $V^{(4)}$, f are real functions of the scalars $\varphi^{\mathcal{A}}$. In addition to this A^1, $A^2_{\mathcal{A}}$, $A^3_{\mathcal{AB}}$ and $A^4_{\mathcal{A}}$ are further functions of $\varphi^{\mathcal{A}}$ that will later be determined by comparing the reduction of this action with the three-dimensional result. As this action is not supersymmetric we could in principle have made a much more general proposal for the couplings that appear. However, it will turn out that (8.71) is sufficiently general to allow for a matching with the three-dimensional theory to be performed. For convenience we note here that performing this calculation one finds that the potential is given in terms of a real function $\mathcal{F}$ by

$$V^{(4)} = 2G^{\mathcal{AB}}\partial_{\mathcal{A}}\mathcal{F}\partial_{\mathcal{B}}\mathcal{F} - 3\mathcal{F}^2\,, \tag{8.73}$$

and that the A functions are given in terms of $\mathcal{F}$ and f by

$$A^1 = \mathcal{F}\,, \qquad A^2_{\mathcal{A}} = \partial_{\mathcal{A}}\mathcal{F}\,, \qquad A^3_{\mathcal{AB}} = D_{\mathcal{A}}\partial_{\mathcal{B}}\mathcal{F} - \tfrac{1}{2}\mathcal{G}_{\mathcal{AB}}\mathcal{F}\,. \qquad A^4_{\mathcal{A}} = \partial_{\mathcal{A}}f\,. \tag{8.74}$$

The action $S^{(4)}_{\rm min}$ given in (8.71) should be used with caution. It was constructed as the minimal functional consistent with four-dimensional Lorentz invariance that yields the three-dimensional action upon interval reduction. Note that this construction does not ensure conservation of the currents coupling to gravitini and gauge fields. Recall from section 1.4 that this is needed in a consistent theory of massless spin one or two particles in the purely bosonic case [31] and also spin 3/2 particles in the supersymmetric case [32]. Furthermore, we point out that the interpretation of (8.71) as a Wilsonian effective action is questionable, since it might not capture the dynamics of all light degrees of freedom. All scalars satisfying Dirichlet boundary conditions have only massive excitations for a finite interval length and do not enter the action (8.71). If they are actually present in the four-dimensional spectrum, however, they become arbitrarily light as the interval grows large.

This puzzle can be solved in the cases in which the Spin(7) quotients admit a weak-coupling limit of the kind discussed below in section 8.5. The outcome of our analysis suggests the following picture. A possible four-dimensional Wilsonian effective action $S^{(4)}_{\rm W}$ completing $S^{(4)}_{\rm Min}$ on a large interval could be given by a $\mathcal{N}=1$ Lagrangian $\mathcal{L}^{(4)}_{\mathcal{N}=1}$ for F-theory on the original Calabi-Yau space Y_4 supplemented by the boundary conditions or a boundary action $\mathcal{L}^{(3)}$. Hence, it takes the form

$$S^{(4)}_{\rm W} = \int_{\mathcal{M}_4}\mathcal{L}^{(4)}_{\mathcal{N}=1} + \int_{\partial\mathcal{M}_4}\mathcal{L}^{(3)}\,. \tag{8.75}$$

The restoration of the Calabi-Yau moduli space from the moduli space of the Spin(7) manifold in the large interval limit is very non-trivial. We have strong evidence to support this claim in the weakly-coupled setups of section 8.5 but we are not able prove that this is a general feature of Spin(7) manifolds obtained from quotients of Calabi-Yau fourfolds.

Let us stress that the action (8.71) neglects the couplings of charged matter that will be present in a general F-theory compactification. Furthermore, we have not displayed the terms of higher order in the fermions. These can be added by making an Ansatz for these couplings and reducing them to three dimensions with the boundary conditions described above. The coefficients are then determined by comparing the zero mode result to a general three-dimensional $\mathcal{N}=1$ theory in which the higher fermionic couplings are known in terms of the three-dimensional $\mathcal{N}=1$ characteristic functions determined by the reduction of the terms in (8.71).

Let us conclude this section by noting that the inclusion of Dirichlet scalar is not the only ambiguity in the uplift from three to four dimensions. One has also to analyze carefully the uplift of three-dimensional scalars, since they can come both from a scalar or a vector in four dimensions. This issue, however, is not specific to the Spin(7) setup and indeed already appears in the more familiar case of F-theory on a Calabi-Yau fourfold. As we will see in the next section, we can solve all such ambiguities in the setup under consideration appealing to the Calabi-Yau geometry that underlies the Spin(7) quotients we study.

8.4.2 Effective action of F-theory on Spin(7) manifolds

Having described the three-dimensional effective theory obtained for the quotient torus fibered Spin(7) geometry in subsections 8.3.2 and 8.3.3 and the details on the interval reduction in subsection 8.4.1 we are now in the position to perform the reduction and read off the couplings of the four-dimensional theory (8.71). Clearly, proposing that the coupling functions take the same form in the four-dimensional theory is a speculative part of the analysis. It amounts on the one hand to sending the size of the interval I to infinity, and on the other hand shrinking the fiber volume. This means that one has to be performing the M-theory to F-theory limit. In supersymmetric F-theory compactifications it has become clear over the last years [206, 137, 234] that many couplings in the three-dimensional theory obtained from M-theory appear to also have an F-theory interpretation. Motivated by these advances we perform a similar oxidation for the Spin(7) compactification. However, it should be stressed that we will only talk about zero modes in the following and many of the subtleties are, in fact, hidden in the treatment of massive modes.

The first step is to implement the F-theory limit explicitly. Note that not all couplings arising in the M-theory reduction are classical from the F-theory perspective on a small compact space. Various couplings can be induced at loop level when integrating out massive Coulomb branch and Kaluza-Klein modes. To extract the classical terms only, we proceed as in section 7.6.1 and we assign suitable scalings to three-dimensional fields. In analogy with (7.155) the correct scalings are

$$v^0 \rightarrow \epsilon v^0 , \qquad v^{\alpha-} \rightarrow \epsilon^{-1/2} v^{\alpha-} , \qquad v^i \rightarrow \epsilon^{1/4} v^i , \qquad r \rightarrow \epsilon^{-3/4} r . \tag{8.76}$$

They ensure precisely that the couplings with intersection numbers (8.59), i.e. $\mathcal{K}_{0\alpha_-\beta_-,\gamma_-}, \mathcal{K}_{0\alpha_-\beta_+,\gamma_+}$ and $\mathcal{K}_{ij\alpha_-\beta_-}, \mathcal{K}_{ij\alpha_+\beta_+}$ are surviving the $\epsilon \to 0$ limit. Translated into the coordinates ϕ^{I-} one thus finds

$$\phi^0 \;\to\; \epsilon^{9/8}\phi^0\;, \qquad \phi^{\alpha-} \;\to\; \epsilon^{-3/8}\phi^{\alpha-}\;, \qquad \phi^i \;\to\; \epsilon^{3/8}\phi^i\;, \qquad \mathcal{V} \;\to\; \epsilon^{-1/2}\mathcal{V}\;. \tag{8.77}$$

Combining these scalings with the coordinate redefinitions (8.63) one extracts the leading terms of all fields. We first introduce the $\phi_{\rm b}^{\alpha-}$ defined as the leading term in (8.63). In the limit the normalization constraint (8.45) translates to the condition

$$N_{\rm b} \equiv \tfrac{1}{3!}\kappa_{\alpha_-\beta_-\gamma_-}\phi_{\rm b}^{\alpha-}\phi_{\rm b}^{\beta-}\phi_{\rm b}^{\gamma-} = 1\;. \tag{8.78}$$

This implies that only $h^{1,1}_-(B_3)-1$ coordinates $\phi_{\rm b}^{\alpha-}$ are independent. The missing degree of freedom is encoded by the base volume $\mathcal{V}_{\rm b}$ arising as leading term in the definition (8.63). After the $\epsilon \to 0$ limit the resulting three-dimensional action can be matched with a the reduction of a four-dimensional theory reduced on an interval of length r with boundary conditions introduced in subsection 8.4.1. This allows us to read off the data of the four-dimensional theory from the three-dimensional action.

We first note that all couplings containing three-dimensional vectors or fermions are formally lifted from three-dimensional to four-dimensional in a Lorentz compatible way. For example, the kinetic terms in (8.14) for the three-dimensional fermions $\chi^{\alpha-}$, which are in the same three-dimensional, $\mathcal{N}=1$ multiplets as the scalars $\phi_{\rm b}^{\alpha-}$, are given by

$$\tfrac{1}{2}\tilde{g}_{\alpha_-\beta_-}\bar{\chi}^{\alpha-}\not{D}\chi^{\beta-}\;. \tag{8.79}$$

These are lifted by completing the $\chi^{\alpha-}$ into four-dimensional fermions and matching $\tilde{g}_{\alpha_-\beta_-}$ with the reduction of the equivalent four-dimensional terms after performing the reduction and Weyl rescaling as well as implementing the $\epsilon \to 0$ limit with (8.77). In this way we can read off

$$\mathcal{G}_{\alpha_-\beta_-} = (\tilde{g}_{\alpha_-\beta_-})_{\epsilon=0} = 4\mathcal{V}_{\rm b}^3\int_{B_3}\xi^{\rm b}_{\alpha_-}\wedge *\xi^{\rm b}_{\beta_-}\;, \tag{8.80}$$

where the four-forms $\xi^{\rm b}_{\alpha_-}$ are given by

$$\xi^{\rm b}_{\alpha_-} = \tfrac{1}{4}\mathcal{V}_{\rm b}^{-\frac{4}{3}}P_{\alpha_-}{}^{\gamma_-}\omega_{\gamma_-}\wedge\omega_{\beta_-}\phi_{\rm b}^{\beta-}\;, \qquad P_{\alpha_-}{}^{\beta_-} = \delta_{\alpha_-}{}^{\beta_-} - \tfrac{1}{3!}\kappa_{\alpha_-\gamma_-\delta_-}\phi_{\rm b}^{\gamma-}\phi_{\rm b}^{\delta-}\phi_{\rm b}^{\beta-}\;. \tag{8.81}$$

The other components of the four-dimensional scalar metric $G_{\mathcal{AB}}$ appearing in (8.71) may then be deduced in a similar way by expanding $\varphi^{\mathcal{A}} = (\mathcal{V}_b, \phi^{\alpha-}, \varphi^{\mathcal{K}}, \zeta_{\alpha_+})$ and making the comparison with (8.14) and (8.46). This gives $\mathcal{G}_{\mathcal{V}_b\mathcal{V}_b} = \frac{4}{6}\mathcal{V}_{\rm b}^{-2}$ and

$$\mathcal{G}_{\mathcal{KL}} = (\tilde{g}_{\mathcal{KL}})_{\epsilon=0} = 4\mathcal{V}_{\rm b}^3\int_{B_3}\xi^{\rm b}_{\mathcal{K}}\wedge *\xi^{\rm b}_{\mathcal{L}}\,, \quad \mathcal{G}^{\alpha_+\beta_+} = (h_{\alpha_+\beta_+})^{-1}_{\epsilon=0} = \left[\frac{1}{2\mathcal{V}_b}\int_{B_3}\omega_{\alpha_+}\wedge *\omega_{\beta_+}\right]^{-1}\,, \tag{8.82}$$

Next we can consider the comparison of the kinetic terms for the scalars ϕ^i with the reduction of the four-dimensional vector kinetic terms. In this way we find that the coupling function f is given by

$$fC_{ij} = (r^2 g_{ij})_{\epsilon=0} = \mathcal{V}_{\rm b}^{2/3}C_{ij}b_S^{\alpha-}\kappa_{\alpha_-\beta_-\gamma_-}\phi_{\rm b}^{\beta-}\phi_{\rm b}^{\gamma-}\;. \tag{8.83}$$

Similarly the reduction of the potential for the four-dimensional theory may be compared with the general three-dimensional $\mathcal{N} = 1$ result (8.15) from which we find (8.73) where the function $\mathcal{F}$ is related to the function F, which determines the potential of the quotiented Calabi-Yau reduction, by

$$\mathcal{F} = (rF)_{\epsilon=0} = \Big(e^{K^{\mathrm{F}}/2} \int_{Y_4} \mathrm{Re}(\Omega) \wedge G_4\Big)_{\mathcal{L}_\sigma} . \tag{8.84}$$

where $K^{\mathrm{F}} = -2\log \mathcal{V}_{\mathrm{b}} - \log \int_{Y_4} \Omega \wedge \bar{\Omega}$. Finally we note that by comparing the fermionic couplings in the reduction of (8.71) with (8.14) we find (8.74).

In the preceding analysis we did not include charged matter. Clearly, in a general F-theory compactification with fluxes chiral matter will be part of the four-dimensional massless spectrum. This matter can become massive when dimensionally reduced on an interval if the scalars ϕ^i_{b} get a vacuum expectation value. This implies that these have to be integrated out in the three-dimensional low-energy effective theory. We have already seen this mechanism at work in six dimensions in the previous chapter. It is also present in the context of F-theory on elliptically fibered Calabi-Yau fourfold. In this case chiral matter generates one-loop corrections to the Chern-Simons terms of the three-dimensional $\mathcal{N} = 2$ action [137, 140, 114], in very much the same spirit as what happens in five dimensions, see chapter 9.

Since we are now considering a three-dimensional $\mathcal{N} = 1$ action the appearance of quantum effects due to massive states will be different. In particular we expect that part of the three-dimensional potential F will admit a one-loop term

$$F \supset F^{\mathrm{class}} + F^{1-\mathrm{loop}} . \tag{8.85}$$

This classical term will lift to the four-dimensional superpotential (8.84) in our simple configurations with only one unbroken non-Abelian gauge group. The one-loop term can be obtained by considering the general Spin(7) potential $\mathcal{F}$ with (8.50), imposing that up-lift conditions (8.66), and keeping the term that vanish in the limit $\epsilon \to 0$. This leads to the identification

$$F^{1-\mathrm{loop}} \overset{?}{=} \tfrac{1}{8}\mathcal{V}^{-2} \int_{Z_4} J \wedge J \wedge G_4 = \tfrac{1}{8}\mathcal{V}^{-1}\Theta_{ij}\phi^i\phi^j . \tag{8.86}$$

It would be very interesting to check this match for an explicit example by computing both the general one-loop contribution in field theory and the flux intersection Θ_{ij} of the form (8.23).

8.5 Quotients in the weak-coupling limit

In this section we discuss the realization of Spin(7) geometries as quotients of elliptically fibered fourfolds in the Sen's weak-coupling limit introduced in section 5.4. In particular we analyze the impact of the holomorphic quotient associated to Sen's limit on the Spin(7) geometry. We will be thus able to establish an explicit connection between the Type IIB orientifold action and a $\mathbb{Z}_2$ symmetry acting on the M-theory geometry. This information will allow us to address the problem of Dirichlet scalars in the interval uplift raised in section 8.4.1.

8.5.1 Sen's limit and the antiholomorphic quotient

Sen's limit has been introduced for a general elliptically fibered Calabi-Yau $(n+1)$-fold in section 5.4: all results quoted there can be applied to the present situation. For our present purposes it is convenient to manipulate the Weierstrass equation in the patch $z \neq 0$, given in (5.22), as follows. In the limit $C \to 0$ equation (5.22) is conveniently rewritten in terms of the new coordinates

$$x = h\tilde{x} \; , \qquad y = h^{\frac{3}{2}}\tilde{y} \; , \tag{8.87}$$

where h is the polynomial on the base that encodes the position of the O7-plane in Sen's limit, see (5.38) and (5.41). Indeed, (5.22) reformulated in terms of $\tilde{x}$, $\tilde{y}$ reads

$$\tilde{y}^2 = \tilde{x}^3 - 3\tilde{x} - 2 \; , \tag{8.88}$$

which is manifestly independent of the base coordinates.

The harmonic one form of the torus $\Omega_1 = \frac{dx}{y}$ is given in terms of these rescaled coordinates by $\Omega_1 = h^{-\frac{1}{2}}\frac{d\tilde{x}}{\tilde{y}}$. The O7-action may then be seen by moving once around $h = 0$ and noting that $\Omega_1 \to -\Omega_1$. The Calabi-Yau threefold Y_3 which is present in the weak-coupling limit is then the double cover of the base B_3 such that Ω_1 becomes single valued. This can be made more explicit as follows. Recall from section 5.4 that the Calabi-Yau threefold Y_3 is described by introducing a new coordinate ξ and a new equation (5.42), which we record here again for convenience,

$$Y_3 \;\; : \xi^2 = h \; . \tag{8.89}$$

Recall also that the holomorphic orientifold involution is given by

$$\sigma_{\rm h} : Y_3 \to Y_3 \; , \qquad \xi \to -\xi \; , \tag{8.90}$$

and has O7-planes at the fixed points given by $h = 0$. Formally lifting Ω_1 from the base to its double cover Y_3 we may then write $\Omega_1 = \frac{d\tilde{x}}{\xi\tilde{y}}$ and see the consistency of the O7-monodromy action $\Omega_1 \to -\Omega_1$ with the map $\xi \to -\xi$.

Next we observe that we can write Ω_1 as $\Omega_1 = dz$ where z is the complex coordinate of the torus. If the two independent one-cycles of the torus are denoted A and B with corresponding real coordinates x_A and x_B, then the complex coordinate z reads $z = x_A + \tau x_B$, with τ the complex structure parameter of the torus. This shows that the action of the holomorphic involution (8.90) induces a reflection R_{AB} of the coordinates of the A and B cycles given by $(x_A, x_B) \to (-x_A, -x_B)$. This formal geometric action on the the torus coordinates encodes the intrinsic parities of the Type IIB fields under the orientifold involution.

As a further step we study these effects in a setups in which the Calabi-Yau fourfold is also quotiented by an antiholomorphic involution σ. By considering the action of the different involutions on the ambient space of the fiber and demanding the invariance of the polynomial which defines the Calabi-Yau fourfold we can deduce the action of σ on the Weierstrass coefficients and the functions which appear in the weak-coupling limit. We have already formulated suitable conditions on f and g in (8.11). When Sen's limit is considered these are supplemented with

$$h_{\sigma_B(p)} = \overline{h_p} \ , \qquad \eta_{\sigma_B(p)} = \overline{\eta_p} \ , \qquad \chi_{\sigma_B(p)} = \overline{\chi_p} \ , \tag{8.91}$$

where p is an arbitrary point on the base B_3. Note that we do not have a rigorous proof that (8.11) and (8.91) must always be satisfied in order to ensure compatibility between the fibration structure and the antiholomorphic involution. Nonetheless, we have found this to be the case in all examples we have constructed using simple involutions on hyper-surfaces in toric ambient spaces.

We now introduce an antiholomorphic involution

$$\sigma_{\rm ah} : Y_3 \to Y_3 \ , \tag{8.92}$$

induced by σ. However, we must note that the action of σ_B on h does not uniquely determine the action of $\sigma_{\rm ah}$ on ξ which can either be $\xi \to \bar{\xi}$ or $\xi \to -\bar{\xi}$. Both choices are related by $\sigma_{\rm h}$ given in (8.90) and without loss of generality we can choose $\sigma_{\rm ah}$ to act as $\xi \to \bar{\xi}$. As a consequence the action of $\sigma_{\rm ah}$ on the uplift of Ω_1 is given by $\Omega_1 \to \bar{\Omega}_1$. Writing Ω_1 in terms of x_A and x_B and combining the action of the two involutions $\sigma_{\rm h}$ and $\sigma_{\rm ah}$ on Ω_1 and τ we find the corresponding actions R_{AB}, R_A, and R_B on the coordinates (x_A, x_B) of the A and B cycles. The set of combined quotients in the weak limit may then be summarized by

$$\begin{aligned} \sigma_{\rm h}:&\ (u_i,\xi) \to (u_i,-\xi) \ , & R_{AB}:&\ (x_A,x_B) \to (-x_A,-x_B), \\ \sigma_{\rm ah}:&\ (u_i,\xi) \to (\sigma_B(u_i),\bar{\xi}) \ , & R_B:&\ (x_A,x_B) \to (x_A,-x_B), \\ \sigma_{\rm h}\sigma_{\rm ah}:&\ (u_i,\xi) \to (\sigma_B(u_i),-\bar{\xi}) \ , & R_A:&\ (x_A,x_B) \to (-x_A,x_B), \end{aligned} \tag{8.93}$$

where u_i denote collectively the coordinates on the base space B_3 and each line lists the action on Y_3 along with the formally induced reflection on an auxiliary T^2. By considering the form of these quotients we see that $\sigma_{\rm h}$ and $\sigma_{\rm ah}$ always commute on bosons and that the dimension of the fixed space of $\sigma_{\rm ah}$ in Y_3 is always the same as the dimension of the fixed space of the product $\sigma_{\rm h}\sigma_{\rm ah}$. We note that in the case (b), in which σ_B has a one-dimensional fixed space, the orbifold singularities of B_3 must also be up-lifted to the double cover Y_3. One can analyze these singularities in local patches analogously to the description given in section 8.2.2.

8.5.2 Intrinsic parities of Type IIB fields from M-theory

Let us now investigate how the geometric reflections R_A, R_B, and R_{AB} in (8.93) can be translated into intrinsic parities of Type IIB fields associated to the corresponding geometric actions $\sigma_{\rm h}$, $\sigma_{\rm ah}$, and $\sigma_{\rm h}\sigma_{\rm ah}$. Recall that M-theory/F-theory duality predicts that the B-cycle is identified, after T-duality to Type IIB, with the fourth direction of spacetime that grows large in the F-theory limit, which will be denoted x^3. Furthermore, we need the results (3.63), (3.64), and (3.67) derived in section 3.4 studying the duality between M-theory on a torus and Type IIB on a circle. By means of the usual adiabatic argument we can extend the validity of those relations to a non-trivial fibration of the torus over the base space B_3.

Type IIB	$\Omega_\mathrm{p}\,(-1)^{F_L}$	$R_3\,(-1)^{F_L}$	$R_3\,\Omega_\mathrm{p}$	M-theory	R_{AB}	R_B	R_A
Φ	$+$	$+$	$+$	g_{AA}	$+$	$+$	$+$
$g_{\mu\nu}$	$+$	$+$	$+$	$g_{\mu\nu}$	$+$	$+$	$+$
$g_{\mu 3}$	$+$	$-$	$-$	$C_{\mu AB}$	$+$	$-$	$-$
g_{33}	$+$	$+$	$+$	g_{BB}	$+$	$+$	$+$
$B_{\mu\nu}$	$-$	$+$	$-$	$C_{\mu\nu A}$	$-$	$+$	$-$
$B_{\mu 3}$	$-$	$-$	$+$	$g_{\mu B}$	$-$	$-$	$+$
C_0	$+$	$-$	$-$	g_{AB}	$+$	$-$	$-$
$C_{\mu\nu}$	$-$	$-$	$+$	$C_{\mu\nu B}$	$-$	$-$	$+$
$C_{\mu 3}$	$-$	$+$	$-$	$g_{\mu A}$	$-$	$+$	$-$
$C_{\mu\nu\rho 3}$	$+$	$+$	$+$	$C_{\mu\nu\rho}$	$+$	$+$	$+$

Table 8.1: Summary of all components of bosonic fields of Type IIB with parities under the transformations $\Omega_\mathrm{p}\,(-)^{F_L}$, $R_3\,(-)^{F_L}$, and $R_3\,\Omega_\mathrm{p}$. For each component the M-theory origin is provided together with its parities under the transformations R_{AB}, R_B, and R_A. By slight abuse of notation indices μ, ν, ρ refer both to the three external non-compact directions of spacetime and to the internal directions along the base space B_3. On the Type IIB side the index 3 refers to the direction that grows large in the F-theory limit. On the M-theory side the labels A, B denote components along the A- and B-cycle of the torus, respectively. The former is the M-theory circle, the latter is the T-duality circle. The components $C_{\mu\nu\rho\sigma}$ of C_4 in Type IIB are not listed in the table as their are not independent by virtue of the ten-dimensional self-duality constraint (2.26).

Let R_3 be the reflection of the x^3 direction on the Type IIB side of the duality. We then have the following correspondence:

$$\begin{array}{ccc} \text{M-theory} & & \text{Type IIB} \\ R_{AB} & \leftrightarrow & \Omega_\mathrm{p}\,(-1)^{F_L} \\ R_B & \leftrightarrow & R_3\,(-1)^{F_L} \\ R_A & \leftrightarrow & R_3\,\Omega_\mathrm{p} \end{array} \tag{8.94}$$

To prove this correspondence we record in table 8.1 all components of Type IIB bosonic fields together with their M-theory origins, inferred from (3.63), (3.64), and (3.67). As we can see, computing the parity of all the Type IIB components using the Type IIB actions $\Omega_\mathrm{p}\,(-)^{F_L}$, $R_3\,(-)^{F_L}$, $R_3\,\Omega_\mathrm{p}$ gives exactly the same result as computing the parity of the associated M-theory fields under the reflections R_{AB}, R_B, R_A. This observation will be the starting point of our detailed analysis of Type IIB setups originating from the weak-coupling limit of quotient Spin(7) manifolds.

8.5.3 Remarks on the geometry of fixed loci in M-theory

It is interesting to comment on the M-theory background that corresponds to the weak-coupling limit we have described. Clearly one could compactify M-theory on Z_8 directly and should recover the above weak-coupling setup as a specific limit in the geometric moduli space. However one may instead follow the prescription above by first going to the Sen limit of Y_4 and then considering the additional quotient by σ. Having done this we will then take a further limit in which the M-theory circle becomes small and may then consider the set of effective quotients in Type IIA. The local geometry near the fixed points of the various involutions can then be analyzed separately.

The holomorphic involution $\sigma_{\rm h}$ has a four-dimensional fixed space on Y_3. Cutting out a patch of the two-dimensional space normal to this fixed locus and considering the T^2 fibers over it we obtain a four-dimensional space that is locally of the form

$$(S^1_A \times S^1_B \times \mathbb{R}^2)/\mathbb{Z}_2 \ , \tag{8.95}$$

where $\mathbb{R}^2$ represents the normal space on Y_3 and S^1_A, S^1_B are independent cycles of the elliptic fiber such that S^1_A is the M-theory circle and S^1_B is the circle along which one applies T-duality to go to F-theory. Let us recall that the geometry of the normal space of a lifted O6-plane in M-theory is asymptotically given by $(S^1_A \times \mathbb{R}^3)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ inverts all coordinates simultaneously. We may then infer that (8.95) signals the presence of an O6-plane localized at a point along the circle S^1_B. This result is well known and is consistent with the fact that in Type IIB the holomorphic action is associated with the presence of O7-planes in the geometry.

Similarly we can consider the fixed-point sets of the antiholomorphic involution. In doing this we will focus on case (a) where the fixed space of σ_B is three-dimensional. It is then convenient to combine the actions $\sigma_{\rm ah}$ and $\sigma_{\rm h}\sigma_{\rm ah}$ with the induced reflections R_B and R_A to form the products $\sigma_{\rm ah}R_A$ and $\sigma_{\rm h}\sigma_{\rm ah}R_B$. The normal space to the fixed-point sets of these total actions is locally given by

$$(S^1_B \times \mathbb{R}^3)/\mathbb{Z}_2 \quad \text{and} \quad (S^1_A \times \mathbb{R}^3)/\mathbb{Z}_2 \ , \tag{8.96}$$

respectively. The corresponding Type IIA objects are then given by a six-dimensional orbifold plane Orb5 and a O6-plane that wraps the S^1_B cycle. We will comment on this setup in more detail in the next section. One can also perform this analysis for the case in which σ_B has a one-dimensional fixed space. The objects that arise in this situation will be discussed in section 8.6.3.

8.5.4 More on Dirichlet scalars

In section 8.4.1 we have pointed out the ambiguity in the interval uplift related to the possibility of adding four-dimensional scalars with Dirichlet boundary conditions without affecting the low-energy theory for three-dimensional zeromodes. We can now revisit this issue from the perspective of the antiholomorphic quotient $\sigma_{\rm ah}$ of the Type IIB weakly coupled setup.

To begin with, note that the interval is realized by taking the coordinate x^3 of the circle that grows large in the F-theory limit and acting with the $\mathbb{Z}_2$-action $R_3 : x^3 \mapsto -x^3$. Its fixed points are mapped to the endpoints of the interval. When the interval is realized as $S^1/\mathbb{Z}_2$ in this way, a scalar with Dirichlet boundary conditions is equivalent to a scalar that has an intrinsic negative parity under the action of R_3, or an R_3-odd scalar for short. The use the terminology R_3-even scalar for scalars that have intrinsic positive parity.

After this preliminary remark we can analyze the uplift of the three-dimensional action for Kähler and complex structure moduli in turn. This will establish a match of the three-dimensional Spin(7) moduli with the R_3-even scalars of the four-dimensional theory. Let us start with the Kähler moduli. In Type IIB language these are given by

$$T_\alpha = \tfrac{1}{2!}\int_{Y_3} \omega_\alpha \wedge J_{\rm b}^2 + i\int_{Y_3} \omega_\alpha \wedge C_4 \ . \tag{8.97}$$

This expression is the analog of (8.36) but now ω_α and $J_{\rm b}$ are understood as $(1,1)$-forms on the double cover Y_3 of the base B_3. Recall the split introduced before (8.55) of $H^2(B_3)$ that translates into a split of $H^2(Y_3)$ into positive and negative subspaces under the action of the antiholomorphic involution $\sigma_{\rm ah}$.

As a result of the formal relation $\sigma_B^*(B_3) = -B_3$, which can be uplifted to $\sigma_{\rm ah}^*(Y_3) = -Y_3$, an expression of the form $\int_{Y_3}\lambda_6$ survives the $\sigma_{\rm ah}$-projection only if λ_6 is negative under $\sigma_{\rm ah}^*$. As far as $\mathrm{Re}\,T_\alpha$ is concerned, we can use the results of section 8.3.3—uplifted to the double cover Y_3—to infer that

$$\sigma_{\rm ah}^*(\omega_{\alpha_+}\wedge J_{\rm b}^2) = +\omega_{\alpha_+}\wedge J_{\rm b}^2\ , \qquad \sigma_{\rm ah}^*(\omega_{\alpha_-}\wedge J_{\rm b}^2) = -\omega_{\alpha_+}\wedge J_{\rm b}^2\ . \tag{8.98}$$

In order to analyze $\mathrm{Im}\,T_\alpha$ geometric data must be supplemented by the intrinsic parity of the Type IIB four-form C_4 under $\sigma_{\rm ah}$. From the results of sections 8.5.1 and 8.5.2 we know that the geometric action of $\sigma_{\rm ah}$ on Y_3 must be accompanied by the reflection R_B in the auxiliary T^2. The latter reflection is in turn equivalent to $R_3(-1)^{F_L}$ in Type IIB language. The intrinsic parity we need is determined by the $(-1)^{F_L}$ factor, so it is negative for C_4. As a result we have effectively

$$\sigma_{\rm ah}^*(\omega_{\alpha_+}\wedge C_4) = -\omega_{\alpha_+}\wedge C_4\ , \qquad \sigma_{\rm ah}^*(\omega_{\alpha_-}\wedge C_4) = +\omega_{\alpha_-}\wedge C_4\ . \tag{8.99}$$

In summary, we find

$$R_3\text{-even}:\ \mathrm{Re}\,T_{\alpha_-},\ \mathrm{Im}\,T_{\alpha_+}\ , \qquad\qquad R_3\text{-odd}:\ \mathrm{Re}\,T_{\alpha_+},\ \mathrm{Im}\,T_{\alpha_-}\ . \tag{8.100}$$

As anticipated above, the R_3-even scalars match exactly with the three-dimensional moduli that survive the σ quotient on the Calabi-Yau fourfold Y_4 on the M-theory side.

Let us now turn to complex structure moduli. From a Type IIB perspective they correspond to complex structure moduli of the threefold Y_3, D7-brane moduli, and the axio-dilaton. The action of the antiholomorphic involution $\sigma_{\rm ah}$ on Y_3 is such that

$$\sigma_{\rm ah}^*\Omega^{3,0} = e^{2i\theta}\,\overline{\Omega^{3,0}}\ . \tag{8.101}$$

This is completely analogous to the corresponding σ-action on the fourfold Y_4. Imposing (8.101) one infers that the R_3-even complex structure moduli span a real subspace of the four-dimensional $\mathcal{N}=1$ moduli space. With similar arguments it is possible to check the correspondence between three-dimensional Spin(7) moduli and four-dimensional R_3-even moduli related to D7-branes and the axio-dilaton.

It is important to highlight the generic presence of R_3-odd scalars. Such scalar degrees of freedom cannot have a constant non-vanishing profile along the x^3 direction, and therefore do not correspond to moduli in the four-dimensional theory. From a four-dimensional perspective on a finite interval such scalars arise only as massive excitations. In conclusion, we can state that the weakly coupled Type IIB picture suggests that the four-dimensional moduli, which are R_3-even, are in one-to-one correspondence with the Spin(7) moduli in the three-dimensional action (8.46). The interpretation of the R_3-odd scalars is instead related to M2-brane states. We will comment further on this issue in section 8.6.4.

8.6 Weak-coupling setups

In this section we introduce Type IIB and Type IIA string theory setups that can arise in the weak-coupling limit of the geometries introduced in section 8.2.2. In subsection 8.6.1 we first discuss the case in which the fixed-point locus of σ_B is three-dimensional, i.e. the case (a) in (8.7). We find that the Type IIB setup contains O5-planes and exotic orbifold five-planes. The case of a one-dimensional fixed-point set of σ_B, case (b) in (8.8), is discussed in section 8.6.2. This yields exotic orientifold three-planes and orbifold three-planes that we describe in detail on a torus background. In both setups our strategy is to start with a proposed Type IIB setting and then stepwise translate the objects which appear into the T-dual Type IIA setting and finally to the geometry of a Spin(7) manifold. That the unusual objects that we have identified preserve mutual supersymmetry in both setups can be checked explicitly in torus examples as shown in section 8.6.3. Collecting these insights we then comment on the supersymmetry restoration in the large interval limit in section 8.6.4.

8.6.1 Weak-coupling setup with five-planes

The first setting under consideration is obtained by examining Type IIB on the background

$$\mathcal{M}_{10}^{\text{IIB}} = (\mathbb{R}^{1,2} \times S^1 \times Y_3)/G \ , \tag{8.102}$$

where $\mathbb{R}^{1,2}$ is three-dimensional Minkowski space, Y_3 is a Calabi-Yau threefold, and the symmetry group G is generated by the transformations[9]

$$\mathcal{O}_1 = \Omega_{\text{p}}\, \sigma_{\text{h}}\, (-1)^{F_L} \ , \qquad\qquad \mathcal{O}_2 = R_3\, \sigma_{\text{ah}}\, (-1)^{F_L} \ . \tag{8.103}$$

Let us remind the reader that Ω_{p} and F_L are the world-sheet parity and the left-moving spacetime fermion number and that R_3 denotes the reflection of the coordinate x^3 along the S^1 in (8.102). This action turns the circle S^1 into the interval $I = S^1/\mathbb{Z}_2$. The geometric maps σ_{h} and σ_{ah} are a holomorphic and an antiholomorphic involution of the Calabi-Yau threefold Y_3, respectively. Both are demanded to be isometries and required to commute on bosons, as we discuss in more detail below. Of course, this specific choice of Type IIB setup is motivated by the considerations of the previous sections. In particular, the geometric actions σ_{h} and σ_{ah} will be identified with the actions introduced in (8.90) and (8.92), while the stringy factors Ω_{p}, $(-1)^{F_L}$ are introduced according to our findings in sections 8.5.1 and 8.5.2.

[9] We follow the conventions of [46].

Since $\sigma_{\rm h}$ is holomorphic its fixed-point set $H_{\sigma_{\rm h}}$ is holomorphically embedded in Y_3. In order to connect to an F-theory setup we will demand in the following that $H_{\sigma_{\rm h}}$ is complex two-dimensional. This ensures that the fixed points of $\mathcal{O}_1$ are O7-planes extending along $\mathbb{R}^{1,2} \times I$ and wrapping $H_{\sigma_{\rm h}}$. To cancel the tadpoles induced by these negative tension objects the setup should also contain D7-branes filling $\mathbb{R}^{1,2} \times I$. The setting obtained by $\mathcal{O}_1$ is known to arise as the weak-coupling limit of F-theory compactified on a Calabi-Yau fourfold [235, 129], as we already recalled in section 8.5.

The action of $\mathcal{O}_2$ is more unusual as it represents a geometric orbifold action combined with a $(-1)^{F_L}$ action. These sorts of exotic orbifolds have been studied in [236, 237, 238, 239, 240, 241]. Let us note also that the presence of the reflection R_3 is necessary in the $\mathcal{O}_2$ action, since an antiholomorphic involution $\sigma_{\rm ah}$ alone is a Pin-odd transformation and hence would not be a symmetry of the chiral Type IIB string theory. In the following we demand that $\sigma_{\rm ah}$ has a real three-dimensional fixed-point set $L_{\sigma_{\rm ah}}$. The space $L_{\sigma_{\rm ah}}$ is a special Lagrangian submanifold due to the properties of $\sigma_{\rm ah}$. This implies that the fixed-point set of $\mathcal{O}_2$ is real six-dimensional including the non-compact three-dimensional spacetime $\mathbb{R}^{1,2}$. The fixed points of $\mathcal{O}_2$ are located at the ends of the interval I. We call the resulting fixed planes X5-planes and will describe their properties in more detail below.

The geometric actions $\sigma_{\rm h}$ and $\sigma_{\rm ah}$ are required to satisfy the properties

$$\sigma_{\rm h}\, R_3 = R_3\, \sigma_{\rm h}\ , \qquad \sigma_{\rm ah}\, R_3 = (-1)^{F_L+F_R}\, R_3\, \sigma_{\rm ah}\ , \qquad \sigma_{\rm h}\sigma_{\rm ah} = (-1)^{F_L+F_R}\, \sigma_{\rm ah}\sigma_{\rm h}\ , \tag{8.104}$$

where the factor $(-1)^{F_L+F_R}$ signals commutation on bosons and anticommutation on ten-dimensional fermions. Under these assumptions one easily computes the algebra of operators $\mathcal{O}_1$, $\mathcal{O}_2$ to be

$$\mathcal{O}_1^2 = \mathcal{O}_2^2 = \mathbb{1}\ , \qquad\qquad \mathcal{O}_1\mathcal{O}_2 = \mathcal{O}_2\mathcal{O}_1\ . \tag{8.105}$$

Consistently quotienting out by $\mathcal{O}_1$ and $\mathcal{O}_2$ implies that one has to also consider the fixed points of the combined action

$$\mathcal{O}_3 \equiv \mathcal{O}_1\mathcal{O}_2 = \Omega_{\rm p}\, R_3\, \sigma_{\rm h}\, \sigma_{\rm ah}\ . \tag{8.106}$$

The fixed-point loci of this action $\mathcal{O}_3$ are O5-planes that fill $\mathbb{R}^{1,2}$ and wrap the three-dimensional special Lagrangian fixed-point set $L_{\sigma_{\rm h}\sigma_{\rm ah}}$ of $\sigma_{\rm h}\, \sigma_{\rm ah}$ in Y_3. As with the O7-planes, these O5-planes also induce a non-trivial tadpole that has to be cancelled. This requires us to include D5-branes into the setup that fill $\mathbb{R}^{1,2}$, localize on I, and wrap a three-cycle in Y_3 homologous to $L_{\sigma_{\rm h}\sigma_{\rm ah}}$. In the following, we will consider only D5-branes directly wrapping $L_{\sigma_{\rm h}\sigma_{\rm ah}}$. A summary of the objects that occur in this setup can be found in table 8.2.

This implies that the Type IIB weak-coupling limit contains the familiar orientifold planes as well as X5-planes. The latter planes have been studied in detail in the literature [236, 237, 238, 239, 240, 241] within a different context and given their prominent role it is worthwhile to recall their main features. The X5-planes can be seen to be the S-dual of an O5-plane with a single D5-brane on top of it. Indeed, since S-duality maps $(-1)^{F_L} \leftrightarrow \Omega_{\rm p}$ in Type IIB we see that the orbifold action maps to that of an O5-plane. The presence of the D5-brane on top of it can be inferred from tadpole cancellation and the presence of a $U(1)$ symmetry supported on the X5-plane which is the S-dual of the gauge symmetry on the D5-brane. The $U(1)$ is part of the twisted sector, which is most easily identified in the Type

symmetry	fixed object	location	tadpoles
$\mathcal{O}_1$	O7	$\mathbb{R}^{1,2} \times I \times H_{\sigma_{\rm h}}$	add D7
$\mathcal{O}_2$	X5	$\mathbb{R}^{1,2} \times L_{\sigma_{\rm ah}}$	no tadpole
$\mathcal{O}_3$	O5	$\mathbb{R}^{1,2} \times L_{\sigma_{\rm h}\sigma_{\rm ah}}$	add D5

Table 8.2: Summary of the symmetry transformations acting on the Type IIB setup (8.102), together with the objects appearing at the associated fixed-point loci, and their location.

IIA dual that is just a simple orbifold as we discuss in more detail below. In fact the local orbifold singularity was studied in a global compact setting which is the orbifold limit of a K3 (which is in turn dual to heterotic on T^4). In this global completion, the $U(1)$ is one of the 16 $U(1)$s arising from the twisted sector of the K3 orbifold limit, or in the geometric regime from dimensionally reducing C_3 on one of the blow-up cycles and sits in a six-dimensional vector multiplet.

Having identified the weak-coupling objects in table 8.2 we now note that they can preserve three-dimensional $\mathcal{N} = 1$ supersymmetry along $\mathbb{R}^{1,2}$. Indeed, compactification on the setup (8.102) before performing the quotient with respect to G yields a theory with eight supercharges. This is reduced to two supercharges by the presence of O7-planes, D7-branes, and X5-planes. The O7-D7 system does not break supersymmetry completely because, in the simple case in which the D7-branes sit on top of the O7-planes, all these object wrap the holomorphic cycle $H_{\sigma_{\rm h}}$ in Y_3. In a similar fashion, the X5-plane and the O5-D5 system do not break supersymmetry completely because they wrap special Lagrangian sub-manifolds $L_{\sigma_{\rm ah}}$, $L_{\sigma_{\rm h}\sigma_{\rm ah}}$. Finally, mutual supersymmetry among these objects can be inferred by noting that the calibration of the special Lagrangian sub-manifolds is adapted by construction to the complex structure with respect to which $H_{\sigma_{\rm h}}$ is holomorphic. We will check mutual supersymmetry explicitly in the case of toroidal models in section 8.6.3.

Let us now follow the various objects to Type IIA string theory and lift them to a geometric Spin(7) setup of F-theory. Firstly, we T-dualize along the x^3 direction, i.e. the direction associated to the interval $I = S^1/\mathbb{Z}_2$. The resulting Type IIA background is

$$\mathcal{M}_{10}^{\rm IIA} = (\mathbb{R}^{1,2} \times \tilde{S}^1 \times Y_3)/\tilde{G} \ , \tag{8.107}$$

where $\tilde{S}^1$ is the T-dual circle and the symmetry group $\tilde{G}$ is generated by the T-duals of $\mathcal{O}_1$ and $\mathcal{O}_2$, given by

$$\tilde{\mathcal{O}}_1 = \Omega_{\rm p} R_3\, \sigma_{\rm h} (-1)^{F_L} \ , \qquad\qquad \tilde{\mathcal{O}}_2 = R_3\, \sigma_{\rm ah} \ , \tag{8.108}$$

respectively. We also record the T-dual of the combined action $\mathcal{O}_3$

$$\tilde{\mathcal{O}}_3 = \Omega_{\rm p}\, \sigma_{\rm h}\, \sigma_{\rm ah}\, (-1)^{F_L} \ . \tag{8.109}$$

These expressions for the T-dual actions will be tested in the explicit toroidal model discussed below.

We realize that both $\tilde{\mathcal{O}}_1$ and $\tilde{\mathcal{O}}_3$ are Type IIA orientifold involutions that admit O6-planes along their fixed-point loci. On the one hand, the O6-planes associated to $\tilde{\mathcal{O}}_1$ span $\mathbb{R}^{1,2}$ and wrap the four-cycle H_{σ_h} in Y_3. On the other hand, the O6-planes arising from $\tilde{\mathcal{O}}_3$ span $\mathbb{R}^{1,2} \times \tilde{I}$, where $\tilde{I} = \tilde{S}^1/\mathbb{Z}_2$ is the T-dual interval, and wrap the three-cycle $L_{\sigma_{\rm h}\sigma_{\rm ah}}$. In contrast $\tilde{\mathcal{O}}_2$ is simply an orbifold action on the compact part of (8.107). Its fixed loci are six-dimensional orbifold planes denoted by Orb5. The fixed-point objects which appear in Type IIA are summarized in table 8.3.

symmetry	fixed object	location	tadpoles
$\tilde{\mathcal{O}}_1$	O6	$\mathbb{R}^{1,2} \times H_{\sigma_{\rm h}}$	add D6
$\tilde{\mathcal{O}}_2$	Orb5	$\mathbb{R}^{1,2} \times L_{\sigma_{\rm ah}}$	no tadpole
$\tilde{\mathcal{O}}_3$	O6	$\mathbb{R}^{1,2} \times \tilde{I} \times L_{\sigma_{\rm h}\sigma_{\rm ah}}$	add D6

Table 8.3: Summary of the symmetry transformations acting on the T-dual Type IIA setup (8.107), together with the objects appearing at the associated fixed-point loci, and their location.

In order to lift these quotients to M-theory we can make use of the correspondence

$$\begin{array}{ccc} \text{M-theory} & & \text{Type IIA} \\ R_{11} & \leftrightarrow & \Omega_{\rm p}\,(-1)^{F_L} \\ \mathcal{C} & \leftrightarrow & \Omega_{\rm p} \end{array} \tag{8.110}$$

where, on the M-theory side, R_{11} is the reflection of the eleventh direction of spacetime and where $\mathcal{C}$ is an involution that acts trivially on spacetime and reverses the sign of the three-form C_3. This correspondence can be checked by testing the action on the M-theory side and on the Type IIA side on all bosonic fields, in a similar spirit as what we have done explicitly in table 8.1 to test (8.94). As a consequence we discover that the quotients (8.108) are descended from M-theory quotients which act as

$$\tilde{\mathcal{O}}_1^M = R_3 R_{11}\,\sigma_{\rm h}\ , \qquad \tilde{\mathcal{O}}_2^M = R_3\,\sigma_{\rm ah}\ , \qquad \tilde{\mathcal{O}}_3^M = R_{11}\,\sigma_{\rm h}\sigma_{\rm ah}\ . \tag{8.111}$$

Identifying the 11 and 3 directions with the A and B cycles of the elliptic fiber respectively, these quotients can then be matched to the quotients appearing in (8.93).

For many applications, such as checking the supersymmetry properties of the setup in section 8.6.3, it turns out to be convenient to introduce the configurations on a six-torus T^6 instead of Y_3. Real coordinates on the ten-dimensional background $\mathbb{R}^{1,2}\times S^1\times T^6$ are denoted by x^m, $m = 0,\ldots,9$. In the internal space T^6 they combine into complex coordinates z_i, $i = 1,2,3$ as $z_1 = x^4 + ix^5$, $z_2 = x^6 + ix^7$, $z_3 = x^8 + ix^9$. We implement the holomorphic involution $\sigma_{\rm h}$ and the antiholomorphic involution $\sigma_{\rm ah}$ as

$$\sigma_{\rm h} : (z_1, z_2, z_3) \to (z_1, z_2, -z_3)\ , \qquad \sigma_{\rm ah} : (z_1, z_2, z_3) \to (\bar{z}_1, \bar{z}_2, \bar{z}_3)\ . \tag{8.112}$$

Hence the actions (8.103) take the form

$$\mathcal{O}_1 = \Omega_{\rm p}\,R_{89}\,(-1)^{F_L}\ , \qquad \mathcal{O}_2 = R_{3579}\,(-1)^{F_L}\ , \qquad \mathcal{O}_3 = \Omega_{\rm p}\,R_{3578}\ , \tag{8.113}$$

where R_m denotes the reflection of the real coordinate x^m, and $R_{m_1\ldots m_N} = R_{m_1}\ldots R_{m_N}$. This implies that the extended fixed-point objects of $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3 = \mathcal{O}_1\mathcal{O}_2$ are extended along the x^m-directions as listed in table 8.4.

symmetry	fixed object	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
$\mathcal{O}_1$	O7	×	×	×	×	×	×	×	×		
$\mathcal{O}_2$	X5	×	×	×		×		×		×	
$\mathcal{O}_3 = \mathcal{O}_1\,\mathcal{O}_2$	O5	×	×	×		×		×			×

Table 8.4: The location of the fixed-point sets of the Type IIB involutions (8.113) are displayed in coordinates x^m for the toroidal model on $\mathbb{R}^{1,2} \times S^1 \times T^6$. The symbol × indicates that the object fills this dimension. In all other directions the objects are at fixed points.

We can now study the dual Type IIA picture obtained by T-duality along x^3. The background is $\mathbb{M}^{2,1} \times \tilde{S}^1 \times T^6$, and the actions on this background read

$$\tilde{\mathcal{O}}_1 = \Omega_{\rm p}\, R_{389}\, (-1)^{F_L} \ , \qquad \tilde{\mathcal{O}}_2 = R_{3579} \ , \qquad \tilde{\mathcal{O}}_3 = \Omega_{\rm p}\, R_{578}\, (-1)^{F_L} \ . \tag{8.114}$$

In this toroidal model one can evaluate explicitly $\tilde{\mathcal{O}}_i = T_3 \mathcal{O}_i T_3^{-1}$, with T_3 being the operator that implements T-duality along the x^3 coordinate, using the rules collected in section 8.6.5. The fixed-point loci of $\tilde{\mathcal{O}}_1$, $\tilde{\mathcal{O}}_2$, and $\tilde{\mathcal{O}}_3$ extend along the real coordinates x^0, x^1, x^2, $\tilde{x}^3$, x^4, ..., x^9 as shown in table 8.5.

symmetry	fixed object	x^0	x^1	x^2	$\tilde{x}^3$	x^4	x^5	x^6	x^7	x^8	x^9
$\tilde{\mathcal{O}}_1$	O6	×	×	×		×	×	×	×		
$\tilde{\mathcal{O}}_2$	Orb5	×	×	×		×		×		×	
$\tilde{\mathcal{O}}_3 = \tilde{\mathcal{O}}_1\,\tilde{\mathcal{O}}_2$	O6	×	×	×	×	×		×			×

Table 8.5: The location of the fixed-point sets of the Type IIA involutions (8.114) are displayed in coordinates x^m for the toroidal model on $\mathbb{R}^{1,2} \times S^1 \times T^6$. The symbol × indicates that the object fills this dimension. In all other directions the objects are at fixed points.

The M-theory lift of this toroidal Type IIA background is completely analogous to the general case discussed in (8.111). For the convenience of the reader we summarize the quotients and objects that lie at the fixed spaces in table 8.6.

Type IIB quotient		Type IIA quotient		M-theory quotient
$\mathcal{O}_1 = \Omega_{\rm p} R_{89} (-1)^{F_L}$	(O7)	$\tilde{\mathcal{O}}_1 = \Omega_{\rm p} R_{389} (-1)^{F_L}$	(O6)	$\sigma_{\rm h} R_{AB} = R_{38911}$
$\mathcal{O}_2 = R_{3579} (-1)^{F_L}$	(X5)	$\tilde{\mathcal{O}}_2 = R_{3579}$	(Orb5)	$\sigma_{\rm ah} R_B = R_{3579}$
$\mathcal{O}_1\,\mathcal{O}_2 = \Omega_{\rm p} R_{3578}$	(O5)	$\tilde{\mathcal{O}}_1\,\tilde{\mathcal{O}}_2 = \Omega_{\rm p} R_{578} (-1)^{F_L}$	(O6)	$\sigma_{\rm h}\sigma_{\rm ah} R_A = R_{57811}$

Table 8.6: Summary of the symmetry transformations modded out in Type IIB, Type IIA and M-theory in the case that σ_B has a three-dimensional fixed space. The individual geometric actions have been introduced in section 8.5.

8.6.2 Weak-coupling setups with three-planes

This section is devoted to the situation in which the fixed-point locus of the antiholomorphic involution on the base manifold is one-dimensional. This is described by case (b) as shown in (8.8). In this case the fixed locus of $\sigma_{\rm ah}$ sits on top of a $\mathbb{Z}_2$ orbifold singularity of Y_3. In the following we refrain from a description of such setups for a general Calabi-Yau threefold, and rather discuss directly the toroidal model. This allows us to identify the localized objects that appear in the weak-coupling limit and to study in section 8.6.3 their mutual supersymmetry properties in a controlled way.

The Type IIB background we analyze is obtained starting from $\mathbb{R}^{1,2} \times S^1 \times T^6/\mathbb{Z}_2$ and taking the quotient with respect to the symmetry group generated by the transformation $\mathcal{O}_1$ defined in (8.113) and by the new transformation $\widehat{\mathcal{O}}_2$, where

$$\mathcal{O}_1 = \Omega_{\rm p}\, R_{89}\, (-1)^{F_L} \ , \qquad \widehat{\mathcal{O}}_2 = R_{3579}\, H\, (-1)^{F_L} \ , \tag{8.115}$$

and where H denotes the holomorphic action

$$H : (z_1, z_2, z_3) \to (z_2, -z_1, z_3) \ . \tag{8.116}$$

In this toroidal model the patch U described in (8.8) is extended to cover the whole of the internal space so that the (z_1, z_2, z_3) coordinates that we describe are identified by $\rho : (z_1, z_2, z_3) \to (-z_1, -z_2, z_3)$.

The presence of the factor R_3 inside $\widehat{\mathcal{O}}_2$ gives rise to the interval $I = S^1/\mathbb{Z}_2$ exactly as in the previous sections. However in this case the action of $\widehat{\mathcal{O}}_2$ is not directly an involution on the (z_1, z_2, z_3) coordinates. Rather the algebra satisfied by $\mathcal{O}_1$, $\widehat{\mathcal{O}}_2$ is given by

$$\mathcal{O}_1^2 = \mathbb{1} \ , \qquad \widehat{\mathcal{O}}_2^4 = \mathbb{1} \ , \qquad \mathcal{O}_1\, \widehat{\mathcal{O}}_2 = \widehat{\mathcal{O}}_2\, \mathcal{O}_1 \ , \tag{8.117}$$

where the operation $\widehat{\mathcal{O}}_2^2$ reproduces the identification $\rho = R_{4567}$.

The full symmetry group acting on the (z_1, z_2, z_3) coordinates of the covering T^6 then consists of the set of transformations

$$\Big\{ \mathbb{1},\, \mathcal{O}_1,\, \widehat{\mathcal{O}}_2,\, \widehat{\mathcal{O}}_2^2,\, \widehat{\mathcal{O}}_2^3,\, \mathcal{O}_1\, \widehat{\mathcal{O}}_2,\, \mathcal{O}_1\, \widehat{\mathcal{O}}_2^2,\, \mathcal{O}_1\, \widehat{\mathcal{O}}_2^3 \Big\} \tag{8.118}$$

with actions summarized in table 8.8. To each non-trivial element we can associate a localized object, as follows.

- $\mathcal{O}_1$: this involution is associated to O7-planes exactly as discussed in the previous section.
- $\widehat{\mathcal{O}}_2$: this transformation contains the factor $(-1)^{F_L}$ and admits a fixed-point locus that is real four-dimensional, fills $\mathbb{R}^{1,2}$, and is localized at the endpoints of the interval. We call the associated objects X3-planes.
- $\widehat{\mathcal{O}}_2^2$: as mentioned above, this is a standard $\mathbb{Z}_2$ orbifold action. Its fixed-point locus is six-dimensional, fills $\mathbb{R}^{1,2}$ and the interval, and will be denoted by Orb5.

- $\widehat{\mathcal{O}}_2^3$: this transformation gives another X3-plane that lies on top of the X3-plane associated to $\widehat{\mathcal{O}}_2$. These two X3-planes are identified under ρ.
- $\mathcal{O}_1\,\widehat{\mathcal{O}}_2$: this action contains a factor Ω_p but its geometric part squares to the identity only up to the $\mathbb{Z}_2$ orbifold action. The associated fixed-point locus is four-dimensional, fills $\mathbb{R}^{1,2}$, and is localized at the endpoints of the interval. We refer to the associated objects as XO3-planes.
- $\mathcal{O}_1\,\widehat{\mathcal{O}}_2^2$: in this case we have a factor $\Omega_\text{p}\,(-1)^{F_L}$ and the geometric action squares to one without invoking the $\mathbb{Z}_2$ orbifold. We thus find standard O3-planes.
- $\mathcal{O}_1\,\widehat{\mathcal{O}}_2^3$: this action gives another XO3-plane that is located on to of the XO3-plane at the fixed points of $\mathcal{O}_1\widehat{\mathcal{O}}_2$. These two XO3-planes are identified under ρ.

The fixed spaces of these quotients and the objects that lie at them are summarized in table 8.7.

symmetry	fixed object	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
$\mathcal{O}_1$	O7	×	×	×	×	×	×	×	×		
$\mathcal{O}_1\,\widehat{\mathcal{O}}_2^2$	O3	×	×	×	×						
$\widehat{\mathcal{O}}_2^2$	Orb5	×	×	×	×					×	×
$\widehat{\mathcal{O}}_2$ & $\widehat{\mathcal{O}}_2^3$	X3	×	×	×						×	
$\mathcal{O}_1\,\widehat{\mathcal{O}}_2$ & $\mathcal{O}_1\,\widehat{\mathcal{O}}_2^3$	XO3	×	×	×							×

Table 8.7: Localized objects in the Type IIB setup with involutions $\mathcal{O}_1$ and $\widehat{\mathcal{O}}_2$ are displayed in coordinates x^m for the toroidal model on $\mathbb{R}^{1,2}\times S^1\times T^6$. The symbol × indicates that the object fills this dimension. In all other directions the objects are at fixed points.

Let us note that the X3-planes encountered here are the analogs of the X5-planes of section 8.6.1, since they arise from an orbifold action dressed with an additional $(-1)^{F_L}$ factor. However, the X3-planes can only exist if they are confined to lie within the Orb5 locus of the $\widehat{\mathcal{O}}_2^2$ action. A natural conjecture for the S-dual of an X3-plane appears to be a system of XO3-planes, as introduced above, with suitable localized three-branes to cancel the tadpole. It would be desirable to study these configurations in more detail.

Having described the Type IIB setup we can apply the rules of section 8.6.5 to determine the T-duals of all actions listed above. The M-theory uplifts are then inferred by using (8.110). The resulting Type IIA actions and the objects that lie at their fixed points together with M-theory symmetries are summarized in table 8.8. One can then make contact with the discussion of section 8.5 by matching the A and B cycles with the 11 and 3 directions, respectively.

Type IIB quotient		Type IIA quotient		M-theory quotient
$\mathcal{O}_1 = \Omega_{\rm p} R_{89}(-1)^{F_L}$	(O7)	$\tilde{\mathcal{O}}_1 = \Omega_{\rm p} R_{389}(-1)^{F_L}$	(O6)	$\sigma_{\rm h} R_{AB} = R_{38911}$
$\widehat{\mathcal{O}}_2^2 = R_{4567}$	(Orb5)	$\hat{\tilde{\mathcal{O}}}_2^2 = R_{4567}$	(Orb5)	$\rho = R_{4567}$
$\mathcal{O}_1\,\widehat{\mathcal{O}}_2^2 = \Omega_{\rm p} R_{456789}(-1)^{F_L}$	(O3)	$\tilde{\mathcal{O}}_1\,\hat{\tilde{\mathcal{O}}}_2^2 = \Omega_{\rm p} R_{3456789}(-1)^{F_L}$	(O2)	$\sigma_{\rm h}\rho R_{AB} = R_{345678911}$
$\widehat{\mathcal{O}}_2 = R_{3579} H (-1)^{F_L}$	(X3)	$\hat{\tilde{\mathcal{O}}}_2 = R_{3579}\, H$	(Orb3)	$\sigma_{\rm ah} R_B = R_{3579} H$
$\widehat{\mathcal{O}}_2^3 = R_{3469} H (-1)^{F_L}$	(X3)	$\hat{\tilde{\mathcal{O}}}_2^3 = R_{3469}\, H$	(Orb3)	$\sigma_{\rm ah}\rho R_B = R_{3469} H$
$\mathcal{O}_1\,\widehat{\mathcal{O}}_2 = \Omega_{\rm p} R_{3578} H$	(XO3)	$\tilde{\mathcal{O}}_1\,\hat{\tilde{\mathcal{O}}}_2 = \Omega_{\rm p} R_{578} H (-1)^{F_L}$	(XO4)	$\sigma_{\rm h}\sigma_{\rm ah} R_A = R_{57811} H$
$\mathcal{O}_1\,\widehat{\mathcal{O}}_2^3 = \Omega_{\rm p} R_{3468} H$	(XO3)	$\tilde{\mathcal{O}}_1\,\hat{\tilde{\mathcal{O}}}_2^3 = \Omega_{\rm p} R_{468} H (-1)^{F_L}$	(XO4)	$\sigma_{\rm h}\sigma_{\rm ah}\rho R_A = R_{46811} H$

Table 8.8: Summary of the symmetry transformations modded out in Type IIB, Type IIA and M-theory in the case that σ_B has a one-dimensional fixed space. The individual geometric actions have been introduced in section 8.5.

8.6.3 Mutual supersymmetry in toroidal setups

This section is devoted to the study of the mutual supersymmetry properties of the localized objects introduced in the above sections 8.6.1 and 8.6.2. Our analysis will be simplified by considering the torus setups of table 8.4 and table 8.7. As a result, we do not perform any additional orbifold quotient and we rather let Y_3 be a simple six-torus, even though this implies a bulk sector with 32 real supercharges. These arguments therefore do not prove the supersymmetry of the setups with more complicated geometries. However, they do demonstrate that the unusual objects that we describe do not automatically break supersymmetry completely either on their own or when combined with the other sorts of fixed objects we consider.

Let us first study the setup of section 8.6.1 with weak-coupling objects listed in table 8.4. We also expect that these localized objects do not break supersymmetry completely, since the for any pair of them the number of different Dirichlet/Neumann directions is a multiple of four. This is a general observation for localized objects intersecting at right angles proven for instance in [46]. As a warm-up for the more involved case of section 8.6.2, we discuss a more explicit way to infer that this setup preserves a finite amount of supersymmetry. To this end, it is useful to combine the two ten-dimensional supersymmetry parameters into an R-symmetry doublet $\epsilon = (\epsilon_L, \epsilon_R)^{\mathsf{T}}$, where the subscripts L, R refer to their world-sheet origin. Operators $\mathcal{O}_i$ are represented as elements of the tensor product of the R-symmetry group with Spin(1,9). One has

$$\mathcal{O}_1 = i\sigma^2 \otimes \Lambda(R_{89}) \ , \qquad \mathcal{O}_2 = -\sigma^3 \otimes \Lambda(R_{3579}) \ , \qquad \mathcal{O}_3 = i\sigma^2 \otimes \Lambda(R_{3578}) \ , \tag{8.119}$$

where the σ's are Pauli matrices, and $\Lambda(M)$ denotes the Spin(1,9) element associated to $M \in SO(1,9)$. Note that $\Omega_{\rm p}$ is realized as σ^1, while $(-1)^{F_L}$ corresponds to $-\sigma^3$. Supersymmetry is preserved if a non-vanishing solution ϵ is found to the equations

$$\mathcal{O}_1\,\epsilon = \epsilon \ , \qquad \mathcal{O}_2\,\epsilon = \epsilon \ . \tag{8.120}$$

The analogous condition with $\mathcal{O}_3$ is not independent. These equations can be studied explicitly recalling that $\Lambda(R_m) = i\Gamma\Gamma_m$ in the light-cone formalism. One indeed finds that the operator

$$\lambda_1(\mathcal{O}_1 - \mathbb{1}) + \lambda_2(\mathcal{O}_2 - \mathbb{1}) \tag{8.121}$$

has a non-trivial kernel of relative dimension 1/4 for $\lambda_1, \lambda_2 \in \mathbb{C}$. Taking into account that ϵ_L, ϵ_R are Majorana spinors, we have proved that the toroidal setup under examination preserves 8 real supercharges. This may then be further broken if the torus is replaced by a Calabi-Yau threefold. Note also that the representation (8.119) can be used to check explicitly the algebra (8.105) on fermionic fields.

With this preparation we can now also analyze the setup introduced in section 8.6.2. The mutual supersymmetry properties of the localized objects listed in table 8.7 can be studied explicitly by representing the actions of $\mathcal{O}_1$ and $\widehat{\mathcal{O}}_2$ on the ten-dimensional supersymmetry parameters. We do not need to consider all other symmetries since they are generated by $\mathcal{O}_1$ and $\widehat{\mathcal{O}}_2$. The action of $\mathcal{O}_1$ was given in (8.119). The action of $\widehat{\mathcal{O}}_2$ reads

$$\widehat{\mathcal{O}}_2 = -\sigma^3 \otimes \Lambda(R_{3579})\, \Lambda(H) \ , \tag{8.122}$$

where

$$\Lambda(R_{3579}) = \Gamma_{3579} \ , \qquad\qquad \Lambda(H) = \frac{1}{2}(\mathbb{1} - \Gamma_{46})(\mathbb{1} - \Gamma_{57}) \ . \tag{8.123}$$

We can thus study the operator

$$\lambda_1(\mathcal{O}_1 - \mathbb{1}) + \lambda_2(\widehat{\mathcal{O}}_2 - \mathbb{1}) \tag{8.124}$$

and show straightforwardly that, for $\lambda_1, \lambda_2 \in \mathbb{C}$, it has non-trivial kernel of relative dimension 1/8, thus proving that our toroidal setup preserves four real supercharges. Note that in this setup the Dirichlet/Neumann direction rule is not applicable, since we have an orbifold action and the geometric transformations under examination do not just consist of reflections. Let us stress again that the amount of preserved supersymmetry will decrease further when replacing the torus by a Calabi-Yau manifold. It would be interesting to investigate the rules for this breaking in this more general situation.

8.6.4 Large-interval limit and supersymmetry restoration

In this section we discuss some properties of the Type IIB setup described above in the limit in which the size of the interval I is sent to infinity. More precisely, we focus on the resulting four-dimensional low-energy effective action and we argue that, for any observer in the bulk of I, such a theory is indistinguishable from the four-dimensional $\mathcal{N} = 1$ effective theory obtained by quotienting Type IIB with respect to $\mathcal{O}_1$ only.

In order to simplify the discussion we suppose that the quotient under the action of G generated by $\mathcal{O}_1$ and $\mathcal{O}_2$ is performed in two steps. In particular, we consider first the quotient under $\mathcal{O}_2$ and later implement $\mathcal{O}_1$, since the latter does not affect the following arguments. We are interested in the dynamics of excitations with wavelength much larger than the typical size of the internal space parametrized by coordinates $x^4, \ldots, x^9$. This size, in turn, is supposed to be large compared to the string scale. As a result, the only states that become light as the interval I decompactifies are states with no winding and with non-vanishing Kaluza-Klein mode along x^3 only.

Such states are conveniently packaged into four-dimensional fields depending on $x^0, \ldots, x^3$ and satisfying Dirichlet or Neumann boundary conditions at the endpoints of the interval. More precisely, invariance under $\mathcal{O}_2$ implies that expansion of the massless fields of Type IIB supergravity onto positive and negative cohomologies of Y_3 under $\sigma_{\rm ah}$ yields four-dimensional fields with definite parity under reflection of x^3. As already noted in section 8.5.4, fields with negative parity satisfy Dirichlet boundary conditions at the endpoints of the interval and for finite interval size cannot be accessed in the low-energy theory, because they always carry at least one unit of Kaluza-Klein momentum along the direction of x^3.

When the size of the interval becomes much larger than the typical wavelength of the excitations we want to study, however, the states associated to four-dimensional fields with Dirichlet boundary conditions become accessible again to the low-energy dynamics. This implies that we can excite fluctuations of all four-dimensional fields, irrespectively of their parity under reflection of x^3.[10] We are thus led to argue that in the limit of infinite interval I the low-energy four-dimensional effective action is the same as the one that would be obtained without performing the quotient with respect to $\mathcal{O}_2$. Thus, in this limit the group G effectively reduces to $\mathcal{O}_1$ only, and we have a Calabi-Yau orientifold that yields a four-dimensional $\mathcal{N}=1$ effective action.

We conclude this section with a short remark about the Type IIA interpretation. The Kaluza-Klein states that become light in the limit on the Type IIB side correspond to winding states on the Type IIA side. Kaluza-Klein states of a four-dimensional field with Neumann or Dirichlet boundary conditions at the endpoint of the interval have the schematic form

$$|\psi, n_3 = N, w_3 = 0\rangle \pm |\psi, n_3 = -N, w_3 = 0\rangle \ , \tag{8.125}$$

respectively. In this expression n_3, w_3 are the Kaluza-Klein level and winding in the x^3 direction, $N \in \mathbb{Z}$, and ψ is a shorthand notation for the oscillator structure of the state. T-duality along x^3 maps such a state to

$$|\psi, \tilde{n}_3 = 0, \tilde{w}_3 = N\rangle \pm |\psi, \tilde{n}_3 = 0, \tilde{w}_3 = -N\rangle \ , \tag{8.126}$$

where $\tilde{n}_3$, $\tilde{w}_3$ denote Kaluza-Klein level and winding along the T-dual coordinate $\tilde{x}^3$.

In the uplift to M-theory it is natural to presume that one finds a linear superposition of M2-brane states with opposite winding on the two-torus spanned by $\tilde{x}_3$ and the M-theory circle x^{11}. The presence of such M2-brane states might help to explain how the moduli space of the Spin(7) manifold with vanishing fiber can be enhanced to the moduli space of the Calabi-Yau fourfold with vanishing fiber. In particular, this requires a complexification of the real Spin(7) moduli space to form a Kähler manifold.

[10] Strictly speaking, only Neumann fields can have a constant VEV. For a Dirichlet field the allowed profile with the minimum energy is of the form $\sin(x^3/r)$, where πr is the length of the interval, and can be considered approximately as a constant VEV in a sufficiently small region in the bulk of the interval.

8.6.5 Some reference formulae

In the previous sections we have described several quotients which are built from a set of fundamental symmetry actions. Let us now collect some reference formulae that are useful for checking many of the results quoted before. The basic building blocks of all symmetry actions we have analyzed in the toroidal setups of sections 8.6.1 and 8.6.2 are $\Omega_{\rm p}$, $(-1)^{F_L}$, and $R_{m_1\dots m_n} = R_{m_1}\dots R_{m_n}$, where R_m describes the parity inversion $x^m \to -x^m$. These satisfy the algebra

$$\begin{gathered}\Omega_{\rm p}^2 = 1 \ , \qquad R_m^2 = 1 \ , \qquad ((-1)^{F_L})^2 = 1 \ , \\ \Omega_{\rm p}(-1)^{F_L} = (-1)^{F_R}\Omega_{\rm p} \ , \quad \Omega_{\rm p} R_m = R_m \Omega_{\rm p} \ , \quad R_m(-1)^{F_L} = (-1)^{F_L} R_m \ , \\ R_m R_n = (-1)^{F_L+F_R} R_n R_m \quad \text{if} \quad n \neq m \ , \end{gathered} \tag{8.127}$$

where F_R is the right-moving spacetime fermion number. Let us mention a subtle point. Defining R_m as a parity inversion implies a definition of the action of R_m on fermions that is only unique up to a phase. Here we have made a choice to discuss $R_m^2 = 1$. This convention is appropriate for the way we describe Op-planes and is consistent with the conventions of [46].[11]

It is also useful to collect the transformation properties of these actions under T-duality,

$$\begin{aligned} T_m(-1)^{F_L}T_m^{-1} &= (-1)^{F_L} \ , & T_m\Omega_{\rm p}T_m^{-1} &= \Omega_{\rm p} R_m \ , \\ T_m R_m T_m^{-1} &= R_m \ , & T_m R_n T_m^{-1} &= R_n(-1)^{F_L} \quad \text{if} \quad n \neq m \ , \end{aligned} \tag{8.128}$$

where T_m represents T-duality in the m direction. Finally, let us record the uplift of these actions from Type IIA to M-theory,

$$R_m \to R_m \ , \qquad (-1)^{F_L} \to R_{11}\mathcal{C} \ , \qquad \Omega \to \mathcal{C} \ . \tag{8.129}$$

Recall that R_{11} is the inversion of the M-theory circle and $\mathcal{C}$ acts by reversing the sign of the M-theory three-form C_3.

8.6.6 Comments on charged matter

The effective action derived in section 8.4.2 does not furnish an explicit description of the charged matter spectrum of F-theory on the class of Spin(7) manifolds under consideration. This is related to the general difficulty, already pointed out in section 5.3, that charged matter becomes massive after the gauge group is broken to the Coulomb branch and is thus automatically integrated out on the M-theory side of the M-theory/F-theory duality.

To get information about charged matter we can alternatively start looking at the weak coupling limit of our F-theory setup. One can engineer charged matter by means of intersecting D7-branes that wrap holomorphic cycles in the threefold Y_3 and have $(1,1)$-type world-volume flux to ensure the presence of four-dimensional chiral fermions. We refer the reader to e.g. [106] for a review. As

[11] Other conventions can lead to $R_m^2 = (-1)^{F_L+F_R}$.

we have seen, the crucial new ingredient is the antiholomorphic involution $\sigma_{\rm h}$ combined with the transformation R_3 to have a symmetry of Type IIB.

We can specialize further and consider a point in moduli space in which the Calabi-Yau threefold Y_3 is realized as a toroidal orbifold. In this toroidal setups the embedding of D7-branes is described by one linear holomorphic equation for the flat complex coordinates of the torus. Information about the charged matter spectrum can be obtained by first principles, by quantizing open strings stretching between D7-branes. We can make some general remarks on the interplay between holomorphically embedded D7-branes and the antiholomorphic involution. First of all, the image branes are also holomorphically embedded, if the antiholomorphic action is linear in the flat coordinates of the torus. Second of all, the world-volume flux of an image brane is still of $(1,1)$-type, but its sign is reversed compared to the original brane. These considerations imply that if we start with a supersymmetric setup that contains only holomorphic branes with $(1,1)$ fluxes, these features are not spoiled by the introduction of image branes under the antiholomorphic involution. Any intersection of any two branes or image branes possesses at least one complex massless scalar. Of course, one has to take into account the projection onto invariant states to determine if supersymmetry is actually present, or if different number of bosonic and fermionic massless states is projected out.

It is possible to argue that the robust features of the charged matter spectrum are insensitive to the details of the full compactification setup, and only depend on the local geometry around the intersection of the two D7-branes. This can be effectively described by looking at a non-compact model with flat D7-branes in $\mathbb{R}^{1,3} \times \mathbb{C}^3$. It captures the neighborhood of a fixed locus on the base B_3. Therefore the antiholomorphic action σ in local coordinates can be taken to correspond for instance to case (a) or case (b) discussed in section 8.2.2. If $\sigma_{\rm h}$ does not square to the identity, its square is included as an additional holomorphic orbifold action, in such a way that $\sigma_{\rm h}^2 = \mathbb{1}$ in the quotient space. It is possible to perform explicitly the projection onto invariant states for the two linear actions of cases (a) and (b). One can then compare the result with the purely orientifold projection without the antiholomorphic involution $\sigma_{\rm h}$ and without R_3. We refrain from a detailed account of the computation, and rather state our findings. For both case (a) and case (b) the same number of bosonic and fermionic degrees of freedom survives the projection. This signals that the charged matter spectrum is $\mathcal{N} = 1$ supersymmetric also after the antiholomorphic orbifold action is taken into account.

It can be checked that, irrespectively of the position of the D7-branes and their images under the action of $\sigma_{\rm h}$, no open string state can be invariant under the action of $\sigma_{\rm h}\, R_3$, but rather that open string states are always swapped in pairs. This seems to prevent an undemocratic truncation of the spectrum in such a way that the same number of bosonic and fermionic degrees of freedom is obtained. This general feature can be related to a mismatch between holomorphic embedding and antiholomorphic involution. On the one hand, charged matter is localized at the intersection of two D7-branes, which is a complex one-dimensional holomorphic subspace of the internal six-torus. On the other hand, the fixed locus of the antiholomorphic involution is either a real one-dimensional subspace (see case (b) in section 8.2.2), or a real three-dimensional subspace incompatible with the holomorphic structure (see case (a)). It is therefore impossible to have the intersection inside the fixed locus of the antiholomorphic involution.

In light of the supersymmetry restoration in the bulk sector argued in section 8.6.4 these findings about localized charged matter are not surprising. In the decompactification limit we therefore expect the full four-dimensional action—bulk fields and charged matter—to be $\mathcal{N} = 1$ supersymmetric. Let us point out that there are many other interesting open questions that can be addressed in toroidal models. For instance, it might be possible to relate closed string twisted sectors of the antiholomorphic orbifold action to resolution modes of the Spin(7) geometry. This might shed some light on geometries for which no resolution can be found in the mathematical literature.

PART III

Tensor towers and Chern-Simons theories

CHAPTER 9

Five-dimensional Chern-Simons terms at one loop

This chapter is devoted to a purely field-theoretical problem: the determination of the one-loop corrections to Chern-Simons levels in five-dimensions. Our findings extend known results in the literature [242, 211] by considering not only the corrections due to massive spin-1/2 fermions, but also to massive spin-3/2 fermions and so-called massive self-dual tensors. The latter are defined in section 9.2 and their importance for the study of tensor theories in six dimensions will be discussed at length in chapter 10. The results of this chapter can find applications both in five-dimensional and in six-dimensional contexts. Some examples are provided in chapter 10.

9.1 An exception to the decoupling paradigm

As we have briefly recalled in the introductory section 1.3, the derivation of a Wilsonian low-energy effective action amounts to integrating out all excitations beyond a chosen cutoff energy scale and obtaining a theory with modified couplings for the remaining degrees of freedom. The corrections to the low energy effective action obtained by integrating out massive fields are organized in an expansion in the inverse mass scale. In the limit of large cutoff scale corrections are typically strongly suppressed and can be neglected. In this case all modes with masses above the cutoff scale become effectively non-dynamical and can be decoupled from the theory. This is the subject of well known results in quantum field theory, such as the Appelquist-Carazzone-Symanzik decoupling theorem [243].

This reasoning, however, breaks down for certain types of couplings. Four-dimensional examples are furnished by Goldstone-Wilczek currents [244] and Wess-Zumino terms [245] generated by integrating out a fermion that becomes massive via Yukawa coupling to a scalar that gets a non-vanishing VEV. They are independent of the fermion mass and have to be included in the low-energy effective action even in the limit in which it is taken to infinity. The couplings we will study in this chapter—gauge and gravitational Chern-Simons couplings in five-dimensional theories—exhibit similar features.

The five-dimensional quantum field theories under consideration will propagate both massless and massive degrees of freedom. As mentioned above, we will study the effects of massive spin-1/2 fermions, spin-3/2 fermions, and self-dual tensors. The underlying common feature of these fields is parity violation: their Lagrangians, discussed in section 9.3.1, are not invariant under parity.[1] In particular, fermions induce parity violation through their mass terms, while tensors violate parity via their kinetic term. Let us stress that the latter is of a non-standard form and is different from the kinetic term for massless tensors in five dimensions, which are dual to massless vectors.

The massive fields are minimally coupled to a massless $U(1)$ gauge field A with field strength F. We aim to derive the corrections to the gauge Chern-Simons term $A \wedge F \wedge F$ and the gravitational Chern-Simons term $A \wedge \mathrm{tr}\,(\mathcal{R} \wedge \mathcal{R})$, where $\mathcal{R}$ is the five-dimensional curvature two-form, induced by integrating out all massive fields. As we will demonstrate, after appropriate overall normalization each of the massive fields yields an integer contribution to the quantum-corrected Chern-Simons couplings. This is consistent with the topological nature of the Chern-Simons couplings that implies that their prefactors are quantized and turn out to be independent of the mass scale of the fields that are integrated out.

The effect we are interested in is one-loop exact, as argued in section 9.2. Indeed, it can be interpreted as a parity anomaly matching in five dimensions, as follows. The original theory, containing both massless and massive fields of the kind listed above, is parity-violating because of the latter. The Wilsonian effective action for massless modes is parity-violating by virtue of a one-loop effect generated integrated out massive fields. The connection between Chern-Simons terms and anomalies is actually richer: in chapter 10 we will explore how five-dimensional Chern-Simons term can encode six-dimensional anomalies. We will also test our one-loop results against the geometric predictions inferred by means of the duality between F-theory and M-theory in chapter 7. Let us point out that these results have also been shown to agree with the genus-one corrections to Chern-Simons terms predicted in heterotic string theory. We refrain from giving here an account of the computation and we rather refer the reader to [246] for more details.

9.2 Summary of the results

Let us start by summarizing the results of this chapter, which will be derived in the next sections. The object of our investigation are five-dimensional theories in which some massive fields are coupled to a $U(1)$ gauge field A_μ and to the metric $g_{\mu\nu}$. In particular, we study how quantum corrections due to massive fields can generate the Chern-Simons couplings

$$S_{AFF} = k_{AFF} \int A \wedge F \wedge F \ , \qquad S_{ARR} = k_{ARR} \int A \wedge \mathrm{tr}\,(\mathcal{R} \wedge \mathcal{R}) \tag{9.1}$$

[1]Note that in five dimensions the reflection of all four spatial coordinates is a transformation belonging to the identity component of the Lorentz group. By parity in five dimensions we mean the reflection of one or three spatial directions.

in the low energy effective action. In these expressions $F = dA$ is the field strength of the $U(1)$ gauge field and $\mathcal{R}$ denotes the curvature two-form built from the metric $g_{\mu\nu}$. More precisely, $\mathcal{R}$ is a $\mathfrak{so}(1,4)$-valued two-form with components

$$\mathcal{R}^a{}_b = \tfrac{1}{2}\, e^a{}_\rho\, e_b{}^\sigma\, R^\rho{}_{\sigma\mu\nu}\, dx^\mu \wedge dx^\nu \ , \tag{9.2}$$

where $e^a{}_\mu$ is the five-dimensional vielbein and $R^\rho{}_{\sigma\mu\nu}$ is the Riemann tensor.

We show that three classes of massive fields are capable of generating the Chern-Simons terms (9.1) in the quantum effective action: massive spin-1/2 fermions ψ, massive self-dual tensors $B_{\mu\nu}$, and massive spin-3/2 fermions ψ_μ. By massive self-dual tensor we mean a complex two-form $B_{\mu\nu}$ that admits a non-standard first order kinetic term $\bar{B} \wedge dB$ together with a mass term $m\bar{B} \wedge *B$. Its free equation of motion thus reads schematically

$$*dB \propto mB \ . \tag{9.3}$$

These tensor fields and their coupling to a $U(1)$ gauge field has been analyzed in [247] and will be re-considered from a six-dimensional perspective in chapter 10. Further details about massive self-dual tensors are given in section 9.3.1. We refer to these fields as self-dual because they can be thought of as the excited Kaluza-Klein modes of a six-dimensional self-dual tensor compactified on a circle.

Spin-1/2 fermions, self-dual tensors, and spin-3/2 fermions can be characterized in terms of associated representations of the massive little group in five dimensions, $SO(4) \cong SU(2) \times SU(2)$. Such representations are labelled by a pair of half-integer spins (j_1, j_2). The correspondence between massive fields and $SO(4)$ representations is summarized in table 9.1.

field	free EOM	$SO(4)$ rep.
spin-1/2 fermion ψ	$(\partial\!\!\!/ - c_{1/2}m)\psi = 0$	$\left(\frac{1}{2},0\right)$ or $\left(0,\frac{1}{2}\right)$
self-dual tensor $B_{\mu\nu}$	$(*d - ic_B m)B = 0$	$(1,0)$ or $(0,1)$
spin-3/2 fermion ψ_μ	$(\gamma^{\rho\mu\nu}\partial_\mu + c_{3/2}m\gamma^{\rho\nu})\psi_\nu = 0$	$\left(\frac{1}{2},1\right)$ or $\left(1,\frac{1}{2}\right)$

Table 9.1: Summary of massive representations considered in this chapter.

We have included the equation of motion that puts each field on-shell in the absence of interactions. The coefficients $c_{1/2}$, c_B, $c_{3/2}$ can take the values ± 1 and determine which $SO(4)$ representation is realized. Note that here and in the following m denotes the mass of the physical one-particle states and is thus taken to be positive. The pairs of representations (j_1, j_2) and (j_2, j_1) are interchanged under parity. Correspondingly, these classes of fields break parity at tree level. From this point of view, let us stress again that the fact that couplings of the form (9.1) are generated in the effective action can be interpreted as a parity anomaly: quantum effects compensate for the parity violation originally induced by these families of massive fields, after they are integrated out.

The following table summarizes our findings for the coefficients k_{AFF}, k_{ARR} of the induced Chern-Simons couplings in (9.1). Coefficients $c_{1/2}$, c_B, $c_{3/2}$ correspond to those in table 9.1. The symbol

q denotes the $U(1)$ charge of the massive fields. It is a dimensionless quantity and its normalization

	spin-1/2 fermion ψ	self-dual tensor $B_{\mu\nu}$	spin-3/2 fermion ψ_μ
$k_{AFF} =$	$-\frac{1}{48\pi^2}\, q^3 \cdot c_{1/2}$	$-\frac{1}{48\pi^2}\, q^3 \cdot (-4\, c_B)$	$-\frac{1}{48\pi^2}\, q^3 \cdot (5\, c_{3/2})$
$k_{ARR} =$	$-\frac{1}{384\pi^2}\, q \cdot c_{1/2}$	$-\frac{1}{384\pi^2}\, q \cdot (8\, c_B)$	$-\frac{1}{384\pi^2}\, q \cdot (-19\, c_{3/2})$

Table 9.2: Summary of the one-loop contributions for various fields.

is fixed by the minimal coupling prescription $\partial_\mu \to \partial_\mu - iqA_\mu$. The derivation of these results is the subject of the upcoming sections. Nonetheless, let us stress here two crucial aspects of the computation. Firstly, k_{AFF} and k_{ARR} are quantum corrected at one-loop only. This is expected by arguments involving locality of the effective action and quantization of the Chern-Simons couplings [242] and is consistent with the interpretation in terms of parity anomalies in five dimensions.

Secondly, our results are derived using a simple quadratic action for the massive fields, which only includes minimal coupling to the gauge field A_μ and the metric $g_{\mu\nu}$. We argue that k_{AFF} and k_{ARR} are indeed insensitive to any fine detail of the interactions. For the k_{AFF} coupling, the effect of some non-minimal interactions is analyzed explicitly in section 9.3.4. It is shown there that such non-minimal couplings do not affect the renormalized value of k_{AFF}. These features are expected for topological couplings such as (9.1) that can be interpreted as parity anomalies.

Note that we refrain from a discussion about the possibility to write down fully consistent interacting theories for the three classes of massive fields under examination. For instance, it is expected that an interacting theory of massive spin-3/2 fermions is only possible in presence of (possibly spontaneously broken) supersymmetry, even though our findings are independent of the precise way it is realized in the five-dimensional action. From this point of view, we do not consider other parity-violating representations of $SO(4)$, such as $(\frac{3}{2}, 0)$ or $(2, 0)$, because no example is known of consistent interacting theories for the corresponding massive fields.

9.3 Feynman diagram computation

In this section we compute the coefficients of the Chern-Simons couplings (9.1) in perturbative quantum field theory. We start by reviewing the actions for the massive spin-1/2 fermion, self-dual tensor, and spin-3/2 fermions minimally coupled to the $U(1)$ gauge field and the metric. We then describe the main points of the Feynman diagram calculations for the gauge and the gravitational Chern-Simons terms. We conclude the section by studying the effect of some non-minimal couplings on the gauge Chern-Simons term.

9.3.1 Minimally coupled massive actions

The Chern-Simons couplings (9.1) can be captured by one-loop computations in a theory where the massive fields considered above are minimally coupled to the $U(1)$ gauge field A_μ and the metric $g_{\mu\nu}$. In this section we briefly review the corresponding actions.

A spin-1/2 fermion is described by a five-dimensional Dirac spinor ψ. In order to couple it to the metric $g_{\mu\nu}$ we have to use the vielbein $e^a{}_\mu$. The action for ψ minimally coupled to the $U(1)$ gauge field A_μ and the vielbein $e^a{}_\mu$ is taken to be

$$S_{1/2} = \int d^5x\, e\left[-\bar{\psi}\gamma^\mu \mathcal{D}_\mu\psi + c_{1/2}m\,\bar{\psi}\psi\right]\ ,\quad c_{1/2} = \pm 1\ , \tag{9.4}$$

where $e = \det e^a{}_\mu$, $\gamma^\mu = \gamma^a e_a{}^\mu$, and where we have introduced the full spacetime and $U(1)$ covariant derivative

$$\mathcal{D}_\mu\psi = \partial_\mu\psi + \tfrac{1}{4}\omega_{\mu ab}\gamma^{ab}\psi - iqA_\mu\psi\ . \tag{9.5}$$

On the right hand side, $\omega_{\mu ab}$ is the Levi-Civita spin connection constructed from the vielbein, and q is the $U(1)$ charge of the fermion ψ. More details about our spacetime and gamma-matrix conventions can be found in appendix A and in particular in section A.2. As stated in section 9.2, m is the positive physical mass and $c_{1/2}$ labels two inequivalent spinor representations of the massive little group $SO(4)$ in five dimensions. Under a parity transformation, the sign of $c_{1/2}$ is reversed.

Let us now turn to massive self-dual tensors in five-dimensions. Their action, including the coupling to a $U(1)$ gauge field, can be written as

$$S_B = \int d^5x\sqrt{-g}\left[-\tfrac{1}{4}ic_B\,\epsilon^{\mu\nu\rho\sigma\tau}\bar{B}_{\mu\nu}\mathcal{D}_\rho B_{\sigma\tau} - \tfrac{1}{2}\,m\bar{B}_{\mu\nu}B^{\mu\nu}\right]\ ,\quad c_B = \pm 1\ . \tag{9.6}$$

The relevant part of the spacetime and $U(1)$ covariant derivative reads

$$\mathcal{D}_{[\rho}B_{\mu\nu]} = \partial_{[\rho}B_{\mu\nu]} - iqA_{[\rho}B_{\mu\nu]}\ . \tag{9.7}$$

Note that $g = \det g_{\mu\nu}$ and that $\epsilon^{\mu\nu\rho\sigma\tau}$ denotes the five-dimensional Levi-Civita tensor. In our conventions, it satisfies $\epsilon^{01234} = -1/\sqrt{-g}$ if $0,\ldots,4$ are curved indices. Note that in this case parity violation is not due to the mass term, but to the kinetic term. The form (9.6) of the action can be argued from purely five-dimensional considerations, but it is most easily derived by means of circle compactification from six dimensions. This reduction will be performed in chapter 10.

Finally, a spin-3/2 fermion is described by a Dirac vector-spinor ψ_μ with action

$$S_{3/2} = \int d^5x\, e\left[-\bar{\psi}_\rho\gamma^{\rho\mu\nu}\mathcal{D}_\mu\psi_\nu - c_{3/2}m\,\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu\right]\ ,\quad c_{3/2} = \pm 1\ , \tag{9.8}$$

where the antisymmetric part of the spacetime and $U(1)$ covariant derivative is given by

$$\mathcal{D}_{[\mu}\psi_{\nu]} = \partial_{[\mu}\psi_{\nu]} + \tfrac{1}{4}\omega_{[\mu|ab}\gamma^{ab}\psi_{\nu]} - iqA_{[\mu}\psi_{\nu]}\ . \tag{9.9}$$

In analogy with the spin-1/2 case, the two inequivalent representations of $SO(4)$ differ by the sign of the mass term.

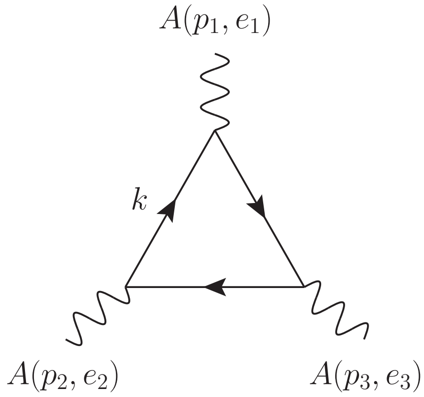


Figure 9.1: One-loop Feynman diagram involved in the computation of the Chern-Simons coefficient k_{AFF}. The external lines are three vectors A with incoming momenta p_1, p_2, p_3 and polarization vectors e_1, e_2, e_3. The internal lines can represent a massive spin-1/2 fermion, a massive self-dual tensor, or a massive spin-3/2 fermion. The loop momentum k flows in the direction of the arrow.

9.3.2 Computation of the $A \wedge F \wedge F$ coupling

The $U(1)$ Chern-Simons coupling $A \wedge F \wedge F$ does not involve the gravitational field. As a consequence, throughout this section we can ignore the coupling of massive fields to gravity and take $g_{\mu\nu} = \eta_{\mu\nu}$. No distinction between flat and curved indices is made. The coupling to A_μ can be treated perturbatively in the framework of quantum field theory on flat spacetime.

The coefficient of the $A \wedge F \wedge F$ term in the quantum effective action can be extracted from the three-point function of the gauge field A_μ. More precisely, we work in momentum space and we denote by Γ_{AAA} the sum of 1PI Feynman diagrams with three external vectors with incoming momenta p_1, p_2, p_3 and polarization vectors e_1, e_2, e_3. The Chern-Simons term

$$k_{AFF} \int A \wedge F \wedge F = -k_{AFF} \int d^5x \, \epsilon^{\mu\nu\rho\sigma\tau} A_\mu \partial_\nu A_\rho \partial_\sigma A_\sigma \tag{9.10}$$

in the effective action corresponds to a contribution to Γ_{AAA} of the form

$$\Gamma_{AAA} \supset i3! \times (-k_{AFF}) \, \epsilon_{\lambda\tau\mu_1\mu_2\mu_3} \, p_1^\lambda \, p_2^\tau \, e_1^{\mu_1} e_2^{\mu_2} e_3^{\mu_3} \ , \tag{9.11}$$

where we have included a factor of i from the Feynman rules and the combinatorial factor 3! to take into account symmetry under permutations of the three vectors. Contributions to Γ_{AAA} different from (9.11) will be ignored. They correspond to higher-derivative and non-local terms in the effective action. As already mentioned, we expect that the right hand side of (9.11) is corrected at one loop only. As shown in section 10.2 our one-loop results pass non-trivial tests in the framework of F-theory.

We can derive Feynman rules using the actions (9.4), (9.8) and (9.6) evaluated in flat spacetime and extract the propagators for massive fields, together with the interaction tri-vertex among two massive fields and one gauge field A_μ. These propagators and vertices are listed in appendix B.2.

At the one-loop level, only one class of diagrams can be built using the interaction vertices at hand. A representative diagram is depicted in figure 9.1. Wiggly lines represent the external vectors, while solid lines represent massive fields. Each class of massive fields contributes separately to the amplitude. To get the full answer, it has to be summed with the analog diagram where the orientation of the loop is reversed. This is equivalent to swapping the labels 1 and 2 on the external legs. Since the relevant structure in (9.11) is invariant under this relabeling, the loop-reversed diagram simply gives an overall additional factor 2.

The denominator of the diagram (which is determined through its propagator factors) is the same for all fields running in the loop. If the labeling of figure 9.1 is adopted, it is given by

$$\mathbb{D} = \frac{1}{k^2+m^2}\frac{1}{(k-p_2)^2+m^2}\frac{1}{(k+p_1)+m^2} \ , \tag{9.12}$$

which is to be completed by a suitable numerator factor $\mathbb{N}$ which particularly encodes information about the vertices and is strongly dependent on the fields running in the loop. In (9.12), the usual Feynman $i\epsilon$ prescription is understood. We make use of Schwinger parametrization to unify denominators, and write

$$\mathbb{D} = \frac{1}{m^6}\int_0^\infty d\alpha \int_0^\infty d\beta \int_0^\infty d\gamma \, e^{-(\alpha+\beta+\gamma)(\ell^2+\Delta)/m^2} \ . \tag{9.13}$$

In this expression, α, β, γ are dimensionless parameters, and we have made use of the shorthand notations

$$\ell = k - yp_2 + zp_1 \ , \quad \Delta = m^2 + 2yzp_1\cdot p_2 + y(1-y)p_1^2 + z(1-z)p_2^2 \ , \tag{9.14}$$

where $y = \beta/(\alpha+\beta+\gamma)$ and $z = \gamma/(\alpha+\beta+\gamma)$. The full diagram is then given by

$$\mathbb{I} = \mathbb{D}\cdot\mathbb{N} = \frac{1}{m^6}\int_0^\infty d\alpha \int_0^\infty d\beta \int_0^\infty d\gamma \int \frac{d^5\ell}{(2\pi)^5} \, e^{-(\alpha+\beta+\gamma)(\ell^2+\Delta)/m^2} \, \mathbb{N} \ , \tag{9.15}$$

where, of course, the numerator is different for different species of massive fields running in the loop. We also note that the sum of the diagram in figure 9.1 with the diagram with the opposite orientation has a distinct symmetry with respect to exchanging the external points. On general grounds, one can show that this symmetries restrict the parity violating part of the integrand in (9.15) at the bilinear level in the external momenta to only depend on the Schwinger parameters in the combination $(\alpha+\beta+\gamma)$. This is a useful consistency check we have applied throughout the computations.

By naive power-counting arguments, we do not expect any infrared divergence in this one-loop diagram, but we cannot exclude the possibility of ultraviolet divergences. If Schwinger parametrization is used, the integral over the loop momentum ℓ contains an exponential factor and (after Wick rotation) is convergent as long as $\alpha+\beta+\gamma$ is strictly positive. Ultraviolet divergences are translated into divergences in the α, β, γ integration, coming from the region where these three parameters are simultaneously small. We regularize the amplitude by cutting out this portion of the α, β, γ integration domain with a step-function: in (9.15) we make the replacement

$$\int_0^\infty d\alpha \int_0^\infty d\beta \int_0^\infty d\gamma \quad \to \quad \int_0^\infty d\alpha \int_0^\infty d\beta \int_0^\infty d\gamma \, \theta(\alpha+\beta+\gamma-\epsilon) \ , \tag{9.16}$$

where $\epsilon > 0$ is the regulator.

Recall from (9.11) that we are only interested in the coefficient of a term with two powers of external momenta contracted with an ϵ-symbol. This allows us to simplify the computation of the diagram.

First of all, only the terms that contain an ϵ-symbol have to be kept in the numerator. If a self-dual tensor runs in the loop, the ϵ-symbol is introduced directly at the level of Feynman rules both in the propagator and in the vertex. When a spinor runs in the loop, the ϵ-symbol is generated by traces of gamma matrices. This follows from the identities

$$\mathrm{tr}\,1 = 4\ , \quad \mathrm{tr}\,\gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5} = 4i\,\epsilon^{\mu_1\mu_2\mu_3\mu_4\mu_5}\ , \quad \mathrm{tr}\,\gamma^{\mu_1\cdots\mu_p} = 0 \text{ for } p = 1,2,3,4. \tag{9.17}$$

We see that only those terms need to be retained that contain an odd number of gamma matrices greater than or equal to five.

Second of all, we can perform a formal power series expansion of (9.15) in p_1, p_2 and we can neglect all terms that are not bilinear in p_1 and p_2. In particular, this implies that we can use the approximation $\Delta \approx m^2$, since all other terms in the exact expression (9.14) for Δ would generate additional powers of external momenta of the form p_1^2, p_2^2, or $p_1 \cdot p_2$.

Finally, by symmetry arguments (not spoiled by our choice of regulator), we can make the following replacements in the numerator under the $\int d^5\ell$ integral:

$$\begin{aligned}&\ell_{\mu_1}\ldots\ell_{\mu_r} \to 0 \text{ if } r \text{ is odd}\ ,\\ &\ell_\mu\ell_\nu \to \tfrac{1}{5}\,\ell^2\,\eta_{\mu\nu}\ , \qquad \ell_{\mu_1}\ell_{\mu_2}\ell_{\mu_3}\ell_{\mu_4} \to \tfrac{1}{35}\,(\ell^2)^2(\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4} + \eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4})\ , \qquad \ldots\end{aligned} \tag{9.18}$$

All tensor integrals in the loop momentum are thus reduced to scalar integrals.

The calculation of the diagram is now straightforward but tedious.[2] After the numerator algebra is performed and the replacements (9.18) are made, the integrals over the loop momentum and the Schwinger parameters are computed using the formulae

$$\int \frac{d^5\ell}{(2\pi)^5}\, e^{-(\alpha+\beta+\gamma)\ell^2/m^2}(\ell^2)^n = \frac{im^{2n+5}}{24\pi^3}\frac{\Gamma(n+5/2)}{(\alpha+\beta+\gamma)^{n+5/2}}\ , \tag{9.19}$$

$$\begin{aligned}&\int_0^\infty d\alpha \int_0^\infty d\beta \int_0^\infty d\gamma\, \theta(\alpha+\beta+\gamma-\epsilon)\, \frac{e^{-(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)^a}\,\alpha^{n_1}\beta^{n_2}\gamma^{n_3} =\\ &\qquad = \frac{\Gamma(1+n_1)\Gamma(1+n_2)\Gamma(1+n_3)}{\Gamma(3+n_1+n_2+n_3)}\Gamma(3+n_1+n_2+n_3-a;\epsilon)\ .\end{aligned} \tag{9.20}$$

We have performed the usual Wick rotation $\ell^0 \to i\ell^0$ in the first integral and have introduced the incomplete gamma function

$$\Gamma(x;\epsilon) = \int_\epsilon^\infty d\tau \tau^{x-1} e^{-x} \tag{9.21}$$

in the second integral.

Let us consider the diagram where the spin-1/2 fermion ψ runs in the loop. By power-counting we expect a quadratic divergence, since the numerator has up to three powers of the loop momentum.

[2] We made use of the *Mathematica* packages *xTensor* of the bundle *xAct* [248] and *GAMMA* [249].

The parity-violating part of the numerator, however, turns out to be of zero-th order in the loop momentum, thus giving a finite result without the need of any regulator.

This does not hold for the diagrams where $B_{\mu\nu}$ and ψ_μ run in the loop. In fact, even though the parity-violating part of the numerator has a better UV behavior than the full diagram, it still contains terms proportional to ℓ^2 or $(\ell^2)^2$. This implies that both diagrams have a divergent piece. In our regularization scheme such divergences appear as coefficients of negative powers of the regulator ϵ in a formal expansion of the diagram.

We can then give the ϵ-expansion for all the three species under consideration: spin-1/2 fermions ψ, tensors $B_{\mu\nu}$, and spin-3/2 fermions ψ_μ,

$$(\text{diagram})_{1/2} = \frac{i}{64\pi^2}\, c_{1/2}\, q^3 \left[+ 4 + \mathcal{O}(\epsilon^{1/2}) \right] , \tag{9.22}$$

$$(\text{diagram})_{B} = \frac{i}{64\pi^2}\, c_{B}\, q^3 \left[+ \frac{15}{\sqrt{\pi}}\epsilon^{-1/2} - 16 + \mathcal{O}(\epsilon^{1/2}) \right] , \tag{9.23}$$

$$(\text{diagram})_{3/2} = \frac{i}{64\pi^2}\, c_{1/2}\, q^3 \left[- \frac{105}{4\sqrt{\pi}}\epsilon^{-3/2} + \frac{15}{4\sqrt{\pi}}\epsilon^{-1/2} + 20 + \mathcal{O}(\epsilon^{1/2}) \right] . \tag{9.24}$$

Note that the factor (-1) for a fermionic loop has been taken into account, but we have not inserted the overall factor 2 due to the diagram with the reversed loop orientation.

In order to extract the physical observable k_{AFF} from these expressions we adopt a minimal subtraction prescription: negative powers of ϵ in the expansion are discarded. This gives the results of table 9.2. In section 9.3.4 we discuss the effect of non-minimal couplings and show how they can be used to cancel divergences.

9.3.3 Computation of the $A \wedge \mathrm{tr}\,(\mathcal{R} \wedge \mathcal{R})$ coupling

Let us now turn to the discussion of the mixed $U(1)$-gravitational Chern-Simons term $A \wedge \mathrm{tr}\,(\mathcal{R} \wedge \mathcal{R})$. To compute its coefficient we treat the coupling of massive fields to gravity perturbatively. The metric is written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \tag{9.25}$$

and computations are performed order by order in a formal power series in $h_{\mu\nu}$ around flat spacetime. Indices $\mu, \nu, \ldots$ are thus raised and lowered with $\eta_{\mu\nu}$ and its inverse and no distinction is made between flat and curved indices. Further details about the expansion in $h_{\mu\nu}$ are collected in appendix B.1.

When $A \wedge \mathrm{tr}\,(\mathcal{R} \wedge \mathcal{R})$ is expanded according to (9.25) terms with arbitrarily high powers of $h_{\mu\nu}$ are generated, because of the non-linear dependence of the Riemann tensor on the metric. Nonetheless, in order to read off the Chern-Simons coupling we can restrict to the lowest order term,

$$k_{ARR} \int A \wedge \mathrm{tr}\,(\mathcal{R} \wedge \mathcal{R}) = \tag{9.26}$$
$$= -\tfrac{1}{2} k_{ARR} \int d^5x\, \epsilon^{\mu_1\mu_2\mu_3\mu_4\mu_5} A_{\mu_1} \partial_\lambda \partial_{\mu_2} h_{\tau\mu_3} \left[\partial^\tau \partial_{\mu_4} h^\lambda{}_{\mu_5} - \partial^\lambda \partial_{\mu_4} h^\tau{}_{\mu_5} \right] + \mathcal{O}(h^3) \ .$$

As a consequence, the constant k_{ARR} can be extracted from the sum of 1PI Feynman diagrams with one vector and two gravitons, denoted Γ_{Ahh}. More precisely, the sought-for Chern-Simons coupling corresponds to the contribution

$$\Gamma_{Ahh} \supset i2! \times \tfrac{1}{2} k_{ARR}\, \epsilon_{\mu_0\mu_1\mu_2\lambda\tau}\, p_1^\lambda p_2^\tau \left(p_{1\,\nu_2} p_{2\,\nu_1} - \eta_{\nu_1\nu_2} p_1 \cdot p_2\right) e_0^{\mu_0} e_1^{\mu_1\nu_1} e_2^{\mu_2\nu_2} \ , \tag{9.27}$$

where p_1, p_2 are the incoming momenta of the gravitons, e_0 is the polarization tensor of the vector, and e_1, e_2 are the symmetric polarization tensors of the gravitons. The prefactor $i2!$ comes from the standard Feynman rule prescriptions. Any term that does not match the structure of the right hand side of (9.27) will be neglected, since it would correspond to higher-derivative and non-local terms in the effective action.

It is interesting to note that the tensor structure in (9.27) is transverse with respect to both the vector and the graviton polarization tensors, i.e. it vanishes if any of the replacements

$$e_0^\mu \to p_0^\mu = -p_1^\mu - p_2^\mu \ , \quad e_1^{\mu\nu} \to a^{(\mu} p_1{}^{\nu)} \ , \quad e_2^{\mu\nu} \to a^{(\mu} p_2{}^{\nu)} \tag{9.28}$$

is made, for arbitrary a^μ. It can be shown that this tensor structure is the only structure with an ϵ-symbol and four powers of external momenta that has this transversality property and is symmetric in the exchange of labels 1 and 2. Its appearance is a consequence of gauge invariance. Transversality with respect to e_0 reflects invariance of (9.26) under $U(1)$ transformations. Transversality with respect to e_1, e_2 derives from invariance of (9.26) under diffeomorphisms. Recall that under an infinitesimal diffeomorphism with parameter ξ^μ we have

$$\delta h_{\mu\nu} = 2\partial_{(\mu}\xi_{\nu)} + \mathcal{O}(h) \ . \tag{9.29}$$

Gauge invariance can be used as a self-consistency check of the Feynman diagram computation. Indeed, we find that the desired contributions to Γ_{Ahh} organize into the structure (9.27) after all relevant diagrams are summed.

The Feynman rules needed in the diagrammatic computation of Γ_{Ahh} are deduced by expanding the actions (9.4), (9.8), (9.6) for the massive fields according to (9.25). This gives interaction vertices of arbitrarily high powers in $h_{\mu\nu}$ but we only need an expansion up to second order in $h_{\mu\nu}$. More precisely, four kinds of vertices are relevant for the calculation of Γ_{Ahh}. If we denote any of the massive fields ψ, $B_{\mu\nu}$, ψ_μ as Φ, we need: the gauge tri-vertex $\bar{\Phi}\Phi A$, already considered in the previous section; the gravitational tri-vertex $\bar{\Phi}\Phi h$; the purely gravitational quadri-vertex $\bar{\Phi}\Phi hh$; the mixed gauge-gravitational quadri-vertex $\bar{\Phi}\Phi A h$. All such vertices are collected in appendix B.2.

The presence of quadri-vertices enlarges the family of one-loop Feynman diagrams that can be built. In particular, we have three different topologies, depicted in figure 9.2. The total amplitude is given by the sum

$$2(a) + (b) + 2(c) \ , \tag{9.30}$$

where diagram (a) is counted twice because of the two possible orientations of the loop, and diagram (c) is counted twice according to which graviton is connected to the mixed quadri-vertex.

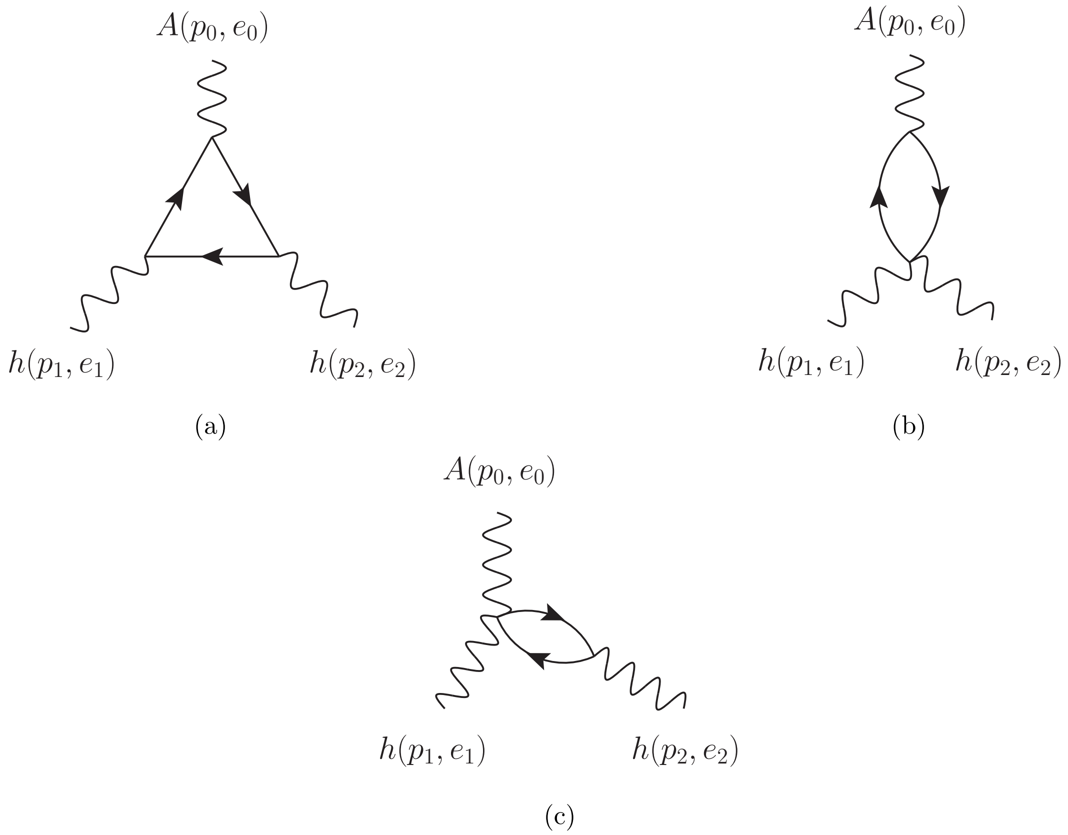


Figure 9.2: One-loop Feynman diagrams involved in the computation of the Chern-Simons coefficient k_{ARR}. The external line on top represents a vector A with incoming momentum p_0 and polarization vector e_0. The other external lines are gravitons h with incoming momenta p_1, p_2 and symmetric polarization tensors e_1, e_2. The internal lines can represent a massive spin-1/2 fermion, a massive self-dual tensor, or a massive spin-3/2 fermion.

For each diagram, denominators can be unified by means of Schwinger parameters. In diagram (a) three parameters are needed, as in the previous section, while diagrams (b) and (c) require only two parameters. Up to minor changes, the methods described in the previous section can be applied straightforwardly to the diagrams at hand. In particular, UV divergences in diagrams (b) and (c) are regulated by means of the replacement

$$\int_0^\infty d\alpha \int_0^\infty d\beta \quad \to \quad \int_0^\infty d\alpha \int_0^\infty d\beta\, \theta(\alpha+\beta-\epsilon) \;, \tag{9.31}$$

where α, β are the Schwinger parameters and ϵ is the regulator. For the sake of completeness, we record the two-parameter analog of the identity (9.20),

$$\int_0^\infty d\alpha \int_0^\infty d\beta\, \theta(\alpha+\beta-\epsilon) \frac{e^{-(\alpha+\beta)}}{(\alpha+\beta)^a} \alpha^{n_1}\beta^{n_2} = \tag{9.32}$$

$$= \frac{\Gamma(1+n_1)\Gamma(1+n_2)}{\Gamma(2+n_1+n_2)} \Gamma(2-a+n_1+n_2;\epsilon) \;. \tag{9.33}$$

Let us stress an important difference between the present computation and the one discussed in the previous section. In the case of the gauge Chern-Simons couplings, the relevant tensor structure (9.11) does not contain any product of external momenta. This allowed us to use the approximation $\Delta \approx m^2$ in the computation of the diagram in (9.15). In the present case, one of the two parts of the gauge invariant tensor structure (9.27) is proportional to $p_1 \cdot p_2$. This implies that we have to keep the $p_1 \cdot p_2$ term inside Δ and expand $e^{(\alpha+\beta+\gamma)\ell^2/m^2}$ (or $e^{(\alpha+\beta)\ell^2/m^2}$) in a power series in the external momenta. This is indeed crucial to obtain the gauge invariant structure (9.27) after all the three diagrams are combined according to (9.30).

As in the case of the gauge Chern-Simons term, the parity violating part of the diagrams has a better UV behavior than expected from naive power-counting. Nevertheless, the diagrams in which the self-dual tensor and the spin-3/2 fermion run in the loop have some divergent parts. After all diagrams are summed according to (9.30) and the total expression is organized in powers of ϵ, the ϵ^0 coefficient is proportional to the gauge-invariant combination (9.27), while negative-power coefficients are not gauge-invariant. This leads us to apply a minimal subtraction prescription and simply drop the unphysical divergent pieces. In this way the results of table 9.2 are obtained.

Let us conclude this section with a side remark. Recall from section 9.3.2 that the relative weight between the diagram for spin-1/2 and spin-3/2 fermion contributions to k_{AFF} is five. This result can be derived straightforwardly from an alternative form of the massive action for a spin-3/2 ψ_μ,

$$S'_{3/2} = \int d^5x\, e\left[-\bar\psi_\rho \gamma^\mu \mathcal{D}_\mu \psi^\rho + c_{3/2} m\, \bar\psi_\rho \psi^\rho\right] \;, \quad c_{3/2} = \pm 1 \;. \tag{9.34}$$

Indeed, when this action is evaluated on a flat background, it gives exactly the same propagator and vertex as the spin-1/2 action (9.4), up to a factor of the metric $\eta_{\mu\nu}$.

Remarkably, the alternative action (9.34) gives also the correct relative weight -19 between the spin-1/2 and the spin-3/2 contributions to k_{ARR}. This claim has been checked against an explicit Feynman diagram computation. To get the correct result is crucial to take into account the corrections to the vertices coming from the Christoffel symbols inside the covariant derivative $\mathcal{D}_\mu \psi^\rho$. Indeed, the vertices generated by the Christoffel symbol contribute a relative factor of -24 that combines with five times the spin-1/2 result to give -19.

This finding resembles a similar result about gravitational anomalies in six dimensions [199]. In order to compute the contribution of a massless chiral spin-3/2 field ψ_μ to gravitational anomalies in six dimensions, one can use two different Lagrangians, proportional to

$$\bar{\psi}_\rho \gamma^{\rho\mu\nu} \nabla_\mu \psi_\nu \qquad \text{or} \qquad \bar{\psi}_\rho \gamma^\mu \nabla_\mu \psi^\rho \ , \tag{9.35}$$

where ∇ denotes the six-dimensional Levi-Civita covariant derivative. It is shown that the difference between these Lagrangians cannot affect the anomalous part of the four-graviton one-loop diagram. Note that if we compactify the six-dimensional Lagrangians (9.35) on a circle, the resulting actions for the massive Kaluza-Klein modes have kinetic and mass terms as given in (9.8) and (9.34), respectively. We are thus led to conjecture that corrections to the five-dimensional Chern-Simons terms (9.1) are insensitive to the precise form of the differential operator in the kinetic term and the corresponding form of the mass term.

9.3.4 Non-minimal couplings and renormalization

The aim of this section is to describe the effect of non-minimal couplings on the Chern-Simons term $A \wedge F \wedge F$. Gravity is decoupled and the metric is taken to be $\eta_{\mu\nu}$. As far as fermions are concerned, we consider Pauli couplings built by contracting a spinor bilinear with the $U(1)$ field strength $F = dA$. In particular, we have analyzed the couplings

$$\mathcal{L}^{\rm nm}_{1/2} = \tfrac{1}{2} i \tilde{q}_{1/2} \, F_{\mu\nu} \, \bar{\psi} \gamma^{\mu\nu} \psi \ , \quad \mathcal{L}^{\rm nm}_{3/2} = \tfrac{1}{2} i \tilde{q}_{3/2} \, F_{\mu\nu} \, \bar{\psi}_\rho \gamma^{\mu\nu\rho\sigma} \psi_\sigma + \tfrac{1}{2} i \tilde{q}'_{3/2} \, F_{\mu\nu} \, \bar{\psi}^\mu \psi^\nu \ . \tag{9.36}$$

For massive self-dual tensors we have studied instead

$$\mathcal{L}^{\rm nm}_B = \tilde{q}_B \, \bar{B}_{\mu\nu} F^{\nu\rho} B_\rho{}^\mu + \tilde{q}'_B \, \bar{B}_{\mu\nu} F^{\nu\rho} B_{\rho\sigma} F^{\sigma\mu} \ . \tag{9.37}$$

The computation of section 9.3.2 can be repeated including these additional vertices. The corresponding Feynman rules can be obtained straightforwardly with the standard prescriptions. Note, however, that the coupling $\tilde{q}'_B$ induces a quadri-vertex and therefore diagrams with a topology as diagram (b) or (c) in figure 9.2 have to be included.

We refrain from a detailed account on the computation. Nonetheless, its outcome is remarkable: all non-minimal couplings $\tilde{q}_{1/2}$ to $\tilde{q}'_B$ drop from the ϵ^0 coefficient of the combination of all diagrams and enter only the coefficients of negative powers in ϵ.

This implies that they can be used to cancel divergences in the spin-3/2 and tensor diagrams. Recall from (9.24) that the triangle diagram with a spin-3/2 fermion running in the loop has two non-vanishing negative powers of ϵ if only the minimal coupling q is switched on. Our computations reveal that turning $\tilde{q}_{3/2}$, $\tilde{q}'_{3/2}$ on does not introduce higher negative powers, i.e. higher divergences, and does not affect the coefficient of the ϵ^0 power. We can thus tune $\tilde{q}_{3/2}$ and $\tilde{q}'_{3/2}$ and cancel divergences without altering the finite part of the diagram.

The same strategy can be applied to tensors. The reader might wonder why we take into account two non-minimal couplings for tensors, if the corresponding diagram has only one divergent part, as can be seen from (9.23). This is necessary since it can be checked that inclusion of the coupling $\tilde{q}_B$ introduces higher divergences that require the introduction of $\tilde{q}'_B$ to be cancelled.

Our findings suggest the interpretation of non-minimal couplings (9.36) and (9.37) as counterterms. Dimensional analysis reinforces this claim, since it shows that non-minimal couplings $\tilde{q}_{1/2}$ to $\tilde{q}'_B$ have negative mass dimension. In the limit in which the masses of ψ, $B_{\mu\nu}$, and ψ_μ tend to infinity and these fields are integrated out, non-minimal couplings are suppressed. A similar counterterm analysis for the gravitational Chern-Simons term is a formidable task and is not addressed in this work. Nevertheless, it is plausible that a similar mechanism can be implemented to cancel all divergences without changing the results of table 9.2.

CHAPTER 10

Abelian tensor towers and five-dimensional supergravities

In this chapter we will discuss several applications of the one-loop corrections to Chern-Simons terms in five dimensions computed in the previous chapter. To begin with, we discuss in detail how a self-dual tensor in six dimensions can be conveniently described in terms of a tower of massive tensors in five dimensions. This observation constitutes the bridge that connects five-dimensional Chern-Simons couplings and six-dimensional physics. We exploit this connection to complete our discussion about the role of one-loop effects in the duality between M-theory and F-theory on a Calabi-Yau threefold, started in chapter 7. Furthermore, we show how quantum-corrected Chern-Simons couplings can provide a useful tool in the exploration of apparently consistent supergravities in five-dimensions.

10.1 A lower-dimensional action for chiral p-forms

Even though we are mainly interested in the dynamics of self-dual tensors in six dimensions, the construction of this section applies straightforwardly to chiral p-forms in $D = 2p + 2$ dimensions, i.e. p-forms with self-dual or antiself-dual field strength. We will therefore develop the formalism for general p. Let us note that, since we consider Lorentz signature, p has to be even.

We have already seen in sections 2.2 and 3.5 two important example of the key role played by chiral p-forms in string theory and M-theory. On the one hand, the massless spectrum of Type IIB superstring theory contains a chiral four-form. On the other hand, the world-volume theory of an M5-brane includes a chiral two-form, i.e. a self-dual tensor. From a field-theoretic point of view, quantization of such fields is a non-trivial task, since it is notoriously hard to impose the duality constraint at the level of the action [250]. Different solutions to this problem have been proposed, based on breaking of manifest Lorentz invariance, introduction of auxiliary fields, or a holographic approach [251, 252, 253, 254, 255].

Our approach circumvents a direct description of chiral p-forms in $D = 2p + 2$ dimensions. We perform a circle compactification that leads us to a $(D-1)$-dimensional action which can be used to study the dynamics of these p-forms. This approach is inspired by observations coming from the study of string and M-theory effective actions. Firstly, six-dimensional (2,0) superconformal field theories for a stack of M5-branes have been conjectured to be equivalent to five-dimensional super Yang-Mills theories [33]. This conjecture has been reviewed briefly chapter 6. Secondly, six-dimensional effective actions of F-theory compactifications with an arbitrary number of chiral tensors can been derived by using the dual five-dimensional M-theory setups, as we have seen in chapter 7. In both frameworks excited Kaluza-Klein modes are essential for the correspondence between the six- and five-dimensional physics.

Our starting point is a D-dimensional pseudoaction, which has to be supplemented by the self-duality constraint at the level of the equations of motion, as usual. One spatial direction is compactified on a circle, and chiral p-forms are expanded onto a Kaluza-Klein tower of $(D-1)$-dimensional p- and $(p-1)$-forms. Both zeromodes and excited modes are retained, and are subject to duality constraints coming from self-duality in D dimensions. These constraints can be implemented in a proper $(D-1)$-dimensional action, which is given explicitly in (10.14) below. The derivation of the next section can be seen, on the one hand, as a generalization of the reduction performed in section 7.4.1 to include all Kaluza-Klein modes, and, on the other hand, as a variant of the compactification of non-chiral p-forms discussed in section 4.2.

Let us conclude this section by pointing out that this formalism can be also useful in the study of systems other than six-dimensional tensors. For instance, it may be applied to the democratic formulation of Type II supergravities [49] or to four-dimensional Maxwell actions with manifest electric-magnetic duality, see e.g. [256].

10.1.1 Derivation of the action

A free chiral p-form $\hat{B}$ in $D = 2p+2$ dimensions (with p even) is subject to the self-duality condition

$$\hat{*}\,\hat{\mathcal{H}} = c_B\,\hat{\mathcal{H}}\ , \tag{10.1}$$

where $c_B = \pm 1$ and $\hat{\mathcal{H}} = d\hat{B}$. This constraint is first-order, and is not easily derived from an action. However, differentiation of (10.1) gives a second-order equation which is readily obtained from the pseudoaction

$$\hat{S} = \int -\tfrac{1}{4}\hat{\mathcal{H}} \wedge \hat{*}\,\hat{\mathcal{H}}\ . \tag{10.2}$$

The prefactor is chosen to have canonical normalization in the following discussion. Note that the pseudoaction formalism can be also applied to setups including several p-forms and their couplings to other fields.

Let us now put the pseudoaction (10.2) on a circle, by means of the standard Kaluza-Klein Ansatz for the metric. We write it as

$$d\hat{s}^2(x,y) = ds^2(x) + r^2(x)[dy - A^0(x)]^2\ , \tag{10.3}$$

where x are the non-compact $D-1$ coordinates, $y \sim y + 2\pi$ is the coordinate along the circle, r is the compactification radius, and A^0 is the Kaluza-Klein vector, with field strength $F^0 = dA^0$. We do not consider the dynamics of gravity in D dimensions, so that the five-dimensional metric, the scalar r, and the vector A^0 are best understood as non-dynamical background fields. It is nonetheless useful and by no means more difficult to keep track of these fields in the computation we are about to perform.

We expand the D-dimensional p-form $\hat{B}$ in Kaluza-Klein modes according to

$$\hat{B} = \sum_{n\in\mathbb{Z}} e^{iny}\big[B_n + A_n \wedge (dy - A^0)\big] \ , \tag{10.4}$$

where B_n, A_n are $(D-1)$-dimensional p-forms and $(p-1)$-forms, respectively, and only depend on the non-compact coordinates x. Our formalism requires $p > 0$, and hence is not applicable to chiral scalars in two dimensions. Note that Kaluza-Klein modes are subject to a reality condition, e.g. $\bar{B}_n \equiv (B_n)^* = B_{-n}$.

Dimensional reduction of the higher-dimensional field strength $\hat{\mathcal{H}}$ is conveniently described in terms of the lower-dimensional field strengths

$$\mathcal{H}_n = \mathcal{D}B_n + A_n \wedge F^0 \ , \quad \mathcal{F}_n = \mathcal{D}A_n + inB_n \ , \tag{10.5}$$

where we have introduced the covariant exterior derivative $\mathcal{D} = d + inA^0$ acting on the nth mode. These field strengths are invariant under the gauge transformations

$$\delta B_n = \mathcal{D}\Lambda_n - \lambda_n \wedge F^0 \ , \quad \delta A_n = \mathcal{D}\lambda_n - in\Lambda_n \ , \tag{10.6}$$

where Λ_n is a p-form and λ_n is a $(p-1)$-form. Let us point out that (10.5) and (10.6) are a special case of (4.21) and (4.22).[1] This has to be expected since we have not implemented the self-duality constraint yet. A straightforward computation shows that the pseudoaction (10.2) is reduced to the sum $\sum_n \tilde{S}_n$, where

$$\tilde{S}_n = \int -\tfrac{1}{4} r\, \bar{\mathcal{H}}_n \wedge * \mathcal{H}_n - \tfrac{1}{4} r^{-1} \bar{\mathcal{F}}_n \wedge * \mathcal{F}_n \ . \tag{10.7}$$

Note that we have omitted and inconsequential prefactor 2π coming from the range of the compact coordinate y. If necessary, it can be straightforwardly reinstated in all following expressions in $(D-1)$ dimensions. Finally, the self-duality constraint (10.1) yields a constraint for each Kaluza-Klein level,

$$r * \mathcal{H}_n = c_B\, \mathcal{F}_n \ , \tag{10.8}$$

In the following, we implement these constraints at the level of the lower-dimensional action. To this end, zeromodes and excited modes are treated differently.

For the sake of simplicity, we will henceforth drop the Kaluza-Klein subscript on zeromodes, $B \equiv B_0$, $A \equiv A_0$. As we can see from (10.6), the shift symmetry of the theory with parameters Λ_n acts trivially on the zeromode A. Because of the self-duality constraint, B and A thus furnish a

[1] To compare these expression, note that the Kaluza-Klein vector has now a minus sign relative to section 4.2 motivated by our F-theory analysis if chapter 7. Note also that we are considering only the case in which p is even.

redundant description of the same degrees of freedom, and no gauge-fixing condition can eliminate this redundancy. Therefore, either A or B has to be eliminated by hand from the action. In the following, we choose to remove B and construct an action in terms of A only.

To achieve this goal, we modify $\tilde{S}_0$ given in (10.7) adding

$$\Delta\tilde{S}_0 = \int \tfrac{1}{2}c_B\, \mathcal{H}\wedge\mathcal{F} - \tfrac{1}{2}c_B\, A^0\wedge\mathcal{F}\wedge\mathcal{F}\ . \tag{10.9}$$

This term is a total derivative as a functional of A, B, A^0, and is such that the sum $\tilde{S}_0 + \Delta\tilde{S}_0$ can be written as a functional of $A, \mathcal{H}, A^0$. Moreover, (10.9) is engineered to get the duality constraint (10.8) for zeromodes upon variation with respect to $\mathcal{H}$, which appears only algebraically. We are thus able to integrate out $\mathcal{H}$ to get a proper $(D-1)$-dimensional action depending on A, A^0 only. It reads

$$S_0 = \int -\tfrac{1}{2}r^{-1}\mathcal{F}\wedge *\mathcal{F} - \tfrac{1}{2}c_B\, A^0\wedge\mathcal{F}\wedge\mathcal{F}\ . \tag{10.10}$$

Note that (10.5) implies $\mathcal{F} = dA$ for $n = 0$. This action is a p-form generalization of the action in section 7.4.1 for the zeromodes of a self-dual tensor.

Let us now turn to the discussion of the self-duality condition for the nth excited modes B_n, A_n. For $n \neq 0$, the shift symmetry with parameter Λ_n in (10.6) acts non-trivially on A_n. As a result, the redundancy of the formalism is simply a manifestation of gauge invariance. Both B_n and A_n are thus allowed to enter the action in the gauge-invariant combination $\mathcal{F}_n$ given in (10.5). The distinctive feature of the $n \neq 0$ case is the identity

$$\mathcal{D}\mathcal{F}_n = in\mathcal{H}_n\ , \tag{10.11}$$

which is immediately derived from (10.5). It allows us to modify $\tilde{S}_n$ in (10.7) by adding

$$\Delta\tilde{S}_n = \int \tfrac{1}{4}c_B\, \bar{\mathcal{H}}_n\wedge\mathcal{F}_n + \tfrac{i}{4n}c_B\, \bar{\mathcal{F}}_n\wedge\mathcal{D}\mathcal{F}_n + \text{c.c.} \tag{10.12}$$

Indeed, this quantity is a total derivative as a functional of A_n, B_n, A^0. However, the total action $\tilde{S}_n + \Delta\tilde{S}_n$ can be seen as a functional of $\mathcal{F}_n, \mathcal{H}_n, A^0$, in which $\mathcal{H}_n$ enters only algebraically. As in the discussion of the zeromodes, the duality constraint (10.8) is implemented through integrating out $\mathcal{H}_n$. We are thus left with the proper action

$$S_n = \int -\tfrac{1}{2}r^{-1}\bar{\mathcal{F}}_n\wedge *\mathcal{F}_n + \tfrac{i}{2n}c_B\, \bar{\mathcal{F}}_n\wedge\mathcal{D}\mathcal{F}_n\ , \tag{10.13}$$

where A_n, B_n only appear through $\mathcal{F}_n$.

We are now in a position to write down the total action in $D-1$ dimensions. It reads

$$S = \int -\tfrac{1}{2}r^{-1}\mathcal{F}\wedge *\mathcal{F} - \tfrac{1}{2}c_B\, A^0\wedge\mathcal{F}\wedge\mathcal{F} + \sum_{n=1}^{\infty}\int -r^{-1}\bar{\mathcal{F}}_n\wedge *\mathcal{F}_n + \tfrac{i}{n}c_B\, \bar{\mathcal{F}}_n\wedge\mathcal{D}\mathcal{F}_n\ . \tag{10.14}$$

Note that we sum (10.13) over positive n only, thanks to the reality conditions on A_n, B_n. The action (10.14) should be contrasted with the action (4.20) for a non-chiral p-form on a circle. At the level of zeromodes, we do not have any kinetic term for the p-form B, but we find a Chern-Simons coupling to the Kaluza-Klein vector, which will be important later. As far as excited modes are concerned, we do not find the expected term $-r\bar{\mathcal{H}}_n \wedge *\mathcal{H}_n$, but rather a non-standard kinetic term for B_n: it is first-order in derivatives and it violates five-dimensional parity. In analogy with (4.20), however, it is worth pointing out that the physical degrees of freedom of excited modes can be described in terms of a massive p-form B_n only. In fact, the gauge symmetry (10.6) can be fixed imposing the condition $A_n = 0$, thus setting $\mathcal{F}_n = inB_n$. As a result, the second integral in (10.14) becomes

$$\sum_{n=1}^{\infty} \int -n^2 r^{-1} \bar{B}_n \wedge *B_n + ic_B\, n\, \bar{B}_n \wedge \mathcal{D}B_n \ . \tag{10.15}$$

The classical mass parameter is $m_n = (n^2 r^{-1})(c_B\, n)^{-1} = c_B\, n\, r^{-1}$. This action for $p = 2$ reproduces the action (9.6) that has been used as starting point for our discussion of one-loop corrections induced by massive tensors in five dimensions.

Note that (10.15) is invariant under local $U(1)$ transformations of the complex p-form B_n gauged by A^0. In [247] this gauging is absent, and therefore it is possible to integrate out the real or imaginary part of B_n consistently. The resulting action is the standard massive Proca action for p-forms and has no explicitly parity-violating terms. By contrast, the gauging in (10.15) introduces parity-odd interactions that are essential for our analysis.

The action (10.15) is expected to be supersymmetrizable in many cases of interest, since our findings are reminiscent of tensor hierarchies in supergravity. For $\mathcal{N} = 2$ models in five dimensions, we refer the reader to e.g. [257]. Note also that (10.15) has strong analogies with the Lagrangian of Kaluza-Klein modes χ_n of a higher-dimensional spin-1/2 fermion on the circle,

$$\mathcal{L}_n^{\text{ferm}} = -\bar{\chi}_n \gamma^\mu \mathcal{D}_\mu \chi_n + m_n \bar{\chi}_n \chi_n + \mathcal{L}_n^{\text{supp}} \ . \tag{10.16}$$

First of all, $\mathcal{D}_\mu = \partial_\mu + inA^0_\mu$ contains minimal coupling to A^0 with charge n. Second of all, the lower-dimensional mass parameter $m_n = c_{1/2} n r^{-1}$ depends on the higher-dimensional chirality $c_{1/2}$. Finally, in $\mathcal{L}_n^{\text{supp}}$ couplings are collected which are suppressed by the mass scale r^{-1}. They are of the same form as the non-minimal Pauli-like couplings discussed in section 9.3.4.

10.2 One-loop Chern-Simons terms and M-theory/F-theory duality

In this section we apply the results of chapter 9 to the context of M-theory/F-theory duality in six dimensions. This has been analyzed in chapter 7, where we noticed that Chern-Simons terms in the five-dimensional action of M-theory reduced on an elliptically fibered threefold fall into two distinct categories, see (7.157). On the one hand, some terms can be straightforwardly reproduced on the F-theory side by means of the classical circle reduction of a suitable six-dimensional pseudoaction. On the other hand, some terms can never be obtained in this way, and in section 7.6.3 we have argued that they are generated at one loop once massive Kaluza-Klein modes and W-bosons of the circle reduction are integrated out. We are now in a position to substantiate this claim.

(1,0) theory		(2,0) theory	
gravity multiplet	$(g_{\mu\nu},\, B^+_{\mu\nu},\, 2\psi^+_\mu)$	gravity multiplet	$(g_{\mu\nu},\, 5B^+_{\mu\nu},\, 4\psi^+_\mu)$
tensor multiplet	$(B^-_{\mu\nu},\, \phi,\, 2\psi^-)$	tensor multiplet	$(B^-_{\mu\nu},\, 5\phi,\, 4\psi^-)$
vector multiplet	$(A_\mu,\, 2\psi^+)$		
hypermultiplet	$(4\phi,\, 2\psi^-)$		

Table 10.1: Schematic form of supersymmetric spectra of $(1,0)$ and $(2,0)$ theories. The symbols $g_{\mu\nu}$, $B_{\mu\nu}$, ψ, ϕ represent the metric, a tensor, a Majorana-Weyl spinor, a real scalar field respectively. The prefactor counts the number of fields of a given species within each multiplet. The superscript $\pm$ denotes (anti)self-duality for the tensors B or chirality for the fermions ψ.

More precisely, our focus will be on the gauge and gravitational Chern-Simons actions in five-dimensional low energy effective supergravity theories with eight or sixteen supercharges. The latter case is not directly related to our discussion of F-theory in six dimensions, but it is interesting since it allows us to study the one-loop structure of the circle reduction of an Abelian (2,0) theory in six dimensions. This constitutes a useful preliminary study for possible generalizations to non-Abelian (2,0) theories, which will be addressed in chapter 11.

10.2.1 Field theory prediction

Let us apply the results of the one-loop computation of chapter 9 to the framework of $(1,0)$ and Abelian $(2,0)$ six-dimensional theories compactified on a circle. The field content of their supersymmetry multiplets is summarized in table 10.1 and features chiral fermions and (anti)self-dual tensors.

The requirement of anomaly cancellation imposes some constraints on the spectrum of these theories. For a review of anomaly cancellation in (1,0) theories, see section 7.3.2. The case of (2,0) theories is simpler. We only have to consider purely gravitational anomalies and the anomaly polynomial takes the form [199]

$$I_8^{(2,0)} = \frac{T-21}{(2\pi)^4}\left[-\frac{1}{192}\,\mathrm{tr}\,\mathcal{R}^4 + \frac{1}{768}\,(\mathrm{tr}\,\mathcal{R}^2)^2\right] , \tag{10.17}$$

where T is the number of (2,0) tensor multiplets. From (10.17) we see that as soon as the coefficient of the irreducible term $\mathrm{tr}\,\mathcal{R}^4$ vanishes, the entire polynomial vanishes as well. In summary, the absence of gravitational anomalies requires

$$(1,0): \qquad H - V = 273 - 29\,T \ , \tag{10.18}$$

$$(2,0): \qquad T = 21 \ , \tag{10.19}$$

where T, V, H are the numbers of tensor multiplets, vector multiplets, and hypermultiplets, respectively.

Upon compactification on a circle, the massive Kaluza-Klein modes of chiral fields are precisely given by the three families of massive fields summarized in table 9.1. More precisely, the excited modes of a symplectic Majorana-Weyl spinor are Dirac spinors and the modes of a (anti)self-dual tensor are massive complex self-dual tensors. We adopt conventions such that a positive chirality in six-dimensions correspond to a positive coefficient $c_{1/2}$, c_B, or $c_{3/2}$ in the mass term for excited Kaluza-Klein modes.

Recall that the Ansatz for the metric reads

$$ds_6^2 = ds^2 + r^2(dy - A^0)^2 \ , \tag{10.20}$$

where r is the circle radius and A^0 is the Kaluza-Klein vector. This choice of the sign of A^0 in the metric Ansatz implies that an excited mode with dependence e^{iny} on the internal coordinate couples minimally to A^0 with $U(1)$ covariant derivative $\partial_\mu + inA^0_\mu$. This has to be contrasted with the minimal coupling prescription $\partial_\mu - iqA_\mu$ used in the loop computation of chapter 9. If we identify A^0 and A, we infer that the electric charge q of chapter 9 is given by $q = -n$ for the n-th Kaluza-Klein mode of any six-dimensional field.

In order to compute k_{AFF} and k_{ARR} defined in (9.1) we just have to sum the contributions of table 9.2 according to the spectra listed in 10.1. For a $(1,0)$ theory, we have

$$\begin{aligned} k^{(1,0)}_{AFF} &= -\frac{1}{48\pi^2}\sum_{n=1}^{\infty}(-n)^3\Big[2(V-H-T) + 2\cdot 5 + (1-T)(-4)\Big] = -\frac{9-T}{24(2\pi)^2} \ , \\ k^{(1,0)}_{ARR} &= -\frac{1}{384\pi^2}\sum_{n=1}^{\infty}(-n)\Big[2(V-H-T) + 2\cdot(-19) + (1-T)(+8)\Big] = \frac{12-T}{24(2\pi)^2} \ , \end{aligned} \tag{10.21}$$

where we made use of the anomaly cancellation condition (10.18) and we employed zeta-function regularizations $\sum n^3 \to \zeta(-3) = 1/120$ and $\sum n \to \zeta(-1) = -1/12$ for the divergent sum over Kaluza-Klein levels. In a similar fashion, for a $(2,0)$ theory we find

$$\begin{aligned} k^{(2,0)}_{AFF} &= -\frac{1}{48\pi^2}\sum_{n=1}^{\infty}(-n)^3\Big[4(-T) + 4\cdot 5 + (5-T)(-4)\Big] = 0 \ , \\ k^{(2,0)}_{ARR} &= -\frac{1}{384\pi^2}\sum_{n=1}^{\infty}(-n)\Big[4(-T) + 4\cdot(-19) + (5-T)(+8)\Big] = \frac{T+3}{96(2\pi)^2} = \frac{1}{4(2\pi)^2} \ , \end{aligned} \tag{10.22}$$

where we recalled $T = 21$ from (10.19).

Let us point out that the connection between six-dimensional anomalies and five-dimensional loop corrections to Chern-Simons coupling can also be seen by means of the following heuristic argument. Recall that six-dimensional anomalies emerge in one-loop diagram with four external massless states. Consider an anomalous four-graviton one-loop amplitude and choose the polarization tensors in the external legs in such a way to extract the component $\langle \hat g_{yy}\hat g_{\mu y}\hat g_{\nu y}\hat g_{\rho y}\rangle$, where $\hat g$ denotes the six-dimensional graviton and y is the compact coordinate. As can be seen from (10.3), this four-point function in six-dimensions is related to $r\langle A^0_\mu A^0_\nu A^0_\rho\rangle$ in five dimensions, once the metric component $\hat g_{yy}$ is replaced by its background value r. In six dimensions, the anomalous part of the amplitude is generated by massless chiral fields running in the loop. In five dimensions, we are thus led to compute the contribution to $\langle A^0_\mu A^0_\nu A^0_\rho\rangle$ coming from all Kaluza-Klein modes of these chiral fields. Similar arguments apply to other one-loop corrected Chern-Simons terms in five dimensions.

10.2.2 F-theory check

We have discussed at length the F-theory realization of (1,0) supergravities in chapter 7. Let us now briefly review how (2,0) theories can fit in a similar context. In the standard F-theory paradigm the fibration is non-trivial and the base space is not Ricci-flat. It is of course possible, however, to follow the same chain of dualities from M-theory to F-theory on a six-dimensional internal space that is a direct product of a torus with a Calabi-Yau two-fold, i.e. a K3 surface. The resulting Type IIB setup is precisely a (2,0) theory with 21 tensor multiplets, in accord with the anomaly cancellation condition (10.19). When this theory is reduced on a circle, it should reproduce the five-dimensional effective action of M-theory on $K3 \times T^2$.

Since we are focussing on Chern-Simons coupling, we only need to consider the topological part of the eleven-dimensional M-theory effective action. This contains both the familiar two-derivative Chern-Simons term in (3.34) and the higher-derivative correction considered in section 7.133. Both terms are conveniently written as

$$S^{(11)}_{\text{top}} = \int \Big[-\frac{1}{6}\frac{1}{(2\pi)^2} C_3 G_4 G_4 - \frac{1}{192}\frac{1}{(2\pi)^4} C_3 \left(\text{tr}R^4 - \frac{1}{4}(\text{tr}R^2)^2 \right) \Big] \,, \tag{10.23}$$

where we have suppressed wedge products for brevity. This form of the action is written in a slightly unusual normalization that is best suited to investigate the integrality properties of Chern-Simons couplings. In particular the M-theory three-form C_3 has mass dimension three. Furthermore, this form of the action is consistent with the fact that $\int G_4/(2\pi)$ is half-integrally quantized and that $\exp\, iS$ gives a well-defined functional in the path integral, once all terms of the effective action and the gravitino functional measure are taken into account [62]. This is crucial to match one-loop computations in field-theory, since the standard Feynman rules are derived by an expansion of $\exp\, iS_{\text{int}}$, fixing the absolute normalization of one-loop induced Chern-Simons terms.

Let X_6 denote the internal space, for us Y_3 or $K3 \times T^2$. The M-theory three-form is expanded on a basis $\{\omega_A\}$ of harmonic two-forms on X_6 as

$$C_3 \supset A^A \wedge \omega_A \,, \tag{10.24}$$

where A^A are five-dimensional vectors. They have mass dimension one and their field strengths $F^A = dA^A$ are such that $\int F^A/(2\pi)$ is integrally quantized. Dimensional reduction of the action (10.23) yields the five-dimensional topological terms [204, 207, 234]

$$S^{\text{CS}} = \frac{1}{(2\pi)^2} \int \Big[-\frac{1}{6}\mathcal{K}_{ABC} A^A F^B F^C + \frac{1}{96} c_A A^A \text{tr}R^2 \Big] \,, \tag{10.25}$$

where we have introduced

$$\mathcal{K}_{ABC} = \int_{X_6} \omega_A \wedge \omega_B \wedge \omega_C \,, \quad c_A = \int_{X_6} \omega_A \wedge c_2(X_6) \,. \tag{10.26}$$

Recall from section 7.5.2 that, if $X_6 = Y_3$, it is essential to perform the shift (7.129), so that the Kaluza-Klein vector on the F-theory side is matched with the linear combination of vectors A^A along the direction of the two-form

$$\omega_0 = \mathrm{PD}([B_2]) + \frac{1}{2}c_1(B_2) \ , \tag{10.27}$$

where $\mathrm{PD}([B_2])$ is the Poincaré dual two form of the divisor class of the base B_2 of the elliptic fibration, and $c_1(B)$ is its first Chern class.[2] The geometry of elliptically fibered Calabi-Yau threefolds ensures

$$\begin{aligned} \mathcal{K}_{000} &= \frac{1}{4}\int_{B_2} c_1(B_2)^2 = \frac{1}{4}(10 - h^{1,1}(B_2)) \ , \\ c_0 &= \int_{B_2} \left[c_2(B_2) + 5c_1(B_2)^2\right] = 4(13 - h^{1,1}(B_2)) \ . \end{aligned} \tag{10.28}$$

This in turn implies that the Chern-Simons sector of M-theory on Y_3 contains the terms

$$S^{\mathrm{CS}} \supset \frac{1}{(2\pi)^2}\int \left[- \frac{10 - h^{1,1}(B_2)}{24} A^0 F^0 F^0 + \frac{13 - h^{1,1}(B_2)}{24} A^0 \mathrm{tr} R^2 \right] \ . \tag{10.29}$$

We just have to recall that the number of tensor multiplets of the $(1,0)$ theory is related to the geometry of Y_3 by

$$h^{1,1}(B_2) = T + 1 \ , \tag{10.30}$$

see section 7.6.1, to recognize a perfect match with the field theory prediction of the previous section.

In the case of compactification of M-theory on $X_6 = K3 \times T^2$, the Kaluza-Klein vector is identified with the vector along the only two-form on the torus, which we denote ω_0. As a result,

$$\mathcal{K}_{000} = 0 \ , \quad c_0 = \int_{K3\times T^2} \omega_0 \times c_2(K3 \times T^2) = \int_{K3} c_2(K3) = 24 \ . \tag{10.31}$$

This implies that the gauge Chern-Simons term is absent, while the gravitational Chern-Simons is given by

$$S^{\mathrm{CS}} \supset \frac{1}{(2\pi)^2}\int \frac{1}{4} A^0 \mathrm{tr} R^2 \ , \tag{10.32}$$

in agreement with the field theory computation.

So far we have focused on Chern-Simons coupling involving only the Kaluza-Klein vectors. There are additional terms in the reduction of M-theory on Y_3 that are interpreted as one-loop effects on the F-theory side. They are of the form

$$k_{0ij}\int A^0 F^i F^j + k_{ijk}\int A^i F^j F^k \ , \tag{10.33}$$

where A^i are the five-dimensional vectors that are lifted to six-dimensional vectors. The index i labels the Cartan generators of the gauge group, since the duality between M-theory and F-theory only works in the Coulomb phase. The coefficients k_{0ij}, k_{ijk} can be computed geometrically and are related to the charged spectrum of the theory, see for instance [211, 137].

To compute the coefficient of these couplings in field theory we need to consider diagrams where all massive fields charged under A^0 and/or A^i run. Those are the Kaluza-Klein zeromodes and excited modes of the fields that acquire a mass after the gauge group is broken by giving a non-vanishing VEV

[2]Strictly speaking one has to pull back $c_1(B)$ to Y_3, but we will suppress the pullback in the following.

to the scalars in the five-dimensional vector multiplets. We do not perform here a similar analysis, but the techniques developed so far can be applied to attack this problem. It has indeed been shown in [211] that the Chern-Simons coefficient k_{ijk} receives one-loop corrections by massive gauge degrees of freedom. Furthermore, [187] contains an analysis of Chern-Simons couplings k_{0ij} in the more general context of a possibly rational—as opposed to holomorphic—zero-section of the elliptically fibered Calabi-Yau threefold.

Let us close this section with a comment about a special case that recently attracted interest [130]. Namely, let us consider an M-theory compactification with $\chi(Y_3) = 0$. When Y_3 is elliptically fibered one can lift the theory to a six-dimensional $(1, 0)$ model. For simplicity, we assume that Y_3 has no gauge group singularities and hence the $(1, 0)$ theory has no vector multiplets, $V = 0$. In this case the Euler number is simply given by $\chi = -60 \int_{B_2} c_1(B)^2 = -60(9 - T)$ and we see that $\chi = 0$ implies $T = 9$. The anomaly cancellation condition (10.18) requires then $H = 12$. Can this model be interpreted as a spontaneously broken $(2, 0)$ theory? Suppose we are given a possibly non-Abelian $(2, 0)$ theory with 21 tensor multiplets, in accord with absence of gravitational anomalies. They correspond to 21 tensor multiplets and 21 hypermultiplets in $(1, 0)$ language, as can be seen from table 10.1. Let us further imagine that the original theory undergoes a spontaneous supersymmetry breaking in such a way that only T tensor multiplets out of 21 and and only H hypermultiplets out of 21 remain massless. In order for the resulting $(1, 0)$ theory to be free of gravitational anomalies, we must have $H = 273 - 29T$. The requirement $0 \leq H \leq 21$ together with the integrality of T determines $T = 9$, $H = 12$ as the only possible breaking pattern. This agrees with the geometric setup with $\chi = 0$. Furthermore, for $T = 9$ we have $k^{(1,0)}_{AFF} = 0$, see (10.21), and the term $A^0 \wedge F^0 \wedge F^0$, which is incompatible with 16 supersymmetries, does not enter the circle reduction of the $(1, 0)$ theory. These might be considered as hints in favor of the spontaneous symmetry breaking scenario. If such breaking is actually possible, and how it may be realized, remains to be investigated.

10.3 Exploring the landscape of five-dimensional supergravities

As another application of the results of chapter 9 we would like to address the following question. Suppose we are given a five-dimensional supergravity theory, in terms of its massless spectrum and couplings in the effective action. Is it possible to determine if this theory can be understood as the effective low-energy description of an anomaly-free six-dimensional supergravity theory on a circle?

This investigation can be motivated by the following considerations. On general grounds, it is an interesting problem to study the constraints that gravity places on low-energy quantum field theories. For instance, even-dimensional chiral theories are subject to the requirement of cancellation of gravitational anomalies. In the spirit of [18, 258, 178, 170], one can maybe look for analogue constraints in odd-dimensional theories by exploring classes of models that cannot be seen as a circle reduction of an anomaly-free even-dimensional theory. More specifically, five-dimensional quantum field theories with coupling to gravity and their relations to six-dimensional theories play an important role in many proposals for the low-energy description of the world-volume theory of a stack of M5-branes. This topic is the subject of chapter 11.

It would be desirable to classify those five-dimensional theories which are consistent at the quantum level. This is a formidable task and therefore it is advantageous to first try to understand a subset of these theories, namely those that come from a circle reduction from six dimensions (see figure 1). Of course, not all consistent five-dimensional theories arise in such a circle compactification. Well-known examples include Calabi-Yau threefold reductions of M-theory that in general do not admit a six-dimensional lift if the threefold is not elliptically fibered, see [204, 176] and chapter 7.

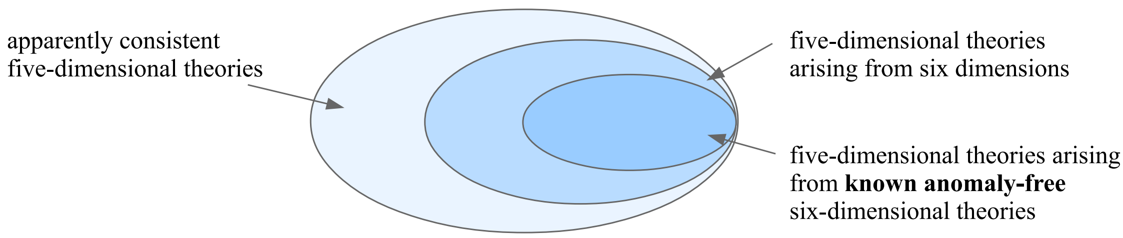


Figure 10.1: Five dimensional effective low-energy theories coupled to gravity which arise through compactification of anomaly-free six-dimensional theories form a subset of all apparently quantum-consistent theories.

Deciding upon this question is generically a highly non-trivial task, for various reasons. On the one hand, in order to extract the low-energy effective action of a six-dimensional theory on a circle one needs not only to perform a classical dimensional reduction, but also to integrate out massive excitations such as Kaluza-Klein modes. Five-dimensional quantum effects due to these massive excitations can make a direct comparison to a possible higher-dimensional action prohibitively difficult. On the other hand, the structure of six-dimensional supergravities is quite rich and is not completely under control. The study of non-Abelian interactions among self-dual tensors, in particular, remains an open problem in the context of $(2,0)$ theories and has been investigated in $(1,0)$ models in the regime where gravity is decoupled [259].

Even if we do not have control over the full class of six-dimensional supergravities, we can still formulate non-trivial conditions for a given five-dimensional theory to be lifted to a specific subset of six-dimensional models. Moreover, there are objects at the quantum level of the theory that are robust under dimensional reduction. Anomalies, and in particular gravitational ones, are examples of such objects, since they are mostly sensitive to more general features of the theory rather than intricate details of the action [199]. In this note, we discuss the possibility to study them using classical and one-loop gauge and gravitational Chern-Simons terms in the theory obtained by compactification on a circle. Reversing the logic, we try to argue that a careful study of Chern-Simons terms in a generic five-dimensional gauge theory allows to obtain non-trivial information about the spectrum (and thus also about the quantum-consistency) of a potential six-dimensional parent theory.

The two setups that we investigate admit eight and sixteen supercharges, respectively. Firstly, we suppose we are given a five-dimensional Abelian action with eight supercharges and we explore the possibility to lift it to a $(1,0)$ theory with simple gauge group. We find that non-trivial necessary conditions can be formulated in terms of the Chern-Simons sector only. Secondly, we take an Abelian theory with sixteen supercharges and we search for a possible lift to an Abelian $(2,0)$ theory. As before, a necessary condition on the Chern-Simons couplings, accompanied by suitable kinetic terms to fix the normalization of the fields, is found.

10.4 Six-dimensional origin of five-dimensional theories

In this section we provide two examples to show that it is possible to quantitatively address the problem of possible six-dimensional origins of a given five-dimensional theory. In particular, one can find explicit constraints on the spectrum and supersymmetry content of the parent six-dimensional theory in terms of the five-dimensional Chern-Simons couplings. Of course our findings based on Chern-Simons terms alone cannot be viewed as a classification of all five-dimensional theories that can arise in a circle compactification in the spirit of figure 10.1. We believe, however, that the content of this section can be seen as a first step towards a systematic analysis of consistency conditions for five-dimensional quantum field theories in the presence of gravity.

10.4.1 $\mathcal{N}=2$ supersymmetric theories

Let us remind the reader that in our notation $\mathcal{N}=2$ supersymmetry corresponds to minimal supersymmetry in five dimensions, i.e. eight real supercharges. We consider minimal supergravity coupled to n Abelian vector multiplets and a number of massless neutral hypermultiplets. The supersymmetric action of such a theory contains the topological couplings

$$S^{(5)}_{CS} = \frac{1}{(2\pi)^2}\int \Big[k_{ABC}\, A^A \wedge F^B \wedge F^C + \kappa_A\, A^A \wedge \mathrm{tr}\,(\mathcal{R}\wedge\mathcal{R}) \Big]\ , \tag{10.34}$$

where A^A, $A=1,\ldots,n+1$ denotes collectively the graviphoton and the vectors from the vector multiplets, $F^A = dA^A$ are the corresponding Abelian field strengths, and $\mathcal{R}$ is the curvature two-form. Supersymmetrizations of the second term are discussed in [202, 260].

If an $\mathcal{N}=2$ theory can be seen as the circle reduction of a six-dimensional theory, it has to come from a $(1,0)$ theory. On the one hand, if the six-dimensional theory had more supersymmetry, we would find more than eight supercharges in five dimensions.[3] On the other hand, it seems impossible to lift the five-dimensional gravitino of an $\mathcal{N}=2$ theory to a consistent, interacting six-dimensional theory with no supersymmetry. Note that a five-dimensional theory with massless $U(1)$ gauge fields can arise as low energy effective action of a possibly non-Abelian six-dimensional theory on a circle. This is what happens when the gauge group is broken to the five-dimensional Coulomb branch by

[3] Here we consider only simple compactifications on a circle. In particular, we do not discuss any compactification mechanism which (partially) breaks supersymmetry.

giving a VEV to the scalars in the five-dimensional vector multiplets. For simplicity, in the following we study the possibility to lift the five-dimensional theory to a non-Abelian $(1,0)$ with simple gauge group G. The generalization to semi-simple G is straightforward. The inclusion of $U(1)$ factors is also possible, but would make the analysis of the six-dimensional action and anomalies more involved.

The first step in the search for a parent six-dimensional theory is to determine if the five-dimensional spectrum can be lifted to six dimensions. Five-dimensional hypermultiplets directly lift to six-dimensional hypermultiplets, which are allowed in the $(1,0)$ theory. To understand the possible lift of the vector sector to six dimensions one has to divide the $n+1$ five-dimensional vector fields A^B into three sets:

- the vector A^0 that lifts to the Kaluza-Klein vector in the reduction of the six-dimensional metric on a circle;
- the vectors A^α, $\alpha = 1,\ldots,T+1$ that lift to components of T six-dimensional tensor multiplets and a single tensor in the supergravity multiplet;
- the vectors A^i, $i = 1,\ldots,\mathrm{rank}(G)$ that lift to Cartan elements of six-dimensional gauge group G.

Furthermore, to allow for a consistent six-dimensional parent theory, the constants k_{ABC} and κ_A in (10.34) have to split in such a way to accommodate the following Chern-Simons terms for the above mentioned classes of vector fields

$$
\begin{aligned}
S^{(5)}_{CS} = \frac{1}{(2\pi)^2}\int\Big[-\frac{1}{2}\Omega_{\alpha\beta}A^0F^\alpha F^\beta + \frac{1}{2}b^\alpha\Omega_{\alpha\beta}C_{ij}\,A^\beta F^i F^j - \frac{1}{8}a^\alpha\Omega_{\alpha\beta}A^\beta \mathrm{tr}\,R^2\Big] \\
+\frac{1}{(2\pi)^2}\int\Big[k_0\,A^0F^0F^0 + k_{ij}\,A^0F^iF^j + k_{ijk}\,A^iF^jF^k + \kappa_0\,A^0\mathrm{tr}\,R^2\Big]\,,
\end{aligned}
\tag{10.35}
$$

where we suppressed wedge products for brevity. As discussed in chapter 7 and for example in [170], only the Chern-Simons terms in the first line can be lifted to a classical six-dimensional action, while the terms in the second line cannot be obtained by classical reduction on a circle. We know from sections 10.2.1 and 10.2.2, however, that they do come from a six-dimensional action as soon as quantum effects are included in the dimensional reduction. It is precisely the interplay between these two subsets of Chern-Simons terms that allows us to formulate necessary conditions for the five-dimensional theory to come from an anomaly-free $(1,0)$ theory.

It is useful to recall from section 7.3 that the constant symmetric matrix $\Omega_{\alpha\beta}$ has signature $(1,T)$ and is identified with the $SO(1,T)$ invariant metric associated to the moduli space $SO(1,T)/SO(T)$ of the scalars in the tensor multiplets in six-dimensions. The matrix C_{ij} is identified with the Cartan matrix of the gauge group G. The constant vectors b^α and a^α are the coefficients of the Green-Schwarz terms that cancel factorizable anomalies, see section 7.3.2. Note also that the vector b^α determines the kinetic term of six-dimensional vectors, as can be seen from (7.41).

As mentioned above, the requirement of anomaly cancellation in the parent $(1,0)$ theory allows us to formulate necessary conditions on the Chern-Simons terms for the lift to six-dimensions to be possible. In the following, we focus on six-dimensional gravitational anomalies, since they do not depend on many details of the charged hypermultiplet spectrum in six dimensions. The conditions for the absence of purely gravitational a anomalies in (1,0) theories have been give in section 7.3.2, but we record them here again for convenience,

$$H - V = 273 - 29\,T \ , \qquad a^\alpha \Omega_{\alpha\beta} a^\beta = 9 - T \ , \tag{10.36}$$

where as usual T, V, H are the number of six-dimensional tensor multiplets, vector multiplets, and hypermultiplets, respectively. To check the first condition in (10.36) directly we would need to know the number of hypermultiplets H in six dimensions. This number, however, is in general different from the number of neutral massless hypermultiplets in five dimensions, since some charged hypermultiplets become massive after breaking of the gauge group, and therefore do not appear in the five-dimensional effective action.

This problem can be circumvented by studying the Chern-Simons terms in (10.35). In particular, the couplings k_0 and κ_0 encode information about the gravitational anomaly cancellation conditions (10.36). To see this, let us first recall from section 10.2.1 that k_0 and κ_0 can be computed explicitly by summing the contributions of all Kaluza-Klein modes of chiral fields in six-dimensions, with result

$$k_0 = \frac{1}{24}(T-9) \ , \qquad \kappa_0 = \frac{1}{24}(12-T) \ . \tag{10.37}$$

These expressions hold under the assumption that the first condition in (10.36) is satisfied, but they only involve the number T of tensor multiplets of the theory, which can be read off from range of the α indices in (10.35). Combining (10.37) with the second condition in (10.36) we get the following necessary conditions for the Chern-Simons terms (10.35) to be lifted to six-dimensional theory free of gravitational anomalies:

$$24\,k_0 = -a^\alpha \Omega_{\alpha\beta} a^\beta = T - 9 \ , \qquad 24\,\kappa_0 = a^\alpha \Omega_{\alpha\beta} a^\beta + 3 = 12 - T \ . \tag{10.38}$$

These equations encode three independent requirements and cannot be trivially satisfied by rescaling A^0 and A^α.

One can formulate similar tests on the Chern-Simons coefficients in (10.35) to check if the candidate parent theory is free of purely gauge anomalies. Such conditions involve a comparison between $b^\alpha \Omega_{\alpha\beta} b^\beta$ and the coupling k_{ijk}, which contains crucial information about the six-dimensional charged hypermultiplet spectrum [211, 137]. While it was only shown for specific examples [137], and not yet in general, that the knowledge of the Chern-Simons coefficients allows to check cancellation of six-dimensional gauge anomalies, we believe that such a statement should hold in general. In a similar way, we suspect that conditions involving $a^\alpha \Omega_{\alpha\beta} b^\beta$ and the Chern-Simons coupling k_{ij} can be used to test if the six-dimensional theory is free of mixed gauge-gravitational anomalies.

10.4.2 $\mathcal{N} = 4$ supersymmetric theories

We can apply the strategy outlined so far also to five-dimensional theories with sixteen supercharges, denoted $\mathcal{N} = 4$. We restrict to the theory of n Abelian vector multiplets coupled to supergravity.

Recall that the $\mathcal{N}=4$ supergravity multiplet contains six vectors. Five of them form the **5** representation of the $SO(5)_R$ R-symmetry group, while the sixth one is a singlet.[4] The singlet will be denoted A^0, and the remaining ones together with the n gauge fields from the vector multiplets are denoted A^A, $A = 1, \dots, n+5$. The collective index A is a fundamental $SO(5,n)$ index. The associated constant metric is denoted η_{AB}. With this notation the topological sector of the action reads

$$S^{(5)}_{CS} = \frac{1}{(2\pi)^2} \int \Big[-\frac{1}{2} \eta_{AB}\, A^0 \wedge F^B \wedge F^C + \kappa_0\, A^0 \wedge \mathrm{tr}\,(R \wedge R) \Big] \,. \tag{10.39}$$

To the best of our knowledge it has not been shown that the gravitational Chern-Simons coupling can be supersymmetrized. We will see, however, that in some circumstances it can be generated at the quantum level from a six-dimensional theory with sixteen supercharges on a circle. We thus expect it to be an admissible coupling in the five-dimensional $\mathcal{N}=4$ action.

In contrast to the $\mathcal{N}=2$ case, the Chern-Simons sector of an $\mathcal{N}=4$ theory is too simple to provide any test that cannot be trivially satisfied by means of rescaling of A^0, A^A. Therefore, we also need to record some kinetic terms in order to fix this ambiguity. This requires some additional notation. Each vector multiplet contributes five scalars to the spectrum. These $5n$ scalars parametrize the coset space $SO(5,n)/SO(5) \times SO(n)$. This is conveniently described in terms of matrices ${L_A}^i$, ${L_A}^I$, where i, I are fundamental indices of $SO(5)$, $SO(n)$ respectively. These matrices satisfy

$$\eta_{AB} = \delta_{ij} {L_A}^i {L_B}^j - \delta_{IJ} {L_A}^I {L_B}^J \ , \quad G_{AB} = \delta_{ij} {L_A}^i {L_B}^j + \delta_{IJ} {L_A}^I {L_B}^J \ , \tag{10.40}$$

where G_{AB} is a non-constant, positive-definite matrix that enters the gauge coupling function. The needed kinetic terms are

$$S^{(5)}_{\mathrm{kin}} = \frac{1}{(2\pi)^2} \int \Big[R * 1 - \frac{1}{2} d\sigma \wedge * d\sigma - \frac{1}{2} e^{2\sigma/\sqrt{6}} G_{AB} F^A \wedge * F^B - \frac{1}{2} e^{-4\sigma/\sqrt{6}} F^0 \wedge * F^0 \Big] \ , \tag{10.41}$$

in which σ is the scalar in the gravity multiplet. The sum $S^{(5)}_{CS} + S^{(5)}_{\mathrm{kin}}$ can be supersymmetrized since it coincides with part of the standard form of the five-dimensional $\mathcal{N}=4$ action as found e.g. in [261], up to field redefinitions.[5]

The five-dimensional $\mathcal{N}=4$ theory under examination can come from circle reduction of a $(2,0)$ or $(1,1)$ theory. Since $(1,1)$ theories are non-chiral, we cannot use anomalies as a check of the quantum consistency of the candidate parent theory. For this reason, in the rest of this section we formulate necessary conditions for the lift of the five-dimensional theory to a $(2,0)$ theory, and we do not give conditions for the lift to a $(1,1)$ theory. Furthermore, since a six-dimensional action for non-Abelian $(2,0)$ is not known, we explore the possibility to lift the five-dimensional theory to an Abelian $(2,0)$ theory.

Recall that in such a theory the only matter multiplets are tensor multiplets. As we have seen in section 10.2.1, cancellation of gravitational anomalies requires a number $T = 21$ of them. This

[4] This structure is fixed by identifying the five-dimensional gravity multiplet.

[5] More precisely, we have performed an overall rescaling of the action, together with the redefinitions $\sigma_{\text{there}} = \sigma_{\text{here}}/\sqrt{2}$, $A^0_{\text{there}} = A^0_{\text{here}}/\sqrt{2}$, $A^A_{\text{there}} = A^A_{\text{here}}/\sqrt{2}$. Our form of the action is best suited for comparison between tree-level and one-loop terms. It is such that the action and the vectors both have period 2π. It has been inferred by deriving $S^{(5)}_{CS}$ from M-theory on $K3 \times T^2$ making use of the effective action discussed in [62].

implies that the five-dimensional theory must have exactly 26 vectors in addition to the singlet A^0. This provides a first elementary check on (10.39). A far less trivial check comes from the gravitational Chern-Simons coupling κ_0. It cannot be generated by reduction of the classical Abelian $(2,0)$ action on a circle, and it is rather generated by one-loop diagrams in which massive Kaluza-Klein modes run in the loop. We can read off the value of this coupling from the results of section 10.2.1,

$$\kappa_0 = \frac{1}{4} \ . \tag{10.42}$$

If in $S^{(5)}_{CS} + S^{(5)}_{\text{kin}}$ a different value of κ_0 appears, the theory cannot be lifted to an Abelian $(2,0)$ theory.

CHAPTER 11

Non-Abelian tensor towers and (2,0) theories

In this chapter we present a proposal for a five-dimensional action designed to capture some features of interacting six-dimensional (2,0) theories. A five-dimensional approach allows us to elude some of the immediate difficulties in formulating a Lagrangian for self-dual tensors. It is inspired by the M-theory/F-theory duality for F-theory vacua in six dimensions, see section 5.3, but it can be also related to the proposal of [33, 34] about (2,0) theories and five-dimensional maximally supersymmetric Yang-Mills theory.

11.1 The search for a five-dimensional Lagrangian description

As we have seen in chapter 6, among the most interesting implications of M-theory and string theory is the existence of interacting superconformal quantum field theories in six dimensions with (2,0) supersymmetry. They are labelled by ADE Dynkin diagrams and reduce to maximally supersymmetric Yang-Mills in five dimensions. Therefore, they are expected to possess some sort of gauge symmetry, even though they do not have any massless vector in their spectrum. Crucially, they have instead massless antiself-dual tensors.

There are some immediate complications that have to be addressed in the search for a Lagrangian description of $(2,0)$ theories. Two separate problems are particularly prominent. Firstly, the naive Lorentz covariant kinetic term for tensors vanishes identically upon imposing the (anti)self-duality constraint. Different solutions to this problem have been proposed, based on breaking of manifest Lorentz invariance, introduction of auxiliary fields, or a holographic approach [251, 247, 262]. Secondly, the 'gauge group' structure of the theory is particularly elusive, as the absence of vectors prevents any naive attempt to write down non-Abelian gauge covariant derivatives. Indeed, (2,0) theories are believed to be connected to the formalism of gerbes, rather than vector bundles. Recent discussions

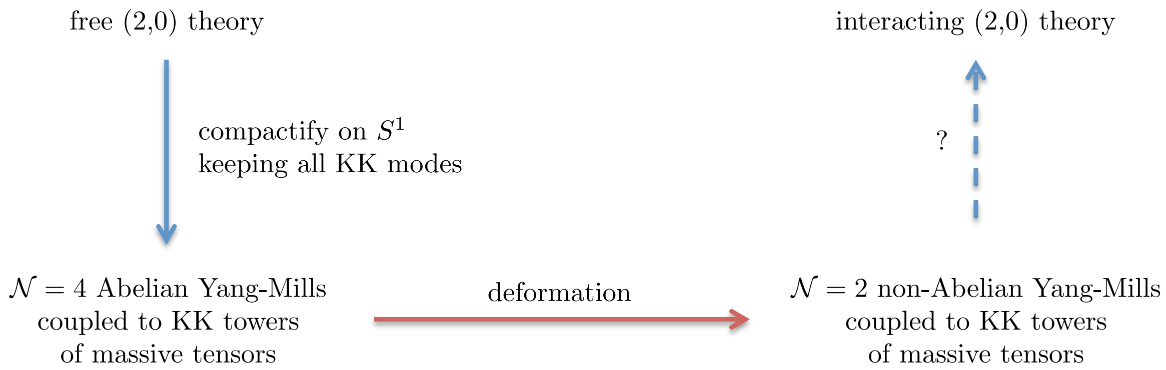


Figure 11.1: Schematic representation of the five-dimensional approach to interacting (2,0) theories followed in this chapter. A free (2,0) theory can be described by a pseudoaction which can be compactified on a circle keeping all Kaluza-Klein modes. This has been done for (anti)self-dual tensors in section 10.1. The resulting five-dimensional theory is maximally supersymmetric Abelian Yang-Mills theory coupled to infinite towers of matter fields, including massive tensors. In this chapter we study a deformation of this theory that preserves all degrees of freedom and switches on a non-Abelian gauging. This deformed theory has manifest $\mathcal{N} = 2$ supersymmetry. It can be the starting point for an indirect exploration of interacting (2,0) theories.

about the various complications in formulating $(2,0)$ theories can be found in [263, 35, 264, 265, 37, 266, 267].

In this chapter we approach $(2,0)$ theories by studying a five-dimensional action for an infinite tower of modes that can be interpreted as Kaluza-Klein states. We propose that using this perspective one can address both the self-duality as well as the non-Abelian gaugings at the level of an action. Our program is summarized in figure 11.1. The dynamics of an Abelian (2,0) theory is trivial and can be captured by a simple quadratic pseudoaction. As we have seen in section 10.1 this can be reduced on a circle keeping all excited Kaluza-Klein modes so that the resulting five-dimensional theory has an infinite number of massive tensor fields, together with massless vectors. The latter are crucial, because they allow us to study deformations of the five-dimensional theory that include some non-Abelian gauge group. The resulting deformed action is hopefully able to capture a subset of the couplings of the sought-for (2,0) theory, or some robust feature thereof.

More precisely, we will write a five-dimensional superconformal action with $\mathcal{N} = 2$ supersymmetry, i.e. with eight supercharges, whose spectrum contains all the expected degrees of freedom of the six-dimensional $(2,0)$ tensor multiplets compactified on a circle. The theory also features an additional $\mathcal{N} = 2$ vector multiplet containing the circle radius and the Kaluza-Klein vector of the six-dimensional metric. The total gauge group is of the form $G \times U(1)$, where G is a simple simply-laced non-Abelian group that is interpreted as the 'gauge group' of the (2,0) theory, while the $U(1)$ factor is associated to the Kaluza-Klein vector. The gauge bosons of G and their supersymmetry partners are contained in $\mathcal{N} = 2$ vector multiplets and hypermultiplets that are neutral under the Kaluza-Klein $U(1)$, and are thus interpreted as zeromodes. The remaining infinite collection of $\mathcal{N} = 2$ tensor multiplets and hypermultiplets are labelled by an integer that also corresponds to their charge under the Kaluza-Klein $U(1)$. They are therefore regarded as excited modes. Our action fits in the general $\mathcal{N} = 2$ superconformal framework of [268, 269] that extends and applies [270, 201, 271, 200, 257].[1] It is crucial, however, that all couplings in our theory are only given in terms of group theoretical constants associated to G and the Kaluza-Klein levels.

The five-dimensional superconformal invariance of the $\mathcal{N} = 2$ action is implemented in a way compatible with a subgroup of the six-dimensional superconformal group. This implies that the additional vector multiplet, containing the circle radius and the Kaluza-Klein vector, has to transform in accord with the six-dimensional line element. However, in order to more directly interpret the $\mathcal{N} = 2$ superconformal action as a Kaluza-Klein theory, one has to fix superconformal invariance. We consider a restriction of the action that preserves $\mathcal{N} = 2$ Poincaré supersymmetry by giving a vacuum expectation value to the entire multiplet containing the circle radius and the Kaluza-Klein vector. After this gauge-fixing the infinite tower of tensor multiplets and hypermultiplets will gain a mass proportional to the Kaluza-Klein scale set by the circle radius. The non-Abelian gaugings and the realization of only half the maximal supersymmetry, however, prevent us from lifting the five-dimensional theory directly to six dimensions.

In two special cases, however, our action has a clear six-dimensional interpretation, thus furnishing a first simple sanity check of our formalism. Firstly, considering zero modes alone the restricted $\mathcal{N} = 2$ action reduces to only maximally supersymmetric Yang-Mills theory with gauge group G. The zero mode sector is automatically invariant under sixteen supercharges and is thus $\mathcal{N} = 4$ supersymmetric. Secondly, if non-Abelian gaugings are switched off the five-dimensional action including all excited modes is again automatically invariant under $\mathcal{N} = 4$ supersymmetry and coincides with the circle compactification of the (2,0) pseudoaction for Abelian tensor multiplets.

The connection between six-dimensional anomalies and five-dimensional one-loop Chern-Simons terms encountered in sections 10.2 and 10.4 hints to the fact that our action can be used to probe anomalies of (2,0) theories. For instance, one can couple the five-dimensional theory to a background vector gauging R-symmetry and use the quantum-corrected Chern-Simons terms for this vector as window on the R-symmetry anomalies of the (2,0) theories. This is related to their conformal anomaly, which has received a lot of attention recently [273, 153, 158, 274, 275, 276]. We refer the reader to [277] for a first step in this application of the action proposed in this chapter, as well as for a discussion of harmonic superspace inspired techniques to achieve R-symmetry and supersymmetry enhancement.

11.2 Supersymmetric spectrum and non-Abelian gauging

This section is devoted to the discussion of the supersymmetric spectrum of the five-dimensional theories of non-Abelian tensors which will be constructed in the following sections. Our starting point consists of a number of tensor multiplets of six-dimensional rigid $(2,0)$ superconformal symmetry. This spectrum is dimensionally reduced on a circle and the resulting $\mathcal{N} = 4$ supermultiplets are described.

[1] Recent progress on the construction of $(1,0)$ superconformal theories in six dimensions can be found in [259, 272].

Moreover, a mechanism for a non-Abelian gauging of tensors is implemented. The decomposition of the $\mathcal{N}=4$ spectrum into $\mathcal{N}=2$ multiplets and the discussion of conformal invariance is relegated to section 11.3.

11.2.1 (2,0) tensor multiplets

Let $\boldsymbol{\mathcal{T}}^I$ be a collection of $(2,0)$ tensor multiplets in six dimensions. The index I plays here the role of a degeneracy index, but will be identified with an adjoint index of a non-Abelian gauge group in subsection 11.2.4. Boldface symbols will be used throughout to denote six-dimensional quantities. The field content of $\boldsymbol{\mathcal{T}}^I$ is given by

$$\boldsymbol{\mathcal{T}}^I = (\mathbf{B}^I_{\boldsymbol{\mu\nu}}, \boldsymbol{\sigma}^{I\,ij}, \boldsymbol{\lambda}^{I\,i}) \ , \tag{11.1}$$

where $\mathbf{B}^I_{\boldsymbol{\mu\nu}}$ is a tensor (two-form), $\boldsymbol{\sigma}^{I\,ij}$ are scalars, $\boldsymbol{\lambda}^{I\,i}$ are spin-1/2 fermions. In our conventions, the supersymmetry parameter is a left-handed Weyl spinor, the tensors have negative chirality, i.e. their field strength $\boldsymbol{\mathcal{H}}^I = d\mathbf{B}^I$ obey the antiself-duality constraint $*\boldsymbol{\mathcal{H}}^I = -\boldsymbol{\mathcal{H}}^I$, and the fermions $\boldsymbol{\lambda}^{I\,i}$ are right-handed Weyl spinors. Indices $i,j = 1,\dots 4$ are indices of the $\mathbf{4}$ representation of $USp(4)_R$, the R-symmetry group of the $(2,0)$ supersymmetry algebra. The tensors $\mathbf{B}^I_{\boldsymbol{\mu\nu}}$ are singlets of $USp(4)_R$, the fermions $\boldsymbol{\lambda}^{I\,i}$ transform in the $\mathbf{4}$ representation, while the scalars $\boldsymbol{\sigma}^{I\,ij}$ belong to the $\mathbf{5}$ representation, i.e. they are antisymmetric and traceless

$$\boldsymbol{\sigma}^{I\,ij} = -\boldsymbol{\sigma}^{I\,ji} \ , \qquad \Omega_{ij}\boldsymbol{\sigma}^{I\,ij} = 0 \ . \tag{11.2}$$

In the last equation Ω_{ij} is the primitive antisymmetric invariant of $USp(4)_R$. We refer the reader to section A.2 in appendix A for our conventions. Tensor multiplets are pseudoreal, i.e. they satisfy

$$(\boldsymbol{\mathcal{T}}^I)^* = \boldsymbol{\mathcal{T}}^I \ : \quad \begin{cases} \bar{\mathbf{B}}^I_{\boldsymbol{\mu\nu}} \equiv (\mathbf{B}^I_{\boldsymbol{\mu\nu}})^* = \mathbf{B}^I_{\boldsymbol{\mu\nu}} \ , \\ \bar{\boldsymbol{\sigma}}^I_{ij} \equiv (\boldsymbol{\sigma}^{I\,ij})^* = \Omega_{ik}\Omega_{jl}\boldsymbol{\sigma}^{I\,kl} \ , \\ \bar{\boldsymbol{\lambda}}^{I\,i} \equiv (\boldsymbol{\lambda}^I_i)^\dagger\boldsymbol{\gamma}^0 = \Omega^{ij}(\boldsymbol{\lambda}^I_j)^\mathsf{T}\mathbf{C} \ . \end{cases} \tag{11.3}$$

The last line encodes the usual symplectic-Majorana condition. The quantities $\boldsymbol{\gamma}^0, \mathbf{C}$ are the timelike gamma matrix and the charge conjugation matrix in six dimensions, respectively.

The $(2,0)$ Poincaré superalgebra can be enlarged to the superconformal algebra $OSp(8^*|4)$ [278, 30]. This requires the introduction of new generators for dilatations, conformal boosts, special supersymmetry transformations, and R-symmetry transformations. The action of these generators on physical fields can be found in [279, 280]. A more detailed discussion the rigid superconformal theory will be given in sections 11.3 and 11.4.1 in the context of $\mathcal{N}=2$ supersymmetry in five dimensions. In this section we just focus on the Weyl weights, which are the charges under dilatations. For the fields in the tensor multiplets $\boldsymbol{\mathcal{T}}^I$ they are collected in Table 11.1.

Let us now discuss the Poincaré supersymmetry transformations and the pseudoaction of a collec-

multiplet	fields	type	comments	$USp(4)_R$	Weyl weight
$\boldsymbol{\mathcal{T}}^I$ massless tensor multiplet	$\mathbf{B}^I_{\mu\nu}$	antiself-dual tensor	pseudoreal	**1**	0
	$\boldsymbol{\sigma}^{I\,ij}$	scalar	pseudoreal	**5**	2
	$\boldsymbol{\lambda}^{I\,i}$	right-handed spinor	pseudoreal	**4**	5/2

Table 11.1: Field content of an on-shell tensor multiplet $\boldsymbol{\mathcal{T}}$ of rigid $(2,0)$ superconformal symmetry in six dimensions. The precise formulation of the reality properties of the fields is found in (11.3).

tion of non-interacting tensor multiplets $\boldsymbol{\mathcal{T}}^I$. The $(2,0)$ supersymmetry transformations read [280][2]

$$\begin{aligned} \delta(\boldsymbol{\epsilon})\mathbf{B}^I_{\mu\nu} &= -\bar{\boldsymbol{\epsilon}}^i\gamma_{\boldsymbol{\mu\nu}}\boldsymbol{\lambda}^I_i \ , \\ \delta(\boldsymbol{\epsilon})\boldsymbol{\lambda}^{I\,i} &= \tfrac{1}{6}\boldsymbol{\mathcal{H}}^I_{\boldsymbol{\mu\nu\rho}}\boldsymbol{\gamma}^{\boldsymbol{\mu\nu\rho}}\boldsymbol{\epsilon}^i + 2\boldsymbol{\gamma}^{\boldsymbol{\mu}}\partial_{\boldsymbol{\mu}}\boldsymbol{\sigma}^{I\,ij}\boldsymbol{\epsilon}_j \ , \\ \delta(\boldsymbol{\epsilon})\boldsymbol{\sigma}^{I\,ij} &= -4\left(\bar{\boldsymbol{\epsilon}}^{[i}\boldsymbol{\lambda}^{I\,j]} + \tfrac{1}{4}\Omega^{ij}\bar{\boldsymbol{\epsilon}}^k\boldsymbol{\lambda}^I_k\right) \ . \end{aligned} \tag{11.4}$$

Recall that the tensor field strength is defined as $\boldsymbol{\mathcal{H}}^I_{\boldsymbol{\mu\nu\rho}} = 3\partial_{[\boldsymbol{\mu}}\boldsymbol{B}^I_{\boldsymbol{\nu\rho}]}$. Note that contraction with $\boldsymbol{\gamma}^{\boldsymbol{\mu\nu\rho}}\boldsymbol{\epsilon}^i$ automatically selects the antiself-dual part of the field strength, because $\boldsymbol{\epsilon}^i$ is a left-handed Weyl spinor in our conventions. The supersymmetry algebra closes only up to the free-field equations of motion for $\boldsymbol{B}^I_{\boldsymbol{\mu\nu}}, \boldsymbol{\lambda}^{I\,i}, \boldsymbol{\sigma}^{I\,ij}$. They can be derived from the following supersymmetric pseudoaction:

$$S^{(6)} = \int d^6x\, d_{IJ}\Big\{ -\tfrac{1}{12}\,\boldsymbol{\mathcal{H}}^{I\,\boldsymbol{\mu\nu\rho}}\boldsymbol{\mathcal{H}}^J_{\boldsymbol{\mu\nu\rho}} - \tfrac{1}{2}\,\partial^{\boldsymbol{\mu}}\boldsymbol{\sigma}^{I\,ij}\partial_{\boldsymbol{\mu}}\boldsymbol{\sigma}^J_{ij} - \tfrac{1}{4}\,\bar{\boldsymbol{\lambda}}^{I\,i}\boldsymbol{\gamma}^{\boldsymbol{\mu}}\partial_{\boldsymbol{\mu}}\boldsymbol{\lambda}^J_i\Big\} \ . \tag{11.5}$$

We stress that this is not a proper action, since the self-duality constraint on the field strengths of tensors cannot be derived from it, and has to be imposed at the level of the equations of motion. In order to write down kinetic terms, the symmetric, positive-definite, constant matrix d_{IJ} has been introduced.

Finding a non-Abelian deformation of the six-dimensional pseudoaction (11.5) is a formidable task. In particular, there are no vectors in the spectrum which could be used as gauge connections. Indeed, $(2,0)$ gauge theories of tensors are conjectured to be a non-Abelian generalization of gerbes, with two-form connections [281, 282, 267] (see also [283]). As mentioned in section 11.1, our strategy is to avoid these difficulties by performing the gauging in the reduced five-dimensional theory.

11.2.2 Compactification on a circle and five-dimensional $\mathcal{N} = 4$ spectrum

We compactify one spatial dimension on a circle using the standard Kaluza-Klein Ansatz for the metric,

$$\mathbf{g}_{\boldsymbol{\mu\nu}}d\mathbf{x}^{\boldsymbol{\mu}}d\mathbf{x}^{\boldsymbol{\nu}} = g_{\mu\nu}dx^\mu dx^\nu + r^2(dy - A^0_\mu dx^\mu)^2 \ . \tag{11.6}$$

On the right hand side $g_{\mu\nu}$ is the five-dimensional metric, r is the radius of the circle, $y \sim y + 2\pi$ is the compact coordinate along the circle, and A^0_μ is the Kaluza-Klein vector with Abelian field

[2] Compared to reference [280], the fields and the supersymmetry parameter have been rescaled by suitable factors to achieve canonical normalization in the pseudoaction below.

strength $F^0 = dA^0$. In the rigid limit, $g_{\mu\nu}$ is the flat Minkowski metric, r is constant and A^0 vanishes. Later on, we will promote these quantities to fields, however, since they will play a crucial role in the superconformal theories of section 11.4.

Upon compactification on a circle, the scalars $\boldsymbol{\sigma}^{I\,ij}$ and the spinors $\boldsymbol{\lambda}^{I\,i}$ give rise to a Kaluza-Klein tower of five-dimensional scalars $\sigma_n^{I\,ij}$ and spinors $\lambda_n^{I\,i}$, where $n \in \mathbb{Z}$. More precisely we write

$$\boldsymbol{\sigma}^{I\,ij} = r^{-1} \sum_{n\in\mathbb{Z}} e^{iny} \sigma_n^{I\,ij} \ , \qquad \boldsymbol{\lambda}^{I\,i} = r^{-1} \sum_{n\in\mathbb{Z}} e^{iny} \lambda_n^{I\,i} \otimes \eta \ , \tag{11.7}$$

where η is a constant two-component spinor. Note that we have included a factor of r^{-1} in the Kaluza-Klein Ansatz, in order to have five-dimensional fields $\sigma^{I\,ij}, \lambda^{I\,i}$ of canonical dimensions 1 and 3/2, respectively. These fields are also the natural variables compatible with the lower-dimensional supersymmetry. As far as the tensors are concerned, reduction of $\mathbf{B}^I_{\boldsymbol{\mu\nu}}$ furnishes both a tower of tensors $B^I_{n\,\mu\nu}$ and of vectors $A^I_{n\,\mu}$ in five dimensions, see section 10.1. We can write

$$\mathbf{B}^I = \sum_{n\in\mathbb{Z}} e^{iny} \left[B_n^I + A_n^I \wedge (dy - A^0) \right] \ . \tag{11.8}$$

As a consequence of the six-dimensional antiself-duality constraint, $B^I_{n\,\mu\nu}$ and $A^I_{n\,\mu}$ do not contain independent degrees of freedom. On the one hand, the antiself-duality constraint can be used to eliminate the tensor zero modes $B^I_{0\,\mu\nu}$ from the spectrum of the five-dimensional theory, keeping the vector zero modes $A^I_\mu \equiv A^I_{0\,\mu}$ only. On the other hand, excited modes $B^I_{n\,\mu\nu}, A^I_{n\,\mu}$ are related by a Stückelberg-like symmetry in the invariant derivative $F^I_n = dA^I_n + inB^I_n$, as in [247] and in section 10.1. In this way $B^I_{n\,\mu\nu}$ can 'eat' $A^I_{n\,\mu}$ and become a massive tensor field in five dimensions. In conclusion, reduction of $\mathbf{B}^I_{\boldsymbol{\mu\nu}}$ yields a massless vector A^I_μ and a tower of complex massive tensors $B^I_{n\,\mu\nu}$. A purely bosonic Lagrangian for $A^I_\mu, B^I_{n\,\mu\nu}$ coupled to the Kaluza-Klein vector A^0_μ has been given in (10.14) and (10.15). For our present purposes it is conveniently written as

$$\begin{aligned} \mathcal{L}_{\text{tens}} \;=\;& d_{IJ}\Big[-\tfrac{1}{4} r^{-1} F^I_{\mu\nu} F^{J\,\mu\nu} - \tfrac{1}{8}\epsilon^{\mu\nu\lambda\rho\sigma} A^0_\mu\, F^I_{\nu\lambda}\, F^J_{\rho\sigma} \Big] \\ &+ \sum_{n=1}^{\infty} d_{IJ}\Big[-\tfrac{1}{2} r^{-1} \bar{F}^I_{n\,\mu\nu} F_n^{J\,\mu\nu} + \tfrac{i}{4n}\epsilon^{\mu\nu\lambda\rho\sigma} \bar{F}^I_{n\,\mu\nu}\, \mathcal{D}^{\text{KK}}_\lambda F^J_{n\,\rho\sigma} \Big] \ . \end{aligned} \tag{11.9}$$

On the right hand side we have introduced the Abelian field strength $F^I = dA^I$ and we have used the Stückelberg gauge-fixed expression for the tensors

$$F^I_{n\,\mu\nu} = inB^I_{n\,\mu\nu} \ . \tag{11.10}$$

It will be convenient to use this rescaled $F^I_{n\,\mu\nu}$ to represent the tensors in the remainder of this work. Indices I, J are contracted with a constant metric d_{IJ}. In section 11.2.4 it will be related to group-theoretical invariants after the degeneracy index I is promoted to a gauge index. We have also made use of the shorthand notation $\mathcal{D}^{\text{KK}}_\mu X_n = \partial_\mu X_n + inA^0_\mu X_n$ for generic Kaluza-Klein modes X_n. More information about this covariant derivative will be given in section 11.2.3.

The main purpose of our work is to provide a supersymmetric non-Abelian generalization of the action (11.9). As a first step, we discuss how five-dimensional fields are organized in $\mathcal{N} = 4$ multiplets.

The R-symmetry group is again $USp(4)_R$, and the transformation properties of the fields under R-symmetry are unaffected by dimensional reduction. The vector zero mode A^I combines with the zero modes $\sigma^{I\,ij} \equiv \sigma_0^{I\,ij}$ and $\lambda^{I\,i} \equiv \lambda_0^{I\,i}$ into a single vector multiplet which we will denote as

$$\mathcal{V}^I = (A_\mu^I, \sigma^{I\,ij}, \lambda^{I\,i}) \ . \tag{11.11}$$

Each massive tensor F_n^I combines with the corresponding excited modes $\sigma_n^{I\,ij}, \lambda_n^{I\,i}$ into a massive tensor multiplet

$$\mathcal{T}_n^I = (F_{n\,\mu\nu}^I, \sigma_n^{I\,ij}, \lambda_n^{I\,i}) \ , \qquad n \in \mathbb{Z}^* \ . \tag{11.12}$$

As a consequence of the reality conditions (11.3) in six dimensions, the vector multiplet is pseudoreal,

$$(\mathcal{V}^I)^* = \mathcal{V}^I \ : \quad \begin{cases} \bar{A}_\mu^I \equiv (A_\mu^I)^* = A_\mu^I \ , \\ \bar{\sigma}_{ij}^I \equiv (\sigma^{I\,ij})^* = \Omega_{ik}\Omega_{jl}\sigma^{I\,kl} \\ \bar{\lambda}^{I\,i} \equiv (\lambda_i^I)^\dagger \gamma^0 = \Omega^{ij}(\lambda_j^I)^\mathsf{T} C \ , \end{cases} \tag{11.13}$$

and the tensor multiplets satisfy

$$(\mathcal{T}_n^I)^* = \mathcal{T}_{-n}^I \ : \quad \begin{cases} \bar{F}_{n\,\mu\nu}^I \equiv (F_{n\,\mu\nu}^I)^* = F_{-n\,\mu\nu}^I \ , \\ \bar{\sigma}_{n\,ij}^I \equiv (\sigma_n^{Iij})^* = \Omega_{ik}\Omega_{jl}\sigma_{-n}^{Ikl} \ , \\ \bar{\lambda}_n^{I\,i} \equiv (\lambda_{n\,i}^I)^\dagger \gamma^0 = \Omega^{ij}(\lambda_{-n\,j}^I)^\mathsf{T} C \ . \end{cases} \tag{11.14}$$

We can thus restrict our attention to positive n only, to avoid a redundant description of the same degrees of freedom. Note that now γ^0, C refer to spinors in five dimensions. Our conventions about five-dimensional spinors are collected in an appendix in section A.2 along with some useful identities. It is interesting to contrast the reality condition for spinors on zero modes and on excited modes: the former is the usual symplectic-Majorana condition, but the latter relates two different symplectic multiplets, λ_n^i and λ_{-n}^i, and imposes no constraint on either of them separately. In this respect λ_n^i is referred to as 'complex.' As discussed in section A.2, every complex symplectic spinor as $\lambda_n^{I\,i}$ is equivalent to a doublet of symplectic-Majorana spinors.

Since there is no known extension of the five-dimensional $\mathcal{N} = 4$ Poincaré superalgebra to a superconformal algebra [278, 30], there is no well-defined notion of Weyl weight for $\mathcal{N} = 4$ supermultiplets. Six-dimensional superconformal $(2, 0)$ symmetry, however, implies a (classical) scaling symmetry of the five-dimensional $\mathcal{N} = 4$ theory. From the metric Ansatz (11.6) we infer that the compactification radius r has scaling weight -1, as will be further discussed in section 11.3.2. The scaling weights of all fields in vector and tensor multiplets can be extracted by comparing the six-dimensional Weyl weights listed in Table 11.1 with the Kaluza-Klein Ansätze (11.7), (11.8). They are found in Table 11.2, together with a summary of $USp(4)_R$ representations.

11.2.3 Mass scale and Kaluza-Klein gauging

Let us analyze in more detail the role played by the compactification radius r and the Kaluza-Klein vector A^0. The $(2, 0)$ theory we started from has no mass scale. (Recall that we consider the deep IR

multiplet	fields	type	comments	$USp(4)_R$	scaling weight
$\mathcal{V}^I$ massless vector multiplet	$A^I_\mu \equiv A^I_{0\,\mu}$	vector	pseudoreal	**1**	0
	$\sigma^{I\,ij} \equiv \sigma_0^{I\,ij}$	scalar	pseudoreal	**5**	1
	$\lambda^{I\,i} \equiv \lambda_0^{I\,i}$	spinor	pseudoreal	**4**	3/2
$\mathcal{T}^I_n$ massive tensor multiplet	$F^I_{n\,\mu\nu}$	tensor	complex	**1**	0
	$\sigma_n^{I\,ij}$	scalar	complex	**5**	1
	$\lambda_n^{I\,i}$	spinor	complex	**4**	3/2

Table 11.2: Field content of $\mathcal{N}=4$ vector multiplets $\mathcal{V}^I$ and tensor multiplets $\mathcal{T}^I_n$ in five dimensions. The precise formulation of the pseudoreality properties of the fields in $\mathcal{V}^I$ is found in (11.13). The last column collects the weights with respect to the five-dimensional scaling symmetry inherited from full six-dimensional conformal invariance.

dynamics where gravity is decoupled.) In contrast, the dimensionally reduced theory has a mass scale set by the inverse of the compactification radius r. In particular, the nth excited modes $F_{n\,\mu\nu}, \sigma_n^{ij}, \lambda_n^i$ have masses proportional to

$$m_n = nr^{-1} \ , \tag{11.15}$$

as can be seen by comparing the mass and kinetic terms for the respective fields as given below. In order to infer this, we recall that $B_{n\,\mu\nu}$ is related with $F_{n\,\mu\nu}$ by the rescaling (11.10). It is worth recalling the role of r in the conjectured equivalence between $(2,0)$ theories and five-dimensional super-Yang-Mills theories [263, 33]. Even if a complete formulation of $(2,0)$ theories in the non-Abelian case is not available, upon compactification on a circle they have to yield super-Yang-Mills in the massless sector, corresponding to the multiplets $\mathcal{V}^I$ in our notation. The Yang-Mills coupling constant in five dimensions has mass dimension $[g]=-1/2$, and is identified with the compactification radius,

$$g^2 = r \ , \tag{11.16}$$

consistently with the fact that $(2,0)$ theories have no tunable parameter and compactification is the only source of a mass scale.

The Kaluza-Klein field can be interpreted as a five-dimensional gauge connection which is needed when a global $U(1)$ symmetry is promoted to a local symmetry. This $U(1)$ symmetry will be denoted $U(1)_{\mathrm{KK}}$. Since it will play a key role in our formulation of the non-Abelian five-dimensional action, let us discuss this symmetry in more detail and introduce some useful notation. $U(1)_{KK}$ originates from constant shifts of the compact coordinate $y' = y - \Lambda$. These can be undone by redefining the nth Kaluza-Klein mode of a field X as $X'_n = e^{in\Lambda}X_n$, as can be seen from (11.8), Thus, the nth Kaluza-Klein mode of any field has electric charge n under $U(1)_{\mathrm{KK}}$. The associated infinitesimal transformation reads

$$\delta_{\mathrm{KK}}(\lambda) X_n = in\lambda X_n \ . \tag{11.17}$$

If we demand

$$\delta_{\mathrm{KK}}(\lambda) A^0_\mu = -\partial_\mu \lambda \ , \tag{11.18}$$

we can gauge $U(1)_{\rm KK}$ by introducing the covariant derivative

$$\mathcal{D}^{\rm KK}_\mu X_n = \partial_\mu X_n + in A^0_\mu X_n \ . \tag{11.19}$$

From a six-dimensional perspective, A^0 is identified with fluctuations of the off-diagonal components of the metric, as can be seen from (11.6). Its gauge transformation (11.18) is just a special case of a six-dimensional diffeomorphism along the circle, and the minimal coupling to X_n (11.19) is required by six-dimensional covariance.

In section 11.4 it will prove useful to rewrite the $U(1)_{\rm KK}$ gauging in terms of real fields. To this end, we exploit the isomorphism $U(1)_{\rm KK} \cong SO(2)_{\rm KK}$ and for any complex field X_n of charge n we introduce the $SO(2)_{\rm KK}$ doublet X^α_n, $\alpha = 1, 2$ via

$$X_n = \tfrac{1}{\sqrt{2}}\big(X^{\alpha=1}_n + iX^{\alpha=2}_n\big) \ . \tag{11.20}$$

Since the action of $U(1)_{\rm KK}$ on X_n is given by $X'_n = e^{in\Lambda}X_n$, the corresponding action of $SO(2)_{\rm KK}$ on X^α_n reads

$$X'^\alpha_n = M^\alpha{}_\beta X^\beta_n \ , \qquad M^\alpha{}_\beta = \begin{pmatrix} \cos n\,\Lambda & -\sin n\,\Lambda \\ \sin n\,\Lambda & \cos n\,\Lambda \end{pmatrix} = \delta^{\alpha\gamma}\left(\delta_{\gamma\beta} - n\Lambda\epsilon_{\gamma\beta} + \mathcal{O}(\Lambda^2)\right) \ . \tag{11.21}$$

The Kaluza-Klein covariant derivative of the doublet X^α_n is therefore

$$\mathcal{D}^{\rm KK}_\mu X^\alpha_n = \partial_\mu X^\alpha_n + n\epsilon_{\beta\gamma}\delta^{\gamma\alpha}\, A^0_\mu X^\beta_n \ , \tag{11.22}$$

where we have chosen the representation

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{11.23}$$

for the antisymmetric invariant of $SO(2)_{\rm KK}$. In the last equations we have implicitly assumed that X_n is a boson. As explained in the appendix section A.2, the same formalism can be applied to symplectic spinors.

As a first application we present the reformulation of the bosonic action (11.9) with $SO(2)_{\rm KK}$ doublets instead of complex fields. Inserting (11.20) for the tensors F^I_n into (11.9) we find

$$\begin{aligned} \mathcal{L}_{\rm tens} \ &= \ d_{IJ}\Big[-\tfrac14 r^{-1} F^I_{\mu\nu}F^{J\,\mu\nu} - \tfrac18 \epsilon^{\mu\nu\lambda\rho\sigma} A^0_\mu\, F^I_{\nu\lambda}\, F^J_{\rho\sigma}\Big] \\ &\quad + \sum_{n=1}^{\infty} d_{IJ}\Big[-\tfrac14 r^{-1}\delta_{\alpha\beta}F^{I\alpha}_{n\,\mu\nu}F^{J\beta\,\mu\nu}_n - \tfrac{1}{8n}\epsilon_{\alpha\beta}\epsilon^{\mu\nu\lambda\rho\sigma}F^{I\alpha}_{n\,\mu\nu}\,\mathcal{D}^{\rm KK}_\lambda F^{J\beta}_{n\,\rho\sigma}\Big] \ , \end{aligned} \tag{11.24}$$

where we have used the identities (A.28). These terms together with the Kaluza-Klein gauging, and the non-Abelian gaugings that we introduce next, turn out to be sufficient to determine the key characteristic data of the complete supersymmetric theory discussed in section 11.4.

11.2.4 Non-Abelian gauge transformation and covariant derivative

In our discussion of the five-dimensional spectrum zero modes and excited modes are treated on a very different footing, at the expense of manifest six-dimensional Lorentz symmetry. However, this enables us to implement a non-Abelian gauging, since we can use massless vectors in five dimensions as gauge connections, and treat all other fields as charged matter. This implementation is the only straightforward gauging compatible with the Kaluza-Klein charges under the assumption that the gauge parameter is neutral under $U(1)_{\rm KK}$. The same strategy has been proposed in the literature in a similar context, see e.g. [35, 267]. Identifying a possible six-dimensional interpretation for this non-democratic gauging is a non-trivial task that will not be addressed in this work.

To define a non-Abelian gauging we first identify the degeneracy index I with the adjoint index of some non-Abelian group G. More precisely, we let I enumerate the elements t_I of a basis of anti-Hermitian generators of the associated Lie algebra, so that $I = 1, \dots |G| \equiv \dim(G)$. We introduce the structure constants and the Cartan-Killing metric by

$$[t_I, t_J] = -f_{IJ}{}^K t_K \ , \qquad d_{IJ} = \mathrm{Tr}(t_I t_J) \ . \tag{11.25}$$

Both $f_{IJ}{}^K$ and d_{IJ} are real. We assume d_{IJ} is non-singular and positive definite, and we use it together with its inverse d^{IJ} to raise and lower adjoint indices. For example, $f_{IJK} = d_{IL} f_{JK}{}^L$. Furthermore, we take f_{IJK} to be completely antisymmetric. The groups under consideration are taken to be of A-D-E type.

In order to realize a non-Abelian gauging of the spectrum (11.11), (11.12), we interpret A^I as a gauge connection, while all other fields will be seen as adjoint matter. More precisely, we postulate the following infinitesimal transformation rules under the action of the non-Abelian gauge group G,

$$\delta_G(\alpha) A^I_\mu = \partial_\mu \alpha^I + f_{JK}{}^I A^J_\mu \alpha^K \ , \qquad \delta_G(\alpha) X^I = -f_{JK}{}^I \alpha^J X^K \ , \tag{11.26}$$

where α is the scalar gauge parameter and X^I is any field among $\sigma^{I\,ij}, \lambda^{I\,i}, \sigma_n^{I\,ij}, \lambda_n^{I\,i}, F^I_{n\,\mu\nu}$ $(n > 0)$. Recalling (11.19), we see that the full $G \times U(1)_{\rm KK}$ covariant derivative of any adjoint field X^I_n with Kaluza-Klein charge n is given by[3]

$$\mathcal{D}_\mu X^I_n = \partial_\mu X^I_n + in\, A^0_\mu X^I_n + f_{JK}{}^I A^J_\mu X^K_n \ . \tag{11.27}$$

We note here that $\mathcal{D}_\mu X^I_n$ has the same charge under $U(1)_{KK}$ as X^I_n itself. The non-Abelian field-strength of A^I reads

$$F^I_{\mu\nu} = 2\partial_{[\mu} A^I_{\nu]} + f_{JK}{}^I A^J_\mu A^K_\nu \ , \tag{11.28}$$

transforms in the adjoint representation, satisfies the Bianchi identity $\mathcal{D}_{[\mu} F^I_{\nu\rho]} = 0$, and enters the commutator of covariant derivatives as specified by

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] X^I_n = in F^0_{\mu\nu} X^I_n + f_{JK}{}^I F^J_{\mu\nu} X^K_n \ . \tag{11.29}$$

The algebra of gauge transformations closes on all fields, according to

$$[\delta_G(\alpha_1), \delta_G(\alpha_2)] = \delta_G(\alpha_3) \ , \qquad \alpha^I_3 = f_{JK}{}^I \alpha^J_1 \alpha^K_2 \ . \tag{11.30}$$

[3] Since we work in flat space, we do not have to introduce a spacetime connection and covariant derivative.

11.3 Spectrum in terms of $\mathcal{N}=2$ superconformal multiplets

Since non-Abelian gaugings of tensor multiplets are not consistent with standard $\mathcal{N}=4$ actions determined in [284, 285, 261, 286], we first consider an $\mathcal{N}=2$ formulation. Upon reduction, we get $\mathcal{N}=2$ vector, tensor and hypermultiplets, and we can exploit the $\mathcal{N}=2$ rigid superconformal formalism of [268, 269].

11.3.1 Splitting of $\mathcal{N}=4$ multiplets

To rewrite the $\mathcal{N}=4$ spectrum in terms of $\mathcal{N}=2$ supermultiplets, we consider the splitting of the original R-symmetry group $USp(4)_R$ according to

$$USp(4)_R \to SU(2)_R \times SU(2) \ , \tag{11.31}$$

where the first factor is the R-symmetry group of the $\mathcal{N}=2$ algebra, and the second factor is an extra global symmetry of the theory. We use indices $a,b=1,2$ for the $\mathbf{2}$ representation of $SU(2)_R$, while indices $\dot{a},\dot{b}=1,2$ refer to the $\mathbf{2}$ representation of $SU(2)$. Under (11.31) the branching rules for the relevant representations of $USp(4)_R$ read

$$\begin{array}{ccccccccc} \mathbf{5} & \to & (\mathbf{1},\mathbf{1})+(\mathbf{2},\mathbf{2}) \ , & & & \mathbf{4} & \to & (\mathbf{2},\mathbf{1})+(\mathbf{1},\mathbf{2}) \ , \\ \sigma_n^{I\,ij} & \to & \phi_n^I \quad , \quad q_n^{I\,a\dot{b}} \quad , & & & \lambda_n^{I\,i} & \to & \chi_n^{I\,a} \quad , \quad \zeta_n^{I\,\dot{b}} \quad , \end{array} \tag{11.32}$$

where the entries in the brackets correspond to the two $SU(2)$'s, and we have introduced the bosonic fields ϕ_n^I, $q_n^{I\,a\dot{b}}$, and the fermionic fields $\chi_n^{I\,a}$, $\zeta_n^{I\,\dot{b}}$ which will be discussed in more detail in the following.

Let us summarize the complete multiplets of rigid $\mathcal{N}=2$ supersymmetry originating from the $\mathcal{N}=4$ spectrum of section 11.2.2. Firstly, we find the vector multiplets

$$\widehat{\mathcal{V}}^I = (A_\mu^I, \phi^I, \chi^{I\,a}, Y_{ab}^I) \equiv (A_{0\,\mu}^I, \phi_0^I, \chi_0^{I\,a}, Y_{0\,ab}^I) \ . \tag{11.33}$$

The vector A_μ^I is still identified with the gauge connection. The real scalar ϕ^I is a singlet $(\mathbf{1},\mathbf{1})$ under $SU(2)_R\times SU(2)$ and originates from $\sigma^{I\,ij}$. The spinor $\chi^{I\,a}$ belongs to the $(\mathbf{2},\mathbf{1})$ representation and comes from the decomposition of $\lambda^{I\,i}$. The scalars $Y_{ab}^I = Y_{ba}^I$ are auxiliary fields of the $\mathcal{N}=2$ superconformal formalism and transform in the $(\mathbf{3},\mathbf{1})$ representation. They would arise from the decomposition of auxiliary fields in the (linearized) off-shell $\mathcal{N}=4$ vector multiplet (see e.g. [287]) that transform in higher irreducible representations of $USp\,(4)_R$. The multiplets $\widehat{\mathcal{V}}^I$ are pseudoreal,

$$(\widehat{\mathcal{V}}^I)^* = \widehat{\mathcal{V}}^I \ : \quad \begin{cases} \bar{A}_\mu^I \equiv (A_\mu^I)^* = A_\mu^I \ , \\ \bar{\phi}^I \equiv (\phi^I)^* = \phi^I \ , \\ \bar{\chi}^{I\,a} \equiv (\chi_a^I)^\dagger\gamma^0 = \epsilon^{ab}(\chi_b^I)^\mathsf{T} C \ , \\ \bar{Y}^{I\,ab} \equiv (Y_{ab}^I)^* = \epsilon^{ac}\epsilon^{bd}Y_{cd}^I \ , \end{cases} \tag{11.34}$$

where ϵ^{ab} is the primitive antisymmetric invariant of $SU(2)_R$.

Secondly, in a completely analogous fashion we have the tensor multiplets

$$\widehat{\mathcal{T}}_n^I = (F^I_{n\,\mu\nu}, \phi^I_n, \chi_n^{I\,a}, Y^I_{n\,ab}) \ , \qquad n > 0 \ , \tag{11.35}$$

with the scalars ϕ^I_n in the $(\mathbf{1},\mathbf{1})$ representation, the spinors $\chi^{I\,a}$ in the $(\mathbf{2},\mathbf{1})$ representation, and the auxiliary fields $Y^I_{n\,ab}$ in the $(\mathbf{3},\mathbf{1})$ representation. In contrast to their counterparts in $\widehat{\mathcal{V}}^I$, all fields in $\widehat{\mathcal{T}}_n$ are complex and will become massive after breaking of conformal invariance, as discussed in more detail below.

Finally, we find the hypermultiplets

$$\widehat{\mathcal{H}}_0^I \equiv \widehat{\mathcal{H}}^I = (q^{I\,a\dot{b}}, \zeta^{I\,\dot{b}}) \equiv (q_0^{I\,a\dot{b}}, \zeta_0^{I\,\dot{b}}) \ , \qquad \widehat{\mathcal{H}}_n^I = (q_n^{I\,a\dot{b}}, \zeta_n^{I\,\dot{b}}) \ , \qquad n > 0 \ . \tag{11.36}$$

They consist of scalars $q_n^{I\,a\dot{b}}$ that are the $(\mathbf{2},\mathbf{2})$ component of $\sigma_n^{I\,ij}$ under the branching (11.32), and of spinors $\zeta_n^{I\,\dot{b}}$ that belong to the $(\mathbf{1},\mathbf{2})$ branch in the reduction of $\lambda_n^{I\,i}$. For $n > 0$ the hypermultiplet is complex and massive (in the broken phase of conformal symmetry). For $n = 0$ it is massless and pseudoreal,

$$(\widehat{\mathcal{H}}^I)^* = \widehat{\mathcal{H}}^I \ : \quad \begin{cases} \bar{q}^I_{a\dot{a}} \equiv (q^{I\,a\dot{a}})^* = \epsilon_{ab}\epsilon_{\dot{a}\dot{b}} q^{I\,b\dot{b}} \ , \\ \bar{\zeta}^{I\,\dot{a}} \equiv (\zeta^I_{\dot{a}})^\dagger \gamma^0 = \epsilon^{\dot{a}\dot{b}} \zeta^{I\,\top}_{\dot{b}} C \ , \end{cases} \tag{11.37}$$

where we have made use of the primitive antisymmetric invariants $\epsilon^{ab}, \epsilon^{\dot{a}\dot{b}}$ of $SU(2)_R$ and $SU(2)$. The Weyl weights of all the fields introduced in this section are collected in Table 11.3, along with a summary of $SU(2)_R \times SU(2)$ representations. The matching of the Weyl weights of $\mathcal{N} = 4$ fields and $\mathcal{N} = 2$ fields will be discussed in the next subsection.

11.3.2 Restoration of five-dimensional conformal symmetry

It is important to clarify the role of conformal symmetry in our discussion. Our goal is a five-dimensional action that is able to capture some crucial ingredients of a non-Abelian $(2,0)$ model. This six-dimensional theory is invariant under rigid conformal transformations [279, 280], i.e. transformations that leave the six-dimensional line-element invariant up to a factor. We refrain from a complete account on the transformation properties of the six-dimensional fields. In our discussion we restrict our attention mostly to the Weyl weights of the fields as listed in Table 11.1.

If we compactify the six-dimensional theory on a circle using (11.6), we expect some generators of the six-dimensional conformal symmetry to be spontaneously broken. The remaining generators are those which act only on the five-dimensional line element. In particular, the Weyl invariance discussed above will be broken, unless we also allow for a rescaling of the compactification radius, i.e. unless we would consider transformations of the form

$$g_{\mu\nu} dx^\mu dx^\nu = ds^2 \ \mapsto\ \Omega^{-2} ds^2 \, , \qquad\qquad r \ \mapsto\ \Omega^{-1} r \, . \tag{11.38}$$

Another way to see that Weyl invariance is compromised in the dimensionally reduced theory is to notice that the multiplets $\widehat{\mathcal{T}}_n, \widehat{\mathcal{H}}_n$ have become massive with masses m_n given in (11.15). Since Weyl invariance is incompatible with massive fields, the Kaluza-Klein masses m_n break conformal invariance

multiplet	fields	type	comments	$SU(2)_R \times SU(2)$	Weyl weight
$\widehat{\mathcal{V}}^I$ massless vector multiplet	$A^I_\mu \equiv A^I_{0\,\mu}$	vector	pseudoreal	$(\mathbf{1},\mathbf{1})$	0
	$\phi^I \equiv \phi^I_0$	scalar	pseudoreal	$(\mathbf{1},\mathbf{1})$	1
	$\chi^{I\,a} \equiv \chi^{I\,a}_0$	spinor	pseudoreal	$(\mathbf{2},\mathbf{1})$	3/2
	$Y^I_{ab} \equiv Y^I_{0\,ab}$	scalar	auxiliary	$(\mathbf{3},\mathbf{1})$	2
$\widehat{\mathcal{H}}^I \equiv \widehat{\mathcal{H}}^I_0$ massless hyperm.	$q^{I\,a\dot{b}} \equiv q^{I\,a\dot{b}}_0$	scalar	pseudoreal	$(\mathbf{2},\mathbf{2})$	3/2
	$\zeta^{I\,\dot{a}} \equiv \zeta^{I\,\dot{a}}_0$	spinor	pseudoreal	$(\mathbf{1},\mathbf{2})$	2
$\widehat{\mathcal{T}}^I_n$ massive tensor multiplet	$F^I_{n\,\mu\nu}$	tensor	complex	$(\mathbf{1},\mathbf{1})$	0
	ϕ^I_n	scalar	complex	$(\mathbf{1},\mathbf{1})$	1
	$\chi^{I\,a}_n$	spinor	complex	$(\mathbf{2},\mathbf{1})$	3/2
	$Y^I_{n\,ab}$	scalar	auxiliary	$(\mathbf{3},\mathbf{1})$	2
$\widehat{\mathcal{H}}^I_n$ massive hyperm.	$q^{I\,a\dot{b}}_n$	scalar	complex	$(\mathbf{2},\mathbf{2})$	3/2
	$\zeta^{I\,\dot{a}}_n$	spinor	complex	$(\mathbf{1},\mathbf{2})$	2
$\widehat{\mathcal{V}}^0$ massless vector multiplet	A^0_μ	vector	pseudoreal	$(\mathbf{1},\mathbf{1})$	0
	ϕ^0	scalar	pseudoreal	$(\mathbf{1},\mathbf{1})$	1
	$\chi^{0\,a}$	spinor	pseudoreal	$(\mathbf{2},\mathbf{1})$	3/2
	Y^0_{ab}	scalar	auxiliary	$(\mathbf{3},\mathbf{1})$	2

Table 11.3: Field content of $\mathcal{N}=2$ vector multiplet $\widehat{\mathcal{V}}$, tensor multiplets $\widehat{\mathcal{T}}_n$ and hypermultiplets $\widehat{\mathcal{H}}, \widehat{\mathcal{H}}_n$ in five dimensions. The additional multiplet $\widehat{\mathcal{V}}^0$ is included. The precise formulation of the pseudoreality properties of the fields in $\widehat{\mathcal{V}}^I, \widehat{\mathcal{H}}^I$ is found in (11.34) and (11.37), respectively. The specification 'massless' or 'massive' applies to the broken phase of conformal symmetry.

explicitly. This can be remedied by allowing them to transform as $m_n \mapsto \Omega\, m_n$ as can be inferred from (11.38).

Note that the $\mathcal{N}=2$ Poincaré supersymmetry algebra does admit a superconformal extension, given by the exceptional superalgebra $F^2(4)$ [278, 30]. This is in contrast with the $\mathcal{N}=4$ case considered before. In practice, five-dimensional rigid superconformal invariance is restored by introducing additional five-dimensional degrees of freedom. Indeed, we can promote the radius r to the scalar component of a full $\mathcal{N}=2$ vector multiplet

$$\widehat{\mathcal{V}}^0 = (A^0_\mu, \phi^0, \chi^{0\,a}, Y^0_{ab}) \;, \tag{11.39}$$

where A^0_μ can be identified with the Kaluza-Klein vector introduced in (11.6). We can combine this vector multiplet with the physical vector multiplets introduced in the last section and denote them collectively as

$$\widehat{\mathcal{V}}^{\hat I} = (\widehat{\mathcal{V}}^0, \widehat{\mathcal{V}}^I) \;, \qquad \hat I = 0, 1\ldots, |G| \;. \tag{11.40}$$

Using the multiplet $\widehat{\mathcal{V}}^0$ we can make the $\mathcal{N}=4$ to $\mathcal{N}=2$ split of the spectrum more explicit. We follow the split (11.32) and we match the scaling weights of Table 11.2 with the Weyl weights of Table 11.3 to infer that the proper map from $\mathcal{N}=4$ to $\mathcal{N}=2$ multiplets is of the form

$$\sigma^{I\,ij}_n \mapsto \begin{pmatrix} \frac{1}{\sqrt2}\epsilon^{ab}\,\phi^I_n & (\phi^0)^{-1/2} q^{I\,a\dot b}_n \\ -(\phi^0)^{-1/2} q^{I\,b\dot a}_n & -\frac{1}{\sqrt2}\epsilon^{\dot a\dot b}\,\phi^I_n \end{pmatrix} \,, \qquad \lambda^{I\,i}_n \mapsto \begin{pmatrix} \chi^{I\,a}_n \\ \sqrt2(\phi^0)^{-1/2}\zeta^{I\,\dot a}_n \end{pmatrix} \,, \qquad n \geq 0 \;. \tag{11.41}$$

Prefactors are chosen for later convenience. Note that the split (11.31) is not unique.

In the action of section 11.4.1 the additional multiplet $\widehat{\mathcal{V}}^0$ will couple to all other multiplets making the action superconformally invariant. To give a direct link with the Kaluza-Klein reduction it will be convenient to return to the broken phase of the superconformal symmetry by setting the additional fields to a fixed value. This requires to set

$$\langle \phi^0 \rangle = \frac{1}{r} = \frac{1}{g^2} \;, \qquad \langle \chi^{0\,a} \rangle = \langle Y^0_{ab} \rangle = \langle A^0_\mu \rangle = 0 \;, \tag{11.42}$$

where g is the gauge coupling of the five-dimensional Yang-Mills theory. It is important to stress that imposing the condition (11.42) corresponds to a restriction of the theory. Indeed not all values of $\chi^{0\,a}, Y^0_{ab}$ and A^0_μ can be mapped by a superconformal transformation to zero. Nevertheless we will show below that a Poincaré supersymmetric theory arises after imposing (11.42). Moreover, the Weyl rescaling (11.38) of r, as dictated by the six-dimensional conformal symmetry, is precisely compatible with the Weyl weight of ϕ^0 in the identification (11.42). In fact, we will show that the five-dimensional action still retains a scaling symmetry if the radius is rescaled as in (11.38).

In the broken phase of conformal symmetry determined by (11.42) the hypermultiplets fields $q^{I\,a\dot b}_n, \zeta^{I\,\dot a}_n$ are not convenient variables, since their mass dimensions are not canonical. As a consequence, we define the rescaled fields

$$h^{I\,a\dot b}_n = g\, q^{I\,a\dot b}_n \;, \qquad \psi^{I\,\dot a}_n = g\,\zeta^{I\,da}_n \;, \qquad n \geq 0 \;, \tag{11.43}$$

in such a way that all scalars have mass dimension and scaling weight 1, and all fermions have mass dimension and scaling weight 3/2.

11.4 Supersymmetric actions and conformal invariance

In this section we introduce a five-dimensional non-Abelian $\mathcal{N} = 2$ supersymmetric action for the Kaluza-Klein spectrum obtained in section 11.3. Our theory will include couplings which are only specified in terms of group theoretical constants and the Kaluza-Klein levels. An $\mathcal{N} = 2$ superconformal action is presented in section 11.4.1, while the Kaluza-Klein sums are made explicit in a restricted action in section 11.4.2. We propose to interpret this theory as an $\mathcal{N} = 2$ subsector of a dimensionally reduced $(2, 0)$ theory.

11.4.1 An $\mathcal{N} = 2$ superconformal action for the Kaluza-Klein spectrum

In the following we introduce an $\mathcal{N} = 2$ superconformal action for the spectrum discussed in section 11.3. Superconformal invariance is retained since we will include the additional vector multiplet $\widehat{\mathcal{V}}^0$, defined in (11.39), containing the radius and the Kaluza-Klein vector. It will be necessary to introduce some additional notation in order to make contact with the general $\mathcal{N} = 2$ superconformal actions introduced in [269]. The fields identified with the Kaluza-Klein zero modes are denoted as in Table 11.3:

$$\text{vector multiplets:} \quad (A^0_\mu, \phi^0, \chi^{0\,a}), \quad (A^I_\mu, \phi^I, \chi^{I\,a}) \qquad \text{hypermultiplets:} \quad (q^{Ia\dot{b}}, \zeta^{I\dot{b}}) \ . \tag{11.44}$$

For the fields identified with excited Kaluza-Klein modes it will be convenient to use the notation (11.20) for complex fields introducing the $SO(2)_{\text{KK}}$ index α. This leads us to the following excited spectrum:

$$\begin{aligned} \text{tensor multiplets:} \quad & \big(F^{\{I\alpha n\}}_{\mu\nu}, \phi^{\{I\alpha n\}}, \chi^{\{I\alpha n\}a}\big) \equiv \big(F^{I\alpha}_{n\,\mu\nu}, \phi^{I\alpha}_n, \chi^{I\alpha\,a}_n\big) \\ \text{hypermultiplets:} \quad & \big(q^{\{I\alpha n\}a\dot{b}}, \zeta^{\{I\alpha n\}\dot{b}}\big) \equiv \big(q^{I\alpha\,a\dot{b}}_n, \zeta^{J\alpha\,\dot{b}}_n\big) \ . \end{aligned} \tag{11.45}$$

The main complication in the notation arises from the multi-index $\{I\alpha n\}$ which labels simultaneously the non-Abelian components $I, J = 1, \ldots, |G|$, the $SO(2)_{\text{KK}}$ labels $\alpha, \beta = 1, 2$, and the Kaluza-Klein levels $n, m \geq 1$. To avoid cluttering of indices in the following expressions we will denote this multi-index by

$$M = \{I\alpha n\} \ , \qquad N = \{J\beta m\} \ . \tag{11.46}$$

A summation over M, N then always amounts to summing over all indices including the infinite tower of Kaluza-Klein modes. We will present the superconformal action as function of the four types of multiplets in (11.44) and (11.45). To do that in an efficient way it is useful to introduce the following index combinations

$$\widehat{I} \equiv (0, I) \qquad \Lambda \equiv (0, I, M) \ , \qquad \mathcal{I} = (I, M) \tag{11.47}$$

This means that $\widehat{I}, \widehat{J}, \ldots$ label all vector multiplets and run over $|G|+1$ values, $\Lambda, \Sigma, \ldots$ run over all tensor and vector multiplets including the Kaluza-Klein tower. The indices $\mathcal{I}, \mathcal{J}, \ldots$ label all hypermultiplets, or vectors and tensor multiplets without $\mathcal{V}^0$. Finally, we also define

$$F^\Lambda_{\mu\nu} \equiv (F^{\widehat{I}}_{\mu\nu}, F^M_{\mu\nu}) \equiv (F^{\widehat{I}}_{\mu\nu}, n\epsilon_{\beta\gamma}\delta^{\gamma\alpha}B^{I\beta}_{n\,\mu\nu}) \ . \tag{11.48}$$

where we have recalled the definition of $F^M_{\mu\nu} = F^{I\alpha}_{n\,\mu\nu}$ as given in (11.10). It is crucial to stress that the Kaluza-Klein interpretation dictates this non-trivial identification of $F^M_{\mu\nu}$ with $B^{I\alpha}_{n\,\mu\nu}$. The important point is that while the $F^M_{\mu\nu}$ admit a rescaling with the Kaluza-Klein level compared to $B^{I\alpha}_{n\,\mu\nu}$, the scalars and fermions in the same multiplet are trivially matched with the $\mathcal{N}=2$ formalism of [269].[4] The non-trivial rescaling of $B^{I\alpha}_{n\,\mu\nu}$ turns out to be consistent with the dimensional reduction of the supersymmetry variations as can be checked for the Abelian six-dimensional theory recorded in section 11.2.1.

We are now in the position to discuss the Lagrangian in detail. The vector-tensor sector of an $\mathcal{N}=2$ superconformal theory can be specified by introducing a constant symmetric object $C_{\Lambda\Sigma\Theta}$, a constant antisymmetric matrix Ω_{MN}, and the gauge parameters $t_{\widehat{K}\Lambda}{}^{\Sigma}$ [269]. The gauge parameters appear in the covariant derivatives

$$\begin{aligned}
\mathcal{D}_\mu \phi^\Sigma &= \partial_\mu \phi^\Sigma + t_{\widehat{K}\Lambda}{}^{\Sigma}\, A_\mu^{\widehat{K}}\, \phi^\Lambda\ ,\\
\mathcal{D}_\mu \chi^{a\Sigma} &= \partial_\mu \chi^{a\Sigma} + t_{\widehat{K}\Lambda}{}^{\Sigma}\, A_\mu^{\widehat{K}}\, \chi^{a\Lambda},\\
\mathcal{D}_\mu F^N_{\nu\rho} &= \partial_\mu F^N_{\nu\rho} + t_{\widehat{K}M}{}^{N}\, A_\mu^{\widehat{K}}\, F^M_{\nu\rho}\ .
\end{aligned} \tag{11.49}$$

Note that strictly speaking only $C_{\widehat{I}\widehat{J}\widehat{K}}$ encodes extra information in addition to $\Omega_{MN}, t_{\widehat{K}\Lambda}{}^{\Sigma}$. This is due to the fact that $C_{M\Lambda\Sigma}$ are given by

$$C_{M\Lambda\Sigma} = t_{(\Lambda\Sigma)}{}^{N}\Omega_{NM}\ , \tag{11.50}$$

where one symmetrizes in the indices Λ, Σ including the usual factor $1/2$. Here we have extended the range of indices on generators $t_{\Lambda\Sigma}{}^{\Theta}$ with the constraints

$$t_{(\Lambda\Sigma)}{}^{\widehat{I}} = 0\ , \qquad t_{M\Sigma}{}^{\Theta} = 0\ , \tag{11.51}$$

implying the absence of gaugings with a tensor index M.

Since we will later propose to use the $\mathcal{N}=2$ superconformal theory to describe the dimensional reduced $(2,0)$ action we aim to use only couplings which are of group theoretic origin. We like to identify a subsector of the theory as $\mathcal{N}=4$ super-Yang-Mills theory. This implies that components of $C_{\widehat{I}\widehat{J}\widehat{L}}$ have to encode the trace $d_{IJ} = C_{0IJ}$. The coupling $C_{000} = k_{\rm c}$ will determine the kinetic term of the auxiliary vector multiplet $\widehat{\mathcal{V}}^0$, and will be left undetermined at the moment. We choose $C_{00I} = 0$. More interesting are the couplings of the tensor multiplets. Here we are guided by (11.50). To determine the gaugings we first note that the fields in $\widehat{\mathcal{V}}^0$ cannot be gauged, such that $t_{\widehat{I}0}{}^{\Lambda} = t_{\widehat{I}\Lambda}{}^{0} = 0$. Comparing the gaugings (11.22), (11.27) with (11.49), we consider the following identification:

$$t_{K\mathcal{I}}{}^{\mathcal{J}} = \begin{pmatrix} t_{KI}{}^{J} & 0 \\ 0 & t_{KM}{}^{N} \end{pmatrix} = \begin{pmatrix} t_{KI}{}^{J} & 0 \\ 0 & t_{K\{I\alpha n\}}{}^{\{J\beta m\}} \end{pmatrix} = \begin{pmatrix} f_{KI}{}^{J} & 0 \\ 0 & f_{KI}{}^{J}\delta^\beta_\alpha \delta^m_n \end{pmatrix}\ , \tag{11.52}$$

and

$$t_{0\mathcal{I}}{}^{\mathcal{J}} = \begin{pmatrix} 0 & 0 \\ 0 & t_{0M}{}^{N} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & t_{0\{I\alpha n\}}{}^{\{J\beta m\}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & n\,\delta^J_I\,\epsilon_{\alpha\gamma}\delta^{\gamma\beta}\delta^m_n \end{pmatrix}\ , \tag{11.53}$$

[4] This implies that compared to [269] one has to adjust the notation, since there $F^M_{\mu\nu}$ and $B^M_{\mu\nu}$ are trivially identified.

where $SO(2)_{\rm KK}$ indices have been raised and lowered using $\delta_{\alpha\beta}$. Here $t_{KI}{}^{J}, t_{IM}{}^{N}$ parametrize the non-Abelian gaugings with the vector zero modes and are thus given by the structure constants of G. The matrix $t_{0\mathcal{I}}{}^{\mathcal{J}}$ encodes the gauging of the massive tensor multiplets with A^0, which is interpreted as charge under the Kaluza-Klein vector. In addition the antisymmetric matrix Ω_{MN} can be read off from the Chern-Simons type kinetic terms of the tensors F^M in (11.24), and is given by

$$\Omega_{MN} = \Omega_{\{I\alpha n\}\{J\beta m\}} = -\frac{2}{n}\, d_{IJ}\epsilon_{\alpha\beta}\delta_{nm}\ , \tag{11.54}$$

where $n, m \geq 1$ as in the range of the multi-indices (11.46). As we can see, $U(1)_{KK} \sim SO(2)_{\rm KK}$ plays a key role in the construction of this object. While the trace d_{IJ} is symmetric, one can use the indices α, β and the antisymmetric $\epsilon_{\alpha\beta}$, corresponding to the complex number i, to introduce Ω_{MN}. Using (11.50) this will also allow us to introduce the symmetric tensor $C_{M\Lambda\Sigma}$ in terms of the antisymmetric structure constants $f_{IJK} = d_{IL}f^{L}{}_{JK}$. To display the result, we introduce the matrix

$$C_{\mathcal{I}\mathcal{J}} = \begin{pmatrix} C_{IJ} & 0 \\ 0 & C_{MN} \end{pmatrix} = \begin{pmatrix} C_{IJ} & 0 \\ 0 & C_{\{I\alpha n\}\{J\beta m\}} \end{pmatrix} = \begin{pmatrix} d_{IJ} & 0 \\ 0 & d_{IJ}\delta_{\alpha\beta}\delta_{nm} \end{pmatrix}\ . \tag{11.55}$$

In summary, taking into account the total symmetry in all three indices, all components of $C_{\Lambda\Sigma\Theta}$ are determined by

$$\begin{aligned} &C_{0\mathcal{I}\mathcal{J}} = C_{\mathcal{I}\mathcal{J}}\ , \qquad C_{000} = k_{\rm c}\ , \qquad C_{MNK} = C_{\{I\alpha n\}\{J\beta m\}K} = -\frac{1}{n} f_{IJK}\,\epsilon_{\alpha\beta}\delta_{nm}\ , \\ &C_{00\mathcal{I}} = C_{IJ\mathcal{K}} = C_{MNP} = 0\ . \end{aligned} \tag{11.56}$$

In evaluating these expressions we have used that $\epsilon_{\alpha\gamma}\delta^{\gamma\delta}\epsilon_{\delta\beta} = -\delta_{\alpha\beta}$.

Let us now include the hypermultiplets into the discussion. In a general $\mathcal{N} = 2$ superconformal theory the hypermultiplets span a hypercomplex manifold. We choose the geometry of the hypercomplex manifold appearing in the reduction to be locally flat space. Since the dimension of this manifold is related to the dimension of the gauge group G, it posses sufficiently many isometries to implement a gauging compatible with (11.49). In coordinates $q^{a\dot{a}\mathcal{I}}$ the metric is given by $C_{\mathcal{I}\mathcal{J}}\epsilon_{ab}\epsilon_{\dot{a}\dot{b}}$, with $C_{\mathcal{I}\mathcal{J}}$ as defined in (11.55).[5] The kinetic term of the fermionic partners $\zeta^{\dot{a}\mathcal{I}}$ is simply given by $C_{\mathcal{I}\mathcal{J}}\epsilon_{\dot{a}\dot{b}}$. The gauging of the hyperscalars and fermions is

$$\mathcal{D}_\mu q^{a\dot{b}\mathcal{J}} = \partial_\mu q^{a\dot{b}\mathcal{J}} + t_{\widehat{K}\mathcal{I}}{}^{\mathcal{J}}\, A^{\widehat{K}}_\mu\, q^{a\dot{b}\mathcal{I}}\ , \qquad \mathcal{D}_\mu\zeta^{\dot{a}\mathcal{J}} = \partial_\mu\zeta^{\dot{a}\mathcal{J}} + t_{\widehat{K}\mathcal{I}}{}^{\mathcal{J}}\, A^{\widehat{K}}_\mu\, \zeta^{\dot{a}\mathcal{I}}\ , \tag{11.57}$$

with constant $t_{\widehat{K}\mathcal{I}}{}^{\mathcal{J}}$ given in (11.52) and (11.53).[6]

Using these definitions we can now display the complete non-Abelian $\mathcal{N} = 2$ superconformal

[5] The three complex structures on this hypercomplex manifold are encoded in the SU(2) triplet $J^{c\dot{c}\mathcal{I}}{}_{d\dot{d}\mathcal{J}(ab)}$, where $J^{c\dot{c}\mathcal{I}}{}_{d\dot{d}\mathcal{J}}{}^{a}{}_{b} = \delta^{\mathcal{I}}_{\mathcal{J}}\delta^{\dot{c}}_{\dot{d}}(2\delta^a_d\delta^c_b - \delta^c_d\delta^a_b)$.

[6] The moment maps generating these gaugings are given by $P_{\widehat{K}ab} = \frac{1}{2}C_{\mathcal{I}\mathcal{J}}\, t_{\widehat{K}\mathcal{L}}{}^{\mathcal{J}}\,\epsilon_{c(a}\, q^{\mathcal{I}}_{b)\dot{c}}\, q^{c\dot{c}\mathcal{L}}$.

Lagrangian,

$$\begin{aligned}
\mathcal{L} = {}& \phi^{\Theta} C_{\Theta\Lambda\Sigma} \Big(-\tfrac{1}{4} F^{\Lambda}_{\mu\nu} F^{\Sigma\,\mu\nu} - \tfrac{1}{2} \bar{\chi}^{\Lambda\,a} \not{\mathcal{D}} \chi^{\Sigma}_{a} - \tfrac{1}{2} \mathcal{D}_{\mu} \phi^{\Lambda} \mathcal{D}^{\mu} \phi^{\Sigma} + Y^{\Lambda}_{ab} Y^{\Sigma\,ab} \Big) \\
& + \tfrac{1}{16} \epsilon^{\mu\nu\lambda\rho\sigma} \Omega_{MN} F^{M}_{\mu\nu} \mathcal{D}_{\lambda} F^{N}_{\rho\sigma} - \tfrac{1}{24} \epsilon^{\mu\nu\lambda\rho\sigma} C_{\widehat{I}\widehat{J}\widehat{K}} A^{\widehat{I}}_{\mu} F^{\widehat{J}}_{\nu\lambda} F^{\widehat{K}}_{\rho\sigma} \\
& - \tfrac{i}{8} C_{\Lambda\Sigma\Theta} \Big(\bar{\chi}^{\Lambda\,a} \gamma^{\mu\nu} F^{\Sigma}_{\mu\nu} \chi^{\Theta}_{a} + 4 \bar{\chi}^{\Lambda\,a} \chi^{b\Sigma} Y^{\Theta}_{ab} \Big) \\
& + \tfrac{i}{4}\, \phi^{\Theta} C_{\Theta\Lambda\Sigma} \Big(t_{[\Upsilon\Omega]}{}^{\Lambda} \bar{\chi}^{\Upsilon\,a} \chi^{\Omega}_{a} \phi^{\Sigma} - 4 t_{(\Upsilon\Omega)}{}^{\Lambda} \bar{\chi}^{\Upsilon\,a} \chi^{\Sigma}_{a} \phi^{\Omega} \Big) \\
& - \tfrac{1}{2} \phi^{\widehat{K}} C_{\widehat{K}MN} t_{\widehat{I}P}{}^{M} t_{\widehat{J}Q}{}^{N} \phi^{\widehat{I}} \phi^{\widehat{J}} \phi^{P} \phi^{Q} \\
& + C_{\mathcal{I}\mathcal{J}} \Big(-\tfrac{1}{2} \mathcal{D}_{\mu} q^{\mathcal{I}\,a\dot{b}} \mathcal{D}^{\mu} q^{\mathcal{J}}_{\ a\dot{b}} - \bar{\zeta}^{\mathcal{I}\,\dot{b}} \not{\mathcal{D}} \zeta^{\mathcal{J}}_{\dot{b}} \Big) \\
& + C_{\mathcal{I}\mathcal{J}} \Big(2i t_{\widehat{K}\mathcal{L}}{}^{\mathcal{I}} q^{\mathcal{L}\,a\dot{b}} \bar{\chi}^{\widehat{K}}_{a} \zeta^{\mathcal{J}}_{\dot{b}} + i \phi^{\widehat{K}} t_{\widehat{K}\mathcal{L}}{}^{\mathcal{I}} \bar{\zeta}^{\mathcal{J}\,\dot{a}} \zeta^{\mathcal{L}}_{\dot{a}} \Big) \\
& + C_{\mathcal{I}\mathcal{J}} \Big(t_{\widehat{K}\mathcal{L}}{}^{\mathcal{J}} q^{\mathcal{L}\,a\dot{c}} q^{\mathcal{I}\,b}{}_{\dot{c}} Y^{\widehat{K}}_{ab} - \tfrac{1}{2} t_{\widehat{I}\mathcal{K}}{}^{\mathcal{I}} t_{\widehat{J}\mathcal{L}}{}^{\mathcal{J}} \phi^{\widehat{I}} \phi^{\widehat{J}} q^{\mathcal{K}a\dot{b}} q^{\mathcal{L}}_{\ a\dot{b}} \Big) \ .
\end{aligned} \tag{11.58}$$

This Lagrangian transforms with weight 5 under Weyl rescalings of the fields with weights listed in Table 11.3. Since the line element has Weyl weight -2 as in (11.38) this implies invariance of the five-dimensional action. Furthermore, the Lagrangian (11.58) is invariant under the supersymmetry transformations parametrized by ϵ_a and the special supersymmetry transformations parametrized by η_a given by[7]

$$\begin{aligned}
\delta\phi^{\Lambda} &= \tfrac{i}{2}\, \bar{\epsilon}^{a} \chi^{\Lambda}_{a} \ , \\
\delta A^{\widehat{I}}_{\mu} &= \tfrac{1}{2}\, \bar{\epsilon}^{a} \gamma_{\mu} \chi^{\widehat{I}}_{a} \ , \\
\delta F^{\Lambda}_{\mu\nu} &= -\bar{\epsilon}^{a} \gamma_{[\mu} \mathcal{D}_{\nu]} \chi^{\Lambda}_{a} + i t_{(\Sigma\Theta)}{}^{\Lambda} \phi^{\Sigma}\, \bar{\epsilon}^{a} \gamma_{\mu\nu} \chi^{\Theta}_{a} + i \bar{\eta}^{a} \gamma_{\mu\nu} \chi^{\Lambda}_{a} \ , \\
\delta\chi^{\Lambda\,a} &= -\tfrac{1}{4} \gamma^{\mu\nu} F^{\Lambda}_{\mu\nu} \epsilon^{a} - \tfrac{i}{2} \not{\mathcal{D}} \phi^{\Lambda} \epsilon^{a} - Y^{\Lambda\,ab} \epsilon_{b} + \tfrac{1}{2}\, t_{(\Sigma\Theta)}{}^{\Lambda} \phi^{\Sigma} \phi^{\Theta} \epsilon^{a} + \phi^{\Lambda} \eta^{a} \ , \\
\delta Y^{\Lambda\,ab} &= -\tfrac{1}{2} \bar{\epsilon}^{(a|} \not{\mathcal{D}} \chi^{\Lambda|b)} - \tfrac{i}{2} \Big(t_{[\Sigma\Theta]}{}^{\Lambda} - 3 t_{(\Sigma\Theta)}{}^{\Lambda} \Big) \phi^{\Sigma}\, \bar{\epsilon}^{(a|} \chi^{\Theta|b)} + \tfrac{i}{2} \bar{\eta}^{(a} \chi^{\Lambda|b)} \ , \\
\delta q^{\mathcal{I}\,a\dot{b}} &= -i \bar{\epsilon}^{a} \zeta^{\mathcal{I}\,\dot{b}} \ , \\
\delta\zeta^{\mathcal{I}\,\dot{b}} &= \tfrac{i}{2} \not{\mathcal{D}} q^{\mathcal{I}\,a\dot{b}} \epsilon_{a} - \tfrac{1}{2} \phi^{\widehat{K}} t_{\widehat{K}\mathcal{J}}{}^{\mathcal{I}} q^{\mathcal{J}\,a\dot{b}} \epsilon_{a} - \tfrac{3}{2} q^{\mathcal{I}\,a\dot{b}} \eta_{a} \ .
\end{aligned} \tag{11.59}$$

These transformation rules are consistent with Weyl rescalings of Table 11.3, if one assigns Weyl weight $-1/2$ to the parameter ϵ^a, and the weight $+1/2$ to η^a. Note that the gamma-matrices with lower indices $\gamma_{\mu_1\ldots\mu_k}$ scale with weight $-k$.

This completes the specification of the five-dimensional superconformal action in terms of the group theory invariants d_{IJ}, f_{IJK}, and the tensors $\delta_{\alpha\beta}, \epsilon_{\alpha\beta}$ for complex fields parameterizing the full Kaluza-Klein tower. The crucial insight is that it is possible to combine the symmetric d_{IJ} and the

[7] The expression for $\delta F^{\Lambda}_{\mu\nu}$ with $\Lambda = \widehat{I}$ is not independent from the expression for $\delta A^{\widehat{I}}_{\mu}$. To check their compatibility, note that the second term in $\delta F^{\widehat{I}}_{\mu\nu}$ vanishes thanks to $t_{(\Lambda\Sigma)}{}^{\widehat{I}} = 0$. In order to get the third term in $\delta F^{\widehat{I}}_{\mu\nu}$, one has to promote $\delta A^{\widehat{I}}_{\mu}$ to its full x-dependent form before taking the covariant derivative. As explained in [269], this is done by means of the prescription $\epsilon^a \mapsto \epsilon^a + i x^{\rho} \gamma_{\rho} \eta^a$. The covariant derivative can thus act on an x-linear term in $\delta A^{\widehat{I}}_{\mu}$ and produce the η-term in $\delta F^{\widehat{I}}_{\mu\nu}$.

antisymmetric $\epsilon_{\alpha\beta}$ to define the antisymmetric Ω_{MN} as in (11.54) for the massive Kaluza-Klein modes which naturally are complex fields. This also permits us to combine the totally antisymmetric f_{IJK} and the antisymmetric $\epsilon_{\alpha\beta}$ to define components of the totally symmetric $C_{\Lambda\Sigma\Theta}$. This implies that the non-Abelian version of the Kaluza-Klein theory fits naturally in the framework of $\mathcal{N}=2$ supersymmetry. Furthermore, superconformal invariance can be implemented by introducing the vector multiplet $\widehat{\mathcal{V}}^0$ defined in (11.39).

To close this section let us comment on the role of the additional multiplet $\widehat{\mathcal{V}}^0$ in more detail. We have found that its kinetic term is determined by the constant k_c. Identifying ϕ^0 with the radius r as in (11.42), one can derive the kinetic term of r after dimensional reduction of a six-dimensional gravity theory. This is complicated by the fact that the proper supersymmetric fields in five dimensions involve rescalings with r as described in detail in [262]. However, the choice

$$k_c = 0 \tag{11.60}$$

is natural from the point of view of $\mathcal{N}=4$ supersymmetry, since a Chern-Simons term $k_c\, A^0 \wedge F^0 \wedge F^0$ is absent in this case. Moreover, $k_c = 0$ is consistent with non-dynamical gravity in six dimensions. In the following discussion we work in the phase with (11.42) implying that k_c drops from the action.

11.4.2 Supersymmetric Kaluza-Klein Lagrangian in the broken phase

We are now in the position to present the $\mathcal{N}=2$ action including all Kaluza-Klein levels. This amounts to restoring the Kaluza-Klein indices for the fields and summing up an infinite tower of multiplets $(B^{I\alpha}_{n\,\mu\nu}, \phi^{I\alpha}_n, \chi^{I\alpha\,a}_n)$ and $(q^{I\alpha\,a\dot{b}}_n, \zeta^{I\alpha\,\dot{b}}_n)$ in (11.58). The resulting action is straightforwardly obtained but rather lengthy due to the fact that both $C_{\Lambda\Sigma\Theta}$ and Ω_{MN} appear in copies labeled by Kaluza-Klein indices. The result simplifies, however, if we set $\widehat{\mathcal{V}}^0$ to the values (11.42), thus moving to the broken phase of conformal invariance. Discussing the resulting action will be the task of this section.

As discussed already in section 11.3, the Abelian vector multiplet $\widehat{\mathcal{V}}^0$ plays a special role in the $\mathcal{N}=2$ spectrum. In a Kaluza-Klein theory $\widehat{\mathcal{V}}^0$ has to be interpreted as part of the gravity multiplet with A^0 being the graviphoton under which all excited Kaluza-Klein modes are charged. We decouple gravity completely by imposing the condition (11.42). As we will argue below, ordinary $\mathcal{N}=2$ supersymmetry is preserved despite the breaking of superconformal invariance. Furthermore, we make use of the rescaled hypermultiplet fields $h^{I\,a\dot{b}}_n, \psi^{I\,\dot{a}}_n$ defined in (11.43).

The resulting Lagrangian including all Kaluza-Klein modes listed in Table 11.3 takes the form

$$\mathcal{L} = \mathcal{L}_0 + \sum_{n=1}^{\infty} \mathrm{Re}\mathcal{L}_n \ , \tag{11.61}$$

where $\mathcal{L}_0$ only involves massless multiplets, while $\mathcal{L}_n$ collects all terms constructed with the nth excited modes. We discuss $\mathcal{L}_0$ and $\mathcal{L}_n$ in turn.

To begin with, let us display the zero mode Lagrangian

$$
\begin{aligned}
g^2\mathcal{L}_0 = d_{IJ}\Big[&- \tfrac{1}{4}F^{I\,\mu\nu}F^J_{\mu\nu} - \tfrac{1}{2}\mathcal{D}^\mu\phi^I\mathcal{D}_\mu\phi^J - \tfrac{1}{2}\mathcal{D}^\mu h^{I\,a\dot b}\mathcal{D}_\mu h^J_{a\dot b} - \tfrac{1}{2}\bar\chi^{I\,a}\not{\mathcal{D}}\chi^J_a - \bar\psi^{I\,\dot a}\not{\mathcal{D}}\psi^J_{\dot a} + Y^{I\,ab}Y^J_{ab}\Big] \\
&+ f_{IJK}\Big[+ \tfrac{i}{2}\phi^I\bar\chi^{J\,a}\chi^K_a - i\phi^I\bar\psi^{J\,\dot a}\psi^K_{\dot a} - 2ih^{I\,a\dot b}\bar\chi^J_a\psi^K_{\dot b} + h^{I\,a\dot c}h^{J\,b}{}_{\dot c}Y^K_{ab}\Big] \\
&- \tfrac{1}{2}f_{IJ}{}^H f_{HKL}\phi^I\phi^K h^{J\,a\dot b}h^L_{a\dot b}\ . \qquad (11.62)
\end{aligned}
$$

We recognize that the terms contracted with the trace d_{IJ} are the kinetic terms of the massless vectors, scalars and fermions, as well as the quadratic term for the auxiliary field. The terms involving the structure constants f_{IJK} are Yukawa-type couplings and a scalar potential quartic in the fields $\phi^I, h^{I\,a\dot b}$. We stress that for the massless fields such quartic coupling are only possible if they also include scalars $h^{I\,a\dot b}$ due to the asymmetry of f_{IJK}. In section 11.5 we will discuss the properties of (11.62) in more detail and relate it to $\mathcal{N}=4$ supersymmetric Yang-Mills theory.

Let us now turn to the discussion of the Lagrangians $\mathcal{L}_n$ in (11.61) for the Kaluza-Klein tower. We insert (11.52)-(11.56) into the action (11.58), impose the condition (11.42), and extract the terms for the Kaluza-Klein level n to find

$$
\begin{aligned}
g^2\mathcal{L}_n = d_{IJ}\Big[&- \frac{1}{2}\bar F^{I\,\mu\nu}_n F^J_{n\,\mu\nu} + \frac{i}{4m_n}\epsilon^{\mu\nu\rho\lambda\sigma}\bar F^I_{n\,\mu\nu}\mathcal{D}_\rho F^J_{n\,\lambda\sigma} \\
&- \mathcal{D}^\mu\bar\phi^I_n\mathcal{D}_\mu\phi^J_n - \mathcal{D}^\mu\bar h^{I\,a\dot b}_n\mathcal{D}_\mu h^J_{n\,a\dot b} - \bar\chi^{I\,a}_n\not{\mathcal{D}}\chi^J_{n\,a} - 2\bar\psi^{I\,\dot a}_n\not{\mathcal{D}}\psi^J_{n\,\dot a} \\
&- m_n^2\bar\phi^I_n\phi^J_n - m_n^2\bar h^{I\,a\dot b}_n h^J_{n\,a\dot b} - m_n\bar\chi^{I\,a}_n\chi^J_{n\,a} - 2m_n\bar\psi^{I\,\dot a}_n\psi^J_{n\,\dot a} + 2\bar Y^{I\,ab}_n Y^J_{n\,ab}\Big] \\
+ \frac{1}{m_n}f_{IJK}\Big[&- \frac{i}{2}\phi^K\bar F^{I\,\mu\nu}_n F^J_{n\,\mu\nu} + i\bar\phi^K_n F^{I\,\mu\nu}F^J_{n\,\mu\nu} - i\phi^K\mathcal{D}^\mu\bar\phi^I_n\mathcal{D}_\mu\phi^J_n + 2i\bar\phi^K_n\mathcal{D}^\mu\phi^I\mathcal{D}_\mu\phi^J_n \\
&- i\phi^K\bar\chi^{I\,a}_n\not{\mathcal{D}}\chi^J_{n\,a} + 2i\bar\phi^K_n\bar\chi^{I\,a}\not{\mathcal{D}}\chi^J_{n\,a} + 2i\phi^K\bar Y^{I\,ab}_n Y^J_{n\,ab} - 4i\bar\phi^K_n Y^{I\,ab}Y^J_{n\,ab} \\
&+ \frac{1}{4}F^I_{\mu\nu}\bar\chi^{J\,a}_n\gamma^{\mu\nu}\chi^K_{n\,a} - \frac{1}{2}\bar F^I_{n\,\mu\nu}\bar\chi^{J\,a}\gamma^{\mu\nu}\chi^K_{n\,a} + Y^{I\,ab}\bar\chi^J_{n\,a}\chi^K_{n\,b} - 2\bar Y^{I\,ab}_n\bar\chi^J_a\chi^K_{n\,b} \\
&- 2im_n\phi^K\bar\chi^{I\,a}_n\chi^J_{n\,a} + 3im_n\bar\phi^K_n\bar\chi^{I\,a}\chi^J_{n\,a} - 4im_n\bar h^{I\,a\dot b}_n\bar\chi^J_a\psi^K_{n\,\dot b} - 2im_n\phi^I\bar\psi^{J\,\dot a}_n\psi^K_{n\,\dot a} \\
&+ 2m_n\bar h^{I\,a\dot c}_n h^{J\,b}_n{}_{\dot c}Y^K_{ab} - 3im_n^2\phi^I\bar\phi^J_n\phi^K_n - 2im_n^2\phi^I\bar h^{J\,a\dot b}_n h^K_{n\,a\dot b}\Big] \\
+ \frac{1}{m_n}f_{IJ}{}^H f_{HKL}\Big[&- 3m_n\phi^I\phi^K\bar\phi^J_n\phi^L_n - m_n\phi^I\phi^K\bar h^{J\,a\dot b}_n h^L_{n\,a\dot b} - \phi^I\phi^K\bar\chi^{J\,a}_n\chi^L_{n\,a} \\
&+ \bar\phi^I_n\phi^J\bar\chi^{K\,a}\chi^L_{n\,a} + 2\bar\phi^I_n\phi^K\bar\chi^{J\,a}\chi^L_{n\,a} - \bar\phi^I_n\phi^K_n\bar\chi^{J\,a}\chi^L_a - \tfrac{1}{2}\bar\phi^I_n\phi^J_n\bar\chi^{K\,a}\chi^L_a\Big] \\
- \frac{i}{m_n}f_{IH}{}^{I_1}&f_{JL}{}^{I_2}f_{KI_1I_2}\phi^I\phi^J\phi^K\bar\phi^H_n\phi^L_n\ . \qquad (11.63)
\end{aligned}
$$

The terms contracted with the trace d_{IJ} are kinetic terms and mass terms for all Kaluza-Klein excited modes. We note that the tensors $B^I_{n\,\mu\nu} = -\frac{i}{n}F^I_{n\,\mu\nu}$ have Chern-Simons kinetic terms and a mass term proportional to n^2. Consistent with a Kaluza-Klein reduction all complex scalars ϕ^I_n, $h^{I\,a\dot b}_n$ with $n>0$ have mass terms proportional to n^2, and all fermions $\chi^{I\,a}_n, \psi^{I\,\dot b}_n$ with $n>0$ have mass terms proportional to n. More interestingly, this Lagrangian contains various terms at the non-Abelian level containing f_{IJK}. These include new kinetic terms for all singlets under the second $SU(2)$ in (11.31), Pauli terms coupling the tensors and gauge fields to the fermions, Yukawa type couplings, and a complicated scalar potential. The full scalar potential and four Fermi terms can only be determined after eliminating the auxiliary fields $Y_n^{I\,ab}$. We will discuss this elimination process in section 11.5.

It is important to stress that the action (11.63) preserves $\mathcal{N} = 2$ supersymmetry but breaks the special supersymmetries parametrized by η^a in (11.59). This can be seen straightforwardly by inspecting the superconformal variations of the fermion in $\widehat{\mathcal{V}}^0$:

$$\delta\chi^{0\,a} = -\tfrac{1}{4}\gamma^{\mu\nu}F^0_{\mu\nu}\epsilon^a - \tfrac{i}{2}\not{\mathcal{D}}\phi^0\epsilon^a - Y^{0\,ab}\epsilon_b + \phi^0\eta^a\,. \tag{11.64}$$

Using the condition (11.42) we realize that the supersymmetry parameter ϵ^a drops from (11.64) which implies that the restricted action is still $\mathcal{N} = 2$ supersymmetric. In contrast η^a appears after imposing (11.42) in the transformation $\delta\chi^{0\,a} = g^{-2}\eta^a$, which implies that $\chi^{0\,a}$ is needed to ensure invariance of the action under special supersymmetry transformations. In other words, the condition (11.42) will break the special supersymmetry transformations parametrized by η^a. The ordinary supersymmetry transformations in the restricted phase are given by

$$\begin{aligned}
\delta A^I_\mu &= \tfrac{1}{2}\,\bar\epsilon^a\gamma_\mu\chi^I_a\ ,\\
\delta\phi^I_n &= \tfrac{i}{2}\,\bar\epsilon^a\chi^I_{n\,a}\ ,\\
\delta F^I_{n\,\mu\nu} &= -\bar\epsilon^a\gamma_{[\mu}\mathcal{D}_{\nu]}\chi^I_{n\,a} - \tfrac{i}{2}f_{JK}{}^I\phi^J_n\,\bar\epsilon^a\gamma_{\mu\nu}\chi^K_a + \tfrac{i}{2}f_{JK}{}^I\phi^J\,\bar\epsilon^a\gamma_{\mu\nu}\chi^K_{n\,a} - \tfrac{1}{2}m_n\,\bar\epsilon^a\gamma_{\mu\nu}\chi^I_{n\,a}\ ,\\
\delta\chi^{I\,a}_n &= -\tfrac{1}{4}\gamma^{\mu\nu}F^I_{n\,\mu\nu}\epsilon^a - \tfrac{i}{2}\not{\mathcal{D}}\phi^I_n\epsilon^a - Y^{I\,ab}_n\epsilon_b + \tfrac{1}{2}f_{JK}{}^I\phi^J\phi^K_n\epsilon^a + \tfrac{i}{2}m_n\phi^I_n\epsilon^a\ ,\\
\delta Y^{I\,ab}_n &= -\tfrac{1}{2}\,\bar\epsilon^{(a|}\not{\mathcal{D}}\chi^{I|b)}_n - if_{JK}{}^I\phi^J_n\,\bar\epsilon^{(a|}\chi^{K|b)} + \tfrac{i}{2}f_{JK}{}^I\phi^J\,\bar\epsilon^{(a|}\chi^{K|b)}_n - \tfrac{1}{2}m_n\,\bar\epsilon^{(a|}\chi^{K|b)}_n\ ,\\
\delta h^{I\,a\dot b}_n &= -i\,\bar\epsilon^a\psi^{I\,\dot b}_n\ ,\\
\delta\psi^{I\,\dot b}_n &= \tfrac{i}{2}\not{\mathcal{D}}h^{I\,a\dot b}_n\epsilon_a - \tfrac{1}{2}f_{JK}{}^I\phi^J h^{K\,a\dot b}_n\epsilon_a - \tfrac{i}{2}m_n h^{I\,a\dot b}_n\epsilon_a\ ,
\end{aligned} \tag{11.65}$$

where $n \geq 0$ labels both zero and excited modes. We close this subsection by pointing out that the Lagrangian (11.61) possesses a scaling symmetry when using the Weyl weights of Table 11.3 and additionally assigning scaling weight $-1/2$ to the gauge coupling constant g, in such a way that m_n has weight $+1$ for any $n > 0$. This can be interpreted as a remnant of the full six-dimensional $(2,0)$ conformal symmetry as discussed in section 11.3.2.

This concludes our discussion of the general $\mathcal{N} = 2$ action for the Kaluza-Klein tower. Our approach can be summarized as follows. While an action for full six-dimensional non-Abelian $(2,0)$ theories is unknown the Abelian free six-dimensional $(2,0)$ theory admits a six-dimensional pseudoaction. It can be compactified on a circle with arbitrary radius yielding a five-dimensional action with $\mathcal{N} = 4$ supersymmetry. We proposed a gauged version of this theory preserving only half, namely $\mathcal{N} = 2$, supersymmetry, by interpreting the zero mode vectors A^I as gauge potentials for the whole Kaluza-Klein tower. In order to argue for a six-dimensional origin of this theory all higher-dimensional symmetries need to be realized or appear in a gauge-fixed phase. Our five-dimensional actions (11.58) and (11.61), however, clearly only realize part of the six-dimensional superconformal $(2,0)$ symmetries manifestly. In particular, we have singled out the zero modes for gauging which seems naively incompatible with six-dimensional Poincaré invariance.

It is precisely the non-Abelian gauging that prevents us to write down an $\mathcal{N}=4$ action. Nevertheless, we regard our Lagrangians as the starting point to give a lower-dimensional Lagrangian formulation for $(2,0)$ theories. In the next subsection 11.5 we concentrate on two special cases in which partial symmetry restoration is achieved.

11.5 Two special cases

We have just proposed a Lagrangian for all Kaluza-Klein modes in an $\mathcal{N}=2$ supersymmetric framework. In particular we made use of complete $\mathcal{N}=2$ vector and tensor multiplets including auxiliary fields $Y_n^{I\,ab}$. These fields appear only algebraically in the Lagrangian and can be eliminated consistently by using their equations of motion. While the action (11.61) is a sum of terms $\mathcal{L}_n$ only involving fields at the Kaluza-Klein level n and zero modes, the elimination of auxiliary fields will induce a non-trivial mixing among excited modes. Despite the fact that it is interesting to investigate this structure in more detail, we will focus here on only two special cases where the computation is straightforward and the lift to $\mathcal{N}=4$ can be performed explicitly.

As a first special case we study the zero mode Lagrangian $\mathcal{L}_0$ given in (11.62), and drop all massive modes. This is motivated physically with the dimensional reduction argument for small radius r where massive Kaluza-Klein modes are dropped, or rather integrated out, that are above a certain energy scale. The equation of motion for the auxiliary fields then simply reads

$$Y^{I\,ab} = -\tfrac{1}{2} f^{I}{}_{JK} h^{J\,a\dot{a}} h^{K\,b}{}_{\dot{a}} \; . \tag{11.66}$$

Inserting (11.66) into (11.62) a quartic potential in h is generated, and the zero mode Lagrangian $\mathcal{L}_0$ takes the form

$$\begin{aligned} g^2\mathcal{L}_{\mathrm{YM}} = d_{IJ}\Big[&-\tfrac{1}{4}F^{I\,\mu\nu}F^{J}_{\mu\nu} - \tfrac{1}{2}\mathcal{D}^{\mu}\phi^{I}\mathcal{D}_{\mu}\phi^{J} - \tfrac{1}{2}\mathcal{D}^{\mu}h^{I\,a\dot{b}}\mathcal{D}_{\mu}h^{J}_{a\dot{b}} - \tfrac{1}{2}\bar{\chi}^{I\,a}\not{\mathcal{D}}\chi^{J}_{a} - \psi^{I\,\dot{a}}\not{\mathcal{D}}\psi^{J}_{\dot{a}}\Big] \\ &+ f_{IJK}\Big[+ \tfrac{i}{2}\phi^{I}\bar{\chi}^{J\,a}\chi^{K}_{a} - i\phi^{I}\bar{\psi}^{J\,\dot{a}}\psi^{K}_{\dot{a}} - 2ih^{I\,a\dot{b}}\bar{\chi}^{J}_{a}\psi^{K}_{\dot{b}}\Big] \\ &+ f_{IJ}{}^{H}f_{HKL}\Big[-\tfrac{1}{4}h^{I\,a\dot{a}}h^{J\,b}{}_{\dot{a}}h^{K}{}_{a}{}^{\dot{b}}h^{L}{}_{b\dot{b}} - \tfrac{1}{2}\phi^{I}\phi^{K}h^{J\,a\dot{b}}h^{L}_{a\dot{b}}\Big] \; . \end{aligned} \tag{11.67}$$

This Lagrangian is a simple rewriting of $\mathcal{N}=4$ super Yang-Mills theory in terms of $\mathcal{N}=2$ multiplets. In order to prove this claim we record the Lagrangian for $\mathcal{N}=4$ super Yang-Mills theory,

$$\begin{aligned} g^2\mathcal{L}^{\mathrm{SYM}}_{\mathcal{N}=4} =& d_{IJ}\Big[-\tfrac{1}{4}F^{I\,\mu\nu}F^{J}_{\mu\nu} - \tfrac{1}{4}\mathcal{D}^{\mu}\sigma^{I\,ij}\mathcal{D}_{\mu}\sigma^{J}{}_{ij} - \tfrac{1}{2}\bar{\lambda}^{I\,i}\not{\mathcal{D}}\lambda^{J}_{i}\Big] \\ &- \tfrac{i}{\sqrt{2}}f_{IJK}\sigma^{I\,ij}\bar{\lambda}^{J}_{i}\lambda^{K}_{j} - \tfrac{1}{16}f_{IJKL}\sigma^{I\,ij}\sigma^{K}{}_{ij}\sigma^{J\,kl}\sigma^{L}{}_{kl} \; . \end{aligned} \tag{11.68}$$

It is invariant under the following $\mathcal{N}=4$ supersymmetry transformations:

$$\begin{aligned} \delta A^{I}_{\mu} &= \tfrac{1}{2}\bar{\epsilon}^{i}\gamma_{\mu}\lambda^{I}_{i} \; , \\ \delta\sigma^{I\,ij} &= -i\sqrt{2}\Big(\bar{\epsilon}^{[i|}\lambda^{J|j]} + \tfrac{1}{4}\Omega^{ij}\bar{\epsilon}^{k}\lambda^{I}_{k}\Big) \\ \delta\lambda^{I\,i} &= -\tfrac{1}{4}F^{I}_{\mu\nu}\gamma^{\mu\nu}\epsilon^{i} - \tfrac{i}{\sqrt{2}}\not{\mathcal{D}}\sigma^{I\,ij}\epsilon_{j} + \tfrac{1}{2}f^{I}{}_{JK}\sigma^{J\,ij}\lambda^{K}_{jk}\epsilon^{k} \; . \end{aligned} \tag{11.69}$$

In order to check that that (11.68) reproduces (11.67) one has to make use of (11.41), (11.42) and (A.20).

As a second special case we consider the Abelian truncation of the full Lagrangian (11.61). This is achieved by dropping all terms constructed with the structure constants f_{IJK}. The equations of motions for the auxiliary fields read simply $Y_n^{I\,ab} = 0$, such that they can be trivially dropped from the Lagrangian. The resulting theory is free and given by

$$\begin{aligned} g^2\mathcal{L}_{\text{free}} = d_{IJ}\Big[&-\tfrac{1}{4}F^{I\,\mu\nu}F^J_{\mu\nu} - \tfrac{1}{2}\partial^\mu\phi^I\partial_\mu\phi^J - \tfrac{1}{2}\partial^\mu h^{I\,a\dot{b}}\partial_\mu h^J_{a\dot{b}} - \tfrac{1}{2}\bar{\chi}^{I\,a}\not{\partial}\chi^J_a - \psi^{I\,\dot{a}}\not{\partial}\psi^J_{\dot{a}}\Big] \\ + \sum_{n=1}^{\infty} d_{IJ}\Big[&-\tfrac{1}{2}\bar{F}^{I\,\mu\nu}_n F^J_{n\,\mu\nu} + \tfrac{i}{4m_n}\epsilon^{\mu\nu\rho\lambda\sigma}\bar{F}^I_{n\,\mu\nu}\partial_\rho F^J_{n\,\lambda\sigma} \\ &- \partial^\mu\bar{\phi}^I_n\partial_\mu\phi^J_n - \partial^\mu h^{I\,a\dot{b}}_n\partial_\mu h^J_{n\,a\dot{b}} - \bar{\chi}^{I\,a}_n\not{\partial}\chi^J_{n\,a} - 2\bar{\psi}^{I\,\dot{a}}_n\not{\partial}\psi^J_{n\,\dot{a}} \\ &- m_n^2\left(\bar{\phi}^I_n\phi^J_n + \bar{h}^{I\,a\dot{b}}_n h^J_{n\,a\dot{b}}\right) - m_n\left(\bar{\chi}^{I\,a}_n\chi^J_{n\,a} + 2\bar{\psi}^{I\,\dot{a}}_n\psi^J_{n\,\dot{a}}\right)\Big]\,. \end{aligned} \tag{11.70}$$

This Lagrangian is the $\mathcal{N}=2$ supersymmetric extension of the purely bosonic Lagrangian (11.9) in the gauge (11.42). In fact, this theory is actually $\mathcal{N}=4$ supersymmetric. This can be seen by comparison to the Lagrangian

$$\begin{aligned} g^2\mathcal{L}^{\text{free}}_{\mathcal{N}=4} = d_{IJ}\Big[&-\tfrac{1}{4}F^{I\,\mu\nu}F^J_{\mu\nu} - \tfrac{1}{4}\partial^\mu\sigma^{I\,ij}\partial_\mu\sigma^J_{ij} - \tfrac{1}{2}\bar{\lambda}^{I\,i}\not{\partial}\lambda^J_i\Big] \\ + \sum_{n=1}^{\infty} d_{IJ}\Big[&-\tfrac{1}{2}\bar{F}^{I\,\mu\nu}_n F^J_{n\,\mu\nu} + \tfrac{i}{4m_n}\epsilon^{\mu\nu\rho\lambda\sigma}\bar{F}^I_{n\,\mu\nu}\partial_\rho F^J_{n\,\lambda\sigma} \\ &- \tfrac{1}{2}\partial^\mu\bar{\sigma}^{I\,ij}_n\partial_\mu\sigma^J_{n\,ij} - \bar{\lambda}^{I\,i}_n\not{\partial}\lambda^J_{n\,i} - \tfrac{1}{2}m_n^2\bar{\sigma}^{I\,ij}_n\sigma^J_{n\,ij} - m_n\bar{\lambda}^{I\,i}_n\lambda^J_{n\,i}\Big]\,. \end{aligned} \tag{11.71}$$

Furthermore, (11.70) or equivalently (11.71) can be obtained by a compactification of the full $(2,0)$ Abelian pseudoaction (11.5) on a circle and therefore admits non-manifest six-dimensional Poincaré invariance. Five-dimensional Kaluza-Klein actions arising from such a compactification have been considered before in [288]. We stress that it is hard to interpret the action (11.70) with the full Kaluza-Klein tower as an effective action for the Coulomb branch of the five-dimensional theory. This is due to the fact that it contains modes of arbitrary high mass m_n that rather should be integrated out above the cutoff scale.

11.6 A possible window on (2,0) conformal anomalies

In this section we would like to point out a possible application of the non-Abelian $\mathcal{N}=2$ action in the broken phase of conformal symmetry given in (11.63). It is based on considerations about anomaly matching for a system of N M5-branes when one brane is moved away from the stack [158]. In what follows we aim at conveying the basic idea that this anomaly matching might be accessible via five-dimensional loop. For a refined discussion and a first computational step in this direction we refer the reader to [277].

Let us consider a single M5-brane in M-theory. The eleven-dimensional Lorentz group $SO(1,10)$ is spontaneously broken to $SO(1,5) \times SO(5)$, with the first factor being the Lorentz on the six-dimensional world-volume of the M5-brane and the second factor being identified with the R-symmetry of the theory. Both the six-dimensional Lorentz group $SO(1,5)$ and the R-symmetry group $SO(5)$ are anomalous as a result of the chiral fields living on the world-volume of the M5-brane. The full eleven-dimensional setup is nonetheless consistent thanks to a subtle anomaly inflow mechanism explained for instance in [148, 289]. In [153] this inflow mechanism was used to consider the case of a stack of N M5-branes. In this case the Lorentz and R-symmetry anomalies cannot be computed directly since we do not have an effective action for the six-dimensional world-volume theory, but can nonetheless be inferred from the bulk contribution. The result is encoded in the anomaly polynomial

$$I_8 = \frac{N}{48}\left[p_2(F_{\mathrm{N}}) - p_2(R) + \frac{1}{4}\left(p_1(R) - p_1(F_{\mathrm{R}})\right)^2\right] + \frac{N^3-N}{24} p_2(F_{\mathrm{R}}) \,, \tag{11.72}$$

where $p_{1,2}(R)$ are Pontryagin classes build from the six-dimensional curvature of the spin connection, while $p_{1,2}(F_{\mathrm{R}})$ are Pontryagin built from the curvature F_{R} of the R-symmetry connection. Let us point out that this connection is not a dynamical field of the theory, but rather a classical background, so that no contradiction arises with the absence of vectors in the massless spectrum of (2,0) theories. From an eleven-dimensional perspective the R-symmetry connection encodes topological data about the normal bundle to the world-volume of the M5-brane stack. From a field theoretic perspective we interpret (11.72) as the gravitational and R-symmetry anomalies of a (2,0) theory with 'gauge group' $G = U(N)$, so that $\mathrm{rank}(G) = N$. This theory is obtained combining a (2,0) theory of type A_{N-1} with an Abelian (2,0) theory, which is associated to the center-of-mass motion in the M5-brane picture. The second term in (11.72) is generated by non-trivial interaction, so that it has to correspond to the A_{N-1} (or equivalently $SU(N)$) part of the 'gauge group.'

Suppose we move one of the M5-branes away from the stack, breaking the 'gauge group' from G to $H \times U(1)$ with $H = U(N-1)$. At the same time R-symmetry is broken from $SO(5)$ to $SO(4)$. At low energies the H factor and the $U(1)$ factor in $H \times U(1)$ decouple, and we thus expect a free theory for the latter. If this were the case, however, the anomaly polynomials before and after moving the brane would not match. More precisely, the first in (11.72) does not change because the total rank of the group is unaffected. The second term, however, changes: only the factor $SU(N-1)$ inside $H \times U(1)$ contributes to this term, which is therefore given by the same expression in (11.72) with N replaced by $N-1$. In summary, the difference of the anomaly polynomials reads

$$I_8|_G - I_8|_{H\times U(1)} = \frac{N^2-N}{8} p_2(F_{\mathrm{R}}) \,. \tag{11.73}$$

This mismatch in anomalies suggests that the effective theory for the $U(1)$ factor in $H \times U(1)$ contains a Wess-Zumino term that has an anomalous variation under R-symmetry transformations in such a way to reproduce the right hand side of (11.73). For an explicit expression of this term see [158].

Upon circle reduction the six-dimensional Wess-Zumino term should yield topological couplings in five dimensions. By analogy with the lesson we have learnt studying M-theory/F-theory duality we expect that these topological couplings are generated in five dimensions by one-loop diagrams in which all relevant massive fields run in the loop. For the case at hand, those would be both excited Kaluza-Klein modes and the tensor analog of W-bosons. This suggests the possibility that, by a suitable coupling of our action (11.63) to a background R-symmetry connection, the anomaly mismatch (11.73) could be extracted (at least to leading order in N) by computing one-loop corrections to five-dimensional couplings. If this program can be substantiated it would show that our action (11.63) correctly encodes the N^3 scaling behavior of (2,0) theories, since the latter is equivalent to the N^2 scaling of the anomaly mismatch (11.73). The interested reader is referred to [277] for some first steps in this direction. A thorough analysis of this subject is a possible interesting direction for future work.

PART IV

Conclusions

CHAPTER 12

Closing remarks

In this last chapter we offer a retrospective look at the setups discussed in this thesis and we provide some outlook for future developments.

12.1 A quick overview

In this work we have investigated various aspects of non-perturbative low-energy physics in the framework of string theory and M-theory. Thanks to the power of the duality between M-theory and F-theory we have been able to determine the effective action of a class of six-dimensional and four-dimensional string compactifications in which the string coupling constant is geometrized and allowed to vary in the internal space in a non-perturbative fashion. This has been possible by means of an indirect approach based on a transdimensional treatment of the physical system under examination. In order to study compactifications to six dimensions we have thus focussed on five-dimensional setups, while three-dimensional models have been the starting point for our exploration of F-theory compactified to four dimensions on a Spin(7) manifold.

This strategy has been inspirational and has been applied to explore the dynamics of six-dimensional theories with self-dual tensors. Our interest in these models comes from the puzzle raised by the low-energy dynamics of a stack of M5-branes in M-theory, which, as we have seen, gives rise to a superconformal (2,0) theory whose dynamics has resisted an explicit description so far. Compactifying one spatial direction on a circle the problem of determining the effective action for a theory with massless self-dual tensors is mapped to the study of massive towers of tensors coupled to five-dimensional massless gauge fields. This approach has also led us to explore in greater detail the dynamics of massive field at the quantum level and to discover an interesting extension of the known results about parity anomaly in five dimensions.

On general grounds an indirect, transdimensional approach to a d-dimensional problem has the obvious drawback of obscuring Lorentz invariance in d dimensions. In some situations, however, this might be a blessing in disguise, as it might open up the possibility to capture symmetries or dynamics that cannot be realized manifestly at the same time as Lorentz symmetry. This would be in the same spirit, for example, of exceptional symmetries in eleven-dimensional supergravity, which are most conveniently analyzed by sacrificing manifest eleven-dimensional Lorentz invariance, see for instance [290] and references therein. From this perspective the setups studied in this work can be considered as examples of a more general program aiming at capturing non-perturbative systems that elude conventional techniques by applying a transdimensional approach.

In our work we have considered the simplest possible situations in which the d-dimensional problem one wants to address is mapped to a $(d-1)$-dimensional problem by means of compactification on a circle or on an interval. In principle, more complicated variants of this strategy are conceivable, relating the d-dimensional setup under examination to a $(d-k)$-dimensional problem by compactification on a k-dimensional space. For example, one might consider a six-dimensional tensor theory compactified to four dimensions on a two-torus. In this case we would be left with the task of coupling a double tower of massive tensors to four-dimensional vectors. Even though this construction is clearly more involved than the circle case it might unveil interesting connections with S-duality in four dimensions, as already pointed out and exploited in [33]. Furthermore, a transdimensional treatment can offer the possibility to explore the effect of non-locality in d dimensions in a controlled way by tuning the couplings of massive Kaluza-Klein modes to the massless field in the lower-dimensional setup.

Before considering these fascinating possibilities, however, there are many open interesting directions that are more directly related to the topics examined in this thesis. These are discussed in the following two sections.

12.2 Exploring F-theory vacua

The compactifications we have considered in part II exemplify in two different ways the fascinating aspects of the duality between F-theory and M-theory. In the case of six-dimensional (1,0) effective actions the constraints coming from supersymmetry and anomaly cancellation offer an enhanced control with respect to four-dimensional F-theory setups. It allows us to appreciate explicitly the subtle interplay between perturbative and non-perturbative physics, quantum corrections and classical geometry. Our analysis relies crucially on the fact that in the five-dimensional $\mathcal{N}=2$ theory all information about the couplings in the vector-tensor sector is encoded in the Chern-Simons coefficients for the vectors. By the same token, in the six-dimensional action the topological Green-Schwarz term can be used to read off the anomaly coefficients a^α, b^α that also govern the kinetic terms for tensor multiplet fields. In summary, topological terms in six and five dimensions are enough to reconstruct the dynamics of vector and tensor multiplets. This is a powerful simplification, since quantum corrections to such topological couplings are restricted and subject to non-renormalization arguments. As a result, they can be computed perturbatively at one-loop without the need to address other difficulties related to higher orders in perturbation theory and the non-renormalizability of the models under examination.

We are confident that in principle also in the hypermultiplet sector of the theory the same match between quantum corrections induced by massive Kaluza-Klein modes on the F-theory side and classical geometry on the M-theory side can be performed. It is nonetheless a hard task to verify this explicitly since we have much less control over five-dimensional quantum corrections in this sector, especially regarding the hypermultiplet moduli space metric. One is led to infer that classical geometry on the M-theory side contains more information about the quantum properties of the UV completion of the five-dimensional theory than can be extracted from a field-theoretical analysis with straightforward techniques.

Incidentally it is interesting to notice that similar considerations prevent us from a straightforward generalization of the landscape analysis of section 10.3 from five to three dimensions. Note in fact that in the three-dimensional theory Chern-Simons terms are also generated at one loop. It was shown in [137, 140] that they capture information about the four-dimensional chiral spectrum and its anomalies. Focussing as in five dimensions on the Coulomb branch, the Chern-Simons terms are specified by a constant matrix Θ_{AB} for the coupling $\int \Theta_{AB}\, A^A \wedge F^B$. These encode both the four-dimensional gaugings of axions, as well as the one-loop contributions from integrated out massive matter. As in five dimensions this matter includes modes that become massive in the Coulomb branch and fields that are Kaluza-Klein modes. However, in contrast to five dimensions one cannot infer all relevant information for the four-dimensional Green-Schwarz mechanism from the Chern-Simons terms alone [291, 140]. The four-dimensional analogs of a^α, b^α introduced in (10.35) do not appear in Chern-Simons terms and one needs to extend the analysis to other non-topological couplings of the effective action. Including these couplings one could proceed in a similar manner as in the five-dimensional case and check if a given three-dimensional theory can effectively arise from a four-dimensional anomaly-free theory.

For the sake of simplicity we have restricted our analysis of six-dimensional (1,0) F-theory vacua to the case of a semi-simple non-Abelian gauge group. The inclusion of Abelian factors makes the anomaly pattern of the theory richer and opens up the possibility to extend the dictionary between anomalies and geometry. We refer the reader to [183] for a detailed discussion. At the level of the effective action obtained via M-theory/F-theory duality the effect of Abelian factors in the gauge group has been analyzed in [187], where the generalization to a rational—as opposed to holomorphic—zero section of the fibration has been discussed too. The one-loop results of chapter 9 have been used to argue that, in the case of a rational zero section, a finite shift in the Chern-Simons terms can be induced by a non-standard mass hierarchy between Kaluza-Klein excited modes and modes that get massive upon gauge symmetry breaking.

F-theory vacua could also be useful in the study of interacting tensor theories in six dimensions. In this work we have advanced a proposal for the study of non-Abelian tensor theories in relation to (2,0) theories. It is nonetheless expected that non-Abelian tensor dynamics can be found also in theories with (1,0) supersymmetry. The superconformal theories of [259, 292] provide an example of (1,0) theories in which tensors are gauged under a non-Abelian group. It would be interesting to realize these model in F-theory and study their M-theory dual in five dimensions. This program is closely related to the recent progress in engineering (1,0) theories with non-Abelian tensor in F-theory: we refer to reader to [293] and references therein. In order to address these setups from the point of view of the M-theory/F-theory duality it would be probably beneficial to generalize the one-loop computations of chapter 9 to the case in which the external vectors are associated to a non-Abelian gauge group. In particular, one might study the one-loop effects induced by minimal coupling to non-Abelian tensors and possibly test them against a geometric prediction. This program has the potential to cast some light on five-dimensional interactions coming from circle reduction of non-Abelian tensor theories.

The other F-theory setup we have studied in this work, four-dimensional compactifications on Spin(7) manifolds, also presents many interesting open problems. Arguably, one of the most pressing issues that deserves a better understanding is the study of Spin(7) resolutions for the geometries we have considered in chapter 8. Let us remind the reader that the constructions of Spin(7) manifolds originally proposed in [216] also included isolated orbifold points coinciding with the fixed points of the antiholomorphic involution. Showing that these can be resolved in a Spin(7)-compatible fashion was a crucial task in [216]. We have not included a study of these modes in this work, but it would be very interesting to understand how they modify the four-dimensional effective theory. In particular, we found that if the antiholomorphic involution has only isolated fixed points on the Calabi-Yau fourfold the torus must be pinched over these points. This suggests an interesting link between the gauge theory dynamics and the singularities that need to be resolved in a Spin(7)-compatible way to obtain a smooth geometry. As for ordinary non-Abelian gauge theory singularities of elliptically fibered Calabi-Yau fourfolds, F-theory might be well-defined on the singular Spin(7) geometry if one can identify the new light states arising near the singularities.

The emergence of an interval in the duality between F-theory and M-theory for Spin(7) manifolds that are quotients of Calabi-Yau fourfolds is one of the main insights of chapter 8. It is most clearly understood in the situations in which the torus fiber is quotiented to a cylinder over the fixed loci of the restriction of the antiholomorphic involution to the base of the elliptic fibration. As discussed in section 8.2.2, however, other geometric setups are possible. The appearance of a Klein bottle as a possible quotient of the torus is a particularly intriguing possibility. It might be possible to exploit, for instance, some of the results of [294] to look for a physical interpretation of this case. Let us point out that maybe compactification on an interval has to be replaced with another prescription such that an $\mathcal{N} = 1$ theory can be obtained in three dimensions. It would be interesting to explore different possibilities, for instance circle Scherk-Schwarz reductions.

Some further insights about F-theory on Spin(7) manifolds can come from the study of the corresponding weakly coupled Type IIB setups, when they exist. Indeed, in chapter 8 we have been able to identify the objects wrapping the singular loci of the antiholomorphic involution. As we have seen, in some cases they have a simple interpretation in Type IIA language as O6-planes. It is known that a Type IIA O6-plane lifts to a smooth geometry in M-theory described by the Atiyah-Hitchin metric. It is thus conceivable that the information obtained from the analysis of weakly coupled setups might provide valuable hints in the search for a Spin(7)-compatible resolution of singularities.

From a more phenomenological perspective the fact that our Spin(7) constructions are based on compactifications that, at a general point in moduli space, preserve only two supercharges means they potentially could be useful for understanding vacua with high-scale supersymmetry breaking in string theory. Although we argued that supersymmetry is restored in the simplest cases we cannot exclude that more complicated constructions can be found where the four-dimensional limit preserves no supersymmetry. Indeed if supersymmetry were completely broken on the boundary of the interval on the M-theory side, for example by fractional branes, it could lead to a scenario where the size of the interval on the F-theory side would interpolate between $\mathcal{N} = 0$ and $\mathcal{N} = 1$ four-dimensional supersymmetry. The non-supersymmetric non-compact limit could be phenomenologically appealing.

It is also intriguing that some of the weakly coupled Spin(7) setups discussed in chapter 8 incorporate objects such as X5-planes. An interesting aspects of the latter is that they support non-BPS but stable states [238, 239, 240]. Recall that the stability of the state is guaranteed as it is the lightest state charged under the $U(1)$ arising from the twisted sector of the X5-plane. It is a particle in Type IIB, similar to a D0-brane in Type IIA, which is confined to lie on the X5-plane. Such a state can be thought of as the S-dual to an open string stretching between the D5-brane and its orientifold image across the O5-plane. The ground state of this string is projected out once the D5-brane sits on top of the O5-plane, and so the lightest state is an excited oscillator. It is interesting that such a stable non-supersymmetric state arises naturally in such setups. In our setups these non-BPS states are localized at the boundaries of the interval, and therefore their phenomenological impact is diluted by the interval length. However, it is conceivable that in alternative constructions one finds these non-BPS states in the bulk such that this dilution does not occur.

In summary, we believe that F-theory, alongside with its numerous and powerful applications to particle physics model building within the framework of the MSSM, is potentially interesting to address different questions ranging from the exploration of the general structure of vacua in string theory to the realization of unconventional setups that might be useful in extending the MSSM paradigm in string phenomenology.

12.3 Uncovering tensor theories

The duality between M-theory and F-theory incorporates naturally a transdimensional treatment of the dynamics. This has been inspirational for our proposal of studying six-dimensional theories with tensors by means of five-dimensional constructions. The main idea underlying our strategy is the consideration of the possibility that, for non-Abelian tensor theories in six-dimensions, a Lagrangian formulation of the dynamics could be incompatible with a manifest realization of all spacetime symmetries.

Ideally, this approach would yield a five dimensional action that in a suitable limit is able to encode correctly the dynamics of tensors and to realize in a non-manifest way all the expected symmetries. A possible way to restore the symmetries that are not manifestly realized in the five dimensional action has been suggested in [277]. Upon circle compactification, a subset of the generators of the six-dimensional superconformal group is spontaneously broken, leaving only a subalgebra of manifestly realized symmetries in the five dimensional action. In order to restore all symmetries one might 'average' over the five-dimensional setups obtained by breaking the superconformal algebra in all possible inequivalent ways. Clearly, it is not straightforward to make precise this notion of 'average' over five-dimensional theories. In [277], however, a first concrete step has been made. By exploiting techniques inspired from harmonic superspace, the R-symmetry group $SU(2)_R$ of the five-dimensional action has been enhanced to the expected R-symmetry group $USp(4)_R$. This can be done by integrating over the coordinates of an auxiliary four-sphere that can be intuitively though of as the space of all possible breakings $USp(4)_R \to SU(2)_R$. It was also argued that supersymmetry might be restored resorting to this framework.

The restoration of Lorentz symmetry seems to be much more challenging. In particular, the undemocratic treatment of zeromodes as gauge connections is an essential ingredient in our construction but seems to be in apparent contrast with six-dimensional Lorentz symmetry. Nonetheless the idea of 'averaging' might be applied to this task as well. Recall that, from a purely five-dimensional perspective, all the dynamics about the vector-tensor sector of the theory is encoded in some invariant tensors. Their components are labelled by collective indices that also contain Kaluza-Klein levels. It is thus conceivable that a suitable integration over the space of such invariant tensors might be a way to restore full six-dimensional Lorentz invariance. Ideally, the same strategy could be used to achieve the full expected superconformal group. Unfortunately we do not have a concrete proposal to implement this strategy, but it can represent an interesting direction for future investigation.

The problem of symmetry restoration is also linked to the role of gravity in this kind of constructions. Even though the usual definition of (2,0) theories implies a decoupling of gravitational degrees of freedom, it might be beneficial to study five-dimensional approaches to matter-coupled (2,0) supergravity theories. This might shed light on the ultimate relevance of the compensating multiplets that are needed to restore five-dimensional superconformal invariance. For instance, we have noticed in section 11.3.2 that the Kaluza-Klein vector fits naturally into a compensating vector multiplet together with the radius of the compactification. It can be interesting to explore what is the relation between this compensator and the gravity multiplet. A five-dimensional approach to six-dimensional (2,0) supergravities is likely to require control over massive gravitons and gravitini. The latter can also play a role in supersymmetry restoration. For instance, it was argued in [130] in a different context that a better understanding of massive spin-3/2 supermultiplets is needed to study off-shell supersymmetry enhancement on Calabi-Yau threefolds with vanishing Euler number.

Another point that deserves a better comprehension is the relation between the Kaluza-Klein inspired approach followed in this work and instantons in maximally supersymmetric Yang-Mills theory, or MSYM for short. As we know from [33, 34] among the non-perturbative states of MSYM there exists a tower of states whose masses have the form $m_n = 4\pi^2 n/g^2$ and are identified with Kaluza-Klein modes of the reduction on a circle with radius $R = g^2/(4\pi^2)$. According to the instanton interpretation, these massive degrees of freedom have a very different status compared to the massless degrees of freedom of the Yang-Mills multiplet, since loosely speaking an instanton can be thought of as a composite object made out of quanta of the gauge field and their superpartners. Our proposal treats zeromodes and excited modes undemocratically at the level of interactions but considers all of them as fundamental fields in the Lagrangian formulation. This perspective might have the advantage of making the study of interactions between massless and massive states easier.

It is a natural question whether the simultaneous inclusion of zeromodes and excited modes in a five-dimensional Lagrangian yields to an overcounting of the massive states of a (2,0) theory on a circle. In order to address this problem we need information over the non-perturbative states of non-Abelian gauge fields coupled to an infinite tower of massive tensors, which lies beyond the scope of this work but might be seen as an interesting future direction. As a preliminary step it could be useful to study in closer detail some non-perturbative aspects of supersymmetric gauge theories coupled to a finite number of massive self-dual tensors. For instance, it might be possible to formulate these theories on suitable curved spaces (e.g. a five-sphere) while keeping a fraction of supersymmetry and exploit localization techniques to compute five-dimensional indices. This sort of calculation might shed light on the 'building blocks' associated to the degrees of freedom of massive tensors coupled to a non-Abelian gauge group.

Since their discovery via string theory and M-theory, superconformal (2,0) theories have proven to be a remarkable theoretical challenge. Nothing seems to be trivial about these theories and one has to face several puzzles in the search for an explicit formulation of their dynamics. It is plausible that all these difficulties will have to be overcome at once to get the desired answer. We can therefore presume that, if we were able to achieve this goal, we would obtain deep insights about interacting quantum field theories. It seems that string theory still has many valuable lessons to teach us.

PART V

Appendices

APPENDIX A

Notation, conventions, and useful identities

This appendix collects our notation and conventions concerning (pseudo)Riemannian geometry, differential forms, five-dimensional spinors. We also present a detailed derivation of the dimensional reduction of the Einstein-Hilbert term on a Ricci-flat compact manifold.

A.1 General conventions for metric, curvature, differential forms

In any number of dimensions we always adopt the mostly plus signature for the spacetime metric. The Christoffel symbols ${\Gamma^\rho}_{\mu\nu}$, the Riemann tensor ${R^\rho}_{\sigma\mu\nu}$, the Ricci tensor $R_{\mu\nu}$, and the Ricci scalar R are defined in terms of the metric $g_{\mu\nu}$ by

$$\begin{aligned} {\Gamma^\rho}_{\mu\nu} &= \tfrac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) \ , \\ {R^\rho}_{\sigma\mu\nu} &= \partial_\mu {\Gamma^\rho}_{\sigma\nu} - \partial_\nu {\Gamma^\rho}_{\sigma\mu} + {\Gamma^\rho}_{\lambda\mu} {\Gamma^\lambda}_{\sigma\nu} - {\Gamma^\rho}_{\lambda\nu} {\Gamma^\lambda}_{\sigma\mu} \ , \\ R_{\mu\nu} &= {R^\rho}_{\mu\rho\nu} \ , \qquad R = g^{\mu\nu} R_{\mu\nu} \ , \end{aligned} \tag{A.1}$$

respectively. Unless otherwise stated, the symbol $\epsilon_{\mu_1 \dots \mu_N}$ always refers to the Levi-Civita tensor in N dimensions. In any local coordinate system $x^0, \dots, x^{N-1}$ it satisfies

$$\epsilon_{01\dots(N-1)} = +\sqrt{-g} \ , \tag{A.2}$$

where g denotes the determinant of the matrix $g_{\mu\nu}$ of the metric components in the chosen local coordinates. A useful identity obeyed by the Levi-Civita tensor is

$$\epsilon^{\mu_1 \dots \mu_{n_1} \rho_1 \dots \rho_{n_2}} \, \epsilon_{\mu_1 \dots \mu_{n_1} \sigma_1 \dots \sigma_{n_2}} = -n_1! \, n_2! \, \delta^{[\rho_1}_{[\sigma_1} \dots \delta^{\rho_{n_2}]}_{\sigma_{n_2}]} \ , \tag{A.3}$$

where indices are raised and lowered with the metric and its inverse and n_1, n_2 are any non-negative integers such that $n_1 + n_2 = N$. Let us stress that we always symmetrize and antisymmetrize indices with weight one, for instance $X_{[\mu\nu]} = \frac{1}{2}(X_{\mu\nu} - X_{\nu\mu})$.

A differential p-form λ is expanded onto a basis of coordinate differentials according to

$$\lambda = \tfrac{1}{p!}\,\lambda_{\mu_1\ldots\mu_p}\,dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\ . \tag{A.4}$$

The wedge product of differential forms is given in components by

$$(\lambda\wedge\lambda')_{\mu_1\ldots\mu_{p+p'}} = \tfrac{(p+p')!}{p!\,p'!}\,\lambda_{[\mu_1\ldots\mu_p}\,\lambda'_{\mu_{p+1}\ldots\mu_{p+p'}]}\ , \tag{A.5}$$

where λ is a p-form and λ' is a p'-form. The exterior differential of a p-form λ satisfies

$$(d\lambda)_{\mu_0\mu_1\ldots\mu_p} = (p+1)!\,\partial_{[\mu_0}\lambda_{\mu_1\ldots\mu_p]}\ . \tag{A.6}$$

We adopt the following definition of the Hodge star operator

$$(*\lambda)_{\nu_1\ldots\nu_q} = \frac{1}{p!}\,\lambda^{\mu_1\ldots\mu_p}\,\epsilon_{\mu_1\ldots\mu_p\nu_1\ldots\nu_q}\ , \tag{A.7}$$

where λ is a p-form and $q = N-p$. As a result, given two p-forms λ and ω, we have

$$\lambda\wedge *\,\omega = \tfrac{1}{p!}\,\lambda^{\mu_1\ldots\mu_p}\,\omega_{\mu_1\ldots\mu_p}\ *1\ , \qquad *1 = \sqrt{-g}\,dx^0\wedge\cdots\wedge dx^{N-1}\ . \tag{A.8}$$

The vielbein $e^a{}_\mu$ (where a is a flat tangent index) is related to the metric by the familiar relation

$$g_{\mu\nu} = \eta_{ab}\,e^a{}_\mu\,e^b{}_\nu\ , \tag{A.9}$$

where η_{ab} is the flat metric diag$(-,+,\ldots,+)$. The torsionless spin connection one-form $\omega^a{}_b = \omega^a{}_{b\mu}dx^\mu$ satisfies $\omega_{ab} = \omega_{[ab]}$ and is determined by the vielbein through

$$de^a + \omega^a{}_b\wedge e^b = 0\ . \tag{A.10}$$

The curvature two-form $\mathcal{R}^a{}_b$ satisfies

$$\mathcal{R}^a{}_b = d\omega^a{}_b + \omega^a{}_c\wedge\omega^c{}_b = \tfrac{1}{2}\,e^a{}_\rho\,e_b{}^\sigma\,R^\rho{}_{\sigma\mu\nu}\,dx^\mu\wedge dx^\nu\ , \tag{A.11}$$

where in the last expression $R^\rho{}_{\sigma\mu\nu}$ is the Riemann tensor and $e_b{}^\sigma$ is the inverse vielbein determined by $e_b{}^\sigma\,e^a{}_\sigma = \delta^a_b$.

A.2 Spinors in five dimensions

Five-dimensional gamma matrices γ^a ($a = 0,\ldots,4$) are constant, complex-valued 4×4 matrices satisfying the anticommutation relation

$$\{\gamma^\alpha,\gamma^b\} = 2\eta^{ab}\,\mathbb{I}\ . \tag{A.12}$$

We use the shorthand notation $\gamma^{a_1\ldots a_p} = \gamma^{[a_1}\ldots\gamma^{a_p]}$, and we choose a representation of gamma matrices such that

$$\gamma^{abcde} = i\,\epsilon^{abcde}\,\mathbb{I}\ . \tag{A.13}$$

We further assume the hermiticity property

$$\gamma^0 \gamma^a (\gamma^0)^{-1} = -(\gamma^a)^\dagger \ . \tag{A.14}$$

The charge conjugation matrix C in five dimensions acts on gamma matrices according to

$$C \gamma^a C^{-1} = +(\gamma^a)^\mathsf{T} \ . \tag{A.15}$$

We use a representation such that C is real and satisfies

$$C^\mathsf{T} = -C = C^{-1} \ . \tag{A.16}$$

In our work we encounter three different kinds of symplectic indices. First of all, we have indices $i, j = 1, \dots, 4$ of the **4** representation of $USp(4)_R$. Secondly we find two different copies of the **2** representation of $SU(2)_R$, labeled by indices $a, b = 1, 2$ and $\dot{a}, \dot{b} = 1, 2$. Each symplectic group is endowed with a primitive antisymmetric invariant: Ω_{ij} for $USp(4)_R$ and $\epsilon_{ab}, \epsilon_{\dot{a}\dot{b}}$ for the two copies of $SU(2)$.

For all symplectic groups we adopt the same conventions regarding the inverse of the antisymmetric invariant, the raising and lowering of indices, and the reality properties. For definiteness, we write down the conventions for $USp(4)_R$. The inverse Ω^{ij} of Ω_{ij} is defined by the relation

$$\Omega_{ik} \Omega^{jk} = \delta_i^j \ . \tag{A.17}$$

Given any object T^i with (at least) one symplectic index, raising and lowering of i are performed according to the NW-SE convention:

$$T^i = \Omega^{ij} T_j \ , \qquad T_i = T^j \Omega_{ji} \ . \tag{A.18}$$

Complex conjugation interchanges upper and lower symplectic indices. The antisymmetric invariant satisfies the reality property

$$(\Omega_{ij})^* = \Omega^{ij} \ . \tag{A.19}$$

An explicit realization of the invariants $\Omega_{ij}, \epsilon_{ab}$ with all required properties is furnished by

$$\Omega_{ij} = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix} = \Omega^{ij} \ , \qquad \epsilon_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon^{ab} \ . \tag{A.20}$$

The second expression can also be applied to $\epsilon_{\dot{a}\dot{b}}, \epsilon^{\dot{a}\dot{b}}$.

Let us now discuss in more detail symplectic spinors, i.e. spinors carrying one of the three kinds of symplectic indices listed above. For definiteness, we write down equations with i, j indices, but the same conventions apply to a, b and $\dot{a}, \dot{b}$ indices. The Dirac bar of a symplectic spinor λ^i is defined according to

$$\bar{\lambda}^i = (\lambda_i)^\dagger \gamma^0 \ . \tag{A.21}$$

If symplectic indices are omitted in a spinor bilinear, a NW-SE contraction is understood,

$$\bar{\lambda}\chi = \bar{\lambda}^i \chi_i \ . \tag{A.22}$$

The Fierz rearrangement formula for anticommuting spinors in five dimensions reads

$$(\bar{\psi}_1\psi_2)(\bar{\psi}_3\psi_4) = -\tfrac{1}{4}(\bar{\psi}_1\psi_4)(\bar{\psi}_3\psi_2) - \tfrac{1}{4}(\bar{\psi}_1\gamma^a\psi_4)(\bar{\psi}_3\gamma_a\psi_2) + \tfrac{1}{8}(\bar{\psi}_1\gamma^{ab}\psi_4)(\bar{\psi}_3\gamma_{ab}\psi_2) \ , \tag{A.23}$$

where spinors $\psi_1, \psi_2, \psi_3, \psi_4$ can carry arbitrary indices and/or labels (e.g. Kaluza-Klein levels). In our conventions, complex conjugation acting on the product of anticommuting variables does not change their order. Therefore, the reality of bilinears is determined by the basic relation [1]

$$(\bar{\lambda}^i \chi_j)^* = \bar{\chi}^j \lambda_i \ . \tag{A.24}$$

The Majorana condition for a symplectic spinor λ^i reads

$$\bar{\lambda}^i = \Omega^{ij}(\lambda_j)^{\mathsf{T}} C \ . \tag{A.25}$$

As a result, if λ^i, χ^j are Majorana, we have the flip property

$$\bar{\lambda}^i \gamma^{\mu_1 \ldots \mu_p} \chi^j = \bar{\chi}^j \gamma^{\mu_p \ldots \mu_1} \lambda^i \ . \tag{A.26}$$

Note that an extra minus sign is needed if the $USp(4)_R$ indices i, j are contracted on both sides according to the NW-SE convention. This implies that $\bar{\lambda}^i \chi_i$ is purely imaginary for real λ^i, χ^i. Any symplectic spinor λ^i can be decomposed in a $SO(2)$ doublet of Majorana symplectic spinors $\lambda^{i\,\alpha}$, $\alpha = 1, 2$:

$$\lambda^i = \tfrac{1}{\sqrt{2}}\big(\lambda^{i\,\alpha=1} + i\lambda^{i\,\alpha=2}\big) \ , \qquad \bar{\lambda}^{i\,\alpha} = \Omega^{ij}(\lambda^\alpha_j)^{\mathsf{T}} C \ . \tag{A.27}$$

Multiplication of λ^i by a $U(1)$ phase is equivalent to an $SO(2)$ rotation of the doublet $\lambda^{i\,\alpha}$. With this understanding, equations (11.21), (11.22) hold also if X is a symplectic spinor.

Let us conclude this section with some identities that are useful in chapter 11. With the definitions (11.20) and (11.23) one infers the following identities to match the $SO(2)$ and the complex notations. They are written with $SU(2)_R$ indices for definiteness, but they hold for arbitrary symplectic indices. One has

$$\begin{aligned}
\delta_{\alpha\beta} x^\alpha y^\beta &= 2\mathrm{Re}(\bar{x}y) \ , & \epsilon_{\alpha\beta} x^\alpha y^\beta &= 2\mathrm{Im}(\bar{x}y) \ , \\
\delta_{\alpha\beta} \bar{\chi}^{a\,\alpha} \lambda^\beta_a &= 2i\mathrm{Im}(\bar{\chi}^a \lambda_a) \ , & \epsilon_{\alpha\beta} \bar{\chi}^{a\,\alpha} \lambda^\beta_a &= -2i\mathrm{Re}(\bar{\chi}^a \lambda_a) \ , \\
\delta_{\alpha\beta} \bar{\psi}^a x^\alpha \lambda^\beta_a &= 2i\mathrm{Im}(\bar{\psi}^a \bar{x} \lambda_a) \ , & \epsilon_{\alpha\beta} \bar{\psi}^a x^\alpha \lambda^\beta_a &= -2i\mathrm{Re}(\bar{\psi}^a \bar{x} \lambda_a) \ ,
\end{aligned} \tag{A.28}$$

where x, y are complex bosonic fields, χ, λ are complex spinors, ψ is a Majorana spinor. The same identities hold when $SU(2)_R$ indices are contracted with a tensor that satisfies a pseudoreality condition (e.g. $Y^{I\,ab}$).

[1] Care has to be taken in raising/lowering indices with Ω in equations involving complex conjugation. For example, moving the index j in (A.24) gives $(\bar{\lambda}^i \chi^j)^* = -\bar{\chi}_j \lambda_i$.

A.3 More details about the reduction of the Einstein-Hilbert term

This section is devoted to a detailed account on the dimensional reduction of the D-dimensional Einstein-Hilbert action to d dimensions on a k-dimensional compact Ricci-flat manifold. For ease of reference we record the Einstein-Hilbert action

$$S = \frac{1}{2}\int_{\mathcal{M}_D} \hat{R}\,\hat{*}1 \ , \tag{A.29}$$

together with the metric Ansatz

$$d\hat{s}^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + g_{mn}(y;X(x))dy^m dy^n \ , \tag{A.30}$$

where x^μ $(\mu = 0,\ldots,d-1)$ are coordinates in external spacetime, y^m $(m = 1,\ldots,k)$ are coordinates on the internal space, and $X^{\mathcal{M}}$ are local coordinates on the moduli space of the internal metric. The notation $g_{mn}(y;X(x))$ signals that the internal metric depends parametrically on the coordinates $X^{\mathcal{M}}$, which are in turn allowed to have a non-trivial dependence on external spacetime. The index $\mathcal{M}$ runs over all massless metric moduli of the compactification.

It is convenient to introduce the shorthand notation

$$\mathcal{H}_{\mathcal{M}\mathcal{N}} = g^{mp}g^{nq}\frac{\partial g_{mn}}{\partial X^{\mathcal{M}}}\frac{\partial g_{pq}}{\partial X^{\mathcal{N}}} \ , \qquad \mathcal{H}_{\mathcal{M}} = g^{mn}\frac{\partial g_{mn}}{\partial X^{\mathcal{M}}} \ . \tag{A.31}$$

Making only use of the chain rule,

$$\partial_\mu g_{mn} = \frac{\partial g_{mn}}{\partial X^{\mathcal{M}}}\,\partial_\mu X^{\mathcal{M}} \ , \tag{A.32}$$

and of the expression of the Ricci scalar in terms of derivative of the metric one can prove the identity

$$\hat{R} = R + \frac{3}{4}\mathcal{H}_{\mathcal{M}\mathcal{N}}\,\partial_\mu X^{\mathcal{M}}\partial^\mu X^{\mathcal{N}} - \frac{1}{4}\mathcal{H}_{\mathcal{M}}\,\mathcal{H}_{\mathcal{N}}\,\partial_\mu X^{\mathcal{M}}\partial^\mu X^{\mathcal{N}} - g^{mn}g^{\mu\nu}\nabla_\mu\left(\frac{\partial g_{mn}}{\partial X^{\mathcal{M}}}\partial_\nu X^{\mathcal{M}}\right) \ . \tag{A.33}$$

In this expression and in the following R denotes the Ricci scalar of the external metric. The internal Ricci scalar is dropped since it vanishes by assumption. Let us remark that all terms involving $\partial_\mu g_{\nu\rho}$ and $\partial_m g_{np}$ in the intermediate steps of the computation are rearranged into the external Ricci scalar, the (vanishing) internal Ricci scalar, and the external covariant derivative ∇_μ.

The last term in (A.33) can be manipulated to give

$$-\,g^{mn}g^{\mu\nu}\nabla_\mu\left(\frac{\partial g_{mn}}{\partial X^{\mathcal{M}}}\partial_\nu X^{\mathcal{M}}\right) = -g^{mn}\frac{\partial^2 g_{mn}}{\partial X^{\mathcal{M}}\partial X^{\mathcal{N}}}\,\partial_\mu X^{\mathcal{M}}\partial^\mu X^{\mathcal{N}} - \mathcal{H}_{\mathcal{M}}\,\nabla^\mu\partial_\mu X^{\mathcal{M}} \ . \tag{A.34}$$

In order to treat the last term we need an integration by parts with respect to integration over external spacetime. We must take into account the fact that the D-dimensional Ricci scalar is multiplied by volume form of the total space, and that the internal volume form depends on external coordinates,

$$\int_{\mathcal{M}_d} d^dx\sqrt{-g}\int_{\mathcal{M}_k} d^ky\sqrt{g}\,\mathcal{H}_{\mathcal{M}}\,\nabla^\mu\partial_\mu X^{\mathcal{M}} = -\int_{\mathcal{M}_d} d^dx\sqrt{-g}\int_{\mathcal{M}_k} d^ky\,\nabla^\mu(\sqrt{g}\,\mathcal{H}_{\mathcal{M}})\partial_\mu X^{\mathcal{M}} \ . \tag{A.35}$$

One can then compute

$$\begin{aligned}\nabla^\mu(\sqrt{g}\,\mathcal{H}_\mathcal{M}) &= \frac{\partial}{\partial X^\mathcal{N}}(\sqrt{g}\,\mathcal{H}_\mathcal{M})\nabla^\mu X^\mathcal{N} \\ &= \sqrt{g}\left(\frac{1}{2}\mathcal{H}_\mathcal{M}\mathcal{H}_\mathcal{N} - \mathcal{H}_{\mathcal{M}\mathcal{N}} + g^{mn}\frac{\partial^2 g_{mn}}{\partial X^\mathcal{M}\partial X^\mathcal{N}}\right)\nabla^\mu X^\mathcal{N} \ , \end{aligned} \tag{A.36}$$

and therefore infer that integration by parts justifies the replacement

$$-\mathcal{H}_\mathcal{M}\nabla^\mu\partial_\mu X^\mathcal{M} \to \left(\frac{1}{2}\mathcal{H}_\mathcal{M}\mathcal{H}_\mathcal{N} - \mathcal{H}_{\mathcal{M}\mathcal{N}} + g^{mn}\frac{\partial^2 g_{mn}}{\partial X^\mathcal{M}\partial X^\mathcal{N}}\right)\partial_\mu X^\mathcal{M}\partial^\mu X^\mathcal{N} \ . \tag{A.37}$$

Remarkably, the term containing the second derivative of the internal metric with respect to the moduli drops out, and we are left with the simple result

$$\hat{R} = R - \frac{1}{4}(\mathcal{H}_{\mathcal{M}\mathcal{N}} - \mathcal{H}_\mathcal{M}\mathcal{H}_\mathcal{N})\,\partial_\mu X^\mathcal{M}\partial^\mu X^\mathcal{N} \ . \tag{A.38}$$

Integrating this quantity against the D-dimensional volume form gives the action

$$S = \frac{1}{2}\int_{\mathcal{M}_d} d^d x\sqrt{-g}\,\mathcal{L} \tag{A.39}$$

with a Lagrangian

$$\mathcal{L} = \mathcal{V}\,R - \frac{1}{4}\partial_\mu X^\mathcal{M}\partial^\mu X^\mathcal{N}\int_{\mathcal{M}_k} d^k y\sqrt{g}(\mathcal{H}_{\mathcal{M}\mathcal{N}} - \mathcal{H}_\mathcal{M}\mathcal{H}_\mathcal{N}) \ , \tag{A.40}$$

where $\mathcal{V}$ is the volume of the internal space, $\mathcal{V} = \int_{\mathcal{M}_k} d^k y\sqrt{g}$.

The next step is a Weyl rescaling of the external metric that casts the Einstein-Hilbert term for d-dimensional gravity in canonical form. For the sake of completeness, we record the transformation rules for the Riemann tensor, the Ricci tensor, and the Ricci scalar under the Weyl rescaling

$$g^{\text{old}}_{\mu\nu} = e^{2\omega}g^{\text{new}}_{\mu\nu} \ . \tag{A.41}$$

One finds, with obvious notation,

$$\begin{aligned}\left[R^\rho{}_{\sigma\mu\nu}\right]^{\text{old}} &= \Big\{R^\rho{}_{\sigma\mu\nu} - 2g_{\sigma[\mu}\left(\nabla^\rho\omega\nabla_{\nu]}\omega - \nabla^\rho\nabla_{\nu]}\omega\right) \\ &\quad + 2\delta^\rho_{[\mu}\left(\nabla_{\nu]}\omega\nabla_\sigma\omega - \nabla_{\nu]}\nabla_\sigma\omega\right) - 2\delta^\rho_{[\mu}g_{\nu]\sigma}\nabla^\lambda\omega\nabla_\lambda\omega\Big\}^{\text{new}} \ , \\ R^{\text{old}}_{\mu\nu} &= \Big\{-g_{\mu\nu}\Big[(d-2)\nabla^\lambda\omega\nabla_\lambda\omega + \nabla^\lambda\nabla_\lambda\omega\Big] + (d-2)\left(\nabla_\mu\omega\nabla_\nu\omega - \nabla_\mu\nabla_\nu\omega\right)\Big\}^{\text{new}} \ , \\ R^{\text{old}} &= e^{-2\omega}\Big\{R - (d-1)(d-2)\nabla^\lambda\omega\nabla_\lambda\omega - 2(d-1)\nabla^\lambda\nabla_\lambda\omega\Big\}^{\text{new}} \ . \end{aligned} \tag{A.42}$$

The Lagrangian (A.40) in terms of the new metric reads

$$\begin{aligned}\mathcal{L} &= e^{\omega(d-2)}\mathcal{V}\Big[R - (d-1)(d-2)\partial_\mu\omega\partial^\mu\omega - 2(d-1)\nabla^\mu\partial_\mu\omega\Big] \\ &\quad - \frac{1}{4}e^{\omega(d-2)}\partial_\mu X^\mathcal{M}\partial^\mu X^\mathcal{N}\int_{\mathcal{M}_k} d^k y\sqrt{g}(\mathcal{H}_{\mathcal{M}\mathcal{N}} - \mathcal{H}_\mathcal{M}\mathcal{H}_\mathcal{N}) \ . \end{aligned} \tag{A.43}$$

As we can see, in order to achieve the canonical Einstein-Hilbert term we have to set

$$\omega = -\frac{1}{d-2} \log \mathcal{V} \ . \tag{A.44}$$

With this assignment the term $\nabla^\mu \partial_\mu \omega$ in (A.43) can be neglected because it is a total derivative. In conclusion, we find

$$\mathcal{L} = R - \frac{d-1}{d-2} \partial_\mu \log \mathcal{V} \, \partial^\mu \log \mathcal{V} - \frac{1}{4\mathcal{V}} \partial_\mu X^{\mathcal{M}} \partial^\mu X^{\mathcal{N}} \int d^k y \sqrt{g} (\mathcal{H}_{\mathcal{M}\mathcal{N}} - \mathcal{H}_{\mathcal{M}} \mathcal{H}_{\mathcal{N}}) \ , \tag{A.45}$$

which corresponds to the result (4.35) given in the main text.

APPENDIX B

Material for the one-loop computation of chapter 9

This appendix collects results that have been useful in the computation of the one-loop corrections to five-dimensional Chern-Simons couplings in chapter 9. As a result we always work in five dimensions in this appendix.

B.1 Gravitational perturbative expansion

In this section we record some useful identities about the gravitational perturbative expansion around flat spacetime. More precisely, we assume a metric of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \tag{B.1}$$

and compute some geometrical quantities derived from the metric in a formal power series in $h_{\mu\nu}$. On the right hand side of the following identities, indices are raised and lowered with the flat metric $\eta_{\mu\nu}$ and its inverse. For instance, $h^{\mu\nu} = h_{\lambda\tau}\eta^{\lambda\mu}\eta^{\tau\nu}$.

The total inverse metric and volume form are given by

$$\begin{aligned} g^{\mu\nu} &= \eta^{\mu\nu} - h^{\mu\nu} + h^{\mu\lambda}h_\lambda{}^\nu + \mathcal{O}(h^3) \ , \\ \sqrt{-g} &= 1 + \tfrac{1}{2}h^\mu{}_\mu + \tfrac{1}{8}(h^\mu{}_\mu h^\nu{}_\nu - 2h^{\mu\nu}h_{\mu\nu}) + \mathcal{O}(h^3) \ . \end{aligned} \tag{B.2}$$

The Christoffel symbols and the Riemann tensor are expanded as

$$\begin{aligned} \Gamma^\rho{}_{\mu\nu} &= \tfrac{1}{2}(\eta^{\rho\sigma} - h^{\rho\sigma})(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) + \mathcal{O}(h^3) \ , \\ g_{\rho\tau}R^\tau{}_{\sigma\mu\nu} &= \Big[-\tfrac{1}{2}\partial_\rho\partial_\mu h_{\sigma\nu} - \tfrac{1}{8}\partial_\lambda h_{\mu\rho}\partial^\lambda h_{\nu\sigma} + \tfrac{1}{8}\partial_\mu h_{\sigma\lambda}\partial_\nu h_\rho{}^\lambda - \tfrac{1}{4}\partial_\rho h_{\mu\lambda}\partial_\nu h_\sigma{}^\lambda + \tfrac{1}{8}\partial_\rho h_{\nu\lambda}\partial_\sigma h_\mu{}^\lambda \\ &\quad + \tfrac{1}{4}\partial_\mu h_{\rho\lambda}\partial^\lambda h_{\nu\sigma} - \tfrac{1}{4}\partial_\rho h_{\nu\lambda}\partial^\lambda h_{\mu\sigma} - (\mu \leftrightarrow \nu)\Big] - (\rho \leftrightarrow \sigma) + \mathcal{O}(h^3) \ . \end{aligned} \tag{B.3}$$

In order to couple spinors to gravity we need to introduce a vielbein $e^a{}_\mu$. It is determined by the metric only up to local Lorentz transformations. We fix this ambiguity by imposing the gauge condition

$$\eta_{ab}\delta^b{}_{[\lambda}e^a{}_{\mu]} = 0 \ , \tag{B.4}$$

where the Kronecker delta plays the role of the vielbein for the flat metric $\eta_{\mu\nu}$. We thus find the following expansions of the vielbein and its inverse,

$$\begin{aligned} e^a{}_\mu &= \eta^{ab}\delta_b{}^\lambda\big[\eta_{\lambda\mu} + \tfrac{1}{2}h_{\lambda\mu} - \tfrac{1}{8}h_{\lambda\tau}h_\mu{}^\tau + \mathcal{O}(h^3)\big] \ , \\ e_a{}^\mu &= \eta_{ab}\delta^b{}_\lambda\big[\eta^{\lambda\mu} - \tfrac{1}{2}h^{\lambda\mu} + \tfrac{3}{8}h^{\lambda\tau}h^\mu{}_\tau + \mathcal{O}(h^3)\big] \ . \end{aligned} \tag{B.5}$$

Finally, let us record the expansion of the flat components of the spin connection, which enter the fermion covariant derivatives:

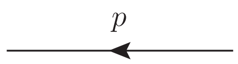
$$\begin{aligned} e_c{}^\tau\omega_{\tau ab} = \delta_c{}^\rho\delta_a{}^\mu\delta_b{}^\nu\big[&- \tfrac{1}{2}\partial_\mu h_{\nu\rho} - \tfrac{1}{2}h_{\nu\tau}\partial^\tau h_{\mu\rho} + \tfrac{1}{2}h_{\rho\tau}\partial_\mu h_\nu{}^\tau \\ &+ \tfrac{1}{2}h_{\nu\tau}\partial_\mu h_\rho{}^\tau + \tfrac{1}{4}h_{\nu\tau}\partial_\rho h_\mu{}^\tau - (\mu \leftrightarrow \nu) + \mathcal{O}(h^3)\big] \ . \end{aligned} \tag{B.6}$$

B.2 Feynman rules

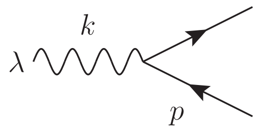
In this section we collect the Feynman rules that can be extracted from the massive actions (9.4), (9.8), (9.6). The propagators are read from the free actions where $A_\mu = 0$ and $g_{\mu\nu} = \eta_{\mu\nu}$. The interaction vertices are obtained by expanding (9.4), (9.8), (9.6) up to second order in the metric perturbation $h_{\mu\nu}$, introduced in (9.25).

In all Feynman rules symmetrization with weight one on graviton polarization indices is understood. Moreover, the momenta of vectors and gravitons are always taken to be entering the vertex, while the momenta of massive fields flow in the same direction as specified by the charge arrow.

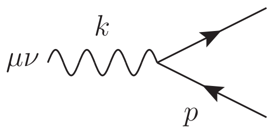
B.2.1 Spin-1/2 fermion



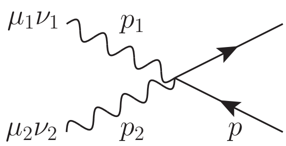
$$= \frac{-\not{p} + i c_{1/2} m}{p^2 + m^2}$$



$$= -q\gamma_\lambda$$



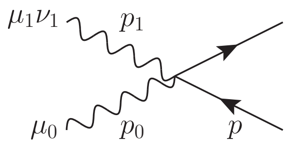
$$= \tfrac{1}{2}\left(i c_{1/2} m + \tfrac{1}{2}\not{P}\right)\eta_{\mu\nu} - \tfrac{1}{4}\gamma_\mu P_\nu \ , \quad P \equiv 2p + k$$



$$= \tfrac{1}{4}\left(i c_{1/2} m + \tfrac{1}{2}\not{P}\right)\left(\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2} - 2\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}\right)$$

$$+ \left[\tfrac{1}{16}\gamma_{\mu_1\mu_2\lambda}p_1^\lambda - \tfrac{1}{8}\gamma_{\mu_1}P_{\nu_1}\eta_{\mu_2\nu_2} + \tfrac{3}{16}\gamma_{\mu_1}P_{\mu_2}\eta_{\nu_1\nu_2} + (1 \leftrightarrow 2)\right] \ ,$$

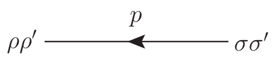
$$P \equiv 2p + p_1 + p_2$$



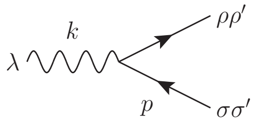
$$= -\tfrac{1}{2}q\eta_{\mu_1\nu_1}\gamma_{\mu_0} + \tfrac{1}{2}q\eta_{\mu_0\mu_1}\gamma_{\nu_1}$$

B.2.2 Massive self-dual tensor

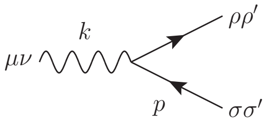
In the following expression antisymmetrization with weight one on tensor polarization indices is understood.



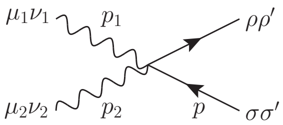
$$= \frac{1}{p^2+m^2}\left\{-ic_B\epsilon_{\rho\rho'\sigma\sigma'\lambda}p^\lambda - 2im\eta_{\rho\sigma}\eta_{\rho'\sigma'} - 4im^{-1}\eta_{\rho\sigma}p_{\rho'}p_{\sigma'}\right\}$$



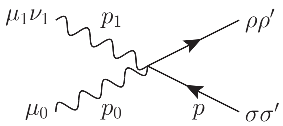
$$= -\tfrac{1}{4}ic_B q\epsilon_{\rho\rho'\sigma\sigma'\lambda}$$



$$= -\tfrac{1}{4}im\eta_{\mu\nu}\eta_{\rho\sigma}\eta_{\rho'\sigma'} + im\eta_{\mu\rho}\eta_{\nu\sigma}\eta_{\rho'\sigma'}$$

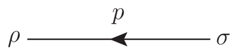


$$= -\tfrac{1}{8}im(\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2} - 2\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2})\eta_{\rho\sigma}\eta_{\rho'\sigma'} - im\eta_{\mu_1\rho}\eta_{\nu_1\sigma}\eta_{\mu_2\rho'}\eta_{\nu_2\sigma'}$$
$$+ \left[\tfrac{1}{2}im\eta_{\mu_1\nu_1}\eta_{\mu_2\rho}\eta_{\nu_2\sigma}\eta_{\rho'\sigma'} - im\eta_{\mu_1\mu_2}\eta_{\nu_1\rho}\eta_{\nu_2\sigma}\eta_{\rho'\sigma'} + (1\leftrightarrow 2)\right]$$

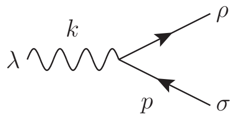


$$= 0$$

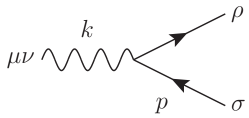
B.2.3 Spin-3/2 fermion



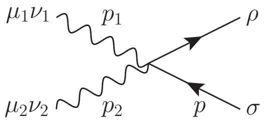
$$= \frac{1}{p^2+m^2}\left\{ \left(\eta_{\rho\sigma} + \frac{p_\rho p_\sigma}{m^2}\right)(-\not{p} + ic_{3/2}m) + \frac{1}{4}\left(\gamma_\rho + \frac{ip_\rho}{c_{3/2}m}\right)(-\not{p} - ic_{3/2}m)\left(\gamma_\sigma + \frac{ip_\sigma}{c_{3/2}m}\right)\right\}$$



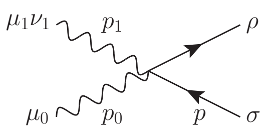
$$= -q\gamma_{\rho\lambda\sigma}$$



$$\begin{aligned}
&= \tfrac{1}{2}\left(\tfrac{1}{2}\gamma_{\rho\lambda\sigma}P^\lambda - ic_{3/2}m\gamma_{\rho\sigma}\right)\eta_{\mu\nu} + \tfrac{1}{4}\gamma_{\rho\sigma\mu}P_\nu \\
&- \tfrac{1}{4}\eta_{\mu\nu}\gamma_\rho k_\sigma + \tfrac{1}{4}\eta_{\mu\nu}\gamma_\sigma k_\rho + \tfrac{1}{4}\eta_{\mu\rho}\gamma_\nu k_\sigma - \tfrac{1}{4}\eta_{\mu\sigma}\gamma_\nu k_\rho - \tfrac{1}{4}\eta_{\mu\rho}\gamma_\sigma k_\nu + \tfrac{1}{4}\eta_{\mu\sigma}\gamma_\rho k_\nu \\
&+ \tfrac{1}{2}\left(\tfrac{1}{2}\gamma_{\sigma\lambda\mu}P^\lambda - ic_{3/2}m\gamma_{\sigma\mu}\right)\eta_{\nu\rho} - \tfrac{1}{2}\left(\tfrac{1}{2}\gamma_{\rho\lambda\mu}P^\lambda - ic_{3/2}m\gamma_{\rho\mu}\right)\eta_{\nu\sigma}\ , \\
&P \equiv 2p + k
\end{aligned}$$



$$\begin{aligned}
&= \tfrac{1}{4}\left(\tfrac{1}{2}\gamma_{\rho\lambda\sigma}P^\lambda - ic_{3/2}m\gamma_{\rho\sigma}\right)(\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2} - 2\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}) \\
&+ \big[-\tfrac{1}{8}\gamma_\rho(p_1+p_2)_\sigma\,(\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2} - 2\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}) \\
&+ \tfrac{3}{8}ic_{3/2}m\gamma_{\sigma\mu_1}\,\eta_{\mu_2\rho}\,\eta_{\nu_1\nu_2} + \tfrac{1}{4}ic_{3/2}m\gamma_{\mu_1\mu_2}\,\eta_{\nu_1\sigma}\,\eta_{\nu_2\rho} \\
&- \tfrac{1}{8}\gamma_{\sigma\mu_1\mu_2}\,P_{\nu_1}\,\eta_{\nu_2\rho} + \tfrac{1}{8}\gamma_{\sigma\mu_1\mu_2}\,P_{\nu_2}\,\eta_{\nu_1\rho} + \tfrac{1}{8}\gamma_{\mu_1\mu_2\lambda}\,P^\lambda\,\eta_{\nu_1\rho}\,\eta_{\nu_2\sigma} - (\rho\leftrightarrow\sigma)\big] \\
&+ \big[-\tfrac{1}{16}\gamma_{\rho\sigma\mu_1\mu_2\lambda}\,p_1^\lambda\,\eta_{\nu_1\nu_2} + \tfrac{1}{8}\gamma_{\rho\sigma\mu_1}\,P_{\nu_1}\,\eta_{\mu_2\nu_2} - \tfrac{3}{16}\gamma_{\rho\sigma\mu_1}\,P_{\mu_2}\,\eta_{\nu_1\nu_2} + (1\leftrightarrow 2)\big] \\
&+ \big[-\tfrac{1}{8}\gamma_\sigma\,p_{1\,\mu_1}\,\eta_{\nu_1\rho}\,\eta_{\mu_2\nu_2} - \tfrac{1}{16}\not{p}_1\,\eta_{\mu_1\mu_2}\,\eta_{\nu_1\sigma}\eta_{\nu_2\rho} + \tfrac{3}{16}\gamma_\sigma\,p_{1\,\mu_2}\,\eta_{\mu_1\rho}\,\eta_{\nu_1\nu_2} \\
&+ \tfrac{5}{16}\gamma_\sigma p_{1\,\mu_1}\,\eta_{\mu_2\rho}\,\eta_{\nu_1\nu_2} - \tfrac{1}{8}\gamma_{\mu_2}p_{1\,\mu_1}\,\eta_{\nu_1\sigma}\,\eta_{\nu_2\rho} - \tfrac{1}{4}\gamma_\sigma p_{1\,\mu_2}\,\eta_{\nu_2\rho}\,\eta_{\mu_1\nu_1} \\
&+ \tfrac{1}{4}\gamma_{\mu_1}p_{1\,\mu_2}\,\eta_{\nu_2\rho}\,\eta_{\nu_1\sigma} + \tfrac{1}{8}\gamma_{\mu_2}p_{1\,\rho}\,\eta_{\mu_1\sigma}\,\eta_{\nu_1\nu_2} - \tfrac{1}{8}\gamma_{\mu_1}p_{1\,\rho}\,\eta_{\nu_1\sigma}\,\eta_{\mu_2\nu_2} \\
&- \tfrac{1}{8}\gamma_{\mu_2}p_{1\,\rho}\,\eta_{\mu_2\sigma}\,\eta_{\mu_1\nu_1} + \tfrac{1}{4}\gamma_{\mu_1}p_{1\,\rho}\,\eta_{\mu_2\sigma}\,\eta_{\nu_1\nu_2} - \tfrac{1}{4}ic_{3/2}m\gamma_{\sigma\mu_1}\,\eta_{\nu_1\rho}\,\eta_{\mu_2\nu_2} \\
&+ \tfrac{3}{16}\gamma_{\sigma\mu_2\lambda}\,P^\lambda\,\eta_{\mu_1\rho}\,\eta_{\nu_1\nu_2} - \tfrac{1}{8}\gamma_{\sigma\mu_1\lambda}\,P^\lambda\,\eta_{\nu_1\rho}\,\eta_{\mu_2\nu_2} - (\rho\leftrightarrow\sigma) + (1\leftrightarrow 2)\big]\ , \\
&P \equiv 2p + p_1 + p_2
\end{aligned}$$



$$= -\tfrac{1}{2}q\eta_{\mu_0\mu_1}\gamma_{\nu_1\rho\sigma} + \tfrac{1}{2}q\eta_{\mu_1\nu_1}\gamma_{\mu_0\rho\sigma} - \tfrac{1}{2}q\eta_{\mu_1\rho}\gamma_{\mu_0\nu_1\sigma} + \tfrac{1}{2}q\eta_{\mu_1\sigma}\gamma_{\mu_0\nu_1\rho}$$

APPENDIX C

Examples of Spin(7) manifolds for F-theory

This appendix collects two explicit examples of elliptically fibered fourfolds endowed with an antiholomorphic involution which is well suited for the Spin(7) construction outlined in section 8.2.1.

C.1 A hypersurface in a $\mathbb{P}_{2,3,1}$ fibration of $\mathbb{P}_{1,1,1,1}$

Let us consider a simple example of the construction described in Section 8.2 in which the Calabi-Yau fourfold Y_4 is described by a polynomial in a toric ambient space constructed by fibering the weighted projective space $\mathbb{P}_{2,3,1}$ over $\mathbb{P}_{1,1,1,1}$. In the language of toric geometry this is described by a reflexive polyhedron with the set of rays given in Table C.1.

vertices	coords.	Q_1	Q_2
$\nu_1 = (\ 1,\ 0,\ 0,\ 0,\ 0)$	x	8	2
$\nu_2 = (\ 0,\ 1,\ 0,\ 0,\ 0)$	y	12	3
$\nu_3 = (-2,\ -3,\ 0,\ 0,\ 0)$	z	0	1
$\nu_4 = (-2,\ -3,\ -1,\ -1,\ -1)$	u_1	1	0
$\nu_5 = (-2,\ -3,\ 1,\ 0,\ 0)$	u_2	1	0
$\nu_6 = (-2,\ -3,\ 0,\ 1,\ 0)$	u_3	1	0
$\nu_7 = (-2,\ -3,\ 0,\ 0,\ 1)$	u_4	1	0

Table C.1: Toric data for a reflexive polyhedron describing a $\mathbb{P}_{2,3,1}$ fibration of $\mathbb{P}_{1,1,1,1}$.

This gives a smooth ambient space in which the Calabi-Yau fourfold will be defined by a homogeneous degree $(24, 6)$ polynomial in the (Q_1, Q_2) identifications. This polynomial may be brought into Weierstrass form where now the coefficients f and g are degree 16 and 24, homogeneous polynomials of the base coordinates $u_1, \ldots, u_4$, respectively. A sufficiently general set of coefficients for these polynomials will then give a smooth Calabi-Yau fourfold. Next we impose a symmetry of this space under the action of the antiholomorphic involution σ where

$$\sigma(u_1, u_2, u_3, u_4, x, y, z) = (\bar{u}_2, -\bar{u}_1, \bar{u}_4, -\bar{u}_3, \bar{x}, \bar{y}, \bar{z})\,. \tag{C.1}$$

This restricts the coefficients of the polynomial. However these coefficients remain general enough that a generic polynomial is still non-singular. The identification σ has no fixed space on the base, as the would-be fixed space $u_1 = u_2 = u_3 = u_4 = 0$ is removed by the Stanley-Reisner ideal. Every point of the base then represents an example of situation (1) as described in section 8.2 and so the Spin(7) holonomy manifold[1] produced upon quotienting by σ is non-singular. This means that no additional resolutions need to be performed.

C.2 A complete intersection in a $\mathbb{P}_{1,1,1,1}$ fibration of $\mathbb{P}_{1,1,2,2}$

Next let us consider a second construction in which the ambient space is formed by fibering $\mathbb{P}_{1,1,1,1}$ over $\mathbb{P}_{1,1,2,2}$. In this case the Calabi-Yau is given by a complete intersection of two polynomials described the following nef-partition in Table C.2.

nef-part.	vertices	coords.	Q_1	Q_2
∇_1	$\nu_1 = (-1, -1, 0, -1, -2, -2)$	y_1	1	0
	$\nu_2 = (0, 0, 0, 1, 0, 0)$	y_2	1	0
	$\nu_3 = (1, 0, 0, 0, 0, 0)$	x_1	1	1
	$\nu_4 = (0, 1, 0, 0, 0, 0)$	x_2	1	1
∇_2	$\nu_5 = (0, 0, 0, 0, 1, 0)$	v_1	2	0
	$\nu_6 = (0, 0, 0, 0, 0, 1)$	v_2	2	0
	$\nu_7 = (-1, -1, -1, 0, 0, 0)$	z_1	0	1
	$\nu_8 = (0, 0, 1, 0, 0, 0)$	z_2	0	1

Table C.2: Toric data for a nef-partition describing a complete intersection in a $\mathbb{P}_{1,1,1,1}$ fibration of $\mathbb{P}_{1,1,2,2}$.

The two polynomials P_1 and P_2 are then associated with the partitions ∇_1 and ∇_2 respectively. These are both degree (4,2) under identifications (Q_1, Q_2).

In this case the base $\mathbb{P}_{1,1,2,2}$ has a complex one-dimensional holomorphic orbifold singularity at $y_1 = y_2 = 0$ before considering any antiholomorphic quotient. This lifts to two separate complex two-dimensional singular spaces in the total ambient space. One, which is associated with the Q_1 identification, lies at $y_1 = y_2 = x_1 = x_2 = 0$ and the other, which is associated with the $Q_1 - Q_2$ identification, lies at $y_1 = y_2 = z_1 = z_2 = 0$.

Let us first consider the singular space which lies at $y_1 = y_2 = x_1 = x_2 = 0$. At this locus the

[1] Note that strictly speaking the quotient manifold is expected to have $SU(4) \times \mathbb{Z}_2$ holonomy.

polynomials can be written as

$$P_1 = a_1 z_1^2 + b_1 z_1 z_2 + c_1 z_2^2 \qquad\qquad P_2 = a_2 z_1^2 + b_2 z_1 z_2 + c_2 z_2^2 \tag{C.2}$$

where $a_{1,2}$ $b_{1,2}$ and $c_{1,2}$ are homogeneous quadratics in v_1 and v_2. The singularities of the ambient space will then intersect both polynomials at the places where one of the roots of P_1 sits on top of one of the roots of P_2. At these points the *resultant* of the pair of polynomials, given by

$$-a_2 b_1 b_2 c_1 + a_1 b_2^2 c_1 + a_2^2 c_1^2 + a_2 b_1^2 c_2 - a_1 b_1 b_2 c_2 - 2a_1 a_2 c_1 c_2 + a_1^2 c_2^2 \, , \tag{C.3}$$

will vanish. This *resultant* is a homogeneous octic in $v_{1,2}$ so gives eight $\mathbb{Z}_2$ singular points on the Calabi-Yau fourfold at which the pair of the polynomials hit the two-dimensional space of singularities in the ambient space.

Next let us consider the singular space which lies at $y_1 = y_2 = z_1 = z_2 = 0$. As before both polynomials will intersect the singularity of the ambient space when the resultant vanishes. This second resultant is a homogeneous quartic in $v_{1,2}$ so gives four $\mathbb{Z}_2$ singular points.

The Calabi-Yau fourfold may have extra singularities associated with the pinching of the torus. To find out where this happens we may make use of the singularity classification described in [295]. This shows that for a generic set of polynomial coefficients the torus pinches with a Type I_1 singularity over the intersection of a homogeneous degree $(72, 0)$ polynomial in the (Q_1, Q_2) identification, with the two polynomials that define the Calabi-Yau. Furthermore we find that this space intersects each of the $\mathbb{Z}_2$ singular points described above.

We now impose a symmetry under the action of the antiholomorphic involution σ defined by,

$$\sigma(y_1, y_2, v_1, v_2, x_1, x_2, z_1, z_2) = (\bar{y}_2, -\bar{y}_1, \bar{v}_2, \bar{v}_1, \bar{x}_2, -\bar{x}_1, \bar{z}_2, \bar{z}_1) \, . \tag{C.4}$$

As before this constrains the coefficients of the polynomials but does not alter the singularity structure of the Calabi-Yau. We note also that in this case σ is not an involution on its own but that the identification Q_1 must be used to make $\sigma^2 = \mathbb{1}$.

The action of σ on the base gives a real one-dimensional fixed line which sits inside the holomorphic orbifold singularity of $\mathbb{P}_{1,1,2,2}$. At most places over this fixed line the torus is unpinched and has no fixed space. It represents an example of situation (2.1) described in Section 8.2. However when the torus pinches over the fixed line of the base the pinched point on the torus becomes fixed under the action of σ and so represents an example of situation (3). In additional, these fixed pinched points on the torus also lie at the eight $\mathbb{Z}_2$ singular points at $y_1 = y_2 = x_1 = x_2 = 0$. By comparison the four $\mathbb{Z}_2$ singular points, which lie at $y_1 = y_2 = z_1 = z_2 = 0$ are not fixed under σ but instead are mapped pairwise into each other.

The quotient of this Calabi-Yau by σ then gives a singular Spin(7) manifold. The presence of these singularities is not a problem in F-theory as this is defined on singular spaces. However in order to use the M-theory duality we have described to find the effective action these singularities must be resolved in an appropriate fashion. It is unclear how one would carry out this resolution or even if such a resolution can be performed at all for this particular Spin(7) manifold. For this reason it would be extremely important to investigate these resolutions further.

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