On Deformations and Quantization in Topological String Theory

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München 2014

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Dissertation
an der Fakultät für Physik
der Ludwig–Maximilians–Universität
München

vorgelegt von
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aus Zürich

München, den 18. Februar 2014

Erstgutachter: Prof. Dr. Ilka Brunner
Zweitgutachter: Prof. Dr. Peter Mayr
Tag der mündlichen Prüfung: 30. April 2014

to my beloved grandmother Lea

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Zusammenfassung

Die Untersuchung der Moduli Räumen von $N = (2,2)$ Superkonformen Feldtheorien und der allgemeineren $N = (2,2)$ Supersymmetrischen Quanten Feldtheorien ist ein langjähriges und vielseitiges Forschungsgebiet. Diese Dissertation konzentriert sich auf gewisse allgemeine Aspekte des erwähnten Studiums, und stellt Entwicklungen von allgemeinen Methoden im Rahmen der Topologischen String Theorie dar. Die vorliegende Arbeit besteht aus zwei Teilen.

Der erste Teil befasst sich mit Aspekten der geschlossenen Topologischen String Theorie und kulminiert in den Inhalt von [52], wo die geometrische Struktur der Topologischen anti-Topologischen Moduli Räumen von $N = (2,2)$ Superkonformen Feldtheorien mit Zentral Ladung $c = 9$, angesichts eines allgemeinen Quantisierung-Rahmens [31, 32] wiederentdeckt wird. Aus dieser Sichtweise erhält man, als Spezialfall, eine klare Einsicht der "holomorphic anomaly equation" von [6]. Diese Arbeit könnte als eine natürliche Erweiterung von früheren Untersuchungen in ähnlicher Richtung betrachtet werden, insbesondere vom grundlegenden Artikel [104].

Der zweite Teil befasst sich mit Aspekten der Untersuchung der Offenen und Geschlossenen Moduli Räumen von Topologischen Konformen Feldtheorien auf Genus Null. Insbesondere, ist hier eine Exposition von [13] enthalten, wo allgemeine Resultate über die Klassifizierung und Berechnung von "bulk-induced" Deformationen von Offenen Topologischen Konformen Feldtheorien erhalten wurden. Letzteres wurde durch eine kohärente algebraische Methode erreicht was sich auf den definierenden L_∞ und A_∞ beteiligten Strukturen bezieht. Teilweise ist die letztere Untersuchung auf beliebige Affine B-twisted Landau Ginzburg Modelle beschränkt. Nachfolgend wird weitere originelle Arbeit dargestellt was die Topologische String-Feld-Theoretische Struktur von B-twisted Landau Ginzburg Modellen vollendet. Insbesondere wird eine "off-shell" Erweiterung der Kapustin-Li Formel von [41, 49] gegeben. Diese "off-shell" Formel bezeichnet einen konsolidierenden Baustein der algebraischen Herangehensweise zur Berechnung des Effektiven Superpotentials von B-twisted Affine Landau Ginzburg Modellen, und kann damit als eine natürliche Entwicklung von der grundlegenden Arbeit [12] betrachtet werden.

Abstract

The study of moduli spaces of $N = (2, 2)$ superconformal field theories and more generally of $N = (2, 2)$ supersymmetric quantum field theories, has been a longstanding, multifaceted area of research. In this thesis we focus on certain selected general aspects of this study and develop general techniques within the framework of topological string theory. This work is naturally divided into two parts.

The first is concerned with aspects of closed topological string theory, and culminates with the content of [52], where the geometrical structure of the topological anti-topological moduli spaces of $N = (2, 2)$ superconformal field theories with central charge $c = 9$ is rediscovered in the light of quantization, within a general framework ([31, 32]). From this point of view, one thus obtains, as a special case, a clear understanding of the holomorphic anomaly equation of [6]. This work can be viewed as a natural continuation of earlier studies in the same direction, most notably the seminal paper [104].

The second part is concerned with aspects of the study of the open and closed moduli space of topological conformal field theories at genus zero. In particular, it contains an exposition of [13], where general results on the classification and computation of bulk-induced deformations of open topological conformal field theories were obtained from a coherent algebraic approach, drawing from the defining L_∞ and A_∞ structures involved. In part, the latter investigation is restricted to arbitrary affine B-twisted Landau Ginzburg models. Subsequently, further original work is presented that completes the topological string field theory structure of B-twisted Landau Ginzburg models, providing in particular an off-shell extension of the Kapustin-Li pairing of [41, 49]. This off-shell pairing constitutes a consolidating building block in the algebraic approach to the computation of the effective superpotential of B-twisted affine Landau Ginzburg models pioneered in [12].

Chapter 1

Introduction

1.1 A rough map of the general context

The present work is concerned with the study of selected aspects of topological string theory. This theory has its conceptual origin in the more general string theory, although it is very interesting in its own right. In particular, from its inception [100], it has provided a rich and convenient tool for the investigation and intuitive understanding of Kähler and especially Calabi-Yau geometry of a priori arbitrary dimension. In the context of string theory, topological string theory naturally arises in critical type II theories with total conformal field theory restricted to a class of the type:

$$CFT_{tot} = CFT_{\mathbb{R}^{1,3}} \otimes CFT_{internal},$$

where $CFT_{\mathbb{R}^{1,3}}$ is given by the left-right supersymmetric sigma-model into $\mathbb{R}^{1,3}$, while $CFT_{internal}$ is an arbitrary $N = (2,2)$ superconformal field theory with central charge $c = 9$.

The most intuitive such models are again sigma-models into a manifold of dimension 6. The requirement of $N = (2,2)$ supersymmetry is that the manifold be Kähler, and that of conformal invariance that it be Calabi-Yau. The latter is a Kähler manifold (M, ω, J) with $c_1(M) = 0$. In particular it admits a nowhere vanishing holomorphic top form Ω.

Part of the aim of string theory in the restricted setting of (1.1) is to have a complete classification of such internal CFTs defined on the sphere S^2. The space of such CFTs should be viewed as the stringy generalization of the space of solutions to $N = 2$ supersymmetric Einstein's equations in the vacuum for a euclidean 6 dimensional manifold. Indeed, in the case of Calabi-Yau manifolds, Yau's theorem [105] ensures that for given Kähler class $[\omega] \in H^{1,1}(M, \mathbb{R})$, there is a unique representative thereof whose associated Kähler metric is Ricci-flat.

The space of internal CFTs is usually studied perturbatively. One starts by considering a particular point in this space and looks to deform the CFT to a neighboring one satisfying the same aforementioned requirements. Such deformations are special kinds of exactly marginal deformations. These are in correspondence with a restricted subset of fields in the original CFT. In fact contrary to the infinite dimensional Hilbert-space of all the states in the CFT, the latter is finite dimensional.

Such a reduction of degrees of freedom calls for a procedure that allows to restrict attention to a "small" subset of the Hilbertspace that contains the exactly marginal deformations of interest. Indeed this procedure exists in the context of $N = (2, 2)$ CFTs and goes by the name of topological twisting [100]. In particular, the latter allows the CFT to descend to a TQFT (topological quantum field theory) and families of CFTs can be safely replaced as families of TQFTs. In fact, to be more precise, these families of TQFTs account only for roughly half of the deformation space. In the case of Calabi-Yau manifolds, the full deformation space is locally a product of complex structure and complexified Kähler structure deformations, and one can construct families of TQFTs corresponding to either factor [102]. The one corresponding to the former is the B-model, while the one corresponding to the latter the A-model. Note in particular that the A-model is sensitive to complexified Kähler structures, that is in the context of CFT, the definition of Kähler manifold is naturally extended to incorporate B-fields in $H^{1,1}(M, \mathbb{R}/\mathbb{Z})$. Non-linear sigma models and their topological twists can be generalized to incorporate a holomorphic superpotential W becoming what are known as general Landau-Ginzburg models, as opposed to the simpler affine Landau Ginzburg models where M is affine space. In the present work we will often resort to general Landau Ginzburg models as illustrative examples for two dimensional TQFT's. In particular we will only consider the B-twist.

The ultimate aim of topological string theory is naturally manifold. Starting from the top, it is the classification of TQFTs arising from topological twisting. Although a priori addressing only partially the classification question of internal CFT's, the latter is in fact widely believed to provide a full answer. This is thanks to mirror symmetry. In strict terms, the latter is an outer automorphism of the $N = (2, 2)$ superconformal algebra, and it allows to interpret a given $N = (2, 2)$ CFT as arising from two seemingly different models. In particular, in the case of Calabi-Yau sigma models, the CFT that corresponds to M is conjectured to be the same as the one corresponding to a so called mirror $\widetilde{M}$ whose A- and B-models are identified with the B- and A-models attached to M respectively.

Moreover, within the class of B-twisted TQFT's, there is a further identification between pairs of models. This time it is between Calabi-Yau sigma models on projective hypersurfaces defined by a polynomial equation $W = 0$ on say $\mathbb{P}^n$, and an orbifolded affine Landau Ginzburg model ([63, 94, 95]) on $\mathbb{C}^{n+1}$, (more precisely $\mathbb{A}^{n+1}$), with superpotential W. This correspondence was given an elegant interpretation in [103] where the respective models were extracted as phases of a more general Gauged Linear sigma model.

Dualities as the ones just described allow, in particular, to study the same moduli space from different perspectives, and simple aspects of one realization aid to the understanding of the more difficult counterparts in the dual models.

The study of moduli spaces, as we tacitly assumed in the above, concerns conformal field theory on the sphere. In string theory one is also interested in quantum surfaces, that is ones of higher genus. Moreover, the study of closed strings is extended to incorporate open strings, which in type II theories can be viewed as giving rise to perturbative excitations of the non-perturbative D-branes.

In order to understand the physical implications of such generalizations, it is worthwhile to adopt an effective field theory viewpoint. Compactifying the effective type II theory down to four dimensions, one obtains an $N = 2$ supergravity theory, and the internal CFT moduli become scalars in vector and hyper multiplets. In particular the TQFT moduli correspond to vector multiplets and the genus zero data one extracts from the TQFT are three point functions, which in the effective four dimensional theory are Yukawa couplings, namely couplings between a scalar boson and two fermions in the same vector-multiplet. From the TQFT perspective, the bosons correspond to certain Neveu-Schwarz highest weight states of the underlying N=(2,2) CFT, while the fermions to the spectral flowed Ramond states. In order to ensure the effective $N = 2$ supersymmetry one should impose certain charge integrality conditions in the microscopic CFT. However we will neglect this in the following. By suitably choosing the internal CFT one can obtain in particular $N = 2$ Yang-Mills theories, for example those of A, D and E type. This is the topic of geometric engineering. Introducing D-branes in more general backgrounds, one can instead obtain more general $N = 1$ Yang-Mills theories. These theories in particular admit a superpotential $\mathcal{W}_{eff}$, which depends both on the open and closed moduli and is the manifestation of the generic obstructions in the deformations of open and closed CFT's. Finally, incorporating higher genus surfaces and restricting attention to closed strings, the higher genus TQFT correlation functions, in fact suitable holomorphic cut-offs thereof, are interpreted as the terms in the coupling of scalars in the vectormultiplet to the gravi-photon field-strength.

From the microscopic internal CFT point view, including open strings, or in other words D-branes, at genus zero (on the disk), implies erecting at each point of the closed TQFT moduli space, a category whose objects are the topological D-branes in the "background" defined by the closed strings, and whose morphisms are the topological open strings. That is, roughly speaking, the moduli space is replaced by a sheaf of categories thereupon. These categories come endowed with extra structure. In particular they are Calabi-Yau A_∞-categories [19]. Suffice it to say at this point, that this structure alone suffices to tackle the classification of open and closed deformations of objects in the fiber category. In particular the structure of the open-closed moduli space in the immediate vicinity of one such object, or D-brane, is encoded in the effective superpotential $\mathcal{W}_{eff}$, which from this microscopic viewpoint is the moduli dependent open three-point function, or equivalently the generating function of the open-closed TCFT, which in addition to the TQFT data incorporates the deforming integrated descendants of the TQFT observables.

Having provided a rough sketch of the general context we can now sharpen the scope and illustrate the aims of the present work

1.2 Aims of the present work

This thesis divides into two parts. The first part comprises chapters 2 and 3, while chapters 4 and 5 constitute the second part.

Chapters 2 and 4 introduce the essential background for the endeavors of chapters 3 and 5 respectively.

In chapter 2 we start in section 2.1 with the definition and characterization of closed two dimensional topological quantum field theories. Sections 2.2 and 2.3 introduce respectively the basic ingredients needed of N=(2,2) CFTs for the construction of the smaller TQFTs, and the general Landau-Ginzburg model as a candidate example for furnishing such N=(2,2) CFTs. Section 2.4 explains the topological twist procedure. In particular we restrict attention to the B-twist, while in section 2.5 we sketch how to obtain the data of the B-twisted general Landau Ginzburg models.

In section 2.6 we introduce the notion of spaces of TQFTs arising as B-twisted CFTs and describe the data controlling the deformation problem. In this way we conclude the sketch of genus zero B-twisted TQFTs. The remaining sections extend to higher genus surfaces. In particular the first of these, section 2.7, reviews the essentials of topological anti-topological fusion which is still genus zero data, but is however crucial to the understanding of twisted theories only at genus higher than zero. In particular, in this section we sketch the logical steps required to come to the conclusion that the moduli space of TQFT's arising from N=(2,2) CFTs of central charge $c = 9$, combined with their CPT conjugates, is a projective special Kähler manifold, though we defer the precise definition to chapter 3, where we study its origin and structure in detail. Finally, in section 2.8 we explain how to couple twisted CFTs to topological gravity to obtain TST (topological string theory), thus paving the way for the definition of the central all encompassing object of closed TST, namely the generating function at all genera of TST scattering amplitudes among marginal deformations. This is an object of the form:

$$Z(u, p),$$

where p is a point on the topological anti-topological moduli space, describing the TQFT within which marginal fields labelled by the parameters u scatter. The properties of this generating function were first investigated in the seminal paper [6]. In particular Z was shown to satisfy a master equation named the “holomorphic anomaly” equation, as it indicates that as one departs from genus zero surfaces, Z acquires dependence on the anti-holomorphic parameters describing the anti-topological theory. Section 2.8.4 is thus dedicated to explain how such a master equation arises in the context of TST.

Finally we pass to the aim of chapter 3 the content of which is [52]. In that paper we showed how the structure possessed by the topological anti-topological moduli space of N=(2,2) CFT's of central charge 9, arises in the context of the quantization of classical phase-spaces. More precisely we study both affine and projective special Kähler manifolds [14, 15]. The former were first discovered as vector-multiplet moduli spaces of rigid N=2 four dimensional gauge theory [34, 88] leading up to [85, 86] where transparent understanding of this structure was revealed. Projective special Kähler manifolds were discovered analogously in the more general N=2 gauged supergravity [21, 22, 89]. Subsequently they were both given precise mathematical characterization both extrinsically [4, 18] and intrinsically [33]. In chapter 3 we rediscover both affine and projective special

Kähler manifolds in the context of Fedosov quantization [31, 32]. In this context $Z(u,p)$ arises naturally as a wave function (representation) of the coherent state corresponding to the point p and as such it will satisfy a master equation which is shown to coincide with the holomorphic anomaly equation of [6]. That $Z(u,p)$ should arise from the quantization of the genus zero moduli space is a natural and expected. Indeed such a verification is essential to understand the precise relation between the genus expansion and quantization, which is at the heart of the string theory proposal for the quantization of gravity. A first derivation of a holomorphic anomaly type equation from a direct quantization procedure, was obtained long ago in [104]. This point of view was revived and improved upon notably in [1, 39, 96]. In chapter 3 we give a first principles derivation of the holomorphic anomaly equation in a much more general context, allowing in particular to extend the results to a general class of quantizable manifolds. Sections 3.1 and 3.1.1 provide the necessary background. In the former we review the definition of quantization and review the simplest case to set conventions for the future sections. In section 3.2 instead we review the quantization procedure of [31, 32]. In section 3.3 we discover affine special Kähler manifolds in this context and quantize them finding the analogue of the holomorphic anomaly equation and subsequently provide its general solution. In section 3.4 we specialize to conic affine special Kähler manifolds. From these one defines projective special Kähler manifolds as certain holomorphic quotients. In particular the projective special Kähler manifolds of interest are quotients of Lorentzian conic special Kähler manifolds. Therefore we explain how to extend the definition of quantization to accommodate "indefinite Hilbertspaces". Of particular curiosity is the discussion of section 3.4.1 where we attempt a guess for the quantum origin of the conic property. Subsequently in section 3.4.5 we finally turn to the quantization of projective special Kähler manifolds and rediscover the holomorphic anomaly equation of [6] having provided its general solution. In this way we conclude the first part of the thesis.

Part two is dedicated to a study of deformations, both open and closed, of open TQFT's of B-twisted models. After extending the discussion of section 2.1 to open and closed TQFT in section 4.1, we immediately pass in section 4.2 to the review of the definition of the open sector, that is D-branes, of B-type in the general untwisted and B-twisted Landau-Ginzburg models. In section 4.3 we provide the bare essentials of the formalism involved in TQFT's arising from $N = (2,2)$ CFT's while in section 4.4 we turn to the properties of open and closed (bulk and boundary) deformations of open TCFT's on the disk. In this way we conclude the introductory chapter 4, with the necessary ingredients to tackle deformations proper. We start chapter 5 by formalizing section 4.4. We thus introduce the relevant properties of A_∞ and L_∞ structures, which govern the deformation problem.

Section 5.2 contains the results of [13]. There we studied bulk-induced deformations of open topological string theory on the disk. That is we considered an object of the D-brane category fibered over a closed string background, together with its given neighborhood non-commutative geometry (A_∞ structure) and deformed it along the base, closed string moduli space. In other words, bulk-induced deformations are lifts of closed deformations to the fibered D-brane categories. In particularly we restrict attention to finitely many objects and use the language of A_∞-algebras rather than categories. Section 5.2 consists of two parts. The first is restricted to arbitrary affine Landau Ginzburg models while the second can be applied to any model. In the former we work off-shell and give the precise map that lifts closed deformations to the open sector. The relevant notion is that of L_∞ morphism. The result of that section can be viewed as a generalization of Kontsevich's deformation quantization [55] to the more general non-commutative affine Landau-Ginzburg setting. In the second part we transport the bulk-induced deformations on-shell. The notion of on- and off-shell is explained in section 5.1. Roughly the former is to the latter what string field theory is to string theory. Having classified the bulk-induced deformations of arbitrary affine Landau-Ginzburg models, albeit neglecting the cyclic structure, we turn to the problem of computing purely open deformations of arbitrary affine Landau-Ginzburg models. In particular we build upon the non-commutative geometric strategy developed in [12] and provide an explicit formula for the off-shell, string field theory, pairing for arbitrary affine Landau Ginzburg models. This pairing correctly reduces to the Kapustin-Li pairing [41, 49] on-shell and can be viewed as completing the string field theory data of affine Landau Ginzburg models. In particular having the pairing is tantamount to having the off-shell effective superpotential $\mathcal{W}_{eff}^{off}$ which generalizes the abstract Chern-Simons action of open string field theory. Transporting $\mathcal{W}_{eff}^{off}$ can then be efficiently achieved using the method of [12].

Chapter 2

Essentials of closed topological string theory

2.1 Closed TQFT basics

In this section we will provide a glimpse of closed $2d$ $TQFT$ (two dimensional topological quantum field theory), at its simplest, that is without decorating it with e.g. spin structure. The emphasis is on showing how from a seemingly general starting point one is instead lead to a very rigid structure. We refer to the original papers [3, 60, 72]. A closed topological quantum field theory is a functor:

$$Z : Cob \longrightarrow Vect_k.$$

An object O in the category Cob is a disjoint union of oriented circles $S^1_\pm$, while morphisms, called bordisms, are boundary-preserving diffeomorphism classes $[\Sigma](O_1, O_2)$ of smooth surfaces $\Sigma(O_1, O_2)$ whose boundary is the oriented disjoint union of the initial and final object:

$$\partial\Sigma(O_1, O_2) = O_1 \sqcup \overline{O}_2,$$

where by $\overline{O}_2$ we have denoted O_2 with opposite orientation. Composition of morphisms:

$$[\Sigma_1][O_1, O_2] \times [\Sigma_2][O_2, O_3] \mapsto [\Sigma_3][O_1, O_3]$$

is given by gluing two representatives along the common boundary. The category $Vect_k$ instead, is the category whose objects are vectorspaces over the field k [1] and whose morphisms are linear maps with the obvious composition. The functor Z preserves coproducts, sending:

$$\sqcup \mapsto \otimes.$$

Now, in order to determine a given TQFT, we need to specify the action on objects and morphisms. Given that Z preserves coproducts, for the former we only need to specify the

[1] For us $k = \mathbb{C}$.

action of Z on the two circles S^1_+ and S^1_-. We will see that $Z(S^1_-)$ is in fact determined by $Z(S^1_+)$. We set:

$$Z(S^1_+) = V$$

and denote $Z(S^1_-) = \overline{V}$. In order for $\overline{V}$ to be determined by V, we need to adjoin the object $\emptyset$ to Cob and accordingly maps to and from it. Clearly $\emptyset$ is the unique unit w.r.t. $\sqcup$, therefore Z must send it to the unique unit w.r.t. $\otimes$, namely k itself:

$$Z(\emptyset) = k.$$

Adjoining $\emptyset$ also clarifies the choice of naming the functor "Z", as its action on a closed bordism yields a number, namely the partition function associated to that bordism. Now we concentrate on morphisms with boundary the disjoint union of two circles. The simplest such morphism is the cylinder $C := S^1_+ \times [0,1]$. Viewed as a map in $Mor(S^1_+, S^1_+)$ it is clearly the identity map, and hence also $Z(C)$. However one can also view C as an element of $Mor(S^1_+ \sqcup S^1_-, \emptyset)$, thus defining the so called evaluation map:

$$ev : V \otimes \overline{V} \longrightarrow k$$

and lastly C can be viewed as a morphism in $Mor(\emptyset, S^1_+ \sqcup S^1_-)$ thus yielding the so called coevaluation map:

$$coev : k \longrightarrow V \otimes \overline{V}.$$

Since C is the identity map:

$$V \cong ev(V, \cdot) \subset \overline{V}^\vee,$$

and likewise replacing S^1_+ with S^1_- and V with $\overline{V}$. Therefore:

$$V \cong \overline{V}^\vee \cong (V^\vee)^\vee.$$

Since the dual is taken in the strictly algebraic sense, V is finite dimensional and:

$$Z(S^1_-) = V^\vee.$$

There is a further morphism in Cob from the circle to the circle, which is independent of the previous ones, namely the cylinder in $Mor(S^1_+, S^1_-)$, which can also be viewed as a map in $Mor(S^1_+ \sqcup S^1_+, \emptyset)$. The action of Z yields a map:

$$\eta : V \otimes V \longrightarrow k$$

called the topological metric. It is easy to show that this is non-degenerate. For example by appropriately composing it with its dual it yields the identity map. So far all of these properties apply invariably to any topological field theory of any dimension including $d = 1$. The peculiarity of $d = 2$ and higher, is the existence of connected morphisms in Cob whose boundaries have more than two disconnected (non-empty) objects. The beauty of two dimensions is the possibility of expressing any such morphism from the previously constructed ones and the "pair of pants". The latter is a morphism in $Mor(S^1_+ \sqcup S^1_+, S^1_+)$. It defines a multiplication:

$$\mu : V \otimes V \longrightarrow V$$

that is associative, because the bordisms in the associativity equation are equivalent. Furthermore m is commutative. This is explained by swapping the incoming circles past each-other and turning the neck close to the outcoming circle to undo the twist and obtain the initial pair of pants. Finally the commutative and associative algebra μ has a unit, namely the disk, viewed as a map:

$$e : k \longrightarrow V$$

since composing $m\mu$ with e yields the identity cylinder. The last ingredient to completely specify a $2d$ TQFT is the dual to the unit, called the trace:

$$\theta : V \longrightarrow k.$$

It is clear that it makes the separate definition of η redundant, since:

$$\eta = \theta \circ \mu.$$

Consequently θ is non-degenerate. To summarize, a closed $2d$ TQFT is equivalent to a finite-dimensional commutative, associative algebra with a non-degenerate trace, that is a *commutative Frobenius algebra*. We remark that, had we allowed for $\mathbb{Z}_2$ graded vectorspaces, we would have obtained a $\mathbb{Z}_2$ graded-commutative Frobenius algebra.

2.2 TQFT essentials of N=(2,2) CFT's

In this section we review the basic ingredients of $\mathcal{N} = (2,2)$ superconformal field theories that we need, to formulate topological conformal field theories (we follow [63]). We assume basic knowledge of conformal field theory ([23]) and thus define an $N = (2,2)$ superconformal field theory as a euclidean two dimensional conformal field theory whose Hilbertspace $\mathcal{H}$ carries a unitary representation of the $N = (2,2)$ super-Virasoro algebra of which we write the holomorphic part:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \\
[L_m, J_n] &= -nJ_{m+n}, \\
[L_m, G^\pm_r] &= \left(\frac{m}{2} - r\right) G^\pm_{m+r}, \\
[J_m, J_n] &= \frac{c}{3} m\delta_{m+n,0}, \\
[J_m, G^\pm_r] &= \pm G^\pm_{m+r}, \\
[G^+_r, G^-_s] &= 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0} \\
[G^+_r, G^+_s] &= [G^-_r, G^-_s] = 0
\end{aligned}$$

where $[\cdot,\cdot]$ denotes the graded commutator. The generators of the above Lie algebra are the modes of the fields:

$$T(z)=\sum_{n\in\mathbb{Z}}L_n z^{-n-2}\qquad G^{\pm}(z)=\sum_{r\in\mathbb{Z}+a}G^{\pm}_r z^{-n-3/2}\qquad J(z)=\sum_{n\in\mathbb{Z}}J_n z^{-n-1}$$

where $a\in\{0,1/2\}$ distinguishes the Ramond from the Neveau-Schwarz sectors. The first current is the holomorphic part of the energy-momentum tensor, the second are the super currents of spin 3/2 and the third is the generator of an affine $u(1)$ symmetry. Its global part is called R-symmetry and rotates the supercharges clockwise or anti-clockwise. By bosonization, the $u(1)$ current can be viewed as:

$$J=i\sqrt{\frac{c}{3}}\partial_z\phi \tag{2.2.1}$$

where ϕ is a free scalar field. Bosonization in this context is the property that, given a CFT whose symmetry algebra contains affine $u(1)$, allows one to split it as the quotient by the $u(1)$ CFT obtained via the Sugawara construction, plus the $u(1)$ CFT. The latter is unique, because there is a unique unitary irreducible representation of the affine $u(1)$ algebra and it is equal to the free boson. This free boson induces a continuous one-parameter group of inner automorphisms of the $N=2$ algebra and contains a subgroup whose generator:

$$U_{1/2}=\exp\left(-\frac{i}{2}\sqrt{\frac{c}{3}}\phi\right) \tag{2.2.2}$$

exchanges the Ramond and Neveau-Schwarz algebras, thus allowing us in particular to recover the representation theory of one from the other. We now consider the unitary highest weight representations of the NS algebra. The starting point for the classification of these is to isolate a minimal set of generators of the NS algebra as a Lie algebra. It is easy to see that one such set is:

$$\{J_{-1},G^+_{-1/2},G^-_{-1/2},J_0,L_0,G^-_{1/2},G^+_{1/2},J_1\}$$

where the Cartan elements J_0,L_0 separate the creation operators on the left from the annihilation operators on the right. A highest weight state $|\phi\rangle$ is then an eigenstate of (L_0,J_0) with eigenvalues (h,q), that is annihilated by $\{G^-_{1/2},G^+_{1/2},J_1\}$. The simplest such state is the $sl(2,\mathbb{C})$ invariant vacuum $|0\rangle_{NS}$ [2], which consequently must also be annihilated by $G^+_{-1/2}$ and $G^-_{-1/2}$. Between a general highest weight state and the vacuum we see however that there are two special choices, a *chiral* highest weight state:

$$G^+_{-1/2}|\phi\rangle_c=0$$

and an *anti-chiral* highest weight state:

$$G^-_{-1/2}|\phi\rangle_a=0.$$

[2] We add the superfluous label indicating that it is in the NS sector, to distinguish it from the state $|0\rangle$ that we will define later on.

Accordingly, there are *chiral* and *antichiral primary fields* ϕ_c and ϕ_a. From the point of view of the Ramond sector they both correspond to the so called *massless BPS representations*, the difference being that from the chiral ring one flows with $U_{1/2}$ and from the antichiral with $U_{-1/2}$. We will restrict our attention throughout to the chiral representations. These in particular satisfy:

$$h = \frac{q}{2}. \tag{2.2.3}$$

We will see shortly that, from this condition and unitarity, it follows that the set of chiral primary fields closes in a graded-commutative ring. Unitarity, in particular, imposes that for a general eigenstate of (L_0, q_0), $h \geq q/2$, thus distinguishing the chiral highest weight states as the ones saturating the bound. Then the product of two chiral primaries

$$\phi_c^1(z)\phi_c^2(w) = \sum_i (z-w)^{h^i - h_c^1 - h_c^2} \psi^i(z)$$

has only regular terms on the right hand side and the only surviving term in the limit $z \to w$ is again a chiral primary. Hence we obtain the so called *chiral ring*. The second constraint on chiral primary fields is:

$$h \leq \frac{c}{6}.$$

This constraint together with the assumption of a non-degenerate vacuum ensures that the chiral ring is finite dimensional. We have thus found within any left (right) $N = 2$ sector of an $N = (2, 2)$ CFT, a candidate for the definition of a closed topological field theory. We are still missing a very important ingredient however, namely a non-degenerate trace. Including the right-moving sector we then have four candidate rings: the left-chiral and right-chiral ring denoted (c, c) as well as the the (a, c), (c, a) and (a, a) ring. Clearly, the (c, c) and (a, a) rings as well as the (c, a) and (a, c) rings are CPT conjugates of each-other, while the remaining pairs of rings are related by mirror symmetry, which we will not discuss in this work.

2.3 The general Landau Ginzburg model

In this subsection we review the basics of the general Landau Ginzburg model (see [46] for an extensive review). We will analyze the minimal constraints for this to flow to an $N = (2, 2)$ CFT. This class of models will serve as our working example throughout. In particular we will often specialize on non-linear sigma models and affine Landau Ginzburg models. For our purposes, the former will serve as physical/geometric motivation for the latter on which we will concentrate in chapter 5.

We introduce holomorphic coordinates (z, θ^+, θ^-), on $N = (2, 2)$ Minkowski space $M^{2|(2,2)}$. These provide charts for a super-Riemann surface $\Sigma^{(2,2)}$. Then the models we consider are built from a vector Φ of spinless chiral superfields:

$$D^+\Phi^i = \overline{D}^+\Phi^i = 0,$$

where D^+, $\overline{D}^+$ are respectively left-moving and right-moving left-invariant vectorfields w.r.t. translations in θ^+ and $\overline{\theta}^+$, dual to the corresponding right-invariant vectorfields, namely the supercharges $\mathcal{Q}^+$ and $\overline{\mathcal{Q}}^+$. In the models of interest, the component fields of Φ are interpreted as maps from Σ to a Riemannian manifold M and their super-partners:

$$\begin{aligned} &X \in C^\infty(\Sigma, M) \\ &\psi \in \Gamma(\Sigma, X^*TM^{1,0} \otimes S) \\ &\overline{\psi} \in \Gamma(\Sigma, X^*TM^{1,0} \otimes \overline{S}) \\ &\chi \in \Gamma(\Sigma, X^*TM^{0,1} \otimes S) \\ &\overline{\chi} \in \Gamma(\Sigma, X^*TM^{0,1} \otimes \overline{S}) \\ &F \in \Gamma(\Sigma, X^*TM^{1,0} \otimes S \otimes \overline{S}) \end{aligned} \tag{2.3.1}$$

where S and $\overline{S}$ are the left and right Weyl spinor bundles. The two models of interest arise as special cases of the general Landau Ginzburg model:

$$S = S_D + S_F = \int_{\Sigma^{(2,2)}} d^2z d^4\theta K(\Phi, \overline{\Phi}) + \int_{\Sigma^{(2,2)}} d^2z \left(d^2\theta^- W(\Phi) + d^2\theta^+ \overline{W}(\overline{\Phi})\right)$$

where $\overline{\Phi}$ is the anti-chiral, hermitian conjugate to Φ, W and K for now are arbitrary functions and, in particular, K is viewed as a Kähler potential due to the symmetry of the theory under transformations $K(\Phi, \overline{\Phi}) \mapsto K(\Phi, \overline{\Phi}) + f(\Phi) + \overline{f}(\overline{\Phi})$ thus defining the manifold M locally as a Kähler manifold. The nonlinear sigma model is recovered by setting $W = 0$, while the affine Landau Ginzburg model by setting M to affine space.

We will continue with the analysis of the general Landau Ginzburg model. So far, by construction we have a rigid $N = (2, 2)$ SUSY theory. In fact, it is supersymmetric provided Σ is flat, otherwise supersymmetry is spoiled by the absence of a covariantly constant spinor. For compact Riemann surfaces this requires genus $g = 1$. Furthermore, a necessary condition for superconformal invariance is the global R- symmetry, which in a fully fledged SCFT would correspond to the currents J and $\overline{J}$. We distinguish between the vector $U(1)_V$ and axial $U(1)_A$ R-symmetries generated by $J + \overline{J}$ and $J - \overline{J}$ respectively. One adds this ingredient by assigning charges ± 1 to $\theta^\pm$ w.r.t. the left-moving R-symmetry, and similarly for the barred coordinates, and assigning some a priori arbitrary vector and axial R-charges q_V^i and q_A^i to Φ^i. Clearly $U(1)_A$ and $U(1)_V$ are both symmetries of S_D, namely S_D has zero R-charges[3]. As for S_F one can realize $U(1)_A$ by assigning zero charges to the chiral superfields, while for the $U(1)_V$ symmetry we see that e.g. $d^2\theta^-$ has charge -2. Hence W must have charge 2, namely it must be a quasi-homogeneous function of Φ of degree 2:

$$W(\lambda^{q_V^i} \Phi^i) = \lambda^2 W(\Phi).$$

For the R-symmetry to be preserved also at the quantum level, the path-integral measure has to be invariant. Thus we resort to the field content (2.3.1). Clearly the $U(1)_V$ symmetry is preserved, while the $U(1)_A$ symmetry is generically anomalous due to a generical

[3] In fact there is a subtlety concerning the four-Fermi term, which is viewed as a an additional perturbation away from the large volume limit.

discrepancy in the number of left-Weyl and right-Weyl zero-modes. This discrepancy is the index of the Dirac operator and is determined via the Atiyah-Singer index theorem to be:

$$k = dim_{\mathbb{C}}(M)(2g-2) + 2\int_{X(\Sigma)} c_1(T^{(1,0)}M). \tag{2.3.2}$$

Thus we observe that this discrepancy depends on the topology of M, of X and of Σ, and that for these fixed, the $U(1)_A$ symmetry is broken to $\mathbb{Z}_k$. Having analyzed the R-symmetry, we will say a few words about scaling symmetry, which together with rotational invariance, is often sufficient to ensure full conformal invariance in two dimensions (see e.g. [23]). What is rescaled is the intrinsic metric on the Riemann surface $h \mapsto \lambda^2 h$. For the action to remain a scalar under this transformation, $z \mapsto \lambda z$ and $\theta^\pm \mapsto \sqrt{\lambda}^{-1}\theta^\pm$ and similarly for the complex conjugates. If we restrict attention to the D-term, and calculate the beta function $\beta(g)$ for the Kähler metric at one-loop, in other words in the large-volume limit, we will find $\beta(g) = 0$, hence scale invariance, if the Ricci tensor $R_{\mu\overline{\nu}}$ vanishes. In fact, one can obtain an exact result to all orders that for genus $g = 1$ is

$$c_1(M) = 0.$$

This reveals an intimate relationship between axial R-symmetry and scaling symmetry that can be understood as a consequence of supersymmetry [4]. If, instead, we restrict attention to the F-term, we see that the measure scales like λ while W is a priori unchanged. This fact indicates that in the renormalization group flow of the Landau Ginzburg theory each field should acquire a scaling dimension. This in turn would affect the D-term. At this point one uses the F-term non-renormalization theorem, that says that the F-term does not get renormalized up to wave-function renormalization, and in particular, any renormalization of the D-term does not affect the F-term [38, 84]. In conclusion, Landau Ginzburg models are not by themselves potential conformal field theories, but it is believed that the IR-fixed point of such a theory under RG flow is one (here we only give the earlier references [47, 51, 68, 95]).

Now, let us analyze what specific properties W should have in order to absorb the factor of λ in the measure after having given weights to the chiral fields. Again we obtain quasi-homogeneity:

$$\lambda W(\Phi^i) = W(\lambda^{\Delta_i}\Phi_i).$$

This can be achieved, as we saw earlier, by setting the total left-right dimension (possible conformal weight) of the chiral primary fields to half of their $U(1)_V$ charge. What this relation tells us, is that given (2.2.3), the Φ^i's are good candidate generators for the (c, c) ring, while the same argument shows that $\overline{\Phi}$ is related to the (a, a) ring. In the next section we will explain how to reduce the general Landau-Ginzburg theory to a TQFT whose space of states V (as we denoted it in 2.1) is the space of (c, c) highest weight states of the hypothetical IR fixed point of the Landau Ginzburg theory. This procedure goes by the name of B-type topological twist.

[4] A proper understanding of this fact requires analysis of the A-model, which we will not pursue (see e.g. [46]).

2.4 B-type topological twist

The aim is twofold: given a $N = (2,2)$ conformal field theory with Hilbert-space $\mathcal{H}$, to express the space of (c,c) primary states as the cohomology of an operator Q on $\mathcal{H}$, and to construct a topological field theory such that

$$Z(S^1_+) =: V = H_Q(\mathcal{H}).$$

We restrict attention to the left-moving sector. We observed that $|\phi\rangle_c$ is annihilated by two nilpotent operators $G^+_{-1/2}$ and $G^-_{1/2}$, that are adjoints of each other. Therefore $\mathcal{H}$ has a Hodge decomposition:

$$\mathcal{H} = Harm \oplus G^+_{-1/2}\mathcal{H} \oplus G^-_{1/2}\mathcal{H}.$$

The space of chiral highest weight states is identified with $Harm$, that is the space of Harmonic states with laplacian $\Delta = [G^+_{-1/2}, G^-_{1/2}] = 2L_0 - J_0$. We can view these as representatives of the cohomology of $Q_L := G^+_{-1/2}$. Then, including the right-moving sector, we obtain that the space of (c,c) states is isomorphic to the cohomology $H_Q(\mathcal{H})$, where:

$$Q = Q_L + Q_R$$

and $Q_R = \overline{G}^+_{-1/2}$. To build a topological field theory out of this data, one requires that the energy-momentum tensor thereof be Q-exact, so that correlation functions in that theory are independent of the two-dimensional metric. The obvious choice is $\tilde{T}$, with left-moving modes:

$$\tilde{L}_n = \frac{1}{2}[Q, G^-_{1/2+n}] = L_n - \frac{(n+1)}{2} J_n.$$

The modes of the new energy-momentum tensor form a Virasoro algebra with vanishing central charge. The spin of Q (and conformal weight) vanishes also, i.e. it becomes a scalar making it a well defined symmetry generator also on curved Riemann surfaces. Given a Riemann surface Σ one chooses an arbitrary generator $\gamma \in H_1(\Sigma)$, then:

$$Q = \oint_\gamma (dz G^+ + d\overline{z}\overline{G}^+).$$

It is straightforward to check, by differentiating w.r.t. the world sheet metric, that the map $L_n \mapsto \tilde{L}_n$ induces the following map on correlation functions:

$$\langle \,\cdot\, \rangle_\Sigma = \langle \,\cdot\, \exp\left(-\frac{1}{2}\int_\Sigma A \wedge (Jdz - \overline{J}d\overline{z})\right)\rangle_{\Sigma,CFT} \tag{2.4.1}$$

where A is the spin connection. These define the twisted theory.

It is important to stress that the exponential in (2.4.1) together with a possible insertion are path ordered, so as to correspond to the simple introduction of the local classical coupling of spin to the spin connection in the path-integral formulation. In order to understand the relationship between the physical and topological correlation functions, we shall consider $\Sigma = S^2$, use bosonization (2.2.1) and integrate by parts. We obtain that the inserted operator equals:

$$\exp\left(-\frac{i}{2}\sqrt{\frac{c}{3}}\int_{S^2} R_{z\bar{z}}\,\phi\,dz\wedge d\bar{z}\right).$$

Now we can use the fact that the topological correlation functions are independent of the world sheet metric and judiciously choose a one-parameter family, with parameter t, of metrics that give S^2 the shape of a cylinder capped off by two hemispheres, so that in the limit $t \to \infty$ the Ricci tensor peaks at the two ends to two delta functions while vanishing elsewhere. Let us consider inserting a chiral primary field ϕ_k in the center of the left hemisphere at e_L, a second one ϕ_j on the cylinder at p and a third ϕ_i at e_R. Then their topological correlator is given by:

$$\lim_{t\to\infty}\left\langle \phi_i(e_R)\exp\left(-\frac{i}{2}\sqrt{\frac{c}{3}}\langle\phi\rangle_{D^R_{t^{-1}}}\right)e^{-iHt_p}\phi_j(p)e^{-iH(t-t_p)}\exp\left(-\frac{i}{2}\sqrt{\frac{c}{3}}\langle\phi\rangle_{D^L_{t^{-1}}}\right)\phi_k(e_L)\right\rangle_{S^2}, \tag{2.4.2}$$

where by $\langle\phi\rangle^L_{t^{-1}}$ we mean the average value of ϕ on the vanishing disk on the left and similarly for the disk on the right. We find the spectral flow operator (2.2.2) acting on incoming and outcoming chiral primary states, thus transforming them to Ramond ground states $|i\rangle$ and $|j\rangle$. This is no surprise, since as it was true for the super-charge Q, the map $L_0 \mapsto \tilde{L}_0$ transforms NS boundary conditions to R boundary conditions. In more compact notation, therefore, we have that the topological three point function is given by:

$$\langle\phi_k\phi_j\phi_i\rangle_{S^2} = \langle k|\phi_j|i\rangle_{S^2} =: C_{ijk}$$

and is obviously independent of the points of insertion. This can either be viewed in the CFT context as a consequence of the $SL(2,\mathbb{C})$ invariance of the vacuum, or more simply due to topological invariance of the twisted theory. In order to see that the twisted theory is indeed a topological field theory as defined in section 2.1, we must assume that the topological metric:

$$\eta_{ij} := \langle j|i\rangle_{S^2}$$

is non-degenerate. Then we have all the other ingredients. For example, the unit is given by:

$$e : \mathbb{C} \ni 1 \mapsto |0\rangle := U_{1/2}|0\rangle_{NS}$$

the trace is given by:

$$\theta : V \ni |i\rangle \mapsto \langle 0|i\rangle_{S^2}$$

and the multiplication is given by:

$$\mu : V\otimes V \ni |i\rangle\otimes|j\rangle \mapsto C^k_{ij}|k\rangle,$$

where C^k_{ij} are the structure constants of the (c,c) ring. In order to assert that the above data defines a 2d closed TQFT, we only need to check that $\eta = \theta\circ\mu$. Indeed:

$$\eta_{ij} = \langle\phi_j\phi_i\rangle = C^k_{ij}\langle 0|k\rangle_{S^2}.$$

With the same effort we obtain:

$$C_{ij}^k = \eta^{kr} C_{rij}.$$

The above however does not suffice to assert that the topological correlation functions on higher genus Riemann-surfaces can be obtained from the genus zero ones via the TQFT prescription. Showing this is tantamount to showing that the correlation functions of the twisted theory enjoy a certain factorization property. It can be shown in particular that (see e.g. [24])

$$\langle \phi_i \phi_j \rangle_{\Sigma_g} = \langle \phi_i | r \rangle_{\Sigma_1} \eta^{rs} \langle s | \phi_j \rangle_{\Sigma_{g-1}}$$

for every genus $g \geq 1$. The way one does this is by choosing a metric on Σ_g that makes the tube linking Σ_1 and Σ_2 infinitely long so that the time-evolution operator along the tube becomes the projector onto the Ramond ground states. Lastly, the correlation functions on the torus satisfy:

$$\begin{aligned} \langle \phi_i \phi_j \rangle_{T^2} &= (-1)^{|l|(|k|+|i|+|j|)} \eta^{lk} \langle k | \phi_i | r \rangle_{S^2} \eta^{rs} \langle s | \phi_j | l \rangle_{S^2} \\ &= (-1)^{|l|(|k|+|i|+|j|)} C_{ir}^l C_{jl}^r. \end{aligned}$$

This is again obtained by making two strands of the torus infinitely long.

2.5 TQFT data of the B-type models

We now go back to the LG-model and we ask under what conditions we can B-twist it. The previous discussion regarded the twisting of $N = 2$ conformal field theories. In this case we don't have a conformal field theory, but we notice that in order to define the twisting, in the end (2.4.1), we only needed $J dz - \overline{J} d\bar{z}$, that is, the Noether current of $U(1)_A$. That is we need $U(1)_A$ to be a symmetry of the theory on flat world-sheets. Therefore, while M has to have $c_1(M) = 0$, in principle W is not required to be quasi-homogeneous, as that is required by $U(1)_V$ invariance. This doesn't mean however that the resulting theory will be topological anyway, in particular the resulting space V could be infinite-dimensional. The spin content with respect to $\tilde{L}_0 - \tilde{\overline{L}}_0$ of the twisted LG theory, defined by (2.3.1), is now changed compared to the untwisted theory. Comparing with (2.3.1), now:

$$\begin{aligned} &X \in C^\infty(\Sigma, M) \\ &\psi \in \Gamma(\Sigma, X^* TM^{1,0} \otimes (T^{1,0}\Sigma)^\vee) \\ &\overline{\psi} \in \Gamma(\Sigma, X^* TM^{1,0} \otimes (T^{0,1}\Sigma)^\vee) \\ &\chi \in \Gamma(\Sigma, X^* TM^{0,1}) \\ &\overline{\chi} \in \Gamma(\Sigma, X^* TM^{0,1}). \end{aligned} \tag{2.5.1}$$

Therefore twisting in this respect really just amounts to twisting the bundles by appropriate spin-bundles. We can however still view them as sections of the previous bundles introducing the coupling to the spin connection in the correlator. If we do that we see that we inherit the $U(1)_A$ anomaly for curved Riemann surfaces, as the additional piece in the action is symmetrical. We shall now sketch how to obtain the closed topological field theory associated to the Landau Ginzburg model. First of all we observe that we can choose representatives of chiral primary fields that depend only on the spinless fields X, $\overline{\psi}$ and χ. The other fields give Q-exact contributions, because on them the action of $[L_0, \cdot]$ is invertible. Therefore, since L_0 is $[Q, \cdot]$ exact, then so will be the space of non spinless fields. It is convenient to express $\overline{\psi}$ and χ in terms of

$$\eta^{\bar{i}} = \chi^{\bar{i}} + \overline{\chi}^{\bar{i}}, \qquad \theta_i = g_{i\bar{i}}(\chi^{\bar{i}} - \overline{\chi}^{\bar{i}}).$$

Therefore a chiral primary field is of the form:

$$O_\alpha = \alpha^{\bar{j}_1,\dots,\bar{j}_n}_{\bar{i}_1,\dots,\bar{i}_m}(X)\eta^{\bar{i}_1}\cdots\eta^{\bar{i}_m}\theta_{i_1}\cdots\theta_{i_n}.$$

The action of the B-supercharge defined by $\delta = \epsilon[Q, \cdot]$ is given by:

$$\delta X^i = 0, \qquad \delta X^{\bar{i}} = \epsilon\eta^{\bar{i}}, \qquad \delta\theta_i = -\epsilon\partial_i W,$$
$$\delta\overline{\psi}^i = 2i\epsilon\overline{\partial}X^i, \qquad \delta\psi^i = -2i\epsilon\partial X^i.$$

By fermionic localization, the path-integral localizes on the fixed points of δ. One localizes in two steps (the original reference is [94], however we follow [41]). First one concentrates on the equation:

$$dX^i = 0$$

whose solutions are constant maps. At this level we can draw a one to one correspondence between the operators O_α and geometrical objects on M. We have the following identification:

$$\eta^{\bar{i}} \leftrightarrow dX^{\bar{i}}, \qquad \theta_i \leftrightarrow \partial_i.$$

Therefore, after the first localization we have the following complex computing V:

$$\left(\Gamma(M, \Lambda^\bullet T^{(1,0)}M \otimes \Lambda^\bullet (T^{(0,1)}M)^\vee), \overline{\partial} + [W, \cdot]\right), \tag{2.5.2}$$

where $[\cdot, \cdot]$ is the extension of the Schouten-Nijenhius bracket to polyvectorfield-valued differential forms. Given two polyvectorfields of the form $\xi = \xi^1 \wedge \cdots \wedge \xi^m$ and $\zeta = \zeta^1 \wedge \cdots \wedge \zeta^n$:

$$[\xi, \zeta] := \sum_{i,j}(-1)^{i+j}[\xi^i, \zeta^j]\xi^1 \wedge \cdots \wedge \widehat{\xi^i} \wedge \cdots \wedge \xi^m \wedge \zeta^1 \wedge \cdots \wedge \widehat{\zeta^j} \wedge \cdots \wedge \zeta^n.$$

While for two polyvectorfield-valued differential forms $\xi\alpha$, $\zeta\beta$:

$$[\xi\alpha, \zeta\beta] = (-1)^{|\alpha||\zeta|}[\xi, \zeta]\alpha \wedge \beta. \tag{2.5.3}$$

The bracket satisfies the graded Jacobi-identity, therefore it endows the complex (2.5.2) with the structure of a graded Lie-algebra. In fact, including the differential, the complex becomes a differential graded Lie-algebra, which simply means that the differential satisfies the Jacobi-identity w.r.t. $[\cdot,\cdot]$. We will describe the importance of this structure in the following section when we consider spaces of topological field theories. For our immediate purposes however we notice that we can already extract the candidate (c,c) ring from (2.5.2). Namely, it is the cup product $\wedge$ descended to cohomology.

Now we proceed to the second step of localization. In particular we will describe how the path-integral localizes on the sphere, allowing us to extract the desired TQFT data. After the first step, the path-integral already reduced the following integral over zero-modes:

$$\langle O_\alpha \rangle_{S^2} = \mathcal{N} \int dX d\eta d\theta \exp\left(-\frac{A}{4}\delta(g^{\bar{i}i}\partial_{\bar{i}}\overline{W}\theta_i)\right) O_\alpha,$$

where A is the surface area of S^2 and $\mathcal{N}$ is an irrelevant normalization factor[5]. As desired, the integral does not depend on the world sheet metric thanks to the exactness of the exponent. The integration of the fermonic fields just extracts the highest degree in polyvectorfields and differential forms and in more geometrical notation one obtains:

$$\langle \alpha \rangle_{S^2} = \mathcal{N} \int_M \Omega \wedge \left(\Omega \vee \exp\left(-\frac{A}{4}\delta(g^{\bar{i}i}\partial_i \wedge \partial_{\bar{i}}\overline{W}\right) \wedge \alpha\right),$$

where Ω is a holomorphic top-form. At this point we distinguish in particular the case $W = 0$. For this, clearly localization on the zero-modes suffices.

In the case $W \neq 0$ it is assumed that W has isolated critical points and that the offshell-complex (2.5.2) simplifies. In particular the cohomology w.r.t. $\overline{\partial}$ vanishes. Then the above integral is evaluated in the IR-limit $A \to \infty$. The exponent is made of two terms: one proportional to the Hessian $H_{ij} = D_i \partial_j W$ of W (where D denotes the Levi-Civita covariant derivative) and one proportional to $|\partial W|^2$, which, as predicted by fermionic localization, localizes the path integral and fields onto the critical locus of W. At this point one assumes that W has non-degenerate critical points, an assumption we will later be able to discard. One can then use steepest descent: expand $|\partial W|^2$ up to second order around the critical locus and evaluate the gaussian integral. One thus expects a sum over the critical locus with an integral inversely proportional to the determinant of H_{ij} or more precisely to the product of the holomorphic and anti-holomorphic hessian arising in the expansion of $|\partial W|^2$. In fact, the denominator will be proportional to a power of A and one picks the highest power in the hessian coming from the first term in the exponent to cancel the A dependence. This leads to the fact that the only non-vanishing correlator is for $\alpha = f$ a function on M and:

$$\langle f \rangle_{S^2} \propto \sum_{p \in Crit W} \frac{f(p)}{|\det H_{ij}(p)|}. \tag{2.5.4}$$

The above is well defined thanks to the assumption that W has non-degenerate critical points. To relax this assumption one writes (2.5.4) in the more invariant form:

$$\langle f \rangle_{S^2} \propto \int_\Gamma \frac{f\,\Omega}{\partial_1 W \wedge \cdots \wedge \partial_n W},$$

[5] Notice the absence of ψ and $\overline{\psi}$. This is due to the fact that on S^2 there are no such zero-modes other than zero.

where Γ is a contour encircling the critical points. Now it is possible to relax the assumption of non-degeneracy by viewing a potential with degenerate critical points as a deformation of a potential with non-degenerate ones such that the asymptotic behavior is unperturbed so that integrating over a large enough contour leaves the result unchanged.

We will restrict attention to the case of isolated critical points. Such is the case for a quasi-homogeneous W. We refer to [37] for the non-degeneracy of the above trace on the cohomology of (2.5.2). It is worth remarking that the residue formula, indicates that the LG model with isolated critical point only "sees" the geometry of the Calabi-Yau in the immediate neighborhood of the critical point. This can be viewed as a consequence of the F-term non-renormalization theorem.

Coming back to the case $W = 0$, the trace is non-degenerate if $c_1(M) = 0$, that is if M is a Calabi-Yau manifold. This follows from the fact that for Calabi-Yau manifolds the trace comes from the analogue of Poincaré duality on Dolbeaux cohomology, that is Serre duality, which asserts that there is a non-degenerate pairing:

$$H^p_{\overline{\partial}}(E) \otimes H^{n-p}_{\overline{\partial}}(E^\vee \otimes K_M) \to \mathbb{C}$$

for a holomorphic bundle E and K_M denotes the canonical bundle. In our case $E = \Lambda^\bullet T^{(1,0)}M$ and we would like $E^\vee \otimes K_M$ to be replaced with E. Indeed $c_1(M) = 0$ iff $K_M = \underline{\mathbb{C}}$, which implies the existence of a nowhere vanishing holomorphic top form. Contracting with it, in turn, gives an isomorphism between $H^p(E)$ and $H^p(E^\vee)$. Indeed via this procedure one recovers the TQFT pairing from the Serre pairing.

We now concentrate on the special case of affine LG models with W having an isolated critical point. The vanishing of Dolbeaux cohomology allows to reduce the complex of zero modes to a smaller one. More precisely the cohomology of $\overline{\partial}$ is all concentrated in degree zero, yielding holomorphic polyvectorfields. Given that, one views (2.5.2) as a double complex and begins computing its cohomology via the spectral sequence whose first page is $H_{\overline{\partial}}$. One finds that, because of the collapse of $H_{\overline{\partial}}$ just explained, the spectral sequence degenerate at the second page. Therefore (2.5.2) is reduced to the simpler complex:

$$\left(\Gamma(X, \Lambda^\bullet T^{1,0}X), [W, \cdot]\right). \tag{2.5.5}$$

The fact that W has an isolated singularity is equivalent ([37]) to the condition that $\partial_1 W, \ldots, \partial_n W$ form a *regular sequence*. In general given a commutative ring R, a sequence of elements $r_1, \ldots, r_k \in R$ is regular if r_j is a non-zero divisor in $R/(r_1, \cdots, r_{j-1})$. In our case the ordering of the sequence doesn't matter. The intuitive way of understanding this definition is that the spaces defined by $\partial_i W = 0$ cut each other transversally, thus minimizing the dimension of their intersection down to an isolated point.

As a consequence of this property, in fact equivalently, the cohomology of (2.5.5) is concentrated in degree zero, and is thus given by the Jacobian ring:

$$\mathrm{Jac}(W) := \frac{\mathbb{C}[x^1, \ldots, x^n]}{(\partial_1 W, \ldots, \partial_n W)}.$$

We now discuss how to build moduli spaces of TQFT's arising from $N = 2$ CFT's.

2.6 Spaces of TQFT's

Here we are interested in investigating the general structure of spaces of TQFT's arising from twisted $N = (2, 2)$ CFT's. One constructs these spaces by starting from one point on them, that is a TQFT, and deforming away from it in all possible directions. In order to understand what this should mean, we recall that TQFT's are the same as commutative Frobenius algebras, and in this context $V = H_Q(\mathcal{H})$. In particular Q is a scalar, and for example, for B-twisted theories it has $U(1)$ charges $(1, 1)$. Deforming such theories should involve somehow deforming $\mathcal{H}$, Q and the Frobenius structure such that along the deformation we still obtain TQFTs. In practice, one keeps $\mathcal{H}$ fixed. We can understand this as keeping the field content of a family of TQFT's fixed at the starting point while the action is perturbed. Moreover, if the $U(1)$ charges are quantized, then the deformed Q must also have, say, charges $(1, 1)$ w.r.t. the starting theory. Therefore, in particular in B-twisted theories, we look a priori for all possible scalar fields Φ of charges $(1, 1)$ such that the following *Maurer Cartan* equation is fulfilled:

$$[Q + \Phi, Q + \Phi] = 0 \tag{2.6.1}$$

however we mod-out by trivial deformations that arise by a change of basis in $\mathcal{H}$. That is we mod out by $Aut(\mathcal{H})$. This is simply the finite as opposed to infinitesimal version, of taking cohomology. We see however that in order to define (2.6.1) we have resorted to the Lie-algebra structure on the space of vertex operators, which together with $[Q, \cdot]$ becomes a DGLA (differential-graded Lie algebra). In practice however one looks for a smaller DGLA that has an isomorphic deformation problem. There is a precise perturbative notion of this, and it leads to the concept of L_∞ quasi-isomorphism. We will introduce and study this structure in chapter 5. In the case of LG-models one such smaller DGLA is (2.5.2). In particular, in the B-model ($W = 0$), the chiral fields have zero $U(1)_V$ charge, therefore $\Phi \in \Gamma(X, T^{1,0}X \otimes (T^{(0,1)}X)^\vee)$ and (2.6.1) becomes:

$$[d\overline{z}^i \wedge \partial_{\bar{i}}, \Phi] + \frac{1}{2}[\Phi, \Phi] = 0. \tag{2.6.2}$$

In the case of the affine LG model, generically $U(1)_V$ invariance is lost, thus one deforms with, a priori, arbitrary elements of (2.5.2). If, however W is a quasi-homogeneous polynomial, the chiral fields are assigned quantized charges, and one deforms again with appropriate ones. In practice one solves (2.6.1) perturbatively. The assumption is that the family of TQFT's connected to a given starting point is smooth, and as such the tangent space at the starting point has the same dimension as the space of TQFT's itself. This notion of dimension of the moduli space is called *virtual dimension*. This assumption is in fact valid for closed TQFT's, in fact as we will sketch later, defining deformations from a CFT point of view, one obtains a smooth moduli space. Then every deformation is a lift of an infinitesimal deformation. If we have a one-parameter family of such deformations $\Phi(t)$, its derivative at $t = 0$ is a non-trivial element in $H_Q(\mathcal{H})$ with appropriate charges. In the case of an affine LG model with isolated critical point, first order deformations trivially

lift to all orders, because the Schouten-Nijenhius bracket of two functions trivially vanishes (more on this in section 5.2.1). For the B-model, the lack of obstructions was proved in [92, 93], see also [5].
From the CFT point of view, from a deformation ϕ, one constructs an F-term deformation by integrating the chiral primary field $\Delta W(\phi)$ with ϕ as lowest component field. Explicitly the other components of the superfield are given by the *descendants* $\phi^{(1)}$ and $\phi^{(2)}$, which in the B-twisted case are:

$$\begin{aligned} \phi^{(1,0)} &:= [G^-_{-1}, \phi] dz, \qquad \phi^{(0,1)} := [\overline{G}^-_{-1}, \phi] d\overline{z} \\ \phi^{(2)} &:= \phi^{(1,1)} = [G^-_{-1}, [\overline{G}^-_{-1}, \phi]] dz \wedge d\overline{z} \end{aligned} \tag{2.6.3}$$

Then:

$$\int d^2 z d^2 \theta^- \Delta W(\phi_i) = \int_{S^2} d^2 z \phi^{(2)}$$

The fact that ϕ is Q-closed then ensures that the integral above is Q-closed, or in other words, that $\Delta\Phi$ is indeed a chiral primary field. If one includes the descendant fields in the space of observables, then one passes from TQFT to TCFT. One can thus view a single TCFT as describing a space of TQFT's.

The F-term perturbs the correlation functions of the TQFT as:

$$\langle \cdot \rangle \mapsto \langle \cdot \, \exp\left(t^i \int d^2 z d^2 \theta^- \Delta W(\Phi_i) + c.c. \right) \rangle = \langle \cdot \, \exp\left(t^i \int d^2 z d^2 \theta^- \Delta W(\Phi_i) \right) \rangle.$$

One can consider more general deformations by arbitrary chiral primaries. Then the *moduli* t^i correspond to a basis of the (c, c) ring and the complex conjugate (a, a) deformations vanish, because they are Q-exact. Notice that just as the corresponding chiral primaries, the moduli are graded-commutative. However the ones corresponding to exactly marginal deformations are commutative. One can now define the t-dependent three point functions $C_{ijk}(t)$ out of which the t-dependent TQFT can be extracted. Using the invariance of the Ramond vacuum (i.e. the Ward identities) under G_{-1}, G_0, G_1 and their right-moving counterparts, one arrives [24] at the following differential equations:

$$\partial_i C_{jkl} = \partial_l C_{jki} \tag{2.6.4}$$

$$\partial_i C_{0kl} = \partial_i \eta_{kl} = 0, \tag{2.6.5}$$

where we have restricted ourselves to commutative moduli. Equation (2.6.4) implies the existence of a function $\mathcal{F}(t)$ such that:

$$C_{ijk}(t) = \partial_i \partial_j \partial_k \mathcal{F}(t),$$

while (2.6.5) is the statement that the topological metric is flat in these so called *special coordinates* t^i. The above equations define the notion of a *Frobenius Manifold*, i.e. a flat family of Frobenius algebras (see [28]). Clearly, the CFT definition of deformation is more refined than that through the Maurer Cartan equation (2.6.1). While the latter is only concerned with deformations of Q, the former also takes into account the deformation of the remaining TQFT structure, namely the trace θ, and the multiplication μ. However given a deformation of Q, we can immediately read off a deformation of θ. We will obtain in particular the metric η written in non-flat coordinates. Similarly, the partial derivatives will be replaced as:

$$\partial_i \longrightarrow \nabla_i$$

where ∇_i is the flat Levi-Civita connection of η. Hence the problem of solving for flat coordinates becomes the one of mapping to Riemann-normal coordinates. This procedure is particularly simple for affine Landau Ginzburg models with an isolated singularity. As discussed previously, there one can choose a one parameter family of potentials $W(x, \tilde{t})$ of the form:

$$W(x, \tilde{t}) = W(x) + \sum_{i=1}^{k} \tilde{t}^i \phi_i(x)$$

where ϕ_i is an arbitrary basis of Jac(W). In the next section we review the structure of topological anti-topological fusion as a preliminary to chapter 3 where in particular we will show how this structure restricted to the exactly marginal sector emerges from quantization.

2.7 Topological anti-topological fusion

As we described in the previous section, the geometry of the moduli space of $TQFT$'s arising from $N = (2, 2)$ CFT's is that of a Frobenius manifold. Crucial to our investigation in chapter 3, is the geometry of the combined (c, c) and (a, a) moduli-spaces. We shall thus briefly review the results of [15] on the more general combined deformation space that includes all chiral primary fields as opposed to only the marginal ones. In the end, however, we will restrict attention only to the latter. As we discussed in section 2.2, given a $N = (2, 2)$ CFT, Ramond ground states can be viewed as either chiral primary states flowed by $U_{1/2}$ or as antichiral primary states flowed by $U_{-1/2}$. We also observed however in the case of the B-type twist, that the topological twist has the same effect as spectral flow at the topological level, but its implementation can be generalized to non conformal theories, as long as an appropriate global R-symmetry is present. Following the computation in (2.4.2), we observe that in general, the analogue of Ramond ground states is obtained by inserting a chiral primary field at the tip of a hemisphere of the infinitely stretched sphere and by including the coupling of the (in the case of the B-twist) axial current to the spin connection. In this way one can obtain a basis of ground state, say, $|i\rangle$. One could also insert (a, a) fields instead, and couple to the axial current with the opposite sign. In this way one obtains a basis that we shall denote by $|\bar{i}\rangle$, where ϕ_i and $\overline{\phi}_i$ are intended as CPT conjugates of each other. These two bases are then related by a CPT matrix. Using the convention of [15]

$$\langle \bar{j}| = \langle i| M^i_{\bar{j}}.$$

CPT is an involution and, since it acts on $\langle\bar{j}|$ by the complex conjugate M^*:

$$MM^* = 1.$$

As a consequence, one obtains a hermitian metric:

$$\langle\bar{j}|i\rangle =: g_{i\bar{j}} = \eta_{ik} M^{k}_{\bar{k}}.$$

At this point one is interested in the dependence of η, g and C and their complex conjugates on (c,c) and (a,a) parameters t^i and $\bar{t}^i$ respectively, i.e. one perturbs by inserting the F term deformation and its conjugate introduced in the previous section. To study this, one introduces a connection A subject to the requirement that its associated covariant derivative D maps states $|i\rangle$ to states orthogonal to all ground states, in the same way as one proceeds in standard quantum mechanics perturbation theory. The path-integral computation on the infinitely extended hemispheres shows that A is in fact in the gauge:

$$A_i = g^{-1}\partial_i g, \qquad A_{\bar{i}} = 0.$$

The vanishing of the term on the right is a consequence of the fact that topological (i.e. purely (c,c)) correlators are independent of $\bar{t}^i$. The result of [15] is that these structures are related by the so called *tt^*-equations*, which can be understood as the requirement that the following two connections:

$$\hat{\nabla}_\alpha := D + \alpha C$$
$$\hat{\overline{\nabla}}_\alpha := \overline{D} + \alpha^{-1}\overline{C}$$

are flat for arbitrary value of the parameter α. These equations extend the flatness of the purely topological sector, for which note that the combined equations (2.6.4, 2.6.5) are equivalent to the flatness of:

$$\nabla_\alpha = \nabla + \alpha C$$

for arbitrary α. The connection $\hat{\nabla}_\alpha$ is called the *Gauss-Manin connection*, and, just as ∇ (or ∇_α), allows to define parallel transported ground states and thus in principle solves the deformation problem. At this point, from this very general construction for arbitrary $N = 2$ theories, one can restrict oneself to the case of TCFT's and in particular to the marginal sector, expecting to recover a metric on half (since we B-twisted, deformation by the remaining twisted F-terms are trivial) of the moduli space of $N = (2,2)$ CFT's. Thus one expects agreement with the general definition of a metric on spaces of 2d QFT's of [106]. We will be solely interested in the case of $N = (2,2)$ CFT's with central charge $c = 9$, which in the case of sigma models corresponds to Calabi-Yau's of complex dimension 3. In that case, the axial anomaly in the trace θ is -6 (see (2.3.2)). Therefore, while the three point function of three marginal chiral primaries with charges $(1,1)$ precisely cancels the anomaly and is thus non-zero, restricted to i,j marginal:

$$C_{0ij} = \eta_{ij} = 0.$$

Thus the only geometrical objects remaining on the sub-bundle of marginal chiral and anti-chiral primaries are:

$$g_{i\bar{j}}, \qquad C_{ijk}.$$

Again for reasons of axial R-symmetry selection rules:

$$g_{0\bar{i}} = 0.$$

Using the above equations and the fact that multiplying with marginal chirals raises charge by one unit, one shows that the restricted tt^*-equations are equivalent to the statement that the data:

$$G_{i\hat{j}} := \frac{g_{i\bar{j}}}{g_{0\bar{0}}}, \qquad C_{ijk}$$

and their conjugates define a *projective special Kähler manifold* and indeed $G_{i\hat{j}}$ coincides with the metric in [106]. We defer the precise definition of projective special Kähler manifold to later sections, as we are going to rediscover it within the structure of quantization. In the next section we will briefly recall the content of the holomorphic anomaly equation generalizing the tt^*-equations to higher genus (quantum) Riemann surfaces.

2.8 Quantum preliminaries and Holomorphic anomaly

First of all, from the string theoretical perspective, $g_{i\bar{j}}$ and C_{ijk} are classical objects, as they describe correlators on genus zero surfaces, and spaces of CFT's on genus zero surfaces in string theory replace the notion of moduli spaces of solutions to Einstein's equations, which in turn describe classical gravity. The recipe for quantum gravity in string theory is then to do string theory on higher genus surfaces. This recipe generalizes the concept that in perturbative quantization a (formal) expansion over Planck's constant is a (formal) expansion over the number of loops in the Feynman diagrams arising in path-integral quantization. The application of this method to the manifold of marginal chiral primaries culminates with the Holomorphic anomaly equation of [6] of which we will describe some salient features in a moment.

First it is worth mentioning the simplest property of special Kähler manifolds which indicates their "predisposition" to be quantized in the non-stringy sense. The simplest property of projective special Kähler manifolds is:

$$G_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log\langle\bar{0}|0\rangle.$$

In other words:

$$\langle\bar{0}|0\rangle = e^{-K}$$

where K is the Kähler potential of G. That is the metric is a representative of the the first Chern class of a hermitian line-bundle with hermitian form $\langle\bar{0}|0\rangle$. This bundle is by definition, the vacuum bundle $\mathcal{L}$. In other words the manifold is a *Hodge manifold*, and by the Kodaira embedding theorem, if compact, can be viewed as a projective variety.

Compactness is not important for our purposes, while as we will see in the future section, quantization itself can be viewed as the study of maps into a direct limit of projective spaces. Moreover, the way we will understand the above structure is as the first order in the expansion of the quantization map into infinite projective space. We shall shortly recall the main steps for the holomorphic anomaly equation. We start with explaining how to couple a topological conformal field theory to gravity. For this we will need to know the dimension of the complex structure moduli space of Riemann surfaces of arbitrary genus.

2.8.1 Preliminaries on $\mathcal{M}_{g,n}$

Just as for the case of Calabi-Yau manifolds discussed above, also here deformations of complex structures are solutions to (2.6.2) and the tangent space to $\mathcal{M}_g$ at a given point, that is a Riemann surface Σ_g is $H^1_{\bar{\partial}}(\Sigma_g, T^{(1,0)}\Sigma_g)$ with dimension denoted as $h^1(\Sigma_g, T^{1,0}\Sigma_g)$. The latter is then the dimension of the tangent space at Σ_g of the moduli space of complex structures $\mathcal{M}_g$. This virtual dimension of $\mathcal{M}_g$ can be computed via the Riemann-Roch theorem. More precisely this gives the Euler characteristic of $T^{1,0}\Sigma_g$:

$$h^1(\Sigma_g, T^{1,0}\Sigma_g) - h^0(\Sigma_g, T^{1,0}\Sigma_g) = 3g - 3.$$

The second term on the left-hand side is the space of global sections of $T^{1,0}\Sigma_g$. Thus holomorphic vector-fields in $h^0(\Sigma_g, T^{1,0}\Sigma_g)$ are precisely the infinitesimal generators of continuous automorphisms of Σ_g. Only Σ_0 and Σ_1 have continuous automorphisms. The former are the Möbius transformations forming a three dimensional Lie-group, while the latter is composed of one generator describing holomorphic translations along the (abelian) elliptic curve.

An object with a continuous family of automorphisms is termed *unstable*. In this case the unstable surfaces are therefore the ones with genus 0 and 1. Stabilizing these surfaces means "freezing" the continuous automorphisms. In the case of the sphere, this requires the introduction of three marked points, while in the case of the elliptic curve, the introduction of one. One then has well defined moduli spaces $\mathcal{M}_{g,n}$ of genus g Riemann surfaces with n marked points, where $n \geq 0$ for $g \geq 2$ while $n \geq 1$ for $g = 1$ and $n \geq 3$ for $g = 0$.

2.8.2 Coupling to gravity

At fixed topological surface Σ, passing from conformal field theory to string theory means integrating over two dimensional metrics while gauging the group $Diff \times Weyl$ of diffeomorphisms times Weyl rescalings of the metric. At first sight, if we start from a topological conformal field theory already, rather than from a general CFT, it seems almost superfluous to integrate over metrics to then gauge away diffeomorphisms etc.. However even in the simplest case of the "sigma model with target a point", one obtains a very interesting result, namely two dimensional gravity itself, which has as action only the topological Gauss-Bonnet term. The complexity in that case all resides in the non-trivial topology of the moduli-space of metrics (by $Diff \times Weyl$). In particular the observables in that theory are cohomology classes of that space (see e.g. [25, 54, 101]). Now, in formulating the coupling to gravity of a general conformal field theory, one can use the BRST technique, whereby one first fixes a gauge for the metric w.r.t. $Diff$, leaving the integral over the moduli space and residual gauge freedom corresponding to conformal transformations combined with Weyl rescalings. At least on the sphere, one can completely fix the metric by diffeomorphisms up to a positive scalar function which can be viewed as the exponential of a so called *Liouville field*. Whether or not one can gauge away Weyl rescalings depends on whether one can gauge the residual conformal transformations and this happens if the central charge of the matter theory is 26. In that case, if we denote by $\mathfrak{g}$ the Virasoro algebra with generators X_i and structure constants f_{ij}^k (for ease of notation) of the matter theory, the BRST charge:

$$Q_{BRST} = c^i X_i + f_{ij}^k c^i c^j b_k$$

squares to zero, where $\{c^i, b_j\} = \delta^i_j$ are the Ghost fields who form a conformal field theory with energy-momentum tensor twice the second term on the right-hand side above, that has indeed central charge -26. All of the above to say, that this procedure is not good in our case of a topological conformal field theory, because there the central charge vanishes. On the other hand this is what made it topological in the first place thus suggesting an easier way. This easier way adopted in e.g. [25] indeed exists and is motivated by the fact that in the case of the "CFT of a point", one could have used a different complex than the BRST complex to obtain the same result. This is because, since in that case the theory is topological, one expects the observables to encode only topological information of the space one is integrating over, and thus one could use the complex computing the cohomology of that space. Since the space is the quotient by a group, it should be a complex computing equivariant cohomology. Thus instead of the BRST complex, ultimately we should use a complex that computes the equivariant cohomology of the matter TCFT Hilbertspace $\mathcal{H}$ by the Virasoro algebra. There are many such complexes as the Weil-complex or Cartan-complex etc..., however as pointed out in [17] under further suggestion, one such complex is given by the BRST complex again, but of the supersymmetrized version of $\mathfrak{g}$:

$$\mathfrak{g} \oplus \mathfrak{g}[1],$$

where [1] denotes parity shift. Therefore, to the b, c system one adds the β, γ system. Together these form a TCFT, and in particular the total central charge is zero. Therefore one can trivially gauge away the Liouville field, and in addition to the bosonic moduli of 2-d metrics, one also has the supersymmetric partners. Thus the path integral over a genus $g \geq 2$ surface, in particular, takes the form:

$$Z = \int \prod_{i=1}^{3g-3} dm^i d\overline{m}^{\bar{j}} \prod_{i=1}^{3g-3} d\hat{m}^i d\overline{\hat{m}}^{\bar{j}} \langle \exp(-S_{matter+Ghosts}(m, \hat{m})) \rangle_{\Sigma}.$$

The integral over fermonic moduli is the highest derivative with respect to them, and in particular, in the same way as:

$$\frac{\partial}{\partial m^i} S = (T, \mu_i)$$

where T is the holomorphic part of the energy momentum tensor,

$$\frac{\partial}{\partial \hat{m}^i} S = (G, \mu_i)$$

where G is again a spin 2 super partner of T, namely the one for which $T = 1/2[Q, G]$ and depends on the chosen twist. In the case of the B-twist the latter is G^- and $\overline{G}^-$ for the complex conjugates. The objects μ_i we have introduced above denote basis elements of $H^1_{\overline{\partial}}(\Sigma, T^{(1,0)}\Sigma)$ commonly called *Beltrami differentials* and $(\,\cdot\,,\,\cdot\,)$ is the obvious pairing. Differentiating $6g-6$ times one gets the desired formula. However, one should notice that G is not a supercurrent of the matter TCFT alone, but the sum of that and the one from the b, c, β, γ system. However, in the special case of $c_{matter} = 9$, the axial anomaly in $\langle\,\cdot\,\rangle_\Sigma$ is precisely cancelled by the measure over fermionic moduli and thus one can safely consider the part of Z that depends only on the matter degrees of freedom. This is also the only case we will really be interested in, in the coming sections, as only in that case is the combined (c, c), (a, a) moduli space a special Kähler manifold. Apart from chapter 3 where we discuss their quantization, the later sections will only be concerned with genus-zero amplitudes even though coupled with open-strings. However we will sketch the reduction in the general case. The coming section could be skipped at this point as it only serves as reference for part two.

2.8.3 S^1-equivariant cohomology: the closed string origin of cyclic cohomology/homology

If not only for the sake of completeness, we add this part because it makes contact with investigations in section 5.3.7, which in particular lead to the definition and construction of the "cyclic" residue formula. This section serves to explain how this formula corresponds to the topological string theory extension of the TQFT trace θ. We suggest to first read part two and return to this section when referenced.

Let us review how the "matter + ghost" system collapses to a much simpler "matter + 1 parameter". For this we go back to the BRST complex of $\mathfrak{g} \oplus \mathfrak{g}[1]$. We have not yet specified what the complex is. BRST can mean various things especially in infinite dimensions. Let us call the total BRST charge of the TCFT "matter + ghost" $Q_{tot} = Q_{matter} + Q_{ghost} = Q + Q_{ghost}$. On the other hand Q_{tot} is exactly Q_{BRST} for $\mathfrak{g} \oplus \mathfrak{g}[1]$. This settled, however, the BRST complex turns out not to be the cohomology of $\mathcal{H} \otimes \mathcal{H}_{ghosts}$, but of $((\mathcal{H} \otimes \mathcal{H}_{ghosts})_{basic}$, which is the restriction of the former on elements annihilated by $(G_{tot} - \overline{G}_{tot})_0$ and $L_0 - \overline{L}_0$. The reason for this extra piece of complexity is discussed in [77] and arises in analyzing the obstruction to defining a globally well-defined measure on the moduli space of punctured surfaces. As explained in [77] and again with renewed clarity in [27] (see also [107] for string field theory perspective), in defining this measure, one first considers the insertion of a state ψ on a Riemann surface Σ. To define this, one takes a limit in time $t \to \infty$. At finite t the state is placed on a circle enclosing a disk of finite size on Σ, that has been removed. There one defines a local coordinate system (z_P) around the center P of the removed disk, with corresponding (local) Virasoro algebra. One thus has a bundle:

$$\mathcal{P}_{g,n} \to \mathcal{M}_{g,n}$$

where $\mathcal{M}_{g,n}$ is the moduli space of Riemann surfaces of genus g with n-marked points, while $\mathcal{P}_{g,n}$ is a refinement thereof taking into account all possible coordinate systems in the vicinity of each separate insertion and then modded out by automorphisms of the Riemann surface with boundary. To define a measure Ω on $\mathcal{M}_{g,n}$, one first defines a measure $\tilde{\Omega}$ on $\mathcal{P}_{g,n}$. Taking the limit in t should induce Ω as:

$$\Omega = s^* \tilde{\Omega}$$

where s is a section of the bundle just described. However such a measure is only well defined provided the fields inserted at the punctures satisfy certain properties. Indeed the desired s by itelf does not exist. Such an s corresponds to a continuously varying coordinate system on Σ, meaning that Σ is parallelizable which is only true if $g = 1$. Instead one has to allow for the coordinate system, that is the section s, to be well defined up to a phase. A phase of rotation of the deleted circles. In other words s is well defined if the phase arbitrariness is absorbed by the field insertions. That is the fields should be rotational invariant. More precisely one finds that they should be annihilated by $L_0 - \overline{L}_0$ and also $G_0 - \overline{G}_0$. Note that $L_0 - \overline{L}_0$ is the generator of rotations, playing the role of L_X, the Lie derivative w.r.t. the generator of rotations X, while $G_0 - \overline{G}_0$ plays the role of ι_X and Q of d. After a lengthy discussion, by analogy, we have thus recovered $(\mathcal{H} \otimes \mathcal{H})_{basic}$. In [35] it is shown that:

$$H((\mathcal{H} \otimes \mathcal{H}_{ghosts})_{basic}, Q) \cong H(\mathcal{H}(0)[\Omega], Q + \Omega(G_0 - \overline{G}_0))$$

where $\mathcal{H}(0)$ is the subspace of $\mathcal{H}$ annihilated by $L_0 - \overline{L}_0$, while Ω is a variable of degree 2. This degree arises because d is of tensor degree 2 higher than ι_X. That is, the right hand side above, is the Cartan-model for S^1-equivariant cohomology of $\mathcal{H}$. The Ω^n's are represented by:

$$\Omega^n \leftrightarrow \frac{1}{2^n}(\gamma_0 - \overline{\gamma}_0)^n c_1(\mathcal{L}_P)$$

where $\mathcal{L}_P$ is the pull-back of the tangent space at the marked point P to $\mathcal{M}_{g,n}$. So, to summarize our rough sketch, in arbitrary dimension, coupling to topological gravity adds to the (c, c) ring say, also the above *contact terms*, which are the observables of topological gravity alone. Moreover we also see that apart from the case $c = 9$, in the more general case topological string theory is ineluctably intertwined with the contact terms because of axial anomaly selection rules.

2.8.4 Holomorphic anomaly

We come back to the case of $c = 9$ TCFT's and consider the stringy prescription for the quantization of the combined (c, c) and (a, a) TCFT moduli space. We will only illustrate the most important steps, as part of the goal of this work is precisely to give a complete derivation of the same result from the point of view of a non-stringy and more non-perturbative quantization approach.

The equation describing the quantum nature of TST on higher genus surfaces, is the holomorphic anomaly equation of [6]. This expresses the dependence of topological string theory amplitudes, say of (c,c) fields, on (a,a) moduli for genus $g \geq 1$ surfaces. We start with genus $g \geq 2$ and compute in the B-twisted case. Let

$$F_g(t,\bar{t}) = \int_{\mathcal{M}_g} \prod_{i=1}^{3g-3} dm^i d\overline{m}^i \langle \prod_{i=1}^{3g-3} (G^-,\mu_i)(\overline{G}^-,\overline{\mu}_i) \exp\left(-S(t,\bar{t},m,\overline{m})\right)\rangle$$

be the generating function of genus $g \geq 2$ connected amplitudes among marginal fields. Then:

$$\frac{\partial}{\partial \bar{t}^{\overline{\alpha}}} F_g = \int_{\mathcal{M}_g} \prod_{i=1}^{3g-3} dm^i d\overline{m}^i \langle \prod_{i=1}^{3g-3} (G^-,\mu_i)(\overline{G}^-,\overline{\mu}_i) \int_{\Sigma} [Q^L,[Q^R,\overline{\phi}^{\overline{\alpha}}]] d^2z \exp\left(-S(t,\bar{t},m,\overline{m})\right)\rangle.$$

Now one uses the fact that the vacuum is annihilated by Q^L and Q^R and therefore when moving e.g. Q^L next to G^- we can replace their product with their commutator and likewise for Q^R and $\overline{G}^-$, so we obtain that alternatively, a pair (G^-,μ_i), $(\overline{G}^-,\overline{\mu}_j)$ is replaced by $2(T,\mu_i)$, $2(\overline{T},\overline{\mu}_j)$ respectively up to a sign. The insertion of the latter two operators in turn corresponds to differentiating with respect to m^i and $\overline{m}^j$. In the end one obtains an equation of the form:

$$\begin{aligned}\frac{\partial}{\partial \bar{t}^{\overline{\alpha}}} F_g &= \int_{\mathcal{M}_g} \overline{\partial}\partial \omega_g(\overline{\phi}^{\overline{\alpha}}) \\ &= \int_{\partial\mathcal{M}_g} \frac{1}{2}(\partial - \overline{\partial})\omega_g(\overline{\phi}^{\overline{\alpha}}).\end{aligned}$$

In order to qualitatively understand the composition of the boundary of moduli space $\partial\mathcal{M}_g$, recall that the 6 independent real moduli corresponding to the introduction of a tube on the surface consist of two moduli for the position of the insertion of each boundary of the tube, as well as two moduli for the twist and length of the tube. There is no boundary arising from the positions of the boundary circles, as Σ is boundary less, and two circles coincide on a co-dimension two subspace of $\mathcal{M}_g$. If we denote by τ the complex modulus of the tube, then the surviving derivative in the integrand is the derivative w.r.t. $\mathrm{Im}(\tau)$, which is the length of the tube. At the boundary $\mathrm{Im}\tau \to \infty$. In particular time-evolution along the cylinder becomes the projector onto Ramond ground states. The point of insertion of $\overline{\phi}^{\alpha}$ is integrated over Σ. The insertions outside of the tube don't contribute, as in the limit $\tau \to \infty$ the action of $1/2(\partial - \overline{\partial})$ vanishes on the time-evolution operator. While when the insertion is on the tube, then the contribution of the tube prior to differentiation by $\mathrm{Im}(\tau)$ diverges as:

$$\lim_{\tau\to\infty} \int_{C_\tau} d^2z\, \eta^{\bar{i}\bar{j}} |\bar{i}\rangle\langle\bar{j}|\phi_{\overline{\alpha}}(z,\overline{z})|\bar{k}\rangle\langle\bar{l}|\eta^{\bar{k}\bar{l}}.$$

Now one decomposes the integral over the cylinder C_τ into an integral over time and one over the space circle, inserting the latter in the three-point correlator. Then upon differentiation by $\mathrm{Im}(\tau)$ and change of basis for the ground states, one obtains:

$$|\beta\rangle C_{\overline{\alpha}\overline{\beta}\overline{\gamma}} g^{\beta\overline{\beta}} g^{\gamma\overline{\gamma}} \langle\gamma|.$$

Notice that we can restrict to marginal (greek) indices due to axial R-symmetry selection rules, albeit only for $c = 9$ theories. What is left to do is identify the degenerating cylinders. There are two types of such cylinders, ones connecting a genus $g-r$ to a genus $r > 1$ surface, and one that degenerating transforms Σ_g into a genus $g-1$ surface. In the end one obtains:

$$\frac{\partial}{\partial \overline{t}^{\overline{\alpha}}} F_g = \frac{1}{2} C_{\overline{\alpha}\overline{\beta}\overline{\gamma}} g^{\beta\overline{\beta}} g^{\gamma\overline{\gamma}} \left(\sum_{r=1}^{g-1} \left(D_\beta F_r D_\gamma F_{g-r} \right) + D_\beta D_\gamma F_{g-1} \right). \qquad (2.8.1)$$

The operators D_β are covariant derivatives introducing marginal chiral fields ϕ_β. As these are introduced they couple to spare moduli in the measure describing their position on the component Σ' of Σ they belong to. In this way:

$$\phi_\beta \to \int_{\Sigma'} \phi_\beta^{(2)}$$

The connections D_β differ from simple partial derivatives in the corresponding moduli t^β, because the marginal field does not commute with the action, instead a contact term is produced upon contraction with the coupling of $U(1)_A$ R-symmetry to the spin connection producing in turn a term proportional to the Euler Characteristic of Σ'. This phenomenon indicates that F_g is a section of $\mathcal{L}^{2-2g} \to M$. Where $\mathcal{L}$ is the Vacuum bundle. This fact indicates that F_g is not exactly a generating function for correlation functions of marginal fields, in fact introducing more marginal fields one obtains contact terms between them that are proportional to Γ, the Levi-Civita connection on the special Kähler manifold (M, G). Hence D is the Levi-Civita covariant derivative twisted by appropriate powers of $\mathcal{L}$. One thus introduces a generating function proper for the correlation functions:

$$C^g_{\alpha_1,\ldots,\alpha_n} = D_{\alpha_1} \cdots D_{\alpha_n} F_g$$

with counting parameters u^{α_i}. One can view these correlation functions as expressing a kind of functorial relation:

$$\mathcal{M}_{g,n} \mapsto S^n T^{1,0} M \otimes \mathcal{L}^{2-2g}.$$

Without going into the details, the functor is nothing but the path-integral and its properties are extracted from the holomorphic anomaly equation, which describes how decomposition of moduli spaces of Riemann surfaces is mapped to the decomposition of correlation functions.

In the end one packages all the data of correlation functions of marginal fields on arbitrary Riemann surfaces, including $g = 0$ and $g = 1$, in a theory $(t, \overline{t})$ dependent generating function. Denoting by p the point on M with coordinates $(t, \bar{t})$, one has introduced the object which we shall denote as:

$$Z_{red}(u, p, \lambda)$$

where the notation *red* will be explained in 3.4.5, while λ counts the genus of the surface where marginal fields are scattered undoing the twist of $\mathcal{L}^{2-2g}$. It is precisely Z_{red} which we will rediscover from the mathematics of the quantization of M and we thus defer the complete analysis of its properties to chapter 3.

It should be noted at this point that we have avoided the discussion of genus $g = 0$ and $g = 1$ surfaces. The holomorphic anomaly equation in this case is one on the moduli space of their stable counterparts, namely $\mathcal{M}_{0,3}$ and $\mathcal{M}_{1,1}$. The equation for the former is part of the tt^*-equations:

$$\partial_{\overline{\alpha}}(\partial_\alpha \partial_\beta \partial_\gamma F_0) = 0$$

while for the latter:

$$\overline{\partial}_{\bar{j}}(\partial_i F_1) = \frac{1}{2}\mathrm{tr} C_i \overline{C}_j - \frac{\mathrm{tr}(-1)^F}{24} G_{i\bar{j}}.$$

While genus zero surfaces define the geometry of M via the tt^*-equations, the above equation for genus $g = 1$ surfaces ensures the integrability of the extension of (2.8.1) to $Z(u, p, \lambda)$. We won't go into further detail, because we will again rediscover these properties in chapter 3, and also because, as we will learn, these properties are more naturally attributed to more general objects describing a larger moduli space $\tilde{M}$. This moduli-space is the total space of $\mathcal{L} \to M$. Vice-versa, M is recovered as the holomorphic quotient of $\tilde{M}$ by the structure group of $\mathcal{L}$ (more on this in section 3.4.5). It turns out, (see e.g. [33]), that $\tilde{M}$ is a so called Lorentzian affine special Kähler manifold. In the following we shall in-fact rediscover projective special Kähler manifolds in steps. First we shall discover affine Riemannian special Kähler manifolds, then extend to the quantization of affine special Kähler manifolds of arbitrary signature, in particular Lorentzian. And finally we shall explain how to extract the quantization of their holomorphic quotients, rederiving the defining holomorphic equation for Z_{red} in section 3.4.5. We start by reviewing the basics of quantization of classical phase spaces.

Chapter 3

The Quantization of Special Kähler Manifolds

3.1 Quantization review

In this section we will review the general principles underlying quantization of classical phase spaces. The basic ingredients are a classical phase space, which we will assume to be a smooth manifold M, a space of quantum states, which by definition is complex projective space $\mathbb{P}^n$ of a priori arbitrary dimension n, and a quantization map. This is a map

$$\phi : M \to \mathbb{P}^n$$

identifying the classical state space (or a portion thereof) as a subset of the space of quantum states[1]. The image of this map, $\phi(M)$, is known as the space of coherent states. Part of the problem of quantization is the classification of such triples. Complementary to that, is the task of transporting the basic invariants of $\mathbb{P}^n$ via ϕ to M. The basic algebraic invariant of interest is the maximal compact subgroup G of $\mathrm{Aut}(\mathbb{P}^n)$. This can also be viewed as the group of automorphisms of $\mathbb{P}^n$ endowed with the pairing:

$$(x, y) := \frac{|\langle x, y\rangle|}{||x||\,||y||},$$

where $\langle \cdot, \cdot \rangle$ denotes a sesquilinear product on $\mathbb{C}^{n+1}$. It is the result of Wigner's Theorem, that G is composed exactly of the unitary and antiunitary transformations of $(\mathbb{C}^{n+1}, \langle \cdot, \cdot \rangle)$. The *space of quantum observables* is the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, while what is known as the *algebra of quantum observables* is its universal enveloping algebra $U(\mathfrak{g})$.

The algebra of classical observables is recovered as follows. Let $F \in U(\mathfrak{g})$, then its classical counterpart is a complex valued function $f : M \to \mathbb{C}$ given by:

$$f(p) = \langle \phi(p), F\phi(p) \rangle.$$

[1] As classical mechanics should in principle be recovered from quantum mechanics, ϕ should be in some sense faithful, e.g. an immersion or even embedding.

The universal enveloping algebra is represented by the so-called star-product $\star$, which by definition must satisfy:

$$(f \star g)(p) := \langle \phi(p), F \cdot G\, \phi(p) \rangle.$$

In the following we will suppress one degree of arbitrariness in the choice of the quantization triple, namely the dimension of projective space. In fact, without loss of generality, we are allowed to consider the direct limit:

$$\mathbb{P}^\infty := \lim_{n \to \infty} \mathbb{P}^n.$$

This in turn can be viewed as $\mathcal{H}/\mathbb{C}^*$, where $\mathcal{H}$ is an infinite dimensional separable Hilbert-space. It is important to remark, as it will be crucial in what follows, that any two such Hilbert-spaces are isomorphic. Before venturing into the more general case, it will be useful to recall the very well known quantization of $\mathbb{R}^{2d}$.

3.1.1 The simplest case: $M = \mathbb{R}^{2d}$

Given the definition of quantization above, a priori there are a multitude of quantization maps ϕ of $\mathbb{R}^{2d}$. However its canonical quantization presupposes a much more rigid structure than that of a smooth manifold. Indeed $\mathbb{R}^{2d}$ is identified with its group of translations Γ, or more precisely with an orbit, e.g. $\Gamma \cdot e$, where e denotes the identity element $e = 0 \in \mathbb{R}^{2d}$. Then the quantization maps reduce to the projective representations:

$$\rho : \Gamma \to \mathrm{Aut}(\mathbb{P}^\infty).$$

As is well known these are in one-to-one correspondence with the family of linear representations:

$$\hat{\rho} : \hat{\Gamma} \to \mathrm{Aut}(\mathcal{H})$$

labeled by a central extension $\hat{\Gamma}$ of Γ. These in turn are fully specified by the choice of a skew-symmetric bilinear form $\omega^{-1} \in \bigwedge^2(\mathrm{Lie}(\Gamma))^*$. Passing to the Lie algebra description altogether, $\mathrm{Lie}(\hat{\Gamma})$ is then specified by the following commutation relations:

$$[\hat{x}^i, \hat{x}^j] = i\omega^{-1}(x^i, x^j).$$

By a slight abuse of notation, we have multiplied the generators by $i = \sqrt{-1}$, so that these will be represented as self-adjoint operators. We will restrict attention to the case where ω^{-1} is non-degenerate. Although this is no real loss of generality in the present case, it will be in the following sections, where the space $(\mathbb{R}^{2d}, \omega^{-1})$ is generalized to a Poisson manifold, while we will be solely interested in the symplectic case[2]. In the non-degenerate case, $\hat{\Gamma}$ is known as a *Heisenberg group*, and these are in fact all equivalent. This simply

[2]Roughly speaking the requirement that the phase-space be symplectic replaces the notion that the quantization map ϕ should be " faithful".

follows from the fact that any non degenerate skew-symmetric matrix can be brought to canonical form ϵ by an invertible matrix Λ, as

$$\Lambda^T \omega \Lambda = \epsilon.$$

It will be useful in the following to introduce further canonical objects: η, the standard euclidean metric, and the complex structure I given by:

$$\eta = I\epsilon.$$

Part of the Stone-von-Neumann-Mackey Theorem states that $\hat{\Gamma}$ has a unique, up to isometry, unitary irreducible and infinite dimensional representation on a separable Hilbert-space. In fact, since infinite dimensional separable Hilbert-spaces are all equivalent, we can view each such $\mathcal{H}$ as furnishing such an irreducible representation. Indeed this is realized as follows. First, presupposing canonical form, split $\mathrm{Lie}_{\mathbb{C}}(\hat{\Gamma})$ into raising and lowering subalgebra spanned by the operators:

$$\hat{x}^i(\eta + i\epsilon)_{ij} \qquad \text{and} \qquad \hat{x}^i(\eta - i\epsilon)_{ij}$$

respectively. The former are commonly known as annihilation while the latter as creation operators. Then choose an orthonormal basis $\{|n\rangle\}$ of $\mathcal{H}$ enumerated by $n \in \mathbb{N}_0^d$. Declare $|0\rangle$ to be the highest weight vector and let the action of $\hat{x}^i(\eta - i\epsilon)_{ij}$ be specified by:

$$\hat{x}^i(\eta - i\epsilon)_{ij}|n\rangle \sim |n + e_j\rangle,$$

where e_j denotes the unit vector in the jth direction and $\sim$ indicates equal up to a suitable unique proportionality factor. At this point we can turn to the representation $\hat{\rho}$:

$$\hat{\rho}(p) = \exp(i\omega_{ij}x^i\hat{x}^j),$$

where $p = (x^1, \ldots, x^{2d})$. Given this representation, it is straightforward to obtain the quantization map:

$$\phi(p) = \hat{\rho}(p)|\psi\rangle, \tag{3.1.1}$$

where $|\psi\rangle$ is an arbitrary, nonzero, state in $\mathcal{H}$. This choice is irrelevant, it can be removed by an automophism of $\mathbb{P}^n$. If we choose $|\psi\rangle = |0\rangle$, we recover the canonical notion of coherent state:

$$|x\rangle = \exp(i\omega_{ij}x^i\hat{x}^j)|0\rangle. \tag{3.1.2}$$

This in particular satisfies:

$$\hat{x}^i(\eta + i\epsilon)_{ij}|x\rangle = x^i(\eta + i\epsilon)_{ij}|x\rangle,$$

namely, it is an eigenstate of the annihilation operators.

For the sake of completeness we will sketch how to extract the star-product in the particular case $M = \mathbb{R}^2$ [3]. While the above notation is convenient as a reference for future sections, we

[3] The more general case $M = \mathbb{R}^{2d}$ is then obtained in a straightforward fashion.

will, solely for this independent appendix to this review section, use the common notation with annihilation operator a and creation operator $a^\dagger$, which up to a factor are equivalent to the ones defined above. Then the canonical coherent states are usually denoted as $|\alpha\rangle$, where $\alpha = (1/\sqrt{2})(y^1 + iy^2)$ and $(y^1, y^2) = x$ are the coordinates of a point $p \in \mathbb{R}^2$. Then, from the defining property:

$$a|\alpha\rangle = \alpha|\alpha\rangle,$$

one can recover the state $|\alpha\rangle$ as:

$$|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_{n\geq 0}\frac{\alpha^n}{\sqrt{n!}}|n\rangle,$$

where we have normalized $|\alpha\rangle$ to 1. Moreover, in the usual notation:

$$\hat{\rho}(p) =: U(\alpha) = \exp(\alpha a^\dagger + \overline{\alpha} a).$$

Then

$$\begin{aligned}
(f \star g)(p) &= \langle\alpha|F \cdot G|\alpha\rangle \\
&= \sum_{n\geq 0}\langle 0|U(-\alpha)FU(\alpha)\ |n\rangle\langle n|\ U(-\alpha)GU(\alpha)|0\rangle \\
&= \sum_{n\geq 0}\frac{1}{n!}\langle 0|U(-\alpha)FU(\alpha)\ (a^\dagger)^n|0\rangle\langle 0|a^n\ U(-\alpha)GU(\alpha)|0\rangle \\
&= \sum_{n\geq 0}\frac{1}{n!}\langle 0|\mathrm{ad}^n_{-a^\dagger}[U(-\alpha)FU(\alpha)]|0\rangle\langle 0|\mathrm{ad}^n_a[U(-\alpha)FU(\alpha)]|0\rangle \\
&= \sum_{n\geq 0}\frac{1}{n!}\left(\frac{\partial^n}{\partial\alpha^n}f\right)(p)\left(\frac{\partial^n}{\partial\overline{\alpha}^n}g\right)(p) \\
&= \left(f\exp\left(\frac{1}{2}\langle\overleftarrow{\nabla}, (\eta^{-1} + i\epsilon^{-1})\ \overrightarrow{\nabla}\rangle\right)g\right)(p).
\end{aligned}$$

In order to appreciate the significance of the star product in physics, one should introduce Planck's constant $\hbar$, which we have implicitly set to 1. The latter is reintroduced precisely by the following change of coordinates: $(y_1, y_2) \mapsto (\sqrt{\hbar}y_1, \sqrt{\hbar}y_2)$. Then the star product reads:

$$(f \star g)(p) = \left(f\exp\left(\frac{\hbar}{2}\langle\overleftarrow{\nabla}, (\eta^{-1} + i\epsilon^{-1})\ \overrightarrow{\nabla}\rangle\right)g\right)(p).$$

Using the above one can now, in particular, recover Hamilton's equations of classical mechanics as the classical limit ($\hbar \to 0$) of Heisenberg's equations.

3.2 A more general case: quantization of symplectic manifolds

We are now faced with the problem of generalizing this beautiful yet very special construction for $\mathbb{R}^{2d}$ to the general case of a symplectic manifold M of dimension $2d$. For this, we follow Fedosov's method [31, 32]. Accordingly we construct quantization maps as follows. First we choose a point $p \in M$ and declare that this be mapped to the point $\phi(p) =: |p\rangle \in \mathbb{P}^\infty$. Next, we declare that any other point p' in the vicinity (to be explained later) of p be mapped to the point:

$$p' \to \phi(p') =: U(p',p)|p\rangle,$$

for some $U(p',p) \in \mathrm{Aut}(\mathbb{P}^\infty)$. In particular it must be continuously connected to the identity, therefore $U(p',p)$ is unitary. In fact, thanks to the QR decomposition of matrices, this is no loss of generality. Let's now erect a (at this point arbitrary) coordinate system $\{x^k\}$ in a neighborhood V_p of p. And let's define the object

$$-A(p) = \frac{\partial}{\partial x^k} U(p',p)\bigg|_{p'=p} dx^k.$$

We can interpret A as a flat connection on a $\mathbb{P}^\infty$-bundle over M. For computational purposes however, it is more convenient to work on the corresponding $\mathcal{H}$-bundle where, by a slight abuse of notation, the connection A is allowed to have holonomies in the centre of $\mathrm{Aut}(\mathbb{P}^\infty)$ namely $\mathbb{C}^*$. That is, A satisfies the Maurer-Cartan equation:

$$dA + A \wedge A \in \Omega^2(M, \mathbb{C}).$$

In other words, A is projectively flat as a connection on the $\mathcal{H}$-bundle. Without loss of generality we can however assume that, as a connection on the $\mathcal{H}$-bundle, A is flat, namely:

$$dA + A \wedge A = 0. \tag{3.2.1}$$

For this we simply have to twist the $\mathcal{H}$-bundle by a hermitian line-bundle with a connection whose curvature precisely cancels that of A. We shall hitherto refer to the state $|p\rangle$ parallel transported by A, viewed as an element in $\mathcal{H}$, as $|p\rangle_A$. Clearly $U(p',p)$ can be written in the form:

$$U(p',p) = \mathcal{P}\exp\left(-\int_\gamma A\right),$$

where $\mathcal{P}$ stands for path-ordered, $\gamma : [0,1] \to M$ is a path with endpoints $\gamma(0) = p$, $\gamma(1) = p'$, and since A is flat the result of the integration only depends on the homotopy class $[\gamma]$. In particular, if we restrict attention to a simply connected, or even better, contractible neighborhood of p, then the result of integration is completely independent of the chosen path and in that case we are allowed to refer to the integral as $\int_p^{p'}$. One could also do this globally if one replaces M with its universal cover altogether.

So far the discussion was very general, in that we have not required any special properties of M other than it be smooth and we have traded the notion of quantization map for that of a flat connection on an $\mathcal{H}$-bundle. The interesting step is now to find a good classification of the solutions to (3.2.1). We will assume at this point that M is symplectic, and we choose V_p to be a Darboux patch, namely a coordinate neighborhood where the symplectic form ω

is flat. Attached to this flat symplectic form we have a corresponding Heisenberg algebra with generators $\hat{x}^i$ and $\mathcal{H}$ is the corresponding irreducible representation. The intuition behind this is to envisage the tangent space T_pM at each point $p \in M$ as a copy of Lie(Γ). In a suitable sense every self-adjoint operator of $\mathcal{H}$ is an element of $U(\mathrm{Lie}(\Gamma))$. In informal terms, this follows from the following decomposition of the projector on the highest weight state $|0\rangle$:

$$|0\rangle\langle 0| = \sum_{\vec{k}\in\mathbb{N}_0^d} (-1)^k \frac{(a^\dagger)^k a^k}{k!} = \sum_{\vec{k}\in\mathbb{N}_0^d} (-1)^k \binom{a^\dagger a}{k}.$$

In the above we have used multi-index notation. This decomposition allows us to expand A as follows:

$$A(p) = i\sum_{l=0}^{\infty} \sum_{i_1\leq\cdots\leq i_l} (\alpha_{i_1,\ldots,i_l,k}(p)\hat{x}^{i_1}\cdots\hat{x}^{i_l} \ + h.c)dx^k. \tag{3.2.2}$$

Equation (3.2.1) thus decomposes into an infinite number of equations. More precisely (3.2.2) is well defined in the topology defined by the seminorms:

$$|\langle\psi_1|\ \cdot\ |\psi_2\rangle| \ \text{ with } \psi_1 \in \mathcal{S}_d, \psi_2 \in \mathcal{H},$$

where by $\mathcal{S}_d \subset \mathcal{H}$ we denote the space of states whose coefficients c_n in the expansion $\psi_1 = \sum_n c_n|n\rangle$ tend to zero as $||n|| \to \infty$, faster than any polynomial of $n \in \mathbb{N}_0^d$ [4]. Thus the notation $\mathcal{S}_d$ is suggestive for Schwarz-space, although this should not be taken literally. At this point we remark that the canonical, Poissonian coherent states of $M = \mathbb{R}^{2d}$ are elements of $\mathcal{S}_d$. What this restriction on the topology implies, is that for the perturbative ansatz (3.2.2) to be well-defined, we should represent the state $|p\rangle_A$ as a wavefunction whose corresponding complete sequence of linear functionals has as corresponding sequence of states, elements of $\mathcal{S}_d$. We will define such wavefunctions in section 3.3.3.

Returning to the infinite sequence of equations encoded in (3.2.1), we will see in the following that each equation specifies a certain geometrical structure on M. The philosophical perspective one could take about the above expansion is that, as we increase the order in perturbation theory we are chiseling step by step, through equation (3.2.1), the geometry of a Darboux patch of M. In particular we assume that at each step in perturbation theory the Darboux patch be smooth, however this does not impose that the limiting structure be. In other words, our initial assumption that M be smooth could in principle be omitted for the limiting geometries. As a check, and for matters of convention, let's recover the simplest case $M = \mathbb{R}^{2d}$ in this formalism. There the perturbation expansion stops at first order:

$$A = i(\alpha_k + \omega_{ik}\hat{x}^i)dx^k.$$

Thus, solving (3.2.1) yields the following two equations:

$$d\alpha = -\frac{1}{2}\omega$$

[4] This space is also known as the space of rapidly decreasing sequences, which can be equipped with a Fréchet topology.

$$\partial_k(\omega_{il}\hat{x}^i)dx^k \wedge dx^l = 0.$$

The second equation is automatically satisfied. Thus, apart from an irrelevant phase:

$$U(p',p) = \mathcal{P}\exp\left(-i\int_p^{p'}\left(-\frac{1}{2}\omega_{ik}x^i + \omega_{ik}\hat{x}^i\right)dx^k\right).$$

3.3 Somewhere in between: special geometries

In this section we shall investigate the geometry of phase-spaces whose associated connection A stops at second order in the perturbative expansion (3.2.2). We will show in section 3.3.2 that these spaces are actually equivalent to the ones whose connection stops at first order. As will become clear in the following sections this class includes affine (Riemannian) special Kähler manifolds.

First we will investigate equation (3.2.1) to second order. Furthermore we will assume that, to first order, A reduces to the flat case. The connection A then takes the form:

$$A = i(\alpha_k + \omega_{ik}\hat{x}^i + D_{ijk}\hat{x}^i\hat{x}^j)dx^k, \tag{3.3.1}$$

where, given that A is hermitian, and without loss of generality, $D_{ijk} \in \mathbb{R}$ and $D_{ijk} = D_{jik}$. Equation (3.2.1) becomes:

$$d\alpha = -\frac{1}{2}\omega$$

$$D_{krl} - D_{lrk} = 0 \tag{3.3.2}$$

$$\partial_k D_{ijl} - \partial_l D_{ijk} - 2(D_{isk}D_{rjl} + D_{jsk}D_{ril})\omega^{sr} = 0. \tag{3.3.3}$$

We now introduce the following object:

$$G^j_{ki} = 2D_{ilk}\omega^{lj}.$$

The symmetry of D_{ijk} in its first two indices translates to:

$$G^l_{ki}\omega_{lj} - G^l_{kj}\omega_{li} = 0,$$

that is:

$$G_k\omega + \omega G_k^T = 0.$$

In other words, G_k is a symplectic matrix. In terms of G_k, equations (3.3.2) and (3.3.3) read:

$$G^i_{lk} - G^i_{kl} = 0$$

$$\partial_k G_l - \partial_l G_k - [G_k, G_l] = 0.$$

We will learn in section 3.3.2 that G is a connection on the tangent bundle of M. Then the first equation is the statement that G is torsion-free, while the second means that G is flat. So to summarize, the second order quantizations correspond to symplectic manifolds with a flat symplectic connection [5].

[5] Recall that a connection that is both compatible with the symplectic form and torsion free is known as a symplectic connection, while if it is not torsion free it is called quasi-symplectic.

3.3.1 Kähler manifolds: holomorphic connections

We now sharpen our analysis to the case where the phase-space M is a complex symplectic manifold, that is, a Kähler manifold, when endowed with the appropriate compatible metric $g = J\omega$, where $J \in \Gamma(M, \mathrm{End}(TM))$ denotes its complex structure. We then ask when it is that the above constructed quantization map is compatible with J. We define compatibility as follows.

Definition 3.3.1. A quantization map $\phi : M \to \mathbb{P}^\infty$ defined by a projectively flat unitary connection A, is compatible with the complex structure of M, if A admits the following decomposition:

$$A = \frac{1}{2}(B - B^\dagger),$$

where B is a holomorphic, projectively flat connection. That is, ϕ induces a holomorphic map to $\mathbb{P}^\infty$.

It is straightforward to check that the above decomposition for A is unique with B given by:

$$B = A_r(\delta^r_k + iJ^r_k)dx^k.$$

We will now check, at first order, what conditions on the geometry of M must be imposed in order for the compatibility condition of definition 3.3.1 to be fulfilled. The holomorphic connection is given by:

$$\begin{aligned} B &= i\left(\alpha_r + \omega_{ir}\hat{x}^i\right)(\delta^r_k + iJ^r_k)dx^k \\ &= i\left((\delta^r_k + iJ^r_k)\alpha_r - (\omega + ig)_{ki}\hat{x}^i\right)dx^k. \end{aligned}$$

Let Ω denote the curvature of B, then the projective flatness condition reads:

$$\left(i\partial_k(\alpha_l + iJ^r_l\alpha_r) - \frac{1}{2}[(g + i\omega)_{ki}\hat{x}^i, (g + i\omega)_{lj}\hat{x}^j]\right)dx^k \wedge dx^l = \Omega \qquad (3.3.4)$$

$$\partial_k(\omega - ig)_{il} - \partial_l(\omega - ig)_{ik} = 0.$$

The second equation reduces to

$$\partial_k g_{il} - \partial_l g_{ik} = 0.$$

That is, there are Darboux coordinates where:

$$g_{il} = \partial_l f_i = \partial_i f_l = \partial_i\partial_l K,$$

where f_i and K are real valued functions on M. In fact it is straightforward to observe that K is a Kähler potential for M. Moreover, while an arbitrary Kähler potential is defined up to holomorphic functions, K is defined only up to linear ones.

To end the above analysis we shall return to equation (3.3.4), which now reduces to:

$$\Omega = -\frac{i}{2}\omega - J^r_l\partial_r\alpha_k dx^k \wedge dx^l,$$

which, in the gauge:

$$\alpha = -\frac{1}{2}J^l_k \partial_l K dx^k \tag{3.3.5}$$

becomes:

$$\Omega = -\frac{i}{2}\omega.$$

We shall denote this gauge for α as canonical. We now assume that a second order quantizable manifold admits a gauge in which B is of the particular form just considered. This is the case exactly when, in the above Darboux coordinates the flat symplectic connection G vanishes. That is the Darboux coordinates are $(d+G)$-flat. Then, in arbitrary coordinates the constraint on the metric reads:

$$d_{d+G}J = 0.$$

We thus obtain exactly the definition of affine special Kähler manifold (see e.g. [33]). In the following section we shall finally show the equivalence of first and second order quantizable spaces. To conclude this section we shall formalize our findings with the following

Theorem 3.3.2. A Kähler manifold with quantization map $\phi : M \to \mathbb{P}^\infty$ whose corresponding flat connection A is first-order in a suitable coordinate system, and is compatible with the complex structure of M, is precisely an affine special Kähler manifold.

3.3.2 Symplectomorphisms as gauge transformations

Here we will show how symplectomorphisms act on the coherent state $|p\rangle$, thus allowing us in particular to transform the flat connection A from special to arbitrary Darboux coordinates. In particular we will show that every second order connection A of the form (3.3.1) can be brought to first order, under a suitable symplectic change of coordinates. From now on, we shall denote by A_s the first order connection A in special Darboux coordinates. Let $\sigma : M \to M$ denote a local symplectomorphism on M, and let $\Sigma : TM \to TM$ denote its differential. Then σ acts on $\mathcal{H}$ via the unitary map:

$$S := \exp(-if - \frac{i}{2}\left(\log(\Sigma)\omega\right)_{ij} \hat{x}^i \hat{x}^j), \tag{3.3.6}$$

where f is an arbitrary real function and the second term is antihermitian if and only if σ is a symplectomorphism. The function f can be included as σ should only act projectively on $\mathcal{H}$. To verify that σ acts via S we simply need to use the fact that $\mathcal{H}$ is an irreducible representation of the Heisenberg algebra, thus reducing the problem to the following single check:

$$S\hat{x}^k S^{-1} = \Sigma^k_l \hat{x}^l.$$

For this we shall consider the one parameter family of symplectomorphisms defined by $\Sigma_t := \exp(t\log(\Sigma))$ and will show that it is in correspondence with $S_t = \exp(-t(if +$

$\frac{i}{2}\left(\log(\Sigma)\omega\right)_{ij}\hat{x}^i\hat{x}^j)$). To this aim we only need to verify that the two families agree in the immediate neighborhood of $t = 0$:

$$\begin{aligned}
\frac{d}{dt}\left(S_t\hat{x}^k S_t^{-1}\right)|_{t=0} &= -\frac{i}{2}(\log(\Sigma)\omega)_{ij}[\hat{x}^i\hat{x}^j, \hat{x}^k] \\
&= -\frac{i}{2}(\log(\Sigma)\omega)_{ij}(i\omega^{ik}\hat{x}^j + i\omega^{jk}\hat{x}^i) \\
&= \frac{1}{2}(\log(\Sigma)^T - \omega^{-1}\log(\Sigma)\omega)\hat{x} \\
&= \log(\Sigma)^k_l\hat{x}^l,
\end{aligned}$$

where, in the last step, we have used the fact that Σ is a symplectomorphism. On the flat connection A, S acts as a gauge transformation. Let's take $A = A_s$, then, under a coordinate transformation:

$$(A_s)_k \mapsto \Sigma^l_k\left(S(A_s)_l S^{-1} + S\partial_l S^{-1}\right). \tag{3.3.7}$$

In order to compute $S\partial_k S^{-1}$ we resort once again to the flows Σ_t and S_t and compare the time derivatives at arbitrary time t:

$$\begin{aligned}
\frac{d}{dt}S_t\partial_k S_t^{-1} &= S_t\partial_k\left(if + \frac{i}{2}\left(\log(\Sigma)\omega\right)_{ij}\hat{x}^i\hat{x}^j\right)S_t^{-1} \\
&= i\partial_k f + \frac{i}{2}(\partial_k(\log(\Sigma))\omega)_{ij}(\Sigma_t)^i_r(\Sigma_t)^j_s\hat{x}^r\hat{x}^s \\
&= i\partial_k f + \frac{i}{2}(\Sigma_t\partial_k(\log(\Sigma))\omega\Sigma_t^T)_{rs}\hat{x}^r\hat{x}^s \\
&= i\partial_k f + \frac{i}{2}(\Sigma_t\partial_k(\log(\Sigma))\Sigma_t^{-1}\omega)_{rs}\hat{x}^r\hat{x}^s \\
&= \frac{d}{dt}\left(it\partial_k f - \frac{i}{2}(\Sigma_t(\partial_k\Sigma_t^{-1})\omega)_{rs}\hat{x}^r\hat{x}^s\right).
\end{aligned}$$

Therefore:

$$\begin{aligned}
(A_s)_k \mapsto\ & iS(\Sigma^l_k\alpha_l + \omega_{il}\Sigma^l_k\hat{x}^i)S^{-1} + \frac{i}{2}\Sigma^l_k\partial_l f - \frac{i}{2}\Sigma^l_k(\Sigma(\partial_l\Sigma^{-1})\omega)_{ij}\hat{x}^i\hat{x}^j \\
&= i\left(\Sigma^l_k(\alpha_l + \partial_l f) + \omega_{ik}\hat{x}^i - \frac{1}{2}\Sigma^l_k(\Sigma(\partial_l\Sigma^{-1})\omega)_{ij}\hat{x}^i\hat{x}^j\right).
\end{aligned} \tag{3.3.8}$$

We thus recovered the general form (3.3.1) and verified that indeed G is a connection on the tangent bundle to M. Moreover, in (3.3.8) we also observe that $i\alpha$ should be viewed as a connection on a line-bundle over M. More precisely, the transformation properties of α under the gauge transformation $\exp(-if)$ show that this line bundle is precisely the pullback of the unitary tautological bundle $\mathcal{H} \to \mathbb{P}^\infty$, or Hopf-fibration, via the quantization map. Henceforth we shall denote the operator S corresponding to the differential Σ as S_Σ.

3.3.3 The "coherent" tangent bundle

In this section we will give an explicit realization of the state $|p\rangle_A$ as a wavefunction $Z_A(u,p)$. The following discussion in fact applies to any Kähler manifold. We wish the wavefunction to correspond to a covariant tensor on M thus allowing us to speak of the state $|p\rangle_A$ as a coordinate independent object. We thus define the wavefunction as follows:

$$Z_A(u,p) = {}_{p,A}\langle u|p\rangle_A,$$

where $|u\rangle_{p,A} \in \mathcal{H}$ is a state that corresponds to a point $u^i\partial_i \in T_pM$. As discussed in section 3.1.1, the correct choice for $|u\rangle_{p,A} \in \mathcal{H}$ that reflects the vector-space structure of T_pM, is that of a coherent-state (3.1.1). In order to make this state covariant with respect to the choice of the flat connection A, we define it through the property:

$$\hat{x}^T(g+i\omega)|u\rangle_{p,A} = u^T(g+i\omega)|u\rangle_{p,A}, \tag{3.3.9}$$

that is $|u\rangle_{p,A}$ is the eigenstate of the annihilation operators $\hat{x}^i(g+i\omega)_{ij}$ defined according to the Kähler structure and coordinate system induced by the flat connection A. Clearly, this state is an element of $\mathcal{S}_d$, hence in the above defined wavefunction realization, our perturbation expansion (3.2.2) is completely well defined.

It is worth remarking here, that under the involution $J \to -J$, $g \to -g$, $\omega \to \omega$ or the involution $J \to J$, $g \to -g$, $\omega \to -\omega$ we would map a positive normed state to a "negative normed state", which is therefore non-existent as an element of a (positive) Hilbert-space. We shall forget this remark until we encounter Lorentzian conic special Kähler manifolds in section 3.4. Here, and until otherwise stated, we will restrict ourselves to Riemannian Kähler manifolds.

The fundamental property of $|u\rangle_{p,A}$, is that under a symplectomorphism with differential Σ and corresponding unitary operator S_Σ, it transforms as follows:

$$|u\rangle_{p,\tilde{A}} \sim S_\Sigma|\Sigma^T u\rangle_{p,A}, \tag{3.3.10}$$

where by $\sim$ we mean equal up to a phase and where $\tilde{A}$ is the gauge transformed connection (3.3.7). Equation (3.3.10) follows immediately from the fact that both the left- and right-hand side satisfy the defining equation (3.3.9) with $g+i\omega$ replaced by $\tilde{g}+i\tilde{\omega}$. This property translates to the following property for the wavefunction:

$$\begin{aligned} Z_{\tilde{A}}(u,p) &= {}_{p,\tilde{A}}\langle u|p\rangle_{\tilde{A}} \\ &= {}_{p,\tilde{A}}\langle u|S_\Sigma S_\Sigma^{-1}|p\rangle_{\tilde{A}} \\ &= {}_{p,\tilde{A}}\langle u|S_\Sigma|p\rangle_A \\ &= {}_{p,A}\langle \Sigma^T u|p\rangle_A \\ &\sim Z_A(\Sigma^T u,p), \end{aligned} \tag{3.3.11}$$

that is, $Z_A(u,p)$ should be viewed as a section of the line-bundle $\pi^*(\mathcal{L}\otimes\mathcal{L}'^\vee) \to TM$ where $\pi : TM \to M$ is the canonical projection of the tangent-bundle, $\mathcal{L}$ is the pullback, under the quantization map, of the tautological line-bundle on $\mathbb{P}^\infty$, and $\mathcal{L}'$ is a, at this point unspecified, unitary line bundle whose introduction is due to the fact that (3.3.10) is an "equation up to a phase". Equivalently, property (3.3.11) says that the object $Z_A(\cdot, p)$ is an element of $\Gamma(M, (\mathcal{L} \otimes \mathcal{L}'^\vee) \otimes S^\bullet TM)$, which is what we set out to achieve. By $S^\bullet$ we have denoted symmetric tensors.

Now we shall give an explicit expression for $|u\rangle_{p,\tilde{A}}$. To this aim we use the fact that for a Kähler manifold, if ω is in canonical form, the map transforming Darboux to Riemann normal coordinates erected at a point $p \in M$ is a symplectomorphism at p. Let's denote the differential from Riemann normal coordinates at p by Λ and corresponding gauge transformation by S_Λ, then:

$$|u\rangle_{p,A} \sim S_\Lambda |\Lambda^T u\rangle_{flat}, \tag{3.3.12}$$

where by $|u\rangle_{flat}$, we denote the canonical, Poissonian coherent state (3.1.2).

For the following we will need to know how the Heisenberg algebra acts on $|u\rangle_{p,A}$ and also how $|u\rangle_{p,A}$ depends on p. Armed with the standard result in the flat case, under the assumption that the state is normalized to 1, we obtain:

$$\hat{x}|u\rangle_{p,A} = \left(\frac{1}{2}(1 - iJ)^T u + \frac{1}{2}(g^{-1} - i\omega^{-1})\vec{\nabla}_u + \frac{1}{4}(1 + iJ)^T u\right)|u\rangle_{p,A}. \tag{3.3.13}$$

It will be convenient in later sections to introduce the following notation:

$$v := \frac{1}{2}(1 - iJ)^T u$$
$$D_v := (1 - iJ)\vec{\nabla}_u.$$

Thus equation (3.3.13) takes the form:

$$\hat{x}|u\rangle_{p,A} = \left(v + \frac{1}{2} g^{-1} D_v + \frac{1}{2}\overline{v}\right)|u\rangle_{p,A}.$$

We will also be needing the following simple identities. First of all:

$${}_{p,A}\langle 0|u\rangle_{p,A} = \exp\left(-\frac{1}{4}||u||^2_{g(p)}\right)$$

and for any state $|\psi\rangle$, $\langle\psi|u\rangle_{p,A}$ is of the form:

$$\langle\psi|u\rangle_{p,A} = \exp\left(-\frac{1}{4}||u||^2_{g(p)}\right) f_\psi((1 - iJ)^T u),$$

where f_ψ is an arbitrary, appropriately normalizable[6], analytic function. Moreover, if we assume that:

$${}_{p,A}\langle 0|\psi\rangle \neq 0,$$

[6] We will discuss normalization conditions shortly.

then we can write f_ψ as the exponential of a power series in u.

Next we turn to the dependence of $|u\rangle_{p,A}$ on p. We shall thus analyse the action of the coordinate vectorfields at p on $|u\rangle_{p,A}$:

$$\begin{aligned}\frac{\partial}{\partial x^k}|u\rangle_{p,A} &= \text{phase}\cdot\left(\frac{\partial}{\partial x^k} - i\beta_k\right) S_\Lambda|\Lambda^T u\rangle_{flat} \\ &= \left(-i\beta_k + (\partial_k S_\Lambda)S_\Lambda^{-1} + (\partial_k\Lambda)^i_r u^r(\Lambda^{-1})^s_i\frac{\partial}{\partial u^s}\right)|u\rangle_{p,A} \\ &= \left(-i\beta_k + \frac{i}{2}(\Lambda(\partial_k\Lambda^{-1})\omega)_{ij}\hat{x}^i\hat{x}^j - u^r(\Lambda(\partial_k\Lambda^{-1}))^s_r\frac{\partial}{\partial u^s}\right)|u\rangle_{p,A}.\end{aligned}$$

In the above we have introduced a connection $i\beta = i\beta_k dx^k$ with $\beta_k \in \mathbb{R}$ on $\mathcal{L}'$. Now we use the fact that Λ is related to the Levi-Civita connection through:

$$\Gamma_k = -\Lambda\partial_k\Lambda^{-1}.$$

Of course the above is valid only at p, moreover, because of that, the connection β_k should not be expected to be flat. Finally we obtain:

$$\left(\frac{\partial}{\partial x^k} + i\beta_k - u^r\Gamma^s_{kr}\frac{\partial}{\partial u^s} + \frac{i}{2}(\Gamma_k\omega)_{ij}\hat{x}^i\hat{x}^j\right)|u\rangle_p = 0 \tag{3.3.14}$$

More explicitly we have:

$$\left(\frac{\partial}{\partial x^k} + i\beta_k - u^r\Gamma^s_{kr}\frac{\partial}{\partial u^s} + \frac{i}{2}(\Gamma_k\omega)_{ij}(v + \frac{1}{2}g^{-1}D_v + \frac{1}{2}\overline{v})^i(v + \frac{1}{2}g^{-1}D_v + \frac{1}{2}\overline{v})^j\right)|u\rangle_p = 0.$$

At this point it is important to notice that as we chose the map Λ to Riemann normal coordinates we could have also chosen a vielbein[7]. The difference reflects itself in the choice of connection β. More generally, the states $|u\rangle_p$ are parallel transported along M (up to a phase) by the lift to the Hilbert bundle of any metric compatible and (quasi-)symplectic connection on TM. Choosing the vielbein instead of the map to Riemann normal coordinates corresponds to the Weitzenböck connection [99], which while not torsion free as the Levi-Civita connection, is flat. To see this we resort to the defining equation (3.3.9). Let B denote a connection on TM. Choose a path $\gamma : [0,1] \to M$. The statement that $|u\rangle_p$ is parallel transported (up to a phase) by the lift of B along γ is:

$$\left|\left[\mathcal{P}\exp\left(\int_0^t \gamma^* B\right)\right]^T u\right\rangle_{\gamma(t)} \sim \mathcal{P}\exp\left(-\frac{i}{2}\int_0^t (\gamma^* B\omega)_{ij}\hat{x}^i\hat{x}^j\right)|u\rangle_{\gamma(0)}.$$

The above is equivalent to:

$$(g(\gamma(t)) - i\omega)_{ij}\hat{x}^j\mathcal{P}\exp\left(\frac{i}{2}\int_0^t (\gamma^* B\omega)_{ij}\hat{x}^i\hat{x}^j\right)|u\rangle_{\gamma(0)}$$

[7]Of course this is valid only locally, however for parallelizable manifolds (e.g. Lie groups) this can hold globally.

$$= (g(\gamma(t)) - i\omega)_{ij} \left(\left[\mathcal{P}\exp\left(\int_0^t \gamma^* B \right) \right]^T u \right)^j \mathcal{P}\exp\left(\frac{i}{2} \int_0^t (\gamma^* B\omega)_{ij} \hat{x}^i \hat{x}^j \right) |u\rangle_{\gamma(0)},$$

which infinitesimally reads:

$$\begin{aligned} 0 &= X^k \left(\partial_k g_{ij} (\hat{x}^j - u^j) - (g - i\omega)_{ij} (B_k)^j_l (\hat{x}^l - u^l) \right) |u\rangle_0 \\ &= X^k \left((\partial_k g_{ij} - (B_k)^l_i g_{lj} - g_{il} (B_k)^l_j) + i((B_k)^l_i \omega_{lj} + \omega_{il} (B_k)^l_j) \right) (\hat{x}^j - u^j) |u\rangle_0. \end{aligned}$$

We thus obtain that whatever the choice of X and thus γ, the above is fulfilled provided B is compatible with both metric and symplectic form.

To end this section we shall discuss the normalization condition on $Z_A(u,p)$. By this we mean the property that if $|p_0\rangle_A$ is normalized to 1 for a given point $p_0 \in M$ so will $|p\rangle_A$ for any other $p \in M$ due to the fact that the flat connection A induces a unitary parallel transport. In order to express this property in terms of $Z_A(u,p)$, recall that in ordinary quantum mechanics, that is the quantum mechanics of $M = \mathbb{R}^{2d}$, the identity operator is expressed in terms of the coherent states as follows:

$$\mathrm{Id} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2n}} |u\rangle\langle u| \, du^1 \wedge \cdots \wedge du^{2d}. \tag{3.3.15}$$

Thus, resorting to (3.3.12), in our case we obtain:

$$\mathrm{Id} = \frac{1}{(2\pi)^d} \int_{T_pM} \sqrt{\det(g_A(p))} \, |u\rangle_{p,A} \, {}_{p,A}\langle u| \, du^1 \wedge \cdots \wedge du^{2d}, \tag{3.3.16}$$

and therefore:

$$1 = \frac{1}{(2\pi)^d} \int_{T_pM} \sqrt{\det(g_A(p))} \, |Z_A(u,p)|^2 \, du^1 \wedge \cdots \wedge du^{2d}. \tag{3.3.17}$$

In the above we have denoted the metric by g_A to emphasize that it is expressed in the coordinate system corresponding to A.

3.3.4 Master equation

At this point we have all the ingredients to formulate the master equation. By this we mean the statement that $|p\rangle_A$ is parallel transported by the flat connection A, expressed as a differential equation for the wavefunction $Z_A(u,p)$. Using (3.3.14) we obtain:

$$\begin{aligned} 0 &= {}_{p,A}\langle u| \frac{\partial}{\partial x^k} + A_k |p\rangle_A \\ &= \frac{\partial}{\partial x^k} Z_A(u,p) - \frac{\partial}{\partial x^k} \left({}_{p,A}\langle u| \right) |p\rangle_A + {}_{p,A}\langle u| A_k |p\rangle_A \\ &= \left(\frac{\partial}{\partial x^k} - i\beta_k - u^r \Gamma^s_{kr} \frac{\partial}{\partial u^s} \right) Z_A(u,p) + {}_{p,A}\langle u| - \frac{i}{2} (\Gamma_k \omega)_{ij} \hat{x}^i \hat{x}^j + A_k |p\rangle_A \end{aligned}$$

$$= \left(\frac{\partial}{\partial x^k} - i\beta_k - u^r \Gamma^s_{kr} \frac{\partial}{\partial u^s} \right) Z_A(u,p) + i\,{}_{p,A}\langle u|\alpha_k + \omega_{ik}\hat{x}^i - \frac{1}{2}((\Gamma_k - G_k)\omega)_{ij}\hat{x}^i\hat{x}^j|p\rangle_A.$$

Resorting to (3.3.13) we thus obtain:

$$\Big(\frac{\partial}{\partial x^k} - u^r \Gamma^s_{kr} \frac{\partial}{\partial u^s} + i(\alpha_k - \beta_k) - i\omega_{ki}(\overline{v} + \frac{1}{2}g^{-1}\overline{D}_v + \frac{1}{2}v)^i$$
$$+\frac{i}{2}C_{kij}(\overline{v} + \frac{1}{2}g^{-1}\overline{D}_v + \frac{1}{2}v)^i(\overline{v} + \frac{1}{2}g^{-1}\overline{D}_v + \frac{1}{2}v)^j \Big) Z_A(u,p) = 0, \qquad (3.3.18)$$

where the tensor C is given by:

$$C_{kij} = ((\Gamma_k - G_k)\omega)_{ij}.$$

Equation (3.3.18) should be viewed as a version of the holomorphic anomaly equation of [6], which is the master equation for the generating function of topological closed-string amplitudes. However at this point this statement is not completely transparent. Indeed in our case the phase space is an affine special Kähler manifold, while what should play the role of a classical phase space in topological string theory is the vector-multiplet moduli space of $\mathcal{N} = (2,2)$ conformal field theories. This has the structure of a projective special Kähler manifold. We shall tackle this geometry in section 3.4.5. The crucial point is that projective special Kähler manifolds can be recovered as quotients of affine conic (however Lorentzian) special Kähler manifolds.

A further remark, that we will clarify in later sections, is that so far Planck's constant $\hbar$ has not manifestly appeared in our quantization scheme. At this point, its introduction would be merely as an arbitrary rescaling of the symplectic form. We will instead see in section 3.4.5 how the notion of Planck's constant arises naturally in the passage from affine to projective geometry.

Before delving into these matters we will analyze the solution to the master equation. We shall denote by $|p\rangle^s$ the coherent state of $p \in M$ and $|u\rangle^s_p$ the coherent state of $u \in T_pM$ in special Darboux coordinates gauge. Then, the master equation reduces to:

$$\Big(\frac{\partial}{\partial x^k} - u^r \Gamma^s_{kr} \frac{\partial}{\partial u^s} + i(\alpha_k - \beta_k) - i\omega_{ki}(\overline{v} + \frac{1}{2}v)^i + \frac{1}{2}(\overline{D}_v)_k$$
$$+\frac{i}{2}(\Gamma_k\omega)_{ij}\overline{v}^i\overline{v}^j + \frac{i}{8}(\Gamma_k\omega)_{ij}(g^{-1}\overline{D}_v + v)^i(g^{-1}\overline{D}_v + v)^j \Big) Z_s(u,p) = 0,$$

where we have used $\omega_{ki}(g^{-1}\overline{D}_v)^i = i(\overline{D})_k$ and the fact that $\Gamma_k\omega$ splits into holomorphic and anti-holomorphic components. This is immediately verified using the explicit formula:

$$(\Gamma_k\omega)_{ij} = \frac{1}{2}\partial_i\partial_k\partial_r K J^r_j.$$

Since C is a tensor, it will split in holomorphic and anti-holomorphic components in any coordinate system. Thus the equation above is valid in general with $(\Gamma_k\omega)_{ij}$ replaced by

C_{kij}. It will be convenient in the following to split the master equation into its holomorphic and anti-holomorphic parts. In the following we shall compute in special Darboux coordinates. The anti-holomorphic part is then given by:

$$(1+iJ)^k_l \left(\frac{\partial}{\partial x^k} - \frac{1}{2}\overline{v}^r \Gamma^s_{kr} (D_v)_s + i(\alpha_k - \beta_k) - \frac{i}{2}\omega_{ki} v^i \right.$$
$$\left. + \frac{1}{2}(\overline{D}_v)_k + \frac{i}{2} C_{kij} \overline{v}^i \overline{v}^j \right) Z_s(u,p) = 0,$$

while the holomorphic part reads:

$$(1-iJ)^k_l \left(\frac{\partial}{\partial x^k} - \frac{1}{2} v^r \Gamma^s_{kr} (\overline{D}_v)_s + i(\alpha_k - \beta_k) - i\omega_{ki} \overline{v}^i \right.$$
$$\left. + \frac{i}{8} C_{kij} (g^{-1}\overline{D}_v + v)^i (g^{-1}\overline{D}_v + v)^j \right) Z_s(u,p) = 0.$$

We shall now assume:

$$_{p,A}\langle 0|p\rangle_A \neq 0. \tag{3.3.19}$$

This is no loss of generality as long as one restricts attention to a small enough neighborhood of p. Then as discussed in section 3.3.3, we are allowed to write the following ansatz for $Z_s(u,p)$:

$$Z_s(u,p) = \exp\left(\sum_{n\geq 0} \sum_{i_1,\ldots,i_n} \frac{1}{n!} C^n_{i_1,\ldots,i_n} \overline{v}^{i_1} \cdots \overline{v}^{i_n} - \frac{1}{4} ||u||^2_{g(p)} \right),$$

where the C^n's are symmetric tensors. For the following we shall need a few identities:

$$\begin{aligned} (\overline{D}_v)_k v^l &= 0 \\ (\overline{D}_v)_k \overline{v}^l &= (1+iJ)^l_k \\ ||u||^2_g &= 2\overline{v}^i v^j g_{ij}. \end{aligned}$$

First we analyze the equation arising from the anti-holomorphic part of the flat connection:

$$\begin{aligned} 0 = & \sum_{n\geq 0} \sum_{i_1,\ldots,i_n} \frac{1}{n!} \overline{v}^k \partial_k C^n_{i_1,\ldots,i_n} \overline{v}^{i_1} \cdots \overline{v}^{i_n} \\ & + \frac{1}{n!} \overline{v}^k \sum_{r=1}^{n} C^n_{i_1,\ldots,i_n} \overline{v}^{i_1} \cdots \frac{i}{2} \partial_k J^{i_r}_s u^s \cdots \overline{v}^{i_n} \\ & + \frac{1}{(n-1)!} C^n_{i_1,\ldots,i_n} \overline{v}^{i_1} \cdots \overline{v}^{i_n} \\ & + i\overline{v}^k (\alpha_k - \beta_k) + \frac{i}{2} C_{kij} \overline{v}^k \overline{v}^i \overline{v}^j. \end{aligned} \tag{3.3.20}$$

Now we use the fact that in a Kähler manifold, the complex structure is parallel transported by the Levi-Civita connection. In components this reads:

$$\partial_k J^r_s + J^t_s \Gamma^r_{kt} - J^r_t \Gamma^t_{ks} = 0.$$

We substitute for $\partial_k J^{i_r}_s$ in (3.3.20). The part of the expression involving the Levi-Civita connection thus becomes:

$$\begin{aligned}
&\frac{i}{2n!}\overline{v}^k \sum_{r=1}^{n} C^n_{i_1,\ldots,i_n}\overline{v}^{i_1}\ldots\left[-J^t_s\Gamma^{i_r}_{kt} + J^{i_r}_t\Gamma^t_{ks}\right]u^s\cdots\overline{v}^{i_n}\\
&= -\frac{1}{n!}\overline{v}^k\Gamma^{i_r}_{ks}\overline{v}^s\sum_{r=1}^{n} C^n_{i_1,\ldots,i_n}\overline{v}^{i_1}\ldots\widehat{\overline{v}^{i_r}}\cdots\overline{v}^{i_n}\\
&= 0.
\end{aligned}$$

We thus obtain the following recursive formula:

$$C^{n+1}_{i_1,\ldots,i_{n+1}}\overline{v}^{i_1}\cdots\overline{v}^{i_{n+1}} = -\overline{v}^k\partial_k C^n_{i_1,\ldots,i_n}\overline{v}^{i_1}\cdots\overline{v}^{i_n}$$

for all $n \geq 3$, while the lower terms yield:

$$\begin{aligned}
C^1_{i_1}\overline{v}^{i_1} &= -\overline{v}^k\partial_k C^0 - i\overline{v}^k(\alpha_k - \beta_k)\\
C^2_{i_1,i_2}\overline{v}^{i_1}\overline{v}^{i_2} &= -\overline{v}^k\partial_k C^1_{i_1}\overline{v}^{i_1}\\
C^3_{i_1,i_2,i_3}\overline{v}^{i_1}\overline{v}^{i_2}\overline{v}^{i_3} &= -\overline{v}^k\partial_k C^2_{i_1,i_2}\overline{v}^{i_1}\overline{v}^{i_2} - iC_{ijk}\overline{v}^i\overline{v}^j\overline{v}^k.
\end{aligned}$$

At this point it is convenient to introduce complex coordinates $(z^1,\ldots,z^d)$ with $d = \dim(M)/2$ that we shall label with greek letters. We shall further denote by $\nabla^{(0,1)}$ the anti-holomorphic part of the Levi-Civita covariant derivative. We now introduce the covariant tensors $\mathcal{C}^n$ defined by:

$$\begin{aligned}
\mathcal{C}^n &= \frac{(-1)^n}{2^n}C_{i_1,\ldots,i_n}(1+iJ)^{i_1}_{j_1}\cdots(1+iJ)^{i_n}_{j_n}dx^{j_1}\cdots dx^{j_n}\\
&= \mathcal{C}^n_{\overline{\mu}_1,\ldots,\overline{\mu}_n}dz^{\overline{\mu}_1}\cdots dz^{\overline{\mu}_n},
\end{aligned}$$

where by "$\cdot$" we denote the symmetrized tensor product. The above equations then take the form:

$$\mathcal{C}^1 = \overline{\partial}\mathcal{C}^0 + i(\alpha_{\overline{\mu}} - \beta_{\overline{\mu}})dz^{\overline{\mu}} \tag{3.3.21}$$

$$\mathcal{C}^3 = \nabla^{(0,1)}\mathcal{C}^2 + iC_{\overline{\mu}_1,\overline{\mu}_2,\overline{\mu}_3}dz^{\overline{\mu}_1}dz^{\overline{\mu}_2}dz^{\overline{\mu}_3} \tag{3.3.22}$$

$$\mathcal{C}^{n+1} = \nabla^{(0,1)}\mathcal{C}^n \quad \forall n \geq 1,\ n \neq 2. \tag{3.3.23}$$

Notice, in particular, that the solution to the master equation is completely determined by a single object, $\mathcal{C}^0$. Clearly the latter is given by:

$${}_p\langle 0|p\rangle = \exp(\mathcal{C}^0(p)) =: c(p).$$

Notice, that thanks to property (3.3.11), this quantity is independent of A. Indeed the above is a section of $\mathcal{L}\otimes\mathcal{L}'^{\vee}\to M$ which (by assumption (3.3.19)) is non-vanishing over the open set under consideration. This section is, in turn, determined by the holomorphic part of the master equation. For its analysis, it is convenient to replace the wavefunction by the inhomogeneous tensor:

$$\mathcal{C} = \exp\left(\sum_{n\geq 0}\frac{(-1)^n}{n!}\mathcal{C}^n\right).$$

In terms of this, the anti-holomorhic part of the master equation reads:

$$\left(\nabla^{(0,1)} + i(\alpha_{\overline{\mu}} - \beta_{\overline{\mu}})dz^{\overline{\mu}} + d\overline{z}^{\overline{\mu}}\iota_{\partial_{\overline{\mu}}} + \frac{i}{2}C_{\overline{\mu\nu\rho}}d\overline{z}^{\overline{\mu}}d\overline{z}^{\overline{\nu}}d\overline{z}^{\overline{\rho}}\right)\mathcal{C} = 0. \tag{3.3.24}$$

The holomorphic part instead becomes:

$$\left(\partial + \frac{i}{2}dz^{\mu}C_{\mu\nu\rho}g^{\nu\overline{\nu}}g^{\rho\overline{\rho}}\iota_{\partial_{\overline{\nu}}}\iota_{\partial_{\overline{\rho}}} + i(\alpha_{\mu} - \beta_{\mu})dz^{\mu} - \frac{i}{2}\omega\right)\mathcal{C} = 0. \tag{3.3.25}$$

The above can be seen as yielding an infinite number of differential equations for the section c.

Now we shall analyse the integrability of the master equation. As we will see, and as is to be expected, this precisely specifies the line-bundle $\mathcal{L}'$. The result of this computation are the following three equations, namely the $(0,2)$, $(1,1)$ and $(2,0)$ components of the underlying Maurer-Cartan equation respectively:

$$\begin{aligned}
\partial_{\overline{\mu}}(\alpha_{\overline{\nu}} - \beta_{\overline{\nu}}) - \partial_{\overline{\nu}}(\alpha_{\overline{\mu}} - \beta_{\overline{\mu}}) &= 0\\
i\partial_{\mu}(\alpha_{\overline{\nu}} - \beta_{\overline{\nu}}) - i\partial_{\overline{\nu}}(\alpha_{\mu} - \beta_{\mu}) &= \frac{1}{2}\left(C_{\mu\rho\sigma}g^{\rho\overline{\rho}}g^{\sigma\overline{\sigma}}C_{\overline{\nu\rho\sigma}} + 2g_{\mu\overline{\nu}}\right)\\
\partial_{\mu}(\alpha_{\nu} - \beta_{\nu}) - \partial_{\nu}(\alpha_{\mu} - \beta_{\mu}) &= 0.
\end{aligned}$$

In particular, choosing α in canonical form (3.3.5), we have:

$$\begin{aligned}
\partial_{\overline{\mu}}\beta_{\overline{\nu}} - \partial_{\overline{\nu}}\beta_{\overline{\mu}} &= 0\\
i\partial_{\mu}\beta_{\overline{\nu}} - i\partial_{\overline{\nu}}\beta_{\mu} &= -\frac{1}{2}C_{\mu\rho\sigma}g^{\rho\overline{\rho}}g^{\sigma\overline{\sigma}}C_{\overline{\nu\rho\sigma}} = -\frac{1}{2}R_{\mu\overline{\nu}} = \frac{i}{2}\rho_{\mu\overline{\nu}}\\
\partial_{\mu}\beta_{\nu} - \partial_{\nu}\beta_{\mu} &= 0,
\end{aligned}$$

where $R_{\mu\overline{\nu}}$ and $\rho_{\mu\overline{\nu}}$ denote the components of the Ricci tensor and Ricci form respectively. See appendix A.1 for the identity used in the second equation. Thus, since smooth line-bundles are completely specified by their first Chern-class we obtain that $\mathcal{L}'$ is isomorphic, as a smooth bundle, to the square-root of the canonical bundle. Moreover β is in canonical form:

$$\beta = i(\partial - \overline{\partial})\chi$$

and up to holomorphic gauge transformations, we can choose:

$$\chi = \frac{1}{4}\log\sqrt{\det g}.$$

We can trivially twist the line-bundle $\mathcal{L}\otimes\mathcal{L}'^{\vee}$ to an anti-holomorphic line-bundle by multiplying $\mathcal{C}$ by an appropriate factor. We thus define:

$$\mathcal{S} := (\det g)^{\frac{1}{8}}\, e^{\frac{K}{2}}\mathcal{C}.$$

Finally, in terms of $\mathcal{S}$, the master equation acquires the form of a "(anti-) holomorphic-anomaly equation":

$$\left(\nabla^{(0,1)} - \overline{\partial}K - \frac{1}{2}\overline{\partial}\log\sqrt{\det g} + d\overline{z}^{\overline{\mu}}\iota_{\partial_{\overline{\mu}}} + \frac{i}{2}C_{\overline{\mu\nu\rho}}d\overline{z}^{\overline{\mu}}d\overline{z}^{\overline{\nu}}d\overline{z}^{\overline{\rho}}\right)\mathcal{S} = 0 \qquad (3.3.26)$$

$$\left(\partial + \frac{i}{2}dz^{\mu}C_{\mu\nu\rho}g^{\nu\overline{\nu}}g^{\rho\overline{\rho}}\iota_{\partial_{\overline{\nu}}}\iota_{\partial_{\overline{\rho}}} - \frac{i}{2}\omega\right)\mathcal{S} = 0. \qquad (3.3.27)$$

Now, we shall consider the section of the holomorphic line-bundle $(\mathcal{L}\otimes\mathcal{L}'^{\vee})_{ahol}$:

$$s(p) := \exp(\mathcal{S}^0).$$

The first order component of the holomorphic part of the master equation reads:

$$\partial_{\mu}s = -\frac{i}{2}C_{\mu\nu\rho}g^{\nu\overline{\nu}}g^{\rho\overline{\rho}}\left(D^{(0,1)}D^{(0,1)}s\right)_{\overline{\nu\rho}}, \qquad (3.3.28)$$

where $D^{(0,1)}$ denotes the anti-holomorphic part of the Levi-Civita connection twisted by the Chern connection on $(\mathcal{L}\otimes\mathcal{L}'^{\vee})_{ahol}$, which is naturally a hermitian line-bundle with hermititan form:

$$h = (\det g)^{-\frac{1}{4}}e^{-K}.$$

In fact, the integrability of the master equation ensures that a solution s to (3.3.28) lifts, through the recursion relations (3.3.21–3.3.23), to a solution of the full master equation. Of course, a priori, there are, if any, more than one solution to the above equation. indeed in the simplest case, i.e. $M = \mathbb{R}^{2d}$, there are infinitely many solutions. This follows immediately from the fact that, in that particular case, the tensor C vanishes and thus any anti-holomorphic section s solves the problem. Recall that this arbitrariness corresponds to the freedom of choosing the image of a particular marked point on M via the quantization map. The canonical quantization of $\mathbb{R}^{2d}$ has as solution $s = 1$. In the next subsection we will construct the general solution for any affine special Kähler manifold.

3.3.5 Constructing the solution I: the role of special coordinates

In this section we will give an explicit expression for the Green's function of the holomorphic anomaly equation, thereby providing its general solution. We start by choosing an arbitrary point $p_0 \in M$ and declare that:

$$p_0 \overset{\phi}{\mapsto} |p_0\rangle_A := |y\rangle_{p_0,A}.$$

Now we ask where a point p, in its Darboux neighborhood, is mapped to. For this we shall consider the action of the annihilation operators on $|p\rangle$

$$\begin{aligned}\hat{x}^T(g+i\omega)_{p_0,A}|p\rangle_A &= \hat{x}^T(g+i\omega)_{p_0,A}U(p,p_0)|y\rangle_{p_0,A}\\ &= U_A(p,p_0)\left(U_A(p_0,p)\hat{x}^T U_A(p,p_0)\right)(g+i\omega)_{p_0,A}|y\rangle_{p_0,A}.\end{aligned}$$

The p dependent operator $\hat{x}^T$ is found by computing its infinitesimal variation:

$$\begin{aligned}\frac{\partial}{\partial x^k}\left(U_A(p,p')\hat{x}^l U_A(p',p)\right|_{p'=p} &= [A_k,\hat{x}^l]\\ &= i[\omega_{ik}\hat{x}^i - D_{ijk}\hat{x}^i\hat{x}^j,\hat{x}^l]\\ &= -\omega_{ik}\omega^{il} - 2D_{ijk}\omega^{jl}\hat{x}^i\\ &= \delta^l_k - G^l_{ki}\hat{x}^i.\end{aligned}$$

Bringing the last term to the left hand side we thus obtain:

$$(\partial_k + G_k)\left(U_A(p,p')\hat{x}U_A(p',p)\right|_{p'=p} = \delta^l_k,$$

therefore:

$$U_s(p_0,p)\hat{x}U_s(p,p_0) = \hat{x} + x^p_{p_0},$$

where $x^p_{p_0}$ is the coordinate vector of p in special Darboux coordinates around p_0. Therefore:

$$\hat{x}^T(g+i\omega)_{p_0,s}|p\rangle_s = (x^p_{p_0}+y)^T(g+i\omega)_{p_0}|p\rangle_s,$$

hence:

$$|p\rangle_s \sim |x^p_{p_0}+y\rangle_{p_0,A_s}.$$

From the point of view of topological string theory, what the above equation means is that the topological conformal field theory corresponding to the point p is related to the one at p_0 by a deformation whose modulus corresponds to the special coordinate vector of p relative to p_0.

Now we shall fix the phase ambiguity. We will need the following identity:

$$\left((\overline{D}_v)_k + \frac{1}{2}(g-i\omega)_{kj}u^j\right)|u\rangle_p = 0$$

and for simplicity, until otherwise stated we shall denote $x^p_{p_0}$ simply by x. Then

$$\begin{aligned}0 &= \left(\frac{\partial}{\partial x^k} + i(\alpha_k-\gamma_k) - i\omega_{ki}\hat{x}^i\right)|x+y\rangle_{p_0,A_s}\\ &= \left(\frac{\partial}{\partial x^k} + i(\alpha_k-\gamma_k) - i\omega_{ki}(v+\frac{1}{2}g^{-1}_{p_0}D_v+\frac{1}{2}\overline{v})^i\right)|x+y\rangle_{p_0,A_s}\end{aligned}$$

$$= i(\alpha_k - \gamma_k - \frac{1}{2}\omega_{kj}(x+y)^j)|x+y\rangle_{p_0,A_s},$$

where $i\gamma_k = \partial_k\theta$ and θ is the phase discrepancy. Also, in the present case, $v = (1/2)(1 - iJ)x$. Thus, since we have chosen α in canonical form:

$$\begin{aligned}\gamma_k &= \alpha_k - \frac{1}{2}\omega_{kj}(x+y)^j \\ &= -\frac{1}{2}J_k^l\partial_l K - \frac{1}{2}\omega_{kj}(x+y)^j.\end{aligned}$$

So we have:

$$|p\rangle_s = \exp\left(\frac{i}{2}\int_{p_0}^{p}(J_k^l\partial_l K + \omega_{kj}(x+y)^j)dx^k\right)|x+y\rangle_{p_0,A_s}.$$

Now we are left to compute the kernel:

$$K(u,p;x+y,p_0) := {}_{p,A_s}\langle u|p\rangle_s^y. \tag{3.3.29}$$

Given the above, the general solution to the master equation over a special Darboux patch is given by:

$$Z(u,p)^f = \int_{T_{p_0}M}\sqrt{\det g_s(p_0)}\,dy^1\wedge\cdots\wedge dy^{2d}\,K(u,p,x_s+y,p_0)\cdot \tag{3.3.30}$$

$$\exp\left(-\frac{1}{4}||y||^2_{g_s(p_0)}\right)f((1+iJ_0)y), \tag{3.3.31}$$

where f is any analytic gaussian-integrable function. The final step is thus to obtain an expression for the states $|u\rangle_{p,s}$ that is valid over an entire Darboux patch, rather than just at a point as we previously defined them in (3.3.12).

3.3.6 Constructing the solution II: revisiting the coherent tangent bundle

In this section we will give a patchwise description of the coherent states $|u\rangle_{p,s}$. The simplest way to find $|u\rangle_p$ is through the defining differential equations (3.3.9, 3.3.13, 3.3.14). It is convenient to express $|u\rangle_{p,s}$ as the wavefunction $\psi_u(q,J) = \langle q|u\rangle_{p,s}$, where J is the complex structure matrix at the point p and $|q\rangle$ is the eigenstate of $\hat{q}$ with vector of eigenvalues q. Here the vector of operators $\hat{q}$ is the upper half of the vector $\hat{x}$ in special Darboux coordinates, that is the position coordinates, rather than the momentum coordinates, which we will not label to avoid confusion with the label p that stands for the point on the manifold on which these wavefunctions are erected. Accordingly we will need to split metric and symplectic form in $d\times d$ blocks:

$$g =: \begin{pmatrix} R_1 & R_2 \\ R_2^T & R_4 \end{pmatrix},\quad \omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then (3.3.9) becomes:

$$\begin{pmatrix} R_1 & R_2 + i \\ R_2^T - i & R_4 \end{pmatrix} \begin{pmatrix} q - u_q \\ -i\nabla_q - u_p \end{pmatrix} \psi_u(q, J) = 0,$$

which is equivalent to:

$$\nabla_q \psi_u = (i\tau(q - u_q) + iu_p)\psi_u, \tag{3.3.32}$$

where

$$\tau = R_4^{-1}(i - R_2^T) = -(R_2 + i)^{-1} R_1$$

is a symmetric matrix called the complex modulus and encodes one-to-one the complex structure $J(p)$. In particular τ is an element of the Siegel upper-half space [73], which we shall denote as $\mathbb{H}_d$. A consequence of which is that:

$$\det \mathrm{Im}\tau > 0.$$

The solution to (3.3.32) is then:

$$\psi_u(q, \tau) = \mathcal{N}(\tau, u) \exp\left(\frac{i}{2}\langle (q - u_q), \tau(q - u_q)\rangle + iu_p q\right),$$

where $\mathcal{N}(\tau, u)$ is yet to be determined. The latter is however constrained by three conditions. The first is the normalization condition on $|u\rangle_{p,s}$ while the second and third are the defining equations (3.3.13, 3.3.14). The first implies:

$$\mathcal{N}(\tau, u) = |\mathcal{N}(\tau, u)| e^{i\theta(\tau, u)}$$

where:

$$\begin{aligned} |\mathcal{N}| &= \left(\int_{\mathbb{R}^d} d^d q \left| \exp\left(\frac{i}{2}\langle (q - u_q), \tau(q - u_q)\rangle + iu_p q\right)\right|^2\right)^{-\frac{1}{2}} \\ &= \pi^{-d/4} \left(\det \mathrm{Im}\tau\right)^{1/4}. \end{aligned}$$

Equation (3.3.13) then reduces to:

$$(\nabla_{u_q} + \overline{\tau}\nabla_{u_p})\theta = -\frac{1}{2}(u_p + \overline{\tau} u_q).$$

Solving for the real and imaginary parts separately we obtain:

$$\theta = -\frac{1}{2}\langle u_q, u_p\rangle + \gamma(\tau).$$

Thus we arrive at the following solution:

$$\psi_u(q, \tau) = \pi^{-d/4} \left(\det \mathrm{Im}\tau\right)^{1/4} \exp\left(i\gamma(\tau) - \frac{i}{2}\langle u_q, u_p\rangle + \frac{i}{2}\langle (q - u_q), \tau(q - u_q)\rangle + iu_p\, q\right),$$

where the only undetermined quantity is the phase $\gamma(\tau)$. The phase is fixed (always up to an irrelevant constant), by equation (3.3.14) and the choice of the connection β on $\mathcal{L}'$. In particular equation (3.3.14) yields:

$$\partial_k\gamma = -\beta_k - \frac{1}{2}\mathrm{tr}\left([(\Gamma_k\omega)_{pp}]\mathrm{Im}\tau\right).$$

Here we remark the similarity of $\psi_u(q,\tau)$ with the wavefunctions discussed in [26]. At this point we have all the ingredients to compute the kernel (3.3.29) of the master equation. We start with the computation of the overlap between a coherent state at p_0, where the complex modulus is τ_1, and one at p with complex modulus τ_2:

$$\begin{aligned}
&{}_{\tau_2}\langle u_2|u_1\rangle_{\tau_1} = \int d^dq\overline{\psi}(q,u_2,\tau_2)\psi(q,u_1,\tau_1) = \\
&(2i)^{d/2}\frac{(\det\mathrm{Im}\tau_1)^{1/4}(\det\mathrm{Im}\tau_2)^{1/4}}{(\det(\tau_1-\overline{\tau_2}))^{1/2}}\cdot \\
&\exp\left(-\frac{i}{2}\langle u_{q,1},z_1\rangle + \frac{i}{2}\langle u_{q,2},\overline{z}_2\rangle - \frac{i}{2}\langle(z_1-\overline{z}_2),(\tau_1-\overline{\tau}_2)^{-1}(z_1-\overline{z}_2)\rangle\right) \\
&\cdot\exp\left(i\int_{p_0}^{p}\left(\beta+\frac{1}{2}\mathrm{tr}((\Gamma_k\omega)_{pp}\mathrm{Im}\tau)dx^k\right)\right),
\end{aligned}$$

where we have introduced the complex coordinates $z = u_p - \tau u_q$. We shall now denote by z_1 the coordinates corresponding to $u_1 = x + y$ and by z_2 the ones corresponding to u, then the kernel is given by:

$$\begin{aligned}
&K(u,p;x+y,p_0) = (2i)^{d/2}\frac{(\det\mathrm{Im}\tau_1)^{1/4}(\det\mathrm{Im}\tau_2)^{1/4}}{(\det(\tau_1-\overline{\tau_2}))^{1/2}}\cdot \\
&\exp\left(-i\int_{p_0}^{p}\left((\alpha-\beta)-\frac{1}{2}\omega_{kj}(x+y)^jdx^k-\frac{1}{2}\mathrm{tr}((\Gamma_k\omega)_{pp}\mathrm{Im}\tau)dx^k\right)\right) \\
&\exp\left(-\frac{1}{4}||u||^2_{g(p)} - \frac{i}{2}\langle u_{q,1},z_1\rangle + \frac{1}{4}\langle\overline{z}_2,R_4(p)\overline{z}_2\rangle - \frac{i}{2}\langle(z_1-\overline{z}_2),(\tau_1-\overline{\tau}_2)^{-1}(z_1-\overline{z}_2)\rangle\right).
\end{aligned}$$

We thus conclude the study of affine Riemannian special Kähler manifolds having provided the general solution (3.3.31) to the master equation (3.3.18).

3.4 Conic special Kähler manifolds

First of all we shall recover the structure of a projective special Kähler manifold in a way best suited for our quantization technique. Before that we shall recall the standard definitions (see e.g. [64] for a comprehensive review).

Definition 3.4.1. (**Projective special Kähler manifold - 1**) A projective special Kähler manifold is a holomorphic quotient of an affine conic special Kähler manifold by its defining holomorphic $\mathbb{C}^*$ action.

Definition 3.4.2. (**Conic special Kähler manifold - 1**) An affine conic special Kähler manifold is an affine special Kähler manifold equipped with a free holomorphic $\mathbb{C}^*$ action whose generating holomorphic vectorfield H is a homothetic Killing vectorfield for the flat symplectic connection:

$$(d+G)H = \pi^{(1,0)} := \frac{1}{2}(1-iJ)^l_k \, dx^k \otimes \partial_l. \tag{3.4.1}$$

Recall, that by definition of affine special Kähler: $d_{d+G}\pi^{(1,0)} = 0$. This ensures the existence of a vectorfield that satisfies the equation above. The restriction here, is that this vectorfield is required to be holomorphic.

We shall now analyse equation (3.4.1) in special Darboux coordinates. We shall introduce the vectorfield X through:

$$H = \pi^{(1,0)}(-iX) = -\frac{i}{2}(1-iJ)X.$$

Without loss of generality, we can choose X real. Then equation (3.4.1) becomes:

$$\partial_i(-JX)^j = \delta^j_i \tag{3.4.2}$$
$$\partial_i X^j = J^j_i. \tag{3.4.3}$$

Thus, up to an irrelevant constant vectorfield, from the second equation we obtain:

$$X^j = \omega^{rj}\partial_r K,$$

which can be recast in the form

$$dK = \iota_X \omega,$$

that is, X is the Hamiltonian vectorfield with Hamiltonian the special Kähler potential. From the first equation we obtain:

$$\begin{aligned}
0 &= (\partial_i J^j_k)X^k \\
&= (\partial_i g_{kr})\omega^{rj}X^k \\
&= \omega^{rj}(X^k\partial_k g_{ij}) \\
&= -\omega^{jr}((L_X g)_{ji} + g([X,\partial_j],\partial_i) + g(\partial_j,[X,\partial_i])) \\
&= -\omega^{jr}((L_X g)_{ji} + \omega_{ji} + \omega_{ij}) \\
&= -\omega^{jr}(L_X g)_{ji}.
\end{aligned}$$

Therefore equation (3.4.2) is equivalent to:

$$L_X g = 0. \tag{3.4.4}$$

Since X determines g, the above is a differential equation for K and defines a particular class of special Kähler metrics, namely the conical ones. Another way to read equation (3.4.4) is as follows:

$$0 = \omega^{kl}\partial_k K \partial_l g_{ij}$$

$$= 2g^{ks}\partial_k K C_{sij}.$$

From this it follows:

$$C_{ijk}H^k = 0.$$

We thus arrive at our best suited definition for conic special Kähler manifolds:

Definition 3.4.3. (**Conic special Kähler manifold - 2**) A conic special Kähler manifold is an affine special Kähler manifold whose symplectic vectorfield X, defined locally through $dK = \iota_X \omega$ is simultaneously a Killing vectorfield for the Kähler metric g.

The existence of the vectorfield X implies that the function K can be extended from a special Darboux patch to any simply connected patch containing it, in particular to a patch which is dense in M.

Now we shall shortly digress to recover the above geometric structure from the point of view of quantization. For this we will have to find yet two other ways to write identity (3.4.4):

$$\begin{aligned} 0 &= \omega^{kl}\partial_k K \partial_l g_{ij} \\ &= \omega^{kl}\partial_k K \partial_l \partial_i \partial_j K \\ &= \partial_i(\omega^{kl} g_{lj} \partial_k K) - \omega^{kl} g_{ik} g_{lj}, \end{aligned}$$

which becomes:

$$\partial_i(J_j^k \partial_k K) = \omega_{ij}$$

and because of the non-degeneracy of ω this is equivalent to:

$$\partial_i(g^{jk}\partial_k K) = \delta_i^j,$$

which integrates to:

$$g^{ik}\partial_k K = x^i,$$

from which:

$$\begin{aligned} \partial_k K &= g_{ki} x^i \\ &= \partial_k(\partial_i K x^i) - \partial_k K. \end{aligned}$$

Therefore, up to an irrelevant constant that can be absorbed in the definition of K:

$$2K = \partial_i K x^i = \partial_i K g^{ij} \partial_j K = ||X||_g^2,$$

meaning in particular that K describes a conic special Kähler manifold if and only if it is homogeneous of degree 2 in special Darboux coordinates.

3.4.1 Digression: a guess for the quantum origin of the conic property

In this section we shall attempt a guess for the quantum origin of the conic property for a special Kähler manifold. It seems as though it comes from the requirement that the special Kähler potential K be the classical counterpart of a quantum Hamiltonian $\hat{K}$ that preserves the space of coherent states. More precisely, $\hat{K}$ should preserve the space of coherent states as a subspace of $\mathcal{H}$, thus with no phase ambiguities, provided α_k is set to:

$$\alpha_k = -\frac{1}{2}\omega_{ik}x^i. \tag{3.4.5}$$

Thus we have the property:

$$K(p) = {}_s\langle p|\hat{K}|p\rangle_s$$

and the statement that $\hat{K}$ preserves the space of coherent states with no phase ambiguities, is:

$$\exp(i\hat{K}t)|p\rangle_s = |\chi_t(p)\rangle_s, \tag{3.4.6}$$

where by χ_t we denote the canonical flow of K. Infinitesimally the above reads:

$$\begin{aligned} i\hat{K}|p\rangle_s &= X^k A_k(p)|p\rangle_s \\ &= -iX^k(\alpha_k + \omega_{ik}\hat{x}^i)|p\rangle_s. \end{aligned} \tag{3.4.7}$$

Applying ${}_s\langle p|$ to the above equation we obtain the desired result:

$$\begin{aligned} K &= \frac{1}{2}X^k\omega_{ik}x^i - {}_s\langle p|\hat{x}^i|p\rangle_s\omega_{ik}X^k \\ &= -\frac{1}{2}\partial_i K x^i + \partial_i K\, {}_s\langle p|\hat{x}^i|p\rangle_s \\ &= \frac{1}{2}\partial_i K x^i. \end{aligned}$$

In the last step we used the fact that ${}_s\langle p|\hat{x}^i|p\rangle_s = x^i$. This follows from:

$$\begin{aligned} \partial_j({}_s\langle p|\hat{x}^i|p\rangle_s) &= {}_s\langle p|[A_j(p), \hat{x}^i]|p\rangle_s \\ &= i\omega_{kj}[\hat{x}^k, \hat{x}^i] \\ &= \delta^i_j. \end{aligned}$$

Now we shall ask under what changes of gauge for α, equation (3.4.7) remains unaltered[8]. Clearly the gauge transformations are reduced to:

$$|p\rangle_s \to \exp(if(p))|p\rangle_s \quad \text{with } f \in C^\infty(M) \quad \text{such that } X \cdot f = 0.$$

[8]Notice that for a conic special Kähler manifold, the gauge (3.4.5) is the same as canonical gauge (3.3.5).

Now, the structure of a conic special Kähler manifold is necessary for (3.4.6) to be fulfilled, but it is by no means sufficient. Indeed, in the simplest case of $\mathbb{R}^{2d}$ one can check that the admissible K's are reduced to quadratic ones. What distinguishes the case $M = \mathbb{R}^{2d}$ is not the connection A, but rather an initial choice $|p_0\rangle_s$ for a definite marked point $p_0 \in M$. Thus turning the argument around, once K is fixed, equation (3.4.6) puts constraints on this choice. Let

$$\Delta(p_2, p_1) := {}_s\langle p_2|p_1\rangle_s,$$

then, one such natural constraint is:

$$\Delta(p_2, p_1) = \Delta(\chi_t(p_2), \chi_t(p_1)).$$

Infinitesimally, the above becomes the following equation:

$$-i(\partial_i K(p_2) - \partial_i K(p_1))\omega^{ik}\frac{\partial}{\partial x_1^k}\Delta(p_2, p_1) = \frac{1}{2}\partial_i K(p_2)(x_2^i - x_1^i)\Delta(p_2, p_1).$$

As stated before, for fixed K this can be viewed as a differential equation for Δ while, for fixed Δ it can be viewed as a constraint on the choice of K. There is in fact an even more elementary constraint on K if we allow M to contain the point at the origin of the coordinate system. Then, indeed, the only homogeneous degree 2 functions are the quadratic ones. Therefore, in the more general case we need to assume that $0 \notin M$. This condition however is but a consequence of the further requirement entailed in definition 3.4.2 that the action of X be free as 0 would clearly be a fixed point. In fact it is the unique fixed point and there the metric is singular.

3.4.2 Coping with negative signature

In this section we shall investigate how the definition of quantization should be modified in the case of a non-Riemannian Kähler manifold. This is of interest since precisely moduli spaces of $N = (2, 2)$ 2-d super conformal field theories are of this type. Recall that in the discussion of the coherent tangent bundle it was crucial that M be Riemannian, otherwise the definition of coherent state would have implied the existence of negative normed states in the Hilbertspace. Let's start with the simplest case, namely again $M = \mathbb{R}^2$, this time however we shall change the Kähler structure as follows: $\omega \to -\omega$, $J \to J$ and $g \to -g$. Under this change, the Heisenberg algebra is changed to:

$$[\hat{x}, \hat{p}] = -i$$

and in terms of the annihilation operator $a = (1/\sqrt{2})(\hat{x} + i\hat{p})$ and creation operator $b = (1/\sqrt{2})(\hat{x} - i\hat{p})$:

$$[a, b] = -1.$$

We shall now define the highest weight state v_0 as before through:

$$av_0 = 0.$$

Then a basis for the highest weight representation is furnished by:

$$v_n = \frac{b^n}{\sqrt{n!}} v_0, \;\; n \in \mathbb{N}_0.$$

At this point we realize the impossibility of finding, in this representation, a positive definite sesquilinear bilinear form with respect to which $\hat{x}$ and $\hat{p}$ are self-adjoint. Indeed given such a bilinear form B, we would have:

$$0 < B(bv_0, bv_0) = B(v_0, abv_0) = B(v_0, [a,b]v_0) = -B(v_0, v_0),$$

which is clearly contradictory. Instead, what we can require is the existence of two bilinear forms B_+ and B_-. We shall require the former to be positive definite and sesquilinear, thus introducing a Hilbertspace topology on the representation. However B_+ will have the property that $\hat{x}$ and $\hat{p}$ are anti-self-adjoint with respect to it. On the other hand B_- is non-degenerate sesquilinear such that $\hat{x}$ and $\hat{p}$ are self-adjoint, but it will be indefinite. Under the normalization $B_+(v_0, v_0) = B_-(v_0, v_0) = 1$ we thus obtain:

$$B_+(v_m, v_n) = \delta_{m,n} \qquad B_-(v_m, v_n) = (-1)^n \delta_{m,n}.$$

Thus $\{v_n\}_{n \in \mathbb{N}_0}$ form an orthonormal basis of the Hilbertspace $\mathcal{H}$. We can define B_- in terms of B_+ as:

$$B_-(\,\cdot\,,\,\cdot\,) = B_+(\,\cdot\,, (-1)^{ba} \cdot).$$

A suggestive way of interpreting $(-1)^{ba}$ is as $(-1)^F$ where F is a "fermion-number" operator. For this we need to introduce the following decomposition of $\mathcal{H}$:

$$\mathcal{H} = \mathcal{H}_{even} \oplus \mathcal{H}_{odd},$$

where $\mathcal{H}_{even}$ is the subspace spanned by v_n with n even and $\mathcal{H}_{odd}$ is defined analogously. Then we define fermion fields ψ_1 and ψ_2 by:

$$\begin{aligned} \psi_1 v_{2n} &= 0 & \psi_1 v_{2n+1} &= v_{2n} \\ \psi_2 v_{2n} &= v_{2n+1} & \psi_2 v_{2n+1} &= 0 \;\; \forall n \in \mathbb{N}_0. \end{aligned}$$

Then ψ_1 and ψ_2 satisfy the following properties:

$$\{\psi_1, \psi_2\} = 1$$

and

$$B_+(\psi_1 \cdot, \cdot) = B_+(\cdot, \psi_2 \cdot).$$

Finally we can define F by:

$$F = \psi_2 \psi_1.$$

On $\mathcal{H}$ we can furthermore define the differential:

$$Q = \psi_1 (ba - F).$$

This allows us to single out v_0, which is the unique invariant state under the $U(1)$ action:

$$R(\theta) := e^{i\theta ba},$$

as the unique representative of Q-cohomology $H^0_Q(\mathcal{H})$.

Now let's turn to coherent states. These are now given by:

$$|\alpha\rangle = \exp(-\alpha b + \overline{\alpha} a)v_0 = \exp\left(\frac{|\alpha|^2}{2}\right)\sum_{n\geq 0}\frac{(-\alpha)^n}{\sqrt{n!}}v_n.$$

Clearly the translation operator is not unitary any more, that is, it is not an isometry with respect to B_+, but it is an isometry with respect to B_-. It is in fact unbounded, but clearly its domain includes the coherent states. Now we shall generalize the above discussion to understand the structure of the coherent tangent bundle for an arbitrary Lorentzian Kähler manifold.

3.4.3 The coherent tangent bundle in the Lorentzian case

In this section we will construct the coherent tangent bundle in the case of a general Lorentzian Kähler manifold. An important result will be that contrary to the Riemannian case, in the present case the Hilbert-bundle is not necessarily trivial, in particular it cannot in general be trivialized on an entire Darboux patch. The caveat stems from the fact that in the Lorenztian case one needs to make a choice of two "negative" directions, and this choice depends non trivially, not only on the symplectic form, but also on the metric, which contrary to the symplectic form, cannot be flat on an entire patch unless the Riemann curvature vanishes. In this section we will show however that if one makes a choice of negative directions at a given point $p_0 \in M$, this choice can be extended to an open neighborhood V_+ containing p_0. Ultimately the detailed choice at p_0 will be irrelevant.

We start by considering a point $p_0 \in M$ and erect a Darboux coordinate system in the neighborhood of p_0 such that at p_0 the metric is the standard Lorentzian metric[9]. The coherent tangent bundle at p_0 will then be the collection of states defined by:

$$\hat{x}^T(\eta + i\epsilon)|u\rangle_{p_0} = u^T(\eta + i\epsilon)|u\rangle_{p_0}.$$

In particular there will be a state $|0\rangle_{p_0}$. The representation of the Heisenberg algebra thus obtained with highest weight $|0\rangle_{p_0}$, as we have observed in the above section, is naturally not a Hilbert space, but rather a vector space equipped with the pairing B_-, with respect to which $\hat{x}^i$ are hermitian, which is defined precisely as in section 3.4.2 with a and b corresponding to the first coordinates x^1 and x^{d+1}. We also observed in the previous section that we can however endow this vectorspace with the structure of a Hilbert space $\mathcal{H}$ with scalar product B_+ with respect to which $\hat{x}^1$ and $\hat{x}^{d+1}$ are anti-hermitian while the rest are hermitian.

[9]This can obviously also be achieved for special Darboux patches in special Kähler manifolds.

Now we ask how large the Darboux neighborhood V_+ of p_0 is allowed to be for the coherent states $|u\rangle_p$ for $p \in V_+$ to belong to the same Hilbert space $\mathcal{H}$. We shall denote by Sp^ϵ the symplectic group with symplectic form ϵ, then the previous question is clearly equivalent to determining the subset:

$$Sp^\epsilon_+ := \{\Lambda \in Sp^\epsilon \mid |0\rangle^\Lambda \in \mathcal{H}\},$$

where $|0\rangle^\Lambda$ is defined through the condition:

$$(\Lambda^T \hat{x})^T (\eta + i\epsilon)|0\rangle^\Lambda = 0. \tag{3.4.8}$$

Then, denoting by V^{max} the maximal Darboux patch containing p_0:

$$V^{max}_+ = \{p \in V^{max} \,|\exists \Lambda \in Sp^\epsilon_+ \;\; s.t. \;\; g(p) = \Lambda \eta \Lambda^T\}.$$

As we have seen explicitly in section 3.3.3, in the Riemannian case, $Sp_+ = Sp$ and therefore $V^{max}_+ = V^{max}$. This is but a consequence of the Stone von Neumann theorem that asserts, in particular, the uniqueness of unitary irreducible representations of the Heisenberg algebra. The Lorentzian case, however, corresponds to non-unitary representations, and indeed, as we will show, $Sp^\epsilon_+ \subsetneq Sp^\epsilon$. We will show however that Sp^ϵ_+ contains an open neighborhood of the identity, a requirement to, at least locally, quantize M. As a concrete representation for $\mathcal{H}$ we choose the usual $L^2(\mathbb{R}^d)$ where $\hat{x}$ acts as:

$$\hat{x} = \tilde{E} \begin{pmatrix} q \\ -i\nabla_q \end{pmatrix},$$

where:

$$\tilde{E} := \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}$$

and:

$$E = \begin{pmatrix} i & 0 \\ 0 & 1_{(d-1 \times d-1)} \end{pmatrix}.$$

Then, resorting to the notation in section 3.3.6, equation (3.4.8) becomes:

$$\begin{pmatrix} R_1 & R_2 + iE^2 \\ R_2^T - iE^2 & R_4 \end{pmatrix} \tilde{E} \begin{pmatrix} q \\ -i\nabla_q \end{pmatrix} \psi_0(q, \Lambda) = 0,$$

with $\psi_0(q, \Lambda)$ the wavefunction corresponding to $|0\rangle^\Lambda$. The above is equally well written as:

$$\nabla_q \psi = i\tilde{\tau} q \psi,$$

where:

$$\tilde{\tau} := -E^{-1}(R_2 + iE^2)^{-1} R_1 E = -E^{-1} R_4^{-1} (R_2^T - iE^2) E,$$

thus:

$$\psi(q) = \mathcal{N} \exp(i\langle q, \tilde{\tau} q\rangle)$$

and $\psi \in L^2(\mathbb{R}^d)$ provided:

$$\operatorname{Im} \tilde{\tau} > 0. \tag{3.4.9}$$

Since the above is an open set, by local continuity of $\Lambda(p)$, $g(\Lambda)$ and $\tilde{\tau}(g)$, V_+^{max} contains an open neighborhood of p_0. One can easily check by way of counterexample that condition (3.4.9) is non trivial and in particular $Sp^\epsilon_+ \subsetneq Sp^\epsilon$.

Consider now the complex modulus:

$$\tau := E\tilde{\tau}E^{-1}.$$

Clearly τ is in the Siegel upper half space. From this we deduce the complete characterization of Sp^ϵ_+:

$$Sp^\epsilon_+ = \{\Lambda \in Sp^\epsilon \,|\operatorname{Im}(E^{-1}\tau(\Lambda)E) > 0\}.$$

At this point we shall study the states $|u\rangle^\Lambda$, which we shall rename $|u\rangle_\tau$. Following the analogous steps for $|0\rangle^\Lambda$ we obtain that the corresponding wavefunction $\psi_u(\tau, q)$ is of the form:

$$\psi_u(\tau, q) = \mathcal{N}(\tau, u) \exp\left(\frac{i}{2}\langle (q - E^{-1}u_q), \tilde{\tau}(q - E^{-1}u_q)\rangle + i\langle E^{-1}u_p, q\rangle\right).$$

Just as the normalization constant in section 3.3.6, here $\mathcal{N}(\tau, u)$ is fixed by three analogous conditions. The first is a normalization condition with respect to B_- instead of B_+:

$$1 = {}_\tau\langle u|(-1)^F|u\rangle_\tau = \int_{\mathbb{R}^d} d^d q\, \overline{\psi}_u(\tau, q)\psi_u(\tau, E^2 q).$$

Solving for $|\mathcal{N}|$ yields:

$$|\mathcal{N}(\tau, u)| = \pi^{-d/4}(\det \operatorname{Im} \tau)^{1/4}.$$

The second condition on $\psi_u(\tau, q)$ is equation (3.3.13) which remains unchanged in the Lorentzian case. As in section 3.3.6, let θ be defined through:

$$\mathcal{N}(\tau, u) = |\mathcal{N}(\tau)|e^{i\theta(\tau, u)}.$$

Then (3.3.13) is equivalent to:

$$\nabla_{u_q}\theta + E^2\overline{\tau}E^2\nabla_{u_p}\theta = -\frac{1}{2}E^2\overline{\tau}u_q - \frac{1}{2}E^2 u_p,$$

the solution to which is:

$$\theta(\tau, u_q, u_p) = -\frac{1}{2}\langle u_p, E^2 u_q\rangle + \gamma(\tau).$$

Therefore in the Lorentzian case, the wavefunction of the coherent state $|u\rangle_\tau$ is given by:

$$\psi_u(\tau, q) = \pi^{-d/4}(\det \operatorname{Im} \tau)^{1/4} \exp(i\gamma(\tau))\cdot$$

$$\exp\left(\frac{i}{2}\langle (q-E^{-1}u_q),\tilde{\tau}(q-E^{-1}u_q)\rangle + i\langle E^{-1}u_p, q\rangle - \frac{i}{2}\langle u_p, E^2 u_q\rangle\right).$$

Using (3.3.14) we can again fix the phase. For M affine special Kähler, on a special Darboux patch we obtain:

$$\partial_k\gamma = -\beta_k - \frac{1}{2}\mathrm{tr}([(\Gamma_k\omega)_{pp}]\,\mathrm{Im}\,\tau\; E^2).$$

At this point we can compute the overlap ${}_{\tau_2}\langle u_2|(-1)^F|u_1\rangle_{\tau_1}$, which, as in the Riemannian case, essentially corresponds to the propagator of the master equation:

$$\begin{aligned}
&{}_{\tau_2}\langle u_2|(-1)^F|u_1\rangle_{\tau_1} := \int_{\mathbb{R}^n}\overline{\psi_{u_2}}(\tau_2,q)\psi_{u_1}(\tau_1,E^2q)\\
&= (2i)^{d/2}\frac{(\det\mathrm{Im}\tau_1)^{1/4}(\det\mathrm{Im}\tau_2)^{1/4}}{(\det(\tau_1-\overline{\tau_2}))^{1/2}}\cdot\\
&\exp\left(-\frac{i}{2}\langle E^2u_{q,1},z_1\rangle + \frac{i}{2}\langle E^2u_{q,2},\overline{z}_2\rangle - \frac{i}{2}\langle E^2(z_1-\overline{z}_2),(\tau_1-\overline{\tau}_2)^{-1}(z_1-\overline{z}_2)\rangle\right)\cdot\\
&\exp\left(i\int_{p_0}^{p}\left(\beta+\frac{1}{2}\mathrm{tr}((\Gamma_k\omega)_{pp}\mathrm{Im}\tau\;E^2)dx^k\right)\right),
\end{aligned}$$

where, as in the Riemannian case, we have introduced the complex coordinates $z = u_p - \tau u_q$. Thus the expression is identical to the one in the Riemannian case with the only difference that the bilinear form on configuration space is now the standard Minkowski bilinear form $\langle E^2\cdot\,,\,\cdot\rangle$ instead of the standard scalar product. Anlogously to the case of Riemannian affine special Kähler manifolds where the propagator is given by (3.3.29), we shall see in the next section that in the Lorentzian case the propagator is given by:

$$\begin{aligned}
&K(u,p,x+y,p_0) := {}_{p,A_s}\langle u|(-1)^F|p\rangle_s^y\\
&= (2i)^{d/2}\frac{(\det\mathrm{Im}\tau_1)^{1/4}(\det\mathrm{Im}\tau_2)^{1/4}}{(\det(\tau_1-\overline{\tau_2}))^{1/2}}\cdot\\
&\exp\left(-i\int_{p_0}^{p}\left((\alpha-\beta)-\frac{1}{2}\omega_{kj}(x+y)^j dx^k - \frac{1}{2}\mathrm{tr}((\Gamma_k\omega)_{pp}\mathrm{Im}\tau\;E^2)dx^k\right)\right)\cdot\\
&\exp\left(-\frac{1}{4}||u||^2_{g(p)}\right)\cdot\\
&\exp\left(-\frac{i}{2}\langle E^2u_{q,1},z_1\rangle + \frac{1}{4}\langle\overline{z}_2,R_4(p)\overline{z}_2\rangle - \frac{i}{2}\langle E^2(z_1-\overline{z}_2),(\tau_1-\overline{\tau}_2)^{-1}(z_1-\overline{z}_2)\rangle\right),
\end{aligned}$$

where we have used the same notation as in section 3.3.6.

3.4.4 Remarks on the quantization of Lorentzian conic special Kähler manifolds

In this section we shall first discuss how the quantization of Riemannian affine special Kähler manifolds translates to the Lorentzian case, show how to project to positive normed states, and then discuss normalization conditions of the wavefunction $Z(u,p)$ thus presenting the form of the general solution to the master equation. We shall develop the first point in the form of a series of remarks:

- To quantize an appropriate Darboux neighborhood (V_+) of $p_0 \in M$, one chooses the Darboux coordinates such that $g(p_0) = \eta$.
- Locally quantization involves a triple $(V_+, \phi, \mathbb{P}^\infty)$, but contrary to the Riemannian case, now $\mathbb{P}^\infty$ is endowed with the pairing

$$(v,w)_- = \frac{|B_-(v,w)|}{\left(B_-(v,v)B_-(w,w)\right)^{1/2}}$$

  and the group of automorphisms of $\mathbb{P}^\infty$ is defined accordingly.
- The flat connection A does thus no longer induce a unitary parallel transport, but rather a parallel transport that is an isometry w.r.t. B_-.
- One must choose generators of the Heisenberg algebra $\hat{x}^i$, such that $\hat{x}^1$ and $\hat{x}^{d+1}$ are anti-hermitian w.r.t. B_+, while the rest are hermitian.
- The form of the operator S_Σ introduced as S in (3.3.6) is left unchanged, and it is now an isometry w.r.t. B_-.
- As a consequence in order for the tensorial property (3.3.11) of $Z(u,p)$ to hold, the definition of the wavefunction must be replaced by:

$$Z(u,p) := {}_{p,A}\langle u|(-1)^F|p\rangle_A.$$

With the above modifications the quantization procedure of affine Lorentzian special Kähler manifolds proceeds without change as the one for Riemannian affine special Kähler manifolds until the end of section 3.3.4 with the only exception of the normalization conditions (3.3.15, 3.3.16, 3.3.17). One last remark regards section 3.4.1 where, in the Lorentzian case, all matrix elements of the form $\langle p_2|\mathcal{O}|p_1\rangle$ must be replaced with $\langle p_2|(-1)^F\mathcal{O}|p_1\rangle$.

Projecting onto "positive normed" states: the coherent horizontal bundle

Let M be a conic special Kähler manifold of dimension $2d$, we shall distinguish between three regions of M:

$$\begin{aligned} M_+ &:= \{p \in M | K(p) > 0\}, \\ M_0 &:= \{p \in M | K(p) = 0\}, \\ M_- &:= \{p \in M | K(p) < 0\}. \end{aligned}$$

As discussed earlier M_0 is singular with a conic singularity approaching $x = 0$. Now we shall concentrate on M_-. There, an orthonormal basis of negative or "timelike directions" in the tangent bundle TM_- is given by the hamiltonian vectorfield X and JX. Indeed:

$$g(X, X) = g(JX, JX) = 2K < 0.$$

Therefore, on the orthogonal complement with respect to g of X and JX, g is positive definite. We thus define the horizontal bundle as:

$$HM := \{V \in TM_- |\ g(V, X) = g(V, JX) = 0\}.$$

In particular HM is the image of a section $P \in \Gamma(M_-, \mathrm{End}(TM_-))$ of projections $P(p)$, which in special coordinates is given by:

$$P^j_k = \delta^j_k - \frac{1}{2}\partial_k \log|K| g^{ij}\partial_i K + \frac{1}{2}J^l_k\partial_l \log|K|\omega^{ij}\partial_i K. \tag{3.4.10}$$

Corresponding to HM there is a quantum counterpart that we shall name coherent horizontal bundle, defined as the sub-bundle of the trivial Hilbert-bundle, given by the image of the section of projection operators $\mathcal{P} \in \Gamma(M_-, \mathrm{End}(\mathcal{H}))$, where $\mathcal{P}(p, A)$ is an orthogonal projection at every point p. This projection is the obvious generalization of the projector onto v_0 of section [3.4.2]. Thus, the action of $\mathcal{P}(p, A_s)$ on the basis $|u\rangle_{p,s}$ is given by:

$$\begin{aligned}\mathcal{P}(p, A_s)|u\rangle_{p,s} &:= \exp\left(-\frac{1}{4}||(1 - P(p))u||^2_{g(p)}\right)|P(p)u\rangle_{p,s}\\ &= \exp\left(-\frac{1}{2K(p)}|g(H(p), u)|^2\right)|P(p)u\rangle_{p,s}.\end{aligned}$$

It is an easy exercise to check that $\mathcal{P}$ is self-adjoint w.r.t B_-.

Normalization conditions and the general solution

Now we shall pass to normalization conditions. In the Lorentzian case, equations (3.3.16, 3.3.17) are modified to:

$$\begin{aligned}(-1)^F = \frac{1}{(2\pi)^n}\int_{T_pM_-} & du^1 \wedge \cdots \wedge du^{2d}\sqrt{\det g}\cdot\\ & \exp\left(\frac{1}{K(p)}|g(H(p), u)|^2\right)|Pu - (1 - P)u\rangle_{p,A\,p,A}\langle u|.\end{aligned}$$

And the normalization is with respect to B_- rather than B_+, therefore:

$$\begin{aligned}1 = \frac{1}{(2\pi)^n}\int_{T_pM_-} & \sqrt{\det g}\exp\left(\frac{1}{K(p)}|g(H(p), u)|^2\right)\cdot\\ & \overline{Z_A}(Pu - (1 - P)u, p)Z_A(u, p)\, du^1 \wedge \cdots \wedge du^{2d}.\end{aligned}$$

It follows that the general solution to the master equation is then given by:

$$Z(u,p)^f = \int_{T_{p_0}M} dy^1 \wedge \cdots \wedge dy^{2d} \sqrt{\det g_s(p_0)} \exp\left(\frac{1}{K(p_0)}|g(H(p_0),y)|^2\right) \cdot$$
$$K_-(u,p,x_s+y,p_0) \exp\left(-\frac{1}{4}||y||^2_{g_s(p_0)}\right) f((1+iJ_0)y), \qquad (3.4.11)$$

where f is an arbitrary normalizable function w.r.t. B_-.

3.4.5 The quantization of projective special Kähler manifolds

In this section we will construct the wavefunction $Z_{red,A}(u,p)$ for an arbitrary projective special Kähler manifold $\tilde{M}$ of dimension $2d$ that arises as a holomorphic quotient of a Lorentzian conic special Kähler manifold M of dimension $2d{+}2$. First of all, it is convenient at this point to introduce complex coordinates and express H in terms of these. We shall stay in the special Darboux coordinate system, and erect corresponding holomorphic coordinates $(z^0,\ldots,z^d)$. Then (3.4.1) becomes:

$$\partial_\mu H^\nu = \delta^\nu_\mu.$$

Therefore:

$$H = z^\mu \partial_\mu,$$

where the vector of complex special coordinates is related to the vector of Darboux coordinates $x = (x_q, x_p)$ via:

$$z = x_p - \tau x_q.$$

The quotient of M by H clearly has as holomorphic functions the ones defined on M of homogeneous degree 0, therefore $\tilde{M}$ can be covered by affine patches as $\tilde{M}_0$ with coordinates $(y^1,\ldots,y^d)$ given by:

$$(z^0, z^1, \ldots, z^d) =: (\lambda, \lambda y^1, \ldots, \lambda y^d),$$

with $\lambda \neq 0$. In this new coordinate system $(\lambda, y^1, \ldots, y^d)$:

$$H = \lambda \frac{\partial}{\partial \lambda}.$$

From now on we shall label the coordinates y and z with (α,β,γ) and (μ,ν,ρ,σ) respectively. Analogously we will label the corresponding real coordinates with non-capital and capital latin letters respectively. At this point we can express the projection P introduced in (3.4.10) in complex coordinates:

$$P^\nu_\mu = \delta^\nu_\mu - \frac{1}{2}\partial_\mu \log|K| g^{\nu\bar{\rho}} \partial_{\bar{\rho}} K + \frac{1}{2} J^\sigma_\mu \partial_\sigma \log|K| \omega^{\nu\bar{\rho}} \partial_{\bar{\rho}} K$$
$$= \delta^\nu_\mu - \frac{1}{2}\partial_\mu \log|K| (g^{\nu\bar{\rho}} - i\omega^{\nu\bar{\rho}}) \partial_{\bar{\rho}} K$$

$$
\begin{aligned}
&= \delta^{\nu}_{\mu} - \partial_{\mu} \log|K| g^{\nu\overline{\rho}} \partial_{\overline{\rho}} K \\
&= \delta^{\nu}_{\mu} - z^{\nu} \partial_{\mu} \log|K| \\
P^{\nu}_{\overline{\mu}} &= 0 \\
P^{\overline{\nu}}_{\mu} &= 0 \\
P^{\overline{\nu}}_{\overline{\mu}} &= \delta^{\overline{\nu}}_{\overline{\mu}} - z^{\overline{\nu}} \partial_{\overline{\mu}} \log|K|.
\end{aligned}
$$

Therefore, in particular, in special coordinates we have the following holomorphic frame for the horizontal bundle:

$$
\begin{aligned}
V_{\alpha} &= \Sigma^{\mu}_{\alpha} P^{\nu}_{\mu} \partial_{\nu} \\
&= \frac{\partial z^{\mu}}{\partial y^{\alpha}} (\delta^{\nu}_{\mu} - z^{\nu} \partial_{\mu} \log|K|) \partial_{\nu} \\
&= \frac{\partial}{\partial y^{\alpha}} - \left(\frac{\partial}{\partial y^{\alpha}} \log|K| \right) z^{\nu} \partial_{\nu} \\
&= \frac{\partial}{\partial y^{\alpha}} - \left(\frac{\partial}{\partial y^{\alpha}} \log|K| \right) \lambda \frac{\partial}{\partial \lambda},
\end{aligned}
$$

where $\Sigma^{\mu}_{\alpha} = dz^{\mu}/dy^{\alpha}$. We can now define the wavefunction reduced to the projective special Kähler manifold:

Definition 3.4.4. The quantization of the holomorphic quotient $\tilde{M}$ is given by the reduced wavefunction:

$$
Z_{red,A}(u,p) :=_{p,A} \langle \Sigma^T u | \mathcal{P}^{\dagger}(p,A)(-1)^F | p \rangle_A.
$$

Therefore:

$$
Z_{red,A}(u,p) := \exp\left(-\frac{1}{4} ||P(p)\Sigma^T u||^2_{g(p)} - \frac{1}{2K(p)} |g(H(p), \Sigma^T u)|^2 \right) \mathcal{C}_{red}(u^i \partial_i),
$$

where

$$
\mathcal{C}_{red} = \iota^*(\mathcal{C} \circ P), \tag{3.4.12}
$$

and by ι we have denoted the inclusion of the level set λ in M. In particular we have

$$
\mathcal{C}_{red} = \exp\left(\sum_{n \geq 0} \frac{(-1)^n}{n!} \mathcal{C}^n_{red} \right),
$$

with, in special coordinates:

$$
(\mathcal{C}^n_{red})_{\overline{\alpha}_1, \ldots, \overline{\alpha}_n} = \mathcal{C}^n_{\overline{\mu}_1, \ldots, \overline{\mu}_n} \left(\frac{\partial z^{\overline{\mu}_1}}{\partial y^{\overline{\alpha}_1}} - z^{\overline{\mu}_1} \frac{\partial}{\partial y^{\overline{\alpha}_1}} \log|K| \right) \cdots \left(\frac{\partial z^{\overline{\mu}_n}}{\partial y^{\overline{\alpha}_n}} - z^{\overline{\mu}_n} \frac{\partial}{\partial y^{\overline{\alpha}_n}} \log|K| \right)
$$

and:

$$
\frac{\partial z^0}{\partial y^{\alpha}} - z^0 \frac{\partial}{\partial y^{\alpha}} \log|K| = -\lambda \frac{\partial}{\partial y^{\alpha}} \log|K|
$$

$$\frac{\partial z^\beta}{\partial y^\alpha} - z^\beta \frac{\partial}{\partial y^\alpha} \log|K| = \lambda \left(\delta^\beta_\alpha - y^\beta \frac{\partial}{\partial y^\alpha} \log|K| \right).$$

We shall extend the y coordinate system to incorporate $y^0 := 1$, and define h through:

$$K(z, \overline{z}) = -|\lambda|^2 h(y, \overline{y}),$$

then:

$$(\mathcal{C}^n_{red})_{\overline{\alpha}_1,\ldots,\overline{\alpha}_n} = \overline{\lambda}^n \, \mathcal{C}^n_{\overline{\mu}_1,\ldots,\overline{\mu}_n} \left(\delta^{\overline{\mu}_1}_{\overline{\alpha}_1} - y^{\overline{\mu}_1} \frac{\partial}{\partial y^{\overline{\alpha}_1}} \log h \right) \cdots \left(\delta^{\overline{\mu}_n}_{\overline{\alpha}_n} - y^{\overline{\mu}_n} \frac{\partial}{\partial y^{\overline{\alpha}_n}} \log h \right).$$

At this stage we can determine the master equation satisfied by $\mathcal{C}_{red}$. We shall proceed analogously to the affine case. Thus we start by collecting the following computational building blocks. The first crucial building block is the Kähler structure on the projective manifold $\tilde{M}$:

$$\begin{aligned} \tilde{g}_{\alpha\overline{\beta}} &= \Sigma^\mu_\alpha P^\rho_\mu g_{\rho\overline{\sigma}} P^{\overline{\sigma}}_{\overline{\nu}} \Sigma^{\overline{\nu}}_{\overline{\beta}} = \overline{\lambda} \Sigma^\mu_\alpha P^\rho_\mu g_{\rho\overline{\beta}} \\ &= \Sigma^\mu_\alpha \left(g_{\mu\overline{\nu}} - \frac{1}{K} \partial_\mu K \partial_{\overline{\nu}} K \right) \Sigma^{\overline{\nu}}_{\overline{\beta}} \\ &= -|\lambda|^2 \left(\partial_\alpha \partial_{\overline{\beta}} h - \frac{1}{h} \partial_\alpha h \partial_{\overline{\beta}} h \right) \\ &= K \partial_\alpha \partial_{\overline{\beta}} \log h. \end{aligned} \tag{3.4.13}$$

The form obtained in the last step shows that $\tilde{g}$ is indeed a Kähler metric, not on the holomorphic quotient of M by the action of H, but rather on the symplectic quotient of M by the action of X where K is constant. Indeed the above precisely defines the Marsden-Weinstein quotient. We thus define the normalized Kähler metric:

$$\hat{g}_{\alpha\overline{\beta}} := -\partial_\alpha \partial_{\overline{\beta}} \log h.$$

We shortly digress to observe that formula (3.4.13) means that the value of K on the corresponding symplectic quotient is related to Planck's constant via:

$$K = -\frac{1}{\hbar}.$$

In other words, Planck's constant precisely labels the choice of symplectic quotient:

$$M_\hbar \sim K^{-1}(-\hbar^{-1})/S^1.$$

Here $\sim$ means homeomorphic.

Now we shall consider the dependence of $\mathcal{P}(p, A)|\Sigma^T u\rangle_{p,A}$ on p. We shall do this in steps. First we shall consider the dependence on p of the canonical coherent state $|\Lambda^T P \Sigma^T u\rangle$, where we have used the same notation as in section 3.3.3. We obtain:

$$\partial_K |\Lambda^T P \Sigma^T u\rangle = \langle \partial_K (\Lambda^T P \Sigma^T) u, \Lambda^{-1} \nabla_{P\Sigma^T u} \rangle |\Lambda^T P \Sigma^T u\rangle$$

$$= u^T \left(\Sigma P^T \Gamma_K + \partial_K(\Sigma P^T) \right) \nabla_{P\Sigma^T u} |\Lambda^T P \Sigma^T u\rangle. \tag{3.4.14}$$

We now introduce the differential $\tilde{\Sigma}$ from y to z coordinates. In particular:

$$\tilde{\Sigma}^\alpha_\mu := \frac{\partial y^\alpha}{\partial z^\mu} = \lambda^{-1} \left(\delta^\alpha_\mu - y^\alpha \delta^0_\mu \right).$$

Then we have:

$$P^T \tilde{\Sigma} \Sigma P^T = P^T.$$

Therefore:

$$\nabla_{P\Sigma^T u} = P^T \tilde{\Sigma} \nabla_u + (1 - P^T) \nabla_{P\Sigma^T u}.$$

Substituting in (3.4.14) we obtain:

$$\begin{aligned} &\partial_K |\Lambda^T P \Sigma^T u\rangle = \\ &u^T \left(\tilde{\Gamma}_K \nabla_u + \left(\Sigma P^T \Gamma_K + \partial_K(\Sigma P^T) \right) (1 - P^T) \nabla_{P\Sigma^T u} \right) |\Lambda^T P \Sigma^T u\rangle. \end{aligned}$$

In the above we have defined the connection:

$$\tilde{\Gamma}_K = \Sigma P^T \Gamma_K P^T \tilde{\Sigma} + \partial_K(\Sigma P^T) P^T \tilde{\Sigma}. \tag{3.4.15}$$

As we show in appendix A.2, the connection $\tilde{\Gamma}$ splits into purely holomorphic and anti-holomorphic components with $\tilde{\Gamma}^{\overline{\gamma}}_{\overline{\alpha}\overline{\beta}} = (\tilde{\Gamma}^\gamma_{\alpha\beta})^*$ and can be expressed in terms of the Levi-Civita connection $\hat{\Gamma}$ of $\hat{g}$ as follows:

$$\begin{aligned} \tilde{\Gamma}^\gamma_{\alpha\beta} &= \hat{\Gamma}^\gamma_{\alpha\beta} + \partial_\alpha \log |K| \delta^\gamma_\beta \\ \tilde{\Gamma}^\gamma_{0\beta} &= \lambda^{-1} \delta^\gamma_\beta, \end{aligned}$$

where we have denoted by 0 the coordinate λ. In particular $\tilde{\Gamma}$ is compatible with the metric $\tilde{g}$.

Now we turn to the dependence on p of the coherent state $\mathcal{P}(p, A)|\Sigma^T u\rangle_{p,A}$ proper. In fact, to tackle the reduced tensor $\mathcal{C}_{red}$ directly we compute:

$$\begin{aligned} &\partial_K \left(\exp\left(\frac{1}{4} ||P(p)\Sigma^T u||^2_{g(p)} \right) |P\Sigma^T u\rangle_{p,A} \right) = \\ &\left(-i\beta_K - \frac{i}{2} (\Gamma_K \omega)_{IJ} \hat{x}^I \hat{x}^J + u^r \tilde{\Gamma}^s_{Kr} \frac{\partial}{\partial u^s} + u^r \left(\left(\Sigma P^T \Gamma_K + \partial_K(\Sigma P^T) \right) (1 - P^T) \right)^S_r \frac{\partial}{\partial (P\Sigma^T u)^S} \right) \cdot \\ &\exp\left(\frac{1}{4} ||P(p)\Sigma^T u||^2_{g(p)} \right) |P\Sigma^T u\rangle_{p,A}. \end{aligned}$$

In the above we have used the metric compatibility of $\tilde{\Gamma}$. At this point we need to compute the action of $\hat{x}$ on the coherent state. However only the components of $\hat{x}$ along the horizontal bundle act naturally as differential operators. We shall now focus our attention on those:

$$\begin{aligned}
&\tilde{\Sigma}^T P\hat{x}\left(\exp\left(\frac{1}{4}||P(p)\Sigma^T u||^2_{g(p)}\right)|P\Sigma^T u\rangle_{p,A}\right)\\
&=\left(\frac{1}{2}\tilde{\Sigma}^T P(1-iJ)^T P\Sigma^T u+\frac{1}{2}\tilde{\Sigma}^T P(g^{-1}-i\omega^{-1})P^T\tilde{\Sigma}\nabla_u\right)\exp\left(\frac{1}{4}||P(p)\Sigma^T u||^2_{g(p)}\right)|P\Sigma^T u\rangle_{p,A}\\
&=\left(\frac{1}{2}(1-i\hat{J})^T u-\frac{1}{2K}(\hat{g}^{-1}-i\hat{\omega}^{-1})\nabla_u\right)\exp\left(-\frac{1}{4}K||u||^2_{\hat{g}(p)}\right)|P\Sigma^T u\rangle_{p,A},
\end{aligned}$$

Now we shall present the explicit form of the master equation:

$$\Sigma {P^T}_{p,A}\langle\Sigma^T u|\mathcal{P}^\dagger(p,A)(-1)^F(d+A)|p\rangle_A=0.$$

Using (3.3.13), we arrive at the master equation for $\mathcal{C}_{red}(u^i\partial_i)$:

$$\begin{aligned}
&\Bigg((\Sigma P^T\nabla_x)_k-u^r(\hat{\Gamma}_k)_r^s\frac{\partial}{\partial u^s}+i(\Sigma P^T(\alpha-\beta))_k\\
&+iK\hat{\omega}_{ik}\left(\frac{1}{2}(1-i\hat{J})^T u-\frac{1}{2K}(\hat{g}^{-1}-i\hat{\omega}^{-1})\nabla_u\right)^i\\
&+\frac{i}{2}C_{kij}\left(\frac{1}{2K}(\hat{g}^{-1}-i\hat{\omega}^{-1})\nabla_u\right)^i\left(\frac{1}{2K}(\hat{g}^{-1}-i\hat{\omega}^{-1})\nabla_u\right)^j\\
&+\frac{i}{2}C_{kij}\left(\frac{1}{2}(1-i\hat{J})^T u\right)^i\left(\frac{1}{2}(1-i\hat{J})^T u\right)^j\\
&-u^r(\Sigma P^T)_k^K\left(\left(\Sigma P^T\Gamma_K+\partial_K(\Sigma P^T)\right)(1-P^T)\right)_r^S\frac{\partial}{\partial(P\Sigma^T u)^S}\Bigg)\mathcal{C}_{red}(u^i\partial_i)=0. \qquad (3.4.16)
\end{aligned}$$

Before expressing the master equation in holomorphic and anti-holomorphic parts, we shall decompose the last term of (3.4.16) in holomorphic and anti-holomoprhic parts. Since $\mathcal{C}_{red}$ has only anti-holomorphic legs, in complex coordinates the holomorphic part is given by:

$$\begin{aligned}
&-(\Sigma P^T)^\sigma_\beta dy^{\overline{\alpha}}\Sigma^{\overline{\mu}}_{\overline{\alpha}}(\partial_\sigma P^{\overline{\nu}}_{\overline{\mu}})(1-P)^{\overline{\rho}}_{\overline{\nu}}\iota_{\partial_{\overline{\rho}}}\mathcal{C}_{red}\\
&=(\Sigma P^T)^\sigma_\beta dy^{\overline{\alpha}}\partial_\sigma\partial_{\overline{\alpha}}\log|K|z^{\overline{\rho}}\iota_{\partial_{\overline{\rho}}}\mathcal{C}_{red}\\
&=dy^{\overline{\alpha}}\hat{g}_{\beta\overline{\alpha}}z^{\overline{\rho}}\iota_{\partial_{\overline{\rho}}}\mathcal{C}_{red},
\end{aligned}$$

while the anti-holomorphic part reads:

$$\begin{aligned}
&-(\Sigma P^T)^{\overline{\sigma}}_{\overline{\beta}}dy^{\overline{\alpha}}\Sigma^{\overline{\mu}}_{\overline{\alpha}}(P^{\overline{\tau}}_{\overline{\mu}}\Gamma^{\overline{\nu}}_{\overline{\sigma}\overline{\tau}}+\partial_{\overline{\sigma}}P^{\overline{\nu}}_{\overline{\mu}})(1-P)^{\overline{\rho}}_{\overline{\nu}}\iota_{\partial_{\overline{\rho}}}\mathcal{C}_{red}\\
&=0.
\end{aligned}$$

The above is a result of the following identity:

$$(P^{\overline{\tau}}_{\overline{\mu}}\Gamma^{\overline{\nu}}_{\overline{\sigma}\overline{\tau}}+\partial_{\overline{\sigma}}P^{\overline{\nu}}_{\overline{\mu}})(1-P)^{\overline{\rho}}_{\overline{\nu}}$$

$$
\begin{aligned}
&= (\Gamma^{\overline{\nu}}_{\overline{\sigma}\overline{\mu}} + \partial_{\overline{\sigma}} P^{\overline{\nu}}_{\overline{\mu}})(1-P)^{\overline{\rho}}_{\overline{\nu}} \\
&= (g^{\overline{\nu}\nu}\partial_{\overline{\sigma}} g_{\nu\overline{\mu}} + \partial_{\overline{\sigma}} P^{\overline{\nu}}_{\overline{\mu}})(1-P)^{\overline{\rho}}_{\overline{\nu}} \\
&= \frac{1}{K}\partial_{\overline{\sigma}}(z^{\nu} g_{\nu\overline{\mu}}) z^{\overline{\rho}} - (\delta^{\overline{\nu}}_{\overline{\sigma}}\partial_{\overline{\mu}}\log|K| + z^{\overline{\nu}}\partial_{\overline{\sigma}}\partial_{\overline{\mu}}\log|K|) z^{\overline{\rho}}\partial_{\overline{\nu}}\log|K| \\
&= \frac{\partial_{\overline{\sigma}}\partial_{\overline{\mu}}K}{K} z^{\overline{\rho}} - \frac{\partial_{\overline{\sigma}}\partial_{\overline{\mu}}K}{K} z^{\overline{\rho}} \\
&= 0.
\end{aligned}
$$

In the first step above we have used the fact that in special coordinates:

$$
z^{\overline{\rho}}\Gamma_{\overline{\rho}} = g^{-1} z^{\overline{\rho}}\partial_{\overline{\rho}} g = 0. \tag{3.4.17}
$$

We are thus left to compute

$$
z^{\overline{\rho}}\iota_{\partial_{\overline{\rho}}}\mathcal{C}_{red}.
$$

For this we need to resort to (3.3.24) using (3.4.17). We obtain:

$$
\begin{aligned}
z^{\overline{\rho}}\iota_{\partial_{\overline{\rho}}}\mathcal{C}_{red} &= \left(\iota^*\left(z^{\overline{\rho}}\iota_{\partial_{\overline{\rho}}}\mathcal{C}\right)\circ P\right) \\
&= \left(\iota^*\left(\left(-z^{\overline{\rho}}\partial_{\overline{\rho}} - i(\alpha-\beta)(\overline{H})\right)\mathcal{C}\right)\circ P\right) \\
&= \left(-\overline{\lambda}\frac{\partial}{\partial\overline{\lambda}} + dy^{\overline{\alpha}}\iota_{\partial_{\overline{\alpha}}} - i(\alpha-\beta)\left(\overline{\lambda}\frac{\partial}{\partial\overline{\lambda}}\right)\right)\mathcal{C}_{red}.
\end{aligned}
$$

In order to isolate the dependence of $\mathcal{C}_{red}$ on λ and $\overline{\lambda}$, we use the fact that in y coordinates $C_{\alpha\beta\gamma}$ is holomorphic homogeneous of degree 2 in λ. Thus, we define the normalized C tensor through:

$$
C_{\alpha\beta\gamma}(\lambda, y) = \lambda^2\hat{C}_{\alpha\beta\gamma}(y).
$$

At this point we have all the ingredients to express the master equation (3.4.16) in holomorphic and anti-holomoprhic parts. The anti-holomorphic part reads:

$$
\begin{aligned}
&\left(\nabla^{(0,1)}_{\overline{\alpha}} - (\partial_{\overline{\alpha}}\log h)\overline{\lambda}\frac{\partial}{\partial\overline{\lambda}} + i(\alpha-\beta)_{\overline{\alpha}} - i(\partial_{\overline{\alpha}}\log h)(\alpha-\beta)\left(\overline{\lambda}\frac{\partial}{\partial\overline{\lambda}}\right)\right. \\
&\left.+\ \iota_{\partial_{\overline{\alpha}}} + \frac{i\overline{\lambda}^2}{2}\hat{C}_{\overline{\alpha}\overline{\beta}\overline{\gamma}} dy^{\overline{\beta}} dy^{\overline{\gamma}}\right)\mathcal{C}_{red} = 0,
\end{aligned} \tag{3.4.18}
$$

while the holomorphic part reads:

$$
\begin{aligned}
&\left(\partial_{\alpha} - (\partial_{\alpha}\log h)\lambda\frac{\partial}{\partial\lambda} + i(\alpha-\beta)_{\alpha} - i(\partial_{\alpha}\log h)(\alpha-\beta)\left(\lambda\frac{\partial}{\partial\lambda}\right)\right. \\
&+ dy^{\overline{\beta}}\hat{g}_{\alpha\overline{\beta}}\left(-\overline{\lambda}\frac{\partial}{\partial\overline{\lambda}} + dy^{\overline{\gamma}}\iota_{\partial_{\overline{\gamma}}} - i(\alpha-\beta)\left(\overline{\lambda}\frac{\partial}{\partial\overline{\lambda}}\right)\right) \\
&\left.-\ iK\hat{\omega}_{\alpha\overline{\beta}} dy^{\overline{\beta}} + \frac{i\lambda^2}{2K^2}\hat{C}_{\alpha\beta\gamma}\hat{g}^{\beta\overline{\beta}}\hat{g}^{\gamma\overline{\gamma}}\iota_{\partial_{\overline{\beta}}}\iota_{\partial_{\overline{\gamma}}}\right)\mathcal{C}_{red} = 0.
\end{aligned}
$$

There are of course two further equations left, inherited from the master equation of the conic affine special Kähler manifold M. We have already made full use of the anti-holomorphic part to express the last term in (3.4.16) as a differential operator on $\mathcal{C}_{red}$. From the holomorphic part we obtain instead:

$$\left(\frac{\partial}{\partial\lambda} + i(\alpha-\beta)\left(\frac{\partial}{\partial\lambda}\right)\right)\mathcal{C}_{red}(u^i\partial_i) = 0, \tag{3.4.19}$$

which in components reads:

$$\left(\frac{\partial}{\partial\lambda} + i(\alpha-\beta)\left(\frac{\partial}{\partial\lambda}\right)\right)\mathcal{C}^0_{red} = 0$$
$$\mathcal{C}^n_{red}(\lambda,\overline{\lambda},y,\overline{y}) = \tilde{\mathcal{C}}^n_{red}(\overline{\lambda},y,\overline{y}) \ \ \forall n \geq 1.$$

With (3.4.19) the holomorphic part of the master equation simplifies to:

$$\left(\partial_\alpha + i(\alpha-\beta)_\alpha + dy^{\overline{\beta}}\hat{g}_{\alpha\overline{\beta}}\left(-\overline{\lambda}\frac{\partial}{\partial\overline{\lambda}} + dy^{\overline{\gamma}}\iota_{\partial_{\overline{\gamma}}}\right) - dy^{\overline{\beta}}\hat{g}_{\alpha\overline{\beta}}\left(i(\alpha-\beta)\left(\overline{\lambda}\frac{\partial}{\partial\overline{\lambda}}\right) - K\right)\right.$$
$$\left. + \frac{i\lambda^2}{2K^2}\hat{C}_{\alpha\beta\gamma}\hat{g}^{\beta\overline{\beta}}\hat{g}^{\gamma\overline{\gamma}}\iota_{\partial_{\overline{\beta}}}\iota_{\partial_{\overline{\gamma}}}\right)\mathcal{C}_{red} = 0. \tag{3.4.20}$$

As a last step we will choose for α and β the gauge adopted in section 3.3.4 and we will express the master equation (3.4.18, 3.4.19, 3.4.20) as an equation for $\mathcal{S}_{red}$, which analogously to $\mathcal{S}$ in section 3.3.4, is defined as:

$$\mathcal{S}_{red} = (\det g)^{\frac{1}{8}}\, e^{\frac{K}{2}}\mathcal{C}_{red}.$$

Noticing that in this gauge:

$$\beta\,(H) = \beta\left(\overline{H}\right) = 0,$$

we obtain that (3.4.18, 3.4.20):

$$\left(\nabla^{(0,1)}_{\overline{\alpha}} - (\partial_{\overline{\alpha}}\log h)\overline{\lambda}\frac{\partial}{\partial\overline{\lambda}} - 2i\beta_{\overline{\alpha}} + \iota_{\partial_{\overline{\alpha}}} + \frac{i\overline{\lambda}^2}{2}\hat{C}_{\overline{\alpha}\overline{\beta}\overline{\gamma}}dy^{\overline{\beta}}dy^{\overline{\gamma}}\right)\mathcal{S}_{red} = 0, \tag{3.4.21}$$

$$\left(\partial_\alpha + dy^{\overline{\beta}}\hat{g}_{\alpha\overline{\beta}}\left(-\overline{\lambda}\frac{\partial}{\partial\overline{\lambda}} + dy^{\overline{\gamma}}\iota_{\partial_{\overline{\gamma}}}\right) + \frac{i\lambda^2}{2K^2}\hat{C}_{\alpha\beta\gamma}\hat{g}^{\beta\overline{\beta}}\hat{g}^{\gamma\overline{\gamma}}\iota_{\partial_{\overline{\beta}}}\iota_{\partial_{\overline{\gamma}}}\right)\mathcal{S}_{red} = 0, \tag{3.4.22}$$

while (3.4.19) becomes:

$$\frac{\partial}{\partial\lambda}\mathcal{S}_{red} = 0. \tag{3.4.23}$$

We have finally arrived at the precise generalization (3.4.21, 3.4.22, 3.4.23) of the holomorphic anomaly equation of [6] while at the same time having provided its general solution (3.4.11, 3.4.12).

3.5 Concluding remarks

In the present paper we have shown how special Kähler manifolds arise from the structure of quantization, and constructed their quantum counterpart. Crucial to our constructions was the central idea developed in [104] and the formalism of [32]. We have shown how a general version of the holomorphic anomaly equation of [6] arises in our construction while at the same time providing its general solution.

The present work needs however to be further developed to understand better the physical, string theoretic, meaning of these solutions. In particular it is still to be understood, from a quantization perspective, how to isolate the analogue of the generating function of closed topological strings in a given D-brane configuration [75] . In this regard, it seems as though a starting point for these developments within this work could be the discussion at the end of section 3.4.1.

Chapter 4

Essentials of open-closed topological conformal field theory

4.1 Open-closed TQFT basics

Here we continue the discussion of section 2.1 to include open strings. We again refer to [60, 72]. An open-closed TQFT is a functor:

$$Z^{oc} : Cob^{oc} \longrightarrow Vect_k.$$

The category Cob^{oc} is an extension of Cob that includes as objects also intervals $I_\pm$ with positive and negative orientations and arbitrary disjoint unions among themselves and with disjoint unions of circles $S^1_\pm$. Accordingly the spaces of morphisms are extended to include bordisms with the corresponding more general boundaries. We will in fact add one more degree of complexity by labeling the boundaries of the intervals with elements $(a, b, c, etc.)$ of a set or more generally of a class $\mathcal{B}$ of boundary conditions, or *branes*. We will refer to such intervals as e.g. I^{ab}_+. Accordingly, composition of morphisms must be compatible with boundary conditions. In order to specify Z^{oc} completely, just as in the closed case, we only need to define its action on a finite number of objects and morphisms. First, let us concentrate on the strips in $Mor(I^{ab}_+, I^{ab}_+)$. We set

$$Z(I^{ab}_+) =: V^{ab}.$$

Then using the same reasoning as for the cylinder in the closed case, we obtain:

$$Z(I^{ab}_-) = (V^{ab})^\vee$$

and maps ev^{ab} and $coev^{ab}$. Moreover, just as in the closed case, we obtain non-degenerate topological metrics which we will collectively call η_o (where o stands for open) with non-vanishing components:

$$\eta_o : V^{ab} \otimes V^{ba} \longrightarrow k$$

We now investigate disks, with three boundary components, in $Mor(I_+^{ab} \sqcup I_+^{ca}, I_+^{cb})$. These define maps that we will collectively call μ_o with non-vanishing components:

$$\mu_o : V^{ab} \otimes V^{ca} \longrightarrow V^{cb}.$$

One can easily check that μ_o is associative, but it is in general not commutative since we do not have the freedom of "twisting the neck" of the outcoming string. Finally, in order to specify the action of Z^{oc} on disks with boundary an arbitrary disjoint union of intervals, we need units and traces:

$$e_o^a : k \longrightarrow V^{aa}, \qquad \theta_o^a := (e_o)^\vee : V^{aa} \longrightarrow k.$$

We will refer to θ_o^a also collectively as θ_o. Again we have the property:

$$\eta_o = \theta_o \circ \mu_o,$$

where the identity is understood as valid whenever composition is compatible with the boundary conditions. Thus, we have described the restriction Z^o of Z^{oc} on the proper subcategory $Cob^o \subset Cob^{oc}$ of purely open strings with only disks as bordisms, and have found that it is equivalent to what is called a *Calabi-Yau category* $\mathcal{O}$ ("o" for open), with class of objects $\mathcal{B}$ and morphism spaces V^{ab}. In later sections we will often consider finite subcategories of $\mathcal{O}$, that is with finitely many objects, and view them as single non-commutative Frobenius algebras. In order to specify Z^{oc} completely, we need to consider a finite set of bordisms with boundary containing both open and closed strings. In particular we consider the cylinder C_{oc}^{aa} in $Mor(I_+^{aa}, S_+^1)$ that describes an open string joining its endpoints and yields a map:

$$bobu : V^{aa} \longrightarrow V,$$

where *bobu* stands for "boundary to bulk". Similarly we can go the other way with the map:

$$bubo : V \longrightarrow V^{aa}.$$

In fact, it can be shown that the data collected so far specifies Z^{oc} completely. However this data is subject to constraints, which describe different decompositions of the same surface. The simplest such constraints come from the decomposition of the disk with one interval as boundary:

$$\theta_o = \theta_c \circ bobu$$

and the dual:

$$e_o = bubo \circ e_c.$$

Then we have the two decompositions of the cylinder in $Mor(I_+^{aa} \sqcup S_+^1, \emptyset)$ yielding the constraint:

$$\eta_c(bobu \otimes 1) = \eta_o(1 \otimes bubo), \tag{4.1.1}$$

which expresses the fact that *bobu* and *bubo* are adjoints of each-other. One can show that there is in fact only one last constraint, namely the Cardy condition. This comes from two different decompositions of the cylinder viewed as a map in $Mor(I_+^{aa} \sqcup I_+^{bb}, \emptyset)$. One decomposition of such a map is given by first joining the ends of each string and then mapping to k with the cylinder in $Mor(S_+^1 \sqcup S_+^1, \emptyset)$. Applying Z^{oc} then yields the map:

$$\eta_c(bobu \otimes bobu).$$

There is however a way of decomposing this map in the purely open channel. This is given by considering the disk in $Mor(I_+^{aa} \sqcup I_+^{ba} \sqcup I_+^{bb}, I_+^{ba})$ and then glueing the incoming I_+^{ba} with the outcoming I_+^{ba} using first coevaluation and then evaluation. Applying Z^{oc} we obtain:

$$ev(\mu_o \otimes 1)(1 \otimes \mu_o \otimes 1)(1 \otimes T \otimes 1)(1 \otimes 1 \otimes coev)$$

where T is a *brading*, namely an isomorphism that swaps the tensor factors. For $\mathbb{Z}_2$-graded vectorspaces (as we discussed earlier for the purely closed case) the swapping is accompanied by a sign. Composing the above maps we obtain:

$$\psi^{aa} \otimes \psi^{bb} \mapsto \sum_i sign(i, \psi^{aa}) e^i(\mu(\psi^{aa}, \mu(e_i, \psi^{bb})))$$

where $\{e_i\}$ is a basis in V^{ba} and $\{e^i\}$ is its corresponding dual basis. So, to summarize:

$$\sum_i sign(i, \psi^{aa}) e^i(\mu(\psi^{aa}, \mu(e_i, \psi^{bb}))) = \eta_c(bobu(\psi^{aa}), bobu(\psi^{bb})). \tag{4.1.2}$$

The above condition should be viewed as yielding a generalization of the Hirzebruch-Riemann-Roch theorem (see [11]). In fact the immediate generalization is given by setting ψ^{aa} and ψ^{bb} to the idempotents (units in fact) e^a and e^b. Indeed in that case the Cardy condition reduces to:

$$\mathrm{tr}_{V^{ba}}((-1)^F) = \eta_c(bobu(e^a), bobu(e^b)),$$

where we have denoted by $(-1)^F$ the operator implementing the brading. In the following sections we will first discuss how some of the structures arise and are generalized in the context of open and closed topological conformal field theories. In particular we will see how, when considering spaces of $TCFT$'s as opposed to isolated points thereof, the category $\mathcal{O}$ becomes a minimal, Calabi-Yau A_∞ category, which apart from associative composition of morphisms has higher composition maps. Crucial for our investigations will be the lift of these on-shell categories to off-shell differential graded categories. In particular our analysis will culminate with explicit generalizations of *bubo* which allows us to transport bulk deformations to the boundary sector and finally we will give an explicit formula for the off-shell differential graded version of η_o for Landau-Ginzburg models, completing the string field theory data of open Landau-Ginzburg models.

4.2 Physical and topological B-type branes

Here we turn to the definition of the open sector in topological conformal field theories. As for the purely closed case, we will use the general B-twisted Landau-Ginzburg model as our illustrative example. In particular in our future investigations we will often restrict ourselves to affine Landau Ginzburg models.

Introducing the open sector in a CFT is tantamount to extending the definition of the CFT to Riemann surfaces with boundaries. The counterpart of the sphere, is now the sphere with a disk removed, that is the disk itself. In a functorial definition of CFT (see [82]) the complex disk with a boundary condition inserted on the boundary, defines a state in the closed string sector called *boundary state*. More precisely, the boundary state is inserted at the origin of the disk, where the in-state Hilbertspace resides, and is propagated to the boundary of the disk. Passing to the topological sector of the theory, e.g. by B-twisting, the boundary state should reduce to $bobu(e_o^a)$. Similarly the TFT disk in $Mor(I_+^a, I_+^b)$ should arise from the infinitely extended strip with left and right boundary conditions inserted. The boundary conditions we are interested in are in particular conformal. From the point of view of the strip where time runs along the infinitely extended direction, conformal means that there is no energy flow across either boundary:

$$T - \overline{T}\big|_{bdry} = 0. \tag{4.2.1}$$

If, as in our case of $N = (2,2)$ superconformal symmetry, the Virasoro algebra is extended to an algebra $\mathcal{A} \oplus \overline{\mathcal{A}}$, then, the energy reflecting condition is generalized to an arbitrary current $W(z)$ as:

$$W - \Omega(\overline{W})\big|_{bdry} = 0$$

where Ω is an automorphism of $\overline{A}$ preserving, in particular, the Virasoro generators. In our case, to the current T we supplement J, G^+ and G^-. The basic automorphisms Ω of interest could act on the supercharges by multiplication by ± 1 and could exchange their R-charges. The latter operation corresponds to the Mirror-automorphism and would implement A-type boundary conditions. We are interested in B-type boundary conditions, and we set the arbitrary multiplicative sign to $+$. Then (4.2.1) is supplemented by:

$$J - \overline{J}\big|_{bdry} = 0, \qquad G^{\pm} - \overline{G}^{\pm}\Big|_{bdry} = 0.$$

Therefore, the surviving diagonal $N = 2$ algebra is generated by $T+\overline{T}$, $J+\overline{J}$ and $G^{\pm}+\overline{G}^{\pm}$. In particular, the theory with boundary has the B-type supersymmetry Q. The conditions at the boundary just discussed are sufficient at the classical level. At the quantum level, one must take into account the so called GSO projection which we could neglect all along in the purely closed case, and simply comes from the fact that in the path-integral one also sums over spin structure, thus giving rise to a $\mathbb{Z}_2$-Haar integral which projects on $\mathbb{Z}_2$-invariants. We already leave the abstract discussion of boundary conditions in $N = (2,2)$ CFT's and jump to the investigation of boundary conditions in general LG-models. We will primarily sketch the approach of [40, 41, 45, 61] and we also refer to seminal work [7] regarding boundaries in affine Landau-Ginzburg models. In that approach one uses string-theory intuition to guess the most general form for boundary interactions. In particular we will use terminology which will become clear within our framework only a posteriori, as the concept of anti-brane.

To start, first of all one concentrates on the D-term and views it as a special case of a $N = (1,1)$ theory. In that more general case one can extend the theory to the infinite strip without imposing boundary conditions, but by adding a certain standard boundary term. In order to add boundary interactions to such a theory, one uses the intuition gained in the study of the type II A/B open string spectrum on flat space-time with Neumann-Neumann boundary conditions. In that case one is particularly interested in the lowest energy states which are viewed as providing infinitesimal deformations of the underlying theory. In particular, as one is interested in deformations of the vacuum, one analyzes the lowest energy states in the NS sector and finds a space-time massless vector boson A and a tachyonic scalar T. At this point the GSO projection will play a prominent role. Depending on how we interpret A and T, the GSO projection will gauge away one of the two. The result is that if we view the Neumann-Neumann boundary conditions as describing open strings stretched from a space-filling brane to itself, then A survives, while if we interpret the open strings as stretching from the space-filling brane to a space-filling anti-brane, then T survives. The physical property of anti-brane that one uses to understand this phenomenon is one we haven't yet described within our framework, suffice it to say, that if the space filling brane comes with multiplicity N, that is with an N-dimensional Chan-Paton space X_0, then to the anti-brane one associates X_1 with reversed parity. That is one considers the composite Chan-Paton space:

$$E = X_0 \oplus X_1$$

and A and T also come with appropriate multiplicity, that is they are generalized to:

$$T = \begin{pmatrix} 0 & J \\ E & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}.$$

Extending this construction from flat space-time to a general D-term, E is generalized to a super-vectorbundle, in particular, locally it is still to be viewed as a multiplicity space, that is it is a locally free module over the coordinate ring, or more precisely the sheaf of functions of the underlying Kähler manifold M. Instead T and A combine to form a *superconnection* $B = T + A$ on $X_0 \oplus X_1$, where A has degree one because it is a gauge field, so in particular of form degree 1, while T is of degree 1, because that is the difference in degrees between X_1 and X_0. It is important to stress that if the theory described by M is a CFT, the A can in principle generate marginal deformations, while T generates relevant deformations. That is the theory ceases to be a CFT but in principle flows to a new CFT where the tachyon has condensed [87] to form a new configuration of boundary conditions, in particular more general Dirichlet branes of lower dimension. The Tachyon is therefore fundamental if we add a super potential W, because the latter will force the brane to be localized at its critical points.

At this point one looks for a coupling of the super connection B to the boundary of the surface, in our case the an infinitely extended strip $\mathbb{S}$. Up to supersymmetrization, this coupling should simply introduce a Wilson line in the Path-integral for each of the two boundaries of $\mathbb{S}$. Imposing a diagonal $N = 1$ supersymmetry one finds(see e.g. [40]) the following improved connection over either boundary:

$$\mathcal{A}_t = \dot{x}^I A_I(x) - \frac{i}{4} F_{IJ}(x)\psi^I\psi^J + \frac{i}{2}\psi^I D_I T(x) + \frac{1}{2}T(x)^2, \tag{4.2.2}$$

where, in the particular case of a Kähler target, I runs over holomorphic and anti-holomorphic coordinates i and $\bar{i}$, while ψ is the linear combination of the corresponding left and right fermions surviving the supersymmetric integration over $\partial\mathbb{S}$. For B-type $N=1$ supersymmetry $\psi^I = (\eta^i, \eta^{\bar{i}})$ where $\eta^i = (\psi^i + \overline{\psi}^i)$ and $\eta^{\bar{i}} = (\chi^{\bar{i}} + \overline{\chi}^{\bar{i}})$. The first and third term are the starting points and the remaining terms one finds by requiring supersymmetry[1]. A supersymmetric $\mathcal{A}_t$ means that the corresponding bulk field, in other words the "would be" boundary state, is supersymmetric. The latter is given by considering the boundary of the circle rather than the boundary of the strip. Thus the bulk field is the Wilson loop, or holonomy:

$$\phi_{Bulk}(\mathcal{A}) = \mathrm{str}\left(P\exp\left(-\int_{S^1} dt\mathcal{A}_t\right)\right)$$

where the supertrace is the canonical one on the fibers of $X_0 \oplus X_1$. In fact under the $N=1$ supersymmetry variation δ_1, $\mathcal{A}_t$ transforms as:

$$\delta_1\mathcal{A}_t = \mathcal{D}_t(-\epsilon_1(T - \psi^I A_I)) \tag{4.2.3}$$

thus leaving a boundary contribution to the supersymmetry variation if the boundary circle is replaced by an interval. Imposing B-type $N=2$ supersymmetry constrains the boundary data as follows:

$$F^{(0,2)} = F^{(2,0)} = 0 \tag{4.2.4}$$

$$T = iQ_b - iQ_b^\dagger \tag{4.2.5}$$

$$D_{\bar{i}}Q_b = D_i Q_b^\dagger = 0. \tag{4.2.6}$$

(4.2.7)

Moreover there is a further constraint on Q that depends on the superpotential, namely:

$$Q_b^2 = W\mathrm{Id} + c. \tag{4.2.8}$$

We shall explain, given the prior constraints, how the latter arises. Beforehand we should interpret the first three constraints. Equation (4.2.4) can be rewritten as:

$$(\overline{\partial} + A^{0,1})^2 = 0.$$

That is, the supervectorbundle $E = X_0 \oplus X_1$ is holomorphic with holomorphic structure given by $A^{0,1}$. Equation (4.2.5) says that the tachyon splits into the tachyons arising from

[1] In fact in [40] a more general connection was constructed, but we will not need it in the remaining discussion

the two separate $N = 1$ supersymmetries of the B-type $N = 2$ supersymmetry, namely Q and $Q^\dagger$ respectively. Equation (4.2.6) stating the holomorphicity of Q, could in fact be generalized. As it stands it will ensure that, given the fourth constraint, the total bulk/boundary supercharge Q_{tot} is a differential. The appearance of W in the constraint is due to the so called Warner term [97]. As we mentioned earlier, in the case of $W = 0$ one can add a canonical boundary term so that the D-term preserves $N = 1$ supersymmetry without resorting to extra, Chan-Paton, degrees of freedom. The same is true if we add a non trivial super potential. However, while in the former case the boundary term continues to be B-type $N = 2$ supersymmetric, in the latter case this is no longer true. Instead, denoting $\delta_Q = i\bar{\epsilon}Q$ one obtains:

$$\delta_Q S = -\int_{\partial\mathbb{S}} dt\bar{\epsilon}\eta^i\partial_i W,$$

which is precisely cancelled by a term arising in the Wilson line whose variation is:

$$\int_{\partial\mathbb{S}} dt\bar{\epsilon}\eta^i\partial_i Q^2$$

provided indeed (4.2.8) is fulfilled. In fact the cancellation of the Warner term ensures cancellation of further terms involving the commutator $[Q^2, Q^\dagger]$, provided however that the integration constant c commute with $Q^\dagger$. In the end, the extra constant is usually omitted, as it is in our case, for example by assuming that W be quasi-homogeneous. The data given so far describes B-type conditions in the still untwisted theory on the infinite strip. However we could also interpret it with no change as B-twisted boundary conditions by modifying the bulk field content to the twisted version (2.5.1). We will work in this topological setting unless otherwise stated.

At this point we introduce boundary topological fields. We do this through the definition of bulk fields. The picture we should have in mind is that defining the bulk topological metric (2.4.2), namely two stretched hemispheres with at each tip the insertion of a bulk field, except that now the left, say, hemisphere is replaced completely by its boundary circle with boundary conditions on it as well as boundary field insertions interpolating between them. From this point of view, the CFT picture of the infinitely extended strip becomes the TQFT picture of a disk with two points removed from the boundary circle. From a path-integral perspective, the boundary circle with boundary fields corresponds to the insertion of a term of the form:

$$\mathrm{str}(O_0(t_0)U_0(t_0,t_1)O_1(t_1)U_1(t_1,t_2)O_2(t_2)\cdots O_n(t_n)U_n(t_n,t_0)), \tag{4.2.9}$$

where $U_k(t_k, t_{k+1}) = P\exp(-\int_{t_k}^{t_{k+1}} dt\mathcal{A}_t^k)$ is the parallel transport operator with respect to the boundary condition $\mathcal{A}_t^k$, and the O_i's are a priori off-shell boundary fields inserted at the point t_i and are defined as local operators of the bulk fields valued in $\mathrm{End}(E_k, E_{k+1})$. It is clear therefore, that from the point of view of the boundary observables, the improved superconnection is a Hamiltonian. Now we want to understand the space of on-shell boundary fields, that is the fields the path-integral localizes onto. Therefore we consider the action of δ_Q on the above insertion. For this, notice that from (4.2.3) and the discussion of the Warner term:

$$\delta_Q U_k(t_k, t_{k+1}) = \left(\int_{t_k}^{t_{k+1}} dt\bar{\epsilon}\eta^i\partial_i W\right) U_k(t_k, t_{k+1}) \\ - \bar{\epsilon}(iQ_b^k - \eta^{\bar{i}}A_{\bar{i}}^k)(t_k)U_k(t_k, t_{k+1}) + U_k(t_k, t_{k+1})\bar{\epsilon}(iQ_b^k - \eta^{\bar{i}}A_{\bar{i}}^k)(t_{k+1}).$$

Hence, applying δ_Q to (4.2.9) we obtain multiplication by the negative Warner term plus a sum of terms each summand of which is equal, up to a sign, to (4.2.9) except for the alternate replacement of one boundary field by:

$$\delta_b O_k = \delta_Q O_k + \bar{\epsilon}((iQ_b^k - \eta^{\bar{i}}A_{\bar{i}}^k)O_k \pm O_k(iQ_b^{k+1} - \eta^{\bar{i}}A_{\bar{i}}^{k+1})).$$

If we restrict attention to a finite number of boundary conditions, we can take their direct sum, which consists of the direct sum of the superbundles and corresponding connections. Then the above reduces to the more transparent equation:

$$\delta_b O_k = \delta_Q O_k + \bar{\epsilon}[iQ_b - \eta^{\bar{i}}A_{\bar{i}}, O_k].$$

We observe that δ_Q-cohomology translates to δ_b-cohomology for the boundary fields. In particular, the path-integral localizes onto boundary observable in δ_b-cohomology. To completely characterize the boundary fields we have to introduce an ingredient in the theory with boundaries that we have chosen to omit so far. The ingredient is locality. That is the equation of motion for bulk fields must be local. More importantly a topological theory where equations of motion are trivial, since the hamiltonian is zero, must be local by definition as this is the ingredient that allows decomposition of topological surfaces to be paralleled by decomposition of correlation functions. Locality is an issue in the present context, because the boundary interaction term is manifestly non-local. One ensures locality by introducing a projector P_{BC} in the path integral that constrains variations of the bulk action, with respect to bulk fields, to be cancelled by variations of the boundary interaction term, so that the Euler Lagrange equations of motion are given solely by considering the local bulk term. This projection therefore defines boundary conditions, and in the case of a vanishing tachyon one obtains Neumann-Neumann boundary conditions motivating the initial construction involving vector massless gauge fields and scalar tachyons. These boundary conditions, in particular, "halve" the number of degrees of freedom of off-shell boundary fields. The relevant boundary condition arising from variation by η^i leads to the identification:

$$\theta_i \sim -i\partial_i Q_b - (iF_{i\bar{j}} + G_{i\bar{j}})\eta^{\bar{j}}, \tag{4.2.10}$$

namely θ_i is expressible in terms of $\overline{\eta}^{\bar{i}}$.

Localization of the path-integral proceeds analogously to the purely bulk case (see section 2.5). One can first localize on zero-modes and consequently onto the critical points of W. Upon localization onto zero-modes, the bulk degrees of freedom of off-shell boundary observables are then reduced from arbitrary polyvectorfield-valued differential forms to differential forms. More precisely, after localizing onto zero-modes we obtain the following complex of off-shell boundary observables:

$$V_o^{off-shell} = (\Gamma(X, \mathrm{End}(E) \otimes \Lambda^\bullet (T^{(0,1)}X)^\vee), D^{(0,1)} + [Q_b, \cdot]). \tag{4.2.11}$$

Notice that the term in the bulk differential involving W has disappeared, because its action is trivial on degree zero polyvectorfields. Is important to notice for further discussion, that the above complex is naturally a bi-complex with differentials $D^{(0,1)}$ and $[Q_b, \cdot]$.

So far we have characterized off-shell and on-shell boundary observables, that is in particular the TQFT datum V^o and we notice that it is naturally a unital non-commutative associative algebra where the product is the cup product, which is the composition of $\wedge$ and the product of matrices in $\mathrm{End}(E)$. To complete the TQFT data we still need maps *bobu*, *bubo* and θ_o. We distinguish between the cases $W = 0$ and $W \neq 0$. In the former case, localizing onto zero-modes leaves an exact boundary interaction term which therefore can be set to zero. One obtains:

$$\theta_o(O_\alpha) \propto \int_X \Omega \wedge \mathrm{str}(\alpha).$$

Thanks to the non-degeneracy of θ_o, we can read off *bobu*:

$$bobu(\alpha) = \mathrm{str}(\Pi \wedge \alpha),$$

while *bubo* can be read off by imposing property (4.1.1). One obtains:

$$bubo(\omega) = P_{BC}(\omega \mathrm{Id}_E) = \int_{T^{(1,0)}X[1]} d\partial_i \cdots d\partial_n \, \Pi \wedge \omega \mathrm{Id}_E,$$

where $\Pi \in \Gamma(X, \mathrm{End}(E) \otimes \Lambda^\bullet T^{(1,0)}X \otimes \Lambda^\bullet (T^{(0,1)}X)^\vee)$ implements the boundary conditions (4.2.10) upon integration over the parity reversed holomorphic tangent bundle $T^{(1,0)}X[1]$:

$$\Pi = (\partial_1 + i\partial_1 Q_b + (iF_{1\overline{j_1}} + G_{1\overline{j_1}})dz^{\overline{j_1}}) \wedge \cdots \wedge (\partial_n + i\partial_n Q_b + (iF_{n\overline{j_n}} + G_{n\overline{j_n}})dz^{\overline{j_n}}).$$

We now proceed to the case $W \neq 0$ and as in the bulk case, restrict attention to W having an isolated singularity and assume that the cohomology of $D^{(0,1)}$ is trivial, that is concentrated in tensor degree zero, so that the cohomology of the total complex (4.2.11) reduces to the second page of the spectral sequence:

$$V_o = H_{[Q_b, \cdot]}(H_{D^{(0,1)}}(V_o^{off-shell})) = (\Gamma(X, \mathrm{End}(E)), [Q_b, \cdot]).$$

After localizing onto zero-modes, the surviving terms in the bulk and boundary terms are both exact in δ_Q and δ_b respectively. The latter is set (flown) to zero while the former is computed in the infrared limit as in the bulk case. Crucial are the boundary conditions (4.2.10) that introduce in particular a term proportional ∂Q_b in the bulk action. This term is the only one to survive localization onto the critical point of W and one obtains

$$\theta_o \propto \pm \int_X \frac{\Omega \, \mathrm{str}(\alpha \partial_1 Q_b \cdots \partial_n Q_b)}{\partial_1 W \cdots \partial_n W}$$

where $\pm$ is a relative sign that we will fix later on in section 5.3.8. The above is called the *Kapustin-Li* trace after the first who gave a physical derivation thereof, while the boundary topological metric η_o is referred to as the *Kapustin-Li pairing.* The remaining $TQFT$ data are:

$$bobu(\alpha) = \pm \operatorname{str}(\alpha \partial_1 Q_b \cdots \partial_n Q_b)$$

and

$$bubo(f) = f \mathrm{Id}_E.$$

The non-degeneracy of (4.2) follows from Serre duality and the Calabi-Yau condition as explained in section 2.5, while the non-degeneracy of the Kapustin-Li pairing was proved in [74]. The only non-trivial constraint to be checked for the above data to constitute a TQFT is the Cardy condition (4.1.2). This was proven for the B-model in [11]. For the affine Landau Ginzburg model with isolated singularity it was proven in [79] and the result was extended within a more general framework to include the case of orbifolded affine Landau Ginzburg models in [8].

So far we have sketched how to extract the open-closed TQFT data from twisted $N = (2,2)$ QFT's by examples and in our case of B-twisted models we were not forced to impose $U(1)_V$ R-symmetry. Although of great importance especially for the Landau Ginzburg/ Calabi-Yau correspondence and thus for an understanding of $N = (2,2)$ CFT's slightly beyond the purely topological sector, we omit this topic as later we will be interested in the general non R-symmetric case. In particular W need not be quasi-homogeneous for our constructions in chapter 5.

4.3 Formalizing TQFT data of B-twisted models

In this section we formalize the structure of TQFT arising from the B-twisted models. We discuss the B-model and the affine Landau Ginzburg models separately (comprehensive reviews for the former are contained in e.g. [2, 40]). To have a proper understanding of the B-model we should introduce the $U(1)_V$ R-symmetry. We resort back to (4.2.2) and we see that if we assign the standard R-charge -1 to η^i, Q must have R-charge 1. To implement this we need a representation:

$$R : \mathbb{C}^* \to \mathrm{Aut}(E).$$

It is customary to use $\mathbb{C}^*$ rather than $U(1)$. This means, that E is not only $\mathbb{Z}_2$ graded but $\mathbb{Z}$ graded. The two gradings are compatible because Q has degree 1 w.r.t. both. In the B-model $Q^2 = 0$, therefore E should be viewed as an arbitrary, bounded for dimensional reasons, complex of holomorphic vector-bundles:

$$(E, Q)$$

The same holds for:

$$(\mathrm{Hom}(E_a, E_b), Q_{ab}).$$

The space of onshell open strings is the cohomology V_o if $V_o^{off-shell}$ defined in (4.2.11). In the context of the TQFT framework outlined in section 4.1, we want to understand this as a morphism space in an additive category. In fact this space turns out to be the space of morphisms:

$$\mathrm{Hom}_{D^b(X)}(\mathcal{E}_a, \mathcal{E}_b[i])$$

in the *bounded derived category of coherent sheaves* $D^b(X)$. We have denoted by $\mathcal{E}$ the sheaf of sections of E while $[i]$ denotes a shift by i units of the complex $\mathcal{E}_b$. The notion of derived category is very general and the starting point is an *abelian* category $\mathcal{A}$, namely an additive category which closes under taking kernels and cokernels. In our case the candidate category would be the category of holomorphic vectorbundles, however this is not abelian. In particular taking the cokernel of two finite rank holomorphic vectorbundles, one obtains a more general object called a *coherent sheaf*. This is precisely a sheaf that locally is the cokernel of two free sheaves. The category of coherent sheaves is abelian, so one considers this. The next step is to consider a category of complexes $C(A)$ of $\mathcal{A}$. In the case of $D^b(X)$ one considers complexes with bounded cohomology. The derived category $D(\mathcal{A})$, has as objects those of $C(\mathcal{A})$. The morphisms instead can be specified as follows. One takes at first morphisms in $C(\mathcal{A})$ and identifies homotopic pairs. Recall that given two complexes $C_1 = (M_1, d_1)$ and $C_2 = (M_2, d_2)$, two maps $f, g : C_1 \to C_2$ are said to be homotopic if there is a homotopy $h : C_1 \to C_2[-1]$:

$$hd_1 + d_2 h = f - g.$$

In particular if the complexes admit a map homotopic to zero, they have the same cohomology. Then one localizes onto quasi-isomorphisms. A quasi-isomorphism is defined as a map of complexes that reduces to an isomorphism on cohomology. Localizing on quasi-isomorphisms proceeds in two steps. First one adjoins formal inverses q^{-1} of quasi-isomorphisms q. Finally one defines a general morphism between two complexes C_1, C_2 to be an arbitrary sequence of homotopy classes of morphisms in $C(\mathcal{A})$ supplemented by the previously described inverses, modulo composition of intermediate morphisms. Under this equivalence relation each morphism can be presented in one of the following two ways:

$$C_1 \overset{q_1^{-1}}{\to} C \overset{f_2}{\to} C_2, \qquad C_1 \overset{f_1}{\to} C \overset{q_2^{-1}}{\to} C_2.$$

If we assume for a moment that $Q = 0$, therefore if we restrict ourselves to mere vectorbundles, then:

$$\begin{aligned} V_o &= H^\bullet_{D^{(0,1)}}(X, \mathrm{Hom}(E_a, E_b)) = \check{H}^\bullet(X, \mathcal{H}om(\mathcal{E}_a, \mathcal{E}_b)) = H^\bullet(X, \mathcal{H}om(\mathcal{E}_a, \mathcal{E}_b)) \\ &=: \mathrm{Ext}^\bullet(\mathcal{E}_a, \mathcal{E}_b) \end{aligned}$$

The second equality is the Čech-Dolbeaux isomorphism which holds provided the base manifold X is Kähler. The third equality is the statement that Čech cohomology is the sheaf cohomology of $\mathcal{H}om(\mathcal{E}_a, \mathcal{E}_b)$. The name Ext is motivated by the fact that $\mathrm{Ext}^i(\mathcal{E}_a, \mathcal{E}_b)$

is the space of i-fold extensions of $\mathcal{E}_b$ by $\mathcal{E}_a$. In particular, in the case $i = 1$, $\mathrm{Ext}^1(\mathcal{E}_a, \mathcal{E}_b)$ is the space of non-trivial short exact sequences of sheaves:

$$0 \to \mathcal{E}_a \to \mathcal{F} \to \mathcal{E}_b \to 0$$

induced by the short exact sequence of vectorbundles $E_a \stackrel{\iota_a}{\to} E_a \oplus E_b \stackrel{p_b}{\to} E_b$. More generally, in the case of a single coherent sheaf, that is of a single object in $\mathcal{A}$ one can include it as an object in $D(\mathcal{A})$ as the degree zero component of a complex which is zero in every other degree. Then one has:

$$\mathrm{Ext}^i(\mathcal{E}_a, \mathcal{E}_b) = \mathrm{Hom}_{D^b(X)}(\mathcal{E}_a, \mathcal{E}_b[i]).$$

In this categorical language, the existence of a non-degenerate pairing in the open TQFT, which follows from Serre duality and the Calabi-Yau condition, translates to the property that $D^b(X)$ has a trivial *Serre functor* $S : D^b(X) \to D^b(X)$. A Serre functor of a derived category, if it exists, is defined by the property:

$$\mathrm{Hom}_{D(\mathcal{A})}(O_a, O_b)^\vee = \mathrm{Hom}_{D(\mathcal{A})}(O_b, S(O_a)).$$

A Serre functor is called trivial if $S = (\,\cdot\,)[i]$ for some i. Precisely in the case of a Calabi-Yau manifold, Serre duality (see section 2.5) says that this is the case and the shift is given by the degree of the holomorphic top form, that is the dimension of X.

Now we turn to the affine Landau Ginzburg model. Recall that in this case

$$O(X) = \mathbb{C}[x^i, \cdots, x^n] =: \mathcal{R}.$$

Therefore, the sheaves of sections of the super-vectorbundles reduce to free $\mathcal{R}$-modules $M_0 \oplus M_1$. In this context we will use throughout the notation:

$$Q_b = D$$

and D is called a *matrix factorization* of W. From now on we will be interested solely in the case of W having an isolated singularity at 0.

The off-shell open string space of zero-modes reduces to:

$$(\mathrm{Hom}((M_0 \oplus M_1)_a, (M_0 \oplus M_1)_b), D_a(\,\cdot\,) \pm (\,\cdot\,)D_b). \tag{4.3.1}$$

Obviously, contrary to the B-model case, $(M_0 \oplus M_1, D)$ is not a complex. One can however view it again as a complex so as to make contact with the formalism described previously for the B-model. Before we do that however, we notice that in this case, defining a category whose set of morphisms is precisely the cohomology of (4.3.1) is straightforward. One starts with an off-shell category

$$MF(W)$$

known as the *category of matrix facotrizations*, whose objects are matrix factorizations $(M_0 \oplus M_1, D)$ and whose set of morphisms is $\mathrm{Hom}((M_0 \oplus M_1)_a, (M_0 \oplus M_1)_b)$. Then the desired category is simply obtained by replacing the sets of morphisms with their cohomology. One thus obtains the *homotopy category of matrix factorizations*

$$[MF(W)].$$

We now briefly sketch the initial basic steps required to make contact with the formalism of the B-model. We notice that we can fit M_0 and M_1 into a short exact sequence:

$$0 \longrightarrow M_1 \xrightarrow{F} M_0 \longrightarrow \text{coker}(F) \to 0$$

where:

$$D = \begin{pmatrix} 0 & F \\ G & 0 \end{pmatrix}$$

To show that it is exact one first has to observe that $dim(M_0) = dim(M_1)$. Indeed $tr(W\text{Id}_{M_0}) = tr(FG) = tr(GF) = tr(W\text{Id}_{M_1})$. Thus we see that this condition requires only $W \neq 0$. Now supposing there is a non-zero element in $v \in \text{Ker}(F)$, then $Wv = 0$, therefore $v = 0$. The above sequence can be turned into an exact sequence of free modules over $S = \mathcal{R}/W$. In that way $Q^2 = 0 \in S$ as in the B-model case and requiring the modules to be free we also recover a local notion of complexes of vectorbundles as for the B-model. As a sequence of S modules the above sequence is clearly not exact anymore. On the contrary $\text{Ker}(F) = \text{Im}(G)$. Therefore to the left of M_1 we can adjoin $\text{Im}(G)$, this however is clearly no longer free, therefore we can adjoin M_0 instead and continue this procedure indefinitely alternating the roles of F and G to obtain:

$$\cdots \xrightarrow{\overline{G}} \overline{M}_1 \xrightarrow{\overline{F}} \overline{M}_0 \xrightarrow{\overline{G}} \overline{M}_1 \xrightarrow{\overline{F}} \overline{M}_0 \longrightarrow \text{coker}(\overline{F}) \longrightarrow 0,$$

where we have overlined modules and maps to indicate that they are defined over S. That is we obtained what is called a 2-periodic resolution of $\text{coker}(\overline{F})$. A theorem by Eisenbud (see [30] and [29] for a concise review of the topic) states that in fact any S module (of finite rank) admits a free resolution which becomes 2-periodic after a finite number of steps. One can then start with the abelian category of S-modules. Out of this one can construct two categories of complexes. One which consists of complexes with bounded cohomology and one which consists of *perfect complexes*, namely complexes that are quasi-isomorphic to bounded complexes of free-modules. One can then pass to the derived categories to obtain $D^b(S)$ and $D^b_{perf}(S)$ respectively. Intuitively we see that a matrix factorization only sees the tail of complexes of free S-modules, infact, it was shown in [9] that $[MF(W)]$ is equivalent to $D^b(S)/D^b_{perf}(S)$. The equivalence and the quotient are intended in the setting of *triangulated cateogories*. This is a notion that formalizes the structure of the derived category, we refer to e.g. [2] for a light introduction to the topic. It was shown in [76] that the notion of category of matrix factorizations can be globalized, by replacing $D^b(S)$ with $D^b(X)$ where X is possibly singular[2] and likewise $D^b_{perf}(S)$ is replaced with $D^b_{perf}(X)$ which consists of complexes of coherent sheaves which are quasi-isomorphic to bounded complexes of holomorphic vectorbundles. The crucial observation is that if X is smooth, $D^b(X) = D^b_{perf}(X)$, therefore the quotient exactly detects the singularities of X and is thus called the *derived category of singularities of* X.

We will now focus on general algebraic structures involved in the study of deformations of open-closed TQFT's arising from general twisted $N = (2, 2)$ QFT's. In particular

[2] We omit the exact restrictions on X and refer to [76].

we will be solely interested in the classical genus zero setting, that is on the disk with both boundary and bulk insertions paralleling the discussion in 2.6. The mathematical structures required to tackle deformations are much richer than those provided by the derived category, indeed the derived category only describes first-order deformations.

4.4 Spaces of open-closed TQFT's on the disk

This section is primarily based on [42], however we also refer to [43, 44] for earlier developments. The aim of this section is to provide a synthetic description of the general algebraic structure of spaces of open and closed TQFT's on the disk arising from twisted $N=(2,2)$ CFT's. The global conformal/holomorphic automorphisms of the $N=(2,2)$ super-disk, are constituted by the global part of a diagonal $N=2$ algebra as discussed at the beginning of the previous section. Therefore, as in the twisted theory we are in the Ramond sector, the vacuum state of the super-disk must be invariant under the global $N=2$ group. We relax invariance on the global R-symmetry, and concentrate on the generators:

$$Q, \qquad \tilde{G}_{-1}, \qquad \tilde{G}_0, \qquad \tilde{G}_1,$$

where $\sim$ indicates a diagonal left- right- generator. In the case of the B-twist $\tilde{G}=G^{+}+\overline{G}^{+}$. The boundary chiral primary fields of the untwisted theory are now identified with $[Q,\cdot]$-cohomology, while boundary deformations of the theory are defined analogously to the bulk case. They are implemented by adding to the bulk-boundary action, an F-term which is the integral of the boundary chiral field whose lowest component is a chiral primary. Denoting the physical fields in Q-cohomology by ψ_a, then the boundary interaction between t_k and t_{k+1} is deformed by the insertion of:

$$P\exp\left(u^a\Delta F(\psi_a)\right) = P\exp\left(u^a\int_{t_k}^{t_{k+1}} dt[\tilde{G}_{-1},\psi_a]\right). \tag{4.4.1}$$

For the boundary descendant one uses the notation:

$$\psi_a^{(1)} = dt[\tilde{G}_{-1},\psi_a]$$

analogous to that for the bulk descendants (2.6.3). The parameters u^a are the boundary moduli. As we shall see in a moment these should not be assumed to be commuting variables. The boundary deformed TQFT is defined on the same space V_o, while the deformed non-commutative Frobenius structure is encoded in:

$$\eta_o(\psi_{a_0},\mu_u(\psi_{a_1}\otimes\psi_{a_2})) = \langle\psi_{a_0}(t_0)\psi_{a_1}(t_1)P\exp\left(u^a\int_{t_1}^{t_n}\psi_a^{(1)}\right)\psi_{a_2}(t_2)\rangle_{disk}. \tag{4.4.2}$$

The points t_0, t_1 and t_2 are chosen at will thanks to the global conformal $SL(2,\mathbb{R})$ invariance of the disk, i.e. of the vacuum. In particular η_o is manifestly flat. We observe that due to the non-commutativity of the boundary algebra, the deformed product μ_u cannot be associative and neither will it be compatible with η_o. Thus boundary deformations do not strictly describe a family of open TQFT's. One can however cure this problem by choosing the moduli u^a to be graded commutative and by replacing ψ_{a_i} by $u^{a_i}\psi_{a_i}$. In this way however one loses all of the non-commutative structure of open amplitudes. We will instead be interested in honest boundary amplitudes. Declaring u^a to be the generators of a free algebra:

$$\mathbb{C}[u^a], \qquad \cdot = \otimes,$$

one can encode the amplitudes as the coefficients in the Taylor expansion in u^a of (4.4.2). Compatibility of the product with the metric is replaced by the more sophisticated Ward identity describing the invariance of the vacuum under $\tilde{G}_{-1}$, $\tilde{G}_0$ and $\tilde{G}_1$:

$$\begin{aligned}&\langle \psi_{a_0}\psi_{a_1} P \int \psi^{(1)}_{a_2} \cdots \int \psi^{(1)}_{a_{m-1}} \psi_{a_m}\rangle \\ &= (-1)^{\tilde{a}_1+\cdots+\tilde{a}_{m-2}} \langle \psi_{a_0} P \int \psi^{(1)}_{a_1} \int \psi^{(1)}_{a_2} \cdots \int \psi^{(1)}_{a_{m-2}} \psi_{a_{m-1}} \psi_{a_m}\rangle,\end{aligned}$$

where we have introduced the tilde degree $\tilde{a} = |a| + 1$. Now one can use the cyclicity property of the topological metric:

$$\omega_{ab} := \eta_o(\psi_a\psi_b) = (-1)^{|a||b|}\eta_o(\psi_b\psi_a) = (-1)^{|a||b|}\omega_{ba},$$

which simply reflects the closure of the circle, to obtain:

$$\begin{aligned}&\langle \psi_{a_0}\psi_{a_1} P \int \psi^{(1)}_{a_2} \cdots \int \psi^{(1)}_{a_{m-1}} \psi_{a_m}\rangle \\ &= (-1)^{\tilde{a}_m(\tilde{a}_1+\cdots+\tilde{a}_{m-2})+|a_m|(\tilde{a}_0+\tilde{a}_{m-1})} \langle \psi_{a_m}\psi_{a_0} P \int \psi^{(1)}_{a_1} \int \psi^{(1)}_{a_2} \cdots \int \psi^{(1)}_{a_{m-2}} \psi_{a_{m-1}}\rangle.\end{aligned}$$

The above identity is also referred to as *ciclicity* of TCFT amplitudes.

Associativity of the boundary deformed product is instead replaced by the more subtle Ward identity describing the invariance of the vacuum under Q:

$$\langle [Q, \psi_{a_0}\psi_{a_1} P \int \psi^{(1)}_{a_2} \cdots \int \psi^{(1)}_{a_{m-1}} \psi_{a_m}]\rangle = 0.$$

The adjoint action of Q alternatively produces terms of the form:

$$[Q, [\tilde{G}_{-1}, \psi_{a_i}]] = \frac{\partial}{\partial t^i}\psi_{a_i}.$$

These terms are part of a product in a nested integral, that is an integral over a simplex. In rewriting the sum of terms given by the action of $[Q, \cdot]$ in terms of the TCFT amplitudes, one obtains various contributions from the boundary of the simplex, in fact also the higher boundaries, that is the boundaries of the faces of the simplex down to vertices, because of singular contact terms corresponding to operator products of descendants. These boundaries correspond to configurations where two or more open string vertices coincide.

The singular, contact, terms imply that as these points meet they "bubble" off to form a new disk giving rise to a lower order amplitude. A careful analysis of this bubbling was carried out in [42]. To describe the resulting Ward identity we shall introduce the Taylor coefficients of the deformed product μ_u:

$$\widetilde{r}_n(\psi_{a_1} \otimes \cdots \otimes \psi_{a_n}) = (-1)^{\tilde{a}_1+\cdots+\tilde{a}_n} \omega^{ab} \langle \psi_b \psi_{a_1} P \int \psi_{a_2}^{(1)} \cdots \int \psi_{a_{n-1}}^{(1)} \psi_{a_n} \rangle \psi_a.$$

The choice of complicating the notation of the higher products by supplementing them with a $\sim$ was made to be consistent with the formalism of [13]. The Ward identities of Q then read:

$$\sum_{m \le n} \sum_{k \le n+1-m} (-1)^{\tilde{a}_1+\cdots+\tilde{a}_{k-1}} r_{n-m+1}(\psi_{a_1} \otimes \cdots \otimes r_m(\psi_{a_k} \otimes \cdots \otimes \psi_{a_{k+m-1}}) \otimes \cdots \otimes \psi_{a_n}) \tag{4.4.3}$$

$$= 0. \tag{4.4.4}$$

The above identity means that the higher products r_n define an A_∞-*algebra*, more precisely a *minimal* A_∞-*algebra*, namely one where the first non vanishing product is r_2. This is a special case of a *strong* A_∞-*algebra* where the first non-vanishing product is r_1, which distinguishes itself from a *weak* A_∞-*algebra* that has a non-vanishing r_0. If we consider an indefinite number of branes, we must speak in terms of A_∞-*categories*.

In terms of the the higher products, the cyclicity condition reads:

$$\omega(\psi_{a_0}, r_n(\psi_{a_1} \otimes \cdots \otimes \psi_{a_n})) = (-1)^{\tilde{a}_n(\tilde{a}_0+\cdots\tilde{a}_{n-1})} \omega(\psi_{a_n}, r_n(\psi_{a_1} \otimes \cdots \otimes \psi_{a_{n-1}})). \tag{4.4.5}$$

In the terminology of [19], a cyclic minimal A_∞-algebra is called a *Calabi-Yau* A_∞-*algebra*. The Calabi-Yau property is mimicked by the non-degeneracy of ω. In the language of A_∞-categories, the existence of a non-degenerate pairing implies an extension to the A_∞-level of the trivial Serre functor in the simpler derived category.

Finally we shall consider including bulk deformations of the open theory thus completing the structure of open-closed TCFT on the disk. The picture that should come to mind is that of a space of closed $TQFT's$ fibered by categories of purely open TQFT's. In including bulk deformations, one notices in particular that amplitudes of the form

$$\langle \psi_{a_0} \phi_i \rangle$$

are allowed as they completely fix $SL(2,\mathbb{R})$ invariance, therefore a general bulk perturbation, perturbs the purely open amplitude by the addition of:

$$\langle \psi_{a_0} t^{i_0} \phi_{i_0} P \int \psi_{a_1}^{(1)} \cdots \int \psi_{a_m}^{(1)} \exp\left(t^i \int_D \phi_i^{(2)} \right) \rangle.$$

Accordingly one has deformed higher products r_n^t. The Ward identities for $\tilde{G}_{-1}$, $\tilde{G}_0$, $\tilde{G}_1$ and Q are now tantamount to the property that the deformed products constitute a, possibly, weak cyclic A_∞ structure. The full bulk-buondary TCFT on the disk is then encoded in the bulk-boundary superpotential:

$$\mathcal{W}_{eff}(u,t) = \sum_n \frac{1}{n+1}\omega(\psi^{a_0}, r_n^t(\psi^{a_1}\otimes\cdots\otimes\psi^{a_n}))u^{a_0}\cdots u^{a_n}$$

The label eff indicates that in an effective field theory description of string theory, $\mathcal{W}_{eff}$ serves as an effective superpotential in a four dimensional $N = 1$ gauge theory. From the stringy perspective, the adjective superpotential is motivated by the fact that branes preserving the scalar Q supercharge precisely correspond to the critical locus of $\mathcal{W}_{eff}$. We will describe this in the following section and in later sections we will gain a better mathematical understanding of this object.

We remark, that the amplitudes corresponding to r_0^t:

$$\langle\psi_a t^{i_0}\phi_{i_0}\exp\left(t^i\int_D\phi_i^{(2)}\right)\rangle = \langle\psi_a\rangle_t \tag{4.4.6}$$

are *tadpoles* and are cancelled in string theory by including unoriented surfaces. The presence of tadpoles implies that, in particular, open three-point functions in the bulk-deformed vacuum, are position dependent. Thus tadpoles signal the breaking of global conformal invariance of the vacuum. Within this framework, tadpoles can be cancelled by solving a Maurer Cartan equation via a weak-A_∞-isomorphism (see section 5.1.1). Regardless of the presence of tadpoles, the crucial fact we learnt in this section is that bulk deformations of open TCFT's are to be viewed as deformations of the corresponding cyclic A_∞-structure. It was proved in [19], that the converse to this statement is also true, namely that the first-order deformations of the minimal A_∞-structure (without the cyclicity requirement) associated to an open TCFT build a closed TCFT. The latter closed TCFT is the universal closed sector of the open TCFT. We will understand this concretely in the case of affine Landau Ginzburg models in section 5.2.

Chapter 5

Investigations on open-closed topological conformal field theory

5.1 Formalizing deformations of TQFT's on the disk

The aim of this section is twofold. Apart from giving a workable mathematical framework for the concepts introduced in the previous section, we will start conveying the link between the off-shell and on-shell descriptions of the deformation theory of open and closed TCFT's. While in the former approach the deformation theory is governed by a minimal A_∞-structure, in the latter it is governed by a simpler *differential graded associative* structure. These are both special cases of (weak) A_∞ structures (see the seminal work of [90, 91] and [53] for a concise review). After having defined these properly we will show how to obtain the on-shell structure from the off-shell structure. The central result is the *minimal model theorem* [70] for which we will present a concise proof. Subsequently we will turn to the formal description of deformations of A_∞ structures themselves. By combining the two we will have thus provided all the necessary ingredients to tackle the problem of classifying and computing open and closed deformations on the disk.

5.1.1 A_∞- and L_∞-algebras

A *weak* A_∞*-algebra* is a ($\mathbb{Z}$- or $\mathbb{Z}_2$-) graded vector space[1] $A = \bigoplus_i A_i$ together with a codifferential ∂ of degree $+1$ on the tensor coalgebra

$$T_A = \bigoplus_{n\geqslant 0} A[1]^{\otimes n}\,.$$

The vectorspace A is a space of open strings. Later on we will keep this notation for the space of off-shell zero modes $V_o^{off-shell}$.

[1] We always work over the field $\mathbb{C}$.

A tensor *coalgebra* is dual to the tensor algebra. Instead of an associative product, it is defined by an associative *coproduct*

$$\Delta(a_1 \otimes \ldots \otimes a_n) = \sum_{j=0}^{n} (a_1 \otimes \ldots \otimes a_j) \otimes (a_{j+1} \otimes \ldots \otimes a_n)\,.$$

A *codifferential* is a special case of a *coderivation*. The latter is an element $\Phi \in \mathrm{End}(T_A)$ that satisfies the co-Leibniz rule w.r.t. Δ:

$$\Delta \circ \Phi = (\Phi \otimes \mathrm{id}_{T_A} + \mathrm{id}_{T_A} \otimes \Phi) \circ \Delta. \tag{5.1.1}$$

We denote the space of coderivations $\mathrm{Coder}(TA[1])$. A codifferential ∂ is a coderivation that squares to zero:

$$\partial^2 = 0\,. \tag{5.1.2}$$

To see that the definition of A_∞-algebra given here is equivalent with the one given in (4.4.4), we decompose the codifferential as

$$\partial = \sum_{m,n \geqslant 0} \partial_m^n \quad \text{where} \quad \partial_m^n \in \mathrm{Hom}(A[1]^{\otimes m}, A[1]^{\otimes n}).$$

The property that ∂ is a coderivation implies that all the homogeneous maps ∂_m^n are determined by the smaller set:

$$r_n = \partial_n^1 : A[1]^{\otimes n} \longrightarrow A[1].$$

Indeed applying the co-Leibniz rule (5.1.1) one finds:

$$\partial_m^n = \sum_{j=0}^{n-1} \mathrm{id}_{A[1]}^{\otimes j} \otimes \partial_{m-n+1}^1 \otimes \mathrm{id}_{A[1]}^{\otimes (n-j-1)}\,. \tag{5.1.3}$$

Thus the A_∞ structure encoded in ∂ is equivalently described by the maps r_m, for which the condition (5.1.2) translates into the bilinear constraints

$$\sum_{i,j \geqslant 0,\, i+j \leqslant n} r_{n-j+1} \circ \left(\mathrm{id}_{A[1]}^{\otimes i} \otimes r_j \otimes \mathrm{id}_{A[1]}^{\otimes (n-i-j)}\right) = 0 \tag{5.1.4}$$

for all $n \geqslant 0$. We thus recover (4.4.4). We write ∂_n for the codifferential determined solely by r_n, and we have the decomposition $\partial = \sum_{n \geqslant 0} \partial_n$.

A weak A_∞-algebra (A, ∂) is *unital* if there exists an element $e \in A_0$ such that $r_2(e \otimes a) = -a$, $r_2(a \otimes e) = (-1)^{\tilde{a}} a$ for all homogeneous $a \in A[1]$, and all other products r_n vanish if applied to a tensor product involving e.

Recall (see 4.2.11) that the off-shell open string space $V_o^{off-shell}$ is naturally endowed with an associative product "·" and it is also a complex whose differential d is compatible with the algebra structure. In the framework of A_∞ algebras, this is the special case of a strong A_∞-algebra whose only non-vanishing higher maps r_n are r_1 and r_2:

$$d = r_1\,, \quad a\cdot b = (-1)^{\tilde{a}} r_2(a\otimes b)$$

Indeed the A_∞ constraints reduce to:

$$d(a\cdot b) = d(a)\cdot b + (-1)^{|a|} a\cdot d(b)\,, \quad (a\cdot b)\cdot c = a\cdot(b\cdot c).$$

In studying the deformations of DGA's we will encounter tadpoles just as in the on-shell case (4.4.6), deforming DGA's to *weak DGA's*, that in addition have a map $r_0 : \mathbb{C}\to A[1]$. Denoting $C := r_0(1)$, this is subject to the A_∞ constraints:

$$dC = 0$$

and deforms d to a curved differential:

$$d^2 = [\cdot, C]$$

unless C is in the center of A.

From the point of view of DGA's, an arbitrary A_∞-algebra is a "DGA up to homotopy". In particular r_2 is in general not associative, but it is only up to a homotopy w.r.t. r_1 defined by r_3. In the case of a minimal A_∞ algebra, r_2 is associative on the nose as $r_1 = 0$.

We now would like to understand how, given an (off-shell) DGA, we can construct an on-shell minimal A_∞-algebra. The latter will be called a *minimal model* for the DGA. For this we need the notion of a map between (weak) A_∞ algebras. One refers to such a map as an A_∞-morphism.

Given two weak A_∞-algebras (A,∂) and (A',∂'), a *(weak) A_∞-morphism* between them is a morphism $F\in \mathrm{Hom}(T_A, T_{A'})$ of degree 0 between the associated codifferential coalgebras, i. e.

$$\Delta\circ F = (F\otimes F)\circ\Delta\,, \quad F\circ\partial = \partial'\circ F\,. \tag{5.1.5}$$

If we decompose

$$F = \sum_{m,n\geqslant 0} F^n_m \quad \text{where} \quad F^0_0 = 1\,, \quad F^n_m \in \mathrm{Hom}(A[1]^{\otimes m}, A'[1]^{\otimes n})$$

then the first equation in (5.1.5) implies that

$$F^n_m = \sum_{j_1+\ldots+j_n=m} F^1_{j_1}\otimes\ldots\otimes F^1_{j_n}$$

can be expressed in terms of the maps F^1_n. We call F a weak A_∞-*isomorphism* if F^1_1 is an isomorphism of vector spaces. A weak A_∞-morphism F between strong A_∞-algebras (A,r_n) and (A',r'_n) is called an A_∞-*quasi-isomorphism* if F^1_1 induces a vector space isomorphism between the cohomologies of r_1 and r'_1, and it is called a *(strong) A_∞-morphism* if $F^n_m = 0$ whenever $n > m$.

We can now tackle the problem of transporting the off-shell DGA on-shell. In fact we will give a proof of the more general *minimal model theorem* that answers the more general question of transporting a strong A_∞-algebra (A,∂) to a quasi-isomorphic minimal A_∞-algebra $(H,\widetilde{\partial})$ on a apace of harmonic representatives H of the cohomology $H_{r_1}(A)$.

For this we need a vectorspace decomposition:

$$A = H \oplus B \oplus L$$

where $B = \mathrm{Im}(r_1)$ and L is a complement of $\mathrm{Ker}(r_1)$ specified by a choice of propagator $G \in \mathrm{End}(A)$ satisfying $\pi_B = r_1 G$ where π_B projects onto B. Then G specifies L through $\pi_L = G r_1$ and lastly H is specified, by exclusion, by the homotopy identity:

$$r_1 G + G r_1 = \mathrm{id}_{A[1]} - \pi_H .$$

Finally we can state the minimal model theorem.

Proposition 5.1.1. Let (A,∂) be a strong A_∞-algebra with r_1-cohomology H. There is a unique coalgebra morphism $F \in \mathrm{Hom}(T_H, T_A)$ and a unique minimal A_∞-structure $\widetilde{\partial} \in \mathrm{Coder}(T_H)$ that satisfy the equations

$$\partial F = F\widetilde{\partial}\,, \tag{5.1.6}$$

$$\widetilde{\partial}^1_1 = 0\,,$$

$$F^1_1 = \iota_H : H[1] \hookrightarrow A[1]\,, \tag{5.1.7}$$

$$F^1_n = -G \sum_{k=2}^{n} \partial^1_k F^k_n\,. \tag{5.1.8}$$

Proof. First we will show that the condition that F be an A_∞-morphism follows from the conditions above, then we show that $\widetilde{\partial}$ is indeed a codifferential. Since F is a coalgebra morphism, (5.1.6) reduces to

$$\partial^1_1 F^1_n + \sum_{l=2}^{n} \partial^1_l F^l_n = \sum_{k=1}^{n-1} F^1_k \widetilde{\partial}^k_n$$

for all $n \geqslant 1$. We rewrite the above set of equations by splitting them into three parts:

$$\pi_H\Big(\sum_{l=2}^{n} \partial^1_l F^l_n\Big) = \widetilde{\partial}^1_n\,, \tag{5.1.9}$$

$$\pi_B\Big(\partial^1_1 F^1_n + \sum_{l=2}^{n} \partial^1_l F^l_n\Big) = 0\,, \tag{5.1.10}$$

$$\pi_L\Big(\sum_{l=2}^{n} \partial^1_l F^l_n - \sum_{k=1}^{n-1} F^1_k \widetilde{\partial}^k_n\Big) = 0\,. \tag{5.1.11}$$

The first immediate observation is that $\widetilde{\partial}$ is uniquely determined by (5.1.9). Equation (5.1.10) follows by employing (5.1.8) and (5.1.7), and to show (5.1.11) we compute

$$\pi_L\Big(\sum_{l=2}^{n}\partial^1_l F^l_n - \sum_{k=1}^{n-1} F^1_k\widetilde{\partial}^k_n\Big) = \pi_L\Big(\sum_{l=2}^{n}\partial^1_l F^l_n + G\sum_{k=2}^{n-1}\sum_{r=2}^{k}\partial^1_r F^r_k\widetilde{\partial}^k_n\Big)$$
$$= \pi_L\Big(\sum_{l=2}^{n}\partial^1_l F^l_n + G\sum_{k=2}^{n-1}\sum_{r=2}^{k}\partial^1_r \partial^r_k F^k_n\Big) = 0\,.$$

In the first step we used (5.1.8), the second step is the induction step which allows us to commute $\widetilde{\partial}$ through F, and in the last step we used $\partial^2 = 0$.

To show that $\widetilde{\partial}^2 = 0$ we note that

$$\sum_{k=1}^{n}\widetilde{\partial}^1_k\widetilde{\partial}^k_n = \sum_{k=1}^{n}\pi_H\Big(\sum_{l=2}^{k}\partial^1_l F^l_k\widetilde{\partial}^k_n\Big) = \sum_{k=1}^{n}\pi_H\Big(\sum_{l=2}^{k}\partial^1_l\partial^l_k F^k_n\Big) = 0\,.$$

□

The minimal model thus constructed is called a Merkulov-type minimal model after [70]. When (A,∂) is a DGA, we see that the minimal model is built out of the basic building blocks Gr_2, namely trilinear vertices. When A is supplemented by an off-shell pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{os} : A\otimes A\to\mathbb{C}$, the higher products of the minimal models can be understood as the trees in the Feynman expansion of the classical limit of a formal Chern-Simons theory with action:

$$S \propto \frac{1}{2}\langle\Psi, d\Psi\rangle_{os} + \frac{1}{3}\langle\Psi,\Psi\cdot\Psi\rangle_{os} = \mathcal{W}^{off-shell}_{eff},$$

where Ψ is known as the *string field*, namely it is an element in $A[u]$ with the formal expansion:

$$\Psi = a_k u^k,$$

where the u^i are the open deformation moduli dual to a_k. The off-shell pairing needs to be non-degenerate on d-cohomology and cyclic describing off-shell amplitudes on the disk. Moreover it has to restrict to the on-shell pairing. Then G is chosen accordingly, giving rise to a cyclic minimal model. In the case of the B-model the on-shell paring η_o lifts trivially off-shell, while in affine Landau Ginzburg models it does not. In some way, while Landau Ginzburg models have a much simpler structure than B-models regarding the algebra of open observables, their complexity is hidden all in the structure of the off-shell pairing which instead is trivial in B-models. In the last section we will provide a formula for the off-shell pairing in arbitrary affine Landau Ginzburg models thus completing their string-field theory description. The correct setting is that of non-commutative geometry where the notion of an off-shell pairing as described above is replaced by that of a flat homologically symplectic form. In Landau Ginzburg models, we still have a homologically symplectic form which however is not flat. This means that the string field theory action is not of the simple Chern-Simons type, but has higher order terms.

After this short aside, we stress in particular, that the fulfillment of cyclicity of the minimal model all resides in the particular choice of G. Finally The Merkulov type minimal model for a DGA clarifies the role of the A_∞ algebra in the deformation theory of branes. If one were to study this directly in analogy to the discussion of 2.6 for closed strings, the equation one would like to solve is:

$$(r_1+\alpha)^2 = r_1\alpha + \frac{1}{2}\{\alpha,\alpha\} = 0, \tag{5.1.12}$$

where α must have the same degree as r_1. Contrary to the case of closed strings which are perturbatively unobstructed, deformations of open strings are generically obstructed. That is, for an arbitrary deformation α

$$\pi_H\{\alpha,\alpha\}$$

will not necessarily vanish, making (5.1.12) inconsistent. Perturbatively, (5.1.12) is precisely solved by a Merkulov type minimal model. One considers a general first-order deformation $\alpha^{(1)} = \alpha_k^{(1)} u_s^k$, where now u_s^k are graded symmetric moduli and clearly $\alpha_k^{(1)}$ furnish a basis of representatives $H_{r_1}(A)$. Then one observes that:

$$\alpha = F_n^1((\alpha^{(1)})^{\otimes n})$$

yields a solution, provided the obstructions vanish. The obstruction is precisely

$$\sum_n \widetilde{\partial}_n^1((\alpha^{(1)})^{\otimes n}).$$

We thus observe that restricting to graded commutative moduli was not an assumption, as the deformation problem is only sensitive to those. More importantly we see, as mentioned in the previous section, that the obstructions are precisely

$$\partial_{u^i}\mathcal{W}_{eff},$$

where $\mathcal{W}_{eff}$ is the generating function of connected on-shell open string amplitudes on the disk, or alternatively, the generating function of connected string field theory Feynman diagrams. We remark that the same argument applies to the deformation related to more general strong A_∞-algebras. However it is worth mentioning that a constructive result of [62] ensures that any strong A_∞-algebra is quasi-isomorphic to a DGA, thus reversing the minimal model theorem. Hence such a DGA is called an *anti-minimal model*. This means, in particular, that given any open TCFT there is always an off-shell model described by a DGA putting on firm ground our explanations by example in section 4.2. There is an analogous result ensuring the same for closed TCFT's. There the A_∞ structure is replaced by a L_∞ structure.

A *(weak) L_∞-algebra* is to a Lie algebra what a (weak) A_∞-algebra is to an associative algebra (hence the names). Thus, given a ($\mathbb{Z}_2$ or $\mathbb{Z}$) graded vectorspace V one replaces the tensor algebra by a graded symmetric tensor algebra S_V:

$$S_V = T_V/(u\otimes v - (-1)^{|u||v|} v\otimes u)$$

for homogeneous u and v, and the product of n homogeneous elements becomes:

$$v_1 \wedge \ldots \wedge v_n \longmapsto \sum_{j=0}^{n} \sum_{\sigma \in \mathrm{Sh}(j,n)} \varepsilon_{\sigma; v_1,\ldots,v_n} (v_{\sigma(1)} \wedge \ldots \wedge v_{\sigma(j)}) \otimes (v_{\sigma(j+1)} \wedge \ldots \wedge v_{\sigma(n)}),$$

where $\mathrm{Sh}(j,n)$ is the set of permutations σ of n elements that satisfy $\sigma(1) < \ldots < \sigma(j)$ and $\sigma(j+1) < \ldots < \sigma(n)$, and the sign $\varepsilon_{\sigma;v_1,\ldots,v_n}$ is defined via $v_{\sigma(1)} \wedge \ldots \wedge v_{\sigma(n)} = \varepsilon_{\sigma;v_1,\ldots,v_n} v_1 \wedge \ldots \wedge v_n$.

With this preparation a *weak L_∞-algebra* is a graded vector space V together with a codifferential $\mathfrak{d}$ of degree $+1$ on S_V. Again, the fact that $\mathfrak{d}$ is a coderivation allows us to equivalently consider a family of maps $\ell_n : V[1]^{\wedge n} \to V[1]$ with constraints coming from the condition $\mathfrak{d}^2 = 0$. For more details we refer e. g. to [56, 57].

In particular the analogous case to a DGA is a *DGLA* (*differential graded Lie algebra*) where the maps ℓ_0 and ℓ_n for $n \geqslant 3$ all vanish:

$$d = \ell_1\,, \quad [u,v] = (-1)^{\tilde{u}} \ell_2(u \wedge v) \tag{5.1.13}$$

and the L_∞ constraints read:

$$\begin{aligned} &[u,v] = (-1)^{|u|\,|v|}[v,u]\,, \quad d([u,v]) = [d(u),v] + (-1)^{|u|}[u,d(v)]\,, \\ &(-1)^{|u|\,|w|}[u,[v,w]] + (-1)^{|u|\,|v|}[v,[w,u]] + (-1)^{|v|\,|w|}[w,[u,v]] = 0\,. \end{aligned}$$

We observe that as the off-shell space of open string zero-modes is naturally a DGA, the off-shell space of closed string zero modes is a DGLA with bracket the Schouten-Nijenhuis bracket (2.5.3). Similarly to the open sector, the on-shell closed TCFT carries a minimal L_∞ structure. Contrary to the case of the open sector where the A_∞ structure could not simply be deduced from a deformation theoretic approach (due to non-commutativity), in the closed TCFT case, we don't need to analyze the Ward identities to understand the presence of an L_∞-structure. Indeed, if we adopt the definition of a TCFT as one arising from a (twisted) $N = (2,2)$ CFT, then the associated TQFT observables are by definition Q-cohomology classes of a larger space of vertex operators, which, once endowed with Q, constitute a DGLA. Moreover, $\mathcal{H}$ is a unitary representation of the Virasoro algebra, therefore one has a canonical choice for the propagator of Q, which coincidentally in that framework is precisely called G/L_0, where G in that context is the conjugate supercharge. Hence $H_{[Q,\cdot]}(\mathrm{End}(\mathcal{H}))$ is automatically endowed with the obvious Merkulov-type L_∞ algebra which is defined in the completely analogous way as its A_∞ cousin.

Finally we can now remark, that in the purely open-deformation theory, the surviving part of the minimal A_∞ structure is precisely the L_∞-algebra that is induced by the projection $T_A \to S_A$. The interesting fact is that not all L_∞-algebras arise in this way. This is in particular valid for a general closed sector.

Our next task is to understand closed string deformations of open TCFT's. In the previous section, we understood that these are deformations of the minimal A_∞-algebra $\tilde{\partial}$ to a, a priori, weak one:

$$\tilde{\partial} + \tilde{\delta},$$

where now we understand that $\tilde{\delta} \in \mathrm{Coder}(T_A)$ of degree 1 and satisfies the Maurer Cartan equation:

$$(\widetilde{\partial} + \widetilde{\delta})^2 = [\widetilde{\partial}, \widetilde{\delta}] + \frac{1}{2}[\widetilde{\delta}, \widetilde{\delta}] = 0. \tag{5.1.14}$$

Notice that $\mathrm{Coder}(T_H)$ is indeed a Lie algebra, but it is not an associative algebra. Studying bulk deformations of open TCFT's means mapping the DGLA of the off-shell bulk sector or its minimal model, to the DGLA

$$(\mathrm{Coder}(T_H), [\widetilde{\partial}, \cdot], [\cdot, \cdot]).$$

Intuitively one might think that the correct notion of map is a map between $DGLA$'s, however this turns out to be too restrictive. Again one has to work in the larger L_∞ setting, and the right notion is that of L_∞-morphism, the definition of which can be deduced from that of its A_∞ cousin. The central very simple result that we will prove momentarily, is that given two arbitrary L_∞-algebras, an L_∞-morphism between them maps solutions of the associated Maurer Cartan equation of the first to solutions of the Maurer Cartan equation of the second. In particular, if the map is a quasi-isomorphism, then it maps the deformations faithfully up to gauge transformations. In detail, for an arbitrary L_∞-algebra (V, ℓ_n) its Maurer-Cartan equation reads

$$\sum_{n \geqslant 1} \frac{1}{n!}\, \ell_n(\delta^{\wedge n}) = 0\,,$$

and we denote by $\mathcal{MC}(V, \ell_n)$ its space of formal power series solutions $\delta \in V_1$ modulo the action of the group generated by *gauge transformations*

$$\delta \longmapsto \delta + \sum_{n \geqslant 1} \frac{1}{(n-1)!}\, \ell_n(\varphi \wedge \delta^{\wedge(n-1)})$$

for all $\varphi \in V_0$. The promised important result of [55, 71] is the following.

Proposition 5.1.2. Let $F : (V, \ell_n) \to (V', \ell'_n)$ be an L_∞-morphism. Then

$$\delta \longmapsto \sum_{n \geqslant 1} \frac{1}{n!}\, F_n(\delta^{\wedge n}) \tag{5.1.15}$$

maps elements in $\mathcal{MC}(V, \ell_n)$ to elements in $\mathcal{MC}(V', \ell'_n)$. Furthermore, if F is an L_∞-quasi-isomorphism, then (5.1.15) is an isomorphism.

Proof. Denote by $\mathfrak{d}$ the coderivation determined by the higher maps ℓ_n. Consider a weak coalgebra morphism $M \in \mathrm{End}(S_V)$ with the properties $M_0^1 = \delta$ and $M_1^1 = \mathrm{id}_{V[1]}$, where δ is a solution to the Maurer-Cartan equation for $\mathfrak{d}$. Then we have $M_0^n = \frac{1}{n!}\delta^{\wedge n}$ and the Maurer-Cartan equation for $\mathfrak{d}$ can be rewritten as

$$(\mathfrak{d}M)_0^1 = 0\,.$$

Now let $\mathfrak{d}'$ denote the coderivation corresponding to the products ℓ'_n. We have

$$(\mathfrak{d}' F M)^1_0 = (F \mathfrak{d} M)^1_0 = \sum_{k \geqslant 1} F^1_k (\mathfrak{d} M)^k_0 = 0$$

where the last equation follows from the fact that $(\mathfrak{d} M)^k_0$ is uniquely determined by $(\mathfrak{d} M)^1_0$. We have thus shown that $\delta' = (FM)^1_0$ solves the Maurer-Cartan equation for $\mathfrak{d}'$. In expanded form this reads

$$\delta' = \sum_{n \geqslant 1} F^1_n M^n_0 = \sum_{n \geqslant 1} \frac{1}{n!} F^1_n (\delta^{\wedge n}) \,.$$

If F is a quasi-isomorphism, then it is an isomorphism between the spaces of first order deformations, and as it admits homotopy inverses, this isomorphism extends to all orders. □

We observe that solving (5.1.14) up to gauge transformations only to first order is the same as computing the cohomology

$$\mathrm{HH}^\bullet(H, \widetilde{\partial}) = H_{[\widetilde{\partial}, \,\cdot\,]}(\mathrm{Coder}(T_H))$$

of the *Hochschild cochain complex* $(\mathrm{Coder}(T_H), [\widetilde{\partial}, \,\cdot\,])$, known as the *Hochschild cohomology*. In this sense Hochschild cohomology $\mathrm{HH}^\bullet(H, \widetilde{\partial})$ classifies deformations of $(H, \widetilde{\partial})$.

There is an important subtlety in the definition of the Hochschild cochain complex on an A_∞ algebra (A, ∂). As we saw in our discussion of equation (5.1.3), coderivations of T_A are isomorphic to collections of multilinear maps. However, one may either consider infinitely or finitely many multilinear maps, so there are actually Hochschild cochain complexes of the *first kind* and of the *second kind* respectively,[2]

$$\mathrm{Coder}(T_A)^{\mathrm{I}} \cong \prod_{n \geqslant 0} \mathrm{Hom}(A[1]^{\otimes n}, A[1]) \,,$$
$$\mathrm{Coder}(T_A)^{\mathrm{II}} \cong \bigoplus_{n \geqslant 0} \mathrm{Hom}(A[1]^{\otimes n}, A[1]) \,.$$

Both complexes are endowed with the same differential $[\partial, \,\cdot\,]$, but they have different invariance properties: Hochschild cohomology of the first kind is invariant under (strong) A_∞-quasi-isomorphisms [58], while Hochschild cohomology of the second kind is invariant under weak A_∞-isomorphisms [78].

In the following we will give a formula for the general solution to (5.1.14). The first part of our strategy is pertinent only to arbitrary affine Landau Ginzburg models, while the latter part is applicable to any open-closed TCFT. We stress that, at first order, of the two cohomologies, it is Hochschild cohomology of the second kind we will be computing.

[2]There is a much deeper origin of the two different kinds of Hochschild complexes, see [78, 80]

5.2 Bulk-deformed Landau-Ginzburg models

In an affine Landau Ginzburg model with potential W, the notion of bulk-induced deformations of the on-shell boundary space $(H, \widetilde{\partial})$ relies on the existence of an L_∞-morphism

$$L : (T_{\text{poly}}, [-W, \cdot\,]_{\text{SN}}, [\,\cdot\,,\,\cdot\,]_{\text{SN}}) \longrightarrow (\text{Coder}(T_H), [\widetilde{\partial}, \cdot\,], [\,\cdot\,,\,\cdot\,]) \tag{5.2.1}$$

from the DG Lie algebra of off-shell bulk fields to the DG Lie algebra of coderivations on the boundary side. A *bulk-induced deformation* is then defined as the image under L of a deformation of the pure bulk theory. In this section we give an explicit construction of L.

The map (5.2.1) splits naturally as the composition of two L_∞-morphisms that we discuss in subsections 5.2.1 and 5.2.2, respectively. The first is an L_∞-quasi-isomorphism

$$(T_{\text{poly}}, [-W, \cdot\,]_{\text{SN}}, [\,\cdot\,,\,\cdot\,]_{\text{SN}}) \longrightarrow (\text{Coder}(T_A), [\partial, \cdot\,], [\,\cdot\,,\,\cdot\,]) \tag{5.2.2}$$

and thus identifies the two deformation problems. It can be viewed as a “weak” version of Kontsevich’s construction for (local) deformation quantisation, or rather its complex cousin. The second L_∞-morphism

$$(\text{Coder}(T_A), [\partial, \cdot\,], [\,\cdot\,,\,\cdot\,]) \longrightarrow (\text{Coder}(T_H), [\widetilde{\partial}, \cdot\,], [\,\cdot\,,\,\cdot\,]) \tag{5.2.3}$$

transports off-shell deformations on-shell and can be viewed as the L_∞-formulation and enhancement of the homological perturbation lemma.[3]

5.2.1 Weak deformation quantisation

The bulk data of the affine Landau Ginzburg model is the ring $\mathcal{R} = \mathbb{C}[x^1, \cdots, x^d]$, a potential $W \in \mathcal{R}$ with an isolated critical point (see the discussion in section 2.3). The boundary data (see section 4.4) is a matrix factorization $D \in \text{Mat}(\mathcal{R}, 2r \times 2r)$.

As follows from our earlier discussion, bulk deformations of B-twisted Landau-Ginzburg models are solutions $\gamma \in T_{\text{poly}}$ of the Maurer-Cartan equation

$$[-W, \gamma]_{\text{SN}} + \frac{1}{2}[\gamma, \gamma]_{\text{SN}} = 0\,, \tag{5.2.4}$$

where the Schouten-Nijenhuis bracket on T_{poly} is given by

$$[\zeta_1 \wedge \ldots \wedge \zeta_m, \xi_1 \wedge \ldots \wedge \xi_n]_{\text{SN}} = \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{i+j} [\zeta_i, \xi_j]$$
$$\cdot\, \zeta_1 \wedge \ldots \wedge \zeta_{i-1} \wedge \zeta_{i+1} \wedge \ldots \wedge \zeta_m \wedge \xi_1 \wedge \ldots \wedge \xi_{j-1} \wedge \xi_{j+1} \wedge \ldots \wedge \xi_n$$

[3] As we are concerned with the Hochschild cochain complex of the second kind, the map (5.2.2) is a quasi-isomorphism, while the map (5.2.3) is generically not. However, if we used the Hochschild cochain complex of the first kind, (5.2.3) would always be a quasi-isomorphism, but generically not (5.2.2). See also remark on Hochschild cohomologies of first and second kind in 5.1.

and

$$[\zeta_1 \wedge \ldots \wedge \zeta_m, f]_{\mathrm{SN}} = \sum_{i=1}^{m} (-1)^i \zeta_i(f) \zeta_1 \wedge \ldots \wedge \zeta_{i-1} \wedge \zeta_{i+1} \wedge \ldots \wedge \zeta_m$$

for $\zeta_i, \xi_j \in \Gamma(X, T^{(1,0)}X)$ and $f \in \Gamma(X, \mathcal{O}_X)$.

We restrict our attention to formal power series in a set of parameters t, i. e. $\gamma = \sum_{i\geqslant 1} t^i \gamma^{(i)}$. This assumption allows to solve (5.2.4) perturbatively. At first order the equation reads

$$[-W, \gamma^{(1)}]_{\mathrm{SN}} = 0\,, \tag{5.2.5}$$

and hence we have, up to gauge transformations, $\gamma^{(1)} \in \mathrm{Jac}(W)$, the on-shell bulk space. As already mentioned in 2.6, one simplicity of affine Landau-Ginzburg models lies in the fact that the solutions of (5.2.5) are automatically solutions of the full Maurer-Cartan equation, because the Schouten-Nijenhuis bracket of two functions vanishes. Having thus fully solved the deformation problem in the bulk sector, the problem of computing bulk-induced deformations reduces to that of transporting bulk deformations appropriately to the boundary sector. The first main result in accomplishing this task is the following, which at its core is a weak version of deformation quantisation.

Theorem 5.2.1. There is an L_∞-quasi-isomorphism from $(T_{\mathrm{poly}}, [-W, \cdot\,]_{\mathrm{SN}}, [\,\cdot\,,\,\cdot\,]_{\mathrm{SN}})$ to $(\mathrm{Coder}(T_A), [\partial, \cdot\,], [\,\cdot\,,\,\cdot\,])$.

To understand this result a crucial role is played by the curved associative algebra

$$(A, \partial_0 + \partial_2) \quad \text{with} \quad \partial_0^1 = -W \cdot e\,,$$

where e denotes the identity matrix of the same size as D. In the following we will write ∂ as $\partial_1 + \partial_2$ to distinguish it from $\partial_0 + \partial_2$. We proceed to give a constructive proof of the above theorem. For the purpose of clarity we shall split it into the following three parts. Notice that we proceed in backward steps.

Lemma 5.2.2. There is an L_∞-quasi-isomorphism

$$(\mathrm{Coder}(T_A), [\partial_0 + \partial_2, \cdot\,], [\,\cdot\,,\,\cdot\,]) \longrightarrow (\mathrm{Coder}(T_A), [\partial_1 + \partial_2, \cdot\,], [\,\cdot\,,\,\cdot\,])\,.$$

Proof. We define a weak coalgebra isomorphism T (for off-shell "tadpole-cancellation", i. e. T is a weak A_∞-isomorphism to a non-curved A_∞-algebra) via its fundamental maps $T_0 = -D$ and $T_1 = \mathrm{id}_{A[1]}$, and we compute

$$\begin{aligned}
[(\partial_1 + \partial_2) \circ T]_0^1 &= [D, -D] + D(-D) = -W \cdot e = [T \circ (\partial_0 + \partial_2)]_0^1\,,\\
[(\partial_1 + \partial_2) \circ T]_1^1 &= [D, \cdot\,] - \partial_2^1(D \otimes \mathrm{id}_{A[1]} + \mathrm{id}_{A[1]} \otimes D) = 0 = [T \circ (\partial_0 + \partial_2)]_1^1\,,\\
[(\partial_1 + \partial_2) \circ T]_2^1 &= \partial_2^1 = [T \circ (\partial_0 + \partial_2)]_2^1\,.
\end{aligned}$$

Hence we have $(\partial_1 + \partial_2) \circ T = T \circ (\partial_0 + \partial_2)$, i. e. T is a weak A_∞-isomorphism from $(A, \partial_0 + \partial_2)$ to $(A, \partial_1 + \partial_2)$. The L_∞-morphism is then given by the inverse adjoint action of T. That it is a quasi-isomorphism follows from the fact that Hochschild cohomology of the second kind is invariant under weak DG isomorphisms [78]. □

The next step in the proof of theorem 5.2.1 is an L_∞-version of Morita equivalence:

Lemma 5.2.3. There is an L_∞-quasi-isomorphism

$$(\mathrm{Coder}(T_R), [\widehat{\partial}_0 + \widehat{\partial}_2, \,\cdot\,], [\,\cdot\,, \,\cdot\,]) \longrightarrow (\mathrm{Coder}(T_A), [\partial_0 + \partial_2, \,\cdot\,], [\,\cdot\,, \,\cdot\,]) \,,$$

where $\widehat{\partial}_0 + \widehat{\partial}_2$ is the codifferential on T_R induced from $\partial_0 + \partial_2$.

Proof. First we construct the cotrace map $C_1^1 = \mathrm{cotr} : \mathrm{Coder}(T_R) \to \mathrm{Coder}(T_A)$, then we show that the coalgebra morphism C defined by $C^1 = C_1^1$ is the desired L_∞-quasi-isomorphism. The cotrace map is a slightly modified version of the case for ungraded algebras, see e. g. [65]: for $\Phi \in \mathrm{Coder}(T_R)$ we define $\mathrm{cotr}(\Phi) \in \mathrm{Coder}(T_A)$ via $\mathrm{cotr}(\Phi)_0^1 = \Phi_0^1 \cdot e$ and

$$\big(\mathrm{cotr}(\Phi)_m^1(a_1 \otimes \ldots \otimes a_m)\big)_{kl} = \sum_{i_1,\ldots,i_{m-1}=1}^{2r} \Phi_m^1((\sigma^{m+1}a_1)_{ki_1} \otimes \ldots \otimes (\sigma a_m)_{i_{m-1}l})$$

where σ is the unique matrix that for homogenous elements $a \in A$ satisfies $\sigma a = (-1)^{|a|} a \sigma$, and $2r$ is the size of D. It is then straightforward to show that

$$[\mathrm{cotr}(\Phi_1), \mathrm{cotr}(\Phi_2)] = \mathrm{cotr}([\Phi_1, \Phi_2]) \,, \tag{5.2.6}$$

i. e. cotr is a map of Lie algebras. In order to show that C is an L_∞-morphism it then suffices to check that cotr is a map of complexes. This however follows immediately from (5.2.6) by noting that

$$\partial_0 + \partial_2 = \mathrm{cotr}(\widehat{\partial}_0 + \widehat{\partial}_2) \,.$$

That C_1^1 is indeed a quasi-isomorphism follows from a spectral sequence argument and will be shown together with the next proposition. □

Now we arrive at the last and hardest step in the proof of theorem 5.2.1.

Proposition 5.2.4. There is an L_∞ quasi-isomorphism from $(T_{\mathrm{poly}}, [-W, \,\cdot\,]_{\mathrm{SN}}, [\,\cdot\,, \,\cdot\,]_{\mathrm{SN}})$ to $(\mathrm{Coder}(T_R), [\widehat{\partial}_0 + \widehat{\partial}_2, \,\cdot\,], [\,\cdot\,, \,\cdot\,])$.

Fortunately we can build on Kontsevich's result on deformation quantisation [**?**], which says that the above is true for the case $W = 0$. More precisely, Kontsevich's result concerns polyvector fields on $\mathbb{R}^d$, but as we are dealing with affine space, his result extends trivially.

Before delving into the proof of proposition 5.2.4, let us briefly recall the aim and method of deformation quantisation. One is interested in quantising a classical system described by a phase space M (a real smooth manifold, for simplicity consider $M = \mathbb{R}^d$) or alternatively rather by its commutative, associative algebra of observables $(C^\infty(M, \mathbb{R}), \,\cdot\,)$. *Quantising* in this context amounts to deforming the multiplication "$\cdot$" to an associative, but not necessarily commutative product "$\star$", in order to pass to the algebra of quantum observables $(C^\infty(M, \mathbb{R})[\![\hbar]\!], \star)$ while postponing the task of representing it on a Hilbert space. Below we will drop formal parameters like $\hbar$ from our notation.

To rephrase the problem in a more compact notation, we denote "$\cdot$" by ∂_2^1 so that the deformation problem becomes that of solving the Maurer-Cartan equation of the DG Lie algebra $(\mathrm{Coder}(T_{C^\infty(M,\mathbb{R})}), [\partial_2, \,\cdot\,], [\,\cdot\,,\,\cdot\,])$. Kontsevich's solution is to construct an L_∞-quasi-isomorphism

$$K : (\Gamma(M, \bigwedge TM), [\,\cdot\,,\,\cdot\,]_{\mathrm{SN}}) \longrightarrow (\mathrm{Coder}(T_{C^\infty(M,\mathbb{R})}), [\partial_2, \,\cdot\,], [\,\cdot\,,\,\cdot\,])\,.$$

Thus by proposition 5.1.2 every perturbatively deformed product "$\star$" originates from a Poisson structure on M, i.e. a degree 2 polyvector field α which satisfies the Maurer-Cartan equation $[\alpha, \alpha]_{\mathrm{SN}} = 0$.

As already observed, we can view our theorem 5.2.1 as a generalisation of deformation quantisation: endowed with a non-trivial differential $[-W, \,\cdot\,]$, the DG Lie algebra of polyvector fields now governs deformations of a DG algebra, and not just a commutative associative one.

The proof of proposition 5.2.4 splits into two parts. First we show that the L_∞-morphism K can be extended to the case $W \neq 0$, then we show that it is still a quasi-isomorphism.

Proof. We start by recalling Kontsevich's construction. The L_∞-morphism K is given by

$$(K_n^1(\gamma_1 \wedge \ldots \wedge \gamma_n))_m^1 = \sum_{\Gamma \in \mathcal{G}(n,m)} w_\Gamma \, U_\Gamma$$

where $\mathcal{G}(n,m)$ denotes the set of certain directed graphs Γ to which in turn we will associate certain weights $w_\Gamma \in \mathbb{R}$ and multilinear maps U_Γ on $R[1]$. To describe these, consider the unit disc D in the complex plane. Choose m marked points $q_{\bar{1}}, \ldots, q_{\bar{m}}$ (which we associate with functions $f_{\bar{1}}, \ldots, f_{\bar{m}}$) on the boundary ∂D and n marked points $p_1, \ldots, p_n$ (which we associate with polyvector fields $\gamma_1, \ldots, \gamma_n$) in the interior. These $m+n$ marked points coincide with the vertices of the graph $\Gamma \in \mathcal{G}(n,m)$.

The possible edges between vertices are constrained by the following rules: (i) for every polyvector field γ_k, there are precisely $\widetilde{\gamma}_k$ edges $e_k^1, \ldots, e_k^{\widetilde{\gamma}_k}$ starting at p_k and ending on distinct marked points different from p_k, (ii) each marked point on the boundary has zero outgoing edges and at least one incoming edge, (iii) the total number of edges is $\dim(C^{n,m}) = 2n + m - 2 \geqslant 0$, where we denote by $C^{n,m}$ the moduli space of the above described marked points on the unit disc with a choice of orientation. Here we run clockwise around the circle and the orientation is well-defined by omitting the point $\mathrm{i} \in \partial D$. This special point is to be viewed as representing the "out-state".

To construct the map $U_\Gamma \in \mathrm{Hom}(R[1]^{\otimes m}, R[1])$ for fixed polyvector fields $\gamma_1, \ldots, \gamma_n$, one views the edges ending on a vertex as the action of the coordinate vector fields on the function associated to the vertex and then takes the product over all such actions. More precisely, if we write

$$\gamma_i = \gamma^{j_{i,1} \ldots j_{i,\widetilde{\gamma}_i}} \partial_{j_{i,1}} \wedge \ldots \wedge \partial_{j_{i,\widetilde{\gamma}_i}}$$

and denote by $\Gamma_{\bullet\to k}$ the set of edges ending on vertex k, then we have

$$U_\Gamma(f_1 \otimes \ldots \otimes f_m) = \sum_I \Big[\prod_{i=1}^{n}\Big(\prod_{e\in\Gamma_{\bullet\to i}} \partial_{I(e)}\Big)\gamma_i^{I(e_i^1)\ldots I(e_i^{\widetilde{\gamma}_i})}\Big]\Big[\prod_{\bar{\jmath}=\bar{1}}^{\bar{m}}\Big(\prod_{e\in\Gamma_{\bullet\to\bar{\jmath}}} \partial_{I(e)}\Big) f_{\bar{\jmath}}\Big]$$

where the sum is over all maps $I : \Gamma_1 \to \{1, \ldots, d\}$.

The weights w_Γ are certain integrals over the moduli space $C^{n,m}$. In order to understand these, consider for every edge e_k^r, the angle map $\varphi_{e_k^r} : \overline{D} \times \overline{D} \to (0, 2\pi]$ measuring the (clockwise) angle between the edge e_k^r and the line connecting p_k to the out-state at i.[4] If we denote by $\iota : C^{n,m} \to D^{\times n} \times \partial D^{\times m}$ the canonical embedding of the moduli space, then the weights are given by

$$w_\Gamma = \frac{1}{(2\pi)^{2n+m-2}} \int_{\iota(\overline{C}^{n,m})} \bigwedge_{k=1}^{n} (\mathrm{d}\varphi_{e_1^k} \wedge \ldots \wedge \mathrm{d}\varphi_{e^k_{\widetilde{\gamma}_k}})\,. \tag{5.2.7}$$

We are now in a position to start with the proof proper. Let $\widehat{\mathfrak{d}}_1^1 = [\widehat{\partial}_0 + \widehat{\partial}_2, \cdot\,]$ and define $\widehat{\mathfrak{d}}_2^1$ via $\widehat{\mathfrak{d}}_2^1(\Phi_1 \wedge \Phi_2) = (-1)^{\widetilde{\Phi}_1}[\Phi_1, \Phi_2]$. We want to show that K continues to be an L_∞-quasi-isomorphism also in the curved case, i. e.

$$K_n^1 l_n^n + K_{n-1}^1 l_n^{n-1} = \widehat{\mathfrak{d}}_1^1 K_n^1 + \widehat{\mathfrak{d}}_2^1 K_n^2\,,$$

where we denote the DG Lie algebra structure on T_{poly} by the maps l_m^n. If we define the coderivation c given by $c^1 = c_1^1 = \{\partial_0, \cdot\,\}$, then by Kontsevich's result the above reduces to

$$K l_1 = cK$$

which in expanded form reads

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^{\sum_{s=1}^{k-1}\widetilde{\gamma}_s}(K_n^1(\gamma_1 \wedge \ldots \wedge [-W, \gamma_k] \wedge \ldots \wedge \gamma_n))_m^1(f_1 \otimes \ldots \otimes f_m)\\ &= \sum_{l=0}^{m}(-1)^l(K_n^1(\gamma_1 \wedge \ldots \wedge \gamma_n))_{m+1}^1(f_1 \otimes \ldots \otimes f_l \otimes (-W) \otimes \ldots \otimes f_m)\,.\end{aligned} \tag{5.2.8}$$

We will now analyse the right-hand side to find that it is the same as the left-hand side; see figure 5.2.1 for a visualisation. Fix a graph $\Gamma \in \mathcal{G}(n, m+1)$ and consider the first summand on the right-hand side of (5.2.8). Pick an edge e_k^r ending on $-W$ and carry the corresponding differential form $\mathrm{d}\phi_{e_k^r}$ to the very left of the integral (5.2.7). This results in picking up a sign $\mu = (-1)^{\sum_{s=1}^{k-1}\widetilde{\gamma}_s}(-1)^{r-1}$. Now consider the l-th term in (5.2.8). This differs from the first term by a sign $(-1)^l$ which corresponds to the determinant of the Jacobian of the map transforming $(q_1, \ldots, q_{m+1}) \mapsto (q_2, \ldots, q_l, q_1, \ldots, q_{m+1})$ contributing

[4] More precisely the angle is measured with respect to the hyperbolic metric, and the edges are the associated geodesics. This however does not influence our discussion.

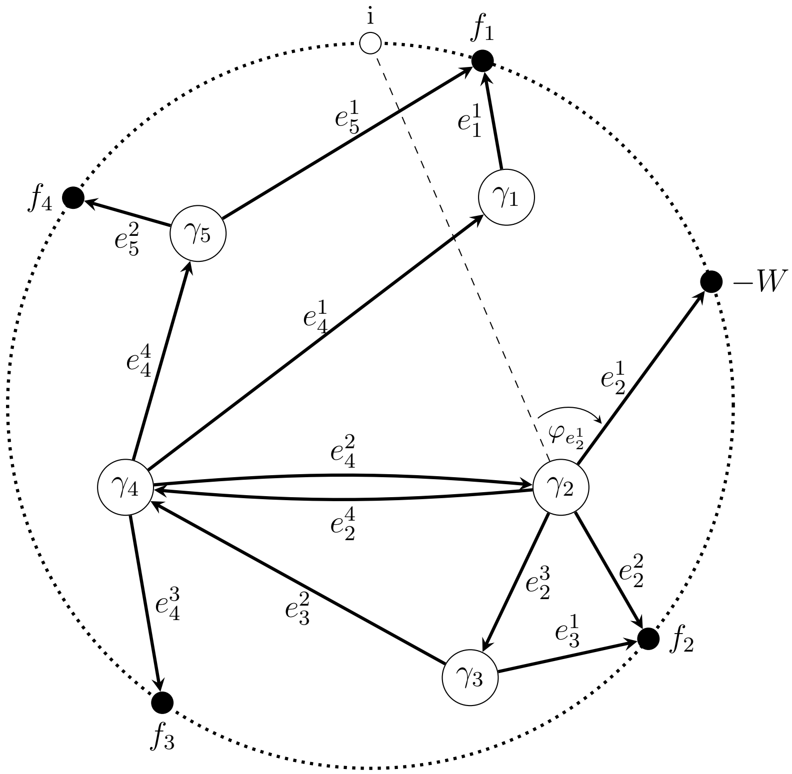

Figure 5.2.1: A graph contributing to $K_5^1(\gamma_1 \wedge \ldots \wedge \gamma_5)_5^1(f_1 \otimes (-W) \otimes f_2 \otimes f_3 \otimes f_4)$ (from [13]).

to the weight. This sign cancels the sign present in the sum, therefore performing the sum over l yields an integral of the angle $\varphi_{e_k^r}$ over $(0, 2\pi]$ which decouples from the rest and yields 2π. This is then absorbed by a 2π in the denominator of the prefactor of (5.2.7). We are then left with an integral over $\overline{C}^{n,m}$. The sign μ is the product of the sign present on the left-hand side of the equation times the sign coming from the Schouten-Nijenhuis bracket, and we see that (5.2.8) indeed holds true.

We will now prove that K_1^1 is a quasi-isomorphism. Consider $\gamma \in T_{\text{poly}}$ of degree $\widetilde{\gamma} = n$. By construction $(K_1^1(\gamma))^1(f_1 \otimes \ldots \otimes f_m)$ is non-zero only for $m = n$ and is given by

$$(K_1^1(\gamma))_n^1(f_1 \otimes \ldots \otimes f_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sign}(\sigma) \gamma^{i_1 \ldots i_n} \prod_{k=1}^{n} \partial_{i_{\sigma(k)}} f_k \, .$$

By the Hochschild-Kostant-Rosenberg theorem K_1^1 is a quasi-isomorphism from $(T_{\text{poly}}, 0)$ to $(\text{Coder}(T_R), [\widehat{\partial}_2, \cdot\,])$.

To show that K_1^1 is also a quasi-isomorphism in our case $W \neq 0$, the strategy is to view $(\text{Coder}(T_R), [\widehat{\partial}_0 + \widehat{\partial}_2, \cdot\,])$ as a bicomplex after choosing appropriate linear combinations of tensor and tilde degrees. We then choose to compute the associated spectral sequence whose first page is $[\widehat{\partial}_2, \cdot\,]$-cohomology. The spectral sequence degenerates at the second page yielding

$$H_{[\widehat{\partial}_0+\widehat{\partial}_2,\,\cdot\,]}(\mathrm{Coder}(T_R)) = \mathrm{HH}^\bullet(R, \widehat{\partial}_0 + \widehat{\partial}_2) = \mathrm{Jac}(W)\,,$$

see appendix B.1 for details. As explained also in [10, 78], the chosen spectral sequence computes Hochschild cohomology of the second kind. This concludes the proof of proposition 5.2.4.

Essentially the same argument is used if we replace $\mathrm{Coder}(T_R)$ with $\mathrm{Coder}(T_A)$ in the setting of lemma 5.2.3. The first page then is classical Morita equivalence whose proof is the same in the $\mathbb{Z}_2$-graded case. □

5.2.2 Transporting bulk deformations on-shell

After having found the solutions to the Maurer-Cartan equation describing bulk-induced deformations of the off-shell open string algebra, we are now faced with the task of transporting them on-shell. We shall do so by constructing an L_∞-morphism

$$(\mathrm{Coder}(T_A), [\partial, \,\cdot\,], [\,\cdot\,,\,\cdot\,]) \longrightarrow (\mathrm{Coder}(T_H), [\widetilde{\partial}, \,\cdot\,], [\,\cdot\,,\,\cdot\,])\,. \tag{5.2.9}$$

A crucial observation is that $(T_H, \widetilde{\partial})$ is a deformation retract of (T_A, ∂). We will start with a general discussion of this notion on the level of complexes and of the natural L_∞-morphism that it gives rise to. Then we will specialise to our case of open string algebras, explicitly construct the associated deformation retract data, and thus arrive at the L_∞-morphism (5.2.9) to transport the off-shell deformation δ from the previous subsection to the on-shell algebra $(H, \widetilde{\partial})$. While we will apply it to the case of Landau-Ginzburg models, we note that our construction of the map (5.2.9) works for arbitrary A_∞-algebras (A, ∂) and their minimal models $(H, \widetilde{\partial})$.

Deformation retractions

A *deformation retraction*

$$(C_2, d_2) \underset{p}{\overset{i}{\rightleftarrows}} (C_1, d_1) \circlearrowleft h \tag{5.2.10}$$

consists of the following data: two complexes (C_1, d_1) and (C_2, d_2), two maps of complexes $i : (C_2, d_2) \to (C_1, d_1)$ and $p : (C_1, d_1) \to (C_2, d_2)$, and a homotopy $h \in \mathrm{End}(C_1)$. These data are subject to the relations

$$pi = \mathrm{id}_{C_2}\,, \quad \mathrm{id}_{C_1} - ip = d_1 h + h d_1\,, \tag{5.2.11}$$

and we refer to (C_2, d_2) as the deformation retract of (C_1, d_1). The homotopy h is said to be in *standard form* if it satisfies

$$hh = hi = ph = 0\,.$$

Given the data (5.2.10), we can construct maps $M_n^1 : \mathrm{End}(C_1)[1]^{\wedge n} \to \mathrm{End}(C_2)[1]$ via

$$M_n^1(a_1 \wedge \ldots \wedge a_n) = \sum_{\sigma \in S_n} \varepsilon_{\sigma;a_1,\ldots,a_n} p a_{\sigma(1)}\, h\, a_{\sigma(2)} \ldots h a_{\sigma(n)} i$$

for all homogeneous $a_1, \ldots, a_n \in \mathrm{End}(C_1)$, where $\varepsilon_{\sigma;a_1,\ldots,a_n}$ denotes the Koszul sign introduced in subsection 5.1.1. We will denote by M the coalgebra morphism $S_{\mathrm{End}(C_1)} \to S_{\mathrm{End}(C_2)}$ uniquely defined from the maps M_n^1. This morphism is the central ingredient of our version of the homological perturbation lemma:

Proposition 5.2.5. $M : (\mathrm{End}(C_1), [d_1, \cdot\,], [\,\cdot\,,\,\cdot\,]) \to (\mathrm{End}(C_2), [d_2, \cdot\,], [\,\cdot\,,\,\cdot\,])$ is an L_∞-quasi-isomorphism.

Proof. It is convenient to define also the collection of maps $S_n^1 : \mathrm{End}(C_1)[1]^{\otimes n} \to \mathrm{End}(C_2)[1]$ by

$$S_n^1(a_1 \otimes \ldots \otimes a_n) = p a_1 h a_2 \ldots h a_n i\,.$$

In fact we are going to prove that the corresponding coalgebra morphism S is an A_∞-quasi-isomorphism and hence M will be the L_∞-morphism induced by S on the commutator algebra.

First we prove that S is an A_∞-morphism, i. e.

$$S_n^1 b_n^n + S_{n-1}^1 b_n^{n-1} = \widetilde{b}_1^1 S_n^1 + \widetilde{b}_2^1 S_n^2 \tag{5.2.12}$$

where b_1^1 and b_2^1 are defined by $b_1^1(a) = [d_1, a]$ and $b_2^1(a_1 \otimes a_2) = (-1)^{\widetilde{a}_1} a_1 a_2$ and similarly $\widetilde{b}_1^1$ and $\widetilde{b}_2^1$. For $n = 1$ the condition (5.2.12) is easily checked,

$$\widetilde{b}_1^1 S_1^1(a) = [d_2, pai] = p[d_1, a]i = S_1^1 b_1^1(a)$$

where we have only used that i and p are maps of complexes.

To prove (5.2.12) for all $n \geqslant 1$ we first compute

$$\begin{aligned}
&\widetilde{b}_1^1 S_n^1(a_1 \otimes \ldots \otimes a_n) = p[d_1, a_1 h a_2 \ldots h a_n]i \\
&= p\left(\sum_{k=1}^{n-1} (-1)^{\sum_{l=1}^{k-1} \widetilde{a}_i} a_1 h \ldots [d_1, a_k h] \ldots h a_n + (-1)^{\sum_{l=1}^{n-1} \widetilde{a}_i} a_1 h a_2 \ldots h[d_1, a_n]\right) i \\
&= S_n^1 b_n^n(a_1 \otimes \ldots \otimes a_n) - p\left(\sum_{k=1}^{n-1} (-1)^{\sum_{l=1}^{k} \widetilde{a}_i} a_1 h \ldots a_k [d_1, h] a_{k+1} \ldots h a_n\right) i
\end{aligned}$$

where we have only used that S_1^1 is a map of complexes. Now we only have to insert the identity $[d_1, h] = \mathrm{id}_{C_1} - ip$ in the above equation to obtain the desired result.

In order to conclude the proof we still need to show that M_1^1 is a quasi-isomorphism. We already know that M_1^1 is a map of complexes and since (C_2, d_2) is a deformation retract of (C_1, d_1), we are left to verify that if $a \in \mathrm{End}(C_1)$ represents a non-trivial element in b_1^1-cohomology, then $pai \neq 0$. Suppose however $pai = 0$, then

$$0 = ipaip = (\mathrm{id}_{C_1} - [d_1, h])a(\mathrm{id}_{C_1} - [d_1, h]) \tag{5.2.13}$$

and hence $a = b_1^1([h, a] - ha[d_1, h])$. This contradicts the assumption on a. □

Deformation retractions from A_∞-algebras

The L_∞-morphism M allows us to transport deformations of (C_1, d_1) to deformations of (C_2, d_2). In our case of interest these two complexes are given by (T_A, ∂) and $(T_H, \widetilde{\partial})$, respectively, and we ask for the additional property that the deformation of $\widetilde{\partial}$ must continue to be an A_∞-structure. Hence we will now explain under which circumstances this is guaranteed to be the case, i. e. when (5.2.9) maps coderivations to coderivations.

Proposition 5.2.6. Assume that

$$(T_{A_2}, \partial_2) \underset{p}{\overset{i}{\rightleftarrows}} (T_{A_1}, \partial_1) \circlearrowleft h \tag{5.2.14}$$

is a deformation retraction where (A_1, ∂_1) and (A_2, ∂_2) are A_∞-algebras, and h is in standard form. Then for $a_1, \dots, a_n \in \operatorname{Coder}(T_{A_1})$, we have $M^1_n(a_1 \wedge \dots \wedge a_n) \in \operatorname{Coder}(T_{A_2})$ for all $n \geqslant 1$.

Proof. Let us introduce some convenient notation: We write $\mathcal{A} = \operatorname{End}(T_{A_1})$, and for an element $f \in \operatorname{End}(T_{A_1})$, let L_f and R_f denote the left and right multiplication by f, respectively. Define the left and right ideals $I_L = \operatorname{Ker}(R_i) \cap \operatorname{Ker}(R_h)$ and $I_R = \operatorname{Ker}(L_p) \cap \operatorname{Ker}(L_h)$. Since h is in standard form, we have $h \in I = I_L \cap I_R$. Finally we define $\pi = ip$, $J = I_L + I_R$ and $\mathcal{B} = \operatorname{lin}_{\mathbb{C}}(\operatorname{id}_{T_{A_1}} - \pi)$.

Now we will show that in fact $S^1_n(a_1 \otimes \dots \otimes a_n)$ is a coderivation. Denote $\Lambda_n = a_1 h a_2 \dots h a_n$ and assume without loss of generality that $a_1, \dots, a_n$ are homogeneous. The crucial observation is that $\Delta\Lambda_n$ admits the decomposition (proved in appendix B.2)

$$\Delta\Lambda_n = ((\operatorname{id}_{T_{A_1}} + \mathcal{B}) \otimes \Lambda_n + \Lambda_n \otimes (\operatorname{id}_{T_{A_1}} + \mathcal{B}) + J \otimes \mathcal{A} + \mathcal{A} \otimes J)\Delta\,, \tag{5.2.15}$$

where we use a short-hand notation where e. g. "$J \otimes \mathcal{A}$" means "some element in $J \otimes \mathcal{A}$". It then follows that $S^1_n(a_1 \otimes \dots \otimes a_n) = p\Lambda_n i$ satisfies

$$\Delta p \Lambda_n i = (p\Lambda_n i \otimes \operatorname{id}_{T_{A_2}} + \operatorname{id}_{T_{A_2}} \otimes p\Lambda_n i)\Delta\,,$$

which says that $S^1_n(a_1 \otimes \dots \otimes a_n)$ is a coderivation. □

We have thus proved that M continues to be an L_∞-morphism when restricted to coderivations. However, it will then generically no longer be a quasi-isomorphism (as we discuss in appendix B.3).

On-shell bulk-induced deformations

Proposition 5.2.6 enables us, given a deformation retraction of A_∞-algebras (5.2.14), to transport deformations of ∂_1 to deformations of ∂_2. To accomplish our aim to construct bulk-deformed open topological string theories for Landau-Ginzburg models, we are now left to specify the deformation retract data

$$(T_H, \widetilde{\partial}) \underset{\bar{F}}{\overset{F}{\rightleftarrows}} (T_A, \partial) \circlearrowleft U \tag{5.2.16}$$

paying attention to the condition that the homotopy U be in standard form. In writing (5.2.16) we have already revealed that in the case at hand the inclusion map is given by the minimal model morphism $F:(H,\widetilde{\partial})\to(A,\partial)$ of proposition 5.1.1. It remains to find its homotopy inverse $\bar{F}$ and the homotopy U itself. This is achieved by the next proposition which constructs $\bar{F}$ and U explicitly.

Proposition 5.2.7. For any A_∞-algebra (A,∂), there exist a unique colagebra morphism $\bar{F}$ and a map U that make (5.2.16) a deformation retraction and satisfy the conditions

$$\begin{aligned}
\Delta U &= \frac{1}{2}(U\otimes(\mathrm{id}_{T_A}+F\bar{F})+(\mathrm{id}_{T_A}+F\bar{F})\otimes U)\Delta\,, && (5.2.17)\\
U_1^1 &= G\,,\\
U_n^1 &= -G\partial_2^1 U_n^2 \text{ for } n\geqslant 2\,, && (5.2.18)\\
\bar{F}_1^1 &= \pi_H\,,\\
\bar{F}_n^1 &= -\pi_H([\partial,U])_n^1 = -\pi_H\partial_2^1 U_n^2\,. && (5.2.19)
\end{aligned}$$

Moreover, these conditions allow for an explicit construction of $\bar{F},U$, and U is in standard form.

Proof. Here we will prove proposition 5.2.7. That $\bar{F}$ must be of the form (5.2.19) follows by applying π_H to

$$\mathrm{id}_{T_A}-F\bar{F}=[\partial,U]\,. \tag{5.2.20}$$

$\bar{F}$ is then automatically an A_∞-morphism by

$$F(\widetilde{\partial}\bar{F}-\bar{F}\partial)=[\partial,F\bar{F}]=-[\partial,\partial U+U\partial]=0$$

and the injectivity of F. Next we show that (5.2.17) is compatible with (5.2.20):

$$\begin{aligned}
&\Delta(\mathrm{id}_{T_A}-F\bar{F})=\Delta(\partial U+U\partial)\\
&=\frac{1}{2}(\partial\otimes\mathrm{id}_{T_A}+\mathrm{id}_{T_A}\otimes\partial)(U\otimes(\mathrm{id}_{T_A}+F\bar{F})+(\mathrm{id}_{T_A}+F\bar{F})\otimes U)\\
&\quad+\frac{1}{2}(U\otimes(\mathrm{id}_{T_A}+F\bar{F})+(\mathrm{id}_{T_A}+F\bar{F})\otimes U)(\partial\otimes\mathrm{id}_{T_A}+\mathrm{id}_{T_A}\otimes\partial)\\
&=\frac{1}{2}(\mathrm{id}_{T_A}-F\bar{F})\otimes(\mathrm{id}_{T_A}+F\bar{F})+\frac{1}{2}(\mathrm{id}_{T_A}+F\bar{F})\otimes(\mathrm{id}_{T_A}-F\bar{F})\\
&\quad+[\partial,(\mathrm{id}_{T_A}+F\bar{F})]\otimes U-U\otimes[\partial,(\mathrm{id}_{T_A}+F\bar{F})]\\
&=(\mathrm{id}_{T_A}-F\bar{F}\otimes F\bar{F})\Delta=\Delta(\mathrm{id}_{T_A}-F\bar{F})
\end{aligned}$$

where in the penultimate step we made use of the fact that $[\partial,F\bar{F}]=0$ and in the last step we used the fact that $F\bar{F}$ is a coalgebra morphism. This calculation shows that if we chose U_n^1 appropriately, condition (5.2.17) ensures that U is a solution of (5.2.20).

Inspection of (5.2.20) reveals that U_1^1 must be a homotopy for ∂_1^1, and we can therefore choose $U_1^1=G$. For $n\geqslant 2$ we observe that U_n^1 of the form (5.2.18) satisfies

$$\pi_B(\partial U+U\partial)_n^1=0\,, \tag{5.2.21}$$

and moreover

$$\pi_L(\partial U + U\partial)^1_n = -\pi_L(F\bar{F})^1_n \tag{5.2.22}$$

holds for $n = 2$. In order to show that this is also true for $n > 2$ we proceed by induction. We start by substituting (5.2.18) into the left-hand side of (5.2.22) to obtain

$$-G\partial^1_2(\partial^2_2 U^2_n + U^2_{n-1}\partial^{n-1}_n + U^2_n\partial^n_n) = -G\partial^1_2(\partial U + U\partial)^2_n = G\partial^1_2(F\bar{F})^2_n = -\pi_L(F\bar{F})^1_n$$

where in the first equality we used the associativity of ∂, i. e. $\partial^1_2\partial^2_3 = 0$. The second equality is the induction step that is well-defined due to (5.2.17), while in the third equality we used $[\partial, F\bar{F}] = 0$ and the definition of F from 5.1.1.

We are thus left to verify that $\bar{F}$ is a left inverse of F and that U is in standard form. First we give the explicit recursive formulas

$$\begin{aligned} U^1_n &= -\frac{1}{2}G\partial^1_2\left(\sum_{l=1}^{n-1}(U^1_l \otimes (\mathrm{id}_{T_A} + F\bar{F})^1_{n-l} + (\mathrm{id}_{T_A} + F\bar{F})^1_{n-l} \otimes U^1_l\right), \\ \bar{F}^1_n &= -\frac{1}{2}\pi_H\partial^1_2\left(\sum_{l=1}^{n-1}(U^1_l \otimes (\mathrm{id}_{T_A} + F\bar{F})^1_{n-l} + (\mathrm{id}_{T_A} + F\bar{F})^1_{n-l} \otimes U^1_l\right). \end{aligned} \tag{5.2.23}$$

The maps U^m_n for $m > 1$ are then completely determined by repeated application of the coproduct Δ. Now we show that $\bar{F}F = \mathrm{id}_{T_A}$. Since $\bar{F}F$ is a coalgebra morphism, we only need to consider the subset of equations

$$(\bar{F}F)^1_n = (\mathrm{id}_{T_H})^1_n\,. \tag{5.2.24}$$

Clearly the above is satisfied for $n = 1$. For $n > 1$, $\bar{F}^k_1 F^n_k$ vanishes at $k = 1$ due to $\mathrm{Im}(F^1_n) \subset L \subset \mathrm{Ker}(\pi_H)$. While for $k > 1$ it vanishes because from (5.2.23) we see that each summand in $\bar{F}^1_n$ has at least one factor proportional to G. However each tensor factor of each summand is in $\mathrm{Im}(F^1_k) \subset H \oplus L = \mathrm{Ker}(G)$.

That U is in standard form, i. e. $\bar{F}U = 0$, $UF = 0$ and $UU = 0$, follows from an argument in direct analogy to the above proof for $\bar{F}$.

The above proof is easily extended to the case where (A, ∂) is an arbitrary A_∞-algebra by replacing the formula for U^1_n with $U^1_n = -G\left(\sum_{k=2}^n \partial^1_k U^k_n\right)$ from which it follows that $\bar{F}^1_n = -\pi_H\left(\sum_{k=2}^n \partial^1_k U^k_n\right)$. □

Now we have arrived at the point to put together all the results obtained in this section. We apply the L_∞-morphism M of proposition 5.2.5 in conjunction with the deformation retraction of proposition 5.2.7. Recall that in the previous subsection we found that for Landau-Ginzburg models off-shell deformations $\delta \in \mathrm{Coder}(T_A)$ are precisely the bulk-induced coderivations determined by

$$\delta^1_0 = \sum_i t_i\phi_i \tag{5.2.25}$$

where $\{\phi_i\}$ is a basis of the bulk space $\mathrm{Jac}(W)$, and t_i are the associated moduli. By proposition 5.1.2 we can use M to map δ to deformations $\widetilde{\delta}$ of the on-shell open string algebra $(H, \widetilde{\partial})$, and proposition 5.2.6 ensures that $\widetilde{\partial} + \widetilde{\delta}$ indeed encodes an A_∞-structure. Thus our final result is the following.

Theorem 5.2.8. The bulk-induced deformations of the on-shell Landau-Ginzburg open string algebra $(H, \widetilde{\partial})$ are given by

$$\widetilde{\delta} = \sum_{n \geqslant 1} \frac{1}{n!} M_n^1(\delta^{\wedge n}) = \sum_{n \geqslant 1} \bar{F}(\delta U)^n \delta F = \bar{F}(\mathrm{id}_{T_A} - \delta U)^{-1} \delta F \,. \tag{5.2.26}$$

By substituting (5.2.25) together with the concrete formulas for $F, \bar{F}$ and U into (5.2.26), one obtains explicit expressions for bulk-deformed A_∞-products on H.

To make sure that the bulk moduli dependent A_∞-structure encoded in $\widetilde{\partial} + \widetilde{\delta}$ immediately describes all amplitudes of bulk-deformed open topological string theory, it has to be shown that also the deformed A_∞-products are cyclic with respect to the Kapustin-Li pairing.

Let us for the moment restrict to those bulk fields that "are seen by the open TFT of the brane D", i. e. those $\phi \in \mathrm{Jac}(W)$ that are not mapped to zero by the bulk-boundary map. We denote an off-shell deformation that arises from such a bulk field by δ_Z. Then the on-shell deformation $\widetilde{\delta}_Z$ takes a particularly simple form in our approach:

$$\widetilde{\delta}_Z = \bar{F}(\mathrm{id}_{T_A} - \delta_Z U)^{-1} \delta_Z F = \bar{F} \delta_Z F = \delta_Z \,,$$

where we have used the fact that the image of F only consists of tensor powers of elements in H and the complement of $\mathrm{Ker}([D, \cdot])$ (see (5.1.7, 5.1.8)), and that U acts as G on one tensor factor of each summand. Then it is straightforward to see that the deformed A_∞-algebra is cyclic with respect to the Kapustin-Li pairing.

The rigidity of the methods used in transporting bulk deformations on-shell may suggest the cyclicity of $\widetilde{\partial} + \widetilde{\delta}$ also in the general case when the off-shell bulk deformation is not of the form δ_Z. However, then the abstract manipulation of $\widetilde{\delta}$ in (5.2.26) is more difficult, and at this point we have no proof that $\widetilde{\partial} + \widetilde{\delta}$ is cyclic.

5.2.3 Concluding remarks

In conclusion, we have provided a general formula for the bulk induced deformations of arbitrary affine Landau Ginzburg open TCFT's. This was achieved completely from first principles and results from a more general construction identifying the off-shell bulk sector as seen from the boundary sector. While half of the methods developed are completely general and can be applied to arbitrary TCFT's there are also shortcomings of the approach as it stands. In particular by construction, we were only concerned with the A_∞ structure while not taking into account cyclicity. This problem can in principle be cured by starting from a more refined deformation theory approach. One where the off-shell closed string complexes are replaced by S^1-equivariant versions computing cyclic Hochschild cohomology instead of Hochschild cohomology. The former is the latter supplemented with the cyclicity constraint (4.4.5). At this point we can at least intuitively understand from 2.8.3, that the modified complex is the one computing topological string theory observables rather than TQFT observables. We won't pursue a direct explanation here, instead we pass to the complementary problem of computing purely open TCFT's in arbitrary affine Landau Ginzburg models. We will see that in that investigation we will also uncover some steps in this direction.

5.3 The off-shell ω for affine LG models

Here we will give a formula for the off-shell paring of arbitrary affine Landau Ginzburg models whose construction was anticipated in [13]. This completes their string field theory data. It will be interesting to note that a small byproduct of our analysis will be a formula for the S^1-equivariant, or topological string theory generalization, of the TQFT trace for arbitrary affine LG models. We will remark that this formula can be immediately generalized for heterotic LG models. For these constructions it is convenient to express cyclic A_∞-algebras in terms of non-commutative geometry.

5.3.1 Non-commutative geometry preliminaries

In this section, the aim is to understand cyclic A_∞ algebras in the language of non-commutative geometry. In particular ω, in this context, is a symplectic or homologically symplectic form. We follow primarily [12, 56].

Let (A, ∂) denote an A_∞-algebra, then define $B := A^\vee$ and the corresponding tensor algebra:

$$O(X_A) := \prod_{n\geq 0} B[1]^{\otimes n}$$

with product $\otimes$. We can view X_A suggestively as the non-commutative space whose ring of functions is formal power series in $B[1]$. The A_∞ structure ∂ acts on $O(X_A)$ via its dual. Let s^k denote a basis for $B[1]$ dual to $\{a_k\}$, then Q is defined through:

$$Q(s^{k_1} \otimes \cdots \otimes s^{k_n})(a_{i_1} \otimes \cdots \otimes a_{i_m}) = (s^{k_1} \otimes \cdots \otimes s^{k_n})(\partial(a_{i_1} \otimes \cdots \otimes a_{i_m}))$$

and it is completely defined by the components:

$$Q^k_{i_1 \dots i_n} = s^k(\partial(a_{i_1} \otimes \cdots \otimes a_{i_m}))$$

We will denote:

$$Q^k = s^k \partial^1.$$

In particular, since Q is a coderivation of T_A, Q is a derivation of $O(X_A)$, $Q \in \mathrm{Der}(O(X_A))$, which in geometric language translates to a vectorfield on X_A. Moreover since ∂ squares to zero, so does Q.

It is now natural to consider differential forms on X_A. In commutative geometry, these can be viewed as a ring of functions of the tangent bundle TX_A with a symmetric product over the base and with the anti-symmetric (wedge) product over the fibers. We thus introduce a further $\mathbb{Z}_2$ grading with parity reversal Π. Then the function just described are elements of $O(\Pi TX_A)$. We can apply this definition with no change to the non-commutative case:

$$O(T[1]X_A) := \prod_{n\geq 0}(B[1] \oplus \Pi B[1])^{\otimes n} =: \prod_{m\geq 0} \Omega^m(X_A)$$

In order to distinguish coordinate one-forms from variables in $B[1]$ we introduce the degree 1 differential $d \in \mathrm{Der}(O(\Pi TX_A))$ uniquely defined by the requirement that it act on $B[1]$ as the parity reversal isomorphism. We will be interested in the space of cyclic differential forms. These are defined as:

$$\Omega^0_{cycl}(\Pi TX_A) := O(\Pi TX_A)/[O(\Pi TX_A), O(\Pi TX_A)] =: \prod_{m\geq 0} \Omega^m_{cycl}(X_A)$$

where the commutator respects the $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading, in particular:

$$\begin{aligned}
[s^i, s^j] &= s^i \otimes s^j - (-1)^{\tilde{a}_i\tilde{a}_j} s^j \otimes s^i \\
[s^i, ds^j] &= s^i \otimes ds^j - (-1)^{\tilde{a}_i\tilde{a}_j} ds^j \otimes s^i \\
[ds^i, ds^j] &= ds^i \otimes ds^j + (-1)^{\tilde{a}_i\tilde{a}_j} ds^j \otimes ds^i
\end{aligned}$$

The complex:

$$\left(\prod_{m\geq 0} \Omega^m_{cycl}(X_A), d\right)$$

is known as the *Karoubi complex* and is to be viewed as the non-commutative version of the de Rahm complex (we refer to section 11.3 of [36] for an explanation of this statement). On $\Omega^\bullet$ and $\Omega^\bullet_{cycl}$ alike, one can act with vectorfields $\xi \in \mathrm{Der}(O(X_A))$ via the contraction defined via:

$$\iota_\xi ds^k = \xi^k$$

and therefore also via the Lie derivative:

$$L_\xi := d\iota_\xi + \iota_\xi d$$

If ξ is of $\mathbb{Z}_2$ degree zero, we can view it dually to degree zero coderivations, as generating the dual to an A_∞ morphism, which in this setting is the pull-back ϕ^* by an automorphism ϕ of X_A. More generally, given two A_∞-algebras (A_1, ∂_1) and (A_2, ∂_2), an A_∞-morphism in $\mathrm{Hom}(T_{A_1}, T_{A_2})$ translates to the pullback by a map $\phi : X_{A_1} \to X_{A_2}$ such that

$$\iota_{Q_1}\phi^* = \phi^*\iota_{Q_2}.$$

Now we turn to the geometric description of the on-shell ω. Thus we consider the A_∞-algebra $(H, \tilde{\partial})$. Then:

Proposition 5.3.1 ([12, 56]). the A_∞ structure $\widetilde{\partial}$ is cyclic w.r.t. ω iff:

$$L_{\widetilde{Q}}\omega = 0$$

where by a slight abuse of notation, $\omega \in \Omega_{cycl}(X_H)$ is defined as:

$$\omega = \omega_{kl}(ds^k ds^l)_c$$

Proof.

$$\begin{aligned} L_{\widetilde{Q}}\omega &= d\iota_{\widetilde{Q}}\omega \\ &= \omega_{kl} d(\widetilde{Q}^k ds^l - (-1)^{\tilde{a}_k} ds^k \widetilde{Q}^l)_c \\ &= (\omega_{kl} - (-1)^{\tilde{a}_k \tilde{a}_l}\omega_{lk}) d(\widetilde{Q}^k ds^l)_c \\ &= 2\omega_{kl} d(\widetilde{Q}^k ds^l)_c \\ &= \sum_n 2\omega_{kl}\widetilde{Q}^k_{i_1\ldots i_n} \sum_{r=1}^n (s^{i_1}\cdots ds^{i_r}\cdots s^{i_n} ds^l)_c \end{aligned}$$

therefore $L_{\widetilde{Q}}\omega = 0$ iff:

$$\omega_{kl}\widetilde{Q}^k_{i_1\ldots i_n} = (-1)^{\tilde{a}_{i_n}(\tilde{a}_l + \cdots + \tilde{a}_{i_{n-1}})}\omega_{ki_n}\widetilde{Q}^k_{li_1\ldots i_{n-1}}$$

□

If we change coordinates with an automorphism ϕ, ω will no longer be flat, instead the coordinate independent notion of cyclicity is that ω is a symplectic form with $\widetilde{Q}$ a symplectic vectorfield. The equivalence of the two definitions of cyclicity is granted by the Darboux theorem:

Theorem 5.3.2. (Darboux) given a symplectic form $\omega \in \Omega^2_{cycl}(X_A)$, there is an automorphism $\phi_D : X_A \to X_A$ such that $\phi_D^*\omega = \omega_0$ where ω_0 is the flat part of ω.

Proof. The automorphism is constructed perturbatively. One grades ω by polynomial degree $\omega = \sum_{n\geq 0}\omega_n$. As a first step one constructs an automorphism that eliminates ω_1. Thanks to the Poincaré lemma provided below, $\omega_1 = d\alpha_1$ for some 1-form α_1. Thanks to the non-degeneracy of ω_0, there is a unique vectorfield ξ_1 such that:

$$\alpha_1 = \iota_{\xi_1}\omega_0.$$

Then we can define a first order approximation $(\phi_D)^*_1$ to ϕ_D^* as the algebra morphism that acts as $(1-\xi)$ on $B[1]$. Indeed $\phi_1^*\omega = \omega_0 + \omega_2^1 + \cdots$ and one can continue with this procedure replacing ω_1 with ω_2^1 and so on. □

Proposition 5.3.3 (Poincaré Lemma). $H^n_{dR}(X_A) = 0$ for $n \geq 1$ and $H^0_{dR}(X_A) = \mathbb{C}$.

Proof. This follows exactly as in the commutative case, due to the fact that on formal power series, one can define an Euler vectorfield E, that is defined by:

$$\iota_E\, ds^k = s^k.$$

Then if we grade $\Omega^\bullet_{cycl} = \prod_{m\geq 0} \Omega^\bullet_{cycl}(m)$ by the sum of tensor and polynomial degree, we have that on $\Omega^\bullet_{cycl}(m)$ the following identity holds for $m \geq 1$:

$$d\left(\frac{1}{m}\iota_E\right) + \frac{1}{m}\iota_E d = 1$$

therefore the identity for $m \geq 1$ is homotopic to zero and the Poincaré lemma follows. □

In the following, we will denote the homotopy induced by the Euler vectorfield by

$$h_E.$$

In [12] a very general method for computing open TCFT's was developed. The idea is always that of working off-shell where the A_∞-algebra reduces to a DGA and then to transport structures computed there on-shell. The aforementioned approach consists of three steps. The first consists of an algorithmic procedure that computes the space of cyclic DGA structures off-shell. Up to coordinate transformations, these are elements of:

$$[\omega] \in H_{L_Q}(d\Omega^1_{cycl}(X_A))$$

however ω is not required to be symplectic, instead one has to require that it be symplectic on-shell, that is when restricted to Q_1-cohomology. In other words it must be *homologically symplectic*. The next two steps in the computation of minimal cyclic A_∞-structures are then straightforward. First one constructs an arbitrary, e.g. Merkulov type minimal model $(H, \widetilde{\partial}')$ with minimal model morphism F'(see section 5.1). Thirdly one applies a Darboux automorphism, as the one constructed in the proof of the Darboux theorem. In the end one obtains $\widetilde{Q}$ through:

$$\iota_{\widetilde{Q}}\phi^*_D\phi^*_{F'} = \phi^*_D\phi^*_{F'}\iota_Q.$$

Or more directly, we can apply $\phi^*_D\phi^*_{F'}$ to the off-shell effective superpotential. In this context, it is now clear that $\mathcal{W}_{eff} \in \Omega^0_{cycl}(X_A)$ is the Hamiltonian of $\widetilde{Q}$ w.r.t. $\widetilde{\omega}$:

$$d\mathcal{W}_{eff} = \iota_{\widetilde{Q}}\widetilde{\omega}$$

therefore:

$$\mathcal{W}_{eff} = h_E\phi^*_D\phi^*_{F'}(\iota_Q\omega) = \phi^*_D\phi^*_{F'}(h_E\iota_Q\omega) = \phi^*_D\phi^*_{F'}\mathcal{W}^{off-shell}_{eff}$$

The only criticism to this procedure, is that the first step in the computation of $\mathcal{W}_{eff}$, that is the computation of the off-shell ω, is by far the rate determining one. Moreover it is desirable to have a more conceptual way of computing the off-shell homologically symplectic form. In the next sections we will obtain an explicit analytic formula for the off-shell ω for arbitrary affine Landau Ginzburg models. In other words we will provide a formula for the string-field theory action $\mathcal{W}^{off-shell}_{eff}$. In this sense ω completes the string field theory data of affine LG models. The next section will serve as a map for later sections, outlining the general strategy, and also contains some preliminaries that will be crucial in the following.

Strategy for off-shell ω in LG models

Let (A, ∂) denote the DGA of an affine Landau Ginzburg model and let $\partial^\vee =: Q$. The formula for the off-shell homologically symplectic form is achieved in steps that can be succinctly summarized in the following sequence:

$$\begin{aligned}(d\Omega^1_{cycl}(X_A), L_{Q_{12}})^* \rightsquigarrow (\Omega^0_{cycl}(X_A)/\mathbb{C}, L_{Q_{12}})^* \longrightarrow (C^\lambda_\bullet(A), b_{12}) \xrightarrow{f_1} (\mathrm{Tot}(CC_{\bullet\bullet}), d_{tot})\\ \xrightarrow{f_2} (\mathcal{B}_\bullet, b_{12} + uB) \xrightarrow{f_3} (\overline{\mathcal{B}}_\bullet, b_{12} + u\overline{B})\\ \xrightarrow{f_4} (\overline{\mathcal{B}}_\bullet, b_{02} + u\overline{B}) \xrightarrow{f_5} (\overline{\mathcal{B}}^Z_\bullet, b_{02} + u\overline{B})\\ \xrightarrow{f_6} (\Gamma(X, \Lambda^\bullet T^\vee X) \otimes \mathbb{C}((u))/(u), -dW \wedge +ud)\\ \xrightarrow{f_7} (\mathbb{C}, 0)\end{aligned} \tag{5.3.1}$$

The subscript 12 on Q and b indicates that they describe DGA's. The first squiggly line indicates that the map is not a map of complexes, rather it is an isomorphism at the level of cohomology. The maps f_1 through f_6 are all quasi-isomorphisms. The maps f_1 through f_3 are independent of the choice of TCFT, and describe quasi-isomorphic complexes computing cyclic Hochschild homology of DGA's. They are all DG generalizations of the ones constructed in [50] for associative algebras. In that context, f_2 and its quasi-inverse had already been constructed in [67]. Extending these results to the DG case turns out to be quite straightforward. The map f_4 is to be understood as the dual of the weak A_∞-isomorphism of lemma 5.2.2, while f_5 and f_6 are dual to the cotrace and Hochschild Kostant Rosenberg maps of lemma 5.2.3 and proposition 5.2.4. The last map f_7 we have termed the cyclic residue, and should be understood as the S^1-equivariant, topological string theory trace as opposed to the TQFT trace (see the discussion in section 2.8.3).

Lastly we give here, for future reference, the prevailing tool for the future computations. This is again the HPL (homological perturbation lemma) (see section 5.2.2). However this time it is only applied in the context of complexes and in addition to the deformed differential on the homotopy retract, we need the full deformed HDR (homotopy retraction data). The full HPL reads:

Theorem 5.3.4. (see e.g. [20]) Given an HDR of complexes:

$$(C_2, d_2) \underset{p}{\overset{i}{\rightleftarrows}} (C_1, d_1) \circlearrowleft h\ ,$$

deforming $d_1 \mapsto d_1 + \delta$ one obtains an HDR

$$(C_2, d_2 + \delta_\infty) \underset{p_\infty}{\overset{i_\infty}{\rightleftarrows}} (C_1, d_1 + \delta) \circlearrowleft h_\infty\ ,$$

where:

$$\delta_\infty = p\delta \sum_{n\geq 0}(h\delta)^n i, \qquad i_\infty = \sum_{n\geq 0}(h\delta)^n i, \qquad p_\infty = p\sum_{n\geq 0}(\delta h)^n, \qquad h_\infty = h\sum_{n\geq 0}(\delta h)^n.$$

We reserve separate sections for different portions of the above sequence and the nomenclature $f_1, \cdots f_7$ will not be used in the following. We start from the top.

5.3.2 From zero-forms to closed two-forms

There is an isomorphism ψ^H at the level of L_Q-cohomology, between the space of closed cyclic two-forms $\Omega^2_{cl}(X_A) = d\Omega^1_{cycl}(X_A)$ and the space of cyclic zero-forms $\Omega^0_{cycl}(X_A)$. This is given by the composition of the following two isomorphisms:

$$H_{L_Q}(\Omega^0_{cycl}(X_A)/k) \xrightarrow{\psi_1^H} H_{L_Q}([O(X_A), O(X_A)]) \xrightarrow{\psi_2^H} H_{L_Q}(\Omega^2_{cl}(X_A)).$$

The latter descends from a map ψ_2 at the chain level. We start by specifying ψ_2:

$$\psi_2 : [b_1, b_2] \mapsto (db_1 db_2)_c$$

Clearly ψ_2 is well defined, as the sign in the commutator and in the cyclization is the same, namely:

$$-(-1)^{\tilde{b_1}\tilde{b_2}}.$$

Moreover it is also clear that ψ_2 is an isomorphism onto its image since $\mathrm{Ker}(d) = \mathbb{C} \subset \mathrm{Ker}([O(X_A), \cdot])$. Furthermore, its image is the whole of $\Omega^2_{cl}(X_A)$ by Poincare's Lemma. We are thus left to show that ψ_2 commutes with L_Q:

$$\begin{aligned}
\psi_2 L_Q([b_1, b_2]) &= \psi_2([L_Q(b_1), b_2] + (-1)^{\tilde{b_1}}[b_1, L_Q(b_2)]) \\
&= (dL_Q(b_1)db_2)_c + (-1)^{\tilde{b_1}}(db_1 dL_Q(b_2))_c \\
&= (L_Q(db_1)db_2)_c + (-1)^{\tilde{b_1}}(db_1 L_Q(db_2))_c \\
&= L_Q(db_1 db_2)_c = L_Q\psi_2([b_1, b_2])
\end{aligned}$$

We now pass to ψ_1^H. This is the connecting homomorphism in cohomology that arises from the natural short exact sequence:

$$0 \longrightarrow ([O(X_A), O(X_A)], L_Q) \xrightarrow{i} (O(X_A)/k, L_Q) \xrightarrow{p} (O(X_A)/k)/[O(X_A), O(X_A)], L_Q) \longrightarrow 0$$

Since $O(X_A)/k$ is acyclic (follows simply from unitality of A see next section), the connecting homomorphism on the associated long exact sequence of L_Q-cohomology groups is an isomorphism and is given by:

$$\psi_1^H = i^{-1} L_Q p^{-1},$$

where as usual i^{-1} and p^{-1} are arbitrary right-inverses of i and p respectively. For p^{-1} we can use the canonical inclusion, while for i^{-1} we can use $(1 - t^\vee)/2$ where $t^\vee$ is the generator of cyclic permutations. Finally:

$$\psi_2^H \psi_1^H(f_c) = \frac{1}{2}\psi_2(1 - t^\vee)L_Q(f)$$

In particular, decomposing f in tensor degree as $f = \sum_i f_i$, given that Q defines a DGA, the only terms contributing to the flat part $\omega_{ab}(ds^a ds^b)_c$ of the cyclic two-form are f_1 and f_2:

$$\omega_{ab}(ds^a ds^b)_c = (f_{a_1} Q^{a_1}_{ab} + (Q^{a_1}_a f_{a_1 b} + (-1)^{\tilde{a}} f_{a a_1} Q^{a_1}_b))(ds^a ds^b)_c$$

It is important to keep the above formula in mind, as it is precisely the component of ω one has to analyze in order to verify whether or not ω is homologically symplectic, that is, symplectic when reduced to $H^\vee = (H_{r_1}(A))^\vee$. In particular, for this purpose we only need to analyze:

$$\omega_{ab}(ds^a ds^b)_c\big|_{H^\vee} = f_{a_1} Q^{a_1}_{ab}(ds^a ds^b)_c\big|_{H^\vee} \tag{5.3.2}$$

At the end of our construction of ω we will indeed see that it is homologically symplectic and that its flat part coincides with the Kapustin-Li pairing [41, 49] when restricted to $H^\vee$. The construction of ω is achieved via the chain of quasi-isomorphisms (5.3.1). In particular we view an element in $Z_{L_Q}(\Omega^0_{cycl}(X_A)/\mathbb{C})$ as a quasi-isomorphism:

$$((\Omega^0_{cycl}(X_A)/\mathbb{C})^\vee, L_Q^\vee) \xrightarrow{\quad f \quad} (\mathbb{C}, 0)$$

Indeed:

$$L_Q f = f L_Q^\vee = 0 f = 0.$$

We will henceforth use the more direct notation

$$((\Omega^0_{cycl}(X_A)/\mathbb{C})^\vee, L_Q^\vee) =: (C^\lambda_\bullet, b),$$

which we characterise in the following section.

5.3.3 Cyclic homology of unital DGA's

In this section we will study various incarnations of complexes computing the same cohomology as $(C^\lambda_\bullet, b)$. Unless more precisely specified, we will refer to them all as cyclic complexes. These will constitute crucial building blocks "interpolating" from $(C^\lambda_\bullet, b)$ to $(\mathbb{C}, 0)$. The study of cyclic homology was introduced in [16]. Here and in the following sections, we adopt the notation:

$$X_n = A[1] \otimes A[1]^{\otimes n}$$

We introduce the following operations on $X_\bullet$:

$$\begin{aligned} t &:= (t^\vee)^\vee \\ b_1 &:= \partial_1 \\ b_2' &:= \partial_2 \\ b_2 &:= \partial_2 + (\partial_2^1 \otimes \mathrm{Id})t \\ N|_{X_n} &:= \sum_{k=0}^{n} t^k \end{aligned} \tag{5.3.3}$$

In terms of these

$$(C^\lambda_\bullet, b) = (X_\bullet/(1-t), b_1 + b_2). \tag{5.3.4}$$

For the following we will need to resort to the basic interrelations among the operators (5.3.3):

$$
\begin{aligned}
b_1(1-t) &= (1-t)b_1 \\
b_1 N &= N b_1 \\
b_2(1-t) &= (1-t)b'_2 \\
b'_2 N &= N b_2
\end{aligned}
$$

Crucial for the construction of f (5.3.1) is a bigger cyclic complex out of which $(C^{\lambda}_{\bullet}, b)$ and other cyclic complexes can be recovered as homotopy retracts. In the case of an arbitrary ($\mathbb{Z}_2$-graded) associative algebra A over an arbitrary field k, but with zero differential b_1, this large complex is defined as the following double complex $CC_{\bullet\bullet}$ [67].

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & X_3 & \xleftarrow{1-t} & X_3 & \xleftarrow{N} & X_3 & \xleftarrow{1-t} & \cdots \\
 & & {\scriptstyle b_2}\downarrow & & {\scriptstyle -b'_2}\downarrow & & {\scriptstyle b_2}\downarrow & & \\
0 & \longleftarrow & X_2 & \xleftarrow{1-t} & X_2 & \xleftarrow{N} & X_2 & \xleftarrow{1-t} & \cdots \\
 & & {\scriptstyle b_2}\downarrow & & {\scriptstyle -b'_2}\downarrow & & {\scriptstyle b_2}\downarrow & & \\
0 & \longleftarrow & X_1 & \xleftarrow{1-t} & X_1 & \xleftarrow{N} & X_1 & \xleftarrow{1-t} & \cdots \\
 & & {\scriptstyle b_2}\downarrow & & {\scriptstyle -b'_2}\downarrow & & {\scriptstyle b_2}\downarrow & & \\
0 & \longleftarrow & X_0 & \xleftarrow{1-t} & X_0 & \xleftarrow{N} & X_0 & \xleftarrow{1-t} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

We now want to extend this to the DG level. We define the following graded module:

$$
\tilde{X}_n := \bigoplus_{m \geq 0} X_m y^{m-n}
$$

where y is a degree -1 variable. Then we see that $b := b_2 + y b_1$ and $b' := b'_2 + y b_1$ are degree -1. In order to make contact with (5.3.4) we will tacitly set

$$
y = 1,
$$

but we will keep the dependence on y explicit to keep track of gradings. We now replace $X_\bullet$ by $\tilde{X}_\bullet$ in the cyclic complex, paying attention to the fact that the new complex is not first quadrant anymore. We thus obtain:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \\
0 & \longleftarrow & \tilde{X}_3 & \xleftarrow{1-t} & \tilde{X}_3 & \xleftarrow{N} & \tilde{X}_3 & \xleftarrow{1-t} & \cdots \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \\
0 & \longleftarrow & \tilde{X}_2 & \xleftarrow{1-t} & \tilde{X}_2 & \xleftarrow{N} & \tilde{X}_2 & \xleftarrow{1-t} & \cdots \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \\
0 & \longleftarrow & \tilde{X}_1 & \xleftarrow{1-t} & \tilde{X}_1 & \xleftarrow{N} & \tilde{X}_1 & \xleftarrow{1-t} & \cdots \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \\
0 & \longleftarrow & \tilde{X}_0 & \xleftarrow{1-t} & \tilde{X}_0 & \xleftarrow{N} & \tilde{X}_0 & \xleftarrow{1-t} & \cdots \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \\
0 & \longleftarrow & \tilde{X}_{-1} & \xleftarrow{1-t} & \tilde{X}_{-1} & \xleftarrow{N} & \tilde{X}_{-1} & \xleftarrow{1-t} & \cdots \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Just as in the purely associative case, we can compute the cohomology of this double (actually triple) complex via a spectral sequence even though it is not first quadrant anymore, provided however, the first page is the cohomology of the horizontal differential. If the field k is of characteristic zero (for us $k = \mathbb{C}$), the latter is all concentrated in degree 0 since then $\mathrm{Ker}(1-t) = \mathrm{Im}(N)$. Thus denoting the horizontal differential by ∂, we see that:

$$H_\partial(CC_{\bullet\bullet}) = (\tilde{X}_\bullet/(1-t), b)$$

hence:

$$H(\mathrm{Tot}(CC_{\bullet\bullet})) = H(\tilde{X}_\bullet/(1-t), b) = H(C^\lambda, b).$$

In particular the total complex of the above double complex and (C^λ, b) are equivalent in the derived category, in fact we will construct explicit quasi-isomorphisms to and fro. Before we do that, we introduce a further complex which is equivalent to the preceding ones in the derived category for which we will subsequently also construct explicit quasi-isomorphisms. This further complex can be defined if A is unital. In that case, the even columns of $CC_{\bullet\bullet}$ are acyclic, i.e. homotopic to zero[5]. We are thus looking for a degree 1 map s:

$$s : \tilde{X}_n \to \tilde{X}_{n+1},$$

such that:

$$s(-b') + (-b')s = 1.$$

[5]Note that this fact is the one that ensures the well posedness of the Bar resolution.

It is clear that:

$$s := e \otimes \cdot$$

is such a homotopy, where e is the unit in $A[1]$ with, recall, $\tilde{e} = 1$. We can thus define a new differential B of degrees $(-2, 1)$ on $CC_{\bullet\bullet}$:

$$B : \tilde{X}_n \to \tilde{X}_{n+1}$$

as the composition of the following maps:

$$\begin{array}{ccccc}
\tilde{X}_{n+1} & \xleftarrow{1-t} & \tilde{X}_{n+1} & & \\
 & & \uparrow{\scriptstyle s} & & \\
 & & \tilde{X}_n & \xleftarrow{N} & \tilde{X}_n
\end{array}$$

In short:

$$B := (1 - t)sN.$$

We hence obtain a quasi-isomorphic double complex $\mathcal{B}_{\bullet\bullet}$:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \\
0 & \longleftarrow & \tilde{X}_3 & \xleftarrow{B} & \tilde{X}_2 & \xleftarrow{B} & \tilde{X}_1 & \xleftarrow{B} & \cdots \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \\
0 & \longleftarrow & \tilde{X}_2 & \xleftarrow{B} & \tilde{X}_1 & \xleftarrow{B} & \tilde{X}_0 & \xleftarrow{B} & \cdots \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \\
0 & \longleftarrow & \tilde{X}_1 & \xleftarrow{B} & \tilde{X}_0 & \xleftarrow{B} & \tilde{X}_{-1} & \xleftarrow{B} & \cdots \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \\
0 & \longleftarrow & \tilde{X}_0 & \xleftarrow{B} & \tilde{X}_{-1} & \xleftarrow{B} & \tilde{X}_{-2} & \xleftarrow{B} & \cdots \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \\
0 & \longleftarrow & \tilde{X}_{-1} & \xleftarrow{B} & \tilde{X}_{-2} & \xleftarrow{B} & \tilde{X}_{-3} & \xleftarrow{B} & \cdots \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Analogously to the Hochschild complex, we can package the above in the corresponding total complex $\mathrm{Tot}(\mathcal{B}_{\bullet\bullet}) =: \mathcal{B}_{\bullet}$, by introducing a variable u with the appropriate degree, such that the differential on the total complex is given by:

$$b + uB$$

From the above we find the degree of u requiring that $b + uB$ have total degree -1. Since B has degrees $(|right|, |up|) = (0, 1)$ and total degree is given by $|tot| = |right| + |up|$, u must have degree $(-2, 0)$, thus correctly recovering the degrees of B on $CC_{\bullet\bullet}$. We place the left-most column in degree 0 w.r.t. u, and as a consequence the columns right to it have to be of negative degree in u. Moreover we can extend uB to act as zero on the left-most column, which therefore must be annihilated by u altogether, since $B \neq 0$ on $\tilde{X}_\bullet$. This means that the underlying module of the total complex is:

$$\tilde{X}_\bullet \otimes k((u^{-1}))/(u)$$

where $k((u))$ stands for Laurent series in u. Finally, the total complex is given by:

$$\mathcal{B}_n = \bigoplus_{k \geq 0} \tilde{X}_{n-2k} u^{-k}.$$

In the work of Kassel [50], explicit quasi-isomorphisms

$$C^\lambda_\bullet \leftrightharpoons \mathrm{Tot}(CC_{\bullet\bullet}) \leftrightharpoons \mathcal{B}_\bullet$$

were constructed for the analogous and simpler case of purely associative algebras. The quasi-isomorphism $\mathcal{B}_\bullet \to \mathrm{Tot}(CC_{\bullet\bullet})$ had previously been constructed in [67]. In the next section we will show, by direct application of the method of [50], that the result can be straightforwardly extended to the differential graded case. This will enable us to close the chain of quasi-isomorphisms (5.3.1).

5.3.4 Relating various realizations of the cyclic complex

The key ingredient will be once again the HPL (homological perturbation lemma) (see section 5.2.5). We start with the quasi-isomorphisms:

$$C^\lambda_\bullet \leftrightharpoons \mathrm{Tot}(CC_{\bullet\bullet})$$

Our first aim is that of expressing the collapse of the first page of the spectral sequence $H_\partial(CC_{\bullet\bullet})$ to degree zero as a homotopy retraction:

$$(\tilde{X}_\bullet/(1-t), 0) \underset{p}{\overset{i}{\rightleftarrows}} (CC_{\bullet\bullet}, \partial) \circlearrowleft h$$

Once this is done, we will deform the complex on the right by a differential δ that corresponds to b on even and $-b'$ on odd columns. The HPL then provides the homotopy retraction data:

$$(\tilde{X}_\bullet/(1-t), \delta_\infty) \underset{p_\infty}{\overset{i_\infty}{\rightleftarrows}} (\mathrm{Tot}(CC_{\bullet\bullet}), \partial + \delta) \circlearrowleft h_\infty$$

The desired result is then obtained realizing that in this case $\delta_\infty = p\delta i$, making the deformed complex on the left precisely C^λ.

Before we start the construction it is convenient to package also $CC_{\bullet\bullet}$ as a total complex, by introducing a variable q of degree -1 to grade the columns. The underlying module of the total complex is then:

$$\tilde{X}_\bullet \oplus k((q))/(q)$$

and the total complex is given by:

$$C_\bullet := \bigoplus_{l\geq 0} \tilde{X}_{\bullet - l} q^{-l}$$

with total differential $\partial + \delta$ defined by:

$$\partial : \tilde{X}_\bullet q^{-2n} \oplus \tilde{X}_\bullet q^{-2n-1} \to \tilde{X}_\bullet q^{-2n+1} \oplus \tilde{X}_\bullet q^{-2n}$$

$$x \mapsto (qN) \oplus (q(1-t))\; x$$

$$\delta : \tilde{X}_\bullet q^{-2n} \oplus \tilde{X}_\bullet q^{-2n-1} \to \tilde{X}_\bullet q^{-2n} \oplus \tilde{X}_\bullet q^{-2n-1}$$

$$x \mapsto (b) \oplus (-b')\; x$$

We now start the construction of i, p and h. We set i to be the canonical inclusion, allowed by the identification $\mathrm{Ker}(1-t) = \mathrm{Im}(N)$. We choose p as the canonical projection, that is:

$$p : X_m \to X_m$$
$$x_m \mapsto \frac{N}{m+1} x_m$$

$$p : X_m q^{-n} \to 0 \;\; \forall n \geq 1$$

To find a possible homotopy h we proceed in steps. On $X_m q^0$ we have the identity:

$$\partial_{odd} h_0 = 1 - ip$$
$$q(1-t)h_0 = 1 - \frac{N}{m+1}$$

A simple solution is given by:

$$h_0 = -\frac{q^{-1}}{m+1} t \frac{dN}{dt}$$

We set:

$$h_{even} = h_0.$$

Then on $X_m q^{-even+2}$ we have:

$$\partial_{odd} h_{even} + h_{odd} \partial_{even} = 1$$

$$1 - \frac{N}{m+1} + h_{odd}N = 1$$

the simplest choice is then:

$$h_{odd} = \frac{q^{-1}}{m+1}$$

Finally we have to check that the choices made above are consistent with the homotopy identity on $X_m q^{-odd}$. Indeed, since h and ∂ are polynomials in t with coefficients in k, they all commute with each other, therefore:

$$\begin{aligned}\partial_{even} h_{odd} + h_{even}\partial_{odd} &= \partial_{odd} h_{even} + h_{odd}\partial_{even} \\ &= 1\end{aligned}$$

It remains to be checked that $\delta_\infty = p\delta i$. Indeed:

$$\begin{aligned}\delta_\infty &= p\delta \sum_{k\geq 0} (h\delta)^k i \\ &= p\delta i + p_0\delta_0 h_{-1}\delta \sum_{k\geq 0} (h\delta)^k i \\ &= p\delta i.\end{aligned}$$

Having constructed the homotopy retraction data, the ingredient we will be particularly interested in, is the quasi-isomorphism:

$$i_\infty = \sum_{k\geq 0} (h\delta)^k i : (C^\lambda, b) \to (\mathrm{Tot}(CC_{\bullet\bullet}), \partial + \delta)$$

We now turn to the construction of the quasi-isomorphisms:

$$\mathcal{B}_\bullet \leftrightharpoons \mathrm{Tot}(CC_{\bullet\bullet}).$$

This time we construct the following homotopy retraction data:

$$(\mathcal{B}_\bullet, b) \underset{p'}{\overset{i'}{\rightleftarrows}} (C_\bullet, \delta) \circlearrowleft h'$$

We set:

$$\begin{aligned} i' : \tilde{X}_{n-2k} u^k &\to \tilde{X}_{n-2k} q^{-2k} \\ \tilde{x}_{n-2k} u^{-k} &\mapsto \tilde{x}_{n-2k} q^{-2k}\end{aligned}$$

and

$$\begin{aligned} p' : \tilde{X}_{n-2k} q^{-2k} \oplus \tilde{X}_{n-2k+1} q^{-2k-1} &\to \tilde{X}_{n-2k} u^{-k} \\ \tilde{x}_{n-2k} q^{-2k} &\mapsto \tilde{x}_{n-2k} u^{-k}\end{aligned}$$

$$\tilde{x}_{n-2k+1}q^{-2k-1} \mapsto 0.$$

This simply means, that we view the columns of $\mathcal{B}_{\bullet\bullet}$ as the even columns of $C_{\bullet\bullet}$. The homotopy h', then, has to satisfy the following identities:

$$\begin{aligned} \delta_{odd}h'_{odd} + h'_{odd}\delta_{odd} &= 1 \\ \delta_{even}h'_{even} + h'_{even}\delta_{even} &= 0 \end{aligned}$$

Therefore we can set:

$$h'_{even} = 0.$$

Now we see that the homotopy identity on the odd columns is simply:

$$(-b')h'_{odd} + h'_{odd}(-b') = 1.$$

Hence, we can set:

$$h'_{odd} = s.$$

At this point we deform:

$$\delta \mapsto \delta + \partial$$

and via the HPL we obtain:

$$\begin{aligned} \partial_\infty &= p'\partial \sum_{k\geq 0}(h'\partial)^k i' \\ &= p'_{even}\partial_{odd} \sum_{k\geq 0}(h'_{odd}\partial_{even})^k i'_{even} \\ &= p'_{even}\partial_{odd}h'_{odd}\partial_{even}i'_{even} \\ &= p'_{even}q^2(1-t)sNi'_{even} \\ &= u(1-t)sN \\ &= uB. \end{aligned}$$

which is the desired result. That is, via the HPL we have obtained the desired quasi-isomorphisms:

$$(\mathcal{B}_\bullet, b + uB) \underset{p'_\infty}{\overset{i'_\infty}{\rightleftarrows}} (C_\bullet, \partial + \delta)$$

The quasi-isomorphism we are particularly interested in is p'_∞, which is given by:

$$\begin{aligned} p'_\infty &:= p' \sum_{k\geq 0}(\partial h')^k \\ &= p'_{even} + p'_{even}\partial_{odd}h'_{odd} \\ &= p'(1 + q(1-t)s) \end{aligned}$$

Finally the composition

$$p'_\infty i_\infty : (C^\lambda, b) \to (\mathcal{B}_\bullet, b + uB)$$

reads:

$$p'_\infty i_\infty = \sum_{k\geq 0}(1+(1-t)s\hat{h}\delta)(\hat{h}\delta)^{2k}u^{-k},$$

where $\hat{h} = qh$. There is one last complex computing cyclic homology we need to analyze, through which the map defining the homological symplectic form in (5.3.1) factorizes. This is the reduced cyclic complex

$$(\overline{\mathcal{B}}_\bullet, b + u\overline{B}) := r(\mathcal{B}_\bullet, b + uB),$$

where r is the canonical projection:

$$\begin{aligned} r : A[1] \otimes A[1]^{\otimes n} &\to A[1] \otimes (A/k)[1]^{\otimes n} \\ a_0 \otimes a_1 \otimes \cdots \otimes a_n &\mapsto a_0 \otimes \overline{a}_1 \otimes \cdots \otimes \overline{a}_n. \end{aligned}$$

In particular $\overline{e} = 0$. One can easily check that b commutes with r, while to find $\overline{B}$ we impose that r is a map of complexes:

$$\begin{aligned} rB(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= rsN(a_0 \otimes a_1 \otimes \cdots \otimes a_n) \\ &= e \otimes N(\overline{a}_0 \otimes \overline{a}_1 \otimes \cdots \otimes \overline{a}_n) \\ &= rsNr(a_0 \otimes a_1 \otimes \cdots \otimes a_n). \end{aligned}$$

Therefore:

$$\overline{B} = rsN$$

That r is a quasi-isomorphism follows from the fact that $\mathcal{B}_\bullet/\overline{\mathcal{B}}_\bullet$ is acyclic which in turn follows from a spectral sequence argument. We refer to [65] for details. In particular we obtain:

$$rp'_\infty i_\infty = \sum_{k\geq 0} r(1+s\hat{h}\delta)(\hat{h}\delta)^{2k}u^{-k}$$

It is worth noticing at this point, that as we passed from $\mathcal{B}$ to $\overline{\mathcal{B}}$ we can also proceed the other way round by adjoining a unit to the non-unital algebra A/k. In so doing one defines cyclic homology and three incarnations of the corresponding cyclic complex for arbitrary non-unital DGA's over k of characteristic zero.

5.3.5 Tadpole cancellation on the cyclic complex

In this section we start specializing to the case of affine Landau-Ginzburg models, where $A = \mathrm{Mat}(2d, \mathbb{C}[x^1, \ldots, x^n])$. What we shall describe here is the dual to the tadpole cancellation map on Hochschild cohomology (see lemma 5.2.2) independently studied in [83] and we will extend it to the reduced cyclic complex. We want to find a quasi-isomorphism $T^\vee$ between the following complexes:

$$(\tilde{X}_\bullet, b_2 + yb_1) \xrightarrow{T^\vee} (\tilde{X}_\bullet, b_2 + y^2 b_0)$$

where $b_1^1 = [D, \cdot]$ and $b_0 = 1 \otimes \partial_0$ where $\partial_0^1 = -W$. We thus resort to the tadpole cancellation map T defined in lemma 5.2.2 and propose:

$$T^\vee = 1 \otimes \tilde{T}$$

where $\tilde{T}_0^1 = yT_0^1 = -yD$ and 1 is the identity on $A[1]$. We will refer to the identity on $TA[1]$ as Id. We only have to check that it is a map of complexes. That it is a quasi-isomorphism is then obvious, because it is an isomorphism. Using lemma 5.2.2, we obtain:

$$\begin{aligned}&(b_2 + y^2 b_0)T^\vee - T^\vee(b_2 + yb_1) = \\ &(\partial_2^1 \otimes \tilde{T})(1+t) - (\partial_2^1 \otimes \mathrm{Id})(1+t)(1 \otimes \tilde{T}) + (\partial_1^1 \otimes Id)(1 \otimes \tilde{T}) = \\ &\partial_2^1(\tilde{T}_0^1 \otimes 1 - 1 \otimes \tilde{T}_0^1) \otimes \tilde{T} + \partial_1^1 \otimes \tilde{T} = 0.\end{aligned}$$

Clearly $T^\vee$ commutes with the reduction r, so in order to show that it extends to the reduced cyclic complex we only need to verify that it commutes with $\overline{B}$. Indeed:

$$[\overline{B}, T^\vee] = [rsN, (1 \otimes \tilde{T})] = r(e \otimes N(1 \otimes \tilde{T})) - (1 \otimes \tilde{T})r(e \otimes N) = 0$$

which follows from the beautiful property:

$$N(1 \otimes \tilde{T}) = \tilde{T}N$$

Therefore $T^\vee$ extends to the following quasi-isomorphism:

$$(\overline{\mathcal{B}}_\bullet, b_2 + yb_1 + u\overline{B}) \xrightarrow{T^\vee} (\overline{\mathcal{B}}_\bullet, b_2 + y^2 b_0 + u\overline{B})$$

We remark that the only subtlety one should be aware of, in order to ensure that $\tilde{T}$ is indeed a quasi-isomorphism is that the cyclic complex was defined as a direct product rather than a direct sum. That is, contrary to the sum case, in our case an element of the complex can have infinitely many non-vanishing components. Therefore, in direct analogy with the dual case of Hochschild cohomology of the second kind, where the direct product was replaced with the direct sum, an isomorphism is not required to respect the natural filtration. This fact allows for isomorphisms such as $\tilde{T}$. Notice furthermore that, contrary to the dual case, we didn't have to impose this infiniteness condition. It arose naturally from defining the cyclic complex as the dual of cyclic zero-forms.

5.3.6 The supertrace and HKR

Here we give a formula for the supertrace "str", the dual of the ($\mathbb{Z}_2$ graded) cotrace map "cotr" introduced in lemma 5.2.3 and show that it is a map of complexes between the reduced cyclic complex of the algebra A and that of the commutative algebra $R = \mathbb{C}[x^1, \dots, x^n] = Z(A)$. We will name the second complex $\overline{\mathcal{B}}_\bullet^Z$, where Z stands for center, and it is endowed (by a slight abuse of notation) with the differential $b_2 + y^2 b_0 + u\overline{B}$. We shall require of the supertrace that it preserve tensor degree and commutes with b_2

separately. Then, from the commutativity of R and the fact that it enters the complex shifted in degree by 1:

$$0 = b_2 \,\mathrm{str}(a_0 \otimes a_1) = (-1)^{\tilde{a}_0}\mathrm{str}([a, b])$$

where $[a, b] = a\cdot b - (-1)^{(\tilde{a}_0+1)(\tilde{a}_1+1)} b\cdot a$ is the graded commutator. Then, up to an irrelevant constant:

$$\mathrm{str}(a_0) = \mathrm{tr}(\sigma a_0)$$

where σ was defined in lemma 5.2.3. In order to define the action of str on higher tensor powers we shall require that it satisfies the following identity:

$$(\partial_2^1 \otimes \mathrm{Id})\mathrm{str} = \mathrm{str}(\partial_2^1 \otimes \mathrm{Id}).$$

The above fixes the formula for str completely by repeated application thereof down to tensor degree 1 to obtain:

$$\mathrm{str}(a_0 \otimes \cdots \otimes a_n) = (-1)^{\sum_{k=0}^{n}\sum_{i=0}^{k}(\tilde{a}_i+1)} \sum_{k_0,\ldots,k_n} (\sigma a_0)_{k_0 k_1} \otimes \cdots \otimes (a_n)_{k_n k_0}$$

It then follows easily that str is a map of complexes:

$$(\overline{\mathcal{B}}_\bullet, b_2 + y^2 b_0 + u\overline{B}) \xrightarrow{\ \mathrm{str}\ } (\overline{\mathcal{B}}^Z_\bullet, b_2 + y^2 b_0 + u\overline{B})$$

That it is also a quasi-isomorphism follows in the same way as in the dual case: by a spectral sequence argument using classical Morita equivalence where the differential is equal to b_2, and the dual of the Hochschild Kostant Rosenberg theorem. At this point we apply the dual ϕ of the Hochschild Kostant Rosenberg quasi-isomorphism of proposition 5.2.4 (see [98]), which maps reduced cyclic chains of R to differential forms. More precisely

$$(\overline{\mathcal{B}}^Z_\bullet, b_2 + y^2 b_0 + u\overline{B}) \xrightarrow{\ \phi\ } (\Gamma(X, \textstyle\bigwedge^\bullet T^\vee X) \otimes \mathbb{C}((u))/(u), -dW \wedge + ud).$$

The latter is given by:

$$\phi(r_0 \otimes r_1 \otimes \cdots \otimes r_n) = \frac{1}{n!} r_0 dr_1 \wedge \cdots \wedge dr_n$$

It is then immediate that ϕ is a map of complexes. In particular:

$$\begin{aligned} \phi b_2 &= 0 \\ \phi b_0 &= -dW \wedge \phi \\ \phi \overline{B} &= d\phi. \end{aligned}$$

We have almost completed the chain of quasi-isomorphisms (5.3.1) needed to define the homological symplectic form ω on the off-shell open string space. The last ingredient is what we shall call the cyclic residue, which we introduce and construct in the following section.

5.3.7 The cyclic residue

In this section we will obtain a formula for a residue defined on cyclic homology. This can be viewed as an S^1-equivariant version of the usual residue of Landau-Ginzburg models, which is defined for W having an isolated singularity. We will show that the method can be extended with no change to an arbitrary regular sequence. This generalization, in turn, corresponds to an equivariant extension of the local residue formula of [37] and can be employed in the case of heterotic $N = (2, 0)$ Landau-Ginzburg models generalizing [69]. We start with the construction of the following quasi-isomorphism:

$$(\mathrm{Jac}(W)\Omega \otimes \mathbb{C}((u))/(u), 0) \xleftarrow[\rho]{} (\Gamma(X, \textstyle\bigwedge^\bullet T^\vee X) \otimes \mathbb{C}((u))/(u), -dW \wedge + ud)$$

where $\Omega = dx^1 \wedge \cdots \wedge dx^n$, under the assumption that $\{\partial_1 W, \ldots, \partial_n W\}$ is a regular sequence. The HPL will lead us to the solution also this time, but we won't construct all of the required HRD data. In fact this would be an arduous task. We will instead pretend we have constructed one. We will then find what properties our desired map ρ should have and these will suffice to determine it completely. We start assuming we have the following HRD:

$$(\mathrm{Jac}(W)\Omega \otimes \mathbb{C}((u))/(u), 0) \underset{p}{\overset{i}{\rightleftarrows}} (\Gamma(X, \textstyle\bigwedge^\bullet T^\vee X) \otimes \mathbb{C}((u))/(u), -dW\wedge) \circlearrowleft h$$

We will require of h, that it decrease form degree by 1 and that it preserve u degree. We will denote the component of h acting on form degree k by h_k. Furthermore, p is concentrated in top degree, that is $p = p_n$. We now deform the differential of the complex on the right by ud. Then:

$$\delta_\infty = p_n(ud) \sum_{k \geq 0} u^k (hd)^k \iota = 0.$$

This follows from $d\iota = 0$. While:

$$\begin{aligned} \rho := p_\infty &= p \sum_{k \geq 0} u^k (dh)^k \\ &= p_n \sum_{k \geq 0} u^k (dh_n)^k \end{aligned}$$

So we notice that we only need to know how h acts on top degree forms in order to solve for ρ. Notice, that without loss of generality, h_k is of the form:

$$h_k = -\iota_{\partial_i} H_k^i$$

where $H_k^i \in \mathrm{End}(\Gamma(X, \bigwedge^k T^\vee X))$. Thus, in top degree n, H_n^i corresponds to a map $\hat{H}^i \in \mathrm{End}(\mathbb{C}[x^1, \ldots, x^n])$. Let $\alpha_n = f dx^1 \wedge \cdots \wedge dx^n$, then the homotopy identity in top degree is:

$$-dW \wedge h_n(\alpha_n) = (1 - ip)\alpha_n.$$

In terms of f, the above reads:

$$\sum_{i=1}^{n} \hat{H}^i(f)\partial_i W = (1-\pi)f$$

where π projects onto $\mathrm{Jac}(W)$. At this point the only computation left is to resort to an arbitrary division algorithm that will define for us $\hat{H}^i$ and π. We will spell out one such algorithm in a moment. It is worth remarking first, that the above construction can be immediately generalized for $(\partial_1 W, \dots, \partial_n W)$ replaced by an arbitrary regular sequence of arbitrary length ($\leq n$). The assumption that it be regular ensures that the Koszul complex is a resolution, namely, cohomology is concentrated in top degree. Finally we shall describe a basic division algorithm. First we need to define an ordering on $\mathbb{C}[x^1, \dots, x^n]$, for example we may choose the following:

- $x^1 > \dots > x^n$
- for two monomials $m_1 = (x^1)^{k_1} \dots (x^n)^{k_n}$, $m_2 = (x^1)^{l_1} \dots (x^n)^{l_n}$, $m_1 > m_2$ if $k_1 + \dots + k_n > l_1 + \dots + l_n$.
- if $k_1 + \dots + k_n = l_1 + \dots + l_n$ we shall set $m_1 > m_2$ if, for the smallest i for which $k_i \neq l_i$, $k_i > l_i$.

Furthermore we have to choose an ordering of the sequence of polynomials $\partial_i W$. This ordering is not the ordering defined above, we will denote it as $\succ$, and we shall choose $\partial_1 W \succ \dots \succ \partial_n W$. Then one proceeds analogously to the Euclidean algorithm for the single variable case. To divide the function f one sets as initial conditions $H^i(f) = 0$ and $\pi(f) = 0$ then starts comparing the top degree monomial of f with the top monomials of $\partial_i W$. If the former is divisible by one of the latter ones, one chooses the smallest such $i = i_{min}$, adds the quotient to $H^{i_{min}}(f)$ and redefines f by subtracting to it this quotient multiplied by $\partial_{i_{min}} W$. Or else, one redefines $\pi(f)$ and f by adding the top degree of f to the former and subtracting it to the latter. One then continues until the updated f is equal to zero. Notice that the ordering of the sequence $\partial_i W$ was implicit in the choice of i_{min}. The decomposition generally depends upon this ordering. It doesn't if and only if the sequence $\partial_i W$ is a so called Gröbner basis for the ideal it generates. This latter property however is not required in our case. We refer to [81] for a particularly simple and concise introduction to the topic of multivariate division. Finally we shall reexpress ρ for future reference:

$$\rho = p \sum_{k \geq 0} u^k (L_{\partial_i} H^i_n)^k.$$

At this point we can compose ρ with the residue:

$$(\mathrm{Jac}(W)\Omega \otimes \mathbb{C}((u))/(u), 0) \xrightarrow{\mathrm{Res}} (\mathbb{C}((u))/(u), 0)$$

to obtain the cyclic residue:

$$\langle \alpha \rangle_c := \frac{1}{(2\pi i)^n} \sum_{k \geq 0} u^k \int_\Gamma \frac{(L_{\partial_i} H^i_n)^k \alpha}{\partial_1 W \cdots \partial_n W}$$

where we view x^i as holomorphic coordinates in $\mathbb{C}^n$ and Γ is a sufficiently small n-cycle encircling the singularity of W.

5.3.8 Collecting formulas: the off-shell pairing

Having succeeded in completing the chain of quasi-isomorphisms (5.3.1) we can now compose them paying attention to keeping only the term constant in u. Then, the composition f of quasi-isomorphisms defined in section 5.3.1 reads:

$$f = \frac{1}{(2\pi i)^n} \sum_{k\geq 0} \int_\Gamma \frac{(L_{\partial_i} H^i_n)^k \phi \circ \mathrm{str}(T^\vee\, r\,(1+s\hat{h}\delta)\,(\hat{h}\delta)^{2k}\,(\cdot))}{\partial_1 W \cdots \partial_n W}$$

Happily, the thus constructed ω reduces to the Kapustin-Li paring when restricted to $H^\vee$. Indeed (5.3.2):

$$\begin{aligned}
&\omega_{ab}(ds^a ds^b)_c\Big|_{H^\vee} \\
&= \frac{1}{(2\pi i)^n} \sum_{k\geq 0} \int_\Gamma \frac{(L_{\partial_i} H^i_n)^k \phi \circ \mathrm{str}(T^\vee\, r\,(1+s\hat{h}\delta)\,(\hat{h}\delta)^{2k}\,\partial_2^1(a\otimes b))}{\partial_1 W \cdots \partial_n W}(ds^a ds^b)_c\Bigg|_{H^\vee} \\
&= \frac{1}{(2\pi i)^n} \int_\Gamma \frac{\phi \circ \mathrm{str}(T^\vee\, \partial_2^1(a\otimes b))}{\partial_1 W \cdots \partial_n W}(ds^a ds^b)_c\Bigg|_{H^\vee} \\
&= \frac{(-1)^{\binom{n}{2}}}{(2\pi i)^n} sign(a,b) \int_\Gamma \frac{\mathrm{tr}(\sigma a\cdot b\, \partial_1 D \cdots \partial_n D) dx^1\wedge\cdots\wedge dx^n}{\partial_1 W \cdots \partial_n W}(ds^a ds^b)_c\Bigg|_{H^\vee}
\end{aligned}$$

where:

$$sign(a,b) = -(-1)^{(n+1)(a+b)+a}.$$

Appendix A

A few identities of Special Geometry

A.1 The Ricci tensor

Here we shall just give the form of the Ricci tensor for an affine special Kähler manifold, as it is needed in section 3.3.4. For the sake of coherence we will compute it in special Darboux coordinates. We shall need the expression for the Christoffel symbols, that reduces to

$$\Gamma^k_{ij} = \frac{1}{2} g^{kr} \partial_r \partial_i \partial_j K$$

and in particular the following identity:

$$\begin{aligned} J^r_i \partial_r \partial_k \partial_l K &= -(\partial_k J^r_i) g_{rl} \\ &= -\left(\partial_k (g_{is} \omega^{sr})\right) g_{rl} \\ &= \partial_k \partial_i \partial_s K J^s_l \ \ . \end{aligned}$$

Equivalently the tensor C, which in special Darboux coordinates reads

$$C_{ijk} = \frac{1}{2} J^r_i \partial_r \partial_j \partial_k K \ \ ,$$

is symmetric. Moreover the fact that $\partial_i \partial_j \partial_k K$ is symmetric implies that C splits into holomorphic and anti-holomorphic parts. From the above, in particular, it follows:

$$\begin{aligned} \Gamma^k_{ki} &= \frac{1}{2} g^{kr} \partial_r \partial_k \partial_i K \\ &= -\frac{1}{2} \omega^{ks} J^r_s \partial_r \partial_k \partial_i K \\ &= -\frac{1}{2} \omega^{ks} \partial_r \partial_k \partial_s J^r_i \\ &= 0, \end{aligned}$$

hence:

$$R_{ij} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ki} + \Gamma^k_{kl} \Gamma^l_{ji} - \Gamma^k_{jl} \Gamma^l_{ki}$$

$$= \partial_k \Gamma^k_{ij} - \Gamma^k_{jl}\Gamma^l_{ki} \ .$$

The first term can be rewritten as follows:

$$\begin{aligned}
\partial_k \Gamma^k_{ij} &= \frac{1}{2}(\partial_k g^{kr})\partial_i\partial_j\partial_r K + \frac{1}{2} g^{kr}\partial_k\partial_i\partial_j\partial_r K \\
&= \frac{1}{2} g^{kr}\partial_k\partial_i\partial_j\partial_r K \\
&= -\frac{1}{2}\partial_i g^{kr}\partial_k\partial_j\partial_r K \\
&= \frac{1}{2}\omega^{ks}\partial_i\partial_s\partial_l K \omega^{lr}\partial_k\partial_j\partial_r K \\
&= 2 g^{ks} C_{isl} g^{lr} C_{kjr} \ .
\end{aligned}$$

The second term in the expression for the Ricci tensor can instead be rewritten as:

$$\begin{aligned}
-\Gamma^k_{jl}\Gamma^l_{ki} &= -\frac{1}{4} g^{kr}\partial_r\partial_j\partial_l K g^{ls}\partial_s\partial_k\partial_i K \\
&= \frac{1}{4} g^{kr} J^t_r J^u_t \partial_u\partial_j\partial_l K g^{ls}\partial_s\partial_k\partial_i K \\
&= \frac{1}{4} g^{kr} J^t_r \partial_t\partial_j\partial_u K J^u_l g^{ls}\partial_s\partial_k\partial_i K \\
&= -\frac{1}{4} g^{kr} J^t_r \partial_t\partial_j\partial_u K g^{lu} J^s_l \partial_s\partial_k\partial_i K \\
&= -g^{kr} C_{rju} g^{lu} C_{lki} \ .
\end{aligned}$$

Thus, finally:

$$R_{ij} = g^{ks} C_{isl} g^{lr} C_{kjr}.$$

A.2 The connection on the horizontal bundle

Here we analyze the connection $\tilde{\Gamma}$ defined in (3.4.15) and express it in terms of the Levi-Civita connection $\hat{\Gamma}$ of $\hat{g}$.

First we will show that $\tilde{\Gamma}$ is compatible with the metric $\tilde{g}$. We thus compute:

$$\begin{aligned}
&\left(\partial_K - u^T \tilde{\Gamma}_K \nabla_u\right) ||u||^2_{\tilde{g}} \\
&= u^T \partial_K \left(\Sigma P^T g P \Sigma^T\right) u - 2u^T \left((\Sigma P^T \Gamma_K P^T \tilde{\Sigma} + \partial_K(\Sigma P^T) P^T \tilde{\Sigma}) \Sigma P^T g P \Sigma^T \right) u \\
&= u^T \partial_K \left(\Sigma P^T g P \Sigma^T\right) u - 2u^T \left(\Sigma P^T \Gamma_K P^T g \Sigma^T + \partial_K(\Sigma P^T) P^T g P \Sigma^T\right) u \\
&= u^T \partial_K \left(\Sigma P^T g P \Sigma^T\right) u - 2u^T \left(\Sigma P^T \Gamma_K g \Sigma^T + \partial_K(\Sigma P^T) g P \Sigma^T\right) u \\
&= 0.
\end{aligned}$$

Now we shall express $\tilde{\Gamma}$ in terms of $\hat{\Gamma}$. We start by expressing the latter using the fact that $\hat{g}$ is Kähler:

$$\begin{aligned}
\hat{\Gamma}^{\gamma}_{\alpha\beta} &= \left(K\tilde{\Sigma}^T P g^{-1} P^T \tilde{\Sigma} \partial_\alpha \left(\frac{1}{K} \Sigma P^T g P \Sigma^T\right)\right)^{\gamma}_{\beta} \\
&= \left(\tilde{\Sigma}^T P g^{-1} \overline{P}^T \overline{\tilde{\Sigma}} \left(\partial_\alpha (\overline{\Sigma P}^T) g P \Sigma^T + \overline{\Sigma P}^T \partial_\alpha g P \Sigma^T + \overline{\Sigma P}^T g \partial_\alpha (P \Sigma^T)\right)\right)^{\gamma}_{\beta} \\
&\quad - \partial_\alpha \log |K| \delta^{\gamma}_{\beta} \\
&= \tilde{\Sigma}^T P g^{-1} \overline{P}^T \overline{\tilde{\Sigma}} \overline{\Sigma} \left(\partial_\alpha \overline{P}^T g P \Sigma^T\right) + \tilde{\Gamma}^{\gamma}_{\alpha\beta} - \partial_\alpha \log |K| \delta^{\gamma}_{\beta} \\
&= \tilde{\Gamma}^{\gamma}_{\alpha\beta} - \partial_\alpha \log |K| \delta^{\gamma}_{\beta}.
\end{aligned}$$

The remaining components of $\tilde{\Gamma}$ are given by:

$$\begin{aligned}
\tilde{\Gamma}^{\gamma}_{\overline{\alpha}\beta} &= \left(\partial_{\overline{\alpha}} (\Sigma P^T) P^T \tilde{\Sigma}\right)^{\gamma}_{\beta} \\
&= \Sigma^{\mu}_{\beta} \partial_{\overline{\alpha}} P^{\nu}_{\mu} \lambda^{-1} (\delta^{\gamma}_{\nu} - y^{\gamma} \delta^{0}_{\nu}) \\
&= \Sigma^{\mu}_{\beta} \partial_{\overline{\alpha}} \partial_\mu \log |K| \lambda^{-1} z^{\nu} (\delta^{\gamma}_{\nu} - y^{\gamma} \delta^{0}_{\nu}) \\
&= 0.
\end{aligned}$$

Similarly:

$$\tilde{\Gamma}^{\gamma}_{\overline{0}\beta} = 0,$$

while

$$\begin{aligned}
\tilde{\Gamma}^{\gamma}_{0\beta} &= \left(\frac{\partial}{\partial \lambda} (\Sigma P^T) P^T \tilde{\Sigma}\right)^{\gamma}_{\beta} \\
&= \frac{\partial}{\partial \lambda} \left(\lambda \delta^{\mu}_{\beta} - z^{\mu} \partial_\beta \log |K|\right) \lambda^{-1} (\delta^{\gamma}_{\mu} - y^{\gamma} \delta^{0}_{\mu}) \\
&= \lambda^{-1} \delta^{\gamma}_{\beta}.
\end{aligned}$$

Finally:

$$\begin{aligned}
\tilde{\Gamma}^{\overline{\gamma}}_{\overline{\alpha}\overline{\beta}} &= (\tilde{\Gamma}^{\gamma}_{\alpha\beta})^* \\
\tilde{\Gamma}^{\overline{\gamma}}_{\overline{0}\overline{\beta}} &= (\tilde{\Gamma}^{\gamma}_{0\beta})^*.
\end{aligned}$$

Appendix B

Details of some proofs

B.1 A spectral sequence

Here we present the spectral sequence computation used in the proof of proposition 5.2.4. First we note that $(\mathrm{Coder}(T_R), [\widehat{\partial}_0 + \widehat{\partial}_2,\, \cdot\,])$ is a mixed complex: $[\partial_0,\, \cdot\,]$ decreases tensor degree by 1 and increases tilde degree by 1, while $[\partial_2,\, \cdot\,]$ increases tensor degree by 1 and increases tilde degree by 1. In order to construct a bicomplex, we organise tensor and tilde degrees as follows

$$
\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \\
\cdots & \xrightarrow{-d_0} & C^3_{0+s} & \xrightarrow{-d_0} & C^2_{1+s} & \xrightarrow{-d_0} & C^1_{0+s} & \xrightarrow{-d_0} & C^0_{1+s} & \xrightarrow{-d_0} & \cdots \\
 & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \\
\cdots & \xrightarrow{d_0} & C^2_{1+s} & \xrightarrow{d_0} & C^1_{0+s} & \xrightarrow{d_0} & C^0_{1+s} & \xrightarrow{d_0} & 0 & \xrightarrow{d_0} & \cdots \\
 & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \\
\cdots & \xrightarrow{-d_0} & C^1_{0+s} & \xrightarrow{-d_0} & C^0_{1+s} & \xrightarrow{-d_0} & 0 & \xrightarrow{-d_0} & 0 & \xrightarrow{-d_0} & \cdots \\
 & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \\
\cdots & \xrightarrow{d_0} & C^0_{1+s} & \xrightarrow{d_0} & 0 & \xrightarrow{d_0} & 0 & \xrightarrow{d_0} & 0 & \xrightarrow{d_0} & \cdots
\end{array}
$$

where $s \in \{0, 1\}$ and C^n_m denotes the subspace of $\mathrm{Coder}(T_R)$ of tilde degree m and tensor degree n, and we write $d_0 = [\widehat{\partial}_0,\, \cdot\,]$ and $d_2 = [\widehat{\partial}_2,\, \cdot\,]$.

We choose the first page of the spectral sequence computing $[\widehat{\partial}_0 + \widehat{\partial}_2,\, \cdot\,]$-cohomology of $\mathrm{Coder}(T_R)$ to be the cohomology of d_2, which is given by replacing $C^{\bullet}_{\bullet}$ above with the image of K^1_1 of appropriate degrees. Since $K^1_1 l^1_1 = c^1_1 K^1_1$, for $s = 1$ the second page vanishes, while for $s = 0$ it is

$$
\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow d_2 & & \uparrow d_2 & & \uparrow d_2 & & \uparrow d_2 & & \\
\cdots & \xrightarrow{-d_0} & 0 & \xrightarrow{-d_0} & 0 & \xrightarrow{-d_0} & 0 & \xrightarrow{-d_0} & \mathrm{Jac}(W) & \xrightarrow{-d_0} & \cdots \\
 & & \uparrow d_2 & & \uparrow d_2 & & \uparrow d_2 & & \uparrow d_2 & & \\
\cdots & \xrightarrow{d_0} & 0 & \xrightarrow{d_0} & 0 & \xrightarrow{d_0} & \mathrm{Jac}(W) & \xrightarrow{d_0} & 0 & \xrightarrow{d_0} & \cdots \\
 & & \uparrow d_2 & & \uparrow d_2 & & \uparrow d_2 & & \uparrow d_2 & & \\
\cdots & \xrightarrow{-d_0} & 0 & \xrightarrow{-d_0} & \mathrm{Jac}(W) & \xrightarrow{-d_0} & 0 & \xrightarrow{-d_0} & 0 & \xrightarrow{-d_0} & \cdots \\
 & & \uparrow d_2 & & \uparrow d_2 & & \uparrow d_2 & & \uparrow d_2 & & \\
\cdots & \xrightarrow{d_0} & \mathrm{Jac}(W) & \xrightarrow{d_0} & 0 & \xrightarrow{d_0} & 0 & \xrightarrow{d_0} & 0 & \xrightarrow{d_0} & \cdots
\end{array}
$$

Here for degree reasons the spectral sequence degenerates, yielding the desired result.

We mention that instead of computing the Hochschild cohomology of $(R, \widehat{\partial}_0 + \widehat{\partial}_2)$ we could also compute that of $(A, \partial_1 + \partial_2)$ which by the existence of the weak isomorphism T in the proof of lemma 5.2.2 is isomorphic to $\mathrm{HH}^\bullet(R, \widehat{\partial}_0 + \widehat{\partial}_2)$[1]. The analogous spectral sequence in the case of $(A, \partial_1 + \partial_2)$ is more involved and degenerates only at the third page.

B.2 Homological perturbation for coalgebras

We continue the proof of proposition 5.2.6. We will establish (5.2.15) by induction. For this it is convenient to consider a sequence $\{a_i\}_{i \in \mathbb{N}} \subset \mathrm{Coder}(T_{A_1})$. Then we have

$$
\begin{aligned}
\Delta\Lambda_{n+1} &= \Delta\Lambda_n h a_{n+1} \\
&= ((\mathrm{id}_{T_{A_1}} + \mathcal{B}) \otimes \Lambda_n + \Lambda_n \otimes (\mathrm{id}_{T_{A_1}} + \mathcal{B}) + J \otimes \mathcal{A} + \mathcal{A} \otimes J) \\
&\quad \cdot \frac{1}{2}\Big(h \otimes (\mathrm{id}_{T_{A_1}} + \pi) + (\mathrm{id}_{T_{A_1}} + \pi) \otimes h\Big)(a_{n+1} \otimes \mathrm{id}_{T_{A_1}} + \mathrm{id}_{T_{A_1}} \otimes a_{n+1})\Delta\,.
\end{aligned}
$$

The computation naturally splits into two steps, one involving the summand $(\mathrm{id}_{T_{A_1}} + \mathcal{B}) \otimes \Lambda_n + \Lambda_n \otimes (\mathrm{id}_{T_{A_1}} + \mathcal{B})$ and the other involving the term $J \otimes \mathcal{A} + \mathcal{A} \otimes J$ in the first factor on the right-hand side above. For the first piece we have

$$
\begin{aligned}
&((\mathrm{id}_{T_{A_1}} + \mathcal{B}) \otimes \Lambda_n + \Lambda_n \otimes (\mathrm{id}_{T_{A_1}} + \mathcal{B})) \\
&\quad \cdot \Big(h \otimes \frac{1}{2}(\mathrm{id}_{T_{A_1}} + \pi) + \frac{1}{2}(\mathrm{id}_{T_{A_1}} + \pi) \otimes h)(a_{n+1} \otimes \mathrm{id}_{T_{A_1}} + \mathrm{id}_{T_{A_1}} \otimes a_{n+1}\Big) \\
&= \Big((\mathrm{id}_{T_{A_1}} + \mathcal{B}) \otimes \Lambda_n + \Lambda_n \otimes (\mathrm{id}_{T_{A_1}} + \mathcal{B})\Big)\Big(h a_{n+1} \otimes \frac{1}{2}(\mathrm{id}_{T_{A_1}} + \pi)
\end{aligned}
$$

[1] Recall the invariance of Hochschild cohomology of the second kind under weak A_∞-isomorphisms.

$$
\begin{aligned}
&+ h \otimes \frac{1}{2}(\mathrm{id}_{T_{A_1}} + \pi)a_{n+1} + (-1)^{\widetilde{a}_{n+1}}\frac{1}{2}(\mathrm{id}_{T_{A_1}} + \pi)a_{n+1} \otimes h \\
&+ \frac{1}{2}(\mathrm{id}_{T_{A_1}} + \pi) \otimes h a_{n+1}\Big) \\
= {}& I_R \otimes \mathcal{A} + I \otimes \mathcal{A} + \mathcal{A} \otimes I_L + (\mathrm{id}_{T_{A_1}} + \mathcal{B}) \otimes \Lambda_n h a_{n+1} \\
&+ \Lambda_n h a_{n+1} \otimes (\mathrm{id}_{T_{A_1}} + \mathcal{B}) + I_L \otimes \mathcal{A} + \mathcal{A} \otimes I + \mathcal{A} \otimes I_R \,,
\end{aligned}
$$

where in the last step we have used that $L_h\mathcal{B} \subset I_L$. This is true because $h(\mathrm{id}_{T_{A_1}} - \pi) = h\partial h$. We have thus proved that the first piece in the computation is of the desired form. For the second piece we have

$$
\begin{aligned}
&(J \otimes \mathcal{A} + \mathcal{A} \otimes J)\Big(h a_{n+1} \otimes \frac{1}{2}(\mathrm{id}_{T_{A_1}} + \pi) + h \otimes \frac{1}{2}(\mathrm{id}_{T_{A_1}} + \pi)a_{n+1} \\
&\quad + (-1)^{\widetilde{a}_{n+1}}\frac{1}{2}(\mathrm{id}_{T_{A_1}} + \pi)a_{n+1} \otimes h + \frac{1}{2}(\mathrm{id}_{T_{A_1}} + \pi) \otimes h a_{n+1}\Big) \\
&= I_R \otimes \mathcal{A} + I \otimes \mathcal{A} + \mathcal{A} \otimes I_L + J \otimes \mathcal{A} + \mathcal{A} \otimes J + I_L \otimes \mathcal{A} + \mathcal{A} \otimes I + \mathcal{A} \otimes I_R
\end{aligned}
$$

which is again of the desired form.

B.3 More on the L_∞-morphism M

This appendix supplements subsection 5.2.2. We give a brief explanation of why M_1^1 : $\mathrm{Coder}(T_{A_1}) \to \mathrm{Coder}(T_{A_2})$ is generically not a quasi-isomorphism. For simplicity we denote $C_{A_i} = \mathrm{Coder}(T_{A_i})$ for $i \in \{1, 2\}$ and define $g = M_1^1 = p(\,\cdot\,)i$. Consider the short exact sequence

$$
0 \longrightarrow \mathrm{Ker}(g) \underset{r}{\overset{f}{\rightleftarrows}} C_{A_1} \underset{s}{\overset{g}{\rightleftarrows}} C_{A_2} \longrightarrow 0 \tag{B.3.1}
$$

where f denotes inclusion. As a sequence of modules, (B.3.1) is split exact with left and right inverses r and s respectively given by $r(\phi)_n^1 = (\phi - \pi\phi\pi)_n^1$ and $s(\psi)_n^1 = (i\psi p)_n^1$ for $\phi \in C_{A_1}$ and $\psi \in C_{A_2}$. If we view $\mathrm{Ker}(g)$, C_{A_1} and C_{A_2} as complexes with the appropriate Hochschild differentials, $f$and g are promoted to maps of complexes and one can study the cohomology of C_{A_2}, i.e. the Hochschild cohomology of (A_2, ∂_2), by analysing the long exact sequence of cohomology groups

$$
\ldots \longrightarrow H(\mathrm{Ker}(g)) \longrightarrow H(C_{A_1}) \longrightarrow H(C_{A_2}) \overset{\delta}{\longrightarrow} H(\mathrm{Ker}(g)) \longrightarrow \ldots \tag{B.3.2}
$$

where the coboundary map δ is given by $\delta(\psi) = r([\partial_1, s(\psi)])$ for $\psi \in C_{A_2}$.

If r, s were maps of complexes, (B.3.1) would be promoted to a split exact sequence of complexes, and in that case (B.3.2) would reduce to a short exact sequence. It is readily seen however that r and s are not necessarily maps of complexes, and the following modification must be made. In general we can truncate (B.3.2) as

$$
0 \longrightarrow H(\mathrm{Ker}(g))/\mathrm{Im}(\delta) \longrightarrow H(C_{A_1}) \longrightarrow H(C_{A_2}) \longrightarrow \mathrm{Im}(\delta) \longrightarrow 0\,. \tag{B.3.3}
$$

By the split property of (B.3.1) we observe that

$$\mathrm{Im}(\delta) = (\mathrm{Im}([\partial_1,\,\cdot\,]) \cap \mathrm{Ker}(g))/\mathrm{Im}([\partial_1,\,\cdot\,])|_{\mathrm{Ker}(g)}\,,$$

and hence (B.3.3) becomes

$$0 \longrightarrow H(C_{A_1}) \cap \mathrm{Ker}(g) \longrightarrow H(C_{A_1}) \longrightarrow H(C_{A_2}) \longrightarrow \mathrm{Im}(\delta) \longrightarrow 0$$

Generically this does not simplify, in contrast to the case of the complexes of endomorphisms $\mathrm{End}(T_{A_i})$. Here the corresponding inverse maps r and s are maps of complexes and the long exact sequence reduces to

$$0 \longrightarrow H(\mathrm{End}(T_{A_1})) \cap \mathrm{Ker}(g) \longrightarrow H(\mathrm{End}(T_{A_1})) \longrightarrow H(\mathrm{End}(T_{A_2})) \longrightarrow 0\,.$$

The computation in (5.2.13) then shows that $H(\mathrm{End}(T_{A_1})) \cap \mathrm{Ker}(g) = 0$ and we recover $H(\mathrm{End}(T_{A_1})) \cong H(\mathrm{End}(T_{A_2}))$, i. e. $M_1^1 : \mathrm{End}(T_{A_1}) \to \mathrm{End}(T_{A_2})$ is a quasi-isomorphism.

Acknowledgements

If I somehow pulled through in these years, it is thanks to the few beautiful people who have constantly supported me. I am grateful to Ilka Brunner for her gentle guidance in the course of these years, and for the many insights she has given me in physics. Furthermore, I will always be indebted to her for having given me reason time and again to rise, all the more so when I least deserved it. My gratitude goes also to Nils Carqueville, a true friend and a mentor in physics as well as in more important matters of life. I wish to thank Andres Collinucci for his warm affection and all the physics discussions I have greatly benefitted from. We now come to the reservoir of patience and wisdom embodied by my great friend Patrick Böhl, which I am afraid I have drawn too much from in these years. I am grateful to have met my very dear friend Jad Halimeh, a bundle of talent, who I wish will see happier times in the near future. I also wish to thank my fellow "journey man" Daniel Plencner for the many interesting discussions, and for whom I foresee a brilliant future in physics. I wish to thank all the further friends at LMU and related, I have had the pleasure of knowing and with whom I have had numerous interesting discussions: Jawad A Deo, Stephan Bauer, Riccardo Borsato, Marin Bukov, Alessandra Cagnazzo, Andrei Constantin, Alexandru Cudreasov, Vittoria Demozzi, Cristiano Germani, Leonhard Horstmeyer, Daniel Krefl, Florian Kurz, Noppadol Mekareeya, Adrian Mertens, Karapet Mkrtchyan, Sebastian Novak, Brian Padden, Cristina Pop, Andrea Prudenziati, Nitin Rughonauth, Raffaele Savelli, Christian Schilling, Maximilian Schmidt Sommerfeld, Leonardo Segato, Sergei Winitzki. Of these I must apologize to Jawad Adeo and Nitin Rughonauth in particular for not being there for them much too often. I also wish to thank Calin Lazaroiu for the few, but very instructive conversations.

Finally, as for any other and more important matter of life, I wish to thank my long time friends and my family; above all my mother and Orsolya Adam.

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