Essays on Consumer Search

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In memory of Jacob, the family first economist.

Dankesagung

“מִכָּל מְלַמְּדַי הִשְׂכַּלְתִּי„
בן זומא, פרק ד' א'

”I have learned from all those who taught me”
Ben Zoma, ch. 4, 1.

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Chapter 1

Introduction

”Even the longest journey must begin with a single step.”
Lao-tzu, from ’The Way of Lao-tzu’

1.1 Overview

In many store that consumers come in there is a fixed price on the price sign. Sellers set prices, leaving consumers little possibility to haggle over them. However, consumers have the choice whether to purchase the item, or not to. For some items it is impossible to avoid purchase, and they have an inelastic demand. Example for such would be an electricity provider, bank account, a car if you have a job requiring one and more. Still, even in such cases consumers can decide to decline an offer and decide to purchase the item elsewhere. Thus, when an item must be purchased, the realistic setting is for seller to pose prices and consumers to decide where they buy. To deal with such situations consumer search models were created, where sellers set fixed prices and consumers search for a satisfactory price.

This dissertation deals, mainly, with one of the most popular search models - the Stahl Search model introduced by Stahl [39]. Its wide literature is reviewed in the next section, including profound empirical evidence, applications and more. Here, three research questions are introduced and shortly discussed, involving this model.

Firstly, the question of heterogeneous sellers is addressed. So far, the literature was mainly considering identical sellers, each of the same size and same influence. However, in reality sellers have different size, influence and this may lead to different behavior. The first of research project chapter addresses this topic, and characterizes equilibria when sellers are heterogeneous, under the reserve price assumption. As a side result, asymmetric equilibria of the original model are found, where sellers do not have to use same strategy.

The chapter afterwards deals with the reserve price assumption. In most search models there is a common assumption on search, where consumers have a reserve price and accept any offer below it. This is a simplifying assumption and in the moment equilibria are asymmetric may not hold. In the research project a trembling hand condition is provided, which ensures existence of a reserve price in all equilibria. Therefore, the results of the previous chapter characterize all equilibria with the relevant trembling hand condition.

Then a chapter about minimum price comes. Formally - what happens when a legislator forbids prices below some exogenously given threshold. An example is the Scottish legislator policy on Alcohol prices. Results are surprising, and suggest that under certain conditions expected price offered by sellers would significantly drop, and sellers would not exploit this law to raise prices. The main reason driving this result is information available to consumers. If the limiting price is set sufficiently high it leaves positive profits for sellers. Then, consumers know that selling at the threshold price leaves positive profit for seller and search further when prices observed are significantly higher. As a result, a pure equilibrium emerges where all sellers select the threshold price purely.

Note that the two chapters dealing with reserve price and minimum price are written for the original Stahl model. However, most of the results hold also when sellers have heterogeneous size as introduced in the extension to the Stahl model.

Lastly, an additional model is discussed in the dissertation - Congestion Games. Prices may be quantity dependent, and sellers set prices depending on the amount of consumers purchasing the good. Each seller has a function telling the price dependent on the number of consumers, exogenously given. When we wish to see how the (finite number of) consumers spread out among sellers we apply the popular model of congestion games. It is used in many economic situations, and an easy way to find equilibria for congestion games is a topic of interest. The result provided serves as an important step and characterizes the congestion games where equiliria and greedy profiles are the same, which makes an equilibrium search much faster and easier.

1.2 Consumer Search Literature

Consumer search is a wide field with a large literature. A good point to start a review of the field can be put down to the Diamond paradox in Diamond [19]. The paradox predicts a market failure, and prices rising to infinity. The model is one similar to many consumer search models. Time is discrete. Consumers search sequentially in one store after another, observing one store each period. They can be ether satisfied with the price and purchase, or search further. Sellers set their prices at the start of each period. Sellers can meet demand and have zero production costs.

If a small search cost is introduced, having the cheapest price is not profitable for a seller. She can raise the price by search cost and still all visiting consumer would decide to purchase the item at her store. As in any finite pricing setting prices are bound above zero, lowest price exists. Therefore, in any equilibrium prices would rise to infinity. Note that it makes no difference whether the lowest price is in support of a single seller or of multiple sellers. This is since adding up to the search cost to seller price would still keep consumers happy. As a result prices would rise to infinity and a complete market failure would be faced.

In the real world search frictions exist in many markets and still prices are finite and transactions take place. Thus diamond paradox gave a great development to consumer search literature, where many attempts were made to solve it. Those led to a multitude of consumer search models starting from the 80s.

One of the first models was introduced by Varian in [42]. The main change from the original paradox is a single shot search. Consumers can search only in a single store, and will purchase there only if price is sufficiently low. Unsatisfied consumers leave the game without purchase. Varian model solves the race to infinity problem. Additionally Varian uses a distinction between two types of consumers - informed and uninformed. The informed consumers know the prices in all stores whereas uninformed can only sample prices. Citing Varian: "One might think of a model where stores advertise their sale prices in the weekly newspaper. Informed consumers read the newspaper and uninformed consumers do not". Then, informed consumers purchase the item at lowest offered store, whereas uninformed consumers sample a single store, if the price of the item in that store is less than some reserve price P_M. If it is above - no purchase will be made and consumer will leave the game with no item. For the model exists a symmetric equilibrium, where all sellers charge below P_M, and use a continuous mixed distribution over prices, without empty intervals or mass points.

Additional work by Baye et al. [7], shows that Varian model of sales with more than two firms have two types of equilibria: a unique symmetric equilibrium, and a continuum of asymmetric equilibria. In contrast, the 2-firm game has a unique equilibrium that is symmetric. For the n-firm case the asymmetric equilibria, with mixed strategies can be ranked by first-order stochastic dominance. This enables one to rule out asymmetric equilibria on economic grounds by constructing a Meta game in which both firms and consumers are players. The unique sub game perfect equilibrium of this Meta game is symmetric. From here follows that asymmetric equilibria yield lower expected utility for sellers, and as they choose the strategy - a symmetric equilibrium is better. Therefore, asymmetric equilibria can be ignored in the Varian model.

The concept in Baye et al. [7] was applied also for other popular games. For example, Baye and Morgan in [8] found a family of additional equilibria to the famous Bertrand game. There, consumers observe all prices and decide where to purchase the item. Classical result implies a race to the bottom and zero profit for the sellers. However, additional equilibria exist, where prices can be unbounded. In such equilibrium any finite payoff vector can be obtained as equilibrium. Unbounded prices seem somewhat artificial, but this paper has an important point. One should look for asymmetric equilibria in search models. Those may reveal additional behavior which can occur in the real world.

A different attempt to deal with the Diamond paradox was done by Burdett and Judd [11]. There non-sequential and noisy search model is discussed. Non sequential implies that several prices are observed at once, when the number observed can be random. Then, a consumer can decide whether to buy or to perform additional non sequential search. This idea appears in many additional papers, for example Burdett and Smith [12] or Carlson and McAfee [14].

An additional type of search used by Burdett and Judd [11] is noisy search. There, not an exact price was observed, but rather a noisy signal. Based on price plus some noise searcher needs to decide whether to purchase the item or not. Since expected terms were used, such an addition does not have too significant impact on the results.

Carlson and McAfee in [14] discuss a discrete model, where consumers have heterogeneous search costs, and firms have heterogeneous production costs. Again, consumers know the prices on the market, but do not know who offers which price. After observing a price, consumers decide whether to search on or not. Sellers set prices at start, and take into consideration consumer search costs.

The models mentioned above deal also with an additional phenomenon - price dispersion. Varian in [42], Burdett and Judd [11] and Carlson and McAfee [14] all find that homogeneous goods are sold for different prices. Even when production costs are the same, price dispersion would occur. This is a very interesting topic, and is discussed widely in the literature. For example, Bazucs and Imre in [6] compare milk prices in Hungarian supermarkets and find price differences. In Table 2, average one liter milk prices per network vary between 184 and 240 HUF. When looking at a banking sector, as done for example by Martin and Saurina in [31], in Table 4 standard deviation of 1.7 on average of 16 was observed on Loan products interest and 1.58 standard deviation on average of 9.54 on deposit interest. Those suggest on significant variation of prices on same products, as milk, or banking products, which are homogeneous products.

Already in Carlson and McAfee [14] some very important insights appear. Firstly, when consumers have a reserve price, no seller would charge a price above it. With a price offer above reserve price consumers would move on, purchasing elsewhere, and only after all the market is searched and all sellers offer too high prices consumers may return. This does not occur in equilibrium as in such case a Bertrand run to the bottom would occur.

An additional point appearing here is the Bayesian structure of equilibrium. Consumers know the strategies played by sellers, but do not know which seller chose which strategy. Additionally, Burdett and Judd [11] adds an important insight - costs are fully rolled down to consumers, and therefore, one can normalize the lowest production cost to zero.

Here an important insight noted by Rothschild in [37] comes in. Symmetry plays an important role in existence of a reserve price. When sellers use same mixed strategy an observed price does not provide any additional information regarding prices in other stores. However, if sellers use different strategies, some observed prices may reveal information regarding prices in additional stores. In such cases additional equilibria, without a reserve price may exist. Unfortunately, such equilibria tend to be complex and fall outside the search model literature. This is to point out that additional equilibria may exist, that fall beyond the scope here, but also tend to be disregarded by the literature. One of the few papers dealing with equilibria without reserve price is a working paper by Olszewski and Wolinsky [35]. There, a search setting with two parameters is used, and only partial information is available. In some cases under this setting equilibria without a reserve price may arise. Therefore, even if disregarded by the literature, one should always be aware that additional equilibria may exist.

An additional insight regarding knowledge is noted by Stahl in [40]. There the 'Stackelberg paradigm' and 'Nash Paradigm' are noted. The conventions differ in whether consumers know the 'market distribution' of actual prices being charged but do not know which store is charging which price. For example, if there are N stores whose symmetric mixed-strategy is to draw a random price from a probability distribution F(p), these N independent random draws give N actual prices, and the frequency distribution of these actual prices is denoted $S(p)$. Under the Nash paradigm only $F(p)$ is known to consumers and under Stackelberg $S(p)$ is known to consumers. Note that $S(p)$ provides additional information to consumers, which was not known to sellers when pricing strategy was chosen. Table 1 in Stahl [40] shows that both conventions are applied in literature. However, Stahl in [40] provides several claims in favor of the Nash Paradigm - for example, existence of equilibria and learning. Notably, the Stackelberg paradigm depends on realizations of prices. This implies different games and different behavior depending on a realization of a mixed strategy. Therefore, such paradigm involves knowledge about the randomization mechanism of a seller, and its outcome, which is a quite unrealistic assumption. Moreover, every time the game is played different realizations occur and different knowledge is available to consumers, making a general statement much more complicated, if not impossible.

The model by Watanabe in [45] discusses a different search model. There, each seller has a limited capacity, and if the turnout is too high, some consumers would not get the good. Therefore, trying to purchase at a lower price may end up in failure. The search is directed which can be described as a simple two-stage price posting game. In the first stage, sellers simultaneously post a price they are willing to sell at, given the capacity. Observing the posted prices and capacities, consumers decide simultaneously which supplier to visit in the second stage. If too many consumers show up at a seller - each of them gets served with the corresponding probability. Two types of sellers are in the market - some with capacity of one and some with a larger exogenously given capacity k_m. Exists a symmetric equilibrium where each seller with high capacity posts the identical price p_m and each single capacity seller posts the identical price p_f, where $p_m > p_f$. Additionally, all consumers use the identical mixing strategies. This is a different type of search to the one introduced before, as each consumer knows all prices. The friction rises due to limited capacity, consumer does not know whether one will be served or not. There is an important point to take from this model, which will repeat itself quite often. Notably Watanabe [45] states that for his model larger sellers charge higher price in equilibrium.

One of the most popular search models was introduced by Stahl in [39]. It extends the Varian model [42] and allows consumers to search in multiple stores. The model has a finite number of sellers and two types of consumers - informed (a share of μ) and uninformed. Informed consumers receive price information from all stores, whereas the uninformed consumers perform a sequential search. First store is visited for free, and any additional store costs c to visit, with perfect recall. This means consumers can always return to a previous offer. Mass of consumers is normalized to one, implying many consumers, where each one of them has no strategic significance. Prices are finite and positive, and consumers are symmetric. Utility for consumer is some finite valuation minus item price and minus search costs. Utility for seller is the expected quantity sold times price. Note that Bertrand setting is obtained when $\mu = 1$ and diamond result obtained when $\mu = 0$.

One of the nice properties of the Stahl model, as introduced in Stahl [39], is the unique symmetric equilibrium. Searchers have a reserve price, denote P_M, and sellers use a continuous distribution on some interval (P_L, P_M). Even when stores number increases to infinity degenerate distributions are not approached. Given the seller number (N) and the shoppers fraction (μ). The price distribution function in equilibrium is given by:

$$F(p, P_M) = 1 - \left(\frac{1-\mu}{N\mu} \left(\frac{P_M}{p} - 1 \right) \right)^{N-1}$$

where N is the number of sellers. Note that prices are always weakly below reserve price, and consumers always buy at the first store visited. It is easy to derive that $c + E(F) = P_M$, as otherwise a different reserve price would be chosen by consumers. Now it can be possible to calculate P_M as a function of c, given μ and N, explicitly or at least numerically.

In this article some asymptotic results are provided. As μ approaches zero the setting is closer to the Diamond setting (here a threshold price would be charged), as μ approaches one prices drop to zero, approaching Bertrand setting. In between the change is monotone, implying lower μ will cause a higher expected and reserve price. Similarly, when search costs are approaches zero Bertrand setting would be approximated, since the pricing information is less and less costly to obtain to consumers, and when search price increase to infinity Diamond setting would be approximated. Again, the changes are monotone, and higher search costs imply higher expected and reserve prices. Additionally, as the number of stores increases mass concentrates on the reserve price, as probability of being the cheapest decreases. This leads to increase in the reserve price. From some parameter dependent N^* the change will be monotone, and under some condition happens for any number of stores.

Stahl model was also put down for many empirical tests. For example, Janssen et al. [27] reveal that the model introduced by Stahl in [39] perform very well and predicts correctly the pricing model of 86 out of 87 tested products. The products tested are online products, sold on the site *shopper.com*, and the pricing strategy is a corresponding distribution from the Stahl model.

In an additional empirical paper, by Baye et al. [5] empirically suggests the existence of the two consumer types predicted by this model. Therefore, this paper will concentrate on the Stahl search model. The article finds that a 60% increase in clicks for sellers with lowest price, even above second lowest. Since not all consumers purchase there, and a significant mass goes to such unique seller, suggests that exist two types of consumers, one informed regarding lowest price and one not. This is in line with the structure of the Stahl model, and serves as an additional point in its favor.

The Stahl model is dealt extensively in the literature, and is a very popular model. Numerous extensions to the Stahl Model were introduced, and the various extensions are dealing with nearly every aspect of the model. Among other aspects mentioned are search scope, knowledge available to consumers and consumer heterogeneity.

Among those are introducing heterogeneous searchers. Example for such extensions are Chen and Zhang [15] and Stahl [40], where the searchers have different cost for each additional store they visit. However, the search cost is heterogeneous.

In [40] some consumers have zero search cost. Other consumers have cost according to some bounded away from zero distribution, $\mu(c)$. Characteristics for NE are provided, and in some cases even a pure NE exist. The model concentrates on seller symmetric NE, and makes a distinction on whether there is a mass of zero search cost consumers or not, and whether the search costs are bounded away from zero. In any NE there are some common characteristics, such as no pure NE and no mass points. In addition no seller charges a price above the reserve price of a consumer with highest search cost. If the distribution of search cost is atomless, than exists a pure NE where shops charge the reserve price purely.

Chen and Zhang in [15] have introduced a mixture between the Varian [42] and Stahl [39] models. There, searchers are of two types - the 'Varian' type, who searches only once, and the Stahl type, who can search in multiple stores. Interestingly, if the Varian type searchers have a similar evaluation to the Stahl type searchers then the equilibrium is similar to the Stahl model. However, if it differs by much this would have large impact on the equilibrium. For example, Stahl type searchers would find it optimal to search in several stores, and prices offered would be higher than in the Stahl model.

Another extension introduced advertisement costs, as discussed, for example by Chioveanu and Zhou [16]. There, firms have different means to present prices, and confuse consumers. The equilibrium suggests similar price dispersion and additional frame dispersion. Some sellers would be very confusing and some very straightforward.

Two additional assumptions of the Stahl model were examined. Literature has an extension model where already the first price is costly, such as Janssen et al. [28]. Such an extension would leave some of the searchers completely out of the market, as those would not search even in a single store. The equilibrium would be adjusted to reflect the change in searchers share, and would be similar to the original model equilibrium. For example, if originally μ was one to four, and a third of the searchers are out of the market, then μ would increase to one third. In all other aspects the equilibria and results remains the same.

Another extension is to limit the possibility to freely return to previously visited stores, such as Janssen and Parakhonyak [29]. Recall that in the original model consumers had free recall, and could always return to the cheapest price observed. This extension has little effect on the model, since in case of a reserve price all sellers would offer prices weakly below it. Therefore, all consumers would purchase at the first store they visit, and perfect recall does not play a significant role in the results.

The literature has discussion regarding the sequential search in the model and looks also at non-sequential search, for example in Janssen and Morags-Gonzales [26]. There, Stahl model is extending allowing consumers can choose the intensity of search, affecting the number of prices observed on each search. The results show that only three types of consumer strategies exist, with low, intermediate and high intensity search. The low intensity implies searching for one price or not searching at all, similar to Janssen et al. [28]. Intermediate strategy is searching once, similar to Varian model [42], and high - for two prices. Pricing strategies are atomless, and increasing in the number of firms.

Unknown production cost is introduced in Janssen et al. [30]. In the original Stahl model it was assumed zero. When the production cost is constant but unknown to consumers. The equilibrium consumer search rule is history dependent. In particular, the reservation price in each search round depends on the prices already observed in previous rounds as consumers update beliefs about the production cost realization on the basis of their price observations. Moreover, if a consumer observes her reservation price in the first search round and, being indifferent, continues to search, then her next rounds' reservation prices are higher than her first round reservation price. Additionally, in equilibrium with a reserve price no seller would set a price above this first round reserve price, and therefore, consumers will search in a single store. Some conditions are provided for existence of such equilibria, which include large number of sellers or few shoppers, in addition to large search costs or small uncertainty for production costs. Lastly, empirical evidence form gasoline sales in Austria are provided which support the results.

Most assumptions of the model introduced by Stahl in [39] are discussed extensively, except one main assumption, used extensively in the literature. This is the focus on symmetric equilibria, where all sellers select an identical strategy. One of reasons is the mathematical complexity: Carlson and McAfee [14] and Rothshild [37] showed that in symmetric equilibria consumer reserve price must exist, and in asymmetric ones it does not have to exist. Reserve price assumption is common in the literature, and therefore, the paper considers only NE with reserve price, yet justifies the rationality behind it. Nevertheless, one should note that additional Equilibria without reserve price may exist, and fall beyond the scope here.

One important assumption is prevailing in much of the literature involving the Stahl model. Namely, those sellers are homogeneous. When one wishes to look beyond, this has several impacts. Firstly, searcher will encounter one of the bigger chains more often and less common stores will be visited by fewer searchers. This effect is less encountered online, yet many products are still sold in regular stores. Some effects may arise when sellers have different sizes, and in the real world seller chains have different sizes, as depicted in figure 2.1.

Among the few papers in this field is Burdett and Smith [12], where only a single large firm exists and all other are single store sellers, yet the search was not sequential. Already there it is noted that the larger firm charges a higher price in expected terms. The model is quite similar to the Stahl model, but has some differences. Firstly, both consumers and sellers are continuum of mass one. One single firm is larger than all others (proportion θ of the continuum), and all other have a single store. All outlets of the larger firm have the same price. Consumers search, and observe ether one or two prices, based on nature random draw. An additional point to note is no recall and no discounting. The results include a consumer reserve price, where all small sellers use the same continuous distribution whereas the larger seller selects reserve price purely. Comparative statics state that observing more prices reduces the reserve price and prices in distribution. Increasing the size of largest firm increases prices, and first order stochastic dominance of the new, higher θ configuration over the old one.

An additional paper is Astorne-Figari and A. Yankelevich [2], where only a model with 2 sellers is concerned. There, Stahl model setting with two sellers, denote A and B, is concerned, with a small addition. Searchers are of two types, those who live near seller A, and those who live near seller B. The share of corresponding searchers is unequal, and, without loss of generality, more searchers live near seller A than those who live near seller B. The main impact is heterogeneous distribution of searchers. It is no longer the case that half of them visit each seller, but an exogenously given share. The results suggest that seller A, the one with 'bigger' share of searchers would sell at an expected higher price. This is since both strategies are mixed, and the distribution of seller A first order stochastic dominates the one of seller B. Additionally, the equilibrium pricing strategy would be continuous, except at the reserve price, where seller A would have a mass point. Not surprisingly, the lowest price in support of both sellers is the same. Market share decreases with the 'near' searcher share and prices increase with an increase in total share of searchers. Those results are in line with the Stahl model, adding an important aspect - also in this model the bigger a seller is, the higher a price would be.

An additional empirical point is relevant. The data collected by Bazucs and Imre in [6] suggest an additional correlation between prices and the size of a chain. In January 2012, the corresponding supermarkets stores number was collected from company profiles, and though it is not the relevant period for the research conducted, it provides a viable signal to the number of stores each chain had during the relevant period. Table 2 in Bazucs and Imre [6], together with this information reveal that some correlation exist between the stores number and the average price. Clearly, some additional effects may be in place here, yet due to such a big impact, correlation between chain size and price seem to be also relevant.

The average milk price, together with relevant stores number, is as follows:

Chain Name	Stores Number in Hungary	Avg. price of milk
InterSpar*	100-200	182 HUF
Cora	Below 20	198 HUF
Match	Below 20	200 HUF
Tesco	200-400	205 HUF
Auchan	Below 20	211 HUF
CBA	Above 500	213 HUF
Plus*	100-200	230 HUF
COOP	Above 500	240 HUF

Please note that after the data for the article by Bazucs and Imre in [6] was collected, and before Jan. 2012, the Plus chain was purchased by Interspar [1]. The relevant stores were not added to the stores number of InterSpar.

Another example is from Drogerie stores in Germany. GKL[2] research found that Schlecker

[1]See http://www.portfolio.hu/en/tool/print/2/14361

[2]Conducted in Jan. 2012, comparing prices in 1700 stores in Germany. Media Report for example here: *http://www.handelsblatt.com/unternehmen/handel-dienstleister/gkl-analyse-schlecker-deutlich-teurer-als-die-konkurrenz/6254086.html*

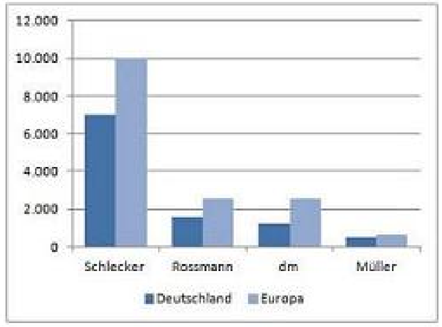

Figure 1.1: Stores number of various Drugstores

is 10-20% more expensive than competitors. Additionally, the number of stores in January 2012 of the various chains in Germany and Europe is given in figure 1.1. Again we see a tendency that the largest chain is the more expensive one. Again, additional factors may have an effect here, but it seems that chain size plays a role when determining prices.

Combining the empirical evidence, it seems that there should be some positive correlation between chain size and price here too. Additionally, some theoretic evidence, for example Astorne-Figari and Yankelevich [2], Burdett and Smith [12] and Watanabe [45] suggest that larger seller would charge higher prices. Yet, an extension to the Stahl model with heterogeneous sellers existing in the literature is surprisingly very limited. Since this model is very popular, it seems an important extension, and should provide with insight how chain goods are priced.

An additional aspect dealt with in the dissertation is minimum pricing. How would the results change if a minimum price was introduced by a legislator. This approach was recently applied in Scotland for alcoholic beverages, as an alternative to taxation. We wish to check whether in more general settings it would be a good idea. Generally, as suggested by Stockwell et al. [41], and Wagenaar et al. [46] such a step should increase prices. There, prices are set at levels higher than market prices of the goods. However, our findings suggest that when the minimum price is set at a market price level, forcing only several sellers to increase their price, rather than setting a bound significantly higher than all market prices, it will be the case that prices drop.

Chapter 2

Heterogeneous Sellers

The Stahl model is one of the most applied consumer search models, with many applications and empirical background. The paper explores an extension where sellers have asymmetries, which is mostly excluded by the literature. Sellers with heterogeneous stores number are introduced, reflecting a typical market structure. As in the original Stahl model, a market consists of several sellers, and consumers, where some face a cost when sequentially searching. The paper shows that no symmetric NE exists in the extension. Additional results suggest that smallest sellers will be the ones offering lowest prices, in line with several real world examples provided in the paper. However, profits remain in most cases fixed per store, making a larger firm more profitable, yet with lower sold quantity. The findings suggest that on some level price dispersion will still exist, together with some level of price stickiness, both observed in reality.

Keywords: Sequential Consumer Search, Oligopoly, Asymmetric NE

JEL Classification Numbers: D43, D83, L13.

2.1 Introduction

Empirical studies, such as Bazucs and Imre [6] or Martin et al. [31], have established that significant price dispersion exists even for homogeneous goods. As the literature suggests, this effect is observed in many market structures and is persistent. One of the explanations for this phenomenon is that consumers search for the cheapest price. Since searching is costly, consumers may settle down for a slightly higher price, explaining price dispersion. In the literature many papers deal with search models, for example

Burdett and Judd [11], Carlson and McAfee [14], Stahl [39], Varian [42] and Watanabe [45]. These models were developed originally in order to provide a solution to the Diamond Paradox [19], which predicted a complete market failure. The search models vary in the scope, the length, the stopping condition or the information revealed during the consumer search. Additional Empirical studies, for example [27], reveal that the model introduced by Stahl in [39] performs very well and predicts correctly the pricing model of 86 out of 87 tested products. Moreover, [5] empirically suggests the existence of the two consumer types predicted by this model. Therefore, this paper will concentrate on the Stahl search model.

An additional Phenomenon that can be observed, for example in Bazucs and Imre [6] and Watanabe [45], is a possible correlation between the price offered by a seller and the number of stores she has. Namely, the more branches a seller has the higher will be the price offered. Despite the fact that the Stahl Model has a variety of extensions, the literature dealing with asymmetries among sellers is not large. This is a very important extension, as in the real world the number of stores a seller has can vary, for example - see table 2.1. Among the few papers in this field is Astorne-Figari and Yankelevitsch [2], where only a model with 2 sellers is concerned. An additional paper is Burdett and Smith [12], where only a single large firm exists and all other are single store sellers. Already in those papers it is noted that the larger firm charges a higher price in expected terms.

This paper investigates whether this is true in a more general setting than in Astorne-Figari and Yankelevitsch [2] and Burdett and Smith [12]. Here the Stahl model is extended, and each seller has a predefined, seller-specific store number. This reflects a more general setting which is closer to reality. Since, generally, sellers vary in size (or popularity), this aspect should be taken into consideration. For example, when comparing discounters, they typically have a different number of stores, as depicted in table 2.1. The site *mysupermarket.co.uk*, compares prices of various supermarkets in UK, suggesting that within-chain prices are similar. That is, a given product costs the same in all supermarkets of chain X. Therefore, different number of stores can play a role in search model setting when prices are not varying within-chain.

The rest of the setting is similar to the original Stahl model [39]. Sellers set the same price for all stores (e.g.: Bank offers for saving accounts). The consumers search sequentially and uniformly among stores, and in this setting it can be different than searching uniformly among sellers. This implies that there is a higher chance for a consumer to turn up at a store belonging to the larger seller. Additionally, if a searcher is unsatisfied with a price, she would refrain from visiting any additional store of the previously visited seller. Similar setting was used by Astorne-Figari and Yankelevitsch [2] and Burdett and Smith [12] in the limited versions of the extension.

An important distinction needs to be made regarding the type of the market. When a seller has several stores, is the pricing strategy done at the store level (store good) or at

Aldi	Netto	Lidl	Norma	Penny	Tengelmann
27	49	36	17	50	89

Figure 2.1: Number of discounter stores in Munich, according to kaufda.de, 5.2011

the headquarters level (chain good). An example for a store good is gasoline sold on fuel stations. The name of the chain usually does not imply that the same price would be encountered on all fuel stations of a given chain. Examples for chain good are banks, or cellphone networks. When one looks for a mortgage, usually a bank offer would be the same in all branches of a given bank. For store goods the original Stahl model is relevant, as observing a price in one store does not imply on prices in all other stores, which is not the case with chain goods, which reveal the price in all stores of a given chain. This can happen, for example, due to binding advertising, or to a strategic decision of the chain. In a store good the number of stores plays no role for the price, and there is no connection between price and chain. In chain good the chain determines the price for all chain stores, making the size of a chain an important factor. The extension in this paper is deals with chain goods, though trough examples provided in the paper, some effects of chain size hold also in store goods case.

The first thing one notice when discussing the extended Stahl model (with heterogeneously sized sellers) is a lack of a symmetric equilibrium, where all sellers use the same strategy. The original Stahl model has a unique symmetric Equilibrium, as shown in Stahl [39] and the literature does not go far beyond it. For comparison, in the Varian search model (see [42]), it is shown in Baye et al. [7] that there are asymmetric equilibria, but those can be ignored. In the Stahl model there might be additional equilibria when different settings are considered. For example, it is shown in Baye and Morgan [8] that one can receive additional equilibria in commonly known games, when the scope is broadened.

2.1.1 Results overview

The main contribution of this paper is introducing the general extension for heterogeneous sellers in the Stahl model, and finding the relevant equilibria. Surprisingly, the extra complexity does not make the equilibrium complex. For example, in the case of a unique smaller seller and unique second smallest seller the results are in line with Astorne-Figari and Yankelevitsch [2], where all but the two smallest sellers select reserve price purely. An additional contribution is dealing with asymmetric equilibria of the model, under a consumer reserve price. Lastly, the paper supports the insight that larger chains, and actually all but smallest chains, would have higher prices for goods.

Results here imply that when there are at least two smallest sellers, all sellers except the smallest sellers select the reserve price purely. The remaining sellers have a similar equilibrium to the original Stahl model, but with a lower portion of uninformed consumers (all those who visit one of the smallest sellers). This extends the result in Burdett and Smith [12], and shows that the lowest price will be obtained in one of the smaller chains. Moreover, in all equilibria found here all consumers buy at the first store they visit and no seller will ever set a price above the reserve price. Additional characteristic of the NE is that expected profits for all sellers are equal to constant times the chain size (store number). An additional point to note is that the equilibria found here extend the known symmetric equilibrium of the original Stahl model and describe its asymmetric equilibria. Note that unlike in the original model, here there is a multitude of equilibria.

In case of a single smallest seller the equilibrium structure slightly changes. Now, it is the smallest and 'second smallest' sellers (the ones with the smallest share except the smallest seller) who mix over the entire interval. Second smallest sellers have a mass point at the reserve price. Note that at least one of the second smallest seller mixes. All larger sellers select the reserve price purely. Note that here the smallest seller has larger profits per store than all other sellers, and all sellers but her have the same lower profit per store. The structure and results are similar to the model introduced in Astorne-Figari and Yankelevich [2], where a model with two heterogeneous sized sellers was discussed.

The Stahl model is dealt extensively in the literature, and is a very popular model. Numerous extensions to the Stahl Model have been introduced, and the various extensions are dealing with nearly every aspect of the model. Among those are introducing heterogeneous searchers. Example for such extensions are Chen and Zhang [15] and Stahl [40], where the searchers have different cost for each additional store they visit. They can differ by the search scope, as discussed in Astorne-Figari and Yankelevich [2], where some stores are near, and thus will be searched first. Another extension introduced advertisement costs, as discussed, for example by Chioveanu and Zhou [16]. There are also models where already the first price is costly, such as Janssen [28], or no possibility to freely return to previously visited store, such as Janssen and Parakhonyak [29]. The literature discusses the sequential search in the model and looks also at non-sequential search, for example in Janssen and Morags-Gonzales [26], or the unknown production cost as shown in Janssen et al. [30]. Most assumptions of the model introduced by Stahl in [39] are discussed extensively, except one main assumption, used extensively in the literature. This is the focus on symmetric equilibria, where all sellers select an identical strategy. One of reasons is the mathematical complexity: Carlson and Mcafee [14] and Rothshild [37] showed that in symmetric equilibria consumer reserve price must exist, and in asymmetric ones it does not have to exist. Reserve price assumption is common in the literature, and therefore, the paper considers only NE with reserve price, yet justifies the rationality behind it. Nevertheless, one should note that additional Equilibria without reserve price may exist, and fall beyond the scope of this paper.

An additional outcome of this model can explain price stickiness, as described for example in Davis and Hamilton [18]. Many equilibria found here have mass points on certain prices. This implies that with some probability the price in the previous round can be the same also in the next round, even though the seller is mixing. In reality it is known that that prices do not change too often and are sticky. The results of this model can provide an insight on why it is so, as prices selected with mass points can remain unchanged during several periods.

The structure of the paper is as follows: first the Stahl model extension is introduced and the difference between the original and extended models is discussed to clarify the nature of the extension. Then characteristics of the equilibria in the model will be provided, firstly in the case with several smallest firms and then with a unique smallest firm. Then the implications of the results are discussed, examples are provided together with some suggestions on how those results can be empirically tested.

2.2 Model

The Stahl model was introduced in Stahl [39]. An extension to it is formally described below. Notation was adjusted to the recent literature on the Stahl model.

There are N sellers, selling an identical good. Seller i owns n_i stores, where n_i is exogenously given and can differ from seller to seller [1]. We look at the case where not all n_i are equal, as the other case is exactly the original model. Production cost is normalized to 0, and it is assumed that seller can meet any demand. Additionally, there are consumers, each of whom wishes to buy a unit of the good. The mass of consumers is normalized to 1. This implies that there are many small consumers, each being strategically insignificant.

Besides the size heterogeneity all sellers are identical. They set their price once at the first stage of the game. If some sellers mix, distributions are selected simultaneously, and only at a later stage realizations take place.

The mass of consumers is normalized to one, implying that there are a large number of strategically insignificant consumers. Consumers are of two types, both value the good at some large value M. A fraction μ of consumers are shoppers, who know where the cheapest price is, and they buy there. In case of a draw they randomize uniformly over all cheapest stores, spreading equally among the cheapest stores. The rest are searchers, who sample prices. Sampling price in the first, randomly and uniformly selected, store is free. It is shown in Janssen et al. [28] that if this not the case then some searchers would avoid purchase, and the results will be adjusted to exclude those searchers. If sampled price is satisfactory - searcher will buy the item. However, if the price is not satisfactory

[1]In the original Stahl Model each seller had a single store.

- the searcher will go on to search in additional stores sequentially, where each additional search has a cost c. The second (or any later) store is randomly and uniformly selected from stores of previously unvisited sellers, and searcher may be satisfied, or search further on. When a searcher is satisfied, she has a perfect and free recall. This implies she will buy the item at the cheapest store she had encountered, randomizing uniformly in case of a draw.

The main difference to the original Stahl Model is that fraction of searchers initially visiting seller i is equal to $\frac{n_i}{\sum_j n_j}$ instead of $1/N$. Further search is also done according to these propensities, and the probability to visit seller i is n_i divided by the sum of the n_i's of sellers the consumer did not visit yet. Thus, at most a single store of a seller is visited by a searcher.

Consumers need to be at both types - informed and uninformed (namely, $0 < \mu < 1$). If there are only shoppers - it is the Bertrand competition setting, e.g. see Baye and Morgan [8], and if there are only searchers the Diamond Paradox, Diamond [19], is encountered, both well studied.

Firstly we make a technical assumption on the model. In order to avoid measure theory problems it is assumed that mixing is possible by setting mass points or by selecting distribution over full measure dense subsets of intervals. This limitation allows all commonly used distributions and finite mixtures between such.

Before going on a couple of very basic results are provided:

- Sellers cannot offer a price above some finite bound M. This has the interpretation of being the maximal valuation of a consumer for the good.
- Searchers accept any price below c. The logic behind it is any price below my further search cost will be accepted, as it is not possible to reduce the cost by searching further.

2.2.1 Extension Idea

The main point of the extension is to catch the fact depicted in table 2.1. Heterogeneous sellers have different number of stores, and due to various reasons keep the price fixed in all of the stores of a given seller. This is not always the case, as some goods do not have to be fixed over all stores of a seller. Therefore, one needs a distinction between **store goods**, that each store set the prices individually, and **chain goods** which have a fixed price in all chains stores.

A good example for the distinction can be understood via two examples for a seller - a bank and a fuel station. Typically all offers of a given bank do not vary among branches. One would get the same offers for a mortgage, credit, interest rate for deposits and other banking products no matter the specific branch of the bank one approaches. Clearly, a different bank would make different offers, but usually it is the case that a specific branch of a bank 'X' on street 'A' would have the same offers as the branch of bank 'X' in street 'B'. When one looks at fuel stations, one sees the exact opposite. Every single fuel station offers station-specific prices, and it is usually the case that fuel station of firm 'Y' on street 'A' would have a different price than a fuel station of the same firm 'Y' on street 'B'.

For the fuel station example, the original Stahl model would suffice, as there being unsatisfied with a specific station does not imply avoiding the seller completely. However, if after visiting one branch of bank 'X' one does not find a satisfactory offer, there is no reason to visit yet another branch of the same bank. Therefore, present here are both aspects of the extension - the probability to encounter each seller is proportional to the number of stores the seller has and the fact that at most a single store of a given seller would be visited by a given searcher.

2.2.2 Game Structure

The game is played between the sellers, searchers and the shoppers. The time line of the game is as follows:

1. Sellers select pricing strategies and searchers set a reserve price denoted P_M. [2]
2. Realizations of prices occur for sellers with mixed strategies.
3. Shoppers go and purchase the item at the cheapest store.
4. Each Searcher selects a store and observes the price in it.
5. If the price observed is weakly below P_M the searcher is satisfied and purchases the item, if not the search continues.
6. All unsatisfied searches select one additional store, pay c and sample the price there.
7. If the price observed is below P_M the searcher is satisfied and purchases the item, if not the search continues
8. ...
9. When the seller observed all stores and observed only prices above P_M she would buy at the cheapest store encountered.

[2] All searchers have the same reserve price.

Following the Bayesian structure of equilibrium from Stahl [39] and Burdett and Judd [11], the knowledge structure is as described below At the time when reserve price and the strategies being are determined the knowledge of the various agents of the game is as follows:

- Sellers have beliefs regarding reserve price set by searchers
- Searchers have beliefs about which strategies (not realizations) are played by sellers.
- Shoppers will know the real price in each store in the moment it is realized.

The probability that seller i sells to the shoppers when offering price p is denoted $\alpha_i(p)$. The expected quantity that seller i sells when offering price p consists of the expected share of searchers that will purchase at her store, plus probability she is the cheapest store multiplied by the fraction of shoppers. This is also the market share of the seller. For seller i let us denote q_i her market share.

Note that the reserve price ensures that the searcher will purchase at the last visited store, unless all stores were searched.

Regarding utilities, note that sellers when making decision can take into consideration only expected utility. This is since their decision is made on the first stage of the game, and cannot be altered afterwards. Therefore, the seller utility is denoted in ex-ante terms, namely, what is the expected utility for the seller when a certain price is chosen. For searchers it is possible to pin down the exact utility, since when a searcher is faced with a price, she can calculate the benefit exactly without uncertainty. Therefore, searcher utility is given in ex-post terms, taken into consideration the exact price paid. Formally, the utilities are as follows:

- Seller i utility is the price seller i charges (p_i) multiplied by the expected quantity sold: $p_i q_i$
- Consumer j utility is a large constant M, from which item price and search costs are subtracted. If she observed n_j prices and in the end purchased the item at price p_j, the utility is: $M - p - (n_j - 1)c$.

The NE of the game, which has a Bayesian form, is as follows:

- No seller can unilaterally adjust the pricing strategy and gain profit in expected terms.
- Reserve price is rational for searchers, and at any price observed they make optimal decision in expected terms.

- Searchers have a reserve price.
- Searchers have beliefs regarding strategies played by sellers.
- Seller pricing strategies and searcher beliefs regarding those strategies coincide.
- Seller belief regarding the reserve price coincide with the actual reserve price.

A rational reserve price P_M has the following properties:

- After observing a price below P_M it is optimal to stop searching.
- After observing a price above P_M it is optimal to continue searching.

Note that the following lemma describes some of the situations where a reserve price of P_M is rational:

Lemma 2.2.1 *Suppose all sellers offer a price weakly below P_M with prob. 1. Additionally, the expected price offered by each of the sellers is weakly above $P_M - c$. Then, P_M is a rational reserve price.*

Proof:

Suppose a searcher observed a price q in the first store. From assumption: $q \leq P_M$.

The expected price in all unvisited stores is a convex combination of expected values, where none of them falls below $P_M - c$. Therefore, the expected price in unvisited stores is at least $P_M - c$.

From here follows that a searcher expects to observe a price of at least $E = P_M - c$ when a search is performed. In such case, the total cost (in expected term) would be $E + c$, or more. Since $E + c = P_M \geq q$, it is always (weakly) better to stop after visiting in the first store and not search in a single additional store.

From similar reasoning it would not be optimal to search in multiple stores. □

Remark 2.2.1 *Note that additional equilibria, without a searcher reserve price, may exist. As Stahl model literature concentrates mainly on consumers with a reserve price, these equilibria fall beyond the scope of this paper.*

Regarding social welfare one can say the following:

Remark 2.2.2 *As the sum of the searcher and seller utilities may differ only in the search cost, any strategy profile where the searchers always purchase the item at the first store visited is socially optimal.*

2.3 Basic Results

The first thing to notice is that no seller would offer a price above the reserve price.

Lemma 2.3.1 *No seller offers a price above P_M in NE.*

Let p be the highest (or supremum) price in support union of all sellers in NE, and suppose $p > P_m$. Such supremum exists as it is assumed that there is a finite bound on prices. Let us distinguish between several cases, based on the number of mass points on p. Remember that in a mixed NE a seller has the same expected profit from **all** prices she has in support.

Consider a seller with price p in support.

Case 1 - no seller has a mass points on p: With probability 1 everyone offers a cheaper price. Searchers would search further, and shoppers would purchase elsewhere. Therefore, in such case the profit of offering price p, or prices arbitrarily close to it is arbitrarily close to 0. A deviation to c would be profitable.

Case 2 - everyone has a mass points on p: There is a positive prob. that all sellers offer price p. This cannot be equilibrium due to undercutting.

Case 3 - some sellers have a mass points on p: Similar to the first case, shoppers would not purchase at a seller offering p. Searchers would search until encountering a price below p. Once such a price is encountered - no searcher would purchase the item at price p. Therefore, no consumer would purchase the item at price p, leading to profit of 0. Again, a deviation to c would be profitable.

To sum it up - for a seller offering a price $p > P_M$ there is a profitable deviation in all cases. □

Corollary 2.3.1 *Any NE is socially optimal. This is since the total utilities of the sellers and consumers sum up to a constant, as long as the searchers buy at the first store they visit.*

Now a proof for the fact that no symmetric NE exists in the model is provided.

Lemma 2.3.2 *No symmetric NE (where all sellers choose the same pricing str.) exists.*

Due to undercutting no pure NE or symmetric NE with mass points can exist. Suppose that exists a mixed strategy symmetric NE. Let the pricing strategy in the symmetric NE be denoted as s, and let F be the distribution function describing the pricing strategy s.

As noted before, no seller sets a price above P_M, as in such case the highest (or sup.) price in the support yields profit 0.

Suppose that two prices p and q are in the support of s with positive density or mass point. Additionally, exist two sellers i, j such that $n_i > n_j$.

Note that due to symmetry of the strategy choice, the probability of a seller to attract shoppers with price p is equal to $(1-F(p))^{N-1}$. The next step is to write down the profits of seller i, for both prices. Those must to be equal, as mixing is possible only between prices that yield the same expected profit:

$$\pi_i(p) = p[(1-F(p))^{N-1}\mu + (1-\mu)n_i/S] = q[(1-F(q))^{N-1}\mu + (1-\mu)n_i/S] = \pi_i(q) \quad (2.1)$$

This contains the expected quantity sold by the seller: prob. to be cheapest times the quantity of shoppers and the quantity of the searchers, which contain only the initial searchers visiting the store (due to corollary 2.3.1).

Similarly, the profit of seller j is as follows:

$$\pi_j(p) = p[(1-F(p))^{N-1}\mu + (1-\mu)n_j/S] = q[(1-F(q))^{N-1}\mu + (1-\mu)n_j/S] = \pi_j(q) \quad (2.2)$$

After subtracting the second equation from the first one obtains the following:

$$((1-\mu)/S)p[n_i - n_j] = ((1-\mu)/S)q[n_i - n_j] \quad (2.3)$$

Note that $(1-\mu)/S \neq 0$, and can be narrowed down. From here, ether $p = q$ or $n_i = n_j$, both cannot occur due to our assumptions. $\square$

Next, a distinction is made regarding the number of smallest sellers, as it has a crucial effect on NE structure.

2.4 Multiple Smallest Sellers

Let the smallest value of the size parameter be denoted as n_m, and the total number of stores $\sum_j n_j$ as S.

Here we make a distinction between two cases - whether there are several sellers with size n_m (smallest), or the smallest seller is unique. In this section the first case is dealt with. The following theorem provides an insight regarding how NE of the model look like:

Theorem 1 *In the Model, with multiple smallest sellers, the NE with reserve price P_M looks as follows:*

- *All the sellers who have a larger number of stores than n_m select the reserve price P_M as a pure strategy.*
- *The agents with n_m have their support between some price P_L and the reserve price P_M, with no mass points except possibly P_M.*
- $P_L = \frac{n_m(1-\mu)}{(1-\mu)n_m+S\mu} P_M$
- *Suppose seller i does not have the interval $(p, p') \subset (P_L, P_M)$ in support. Then the only price above p seller i has in support is P_M.*
- *Profit of seller i is $Const \cdot n_i$.*
- *Let I be an interval to the left of P_M. If sellers i, j both have all of I in support, then both use the same distribution over I.*

Proof shifted to the appendix.

2.4.1 Equilibrium Distribution

In this subsection we will look into the equilibrium described by the theorem. Firstly, the equilibrium strategy for sellers that are not the smallest one is selecting reserve price purely. Sellers with smallest size have a wider choice. Below is a description of all possible equilibrium strategies for smallest sellers.

From theorem 1, any NE of the Stahl model with a reserve price there are at most three groups of strategies, as follows:

1. Top: A group of sellers (possibly empty) that select P_M as a pure strategy
2. Bottom: At least two sellers who have the full support of $[P_L, P_M]$ with some NE dependent continuous full support distr. function F.
3. Middle: A group of sellers (possibly empty) with an individual cutoff price, such that below the cutoff price the distribution used is the same F as the Bottom group. Above the cutoff price seller has a mass point at the reserve price.

Additionally, all sellers have the same profit of $n_i P_M(1-\mu)/S$ and have P_M in support.

Illustration of the three types of strategies can be seen on figure 2.2.

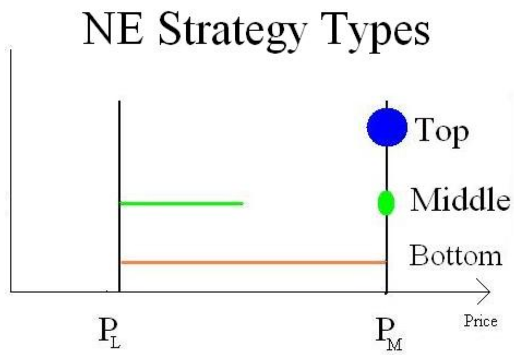

Figure 2.2: The three types of strategies available in a NE of the model

This distribution possibilities follow directly from the fact that sellers must have same distribution on intervals (lemma A.1.9 and corollaries) and that any point in between P_L and P_M needs to be in support of at least two sellers (corollary A.1.2).

Remark 2.4.1 *For any combination where the middle group is empty and the bottom group has at least two sellers exists a corresponding NE. To see this, simply adjust the shoppers share to reflect the game when only searchers visiting the mixing sellers are in the game.*

Remark 2.4.2 *Note that this result extends the original model equilibrium, as all sellers can have the same size in the model discussed here. In the case where all sellers have the same size and choose the middle strategy the resulting equilibrium would be symmetric and identical to the original model equilibrium. When some choose middle or bottom strategies an asymmetric equilibrium of the original model is obtained.*

Now it is possible to elaborate on the structure of F function which is used in equilibrium by sellers, and what reserve price can be used. Suppose that in equilibrium we have B sellers with 'bottom' strategy (mixing over entire support), T sellers with 'top' strategy (pure reserve price) and $M = \{1, 2, \dots m\}$ sellers with 'middle' strategy (cutoff price strategy), with the corresponding cutoff prices of $cp_1, cp_2, \dots cp_m$ and corresponding mass points at the reserve price are with mass of $a_1, a_2, \dots a_m$.

Let the set of sellers with cutoff point above some price p be denoted as $L(p)$.

From the structure of the equilibrium all sellers have equal profit per store, and since all have the size n_m they have equal profit. Additionally, all sellers have P_M in support and the reserve price attracts no shoppers. Therefore, the profit for all sellers is:

$$\pi = n_m P_M (1 - \mu)/S \tag{2.4}$$

For any price p the expected profit needs to be equal to the expression above. At price p seller i has a certain probability $\alpha_i(p)$ to attract shoppers, if she is the cheapest. This can be calculated as follows:

- For each seller $j \neq i$, calculate the probability that j offers a price above p
- Multiply these probabilities
- All larger sellers select P_M purely and therefore do not have an effect.

Let p be a price in (P_L, P_M). For group B this probability is clear and equal to $1 - F(p)$. For group T - it is zero. For group M we need to distinguish between two cases: ether the seller is in $L(p)$ and the probability is $1 - F(p)$, or not, and then it is equal $a(p)$. Combining the cases we get that the expression for the expected profit is as follows:

$$\pi = n_m P_M(1-\mu)/S = p[n_m(1-\mu)/S + \mu(\prod_{j \in B \cup L(p)} (1 - F(p)) \prod_{j \in M \setminus L(p)} (a_j))] \tag{2.5}$$

As the F function is the same we can simplify and get:

$$p[n_m(1-\mu)/S + \mu((1 - F(p))^{b+|L(p)|} \prod_{j \in M \setminus L(p)} a_j)] = P_M n_m(1-\mu)/S \tag{2.6}$$

Extracting $F(p)$ form this equation will yield:

$$F(p) = 1 - \sqrt[b+|L(p)|]{\frac{n_m}{S\mu}(\frac{P_M}{p} - 1)\frac{1-\mu}{\prod_{j \in M \setminus L(p)} a_j}} \tag{2.7}$$

Note that at if seller j has the cutoff price at p, it must be the case that $a_j(p) = 1 - F(p)$, and therefore F will be continuous. It will also be differentiable at all points that are not cutoff prices. Therefore, it is still possible to calculate the density and in some cases expected value explicitly, as the number of cutoff prices is finite. The last step, based on lemma 2.2.1, require finding the expected value $E(F)$, and setting the reserve price at $E(F) + c$. This step is technical and the expressions involved cannot be generally calculated, thus it is not done here for the general case, yet examples with some specific cases are provided.

2.4.2 Equilibria Example

Consider 3 sellers, and consumers where 1/6 are shoppers. 1/2 of the consumers are searchers initially visiting one of the stores, and 1/6 of the consumers are searchers initially visiting each of the two others. The corresponding number of stores is, for example, 3, 1, 1.

The following asymmetric NE exists:

- The searchers have a reserve price of $P_M = c/(1 - ln2) > c$
- The seller with the larger store number offers the reserve price as a pure strategy.
- The other two sellers use the same continuous distribution function on $[P_M/2, P_M]$
- The distribution function for the two mixing sellers is $F(p) = 2 - P_M/p$.

One needs to check that no seller wishes to deviate. Firstly, note that prices of above P_M or below $P_L = P_M/2$ are not profitable for all sellers. Already at P_L there is a prob. 1 to sell to shoppers and there is no need in a further discount. Prices above P_M would leave the seller with sold quantity of zero.

The profit for the pure str. seller is $P_M/2$ and for the other two is $P_M/6$ when offering the price P_M.

The profit of the mixed str. seller when she offers a price $p \in [P_L, P_M)$, is as follows:

$$p(\frac{1}{6} + \frac{1 - F(p)}{6}) = \frac{p}{6}(2 - 2 + \frac{P_M}{p}) = \frac{P_M}{6} \tag{2.8}$$

Therefore, the mixing agents are indeed indifferent between the prices in the interval.

Lastly, one needs to show that the pure str. agent would not deviate to a lower price. His profit when offering a lower price p is:

$$p(\frac{1}{2} + \frac{(1 - F(p))^2}{6}) \leq p(\frac{1}{2} + \frac{1 - F(p)}{6}) < p(\frac{1}{2} + \frac{1 - F(p)}{2}) = \frac{P_M}{2} \tag{2.9}$$

The first inequality due to $F(p)$ being between 0 and 1. The second strict inequality due to $1 - F(p)$ being strictly positive when $p < P_M$. The third equality follows from equation 2.8.

It is clearly visible that the pure str. agent has no incentive to deviate to a different str., and therefore it is a NE.

Note that the expected price of each seller is at least $P_M - c$. Additionally, no seller offers a price above P_M. Therefore, the reserve price P_M is rational from lemma 2.2.1.

2.4.3 Additional Examples

It is possible to construct additional examples for the original model as follows: Add to a symmetric NE setting an additional seller that charges purely reserve price. Then, by adjusting the searchers fractions, similarly to the example provided above a NE will be obtained. Exists a NE where the sellers with the lowest store number ignore the searchers visiting one of the larger sellers and obtain among them a symmetric (for example) NE, and the other sellers set a pure strategy of P_M. The way to show that the profiles are NE are similar to the way that the example above were shown, for example, by having more than one agent selecting the reserve price as a pure strategy, or a seller having a cutoff price.

2.5 Unique Smallest Seller

So far the case where the smallest seller was unique was omitted. This case is dealt here. In this structure exist NE, however these slightly differ in structure from the previous case. In Astorne-Figari and Yankelevich [2] Stahl model with two different sized sellers is discussed, but here it is done in a more general setting with more sellers. The NE below stands in line with two sellers behave according to the results in Astorne-Figari and Yankelevich [2], while the rest offer the reserve price purely.

Let the smallest seller be denoted as m and (one of) the second smallest seller as j, with corresponding store numbers n_m and n_j. Additionally the shares of consumers that would visit seller i as searchers (note corollary 2.3.1) are denoted as follows:

$$Src_i = \frac{n_i}{S}(1-\mu) \tag{2.10}$$

Proposition 2.5.1 *There exists a NE with a reserve price P_M such that:*

1. *All sellers except m and j select P_M purely.*
2. *The lowest price in support is $P_L = P_M(\frac{Src_j}{\mu+Src_j})$.*
3. *m and j mix on the entire interval (P_L, P_M).*
4. *The distributions are as provided below, and j has a mass point at P_M.*

$$F_m(p) = 1 - \frac{Src_j}{\mu}(\frac{P_M}{p} - 1) \tag{2.11}$$

$$F_j(p) = (1 - \frac{P_L}{p})(1 + \frac{Src_m}{\mu}) \tag{2.12}$$

$$P_M = \frac{c}{1 - ln(\frac{Src_j+\mu}{Src_j}) \cdot \frac{Src_j}{\mu}} \tag{2.13}$$

$$F_j(p) \leq F_m(p) \forall p \tag{2.14}$$

Proof:

First see that there is no deviation to sellers m and j:

It is easy to verify that there is a constant profit for sellers m and j in the interval, since:

$$\pi_m(p) = ((1 - F_j(p))\mu + Src_m) \tag{2.15}$$
$$\pi_j(p) = ((1 - F_m(p))\mu + Src_j) \tag{2.16}$$

Market share consists only of searchers initially visiting the stores of a given seller. Probability to attract shoppers is equal to the probability that the other seller (m or j) is above, equal to one minus the corresponding distribution.

Offering prices below P_L is not profitable, as already in P_L one attracts the shoppers w.p.1. Prices above P_M will not be offered due to corollary 2.3.1. Therefore, m and j have no profitable deviation.

Seller k who is not m or j would similarly refrain from selecting prices below P_L or above P_M. Deviating to a price $p \in (P_L, P_M)$ would yield the following profit:

$$\begin{aligned}\pi_k(p) = p((1 - F_m(p))(1 - F_j(p))\mu + Src_k) <\\ p((1 - F_m(p))\mu) + p(Src_k)\end{aligned}$$

Seller j has price p in support and therefore:

$$\pi_j = p((1 - F_m(p))\mu + Src_j) = P_M Src_j \tag{2.17}$$

Combining the equations, the following expression is obtained:

$$\pi_k(p) < \pi_j - Src_j p + pSrc_k = P_M Src_j + p(Src_k - Src_j) \tag{2.18}$$

Note that the size of k is at least the size of j, implying that $Src_k \geq Src_j$. Additionally, note that $p < P_M$. Therefore, the profit of seller k when offering price $p \in (P_L, P_M)$ is below $P_M Src_k$. However, this profit is obtained by k when offering P_M, and therefore, has no profitable deviation from P_M.

Lastly, the reserve price is rational. Note that if one compares the derivatives of F_m and F_j:

$$f_m = \frac{Src_j P_m}{\mu p^2} \tag{2.19}$$
$$f_j = \frac{(\mu + Src_m) P_L}{\mu p^2} \tag{2.20}$$

Using the facts that $Src_m < Src_j$ and $P_M Src_j = P_L(\mu + Src_j)$ it is easy to see that $f_m(p) > f_j(p)$ for any price in (P_L, P_M). As the distribution of j has a mass point at the maximal price of P_M, the expected value of F_m is smaller than the one of F_j.

From Lemma 2.2.1 in order for the reserve price to be rational $E(F_m)$ needs to be at least $P_M - c$, which occurs with equality, due to the structure of P_M:

$$E(F_m(p)) = \int_{P_L} P_M p f_m(p) = \frac{Src_j P_m}{\mu} \int_{P_L} P_M \frac{1}{p} = \frac{Src_j P_m}{\mu} \cdot ln\left(\frac{Src_j + \mu}{Src_j}\right) = P_m - c \tag{2.21}$$

□

Note that here all sellers but m have the same profit per store and a mass point at P_M. The smallest seller has a larger profit per store, and offers more generous discounts. Additional equilibria may exist where several of the smallest sellers after m also mix, with identical distribution (but different than the distr. of m). This comes in line with the results pointed out in Astorne-Figari and Yankelevich [2]. The next step is to denote the general structure of NE in such case. It is given in the theorem below:

Theorem 2 *In the case of a unique smallest seller the NE with a reserve price P_M of the game look as follows:*

- *All sellers with size above n_j select the reserve price purely.*
- *The lowest price in the support union is $P_L = P_M(\frac{Src_j}{\mu+Src_j})$.*
- *Seller m mixes with a continuous, dense distr. function F_m over (P_L, P_M).*
- *Some sellers with size n_j also mix over the entire interval with a continuous dense F_j, such that $F_j(p) < F_m(p)$ for all $p \in (P_L, P_M)$, and in addition have a mass point at P_M.*
- *All sellers except m have the same profit per store, and m has a higher profit per store.*
- *Let I be an interval to the left of P_M. If sellers $i, j \neq m$ both have all of I in support, then both use the same distribution over I.*

Proof shifted to the appendix.

The calculation of the equilibrium distribution is done in similar lines to proposition 2.5.1 and subsection 2.4.1

Remark 2.5.1 *The strategy choice for sellers with second smallest size is similar to smallest sellers in previous case. However, all such sellers will have a mass point at the reserve price.*

2.6 Discussion

Here is a short discussion over the model and results. First, the structure and economic motivation on the results is provided. Afterward a couple of situations are shown where importance of the extension becomes clear. Then, a couple of empirical tests are suggested to verify whether the results hold in the lab. Lastly, some points to future research are suggested.

2.6.1 NE Structure

The structure of the NE is in line with the results in Astorne-Figari and Yankelevich [2] and Burdett and Smith [12]. Here a wider setting for seller sizes is applied. In this extension we see that indeed larger firms find no incentive to compete on shoppers. The reason behind it can be seen easily from the profit structure. The profit of seller i is as follows:

$$\pi_i(p) = p\alpha_i(p) + p(1-\mu)n_i/S \tag{2.22}$$

Where α is the probability to attract shoppers when offering price p.

The profit consists of two components - expected profit from shoppers and profit from searchers. Setting a lower price has two effects - on one side it increases the probability to attract shoppers (higher α), but on the other it reduces the information rent from the searchers (due to lower p). Note that the first, positive, effect is more size independent (probability to be cheapest increases similarly no matter your size), whereas the second effect is size dependent and is more significant for larger sellers. Therefore, a larger seller will find it less attractive to offer discounts. In the case of at least two smallest sellers, these will compete one with the other, and larger sellers will not even bother to enter the 'shoppers market', by sticking to the reserve price. In the case of a unique smallest seller, she will compete for the shoppers with some of the 'second smallest' sellers. Sellers above that 'second smallest' size will stay out.

The three types of strategies for smallest sellers have some economic motivation. The mixing seller wishes to compete over the shoppers when the pure reserve price seller does not to bother with the shoppers. That kind of behavior is common in the economic world, and not in all cases all will compete as predicted by the symmetric NE. If only a single seller decides to compete, she will have monopolistic profits, which would attract additional competitors, and therefore, in NE at least two sellers will compete for the shoppers. As suggested in Astorne-Figari and Yankelevich [2], in the case of a smallest unique seller (their setting is of two seller with different size), the smaller seller will offer lower prices with higher probability, due to lower share of searchers visiting them. Therefore, in a competition with second smallest firms smallest seller has an advantage.

The cutoff price is for sellers that do not wish to be bothered with small probabilities. There are several effects that may cause a seller to refrain from sufficiently low probability events, for example see Barron and Yechiam [4]. Such seller will compete for shoppers, but only at prices that yield the benefit of getting the shoppers from high enough probability. When the probability to attract shoppers is lower than this individual threshold, the seller prefers to refrain from the shoppers market and select the reserve price with mass point instead.

These three strategies in addition to the size implication explain the behavior of sellers in NE of this model.

2.6.2 Real World Examples

First example uses the data from Table 2 in from Bazucs and Imre [6]. This paper discusses the pricing of a homogeneous good (milk) in 8 discounters in Hungary over a period of 5 years (2004-2008). The stores number data was not available in the paper, and current stores number was obtained from company profiles (Jan 2012), with two exceptions. One of the supermarket chains, PLUS, was purchased by another, InterSpar, after the relevant period [3]. Therefore, from the current number of stores by InterSpar the number of PLUS stores (around 170 stores) were subtracted. The current number of stores serves as an indicator to the number of stores in the research period, and is divided into several distinct groups by size. This provides enough insight, and the idea that smaller chains are usually cheaper. Note that there can be additional factors (such as location of stores) affecting the price, however, one factor can indeed be the chain size. The table is as follows:

Chain Name	Stores Number in Hungary	Avg. price of milk
InterSpar*	100-200	182 HUF
Cora	Below 20	198 HUF
Match	Below 20	200 HUF
Tesco	200-400	205 HUF
Auchan	Below 20	211 HUF
CBA	Above 500	213 HUF
Plus*	100-200	230 HUF
COOP	Above 500	240 HUF

Another example is from Drogerie stores in Germany. GKL[4] research found that Schlecker is 10-20% more expensive than competitors. Additionally, the number of stores in January 2012 of the various chains, as collected by GKL, in Germany and Europe is given in figure

[3]For example, see $http://www.portfolio.hu/en/tool/print/2/14361$

[4]Conducted in Jan. 2012 by GKL, comparing prices in 1700 stores in Germany. Media Report for example here: $http://www.handelsblatt.com/unternehmen/handel-dienstleister/gkl-analyse-schlecker-deutlich-teurer-als-die-konkurrenz/6254086.html$

1.1. Again we see a tendency that the largest chain is the more expensive one. Again, additional factors may have an effect here, but it seems that chain size plays a role when determining prices.

To conclude, it seems that there should be some positive correlation between chain size and price here too. Similar effect was observed in a more limited setting in Astorne-Figari and Yankelevich [2] and Burdett and Smith [12]. An additional empirical research checking this connection explicitly should be able to determine how strong it is.

2.6.3 Empirical tests and Policy Suggestions

The structure of NE allows running several empirical tests on a database containing pricing and chain size data. In most NE found here, some sellers select the reserve price with a mass point. This implies that the reserve price will be more commonly selected. Similarly, larger discounts will be rarer, as the reserve price will be more common. Additionally, one should see correlation between chain size and price. Moreover, in a dynamic setting there should be some price sticking, as mass points exist.

When examining the probability for a consumer to encounter a lower price, it is clearly visible that the larger is the variance in store sizes, and the rarer the smallest stores are, the closer expected price paid is to the reserve price. This is an additional factor to examine, and it suggests an interesting policy decision. If the regulator wishes to reduce goods prices, reducing the variance among the selling firms can reduce the price, as exists a NE where less sellers select the reserve price purely.

2.6.4 Future Research

The results here open several important questions, which leave place for a fruitful future research. Firstly, the assumption here is that a reserve price exists. There may be additional NE without a reserve price, and an interesting question is whether such exist and how do these look like. This will allow fully characterizing all NE of the model and fully explaining behavior of sellers. An additional question is combined with the determination of the reserve price. What is the full set of reserve prices under a certain setting, as here only a lemma provides a sufficient condition for its rationality. Moreover, which reserve price will the consumers set in order to minimize their price. Similarly, how would sellers react to increase the reserve price - by sticking to reserve price, or mixing over entire interval, or perhaps a mixture of all three strategies including some specific cutoff prices? This question of consumer welfare and seller welfare will provide an important insight on behavior of these groups, and can provide a policy decision for a regulator in order to induce lower or higher prices.

Additional extensions can be imposing price to be similar in all stores of a seller but not identical. On the consumer side introducing reputation effects, where a high price has a negative effect, but not total disregard, on future search probability. What would be the equilibria when these, or other extensions are introduced.

The Stahl model is a very important tool and the model is being used and applied in numerous papers. The author hopes that this paper provides an additional important insight which will make the Stahl model more applicable and more realistic.

Chapter 3

Reserve Price in Search Models

Never spend more for an acquisition than you have to.
- Ferengi rules of Acquisition, rule #3.

Abstract

Search models are used in a variety of fields. One of those is consumer search and in it the Stahl model is one of the most popular search models. The literature in search models concentrates mainly on equilibria with a consumer reserve price. This is a simplifying condition, which narrows down severely the freedom of the consumer. For example, once the model has a finite number of sellers who select different strategies a situation may arise where reserve price may not exist. This paper addresses the possibility of equilibria without reserve price, when sellers can use asymmetric strategies.

Here a condition is given which ensures existence of reserve price in all equilibria. The condition involves prices where sellers set mass points, and undercutting those prices. The condition states that if a searcher is satisfied with such price, she should be satisfied also when a small discount is offered to that price. Assuming this, it is possible to concentrate only on equilibria with reserve price, and investigate also situations where the sellers are heterogeneous, or equilibria are not symmetric.

3.1 Introduction

Search models are an extensive and important field, with a large literature. The field, originally developed due to diamond paradox, Diamond [19], had since grown and covers many topics, such as consumer search, for example Stahl [39], Varian [42], and many other areas. One of the most popular models applied in the search literature is a model introduced by Stahl [39]. It describes a simple consumer search with two types of consumers, one with an endogenous reserve price. Additional research, for example Janssen et al. [27], shows that Stahl model performs very well in reality and predicts pricing of goods very well. Therefore, this paper concentrates on the Stahl model. In the original paper by Stahl [39], it is explicitly noted that the paper concentrates on symmetric strategy choices of sellers.

Recently there develops literature on an extension of the Stahl model, where the sellers are heterogeneous, for example Astorne-Figari and Yankelevitsch [2], or Burdett and Smith [12]. One of the results of such models is that a symmetric NE is no longer possible when the sellers are heterogeneous. This is a very important extension, as in reality sellers vary in size and popularity. Therefore, it is important to note that asymmetric NE plays an important role in this extension.

However, once asymmetric NE arises, it creates an additional complexity. Throughout the search literature there is a crucial assumption of a searcher reserve price, or of seller symmetry, where sellers use same pricing strategy. If sellers use the same strategy then clearly reserve price exists (see Rothschild [37]). Reserve price is a setting where consumers set a threshold price, which determines whether the good is purchased. The condition is simple - a transaction takes place if and only if the price is weakly (or strictly) below the threshold price. This allows easier analysis of the game and is used in proving many of the results. However, as noted in Rothshild [37], once an asymmetric strategies are in place, additional NE, without a consumer reserve price, may exist. The reason behind it is simple - once a price was revealed to the searcher, she can interpret this price and gain information regarding prices in other stores. The price observed combined with the initial Bayesian beliefs of the searcher, lead to beliefs update regarding strategies used by other sellers. This can lead to situations where no reserve price exists, which are very complex to analyze.

One can wonder whether in search models equilibria without reserve price prevail. A working paper by Olszewski and Wolinsky [35], provide a search model with equilibria that have no reserve price. Clearly, their model differs significantly from the Stahl search model. Yet if certain search models have no reserve price equilibria, it may be the case that also Stahl search model has such equilibria.

This paper provides a condition on the game which ensures existence of a reserve price.

The result has a very important application allowing the search literature to remain in the reserve price world, once this condition is met. Here a wider set of possible strategies is discussed, satisfying a reasonable condition. The main property of the condition is allowing free undercut for sellers, namely, at certain prices it is possible to offer arbitrarily small discounts without losing consumers, and therefore can be seen as a trembling hand perfection condition. The paper shows that in this space of consumers search strategies, all NE have a reserve price.

Similar result can hold for many other search models, which have a similar problem, where asymmetries can lead to asymmetric choice of strategies and to additional equilibria, without reserve price. Then an observed price can imply additional insights on the market. However, similar results can be made for additional search models, limiting the possibility for a NE without a reserve price.

The structure of the paper is as follows: first the Stahl model is introduced. Then the problem of no reserve price is illustrated in a couple of examples. Afterward, 'undercut proof' condition is introduced and lastly, it is shown that such a condition is a sufficient one for reserve price to exist in NE.

3.2 Model

The model is similar to the Stahl model, introduced by Stahl in Stahl [39].

In the model there are N sellers, selling an identical good. Seller i owns n_i store. The production cost is normalized to 0, and seller can meet demand. Additionally, there are consumers, each of whom wishes to buy a unit of the good. The mass of consumers is normalized to 1. This implies that there are many small, strategically insignificant consumers.

The sellers set their price at the first stage of the game. Sellers cannot price discriminate. If the seller mixes then the distribution is selected simultaneously, and only at a later stage the realizations take place.

The consumers are of two types, both evaluate the good at some large finite value M, known to sellers. A fraction μ of consumers are shoppers, who know where the cheapest price is, and buy there. In case of a draw they randomize over all cheapest stores, spreading equally among them. The rest are searchers, who sample prices sequentially. Sampling price in the first, randomly and uniformly selected, **store** [1] is free. The second (or any later) store is randomly and uniformly selected from stores of previously unvisited sellers, and bears a cost c. The searcher may be satisfied, or search further on. When a searcher

[1] note that it is not the same as 'uniformly selected seller'

is satisfied, she has a perfect and free recall. This implies she will buy the item at the cheapest store she had encountered, randomizing in case of a draw.

The reason for the two types of consumers is given already in the original paper introducing the model, Stahl [39]. If only shoppers exist ($\mu = 0$) then the setting is of Bertrand competition, and if only shoppers exist ($\mu = 1$) then the Diamond paradox, Diamond [19] is encountered. In the former setting the sellers will drive the price down to zero and in the latter up to infinity.

Before going on, a technical assumption is required:

Remark 3.2.1 *To avoid measure theory problems it is assumed that mixing is possible by setting mass points or by selecting distribution over full measure dense subsets of intervals.*

Additionally, it is easy to see the following two basic results:

- Sellers do not offer a price above some finite bound M in any NE. This is due to the maximal valuation of a consumer for the good.
- Searchers accept any price below c. The logic behind it is any price below my further search cost will be accepted, as it is not possible to reduce the cost by searching further.

3.2.1 Knowledge and Beliefs

Searchers have beliefs regarding the prices sellers set. For each possible (pure or mixed) strategy s of the model a belief is attached, stating how many stores are priced according to this strategy, denoted $n(s)$ (clearly the sum of $n(s)$ over all possible strategies s is N, the number of stores). Searchers do not know beforehand which realizations took place, nor which seller chose which strategy. Each strategy has an expected price, denoted $e(s)$. This structure is similar to the one applied in Stahl [39, 40].

Suppose the searcher observes the price p. Let the probability that this price p came from strategy s be denoted as $prob(p, s)$. To calculate $prob(p, s)$ first calculate chance that s is selected by some seller, according to searchers beliefs. Additionally the probability that p is the realization of strategy s (relevant for mixed strategies) is weighted in. One needs to note that if some strategies (with positive $n(s)$) have a mass point on p only those will be considered, otherwise the densities will play a role. Formally:

$$prob(p, s) = \frac{n(s)f(p)}{\sum_{p \in s'} n(s')f(p)} \tag{3.1}$$

Now, if the searcher thinks that strategy s was selected, searching further will yield (in expected terms) the expected price in stores belonging to other sellers. Namely, it is the expected price, only that $n(s)$ is now lower by one (as s was observed in one of the stores). The expected size of seller using strategy s is denoted $EN(s)$. Then the calculation of the expected price searching is as follows:

$$EPS = \frac{\sum_{s':n(s')>0,s'\neq s} n(s')e(s') + (n(s) - EN(s))^{+})e(s)}{\sum_{s'} n(s')} \tag{3.2}$$

Clearly if $n(s) < EN(s)$ is zero or below the corresponding element would be zero, or this situation would not be considered at all. For example, if only a single seller uses a price already observed elsewhere, then this seller would not be calculated for future visits.

In the process of searching the structure of beliefs can be probabilistic and more complicated, based on beliefs from which strategies previously observed prices came. Obviously, searchers search further only when the expected price in a search is at least c lower than the lowest price observed so far. Below is an example of how to calculate an expected search price, and additionally illustrates that sometimes no reserve price exists.

3.2.2 Satisfaction Sets

When one goes beyond reserve price, one needs to describe searchers behavior. A natural way is obtained by introducing satisfaction sets. Those describe stopping condition, and are a natural extension to reserve price. This is the most general setting for a stopping condition, allowing pinning down exactly all condition which make a searcher satisfied.

Searchers in the course of a search observe prices, and after some price vectors observed the search ends. Let the search ending vector sets be defined as buying satisfaction sets. Formally, a satisfaction set BS_k consists of all vectors of length k, such that after observing the k prices denoted in the vector, in the corresponding order, the searcher is satisfied, and does not search further. Note that since any price below c is satisfactory BS_k is non empty for all k.

Let v be a vector in BS_k for some $k \geq 1$. The lowest coordinate value in v is denoted as v_m. The set of coordinates in v with v_m will be denoted $Min(v)$. From the definitions v_m is the price paid by the searcher after visiting k stores, observing the prices vector v and being satisfied.

For example, suppose a vector $v = \{8, 5, 9\} \in BS_3$, has $v_m = 5$. v implies the following: the searcher visited 3 stores. In the first observed the unsatisfactory price of 8. In the second the price of 5 was observed, and still was unsatisfactory. After a third search the price of 9 was observed, making the price 5 satisfactory. Such an example can arise when the last visited seller mixes between 9 and some extremely low prices. After seeing that the realization was 9, a previously unsatisfactory price becomes attractive.

Let the supremum element of BS_1 (the first satisfaction set) be denoted as P_M. Let P_S denote the maximal (supremum) price which is in the support of a seller in a given NE. Implying - in NE no one offers a price above P_S. From an assumption no price above M can be set, and therefore, P_S and P_M are well defined and finite.

Note the following property. Let P_{min} be the lowest (infimum) price that is in the support of a strategy in NE. Any price below P_{min} is in BS_1. Namely, any price below the lowest price offered in NE is satisfactory. This follows directly from the Bayesian structure of the beliefs: such price is satisfactory as any additional search will end up with a weakly higher price.

3.2.3 Game Structure

Now after discussing the knowledge and behavior of searchers, we can turn to the structure of the game.

The game is played between sellers, searchers and shoppers. The time line of the game is as follows:

At the first stage, the sellers set their pricing strategies and simultaneously searchers set their common satisfaction sets. Then, if some sellers used a mixed strategy, realization of the mixed strategy is taking place. At the second stage the shoppers observe all the realized prices and purchase the item at the cheapest store. At the third stage, searchers sample randomly (uniformly) selected price. If a searcher is satisfied she purchases the item. If not - she pays c and observes another price, and so forth until ether she is satisfied or sampled all prices. When the searcher observed all stores and observed only unsatisfactory prices she would buy at the cheapest store encountered.

Additionally, when setting the prices and satisfaction sets the following beliefs are taken into consideration: Searchers have beliefs about which strategies were actually played by the sellers and about sellers' sizes. Sellers have beliefs regarding their competitors' strategies and regarding the satisfaction sets. Shoppers will know the real price in each store in the moment it is realized.

The probability that seller i sells to the shoppers when offering price p is denoted $\alpha_i(p)$. Let q be defined as the expected quantity that seller i sells when offering price p. It consists of the expected share of searchers that will purchase at her store, plus the probability she is the cheapest store times μ. It is also the expected market share of the seller.

The seller utility is the price charged multiplied by the expected quantity sold (equals to pq). Since she is setting price at the early stage of the game, she can only has ex ante utility, reflecting the expected income. The consumer utility is a large constant M, from which the price paid for the item and the search costs are subtracted.

The NE of the game has a Bayesian form and is as follows:

- The searchers beliefs coincide with the actual strategies played by sellers.
- The satisfaction sets are rational for the searchers, and they cannot profitably deviate to different satisfaction sets profitably in expected terms.
- Sellers know that searchers are rational.
- No seller can unilaterally adjust the pricing strategy and gain profit in expected terms.
- Searchers beliefs coincide with actual strategies and sizes of sellers
- As usually in NE, there is a common knowledge of rationality.

Remark 3.2.2 *As the sum of the searcher and seller utilities may differ only in the search cost, any strategy profile where the searchers always purchase the item at the first store visited is socially optimal.*

3.2.4 Undercut Proof

Lastly, an undercutting condition is defined, which will lead to existence of a reserve price.

Definition 3.2.1 *Let σ be a strategy profile of the sellers. Let MP denote the set of prices where some sellers have set a mass point.*

Definition 3.2.2 *Let v be a price vector in BS_k with $v_m = p$. Suppose exists $\varepsilon > 0$ such that when subtracting any number smaller than ε from any single coordinate in $Min(v)$ will keep the resulting vector in BS_k. Then v is denoted ε-undercut proof.*

Definition 3.2.3 *Fix a natural number k. Suppose that for any price p in MP exists $\varepsilon_p > 0$ such that for any vector $v \in BS_k$ with $v_m = p$, the vector v is ε_p-undercut proof. Then BS_k will be denoted as undercut proof.*

The condition in the last definition says the following: if a certain vector is satisfactory, then reducing one of the lowest elements in it by a small amount will not affect the decision of the searchers. The main implication of this assumption is simple - for any price p, with a mass point on it, it is possible to undercut it without losing any searchers. This condition can be interpreted as a trebling hand perfection condition. A seller sets a certain price with a mass point, but due to a trembling hand the price is set slightly lower. Consumers, which expect such a possibility, believe that for some small ε, the price is selected by someone who trembled.

Trembling is important only on mass points, as there is a mass of probability. Trembling on prices with regular continuous distributions have zero probability to occur, and therefore, are far less significant than trembling on mass points. A good comparison would be first order and second order 'trembling', where the first order is on mass points and the second order over continuous distributions, where any given price is selected with prob. zero (in comparison to mass points).

As the number of mass points with mass above any positive number is finite, it is possible to take an interval without any significant mass points. The searcher observing such a price will think that it is much more probable that a seller trembled rather than such a price was selected by a distribution without mass points. To sum it up, the condition states that for a vector v and a small positive $\delta < \varepsilon_p$:

$$\begin{array}{rcl} & Min(v) = p \in MP & \\ v = & (v_1, v_2, \dots v_{i-1}, p, v_{i+1}, \dots v_k) & \in BS^k \\ & \Downarrow & \\ v - \delta e_i & = (v_1, v_2, \dots v_{i-1}, p - \delta, v_{i+1}, \dots v_k) & \in BS^k \end{array}$$

Remark 3.2.3 *Reserve price condition is a specific case of undercut proof, as under reserve price if a price p is satisfactory then also any price below p is satisfactory.*

3.3 No Reserve Price Examples

Firstly, consider a case where, in a constellation which is not an equilibrium a situation without a reserve price may arise. It will additionally serve as an example to the relevant calculations.

Example 3.3.1 *Consider a case where three sellers each own a single store. Suppose the search cost c is 0.9 and pricing strategies, equally probable from searchers beliefs, are as follows:*

1. Uniform in [1, 9], exp. value of 5

2. Uniform in [5, 9], exp. value of 7

3. Pure strategy of 7.

After observing the price of 7 the searcher is certain with prob. 1 that she had encountered the third strategy seller. An additional search will yield the average between the expected values of the two strategies: 6, making an additional search worthy.

After observing the price of 7.1 the searcher knows that she had encountered one of the mixed strategies, and due to a distribution likelihood ratio - twice more probable that it is the second strategy. Therefore, with probability 1/3 it is the first str. and probability 2/3 the second str.

If the first strategy was encountered, then an additional search will end up in ether second or third strategy - both with expected price of 7.

If it is the second strategy, then an additional search will end up with expected price of 5 or of 7, as both can occur with equal probability (due to the beliefs) expected price in an additional search in this case is 6.

Combining the two possibilities, when taking into account that the second case is twice more probable, the expected price in an additional search is $(2 \cdot 6 + 7)/3 = 6.333$, making another search not profitable.

3.4 Example for ne reserve price Equilibrium

Consider a market with 2 sellers and mass 1 of consumers. The consumers, similarly to Stahl model, are of two types - informed (share μ) and uninformed. The two sellers set their prices sequentially (first one seller chooses price and it is realized and then the second one), at the first stage of the game. Then, informed consumers (shoppers) visit the cheapest store and purchase the item there. Uninformed consumers (searchers) split equally among the sellers and observe the price in visited store. If the price is satisfactory - the uninformed consumer would purchase the item. If the price is not satisfactory - searcher will visit the other store, pay c and observe the other price. Then, she would buy at the cheapest store.

Consider the following limitation: sellers are allowed to choose one of three prices - low P_L, medium P_M and high P_H. Additionally, if both sellers choose the same price (P_M or P_H), exists a regulator which forces both sellers to sell at P_L. This could be due to a regulator that prevents 'collusive behavior', and punishes both sellers if they seem to collude.

Similarly to the Stahl model searchers have beliefs regarding strategies sellers chose, and search further only if the expected price believed to be at the unvisited store is lower than current price minus c.

The utility for seller is her expected market share times the price charged. The utility for consumer is a large constant minus price paid and search costs.

Remark 3.4.1 *The first acting seller has no incentive to mix, as the second seller will observed the realization before a decision is made. Therefore, the first seller would mix only in case of indifference.*

Consider prices as follows:

$$
\begin{aligned}
P_L &< P_M - c \\
P_M + c &= P_H \\
P_L/2 &< P_M(1+\mu)/2 \\
P_L/2 &< P_H(1-\mu)/2
\end{aligned}
$$

The first condition states that if searchers believe someone chose P_L purely, they would look for such price. The second condition states that if no seller is believed to select P_L no search would occur. The last two state that if a regulator is required to intervene, it is a significant loss for both sellers, and a different strategy should be chosen. That is (P_H, P_M) is a better profile for both agents than (P_H, P_H) or (P_M, P_M).

Now we can look at the game form the sight of sellers.

Clearly, if the first seller chose P_L the second seller must choose P_L. Otherwise, all consumers would purchase the item at the first seller. Then the utility for both sellers would be $P_L/2$.

If the first seller chose P_M, the second seller would not choose P_M due to regulator behavior. In the case that second seller chooses P_L she gets all the market. In such case the utilities would be $(0, P_L)$. If second seller chose P_H, all consumers would be satisfied with their first visited store. In such case the utilities would be: $(P_M(\mu+(1-\mu)/2), P_H(1-\mu)/2)$.

Clearly, if $P_L > P_H(1-\mu)/2$ the first seller would avoid such strategy.

Lastly, if the first seller selects P_H, the second seller can choose P_L or P_M. In the first case, as before, the utilities would be $(0, P_L)$ and in the second $(P_H(1-\mu)/2), P_M(\mu+(1-\mu)/2))$.

Again, if $P_L > P_M(1+\mu)/2$ the first seller would avoid such strategy.

Based on the utilities a pure SPE is possible, by simply comparing all actions and preferences and choosing the best one.

3.4.1 Indifference Case

Let us consider the case where most of the second seller utilities are the same. Namely:

$$\pi = P_L = P_M(1+\mu)/2 = P_H(1-\mu)/2 \tag{3.3}$$

Namely, when the first seller chose P_H or P_M the second seller is indifferent between her options.

Now the second seller can mix between two actions: Suppose first seller chose P_M - mixing between P_L and P_H. The probability selecting P_L (denote λ)needs to be sufficiently low, as otherwise the first seller would prefer selecting P_L.

Note that the expected price offered by the second seller could be below $P_M - c$. Then, after observing P_M searching further is better (first seller observed), but after observing P_H stopping is better (second seller was observed).

In the case the first seller selects P_H a pure P_L can be chosen, due to second seller indifference. And, if first seller chose P_L, the second seller would also choose P_L. Then, if $\lambda < 0.5$ the first seller would choose P_M.

To sum it up, if 3.3 holds and $\lambda P_L + (1-\lambda)P_H < P_M + c$ where $\lambda < 0.5$ the following is an equilibrium:

- If first seller selected P_L or P_H, second seller chooses P_L purely.
- If first seller selected P_M, second seller mixes between P_L (prob λ) and P_H (prob. $1-\lambda$).
- First seller anticipating this selects P_M.
- Searchers find P_M not satisfactory whereas the other two prices are satisfactory.

3.4.2 Numeric Example

Consider the previous example with the following parameters:

$$\mu = 0.02$$
$$P_L = 24.99$$
$$P_M = 49$$
$$P_H = 51$$
$$c = 2$$
$$\lambda = 2/13$$

Note that 3.3 holds, and $\pi = 24.99$. Additionally, expected price second seller offers when mixing is (2*24,99+11*51)/13=46.99. Therefore, after observing P_M searchers would search further. The first seller is expected to receive $11/13\pi$, which is better than the zero (if she selects P_H) or $P_L/2$ (if she selects P_L). The profit for the second seller is π, as long as the first seller does not select P_L.

Therefore, this example demonstrates a possible equilibrium without a reserve price. Clearly, several pure equilibria exist (for example, where both sellers always choose P_L).

So, regarding reserve price - it might be the case that it does not exist. However, the literature concentrates on NE with a reserve price. This paper will provide a basic condition for existence of a reserve price, which will involve a specific type of trembling hand perfection. The result states that even equilibria with non-credible threats by searchers would need to satisfy this condition. This is since only behavior of sellers is analyzed, and deviations that yield them direct profit are proposed.

3.5 Results

The main result of the paper is as follows:

Theorem 3 *Suppose that all BS_k are undercut proof. Then all NE of the model have reserve price of P_M, up to zero measure adjustments. Additionally, P_M is the only price which may have mass points in NE.*

The main part of the proof shifted to the appendix.

Here, let me address the zero measure adjustments, for mathematic completeness. It might be the case that all prices below some P_M are accepted, except several prices of measure zero. It is clear that such prices will affect the equilibrium with probability zero, as the probability for such 'unsupported' price to be selected is zero, as mass points may exist only at P_M. For example, consider a continuous distribution on $[a, b]$ with positive density everywhere in the interval. Now, if $c \in [a, b]$ is excluded from the distribution, the probability for the change to make a difference is zero. Therefore, the zero measure adjustments do not have any effect on the equilibrium, with certainty (prob. 1).

3.6 Summary

The result presented here allows extending the consumer search models and allowing dealing with a finite number of sellers. There some NE without a reserve price may exist, but in very limited setting. The condition presented here allows eliminating the possibility of NE without a reserve price, allowing a simple analysis of many search models.

The condition is a natural introduction of trembling hand perfection. When sellers can tremble, or even just believed to be, it is sufficient. Sellers, by such trembling mistake, offer prices just below what they intended, when selecting prices with mass points, as those are more significant than continuous distribution. When one adds to beliefs such trembling hand perfection, the only equilibria possible in the Stahl model are ones with reserve price. Note that a trembling hand upwards is less plausible, and can be a strategic behavior of a seller to gain a bit more profit. Note that this is a one direction trembling, which gives a slightly better price offer to consumers. Therefore, consumers are inclined to accept this type of trembling. Trembling to a higher price would be less accepted, as then consumers are the ones paying the cost of the trembling, and giving a slightly higher benefit for sellers.

Further research can deal with additional search models and extensions thereof in various fields. Testing whether such a condition is a sufficient one also in other areas. This will allow making the search models more extended and realistic. An additional further research is to identify NE without a reserve price and find a characterizing condition on when those are possible.

Chapter 4

Minimum Price

«Хотели как лучше, а получилось как всегда»

„We wanted the best but it turned out as always“

- V. Chernomyrdin, Prime Minister of Russia (1992-1998)

Abstract

This paper investigates the effects of a low bound price. To do so, a popular and empirically proven model (Stahl (89') [39]) is used. The model is extended to include an exogenously given bound on prices sellers can offer, excluding prices below such bound. The finding are rather surprising - when the bound is set sufficiently high expected price offered (EPO) by sellers drops significantly. The result seem to be robust in the parameters of the model, and driven by the information provided to consumers by such legislation step: when the limitation is set at sufficiently high levels all consumers anticipate the bound price, and searchers reject any price above it. As a result sellers offer the bound price as a pure strategy.

4.1 Introduction

Despite having free markets, it is often the case that governments intervene in trade. The most common example of such is taxation, where certain goods are taxed. This is done despite the known results on welfare impact of such interventions. However, in some cases, e.g. Alcohol or Tobacco products, the government sees a reason to reduce the consumption by imposing a tax. One of the reasons behind it is noted by Cnossen and Sijbren in [13], table 8, an impressive literature connects higher price to lower consumption of alcohol, in coordination with common belief, and helps to reduce many of the high costs of Alcohol consumption - over 120 bn Euros in 2003 only.

An additional new step of intervention is suggested by the Scottish government. Scotland proposes a minimum price per alcohol unit (10 ml) of 0,40£. Meng et al. in a report by Sheffield university [32], perform a research that concludes that such a step would increase prices on alcohol. Clearly, in cases when this bound falls below the market price it has little effect and it would increase prices when it falls above market price level. As shown in Table 2.1 in Meng et al. [32], the latter happens in many cases, but not in all. Here we investigate what happens in the interim case, when the minimum price is set at market price level. Namely, some retailers sold products below this price, and similar products were also sold above the limit. Clearly, the interesting case is when the average market price is above the limitation. The prices are taken from English supermarket, however due to the close relation between England and Scotland, and a similar step being planned by the English government, it is a relevant example.

To emphasize, suggested minimum prices would be as follows:

- Liter of 5% beer would cost at least 2£(around 2,5€).
- A 750 ml. bottle of 15% wine - 4.5£(around 5,6€).
- Half liter of 40% Vodka - 8£(around 10€).

To add to the relevancy of the question, a product example is provided. Note that 4x440ml of Bavaria beer 2.8% lager beer has suggested minimum price is 1.97. On 25.10.2013, prices were varying between 1.79 and 2.99 £, as depicted on figure 4.1.[1]. Therefore, the suggested limitation falls in the relevant area - some sell above and some sell below, with average price above the limitation.

The existing empirical literature, for example, Stockwell et al. [41] and by Meng et al. [32], imply that introducing minimum price would increase the expected prices of the relevant goods. However, there is one main aspect to look at. The minimum price imposed by the

[1]http://www.mysupermarket.co.uk

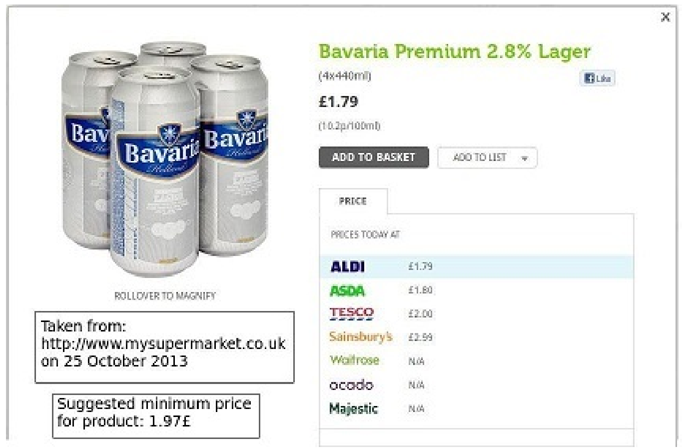

Figure 4.1: Price of 4 pack Bavaria Lager beer from mysupermarket.co.uk

Scottish legislator is not necessarily higher than the previous market price of the good, as noted by Meng et al. [32] table 2.1. In case when the limitation is set above market price, it is clear that prices would rise, as a result of such step. Similarly, if the minimum price is set too low it would have no effect on pricing as there is no need for anyone to adjust price. However, this paper wishes to investigate what happens when minimum price is set moderately. This implies setting the minimum price at some existing market price. The question asked here is what happens in the interim case, where a minimum price is set at price levels which are in equilibrium support, as seen in the example at figure 4.1. The model analyzed here has mixed strategy equilibrium. Therefore, it leaves a range of prices in the relevant interval.

This paper takes a deeper look into introducing lower bounds on price, keeping positive profit for sellers in order to keep them on the market. This is done by analyzing a popular search model, and comparing its equilibrium to one where a lower bound price is introduced. The model analyzed is perhaps one of the most popular search models in the literature - the Stahl Search model, introduced by Stahl [39]. This simple model has good tractable solution, and therefore has a wide literature. Moreover, Janssen et al. [27] find that this model indeed predicts product pricing of many goods, and finds that pricing of 86 from the 87 tested products is in line with the Stahl Model. Additionally, Baye et al. [5] points out that the consumer types suggested by the model are empirically profound. Therefore, such an important model provides a good indicator for checking introduction of lower bounds on prices.

4.1.1 Results Overview

At the first glance, the findings are rather surprising - when the lower bound is chosen at a certain level, not only that prices sink, but selecting the lowest allowed price by all sellers is equilibrium. In an example it reduces the expected price offered (EPO) by approximately 13%! Most importantly, the minimum price minimizing the EPO depends only on parameters of consumer side, and will not be affected by entry of additional sellers. This would keep the lower bound fixed trough changes on the supply side. Additionally, in the case of two sellers it would be a unique equilibrium. Moreover, informed consumers are indifferent and all the benefits are to the uninformed consumers.

Comparative statics suggest that information drives the results. The more uninformed consumers are the larger is the difference to the original Stahl model equilibrium. Those receive a credible signal on what are the prices on the market, and take it into consideration when checking prices. Note that since the minimum price ensures positive profit for sellers, it is profitable for sellers to sell already at the minimum price. This would not hold if the minimum price would be replaced with a governmental tax.

When the minimum price is set sufficiently high the structure of the equilibrium is a pure one, at the minimum price level. This implies that the entire market has the same minimum price, in the spirit of Bertrand competition (for more on Bertrand competition see, for example, Baye and Morgan [8]). Minimum price at lower levels will impose a mass point at the minimum price, and some continuous distribution with lower prices than in the original Stahl model equilibrium. The implication is clear - as long as the minimum price is not set sufficiently high or sufficiently low it will reduce market prices, instead of increasing them. Moreover, pricing competition would be reduced since all sellers would use the same pure strategy.

The reasoning behind such behavior is simple. When the limitation is set at sufficiently high levels, but below the expected price in equilibrium, the signal is credible. Consumers believe this limitation plays a significant role, and as a result prices would be at the bound level or close to it. As a result, also the uninformed consumers receive a signal on what are the prices in the economy, and expect the bound price to be offered. Thus, searchers set their reserve price sufficiently close to the bound price. Sellers anticipating such a behavior would find it profitable to offer the bound price purely, in order to compete for the informed consumers, in a Bertrand competition. Therefore, pure equilibrium at the bound price prevails. When the limitation is set at lower levels this effect diminishes. Now searchers believe that sellers are less probable to be affected by such legislation. As a result, reserve price would be set at a higher level. From sellers' perspective, a mass point on the bound level would be set, with some probability mass above it, but EPO remains below the one in original Stahl model.

As a result the expected price in the market drops when such minimum price is introduced in a market with two or three sellers, and according to Stahl [39], also could hold for a larger number of sellers. This is due to a conjecture from Stahl [39] that reserve price (and thus also EPO) rises with the number of sellers. A table with an example for such increase is provided in Stahl [39]. The logic for such increase is simple - more consumers reduce the chance to be the cheapest seller, and therefore decrease the motivation to offer lower prices.

This paper suggestion is clear: one should be careful when setting a minimum price, as it can reduce prices and actually increase demand. Consider the example in figure 4.1. If the legislator sets the lowest price at 1.97 £, consumers would be aware of this limitation. They would expect the price not to be too far above the limitation. Therefore, 2.99 £offered by Sainsbury's would not be an acceptable price for consumers. The seller anticipating it would need to reduce price closer to the limitation price.

The structure of the paper is as follows: First the Stahl model is introduced, and relevant results on it are provided. Then, minimal price is imposed on the Stahl model, followed by results and an example where prices are lower due to such a step. Afterwards, the specific case with two sellers is looked upon. Lastly, some additional characteristics of the price reducing equilibrium are provided, followed by a short discussion. Larger proofs are shifted to the appendix.

4.2 Stahl Model

The Stahl model, as introduced in Stahl [39] is formally described below. Notation was adjusted to the recent literature on the Stahl model.

There are N sellers, selling an identical good. Each seller owns a single store. The production cost is normalized to 0, and assume that seller can meet demand. Additionally, there are consumers, each of whom wishes to buy a unit of the good, evaluating it at some high M. The mass of consumers is normalized to 1. This implies that there are many small consumers, each of which is strategically insignificant.

The sellers are identical, and set their price once at the first stage of the game. If the seller mixes then the distribution is selected simultaneously, and only at a later stage the realizations take place.

The consumers are of two types. A fraction μ of consumers are shoppers, who know where the cheapest price is, and they buy at the cheapest store. In case of a draw they randomize over all cheapest stores, uniformly. The rest are searchers, who sample prices. Sampling price in the first, randomly and uniformly selected, store is free. It is shown in Janssen et al. [28] that if it is not the case then some searchers would avoid purchase and the rest would behave as in the original model. If observed price is satisfactory - the searcher will buy there. However, if the price is not satisfactory - the searcher will go on to search in additional stores sequentially, where each additional visit has a cost c. The second (or any later) store is randomly selected from the previously unvisited stores, uniformly. The searcher may be satisfied, or search further on. When a searcher is satisfied, she has a perfect and free recall. This implies she will buy the item at the cheapest store she had encountered, randomizing uniformly in case of a draw. The searchers have an endogenous reserve price, P_M, which determines when they are satisfied.

The consumers need to be at both types (namely, $0 < \mu < 1$). If there are only shoppers - it is the Bertrand competition setting, (see Baye and Morgan [8]), and if there are only searchers the Diamond Paradox (Diamond [19]) is encountered, both well studied.

Before going on, make a technical assumption on the model. In order to avoid measure theory problems it is assumed that mixing is possible by setting mass points or by selecting distribution over full measure dense subsets of intervals. This limitation allows all of the commonly used distributions and finite combinations between such.

Additionally, a couple of very basic results are introduced:

- Sellers cannot offer a price above some finite bound M. This has the interpretation of being the maximal valuation of a consumer for the good.
- Searchers accept any price below c. The logic behind it is any price below my further search cost will be accepted, as it is not possible to reduce the cost by searching further.

4.2.1 Game Structure

The game is played between the sellers, searchers and the shoppers. The time line of the game is as follows:

At the first stage, sellers set their pricing strategies and simultaneously searchers set their common reserve price. Then, if some sellers used a mixed strategy, realization of mixed strategies is taking place. At the second stage shoppers observe all the realized prices and purchase item at the cheapest store. At the third stage, searchers sample a price randomly (uniformly) selected. If a searcher is satisfied she purchases the item. If not - she pays c and observes another price, and so forth until ether she is satisfied or sampled all prices. When the searcher observed all stores and observed only unsatisfactory prices she would buy at the cheapest store encountered.

When reserve price and pricing strategies are being determined the knowledge of the various agents of the game is as follows:

- Sellers have beliefs regarding the reserve price set by searchers

- Searchers have beliefs about which pricing strategies were actually played by the sellers.
- Shoppers will know the real price in each store in the moment it is realized.

The probability that seller i sells to shoppers when offering price p is denoted $\alpha_i(p)$. Let q denote the expected quantity that seller i sells when offering price p. The expected quantity sold by a seller consists of the expected share of searchers that will purchase at her store, plus the probability she is the cheapest store multiplied by fraction of shoppers (μ), and is also the expected market share of the seller.

Note that the reserve price ensures that the searcher will purchase at the last visited store, unless all stores were searched.

4.2.2 Utilities and Equilibrium

As the sellers set their price at the start of the game, they examine the expected utility that would be obtained due to their pricing strategy. Searcher decision whether to purchase the item or search further is taken on the later step of the game, when exact utility is available. Therefore, sellers have an ex ante expected utility, whereas searchers and shoppers have ex post utility, as follows:

- Seller utility is price charged multiplied by expected quantity sold.
- Consumer utility is a large constant M, from which item price and search costs are subtracted.

Note that since sellers make their decision at the first stage of the game their utility can be expressed only in expected terms. Consumers know how much they pay for the good when they purchase it. Therefore, for searchers there is an exact expression for utility.

The NE of the game has a Bayesian structure, and is as follows:

- Searchers have a reserve price (P_M).
- Searchers beliefs coincide with seller strategies played.
- Seller have beliefs regarding the reserve price which coincide with P_M.
- Reserve price is rational for the searchers

- No seller can unilaterally adjust the pricing strategy and gain profit in expected terms.

Remark 4.2.1 *As the sum of the searcher and seller utilities may differ only in the search cost, any strategy profile where all searchers always purchase the item at the first store visited is socially optimal.*

The original article also pins down the unique symmetric equilibrium of the model:

Theorem 4 *Stahl, (89') [39]*
In the model described above exists a unique symmetric NE, where all sellers use a continuous distribution function F on the support between some P_L and P_M, which is the reserve price.

An additional important result involves the equilibrium price distribution. Since a seller is indifferent between all strategies in support when mixing she must have equal profit for all prices she offers. Thus in the original model equilibrium the profit equality implies:

$$P\left((1-F(p)^{n-1})\mu + \frac{1-\mu}{n}\right) = P_M\frac{1-\mu}{n}$$

From here follows that:

$$F = 1 - \sqrt[n-1]{\frac{1-\mu}{n\mu}(\frac{P_M}{P} - 1)} \tag{4.1}$$

Since $F(P_L) = 0$ we get that:

$$\frac{1-\mu}{n\mu}(\frac{P_M}{P_L} - 1) = 1$$

Which in turn implies that:

$$P_L = P_M\frac{1+(n-1)\mu}{1-\mu}$$

An additional connection exists between the reserve price P_M and the distribution F:

Lemma 4.2.1 *In a seller symmetric equilibrium the reserve price is exactly c above the expected price of F.*

Proof:

Firstly note that in a symmetric equilibrium observed price does not reveal any additional information about prices in other stores. Therefore, the decision whether to search further or not depend only on the believed expected price of sellers. If this expected price is sufficiently lower than observed prices - search will go on. If not - the searcher would be satisfied.

If $P_M - E(F)$ would be above c - searchers would not be satisfied with price offers of P_M, and search further. If it would be below c - searchers would be better off accepting prices up to $E(F) + c$. Since sellers know this, they would offer $E(F) + c$ and expect the searchers to accept such price. The claim of searchers to reject prices below $E(F) + c$ is not credible. $\square$

One can be surprised regarding such equilibrium structure, and mixed seller strategies. The economic intuition behind it is quite simple. Low prices have high probability to attract informed consumers, and will have a higher market share. Higher price is attractive to extract information rent form the uninformed consumers. In equilibrium these two motivations have equal weight, and therefore a mixed equilibrium prevails.

4.3 Minimum Price

Up to here we were discussing the original Stahl model with its known results. Now we wish to introduce a price limitation in the model. Consider a legislator, who wishes to introduce a minimum price, below which sales are forbidden. For that let us consider the implications of such a step in the Stahl search model.

Let us denote the Stahl model with a minimum price limitation as the limited model. Additionally, fix c, μ and N. Let us reserve to P_M as the reserve price in the corresponding original Stahl model equilibrium, and F the equilibrium distribution.

Definition 4.3.1 *Let us define the price P^* as follows:*

$$P^* = c\frac{1-\mu}{\mu}$$

Theorem 5 *Consider the Stahl model with the parameters N, c and μ. Let P_C be a price weakly above P^*. Suppose that a minimum price of P_C is imposed. In the limited model exists a unique pure strategy equilibrium, where all sellers select P_C purely, and $P_C + c$ is the searcher reserve price.*

Proof shifted to the appendix.

Remark 4.3.1 *Note that any equilibrium where $P_C > P_L$ must have mass points at P_C. The reason is simple - without mass points at P_C there is no motivation to go below this* *price, as it already attracts shoppers with certainty. Therefore, such equilibrium would prevail also without a cutoff price. However, we know that such equilibrium must have P_L in support which is not possible.*

This theorem provides the first important result. If the minimum price is set sufficiently high we will receive a pure equilibrium where all sellers select reserve price purely. Searchers treat such a signal as important for the market and anticipate that prices should be around the minimum price. Sellers anticipating it, and therefore do not offer prices higher than $P_C + c$. However, the share of informed consumers is sufficiently high to make P_C more attractive than $P_C + c$, making a pure equilibrium. This is since at P_C also informed consumers would purchase the item in my store, and at $P_C + c$ only uninformed consumers originally visiting my store would buy there.

4.3.1 Lower Prices

An important question is whether price offers are lower due to this limitation. This question is dealt with next.

Definition 4.3.2 *Let us denote the ratio between the expected price offered (EPO) in the original model and P^* as β. If $\beta > 1$ the expected price offered in the limited model is lower than in the original model.*

The following lemma suggests a condition which ensures that the expected price offered in the limited model.

Lemma 4.3.1 *The expected price offered in the limited model with minimum price of P^* is lower than the one in the original model iff $P_M > c/\mu$.*

Proof:

Note that the expected price offered in the original model is given by $P_M - c$, and $P^* = c\frac{1-\mu}{\mu}$. Therefore:

$$\beta = \frac{\mu(P_M - c)}{c(1-\mu)}$$

The expression is larger than one iff:

$$\mu(P_M - c) > c(1-\mu)$$

after simplifying we obtain the required condition. $\square$

Note that this condition involves an endogenous parameter P_M. When it could be explicitly found, a different, exogenous condition could be provided. However, this is a condition which may be true or false, depending on the parameters of the model. As noted by Stahl in [39], there is a monotone connection between P_M and μ. Lower μ (less informed consumers) imposes higher reserve price, as the motivation to attract shoppers decrease.

As a conclusion, if in an equilibrium of the original Stahl Model it is the case that $P_M(N, \mu, c) > c/\mu$ then it is possible to reduce the payment of uninformed consumers by introducing a minimum price of P^*. The uninformed consumers sample a store randomly, and in expected terms will observe the expected price sellers offer. Therefore, uninformed consumers benefit from such a step.

Note that if this condition does not hold, the reserve price is rather low in comparison to P^*, as P^* would be above $P_M - c$. Only then any price limitation which can be possible would not reduce EPO. Therefore, initially the information rent was rather low.

Below a numeric example with 2 sellers is provided. It is suggested in Table 1 in Stahl [39] that the reserve price increases with N. Therefore, the case with 2 sellers probably has the lowest reserve price. The reason behind such a motivation is simple - when more sellers are in competition, it is harder to be the cheapest seller decreases. Therefore, there is less motivation to offer discounts, which drive the prices up.

4.3.2 Example

Example 4.3.1 *Consider the Stahl model with 2 sellers, search costs of c and $\mu = 1/3$.*

In the example $\mu = 1/3$, such that $(1-\mu)/N = \mu$. Original model equilibrium distribution is $F(p) = 2 - P_M/P$ and $P_L = P_M/2$. The expected price of a seller would be then $E = P_M(ln2)$. Since $E + c = P_M$ we get that $c = P_M(1 - ln(2)) \approx 0.3P_M$.

Comparing the expected price offered and $c(1-\mu)/\mu = 2c$ we get that such an equilibrium would be more profitable for consumers, as prices offered in the equilibrium with minimum price set at $c(1-\mu)/\mu$ are lower than E.

Applying the definition of β it is visible that expected price offered is about 13% less than in the original model equilibrium:

$$\beta = \frac{P_M - c}{c(\frac{1-\mu}{\mu})} + \frac{P_M \ln(2)}{2P_M(1 - \ln(2))} \approx \frac{69}{61} \approx 1.13$$

Therefore, in this case the condition holds and the expected price offered is indeed lower, and is lower by about 13%.

4.4 Two Sellers

In the case of two sellers it is possible to provide some additional results. The main reason behind it is the ability to calculate the expected value of original model equilibrium distribution F. Remember from equation (4.1), that in the general case it is:

$$F = 1 - \sqrt[n-1]{\frac{1-\mu}{n\mu}(\frac{P_M}{P} - 1)} \tag{4.2}$$

And there is no general explicit expression for the expected value of F. However, in the case of two sellers, F looks as follows:

$$F = 1 - \frac{1-\mu}{2\mu}(\frac{P_M}{P} - 1)$$

In such case the expected value is given by:

$$P_M \frac{1-\mu}{2\mu} \ln\left(\frac{1+\mu}{1-\mu}\right)$$

Let us denote $\ln \frac{1+\mu}{1-\mu}$ as κ.

Since $E(F) + c = P_M$, we can obtain a value for the reserve price:

$$P_M = c\frac{2\mu}{2\mu - (1-\mu)\kappa} \tag{4.3}$$

From here it is possible to obtain several additional results. Firstly, a lemma shows that for any two seller Stahl model introducing a low price bound would reduce prices.

Lemma 4.4.1 *The condition from lemma 4.3.1 holds for the case of two sellers.*

Proof shifted to appendix.

Remark 4.4.1 *Stahl [39] argues in table (1) that increasing the number of sellers increases the reserve price (for fixed μ, c). If this is true, then introducing P^* will have the positive implication for searchers, and the pure equilibrium will hold.*

Proposition 4.4.1 *Consider a two seller Stahl model, with a limitation is imposed at price P^*. In such case the equilibrium, as shown in theorem 5 is unique, and no additional mixed or asymmetric equilibria exist.*

Proof of proposition is shifted to the appendix.

A very important question is whether shoppers benefit or not from such equilibrium. The surprising answer is that when the lowest possible they are indifferent when P^* is selected.

Lemma 4.4.2 *Let $N = 2$. Suppose sellers mix independently. Then the expected offer the shoppers observe is exactly equal to P^*.*

Proof:

This follows directly from the expected value of minimum value from two iid variables with the distribution $F = 1 - \frac{1-\mu}{2\mu}(\frac{P_M}{P} - 1)$. Remember that $\min(F_1, F_2)$ is distributed with $1 - (1 - F)^2$.

Then, the expected price shoppers' encounter is given by the distribution:

$$G(p) = 1 - (\frac{1-\mu}{2\mu}\frac{P_m}{p} - 1)^2 \tag{4.4}$$

The expected value of such a distribution is:

$$E(G) = 2(\frac{1-\mu}{2\mu})^2 P_M(\frac{2\mu}{1-\mu} - \log(\frac{1+\mu}{1-\mu})) \tag{4.5}$$

Setting the value of P_M from equation 4.3 into the equation leads to $E(G) = c\mu/(1-\mu)$. $\square$

The two seller case allows us to perform some additional analysis. When $P_C < P^*$ we have equilibria of a different form, as given by corollary C.0.4. There, sellers set a mass point with mass $\rho(P_C)$ on the minimum price P_C, and a continuous distribution over a parameter and P_C dependent interval (P_N, P_M) where P_N is strictly larger than P_C. An important result for comparative statics is below, explaining how does the reserve price (and thus also the expected price offered which is c below P_M) change:

Lemma 4.4.3 *In such equilibria P_M is decreasing in ρ.*

Proof shifted to the appendix.

Thus, increasing P_C between P_L up to P^* increases ρ and slowly decreases the EPO, up to a level where it reaches the lowest value at P^*. Higher values keep the pure equilibrium but with a higher EPO, as sketched in figure 4.4.1.

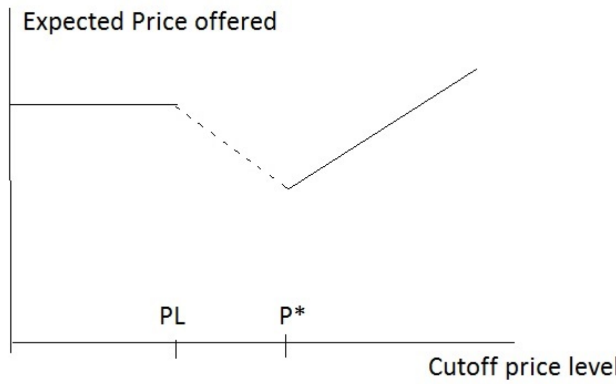

Figure 4.2: change in EPO as func. of minimum price(2 sellers)

Corollary 4.4.1 *Consider the Stahl model with two sellers. For any minimum price in the interval $(P_L, P_M - C)$, the resulting EPO would be lower than in the original equilibrium.*

Following corollary C.0.5, larger mass point at minimum price implies minimum price closer to P^*. Since it is strictly monotone, the equilibrium would be unique. Note that there will be no jump at P_L, due to the continuous nature of the change from the original equilibrium when ρ is small. Now, it is possible to sketch equilibrium behavior as a function of the minimum price, as done in figure 4.2 for the case of two sellers. Note that the slope between P_L and P^* can be of a different nature, as it is only a sketch.

4.4.1 Two Sellers Summary

So, for two sellers some strong results are available. Firstly, when limitation is set at P^*, it is always a good idea to introduce a minimum price if consumers benefit is before the eyes of the decision maker. The resulting equilibrium is unique. If one is concerned about the informed consumers - those are indifferent between the two possibilities and therefore, in total consumers are better off.

Additionally, no matter where the minimum price is set, EPO would drop. It is not only the specifically picked P^*, but any price in the range $(P_L, P_M - c)$ that have such an effect. The effect is global over the entire range of prices. Any level of minimum price below original EPO where it is not possible to use the original equilibrium will not increase market prices. Clearly, it is maximized at P^*, and is lower the further we are from it, but it still prevails. When the limitation is set above P^* we will have a pure Bertrand equilibrium, where all sellers set the limitation price. When the limitation is at p, where

$p < P^*$, then there would be a mass point at p with some additional distribution mass between two prices strictly above p and below original model reserve price.

4.5 More than two sellers

When more than two sellers are involved, the picture starts to be more complex. Firstly, asymmetric equilibria are possible, as noted by Astone-Figari and Yankelevitch [2]. The comparison so far was on symmetric equilibria, and additional ones can be completely different than the ones introduced before.

However, even in the symmetric equilibrium difficulties arise. An additional complexity rises when one tries to calculate the expected value of the original model price distribution. Remember that from equation (4.1):

$$F = 1 - \sqrt[n-1]{\frac{1-\mu}{n\mu}(\frac{P_M}{P} - 1)}$$

Unfortunately, there is no general explicit expression for the expected value of F, which would cover all possible number of sellers. Therefore, most of the results cannot be proven for a general number of sellers.

4.5.1 Three Sellers

It is possible to calculate explicitly the expected value denoted in equation (4.1) when the number of sellers is three.

Using Matlab and calculating the reserve price P_M for the case of 3 sellers, the following expression was obtained:

$$\frac{E(F)}{P_M} = \arctan\left(\sqrt{\frac{3\mu}{1-\mu}}\right) \cdot \sqrt{\frac{1-\mu}{3\mu}}$$

Note that $E(F) = P_M - C$, and from here follows:

$$P_M = \frac{c}{1 - \frac{E(F)}{P_M}} = \frac{c}{1 - \arctan\left(\sqrt{\frac{3\mu}{1-\mu}}\right) \cdot \sqrt{\frac{1-\mu}{3\mu}}}$$

As we have found, in two sellers case the reserve price was given by:

$$c = P_M(1 - \frac{1-\mu}{2\mu}\ln\frac{1+\mu}{1-\mu})$$

Figure 4.3 states the difference between reserve prices for a given μ with 2 and 3 sellers is provided below. It clearly shows that the reserve price for 3 sellers is higher for any $\mu \in (0,1)$. Additionally, lower μ increases the difference in reserve price. From here follows:

Lemma 4.5.1 *For the case of 3 sellers imposing a lower bound on price at the level of P^* would reduce the expected price offered. The percentage prices drop by slightly more than what was with 2 sellers.*

Fix $\mu \in (0,1)$ and c. Let P_2 be the reserve price in the corresponding original Stahl model with 2 sellers, and P_3 with three sellers. Then, $P_3 > P_2$. This fact is immediate from figure 4.3.

Note that since P^* is independent in the number of sellers and expected price offered is $P_M - c$, higher P_M implies more beneficial equilibrium for searchers.

Corollary 4.5.1 *The condition from lemma (4.3.1) holds also for three sellers.*

Since the condition is $P_M > c/\mu$, if it holds for P_2, it would also hold for P_3. This is in line with Table 1 in Stahl [39], suggesting that reserve price is rising with the number of sellers.

An additional result available for three sellers is shoppers' indifference:

Lemma 4.5.2 *Consider the original three seller Stahl model. Informed consumers expected price is exactly P^*.*

Similar to the two seller case, applying the expected value of minimum of three F distr. variables would yield the result.

Remember that $P^* = c(1-\mu)/\mu$. Thus, we can calculate:

$$P^* = P_M(1 - \arctan\left(\sqrt{\frac{3\mu}{1-\mu}}\right))$$

Additionally, the distribution of shoppers price is given by $1-(1-F)^3$. Calculating the expected value would yield P^*. $\square$

Remark 4.5.1 *One can repeat the exercise with any desired number of sellers (n) and a desired share of informed consumers (μ). Then, a numeric calculation, or, perhaps, in some cases even an analytic expression, for the expected value of F can be found. Then it is possible to calculate P_M and verify whether it is above $P^* + c$.*

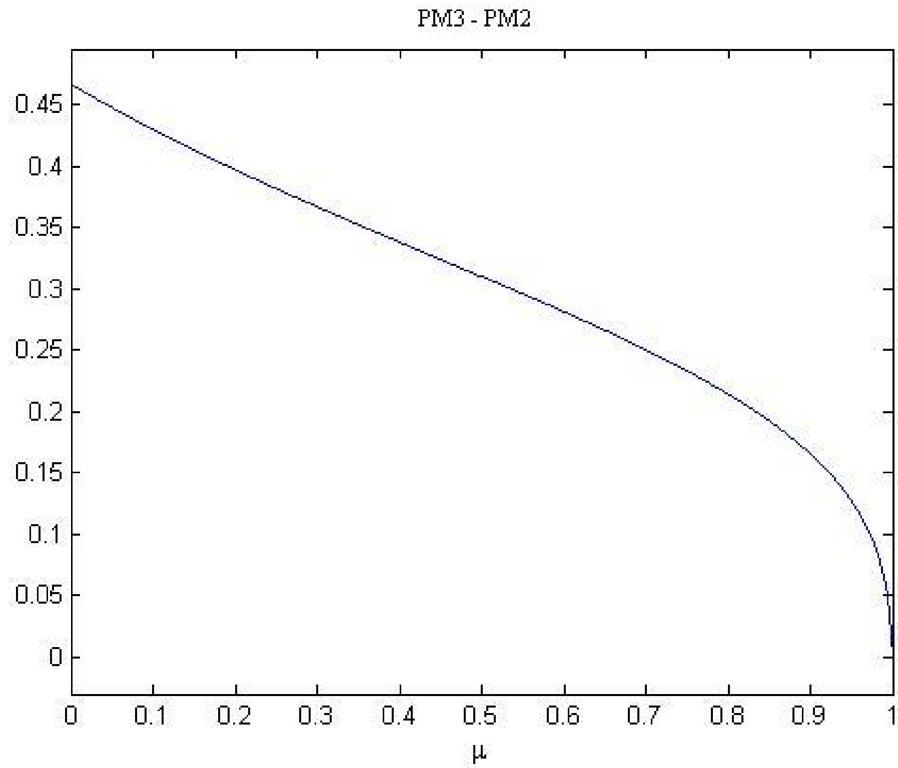

Figure 4.3: P_M with 3 sellers minus P_M with 2 sellers as func. of μ, in c units

4.6 Discussion

A very important question is on the intuition for such a result. A hint can be provided from μ comparative static. As Figure 4.4 suggests, the lower the share of informed consumers, the higher is the effect of introducing such a bound. Additionally, if the price limitation is set **above** P^*, based on theorem 5 - the pure equilibrium would still exist and is unique. Therefore, no additional gain for consumers can be obtained.

Note that if the limitation is set at P^* the information gain of shoppers is exactly zero. They do not get better offers due to their knowledge, but get the same offer as all consumers. Moreover, they get the same offer as before (in expected terms). Therefore, a possible explanation to this phenomenon is information. The law provides additional information to the uninformed consumers regarding what is cheap and what is not. Using this information searchers form more informed beliefs and get a better deal when purchasing the item.

Additionally, the signal needs to be sufficiently credible. If the bound is set too low, no consumer would believe that sellers would go THAT low on pricing. For example, a low bound set below the support of the original Stahl model would probably have zero effect on results, as all sellers would keep on playing the original model equilibrium. Therefore,

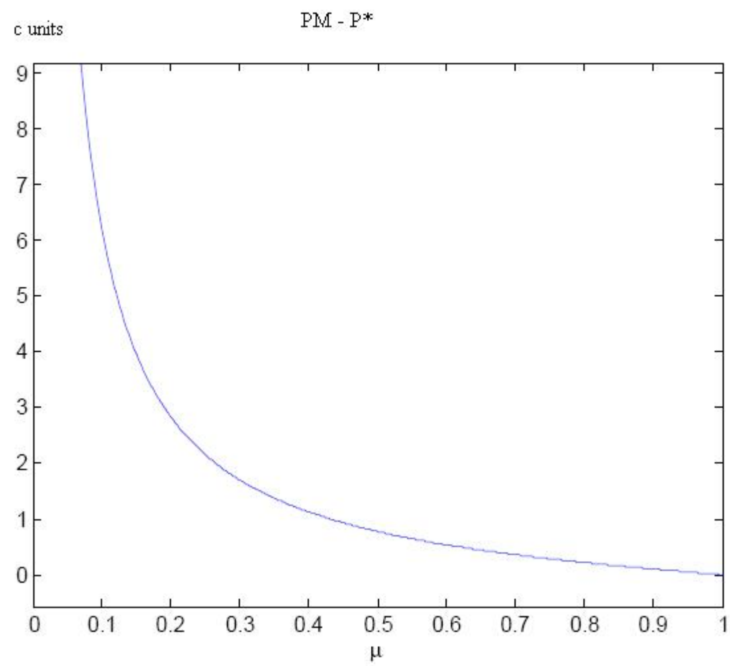

Figure 4.4: $P_M - P^*$ in c units as func. of μ (2 sellers)

this bound needs to be set sufficiently high in order to be credible.

As a result searchers set their reserve price sufficiently close to the bound price. Sellers anticipate it, and since no seller wishes to be above the reserve price, the price dispersion is lower. Additionally, since there is a motivation to attract shoppers and be cheapest, a seller would compare the prices scope available to her. If other sellers have sufficient mass at the bound price, she would not want to deviate from it too. Since the information rent from searchers is low, due to their lower reserve price, it is more attractive to compete for the informed consumers.

Therefore, as an outcome we receive the pure equilibrium.

When the bound price is set at a lower level, this effect diminishes and there are mass points at P_C. The mass on P_C decreases with the bound. Still, the expected price and reserve price are below the original model equilibrium.

4.6.1 Model Relevancy

The following characteristics are important for the model and results:

- Homogeneous goods with same production costs

- Fixed demand which does not depend on the price
- Information asymmetries among consumers
- Small number of big players
- Large price dispersion
- Reserve price is above c/μ.

The first three characteristics are basic for the Stahl model. Firstly, in the Stahl model all consumers end up purchasing the good, no matter the prices. This could reflect a market of essential goods, such as electricity supply, bank account, or as seen by many, cellphone and Internet connection. One may argue that Alcohol is not best described, but as found by Jannsen et al. in [27], the Stahl model is significant also for other markets. Additionally, the Stahl model involves a homogeneous good and some asymmetry among consumers. The next point is due to certain results in case of several sellers. This can motivate a covert step to reduce profits in an oligopolistic market. If the market does not have big players or no big ability to fight politicians, then a more harsh approach can be used, as severe taxation. If there are big players with policy influence, then perhaps such a step can be used to reduce their revenue. The next point elaborates on the price reduction. Large price dispersion implies large information rents which would be absent in the pure equilibrium, and imposes also the last, endogenous point.

The last point is a condition for the new equilibrium to reduce prices, as seem in lemma 4.3.1. Unfortunately, it is not possible to give a general analytic expression to P_M, and therefore it involves this endogenous parameter. If this condition does not hold then the lowest possible price reduction is too close to the original model reserve price. Therefore, it is not possible to benefit consumers. However, as shown in lemmas 4.4.1 and 4.5.1 it holds for two and three sellers. Note that it may also generally hold, as table 1 in Stahl [39] suggests that the reserve price increases with N. If it is indeed the case, then the condition depicted in lemma 4.3.1 will hold for any number of sellers, with similar logic to the one presented in lemma 4.5.1.

The strongest result is presented in lemma 4.4.1. The effect is prevailing over an interval of prices. Moreover, it is the maximal possible interval, since minimum price below P_L allow original equilibrium and prices above $P_M - C$ only allow prices above EPO. Therefore, setting a minimum price at a market price level has an opposite effect to a possible original intuition.

4.7 Summary

This paper studies the impact of a lower bound on the price. If such lower bound is introduced, it may possible to reduce price significantly, and the equilibrium would be unique. Clearly, when the price limitation is set below the lowest price on equilibrium it would have no effect, and when it is set above the expected price in equilibrium the prices would be higher. In the intermediate cases the effect is rather surprising at first sight. In the case with 2 sellers, for the entire interval the expected price in equilibrium with minimum price is lower. Thus, when the lowest allowed price set at any relevant price level, expected price offered by sellers gets significantly lower. Still, all sellers have a certain positive, though lower, profit. Therefore, it is not expected that due to such change sellers will go bankrupt. The lowest EPO is obtained when the limitation is set at the level of P^*, and it reduces prices for the case of 2 sellers, and, judging by Table 1 in Stahl [39] probably also with a general number of sellers.

The price reduction is increasing in the amount of uninformed consumers, suggesting that the reason driving the result is providing uninformed consumers with valuable information. This helps consumers to define what is cheap, and what can be expected, due to such clear legislation. The fact that at the level price P^* informed consumers get the same offer (in expected terms) strengthens this intuition. Lastly, when price limit is set above the critical level of the pure NE, the pure equilibrium still prevails, but with a higher price offered by sellers one to one. Thus, when the information rent is zero, increasing the limiting price has only negative effect on consumers, due to a ban on lower prices.

If the original Stahl model equilibrium uses prices between P_L and P_M (with EPO at $P_M - c$), then, in the case of two sellers setting the minimum price at levels between P_L and $P_M - c$ prices would drop. The drop in prices would be maximized at $P^* = c\frac{1-\mu}{\mu}$. On both sides of P^* it changes continuously. Therefore, one must be careful when setting a minimum price, as it may have an effect opposite to the intention of the legislator. If the limitation is set weakly above P^* the resulting equilibrium would be a pure one, when all sellers set their price at the limitation price. When the limitation is set lower, sellers share a mixed strategy: P_C would have a mass point, and a distribution between two prices strictly above P_C. However, EPO still remains below the level of the original equilibrium.

Additional effect of introducing a low bound price is limiting predatory pricing, for example see Snider [38] and Bolton and Scharfstein [10]. This would allow additional players to enter the market on the seller side increasing competition. In the real world many markets are oligopolistic and deter new entry, which would not be possible once low bound prices are introduced. This will impose an additional positive impact on many markets, and make them far less oligopolistic. This could add an additional discount in pricing, due to higher number of entrants, which fall beyond the scope of this paper.

4.7.1 Future Research

This paper is, to the author's knowledge, one of the first papers suggesting that imposing a minimum price would reduce prices. Opening a new door often adds many questions and new directions. Some of them are introduced below.

A natural next step is taking these results to the lab, or the real world. Empirically compare offered prices in the original and price limited cases and verify the results here. Then it would be politically possible to offer such measures also in the real world, as today no policy maker would consider such step as price reducing.

Additional theoretic steps that provide interest include comparative statics. How changes in μ and N affect the results, and price reduction in the most general case. The results here suggest larger benefit when μ is lower for 2 or 3 sellers, as does shifting from 2 to 3 sellers. General result for any number of sellers would be very useful, yet the general expression may be not analytic. An additional robustness check would be to look at possible asymmetric equilibria of the model, and verify whether in all of them price would be reduced.

An additional important step, is checking the case of heterogeneous searchers, in a model similar to Stahl [40], or heterogeneous sellers, in a similar model to Astorne-Figari and Yankelevich [2]. One can examine a completely different search model, for example the one from Varian [42], and verify whether such bounds reduce prices there.

Chapter 5

Congestion Games - Greedy and NE

Its traffic. There are just too many cars for the given amount of road.

- Jerry Seinfeld

Joint work with prof. Rann Smorodinsky. An older version of this chapter was published in Arxiv.org, and this version was published in Economic Letters

Abstract

Rosenthal [36] introduced the class of congestion games and proved that they always possess a Nash equilibrium (NE) in pure strategies. Fotakis et al. [21] introduce the notion of a greedy strategy tuple, where players sequentially and irrevocably choose a strategy that is a best response to the choice of strategies by former players. Whereas the former solution concept is driven by strong assumptions on the rationality of the players and the common knowledge thereof, the latter assumes very little rationality on the players' behavior. In cases when the two concepts coincide we can forecast outcomes with greater confidence. Fotakis [22] shows that in a certain family of congestion games, the so-called *Extension Parallel* congestion Games, greedy behavior leads to a NE (and so one solution concept forms a subset of the other). In this paper we show that in fact the two solution concepts are equivalent for extension parallel congestion games and furthermore we show this is a unique class of games for which such an equivalence holds.

JEL classification: C72

Key words: Congestion games, Equilibrium, Greediness.

5.1 Introduction to Congestion Games

5.1.1 Congestion Games Literature

To our knowledge, the first time the model of congestion games was used in Wardrop [44], to model traffic routing. In this model each agent walks on a graph. The roads are represented by the resources and the strategies for an agent are the simple paths in the network between two nodes on the graph. Selecting road has a cost, which is the delay experienced by travelers on this road. This delay is a function of the road and the number of agents selecting this specific road and is independent on the identity of agents. Rosenthal in [36] showed that such game possess a Pure NE, implicitly using the existence of Exact Potential, formally defined by Monderer and Shapley in [34] who introduced the notation of congestion games. In Literature there are also references to a subset of Congestion Games - Simple Congestion Games (SCG) where each agent selects a single resource. Among other papers this model is dealt in [23], [25] and [44].

Congestion Games have many interesting properties. For example, those games have Exact Potential, as shown in Monderer and Shapley [34]. Moreover, any game that posses an Exact Potential is isomorphic to Congestion Games. A different proof to this is given in Voorneveld et al. [43]. In those two papers it is shown that existence of an Exact Potential leads to the existence of a Pure NE. In Voorneveld et al. [43] the equivalence between Strong NE, NE and Exact Potential Maximizer strategies is proven. Monderer and Shapley [34] define the Finite Improvement Path (FIP) Property. They show that Exact Potential leads to the FIP property which leads to existence of a Pure NE.

5.1.2 Equilibrium Literature

Congestion games were introduced by Rosenthal [36] , who proved that any congestion game has a Nash equilibrium in pure strategies. In spite of this fact there is still valid concern about the prevalence of Nash equilibrium in reality. There are two classical criticisms over the validity of a Nash equilibrium profile as a *solution concept* which can be made - one that is computationally driven and another that is rationality driven.

The latter criticism, for example see Aumann and Brandenburger [3], is based on the fact that for an equilibrium to prevail players must have common knowledge of rationality, a condition typically unrealistic. The former criticism argues that the existence of a pure Nash equilibrium does not imply it is computationally simple to find such equilibrium. In particular, whenever the strategy space is rich this may be a challenging endeavor. Notably, there is a wide literature dealing with complexity and convergence to a NE or socially best strategy profile in congestion games. Among those a paper by Christodoulou et al. [17], analyzing a best walk converges to a best strategy profile, and Bilò et al. [9] analyzing linear congestion games.

In general, finding a NE may require searching overall strategy tuples, whose number can grow exponentially with the number of players. However, as congestion games have the *finite improvement property* one could suspect that it may be easier to find such an equilibrium.[1] However, it turns out that improvement paths can be exponentially long, as demonstrated by Ieong et al. [25] and Fabrikant et al. [20]. Fabrikant et al. [20] provide an algorithm that finds Nash equilibrium in polynomial time, for an important subset of congestion games. This is done via a reduction to a flow problem, yet leaves little insight regarding the nature of the NE.

Fotakis et al. [21] introduce the notion of a *greedy strategy profile*. Let us consider a dynamic setting with the players joining the game sequentially. Each player, upon arrival, irrevocably chooses a best response strategy to the choice of strategies of the previous players, while ignoring subsequent players. The resulting strategy profile is called a *greedy strategy profile*. Let us denote by $Z(G)$ the set of all greedy strategy profiles. Note the two degrees of freedom in the process - the order of the agents and the tie breaking rule in case of indifference among several options.[2]

Formally, $s \in Z(G)$, if there exists a permutation $\pi : N \to N$ (one-to-one and onto) of the players ($\pi(i)$ denotes the order of i) such that for any player i who chooses strategy s^i we have $\sum_{r \in s^i} \mu_r(c(s^i_\pi)_r + 1) \geq \sum_{r \in t} \mu_r(c(s^i_\pi)_r + 1) \quad \forall t \in \Sigma$, where $c(s^i_\pi)_r = |\{j : r \in s^{\pi(j)}, \pi(j) < \pi(i)\}|$ is the number players preceding i according to the permutation π whose strategy includes resource r. Clearly $Z(G) \neq \emptyset$, and typically $Z(G)$ may contain many such profiles, as generally a player may be indifferent between some choices. Let τ be a tie breaking rule, which prescribes a unique choice whenever a player is indifferent between several options. Together, π and τ impose a unique strategy profile on a game G.

In contrast with the rationality assumption underlying the notion of Nash equilibrium, the rationality requirement from a greedy profile is minimal. Players do best reply, but do so while ignoring anything they do not observe. In addition, calculating a greedy strategy profile is a much less demanding task than calculating Nash equilibrium. Hypothetically, whenever $NE(G) = Z(G)$ the prevalence of an equilibrium in a game is much more likely and referring to it as a solution concept has stronger foundations. This is because the

[1] The finite improvement property asserts that if players sequentially improve their utility by unilateral strategy changes then this process is finite and must end in a Nash equilibrium profile, see, Monderer and Shapley [34].

[2] In fact, previous papers (e.g., [1, 20, 21, 22] introduce two alternative concepts for greedy behavior. In one, agents are present in the game and sequentially best-reply to the current game. In the other, which is the one we adopt here, agents are initially absent, yet arrive sequentially and, as before, take a best reply to the game being played by the subset of players preceding them. Whereas Fotakis et al refer to the former dynamics as a "greedy best response" we use this term for the latter.

identity of the two sets suggests that the rationality assumption underlying an equilibrium profile is weak and the complexity of finding such equilibrium is typically linear with respect to the number of players. This motivates us to study the relationship between the sets $NE(G)$ and $Z(G)$.

Fotakis et al. [21] have already shown that $Z(G) \subset NE(G)$ for simple congestion games, where $\Sigma = R$. Fotakis [22] showed that in a class of congestion games that satisfy two conditions $Z(G) \subset NE(G)$. The two conditions are: (1) the game form is that of 'extension-parallel graph', namely one can map the resources to the set of edges in an extension-parallel graph and the strategies are the set of paths leading from a certain node in the graph (designated as the source node) to another node (designated as the target node);[3] and (2) the resource payoff functions satisfy a property referred to as the 'Common Best Reply' requirement, met in symmetric congestion game, where agents are symmetric, like in our case. Ackerman et al. [1] observe that some greedy best response sequences converge very fast to a NE, when the strategy structure is that of a matroid.

Our Contribution

This paper characterizes the game forms for which $Z(G)$ and $NE(G)$ coincide. In particular, our main result argues that a necessary and sufficient condition for these two solution concepts to coincide is that the game form is that of 'extension-parallel graph', as in Fotakis [22]. These results extend the state of the art knowledge in two ways. First, it is shown that for such game form not only is every greedy profile Nash equilibrium but also vice versa. In addition, we show that for such equivalence to hold for a given game form it must be the case that the game form is of a certain class, namely an 'extension-parallel graph'. In particular, given a game form not satisfying this condition, we show how to construct resource payoff functions such that the set of NE profiles and greedy profiles will not coincide.

The 'extension-parallel graph' game form is also the necessary and sufficient condition for the set of NE profiles to coincide with the set of strong equilibrium profiles, as shown by Holzman and Law Yone [23] and [24][4]. Combining the results in Holzman and Law Yone [23] and [24] and our contribution, we obtain equivalence for 'extension-parallel graph'

[3]A network D is called *Extension-Parallel* if there exists a finite sequence of networks $D_1, D_2, \dots D_m$ with $D_m = D$ in which every D_i satisfies one of the following: (1) it is a single arc network, (2) it is generated from some $D_j, j < i$ by adding a new node and adding an edge from the new node to the source or the target of D_j; and (3) it is generated from D_j and D_k with $j, k < i$ by identifying the sources and targets of the two while keeping all other nodes and edges (for more details we refer the reader to extension parallel graph, as defined in Holzman and Law Yone [24] and Milchteich [33]). Additionally, Holzman and Law Yone [24] show that extension-parallel graph congestion games are equivalent to tree representable congestion games.

[4]Formally, Holzman and Law Yone [24] introduce a notion of 'tree representable' game forms and later show that it is equivalent to extension parallel game forms.

game forms (or tree representable game forms) between greedy profiles and strong NE. Moreover, if the game form is non tree representable, then one can find payoffs where the equivalence will not hold.

5.1.3 Congestion Games Model

Congestion games are a simple, yet powerful model. On one hand it is applicable in many situations, and on the other has many nice properties. This combination lead to congestion games forming a natural class of games that is useful in modeling many realistic settings, such as traffic and communication networks, routing, load balancing and more, with a wide literature.

Informally, in a congestion game there is a set of resources R and each agent need to choose one of allowed subsets from R. For example, consider a set of agents that wish to get from one city to another, and resources are the relevant road segments. Strategies would be segment combinations which start at the source city and end in the target city. Utility depends on the resources chosen and the number of agents choosing each of the resources you did. For example, if too many people choose a specific road segment, a traffic jam will occur, reducing the utility of all users of this road segment. Here, a symmetric setting of congestion games is discussed, where all agents share the same utilities and strategy sets.

Formally, a (symmetric) congestion game is a 4-tuple $(N, R, \Sigma, \{\mu_r\}_{r \in R})$, where N is a finite set of players, R is a finite set of resources, $\Sigma \subset 2^R$ is the set of players' strategies, and for any $r \in R$, $\mu_r : N \to \mathbb{R}$ is the resource's payoff function. A strategy of player i is a choice of a subset of resources, $s^i \in \Sigma$. For any strategy tuple $s = (s^i)_{i \in N} \in \Sigma^N$ let $c(s)_r = |\{i \in N : r \in s^i\}|$ denote the number of players that utilize r (a.k.a. the congestion of the resource r) and denote by $c(s) = (c(s)_1, \dots c(s)_r)$ the congestion vector. The utility of a player is the total payoff for the resources she utilizes. Formally, $U^i(s) = \Sigma_{r \in s^i} \mu_r(c(s)_r)$.

A congestion game is *monotone* if for any $1 \leq k < l \leq N$ and $r \in R$, $\mu_r(k) \geq \mu_r(l)$. This implies the more used a resource is the less attractive it is. Monotone congestion games widely prevail in modeling traffic and communication problems, production resource allocation and more. In *single-signed* congestion games payoffs are either all positive or all negative. Typically, whenever monotone congestion games are used for modeling, they are assumed single-signed.

A *congestion game form* is a pair $F = \{R, \Sigma\}$, composed of the set of resources and a set of strategies (subsets of R). For any congestion game $G = (N, R, \Sigma, \{\mu_r\}_{r \in R})$ let $F(G) = (R, \Sigma)$ denote the corresponding game form. Given a congestion game form F, let $\mathcal{G}(F) = \{G : (F(G) = F) \wedge (G \text{ monotone})\}$ denote the class of all monotone congestion games with the game form F.

A congestion game is called simple if the set of strategies and resources coincide. Namely - each agent need to choose exactly one resource from the available resources. An example for such a situation would be choosing a city to open a single store, where the resources are the possible cities.

We say that a strategy set, $\Sigma \subset 2^R$, is *subset free* if for any $s \neq t \in \Sigma$ we have $s \not\subset t$ and $t \not\subset s$. Thus, a *a subset free Congestion Game (Form)* is a Congestion Game (Form) with a subset free strategy space. For any equilibrium analysis of single-signed monotone congestion games the assumption of subset free strategy sets is without loss of generality. In particular, note that in such games for any pair of strategies $s \subset t$ in Σ either s is dominated by t (in case resource payoffs are all positive) or t is dominated by s (in case resource payoffs are all negative) and so after deletion of dominated strategies we are left with subset free sets.

As usual, a profile $s \in \Sigma^N$ is a pure NE of G, if for each player i, for each strategy $t^i \in \Sigma$, $U^i(s^i, s^{-i}) \geq U^i(t^i, s^{-i})$, where s^{-i} is the vector of strategies of all players but i. Informally, a set of strategies is Nash equilibrium (NE) if no player can do better by unilaterally deviating. The set of all pure NE of a congestion game G will be denoted by $NE(G)$.

5.1.4 Congestion Games Properties

In this section the potential and finite improvement properties will be explained. As shown in [34], exact potential is the strongest property, which also implies the finite improvement property. In turn, finite improvement property implies existence of a pure NE.

Definition 5.1.1 *Let G be a congestion game. Consider a pure strategy profile s and agent i. Suppose that i does not play best reply in s. Thus, exist a strategy t_i yielding to i a higher utility, when all other agents play s_{-i}. The strategy profile s_{-i}, t_i is denoted as a improving profile to s.*

Definition 5.1.2 *Let $s_1, s_2, \ldots$ be a sequence of improving profiles, such that for each $k < 1$, the profile s_k is an improving profile to s_{k-1}. Then the sequence will be denoted as improvement path.*

Definition 5.1.3 *Finite improvement property states that any improvement path is of a finite length. That is, it is not possible to obtain an improvement path of infinite length in game G.*

Clearly, if a game has no pure NE an improvement path will always be infinite, as in any pure strategy profile at least one agent will have a profitable deviation leading to an improving profile.

An *Exact Potential* is a function $\mathbb{P} : S \to \mathbb{R}$ satisfying:

$$\begin{gathered} \mathbb{P}(s) - \mathbb{P}(s_{-i}, t_i) = U_i(s) - U_i(s_{-i}, t_i) \\ \forall i \in N, \forall t_i \in S_i, \forall s \in S_1 \times S_2 \ldots \times S_n \end{gathered} \tag{5.1}$$

Games with an exact potential function are called *Potential Games*, and have a pure NE. This was shown by [34]. An example will illustrate the importance of a potential. Consider a 2 player game where one agent has 3 strategies and the other has 2. Potential is a number associated with each strategy profile, as depicted at the table below.

G^C	L	R
A	P_1	P_2
B	P_3	P_4
C	P_5	P_6

An additional property of an exact potential is as follows: Consider the strategy profiles $\{A, L\}$ and $\{C, L\}$. The associated potential values are P_1 and P_5. The strategy profile $\{C, L\}$ is obtained from the strategy profile $\{A, L\}$ and a deviation of the row agent from A to C. Let us denote the utility of the row agent in the two profiles as U_{AL} and U_{CL}. If P is an exact potential, than it must hold:

$$U_{AL} - U_{CL} = P_1 - P_5 \tag{5.2}$$

The difference in potential is equal to the difference in utility of the deviating agent.

More generally:

Definition 5.1.4 *Let G be a game and let s, t be two pure strategy profiles of the game. If in s and t exactly one agent selects a different strategy then those strategy profiles will be called adjacent.*

A function P is an exact potential for game G only if for any two adjacent strategy profiles s, t, where $s_i \neq t_i$, we have that:

$$U_i(s) - U_i(t) = P(s) - P(t) \tag{5.3}$$

From here it is easy to see that, for example, a maximal value of a potential is a pure NE.

5.2 Introduction

Congestion game form a natural class of games that are useful in modeling many realistic settings, such as traffic and communication networks, routing, load balancing and more. A symmetric congestion game is a 4-tuple $(N, R, \Sigma, \{\mu_r\}_{r \in R})$, where N is a finite set of players, R is a finite set of resources, $\Sigma \subset 2^R$ is the set of players' strategies, and for any $r \in R$, $\mu_r : N \to \mathbb{R}$ is the resource's payoff function. A strategy of player i is a choice of a subset of resources, $s^i \in \Sigma$. For any strategy tuple $s = (s^i)_{i \in N} \in \Sigma^N$ let $c(s)_r = |\{i \in N : r \in s^i\}|$ denote the number of players that utilize r (a.k.a. the congestion of the resource r) and denote by $c(s) = (c(s)_1, \dots c(s)_r)$ the congestion vector. The utility of a player is the total payoff for the resources she utilizes. Formally, $U^i(s) = \Sigma_{r \in s^i} \mu_r(c(s)_r)$.

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Congestion games were introduced by Rosenthal [36] , who proved that any congestion game has a Nash equilibrium in pure strategies. Albeit natural, the notion of a Nash equilibrium has received much criticism as a realistic outcome of a game. More particularly, for an arbitrary game, an epistemic analysis shows that the conditions needed for such an outcome to prevail are quite strong. In fact, no less than common knowledge of the full game structure on the one hand and common knowledge of rationality on the other hand are required.[5]. As for congestion games, which are well known to obtain an exact potential, one might suspect less is needed. However, this is not known and it may be the case that a Nash equilibrium can be supported by a hierarchy of beliefs regarding players' rationality of finite order.

Fotakis et al. [21] introduce the notion of a *greedy strategy profile.* Let us consider a dynamic setting with the players joining the game sequentially. Each player, upon arrival, irrevocably chooses a best reply strategy, given the choice of strategies of the previous players, while ignoring subsequent players. The resulting strategy profile is called a *greedy strategy profile.*[6]

Let us denote by $Z(G)$ the set of all greedy strategy profiles. This set is not necessarily a singleton due to the two degrees of freedom in the process - the order of the players and the tie breaking rule in case of indifference among several options. Formally, $s \in Z(G)$, if there exists a permutation $\pi : N \to N$ (one-to-one and onto) of the players ($\pi(i)$ denotes the order of i) such that for any player i who chooses strategy s^i we have $\sum_{r \in s^i} \mu_r(c(s^i_\pi)_r + 1) \geq \sum_{r \in t} \mu_r(c(s^i_\pi)_r + 1) \quad \forall t \in \Sigma$, where $c(s^i_\pi)_r = |\{j : r \in s^{\pi(j)}, \pi(j) < \pi(i)\}|$ is the number players preceding i, according to the permutation π, whose strategy includes resource r. Clearly $Z(G) \neq \emptyset$, and typically $Z(G)$ may contain many such profiles, as generally a player may be indifferent between some choices. Let τ be a tie breaking rule, which prescribes a unique choice whenever a player is indifferent between several options. Together, π and τ impose a unique greedy strategy profile on a game G.

In contrast with the rationality assumption underlying the notion of Nash equilibrium, the rationality requirement from a greedy profile is minimal. Indeed players are only assumed to be rational, know their own payoffs and observe the choice of actions by some subset of players (their predecessors). Beyond that nothing is assumed. In particular players need not know whether others are rational or in fact if there any other players in the game beyond the subset of players they observe.

Fotakis et al. [21] have already shown that $Z(G) \subset NE(G)$ for simple congestion games, where $\Sigma = R$ and Fotakis [22] extends this congestion games over 'extension-parallel graphs' (in the sequel we formally define this notion) for which the resource payoff func-

[5] for example, see Aumann and Brandenburger [3]

[6] There are two similar yet different notions of greediness in the literature on congestion games. The first notion models a situation where players are present in the game and sequentially best-reply to the current game (e.g., Fabrikant et al. [20] and Fotakis [22]). The second, which is the one we adopt here, players are initially absent, yet arrive sequentially and, as before, take a best reply to the game being played by the subset of players preceding them. Fotakis et al [21] refer to the latter dynamics as a "greedy best response".

tions satisfy a certain property he calls the 'Common Best Reply' (which is trivially satisfied in symmetric congestion games which is the focus of our work). Ackerman et al. [1] observe that some greedy best response sequences converge very fast to a NE, when the strategy structure is that of a Matroid.

5.2.1 Our Contribution

This paper characterizes the game forms for which $Z(G)$ and $NE(G)$ coincide. In particular, our main result argues that a necessary and sufficient condition for these two solution concepts to coincide is that the game form is that identified in the work of Fotakis [22], namely 'extension-parallel graphs'. Thus, the marginal contribution of this work over the existing literature is two-fold. First, it is shown that for the game forms in discussion not only is every greedy profile a Nash equilibrium but also vice versa. In addition, we show that for such equivalence to hold for a given game form it must be the case that the game form is of a certain class, namely a 'extension-parallel graph'. In particular, given a game form not satisfying this condition, we show how to construct resource payoff functions such that the the set of NE profiles and greedy profiles will not coincide.

Our technical observation regarding necessary and sufficient conditions for which for which $Z(G)$ and $NE(G)$ shed some light on two aspects of Nash equilibrium in such congestion games:

- The prevalence of a Nash equilibrium outcome - As discussed above the epistemic conditions for the prevalence of an outcome in $Z(G)$ are quite weak compared with those required for the prevalence of an outcome in $NE(G)$ for an arbitrary game, or in fact an arbitrary congestion game. Given our results, in the class of games we study a Nash equilibrium is supported by weak epistemic requirements.[7]
- The speed of convergence to a Nash equilibrium outcome - The number of steps to compute a Nash equilibrium outcome in the class of games we discuss is N (the number of players). This is strictly faster than polynomial or even exponential results (in the number of players) that hold for larger classes of games. We refer the reader to Fotakis [22] who discusses this in more depth.[8] For more detailed results on the speed of convergence in broader classes of games we refer the reader to Ieong et al. [25], Ackerman et al. [1] and Fabrikant et al. [20]. Our results

[7]One should make a distinction between the rationality assumptions needed to make a NE stable which are typically weak (rationality and knowledge of own payoffs suffices for that) with the rationality assumptions need for players to reach a NE outcome via an introspective analysis, which is what we refer to as the *prevalence* of a Nash equilibrium outcome.

[8]Fotakis [22] focuses on players that best reply within a game, as opposed to our model where players join the game and best-respond. For the related discussion of the speed of convergence this hardly matters.

are discouraging for those who assume that such speedy convergence occurs for a broader class of games than those played over extension-parallel game forms.

Interestingly, Holzman and Law Yone [24], prove that the 'extension-parallel graph' game form is also the necessary and sufficient condition for the set of NE profiles to coincide with the set of strong equilibrium profiles.[9] Combining these results with our contribution we obtain equivalence between greedy profiles and strong NE for for the class of 'extension-parallel graph' game forms. Moreover, for game forms outside this class there are resource payoffs function for which this equivalence no longer holds.

The structure of the article is as follows: Section 2 is a 'warm-up' section with a variety of examples that demonstrate that without any restrictions on the game form there is no structural connection between the sets $NE(G)$ and $Z(G)$. Section 3 formalizes the notion of extension parallel games, and discusses the characteristics of this class. Then in section 4 we present and prove the main result, namely equivalence between $NE(G)$ and $Z(G)$ for extension parallel congestion games.

5.3 Examples

Here we provide several examples for the various relations between $Z(G)$ and $NE(G)$. As we shall demonstrate those can differ depending on the game in question.

Example 5.3.1 $NE(G) \cap Z(G) = \emptyset$ - *Greedy profiles and equilibria are mutually exclusive.*

Consider the congestion game with two players and three resources, with resource payoff functions as depicted on table 5.1(a). The strategy space consists of all possible two resource combinations.

The unique greedy profile (up to player permutation) is (AB, AC, BC). Note this is not a Nash equilibrium since player 1 can profitably deviate from AB to AC, increasing her utility from 8+4 to 8+5. On the other hand the unique Nash equilibrium (up to renaming of players) is (AC, AC, BC), and obtained after this deviation.[10]

[9] Formally, Holzman and Law Yone [24] introduce a notion of 'tree representable' game forms which are equivalent to extension parallel game forms.

[10] Note that this (and next) example shows that in Matroid Games (see Ackerman et al. [1]) the equality $Z(G) = NE(G)$ does not generally hold. The game is a Matroid game, since a combination is a strategy iff it has two resources, which can be a Matroid base.

Example 5.3.2 $Z(G) \subsetneq NE(G)$ - *Greedy profiles strictly contained in NE profiles.*

Consider the congestion game with two players and four resources, with resource payoff functions as depicted on table 5.1(b). Assume, furthermore, that each strategy must contain one of the resources A,B and one of the resources C,D.

Clearly the unique greedy profile is (AC, BD), which is also a Nash equilibrium. However, there is an additional Nash equilibrium profile (AD, BC).

Example 5.3.3 $NE(G) \subsetneq Z(G)$ - *Greedy profiles strictly contain NE profiles.*

Consider the congestion game with two players and three resources, with resource payoff functions as depicted on table 5.1(c). Each player must choose a pair of resources.

The greedy profiles are AB, BC and AC, BC. The first one is the unique pure NE of the game (up to player permutation).

Example 5.3.4 $Z(G) \cap NE(G) \neq \emptyset$, $Z(G) \setminus NE(G) \neq \emptyset$ *and* $NE(G) \setminus Z(G) \neq \emptyset$.

Consider the congestion game with two players and five resources, with resource payoff functions as depicted on table 5.1(d). The set of strategies is composed of the following subsets of resources $\{AB, AC, DB, E\}$.

If we express this game in the standard bi-matrix form we get:

Game	AB	AC	DB	E
AB	-15, -15	-6, **-10**	-11, -12	**-2**, **-10**
AC	**-10**, -6	-105, -105	**-6**, **-3**	-6, -10
DB	-12, -11	**-3**, **-6**	-110, -110	-3, -10
E	**-10**, **-2**	-10, -6	-10, -3	-100, -100

It is now easy to verify that the only two Nash equilibria of this game are (AB, E) and (AC, DB). The greedy profiles, on the other hand, are (AB, E) and (AB, AC).

Example 5.3.5 $NE(G) = Z(G)$ - *Equivalence*

This holds in any simple congestion game with generic resource payoffs (follows from Fotakis [21]).

Figure 5.1: Resource payoff functions for examples of chapter 2

# of players / res.	A	B	C
1	10	10	8
2	8	4	6
3	1	1	5

(a) Example 2.1

# of players / res.	A	B	C	D
1	40	30	20	15
2	10	11	12	13

(b) Example 2.2

# of players / res.	A	B	C
1	10	8	8
2	1	7	6

(c) Example 2.3

# of players / res.	A	B	C	D	E
1	-1	-1	-5	-2	-10
2	-5	-10	-100	-100	-100

(d) Example 2.4

5.4 Extension Parallel Games

A network D is called *Extension-Parallel* if there exists a finite sequence of networks $D_1, D_2, \dots D_m$ with $D_m = D$, each with a node designated as a 'source' node and another node designated as a 'target' node. Each network D_i satisfies one of the following:

1. it is a two-node, single arc network with one node designated source and the other designated target.
2. it is generated from some D_j, where $j < i$ by adding a new node and adding an edge from the new node either to the source (in which case the new node will be designated source while the target node is left unchanged) or to the target (in which case the new node will be designated target while the source node is left unchanged) of D_j ;
3. it is generated from D_j and D_k with $j, k < i$ by merging the sources and targets of the two while keeping all other nodes and edges

Consider a congestion game form F constructed from network D as follows: The resources of the game are the edges of D, the strategies are the paths from the source to the target and the number of players is a finite number N.

Then F is an extension parallel congestion game form, and any game in $\mathcal{G}(\mathcal{F})$ will be called extension parallel congestion game (EPCG).[11] The following topological observation for extension parallel game forms will play an important role in the sequel: a pair of resources A, B and a triplet of strategies s_1, s_2, s_3 constitutes a *bad configuration* if the following holds: $A, B \in s_1$, $A \in s_2 \setminus s_3$ and $B \in s_3 \setminus s_2$. Holzman and Law Yone [24] observed:

[11]To learn more on extension parallel graphs we refer the reader to Holzman and Law Yone [24] and Milchtaich [33].

Theorem 6 *A congestion game form does not have a bad configuration if and only if it is extension parallel.*

In particular, verifying whether a game form is extension parallel can be done by iterating over all possible pairs of resources and triplets of strategies, which is polynomial in both factors.

5.5 Main Result

Our main result links extension parallel congestion games with the equivalence of the two solution concepts based on greediness and equilibrium:

Theorem 7 *Let F be a subset-free Congestion Game Form. F is extension parallel iff for any congestion game $G \in \mathcal{G}(F)$, we have that $Z(G) = NE(G)$.*

We split the proof of Theorem 7 into two propositions, one showing that extension parallel condition is sufficient for the desired equivalence and the other showing it is necessary.

5.5.1 Extension Parallel is Sufficient

Let $BR(\sigma)$ denote the set of players who play their best response strategies in σ, and recall Lemma 1 from Fotakis [22]:

Lemma 5.5.1 *(Fotakis, [22]) Let σ be a profile of EPCG. Let σ'_i be one of i's best replies to σ_{-i}. Then $j \in BR(\sigma) \implies j \in BR(\sigma'_i, \sigma_{-i})$.*

Lemma 5.5.2 *If G is a EPCG then $Z(G) \subset NE(G)$.*[12]

Proof:

We proceed by induction on the number of players. The claim is trivial for any single player game. Assume it holds for $n-1$ players and let $G = (n, R, \Sigma, \{\mu_r\}_{r \in R})$ be an EPCG. Consider the auxiliary game $G' = (n, R' = R \cup \{r'\}, \Sigma \cup \{r'\}, \{\mu_r\}_{r \in R'})$ with $\mu_{r'}(1) < \mu_r(n)$ for all $r \in R$. Note that the choice of resource r' is never a best reply and

[12]Theorem 1 in Fotakis [22] proves a similar result for the scenario where all players are initially present in the game and best-respond sequentially.

so any $NE(G) = NE(G')$ for any set of players. Let $\sigma \in Z(G)$ and let $\sigma' = (\sigma'_i, \sigma_{-i})$, with i being the last player to move when forming σ and $\sigma'_i = r'$. By the induction hypothesis and the structure of G' each player $j \neq i$ best replies to σ'_{-j}. Now let i move to the strategy σ_i, which is his best reply to $\sigma'_{-i} = \sigma_{-i}$. By Lemma 5.5.1 all others are still best replying and so $\sigma \in NE(G') = NE(G)$.

□

Additionally, for the other direction, we can prove the following lemma:

Lemma 5.5.3 *If G is a EPCG then $NE(G) \subset Z(G)$.*

Proof:

We proceed by induction on the number of players. The claim is trivial for any single player game. Let us assume it holds for $n-1$ players and show that it must also hold for n players. Let $s = (s^1, \ldots s^n) \in NE(G)$ and assume, without loss of generality, that $U^n(s) \leq U^i(s)\ \forall i$.

Let us add to the game G a new resource which will contribute one more strategy, denoted s^l, composed of that singleton resource. In particular, s^l has an empty intersection with all strategies of G. The payoff for selecting this strategy as a unique player is $U^n(s)$, and less when more players select it.

Let us denote the extended game as G' . Additionally, for any n player congestion game H let us denote by $P(H)$ the game played by players $1, \ldots, n-1$ over the same game form and payoffs. Also, let $P(s)$ denote the strategy tuple for these players induced from the n-player strategy profile s by deleting the strategy of player n.

Note that $s \in NE(G) \implies s \in NE(G')$. This is so as the payoff is the newly added strategy is weakly dominated strategy. In addition, player n is indifferent between s^n and s^l and so the strategy s^l is a best reply for n. By Lemma 5.5.1, the strategy profile $s' = (s^1, \ldots s^{n-1}, s^l)$ is a NE of the game G'. Removing player n whose strategy does no intersect any of the other players' strategy yields $P(s) \in NE(P(G'))$. Now delete s^l to conclude that $P(s) \in NE(P(G))$

By the induction hypotheses there exists a greedy behavior which leads to the strategy profile $P(s)$ in $P(G)$. Let us denote the corresponding ordering rule as π' and the corresponding tie breaking rule as τ'. Consider π as the ordering rule π' with the addition to have n as the last player. Additionally, let τ be the tie breaking τ' for all players except n and for player n let it be an arbitrary tie breaking rule which chooses s^n whenever it is a best reply. Since s was a NE of G, s^n is one of the best reply strategies of player n to $P(s)$. The new greedy algorithm clearly ends in s and so $s \in Z(G)$. □

Lemmas 5.5.2 and 5.5.3 constitute the proof of:

Proposition 5.5.1 *Let F be an extension parallel* [13] *game form. If $G \in \mathcal{G}(F)$ then $Z(G) = NE(G)$.*

5.5.2 Extension Parallel is Necessary

Lemma 5.5.4 *If for any $G \in \mathcal{G}(F)$, $Z(G) = NE(G)$ then for any 3 distinct strategies, $s^1, s^2, s^3 \in F$ and any 3 distinct resources $A, C, E \in R$ the following structure cannot occur:*

$$s^1 \cap \{A, C, E\} = \{A, C\}$$
$$s^2 \cap \{A, C, E\} = \{C, E\}$$
$$s^3 \cap \{A, C, E\} = \{A, E\}$$

Proof:

Consider a game form for which such a structure occurs and consider a two player game with the following payoff functions (M is arbitrarily chosen to satisfy $\frac{2|R|}{M} < 1$):[14]

# of players / resource	A	C	E	other resources
1	10	9	8	$\frac{1}{M}$
2	1	6	7	$\frac{1}{2M}$

Note that the pair (s^2, s^3) constitutes a Nash equilibrium which cannot be an outcome of a greedy behavior and so $Z(G) \neq NE(G)$ and so a contradiction is reached. $\square$

Let $F = (R, \Sigma)$ be a game form that is not extension parallel. By Theorem 6 there must exist two resources, $A, C \in R$ and three strategies, $s_1, s_2, s_3 \in \Sigma$ such that $A \in (s^1 \cap s^3) \setminus s^2$ and $C \in (s^1 \cap s^2) \setminus s^3$.

Lemma 5.5.5 *Suppose F is not extension parallel and that for any $G \in \mathcal{G}(F)$, $Z(G) = NE(G)$ then $(s^2 \cap s^3) \setminus s^1 = \emptyset$.*

Proof:

[13] and thus subset-free

[14] The game here in the spirit of example 5.3.3.

Suppose that there exists some resource $E \in (s^2 \cap s^3) \setminus s^1$. Then we have that:

$$s^1 \cap \{A, C, E\} = \{A, C\}$$
$$s^2 \cap \{A, C, E\} = \{C, E\}$$
$$s^3 \cap \{A, C, E\} = \{A, E\}$$

thus contradicting Lemma 5.5.4. $\square$

Recall that F is assumed subset-free. Thus, there must exist resources B and D that satisfy $B \in s^2 \setminus s^1$ and $D \in s^3 \setminus s^1$. An immediate corollary of Lemma 5.5.5 is:

Corollary 5.5.1 *Suppose F is not extension parallel. If for any $G \in \mathcal{G}(F)$, $Z(G) = NE(G)$ then $B \neq D$.*

Lemma 5.5.6 *Suppose F is not extension parallel. If for any $G \in \mathcal{G}(F)$, $Z(G) = NE(G)$ then there exists a strategy $s^4 \neq s^2$ such that $s^4 \subset s^2 \cup (s^3 \setminus s^1)$.*

Proof:

Let R denote resource set in the game form F. Let us consider a two player game with the following resource payoff functions (with M satisfying $M \geq |R|^9$):

$r \in$	$s^1 \cap s^2 \cap s^3$	$(s^1 \cap s^2) \setminus s^3$	$(s^1 \cap s^3) \setminus s^2$	$s^1 \setminus (s^2 \cup s^3)$	$s^2 \setminus (s^1 \cup s^3)$	$s^3 \setminus (s^2 \cup s^1)$	$(s^1 \cup s^2 \cup s^3)^c$
$\mu_r(1)$	$-1/M^2$	$-1/M^2$	$-1/\lvert R\rvert^5$	$-1/\lvert R\rvert^5$	$-1/\lvert R\rvert$	$-1/\lvert R\rvert^4$	$-M$
$\mu_r(2)$	$-1/\lvert R\rvert$	$-\lvert R\rvert^6$	$-2M$	$-2M$	$-2M$	$-2M$	$-2M$

By Lemma 5.5.5 $(s^2 \cap s^3) \setminus s^1 = \emptyset$ and so the family of sets in the first row constitutes a partition of the resource set and hence a monotone congestion game is well defined.

Note that from subset-freeness, there is no strategy which is a subset of s^1. Therefore, the first player in a greedy profile must select the strategy s^1, as all other resources yield the payoff $-1/|R|^4$ or less. The second greedy player will select resources only in $s^2 \cup (s^3 \setminus s^1)$, as all other resources have payoff of $-M$ or less.

Assume the statement of the lemma is incorrect and that the only available strategy for the second greedy player is s^2. If the second greedy player chose s^2, the first greedy player has a profitable deviation. She can deviate from s^1 to s^3, avoiding the high negative payoff on resources in $(s^1 \cap s^2) \setminus s^3$ (e.g., on resource C). This implies that $Z(G) \neq NE(G)$ and a contradiction is reached. $\square$

Coupled with subset-freeness (in particular $s^4 \not\subset s^2$) we can conclude:

Corollary 5.5.2 *Suppose F is not extension parallel, but subset-free and for any $G \in \mathcal{G}(F)$ $Z(G) = NE(G)$ then $s^4 \cap (s^3 \setminus (s^1 \cup s^2)) \neq \emptyset$.*

Without loss of generality let $D \in s^4 \cap s^3 \setminus (s^1 \cup s^2)$.

Proposition 5.5.2 *If F is subset-free and for any $G \in \mathcal{G}(F)$ $Z(G) = NE(G)$ then F is extension parallel.*

Proof:

Suppose F is not extension parallel then by Lemmas 5.5.5 and 5.5.6, there exist 4 distinct strategies $s^1, \ldots, s^4$ and 4 distinct resources, A, B, C, D satisfying:

- $A \in (s^1 \cap s^3) \setminus (s^2 \cup s^4)$,
- $B \in s^2 \setminus s^1$,
- $C \in (s^1 \cap s^2) \setminus s^3$,
- $D \in s^4 \cap (s^3 \setminus (s^1 \cup s^2))$,
- $s^4 \subset s^2 \cup (s^3 \setminus s^1)$.

By subset-freeness there exists a resource $E \in s^4 \setminus s^3$, which, in turn, implies that $E \in s^2 \setminus s^3$. If $E \in s^1$ then we have the following structure:

$$\begin{aligned} s^1 \cap \{A, D, E\} &= \{A, E\} \\ s^3 \cap \{A, D, E\} &= \{A, D\} \\ s^4 \cap \{A, D, E\} &= \{D, E\} \end{aligned}$$

thus contradicting Lemma 5.5.4. Therefore, $E \in s^4 \setminus s^3$ implies $E \notin s^1$ and so $s^4 \cap s^1 \subset s^3$. In particular we conclude that $C \notin s^4$ and the following holds:

$$\begin{aligned} s^1 \cap \{A, C, D, E\} &= \{A, C\} \\ s^2 \cap \{A, C, D, E\} &= \{E, C\} \\ s^3 \cap \{A, C, D, E\} &= \{A, D\} \\ s^4 \cap \{A, C, D, E\} &= \{E, D\} \end{aligned}$$

Consider the two player game with the following resource payoff functions (where M is satisfies $\frac{2|R|}{M} < 1$):[15]

[15]This game is in the spirit Example 5.3.2.

# of players / resource	A	E	C	D	other resources
1	40	30	20	15	$\frac{1}{M}$
2	10	11	12	13	$\frac{1}{2M}$

Note that (s^2, s^3) is a Nash equilibrium which cannot be obtained as a greedy profile and so a contradiction is reached. $\square$

The proof of Theorem 7 follows from Propositions 5.5.2 and 5.5.1.

5.6 Summary

We characterize settings where the set of pure Nash equilibria coincides with the set of greedy strategy profiles. We conclude that in such cases a Nash equilibrium forms a viable solution concept as it emerges from very weak rationality assumptions and does not hinge on common knowledge of rationality. In addition, the computational complexity of finding an equilibrium in such games is low.

Appendix A

Heterogeneous Sellers - Omitted Proofs

I have discovered a truly marvelous demonstration
of this proposition that this margin is too narrow to contain.
- Pierre de Fermat, (Nagell 1951, p. 252).

A.1 Proof of Theorem 1

Some of the results will be applicable also for the proof of Theorem 2.

Remark A.1.1 *Remember that in any NE, if seller i is using mixed strategy, she is indifferent among all strategies with positive density. Thus, seller i would receive the same expected profit from all pricing strategies with positive density.*

A.1.1 Mass Points and Highest offered Price

Lemma A.1.1 *In any NE there are no mass points at any price below P_M that can attract shoppers with positive prob.*

Suppose $q < P_M$ and seller i has a mass point at q. Additionally, there is a positive probability that a shopper would purchase the item at price q or above.

Firstly note that due to undercutting no other seller $j \neq i$ has a mass point at price q.

Note that at price q the chance to attract shoppers drops discontinuously for any seller $j \neq i$. Therefore, if seller j has the interval $(q, q+\varepsilon)$, for some small $\varepsilon > 0$ in support, she can deviate to q and increase the probability to attract shoppers. Since $q < P_M$ relevant prices attract searchers, and the share of searchers would not be affected. Therefore, such a deviation is profitable for seller j and would not occur in equilibrium.

From here follows that exists $\varepsilon > 0$, such that in $(q, q+\varepsilon)$ no seller except i has in support. Additionally, there are no mass points at q, except the one i has. Thus, seller $j \neq i$ would never offer a price in $[q, q+\varepsilon)$. From here follows that seller i can set the mass point not on q, but rather on a higher price in $(q, q+\varepsilon)$, with the same probability to attract shoppers, and higher expected profit. This contradicts the fact that initially we had a NE. □

Lemma A.1.2 *In any NE all sellers have P_M as the supremum point of their strategy support.*

From corollary 2.3.1 it cannot be higher than P_M.

Suppose that the supremum price in support of seller i is $p < P_M$. Additionally, suppose that seller i has the (weakly) lowest support supremum. For any price above p and below P_M the probability to sell to shoppers is 0. Therefore, in equilibrium no seller would have a price $q \in (p, P_M)$ in their support.

All sellers cannot have a mass point at p, as in such case undercutting would be profitable. From lemma A.1.1 seller i has no mass point at price p. From here follows that probability to sell to shoppers at price p is 0 for seller $j \neq i$. From here follows that seller j would have no mass point at p, as P_M yields higher profit. Therefore, a deviation exists to seller i, where i selects prices arbitrarily close to P_M instead of prices arbitrarily close to p is profitable. □

This lemma combined with corollary 2.3.1 and lemma A.1.1 show that there can be no mass points at any price except for P_M.

A.1.2 Single Interval

Lemma A.1.3 *In any NE exists an interval I such that the union of the seller strategies is contained in I and dense in it.*

Suppose exists an interval $[a, b]$ (where $a < b < P_M$) such that sellers select prices only below a and above b, and exist prices both below a and above b in support union. Let p

be the highest price below a that is in the support union of the sellers. Let us examine a deviation from p and prices just below it to b.

Note that there are no mass points in the relevant price regions and distribution mass of all sellers is arbitrarily small. Since the probability for someone to select a price arbitrarily close to p and below p is arbitrarily small, the decrease in probability to sell to shoppers is arbitrarily small.

The searchers behavior does not change, as the prices are below P_M.

The profit form raising the price is much higher than such arbitrarily small loss, as it is at least $n_i(b-p)(1-\mu)/n$, as the searchers pay strictly more after the deviation. Therefore, if the support is not continuous there is a profitable deviation, contradicting the NE condition. □

Corollary A.1.1 *In any NE exists an interval $I = [A, P_M]$, such that any NE strategy profile the sellers randomize continuously over I, and possibly some sellers set mass points at P_M, where A is some positive number.*

Lemma A.1.4 *The previous lemma holds also for two sellers. Meaning - any interval has a non empty intersection with the support of at least two sellers.*

Suppose that all points in an interval $[p, p']$ where $(p < p' < P_M)$ are selected by a unique seller denoted i. Then exists a profitable deviation for her would be to set a mass point at p' instead of selecting the original distribution over the interval. □

Corollary A.1.2 *In any NE any interval between A and P_M has points in the support of at least two sellers.*

A.1.3 Multiple Smallest Firms PPB

Let the profit divided by the searchers fraction be denoted as the 'Profit per Branch' (PPB), and denote it as $\hat{\pi}$. The PPB measures the profit the seller gets divided by her stores number. The main point is that in NE each seller has the fixed PPB when mixing, over all strategies with positive density.

Let F_i be the distribution describing the equilibrium mixed strategy of seller i.

Lemma A.1.5 *N-1 sellers have the same PPB*

If all sellers have a mass point at P_M undercutting is possible. Therefore, in any NE at least one seller does not have a mass point at P_M.

If a seller does not have a mass point at P_M she would offer a price below P_M with prob. 1. Thus, all sellers who have a mass point at P_M have the probability of 0 to sell to shoppers at that price.

Therefore PPB for sellers with a mass point at P_M is:

$$\hat{\pi} = \pi/n_i = P_M(1-\mu)/S \tag{A.1}$$

As the support supremum of all sellers is P_M, from lemma A.1.2, one of the two cases must hold in NE, Remember that if all have mass points at P_M undercutting is possible:

- At least two sellers do not have a mass point at P_M
- A single seller does not have a mass point at P_M

In the latter case the lemma holds, as all sellers with mass point at P_M have the same PPB.

Let us concentrate on the first case, and denote two of the sellers without mass point at P_M as i and j. Seller i probability to attract shoppers is the product:

$$\prod_{k \neq i}(1 - F_k(p)) \leq 1 - F_j(p) \tag{A.2}$$

As p goes to P_M, $1-F_j$ goes to 0. Similarly, for seller j it is bounded by $1-F_i$. Therefore, in this case, for prices arbitrarily close to P_M the probability to attract shoppers is arbitrarily close to zero. Therefore, the PPB of all sellers is

$$\hat{\pi} = \pi/n_i = P_M(1-\mu)/S \tag{A.3}$$

□

Corollary A.1.3 *If at least two sellers do not have a mass point at P_M all sellers have the same PPB.*

Corollary A.1.4 *If only one seller, denote i, does not have a mass point at P_M, she would not have PPB lower than all other sellers.*

Seller i can always deviate to P_M and obtain PPB of at least:

$$\hat{\pi} = \pi / n_i = P_M(1-\mu)/S \tag{A.4}$$

This profit is obtained only from searchers, and additional profit can be obtained from shoppers.

Lemma A.1.6 *Suppose smallest seller is not unique. In any NE all sellers have the same PPB. That is, $\frac{\pi_i}{n_i}$ is equal to all sellers.*

From lemma A.1.5, we can concentrate on the case when seller i is the only seller who does not have a mass point at P_M, and has a higher PPB than all other sellers.

Let p_i be the lowest price in the support of i. Additionally, exists a seller denoted j with $n_j \leq n_i$[1]. Since the lowest n_i is not unique (from assumption) such j always exists.

Remember that p_i the lowest (infimum) price seller i can offer. The profit of seller i offering p_i is as follows (Note: all searchers visit exactly one store, from corollary 2.3.1):

$$\pi_i(p_i) = p_i((1 - F_j(p)) \prod_{k \neq i,j} (1 - F_k(p_i)))\mu + (1-\mu)n_i/S) \tag{A.5}$$

The profit of seller j:

$$\pi_j(p_i) = p_i((1 - F_i(p)) \prod_{k \neq i,j} (1 - F_k(p_i)))\mu + (1-\mu)n_j/S) \tag{A.6}$$

If one calculates the PPB for the two sellers, it will be equal to:

$$\hat{\pi}_i = p_i(1 - F_j(p))[\prod_{k \neq i,j} (1 - F_k(p_i))\frac{\mu}{n_i}] + (1-\mu)p_i/S \tag{A.7}$$

$$\hat{\pi}_j = p_i(1 - F_i(p))[\prod_{k \neq i,j} (1 - F_k(p_i))\frac{\mu}{n_j}] + (1-\mu)p_i/S \tag{A.8}$$

Since $0 = F_i(p_i) \leq F_j(p_i)$ and $1/n_i \leq 1/n_j$ one gets that the expression $\hat{\pi}_j$ is weakly higher than the of $\hat{\pi}_i$.

Therefore, PPB of seller j is weakly higher, contradicting our assumption.

Therefore, all sellers must have the same PPB. □

Definition A.1.1 *Let P_L be the lowest possible price which can be offered in equilibrium.*

[1] Here is the only place where the multiple smallest sellers assumption is required

Remember that no mass points are used at prices below P_M. Since P_M is the highest price, it is clear that $P_L < P_M$. Therefore, P_L attracts shoppers with prob. 1. Thus, if P_L is offered by seller i, P_L needs to satisfy:

$$P_L(\mu/n_i + (1-\mu)/S) = P_M(1-\mu)/S) \tag{A.9}$$

Lemma A.1.7 *In NE only sellers with lowest stores number would have P_L in support.*

Let i, j be two sellers satisfying $n_i < n_j$.

Suppose that seller j is the seller with P_L in support. Therefore, PPB of seller j is:

$$\hat{\pi}_j(P_L) = P_L(\mu/n_j + (1-\mu)/S) = P_M(1-\mu)/S \tag{A.10}$$

Since $n_i < n_j$, if seller i would offer P_L, she would have the PPB of:

$$\hat{\pi}_i(P_L) = P_L(\mu/n_i + (1-\mu)/S) \tag{A.11}$$

Since $n_i < n_j$, we get that PPB of seller i would be larger than the one of seller j. Since in any NE PPB of i and j are equal, seller i does not have P_L in support. However, she can deviate to P_L and increase her PPB (and clearly profit), contradicting the NE assumption. □

A.1.4 Serving Shoppers

Definition A.1.2 *Let $\alpha_i(p)$ be denoted as the probability that p is the cheapest price, if seller i selects it. Explicitly: what is the probability of seller i to sell to shoppers given she selects price p. As the distribution is with no mass points except (maybe) P_M, one can define $\alpha_i(p)$ as the product of 'Probability that seller j sets price above p', which is denoted as $\beta_j(p)$. Formally:*

$$\beta_j(p) = 1 - F_j(p) \tag{A.12}$$

$$\alpha_j(p) = \prod_{j \neq i} \beta_j(p) \tag{A.13}$$

Lemma A.1.8 *Suppose seller i has price p in support. Then, for each seller j, $\alpha_i(p)/n_i \geq \alpha_j(p)/n_j$.*

Note that the PPB of seller i is given by:

$$\hat{\pi}_i(p) = p(\mu\alpha_i(p)/n_i + (1-\mu)/S) \tag{A.14}$$

Suppose that $\alpha_i(p)/n_i < \alpha_j(p)/n_j$. Then, if seller j offers price p her PPB would be:

$$\hat{\pi}_j(p) = p(\mu\alpha_j(p)/n_j + (1-\mu)/S) > p(\mu\alpha_i(p)/n_i + (1-\mu)/S) = \hat{\pi}_i(p) \tag{A.15}$$

If seller j would have price p in support she would get a higher PPB than i, contradicting lemma A.1.6. If seller j does not have p in support she can deviate and increase her PPB, which is equal to the PPB of seller i. □

Lemma A.1.9 *Consider a NE str. profile with a reserve price P_M. Let I be an interval which does not include P_M. Suppose seller i has all of interval I in her support, and seller j does not. Then in the support of seller j there are no prices above I except P_M.*

The probability to attract shoppers is as follows:

$$\alpha_i(p) = (1 - F_j(p)) \prod_{k \neq i,j} (1 - F_k(p)) \tag{A.16}$$

Writing out and comparing the α divided by stores number yields:

$$\frac{\alpha_i(p)}{n_i} = \prod_{k \neq i,j} (1 - F_k(p)) \frac{(1 - F_j(p))}{n_i} \tag{A.17}$$

$$\frac{\alpha_j(p)}{n_j} = \prod_{k \neq i,j} (1 - F_k(p)) \frac{(1 - F_i(p))}{n_j} \tag{A.18}$$

From lemma A.1.8 the first expression is weakly larger than the second, as only the highest α/n can have the price in support. This implies that:

$$\frac{(1 - F_j(p))}{n_i} \geq \frac{(1 - F_i(p))}{n_j} \tag{A.19}$$

It follows that in I, seller i has the highest α/n. Assume that exists a price $p' < P_M$ which is the lowest price above I can be in support of seller j. This implies that at price p' seller j has a maximal α/n. Note that since the prices between I and p' cannot not selected by j, $F_j(p') = F_j(p)$ as j does not select prices in between. F_i however, had increased in I, as seller i has some mass over I. This implies that inequality (A.19) holds also for p' as it does for prices in I, as an element on the left hand side was increased, and right hand side remained constant. Thus, at p' it holds strictly.

$$\frac{(1 - F_j(p'))}{n_i} > \frac{(1 - F_i(p'))}{n_j} \tag{A.20}$$

Remark A.1.2 *The reason that the result holds for prices below P_M is that in the course of the proof a division by $\prod_{k\neq i,j}(1-F_k(p))$ was applied, and it needs to be positive. This happens at any price below P_M.*

The inequality above implies that seller j cannot have density on price p', as seller i has a higher α/n, for $p < P_M$. Thus, if seller j does not select some interval in support union, she will not select any price above it except possibly P_M. $\square$

Corollary A.1.5 *In any NE of the game, all sellers that do not have the lowest store number will select P_M as a pure strategy.*

This is since such sellers cannot offer the price P_L with sufficiently high PPB, as required by lemma A.1.7.

Corollary A.1.6 *The last lemma and last corollary holds for any two sellers with the same PPB, and for any seller who does not have the smallest stores number. This holds even if the smallest seller is unique.*

Corollary A.1.7 *Consider a NE with P_M as a reserve price, with a single smallest seller. Let I be an interval which does not include P_M. Consider two sellers i, j with same size above n_m. Suppose seller i, j have all of I in their support. They must do so with the same density.*

Note that the PPB equality must hold for those two sellers, and therefore α_i and α_j in the interval cannot differ. If it was the case, the one with lower density could not select some prices in I, as she would need to jump to P_M.

From the lemmas so far we can conclude the following:

- All sellers with size above n_m select P_M purely.
- All sellers have same profit per store.
- Searchers buy at the first store they visit, as all sellers always offer prices weakly below P_M..
- If seller i does not have the interval $(p, p') \subset (P_L, P_M)$ in support, then the only price above p seller i has in support is P_M. (follows from lemma A.1.9.
- Let I be an interval which does not include P_M. If two sellers have in support all of interval I then both do so with the same density.

- Prices in (P_L, P_M) need to be in support of at least two sellers.

Which completes the proof of Theorem 1

A.2 Proof of Theorem 2

In this section I provide the proof of Theorem 2. Note that some of the results for theorem 1 hold also here. Specifically, the ones in subsections A.1.1 and A.1.2.

As before the smallest store number of a seller is denoted as n_m, and is the parameter of seller m. The next smallest size is denoted n_j and is the parameter of seller j.

Following lemma A.1.6 ether all sellers have except m the same PPB, when m has a weakly higher PPB. From corollary A.1.6 we have that among sellers with same PPB only sellers with smallest size would offer prices below P_M. Therefore, all sellers with size above n_j (second smallest size) offer P_M purely. Additionally, P_L is determined by second smallest sellers, as at least two sellers must 'cover' every interval (corollaries A.1.2 and A.1.6).

Let P_L denote the lowest price in the support union.

Lemma A.2.1 *In any NE the PPB of all sellers except m is equal to $P_M\frac{1-\mu}{S}$. The PPB of seller m is strictly higher. Additionally, all sellers except m have mass points at P_M.*

PPB (profit per branch) of seller i is the profit of seller i divided by the n_i, which is a constant for a given seller.

Here also corollary A.1.2 holds. Thus, some sellers mix, and at least two have at the support the price of P_L, where some seller $i \neq m$ is one of those. Comparing possible PPB when offering price P_L of sellers i and m yield the following equations:

$$PPB_i = P_L(\mu/n_i + (1-\mu)/S) \tag{A.21}$$

$$PPB_m = P_L(\mu/n_m + (1-\mu)/S) \tag{A.22}$$

Since $n_m < n_i$ from definition, it is clear that if m offers this price she will have a higher PPB. This implies that if m offers P_L then she must have a higher PPB than all other sellers.

If m does not have P_L in support, she can deviate to P_L and increase her PPB above the one i has. Therefore, in equilibrium the PPB of seller m is strictly larger than the PPB of all other sellers.

Note that if at least two sellers i, j do not have a mass point at P_M, then all sellers have same PPB. As the price increases to P_M, seller's i probability to attract shoppers is the product:

$$\prod_{k \neq i} (1 - F_k(p)) \leq 1 - F(j) \tag{A.23}$$

As p goes to P_M, $1-F_j$ goes to 0. Similarly, for seller j it is bounded by $1-F_i$. Therefore, exactly one seller does not have a mass point at P_M. □

Lemma A.2.2 *m has higher probability to offer discount. Namely, $F_j < F_m$ in (P_L, P_M).*

Proof:

Note that:

$$\alpha_i(p) = \prod_{k \neq i} (1 - F_k(p)) = \frac{\prod (1 - F_k(p))}{1 - F_i(p)} \tag{A.24}$$

Therefore, the larger α will have the larger F. Note that j sells to no shoppers at price P_M and m sells to all shoppers at price P_L. When we write the profit expressions for the sellers j and m we get the following:

$$\pi_m(p) = p(\mu\alpha_m(p) + \frac{n_m(1-\mu)}{S}) = P_L \left(\mu + \frac{n_m(1-\mu)}{S} \right) \tag{A.25}$$

$$\pi_j(p) = p(\mu\alpha_j(p) + \frac{n_j(1-\mu)}{S}) = P_M \left(\frac{n_j(1-\mu)}{S} \right) \tag{A.26}$$

For simplicity, I denote $Src_i = (1-\mu)n_i/S$

Extracting the expressions for α_m and α_j the following equation is obtained:

$$\alpha_m = \frac{(P_L - p)Src_m + P_L\mu}{p\mu} \tag{A.27}$$

$$\alpha_j = \frac{P_M - p}{p\mu} Src_j \tag{A.28}$$

Comparing $\alpha \cdot p$ we get that:

$$p \cdot \alpha_m = \frac{(P_L - p)Src_m + P_L\mu}{\mu} = Const_m - p \cdot Src_m \tag{A.29}$$

$$p \cdot \alpha_j = \frac{P_M - p}{\mu} Src_j = Const_j - p \cdot Src_j \tag{A.30}$$

Note that $\alpha_m(P_L) = \alpha_j(P_L) = 1$. Obtaining the derivatives and comparing, one gets that $\alpha_m > \alpha_j$ for prices in (P_L, P_M), since $Src_m < Src_j$. Since:

$$\alpha_j(p) \cdot (1 - F_j(p)) = \alpha_m(p) \cdot (1 - F_m(p)) \tag{A.31}$$

we obtain that $(1 - F_j(p)) > (1 - F_m(p))$ or $F_j(p) < F_m(p)$, as required. $\square$

Combining the lemmas we obtain Theorem 2.

Appendix B

Reserve Price Condition - Omitted Proofs

In this appendix Theorem 3 is proven. The theorem will be proved in a sequence of lemmas. Firstly, it is shown that no price above P_M is used and P_M cannot be an isolated point of BS_1. Following step is showing that no mass point exist, except possibly at P_M, and then that no interval holes can exist in equilibrium. All lemmas refer to a NE strategy profile of the game.

Let P_S be the supremum price in the support union of all sellers.

Lemma B.0.3 *In NE no seller has prices above P_M in support: $P_S \leq P_M$*

Suppose $P_S > P_M$. Let us distinguish between several cases:

Case 1: All sellers have mass points on P_S. Due to undercutting this cannot be an equilibrium - with positive prob. get all of the market instead of only $1/n$ of it.
Case 2: Some sellers have mass points on P_S. Undercutting here is also a profitable deviation. The only case when a seller has non zero profit is when some searchers, after observing P_S by $k > 1$ sellers are satisfied with P_S. Due to undercutting proof condition it is possible to undercut P_S and get all of these searchers, and not just $1/k$ of them.
Case 3: No seller has a mass points on P_S. Then, P_S has zero profit, and prices arbitrarily close to P_S have profit arbitrarily close to zero. This is since all sellers offer price bellow P_S w.p. 1, no shoppers would purchase at this price, and searchers would search on and find a cheaper price w.p. 1. Similarly, with prices arbitrarily close below P_S have prob. arbitrarily close to one and profit arbitrarily close to zero. Therefore, a deviation to c would be profitable, as it ensures positive profit. □

Corollary B.0.1 *In NE, for any price p with positive prob. to attract shoppers, at most a single seller has a mass point at p.*

If several sellers have a mass point at some price p, there is a positive prob. for multiple sellers to select p. Thus, undercutting p is possible and the shoppers share will increase discontinuously. Due to the undercutting condition no searchers will be lost. The prob. to satisfy shoppers is strictly higher. The result is that undercutting is a profitable deviation.

Lemma B.0.4 *If $P_M \in BS_1$ and is selected by some sellers it is not an isolated point of BS_1.*

If it is selected with a mass point by some sellers it cannot be an isolated point of BS^1 due to undercut proof. If it is selected without mass points (continuously), it must be a part of an interval, and cannot be isolated. □

Let the maximal (supremum) price in the sellers support which is not in BS_1 (if exists) be denoted as P_{src}. Additionally, let the maximal (supremum) price which has a positive probability to attract shoppers be denoted as P_{shop}.

Lemma B.0.5 $P_{shop} = P_M$ *In words: P_M is the supremum point of strategy support for each seller.*

From lemma B.0.3 it cannot be higher than P_M. Suppose that the supremum price of seller i is $p < P_M$, an is the lowest supremum among all sellers. Formally, $P_{shop} = p < P_M$. For any price above p and below P_M the probability to sell to shoppers is zero.

Firstly, note that no seller would select prices in (p, P_M). The probability to sell to shoppers in this interval is zero, and only uninformed consumers would purchase for a price in (p, P_M). Each seller would maximize her profit when selecting P_M and not prices in (p, P_M).

Consider the case that P_M is not an isolated point of BS^1, and some prices just below P_M are in BS^1.

Let us distinguish between several cases regarding mass points on p:

Case 1: seller i has a mass point at p: Due to corollary B.0.1 no other seller has a mass point on p. Since no other seller has (p, P_M) in support i can increase her price from p into (p, P_M) and gain more profit, without loosing any market share.
Case 2: Seller(s) that are not i have mass points at p: Similarly, one of those sellers can deviate profitably into (p, P_M) instead of offering p.

Case 3: No seller has mass points at p: If there are no mass points at p, seller i can deviate to a price just below P_M instead of p and prices just below p an gain profit due to a higher price.

If P_M is not an isolated point of BS^1 no seller selects it. Since no seller selects prices in (p, P_M), all sellers have p as the support supremum of their strategy. In this case some sellers have no mass point on p, and therefore, it has prob. zero to attract shoppers. Similarly to the cases above a deviation to P_M is profitable, ether from a mass point at p, or if none exists from some interval just below p to P_M.

Therefore, in all cases a profitable deviation exists. $\square$

Corollary B.0.2 *Let p be a price below P_M. At most one seller has a mass point at p.*

Follows directly from Lemma B.0.5 and Corollary B.0.1.

From the construction, BS_1 is dense in some interval of the form $(a, P_M]$. The highest interval in BS_1 is defined as follows:

Definition B.0.1 *Let I be the interval of the form $(a, b]$ such that $b = P_M$ and a is the highest number below P_M such that exists ε satisfying: $(a - \varepsilon, a) \cap BS = \emptyset$.*

This allows I to cover some prices that are not covered in BS_1, as noted in the theorem.

Lemma B.0.6 *P_{src} is below I.*

Suppose $P_{src} \in I$ and in support of seller i. From construction P_{src} is not the infimum of I. Therefore, some prices below P_{src} are in I and BS^1. Thus there is a positive probability for some sellers to select prices below P_{src}, and therefore, some of the searchers initially visiting i may purchase in a different store. Thus, a deviation to a price just below p, which is in BS_1 from construction, would be a profitable deviation. The loss in price is arbitrarily small, and increase in searchers purchasing at seller i would overweight it. $\square$

Lemma B.0.7 *If there are mass points in I, these must be at P_M.*

Suppose that $p \neq P_M \in I$ and seller i has a mass point on it. From corollary B.0.2, No other seller has a mass point on I. Additionally, from lemma B.0.6 all unsatisfied searchers after first visit observe a price below I, implying that additional unsatisfied searchers will not purchase for p.

For seller $j \neq i$ the probability to attract shoppers decrease discontinuously at p. Therefore, in equilibrium seller j would not have prices just above p in support. As a result i can deviate and increase the price from p to a slightly higher price in BS^1, without losing any market share, reaching a contradiction. $\square$

Lemma B.0.8 *No seller has prices below the interval I in support.*

Let the maximal (supremum) price that can be offered by sellers below I denoted as P_I.

Note that no seller has prices above P_I and below I in support. From corollary B.0.2 at most a single seller has a mass point at P_I. Suppose seller i is a seller with P_I in support, and if there is a mass point on P_I she is the seller with the mass point.

Seller i can deviate into the interval I, arbitrarily close to the infimum of I, instead of selecting P_I or prices just below it. The loss of shoppers is arbitrarily small, as no other seller has any mass on any prices between P_I and P_M from lemma B.0.7. The price is higher, and the share of searchers will be weakly higher (or equal if P_I is in BS^1). The latter is due to the fact that all initial searchers now purchase at seller i, and previously some could observe a lower price and never return. Note that due to lemma B.0.6 all unsatisfied searchers after first visit observe a price of at most P_I, implying that additional unsatisfied searchers will not purchase at i. $\square$

Any price below I would be below the seller support union and in BS^1. This completes the proof of the theorem.

Appendix C

Minimum Price - Omitted Proofs

We begin with the proof for theorem 5.

Theorem 8 *If prices below P^* are forbidden, exists a unique pure strategy equilibrium (yet possibly some additional mixed ones), where all sellers select P^* purely, and $P^* + c$ is the reserve price.*

Proof:

The main point in proving the theorem is to show that no seller can deviate. Beforehand note that reserve price of $P^* + c$ is rational, as the expected seller price in equilibrium is P^*. Therefore, if a seller offers above $P^* + c$ it is worthy to search on and encounter P^* at the next store. If offered below $P^* + c$ then an additional search is not worthy.

The profit for a seller in equilibrium is as follows:

$$\pi(P^*) = P^*/N$$

Sellers cannot deviate to a lower price. If a seller deviates to a higher price then there are two possibilities:

- If the price is weakly below $P^* + c$ then the seller will sell only to searchers initially visiting her store
- If the price is above $P^* + c$ then nobody will purchase at the store.

In the latter case the profit is zero, and in the former case the profit is maximized when the price offered is exactly $P^* + c$. The profit when offering $P^* + c$ is:

$$\pi(P^* + c) = (P^* + c)(1 - \mu)/N$$

Therefore, the deviation is not profitable iff $\pi(P^*) \geq \pi(P^* + c)$, which implies:

$$\begin{aligned} P^*/N &\geq (P^* + c)(1-\mu)/N \\ P^* &\geq (P^* + c)(1-\mu) \\ P^*\mu &\geq c(1-\mu) \\ P^* &\geq c\frac{1-\mu}{\mu} \end{aligned}$$

The last inequality holds due to the definition of P^*. Therefore, any minimum price weakly above P^* will also work.

Note that any price other than P^* cannot have positive mass in equilibrium strategy distribution. This is due to undercutting.

□

Next the proof for lemma 4.4.1 is provided.

Lemma C.0.9 *The condition from lemma 4.3.1 holds for the case of two sellers.*

Proof:

From equation 4.3, it follows that:

$$\begin{aligned} f(p) =& \quad \tfrac{1-\mu}{2\mu}\tfrac{P_M}{P^2} \\ E(F(P)) =& \quad P_M - c = \tfrac{1-\mu}{2\mu}\kappa P_M \\ c =& \quad P_M(1 - \tfrac{1-\mu}{2\mu}\kappa) \end{aligned}$$

The required condition is $\mu P_M > c$. This will hold iff:

$$\frac{2\mu - (1-\mu)\kappa}{2\mu} < \mu \tag{C.1}$$

Elaborating the expression yields:

$$\begin{aligned} \frac{2\mu - (1-\mu)\kappa}{2\mu} &< \mu \\ 2\mu - (1-\mu)\kappa &< 2\mu^2 \\ 2\mu - 2\mu^2 &< (1-\mu)\kappa \end{aligned}$$

This is equivalent to: $\kappa > 2\mu$. Note that both expressions are 0 when $\mu = 0$ and both expressions are increasing in μ when $\mu \in (0, 1)$. Comparing the derivatives one sees that:

$$\kappa' = \frac{1}{1+\mu} + \frac{1}{1-\mu} = \frac{2}{1-\mu^2} > 2 = (2\mu)' \forall \mu \in (0,1) \tag{C.2}$$

Thus κ raises more steep than 2μ. Since equality is obtained at 0, it follows that in the range $\kappa > 2\mu$.

□

Lastly, the proof for proposition 4.4.1 is provided:

Proposition C.0.1 *Consider a two seller Stahl model. Suppose a limitation is imposed at price P^*. Then the equilibrium, as shown in theorem 5 is unique, and no additional mixed or asymmetric equilibria exist.*

Proof:

Note that the only additional equilibria where a ban on low prices plays a role are ones where sellers set mass points on the price P^*. Let the mass on the price P^* be denoted ρ. Additionally, as before no seller would offer a price above the reserve price.

The profit of a seller offering the price of P^* would be:

- If the other seller sets price P^* - profit of $P^*/2$
- In any other case profit of $P^*(\mu + (1-\mu)/2)$.

Note that due to undercutting at no price except P^* would be a mass point. Additionally, the reserve price would be the highest price in seller strategy support.

From here follows that seller strategy must be an isolated point at P^*, and then an interval from some P_N up to P_M. This is since at P_N, the next offered price above P^* the profit is:

- If the other seller sets price P^* - profit of $P_N(1-\mu))/2$
- In any other case profit of $P_N(\mu + (1-\mu)/2)$.

Since $(1-\mu)/2 < 1/2$, we get that $P_N > P^* + \varepsilon$ for some positive ε.

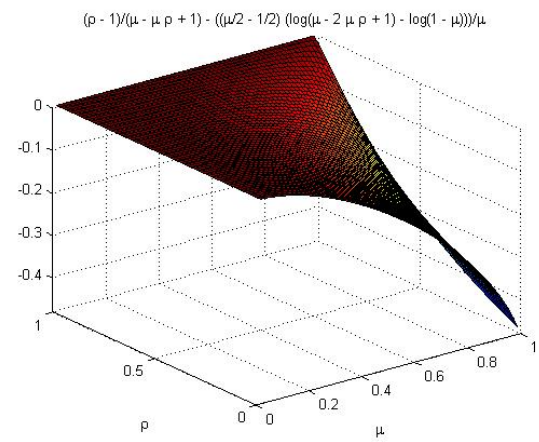

Figure C.1: The difference between LHS and RHS of eq. C.4.

From equal profit when mixing one can get the distribution for the interval (P_N, P_M):

$$F(P) = 1 - \frac{1-\mu}{2\mu}(\frac{P_M}{P} - 1)$$

Let us denote the mass at P_L as ρ. From here, $F(P_N) = \rho$, the ratio of P_N and P_M is as follows:

$$\frac{P_M}{P_N} = \frac{1 + \mu - 2\mu\rho}{1-\mu}.$$

Combining continuous distribution with mass point, yields the expected value as follows:

$$E(F) = \rho P^* + \frac{1-\mu}{2\mu} \log \frac{P_M}{P_N} \tag{C.3}$$

Remember that $E(F) = P_M - c$ and $P^* = c\frac{1-\mu}{\mu}$. Thus, once can replace c and P^* with P_M multiplied by corresponding elements. Note that:

$$\begin{aligned} P_M(1-\mu)/2 = P^*(1-\rho/2)\mu + (1-\mu/2) \\ P_M = P^* \frac{1+\mu-\mu\rho}{1-\mu} \\ P_M = c\frac{P_M}{P^*}\frac{P^*}{c} \end{aligned}$$

Replacing P^* and c with fractions of P_M would cause P_M to be narrowed down throughout the expression, leaving the equality $E(F) = P_M - c$ as follows:

$$\frac{1-\mu}{2\mu}(\log(1 + \mu - 2\mu\rho) - \log(1 - mu)) = \frac{1-\rho}{1+\mu-\mu\rho} \tag{C.4}$$

This equation does not have a solution for $\mu, \rho \in (0, 1)$. For the value of the expression depicted at equation (C.4) (left side minus right side) see Figure C.1.

□

and the corresponding corollary:

Corollary C.0.3 *If lowest price bound set at a level above P^* the pure NE would be unique.*

The main change is that the ratio between the used P^* and c would be higher. This would increase the right hand side on equation C.4, and it would remain without a solution. This is visible via the logarithm expression than needs now to be even larger than for the P^* minimum price.

Corollary C.0.4 *In the case that $P_C < P^*$ additional equilibria are possible. For each price for $P_C \in (P_L, P^*)$ exists an equilibrium where sellers have a mass point on P_C, and then a continuous distribution over an interval (P_N, P_M) where P_M is the reserve price and P_N is a price strictly larger than P_C.*

When going to the other direction and subtracting form the right hand side the equation C.4 would have a solution, implying an equilibrium of the given form. An additional insight is important:

Corollary C.0.5 *The larger the ρ satisfying the equation the closer is the minimum price to P^*.*

Follows directly from the fact that the expression is decreasing as ρ goes further from 1.

The last lemma to be proven is 4.4.3: We have shown that in equilibria when P_C is set below P^*, P_C has a mass point of ρ. Note that lower P_C implies lower ρ.

Lemma C.0.10 *P_M is decreasing in ρ. That is, higher ρ implies lower P_M.*

Proof:

In equation C.3 from the appendix the expression for the expected price offered is calculated as a function of P_M, μ and ρ. Suppose for the moment that P_M remains constant. Then, the derivative over ρ is depicted in figure C.2.

From figure C.2, the expected value is decreasing with ρ. Remember that $E = P_M - c$. Therefore, if P_M remains constant the difference between P_M and the expected price offered would exceed c. From here, P_M must be decreasing in ρ. $\square$

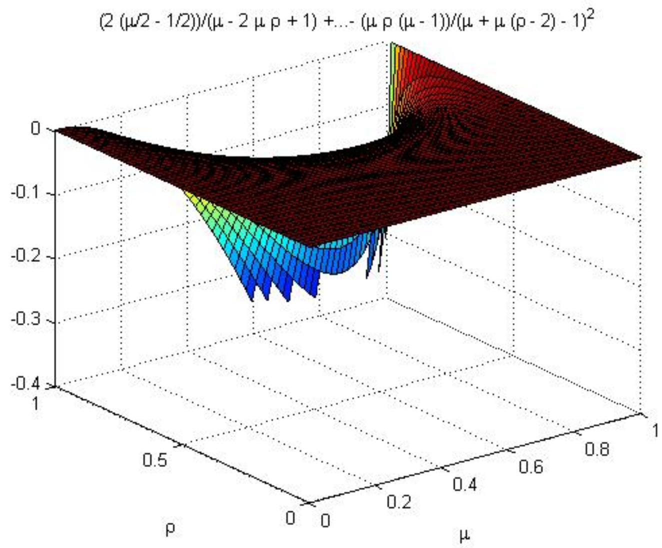

Figure C.2: change in EPO as func. of μ, ρ (2 sellers)

Appendix D

Bibliography

[1] Ackermann H, H. Röglin, B. Vocking, On the Impact of Combinatorial Structure on Congestion Games, Journal of the ACM (JACM) 55 (6), article 25 (08')

[2] Astorne-Figari C., A. Yankelevich, *Asymmetric Sequential Search*, Available at SSRN: http://ssrn.com/abstract=1622809

[3] Aumann R., A. Brandenburger, *Epistemic Conditions for Nash Equilibrium*, Econometrica, Vol. 63(5), 1161-1180 (95')

[4] Barron G., E. Yechiam, *The coexistence of overestimation and underweighting of rare events and the contingent recency effect*, Judgment and Decision Making, Vol. 4(6), 447-460 (09')

[5] Baye M., J. Rupert, J. Gatti, P. Kattuman, J. Morgan, *Clicks, Discontinuities and Firm Demand Online*, Journal of Economics and Management Strategy, vol. 18(4) 935–975 (09')

[6] Bakucs L.Z., F. Imre, *Milk Retail Sales Patterns in a Traditional Economy. The Case of Hungary*, Agricultural Economics Society, 83rd Annual Conference (09')

[7] Baye M., D. Kovenock, C. de Vries, *It takes two to Tango: Equilibrium in a model of sales*, Games and Economic Behavior, vol. 4, 493-510 (92')

[8] Baye M., J. Morgan, *A folk Theorem in one shot Bertrand Games*, Economic Letters, vol. 65, 59-65 (99')

[9] Bilò V., A. Fanelli, M. Flammini, L. Moscardelli, *Performance of One-Round Walks in Linear Congestion Games*, Theory of Computing Systems, Volume 49(1), 24-45 (11')

[10] Bolton P., J. Scharfstein, *A Theory of Predation Based on Agency Problems in Financial Contracting*, American Economic Review. vol. 80(1), 93-107 (90')

[11] Burdett K., K. Judd, *Equilibrium Price Dispersion*, Econometrica, vol. 51, 955-970 (83')

[12] Burdett K., E. Smith, *Price Distribution and Competition*, Economics Letters, vol. 106 180-183 (10')

[13] Cnossen, Sijbren *Alcohol taxation and regulation in the EU*, CESifo working paper, No. 1821 (06')

[14] Carlson J., R. P. McAfee, *Discrete Equilibrium Price Dispersion*, The Journal of Political Economy, Vol. 91(3), 480-493 (83')

[15] Chen Y., T. Zhang, *Equilibrium price dispersion with heterogeneous searchers*, MPRA paper 18757, (09')

[16] Chioveanu I., J. Zhou, *Price Competition and Consumer Confusion*, MPRA paper 17340 (09')

[17] Christodoulou G., V.S. Mirrokni A. Sidiropoulos, *Convergence and approximation in potential games*, STACS 2006, Lecture Notes in Computer Science Volume 3884, 349-360 (06')

[18] Davis M.C., J.D. Hamilton, *Why are prices sticky? The dynamics of wholesale gasoline prices* Working Paper 9741, NBER working papers series, (03')

[19] Diamond P., *A Model of Price Adjustment*, Journal of Economic Theory, vol. 3, 156-168 (71')

[20] Fabrikant A, C. Papadimitriou, K. Talwar, *The complexity of pure Nash equilibria*, The 36th Annnual Association for Computing Machinery Symposium on Theory of Computation, 604-612 (04)

[21] Fotakis D, S. Kontogiannis, P. Spirakis, 2005, *Symmetry in Network Congestion Games: Pure equilibria and anarchy cost*, Workshop on Approximation and Online Algorithms, Lecture Notes in Computer Science 3879, 161-175 (05')

[22] Fotakis D., Congestion games with linearly independant paths: convergence time and price of Anarchy, Theory of Computing Systems 47, 113-136 (10')

[23] Holzman R, N. Law-Yone, Strong Equilibrium in Congestion Games, Games and Economic Behavior 21, 85-101 (97')

[24] Holzman R, N. Law-Yone, Network Structure and strong equilibrium in route selection games, Mathematical social sciences 46, 193-205 (03')

[25] Ieong S., R. McGrew, E. Nudelman, Y. Shoham, Q. Sun, *Fast and Compact: A Simple class of congestion games*, Proceedings of the Twentieth National Conference on Artificial Intelligence. (05')

[26] Janssen M., J. Morags-Gonzales, *Strategic Pricing, Consumer Search and the Number of Firms*, Review of Economic Studies, vol. 71, 1089-1118 (04')

[27] Janssen M., J. Moraga-Gonzales, M. Wildenbeest, *Consumer Search and Oligopolistic Pricing: an Empiric Investigation*, Tinbergen Institute Discussion Papers 04-071/1, Tinbergen Institute (04')

[28] Janssen M., J. Moraga-Gonzales, M. Wildenbeest, *Truly costly sequential search and oligopolistic pricing*, International Journal of Indistrial Organisation, vol. 23 451-466 (05')

[29] Janssen M., A. Parakhonyak, *Consumer Search with Costly Recall*, Tinbergen Institure Discussion Paper TI-2008-002/1 (08')

[30] Janssen M., P. Pichler, S. Weidenholzer ,*Sequential Search with Incompletely Informed Consumers: Theory and Evidense form Retail Gasoline Market*, Vienna Economics Papers 0914, University of Vienna (09')

[31] Martin A., J. Saurina, V. Salas, *Interest Rate Dispersion in Deposit and Loan Markets*Banko de Espana Working Papers 0506, (05')

[32] Meng Y., D. Hill-McManus, A. Brennan *Model-based appraisal of alcohol minimum pricing and off-licensed trade discount bans in Scotland using the Sheffield alcohol policy model*, 2nd update, Jan. 2012, Sheffield University, availible at:
http://www.sheffield.ac.uk/polopoly_fs/1.150021!/file/scotlandupdatejan2012.pdf.

[33] Milchtaich I., *Network Topology and the efficiency of equiliblria*, Games and Economic Behavior 57, 321-346 (06')

[34] Monderer D., L. S. Shapley, *Potential Games*, Games and Economic Behavior 14, 124-143 (96').

[35] Olszewski W., A. Wolinsky, *Search with Partially Informed Stopping Decisions*, working paper (13'), availible at: http://faculty.wcas.northwestern.edu/ wol737/Search.pdf

[36] Rosental R.W., A class of Games Posessing Pure Strategy Nash Equilibria, International Journal of Game Theory, vol. 2, 65-67 (73')

[37] Rothshild M., *Models of Market Organization with Imperfect Information: A survey*, The Journal of Political Economy, vol. 81(6), 1283-1308 (73')

[38] Snider C., *Predatory Incentives and Predation Policy: The American Airlines Case*, (08'). Availible at:
http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.143.7724

[39] Stahl D., *Oligopolistic Prices with Sequential Consumer Search*, The American Economic Review, vol. 79, 700-712 (89')

[40] Stahl D., *Oligopolistic Pricing with Hetrogeneous consumer search*, International Journal of Indistrial Organisation, vol. 14 243-268 (96')

[41] Stockwell T., M. C. Auld, J. Zhao1, G. Martin, *Does minimum pricing reduce alcohol consumption? The experience of a Canadian province*, Addiction Volume 107(5), 912–920, (12')

[42] Varian H., *A Model of Sales*, The American Economic Review, vol. 70, 651-659 (80')

[43] Voorneveld M., P. Borm, F. van Megen, S. Tijs - "Congestion Games and Potential Reonsidered", International Game Theory Review 1, 283-299, (99')

[44] Wardrop J.G. - "Some theoretical aspects of road traffic research", Proceedings of the Institution of Civil Engineers, Part II, Vol. 1 , 325-378, (52')

[45] Watanabe M., *A Model of Merchants*, Journal of Economic Theory, vol. 145, 1865-1889 (10')

[46] Wagenaar A.C. , M.J. Salois, K.A. Komro *Effects of beverage alcohol price and tax level on drinking*, Addiction 2009, 104(2), 179-190 (09')

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