# Non-Perturbative Gravity at different Length Scales

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# Abstract

In this thesis, we investigate different aspects of gravity as an effective field theory. Building on the arguments of self-completeness of Einstein gravity, we argue that any sensible theory, which does not propagate negative-norm states and reduces to General Relativity in the low energy limit is self-complete. Due to black hole formation in high energy scattering experiments, distances smaller than the Planck scale are shielded from any accessibility. Degrees of freedom with masses larger than the Planck mass are mapped to large classical black holes which are described by the already existing infrared theory. Since high energy (UV) modifications of gravity which are ghost-free can only produce stronger gravitational interactions than Einstein gravity, the black hole shielding is even more efficient in such theories. In this light, we argue that conventional attempts of a Wilsonian UV completion are severely constrained. Furthermore, we investigate the quantum picture for black holes which emerges in the low energy description put forward by Dvali and Gomez in which black holes are described as Bose-Einstein condensates of many weakly coupled gravitons. Specifically, we investigate a non-relativistic toy model which mimics certain aspects of the graviton condensate picture. This toy model describes the collapse of a condensate of attractive bosons which emits particles due to incoherent scattering. We show that it is possible that the evolution of the condensate follows the critical point which is accompanied by the appearance of a light mode.

Another aspect of gravitational interactions concerns the question whether quantum gravity breaks global symmetries. Arguments relying on the no hair theorem and wormhole solutions suggest that global symmetries can be violated. In this thesis, we parametrize such effects in terms of an effective field theory description of three-form fields. We investigate the possible implications for the axion solution of the strong CP problem. Since the axion is the (pseudo-) Goldstone boson of a broken $U(1)$ global symmetry, quantum gravitational global symmetry violations could reinstate the CP problem even in the presence of the axion. We show that in the presence of massless neutrinos possible conflicts with the axion solution can be resolved. Demanding a viable axion solution of the strong CP problem, we derive new bounds on neutrino masses. In addition, we investigate the QCD vacuum energy screening mechanism for light quarks. It is well-known that the $\theta$-dependence of the QCD vacuum vanishes linearly with the lightest quark mass. By an analogy with Schwinger pair creation in a strong electric field, we consider vacuum screening by $\eta'$ bubble nucleation. We find that using the standard instanton approximation for the $\eta'$ potential, the linear dependence is not recovered. We take this as an indication for the non-analyticity of the QCD vacuum energy proposed by Witten.

In the last part of this thesis, we are concerned with gravitational effects on cosmological scales. The recent Planck data indicate that one of the best motivated dark matter candidates, the axion, is in conflict with bounds on isocurvature perturbations. We show that the isocurvature fluctuations can be efficiently suppressed when introducing a non-minimal kinetic coupling for the axion field during inflation. Thus, the axion can be a viable dark matter candidate for a large range of parameters. We show that the same coupling allows for the Standard Model Higgs to drive inflation and the dark matter density to be produced by the axion.

Gravitational effects on large scales would also be sensitive to a possible mass for the graviton. However, such a modification has been known to be plagued by inconsistencies. In light of the recent proposal of a ghost-free theory of massive gravity by de Rham, Gabadadze and Tolley, we investigate the cubic order interactions of this theory in terms of helicities of a massive spin-2 particle. We find that it is not possible to truncate the action at cubic order without introducing higher derivative terms strongly coupled at scale $\Lambda_5$. Additionally, we consider possible cubic interaction terms for a massive spin-2 particle on a Minkowski background. We derive the unique interaction terms which are free of higher derivatives.

# Zusammenfassung

In dieser Dissertation untersuchen wir verschiedene Aspekte der als effektive Feldtheorie aufgefassten Quantengravitation. Basierend auf Argumenten der Selbstkomplettierung von Einstein-Gravitation zeigen wir, dass jede Theorie, die keine propagierenden Zustände mit negativer Norm aufweist und die sich zudem auf Einstein-Gravitation im Niederenergie-Limes reduziert, selbstkomplett ist. Durch die Formierung von schwarzen Löchern in Streuexperimenten bei hohen Energien, sind Distanzen, die kleiner als die Planck Skala sind, von jedwedem Zugriff abgeschirmt. Freiheitsgrade mit einer Masse, die größer als die Planck-Masse ist, werden hierbei auf klassische schwarze Löcher abgebildet, die wiederum durch die schon existierende Infrarot-Theorie beschrieben sind. In Modifikationen von Geist-freier Gravitation bei hohen Energien ist die Abschirmung durch schwarze Löcher sogar noch effizienter, da solche, mit Einstein-Gravitation verglichen, stärkere gravitative Wechselwirkungen erzeugen. In Angesicht dessen sind konventionelle Versuche der UV Komplettierung im Wilsonschen Sinne äußerst eingeschränkt. Weiterhin betrachten wir das von Dvali und Gomez vorgeschlagene Quantenbild schwarzer Löcher gemäß dem schwarze Löcher als ein Bose-Einstein Kondensat von vielen schwach gekoppelten Gravitonen aufgefasst werden können. Im Speziellen untersuchen wir ein nicht-relativistisches Analogon, welches gewisse Aspekte dieses Graviton- Kondensat-Bildes widerspiegelt. In diesem Beispielsystem kollabiert ein Kondensat attraktiver Bosonen unter durch inkohärente Streuprozesse verursachter Aussendung von Teilchen. Wir zeigen, dass sich das Kondensat während seiner Zeitentwicklung am kritischen Punkt aufhalten kann mit dem hiermit verbundenen Auftreten einer leichten Mode.

Ein weiterer Aspekt der gravitativen Wechselwirkung betrifft die Frage, in wie weit Quantengravitation globale Symmetrien bricht. Betrachtungen, die auf dem No-Hair Theorem wie auch Wurmloch-Lösungen beruhen, legen nahe, dass globale Symmetrien verletzt werden können. In der vorliegenden Arbeit parametrisieren wir solche Effekte in einer effektiven Feldtheorie von Dreiform-Feldern und untersuchen die möglichen Implikationen für die Axionlösung des starken CP-Problems. Da das Axion ein (Pseudo) Goldstone-Boson einer gebrochenen, globalen U(1) Symmetrie ist, könnten solche, auf der Quantengravitation beruhenden, Symmetriebrechungen das CP Problem wieder einführen, trotz der Anwesenheit eines Axions. Wir zeigen, dass in der Gegenwart eines massenlosen Neutrinos die möglichen Konflikte mit der Axionlösung aufgelöst werden

können. Aus der Voraussetzung, dass eine valide Axionlösung des CP-Problems existiert, leiten wir neue Grenzen für die Neutrinomasse her. Weiterhin untersuchen wir den Abschirmungsmechanismus, der durch die Vakuumenergie für leichte Quarks in der QCD verursacht wird. Es ist bekannt, dass die $\theta$-Abhängigkeit des QCD-Vakuums mit der leichtesten Quarkmasse linear verschwindet. Analog zur Schwinger-Paarerzeugung in einem starken elektrischen Feld betrachten wir die Vakuumsabschirmung durch die Blasenbildung von $\eta'$-Vakua. Unter Verwendung des Potentials für $\eta'$, das durch Standard-Instanton-Rechnungen gegeben ist, können wir die lineare Abhängigkeit nicht reproduzieren. Wir betrachten dies als Indikation für die von Witten vorgeschlagene Nicht-Analytizität der QCD-Vakuumenergie.

Im letzten Teil der Arbeit betrachten wir Gravitationseffekte auf kosmologischen Skalen. Neuste Daten des Planck Satelliten weisen darauf hin, dass einer der vielversprechensten Kandidaten für die Dunkle Materie, das Axion, in Konflikt ist mit durch adiabatischen Fluktuationen vorgegebenen Grenzen. Wir zeigen, dass diese Fluktuationen effizient unterdrückt werden können, wenn für das Axionenfeld während der Inflation ein nicht-minimaler kinetischer Kopplungsterm eingeführt wird. Auf Grundlage dessen kann das Axion als möglicher Kandidat für dunkle Materie erhalten bleiben. Weiterhin zeigen wir, dass ein solcher kinetischer Term dem Higgs-Teilchen des Standardmodells erlaubt die Inflation anzutreiben und dass die Dichte der dunklen Materie tatsächlich vom Axion erzeugt werden kann.

Gravitationseffekte auf großen Skalen sind zudem potentiell von einer möglich endlichen Masse des Gravitons abhängig. Jedoch ist bekannt, dass solche Modifikationen der Gravitation zu Inkonsistenzen führen können. Unter Betrachtung des kürzlich von de Rham, Gabadadze and Tolley vorgeschlagenen Szenarios einer Geist-freien Theorie der massiven Gravitation untersuchen wir die Wechselwirkungen kubischer Ordnung in dieser Theorie, ausgedrückt in den Helizitäten eines massiven Spin-2 Teilchens. Wir finden, dass es nicht möglich ist, die entsprechende Wirkung in kubischer Ordnung zu trunkieren, ohne dass höhere Ableitungsterme eingeführt werden müssen, die auf der Skala $\Lambda_5$ stark gekoppelt sind. Im Weiteren betrachten wir mögliche kubische Wechselwirkungsterme für massive Spin-2 Teilchen auf einem Minkowski-Hintergrund. Wir leiten die eindeutig bestimmten Wechselwirkungen her, welche, ausgedrückt in Helizitäten, frei von höheren Ableitungen sind.

# Contents

# Chapter 1

# Introduction

After almost 100 years of its discovery, General Relativity is still a theory of many puzzles. Whereas all other known fundamental interactions, the strong, weak and electrodynamic forces, can be successfully described within the framework of quantum field theory, gravity has so far eluded a formulation as a fundamental quantum field theory. In this respect, conventionally the most outstanding problems have been considered the problem of ultraviolet (UV) divergences leading to the non-renormalizability of the theory, and the problem of unitarity violation in high energy scattering amplitudes. Efforts to reconcile gravity with the notion of renormalizability have been numerous and amongst others led to the development of string theory and supergravity. One of the most cited problems of non-renormalizability is the inability to make a definite prediction for measurements. Computing quantum corrections in such non-renormalizable theories requires the inclusion of an infinite series of counter-terms all added with a parameter whose value has to be determined from experiment. It was therefore long conventional believe that such theories cannot make definite predicitions and should therefore be disregarded.

From a modern perspective, however, renormalizability is no longer considered to be the sacred criterion for a theory to prove its predictive power. Experiments can only measure parameters up to a certain accuracy and up to a certain energy scale. Therefore, as long as a theory can be organized in such a way that one can quantify the error that comes with its predictions, it should be considered a viable physical approach. This is the underlying philosophy of effective field theory (EFT).

In general, the logic of effective field theory is the following. In order to describe physics at a given scale, one needs to consider degrees of freedom which are propagating and interacting at the specific scale of interest. For example, in order to describe the weak beta decay with characteristic energy scales of order 10 MeV, one can consider Fermi's famous theory of a four-fermion interaction. This theory is non-renormalizable, but nevertheless it describes the beta decay to high accuracy. Today, it is well-known that the underlying fundamental theory is the theory of weak interactions. In this fundamental, renormalizable theory, the point-like four-fermion interaction is mediated by the $W$-boson of mass $m_W \simeq 80$ GeV and it reproduces Fermi's point-like interactions when evaluated at energies much below $m_W$. Such a large separation of scales illustrates the philosophy of effective field that physics of small scales $\sim M^{-1}$ decouples and influences physics at large scales $E^{-1}$ only through corrections which can be organized in terms of a power series in $E/M$.

Starting from a low energy effective field theory, as one moves to higher energy scales the expansion in terms of $E/M$ breaks down at the point when the energy becomes comparable to the scale $E \sim M$ of the underlying, possibly unknown theory. The physics of small length scales becomes important at this point and one has to find a way to resolve it. Usually, this is done by intergrating in new, "heavy" degrees of freedom with masses of order of $M$, which then become dynamical degrees of freedom. Starting from Fermi's theory this would correspond to integrating in the $W$-boson. This process of integrating in new degrees of freedom in order to resolve small distances is referred to as a Wilsonian "UV"-completion.[1] It is important to note that in such an effective low energy description the theory can nevertheless describe quantum processes. For example, it is perfectly possible and legitimate to compute quantum corrections via loop diagrams in an effective field theory, as long as one takes into account that physics beyond a certain energy scale (the cutoff) is already implicitly accounted for in the definition of the coupling constants of the theory. In turn, such quantum corrections can be organized in terms of powers of $E/M$ as well.

## 1.1 Self-Completion of Gravity and Black Hole N-Portrait

Let us come back to gravity. It is well-established that General Relativity describes the dynamics of the solar system with very high accuracy. Therefore, it is sensible to assume that it can be treated as an effective field theory at low energies. In this context, General Relativity can only reliably describe gravitational interactions up to the Planck scale $M_P$ which sets the intrinsic strong coupling scale of gravity. After this scale, new physics is required to take over. This is the road which had been mainly taken to find a possible theory of quantum gravity. It is, however, not the only one. In fact, General Relativity already hints towards the route of escape: the formation of black holes. Black holes are produced whenever an energy $E$ is compacted within a region of space that is smaller than the corresponding Schwarzschild radius $R_S(E)$. Thus, the effective field theory of gravity predicts black hole formation in high energy collisions [1–6]. In this light, it has been suggested that gravity might not need a UV-completion in the Wilsonian sense [3, 7]. In [7], Dvali and Gomez argued that, in fact, the only outcome of a scattering experiment with energies $E$ larger than the Planck scale $M_P$, and impact parameter smaller than $L_P \equiv M_P^{-1}$, is a black hole of mass $M = E$ and size given by its Schwarzschild radius $R_S = 2M/M_P^2$. For energies $E \gg M_P$, this black hole is a well-defined classical object fully described by low energy physics. Therefore, also the stipulated unitarity violation in trans-Planckian scatterings is avoided [7]. Furthermore, it is for Heisenberg's uncertainty principle that in quantum field theory short distance scales can only be probed by high energy scattering experiments. Thus,

[1]Notice that in terms of scattering amplitudes which similarly can be organized in a series $E/M$ in an effective field theory, the breakdown of this expansion is accompanied by a breakdown of unitarity.

to probe sub-Planckian distance scales becomes impossible as the momentum transfer needed inevitably leads to black hole formation [7].

Classical field configurations cannot probe distances smaller than the characteristic wavelength of their constituents which typically sets the size of the configuration. Since black holes of mass $M \gg M_P$ are such classical objects, they cannot probe physics at distances smaller than its size $R_S$. One can lower the momentum transfer $p$ of the scattering experiment and, by continuity, one finds that for any scattering experiment with $p > M_P$ a black hole is the unique outcome [7]. Therefore, Einstein gravity (General Relativity) can be considered self-complete in the UV. This is, however, a different notion of UV-completeness than in the Wilsonian approach. Instead of having to integrate in new heavy ($M > M_P$) degrees of freedom at energies larger than $M_P$, in gravity high-energy states correspond to classical configurations which can indeed be described by the low energy physics of the theory. In consequence, this UV-IR mapping abolishes the need of a UV-completion in the Wilsonian sense [7]. Nonetheless, the effective theory of gravity is a theory of *quantum* gravity. It describes the gravitational interactions in terms of an exchange of gravitons, which are the quanta of the field configuration and can thus be quantized in a standard effective field theory treatment with the only difference that there is no need to integrate in new physics at high energies.

In this thesis, we build on the work of [7] and show that the self-completeness of gravity is expected to hold for any sensible theory which reduces to General Relativity in the low energy approximation [8]. Our argument is based on the fact that in any modification of gravity which propagates only positive norm states, the gravitational interaction can only become stronger compared to General Relativity. We argue that the scale of the onset of strong coupling $M_*$ always coincides with the point of black hole formation [8]. Therefore, the black hole formation sets in for momentum transfer $p = M_*$ even smaller than $M_P$. Furthermore, the implications arising from the self-completeness due to black hole formation for conventional (Wilsonian) attempts of UV completion is investigated; in particular, in the context of the so-called Asymptotic Safety scenario [9, 10]. We find that there cannot exist any UV degrees of freedom which could induce such a behaviour as these states should correspond to black holes [8].

Having established that the theory of gravity is indeed self-complete in the UV, the low energy theory should also provide a quantum picture of black hole physics. Two of the long-standing puzzles of black holes are the origin of entropy [11–13] and the information paradox [14]. In 1972 Bekenstein argued that black holes should carry entropy in order not to violate the second law of thermodynamics [12]. Following his arguments, the entropy is given by the area of the black hole horizon. On the other hand, a statistical mechanics interpretation of the entropy in terms of the number of different micro-states of a black hole seemed elusive until Strominger and Vafa [15] computed the entropy of five-dimensional extremal black holes in $N = 4$ supersymmetry by counting the degeneracy of the corresponding BPS states. It is, however, still unclear how to find a similar microscopic origin for four-dimensional Schwarzschild black holes. The information paradox arises in the context of unitarity violation in the process of black hole evaporation. Hawking showed that due the quantum effects in a black hole spacetime, black holes can emit particles with a spectrum as if they were black bodies with a temperature given by their inverse Schwarzschild radius $T_H \simeq R_S^{-1}$. The information paradox can be described as the problem of a non-unitary evolution of the process of black hole formation and its subsequent quantum evaporation [16]. One could envision to prepare a pure state which one has under full control (information) and which undergoes such an evolution. Since in the semiclassical computation Hawking used the spectrum of emitted particles is thermal, it is described by a mixed state which reveals no information about the original state. Hence, such an evolution cannot be unitary. Since its first appearance, the information paradox has been reformulated in various ways, see, e.g., [17], but the essential aspect in all these treatises is that the evolution of black hole formation and subsequent evaporation seems not to be described by a unitary matrix. However, unitary time-evolution is such a profound principle that it is believed to be valid, and therefore a quantum theory of gravity better provides a resolution. Additionally, considering the fact that the information paradox is a result of applying Hawking's semi-classical computation to black holes of finite mass and finite life-time, it seems that indeed a quantum theory for black holes should be able to resolve the issue.

Recently, Dvali and Gomez put forward a microscopic picture of black holes [18–20] in which black holes are described as Bose-Einstein condensates of $N \gg 1$ gravitons of wavelength of order of the Schwarzschild radius $\lambda = R_S$. The gravitons are weakly interacting with strength $\sim 1/(\lambda^2 M_P^2)$ but due to the collective effect of these many gravitons, the condensate is self-bound. Due to the self-interaction the condensate, however, is leaky and looses particles. In addition, it was argued that this graviton condensate is at the critical point of a quantum phase transition [19]. Quantum phase transitions are necessarily accompanied by large quantum correlations [21], and therefore a semiclassical treatment is inappropriate. In addition, at the critical point, light modes appear which can deform the condensate. Combining these various aspects, the description leads to a simple microscopic understanding of the aforementioned black hole phenomena. The entropy of a black hole can be understood in terms of the light modes which appear at the quantum phase transition and account for the degenerate microstates of the black hole. Hawking evaporation in turn is described by quantum depletion (leakiness) of the condensate. Semiclassical physics in this language corresponds to the limit $N \to \infty$ and corrections are of the form $1/N$ [18, 20]. This could entail the resolution to the information paradox since quantum hair are now important at $1/N$ [20] allowing the information to be retrieved during the evaporation process.

In order to gain a fully fledged quantum picture of black holes, one would need to model the full relativistic theory of graviton condensates which is extremely complex. In this thesis, instead, we want to find a simpler (non-relativistic) toy model which mimics a particular aspect of the graviton condensate picture proposed by Dvali and Gomez. Specifically, we investigate the properties of a non-relativistic Bose condensate which loses particles during its collapse and can be considered as such a toy model for black hole evaporation. We find that it is possible that the condensate always stays at the critical point [22, 23]. When including an external trapping potential in order to mimic the gravitational self-trapping, light modes appear at the critical point.

## 1.2 Gravitational Axion Anomaly and $\eta'$ Bubbles

When treating gravity as an effective field theory, it is also possible to couple it to the Standard Model particle physics and its extensions. An interesting question is how gravitational interactions influence symmetries of other interactions. In particular, a subject of debate has been the gravitational influence on global symmetries. For example, it has been argued that in the low energy effective field theory $U(1)$ symmetry violating operators may be induced, either by quantum corrections or non-perturbative effects [24]. A simplified picture of these effects is due to the no-hair theorem (see e.g. [25]) according to which black holes do not carry any global charges. It is therefore possible to eliminate global charges from our universe by throwing them into a black hole. Since radiative corrections have to include also virtual black holes, global symmetry violating operators will in principle be induced. Furthermore, wormhole solutions might allow for the elimination of a global charge as well [26–30]. In contrast, local symmetries are associated with flux lines which can be measured by an observer at infinity. Therefore, black holes and other non-perturbative solutions cannot destroy local charges as such an observer can continuously monitor the Gaussian flux at infinity. These are the only hair commonly associated with black holes.[2]

The violation of global symmetries, if present, has profound consequences for theories which involve Goldstone or pseudo-Goldstone particles. Probably, the most famous example for such a theory is the Peccei-Quinn (PQ) solution to the strong CP problem [31, 32]. The strong CP problem arises from the non-trivial vacuum structure of quantum chromodynamics (QCD) due to non-perturbative configurations (instantons). This is reflected by the appearance of an angular parameter $\theta$ whose magnitude sets the strength of the violation of the symmetry of simultaneous parity transformation and charge conjugation (CP). The observational bound on CP violation of the strong interactions (QCD) is $\theta \lesssim 10^{-9}$ [33]. Considering that this parameter could, a priori, take any values within the interval $[-\pi, \pi]$, but instead is almost zero, leads to the conclusion that a dynamical mechanism may be at work. In order to implement this idea, one introduces a dynamical field, the axion, which is the Goldstone boson of a broken $U(1)_{PQ}$ symmetry [31, 32]. The axion gains a potential through interactions with QCD instantons [34] and its minimum is such that the effective CP violating $\theta$-parameter (which is now a combination of the original $\theta$-parameter and the axion field) is zero.

If gravitational effects violate global symmetries, the axion solution could be at risk [24, 35–39]. The contribution due to the gravitational anomaly shifts the minimum of the axion potential and depending on the strength of the gravitational symmetry breaking, CP violation could largely exceed experimental bounds. Even though the exact form of the global symmetry violating operators is not known at present, it can be argued that their coupling parameter would need to be a very tiny number ($\mathcal{O}(10^{-54})$) if they are not to interfere with the axion solution for the strong CP problem [24].

[2] In fact, in the quantum $N$-portrait put forward by Dvali and Gomez, it is argued that a black hole can carry quantum hair as an order $1/N$ effect [18]. Therefore, global symmetries might not be violated by black holes after all.

To study this problem, it is convenient to formulate the strong CP problem and the axion solution in a dual description in terms of three-form gauge fields. In this description, the CP violation of QCD is signalled by the appearance of a constant electric four-form background field in the vacuum [40, 41]. The axion solution then corresponds to putting the three-form into a Higgs phase such that its long-range correlations and hence the electric field vanish. In the dual description, gravitational effects can be parametrized by an additional gravitational three-form which couples to the axion [42]. Due to this coupling, the QCD electric four-form field is no longer screened in the vacuum [8].

In this thesis, we want to argue that a possible resolution to this problem can be achieved by considering the neutrino lepton number of the Standard Model along the lines of our work in [43]. In the case of massless neutrinos, the neutrino lepton number is a conserved global $U(1)$ charge, and thus it is expected to be anomalous under the aforementioned gravitational corrections. In the three-form description the anomalous current induces a mass term for the gravitational three-form, which leads to a screening of the gravitational four-form field. Thus in the massless neutrino scenario the axion solution is saved. In contrast, the presence of non-zero neutrino masses suppresses this screening. We show that the condition that the axion still provides a viable solution to the strong CP problem provides a new theoretical bound on the lightest neutrino mass [43].

Additionally, it is well-known that in the presence of massless quarks, the $\theta$-dependence of the vacuum becomes unphysical in the sense that $\theta$ then shifts under a chiral symmetry transformation of the quark phases. Thus vanishing quark masses protect the CP symmetry. For non-zero quark masses, it is then expected that the induced CP violation is proportional to some power of the lightest quark mass $m_q$. This dependence was computed in [44] using current algebra methods and was found to be linear in the mass $\sim m_q$. In this thesis, we will consider a different approach to explain this dependence. Analogously to Schwinger pair creation [45] which is responsible for screening a constant electric field in the two-dimensional Schwinger model [46, 47], we study a screening mechanism for the QCD four-form electric field by nucleation of $\eta'$ bubbles [48]. The $\eta'$ meson is the pseudo-Goldstone boson of the anomalous $U(1)$ chiral symmetry and is sourced by the QCD $\theta$-term. The correct $\theta$-dependence of [44] is not recovered which we take as an indication for the postulated non-analyticity of the vacuum energy [49, 50].

## 1.3 Axion Dark Matter

Another area of physics where gravity plays an essential role is cosmology. With the advance of high precision cosmology gravitational interactions are accessible to measurements also on large scales. The observation of the cosmic microwave background radiation (CMB) has allowed to determine a detailed picture of the history of the universe and precise measurements of cosmological parameters have established a conclusive theoretical model of the universe: prior to the hot Big Bang, the universe underwent a period of accelerated expansion during which all inhomogeneities were diluted. This period of inflation also explains the structure of the universe as originating from quantum fluctuations. The evolution of the universe after inflation is well described by the so-called ΛCDM model which explains, for example, the formation of light elements during Big Bang Nucleosynthesis, the decoupling of photons after recombination leading to today's observable CMB, the observed [51, 52] late acceleration of the universe and the formation of large-scale structures in galaxies. In this scenario, the late-time acceleration is ascribed to a cosmological constant $\Lambda$ which governs the energy-density of the universe today. Structures are formed due to initial perturbations in the density of the so-called dark matter which can be described by a mainly gravitationally interacting pressureless fluid. Due to the gravitational attraction, these initial perturbations could grow and eventually provided a deep enough potential for the baryonic matter to overcome its own pressure and collapse as well. The recent measurements of the Planck satellite have confirmed this picture with unpreceeded precision. The observed structure of the CMB fits nicely within the simplest classes of inflation [53]. The energy density of the universe is distributed between matter making up for about 30% (of which about 85% is given by dark matter), and the cosmological constant, or Dark Energy, which constitutes about 70% [54]. Accordingly, the overall energy content of the universe is made up to only about 5% from ordinary matter described within the Standard Model of particle physics. This is indeed a very puzzling fact which at the same time opens up a potential playground for particle models beyond the Standard Model.

Within the unknown components, dark matter is probably the most straightforward to tackle within the realm of particle physics. The most immediate requirement a viable dark matter candidate has to fulfil is that its interactions with matter by forces other than gravity are strongly suppressed. A perfect candidate seems to be the axion [55–57]. Because of it being a pseudo-Goldstone boson of a broken $U(1)$ symmetry, its interactions are suppressed by the symmetry breaking scale $f_a$. In order to be compatible with astrophysical observation, such as, for example, from star cooling [58], this scale has to be at least larger than $f_a > 10^9$ GeV which renders the coupling to ordinary matter very weak. In the simplest scenario for axion dark matter production, the so-called misalignment mechanism [59], the symmetry is broken before inflation and the expectation value of the axion field $a$ settles to a homogeneous value $a_i \sim f_a$. As the universe cools down due to the expansion, a mass term is generated for the axion. Relaxing towards the minimum of its potential, the axion field starts oscillating thereby producing non-relativistic dark matter particles with a density proportional to $a_i^2$ [59]. The correct dark matter abundance can be obtained for any $f_a > 10^{10}$ GeV [56, 60, 61]. However, during inflation, the massless axion experiences quantum fluctuations around $a_i$ which induce fluctuations in the dark matter density of order $\delta a/a_i \sim H_I/a_i$, where $H_I$ is the Hubble scale. These perturbations persist after inflation and could be detected in the CMB in terms of isocurvature perturbations [62].

So far, however, neither Planck nor any of the previous missions have detected substantial isocurvature perturbations in the CMB. This puts a severe constraint on dark matter models with a dominant axion component [56]. In this work, we present a possibility to soften the isocurvature constraint on the axion by introducing a non-minimal kinetic coupling on the inflating background as put forward in our work [63]. This coupling effectively suppresses the isocurvature perturbations in the axion dark matter density which are generated during inflation. In addition, the Standard Model Higgs boson can be considered a possible candidate for the inflaton. However, constraints from accelerator experiments and cosmology have excluded a Higgs with a non-minimal coupling as an inflaton [64, 65]. Along the lines of earlier work on Higgs inflation [65], we consider a particular model in which inflation is enforced by the Standard Model Higgs field and dark matter is produced by the axion when both fields are non-minimally coupled on a de Sitter background.

## 1.4 Massive Gravity

Lastly, let us come back to gravity. In terms of an effective field theory, gravity is described by the interactions of a massless spin-2 particle. Above we have discussed the effects of possible UV modifications of gravity. Another interesting aspect is to consider infrared (IR) modifications of gravity which have attracted much interest over the past years as they may provide an alternative explanation for the late-time accelerated cosmic expansion, for a review see, e.g., [66].

The most obvious way to modify gravity in the IR is to give a small mass to the graviton. On the linear level, a mass term for a spin-2 particle uniquely leads to the Fierz-Pauli action [67]. Albeit seemingly a small perturbation to the massless theory, a mass term, however small, has profound consequences for the theory. In quantum field theory, degrees of freedom and correspondingly their one-particle states are labelled according to their representation of the Poincaré group, specifically according to their mass and spin. In consequence, a massless spin-2 particle which describes two degrees of freedom, namely, the two helicity-2 polarizations, lives in a different representation than the massive spin-2 particle which describes 5 degrees of freedom, i.e., two helicity-2, two helicity-1 and one helicity-0 polarization. In this light, it is far from trivial that the introduction of a mass term for the graviton can be regarded a small perturbation. Indeed, taking the massless limit, the helicity-0 polarization does not decouple from interactions with external sources and gives rise to the so-called vDVZ discontinuity [68, 69]. This discontinuity leads to a discrepancy between the predictions of General Relativity and the ones of the linear theory in the massless limit. However, Vainshtein [70] argued that around sources the linearised theory of massive gravity breaks down at scales proportional to the fourth power of the inverse mass. Therefore, it is expected that General Relativity is recovered in the massless limit only when the full nonlinear theory is taken into account.

The inclusion of interactions of the full nonlinear theory of General Relativity, however, was shown to lead to inconsistencies [71], as they invoke the appearance of a sixth propagating mode with negative kinetic term – the notorious Boulware-Deser ghost [71]. However, recently, de Rham, Gabadadze and Tolley [72] were able to construct a nonlinear realization of Massive Gravity which, as was subsequently argued [72–75], propagates only five degrees of freedom, thus avoiding the Boulware-Deser ghost.

In this thesis, we further elucidate certain aspects of theories of interacting massive spin-2 particles as put forward by us in [76]. For energies much larger than the mass of the spin-2 particle, the representation of the Poincaré group decomposes into its helicity subgroups. Since possible instabilities in terms of ghost modes are high energy effects, the analysis in terms of helicity components should allow us to gain further insight. We first investigate the addition of cubic nonderivative interaction terms to the cubic Einsteinian derivative interaction. Second, we treat the massive spin-2 particle as a genuine effective field theory of a massive spin-2 particle without any reference to gravity and construct a cubic interaction which differs in the derivative structure from the Einsteinian cubic term, but which nonetheless propagates five degrees of freedom [76]. We find that it differs in structure from the one proposed by [72] and thus, conclude that the Lagrangian structure of such theories is not uniquely defined.

## 1.5 Outline

The outline of this thesis is as follows. In chapter 2 we study the field theoretical description of a generic theory of gravity flowing to Einstein General Relativity in the IR. We first recap the basics of gravitaional interactions in terms of EFT and give a short summary of our findings. We then briefly review the basic findings of [7] who argued that sub-Planckian distances are unobservable in General Relativity due to black hole formation. Based on our work in [8], we then argue that in any ghost-free theory of gravity trans-Planckian propagating quantum degrees of freedom cannot exist and that this puts a severe constraint on any attempt of conventional Wilsonian UV-completion of trans-Planckian gravity.

Chapter 3 is devoted to the study of black holes in terms of Bose-Einstein condensates. At first the idea of the black hole–condensate correspondence of [18, 19] is introduced. Then follows a brief introduction to the theory of Bose-Einstein condensates, after which we discuss a condensed matter toy model for a black hole which features a collapse with simultaneous evaporation and the appearance of light modes. This part will be published in a forthcoming paper [23].

Chapters 4 and 5 discuss aspects of axion physics. In chapter 4, we begin with a summary of the strong CP problem [1, 2] and the axion solution [31, 32]. In order to investigate possible gravitational effects on the axion solution, we employ a dual analysis in terms of three-form fields and by coupling the neutrino lepton number current to gravity, we derive a bound on the neutrino mass by requiring the $\theta$-angle to comply with observational bounds. Furthermore, we investigate the screening of the QCD vacuum energy in the presence of massive quarks. The role of the axion as a dark matter candidate is reviewed in chapter 5, where we suggest a possible mechanism to circumvent current clashes with bounds from isocurvature perturbations [63].

Chapter 6 deals with massive gravity. We first review the status of theories of massive gravity. Following an introduction to the helicity decomposition, we utilize this method as a means to investigate the consistency of such theories. Additionally, we derive a possible cubic interaction term for a massive spin-2 particle which propagates five degrees of freedom [76].

# Chapter 2

# Physics of Trans-Planckian Gravity

Einstein gravity predicts the formation of black holes in high energy collisions. This fact opens up a possible way of a non-Wilsonian UV-completion of gravity. In such a picture, sub-Planckian distances are unobservable as a matter of principle. Degrees of freedom with masses exceeding the Planck scale are large classical black holes. These are in turn described by IR degrees of freedom which are accounted for by Einstein gravity. Since in the weakly coupled regime, new gravitational degrees of freedom always act attractively, Einstein gravity is the theory with the "weakest" gravitational interaction. Using this argument, it is suggested that in a ghost-free theory of gravity trans-Planckian propagating quantum degrees of freedom cannot exist [7]. This UV/IR correspondence puts a severe constraint on any attempt of conventional Wilsonian UV-completion of trans-Planckian gravity [8].

In this chapter, we extend the work of [7] and study the self-completeness of Einstein gravity and its implications. We give a general discussion of gravitational interactions from an effective field theory perspective in section 2.1, where we also introduce the concept of a black hole. In section 2.2, we elaborate on the arguments of [7] that sub-Planckian distances are not accessible in Einstein gravity. Futhermore we argue in section 2.3 that Einstein's theory indeed describes the weakest gravitational interaction [77]. These arguments together with black hole formation in trans-Planckian scattering experiments lead us to the conclusion that in any UV modification of gravity sub-Planckian distances are shielded as we discuss in section 2.4. Section 2.5 concludes this chapter.

## 2.1 Gravitational Interactions

Gravitational interactions are described by Einstein's theory of General Relativity. Its field theoretical formulation is given in terms of the Einstein-Hilbert action coupled to matter

$$S = \int d^4x \sqrt{-g}\,\frac{1}{16\pi\, G_N} R + \int d^4x \sqrt{-g}\,\mathcal{L}_m\ . \tag{2.1}$$

Here $\mathcal{L}_m$ denotes the matter Lagrangian, $G_N$ is the Newton constant and $R$ the Ricci scalar obained from full contractions of the Riemann tensor $R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$. The Cristoffel symbols $\Gamma$ are defined in terms of the metric $g_{\mu\nu}$ by $\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$ and $g$ is the determinant of the metric. This action is invariant under diffeomorphisms. Variation with respect to $g_{\mu\nu}$ yields Einstein's equations which relate matter, given by the energy-momentum tensor $T_{\mu\nu}$, to geometry $g_{\mu\nu}$,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G_N T_{\mu\nu}\ , \tag{2.2}$$

where $G_{\mu\nu}$ is called the Einstein tensor.

In vacuum $T_{\mu\nu} = 0$, e.g., outside a source, the simplest non-trivial spherically symmetric solution is given by the famous Schwarzschild solution

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = -\left(1 - \frac{2G_N M}{r}\right) dt^2 + \left(1 - \frac{2G_N M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\ . \tag{2.3}$$

This solution also describes one of the most intriguing objects in physics: black holes. The coordinate singularity at the Schwarzschild radius $r = 2G_N M \equiv R_S$ denotes a boundary (the black hole horizon) of region of spacetime from which nothing, not even light, can escape. From the above equation it follows that whenever a given source of mass $M$ is localized within a radius smaller than its Schwarzschild radius $R_S$, the Schwarzschild solution outside the source contains a horizon and, hence, the spacetime a black hole. This statement can be made rigorous by the so-called Hoop conjecture [78] which specifies that whenever a given (Lorentz-invariant, cf. section 2.2.1) energy is contained within a region around which a hoop of radius $R_S = 2G_N E$ can be placed, the energy density collapses to form a black hole.

Black holes have a number of peculiar properties of which we will only mention two here; for an introduction to black hole physics see [79]. First, it was observed by Bekenstein that black holes need to carry entropy in order to avoid a violation of the second law of thermodynamics [11]. The entropy is related to the area of its horizon $A = 4\pi R_S^2$ by $S_{BH} = A/(4\hbar)$. Second, Hawking showed that black holes evaporate by investigating perturbative quantization about a black hole geometry [14]. In a nutshell, Hawking radiation is caused by particle production in a non-stationary spacetime. In curved spacetime, the notion of the vacuum is observer dependent and the notion of particles and anti-particles for different observers is related by a Bogoliubov transformation. Accordingly, an observer in the far past, before the black hole collapsed, would define a different vacuum state than a observer in the far future with the black hole spacetime. The Bogoliubov transformation relating the two vacua then creates particles with a thermal spectrum for the observer in the far future. The evaporation follows a Stefan-Boltzmann law with temperature $T_H \simeq R_S^{-1}$. The mass $M$ of the black hole decreases according to

$$\frac{dM}{dt} \sim \frac{\hbar}{G_N^2 M^2}\ . \tag{2.4}$$

For a full treatment of Hawing radiation see, e.g. [79] or Hawking's original computation [14].

From the quantum field theory point of view, the first term of the action (2.1) describes the purely gravitational theory which embodies the self-interactions of a massless spin-2 particle $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric. In lowest order of a derivative expansion, Einstein's gravity (General Relativity) can be shown to be the unique theory of a single interacting massless spin-2 graviton [80–82]. The reduced Planck mass, $M_P \equiv \sqrt{1/(16\pi G_N)} \sim 2.4 \times 10^{18}\mathrm{GeV}$, and the corresponding Planck length, $L_P \equiv M_P^{-1} \sim 10^{-32}\mathrm{cm}$,[1] play a central role in Einstein gravity. For example, from the field theoretical point of view, $M_P$ sets the interaction strength of the canonically-normalized graviton expanded around Minkowski corresponding to the term

$$\frac{1}{M_P}\, h_{\mu\nu}\, T^{\mu\nu} \,. \tag{2.5}$$

Here, $T_{\mu\nu}$ is an arbitrary conserved energy-momentum source. A very special property of gravity is that also self-interactions are regulated by the coupling (2.5), where in this case $T_{\mu\nu}$ is the energy-momentum tensor of the graviton evaluated to a given non-linear order in $h_{\mu\nu}$. In [80], it was shown that one can uniquely reconstruct the action (2.1) by resumming all orders of the graviton self-interactions given by (2.5).

In General Relativity, all energy-momentum sources universally couple to gravity. At the linearized level, one can thus define an effective dimensionless parameter describing the strength of the gravitational interaction for any elementary process of characteristic momentum transfer $p$,

$$\alpha_{\mathrm{Ein}}(p^2) \equiv 16\pi\, G_N\, p^2 = \frac{p^2}{M_P^2} \,. \tag{2.6}$$

Here and throughout this chapter, we consider only asymptotically flat spaces on which the gravitational interactions (2.1) can be expanded in terms of linear gravitons up to the strong coupling scale of the theory. Note that in this way one can construct gauge invariant (i.e. with respect to the background metric diffeomorphism invariant) global and local operators such as the S-Matrix [4] and/or the scattering amplitude $A(p)$ of a scattering process prepared at spatial infinity.

Parametrizing the gravitational coupling according to (2.6) immediately reveals why gravity is weak in low energy (infrared (IR)) processes characterized by $p \ll M_P$ and therefore $\alpha_{\mathrm{Ein}} \ll 1$. Due to the fact that gravity couples universally with (2.6), Einstein gravity admits a universal strong coupling scale $M_P$. Indeed the energy-dependence of $\alpha_{\mathrm{Ein}}$ is the source of the non-renormalizability of Einstein's gravity and the reason why gravitational amplitudes violate perturbative unitarity above the scale $M_P$.

The coupling parametrization (2.6) is equally applicable to extensions of Einstein gravity in which gravity is mediated by additional degrees of freedom as long as they still obey the Strong Equivalence Principle [79]. If they do, the coupling remains universal and an equivalent of (2.6), denoted by $\alpha_{\mathrm{grav}}$ can be defined. For our purposes, it is then useful to parametrize the notion of the gravitational strength $\alpha_{\mathrm{grav}}$ as well as

[1] From now on we set $\hbar = 1$.

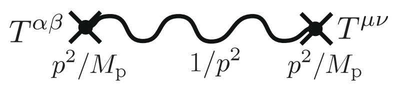


FIGURE 2.1: In the linear regime, the gravitational interaction between two sources is mediated by a graviton. The amplitude of such a process is $A \sim \alpha_{\rm grav}(p^2)$.

its UV-completion by the behaviour of gravitational scattering amplitudes. Consider a scattering on asymptotically flat space among two conserved *external* sources,[2] $T_{\mu\nu}$ and $\tau_{\mu\nu}$, with characteristic momentum-transfer $p$, see also Fig. 2.1. The scattering amplitude can be written as

$$A(p) \;=\; \frac{\alpha_{\rm grav}(p)}{(p^2)^2}\left(T_{\mu\nu}\tau^{\mu\nu} \;+\; b(p)\,T^{\mu}_{\mu}\tau^{\nu}_{\nu}\right)\,. \tag{2.7}$$

This equation defines $\alpha_{\rm grav}$ which at this point is merely a useful parametrization of the gravitational strength. Notice that in any theory in which gravitational interactions are mediated by spin-2 states, the parameter $b(p)$ is generically of order one but might depend on $p$ in a nontrivial way. In principle, $b(p)$ can, however, take larger values if contributions from spin-0 dominate. Such a case can be easily incorporated in our following discussion, but is not of our primary interest. Moreover, the dependence on $b(p)$ can be eliminated by taking at least one of the sources to be relativistic, say $\tau^{\mu}_{\mu} \;=\; 0$.

Universally, the scale of strong gravity can be defined as the lowest energy scale $M_*$ for which

$$\alpha_{\rm grav}(p \equiv M_*) \;=\; 1\,. \tag{2.8}$$

In Einstein gravity, $M_* \;=\; M_P$, whereas in general, $M_*$ can be arbitrarily lower though *never* higher than $M_P$ [77] as we will see below. In any given theory, (in a slight abuse of notation) we refer to the region of energies $p \;\gg\; M_*$ as the trans-Planckian region (or the UV) and to the corresponding length scales $L \;\ll\; L_* \;\equiv\; M_*^{-1}$ as sub-Planckian distances.

Now, in quantum field theory, physics at any given length scale can be described in terms of propagating quantum degrees of freedom. In this sense, all existing states of the theory (including the classical ones) are in principle accounted for as states of degrees of freedom which are propagating at the specific length scales of interest. Of course, as one moves from scale to scale, the notion of elementary propagating degrees of freedom can change, e.g., some may become composites of more fundamental ones, but at any scale there always exist some. This is precisely the concept of effective field theory (EFT).

In this sense, resolving a distance scale $L$ means that one integrates in propagating degrees of freedom of mass (energy) $1/L$ which can be treated as elementary at distances $L$. For instance, one should be able to describe interactions of these degrees of freedom within the space-time interval of size $L$. All known non-gravitational UV-completions

[2] For a brief discussion on the notion of external sources in gravity see appendix A. Furthermore, throughout this chapter, we are only interested in sources that do not violate null energy conditions.

are based on this fundamental notion. By extending this concept to UV-completions of gravity beyond the Planck length $L \ll L_P$ (or more general $L \ll L_*$), one would need to integrate in trans-Planckian degrees of freedom of mass $m = 1/L \gg M_P$. However, it was suggested in [7] that in a theory which is reduces to Einstein gravity at low energies, trans-Planckian propagating degrees of freedom cannot exist. Instead, any such degree of freedom becomes a classical state with smallest size $R_S \sim 2L_P^2/L$; that is it becomes a black hole with Schwarzschild radius $R_S$ corresponding to the mass $1/L$. This classical state is no longer an independent UV entity and is fully described by already existing IR degrees of freedom, such as the massless graviton. Thus, the would-be trans-Planckian states carry no information about the trans-Planckian physics and decouple from quantum processes, just as classical objects should do.[3] Therefore, Einstein gravity self-completes itself in the deep UV by mapping would-be trans-Planckian degrees of freedom to classical IR states with typical energies $1/R_S$ [7]. In particular, this can be understood as the field theoretic manifestation of the fact that in Einstein gravity the Planck length is the shortest length-scale of nature. This is furthermore the underlying reason for the so-called Generalized Uncertainty Principle [83–88].[4] A similar notion also exists in string theory where it can be argued that the fundamental string length as well sets a limit on the shortest distance which is possible to be probed [90–96].

The formation of black hole as an outcome of trans-Planckian collisions is a natural expectation, see e.g. [1, 2]. The discovery of low scale quantum gravity scenarios [97, 98] promoted this possibility to a potentially experimentally-observable phenomenon. Indeed, black hole formation in high energy scatterings at particle colliders was predicted in [98] (for subsequent work in this direction see [99–101]). In [3], this feature of gravity was formulated in terms of the "Asymptotic Darkness" scenario as the unique outcome of trans-Planckian scattering at small impact parameters. In the following, we will furthermore argue that a black hole is the *only* output of *any* trans-Planckian scattering process in *any* healthy theory of gravity. In other words, we will show that there is no contribution from sub-Planckian distance physics in any high (or low) energy scattering process.

It is interesting to address the viability of attempts of conventional (Wilsonian) UV-completions of Einstein gravity in the trans-Planckian domain in the context of the aforementioned self-completeness of gravity. In particular, the proposed self-completeness has important consequences for cases in which gravity is assumed to become weaker in the deep UV, an example of which is the Asymptotic Safety scenario [9, 10]. The mapping of trans-Planckian gravity to classical IR gravity is in conflict with UV-completions of gravity by asymptotically safe behaviour: There simply are no "UV" degrees of freedom which could trigger such a behaviour. At best, the fixed point behaviour of this scenario is fictitious and a relic of the technique used for computing the renormalization group flow of gravity, cf. [102].

[3] In fact, these states can be described as a self-bound Bose condensate of $N \simeq R_S^2/L_P^2$ gravitons with wavelength $\sim R_S$ and interaction strength $\alpha \simeq 1/N$ [18] as we discuss in chapter 3.

[4] In [89], this obstruction to probe short distances has been suggested to be related to a kind of locality bound, where below that scale the local quantum field theory no longer captures all dynamics.

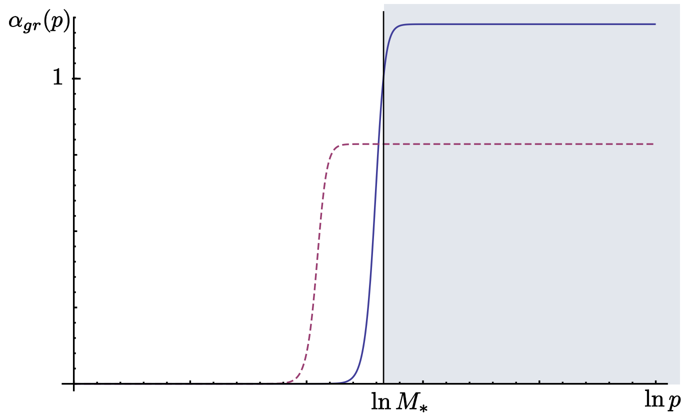


FIGURE 2.2: Momentum-scale dependence of $\alpha_{grav}$. The dashed line shows a running of the gravitational coupling where gravity becomes weaker in the weakly coupled regime. In a ghost free theory this cannot happen. The solid line represents a typical running of $\alpha_{grav}$ usually found within the Asymptotic Safety scenario. Here, gravity first hits the strong coupling ($\alpha_{grav} = 1$) at scale $M_*$, before turning over to the fixed point scaling. The shaded region indicates the regime in which black hole formation takes place and which hence cannot be probed by experiments.

In order to see this, we first argue that gravity cannot become weaker than in pure Einstein gravity before reaching the strong coupling scale by requiring the absence of negative norm states. To be more precise, as shown in [77], in any ghost-free theory in the weak-coupling domain, $\alpha_{grav}(p)$ must satisfy,

$$\alpha_{grav}(p) \geq \alpha_{Ein}(p)\,, \tag{2.9}$$

and the quantity $\alpha_{grav}(p)/\alpha_{Ein}(p)$ must be a non-decreasing function of $p^2$, at least until $\alpha_{grav}(p)$ becomes of order one. In other words, a weakening of gravity cannot set in while $\alpha_{grav}(p) \ll 1$ (for a pictorial representation see Fig. 2.2) unless there are negative norm states in the spectrum. Thus, the gravitational coupling first has to reach the strong coupling point $M_*$ before its strength can start to decrease. However beyond $M_*$, one is in the trans-Planckian domain which is mapped onto classical IR gravity by the formation of classical black holes. Thus, gravity cannot display Asymptotic Safety in any well-defined physical sense; due to the black hole barrier distances shorter than $1/M_*$ can never be probed in principle. For instance, scattering cross sections with center of mass energies $E$ are dominated by black hole production for $E \gg M_P$ and impact parameter $\sim E$. This cross section can be estimated by the geometric cross section of a black hole $\sigma \sim E^2/M_P^4$ which grows with increasing energy [3]. This energy dependence of $\sigma$ is hard to reconcile with the notion of Asymptotic Safety, or with a weakening of UV-gravity in general. This result agrees with a complementary proof of the impossibility of Asymptotic Safety in a theory of gravity containing black holes [103]. In [103], Shomer showed that any UV fixed point at which gravity becomes weaker, as for instance postulated in the Asymptotic Safety scenario, is incompatible with the Bekenstein-Hawking entropy of black holes. A similar observation that the BH barrier prevents probing the fixed point behavior of Asymptotic Safety has been made in [5].

To summarize, the self-completeness of gravity [7] raises the question whether Wilsonian

UV-completions of trans-Planckian gravity are viable or even physically motivated in the light of the inaccessibility of these scales in scattering experiments.

## 2.2 Non-existence of Sub-Planckian Distances in Einstein Gravity

In this section, the notion of gravitational self-completion as put forward in [7] is explained. Accordingly, the non-existence of trans-Planckian physics in Einstein gravity, in the sense of probing distances $L \ll L_P$, is discussed.

Let us briefly clarify that the statement about the impossibility of probing short distance scales is Lorentz invariant as seen from an observer at spatial infinity. The distances (and energies) here refer to the distances (and energies) measured in the center of mass reference frame. In this frame, one may also use the seemingly non-relativistic relation that shorter distances are measured by higher energies, i.e. $E \sim 1/L$. Of course, a boost will change the values of $L$ and $E$ accordingly but not their relation. However, the reader might be worried that in the highly nondynamical gravitational background produced by the colliding sources, the definition of length should include some notion of the local spacetime. In this case, we will always refer to a "length" as the instantaneous local invariant length measured by an ADM observer [79], and in this case, the four dimensional metric is split in a $3+1$ slicing according to

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i - N^i dt)(dx^i - N^i dt) \ . \tag{2.10}$$

Since we are only concerned with s-wave (spherical) scatterings, one may then choose coordinates at a fixed time such that they define the following (instantaneous) three-dimensional metric [78]

$$g_{ij}\Big|_{t=\text{const}} = \phi^4(r)\delta_{ij} \ , \tag{2.11}$$

where $\delta_{ij}$ is the Kronecker delta and $\phi(r)$ is a scaling function. In this way the invariant length is given by

$$L(r_0) \equiv 4\pi \int_0^{r_0} dr \phi^2(r) \ , \tag{2.12}$$

where $r_0$ is the coordinate radius we would like to measure.

From now on, we assume without loss of generality that when considering distances $L$, we implicitly assume the above definitions.

### 2.2.1 Field theoretical Hoop Conjecture

In quantum field theory, any measurement that attempts to resolve a distance scale $L$ has to excite degrees of freedom of energy $1/L$ within a box of size $L$. The explicit realization of such a measurement is to set up a scattering experiment which involves at least two particles. These particles are then boosted in such a way that their (Lorentz-invariant) center of mass energy exceeds $1/L$ and that their impact parameter is less than $L$. For $L \ll L_P$, any attempt of this measurement leads to the formation of a classical black hole (see also [3, 90–96, 98–101]). Note that by itself none of the involved boosted particles is a black hole even when boosted to energies $\gg M_P$ since there is no graviton exchange involved. Instead, their correct description is given by so-called Aichelburg-Sexl geometries [104]. The fact that the outcome of such an experiment will inevitably produce a black hole can be regarded as the field-theoretical interpretation of Thorne's hoop conjecture [105]. It states that a black hole with horizon forms when, and only when, a mass $M$ is compacted into a region whose circumference in every direction is less than its Schwarzschild horizon $R_S(M) = 2M/M_P^2$.[5] Hence, for the scattering experiment above it implies that a black hole forms any time the transfer energy is localized (dynamically) within the Schwarzschild radius $R_S(1/L)$ corresponding to the center of mass energy $E = 1/L$ . Thus, any attempt of resolving sub-Planckian distances will lead to the formation of a macroscopic black hole of horizon size $2L_P^2/L$, which itself can only probe large distances.

This observation leads to two important conclusions. First, an elementary state with mass $M > M_P$ cannot exist because its Compton wavelength $\lambda_c \lesssim 1/L_P$ is smaller than its corresponding Schwarzschild radius $R_S(M > M_P) > 2L_p$. According to the Hoop conjecture it will form a black hole. Second by virtue of the black hole barrier, no sub-Planckian distances may ever be probed. Therefore, to talk about these distances is meaningless from a physical perspective.

The previous discussion has been based on a classical analysis. Thus, one might wonder whether quantum mechanical arguments could lead to a different conclusion. It has been argued in [3] that a scattering experiment of transfer energy $E \gg M_P$ with impact parameter $L \ll L_P$, may indeed produce elementary particles as an outcome with (quantum) probability $e^{-E^2 L_P^2}$. The key observation here is that such a small probability is due to the production of a virtual black hole. This conclusion can be drawn by noticing that the factor $E^2 L_P^2 \sim S$ is proportional to the Bekenstein-Hawking black hole entropy $S = \pi R_S^2 / L_P^2$, and therefore the suppression $e^{-S}$ represents the Boltzmann suppression of the evaporation of a classical black hole. In other words, the produced particles can be interpreted as the result of a black hole which formed during the collision and subsequently evaporated into elementary particles. Because the Compton wavelength of the emitted elementary particle is larger than the Planck length, this implies that again no sub-Planckian distances are probed [7].

## 2.3 Einstein Gravity is the Weakest Gravity

We have seen in the previous section that because of the black hole barrier sub-Planckian distances are unphysical. Therefore, the only sense in which one can think about gravity at trans-Planckian energies is in terms of IR gravity. This fact eliminates the need of a

[5] This version of the conjecture is of course very vague as it implies the existence of an omniscient observer who can define a global event horizon. However, although this conjecture can be generalized by introducing a local definition of horizons, i.e. closed trapped surfaces (see [6] and references therein, [106]), we are only interested in the point of view of asymptotic observers in flat space (S-matrix) where the above formulation of the conjecture is applicable.

UV-completion of Einstein's theory in the Wilsonian sense, which could be, e.g., due to an improved behaviour of the graviton propagator for large $p$.

In this section we show, following the reasoning of [77], that any modification of gravity that does not propagate ghost degrees of freedom in the weak regime always produces a stronger gravitational attraction. In consequence, modifying the theory of gravity can only lead to black hole production at energies lower than $M_P$ making the black hole barrier even more efficient.

For a scattering process of particles with characteristic momentum transfer $\sim p$ and a center of mass energy $E \sim p$, weak gravity is defined as the condition

$$\alpha_{grav}(p) \ll 1, \tag{2.13}$$

where $\alpha_{grav}(p)$ is given by (2.7). For example, in the pion-nucleon scattering at QCD scale energies, Einsteinian gravity is weak. In this regime, consider a *one-graviton* exchange process between two energy-momentum sources $T_{\mu\nu}$ and $\tau_{\mu\nu}$. From (2.7) it follows that the amplitude of this process in momentum space is given by

$$A(p) \simeq T^{\mu\nu}(p)\Delta_{\mu\nu,\alpha\beta}(p)\tau^{\alpha\beta}(p), \tag{2.14}$$

where $T_{\mu\nu}(p)$ and $\tau_{\alpha\beta}(p)$ are the Fourier-transforms of the sources, and $\Delta_{\mu\nu,\alpha\beta}(p)$ is the graviton propagator in momentum space.

In General Relativity, in which the gravitational force is mediated by a single massless spin-2 particle, the tensorial structure of $A(p)$ is uniquely fixed by

$$A_{massless}(p) = G_N \frac{T_{\mu\nu}(p)\tau^{\mu\nu}(p) - \frac{1}{2}T^{\mu}_{\mu}(p)\tau^{\nu}_{\nu}(p)}{p^2} \,. \tag{2.15}$$

However, if gravity deviates from Einstein's theory in the UV (or IR), the structure of $A(p)$ will be different. Nonetheless, it is still *extremely* restricted. This follows directly from the spectral representation of the graviton propagator whose general ghost-free structure in the weak coupling regime is given by

$$\begin{aligned} A(p) &= T^{\mu\nu}\Delta_{\mu\nu,\alpha\beta}\tau^{\alpha\beta} = \\ &= \frac{1}{M_P^2}\left(\frac{T_{\mu\nu}\tau^{\mu\nu} - \frac{1}{2}T^{\mu}_{\mu}\tau^{\nu}_{\nu}}{p^2} + \int_0^\infty ds\rho_2(s)\frac{T_{\mu\nu}\tau^{\mu\nu} - \frac{1}{3}T^{\mu}_{\mu}\tau^{\nu}_{\nu}}{p^2+s} + \int_0^\infty ds\rho_0(s)\frac{T^{\mu}_{\mu}\tau^{\nu}_{\nu}}{p^2+s}\right)\,, \end{aligned} \tag{2.16}$$

where we have separated the contributions from the massless spin-2, the massive spin-2 and the spin-0 poles. It is crucial to note that the absence of ghosts demands $\rho_2(s) \geq 0$ as well as $\rho_0(s) \geq 0$, $\forall s$.

In order to understand the meaning of $\rho_2$ and $\rho_0$, let us consider the ADM decomposition [79] of the metric according to which the graviton can be decomposed into a spin 2 field $h_{ij}$ (the spatial metric), a scalar $N$ (the lapse) and a vector $N^i$ (the shift).[6] In

[6] $i, j, ...$ are 3-dimensional indices (with a positive defined metric) and $\alpha, \beta, ...$ are the 4-dimensional indices

the transverse-traceless gauge, which can always be taken because of the linearized diffeomorphism group, the kinetic term of the spin-2 part looks like $(\partial_\alpha h_{ij})(\partial^\alpha h^{ij})$. This kinetic term has *no* sign ambiguities, and depending on the choice of the 4-dimensional signatures, its sign determines whether $h_{ij}$ is a propagating ghost or not. This sign is encoded in $\rho_2$ in (2.16). Of course, in contrast to GR, the trace $h_{ii}$, for example, which is a scalar degree of freedom, can propagate. The sign of its kinetic term is determined by $\rho_0$. The tensorial structure is fixed by the requirement of the absence of ghosts as well.

From (2.16), one is lead to a powerful conclusion: The running of $\alpha_{grav}(p)$ (or equivalently $G_N(p)$) can be understood in terms of $\rho_2(s)$ and $\rho_0(s)$, and the positivity requirement automatically excludes a weakening of gravity in the weakly-coupled regime [77, 107, 108]. Indeed, using the spectral decomposition (2.16) we can represent $\alpha_{grav}(p)$ in the following form,

$$\frac{\alpha_{grav}(p)}{\alpha_{Ein}(p)} \simeq 1 + p^2 \int_0^\infty ds\, \frac{\rho_2(s)}{p^2+s}\,, \tag{2.17}$$

where $\alpha_{Ein}(p) \equiv p^2/M_P^2$ is the strength of pure-Einstein gravity and at least one sources is relativistic. Due to the positivity of $\rho_2(s)$, $\frac{\alpha_{grav}(p)}{\alpha_{Ein}(p)}$ is a never decreasing function larger than one,

$$\frac{\alpha_{grav}(p)}{\alpha_{Ein}(p)} \geq 1 \quad \text{and} \quad \frac{d}{dp^2}\left(\frac{\alpha_{grav}(p)}{\alpha_{Ein}(p)}\right) \geq 0\,. \tag{2.18}$$

Thus, in the weakly coupled regime, gravity can never become weaker. In other words, Einsteinian gravity is the weakest among all possible gravity theories that flow to Einstein gravity with a given $G_N$ in the IR. A direct consequence of this fact is that, in the weak gravity regime, any modification of Einstein gravity produces (for a given mass) black holes of size $R_H \geq R_S$, where $R_S = 2G_N M$ is the Schwarzschild horizon [77].

The physical meaning of the above statement is clear. Equation (2.16) shows that the gravitational force mediated by positive norm particles is always attractive. Thus, the weakest gravitational coupling at any scale is the one that is mediated by the minimal number of messengers; this is Einstein gravity mediated by a single massless spin-2 graviton. Furthermore, the positivity of $\rho_2(s)$ and $\rho_0(s)$ requires the strong coupling scale $M_*$ of any UV modification of gravity to be lower than the strong coupling scale of pure Einsteinian gravity,

$$M_* \leq M_P\,. \tag{2.19}$$

That this inequality is a direct consequence of (2.18) can be easily seen as follows. The strong coupling scale is given by the minimal scale at which some scattering amplitudes become of order one. To put a bound on this scale, consider a non-relativistic particle of mass $M_v$ which produces a gravitational potential $h_{00}(r)$. This potential can now be probed by an external static non-relativistic source $\tau_{\mu\nu} = \delta^0_\mu \delta^0_\nu\, \delta^3(r-r')\, m$. The strength of this interaction is set by the amplitude

$$\mathcal{A} = \int_0^\infty \frac{h_{00}(r')}{M_P} \delta^3(r-r')\, m\, d^3r' = \frac{h_{00}(r)}{M_P} m\,. \tag{2.20}$$

Whenever $\mathcal{A}/m \sim 1$, unitarity is violated since the probability of the scattering process to happen per unit time and probe mass is of order one. Thus, for a given mass $M_v$ of the particle, there is always a radius $r_v$ at which the unitarity bound of the theory is violated, or in other words

$$2\int_0^\infty ds \, \frac{\rho(s)}{M_P^2} \frac{e^{-\sqrt{s}r_v}}{r_v} M_v = 1 \ . \tag{2.21}$$

Here, we used the fact that one can spectrally decompose $h_{00}(r)$ by using the real space representation of (2.16).

According to the Heisenberg principle, $r_v$ cannot be smaller than the Compton wavelength of a particle with mass $M_v$, i.e. $r_v \geq M_v^{-1}$. Therefore, the minimal $M_v$ is obtained by inverting

$$2\int_0^\infty ds \, \rho(s) e^{-\frac{\sqrt{s}}{M_v}} \left(\frac{M_v}{M_P}\right)^2 = 1 \ , \tag{2.22}$$

which gives

$$M_v = \frac{M_P}{\sqrt{I(M_v)}} \leq M_P \ . \tag{2.23}$$

Here we made use of the fact that

$$I(M_v) \equiv 2\int_0^\infty ds \rho(s) e^{-\frac{\sqrt{s}}{M_v}} \geq 1 \ , \tag{2.24}$$

because $\rho(s) \geq 0$ for any ghost-free theory. This in turn implies that any ghost free theory of gravity can only produce a stronger gravitational field than the one produced in General Relativity for which $\rho(s) = \delta(s)$. As this pole exists in any UV-modification of gravity, it is always included in (2.24). By definition the strong coupling scale of the theory then obeys $M_* = \min_v M_v \leq M_P$.[7] Thus, we have successfully arrived at (2.19).

An example of a healthy modification of Einstein gravity with the property (2.19) is provided by Kaluza-Klein theories in which gravity becomes higher-dimensional above a compactification scale $M_c \equiv 1/R_c$. For example, the 5-dimensional case corresponds to a particular form of (2.16) with

$$\rho_2(s) \, = \sum_n \delta(s - (nM_c)^2) \, , \quad \rho_0(s) \, = \delta(s) \, . \tag{2.25}$$

For energies $p \gg M_c$, (ignoring tensorial structures) the one-graviton exchange amplitude takes the form

$$A(p) \propto \frac{1}{M_* \sqrt{p^2}}, \tag{2.26}$$

where $M_*$ is the 5-dimensional Planck mass. We can recast this in terms of the usual 4-dimensional propagator but with a momentum dependent Newton constant $G_N(p)$

$$A(p) \propto G_N(p) \frac{1}{p^2}, \tag{2.27}$$

[7] A consistency check of (2.23) is obtained by considering that in Einstein gravity, in which $I = 1$, the strong coupling scale is the Planck scale $M_P$.

where $G_N(p) \equiv \frac{\sqrt{p^2}}{M_*^3}$. At $p = M_c$, the "running" Newtonian coupling must match the four-dimensional Newton's constant. This matching gives the well-known geometric relation between the four- and five-dimensional Planck scales

$$M_* = (M_P^2 M_c)^{\frac{1}{3}} \leq M_P. \tag{2.28}$$

In this theory, gravity becomes strong at a scale $M_* \leq M_P$ due to the fact that the compactification scale is smaller than the four-dimensional Planck scale, $M_c < M_P$.

It is important to note that (2.18) is independent of the precise change of the laws of gravity. Hence, the only hope for gravity to become weaker in the UV is to first reach the strong coupling regime, i.e. the scale $M_*$, and only then turn around its strength. The next section investigates whether such a behaviour can be physically meaningful.

## 2.4 Trans-Planckian Gravity is IR Gravity

In the previous section, we argued that gravity cannot become weaker than GR as long as the interactions are in the weak coupling regime. Here, we want to consider the possibility that $\alpha_{grav}$ starts to decrease in the strong coupling regime of gravity, i.e. for $p \gg M_*$. We will argue that such a decrease is unphysical, because the region of $p \gg M_*$ is protected by the black hole barrier and thus it is impossible to probe length scales $L \ll L_*$. Consequently, an asymptotic weakening of gravity at such distances is physically meaningless, in the sense that there are no new degrees of freedom which can be integrated in to restore unitarity by a softening of the scattering amplitudes. Quite the contrary, since gravity in this regime is fully controlled by large distance classical dynamics, unitarity is restored in a non-Wilsonian way by classical states. Put differently, there are no perturbative states with masses above $M_*$ that can be excited due to the fact that the Compton wavelength of such a particle would be smaller than its black hole horizon. Rather, the only well-defined meaning of any such state is that of a non-perturbative classical object, probing at best distances of order of its own black hole horizon. Due to the positivity of the spectral representation (2.16), these distances, however, are always larger than $L_*$.

To see this in detail, consider, for instance, an effective Lagrangian where the UV physics has been integrated out in a theory in which high energy degrees of freedom are perturbative states. In the Wilsonian picture, the information about high energy physics at scales $E = M \gg M_P$ is carried by *propagating* quantum degrees of freedom of mass $\sim M$. For example, integrating out such particles in the theory of a $U(1)$ gauge field $A_\mu$ with field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ results in a series of operators of the form

$$g\, F_{\mu\nu} \frac{1}{M^2 + \Box} F^{\mu\nu}\, F_{\alpha\beta} \frac{1}{M^2 + \Box} F^{\alpha\beta} + \ldots, \tag{2.29}$$

where $g$ is some effective coupling of order one, which is valid for $p \ll M$. Note that the apparent nonlocality indicated by the d'Alembert operator $\Box$ in the denominator is in fact a remnant of integrating out the heavy degree of freedom of mass $M$. The full series still describes a perfectly local quantum field theory.

For comparison, consider now a field theory where the high energy (trans-Planckian) states are classical black holes. Integrating out these classical objects, one obtains a series of operators that is given by the leading order (the low energy approximation) of (2.29) times a Boltzmann suppression,

$$g\, e^{-\frac{M}{M_P}}\, F_{\mu\nu} \frac{1}{M^2} F^{\mu\nu}\, F_{\alpha\beta} \frac{1}{M^2} F^{\alpha\beta} + \dots . \tag{2.30}$$

The above form is due to the fact that black holes, which have been integrated out, scatter into photons by evaporation. Extrapolating the properties of on-shell black holes one expects virtual black holes to be thermodynamical objects as well, and therefore their contribution to the scattering amplitude should be at least Boltzmann suppressed. This produces the exponentially small factor $e^{-\frac{M}{M_P}}$ in (2.30). However, while the operators in (2.29) incorporate propagating degrees of freedom that show up at the next to leading order in an expansion in $\Box/M^2$, in (2.30), objects of mass larger than $M_P$ are not propagating. Therefore, the operators in (2.30) differ substantially from the ones in (2.29). In fact, the operators in (2.30) do not carry any more information about energy scale above $M_P$ than any other operator obtained by integrating out a classical (or solitonic) object of mass $M$.

To summarize, in a Wilsonian UV completion as, for example, given by (2.29), one can in principle read off the structure of higher dimensional operators by performing very precise measurements at low energies and thus decode physics at distances $\sim M^{-1}$. However, in UV completions of the form (2.30), imprints of sub-Planckian distance physics cannot be detected, even in infinitely precise low energy experiments. Hence, scales smaller than the strong coupling scale of the gravitational theory can never be resolved. One can think of this phenomenon as a high energy generalization of the Heisenberg uncertainty principle which forbids probing distances smaller than the inverse momentum transferred in a measurement, $\Delta x \Delta p \gtrsim 1$, for a pictorial representation see Fig. 2.3 and for previous literature we refer to [83–88].

### 2.4.1 Trans-Planckian Pole in Einstein gravity

Let us now consider a concrete example in which one attempts to add an extra propagating degree of freedom to the massless graviton $h_{\mu\nu}$ in the trans-Planckian region, cf. [7]. Let the extra degree of freedom be given by a scalar graviton $\phi$ of mass $m$. The metric seen by an external probe-source is then given by

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_P} + \eta_{\mu\nu} \frac{\phi}{M_P} . \tag{2.31}$$

The following discussion will show that this addition is meaningful only as long as $m \lesssim M_P$. It becomes meaningless for $m$ becoming trans-Planckian, i.e. for $m > M_P$.

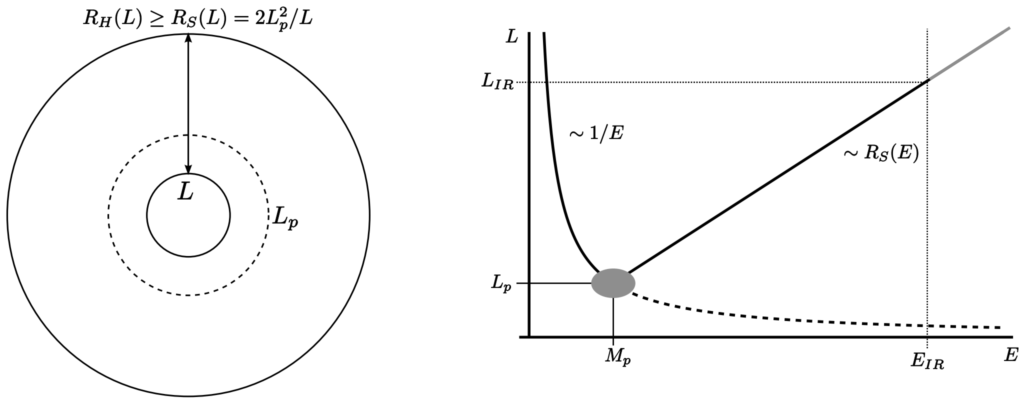


FIGURE 2.3: Trans-Planckian distances are shielded by a black hole barrier. Probing poles at $p^2 = L^{-2} \ll M_P^2$ one has to localize energy of order $L^{-1}$ within the distance $L$. The corresponding black hole horizon of this energy, $R_H(L) \geq R_S(L) = 2L_P^2/L$, shields the sub-Planckian region ($L < L_P$) from being probed by any physical experiment. The sub-Planckian distance $L$ is mapped to the macroscopic distance $R_H(L)$. On the right-hand-side we show a qualitative plot of the energy-distance relation. The grey "blob" around the Planck scale indicates that at the Planck scale itself we don't know how the precise relation between engery and distances is. Also, there is an uncertainty about the far IR black holes, i.e. for energies $E_{IR} = 2L_P^2/L_{IR}$, as we cannot exclude the possibility that at scales $L \gg L_{IR}$ gravity is modified.

However, let us first take a look at Einstein gravity. At large distances, the dynamics of the massless spin-2 graviton is described by Einstein's equation (2.2),

$$G_{\mu\nu} \,=\, 8\pi G_N\, T_{\mu\nu}\,. \tag{2.32}$$

In the weak field limit, the metric and $G_{\mu\nu}$ can be expanded in powers of the dimensionless graviton field

$$g_{\mu\nu} \,=\, \eta_{\mu\nu} \,+\, h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu} + \dots\,, \tag{2.33}$$

where we have absorbed the Planck mass from (2.31) into the definition of $h_{\mu\nu}$. We keep track of the number of gravitons interacting with one another or the source by the use of a superscript denoting the order of the interaction which is equivalent to an expansion in powers of $G_N$. The equations of motions (2.32) to first order are given by

$$\Box\, h^{(1)}_{\mu\nu} \,=\, -\,16\pi\, G_N\,(T^{\text{source}}_{\mu\nu} \,-\, \frac{1}{2}\eta_{\mu\nu}\, T^{\text{source}}\,), \tag{2.34}$$

where $h \,\equiv\, h^{\mu}_{\mu}$, $T \,\equiv\, T^{\alpha}_{\alpha}$, and the harmonic gauge $g^{\alpha\beta}\Gamma^{\gamma}_{\alpha\beta} \,=\, 0$ is employed. Indices are raised and lowered with the background Minkowski metric $\eta_{\mu\nu}$. To linear order, the gauge condition is given by $\partial^{\mu}h^{(1)}_{\mu\nu} \,=\, \frac{1}{2}\partial^{(1)}_{\nu}h$ and the only contribution to $T_{\mu\nu}$ is coming from the energy-momentum tensor of the external source, which is taken to be a static pointlike mass $M$ with $T^{\text{source}}_{\mu\nu} \,=\, \delta^0_{\mu}\delta^0_{\nu}\,\delta^3(r)\,M$. This gives the standard first order result for the metric perturbation,

$$\frac{h^{(1)}_{\mu\nu}}{M_P} \,=\, \delta_{\mu\nu}\,\frac{R_S}{r}\,, \tag{2.35}$$

where $R_S \,=\, 2G_N M$ is the Schwarzschild horizon of the corresponding mass $M$ black hole of mass $M$. This can be compared to the full solution (2.3) and it is clear that

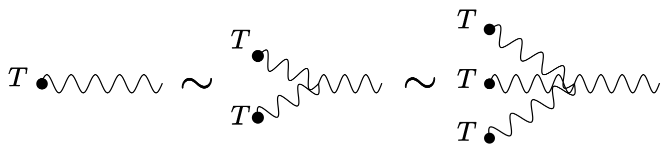


FIGURE 2.4: Gravitational field produced by a source $T$. The wiggled lines represent the emitted gravitons $h_{\mu\nu}$. At the horizon the trilinear and higher order interactions are of the same order as the one-particle exchange. Hence, to obtain any meaningful result the series has to be resummed.

for $R_S/r \ll 1$ the above solution reproduces it.[8] Note that to this order, the signal of approaching the horizon is that $h^{(1)}_{\mu\nu}$ becomes of order one. At the same time, by consistency, the proximity of the horizon is signalled by the second and higher order perturbations in $G_N$ becoming of order one, i.e. the contributions from the non-linear coupling of the graviton to the source are becoming as important as the ones from the linear coupling to the source. Hence, the series has to be resummed, see also Fig. 2.4. This signals the formation of a horizon [109, 110].

Notice, despite the corrections to the metric becoming of order one, the characteristic momenta flowing through the graviton vertices are of order $1/R_S$, and thus, as long as $R_S \gg L_P$, the near horizon geometry is not a probe of Planckian physics. For such sources, gravity is in a weakly-coupled, $\alpha_{\text{grav}} \ll 1$, albeit nonlinear regime. There is an important distinction between the nonlinear regime, characterized by the amplitude of the metric perturbation being order one, $h_{\mu\nu} \sim 1$, which can perfectly well happen while weakly coupled in the sense $\alpha_{\text{grav}} \ll 1$, and the regime of strong coupling where $\alpha_{\text{grav}} \geq 1$. Entering into the nonlinear regime from the linear one simply means that the expansion of the metric $g_{\mu\nu}$ in terms of $h_{\mu\nu}$ breaks down and one has to work with the full metric or equivalently resum. Instead, if $\alpha_{grav} \geq 1$, the effective field theory frame work breaks down and one has to include higher order curvature operators such as, for instance, $R^2$ in the Lagrangian since they are no longer smaller than $R$ itself.

In order to find non-linear corrections, we have to expand (2.32) to second order in $h_{\mu\nu}$, which effectively takes into account the interaction of the graviton with its own energy-momentum tensor $T_{\mu\nu}(h)$. To be fully consistent one would also need to include the corrections to the energy-momentum tensor of the source $T^{(1)}_{\mu\nu}$. Yet, as shown in appendix A, they are negligible at this order.

The equations of motion for the graviton at second order are given by

$$G^{(1)}_{\mu\nu}[h^{(2)}] + G^{(2)}_{\mu\nu}[h^{(1)}] = 0 \,, \tag{2.36}$$

where $G^{(1)}_{\mu\nu}[h^{(2)}]$ is the Einstein tensor expanded to first order in the metric perturbation evaluated for the second order perturbation $h^{(2)}_{\mu\nu}$. Similarly, $G^{(2)}_{\mu\nu}[h^{(1)}]$ denotes the quadratic part of the expanded Einstein tensor evaluated for the first order metric perturbation. We will consider the latter to be sourcing the second order perturbation

[8]It is furthermore instructive to remember that the Newtonian potential $\Phi(r)$ can be recovered from the metric by the identification $\Phi = g_{00} - 1 = h_{00}$.

$h^{(2)}_{\mu\nu}$ and rewrite this contribution to the equations of motion as an graviton energy momentum tensor given by

$$\begin{aligned}8\pi G_N\, T^{(2)}_{\mu\nu}[h^{(1)}] = \quad & -\frac{1}{2}\, h^{(1)\,\alpha\beta}\left(\partial_\mu\partial_\nu h^{(1)}_{\alpha\beta} + \partial_\alpha\partial_\beta h^{(1)}_{\mu\nu} - \partial_\alpha(\partial_\nu h^{(1)}_{\mu\beta} + \partial_\mu h^{(1)}_{\nu\beta})\right)\\ & -\frac{1}{2}\partial_\alpha h^{(1)}_{\beta\nu}\partial^\alpha h^{(1)\,\beta}_{\mu} + \frac{1}{2}\partial_\alpha h^{(1)}_{\beta\nu}\partial^\beta h^{(1)\,\alpha}_{\mu} - \frac{1}{4}\,\partial_\mu\, h^{(1)}_{\alpha\beta}\,\partial_\nu\, h^{(1)\,\alpha\beta}\\ & -\frac{1}{4}\,\eta_{\mu\nu}\,\left(\frac{1}{2}\partial_\alpha\, h^{(1)}_{\beta\gamma}\,\partial^\beta\, h^{(1)\,\alpha\gamma} - \frac{3}{2}\,\partial_\alpha\, h^{(1)}_{\beta\gamma}\,\partial^\alpha\, h^{(1)\,\beta\gamma})\right)\\ & + \frac{1}{4} h^{(1)}_{\mu\nu}\,\Box\, h^{(1)}\,. \end{aligned} \tag{2.37}$$

These equations yield the standard corrections to the metric at second order in $G_N$ [109, 110]. For instance, the zero components are given by

$$\frac{h^{(2)}_{00}}{M_P} = -\frac{1}{2}\,\frac{R_S^2}{r^2} \qquad \text{and} \qquad \frac{h^{(1)}_{00}}{M_P} = \frac{R_S}{r}\left(1 + a\sqrt{\frac{R_S}{R_c}}^{\,3}\right), \tag{2.38}$$

where $a$ is a factor of order 1 and $R_c$ is the radius of the matter distribution of the mass of the source. Taking into account the backreaction of the gravitational field of the source on itself gives a small shift in the "effective" gravitational mass of the source particle, which can be safely neglected. The corrections (2.38) are the manifestation of the fact that at the horizon, i.e. $r = R_S$, the expansion of the metric in powers of $R_S/r$ breaks down, and the series has to be resummed.

Let us now add the extra massive scalar graviton $\phi$ from (2.31) to the spectrum of the theory. It does not change the gravitational field of the massless spin-2 graviton $h_{\mu\nu}$ at first order, since it is only sourced by $T^{\text{source}}_{\mu\nu}$ so that (2.35) still holds. The novelty due to the presence of the massive scalar graviton, which couples to the same static external source via $g^{\mu\nu}T_{\mu\nu}$, is that at second order in $G_N$, $h^{(2)}_{\mu\nu}$ gets additional corrections from the coupling to the energy momentum tensor of $\phi$ which is

$$T^{\phi}_{\mu\nu}(\phi) \;=\; \partial_\mu\phi\partial_\nu\phi \;-\; \frac{1}{2}\,\eta_{\mu\nu}(\partial_\alpha\phi\partial^\alpha\phi \;+\; m^2\phi^2)\,. \tag{2.39}$$

These corrections are accounted for by including the contributions from (2.39) evaluated for the first oder solution $\phi^{(1)} = e^{-mr}(R_S/r)$ on the right hand side of (2.36). Obviously, this contribution gives only an exponentially-supressed correction to $h^{(2)}_{\mu\nu}$.

In contrast, power-law-suppressed corrections can appear if there are couplings between $\phi$ and $h$ of the form,

$$\frac{\phi\partial^n h^k}{M_P^{n+k-3}}\,, \tag{2.40}$$

where the tensorial structure is not disclosed. Such contributions can arise, for example, in non-minimally coupled gravity, and they may induce an effective source for $\phi$,

$$(\Box\, + m^2\,)\,\phi \;=\; \frac{(\partial^n h^k)}{M_P^{n+k-3}} \;+\; \dots\,. \tag{2.41}$$

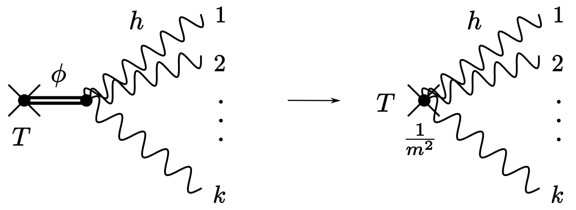


FIGURE 2.5: A heavy scalar (double line) is mediating the interaction between a source $T$ and $k$ gravitons (wiggled lines). Integrating-out this scalar at tree-level will induce an effective point-like interaction between the source and $k$ gravitons.

This can give corrections to $\phi$ which are not exponentially suppressed, but only by powers of $(m\,r)^{-1}$ and $(M_P\,r)^{-1}$. For example, evaluating the right hand side of (2.41) for $h = h^{(1)}$ and $r \gg m^{-1}$ can give corrections of the order (subject to cancellations in the tensorial structure)

$$\frac{\phi^{(k)}}{M_P} \sim \frac{R_S^k}{r^k} \frac{1}{(M_P\,r)^{n-2}(m\,r)^2}\,. \tag{2.42}$$

The reason why these correction are not exponentially suppressed is that they arise from short range processes which do not require the propagation of virtual $\phi$-quanta over distances larger than their Compton wavelengths. In other words, these corrections can be viewed as corrections to the metric in form of non-linear powers of exclusively massless gravitons, which appear as a result of a tree-level integrating out of a heavy scalar graviton of mass $m$ (see also Fig. 2.5) leading to

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_P} + \eta_{\mu\nu}\frac{(\partial^n h^k)}{M_P^{n+k-3} m^2} + \dots\,. \tag{2.43}$$

To summarize, we have seen that corrections coming from a heavy gravitational degree of freedom to the Einstein metric at distances larger than its Compton wavelengths are suppressed either exponentially, or by inverse powers of its mass $m$ and thus cannot significantly affect Einsteinian gravitational dynamics at distances $r \gg m^{-1}$. For example, they cannot interfere with the formation of black holes with Schwarzschild radius $R_S \gg M_P^{-1}$. This is in full accordance with the notion of decoupling of heavy states at low energies [111].[9] Although a heavy quantum state gives negligible corrections to the metric at large distances, for $m \lesssim M_P$, these corrections are still measurable. Thus signatures of new gravitational physics at scales $m^{-1}$ can, in principle, be probed at much larger scales $r \gg m^{-1}$ by precision measurements. However, if $m \gg M_P$ this is not true because the new degree of freedom is no longer a perturbative state. Instead, it is a macroscopic black hole which does not carry any information about the UV physics. Take, for example, the massive scalar graviton $\phi$ of mass $m$ introduced in (2.31). Once $m \gg M_P$, $\phi$ can no longer be treated perturbatively since its Compton wavelength $m^{-1}$ is smaller than its corresponding Schwarzschild radius $R_S$. In order to understand this, it suffices to examine the gravitational field produced by the non-relativistic particle $\phi$ by simply replacing the mass of the source $M$ by the mass $m$ in equation (2.34). The analysis following (2.34) immediately shows that the "particle" $\phi$ develops a horizon of

[9] We rely on the decoupling theorem, i.e., the theory at low energies does not give us any information on the theory at high energies.

size $R_\phi = 2m/M_P^2$ which is larger than $m^{-1}$. For this reason, it has to be considered as a fully legitimate classical black hole. Consequently, the perturbative analysis, in which we considered contributions of virtual $\phi$ quanta, is no longer applicable. Instead, one must take into account that $\phi$ represents a black hole, and therefore any contact interaction resulting from its integrating out must be exponentially suppressed by at least an entropy factor $e^{-S}$, see section 2.2.1. As a result the effective operator obtained by integrating-out $\phi$, e.g. (2.43), can no longer feature a power-law suppressed form. In other words, by becoming trans-Planckian, $\phi$ cannot carry any information other than what is carried by a large IR black hole of the same mass. Therefore, any particle with trans-Planckian mass has to be integrated out as an ordinary classical black hole of the same mass.

We are then led to the conclusion that given the fact that any degree of freedom with mass $m \gg M_P$ is a classical object, it becomes obvious that – no matter how sophisticated – there is *no* process which can probe trans-Planckian physics [7]. This includes also processes like black hole evaporation, primordial quantum fluctuation and scattering experiments. Note that the non-accessibility of sub-Planckian distances is even more efficient than what we described before. In fact, as shown in [4], before the black hole formation an eikonal barrier may form. In this case, eikonal amplitudes which describe the exchange of many soft gravitons in terms of ladder diagrams become important preventing hard energy transfers through a single graviton line which would encode information about short distance physics even before $p \sim M_P$.

### 2.4.2 Trans-Planckian Poles in general Theories of Gravity

In this section, the results of the previous section are extended to theories of gravity in which the strong gravity scale is $M_* < M_p$ following our work in [8]. One can show that for these cases the smallest distance which can be probed are larger than in Einstein gravity. The black hole barrier is more efficient since there are no perturbative elementary states with mass bigger than $M_*$. In order to show this the one-graviton exchange analysis is still a good approximation up to the strong coupling scale $M_*$. In fact, although large classical black hole are self-bound states, individual gravitons in the field interact weakly. The binding potential is a collective effect of a large number of gravitons $N \gg 1$ which together produce a strong gravitational field, cf. chapter 3. Therefore, they can be treated separately as weakly coupled. The proof is a direct consequence of the fact that the scale of strong coupling $M_*$ defined by (2.8) also sets the upper bound on the center of mass energy and the inverse impact parameter above which the black hole formation starts.

The most straightforward way to show this is to start from the opposite end. Let us take a large classical black hole of mass $M$ and horizon $R_H$. The only condition on $R_H$ is that for momenta $p = R_H^{-1}$ gravity is in the weakly-coupled regime, $\alpha_{grav}(p = R_H^{-1}) \ll 1$. The relation between the horizon $R_S$ and $M$ can be found from the condition that $h_{00}(R_H)/M_P = 1$. To be completely general, we employ a spectral representation of $h_{00}$ (2.16) with spectral density $\rho(s) \geq 0$ to account for the additional degrees of freedom in the theory. The condition of having a horizon at $R_H$ can be written as

$$\frac{h_{00}(R_H)}{M_P} = 2\int_0^\infty ds \frac{\rho(s)}{M_P^2}\frac{e^{-\sqrt{s}R_H}}{R_H} M = 1. \tag{2.44}$$

Notice the Yukawa suppression for each massive state in the spectrum. The effective gravitational coupling strength in Fourier space $\alpha_{grav}(p)$ can be represented by,

$$\alpha_{grav}(p) = \frac{(p^2)^2}{M_P^2}\int_0^\infty ds \frac{\rho(s)}{p^2+s}\,. \tag{2.45}$$

One can now start decreasing the mass of the black hole, until the horizon and the inverse mass cross at $R_H = M^{-1}$. We shall denote the corresponding mass by $M_* \equiv L_*^{-1}$. Black holes heavier than $M_*$ are in the classical regime. Accordingly, scattering processes with center of mass energy $M \gg M_*$ and impact parameter $\ll R_H$ will lead to classical black hole formation. The crucial point is that strong coupling is reached around the energy $M_*$ [8]. As a consequence, there is no window above $M_*$ in which one can probe $\alpha_{grav}(p)$ without encountering black hole formation. One can see this easily by evaluating $\alpha_{grav}$ for momenta $p = M_*$ which is found to be of the same order as the quantity $h_{00}(M_*^{-1})/M_P$:

$$\alpha_{grav}(M_*) = \frac{M_*^2}{M_P^2}\int_0^\infty ds \frac{\rho(s)}{1+s/M_*^2} \sim \frac{h_{00}(L_*)}{M_P} = 2\frac{M_*^2}{M_P^2}\int_0^\infty ds\,\rho(s)\,e^{-\sqrt{s}/M_*} = 1. \tag{2.46}$$

The approximate equality follows from the fact that $\rho(s)$ is a positive definite function which gets exponentially cut off by a Boltzmann factor $\mathrm{e}^{-\sqrt{s}R_H}$ for poles $s > M_*$ because they correspond to black holes. Here $R_H$ is the horizon of a classical black hole of mass $M = \sqrt{s}$ determined from (2.44). Accordingly, the integration effectively is cut off at $s = M_*^2$, and therfore, the difference between $e^{-\sqrt{s}/M_*}$ and $(1+s/M_*^2)^{-1}$ is negligible rendering the integrands to be of the same order of magnitude.

### 2.4.3 On the Weakening of Gravity at the strong-coupling Scale

Summing up the previous findings we are lead to the following picture: By gradually increasing the momentum transfer $p$ in a scattering experiment one can probe stronger gravitational couplings. By the time the momentum transfer reaches the scale $M_*$, where gravity becomes strongly coupled, black hole formation starts to take over. Any further attempt of increasing $p$ will result in the formation of larger classical black holes. The region beyond $M_*$ is thus outside of the reach of physical experiments in principle. Therefore, any weakening of $\alpha_{grav}(p)$ for $p \gg M_*$ has no clear physical meaning as it can never be probed. $M_*$ is only an upper bound on the threshold scale of black hole formation, being $M_*$. Therefore our proof is insensitive to the details of the theory and valid for any effective field theory of gravity with a cutoff scale $M_* < M_P$. Approaching the threshold of black hole formation from the weakly coupled linear domain, the one-particle exchange is a good approximation. The scale at which it breaks down coincides with the scale of black hole formation and strong coupling. In this way, a necessary connection between the strong coupling and the threshold of black hole formation emerges, which discloses the impossibility of probing physics at distances shorter than $L_*$. We shall now illustrate our general conclusion on two examples [8].

**An attempt of asymptotically safe gravity in four dimensions** Consider a theory where Einstein gravity is valid all the way up to the Planck scale. In this theory $M_* \equiv M_P$. In the deep UV regime $p \to \infty$ the theory is modified in such a way that the gravitational coupling approaches a fixed point scaling, i.e. $\alpha_{grav} \to \alpha_\infty = const$, as proposed in the Asymptotic Safety scenario [9, 10]. In order to investigate whether this behaviour could have a well-defined physical meaning, one can define an interpolating propagator of the form

$$\Delta(p) = \frac{1}{M_P^2 p^2} \frac{1}{1 + \frac{p^2}{\alpha_\infty M_P^2}}, \tag{2.47}$$

which connects the IR propagator of Einstein gravity $\Delta(p) = \frac{1}{M_P^2 p^2}$ to the stipulated fixed point behaviour $\Delta(p) \to \frac{\alpha_\infty}{p^4}$ for $p \gg \sqrt{\alpha_\infty} M_P$ in the deep UV. In the UV limit, one finds $\alpha_{\text{grav}}(p) = 16\pi G_N(p) p^2 \simeq \alpha_\infty > 1$ and $\alpha_\infty$ is constant. In order to probe distances $r \sim \frac{1}{\sqrt{\alpha_\infty} M_P}$, the center of mass energy needs to be of the order $E \sim \sqrt{\alpha_\infty} M_P$ and the momentum transfer $p \sim E$.

This example is similar to the scenario of an additional graviton of trans-Planckian mass $m$ which we considered in (2.31) and below. The only difference is that now the trans-Planckian state has a negative norm. Let us ignore this sign for a moment since it does not affect our argument about the impossibility of resolving the heavy mass pole. Black hole formation cannot be influenced by the would-be asymptotically safe behavior in the deep UV since for the dynamics of the formation of a black hole of size $R_H$ corresponding to $E \sim \sqrt{\alpha_\infty} M_P$, the ghost pole is decoupled and therefore irrelevant. As a consequence, any attempt of probing the length scales $L = \sqrt{\alpha_\infty}^{-1} L_P$ which correspond to the fixed point regime results in the formation of a black hole of macroscopic size $R_H \simeq 2 L_P \sqrt{\alpha_\infty}$. In this case, the black hole horizon is determined by [8]

$$h_{00}(R_H) = 2 \frac{\sqrt{\alpha_\infty}}{M_P} \frac{1}{R_H} \left[ 1 - \alpha_\infty e^{-\sqrt{\alpha_\infty} M_P R_H} \right] = 1 \,, \tag{2.48}$$

It is apparent that the existence of the heavy ghost pole at $\sqrt{\alpha_\infty} M_P$ only affects the value of the black hole horizon $R_H$ with exponentially weak corrections. Accordingly, in an attempt to probe distances smaller than the Planck length $L_P$, a black hole with radius $R_H \simeq 2 \frac{\sqrt{\alpha_\infty}}{M_P} > M_P^{-1}$ will be produced rendering the penetration of the trans-Planckian region impossible. Asymptotic Safety is thus rendered irrelevant before it had any chance to influence gravitational physics.

To conclude, the existence of the ghost pole, which was assumed to be responsible for the Asymptotic Safety behavior, is rendered meaningless. Moreover, the UV-IR connection of gravity indicates that it should not have been included in the first place. Indeed, as a result of the black hole barrier, any physically sensible trans-Planckian state is mapped to a macroscopic object from the IR region. However, in a consistent theory of gravity there are no negative energy classical states and the ghost pole simply cannot have any

IR counterpart. Thus it should be excluded as a conseequence of the self-consistency of the theory.

**Asymptotically safe gravity with a lower cut-off scale** Next, we wish to consider an extension to the previous example in which gravity becomes strong at a scale $M_5 < M_P$. This happens whenever new (positive norm) gravitons open up at some intermediate energies. A good example of this property is provided by five dimensional Kaluza-Klein theories [112, 113], in which gravity becomes higher-dimensional above the compactification scale $M_c = R_c^{-1}$, cf. (2.25) and below. At short distances $r < R_c$ gravity can probe the extra dimension and becomes strong at distances of the five-dimensional Planck length $L_5 = (R_c/M_P^2)^{\frac{1}{3}}$.

Due to the fact that at high energies gravity can probe the extra dimension, four dimensional gravity becomes "weaker" at these energies. Such a behaviour could be thought of to be similar to the Asymptotic Safety fixed point scenario, but the underlying reason for its weakening is different. Once gravity can penetrate the extra dimension, the gravitational flux lines will also extend into this dimension. Therefore, the gravitational potential at these scales is determined by a five dimensional Gauss law which gives a gravitational potential between two probes with masses $M$ and $m$

$$V(r) \propto \frac{Mm}{M_5^3}\frac{1}{r^2}. \tag{2.49}$$

Here $M_5 \equiv L_5^{-1}$ is the five dimensional Planck scale. The potential with its $\propto 1/r^2$ behaviour falls of faster than the usual four dimensional potential which obeys $\propto 1/r$.

From the point of view of the four dimensional theory, the underlying five dimensional theory is imprinted into the tower of massive scalar Kaluza Klein states with masses $m_n = n^2/R_c^2$, where $n$ is an integer number. These states couple universally to matter, because they are a result of the compactification of the fifth dimension. They also contribute to the scattering amplitudes of gravity and modify the propagator of these theories in four dimensions according to

$$\Delta(p) = \left\{ \sum_{n=1}^{(M_P/M_5)^2} \frac{1}{p^2 + \frac{n^2}{R_c^2}} \right\}, \tag{2.50}$$

where $R_c M_5^3 \equiv M_P^2$. This means that above the scale $\frac{1}{R_c}$ there is a tower of massive gravitons, which makes gravity strong already at scale $M_5$ instead of $M_P$. Consequently, the shortest observable length scale in this theory is $L_5 \equiv M_5^{-1}$.

Consider now such a theory equipped with a gravitational fixed point at scales $p \gg M_5$. The corresponding interpolating propagator is given by

$$\Delta(p) = \left\{ \sum_{n=1}^{(M_P/M_5)^2} \frac{1}{p^2 + \frac{n^2}{R_c^2}} \right\} \frac{1}{1 + \frac{p^2}{\alpha_\infty M_P^2}}. \tag{2.51}$$

As in (2.47), there is a trans-Planckian ghost pole which mimics the fixed point behaviour. Additionally, there exists a black hole barrier at the strong coupling scale $M_5$. Correspondingly, also for this case, the ghost pole cannot be probed and remains unphysical [8]. Indeed for energies required to probe the ghost pole, $E \sim \sqrt{\alpha_\infty} M_P$, the black hole horizon is macroscopic $R_H \simeq \left(\alpha_\infty R_c/M_5^3\right)^{\frac{1}{4}} \gg M_5^{-1}$, and the associated state belongs to the classical gravity region.

We have seen that also in this example Asymptotic Safety has no physical meaning. The black hole barrier, which maps the trans-Planckian region to classical IR gravity, precludes probing distances where the fixed point behaviour might become relevant.

### 2.4.4 Continuum Tails of Trans-Planckian Physics

After having discussed examples of isolated poles in the graviton propagator, we wish to extend this discussion to the case where one includes a continuum of states [7]. Such states could result in sub-leading corrections to the one particle exchange diagrams represented by the decomposition (2.16), which seemingly may be probed in the deep IR. In particular, let us focus on sub-leading corrections that would make gravity slightly weaker.

For clarity, let us consider General Relativity with a strong gravity scale $M_P$ and a perturbation to the Newtonian potential of the type

$$V(r) = G_N \frac{mM}{r}\left(1 - \frac{L^2}{r^2} + \mathcal{O}\left(\frac{L^3}{r^3}\right)\right) . \tag{2.52}$$

In fact, this potential has been considered, for example, as the correction to the Newtonian physics within the Asymptotic Safety scenario for gravity [114]. The 1-loop correction to the Schwarzschild metric in an effective field theory approach studied in [115, 116] was also found to be of this type.

The negative contribution $\sim L^2/r^3$ can be understood as the result of an exchange of a continuum tower of ghost states. This can be easily seen when rewriting (2.52) as, cf. [117],

$$\frac{V(r)}{m} \simeq G_N M\left[\frac{1}{r} - L^2 \int_0^\infty d\tilde{m}\frac{e^{-\tilde{m}r}}{r}\tilde{m}\right] . \tag{2.53}$$

The second term is nothing else than a sum over a continuum of massive particles. Going to Fourier space one readily obtains

$$\frac{V(p)}{m} \simeq G_N M\left[\frac{1}{p^2} - \frac{L^2}{2}\int_0^\infty ds\frac{1}{p^2+s}\right] . \tag{2.54}$$

In other words, the potential (2.52) can be understood in terms of the exchange of a massless graviton and an infinite tower of equally distributed massive ghosts with a constant spectral density $\rho(s) = L^2/2$. Performing the integral in (2.54), one can think of this potential as the usual Newtonian potential $V(r) \propto G_N/r$ where Newton's constant, $G_N$, is no longer constant but momentum dependent corresponding to

$$G_N(p) = G_N[1 - L^2p^2 \ln p] \; . \tag{2.55}$$

Such a behaviour would make gravity weaker at high energies.

If $L \gg L_P$, the contribution from the ghost tower dominates at scales much larger than $L_P$, and the theory is already unreasonable in the IR. Under these circumstances, the continuum correction (2.52) cannot exist in a consistent theory. In the opposite limit, if $L \ll L_P$, for any classical black hole the correction from the ghost tower is subdominant to the first non-linear correction from the massless graviton, which according to (2.38) goes as $\sim R_s^2/r^2$. Thus the ghost tower cannot affect the black hole formation and the black hole barrier cannot be altered. Consequently, the trans-Planckian members of the continuum are also shielded by the black hole barrier making them either unphysical or simply inconsistent [8]. Hence, there is no domain in which the potential (2.52) is a sensible description of physics.

A seeming way out of the impossibility of probing UV distances would be to construct a scattering experiment with center of mass energy $E < M_P^2 L$ and impact parameter $b < L$, i.e. momentum transfer $p > M_P^2 L$. In this region, from the expanded potential (2.52) it seems that black holes cannot form. However, such an experiment cannot be performed in principle. As the impact parameter needed is smaller than the Compton wavelength of the particle of mass $M = E$, these center of mass energies can never probe distances $\sim b$. According to the Heisenberg principle, energies $E$ can only probe distances $\sim 1/E$. Hence, the minimal distance such an experiment can probe is $E^{-1} > (L_P/L)L_P$ which greatly exceeds $L_P$. Once again, the Planck scale turns out to be impenetrable.

### 2.4.5 Sub-Planckian Experiments

In this section, we consider the question of whether trans-Planckian physics can be observed by preparing a scattering experiment at scales in which possible modifications of gravity are already important [8]. Such a scenario was proposed in [118]. The authors envisioned an experiment prepared at distances shorter than the strong gravity scale. They investigated whether the hoop conjecture may be violated thus enabling a resolution of sub-Planckian scales without the black hole barrier being able to interfere.

Suppose the experiment could indeed be set up without black hole formation such that an asymptotic observer can detect some output from the experiment. This then implies that a degree of freedom with energy larger than the Planck mass has to leave a sphere of radius $L_P$ before reaching the detector. The degree of freedom has to have a mass $M > L_P^{-1}$ as otherwise it is not a probe of this small distance region and could not have been localized within it. As soon as it crosses the radius $L_P$, a black hole forms since then gravity is in the weakly coupled regime and thus described by Einstein gravity. However, if there was no black hole before, this would be in contradiction to the conservation of energy which is manifest on an asymptotically flat background. An asymptotic observer may now draw a sphere around the region of the experiment and continuously monitor the energy inside the sphere by measuring the Gaussian flux at infinity. Therefore, the only way of conserving the flux at infinity is to exclude the absence of the black hole during the setup and course of the experiment. We thus conclude that *at best* the experiment was prepared inside of an already existing black hole.

Taking a slightly different perspective, one can say that energy cannot be localized within a distance $L \ll L_P$ when information (encoded in energy) crosses outside this region without causing a surrounding gravitational field of a black hole. Imagine for example an extreme case in which the gravitational force vanishes at some scale $L \ll L_P$ and take a spherical shell placed entirely inside this region. Although this shell has positive energy, it does not gravitate as long as its radius $R \ll L$. Naively, one would conclude that there is no gravitational field outside the sphere such that an asymptotic observer would see only flat space. Now, let the shell communicate with an outside observer by expanding and crossing outside the $L$ sphere. At some point, this sphere will cross into a $R > L$ region. Since the energy of the shell is trans-Planckian, it has to form a black hole. However, the black hole cannot appear out of nothing because of the conservation of the Gaussian flux at infinity. Consequently, a black hole must have been formed from the very beginning when the experiment has been set up.

### 2.4.6 Infrared Scales

For definiteness, we limited the treatment of self-completeness of gravity to theories which flow to Einstein gravity on asymptotically-flat spaces in the deep IR. Nonetheless, a theory may contain an infrared scale $L_{IR}$ beyond which this assumption breaks down. For example, this could be the scale of a small background curvature like the cosmological constant, or something more profound. Since the existence of such a scale might modify the properties of black holes, the connection between deep-UV gravity and classical IR black holes can also be affected. Nonetheless, one can still expect the above conclusions to holds in the energy interval between $1/L_{IR}$ and $L_{IR}/(2L_P^2)$, the latter value being set by the mass of an Einsteinian black hole with Schwarzschild radius equal to $L_{IR}$, see also Fig. 2.3.

If $L_{IR}$ is a curvature radius produced by a positive cosmological constant, we expect the concept of a minimal length to be unaffected. If anything, positive curvature makes it harder to probe short distances, since for a given mass the effective Schwarzschild radius is increased on de Sitter. For example, the time component of a static metric in Schwarzschild coordinates is then given by

$$g_{00} = 1 - 2G_N \frac{M}{r} - \frac{r^2}{L_{IR}} \,. \tag{2.56}$$

For the observed cosmological constant, $L_{IR} = 10^{28}$cm, the deviation from the flat space case only appears for energies comparable to the mass of the observable Universe and can thus safely be ignored. If, on the other hand, $L_{IR}$ is related to a negative cosmological constant, the issue is more subtle and one may consider the concept of

AdS/CFT correspondence [119–121]. This poses an interesting problem on its own but is beyond the scope of this thesis.

## 2.5 Summary

In [7], it was argued that quantum gravity might be fully described by light degrees of freedom and that in this sense Einstein gravity can be considered self-complete. Following this idea, we have shown that this self-completeness property, which is built-in in Einstein gravity, persists for a wide class of its deformations. The basic reason for this universal self-completeness is the non-existence of trans-Planckian propagating degrees of freedom. Any would-be trans-Planckian pole is mapped to a classical IR state which is described by low-energy degrees of freedom of the IR theory.

This remains true in the presence of black hole evaporation if one assumes that it is fully described by low energy physics. For a more detailed account on this matter we refer to chapter 3. This assumption is backed by noticing that, in order to see trans-Planckian corrections to Hawking evaporation, one should integrate in an operator of mass larger than the Planck scale. However, this operator defines a particle of Compton wavelength smaller than the black hole horizon in the weakly coupled region, so that such an operator can only integrate in other black holes, which are again classical objects.

As a consequence, we have seen that the same properties that make Einsteinian gravity self-complete in the deep UV also render many attempts of a conventional UV-completion in the trans-Planckian region physically meaningless. We have focused on the class of attempted UV-completions which are based on the ideas of an asymptotic weakening or of Asymptotic Safety. We have shown that in Einstein gravity and its ghost-free deformations there is essentially no energy interval in which such ideas can be realized in a physically clear way. We have found that, in both cases, the necessary condition is that a weakening (or safety) can only take place within the strong gravity domain. Additionally, we have shown that in ghost-free extensions of Einstein gravity this domain includes the regime where gravity starts to be mapped to the IR region due to black hole formation. In other words, there is no interval of distances in which gravity may be strongly-coupled but not shielded by the black hole barrier. Therefore, such UV completions could only be meaningful if they could be mapped to IR physics. However, the states responsible for an asymptotic weakening have negative norm and hence, they cannot be mapped to well-defined IR states. We did not address the question of a connection between the self-completeness of gravity and a string theoretic completion. This is an interesting problem on its own and we refer the reader to [7] and references therein.

# Chapter 3

# Black Holes as Bose-Einstein-Condensates

In the previous chapter 2, we have elaborated that Einstein's theory of General Relativity is self-complete in a non-Wilsonian way. Due to black hole formation in high-energy scattering, distances shorter than the Planck length $L_P$ (or $L_*$ in a theory with lower cutoff) cannot be probed. Unitarity violation is avoided as the decay of a black hole in highly energetic states is exponentially suppressed. If gravity is indeed self-complete in this sense, also black hole physics must be describable within Einstein's theory when treated on a quantum level.

Such a picture has been put forward in [18–20, 122] where Dvali and Gomez argued that black holes can be described as Bose-Einstein condensates of $N \gg 1$ self-bound gravitons of wavelength of order of the Schwarzschild radius $\lambda = R_S$. Furthermore, this condensate is thought to be at the critical point of a quantum phase transition which is usually accompanied by large quantum correlations necessitating a quantum description beyond a mean-field treatment. As a result, if the black hole is described as such a state, some of its properties cannot be recovered in the semi-classical treatment of General Relativity. The corresponding microscopic theory of self-bound $N$ gravitons can provide a framework within which some of the mysterious properties of black hole physics such as the information paradox [14] and the origin of entropy [12] can be addressed.

In this chapter, we want to investigate certain properties of the black hole condensate picture of [18, 19] by using a toy model inspired by condensed matter physics. In section 3.1, the quantum N-portrait of black holes [18] is reviewed and the notation is set up. Section 3.2 serves as a simplified introduction to the theory of Bose-Einstein condensates and the notion of the black hole as a condensate at a critical point [19] is introduced. We establish a toy model of a collapsing condensate which stays at the critical point throughout its evolution in section 3.3. We also derive the Bogoliubov spectrum for the lowest excitations around such a solution.

## 3.1 Self-bound Graviton Condensates

Let us review the most important ideas of the so-called quantum N-portrait of black holes [18]. Its description rests on the fact that one should consider a black hole to be a bound state of $N \gg 1$ gravitons of wavelength $\lambda = \sqrt{N} L_P$ interacting weakly with strength $\alpha = 1/N$.

The occupation number $N$ can be used as a measure of classicality [123]. Quantum mechanically, (semi)-classical objects are then described by states with a large $N \gg 1$. In quantum field theory the occupation number $N$ counts the number of quanta in a given field configuration. Examples are the number of photons in an electric field or in the present case the number of gravitons in a gravitational field.

The gravitational field produced by a source of mass $M$ and size $R$ is given to leading order by $\phi = -R_S/r$ for $r \geq R$ and $\phi = R_S/2(r^2/R^3 - 3/R)$ inside $R$, where $R_S = 2\hbar M/M_P^2$ is the Schwarzschild radius. The occupation number of gravitons in this field is [18]

$$N \simeq \frac{1}{\hbar} M R_S \simeq \frac{M^2}{M_P^2} \,. \tag{3.1}$$

The measure of classicality $N$ then implies that for large masses $M \gg M_P$ the gravitational field is classical consistent with the expectation that macroscopic gravity is well described by General Relativity.

In terms of individual gravitons, the energy of the gravitational field is given by

$$E_G \sim \frac{M R_S}{R} \sim \sum_\lambda \hbar N_\lambda \lambda^{-1} \,, \tag{3.2}$$

where $N_\lambda$ counts the number of gravitons of energy $\hbar\lambda^{-1}$. The maximum number of quanta in this field configuration is obtained when the distribution is peaked at $\lambda \sim R$. The assumption that the energy is distributed equally among the gravitons leads therefore to the proclamation that to leading order the gravitational field consists of $N \sim E_G R/\hbar = M^2/M_P^2$ gravitons of wavelength $R$ [18, 124].

For a given size $R$ of the source, the number of constituent gravitons [18] is maximized by a black hole with $R = R_S$, as this is the minimal radius the energy $M$ can be compressed into. Increasing the number of gravitons $N$ (and therefore $M$) in a black hole invariably enlarges its radius $R_S$. For this reason, black holes comprise the most densely packed states of gravitons. According to (3.2), at this point the gravitational energy becomes equal to the energy of the source for a black hole which consequently forms a self-sustained system of $N$ gravitons. Independently of the composition of the source, $N$ becomes a universal quantity for the black hole which describes its dynamical characteristics [18].[1] In the spirit of Birkhoff's theorem, $N$ assumes the role of the mass $M$ as the universal defining parameter of the black hole. Interestingly, the occupation

[1]Notice that in case of charged or rotating black holes, $N$ does of course not fully describe the black hole, but there will be additional parameters like, for example, the number of charged constituents $N_B$. However, here we are only concerned with Schwarzschild black holes.

number $N$ also captures the quantum hair associated with the black hole, and rather than being exponentially weak, these hair are only suppressed by factors $1/N$ [20].

To summarize the quantum N-Portrait, we recap the defining features of a black hole of mass $M$ described as a self-sustained graviton condensate [18]. The occupation number of gravitons in the field is defined

$$N = \frac{M^2}{M_P^2} = \frac{R_S^2}{L_P^2} , \tag{3.3}$$

where the Planck length is defined by $L_P = 1/(\hbar M_P)$. The energy or mass of the black hole is determined by the $N$ constituent gravitons [18]

$$M = \sqrt{N}\frac{\hbar}{L_P} = \sqrt{N} M_P , \tag{3.4}$$

which are to leading order of wavelength

$$\lambda = R_S = \sqrt{N} L_P . \tag{3.5}$$

These gravitons are weakly interacting with a dimensionless coupling strength given by [18]

$$\alpha = \frac{L_P^2}{\lambda^2} = \frac{1}{N} . \tag{3.6}$$

Although for large $N$ the individual coupling strength of gravitons is very small, such a bound state can exist for arbitrary $N$ due to the collective potential produced by the $N-1$ other gravitons. This collective binding potential can be described by a Hartree approximation to leading order. Because of the weak interaction strength of the individual gravitons, one can assume that the wave function of the $N$-particle system $\Psi(r)$ is well approximated by a product of the single-particle wave functions $\psi_i(r)$

$$\Psi(r) = \prod_{i=1}^{N} \psi_i(r) . \tag{3.7}$$

In this approximation one graviton is subject to a collective potential which is given by the sum over all individual contributions [18]

$$V(r)\big|_{r \gtrsim \lambda} = -\sum_{i=1}^{N} \hbar\alpha\frac{1}{r} = -N\hbar\alpha\frac{1}{r} . \tag{3.8}$$

Note that a particle of wavelength $\lambda$ effectively does not feel a potential varying on scales smaller than its own wavelength. For a graviton of wavelength $\lambda$, this potential becomes maximally deep at a distance $r = \lambda$. It then follows that such a bound state cannot provide any probe of trans-Planckian physics since its constituents are gravitons of wavelength $\lambda = R_S \gtrsim L_P$. In this regard it can be stipulated that trans-Planckian physics is physics of large occupation number $N$. Probing short distances $L \ll L_P$ results in the formation of a macroscopic black hole with occupation number $N \gg 1$. Independently of whether the source is a classical macroscopic object which itself has a large occupation number or whether the black hole is produced from a two particle scattering experiment, the outcome is universally defined by the center of mass energy $M$ determining the number of gravitons in the field $N = M^2/M_P^2$.

Notice that the bound state, however, is leaky. The escape energy of one graviton is given by $E_{esc} = \hbar\lambda^{-1}$ which is exactly the energy of a single graviton in the condensate. Thus any scattering with non-zero momentum transfer among the gravitons will take one graviton out of the condensate and into the continuum [18]. Following such an event the black hole consists of $N-1$ gravitons which redistribute their energy in such a way that, again, the black hole is a bound state of now $N' = N-1$ gravitons of wavelength $\lambda' = \sqrt{N-1}L_P$ interacting with strength $\alpha' = 1/(N-1)$ according to (3.6). This is one of the pecularities of gravity and should be contrasted with the cold atoms scenario discussed in 3.2; due to the momentum dependent coupling strength $\alpha$ for any $N \gtrsim 1$ such a bound state of gravitons exist. The evaporation process will continue in a self-similar way until the black hole condensate disappears.

### 3.1.1 Leakiness and Hawking Evaporation

From phase space arguments it follows that the most probable scattering is a $2 \to 2$ scattering of gravitons during which one of them gains the escape energy $\sim \hbar\lambda^{-1}$. The rate for such a process is given by [18]

$$\Gamma \simeq \binom{N}{2}\alpha^2 \frac{\hbar}{\sqrt{N}L_P} \simeq N^2 \frac{1}{N^2}\frac{\hbar}{\sqrt{N}L_P} , \tag{3.9}$$

where the first factor is a combinatorical factor from the fact that we have $N$ gravitons of which two should scatter. Of course, also three or more gravitons can scatter and take a graviton into the continuum which is of higher energy, but these processes are suppressed by higher powers of $1/N$. For example, the next-to-leading order process of scattering a graviton of energy $\sim 2\hbar\lambda^{-1}$ into the continuum happens via a consecutive scattering of three gravitons which yields $\Gamma \simeq 2\hbar N^3/(N^4\sqrt{N}L_P)$. The total emission rate is thus an expansion in $1/N$ and to leading order determined by

$$\Gamma = \frac{\hbar}{\sqrt{N}L_P} + \frac{\hbar}{L_P}\mathcal{O}(\frac{1}{\sqrt{N}^3}) \tag{3.10}$$

which sets the characteristic time scale $\Delta t = \hbar\Gamma^{-1}$ during which one graviton of wavelength $\sqrt{N}L_P$ is radiated away. The rate of depletion following from this analysis is [18]

$$\frac{dN}{dt} = -\frac{1}{\sqrt{N}L_P} \tag{3.11}$$

and can be recast into the more familiar evaporation rate for a black hole by taking into account (3.4)

$$\frac{dM}{dt} = -\frac{\hbar}{NL_P^2} . \tag{3.12}$$

One can then define a temperature for large $N$, $T = \hbar/(\sqrt{N}L_P)$, which reproduces the thermodynamic temperature for a black hole of mass $M$, $T_B \simeq \hbar/(G_N M) \sim M_P^2/M$ (see [13, 14, 125] and references therein), where $G_N$ is the Newton constant.

Finally, in order to make a connection with Hawking radiation, one may take the semi-classical limit in which Hawking performed his computation [14]. In the double-scaling limit

$$N \to \infty, \quad L_P \to 0, \quad L \equiv \sqrt{N}L_P = \text{fixed}, \quad \hbar \neq 0 \, , \tag{3.13}$$

one recovers the Stefan-Boltzmann law for the depletion of the leaky condensate

$$\frac{dM}{dt} \sim -\frac{1}{\hbar}\frac{M_P^4}{M^2} \sim -\frac{1}{\hbar^3}R_S^2 T_B^4 \, . \tag{3.14}$$

Note that $\frac{dM}{dt}$ does not diverge in the classical limit $\hbar \to 0$ but rather goes to zero, because $M_P^2 = \hbar G_N^{-1} \to 0$ as $\hbar \to 0$. The black hole temperature therefore scales as $T_B \sim \hbar$ when all other classical quantities remain constant in the limit $\hbar \to 0$.

The most probable wavelength of an emitted graviton is $\lambda \sim R_S$ which is in agreement with the thermal Hawking spectrum. The exponential suppression of high frequency modes can be argued to arise from a cascade of $k$ scattering events that are necessary to produce a highly energetic graviton of wavelength $k^{-1}R_S$ [18].

## 3.2 Condensates at the critical Point

The second important observation made by Dvali and Gomez is the fact that the graviton Bose-Einstein condensates lives right at a critical point of a quantum phase transition.[2] Although two gravitons individually interact very weakly with $\alpha \sim 1/N$, cf. (3.6), the effective potential each graviton feels is made up from $N$ gravitons resulting in a collective strength $\alpha N = 1$. Remarkably, this condition characterizes the point of a quantum phase transition in a simple nonrelativistic bosonic system, see, e.g., [21, 128].

The suggested analogy to nonrelativistic Bose-Einstein condensates at the critical point could help to resolve some of the mysteries of the semi-classical treatment of black hole physics. A fundamental aspect of a quantum phase transition is the existence of large quantum correlations, which are manifest in the appearance of almost gapless collective excitations (the Bogoliubov modes). In addition to being responsible for the instability of the condensate at the critical point, they account for its quantum depletion and the (near) degeneracy of the quantum states of the condensate. These properties, if carried over to the black hole description, could provide a microscopic explanation for such phenomena as black hole evaporation, entropy and also holography. In the semi-classical limit, however, these features become infinitely hard to resolve as the number of constituents $N$ becomes infinite.

[2] We will use the terms critical point and bifurcation point invariably as we want to be able to make a connection between the one-dimensional cases studied in [126, 127] to the three-dimensional situation we will establish in section 3.3.

For a bound state of size $R \sim \lambda$ consisting of $N$ gravitons of wavelength $\lambda$ the total kinetic energy is given by

$$\langle E_k \rangle = N\frac{\hbar}{\lambda} \, , \tag{3.15}$$

and its potential energy reads

$$\langle V \rangle = -\frac{\hbar N^2 \alpha}{\lambda} \, . \tag{3.16}$$

Minimizing the total energy $\langle E_k + V \rangle$ leads to

$$\frac{d\,\langle E_k + V \rangle}{dR} = (1 - \alpha N)\frac{\hbar}{R} = 0 \, . \tag{3.17}$$

It follows that the graviton condensate can form a self-sustained bound state if $\alpha N = 1$. This is what was called maximal packing before. The size of the condensate can be obtained from the virial theorem $\langle E_k \rangle \sim \langle V \rangle$, and it is given by the wavelength of the gravitons $\lambda = \sqrt{N} L_P$ recovering (3.5).

To establish a connection between the graviton condensate and nonrelativistic Bose-Einstein condensates encountered in condensed matter physics, it is instructive to first briefly recap some basic facts about Bose-Einstein condensates. A nonrelativistic interacting Bose gas is described by a many-body Hamiltonian of the form

$$\begin{aligned} \hat{H} &= \int d^3r \hat{\Psi}^\dagger(\vec{r}) \left[ \frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\vec{r}) \right] \hat{\Psi}(\vec{r}) \\ &\quad + \frac{1}{2}\int d^3r d^3r' \hat{\Psi}^\dagger(\vec{r}) \hat{\Psi}^\dagger(\vec{r}') V(\vec{r} - \vec{r}') \hat{\Psi}(\vec{r}) \hat{\Psi}(\vec{r}') \, , \end{aligned} \tag{3.18}$$

where $\hat{\Psi}^\dagger(\vec{r})$ and $\hat{\Psi}(\vec{r})$ are the creation and annihilation operators of bosons of mass $m$, and $V_{ext}(\vec{r})$ is an external potential which we will set to zero in this section. The potential $V(\vec{r} - \vec{r}')$ describes the interaction of two bosons at positions $\vec{r}$ and $\vec{r}'$. The density is normalized according to

$$\int d^3r \langle \hat{\Psi}^\dagger(\vec{r}) \hat{\Psi}(\vec{r}) \rangle = \int d^3r \langle \hat{N}(\vec{r}) \rangle = N \, , \tag{3.19}$$

where $\hat{N}(\vec{r})$ is the number density operator and $N$ the total number of bosons.

The real-space field operator can be expanded in the basis of the single particle operators $\hat{a}_\alpha$ which span the Fock space of the noninteracting theory

$$\begin{aligned} \hat{\Psi}(\vec{r}) &= \sum_\alpha \psi_\alpha(\vec{r}) \hat{a}_\alpha \\ \hat{\Psi}^\dagger(\vec{r}) &= \sum_\alpha \psi^*_\alpha(\vec{r}) \hat{a}^\dagger_\alpha \, , \end{aligned} \tag{3.20}$$

where the $\psi_\alpha(\vec{r})$ are the one-particle wave functions. The operators $\hat{a}_\alpha$ obey the standard commutation relations [129]

$$[\hat{a}_\alpha, \hat{a}_\beta] = [\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta] = 0 \, , \quad [\hat{a}_\alpha, \hat{a}^\dagger_\beta] = \hbar\, \delta_{\alpha,\beta} \, . \tag{3.21}$$

Bose-Einstein condensation is signalled by one state (which we denote w.l.o.g. by $\hat{a}_0^\dagger$) becoming macroscopically occupied with occupation number $N_0$ such that in the thermodynamic limit $N \to \infty$ the ratio $\frac{N_0}{N} \neq 0$. In this case states which have occupation numbers $N_0 \pm 1$ and $N_0$ are indistinguishable and it can be shown (see e.g. [129]) that the creation and annihilation operator for such a state commute. One can therefore replace them by an ordinary $c$-number $\hat{a}_0 = \hat{a}_0^\dagger \simeq \sqrt{N_0}$. The classical wave function $\psi_0(\vec{r})$ takes the role of an order parameter and the global $U(1)$ phase symmetry of $\hat{\Psi}(\vec{r})$ is spontaneously broken. The field operator can then be expanded around its macroscopic expectation value $\psi_0 = \langle \hat{\Psi}(\vec{r}) \rangle$

$$\hat{\Psi}(\vec{r}) = \psi_0(\vec{r}) + \delta\hat{\psi}(\vec{r}) \; . \tag{3.22}$$

The density of the condensate is given by $n_0(\vec{r},t) = |\psi_0(\vec{r})|^2$ and normalized to $\int d^3r |\psi_0(\vec{r})|^2 = N_0 \simeq N$. For a homogeneous condensate in a volume $V$ the wave function of the condensate is $\phi_0(\vec{r}) = \sqrt{N_0/V} \simeq \sqrt{N/V}$.

Upon promoting the field operators $\hat{\Psi}(\vec{r})$ to be time-dependent according to the Heisenberg picture, the evolution of $\hat{\Psi}(\vec{r},t)$ is given by the Heisenberg equation of motion

$$i\hbar\partial_t \hat{\Psi}(\vec{r},t) = [\hat{\Psi}(\vec{r},t), \hat{H}] = \left[ -\frac{\hbar^2\nabla^2}{2m} + V_{\text{ext}} + \int d^3r' \hat{\Psi}(\vec{r'},t) V(\vec{r}-\vec{r}') \hat{\Psi}(\vec{r}',t) \right] \hat{\Psi}(\vec{r},t) \; . \tag{3.23}$$

This equation is so far exact. If $\langle \delta\hat{\psi}(\vec{r},t) \rangle$, or equivalently the relative depletion of the condensate $(N - N_0)/N_0$, is small, the ground state of the condensate can be well approximated by setting $\hat{\Psi}(\vec{r},t) \simeq \psi_0(\vec{r},t)$. Since $V(\vec{r}-\vec{r}')$ shall denote a short-range interaction potential, in the low-energy limit it may be replaced with a contact interaction $V_0\delta(\vec{r}-\vec{r}')$. In the weak coupling limit, the interaction is determined by the s-wave scattering length $a$ and $V_0 = 4\pi\hbar^2 a/m$. It is attractive for $a < 0$ and repulsive for $a > 0$. This approximation holds as long as the momenta of the bosons are small compared to the inverse range of the interatomic interaction, which is the case for $|a|^3 N/V \ll 1$ at zero temperature.

Inserting this mean-field approximation yields the time-dependent Gross-Pitaevskii equation [130, 131]

$$i\partial_t \psi_0(\vec{r},t) = \left[ -\frac{\nabla^2}{2m} + V_{\text{ext}} + V_0 |\psi_0(\vec{r},t)|^2 \right] \psi_0(\vec{r},t) \; , \tag{3.24}$$

where we set $\hbar = 1$ from now on.

The Gross-Pitaevskii equation can equally be derived from an action principle within a field theoretical approach. The real-time action for a gas of nonrelativistic bosonic particles is

$$S[\Psi^*, \Psi] = \int_0^\infty dt \int d^3r \left\{ \Psi^*(\vec{r},t) \left( i\partial_t - \frac{\nabla^2}{2m} + V_{\text{ext}} - \mu \right) \Psi(\vec{r},t) + \frac{1}{2} V_0 |\Psi(\vec{r},t)|^4 \right\} \; , \tag{3.25}$$

where we switched to the grand-canonical ensemble $\hat{E} = \hat{H} - \mu\hat{N}$. The chemical potential is defined by $\mu = \frac{\partial E}{\partial N}$. The mean-field equation of motion for the condensate wave function $\Psi(\vec{r},t) = \psi_{GC0}(\vec{r},t)$ can be derived by varying (3.25) with respect to $\psi_{GC0}(\vec{r},t)$. To obtain (3.24) from this one needs to solve the constraint from the chemical potential and perform the field redefinition $\Psi(\vec{r},t) \to e^{i\mu t}\Psi(\vec{r},t)$. In the following, we drop the subscript differentiating between the canonical and the grand-canonical formalism and keep in mind that one can always change between the descriptions using the above recipe.

For the time-independent condensate ground state, which corresponds to the saddle point of the action (3.25), $\mu$ can be determined by the condition that the linear fluctuations about $\psi_{GC0}(\vec{r})$ vanish, i.e. $\langle \delta\psi_{GC}(\vec{r},t)\rangle = 0$. This leads to the time-independent Gross-Pitaevskii equation [130, 131]

$$\mu\psi_0(\vec{r}) = \left[-\frac{\nabla^2}{2m} + V_{\text{ext}}(\vec{r}) + V_0|\psi_0(\vec{r})|^2\right]\psi_0(\vec{r}) \; . \tag{3.26}$$

For attractive interactions in three dimensions the Bose-Einstein condensate can only be stabilized for a nonzero trapping potential $V_{\text{ext}}$ or within a box of size $L$. Nevertheless, the condensate is only metastable, because the attractive force lowers the interaction energy if the density grows around its center which can eventually overcome the kinetic pressure of its gradient.

### 3.2.1 Bogoliubov Excitations

Once one has obtained the stationary condensate solution $\psi_0(\vec{r})$ of (3.24), one can analyse the effect of quantum fluctuations about this solution. To this end the field is decomposed according to $\Psi(\vec{r},t) = \psi_0(\vec{r}) + \delta\psi(\vec{r},t)$. Separating positive and negative frequency $\omega$ excitations, $\delta\psi(\vec{r},t)$ can be written as

$$\delta\psi(\vec{r},t) = u(\vec{r})e^{-i\omega t} + v^*(\vec{r})e^{i\omega t} \; . \tag{3.27}$$

Inserting this decomposition into (3.25) and using (3.26), one can derive the equations of motion for $u(\vec{r})$ and $v(\vec{r})$

$$\omega u(\vec{r}) = [H_0 - \mu + 2V_0\psi_0^2(\vec{r})]u(\vec{r}) + V_0\psi_0^2(\vec{r})v(\vec{r}) \tag{3.28}$$
$$-\omega v(\vec{r}) = [H_0 - \mu + 2V_0\psi_0^2(\vec{r})]v(\vec{r}) + V_0\psi_0^2(\vec{r})u(\vec{r}) \; , \tag{3.29}$$

where $H_0 = -\frac{\nabla^2}{2m} + V_{\text{ext}}(\vec{r})$. These are the celebrated Bogoliubov-de Gennes equations [132].

In an operator approach they can equivalently be derived by decomposing the fluctuation operator

$$\delta\hat{\psi}(\vec{r}) = \sum_i (u_i(\vec{r})\hat{\alpha}_i(t) + v_i^*(\vec{r})\hat{\alpha}_i^\dagger(t)) \; , \tag{3.30}$$

and deriving the Heisenberg equation of motion (3.24) for $\hat{\Psi}(\vec{r},t) = e^{-i\mu t}(\psi_0(\vec{r}) + \delta\hat{\psi}(\vec{r},t))$. The Hamiltonian for the fluctuations is diagonal in the basis of the Bogoliubov modes $\hat{\alpha}, \hat{\alpha}^\dagger$ where the coefficients $u(\vec{r})$ and $v(\vec{r})$ fulfill (3.28) and (3.29). For standard commutation relations for $\hat{a}_i$ and $\hat{a}_i^\dagger$, the $u(\vec{r})$ and $v(\vec{r})$ are normalized according to

$$\int d^3r \left[u_i(\vec{r})u_j^*(\vec{r}) - v_i^*(\vec{r})v_j(\vec{r})\right] = \delta_{ij} \ . \tag{3.31}$$

For a homogeneous condensate with periodic boundary conditions, the single particle wave functions are plane waves $\psi_\alpha(\vec{r}) = \frac{1}{\sqrt{L^3}}e^{i\vec{k}\vec{r}}$, with $\vec{k} = 2\pi\vec{n}/L$ and $n_i = 0, \pm 1, \pm 2, \ldots$. The ground state wave function is constant, $\psi_0(\vec{r}) = \sqrt{\frac{N}{L^3}} = \psi_0$, and the chemical potential is given by $\mu = V_0|\psi_0|^2$. The spectrum of the Bogoliubov excitations is then

$$\omega_{\vec{k}} = \sqrt{\frac{\vec{k}^2}{2m}\left(\frac{\vec{k}^2}{2m} + 2V_0|\psi_0|^2\right)} \ , \tag{3.32}$$

and $\hat{H} = \sum_{\vec{k}} \omega_{\vec{k}} \hat{\alpha}_{\vec{k}}^\dagger \hat{\alpha}_{\vec{k}}$. Now, one can estimate the number of particles depleted from the condensate into excited states. It is

$$N - N_0 = N' = \langle \delta\hat{\psi}^\dagger(\vec{r})\delta\hat{\psi}(\vec{r})\rangle = \sum_{\vec{k}} |v_{\vec{k}}|^2 \ , \tag{3.33}$$

which gives

$$N' = \sum_{\vec{k}} \frac{1}{2}\left(\frac{\frac{\vec{k}^2}{2m} + V_0|\psi_0|^2}{\omega_{\vec{k}}} - 1\right) \tag{3.34}$$

for a homogeneous condensate with $\omega_{\vec{k}}$ given by (3.32).

For attractive interactions the depletion of a homogeneous Bose-Einstein condensate in a box of size $L^3$ diverges for a certain value of the coupling strength $V_0 N_0 \simeq \frac{L}{m}$. At this point the first excitation becomes gapless and consequently the ground state unstable. In one dimension this value corresponds to the critical point of a quantum phase transition, where the uniform ground state develops into a bright soliton; see for example [21, 133] and references therein. Interestingly, the divergence of the depletion is an artifact of the mean field approximation used in deriving (3.34). It was shown in [133] using an exact diagonalization method that for a one-dimensional Bose gas although the depletion becomes maximal it remains finite at the phase transition point. Additionally the lowest Bogoliubov excitation is gapped of order $\sim N^{-\frac{1}{3}}$ and the Goldstone mode in turn becomes gapless as $\sim 1/N$. The results of mean field theory becomes only exact in the limit $N \to \infty$.

Precisely this is the key point for the black hole correspondence. In (3.10) and below we argued along the lines of [18] that the black hole semi-classical description is only correct up to $1/N$ corrections and that many puzzling properties of black holes thus are due to the inexact description in semi-classical physics [18–20]. The same happens for the critical point of a quantum phase transition. Mean field theory corresponding to a semi-classical treatment cannot capture the correct physics of the phase transition and is always corrected by factors of $\mathcal{O}(1/N)$. If a black hole could indeed be described as a Bose-Einstein condensate at the point of a quantum phase transition, it is clear that quantum fluctuations are important at order $1/N$ instead of the usually assumed exponential suppression $e^{-N}$ [134, 135]. These corrections could provide a mechanism to resolve the information paradox [14] as their accumulated effect tends to be of order one over the lifetime of the black hole.

Inspired by the work [18, 19], the two papers [126, 127] investigated further properties of a one dimensional Bose-Einstein condensate at the critical point. It was shown that quantum correlations are important close to the critical point also for large $N$ [126]. Furthermore it was found [126] that the fluctuation entanglement of modes is peaked at the critical point and is long-range. In [127], it was established that the instabilities at the critical point lead to a logarithmic quantum break time (see references in [127]), i.e. the time it takes to depart $\mathcal{O}(1)$ from mean field theory, which can enhance the effects of the $1/N$ quantum corrections quickly. The quantum break time was connected to the so-called scrambling time [136]. It was argued that if these results could be carried over to the black hole picture, black holes would behave according to the fast scrambling conjecture [136, 137].

### 3.2.2 Black Holes at the critical Point

The energy functional of a localised Bose-Einstein condensate, say of Gaussian form of width $L$, can be approximated by

$$\langle H \rangle = \frac{1}{2m}\frac{N}{L^2} - \frac{|V_0|}{2}\frac{N^2}{L^3}\,, \tag{3.35}$$

using the normalisation of the condensate wave function (3.19). This functional has an extremum at $|V_0|N \simeq L/m$. If the collective potential energy $\sim V_0 N$ is larger than this, the condensate collapses.

The black hole equations (3.15) and (3.16) can be recovered from (3.35) via the replacement

$$m \to \frac{1}{L}\,, \quad \text{and} \quad V_0 \to -\alpha L^2\,. \tag{3.36}$$

Around such a condensate, the excitations are given by (3.32) which in terms of the black hole variables can be written as [19]

$$\omega_{\vec{n}} = \frac{1}{\sqrt{N}L_P}\sqrt{\vec{n}^2(\vec{n}^2 - \alpha N)}\,. \tag{3.37}$$

Here $\vec{n}$ denotes the unit vector of the momentum in $\vec{k}$-direction defined above (3.32). The depletion is determined by (3.33) and yields in the black hole picture [19]

$$N' = \sum_{\vec{n}} \frac{1}{2}\left(\frac{\vec{n}^2 - \frac{1}{2}\alpha N}{\sqrt{\vec{n}^2(\vec{n}^2 - \alpha N)}} - 1\right)\,. \tag{3.38}$$

In this picture the critical point corresponds to $\alpha = 1/N$ which was precisely what was postulated as interaction strength in (3.6). Taking into account $1/N$ corrections, it was shown in [19] that the depletion at the critical point obeys

$$N' \simeq |v_1|^2 \simeq \sqrt{N} \,, \tag{3.39}$$

and the energy gap of the first Bogoliubov mode goes as

$$\omega_1 \simeq \frac{1}{NL_P} \,. \tag{3.40}$$

Since the depletion decreases for higher momentum modes as inverse $\vec{n}^2$, it is sufficient to consider the first mode to gain qualitative insight.

The evaporation law (3.11) can be derived by assuming that the depleted bosons are coupled to the continuum and can leave the condensate. The depletion of a black hole is given by (3.38) and it takes the time $\delta t = \Gamma^{-1}$, see (3.9), to scatter one pair. Therefore to scatter $\sqrt{N}$ gravitons takes $\sqrt{N}\delta t$ and the depletion leads to the same evaporation law as (3.11).

One of the important differences between the black hole condensate and generic cold atom condensates lies within the coupling $\alpha$ and $V_0$. For cold atomic gases, $V_0$ is an external parameter which depends on the interactions of the specific atoms but is independent of the number of bosons $N$. Therefore, the criticality condition $V_0 N \simeq \frac{L}{m}$ can only be achieved for one specific value of $N$ for a given $V_0$.

In gravity, however, the situation is different as $\alpha$ depends precisely in such a way on $N$ that (to leading order) the criticality condition $\alpha N = 1$ can be satisfied for any $N$ [19]. Therefore, a black hole can be understood as a Bose-Einstein condensate at the critical point of a phase transition. It always remains critical as the evaporation of one graviton $N \to N-1$ takes the black hole to another critical condensate by readjusting its size. In this sense, simultaneous depletion and collapse take care of the criticality condition.

The collective, almost gapless quantum excitations of the condensate (3.37) can provide the quantum holographic degrees of freedom which were suggested to be responsible for the black hole entropy in [18]. At the critical point, they become almost gapless. If there exist $N$ such modes which are maximally gapped as $\frac{1}{N}$, there are $N^N$ black hole microstates which are indistinguishable. Therefore, the entropy of a macroscopic black hole described by the number of constituents $N$ scales as $S \simeq N \log N$ [19] and the leading order thus reproduces the Bekenstein entropy $\sim R_S^2/L_P^2$ [12].

## 3.3 A collapsing Condensate as a Black Hole Toy Model

In order to further establish the analogy between the cold atomic system and the graviton condensate, we propose a toy model describing a collapsing Bose-Einstein condensate in three (spatial) dimensions which constantly emits particles during its evolution. In the context of cold atoms such a system has previously been studied, e.g., in [138]. To make a connection with black hole physics, we demand that the Compton wavelength of the bosons is determined by the size of the condensate. Such a system makes it feasible to model a condensate which collapses and simultaneously evaporates while always staying at the critical point. It thus provides a nice playground to study other postulated features of the black hole condensate picture such as the appearance of nearly gapless Bogoliubov modes, which we will compute in the last part of this chapter.

### 3.3.1 Collapsing Bose-Einstein Condensate

In this subsection the equations describing the dynamical evolution of an attractive Bose-Einstein condensate will be introduced. These have been derived in [138] which we will follow closely; see also references therein for earlier work.

The contact interaction in (3.25) is a valid description given by the mean-field Bogoliubov approximation, which is appropriate at low temperatures and considers the interactions to be dominated by two-body s-wave scattering. It does, however, not capture some of the physics that can occur within an inhomogeneous condensate. One aspect is that the incoherent elastic scattering of two particles can lift one of the particles out of the condensate. Notice that such a process is forbidden in a homogeneous condensate by momentum conservation. In an inhomogeneous condensate, however, these kind of scattering events lead to an actual leakage of the condensate.

The rate of the loss of atoms of a condensate generically is given by the imaginary part of the interaction energy [138]

$$\frac{dN_0(t)}{d\,t} = 2\mathrm{Im}[E_{\rm int}(t)]\ . \tag{3.41}$$

Within the Bogoliubov approximation the interaction energy is given by

$$E_{\rm int}(t) = \frac{V_0}{2}\int d^3r|\psi_0(\vec{r},t)|^2\ , \tag{3.42}$$

which is real and thus cannot describe the emission of bosons from the condensate. It can be shown that the two-body interaction matrix $T^{2B} = V_0$ used in the Bogoliubov approximation should be replaced by the many-body interaction matrix $T^{MB}$ which describes also incoherent collisions. The resulting effective action is given by the so-called Caldeira-Leggett model [139] featuring the many-body interaction matrix which acquires an imaginary part [140, 141].

In momentum space, the two-body interaction is

$$E_{\rm int} = \frac{1}{2}\prod_{i=1}^{4}\int\frac{d^3k_i}{(2\pi)^3}\psi_0^*(\vec{k}_4)\psi_0^*(\vec{k}_3)(2\pi)^3\delta(\vec{k}_4+\vec{k}_3-\vec{k}_2-\vec{k}_1)V_0\psi_0(\vec{k}_2)\psi_0(\vec{k}_1)\ , \tag{3.43}$$

whereas the imaginary part of the many-body interaction reads [140, 141]

$$\begin{aligned}&\mathrm{Im}[T^{\rm MB}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)] = \\ &-2\pi V_0^2\int\frac{d^3k}{(2\pi)^3}\delta(\epsilon(\vec{k}_2)+\epsilon(\vec{k}_1)-\epsilon(\vec{k})-\epsilon(\vec{k}_1+\vec{k}_2-\vec{k}))\psi_0^*(\vec{k}_4+\vec{k}_3-\vec{k},t)\psi_0(\vec{k}_1+\vec{k}_2-\vec{k},t)\ ,\end{aligned} \tag{3.44}$$

where $\epsilon(\vec{k}_i) = \frac{\vec{k}_i^2}{2m}$ and $\vec{k}$ is the momentum of the ejected boson. Such a process can be understood diagrammatically by using the optical theorem for the process displayed in Fig. 3.1, i.e. it is the result of an effective three-body interaction which acquires an imaginary part.

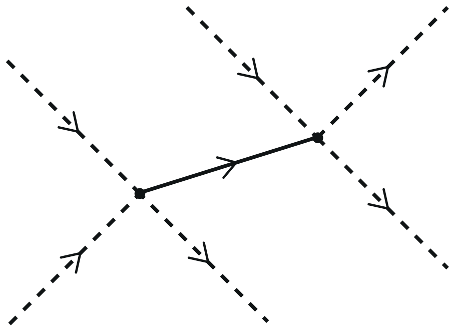

FIGURE 3.1: The effective three-body scattering responsible for the evaporation of the condensate. Using the optical theorem this process describes the incoherent scattering of two condensate atoms (dashed lines) which takes one of them out of the condensate (solid line).

The collapse dynamics of the Bose-Einstein condensate can be described by a variational approach. When considering a harmonic trapping potential of frequency $\omega_0$, a Gaussian wave function with time-dependent width $q(t)$ turns out to provide a good approximation [128, 142, 143]. The ansatz is given by

$$\psi_0(\vec{r}, t) = \sqrt{N_0} \left( \frac{1}{\pi q(t)^2} \right)^{\frac{3}{4}} e^{-\frac{r^2}{2q^2(t)}(1 - i\, m q(t) \dot{q}(t))} , \tag{3.45}$$

where the dot denotes the derivative with respect to $t$. Note that the spatial dependence of the phase ensures the matter flow towards the center of the collapsing cloud. The equations of motion for the variational parameters are obtained from the variation of the energy functional $\langle \hat{H} \rangle$ (3.18) which yields

$$\frac{\delta \langle \hat{H} \rangle}{\delta q(t)} = 0 \quad \Rightarrow \quad m \frac{d^2 q(t)}{dt^2} = -\frac{d}{dq(t)} V(q(t); N_0(t)) , \tag{3.46}$$

where

$$V(q(t), t) = \frac{3}{2mq(t)^2} + \frac{3}{2} m \omega_0^2 q(t)^2 + \sqrt{\frac{2}{\pi}} \frac{V_0 N_0(t)}{mq(t)^3} . \tag{3.47}$$

These equations are then coupled to the equation for the depletion rate of $N_0(t)$ (3.41) which finally yields [138]

$$\frac{dN_0(t)}{dt} = \frac{\sqrt{5}}{8\pi^3} \frac{V_0^2 m N_0(t)^3}{q(t)^4} . \tag{3.48}$$

### 3.3.2 Collapsing Black Hole Toy Model

To establish a connection to black hole physics we make the replacement $m \to L^{-1} = q(t)^{-1}$ (3.36) and investigate the resulting dynamics. Subject to this replacement, the trial wave function (3.45) becomes

$$\psi_0(\vec{r},t) = \sqrt{N_0}\left(\frac{1}{\pi q(t)^2}\right)^{\frac{3}{4}} e^{-\frac{r^2}{2q^2(t)}(1-i\dot{q}(t))} \; . \tag{3.49}$$

The variational energy functional $\langle \hat{H} \rangle$ yields the effective potential (3.47)

$$\frac{dV(q(t),t)}{dq(t)} = -\frac{1}{2q(t)} - \frac{1}{2}\omega_0^2 q(t) - \frac{1}{2}\frac{\dot{q}(t)^2}{q(t)^2} + \frac{1}{\sqrt{2\sqrt{2}\pi^{\frac{3}{2}}}}\frac{|V_0|N_0(t)}{q(t)^3} \; , \tag{3.50}$$

and the depletion rate (3.48) is now

$$\frac{dN_0(t)}{dt} = \frac{\sqrt{5}}{8\pi^3}\frac{V_0^2 N_0(t)^3}{q(t)^5} \; . \tag{3.51}$$

For simplicity, we first consider the untrapped gas $w_0 = 0$. It turns out to be convenient to rescale the parameters such that one is only working with dimensionless quantities. This is achieved by the following rescaling

$$q(t) = \left(\frac{1}{2\pi^2}\right)^{\frac{1}{4}}\sqrt{|V_0|}\,\tilde{q}(t) \; , \quad t = \left(\frac{2}{\pi^3}\right)^{\frac{1}{4}}\sqrt{|V_0|}\,\tilde{t} \; . \tag{3.52}$$

The resulting evolution equations are

$$\begin{aligned} \tilde{q}'' &= -\frac{N}{\tilde{q}} + \frac{1}{\tilde{q}}(1+\frac{\tilde{q}'}{2}) \\ N' &= -\sqrt{\frac{5}{8}}\frac{N^3}{\tilde{q}^5} \; , \end{aligned} \tag{3.53}$$

where the prime denotes the derivative with respect to the rescaled time $\tilde{t}$.

The effective potential can then be obtained from the integration of the dimensionless version of (3.50) which yields

$$V_{\text{eff}}(\tilde{q}) = -\frac{1}{2}\frac{N}{\tilde{q}^2} - (1+\tilde{q}'^2)\text{Log}\tilde{q} \; . \tag{3.54}$$

From this expression (see also Fig. 3.2), it is apparent that without a trapping potential there exists only one extremum given by the maximum at

$$\tilde{q}_{\text{ext}} = \sqrt{\frac{2N}{2+\tilde{q}'^2}} \; . \tag{3.55}$$

To establish the analogy between this collapse and an evaporating black hole, we would like to find a solution for $\tilde{q}(\tilde{t})$ and $N(\tilde{t})$ which behaves as $\tilde{q} \sim \sqrt{N}$. It turns out that such a behaviour can be found from an ansatz where $\tilde{q}'' = 0$. Consider a condensate of intial width $\tilde{q}_0$ which obeys

$$\tilde{q}(\tilde{t}) = \tilde{q}_0 - v\tilde{t} \; , \tag{3.56}$$

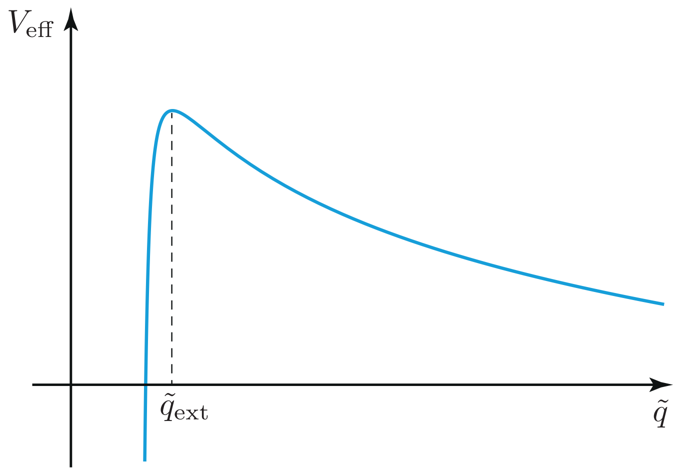


FIGURE 3.2: The effective potential (3.54) is displayed in arbitrary units for a given $N$ and $\tilde{q}'$. The maximum of the potential corresponds to the critical point of the collapse.

with $v = \tilde{q}'$ constituting the velocity of the width. Inserting this ansatz into the first equation in (3.53) and using that $\tilde{q}'' = 0$ yields the algebraic expression for $N(\tilde{t})$ in terms of $\tilde{q}$,

$$N(\tilde{t}) = (1 + \frac{v^2}{2})\tilde{q}(\tilde{t})^2 \ . \tag{3.57}$$

However, to find a consistent expression, $v$ has to solve a constraint which can be obtained from inserting (3.56) and (3.57) into the second equation of (3.53). This generates an algebraic constraint for $v$ which has two solutions

$$v_1 \simeq 0.5 \ , \quad v_2 \simeq 1.24 \ . \tag{3.58}$$

Each of these solutions for $v$ fixes the necessary initial condition on $\tilde{q}_0$ for a given $N_0$

$$\tilde{q}_0 = \sqrt{\frac{2N_0}{2+v^2}} \ . \tag{3.59}$$

In consequence the two solution for $v_1$ or $v_2$ obey $N \sim \tilde{q}^2$ during the collapse. Physically this implies that for both solutions the condensate sits always on maximum of the effective potential (3.55) which characterizes a critical point in three dimensions. We have thus realized a model featuring a self-similar collapse at the critical point. A pictorial representation of such a critical solution is displayed in Fig. 3.3.

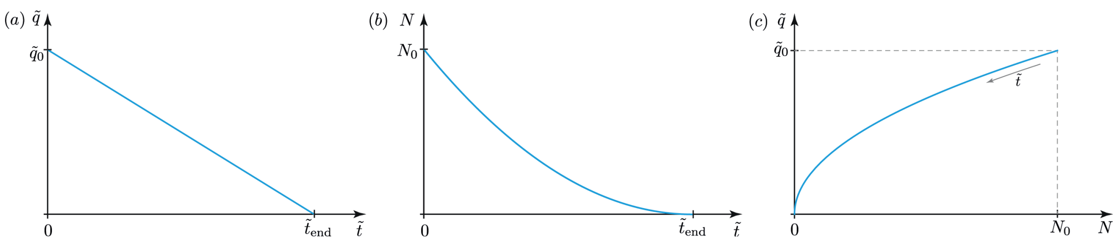


FIGURE 3.3: The critical solution (3.56) and (3.57) for $v_1$ is displayed.

Note, however, the analogy is not perfect since the depletion rate $N'$ of this collapse does not speed up for decreasing $N$ as it would be expected for a black hole (3.11). Instead, it rather slows down for small $N$ according to

$$N' \sim \sqrt{N}\,. \tag{3.60}$$

This also leads to a discrepancy of the lifetime of such a collapsing condensate

$$\tilde{t}_{\mathrm{end}} = \frac{\tilde{q}_0}{v} = \frac{1}{v}\sqrt{\frac{2N_0}{1+2v^2}} \sim \sqrt{N} \tag{3.61}$$

and a black hole which goes as $t_{\mathrm{BH}} \sim M^3 \sim N^{\frac{3}{2}}$.

Considering that our model originally described a non-relativistic contact interaction it does not come as a surprise that we cannot reproduce all the features of black hole dynamics which are given by relativistic long-range interactions. However, it is worth noting that we have achieved to model a self-similar collapse which always stays at the critical point.

Interestingly, even detuning the initial conditions by choosing a $\tilde{q}_0$ too large for the number of particles $N_0$, the evolution will bring the collapse close to the $N \sim \tilde{q}^2$ behaviour at late times, as has been verified numerically [22].

### 3.3.3 Trapped Collapse

So far we neglected the fact that the interactions in gravity are long-range which can lead to self-trapping of matter. Obviously such an effect is not present in a system with short-range interactions. As it turns out, however, the introduction of an external trapping potential allows to mimic the effect of self-trapping and thus to further the analogy between the black hole and the condensates.

Including a harmonic trap with frequency $\omega_0$, the effective potential (3.50) (see Fig. 3.4) possesses now two extrema, one maximum and a new minimum, where the latter corresponds to a metastable solution. The solutions for the position of the maximum and the minimum depend on the ratio of the self-interaction strength $V_0$ to the trapping frequency $\omega_0$ and the number of particles $N$ characterized by the dimensionless parameter

$$\gamma = \frac{N V_0 m^{\frac{3}{2}}}{(4\pi\sqrt{\omega_0})} \tag{3.62}$$

for the nonrelativistic condensate. Increasing the magnitude of $\gamma$ (for an attractive condensate it is negative), one finds that at a critical coupling $\gamma_{\mathrm{cr}}$ the two solutions coincide. After this point there exist no stable solutions and the condensate will always collapse. Such a point is called a bifurcation (see Fig. 3.5). Note that there are indications (see [127] and references therein) that a bifurcation point is essential in obtaining the light modes which are thought to be responsible for the black hole entropy and other quantum properties such as fast scrambling.

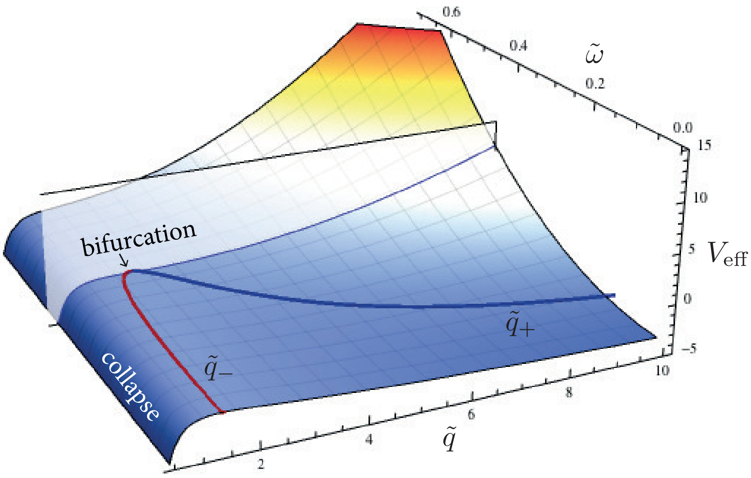


FIGURE 3.4: The effective potential (3.64) is displayed in arbitrary units for a given $N$ and $\tilde{q}'$ in dependence on the width of the condensate $\tilde{q}$ and the trapping frequency $\tilde{\omega}$. Increasing $\tilde{\omega}$ the two extrema as indicated by $\tilde{q}_-$ and $\tilde{q}_+$ merge into a saddle point (bifurcation point). A further increase in $\omega$ leaves no real static solution and the condensate is always unstable.

Starting from (3.50) we follow the same steps as in the previous section to find the evolution of the condensate. While the equation of motion for $N$ (3.51) is unchanged, the evolution of $q$ is now determined by (3.50) with $w \neq 0$. Rescaling the variables according to

$$q = \frac{1}{\omega_c}\tilde{q}\ , \quad N = \frac{\sqrt{2\pi^3}}{|V_0|\omega_c^2}\tilde{N}\ , \quad \omega = \omega_c\tilde{\omega}\ , \quad t = \frac{\sqrt{2}}{\omega_c}\tilde{t}\ , \tag{3.63}$$

with $\omega_c$ representing some arbitrary scale yields the effective potential

$$V_{\rm eff}(\tilde{q}) = -\frac{1}{2}\frac{\tilde{N}}{\tilde{q}^2} - (1 + \frac{\tilde{q}'^2}{2})\mathrm{Log}\tilde{q} + \frac{1}{2}\tilde{\omega}^2\tilde{q}^2\ . \tag{3.64}$$

The local extrema are now given by

$$\tilde{q}_\pm = \frac{1}{2\tilde{\omega}}\sqrt{2 + \tilde{q}'^2 \pm \sqrt{(2+\tilde{q}'^2)^2 - 16\tilde{w}^2\tilde{N}}}\ . \tag{3.65}$$

There exist two real solutions for $\tilde{N} > (\frac{2+\tilde{q}'^2}{4\tilde{w}})^2$, while for $\tilde{N} < (\frac{2+\tilde{q}'^2}{4\tilde{w}})^2$ there are no real solutions. At

$$\tilde{N} = (\frac{2+\tilde{q}'^2}{4\tilde{w}})^2 \tag{3.66}$$

the two solutions coincide defining the aforementioned bifurcation point. Solutions with $\tilde{q}'' = 0$ (which guarantees that $\tilde{q}(\tilde{t})$ is a monotonously decreasing function) are then of the generic form

$$\begin{aligned}
\tilde{q}(\tilde{t}) &= \tilde{q}_0 - v\tilde{t}\ ,\\
\tilde{N}(\tilde{t}) &= \frac{2\tilde{q}_0\tilde{N}_0}{f(\tilde{t})}\tilde{q}^2\ ,\\
\tilde{\omega}(\tilde{t}) &= \frac{1}{\tilde{q}(\tilde{t})}\sqrt{1 + \frac{v^2}{2} - \frac{4\tilde{N}_0\tilde{q}_0^2}{\sqrt{2f(\tilde{t})}}}\ ,
\end{aligned} \tag{3.67}$$

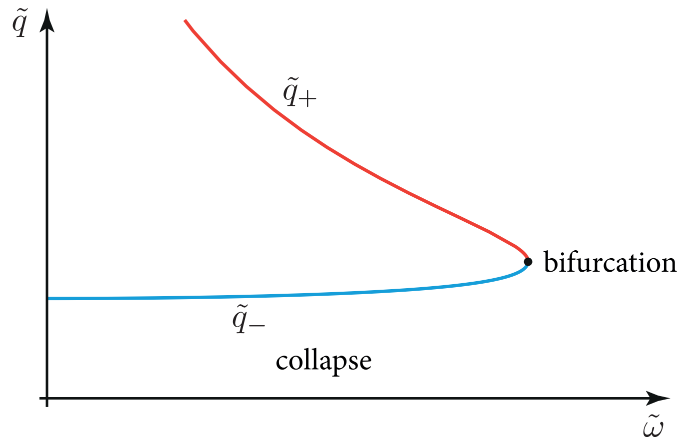


FIGURE 3.5: Solutions to the effective potential for the condensate as a function of $\tilde{\omega}$ for fixed $\tilde{N}$ and $\tilde{q}'$. At a critical value of $\tilde{\omega}$ the two solutions merge into a tangent bifurcation. After this point there exist no real stable solutions.

where $f(\tilde{t})$ is a fourth order polynomial in $\tilde{t}$ which is obtained by inserting the ansatz for $\tilde{q}$ into the equation of motion for $\tilde{N}$. Our goal is to find solutions representing a collapse which continuously stay at the bifurcation point. To realize such solutions we need to make the dimensionless trap frequency $\tilde{\omega}$ time-dependent. Keeping in mind that we are merely modelling a self-trapping potential with an external trap it is not surprising that the trap has to change during the collapse.

The condition that the collapse takes place at the bifurcation is equivalent to demanding

$$\left.\frac{d^2V_{\text{eff}}}{d\tilde{q}^2}\right|_{\text{sol}} = 0 \; . \tag{3.68}$$

In order to obtain a consistent solution the collapse velocity $v$ has to obey

$$v_1 \simeq 0.1 \; , \qquad v_2 \simeq 3 \; . \tag{3.69}$$

The resulting solutions for the three paramters are

$$\tilde{q}_1 \simeq 1.41\sqrt{\tilde{N}_0} - 0.1\tilde{t} \quad \tilde{N}_1 \simeq 0.5\tilde{q}_1^2 \quad \tilde{\omega}_1 = \frac{0.71}{\tilde{q}_1} \tag{3.70}$$

$$\tilde{q}_2 \simeq 0.6\sqrt{\tilde{N}_0} - 3\tilde{t} \quad \tilde{N}_2 \simeq 2.76\tilde{q}_2^2 \quad \tilde{\omega}_2 = \frac{1.66}{\tilde{q}_2} \; . \tag{3.71}$$

Note that these solutions (see Fig. 3.6) still have the property that $\tilde{q} \sim \sqrt{\tilde{N}}$. All the properties, cf. (3.60) and (3.61) we found in the previous section are preserved in this model.

### 3.3.4 Excitation Spectrum

It has been argued in [19] that the collective excitations about the condensate ground state could be responsible for the entropy of a black hole. To investigate this conjecture

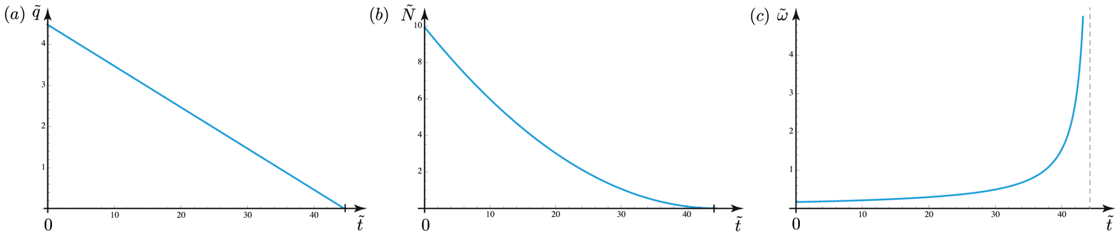


FIGURE 3.6: The critical solution (3.71) is displayed.

further, we make use of the cold atoms analogy and analyse fluctuations about the collapsing solution. The spectrum is obtained from the Bogoliubov-de Gennes equations (3.32) which we solve in three dimensions for a spherically symmetric condensate.

The collapsing solution corresponds to the condensate sitting at the maximum of the effective potential (3.54) or the bifurcation point (3.64) with condensate width $q_{\text{ext}}$. While we have argued that due to self-trapping because of long-range effects a more appropriate black hole toy model can be found by introducing a variable trapping potential 3.3.3, such a setup also provides a better controlled numerical implementation as one can approach the critical point from a metastable solution.

Let us the investigate the excitations about such a solution. We will assume that the collapse occurs quasi-stationary and therefore, we can at each time-step treat the Gaussian condensate wave function with a given $q$, $N$ and $\omega$ as a stationary solution. The excitation spectrum can be determined by the Bogoliubov-de Gennes equations (3.28) and (3.29). The Hamiltonian and the condensate are spherically symmetric. The Bogoliubov modes, therefore, can be decomposed into spherical harmonics $Y_{lm}$ according to

$$\begin{aligned} u(\vec{r}) &= \sum_{lm} Y_{lm}(\theta,\varphi)u_l(r) \ , \\ v(\vec{r}) &= \sum_{lm} Y_{lm}(\theta,\varphi)v_l(r) \ . \end{aligned} \tag{3.72}$$

Carefully evaluating the expressions (3.28) and (3.29), one is left with an effective one-dimensional problem which can be solved by numerically diagonalizing the equations. The numerical implementation of the correct boundary conditions is simplified by the use of the rescaled Bogoliubov modes $u_l(r) = \tilde{u}_l/r$ and $v_l(r) = \tilde{v}_l/r$. We then obtain a system of coupled equations which can be written in terms of the matrix equation

$$\begin{pmatrix} \tilde{H}_{0l} - \mu + 2V_0\psi_0^2 & V_0\psi_0^2 \\ -V_0\psi_0^{*2} & -\tilde{H}_{0l} + \mu - 2V_0\psi_0^2 \end{pmatrix} \begin{pmatrix} \tilde{u}_l \\ \tilde{v}_l \end{pmatrix} = \omega_{nl} \begin{pmatrix} \tilde{u}_l \\ \tilde{v}_l \end{pmatrix} \ , \tag{3.73}$$

where we suppressed the dependence on the radial coordinate. The Hamiltonian which acts on $\tilde{u}_l$ and $\tilde{v}_l$ is given by

$$\tilde{H}_{0l} = -\frac{\partial_r^2}{2m} + \frac{1}{2}m\omega^2 r^2 + \frac{l(l+1)}{r^2} \ . \tag{3.74}$$

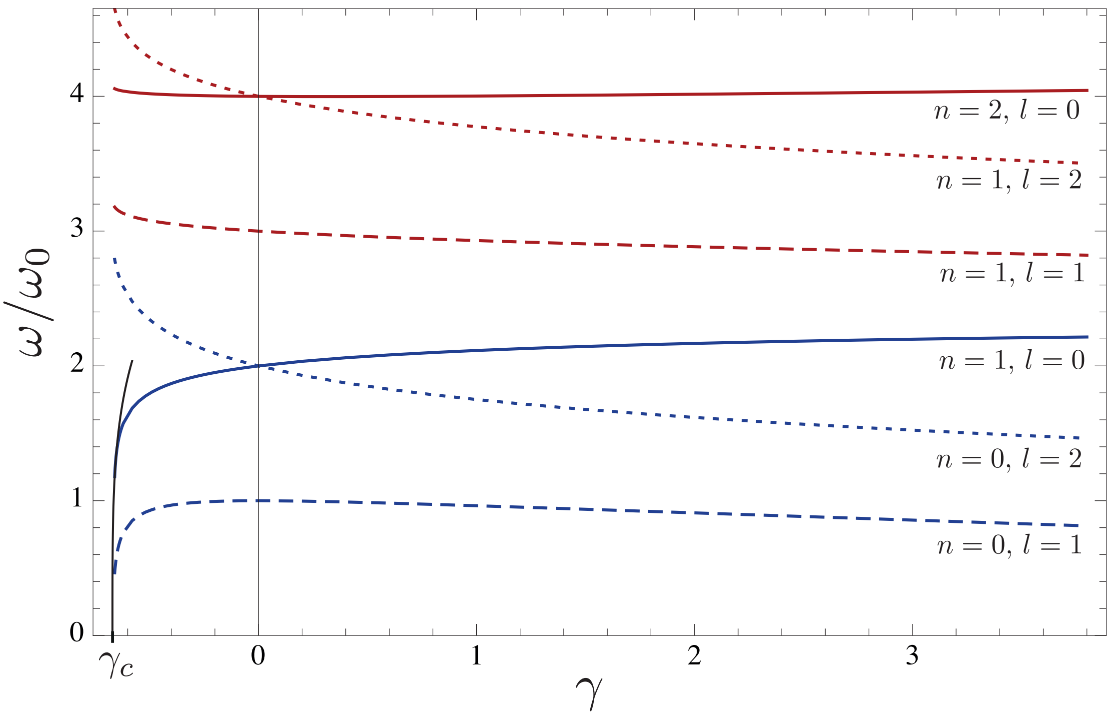


FIGURE 3.7: The Bogoliubov spectrum $\omega$ for a Bose-Einstein condensate in a spherically symmetric harmonic trap is shown in units of the trapping frequency $\omega_0$ as a function of the dimensionless parameter $\gamma$. At the bifurcation point $\gamma_{\text{crit}}$, the $n = 1, l = 0$ excitation becomes gapless and the condensate develops an instability.

We use a straightforward linear discretization of the radial coordinate dependence where the numerical grid has a finite extent that is much larger than the ground state width of the harmonic trapping potential, $r_{\text{max}} \gg 1/\sqrt{m\omega}$. The differential operator $\partial_r^2$ is implemented using a finite difference method. The boundary conditions for the Bogoliubov modes are given by $\tilde{u}(r)|_{r=0} = 0$ and $\tilde{u}(r)|_{r\to\infty} = 0$ and similar for $\tilde{v}$. Note that it is important to correctly implement the (anti) symmetries of $u(r)$ and $v(r)$ around $r = 0$ for even (odd) $l$ as appropriate for the three dimensional problem.

As a check of our numerical procedure, we first compute the spectrum for a spherically symmetric nonrelativistic gas of bosons with mass $m$ subject to an external harmonic trapping potential and compare it to the literature [144, 145]. In order to determine the ground state wavefunction for a given interaction strength $V_0$ we used the variational ansatz (3.45) with the condensate width $q$ as a variational parameter and determine the energy functional $E(q) \equiv \langle \hat{H} \rangle$ given by (3.18). Above a critical interaction strength $\gamma > \gamma_{\text{crit}}$, the variational energy $E(q)$ always features a local minimum at finite $q$ corresponding to a metastable condensate, see Fig. 3.4. This local minimum becomes arbitrary shallow as the bifurcation point is approached. It describes a stable state without matter flow and thus there is no spatial dependence of the phase and one should set $\dot{q} = 0$. Finally, the chemical potential is determined by $\mu \simeq \langle \hat{H} \rangle / N$. This sets the stage for solving the Bogoliubov-de Gennes equations (3.73).

In Fig. 3.7 we show the first excitation energies for the lowest angular momentum modes $l = 0, 1, 2$ as a function of the dimensionless parameter $\gamma$ defined in (3.62) in units of the trapping frequency denoted by $\omega_0$. For $\gamma = 0$, there is no inter-particle interaction, and the Gaussian wave function is the exact solution and accordingly the excitation energies are integer multiples of the trapping frequency $\omega_0$. The $n = 0, l = 1$ mode with its p-wave symmetry is the so-called sloshing mode which exists no matter the interaction strength. It describes the oscillation of the condensate cloud as a whole in the harmonic trap and thus is always determined by the trapping frequency. Deviations from this exact statement in our excitation spectrum are due to the variational wave function not being an exact ground state for nonzero self-interactions $\gamma \neq 0$. The $n = 0, l = 0$ mode in turn is the Goldstone mode of the broken global $U(1)$ phase symmetry of the condensate wave function and is always gapless.

The bifurcation point of the energy functional is given in the variational approach with one Gaussian trial wave function by $\gamma_{\text{crit}} = 0.67$ [144, 145]. Approaching this point the Bogoliubov approximation breaks down which is signalled by the softening of the $n = 1, l = 0$ mode which physically represents the so-called breathing mode of the condensate. It has been shown [144, 145] that within the Bogoliubov approximation the gap vanishes according to a power law $\omega_{10} \sim (\gamma - \gamma_{\text{crit}})^{\xi}$ with the mean field critical exponent $\xi = 1/4$. From the excitation energies close to the critical point we are able to recover this power law with reasonable accuracy which attests the strength and stability of our numerical method. Note that we are, however, not able to follow the excitation spectrum exactly up to the point where the Bogolubov modes become gapless. This is due to our expanding about a variationally obtained ground state which is not the exact ground state of the system. Furthermore, our code could be improved by using more grid points but for computational feasibility we shall be satisfied with the above outcome.

Confident that our code gives us the qualitatively correct Bogoliubov spectrum, we can now turn our attention to the toy model for the collapse of a black hole. In order to do so, we will make the replacement $m \to 1/q$ in (3.74) and in the energy functional $\langle \hat{H} \rangle$ given by (3.18). Furthermore, $\gamma$ has to be adjusted to the black hole toy model which yields

$$\gamma = \frac{V_0 N \omega_0^2}{(2\pi)^{\frac{3}{2}}} \, . \tag{3.75}$$

The collapsing condensate solutions (3.70) and (3.71) are tuned such that they are always at the bifurcation point of the energy functional. For each solution (3.70) and (3.71), therefore, $\gamma(t) = \gamma_{\text{crit}}$ is constant during the collapse and given by

$$\gamma_{\text{crit1}} \simeq -0.126 \, , \quad \gamma_{\text{crit2}} = -3.8 \, . \tag{3.76}$$

In order to obtain these values, one has to go back to the dimensionful quantities $q$, $w$ and $N$.

It is thus sufficient to compute the spectrum once for $\gamma_{\text{crit}}$, because it will remain the same during the collapse within the Bogoliubov approximation. Exactly this self-similar behaviour of the Bogolibov modes is expected for the black hole as argued in section 3.2.1. Numerically we compute the Bogoliubov spectrum for $\gamma > \gamma_{\text{crit}}$ and approach the critical point from this side. This allows us to gain qualitative insight into the

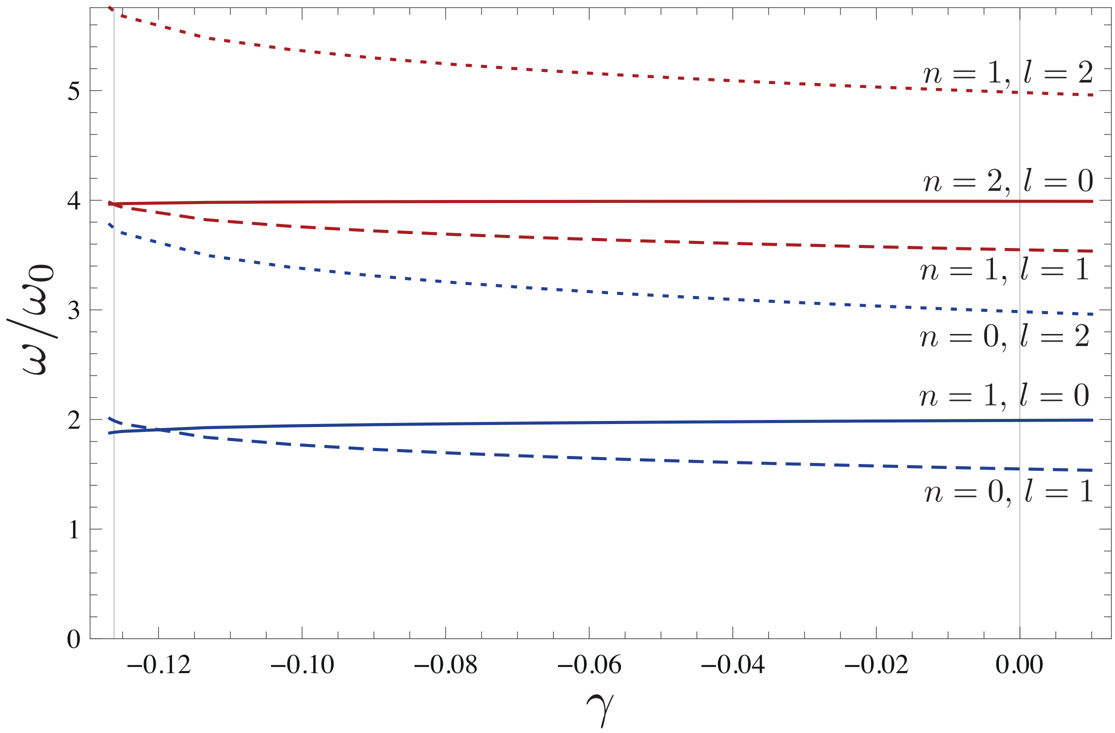


FIGURE 3.8: The Bogoliubov spectrum $\omega$ for the collapsing condensate solution (3.70) in a spherically symmetric harmonic trap is shown in units of the trapping frequency $\omega_0$ as a function of the dimensionless parameter $\gamma$. The $n = 1, l = 0$ mode becomes lighter as one approaches the critical $\gamma$. Due to the rather poor variational ground state approximation, it does not become gapless.

behaviour of the Bogoliubov modes when approaching the critical point. As before, above the critical interaction strength defined by $\gamma$ the variational energy, $E(q)$ has one minimum at $q_{\text{min}}$. This determines the variational ground state wave function around which we expand. In contrast to before, we choose the phase velocity $\dot{q}$ such that it coincides with the value given for the respective critical collapse solution $v_1$ or $v_2$ (3.69). This choice ensures that for $\gamma_{\text{crit}}$ we arrive at the wave function determined by the solutions (3.70) and (3.71).

The results for $v_1 = 0.1$ are displayed in Fig. 3.8. The spectrum has the same qualitative features as the one obtained for a trapped nonrelativistic Bose gas shown in Fig. 3.7. There are, however, differences. For instance, the Bogoliubov excitations without interactions are no longer integer multiples of the trapping frequencies. This is not surprising as in our ansatz the mass is replaced by the inverse size of the condensate and thus the Hamiltonian is given by

$$H = -\frac{q\nabla^2}{2} + \frac{1}{2}\frac{1}{q}\omega^2 r^2 + V_0\psi_0^2 \,. \tag{3.77}$$

This Hamiltonian, for $V_0 = 0$, has a ground state solution which is given by a Gaussian wavefunction (3.45) of width $\bar{q} = \sqrt{q/\omega}$, not $q$. The variational wave function (3.45) is thus even for zero interaction not the exact ground state of the system. As a consequence the Bogoliubov modes are no longer integer multiple frequencies of the trapping frequency $\omega_0$. For all modes apart from the sloshing mode $n = 0, l = 1$ this effect is, however, not very strong. Turning on self-interactions of the bosons, the trial wave function (3.45) becomes less accurate and the quantitative spectrum less accurate. Nonetheless, the qualitative features of the excitations are still visible. Approaching the critical collapse solution $\gamma_{\rm crit1}$, the breathing mode $n = 1, l = 0$ becomes light whereas the others start to become more gapped. We expect that by using a more appropriate trial wave function for the ground state, one recovers the gapless breathing mode at the critical point similar to the case of a trapped spherically symmetric Bose-Einstein gas.

## 3.4 Summary

In this chapter, we considered certain aspects of the black hole condensate picture proposed in [18]. By relying on a simple model borrowed from cold atomic physics, we were able to reproduce specific features expected from the black hole picture. In particular, we considered a condensate which due to incoherent scattering loses particles while it collapses. It was shown that it is possible to find a solution for the time evolution of the condensate such that it always stays at the critical point. In order to model the self-trapping behaviour of gravitons, a time-dependent trapping potential was introduced and we showed that, within the limits of our approximation, at the critical point a light mode appears.

# Chapter 4

# Gravitational Axion Anomaly and $\eta'$ bubbles

It is a long held believe that gravitational interactions intrinsically violate global symmetries [24]. The simplest argument comes from considerations about black holes. Due to the no-hair theorem [25], black holes can only carry gauge charges. Therefore, it is argued that by throwing a global charge into a black hole, it can be removed from our universe. At present, the exact form of the symmetry violating operators is unknown, however, it can be argued that their coupling strengths need to be very small if they are not to interfere with known global symmetries. One context in which a global symmetry, or better a Goldstone boson of a global symmetry breaking, arises is the axion solution to the strong CP problem. The symmetry breaking by quantum gravity corrections could potentially lead to the loss of viability of the axion solution. One can, however, envision a scenario in which the gravitational anomaly is coupled to an extra $U(1)$ symmetry which in turn could soften the constraints on the smallness of the gravitational anomaly couplings.

The strong CP problem is closely related to the nontrivial vacuum structure of QCD. Due to non-perturbative effects, the vacuum energy becomes dependent on an angular parameter [34]. Witten has argued in [49] that the vacuum energy is a multivalued function with nonanalyticities around $\theta = \pi$. In addition, it is known that by the presence of massless quarks this $\theta$-dependence is screened and that for small masses $m_q$ the $\theta$-dependence vanishes linearly in $m_q$. By treating the screening of the vacuum energy as a result of Schwinger pair creation (to be precise, by bubble nucleation), we gain insight into the structure of the vacuum energy.

In section 4.1, we discuss the CP problem. The solution for it in terms of the Peccei Quinn axion is presented in section 4.2. In section 4.3, we reformulate the CP problem and the axion solution in the dual three-form language. Section 4.4 discusses how the aforementioned gravitational operators can be parametrized in the dual description and a resolution of the problem of quantum gravitational effects on the axion solution is presented. In the absence of other influences this allows us the possibility to derive a bound on neutrino masses. In the last section 4.5, we briefly consider the potential non-analyticity of the QCD vacuum energy by investigating the screening in the presence of a light quark in terms of an analogue to Schwinger pair creation.

## 4.1 The strong CP Problem

### 4.1.1 Axial Anomaly

The theory of strong interactions is described by the QCD Lagrangian which takes the following form

$$\mathcal{L} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{\mu\nu\, a} + \sum_{f=1}^{3} i\bar{\psi}_f \not{\mathcal{D}} \psi_f + \sum_f m_f \bar{\psi}_f \psi_f \ , \tag{4.1}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu$ is the field strength of the gluon field $A^a_\mu$ and $g$ is the strong coupling constant. The superscript $a$ denotes the gauge group index. The structure constant of the gauge group $SU(3)$, $f^{abc}$, is defined by $[T^a, T^b] \equiv i f^{abc} T^c$, where the $T^a$s are the generators of $SU(3)$. Quarks are represented by Dirac spinors $\psi_f$, where $f$ denotes the flavour, and $m_f$ denotes their masses. The covariant derivative is given by $\mathcal{D}_\mu = \partial_\mu - i A^a_\mu T^a$ and $\not{\mathcal{D}} = \mathcal{D}_\mu \gamma^\mu$ with the Dirac gamma matrices $\gamma^\mu$. The Lagrangian is invariant under the gauge transformations $A_\mu \to U A_\mu A^{-1} + iU\partial_\mu U^{-1}$, where $U = \exp i\Lambda^a T^a$ is an element of the gauge group $SU(3)$.

In the chiral limit $m_f \to 0$, assuming there are three flavours of quarks,[1] the theory (4.1) has the following (classically) conserved currents

$$\begin{aligned} j^\mu &= \bar{\psi}\gamma^\mu \bar{\psi} \ , \qquad j^{\mu a} = \bar{\psi}\gamma^\mu \tau^a \psi \ , \\ j^{\mu 5} &= \bar{\psi}\gamma^\mu \gamma^5 \bar{\psi} \ , \quad j^{\mu 5 a} = \bar{\psi}\gamma^\mu \gamma^5 \tau^a \psi \ , \end{aligned} \tag{4.2}$$

where the $\tau^a$ are the generators of the flavour $SU(3)$ and $\psi$ is a column of $\psi_f$ and the $\tau^a$ act on these. The conservation law tells us $\partial_\mu j^{\mu i} = 0$.

Actually, however, only the vector currents $j^\mu$ and $j^{\mu a}$ are conserved and correspond to symmetries of the strong interactions, namely, the baryon number symmetry associated with $j^\mu$ and the isospin symmetry $j^{\mu a}$. The axial symmetries $j^{\mu 5}$ and $j^{\mu 5 a}$ do not have any observed counterpart in the strong interactions. In the chiral limit, the latter is spontaneously broken by the formation of quark-antiquark condensates with $\langle \bar{\psi}\psi \rangle \neq 0$. As with any spontaneously broken global symmetry, there is a massless Goldstone boson associated with each broken symmetry generator. In nature, however, the masses of the three lightest quarks are not zero but small. Therefore, there should exist $3^2 - 1 = 8$ light pseudo-Goldstone bosons in the spectrum from the breaking of $j^{\mu 5 a}$. These are

[1] In QCD there exist three light quark flavours, the u, d and s quark. Therefore, the limit of three massless quark flavours is a viable first approximation albeit a very crude one. The symmetry between these three flavours which approximately persists for nonzero masses is the foundation of the eight fold way [146].

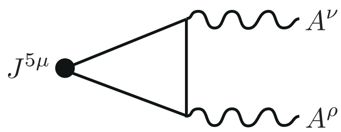


FIGURE 4.1: The triangle diagram which gives an anomalous contribution to the axial current $J_5^\mu$.

the $\pi$, $K$ and $\eta$ mesons. In principle, one would expect that the same is true for the singlet axial $U_A(1)$ symmetry with current $j^{\mu 5}$ requiring the existence of a ninth ($SU(3)$ flavour singlet) light Goldstone boson of a mass close to the pion mass $m_\pi \simeq 135\,\text{MeV}$. Such a particle, however, has not been observed in the spectrum. Instead, there is a flavour singlet meson of mass $m_{\eta'} \simeq 1\text{GeV}$. In the early 1970's, this problem was dubbed the $U_A(1)$ problem [147]. Somehow QCD cannot possess a $U_A(1)$ symmetry on the full quantum level.

Indeed, that the current $j^{\mu 5}$ is anomalous as was first shown by [148–150] by the analysis of the triangle diagram shown in Fig. 4.1 which leads to a violation of the $U(1)_A$ symmetry: the Adler-Bell-Jackiw anomaly [148–150]. It gives a nontrivial contribution to the divergence of the axial current $j^{\mu 5}$ (4.2) and yields

$$\partial_\mu j^{\mu 5} = \frac{g^2 N_f}{32\pi^2} F^{\mu\nu a} \tilde{F}^a_{\mu\nu} \, , \tag{4.3}$$

where $\tilde{F}^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta\, a}$ is the dual field strength, $N_f$ is the number of flavours. According to Noether's theorem, this term induces a change in the action (4.1) under the symmetry transformation

$$\psi_f \to e^{i\gamma^5 \alpha} \psi_f \quad \text{and} \quad \bar{\psi}_\mathrm{f} \to \bar{\psi}_\mathrm{f} \mathrm{e}^{\mathrm{i}\gamma^5 \alpha} \, , \tag{4.4}$$

which for infinitesimal $\alpha$ is given by

$$\delta S = \alpha \int d^4 x \partial_\mu j^{\mu 5} = \int d^4 x \alpha \frac{g^2 N_f}{32\pi^2} F^{\mu\nu a} \tilde{F}^a_{\mu\nu} \, . \tag{4.5}$$

The 1-loop anomaly contribution (4.3) is actually exact [151, 152], as can be shown by deriving the anomaly in the path integral formalism. In this formulation, (4.3) shows up, because the fermionic measure $\mathcal{D}\psi\mathcal{D}\bar{\psi}$ in the path integral $Z = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A_\mu e^{iS}$ is not invariant under (4.4) [152].

Naively, one would think that (4.5) is still equal to zero as the term $F^{\mu\nu\, a}\tilde{F}^a_{\mu\nu} = \partial_\mu K^\mu$ with $K_\mu = \epsilon^{\mu\alpha\beta\gamma} A^a_\alpha (F^a_{\beta\gamma} - \frac{g}{3} f^{abc} A^b_\beta A^c_\gamma)$ is a total derivative [153]. However, 't Hooft showed [154] that instanton configurations can provide non-zero contributions to the integral

$$\int d^4 x \frac{g^2 N_f}{32\pi^2} F^{\mu\nu a} \tilde{F}^a_{\mu\nu} = \nu \, . \tag{4.6}$$

Here, $\nu$ is the topological charge [155] which takes integer values for finite action field configurations such as instantons. Later, Witten argued in [49] that the original derivation by 't Hooft [154] cannot be the complete story. For example, the boundary conditions imposed on $A^a_\mu$ being a pure gauge term $A^a_\mu \sim U\partial_\mu U^{-1}$ is not appropriate for a strongly-interacting theory. We will discuss this issue in more detail in section 4.5, but we want to ascertain that there exist non-zero contributions to (4.3) [50, 156] such that the $U_A(1)$ symmetry is anomalous in QCD.

To summarize, due to the axial anomaly which is signalled by (4.3) having non-zero matrix elements at zero momentum, there is no $U_A(1)$ symmetry in the strong interactions and therefore no Goldstone boson is expected in the chiral limit.

### 4.1.2 Strong CP Problem

With the resolution of the $U(1)_A$ problem another one appeared. The same instantons that are responsible for the $U(1)_A$ not being a symmetry also induce a nontrivial vacuum structure in QCD. There exists a continuous set of vacua labelled by the real parameter $\theta$, which is periodic $0 \leq \theta \leq 2\pi$. For an introduction into the subject, see for example [157]. This $\theta$-dependence is reflected by an additional term appearing in the path integral of pure Yang-Mills theory which yields

$$Z_\theta(J) = \int \mathcal{D}A \exp \int d^4x \left[ -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a} - \frac{g^2\theta}{32\pi^2} F^{\mu\nu a} \tilde{F}^a_{\mu\nu} + J^{\mu a} A^a_\mu \right] . \tag{4.7}$$

Most importantly, the extra term is responsible for the strong interactions to violate the combined CP symmetry of parity and charge conjugation. CP violation of these interactions comes with a strength proportional to the vacuum angle $\theta$. The effect of the CP-violation can, for instance, be characterized by the ratio [44]

$$\frac{\langle g^2 F^{\mu\nu a} \tilde{F}^a_{\mu\nu} \rangle}{\langle g^2 F^{\mu\nu a} F^a_{\mu\nu} \rangle} . \tag{4.8}$$

On the experimental side, the most stringent bound on CP violating effects in QCD arises from the neutron electric dipole moment $d_n$, which is constrained experimentally by $|d_n| < 3 \times 10^{-26}\mathrm{e\,cm}$ [33]. The electric dipole moment of the neutron, in turn, is connected to the vacuum angle $\theta$ via (4.8) as $d_n \sim m_q\theta$. Hence, in order to comply with the experimental bounds, $\theta$ has to be very small, $\theta < 10^{-9}$ [44, 158]. Taking into account the wide range of $0 \leq \theta < 2\pi$ which $\theta$ could a priori take, naturally the question arises why the value of $\theta$ in the vacuum we live in is so small. This is called the *strong CP problem.*

Nevertheless, given certain circumstances $\theta$ can be an intrinsically unobservable parameter and the strong interactions then conserve CP. This is, for example, the case if there exist massless fermions in the theory as discussed in the previous section 4.1.1. Because of the axial anomaly, the chiral transformation (4.4) then induces a change of the $\theta$ vacuum due to the anomaly [159]. The $\theta$-parameter is given by

$$\theta \to \theta' = \theta - N_f \alpha . \tag{4.9}$$

In consequence, in a theory with massless quarks the $\theta$-dependence is rendered unobservable and CP is not violated by the strong interactions.

Standard current algebra results, however, strongly disfavour zero masses for the lightest quark (the u-quark) [160–162] even though experimentally it is not yet ruled out that $m_u = 0$ [163]. The mass term in (4.1) (cf. [164]),

$$\mathcal{L}_m = \sum_f |m_f| \bar{\psi}_f e^{-i\phi_f \gamma^5} \psi_f, \tag{4.10}$$

has in general complex quark masses $m_f = |m_f| e^{i\phi_f}$. The phase $\phi_f$ of the complex mass can be removed by a redefinition of the fields $\psi_f$ in the path integral. This is equivalent to a chiral transformation (4.4) with parameter $\alpha = \phi_f/2$ on the path integral which shifts the vacuum angle (4.9) to

$$\theta' = \theta - \sum_f \frac{1}{2}\phi_f = \theta - \arg\det \mathrm{M}\ , \tag{4.11}$$

where $\det M$ is the determinant of the quark mass matrix. The angle $\theta'$ can be shifted between the quark masses and the term $F\tilde{F}$ but it cannot be removed completely. For this reason $\theta$ remains an observable parameter. For a nonzero value, it renders the strong interactions CP-violating and thus one faces again the strong CP problem.

Interestingly, the above presentation already reveals one solution to the CP problem, namely, having at least one massless quark in the theory, for example $m_u = 0$. In this case, $\det M = 0$ and the quark mass terms become invariant under the chiral transformation (4.4) of the u-quark allowing the angle $\theta'$ to be rotated away. However, it needs rather convoluted theories in order to accommodate a zero mass u-quark, see e.g. [160–162].

## 4.2 Introducing the Axion

The lesson learned from the previous section is that whenever there is a global chiral $U(1)$ symmetry (4.4) which is explicitly broken by color axial anomalies at the quantum level, the $\theta$-parameter[2] becomes unobservable. In [31], Peccei and Quinn followed this line of reasoning and found a dynamic solution to the strong CP problem. They constructed a Lagrangian which is invariant under an additional $U(1)_{PQ}$ by introducing two instead of one Higgs field into the Standard Model. This $U(1)_{PQ}$ symmetry is spontaneously broken at a scale $f_a$ and as a result a Goldstone boson, the so-called axion $a(x)$, appears in the theory. As a matter of fact, the axion will not be massless due to the explicit breaking of $U(1)_{PQ}$ by the axial anomaly. Being the Goldstone boson of the broken $U(1)_{PQ}$, the axion transforms according to

$$a(x) \to a(x) + \alpha f_a\ , \tag{4.12}$$

[2] In the following we assume that we have performed a chiral rotation such that the quark masses are real and all the CP violating phases have been shifted into the coefficient of $F\tilde{F}$.

where the symmetry breaking scale $f_a$ is given by the vacuum expectation value of the corresponding complex scalar (Higgs) field.

From an effective field theory point of view, it is not necessary to specify the underlying dynamical origin of the axion field in order to solve the strong CP problem. The only requirement is that in the low energy approximation the Lagrangian of the axion has to reduce to the following form

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu a}F^a_{\mu\nu} + \theta\frac{g^2}{32\pi^2}F^{\mu\nu a}\tilde{F}^a_{\mu\nu} - \frac{1}{2}\partial_\mu a\partial^\mu a + \mathcal{L}(\partial_\mu a/f_a;\psi_f) + \frac{a}{f_a}\frac{g^2}{32\pi^2}F^{\mu\nu a}\tilde{F}^a_{\mu\nu}\ . \tag{4.13}$$

The last term, which originates from the axial anomaly, not only violates the shift symmetry of the axion but also introduces a potential for it. In the vacuum, the dynamical field $a$ will settle into the minimum of its potential which is given by [31, 165]

$$\langle\frac{\partial V_{\text{axion}}}{\partial a}\rangle = -\frac{1}{f_a}\frac{g^2}{32\pi^2}\langle F^{\mu\nu a}\tilde{F}^a_{\mu\nu}\rangle\bigg|_{\langle a\rangle = -f_a\theta} = 0\ . \tag{4.14}$$

The mass of the axion is then determined by

$$m_a = \langle\frac{\partial^2 V_{\text{axion}}}{\partial a^2}\rangle = -\frac{1}{f_a}\frac{g^2}{32\pi^2}\frac{\partial}{\partial a}\langle F^{\mu\nu a}\tilde{F}^a_{\mu\nu}\rangle\bigg|_{\langle a\rangle = -f_a\theta}\ . \tag{4.15}$$

The minimum of the potential (4.14) lies precisely at the point were the $\theta$-term is cancelled by the axion. This is the celebrated solution to the strong CP problem first proposed by Peccei and Quinn [31].

In their original model, Peccei and Quinn constructed the symmetry breaking scale such that it coincides with the electroweak scale, $f_a \sim \sqrt{G_f}^{-1}$ which resulted in too strong couplings to other particles. An axion with such a low breaking scale has long since been excluded [166, 167]. In contrast, so-called invisible axion models with $f_a \gg \sqrt{G_f}^{-1}$, see e.g. [168–170], have such weak couplings to ordinary matter that they are perfectly viable although they make direct axion searches difficult, see, e.g. [171–173].

The essence of the axion solution to the strong CP problem can be summarized as follows. By promoting the $\theta$-parameter into a dynamical axion field[3] which is the pseudo Goldstone boson of the broken $U(1)_{PQ}$ and sourced by $F\tilde{F}$, one guarantees that the vacuum of the theory is at the minimum of the axion potential (4.14) corresponding to $\langle F\tilde{F}\rangle = 0$. Thus CP is unbroken. Individual axion models only differ by the underlying dynamics that generate the effective Lagrangian (4.13). However, the precise origin of the axion is unimportant for the solution of the strong CP problem.

## 4.3 Dual Description

The strong CP problem and the appearance of the $\theta$ parameter can be reformulated in terms of a massless three-form field that mediates a constant long-range four-form

[3]Essentially, one makes the replacement $\theta \to a/f_a$.

electric field in the vacuum [40, 174]. This electric field is CP-violating and its value is equal to $\langle F\tilde{F}\rangle$. Therefore a nonzero constant electric four-form field corresponds to a nonzero $\theta$-angle. This is analogous to the existence of a constant electric field in two dimensional electrodynamics (the Schwinger model) [45–47], where the electric field is also related to the appearance of a periodic parameter $\theta$.

In the dual description, the solution to the strong CP problem either by the axion or by massless quarks corresponds to dynamically creating a mass gap for the three-form gauge field, i.e. to putting it into a Higgs phase.

### 4.3.1 The strong CP Problem in three-form Language

The QCD strong CP problem is easily formulated in the three-form language (see, e.g., [42, 175, 176]). The $\theta$-term of (4.7) can be rewritten in terms of the Chern-Simons three-form

$$C_{\alpha\beta\gamma} = A^a_\alpha (F^a_{\beta\gamma} - \frac{g}{3} f^{abc} A^b_\beta A^c_\gamma) \,, \tag{4.16}$$

whose field strength is given by the aforementioned four-form electric field $E_{\alpha\beta\gamma\delta} = \partial_{[\alpha} C_{\beta\gamma\delta]}$. The resulting Lagrangian reads

$$\mathcal{L}_\theta = \theta \frac{g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu a} = \theta E_{\alpha\beta\gamma\delta} \epsilon^{\alpha\beta\gamma\delta} = \theta E \,. \tag{4.17}$$

Under an infinitesimal $SU(3)$ gauge transformation of QCD given by $A^a_\mu \to A^a_\mu + \frac{1}{g}\partial_\mu \omega^a + i f^{abc} \omega^b A^c_\mu$, the Chern-Simons three-form shifts according to $C_{\alpha\beta\gamma} \to C_{\alpha\beta\gamma} + \partial_{[\alpha} A^a_\beta \partial_{\gamma]} \omega^a$.

The composite Chern-Simons term $C_{\alpha\beta\gamma}$ becomes a fundamental variable at low energies $\ll \Lambda_{QCD}$ in the sense that it describes the relevant weakly-coupled degrees of freedom. It is known that it propagates long-range correlations if the topological susceptibility $\chi$ is nonzero [40], which can be seen from the associated Chern-Simons current $K_\mu = \epsilon_{\mu\alpha\beta\gamma} C^{\alpha\beta\gamma}$ having a pole at zero momentum [41]

$$\chi = \lim_{q\to 0} q^\mu q^\nu \int d^4x e^{iqx} \langle 0|K_\mu(x) K_\nu(0)|0\rangle = \text{const} \,. \tag{4.18}$$

The correlator of the massless three-form then obeys

$$\langle C, C\rangle_{q\to 0} \propto \frac{1}{q^2} \,. \tag{4.19}$$

It immediately follows from a Fourier analysis that this behaviour induces a constant topological susceptibility, i.e. a constant electric four-form field, in the vacuum since $F\tilde{F} = E = dC \sim qC$. Thus one finds

$$\langle E, E\rangle = \langle F\tilde{F}, F\tilde{F}\rangle_{q\to 0} = \text{const} \,. \tag{4.20}$$

At low energies, the effective Lagrangian for the three-form can be written as

$$\mathcal{L}_{3-\text{form}} = \theta E + K(E) + \dots \,, \tag{4.21}$$

where $K(E)$ includes the kinetic term for $C$ and its exact form is unknown in QCD. The first term (in a low momentum and large $N$ approximation, cf. [42]) will generically be $\propto E^2$. The solution to the associated equation of motion is given by a constant electric field $E$ which is sourced by $\theta$. Therefore, the existence of a constant electric field and the $\theta$-dependence of the theory are intimately linked.

### 4.3.2 Higgsing the four-form electric Field

The dynamical solutions to the strong CP problem, either a zero mass quark or the introduction of an axion field, proposed in section 4.2 can be treated on a common basis in the dual description. The basic idea relies on the fact that an efficient way of screening a constant electric field is by simply putting it into a Higgs phase.

Introducing an anomalous $U(1)_{PQ}$ symmetry – be it the chiral symmetry (4.4) when introducing massless quarks or an additional one via the axion solution – yields a coupling to the three-form field given by $\Lambda^{-2}\epsilon^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma}\frac{\partial_\delta\partial_\mu}{\Box}J_5^\mu$. Here, $J_5^\mu$ denotes the anomalous current of the $U(1)_{PQ}$, see, e.g., [175], whose divergence is given by $E$

$$\partial_\mu J_5^\mu = \alpha_A F\tilde{F} = \alpha_A E \; , \tag{4.22}$$

where $\alpha_A$ is a numerical coupling constant. Note that we dropped Lorentz and group indices in order to clarify the equations. Taking into account the coupling generated via the anomaly Fig. 4.1, the effective Lagrangian (4.21) obtains an additional term and reads

$$\frac{E^2}{2\cdot 4!\Lambda^4} + \Lambda^{-2}\alpha_A E\frac{\partial_\mu}{\Box}J_5^\mu \; . \tag{4.23}$$

Variation with respect to the Chern-Simons field and using (4.22) on the equation of motion yields

$$\Box E = \alpha_A \Lambda^2 E \; . \tag{4.24}$$

The electric field is in a Higgs phase and its pole at zero momentum (4.18) is removed. This demonstrates how the anomalous $U(1)_{PQ}$ symmetry generically puts the electric four-form field into a Higgs phase and thus solves the strong CP problem.

For the axion solution the divergence of the current is related to $a$ via

$$J_5^\mu = f_a \partial_\mu a \; . \tag{4.25}$$

Therefore, the Lagrangian describing the dynamics of the three-form field $C_{\alpha\beta\gamma}$ and the axion is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu a\partial^\mu a + \frac{a}{f_a}E_{\alpha\beta\gamma\delta}\epsilon^{\alpha\beta\gamma\delta} - \frac{1}{2\cdot 4!\Lambda^4}E_{\alpha\beta\gamma\delta}E^{\alpha\beta\gamma\delta} \; . \tag{4.26}$$

The gauge symmetry $C_{\alpha\beta\gamma} \to C_{\alpha\beta\gamma} + \partial_{[\alpha}\Omega_{\beta\gamma]}$, which is unbroken in the Higgs phase, guarantees that the couplings in the dual description are such that they respect the shift symmetry of $a$. Note that this description is, at low energies, completely equivalent to (4.13). The potential term of $a$ can be recovered by integrating out $C_{\alpha\beta\gamma}$ and deriving the equation of motion for $a$

$$\Box a + m_a^2(a - k_0) = 0 \; , \tag{4.27}$$

where $k_0$ is an integration constant. Gauging the dual theory has created a potential term for the axion which automatically requires $\langle a \rangle = 0$ in the vacuum. This is equivalent to the electric four-form being screened in the Higgs phase $E = 0$ (4.14). Thus, we have made the connection between the axion and the $\langle F\tilde{F} \rangle$ vacuum expectation value being zero, as the latter is given by $\langle E \rangle$.

Below (4.21), we mentioned that the exact form of $K(E)$ is unknown; therefore, one should allow for a generic form of $K(E)$ in (4.26). It turns out that its form is closely related to the axion potential $V(a)$ which is obtained after integrating out $E$ in (4.26)

$$\mathcal{L}_a = \frac{1}{2}\partial_\mu a \partial^\mu a - V(a) \; . \tag{4.28}$$

The equivalence is then established by choosing $K(E)$ such that

$$\frac{dV(a)}{da} = -\frac{1}{f_a}E = -\frac{1}{f_a}\mathrm{inv}K'\left(m_a^2(k_0 - a)\right) \; , \tag{4.29}$$

where "inv" denotes the inverse function and the prime differentiation with respect to the argument. Thus a generic potential $V(a)$ can be represented in the dual description via

$$V(k_0 - a) = -\frac{1}{f_a}\int da \; \mathrm{inv}K'\left(m_a^2(k_0 - a)\right) \; . \tag{4.30}$$

An essential property of this formalism is that at the extrema of the axion potential (4.29), the four-form electric field strength $E$ vanishes for any function $K$. Again, this is simply due to the fact that in the Higgs phase the field $E$ is screened. Nonetheless, this is precisely the way the axion solves the strong CP problem in QCD as we shall see in the next section.

The effective potential for the axion is then related to the $\theta$-parameter via $a \leftrightarrow \theta$ and for $K(E) = E^2$, one finds the well-known relation [50, 156]

$$\frac{dV(a)}{da} \leftrightarrow \frac{dE(\theta)}{d\theta} \propto \langle F\tilde{F} \rangle \; . \tag{4.31}$$

In section 4.1, we have seen that the strong CP problem can also be solved by introducing massless quarks which are charged under the chiral symmetry (4.4). In the phenomenological description outlined above, this solution can be understood in terms of the $\eta'$ meson. The $\eta'$ meson is now the pseudo-Goldstone boson of the broken $U(1)$ and related to the current via $J_5^\mu = \Lambda \partial_\mu \eta'$ similar to (4.25). Basically, one only has to replace the axion field $a$ in (4.26) and everything else follows. The electric four-form field is screened in precisely the same way and physics becomes $\theta$-independent.

Thus, the essence of the solution to the strong CP problem lies in finding a way to screen an electric four-form field strength via Higgsing of its gauge field.

For completeness, we would like to mention that one could formulate all of the above in terms of the fully dual picture, where also the axion is dualized and replaced by a two-form field $B_{\mu\nu}$. After the duality transformation of the axion $a \leftrightarrow B_{\mu\nu}$ and the gauge field $A_\mu \leftrightarrow C_{\alpha\beta\gamma}$, their interaction is described by

$$\mathcal{L} = \frac{1}{f_a^2}(P_{\alpha\beta\gamma} - C_{\alpha\beta\gamma})^2 - \frac{1}{2 \cdot 4!\Lambda^4} E_{\alpha\beta\gamma\delta} E^{\alpha\beta\gamma\delta} \,, \tag{4.32}$$

where $m$ is a dimensionful "coupling" constant which is chosen such that it is equal to the mass of the axion later on. $P_{\alpha\beta\gamma} = \partial_{[\alpha} B_{\beta\gamma]}$ is the two-form field strength. This Lagrangian is invariant under the gauge transformation $C_{\alpha\beta\gamma} \to C_{\alpha\beta\gamma} + \partial_{[\alpha}\Omega_{\beta\gamma]}$ and $B_{\mu\nu} \to B_{\mu\nu} + \Omega_{\mu\nu}$, where $\Omega_{\beta\gamma}$ is an arbitrary two-form. Thus, in the dual picture, the $U(1)$ global symmetry is promoted to a local gauge symmetry. This must have necessarily happened in order not to increase the number of degrees of freedom in the dual theory. The Lagrangian (4.32) describes the Higgs phase of the gauge field $C_{\alpha\beta\gamma}$, which gains a mass in a gauge-invariant way by "eating" the two-form $B_{\mu\nu}$.

## 4.4 Potential Issues of the Axion Solution

The aim of this chapter is to investigate two controversies surrounding the axion that are discussed in the literature. The first is the impact of quantum gravity corrections on the axion solution. The second concerns possible issues with the derivation of the axion potential in the instanton gas approximation.

### 4.4.1 Gravity Effects as an extra Three-Form

It is a long standing question whether quantum gravity effects could lead to possible non-CP conserving corrections, see e.g., [24]. Since perturbative gravitational interactions are known to be CP conserving, only nonperturbative effects could introduce CP violating operators. Arguments that nonperturbative quantum gravity could undo the axion solution are generically based on the appearance of gravitational or world-sheet instantons and wormhole solutions, see, e.g., [24, 26–30, 35–39] and references therein. Support for these presumptions comes from the general believe that black holes or wormhole solutions can absorb global charges. In consequence, these charges become inaccessible to an observer in our universe. Hence, quantum gravity is expected to break global symmetries. Local symmetries, on the other hand, are thought to be conserved by quantum gravity. The reason is that since local charges can always be monitored by measuring the Gaussian flux at infinity, they cannot simply disappear due to quantum gravity effects in the vicinity of a wormhole or a black hole. For the (non-) violation of global symmetries in quantum gravity, however, no consistent picture has emerged so far.

Naively, one could think that, since in the dual description of the axion in terms of the two-form field $B_{\mu\nu}$ the global $U(1)$ symmetry was promoted to a gauge symmetry, the axion solution cannot be overthrown by quantum gravity corrections. Of course, this cannot be the full story as it was shown in [24, 35–39] that there exist operators which could lead to the destruction of the strong CP solution in the axion scalar field language. It is, therefore, inevitable that also in the dual description analogous operators will appear. This brings to light the strength of the dual description (the three-form language) (4.32): Since quantum gravity corrections are not expected to violate the gauge symmetry, they must enter the theory (4.32) in a gauge invariant way. In the three-form language, the axion solution amounts to higgsing the three-form which is responsible for the CP violating electric field $E$ in the vacuum. In order to see CP violating operators in the dual picture, gravity must thus induce an independent gauge invariant mass term for the two-form $B_{\mu\nu}$. If gravity provides an additional three-form field coupled to $B_{\mu\nu}$, this is indeed what happens.

Coupling a second three form $G_{\alpha\beta\gamma}$ to the axion $B_{\mu\nu}$ in (4.32) [42] in such a way that it also shifts under gauge symmetries of $G_{\alpha\beta\gamma}$, i.e. $B_{\mu\nu} \to B_{\mu\nu} + \Upsilon_{\mu\nu}$ under $G_{\alpha\beta\gamma} \to G_{\alpha\beta\gamma} + \partial_{[\alpha}\Upsilon_{\beta\gamma]}$, introduces the Lagrangian [42]

$$\mathcal{L} = \frac{1}{f_a^2}(\mathrm{d}B - \alpha_A C - \alpha_G G)^2 + \frac{1}{48\Lambda^4}E^2 + \frac{1}{48m_G^4}E_G^2 \,. \tag{4.33}$$

Here, we have suppressed Lorentz indices and $E_G = \epsilon^{\alpha\beta\gamma\delta}E_{G\alpha\beta\gamma\delta}$ is the field strength of $G_{\alpha\beta\gamma}$ and $\mathrm{d}G = \partial_{[\mu}G_{\alpha\beta\gamma]}$ defines the exterior derivative. The factors $\alpha_A$ and $\alpha_G$ define the relative strengths of the coupling of the axion to QCD and gravity and the scale $m_G$ is the scale at which the gravitational anomaly enters. The axion can now only give mass to the combination $C_1 = \alpha_A C + \alpha_G G$, but not to $C_2 = \alpha_A C - \alpha_G G$ which is orthogonal to $C_1$. Using these three-form fields, one finds that only the field strength of the former $E_1 = \mathrm{d}C_1$ is in the Higgs phase and has vanishing vacuum expectation value. The field strength of the latter $E_2 = \mathrm{d}C_2$, however, produces a long-range electric field in the vacuum. The axion solution to the strong CP problem is, thus, spoiled [42].

Performing a duality transformation on (4.33), one can go back to the standard scalar field description of the axion. The axion potential (4.29) is now determined by both electric fields, the one produced by the axial anomaly of QCD and the one from gravity. As a result, the minimum of the axion potential,

$$\frac{dV(a)}{da} = -\frac{1}{f_a^2}(\alpha_A E + \alpha_G E_G) = 0 \,, \tag{4.34}$$

no longer coincides with $E$.[4] Generally speaking, if the number of three-forms is larger than the number of potential axion fields, there will always be at least one long-range electric four-form field in the vacuum [42]. Note, however, that the axion still solves the strong CP problem within the observational bounds if the gravitational mixing with the axion is strongly suppressed relative to the mixing with QCD, more precisely if $\frac{\alpha_G}{\alpha_A} \ll 10^{-9}$.

[4]Notice that such an additional three-form can be considered as due to higher dimensional operators of the form $(P_{\alpha\beta\gamma} - C_{\alpha\beta\gamma})\frac{m^2\Box}{\Box+m^2}\Pi_\mu^{[\alpha}(P^{\mu\beta\gamma]} - C^{\mu\beta\gamma]})$, which is what one obtains after integrating out $G_{\alpha\beta\gamma}$ in (4.33).

If gravity is to interfere with the axion solution, it has thus to provide an additional three-form. The Chern-Simons term in gravity, $G_{\alpha\beta\gamma} = \frac{1}{12}\Gamma^{\rho}_{\sigma[\alpha}\partial_\beta\Gamma^{\sigma}_{\gamma]\rho}$ with $\Gamma^{\rho}_{\mu\nu}$ the Christoffel connection, is a potential candidate. The topological invariant of the Riemann tensor and its dual $R\tilde{R}$ are related to $G_{\alpha\beta\gamma}$ (cf. (4.17)) via

$$R\tilde{R} = \epsilon^{\alpha\beta\gamma\delta} R^{\rho}_{\sigma\alpha\beta} R^{\sigma}_{\rho\gamma\delta} = \frac{1}{3}\epsilon^{\alpha\beta\gamma\delta}\partial_{[\alpha} G_{\beta\gamma\delta]} = E_G \, . \tag{4.35}$$

The gauge symmetry here are coordinate transformations, cf. chapter 2, under which $G$ shifts as an exterior derivative of a two-form $\Upsilon_{\beta\gamma}$, $G \to G + \mathrm{d}\Upsilon$.

In order for gravity to interfere with the axion solution, two conditions must be satisfied: First, the correlator of $G_{\alpha\beta\gamma}$ must develop a massless pole in the absence of the axion

$$\langle R\tilde{R}, R\tilde{R}\rangle_{q\to 0} = \text{const} \, , \tag{4.36}$$

which implies that there is a nonzero long-range electric four-form field $E_G \neq 0$ in the vacuum. Second, the coupling $aR\tilde{R}$ which is equivalent to $\mathrm{d}BG$ in (4.33) must be generated. This could, for example, happen through the gravitational anomaly [177].

We will not be concerned with the actual origin of such a gravitational three-form and refer to the literature about possible violations of global symmetries due to quantum gravity effects [24]. We rather want to consider what implications these effects have for the axion solution and what can be done about it.

### 4.4.2 A Way out

Coupling additional global symmetries to the gravitational anomaly [177–179] can avoid the breakdown of the axion solution [42]. In such a case, the gravitational anomaly leads to the nonconservation of the corresponding current $\partial_\mu J^\mu_g \propto R\tilde{R}$, and the corresonding $\theta_G \sim E_G$ can be rotated away in the same way as $\theta$ in QCD by introducing a $U(1)_{PQ}$. In the three-form language, this amounts to introducing an additional two-form $B_{G\mu\nu}$ which does not couple to the QCD three-form $C_{\alpha\beta\gamma}$ but only to the gravitational $G_{\alpha\beta\gamma}$ [42] yielding the Lagrangian

$$\mathcal{L} = \frac{1}{2\cdot 4!\Lambda^4}E^2 + \frac{1}{2\cdot 4! m_G^4}E_G^2 + \frac{1}{f_a^2}(\alpha_A C + \alpha_G G + \mathrm{d}B)^2 + m_G^{-2}(G + \mathrm{d}B_G)^2 \, . \tag{4.37}$$

The additional two-form higgses the field $E_G$ and therefore, both $E$ and $E_G$ are screened in the vacuum as can easily be verified from the equations of motion.

Interestingly, with the massless neutrinos, the Standard Model provides a perfect candidate for an anomalous gravity coupling which does not couple to the axial anomaly. In the following, we want to argue along the lines of [43] that the neutrino with the neutrino lepton number serving as the anomalous $U(1)$ can protect the axion solution against quantum gravity corrections. For finite neutrino masses [180], in turn, this mechanism puts a bound on the neutrino mass [43] when other contributions can be discarded. This issue is discussed in section 4.4.3.

To wrap things up, assuming the axion is anomalous with respect to both the triangle anomaly and gravity, the divergence of its corresponding $U(1)_{PQ}$ current (4.25) is sourced by the four-form fields $E$ and $E_G$

$$\partial_\mu J_5^\mu = \alpha_A E + \alpha_G E_G \ , \tag{4.38}$$

where $\alpha_A$ and $\alpha_G$ are numerical coupling constants and the latter is a parameter determined by the nature of the gravitational anomaly. Its value is currently undetermined and we shall treat it as a free parameter. We saw in (4.33) that the axion can only put one electric field into a Higgs phase; the other one still has a pole at $q^2 = 0$.

However, in addition there is now an extra global $U(1)$, the neutrino lepton number symmetry, $\nu_L \to e^{i\alpha_\nu}\nu_L$. Its current $J^\mu_{\nu_L} = \bar{\nu}_L \gamma^\mu \nu_L$ is anomalous with respect to gravity, but not QCD,

$$\partial_\mu J^\mu_{\nu_L} = \alpha_G R\tilde{R} = \alpha_G E_G \ . \tag{4.39}$$

The effective Lagrangian induced by these anomalies is

$$\mathcal{L} = \frac{1}{2\cdot 4!\Lambda^4}E^2 + \frac{1}{2\cdot 4! m_G^4}E_G^2 + \frac{\alpha_A}{f_a^2}E\frac{\partial_\mu}{\Box}J_5^\mu + \frac{\alpha_G}{m_G^2}E_G\frac{\partial_\mu}{\Box}J_5^\mu + \frac{\alpha_G}{m_G^2}E_G\frac{\partial_\mu}{\Box}J^\mu_{\nu_L} \ . \tag{4.40}$$

This Lagrangian is equivalent to (4.37) (cf. also (4.23) in section 4.3.2). Thus, there are no massless modes in $E$ or $E_G$.

### 4.4.3 Neutrino Masses

Observations of neutrino oscillations [180] have established that neutrinos are not massless but possess a tiny mass which is experimentally bounded by $\sum m_\nu \lesssim 0.3\text{eV}$ [181], where the sum is over all neutrino flavours. Following our work in [43], we want to analyse how these affect the CP violation scenarios presented before.

Due to the tiny neutrino mass, the neutrino lepton number is explicitly broken. This introduces an additional factor to the divergence of the current (4.39)

$$\partial_\mu J^\mu_{\nu_L} = \alpha_G R\tilde{R} + 2m_\nu \bar{\nu}_L\gamma^5\nu_L \ . \tag{4.41}$$

Analogously to (4.25), the current is identified with a pseudo-scalar $\eta_\nu$ which is the pseudo-Goldstone boson of the broken symmetry transformation given by

$$\eta_\nu = \frac{1}{m_G^2} 2\bar{\nu}_L\gamma^5\nu_L \quad \text{and} \quad J^\mu_{\nu_L} = m_G \partial^\mu \eta_\nu \ . \tag{4.42}$$

The dynamics of the theory is now governed by the Lagrangian (4.40) with an additional mass term $\frac{1}{2}m_\nu m_G\eta_\nu^2$. Replacing the currents with their corresponding pseudo-Goldstone bosons (4.25) and (4.42) yields [43]

$$\begin{aligned}\mathcal{L} &= \frac{1}{2\cdot 4!\Lambda^4}E^2 + \frac{1}{2\cdot 4! m_G^4}E_G^2 - f_a a\frac{E_G}{m_G^2} - \eta_\nu\frac{E_G}{m_G} - \frac{a}{f_a}E \\ &+\frac{1}{2}\partial_\mu a\partial^\mu a + \frac{1}{2}\partial_\mu\eta_\nu\partial^\mu\eta_\nu - \frac{1}{2}m_\nu m_G\eta_\nu^2\,. \end{aligned} \tag{4.43}$$

The equations of motion for $C$ and $G$ are (in the following we drop indices and work in a coordinate free notation)

$$\begin{aligned} \mathrm{d}\left(\frac{E}{\Lambda_{QCD}^2} + \Lambda_{QCD}^2\frac{a}{f_a}\right) &= 0 \\ \mathrm{d}\left(\frac{E_G}{m_G^2} + m_G^2\frac{a}{f_a} + m_G\eta_\nu\right) &= 0\,, \end{aligned} \tag{4.44}$$

and the ones for $a$ and $\eta_\nu$ read

$$\begin{aligned} f_a\Box a &= -E_G - E \\ \Box\eta_\nu + m_\nu m_G\eta_\nu &= -\frac{E_G}{m_G}\,. \end{aligned} \tag{4.45}$$

Their solution gives the following expressions [43]

$$\begin{aligned} E &= \Lambda_{QCD}^4\frac{m_\nu m_G^4(-\alpha+\beta)}{m_G^4 m_\nu + \Lambda_{QCD}^4(-m_G+m_\nu)} \\ E_G &= m_G^4\frac{m_\nu\Lambda_{QCD}^4(\alpha-\beta)}{m_G^4 m_\nu + \Lambda_{QCD}^4(-m_G+m_\nu)}\,, \end{aligned} \tag{4.46}$$

where $\alpha$ and $\beta$ are dimensionless integration constants of (4.44), i.e. $E = -\Lambda_{QCD}^4\frac{a}{f_a} + \Lambda_{QCD}^4\beta$ and $E_G = -m_G^4\frac{a}{f_a} - m_G\eta_\nu + \alpha m_G^4$.

Let us parametrize this result by the ratio of the neutrino mass to the gravitational scale by defining $\epsilon = \frac{m_\nu}{m_\nu - m_G}$. The solution for the QCD electric four-form field $E_G$ then depends on $\epsilon$ as follows [5]

$$E = \Lambda_{QCD}^4\frac{\epsilon}{\epsilon + \frac{\Lambda_{QCD}^4}{m_G^4}}\,. \tag{4.47}$$

The limit of massless neutrinos, $\epsilon\to 0$ leads to a vanishing of $E$ (as well as $E_G = 0$). Thus, we recover the result of section 4.4.2. Considering a neutrino mass $m_\nu$ much smaller than the gravitational scale $m_G$, $m_\nu \ll m_G$, $\epsilon \simeq -\frac{m_\nu}{m_G}$, the electric field of the QCD vacuum is instead

$$E = -\Lambda_{QCD}^4\frac{m_\nu m_G^3}{\Lambda_{QCD}^4 - m_\nu m_G^3}\,. \tag{4.48}$$

[5]Note that we are not concerned with the actual value the gravitational electric field takes in the vacuum as it is not constrained by measurements.

To be compatible with observations (e.g. the electric dipole moment discussed below (4.8)), the electric field $E = \theta\chi$ must be less than $E < 10^{-9}\Lambda_{QCD}^4$.

It is instructive to look at the possible limits on $E$. If the denominator in (4.48) is dominated by $m_\nu m_G^3$, there is no screening and $E \sim \Lambda_{QCD}^2$. On the other hand, if $\Lambda_{QCD}^4 \gg m_\nu m_G^3$, one obtains a bound on the neutrino mass which is [43]

$$m_\nu \lesssim 10^{-9}\frac{\Lambda_{QCD}^4}{m_G^3} . \tag{4.49}$$

In turn, measuring the neutrino mass would introduce an upper bound on the gravitational scale $m_G$ with which the anomaly enters if neutrinos are to solve the gravitational CP violation. Experimental searches currently focus on the mass range $0.2\,\text{eV} < m_\nu < 2\,\text{eV}$ which would give the bound $m_G \lesssim 0.2\,\text{GeV}$ if detected.

## 4.5 Nonanalyticity of the Axion Potential

In [49], Witten made the compelling argument that the $\theta$-dependence of the QCD vacuum is not necessarily described correctly by the standard instanton gas approximation [34]. In weak coupling, the $\theta$-dependence can reliably be computed via instanton effects which are of order $e^{-8\pi^2/g^2}e^{\pm i\theta}$. However, the imposed boundary condition of the gauge field approaching a pure gauge at infinity

$$A_\mu \to U\partial_\mu U^{-1}, \quad \text{for } \ |x| \to \infty , \tag{4.50}$$

is not sensible in an unbroken, asymptotically free theory which exhibits confinement. Furthermore, as in such theories instantons of all sizes are relevant, one cannot compute the $\theta$-dependence reliably in an instanton gas approximation due to infrared divergences for large instanton contributions.

A different approach can be taken by considering the large $N$ limit of an $SU(N)$ gauge theory [182]. This constitutes a well-suited method to understand a gauge theory with only a small number of light quarks in four dimensions. The large $N$ limit is defined by taking $N \to \infty$ while keeping the 't Hooft coupling $\lambda = g^2 N$ fixed.[6] In this limit, the instantons of the classical action are expected to contribute only at order $e^{-8\pi^2 N/\lambda}$. There are, however, arguments that the $\theta$-dependence of the QCD vacuum is present in leading order of a $1/N$ expansion. For example, in a two-dimensional solvable $CP^{N-1}$ sigma model, it can be shown that some of the properties usually attributed to the effect of instantons, such as, e.g., the existence of a $\theta$ angle and the resolution of the $U(1)$ problem, appear to leading order in $1/N$ [49]. In these theories, the topological charge is not quantized and therefore, there are no instantons for the classical action to begin with.

[6] This limit can be compared to the semi-classical limit we are considering in chapter 3. Black hole semi-classics can be understood as a large $N$ limit in 't Hooft's sense.

Even though the $\theta$-dependence of the vacuum energy $E(\theta)$ cannot be calculated exactly, there are still generic properties which can be established in the large $N$-limit [183, 184]:

$$\begin{aligned} E(\theta) &= N^2\Lambda^2 F(\frac{\theta}{N}) \ , \\ E(\theta) &= E(\theta + 2\pi) \ . \end{aligned} \tag{4.51}$$

These conditions can only be fulfilled if either $E(\theta)$ is a constant or a multivalued function [183]. If the latter is the case, there exists a family of vacua labelled by the integer number $k$ with energy

$$E_k(\theta) = N^2\Lambda^4 F(\frac{\theta + 2\pi k}{N}) \ . \tag{4.52}$$

The true vacuum for any value of $\theta$ is found by minimization of $E_k(\theta)$ with respect to $k$

$$E(\theta) = \min_k N^2\Lambda^4 F(\frac{\theta + 2\pi k}{N}) \ \simeq \Lambda^4 \min_k (\theta + 2\pi k)^2 + \mathcal{O}(\frac{1}{N}). \tag{4.53}$$

This function is periodic in $\theta \to \theta + 2\pi$, but at integer multiples of $\pi$ there is a jump between two branches of different $k$.

Introducing quarks into the theory, the vacuum energy $E(\theta)$ for pure gluodynamics without quarks becomes a potential for the $\eta'$ meson and its mass is given by the Witten-Veneziano formula [50, 156]

$$m_{\eta'}^2 = \frac{4N_f}{f_\pi^2}\left(\frac{d^2E}{d\theta^2}\right)_{\theta=0}^{\text{no quarks}} \ , \tag{4.54}$$

which was derived in the large $N$ limit. The above arguments suggest that the $\theta$-dependence of the vacuum energy

$$E(\theta) = -m_q\Lambda^3 \cos\theta \tag{4.55}$$

which is derived from a standard instanton gas approximation, is not fully correct.

### 4.5.1 Schwinger Pair Creation and $\eta'$ Bubble Nucleation

In the following, we want to add another piece of evidence in favour of the non-analyticity of the QCD vacuum $\theta$-dependence. To this end, we draw an analogy between the screening of integer charges in the two dimensional Schwinger model and the screening of the $\theta$-term in a QCD-like theory with light quarks.

The Schwinger model [45] describes two dimensional quantum electrodynamics with massless fermions. It has interesting similarities to four dimensional Yang-Mills theory. For example, the theory contains no asymptotic states of free fermions and local charge conservation is spontaneously broken [46]. The Lagrangian of this model is

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \ , \tag{4.56}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the photon field strength. This theory is exactly solvable and has a number of interesting features [45, 185]. The theory is in a Higgs phase and therefore charge screening occurs, i.e. there is no long-range correlator between widely seperated charges. Additionally, global chiral symmetry breaking occurs, but, similarly to QCD, due to the existence of an axial anomaly there is no Goldstone boson.

As explained below, adding a mass term

$$m\bar{\psi}\psi \tag{4.57}$$

introduces a dependence of the vacuum energy on an angular variable $\theta$ [47]. Furthermore, it can be shown [47] that for a generic charge $Q$ a long-range force is present. However, most importantly, if $Q$ is an integer multiple of $e$, the long-range force vanishes. This can be seen from the interaction energy for widely separated (of distance L) charges

$$E = \left( E(\theta - \frac{2\pi Q}{e}) - E(\theta) \right) L \, . \tag{4.58}$$

Since $\theta$ is periodic in $2\pi$, the interaction energy is periodic in $Q$. Specifically, whenever $Q$ is an integer multiple of $e$, the energy is zero indicating the absence of a long-range electric field.

The periodicity in $Q$ can be understood as a screening by pair production [46]. The Lagrangian (4.56) has a solution for the electric field strength $F_{01}$ (we have chosen the gauge $A_1 = 0$ such that this is the only independent component of $F_{\mu\nu}$) given by

$$F_{01} = e\partial_1^{-1} j_0 + F \, , \tag{4.59}$$

where $j_0 = \psi^\dagger \psi$ is the current density of the fermions and $F$ is an arbitrary constant background electric field, cf. section 4.2. For certain values of $F$, it is energetically favourable to produce a quark-antiquark pair separated by the distance $L$ [46]. The gain in electrostatic energy for such a process is

$$\Delta E = \frac{1}{2} L \left( (F \pm e)^2 - F^2 \right) \, . \tag{4.60}$$

For $(F \pm e)^2 - F^2 \leq 0$, the energy gain can be made arbitrarily large by increasing the distance of the pair produced. Thus, it can always overcome the energy needed to materialize two electrons out of the vacuum which is $\sim 2m$. While for $|F| \geq \frac{1}{2}e$, the vacuum energy can always be lowered by producing a pair of opposite charge, the electric field will be unscreened for $|F| \leq \frac{1}{2}e$. The parameter $\theta$ with its periodicity of $2\pi$ then depends on $F$ as

$$\theta = 2\pi \frac{F}{e} \, . \tag{4.61}$$

The probability of pair creation is of order one for the critical electric field $F = m^2/e$. Therefore, the $\theta$-dependence is suppressed by order $\sim m^2$ with the lightest fermion mass.

Let us then apply a similar argument to four dimensional QCD with light quarks. In the presence of quarks, the vacuum energy $E(\theta)$ effectively becomes a function of the $\eta'$ meson, $E(\theta + \eta'/\Lambda_{QCD})$, cf. (4.54) and (4.55). For massless fermions the $\theta$ parameter is unphysical, the potential for $\eta'$ becomes flat and its expectation value arbitrary. This constitutes the massless quark solution to the strong CP problem. In this case $\theta$ can always be adjusted such that $E(\theta + \eta'/\Lambda_{QCD})$ is minimal. Turning on quark masses, $\theta$ becomes a physical parameter. It is, however, still $2\pi$ periodic and therefore the potential for $\eta'$ is periodic in $2\pi\Lambda_{QCD}$. This is very similar to the Schwinger case where the charge $Q$ is periodic as in (4.58).

We would like to further the analogy and take the periodicity of $E$ in terms of $\eta'$ as a result of bubble nucleation with different $\eta'$ vacua. To simplify the discussion, let us go back to the three-form picture established in section 4.3. In this language, the $\theta$-dependence of the theory can be understood (similar to the two dimensional Schwinger model) as the appearance of a constant electric background field which we called $E$. It is given by

$$\langle F\tilde{F}\rangle \sim \theta E \,. \tag{4.62}$$

In section 4.3.2 it was shown that if one screens the electric field $E$, the theory is independent of $\theta$. We know that for massless quarks it is indeed zero. Thus, the background field $E$ must depend on the quark masses in such a way that it vanishes together with the lightest quark mass (4.9). The dependence on the lightest quark masses has been calculated a long time ago using current algebra methods [44]. The electric field is of first order in the lighest quark mass

$$\langle F\tilde{F}\rangle \sim \theta m_q \Lambda_{QCD}^3 \,. \tag{4.63}$$

Employing the analogy to the Schwinger model, we consider a screening mechanism by bubble nucleation as proposed above [186]. Analogously to (4.58) and (4.60), the total energy of the vacuum is lowered when $\eta'$ takes a smaller value, as this screens the electric field $E$. The energy gain of a bubble of radius $R$ which has a value of $\eta'$ lower by $2\pi\Lambda_{QCD}$ in the inside compared to an outside value of $\eta'$ is

$$\Delta E_{\text{vac}} = \int d^3x \left( \left(\frac{E}{\Lambda_{QCD}^2} - 2\pi\Lambda_{QCD}^2\right)^2 - \frac{E^2}{\Lambda_{QCD}^4} \right) \simeq -\frac{8}{3}\pi^2 R^3 E \,, \tag{4.64}$$

where we have assumed $E \geq \Lambda_{QCD}^4$. This energy gain has to be compared to the cost of producing such a bubble, which depends on the bubble size in contrast to the two dimensional Schwinger case where the energy cost is always $2m$. Thus, one has to compute the wall tension. Let us assume the potential for $\eta'$ is given by the one obtained from standard instanton calculus (4.55)

$$V(\eta') = m_q \Lambda_{QCD}^3 \cos\frac{\eta'}{\Lambda} \,, \tag{4.65}$$

which sets the mass for $\eta'$ to $m_{\eta'}^2 = m_q\Lambda_{QCD}$. The domain wall solution for such a potential is well-known (see e.g. [157]) and reads

$$\eta'_w(z) = 2(\arcsin\tanh m_{\eta'} z + \frac{\pi}{2}) \,, \tag{4.66}$$

where $\eta'$ is chosen to vary along the $z$-direction. The typical width of such a soliton is $m_{\eta'}^{-1}$. The wall tension is given by

$$T_W = \frac{H}{A} = \int dz \left[ \left( \frac{d\eta'_w(z)}{dz} \right)^2 + V(\eta'_w) \right] \simeq 8 m_{\eta'} \Lambda_{QCD}^2 \,, \tag{4.67}$$

with $A = \int dxdy$ the area of the wall. The energy stored in such a wall for a bubble of size $R$ is then

$$E_W = T_W 4\pi R^2 = 32\pi m_{\eta'} \Lambda_{QCD}^2 R^2 \,. \tag{4.68}$$

Since the energy of a spontaneously nucleated bubble must vanish, the critical radius of the bubble is determined by

$$R_{\text{crit}} = 3 \frac{T_W}{4\pi E \Lambda_{QCD}} \simeq \frac{12 m_{\eta'} \Lambda_{QCD}^2}{E} \,. \tag{4.69}$$

In order for the bubble nucleation process to be efficient enough to screen the vacuum, the probability of nucleation of a critical bubble must be of order $\sim 1$. The probability of nucleation of a bubble $\Gamma \sim e^{-S_{\text{crit}}}$ is given by its Euclidean action $S_{\text{crit}} = H m_{\eta'}^{-1}$. For $S_{\text{crit}} \sim 1$, the electric field $E$ has to have at least the value [48]

$$E \geq \frac{T_W^{\frac{3}{2}}}{\Lambda_{QCD} m_{\eta'}} \simeq m_{\eta'} \Lambda_{QCD}^3 = m_q^{\frac{1}{2}} \Lambda_{QCD}^{\frac{7}{2}} \,. \tag{4.70}$$

This result is in contradiction to the dependency of the electric field on the lightest quark mass derived in [44], see (4.63).

How could such a discrepancy arise? In the two dimensional Schwinger model, the screening of the electric field in the vacuum can be described by pair production of elementary charges. Therefore, one could expect that the analogue of pair creation, namely, the nucleation of bubbles with values of lower $\eta'$ would be able to produce the correct dependence on the lightest quark mass. This seems not to be the case. In order to describe the bubble nucleation process, we assumed that the potential for $\eta'$, which is given by the vaccum energy $E(\theta)$, is given by (4.55). However, as we have outlined above, there are indications that the vacuum energy functional of QCD is not given by (4.55), but instead by (4.53) which is a nonanalytic function of $\theta$. In this case our analysis is invalidated. Therefore, we take the discrepancy between our result (4.70) and the result (4.63) as a further indication for the nonanalytic behaviour of $E(\theta)$.

Finally, let us mention that in the three-form language of section 4.3, the correct dependence on the lightest quark mass (4.63) can be obtained. To see this, let us remind ourselves that the effective Lagrangian for the $\eta'$ field is

$$\mathcal{L} = \frac{1}{2} \frac{E^2}{\Lambda^4} + \frac{Z}{\Lambda^3} \epsilon^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma} \partial_\delta \eta' + \frac{1}{2} \partial_\alpha \eta' \partial^\alpha \eta' + \frac{1}{2} m_{\eta'}^2 \eta'^2 \,, \tag{4.71}$$

where $Z$ is a constant of dimension $[m]^2$ and $\Lambda = \Lambda_{QCD}$. The equations of motions for $C_{\alpha\beta\gamma}$ and $\eta'$ are

$$\begin{aligned}
\partial_\alpha(E - Z\Lambda\eta') &= 0 \\
\partial^2\eta' + m_{\eta'}^2\eta' + \frac{Z}{\Lambda^2}E &= 0 \\
\Rightarrow E &= Z\Lambda(\eta' + {\eta'}_0) \qquad \text{and} \qquad E \neq \frac{dV_{\eta'}}{d\eta'} \\
\Rightarrow \eta' &= \frac{-Z^2}{\Lambda^2 m_{\eta'}^2 + Z^2}{\eta'}_0 \qquad \text{and} \qquad E \simeq \frac{m_{\eta'}^2\Lambda^2}{Z^2}\eta'_0 \propto m_q\ ,
\end{aligned} \tag{4.72}$$

where $\eta'_0$ is an arbitrary integration constant. Thus, the electric field is suppressed linearly by the lightest quark mass as expected from (4.63).

## 4.6 Summary

This chapter contained a brief introduction to axion physics and potential problems it faces. We first discussed the influence of quantum gravity corrections on the axion solution in terms of a dual description. We showed that in terms of three-form gauge fields gravitational corrections can be parametrized as an additional three-form which is coupled to the axion by an anomalous current. By introducing an additional $U(1)$ symmetry whose current is anomalous with respect to gravity and which is provided by the neutrino lepton number, we showed that a possible threat for the success of the axion solution can be avoided. For the case of non-zero neutrino masses, we derived a bound on these masses which depends on the gravitational anomaly scale by requiring that the axion solution is viable within the experimental bounds. In addition, we analyzed the screening of the QCD vacuum electric field in the presence of light quarks in terms of bubble nucleation of different $\eta'$ vacua. We found that using the standard instanton approximated vacuum energy potential does not reproduce the correct result.

# Chapter 5

# Axion Dark Matter and non-minimal Couplings to Gravity

With the recent measurements of the cosmic microwave background by the Planck satellite [187], the theoretical framework for cosmology has been put to further tests. Remarkably, the standard six-parameter ΛCDM model for cosmology provides an excellent fit for the data. Parameter values defining this model have been measured precisely and the composition of the energy density of the universe today is determined by the dark energy density $\Omega_\Lambda \simeq 0.69$, the dark matter density $\Omega_{\mathrm{CDM}} \simeq 0.26$, and the baryonic energy density $\Omega_b \simeq 0.05$. Due to the absence of substantial non-Gaussianities, observations are consistent with the simplest models of inflation which produce an almost scale-invariant adiabatic power spectrum. In light of the constraints on isocurvature perturbations, the bounds on the most generic models of dark matter based on QCD axions have been tightened.

In order to obtain the correct dark matter abundance from axions, while at the same time avoiding an overproduction of isocurvature fluctuations, the Hubble scale during inflation needs to be very low. In this chapter we argue that, by considering non-minimal kinetic couplings of scalar fields to gravity, this picture can be changed and the axion can account for the observed dark matter density while avoiding an overproduction of isocurvature fluctuations. Furthermore, we show that the particle content of the Standard Model can provide for a successful inflationary scenario while the dark matter density is generated by axion particles.

The outline of this chapter is as follows. In section 5.1, we recapitulate the basics of inflation, whereas the concept of dark matter is introduced in section 5.2. In the same section, we also introduce the so-called misalignment mechanism for axion dark matter and review the latest bounds from cosmological observations. A mechanism to ease the conflict between axion dark matter and observational bounds is proposed in section 5.3. The last section 5.4 is devoted to a possible implementation of cosmology within the Standard Model.

## 5.1 The Inflationary Universe

The theory of inflation is one the cornerstones of modern cosmology. It describes the accelerated expansion of our universe before the standard hot big bang scenario [188]. Introduced initially [189–192] to solve such long-standing puzzles as the horizon problem, it was soon realized that it could not only explain the homogeneity of the observed cosmic microwave background but also explain the origin of the perturbations in the universe [193–195]. An isotropic and homogeneous universe is described in General Relativity by the Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = -dt^2 + R(t)^2 \left( \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right) . \tag{5.1}$$

Here, $t$ is the time-coordinate, $(r, \theta, \phi)$ are the spatial three dimensional spherical coordinates and $R(t)$ is the scale factor. The parameter $k$ determines the curvature of the universe and we assume it to be zero in the following as supported by observations [54]. The $(r, \theta, \phi)$ are so-called comoving coordinates since in an FRW background without external forces a particle at rest remains at the same coordinate position during the evolution of the universe. Physical length scales are measured by multiplying the comoving length by the scale factor $R(t)$.

The evolution of the universe is sourced by the energy density of the material it encloses. For the FRW metric, this energy density is given by an isotropic fluid whose equation of motion in the background (5.1) is

$$\dot{\rho} + 3H(\rho + p) = 0 . \tag{5.2}$$

Here, $H = \frac{\dot{R}}{R}$ is the Hubble constant and $\rho$ and $p$ are the density and the pressure of the fluid. From the Einstein equations follow the Friedmann equations

$$H^2 = \frac{1}{3M_P^2}\rho , \tag{5.3}$$

$$\frac{\ddot{R}}{R} = -\frac{1}{3M_P^2}(\rho + 3p) . \tag{5.4}$$

There are three important cosmological solutions to these equations. First, if the fluid consists of non-relativistic particles, called matter, the pressure is effectively zero and $\rho \propto R^{-3}$ and $R \propto t^{\frac{2}{3}}$. Second, for relativistic particles, called radiation, $p = \rho/2$ and $\rho \propto R^{-4}$ and $R \propto t^{\frac{1}{2}}$. The third solution describes a cosmological constant background fluid which has negative pressure $p = -\rho$ and $\rho$ is constant. In such a background, the scale factor, however, grows exponentially. Since we have set $k = 0$, (5.3) defines the critical density $\rho_c$ for which the universe is flat. Note that if we had not set $k = 0$, there would have appeared a term $\sim k/R^2(t)$ on the right-hand side of (5.3). Finally, the density parameter $\Omega$ is defined as the ratio of the density of a given fluid to the critical density

$$\Omega = \frac{\rho}{\rho_c} . \tag{5.5}$$

In a flat FRW universe, the total energy density is $\Omega_{\text{tot}} = 1$.

The standard hot big bang scenario explains the evolution of the universe from an almost isotropic and homogeneous state of high temperature and density to the observed universe today. As the temperature dropped to $T \sim 100\text{GeV}$ due to the universe's expansion, the electro-weak phase transition took place and gauge bosons became massive. Quarks and gluons were bound to protons and neutrons ($T \sim 200\text{MeV}$), and the light elements were formed (nucleosynthesis at $T \sim 0.05\text{MeV}$). At the point of matter-radiation equality ($T \sim \text{eV}$) the initially dominating radiation density became equal to the non-relativistic matter density which prevails today. Shortly afterwards, electrons were bound to nuclei and the photons decoupled. These photons have been free streaming ever since and are imprinted in the cosmic microwave background radiation. Galaxies and clusters formed from small inhomogeneities due to gravitational collapse creating today's structured universe.

The hot big bang scenario gives a successful description of the evolution outlined above. To comply with observations, however, the initial conditions need to be extremely fine-tuned. For instance, the universe today is observed to be flat $|\Omega - 1| \simeq 0$ to the percent level [54]. However, if it is not exactly flat today it would have needed to fulfill the bound $|\Omega - 1| \leq \mathcal{O}(10^{-27})$ at $T \sim 100\text{GeV}$, since the difference $|\Omega - 1|$ grows in time. Additionally, the big bang scenario cannot explain the observed low relic density [196]. Assuming that the Standard Model gauge group $SU(3) \times SU(2) \times U(1)$ originates from a Grand Unified Theory (GUT), there must have been a phase transition in the early universe which broke the GUT gauge group down to the Standard Model. During such a phase transition topological defects, such as monopoles, are usually produced. Standard computations estimate a density for them which is much larger than the observational bounds [196]. Most importantly, the hot big bang scenario cannot explain the isotropy observed in the cosmic microwave radiation. This isotropy exists on such large scales that in a matter- or matter-radiation dominated universe they would have never been in causal contact and the radiation could not have thermalized. Additionally, also the observed perturbations in the CMB could not have been created by any causal physics.

These issues have been solved within the theory of inflation. For an introdution see, e.g., [188, 197]. During inflation the universe undergoes an accelerated expansion with scale parameter $\ddot{R} > 0$ which requires a negative pressure $p < -\rho/3$. The simplest fluid with such an equation of state is a cosmological constant with $p = -\rho$. This leads to an exponential expansion $R(t) \propto e^{Ht}$ with a constant Hubble scale $H$. The expansion immediately solves the horizon and flatness problem as well as the problem of relic abundance. Because of the exponentially growing size of the universe, all energy densities are diminished since the density of the cosmological constant is constant (in the general case for $p < -\rho/3$ the energy density of the inflating background falls off at most as $R(t)^{-2}$). This includes also the energy density stored in the curvature. As inflation corresponds to a decreasing comoving Hubble length $H^{-1}/R(t)$, regions which are now far from causal contact could have been correlated before or during inflation. Generically, one needs about $N \simeq 60$ e-foldings, i.e. during the time $\delta t$ inflation lasts the scale factor has to increase by a factor of about $e^{60}$ in order to solve the horizon problem.

Standard inflationary scenarios are usually governed by a scalar field $\phi$ coupled to gravity which has a density and pressure given by

$$\begin{aligned} \rho_\phi &= \frac{1}{2}\dot{\phi}^2 + V(\phi) \\ p_\phi &= \frac{1}{2}\dot{\phi}^2 - V(\phi) \, . \end{aligned} \tag{5.6}$$

The Friedmann equation becomes

$$H^2 = \frac{1}{3M_P^2}\left[\dot{\phi}^2 + V(\phi)\right] \, , \tag{5.7}$$

and the evolution of $\phi$ obeys

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi) \, , \tag{5.8}$$

where the prime denotes the derivative with respect to $\phi$. Notice that there is no gradient term present in the equation of motion for $\phi$. In order to drive an accelerated expansion of an FRW universe, the background energy density has to be sufficiently smooth. Additionally, any spatial gradient is diluted away exponentially fast during the period of inflation. In order for the scalar field to obey the condition $p < -\rho/3$ its evolution has to be dominated by its potential, i.e. $\dot{\phi} < V(\phi)$ which leads to the so-called slow-roll approximation. Finally, the potential of the scalar field should have a minimum so that inflation can find its end.

The evolution equations (5.7) and (5.8) in the slow-roll approximation reduce to

$$\begin{aligned} H^2 &= \frac{V(\phi)}{3M_P^2} \, , \\ 3H\dot{\phi} &= -V'(\phi) \, . \end{aligned} \tag{5.9}$$

Defining the slow-roll parameters for a minimally coupled scalar field

$$\epsilon = \frac{M_p^2}{2}\left(\frac{V'}{V}\right)^2 \, , \tag{5.10}$$

$$\eta = \frac{M_p^2 V''}{V} \, , \tag{5.11}$$

which measure the slope and the curvature of the potential, the necessary conditions for the approximation to be applicable are

$$\epsilon \ll 1 \, , \quad |\eta| \ll 1 \, . \tag{5.12}$$

One of the intriguing properties of inflation is the existence of an attractor solution [198]. Thus, if the potential is able to support inflation, in time all solutions will converge to this inflating solution.

Once the solution $\phi(t)$ approaches the minimum of the potential $V$, the slow roll conditions (5.12) become violated and the field starts oscillating. Couplings of the inflaton field to matter then transfer its energy density to radiation, a process which is called reheating. A detailed account on this subject can be found in [197].

The great success of inflation rests on its capability to predict the origin of cosmic perturbations which we observe in the CMB and which are thought to be responsible for the formation of structure in our universe. These have first been computed by Chibisov and Mukhanov in [193]. During inflation, the scalar field $\phi$ fluctuates quantum mechanically. These fluctuations are stretched to super-horizon scales with unchanged amplitudes as they become frozen. Thus their values are universally determined by their values at horizon crossing. Let us briefly review how the resulting power spectrum of the CMB is generated. For an excellent introduction we refer to, e.g., [197, 199].

The equations of motions of the scalar field perturbations $\phi = \phi_0 + \delta\phi$ in Fourier space are given by (see for example [197, 199])

$$\delta\ddot{\phi}_k + 2RH\delta\dot{\phi}_k + k^2\delta\phi_k = 0 \; . \tag{5.13}$$

Note that this equation has been derived in the slow roll approximation. While the mean of a given Fourier mode is zero $\langle\delta\phi_k\rangle = 0$, the variance of a given mode is nonzero yet uncorrelated with others

$$\langle\delta\phi_k\delta\phi^*_{k'}\rangle \equiv (2\pi)^3 P_{\delta\phi}(k)\delta^3(\vec{k} - \vec{k}') \; . \tag{5.14}$$

The power spectrum $k^3 P_{\delta\phi}(k)$ is scale-invariant if $k^3 P_{\delta\phi}(k) = const$ which is the case to lowest order in the slow roll approximation (5.12).

Through the coupling to gravity, the fluctuations in the inflaton field are transferred to metric perturbations with a power spectrum determined by [197]

$$P_\Phi = \frac{1}{8\pi^2 M_P^2}\frac{H^2}{\epsilon} \; . \tag{5.15}$$

It has been measured by Planck to be $P_\Phi = 2.21 \times 10^{-9}$ [53] and we have used the convention to rescale $P_\Phi$ by $k^3$ since it is scale-invariant. While in the exact slow-roll limit, the spectrum is scale-invariant, the inclusion of a finite non-zero $\epsilon$ leads to deviations from this flatness which is traditionally parametrized by the so-called spectral index $n_s$ according to

$$P_\Phi = \frac{1}{8\pi^2 M_P^2}\frac{H^2}{\epsilon}\left(\frac{k}{k_0}\right)^{n_s-1} \; . \tag{5.16}$$

The spectrum is called blue-tilted for $n_s > 1$ and red-tilted for $n_s < 1$. The value $n_s = 1$ corresponds to exact scale invariance. In the standard slow roll approximation for a scalar field, the spectral index is given by the slow roll parameters and reads

$$n_s - 1 = 2\eta - 6\epsilon \; . \tag{5.17}$$

The Planck satellite [53] has measured its value to high precision and found $n_s = 0.96 \pm 0.073$ in excellent agreement with the calculations of Chibisov and Mukhanov [193].

## 5.2 Dark Matter

The existence of dark matter – a kind of matter which does not, or only very weakly, interact with photons – has become well-established through observational evidence. Beginning with the observation of galactic rotation curves by Zwicky in the early 1930s [200], later observations confirmed that luminous objects in galaxies and galaxy clusters move faster than expectations from determining the mass by the standard luminosity relation suggest. Applying the virial theorem to galaxies or galaxy cluster, one presumes that outside the visible matter, the velocities $v$ of objects on stable Keplerian orbits should fall off as $v \sim \sqrt{1/r}$ with distance $r$. Instead, however, one observes constant velocity dispersions indicating the existence of a halo of gravitating non-luminous matter. By now a vast amount of evidence for the existence of such dark matter has accumulated. Galactic rotation curves [201–203], gravitational leansing data [204–206] and lately precise information from fitting cosmological parameters to the observations of the CMB [54] have led to the paradigm of dark matter.

With baryonic matter only accounting for an energy density of $\Omega_b = 0.0486$ [54], the total amount of dark matter in our universe is estimated to be $\Omega_{CDM} = 0.26472$ [54] which is about $80 - 85\%$ of all the matter in the universe. Information from large-scale structure formation gives further insight into the nature of dark matter. In order to form the observed structure of our universe [207], dark matter is favoured to have been non-relativistic at the time of structure formation. This leads to the model of cold dark matter. Although dark matter provides a consistent framework to explain observations, it has, so far, only been evident through gravitational interactions. It is thus not excluded that instead of postulating the existence of dark matter, one needs to modify gravity on the appropriate scales. Hitherto, no compelling model of such a modification has been suggested which could address all the evidence for dark matter [208, 209]. For an introductory review on dark matter see [210].

Having established the existence of dark matter, the immediate question is what its particle physics origin could be. Possible candidates must be stable on cosmological time scales in order not to have decayed by now. Furthermore, they must interact only very weakly with radiation and produce the correct relic density. Candidates include primordial black holes, weakly-interacting massive particles (WIMPs), sterile neutrinos and axions. Research is going on in many of these directions, especially since beyond the Standard Model physics predicts a plethora of possible WIMP candidates. In this thesis, however, we will only consider the axion.

### 5.2.1 Axionic Dark Matter via the Misalignment Mechanism

The axion introduced in chapter 4, cf. (4.13), seems to be the perfect dark matter candidate: it naturally appears in the solution to the strong CP problem and its couplings to ordinary matter are highly suppressed as indicated by the bound on the scale of symmetry breaking from astrophysical observations which is $f_a > 10^9\text{GeV}$ [58]. These auspicious premises have led to large activities in the field of axionic dark matter since the early 1980s [211–213]. One can generically distinguish between three scenarios of axion production in the universe providing for the dark matter abundance (for an extensive review see [57]): first by thermal production, second by topological effects via breaking of the PQ symmetry after inflation, or third from the so-called vacuum misalignment mechanism where the initially arbitrary axion field amplitude is forced to coherent oscillations about the minimum of its potential once QCD effects become important.

It has been shown [214, 215] that a thermally produced axion[1], cannot contribute the dominant dark matter component of the universe and satisfy the experimental and theoretical bounds simultaneously. The topological production of axions occurs if the PQ symmetry is restored after inflation, i.e. if the reheating temperature $T_{RH}$ is larger than the temperature of the PQ-phase transition $T_{PQ}$. The symmetry breaking of the PQ symmetry after (or during) reheating generates axionic strings [216, 217]. These strings are usually assumed to decay into axions before their energy density dominates the universe. The axions produced in this way become non-relativistic after the QCD phase transition when the axion mass develops and could contribute to the dark matter density of the universe. This scenario, however, comes with a caveat: After the QCD phase transition there exist $N$ degenerate vacua associated with the color anomaly. Therefore, domain walls between these vacua form and quickly start to dominate the energy density of the universe [60, 218]. Domain walls are gravitationally repulsive and thus lead to an accelerated expansion of the universe. Finally, the structures produced in this scenarios are different from those observed. Therefore, such scenarios are generally disfavoured [60, 218].

This leaves the misalignment mechanism which also provides the most natural production mechansim as it always contributes to the dark matter density. In this scenario, the underlying U(1) symmetry is broken before or at the beginning of inflation at the scale $f_a$ and is not restored during or after inflation. The corresponding Higgs field settles into its minimum and the axion angle $\Theta_i = \frac{a_i}{f_a}$ assumes a constant value between $[-\pi, \pi]$. The value of $\Theta_i$ is constant only within one causally connected region, but can vary between different causal patches. Nevertheless, after inflation each of these patches will have been stretched out such that in our observable universe only one initial angle prevails.

The equation of motion for the axion field $a$ in an expanding FRW background (5.1) is given by

$$\ddot{a} + 3H\dot{a} + m_a^2 a = 0 \; . \tag{5.18}$$

Note, that only the zero-mode contributes significantly to the dark matter abundance, see, e.g., [219], as the energy of the spatial gradient decays much faster. The mass of the axion is produced by non-perturbative effects which are temperature dependent [220]; for the case of zero temperature see section 4.5 and 4.2. Furthermore, at first $H \gg m_a$ which is why the axion field amplitude initially remains constant, $a = a_i = \Theta_i f_a$. Once the temperature of the universe reaches the QCD scale, the instanton potential for the axion will start to build up. The axion will then roll towards the minimum of the potential at $\frac{a}{f_a} = \Theta = 0$ and will start to oscillate with an amplitude $\sim \Theta_i$ when the

[1]These would, in fact, constitute warm dark matter.

temperature $T_{\rm osc}$ is reached. These coherent oscillations about $\Theta = 0$ form a Bose-Einstein-condensate of nonrelativistic axion particles and contribute to the dark matter density of the universe. At the time when $m_a(T_{\rm osc}) \approx 3H(T_{\rm osc})$, the energy density can be determined from the virial theorem [57] which yields

$$\rho_a(T_{\rm osc}) \simeq m_a(T_{\rm osc})^2 a_i^2 \simeq \frac{1}{2} m_a(T_{\rm osc})^2 \Theta_i^2 f_a^2. \tag{5.19}$$

The mass of the axion is produced by nonperturbative, temperature dependent QCD effects. Most computations of the axion potential are performed in the dilute instanton gas approximation [56, 221], where the biggest uncertainty in the mass of the axion appears around the QCD phase transition. Due to strong correlations and strong coupling at the phase transition, the computation of the potential becomes challenging. For an updated computation of the axion mass see [221]. Note that the issues concerning the standard instanton derivation (see chapter 4) and possible non-analyticities of the axion potential can be disregarded in our discussion. We can use the potential derived in [221] since the axion phase $\Theta_i$ needed later on is small enough such that the axionic potential is well approximated by the mass square $m_a^2(T)$. This eliminates the need to know the full axion potential as also the non-analytic potential can be approximated by $m_a^2(T)$ when expanded around zero. As the axion mass (potential) is strongly temperature dependent, we will consider two different regimes in the following. In the first regime oscillations start at a temperature above the QCD phase transition, i.e. $T_{osc} \gtrsim \Lambda_{QCD}$, while in the second they start after the QCD phase transition occured, i.e. $T_{osc} \lesssim \Lambda_{QCD}$. The oscillation temperature is given by [221, 222],

$$T_{osc} \simeq \begin{cases} 9.16 \times 10^2 MeV \left(\frac{f_a}{10^{12} GeV}\right)^{-0.184}, \\ 6.65 \times 10^4 MeV \left(\frac{f_a}{10^{12} GeV}\right)^{-0.5}. \end{cases} \tag{5.20}$$

One finds [221, 222] that depending on the value of $f_a$ the axion starts oscillating earlier for smaller $f_a$, or later (at lower temperatures) for larger $f_a$. Once $m_a \gg H$, it follows from (5.18) that the energy density of axions per comoving volume $\rho_a(t)/R(t)^3$ is conserved. The axion energy density of today is [213]

$$\rho_a(T_0) = \rho_a(T_{\rm osc}) \frac{m_a(T_{\rm osc})}{m_a(T)} \left(\frac{R(T_{\rm osc})}{R(T)}\right)^3, \tag{5.21}$$

where $T_0$ denotes the temperature of the universe today and $R(T)$ is the scale factor at a given temperature $T$ or equivalently at a given time $t$. The mass of the axion today at $T = T_0$ is given by $m_a^2(T_0) \equiv m_a^2 = m_u m_d f_\pi^2 m_\pi^2/((m_u + m_d)^2 f_a^2) \simeq \Lambda_{QCD}^4/f_a^2$. Note that we are not concerned with scenarios [223, 224] which include a short period of late inflation which would dilute the axion density (5.21) more efficiently.

The energy density of axions today can be expressed in terms of the initial misalignment angle $\Theta_i$ and the axion decay constant $f_a$. It reads [56, 222]

$$\Omega_a h^2 \simeq \begin{cases} 0.195\ \Theta_i^2 \left(\frac{f_a}{10^{12}\mathrm{GeV}}\right)^{1.184} & ; (f_a < f_\Lambda), \\ 0.0035345\ \Theta_i^2 \left(\frac{f_a}{10^{12}\mathrm{GeV}}\right)^{1.5} & ; (f_a > f_\Lambda)\ , \end{cases} \tag{5.22}$$

where $f_\Lambda \equiv 3.6 \times 10^{17}$GeV and $h = H_0/(100\mathrm{km}\, s^{-1}\mathrm{Mpc}^{-1})$ is a numerical factor which accounts for the uncertainties in the measurements of the Hubble parameter today $H_0 = (67.3 \pm 1.2)\mathrm{km}\, s^{-1}\mathrm{Mpc}^{-1}$. The upper expression corresponds to $T_{\mathrm{osc}} > \Lambda_{\mathrm{QCD}}$ while the lower applies for $T_{\mathrm{osc}} < \Lambda_{\mathrm{QCD}}$. The dark matter density parameter determined by Planck measurements is $\Omega_{CDM} h^2 = 0.1199 \pm 0.0027$. Fluctuations $\sqrt{\langle \delta\Theta_i{}^2 \rangle}$ in $\Theta_i$ which could be, in principle, induced during inflation contribute to the dark matter energy density. However, for the values of $\Theta_i$ that we consider, they are negligible. Nonetheless, these fluctuations have an observational effect which we will discuss in the next section. Corrections to (5.22) from anharmonicities are non-negligible only for $\Theta_i > \pi$, i.e. for axion cold dark matter in the $f_a < 5 \times 10^{11}$GeV range. These do not change our conclusions qualitatively and for the sake of the argument we assume $f_a > 10^{11}$ GeV.

Equation (5.22) can be recast in terms of the more intuitive ratio of the dark matter density $\rho^a_{\mathrm{DM}}$ provided by axions to the total observed dark matter energy density $\rho^{\mathrm{obs}}_{\mathrm{DM}}$ [54]

$$\frac{\rho^a_{\mathrm{DM}}}{\rho^{\mathrm{obs}}_{\mathrm{DM}}} \simeq \Theta_i^2 \begin{cases} 1.7\ \left(\frac{f_a}{10^{12}\mathrm{GeV}}\right)^{1.184} & ; (f_a < f_\Lambda) \\ 8 \times 10^5 \left(\frac{f_a}{10^{17}\mathrm{GeV}}\right)^{1.5} & ; (f_a > f_\Lambda)\ , \end{cases} \tag{5.23}$$

where $\rho^{\mathrm{obs}}_{\mathrm{DM}} = 1.3$ keV/cm$^3$ [54]. Note that for each value of $f_a \gtrsim 10^{12}$ GeV there is an initial condition for which axions can account for all the dark matter. We denote it by $\Theta_i(f_a)$.

### 5.2.2 Isocurvature Constraints

The main additional constraint on the misalignment mechanism comes from isocurvature perturbations. During inflation the axion is essentially massless since $H_I \gg m_a$. Therefore, the axion will be subject to quantum fluctuations in the quasi-de Sitter spacetime like any light scalar field in de Sitter. The amplitude of these quantum fluctuations is of the order of the Hubble scale (cf. section 5.1) with an almost scale invariant spectrum

$$\left\langle |\delta^2 a(k)| \right\rangle = \frac{H_I^2}{4\pi^2} \frac{2\pi}{k^3} \quad \Leftrightarrow \quad \sigma_\Theta = \frac{H_I}{2\pi f_a}\ , \tag{5.24}$$

where $\sigma_\Theta = \sqrt{\langle \delta\Theta_i{}^2 \rangle}$. The underlying physics also produces perturbations in the inflaton seeding the structure of today's universe. Since the axion potential is flat during inflation and the gradient energy is negligible, see, e.g., [219], these fluctuations, which are called isocurvature or entropy perturbations (cf. [62]), do not perturb the total energy density of the universe, but only the number density of axions.

The gauge-invariant entropy perturbations can be defined as [62][2]

$$S = \frac{\delta n_a}{n_a} - \frac{\delta n_\gamma}{n_\gamma} = \delta_a - \frac{3}{4}\delta_\gamma, \tag{5.25}$$

where $n_a$ and $n_\gamma$ denote the axion and the radiation number density and $\delta_a \equiv \frac{\delta\rho_a}{\rho_a}$ and $\delta_\gamma \equiv \frac{\delta\rho_\gamma}{\rho_\gamma}$ the energy density contrast of the axion field and radiation, respectively. Since all other matter is tightly connected to the radiation energy density (as they are assumed to be in thermal equilibrium), $\delta_b = \frac{3}{4}\delta_\nu = \frac{3}{4}\delta_\gamma$ [62], we will only explicitly consider the entropy perturbation between the axion and radiation. After the perturbations have left the horizon, they remain constant. This is true for both the adiabatic and the curvature perturbations (see, e.g., [62, 225]). The isocurvature mode characterizes the relative number density perturbation of different fluids on constant curvature hypersurfaces, i.e. constant energy density hypersurfaces. Therefore, the density perturbations of the axion and radiation are related by

$$\begin{aligned} \delta\rho_a + \delta\rho_\gamma &= 0 \\ \rho_a\delta_a &= -\rho_\gamma\delta_\gamma. \end{aligned} \tag{5.26}$$

Hence, during radiation domination ($\rho_\gamma \gg \rho_a$), $\delta_a \gg \delta_\gamma$ and the initial condition for the entropy perturbation (5.25) is $S_i = \delta_a = \delta n_a/n_a$. The entropy perturbation, $S$, remains constant as long as it is outside the horizon [62]. When these perturbation reenter the horizon, however, the universe is matter dominated and the perturbations, initially set by the axion fluid, are converted into temperature perturbations,

$$\begin{aligned} S_\star &= -\frac{3}{4}\delta_\gamma = -3\frac{\delta T}{T} \\ \Rightarrow \quad \frac{\delta T}{T}\bigg|_{\text{iso}} &= -\frac{1}{3}\delta_{a_i}. \end{aligned} \tag{5.27}$$

They can be detected in the cosmic microwave background (CMB). Observationally, they can be disentangled from the adiabatic perturbations generated by the inflaton, since the isocurvature perturbations shift the acoustic peaks towards smaller scales [226]. So far, however, no isocurvature perturbations have been observed [53]. The relevant bound on isocurvature perturbations of dark matter is best expressed as a bound on the ratio $\alpha$ of the isocurvature perturbations, $P_{\text{iso}} = \langle(\delta T/T)^2\rangle_{\text{iso}}$, to the total temperature perturbations of the CMB, $P_{\text{tot}} \simeq P_{\text{ad}} = \langle(\delta T/T)^2\rangle_{\text{ad}}$,

$$\alpha \equiv \frac{P_{iso}}{P_{ad}} \simeq \frac{H_I^2}{P_s \pi^2 f_a^2 \Theta_i^2} < 0.039\,(95\%\text{C.L.}), \tag{5.28}$$

where $P_s = 2.2 \times 10^{-9}$ is the power spectrum of the CMB measured by Planck [53].

The bound (5.28) together with the measured dark matter density (5.23) constrains the parameters of the axionic dark matter model. One can translate (5.28) to a bound on

[2]We work in comoving coordinates, so that all quantities have to be considered in the comoving gauge.

the Hubble scale during inflation by inserting $\Theta_i(f_a)$ given by (5.23) which yields

$$H_I < \pi\sqrt{\alpha P_s} f_a \Theta_i(f_a) \,. \tag{5.29}$$

This bound has been much discussed in the literature, see e.g. [56, 222]. If the axion accounts for all the dark matter, one finds

$$H_I < \begin{cases} 2.3\times 10^7 \left(\frac{f_a}{10^{12}\mathrm{GeV}}\right)^{0.41} \ \mathrm{GeV}\,; & (f_a < f_\Lambda) \\ 3.2\times 10^9 \left(\frac{f_a}{10^{17}\mathrm{GeV}}\right)^{0.25} \ \mathrm{GeV}\,; & (f_a > f_\Lambda)\,. \end{cases} \tag{5.30}$$

As explained in chapter 2, all physical scales must be sub-Planckian. Thus, in order to have a sensible effective field theory for the axion, the scale of the effective field theory (here the symmetry breaking scale) has to be smaller than the Planck scale, $f_a < M_P$. This constrains the Hubble scale during inflation to be smaller than $H_I \lesssim 10^{10}$GeV, see also [56].

Generically, the simplest models of inflation, e.g., chaotic single field slow-roll scenarios, have a Hubble scale during inflation of the order $H_I \simeq 10^{13}$GeV. This can be easily checked from (5.15) and (5.17) when taking into account the observational data from Planck which measured the power spectrum $P_s = 2.2\times 10^{-9}$ and the spectral index $n_s = 0.9603 \pm 0.0073$ [53]. For most single field scenarios, in particular those with a monomial potential,[3] $\epsilon \sim \mathcal{O}(10^{-3})$ which gives a Hubble rate $\simeq 10^{13}$GeV.

Considering slow-roll scenarios, only those with a hierarchy between the two slow-roll parameters $\eta \gg \epsilon$ can provide a sufficiently low Hubble scale $H \lesssim 10^{10}$ GeV. It is only then that the axion isocurvature bound is avoided. Possible realizations are, for example, natural inflation scenarios [227], where $n_s$ receives the dominant contribution from a tachyonic mass. However, these scenarios generically need trans-Planckian *physical* scales in order to match observations [53].

Finally, generic multifield scenarios, which could in principle allow for a low Hubble scale, are in conflict with the absence of non-Gaussianities $f_{NL} \lesssim 2.7 \pm 5.7$ [53]. The bound could, nevertheless, be avoided if CMB anisotropies are transferred from isocurvature perturbations of a light scalar which decays to radiation at the end of inflation, e.g., to the curvaton [228]. We will not discuss such models here.

With this we conclude that in generic single field slow-roll inflation, a low Hubble scale is disfavoured and thus these scenarios are in tension with the proposed models of axionic dark matter.

[3]For a potential $V \propto \phi^n$, $\epsilon = \frac{n^2}{2}\frac{M_P^2}{\phi^2} = \frac{n}{2(n-1)}$.

## 5.3 Saving the Dark Matter Axion

In the last section, we saw that the constraints on the axion dark matter scenario which are in conflict with observations are rooted in the large amplitude of quantum fluctuations of the axion during the de Sitter expansion. This section presents two possible ways to suppress these fluctuations. First, we consider a specific model which generically has a low inflationary Hubble scale. Second, we effectively suppress perturbations in the axion field by changing its kinetic term on de Sitter. In this way, the amplitude of the axion field, which imprints its perturbations into temperature fluctuations, is smaller than the canonically normalized field which receives quantum fluctuations during inflation.

### 5.3.1 Small Scale Inflation

In order to obtain a small scale inflationary scenario, we consider the model of [229], which can be understood as a modification of natural inflation [227]. In this model all physical scales are sub-Planckian. First, let us familiarize ourselves with the gravitationally enhanced friction mechanism (GEF) of [65]. The idea of the GEF model is to consider an effective field theory on de Sitter which includes an additional nonrenormalizable coupling in contrast to General Relativity. In the Einstein frame, the Lagrangian is given by [65]

$$\frac{1}{2}\int d^4x\sqrt{-g}\left[M_P^2 R - \left(g^{\alpha\beta} - \frac{G^{\alpha\beta}}{M_\phi^2}\right)\partial_\alpha\phi\partial_\beta\phi - 2V(\phi)\right] . \tag{5.31}$$

We conveniently work in the Einstein frame, as it is the frame in which the graviton has a canonical kinetic term. $G^{\alpha\beta}$ is the Einstein tensor which on a quasi de Sitter spacetime takes the form $G^{\alpha\beta} \simeq -3g^{\alpha\beta}H^2$. Notice that on FRW backgrounds which are driven by a fluid with an equation of state which obeys

$$p \leq -\frac{1}{6}\rho \, , \tag{5.32}$$

the Einstein tensor $G_{\mu\nu}$ defines a causal structure for the kinetic term. This is the case for all inflating backgrounds.

It is important to note that the strong coupling scale on such a background is not given by $M_\phi$, but rather by $\Lambda_{\rm GEF}^3 = (1+3\frac{H^2}{M_\phi^2})M^2 M_P$. Thus, the effective field theory considered is an expansion of higher-dimensional operators which are suppressed by $\Lambda_{\rm GEF}$ on this background. The inflationary scenario considered is slow roll, i.e. all derivatives are "small". Thus only the lowest order in derivatives of the scalar $\phi$ are considered. The Lagrangian (5.31) is ghost-free, meaning that all equations of motions contain only two time derivatives of the fields. In fact, in the decoupling limit, where one first integrates out the equation of motion of gravity and then sends $M_P \to \infty$ keeping $M_P M^2$ and all other scales fixed, one recovers the quartic Galileon term for the scalar field [230, 231] which is known to reduce to an equation of motion without higher time derivatives.

During inflation, the kinetic term of the field $\phi$ is effectively renormalized by

$$\gamma^2 = 1 + \frac{3H_I^2}{M_\phi^2} \ . \tag{5.33}$$

The scalar field $\phi$ must then be canonically normalized to $\bar{\phi} = \gamma\phi$, and the potential changes according to $V(\phi) \to V(\bar{\phi}/\gamma)$. In the high friction regime $H_I \gg M_\phi$ [229], the curvatures of the potential are suppressed by factors of $\gamma$, e.g. $\partial_{\bar{\phi}} V = V'/\gamma \ll V'$, where the prime denotes derivatives w.r.t. $\phi$. Therefore, potentials that are steep in $\phi$ can be flat in $\bar{\phi}$ and at the same time weakly coupled [65, 232]. This is the gravitationally enhanced friction mechanism (GEF) explained in [229] and [233]. In these models, the power spectrum of fluctuations and their spectral index are given by

$$P_s = \frac{H_I^2}{8\pi^2 M_p^2}\frac{1}{\epsilon} \quad \text{and} \quad n_s - 1 = -8\epsilon + 2\eta, \tag{5.34}$$

where

$$\epsilon = \frac{V'^2 M_p^2}{2V^2}\frac{1}{\gamma^2} \quad ; \quad \eta = \frac{V'' M_p^2}{V}\frac{1}{\gamma^2} \tag{5.35}$$

are the slow-roll parameters of the theory, which satisfy $\eta, \epsilon \ll 1$. Note that they are different from those in standard slow roll inflationary theories, cf. (5.9).

Let us return to the original proposal of this section, namely considering a natural inflation scenario equipped with the above described additional gravitational coupling on de Sitter. The potential for the scalar field is given by

$$V(\phi) = \Lambda_h^4(2 - \cos\phi/f_h)). \tag{5.36}$$

Such a potential can be understood as that of a hidden (additional) axion, characterized by a decay constant $f_h$, which is the inflaton. The potential is induced by instanton effects of a hidden gauge group with strong-coupling scale $\Lambda_h$. In the small field case ($\phi/f_h \ll 1$), one finds

$$V(\phi) \simeq \Lambda_h^4(2 - \phi^2/2f_h^2) \ . \tag{5.37}$$

For such a potential (5.34) directly implies

$$1 - n_s \simeq 2\eta = \frac{M_p^2}{f_h^2}\frac{1}{\gamma^2}. \tag{5.38}$$

As Planck measures $1 - n_s \simeq 0.04$, these models only allow for $f_h \ll M_P$ in the high friction limit $\gamma \gg 1$. Otherwise, these models are excluded on the grounds that no physical scale should be larger than the Planck scale.

The power spectrum relates $H_I$ and $\phi/f_h$ via $\epsilon = \phi^2 M_P^2/f_h^4$,

$$H_I = \pi\sqrt{P_s(1-n_s)/2}\, M_p \frac{\phi}{f_h} \simeq 5\times 10^{13}\frac{\phi}{f_h}\text{GeV}. \tag{5.39}$$

Thus, by considering a small ratio $\phi/f_h$, one can have a low scale inflationary scenario with $H_I < 10^{10}$GeV. In this model, axion dark matter created via the misalignment mechanism [56, 222] is in agreement with isocurvature bounds from Planck [53]. Furthermore, since the model is single-field, non-Gaussianities are negligible [234].

### 5.3.2 Suppressing Isocurvature Perturbations

The isocurvature perturbations are given by the ratio $\delta a/a_i$ at the time the axion starts oscillating, i.e. $H \simeq m_a$. The perturbations $\delta a$ are generated during inflation when the scalar field acquires quantum fluctuations due to the de Sitter background. They subsequently leave the horizon and become frozen. In order to compute the induced quantum fluctuations, one has to consider the canonically normalized field. During inflation, the canonically normalized field is $\bar{a} = \sqrt{\gamma}a$, where $\gamma$ is given by (5.33). It receives quantum fluctuations of the order of the Hubble scale, $\delta\bar{a} = \frac{H_I}{2\pi}$. In terms of the field $a$, which is introduced in the Lagrangian (5.31), this yields $\delta a = \frac{H_I}{\sqrt{\gamma}}$. Once the axion field starts oscillating, these perturbations become gravitationally coupled to the radiation fluid as in (5.25). At this time, inflation has long passed, the universe is radiation dominated and $H_I \ll M$. This means that the initial conditions for the axionic entropy perturbations are set by the canonically normalized field $a$. Thus, the correct initial condition for $S_i$ is given by

$$S_i = \frac{\delta a}{a_i} = \frac{H_I}{2\pi\sqrt{\gamma}f_a\Theta_i}, \tag{5.40}$$

where we used $a_i = f_a\Theta_i$. Essentially, this mechanism is equivalent to enhancing the decay constant of the axion, $f_a \to f_a\sqrt{(1+3\frac{H^2}{M_a^2})}$, during inflation only.[4]

Instead of a bound on the Hubble scale as in (5.29), one now obtains a relation between the suppression scale $M_a$ and the isocurvature ratio (5.28) which reads

$$f_a^2\Theta_i^2 = \frac{M_a^2}{A_s\pi^2\alpha}. \tag{5.41}$$

In contrast to before, this is *independent of the scale of inflation* and thus abolishes the need for a small scale inflationary scenario. Note that the constraint (5.29) has changed to $f_a \gg \dfrac{M_a}{\Theta_i}$. Taking into account the required dark matter abundance, $M_a$ can be determined as a function of $f_a$ and the measured parameters $P_s, \alpha$ and $\Omega_c h^2$. The latter is the solution to the equations

$$\Omega_c h^2 \geq \begin{cases} \frac{0.195M_a^2}{\pi^2\alpha f_a^2 A_s}\left(\frac{f_a}{10^{12}\mathrm{GeV}}\right)^{1.184} ; & (f_a \lesssim f_\Lambda) \\ \frac{0.0035345M_a^2}{\pi^2\alpha f_a^2 A_s}\left(\frac{f_a}{10^{12}\mathrm{GeV}}\right)^{1.5} ; & (f_a \gtrsim f_\Lambda)\,. \end{cases} \tag{5.42}$$

[4]This mechanism is similar in spirit to a time-varying axion decay constant $f_a$ proposed by Linde [191] in order to suppress the isocurvature perturbations of the axion.

Using the Planck data [53] the upper bound on $M_a$ is

$$M_a \leq M_a^{\max} = \begin{cases} 4.0 \times 10^7 \left(\frac{f_a}{10^{12}\mathrm{GeV}}\right)^{0.41} \mathrm{GeV}\ ; & (f_a \lesssim f_\Lambda), \\ 5.2 \times 10^9 \left(\frac{f_a}{10^{17}\mathrm{GeV}}\right)^{0.25} \mathrm{GeV}\ ; & (f_a \gtrsim f_\Lambda)\ . \end{cases} \tag{5.43}$$

Imposing $f_a < M_p$, we get $M_a < 1.2 \times 10^{10}$GeV. In our scenario $\Theta_i$ is small as long as $f_a > 10^{11}$ GeV and anharmonic effects can be safely neglected which justifies our assumptions a posteriori. Indeed, for $f_a \sim 4 \times 10^{16}$ GeV we have $\Theta_i = 1.5 \times 10^{-3}$.

Another way of softening the cosmological bounds on the axion was proposed in [235]. There, the possibility that the QCD sector becomes strongly coupled during inflation is discussed. This is achieved by letting the QCD gauge coupling to be determined by the vacuum expectation value of some scalar field, e.g. the inflaton or a dilaton. Such a coupling arises naturally in superstring or supergravity theories where the gauge coupling is determined by the following term in the Lagrangian

$$f(\phi/M_p) F^a_{\mu\nu} F^{a\mu\nu}, \tag{5.44}$$

where $f$ is some function of the field $\phi$ and $F^a_{\mu\nu}$ is the QCD field strength tensor. If $f(\phi/M_p)$ is small during inflation, the scale $\Lambda_{\mathrm{QCD_{eff}}} \propto \exp^{-\alpha f(\phi/M_p)} M_p$ with $\alpha$ being a numerical factor, can become large and instantons can produce a mass $m_a \sim H_I$ for the axion. Hence, the axion will settle into its minimum at $a = 0$ during inflation. Nevertheless, it can still account for the dark matter density of our universe since during reheating with temperature $T_R$ (when the gauge coupling has settled to its low energy value and the axion has no longer a potential) thermal effects will induce perturbations in the axion field. For temperatures $T_R > T > \Lambda_{\mathrm{QCD}}$, when $m_a = 0$, one can assume a random walk for the axion perturbations of $\Delta a \sim T$ per Hubble time. During the interval $[T_R - \Lambda_{\mathrm{QCD}}]$, the deviation can be at most $\Delta a \sim T_R$. Therefore, the typical energy scale of the coherent oscillations is given by $\rho_a \sim \frac{1}{2} m_a^2 T_R^2$. With this mechanism, the bounds on the isocurvature perturbations are easily circumvented, as the axion is no longer light during inflation and hence does not receive quantum fluctuations.

## 5.4 Inflation and Dark Matter from the Standard Model

This section considers a setup where on de Sitter the gravitational couplings to a scalar field are generically of the form (5.31). Within this setup, it is possible that the Standard Model of particle physics (with the addition of the QCD axion) provides the particle content needed to explain both inflation and the dark matter abundance.

So far, no conclusive candidate for the scalar inflaton field has emerged. The Standard Model itself includes only one scalar particle, the Higgs bosons [236–240]. In the Standard Model, the Higgs is introduced to unitarize the scattering amplitudes of $W$-bosons. In this description, the $W$-particles are massive because of spontaneous symmetry breaking in the Higgs sector. The Goldstone boson from the electro-weak symmetry breaking combines with the $W$ vector boson to become its longitudinal polarization; it is "eaten" by the $W$-boson to acquire a mass. In addition, the Higgs particle provides a generic way of giving mass to the Standard Model fermions. The discovery of this particle has long been considered the holy grail of particle physics until recently, the first hints of its discovery appeared [241, 242]. Experiments at the LHC detected a signal of a scalar particle with mass of $\sim 125-126\,\text{GeV}$ [241, 242]. Although it is still not exactly proven that it is the Standard Model Higgs boson, the existence of a scalar particle seems to be established.

As shown in [65], the Higgs boson $h$ with action (5.31) and potential

$$V(h) = \frac{\lambda}{4}(|h|^2 - v^2)^2\,, \tag{5.45}$$

where $v = 246\,\text{GeV}$ is the Higgs boson vacuum expectation value and $\lambda$ is the Higgs self-coupling, leads to a successful model of inflation. In the following, we assume that during inflation $|h| \gg v$ and therefore, the potential is approximately $V(h) \simeq \lambda/4\; h^4$. This is a reasonable assumption since in order to drive inflation the amplitude of the Higgs field must be $|h| \sim 0.01\lambda^{-\frac{1}{4}} M_P$ at that time. Note that canonical inflationary models with a potential $\lambda\phi^4$ are ruled out by Planck because they result in too much gravitational wave emission [53]. However, in the GEF mechanism, the amplitudes of gravitational waves are suppressed and the bounds can be evaded [229, 230].

Equations (5.34) predict $H_I = 2\pi M_p\sqrt{2P_s(1-n_s)/5} = 9\times 10^{13}$ GeV, while $M_h$ remains an open parameter. To determine a bound on $M_h$ we have to complement them with the Friedman equation

$$H_I^2 \simeq \frac{V}{3M_P^2} = \frac{\lambda\phi^4}{12M_P^2}, \tag{5.46}$$

which during slow-roll, $\epsilon \simeq 8M_P^2/(\phi^2\gamma)$, then leads to

$$M_h = 4.0\, M_P(1-n_s)^{\frac{5}{4}} P_s^{\frac{3}{4}} \lambda^{-\frac{1}{4}} \simeq \lambda^{-\frac{1}{4}} 5.5\times 10^{10}\text{GeV}. \tag{5.47}$$

The recent measurement of ATLAS and CMS of the Higgs boson mass, $m_h \approx 126$ GeV [241, 242], give $\lambda = 0.26$. However, $\lambda$ runs from the electroweak scale to the scale $\sim H_I$ where our formulas apply. As an order of magnitude estimate of the values of $\lambda$ during inflation, we considered the Standard Model renormalization group equations up to the scale $H_I$, for a recent computation see e.g. [243]. In order to avoid the electroweak instability problem (see e.g. and [244] references therein), we consider values of the top mass $m_t \simeq 171\text{GeV}$ at the electroweak scale and values of the strong coupling constant $\alpha_s = 0.1184$. This is within the $3\sigma$ range of the measured value. Of course, new physics can play an important role in the running, but this is a source of uncertainties that we cannot address. The additional non-minimal coupling itself influences the running (at scales $> M_h$) on de Sitter. We expect that it softens the running such that $\lambda > 0$ even for different values of $m_t$, but this important aspect is left for future work. All in all, the assumption $\lambda(H_I) > 0$ allows to consider $\lambda \sim 0.01$ as an order-of-magnitude estimate.

If our model is responsible for having Higgs inflation and QCD axions as dark matter, we would ideally only tolerate a small hierarchy between $M_a$ and $M_h$. The ratio $M_a/M_h$ is not fixed by our model and Planck data, but it is bounded from above because of the

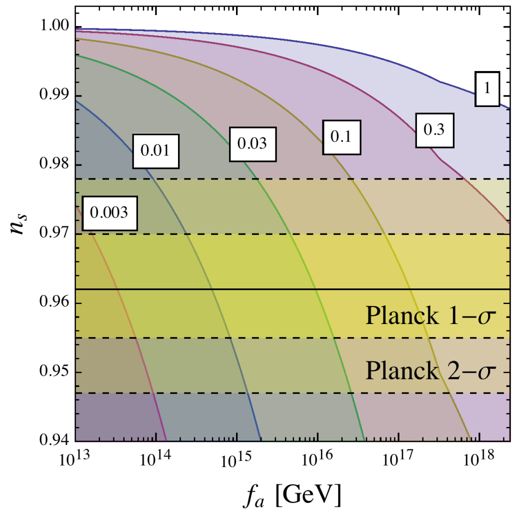


FIGURE 5.1: Isocontours of $M_a^{\max}/M_h$, i.e. the maximum value of $M_a/M_h$ in our model of Higgs inflation and axion dark matter allowed by Planck constraints on isocurvature perturbations. We have used $\lambda(H_I) = 0.01$. The spectral index $n_s$ measured by Planck is shown as yellow bands for 1 and $2\sigma$ error. The area outside is disfavored at 95% C.L.

upper limit for $M_a$, c.f. (5.43). The upper limit $M_a^{\max}/M_h$ depends on the value of $f_a$ and generally decreases with decreasing $f_a$. In Fig. 5.1 we show isocontours of $M_a^{\max}/M_h$ in the $n_s - f_a$ plane. We see that natural values ($M_a^{\max}/M_h \sim 0.1$) are possible for the highest meaningful values of $f_a \sim M_p$. Even for values of $f_a \sim 10^{16}$ GeV, we get quite acceptable ratios of $0.01 < M_a^{\max}/M_h \lesssim 0.03$. These numbers are relatively sensitive to uncertainties on the measured value of $n_s$ but not on $\lambda$, because it enters mainly through $M_h$, which scales as $(1-n_s)^{5/4}/\lambda^{1/4}$ (see (5.47)). Therefore, even though for a complete picture one has to perform the full RG analysis, one can be confident that this will not change the main conclusions.

## 5.5 Summary

The axion is one of the best motivated dark matter candidates at hand. Due to its nature of a pseudo-Goldstone particle in the Peccei-Quinn solution to the strong CP problem, its couplings to other matter are suppressed by the breaking scale of the stipulated $U(1)_{PQ}$ symmetry. If it can be produced in the early universe, it could account for parts, or even all, of the observed dark matter density today. One such production mechanism is the so-called misalignment mechanism in which the axion density originates from an initial random value the axion field asssumes after symmetry breaking. Once QCD effects become important, the axion acquires a mass and while relaxing towards the minimum of its potential, it produces non-relativistic axion particles. However, during inflation the axion is light and thus sensitive to quantum fluctuations. Depending on the inflationary energy scale $\sim H_I$, the generated isocurvature perturbations are in conflict with observations which put a severe constraint on axion dark matter scenarios.

In this chapter, we have shown that by changing the kinetic coupling of the axion on inflationary backgrounds the tension between observation and predictions can be eased. We found that it is possible to produce the correct amount of dark matter without any restriction from the scale of inflation. In addition, we pointed out that with the latest Planck data, possible low scale inflationary models which could avoid the bounds without any additional coupling for the axion are rare. An example of such a model was presented in section 5.3. Lastly, we tried to push our model such that it can accommodate inflation and dark matter within the realm of Standard Model physics (including the axion) and found that it is possible within a common framework of non-minimal kinetic couplings.

# Chapter 6

# Massive Gravity

In relativistic quantum field theory on Minkowski space, one particle states can be labelled according to their representation of the Poincaré group. The Casimir operators, which are those operators that commute with all generators of the group transformations, classify the invariant mass $m^2$ and the spin $s$. Each irreducible representation is then uniquely labelled by $m$ and $s$.

In principle, this classification is most useful when considering the eigenstates of the full Hamiltonian. However, in most cases, the exact diagonalization of the interacting Hamiltonian is extremely involved and not practicable. Instead, one introduces the concept of asymptotic states which are eigenstates of the quadratic part of the Hamiltonian. This is a particular useful representation since interactions are considered to be localized in time and space, and hence the measured particle eigenstates are asymptotically equivalent to the asymptotic states. This also brings about the concept of the S-matrix which is defined as the transition amplitude of some asymptotic initial state $|in\rangle$ to an asymptotic final state $|out\rangle$ induced by the Hamiltonian of the system. The square of the S-matrix can be interpreted as the probability of transition

$$P(in, out) = |\langle out|S|in\rangle|^2 \quad \Rightarrow \quad |\langle out|S|in\rangle| \leq 1 \ . \tag{6.1}$$

It immediately follows that for a fundamental theory the S-matrix should be a unitary operator and each single scattering process must have probability smaller than one.

As discussed in the introduction 1, effective field theories are very useful to describe physics at a certain scale considering only the relevant operators at this scale. Usually this can be thought of as having integrated out heavy physics in the path integral down to a certain scale $\Lambda$ which gives a low energy approximation of the theory. This procedure generically will introduce operators which lead to a nonunitary S-matrix at energies larger than the scale $\Lambda$. However, as the effective theory can only describe physics accurately at energies much lower than $\Lambda$, this seeming unitarity violation will need to be taken care of only when considering the full fundamental theory. In a standard treatment, at these high energy scales, new degrees of freedom will enter the theory (they are integrated in) and thus provide a viable ultraviolet (UV) theory. This is what was called a UV completion in the Wilsonian sense. However, as we have discussed in chapter 2 for gravity there might be other ways to cure the apparent unitarity violation of an effective field theory [7, 8, 245].

It is interesting that the Standard Model of particle physics[1] which describes the interactions of fundamental massive and massless spin-0 and spin-1/2 fields together with a fundamental massless spin-1 field can be considered a fundamental theory that does not violate unitarity at any scale. Gravity on the other hand has to be considered an effective field theory; although it might not be necessary to find a UV completion in the Wilsonian sense, see chapter 2. The field theoretical treatment describes gravity as the interaction of a massless spin-2 particle with itself and universally with all other particles determined by the laws of General Relativity. It can be shown that General Relativity is *essentially* the unique theory of an interacting massless spin-2 particle [80–82], i.e. 2 degrees of freedom. By essentially we mean in the lowest order of a momentum expansion. An interesting question to ask is then whether interactions of a *massive* spin-2 particle obey such a uniqueness theorem and whether one can formulate a consistent interacting theory of a massive spin-2 particle. The latter question is of an even more critical importance and has been of much interest over the past 50 years [66]. This question will be at the center of this chapter.

Let us, however, briefly mention that, apart from these theoretical motivations, an interacting massive spin-2 field would also be an interesting possibility to provide for an infrared (IR) modification of gravity. For instance, at first, it seems that it might not be possible to distinguish a very small graviton mass from a zero mass experimentally.Therefore at long distance scales, where the mass term becomes important, the gravitational laws derived from General Relativity could be modified while solar system experiments would not be affected. It is hoped that this could, for example, explain the accelerated expansion of the universe recently indicated by the measurements of the redshift of supernovae [51, 52], large-scale structure [246], baryon acoustic oscillations [247] and the CMB [54]. If General Relativity is to describe also large scale gravitation, this accelerated expansion has to be attributed to the existence of some sort of dark energy which constitutes a little less than 70% of the universe's energy density [54]. It can either be explained by a mere cosmological constant which is added to the Einstein-Hilbert action, or by some sort of dynamical dark energy, which can be explained, for example, by a scalar field with a negative pressure [248, 249]. Another possibility is to try the aforementioned modification of General Relativity in the infrared, which, for specific deformations, can lead to self-accelerating solutions [250–253]. An extensively studied example of this mechanism is $f(R)$ theory, where $f$ is a function of the Ricci scalar $R$ [252]. These theories can also be considered to provide a particular model of inflation [197, 254]. The addition of a small mass term for the graviton as a possible IR modification however comes with a caveat. Considering that a massive spin-2 particle propagates five degrees of freedom, namely its five polarizations (two helicity-2, two helicity-1 and one helicity-0 modes) in contrast to a massless spin-2 particle which only propagates two polarizations, it is far from clear that in the limit of $m \to 0$ the predictions of the massive theory recover the massless one. In order for this to happen, the additional polarizations would need to decouple in this limit.

[1]The Standard Model does not include the gravitational interaction.

In this chapter, we will in particular focus on the development of Lorentz-invariant theories of massive gravity. Specifically, we will concentrate on the general question of whether consistent – by consistent we mean the absence of additional ghost-like degrees of freedom – interacting theories of massive spin-2 particles can exist. From a technical point of view our study complements the analyses which have been done previously in the literature.

The outline of the chapter is as follows. After reviewing recent developments of massive gravity in section 6.1, we recapitulate the findings of Fierz and Pauli and Boulware and Deser in sections 6.2 and 6.3. We also provide a short analysis of the number of degrees of freedom. Section 6.4 introduces the concept of ghosts from higher derivatives. The Stückelberg formalism in general and specifically for massive gravity is presented in section 6.5. This section concludes with the introduction of the de Rham-Gabadadze-Tolley massive gravity. In section 6.6, we discuss our findings for massive spin-2 particles in terms of helicities. The last section 6.8 deals with the construction of a cubic interaction Lagrangian for a massive spin-2 particle which is ghost-free in terms of helicities.

## 6.1 Developments in Massive Gravity

In 1939, Fierz and Pauli studied the wave equations for a massive spin-2 particle on a Minkowski background [67] and wrote down their unique action [255] at linear order from which consistent wave equations can be obtained. When about thirty years later, van Dam, Veltman [68] and independently Zakharov [69] computed certain predictions in a linear theory of massive gravity in the limit $m \to 0$, one of their profound observations was that the bending of light by the Sun and the perihelion precession of Mercury deviate as much as 25% from the ones obtained in General Relativity. Whereas the additional two vector polarizations which appear in massive gravity decouple in the zero mass limit, the scalar does not, and its coupling to the trace of the energy-momentum tensor, $T^{\mu}_{\mu}$, contributes an additional attractive force between sources. This is called the vDVZ discontinuity. The observational bounds on the bending of light and the perihelion precession agree with the predictions of massless gravity up to 10% (see e.g. [256]), and hence, one could think that even a tiny mass for the graviton is excluded and $m = 0$ hence an exact equality.

However, in 1972 Vainshtein [70] pointed out that the linear approximation does break down in the limit of zero mass and nonlinearities become important at distances $r_V = (2\frac{M}{M_P^2}m^{-4})^{\frac{1}{5}}$ when approaching a source from infinity ($M$ is the mass of the source, $M_P$ the Planck mass, and $m$ the mass of the graviton). Therefore, at least at solar system or galactic scales, nonlinearities are always important for small masses $m$ and experiments do not completely rule out a small graviton mass. Note that it has been shown that in specific models, the nonlinearities can indeed lead to a continuous behaviour in the massless limit [257–259].

Still, the nonlinear extension of massive gravity has been found to be plagued by inconsistencies. In 1972, Boulware and Deser [71] concluded based on a Hamiltonian analysis that "in the massive version of the full Einstein theory, there are necessarily six rather than five degrees of freedom". The additional degree of freedom has a wrong-sign kinetic term on a non-trivial background and thus represents a ghost-like instability. On the level of the action, the appearance of the ghost is signalled by a cubic sixth derivative interaction term for the helicity zero component. The sixth derivative term was found in terms of the leading singularity of the graviton propagator to be $\sim m^{-4}$ in [257]. The appearance of the ghost was shown later in terms of Stückelberg fields in [260]. The latter analysis paved the way towards a new understanding of massive gravity in terms of an effective field theory.

An important insight of [260] was that the exceptionally low cutoff of the nonlinear Fierz-Pauli theory, which is of the order of $(10^{11}\text{km})^{-1}$ could be raised when adding higher order potential terms. This was important as otherwise solar system physics could not be described by these effective theories without the knowledge of the full UV completion. One aspect which cannot be avoided is that these theories still become strongly coupled at $(1000\text{km})^{-1}$. In this light it poses an interesting question whether they can describe the tabletop experiments on earth.

The effective field theory language and the possibility of adding higher order terms in order to raise the cutoff of the theory have triggered a revival of massive gravity. After Creminelli et. al. [261] suggested that ghost-like instabilities are unavoidable, a series of papers tried to construct manifestly stable theories of massive gravity [262, 263]. On a Minkowski background, these nonlinear theories were explicitly resummed in [72] and found to describe five degrees of freedom and, thus, to avoid the Boulware-Deser ghost in the decoupling limit to all orders. It was also shown that the Hamiltonian constraint can be maintained in these theories away from the decoupling limit up to and including fourth order nonlinearities. On the Hamiltonian level it was then argued in [73] that there exist enough constraints to eliminate the ghost degree of freedom in the full nonlinear theory suggested by de Rham, Gabadadze and Tolley. Later these findings were confirmed also in the Stückelberg language [75].

A slightly different approach to the concept of massive gravity is given by the so-called gravitational Higgs mechanism, see for example [2, 264–266] and references therein. Within this approach, it was shown that the ghost reappears at fourth order [267] but can be avoided under certain conditions.

## 6.2 The Fierz-Pauli Action

The action for a free massive spin-2 field described by a symmetric two tensor $h_{\mu\nu}$ on a Minkowski background is determined by the Fierz-Pauli action [67]

$$\begin{aligned} S = \int d^4x\mathcal{L} \;&=\; \int d^4x \bigg( \partial_\mu h^{\mu\nu}\partial_\nu h - \partial_\mu h^{\rho\sigma}\partial_\rho h^\mu_\sigma + \frac{1}{2}\partial_\mu h^{\rho\sigma}\partial^\mu h_{\rho\sigma} - \frac{1}{2}\partial_\mu h \partial^\mu h \\ &\quad -\frac{1}{2}m^2(h^{\mu\nu}h_{\mu\nu} - h^2)\bigg)\,, \end{aligned} \tag{6.2}$$

where $h \equiv h^\mu_\mu$ is the trace of the graviton. This action uniquely describes the five propagating helicities which make up a massive spin-2 particle. The relative coefficient of the mass term of $-1$ between $h^{\mu\nu}h_{\mu\nu}$ and $h^2$ is important to eliminate the additional sixth degree of freedom which would otherwise propagate with a wrong-sign kinetic term [255]. The number of degrees of freedom for (6.2) can be easily counted when taking into account the Bianchi identities which yield two constraint equations

$$\begin{aligned} \partial_\mu h^{\mu\nu} &= \partial_\nu h \,, \\ h &= 0 \,, \end{aligned} \tag{6.3}$$

of which the first one can be obtained by acting with $\partial_\mu$ on the equations of motion and the second one by reinserting the first one into (6.2) and then taking the trace. Equations (6.3) give five constraints which eliminate five of the ten components of the symmetric tensor $h_{\mu\nu}$. If departing from the Fierz-Pauli mass term, the second equation of (6.3) turns into an equation of motion for the trace $h$ which hence becomes propagating.

Let us examine the action (6.2) in more detail. For $m = 0$, it describes linearized Einstein gravity and is invariant under linearized diffeomorphisms, $h_{\mu\nu} \to h_{\mu\nu} + \frac{1}{2}(\partial_\mu\xi_\nu + \partial_\nu\xi_\mu)$, where $\xi_\mu(x)$ defines the linear coordinate transformation. The gauge redundancy fixes the relative coefficients of the two-derivative terms. In order to count degrees of freedom, one can also employ a Hamiltonian analysis. After having integrated (6.2) by parts such that $h_{00}$ and $h_{0i}$ do not appear with time derivatives, the canonical momenta of the Lagrangian (6.2) are

$$\pi_{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \dot{h}_{ij} - \dot{h}_{ii}\delta_{ij} - 2\partial_{(i}h_{j)0} \,, \tag{6.4}$$

where we use the symmetrization convention $a_{(\mu}b_{\nu)} = \frac{1}{2}(a_\mu b_\mu + a_\nu b_\mu)$. The other canonical momenta ($\pi_{00}$ and $\pi_{0i}$) are zero due to the integration by parts. Inverting (6.4), one obtains

$$\dot{h}_{ij} = \pi_{ij} - \pi_{kk}\delta_{ij} + 2\partial_{(i}h_{j)0}. \tag{6.5}$$

Performing the Legendre transformation and rewriting the Lagrangian in terms of the canonical momenta yields

$$\begin{aligned} \mathcal{L} &= \pi_{ij}\dot{h}_{ij} - \mathcal{H} + 2h_{0i}\partial_j\pi_{ij} + h_{00}(\nabla^2 h_{ii} - \partial_i\partial_j h_{ij}) \,, \\ \text{where} \quad \mathcal{H} &= \frac{1}{2}\pi_{ij}^2 - \frac{1}{4}\pi_{ii}^2 + \frac{1}{2}\partial_k h_{ij}\partial_k h_{ij} - \partial_i h_{jk}\partial_j h_{ik} + \partial_i h_{ij}\partial_j hkk - \frac{1}{2}\partial_i h_{jj}\partial_i h_{kk}. \end{aligned} \tag{6.6}$$

The variables $h_{00}$ and $h_{0i}$ appear only linearly in terms without time-derivatives. Therefore, they are Lagrange multipliers which give the constraint equations $\nabla^2 h_{ii} - \partial_i\partial_j h_{ij} = 0$ and $\partial_j\pi_{ij} = 0$. These constraints commute, in the sense of Poisson brackets, with the Hamiltonian. Thus, they are first class constraints. For an introduction to constraint systems see for example [268, 269]. The appearance of first class constraints is characteristic for theories with a gauge symmetry. The constraints together with the induced gauge transformations reduce the physical phase space to a four dimensional hypersurface, which is described by the canonical coordinates of the two physical polarizations of the massless spin-2 graviton and their conjugate momenta.

Adding a mass term to the analysis changes the Hamiltonian and the Lagrangian of (6.6) in the following way

$$\begin{aligned}
\mathcal{L} &= \pi_{ij}\dot{h}_{ij} - \mathcal{H} + m^2 h_{0i}^2 + 2h_{0i}\partial_j\pi_{ij} + h_{00}(\nabla^2 h_{ii} - \partial_i\partial_j h_{ij} - m^2 h_{ii}) \,, \\
\text{where} \quad \mathcal{H} &= \frac{1}{2}\pi_{ij}^2 - \frac{1}{4}\pi_{ii}^2 + \frac{1}{2}\partial_k h_{ij}\partial_k h_{ij} - \partial_i h_{jk}\partial_j h_{ik} + \partial_i h_{ij}\partial_j hkk \\
&\quad -\frac{1}{2}\partial_i h_{jj}\partial_i h_{kk} + \frac{1}{2}(h_{ij}h_{ij} - h_{ii}^2).
\end{aligned} \tag{6.7}$$

Note that the conjugate momenta are unaffected by the additional mass term. However, the structure of the Lagrangian is different and $h_{0i}$ is no longer a Lagrange multiplier. Nevertheless, it is still non-dynamical and its equation of motion yields the algebraic relation

$$h_{0i} = -\frac{1}{m^2}\partial_i\pi_{ij}. \tag{6.8}$$

$h_{00}$ remains a Lagrange multiplier and it enforces the constraint

$$\nabla^2 h_{ii} - \partial_i\partial_j h_{ij} - m^2 h_{ii} = 0 \,, \tag{6.9}$$

which is now a second class constraint. The resulting secondary constraint arises from the fact that the constraint is conserved in time, i.e. it commutes with the Hamiltonian. Since $h_{0i}$ is determined by (6.8) and $h_{00}$ gives two second class constraints (one primary and one secondary), the resulting physical phase space is then ten dimensional and describes the five physical polarizations of the massive spin-2 particle and their conjugate momenta. Departing from the Fierz-Pauli mass term introduces nonlinearities in $h_{00}$ and the constraint which fixes the trace $h_{ii}$ to zero for (6.6) is lost which leads to either a tachyonic or ghost-like sixth degree of freedom [71, 255].

Let us briefly mention coupling to sources. Adding a source term to the Lagrangian (6.2) of the form $h_{\mu\nu}T^{\mu\nu}$ does not change the linear constraint analysis. No matter whether the source is conserved, $\partial_\mu T^{\mu\nu} = 0$, or not, the source coupling will only introduce $h_{00}$ and $h_{0i}$ linearly and without time derivatives. Therefore, it will not affect the number of constraints. Note that the same holds true for any linear coupling of $h_{\mu\nu}$ to sources.

## 6.3 Nonlinear Interactions

First let us consider the case of General Relativity and analyze the constraint structure that arises when introducing nonlinear interactions. Using the ADM formalism [270, 271], the full action can be written as

$$S = \int d^4x\sqrt{-g}\, R = \int d^4x(\pi_{ij}\dot{\gamma}_{ij} - NR^{(0)} - N_i R^i - 2(\pi^{ij}N_j - \frac{1}{2}\pi N^i + N^{|i}\sqrt{\gamma})_{|j}), \tag{6.10}$$

where $\gamma_{ij} \equiv g_{ij}$, $N \equiv (-g^{00})^{-\frac{1}{2}}$, $N_i \equiv g_{0i}$, the R's are functions of the spatial metric $\gamma_{ij}$ and its conjugate momentum $\pi_{ij}$, but do not depend on $N$ or $N_i$. R is the four dimensional Ricci scalar and $-R^{(0)} \equiv {}^3R + \gamma^{-\frac{1}{2}}(\frac{1}{2}\pi^2 - \pi_{ij}\pi^{ij})$ and ${}^3R$ is the three dimensional Ricci scalar with respect to the metric $\gamma_{ij}$. $R^i = -2\pi^{ij}_{|j}$, where the bar "|"

denotes covariant differentiation with respect to the spatial metric $\gamma_{ij}$. In the massless theory, $N$ and $N_i$ are Lagrange multiplier which enforce first class constraints on the system, thereby eliminating four (and correspondingly eight phase space) degrees of freedom. This corresponds to two propagating helicities of the massless spin-2 particle. To linear order $N = 1 - \frac{1}{2}h_{00}$ and $N_i = h_{0i}$ and one recovers the result of the previous section.

Introducing a mass term in the full nonlinear theory of General Relativity leads to immediate trouble, as it reintroduces the sixth degree of freedom which could be tuned away in the Fierz-Pauli theory as we will see. Expanded around a background metric, $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$, the Fierz-Pauli mass term $f = (h_{\mu\nu}h^{\mu\nu} - h^2)$ can be expressed in terms of $N_i$ and the nonlinear $N$ [71],

$$f = h_{ij}^2 - h_{ii}^2 - 2N_i^2 + 2h_{ii}(1 - N^2 - N_i g^{(0)ij} N_j). \tag{6.11}$$

In contrast to the linear case, now $N$ (which to linear order is equivalent to $h_{00}$) appears quadratically and neither $N$ nor $N_i$ are Lagrange multipliers. They are both determined by the constraint equations they yield and as a result the trace $h_{ii}$ is no longer constrained to be nondynamical. Therefore, there are now six degrees of freedom propagating.[2]

To this end, we have recovered the result of Boulware and Deser who concluded in 1972 that adding a mass term to General Relativity unavoidably leads to a sixth propagating mode, the so-called Boulware-Deser ghost. This conclusion has been challenged in recent years by reinvestigations of the theory of massive gravity in terms of an effective field theory language [72, 73, 75, 260]. In order to better understand these developments, we will introduce the effective field theory language and the Stückelberg mechanism in the next section. But first, let us quickly review the connection between ghosts and higher derivative terms.

## 6.4 Ghosts from higher Derivatives

In the following, massive gravity will be analyzed in terms of effective field theory degrees of freedom. In this language, the Boulware-Deser ghost will show up as a higher-derivative term on the Stückelberg fields. Therefore, let us briefly recap the arguments of Ostrogradski (see for example [272–274] and referenes therein) which explain why higher derivative kinetic terms imply additional ghost-like degrees of freedom. Consider the following toy Lagrangian for a scalar field $\phi$ (cf. [261])

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2\Lambda^2}(\Box\phi)^2 - V(\phi) \, , \tag{6.12}$$

where $\Lambda$ denotes some mass parameter which determines the cutoff of the effective field theory. First consider only the time-dependent part of $\phi$ by going to Fourier space and

[2] As a side remark, let us mention that the requirement of adding three additional degrees of freedom to the massless Einstein-Hilbert Lagrangian in order to obtain a massive spin-2 representation necessitates the introduction of a background metric $g^{(0)}_{\mu\nu}$ for the mass term.

absorbing the $\vec{k}$-dependence into the potential $V_k(\phi_k)$,

$$\mathcal{L} = -\frac{1}{2}\dot{\phi}_k^2 - \frac{1}{2\Lambda^2}(\ddot{\phi}_k)^2 - V_k(\phi_k). \tag{6.13}$$

In order to perform a Hamiltonian analysis for this Lagrangian according to the Ostrogradski prescription one introduces the canonical variables $\phi_1 = \phi$ and $\phi_2 = \dot{\phi}$ and we omit the subscripts $k$. The conjugate momenta are then $\Pi_1 = \frac{\delta\mathcal{L}}{\delta\dot{\phi}} = \frac{1}{\Lambda^2}\partial_t^3\phi - \dot{\phi}$ and $\Pi_2 = -\frac{1}{\Lambda^2}\ddot{\phi}$ and the Hamiltonian reads

$$\mathcal{H} = -\mathcal{L} + \Pi_1\dot{\phi_1} + \Pi_2\dot{\phi_2} = -\frac{1}{2\Lambda^2}\Pi_2^2 + \Pi_1\phi_2 + \frac{1}{2}\phi_2^2 + V(\phi_1). \tag{6.14}$$

This Hamiltonian exhibits a linear instability in $\Pi_1$. Choosing negative $\Pi_1$, one can arbitrarily lower the energy of the system and there is no well-defined vacuum state in the theory. This is the essence of Ostrogradski's theorem which states that higher-derivative Lagrangians which are not degenerate exhibit a linear instability in the Hamilitonian.

In terms of a Lagrangian analysis, the ghost instability can be recovered by rewriting the Lagrangian (6.12) in terms of two scalar fields of which one now has a wrong-sign kinetic term

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}(\partial_\mu\phi)^2 - \partial_\mu\chi\partial^\mu\phi - \frac{1}{2}\Lambda^2\chi^2 - V_{\text{int}}(\phi)\\ &= -\frac{1}{2}(\partial_\mu\phi')^2 + \frac{1}{2}(\partial_\mu\chi')^2 - \frac{1}{2}\Lambda^2\chi'^2 - V_{\text{int}}(\phi',\chi')\ ,\end{aligned} \tag{6.15}$$

where in the last line we have diagonalized the kinetic terms and canonically normalized the fields. The first line reduces to (6.12) when integrating out the field $\chi$ via its equations of motion. Of the two fields one has positive energies and the other negative ones, therefore a vacuum state defined as the state of zero particles $\phi$ and $\chi$ and with energy zero will no longer be stable once the interactions are switched on. To be precise, the vacuum can spontaneously create $\phi$ and $\chi$ particles on-shell without violating energy conservation. This is in contrast to theories were all particles have a kinetic term with the correct sign. The fact that for healthy theories the S-matrix overlap between a single particle in-state and a multiparticle out-state vanishes asymptotically crucially depends on the fact that the energy of these states is strictly positive and hence the multiparticle state always has larger energy than the single particle state, see e.g. chapter 5 of [164]. This decay of the vacuum with zero particles into a state with infinitely many $\phi$- and $\chi$-quanta can occur infinitely fast and therefore violates unitarity, as the phase space of final states is infinite, which is equivalent to saying that the density of states around the vacuum is infinite.

Notice that this is actually different to a particle with a tachyonic mass, i.e. a negative mass squared. Even though particle creation from the vacuum can occur in much the same way, the phase space of the final state is not infinite. Rather it is restricted to momenta smaller than the tachyonic mass. On the classical level, this can be seen by the fact that the perturbations grow exponentially as $e^{mt}$ so that the time scale of the decay is bounded by $m^{-1}$.

To conclude the discussion on ghosts from higher derivatives, we want to point out that the above arguments strictly speaking are only valid for fundamental field theories. In an interacting effective field theory, the higher derivative term in (6.12) has to be considered as an operator appearing in an expansion about the scale $\Lambda$. However, the ghost mass in (6.15) is $\Lambda^2$, and thus the system allows for the possibility that new physics entering at scales $\Lambda$ might cure the ghost instability.

## 6.5 Stückelberg Formalism and Massive Gravity

This section provides an introduction to the Stückelberg formalism [275] and will outline the effective field theory developments in massive gravity which have occured over the past decade [260].

### 6.5.1 Stückelberg for massive Gauge Fields

First, let us review the basics of the Stückelberg trick by means of massive electrodynamics. Consider the Proca action

$$S = \int d^4x(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu + A_\mu J^\mu), \tag{6.16}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $J^\mu$ is a source which a priori need not be consereved. For $m = 0$ and the source now being conserved, this action is invariant under the gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$, where $\Lambda$ is an arbitrary function of the spacetime coordinates. Then only two of the four components of $A_\mu$ are propagating. Instead, for $m \neq 0$ the action (6.16) is no longer gauge invariant and it describes three dynamical degrees of freedom, a massive spin-1 particle, because the timelike component of $A_\mu$ can be eliminated via the constraint $\partial_\mu A_\mu = 0$ which follows from the divergence of the equation of motion of (6.16) without sources. Taking the limit $m \to 0$ on the level of the Lagrangian, one naively looses one degree of freedom. Nonetheless this limit is consistent since it can be shown that the correlation function between two consvered sources approaches the massless one for $m \to 0$.

The Stückelberg trick is to include an extra scalar field $\phi$ via the transformation $A_\mu \to A_\mu + \frac{1}{m}\partial_\mu \phi$ and to define the gauge transformation $\delta A_\mu = \partial_\mu \Lambda$ and simultaneously $\delta\phi = -m\Lambda$. The action is then given by

$$S = \int d^4x(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu - mA_\mu\partial^\mu\phi - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + A_\mu J^\mu - \frac{1}{m}\phi\,\partial_\mu J^\mu). \tag{6.17}$$

Note that we introduced the longitudinal scalar $\phi$ in such a way that its kinetic term is already canonically normalized. $A_\mu$ carries now two degrees of freedom and the scalar $\phi$ one. One can recover (6.16) when choosing the unitary gauge $\phi = 0$ which means that (6.16) and (6.17) describe the same physical theories. It is then possible to take the limit $m \to 0$ on the level of the Lagrangian (6.17) without losing degrees of freedom. In this language, one easily sees that the additional scalar decouples if the source is conserved or the non-conserved part goes at least faster than $\sim m$ to zero. The vector and the scalar decouple and we are left with massless electrodynamics and a free, decoupled scalar. The corresponding Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + A_\mu J^\mu \tag{6.18}$$

with gauge symmetries $\delta A_\mu = \partial_\mu \Lambda$ and $\delta\phi = 0$.

In the same way, one can restore gauge invariance in a massive non-abelian gauge theory [260] with Lagrangian

$$\mathcal{L} = -\frac{1}{g^2}\mathrm{Tr}F^2 + \frac{m^2}{g^2}\mathrm{Tr}A^2\ , \tag{6.19}$$

where $F_{\mu\nu}$ is the field strength of the non-abelian field $A_\mu$. Performing a pseudo gauge transformation $A_\mu \to UA_\mu U^\dagger + iU\partial_\mu U^\dagger$, one can introduce the Stückelberg fields $U = e^{i\pi}$. The Lagrangian turns into

$$\mathcal{L} = -\frac{1}{g^2}\mathrm{Tr}F^2 + \frac{m^2}{g^2}\mathrm{Tr}|D_\mu U|^2 \tag{6.20}$$

which transform under the new symmetry $U \to U\Lambda^\dagger$ and $A_\mu \to \Lambda A_\mu \Lambda^\dagger + i\Lambda\partial_\mu\Lambda^\dagger$. Note that the covariant derivative is defined as $D_\mu U \equiv \partial_\mu U - iUA_\mu$. One can switch between the description of (6.19) and (6.20) by going to the unitary gauge where $U = 1$.

We have seen that the physics of the unitary gauge Lagrangian and the Lagrangian with Stückelberg fields is the same even though in contrast to the first, the latter one is invariant under an additional gauge transformation. This reveals the fact that gauge invariance is not a fundamental principle, but rather a redundancy of description. Introducing extra fields with an appropriate set of gauge transformations, one can make any Lagrangian gauge invariant.

The Stückelberg description has the following advantages in terms of effective field theory. Firstly, at energies much larger than the mass $m$, the Stückelberg fields become the longitudinal components of the gauge boson. In the Stückelberg description, it is apparent that these become strongly coupled at scales $\sim 4\pi m/g$. The relevant interactions arise from the mass term and are $\sim (\frac{g}{m})^{n-2}\partial^2\pi^n$, where $n \in \mathbb{N}$. This is obscured in the unitary gauge description and it takes more careful effort to recover these results, for example, by explicitly considering the polarization tensors. Secondly, the Stückelberg fields allow for simple power counting arguments in order to determine which non-gauge-invariant operators can be generated by radiative corrections. Lastly, since it is the longitudinal modes which become strongly coupled, the Stückelberg picture provides a straightforward approach in order to search for possible UV completions.

### 6.5.2 Stückelberg for Massive Gravity

The effective field theory approach of the Stückelberg formalism, which was put forward in [72, 75, 260–263], can be useful for understanding massive gravity. As a warm-up exercise let us start with the linearized theory and see how some of the peculiarities of the Fierz-Pauli theory show up in terms of the Stückelberg fields. The background metric is assumed to be flat Minkowski space $\eta_{\mu\nu}$ and we expand the full metric in terms of $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

The Fierz-Pauli action (6.2) can now be made gauge invariant in a similar way as the massive photon in the previous section by introducing the Stückelberg fields $A_\mu$ via the pseudo gauge transformation $h_{\mu\nu} \to h_{\mu\nu} + \frac{1}{m}(\partial_\mu A_\nu + \partial_\nu A_\mu)$. The resulting action reads

$$\begin{aligned}\mathcal{L} &= \partial_\mu h^{\mu\nu}\partial_\nu h - \partial_\mu h^{\rho\sigma}\partial_\rho h^\mu_\sigma + \frac{1}{2}\partial_\mu h^{\rho\sigma}\partial^\mu h_{\rho\sigma} - \frac{1}{2}\partial_\mu h \partial^\mu h - \frac{1}{2}m^2(h^{\mu\nu}h_{\mu\nu} - h^2) \\ &\quad -\frac{1}{2}F^{\mu\nu}F_{\mu\nu} - 2m(h_{\mu\nu}\partial^\mu A^\nu - h\partial_\mu A^\mu) + h_{\mu\nu}T^{\mu\nu}, \end{aligned} \tag{6.21}$$

and is invariant under the transformations $\delta h_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ and $\delta A_\mu = -m\xi_\mu$, where $\xi_\mu(x)$ is an arbitrary four vector and a function of the spacetime coordinates and $T^{\mu\nu}$ is a conserved source. In order to fully appreciate the benefits of the Stückelberg decomposition, the vector field can be decomposed into a transverse spin-1 and a spin-0 mode according to $A_\mu \to A_\mu + \frac{1}{m}\partial_\mu \phi$. This yields an additional symmetry $\delta A_\mu = \partial_\mu \Sigma$ and $\delta\phi = -m\Sigma$, where $\Sigma$ is an arbitrary scalar-valued function, and the Lagrangian obtains additional terms

$$\begin{aligned}\mathcal{L} &= \partial_\mu h^{\mu\nu}\partial_\nu h - \partial_\mu h^{\rho\sigma}\partial_\rho h^\mu_\sigma + \frac{1}{2}\partial_\mu h^{\rho\sigma}\partial^\mu h_{\rho\sigma} - \frac{1}{2}\partial_\mu h \partial^\mu h - \frac{1}{2}m^2(h^{\mu\nu}h_{\mu\nu} - h^2) \\ &\quad -\frac{1}{2}F^{\mu\nu}F_{\mu\nu} - 2m(h_{\mu\nu}\partial^\mu A^\nu - h\partial_\mu A^\mu) - 2(h_{\mu\nu}\partial^\mu\partial^\nu\phi - h\Box\phi) + h_{\mu\nu}T^{\mu\nu} \, . \end{aligned} \tag{6.22}$$

Note that the spin-2 and the spin-0 degree of freedom are coupled in the $m \to 0$ limit. Diagonalizing the kinetic terms via $h_{\mu\nu} \to h_{\mu\nu} + \eta_{\mu\nu}\phi$, one finds that the scalar receives an additional coupling to the trace of the energy momentum tensor of the source $\sim \phi T^\mu_\mu$. This reveals the origin of the vDVZ discontinuity [68, 69] which lies in the appearance of an additional scalar mode. The diagonalization of the kinetic terms will play an important role later on when we analyze the helicity decomposition in section 6.6. Note that in terms of the Stückelberg fields it can be shown that a detuning of the relative coefficient of the Fierz-Pauli mass term leads to terms $\sim \partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi$ in the Lagrangian which yield higher derivatives on the equation of motion. These indicate the appearance of an additional ghost-like degree of freedom.

Let us now continue the analysis of the fully interacting theory (6.10) and (6.11) and assume a generic background metric $g^{(0)}_{\mu\nu}$. The Lagrangian is given by [71]

$$\mathcal{L} = \sqrt{-g}R - \frac{1}{4}\sqrt{-g^{(0)}}g^{(0)}_{\mu\nu}g^{(0)}_{\alpha\beta}(h_{\mu\alpha}h_{\nu\beta} - h_{\mu\nu}h_{\alpha\beta}). \tag{6.23}$$

The metric of the Ricci tensor $R$ is $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$. In order to achieve a gauge, i.e. diffeomorphism, invariant Lagrangian, the Goldstone (Stückelberg) fields have to be introduced in such a way that the nonlinear diffeomorphisms are respected by the action (6.23) [260]. In principle, one could just proceed similar to the non-abelian gauge theory example and introduce the Goldstone fields via a pseudo gauge transformation of the metric. This approach, however, has the drawback that it will introduce an infinite power expansion of $h_{\mu\nu}$ in the Stückelberg language. In the following, we discuss a slightly different path to the Stückelberg description taken in [66, 72, 262, 263] which is better suited to keep track of the powers of $h_{\mu\nu}$. In this approach, the full metric $g_{\mu\nu}$ still transforms covariantly under general coordinate transformations (diffeomorphisms) and the Stückelberg fields enter only through the background metric via $g^{(0)}_{\mu\nu}(x) \to g^{(0)}_{\alpha\beta}(Y(x))\partial_\mu Y^\alpha \partial_\nu Y^\beta$, where the $Y^\alpha$ define the coordinate transformation. Then, defining $H_{\mu\nu} \equiv g_{\mu\nu} - g^{(0)}_{\alpha\beta}(Y(x))\partial_\mu Y^\alpha \partial_\nu Y^\beta$ and replacing $h_{\mu\nu}$ by $H_{\mu\nu}$ in the action, (6.23) becomes invariant under diffeomorphisms $f(x)$ of $g_{\mu\nu}$ if the $Y^\alpha$ transform as scalars $Y^\alpha(x) \to Y^\alpha(f(x))$.

One can expand the coordinate transformation around the identity $Y^\alpha = x^\alpha - \pi^\alpha$, and on a Minkowski background one finds

$$\begin{aligned} H_{\mu\nu} &= h_{\mu\nu} + \partial_\mu \pi_\nu + \partial_\nu \pi_\mu - \partial_\mu \pi^\alpha \partial_\nu \pi_\alpha \\ H_{\mu\nu} &= h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu + 2\partial_\mu\partial_\nu\phi - \partial_\mu A^\alpha \partial_\nu A_\alpha - \partial_\mu A^\alpha \partial_\nu \partial_\alpha \phi - \partial_\mu \partial^\alpha \phi \partial_\nu \partial_\alpha \phi \ , \end{aligned} \tag{6.24}$$

where the replacement $\pi^\alpha = \eta^{\alpha\beta}(A_\beta + \partial_\beta \phi)$ was made in the last line and $\pi_\mu = \eta_{\mu\alpha}\pi^\alpha$.[3] The infinitesimal symmetry transformation under which $g_{\mu\nu}$ and also $H_{\mu\nu}$ transform as covariant tensors are

$$\begin{aligned} \delta h_{\mu\nu} &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \mathcal{L}_\xi h_{\mu\nu} \ , \\ \delta A_\alpha &= \partial_\alpha \Lambda - \xi_\alpha + \xi^\mu \partial_\mu A_\alpha \ , \\ \delta\phi &= -\Lambda \ , \end{aligned} \tag{6.25}$$

where $\mathcal{L}_\xi$ is the Lie derivative. This kind of Stückelberg prescription allows to construct a Lagrangian with mass terms which is manifestly invariant under general coordinate transformations. If one then replaces $h_{\mu\nu}$ with $H_{\mu\nu}$ (6.24) in (6.23), one obtains a gauge invariant mass term for the graviton given by

$$S = \int d^4x \left( \frac{1}{2} M_P^2 \sqrt{-g} R - \frac{1}{4} M_P^2 m^2 \sqrt{-g} g^{\mu\nu} g^{\alpha\beta} (H_{\mu\alpha} H_{\nu\beta} - H_{\mu\nu} H_{\alpha\beta}) \right) . \tag{6.26}$$

Note that contractions of the full metric with "covariant" metric perturbations $g^{\mu\nu}H_{\mu\nu}$ are invariant under diffeomorphisms. In terms of the Stückelberg decomposition (6.24), the scalar $\phi$ does not have a kinetic term when simply expanding $H_{\mu\nu}$ in (6.26). It is kinetically mixed with the helicity-2 component $h_{\mu\nu}$ via $m^2 M_P^2(\partial_\mu\partial_\nu\phi h^{\mu\nu} - \Box\phi h)$. Performing a canonical transformation $h_{\mu\nu} = \hat{h}_{\mu\nu} + m^2 \eta_{\mu\nu}\phi$, where $\hat{h}_{\mu\nu}$ now denotes the helicity-2 component, and canonically normalizing the fields, $\hat{h}^c_{\mu\nu} = M_P \hat{h}_{\mu\nu}$ and $A^c_\mu = m M_p A_\mu$ and $\phi^c = m^2 M_P \phi$, one can expand the Lagrangian (6.26) in terms of the Stückelberg fields. In the following, we drop the superscript $c$ which denotes canonical normalization of the fields in order to avoid cluttering up our notation. It is the assumed that the fields are canonically normalized. The Lagrangian includes

[3]Note that in contrast to (6.21) the fields in the decomposition (6.24) are not yet canonically normalized, for example $\phi$ has mass dimensions $[m^2]$.

interaction terms of the sort $\frac{(\partial^2\phi)^3}{m^4 M_P}, \frac{(\partial^2\phi)^4}{m^6 M_P^2}, \frac{\partial^2\phi\partial A\partial A}{m^2 M_P}$ which become strongly coupled at the scales $\Lambda_5 = (m^4 M_P)^{\frac{1}{5}}$, $\Lambda_4 = (m^3 M_P)^{\frac{1}{4}}$ and $\Lambda_3 = (m^2 M_P)^{\frac{1}{3}}$. The first term $\frac{(\partial^2\phi)^3}{m^4 M_P}$ violates unitarity at the scale $\Lambda_5$ which is of the order $(10^{11}\text{km})^{-1}$ and leads to a higher derivative kinetic term on a nontrivial background for the scalar $\phi$. In [261] this term was considered to be an indication of a ghost degree of freedom reappearing in the interacting theory, in analogy to what was found by Boulware and Deser [71]. Their argument was the following. If one expands $\phi$ around a nontrivial background $\phi = \bar{\phi} + \varphi$, the induced kinetic term schematically reads

$$-(\partial\varphi)^2 + \frac{(\partial^2\bar{\phi})}{\Lambda_5^5}(\partial^2\varphi)^2, \tag{6.27}$$

which leads to an additional ghost-like degree of freedom of mass $m_{\text{ghost}} \sim \frac{\Lambda_5^5}{\partial^2\bar{\phi}}$. Within the effective field theory, however, $\partial^2\bar{\phi} \lesssim \Lambda_5$ which means that the mass of the ghost is always above the cutoff and the ghost cannot be excited.

Since it is the self-interactions of $\phi$ which become strongly coupled at the lowest scale $\Lambda_5$, operators generated by radiative corrections will, in general, be of the form $\frac{\partial^q(\partial^2\phi)^p}{\Lambda_5^{3p+q-4}}$ with $p, q \in \mathbb{N}$ [260]. These are the only ones allowed for by the symmetries of $\phi$. In unitary gauge, they correspond to terms like $c_{p,q}\partial^q h^p$ with $c_{p,q} \sim \Lambda_5^{-3p-q+4} M_P^p m^{2p}$. On the above background, $\bar{\phi} \sim \frac{M}{M_P}\frac{1}{r}$, and one therefore enters the regime where quantum corrections become important at distance scale $r_* \sim \left(\frac{M}{M_P}\right)^{\frac{1}{3}}\frac{1}{\Lambda_5}$ around the source. This is actually the largest radius derivable from the induced operators and it is exactly the scale at which the mass of the stipulated ghost drops below the cutoff. The conclusion that the higher derivatives (6.27) in this theory induce a ghost degree of freedom in the physical spectrum is, thus, no longer obvious. It is nevertheless true that the UV completion of the theory will have to take care of the higher derivative structure in order to ensure that there is no ghost propagating. Otherwise, the theory will be plagued by the linear instability we have discussed in section 6.4. Notably, the scale $r_*$, at which the effective field theory breaks down, is parametrically larger than the Vainshtein scale $r_V = \left(\frac{M}{M_P}\right)^{\frac{1}{5}}\frac{1}{\Lambda_5}$ implying that there is no region around a classical source where General Relativity is recovered within the effective theory.

In fact, the scale $\Lambda_5$ and all its interactions can be removed by adding higher order potential terms to the Lagrangian [72, 260–263]. This becomes apparent when considering generic potential terms $\mathcal{U}_i(g, H)$ which are of order $i$ in the covariant metric perturbations $H_{\mu\nu}$ [262],

$$S = \int d^4x \left(\frac{1}{2}M_P^2\sqrt{-g}R - \frac{1}{4}M_P^2 m^2\sqrt{-g}g^{\mu\nu}g^{\alpha\beta}(\mathcal{U}_2(g,H) + \mathcal{U}_3(g,H) + \mathcal{U}_4(g,H) + \ldots)\right), \tag{6.28}$$

where $\mathcal{U}_2(g,H) = H_{\mu\nu}^2 - H_\mu^{\mu\nu}$, $\mathcal{U}_3(g,H) = c_1 H_{\mu\nu}^3 + c_2 H H_{\mu\nu}^2 + c_3 H^3$ and so forth. The dots indicate terms of higher order in $H_{\mu\nu}$. To each order, one can then choose the coefficients such that the interactions of order $\Lambda_5$ cancel ($c_1 = 2c_3 + \frac{1}{2}$, $c_2 = -3c_3 - \frac{1}{2}$ and $c_3$ remains a free parameter) [261, 262]. Indeed, the coefficients of the potential terms $\mathcal{U}_i(g, H)$ can be chosen such that the lowest scale at which interactions become strongly coupled is $\Lambda_3$ [72, 261–263]. De Rham, Gabadadze and Tolley [72] showed that one can explicitly resum all the nonlinear terms of the effective field theory of gravity which is constructed as in (6.28). Specifically, the Lagrangian of the theory can be written in terms of the covariant metric perturbation $H_{\mu\nu}$ as

$$\mathcal{L} = \frac{M_P^2}{2}\sqrt{-g}\left(R - \frac{m^2}{4}\mathcal{U}(g,H)\right) , \tag{6.29}$$

where the potential is defined by

$$\begin{aligned} \mathcal{U}(g,H) &= -4\left(\langle\mathcal{K}\rangle^2 - \langle\mathcal{K}^2\rangle\right) = -4\left(\sum_{n\leq 1} d_n\langle H^n\rangle\right)^2 - 8\sum_{n\leq 2} d_n\langle H\rangle \\ \text{with} \qquad & \mathcal{K}^\mu_\nu(g,H) = \delta^\mu_\nu - \sqrt{\delta^\mu_\nu - H^\mu_\nu} \quad \text{and} \quad d_n = \frac{(2n)!}{(1-2n)(n!)^2 4^n} \, . \end{aligned} \tag{6.30}$$

Remember that indices of $H_{\mu\nu}$ are raised with the full metric $g^{\mu\nu}$, as are the ones of $\mathcal{K}_{\mu\nu}$. Angle brackets represent the trace with respect to the full metric such that $\langle H\rangle = g^{\mu\nu}H_{\mu\nu}$ and $\langle H^2\rangle = g^{\mu\alpha}g^{\nu\beta}H_{\mu\nu}H_{\alpha\beta}$. This Lagrangian, which describes the theory referred to as de Rahm-Gabadadze-Tolley (dRGT) massive gravity, exactly recovers the coefficients needed for (6.28) to cancel all strong coupling scales lower than $\Lambda_3$. In [72] it was demonstrated that these theories are free of higher derivative interactions in terms of the Stückelberg fields in the so-called decoupling limit. In this limit one chooses to send $M_P \to \infty$ and $m \to 0$ such that $\Lambda_3 = \sqrt[3]{m^2 M_P} = const$. The only relevant interactions in this limit are the ones among the helicity-2 component $h_{\mu\nu}$ and the helicity-0 mode $\phi$ of the Stückelberg decomposition (6.24). Following [72, 262], the resulting Lagrangian is

$$\mathcal{L}_{\Lambda_3} = -\frac{1}{4}h^{\mu\nu}(\hat{\mathcal{E}}h)_{\mu\nu} + h_{\mu\nu}X^{\mu\nu} , \tag{6.31}$$

where $X_{\mu\nu} = \frac{1}{2}\Lambda_3^3[\Pi\eta_{\mu\nu} - \Pi_{\mu\nu} + \Pi^2_{\mu\nu} - \Pi\Pi_{\mu\nu} + \frac{1}{2}(\Pi^2 - \Pi^2_{\alpha\beta})\eta_{\mu\nu}]$, $\Pi_{\mu\nu} = \partial_\mu\partial_\nu\phi$, and we use the suggestive notation $\Pi = \eta^{\mu\nu}\Pi_{\mu\nu}$, $\Pi^2_{\alpha\beta} = \Pi^\rho_\alpha\Pi_{\rho\beta}$ and so forth.

Already in [72] it was hinted that the theory (6.29) might possess the correct number of degrees of freedom even away from the decoupling limit based on the argument that the Hamiltonian constraint is maintained up to quintic order in the expansion. In [73] Hassan and Rosen performed a full Hamiltonian analysis following [72] and found that in terms of an ADM analysis [270, 271] the constraints from $N$ and $N_i$ are not independent. Therefore, if one solves for the latter constraint first and subsequently reinserts this into the Hamiltonian, $N$ becomes a Lagrange multiplier, thus eliminating the dangerous sixth degree of freedom.

## 6.6 Helicity Analysis of Massive Gravity

A different method to investigate the degrees of freedom of a massive spin-2 particle is to decompose the field into its helicity components as we have put forward in [76]. This can be done whenever the considered theory is Poincaré invariant, local and weakly coupled in at least some energy interval on an (asymptotic) Minkowski background. Therefore, the decomposition in terms of helicities of a massive particle is sensible whenever the energies considered are much larger than the mass of the particle, $\frac{E}{m} \gg 1$. At these energies, the massive spin-2 representation of the Poincaré group decomposes into the direct sum of irreducible helicity representations. In order to fully appreciate the usefulness of the helicity decomposition recall that ghost instabilities are instabilities which can occur on arbitrarily small time scales $t \sim \frac{1}{E}$, signifying that they are UV instabilities. Accordingly, the time-scale could only be limited by an effective field theory cutoff $\Lambda$ which sets an upper bound on the energy scales that can be reliably considered in the effective theory. As a consequence, helicity degrees of freedom are an appropriate means of testing for instabilities in a theory at energy scales $m \ll E \ll \Lambda$, see for example [76]. Additionally, they directly reveal whether the degrees of freedom truly form a spin-2 representation.

### 6.6.1 The Helicity Decomposition

In the previous section 6.5.1, massive gauge theories have proven useful to understand certain techniques which also be applied to massive gravity/ massive spin-2 theories . The same is true for understanding the helicity decomposition. Consider a free massive spin-1 field $A_\mu$ with the Lagrangian (6.16). Quantizing this theory leads to the field operator $A_\mu$ given by (see e.g. [276])

$$A^\mu(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2E_k}} \sum_{\lambda=1}^{3} \left( \epsilon^{*\mu}(\lambda, k) a^\dagger_{k\lambda} e^{ik\cdot x} + \epsilon^\mu(\lambda, k) a_{k\lambda} e^{-ik\cdot x} \right) , \tag{6.32}$$

where $a_k$ and $a^\dagger_k$ are the usual creation and annihilation operators with standard commutation relations and the $\epsilon_\mu(k, \lambda)$ are the polarization vectors of which $1, 2$ are transversal to the three-momentum $\vec{k}$ and 3 is in the direction of $\vec{k}$. Note that in principle one needs four independent polarization vectors to span the field space for a generic four-vector $A_\mu$, but the constraint $\partial_\mu A^\mu = 0$ eliminates one of them. One can choose the polarization vectors $\epsilon^\mu(\lambda, k)$ such that the corresponding creation operators $a^\dagger_{k\lambda}$ create a helicity eigenstate particle of helicity $-1, 0, 1$. This is possible since the Hamiltonian,

$$H = \sum_{\lambda=-,0,+} \int d^3k \omega_k a^\dagger_{k\lambda} a_{k\lambda}, \tag{6.33}$$

is diagonal in the helicity basis, i.e. the Hamiltonian commutes with the helicity operator $\hat{\Lambda} = \int d^3k (a^\dagger_{k+} a_{k+} - a^\dagger_{k-} a_{k-})$. "+" and "–" denote the polarizations corresponding to the helicities +1 and -1. They are linear combinations of the original $1, 2$ polarization and hence transversal. The longitudinal polarization 3 is equal the helicity-0 polarization.

For the helicity decomposition it is important that the 0-polarization evaluated on a state with momentum $\vec{k}$ with $|\vec{k}| \gg m$ approaches $k^\mu/m$ evalutated for the same state.

More precisely the difference

$$(\epsilon^{*\mu}(3,k) - \frac{k^\mu}{m}) = (\frac{|\vec{k}|}{m}, \frac{\vec{k}}{|\vec{k}|}\frac{k_0}{m})^T - (\frac{k_0}{m}, \frac{\vec{k}}{m})^T \simeq \frac{1}{2}\frac{m}{|\vec{k}|}(-1, \frac{\vec{k}}{|\vec{k}|})^T \tag{6.34}$$

should approach zero in the massless limit which it does indeed.

Finally, let us introduce the helicity decomposition for a massive vector field $A_\mu$ which is split into two fields, another 4-vector $\tilde{A}_\mu$ and a scalar $\phi$, in the following way

$$A_\mu = \tilde{A}_\mu + \frac{1}{m}\partial_\mu\phi \ . \tag{6.35}$$

The fields $\tilde{A}_\mu$ and $\phi$ enjoy a common gauge symmetry $\tilde{A}_\mu \to \tilde{A}_\mu + \partial_\mu\Lambda$ and $\phi \to \phi - \Lambda$. This ensures that there are still only three degrees of freedom propagating. The linear Lagrangian is given by (6.17) which we display here again for convenience

$$-\frac{1}{4}\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} - \frac{1}{2}m^2\tilde{A}_\mu\tilde{A}^\mu - m\tilde{A}_\mu\partial^\mu\phi - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi. \tag{6.36}$$

In fact (6.35) is equivalent to the linear Stückelberg decomposition introduced in (6.17). However, its motivation is different. (6.35) is constructed in such a way that there exists a gauge of $\tilde{A}_\mu$ and $\phi$ in which $\phi$ captures solely the helicity-0 component and $\tilde{A}_\mu$ the helicity-1 components of $A_\mu$ in the high energy limit. In such a gauge, such as, for example, the Coulomb gauge ($\tilde{A}_0 = 0$ and $\nabla\tilde{\vec{A}} = 0$), $\tilde{A}_\mu$ carries only the polarizations $\epsilon_\mu(+,k)$ and $\epsilon_\mu(-,k)$. The field operator $\tilde{A}_\mu$ is hence given by (6.32) where the summation now runs only over 1 and 2, as if it were a massless vector field. The mixing between $\tilde{A}_\mu$ and $\phi$ vanishes for this choice because of the transversality of the polarization vectors. The field operator for $\phi$ can then be chosen to be

$$\phi = \int \frac{d^3k}{\sqrt{(2\pi)^3 2E_k}}((ia_k)^\dagger e^{ik\cdot x} + (ia_k)e^{-ik\cdot x}), \tag{6.37}$$

where $a_k = a_{k3}$ of the spin-1 field. Equation (6.34) then tells us that, in the limit $|\vec{k}| \gg m$, $\frac{1}{m}\partial_\mu\phi$ describes exactly the helicity-0 polarization of the massive vector field $A_\mu$. This can be seen by considering the difference between the longitudinal polarization, $A^l_\mu$, and the scalar part of the decomposition $\frac{1}{m}\partial_\mu\phi$ acting on a state of momentum $k$

$$(A^l_\mu - \frac{1}{m}\partial_\mu\phi)|k\rangle \propto (\epsilon^{*\mu}(3,k) - \frac{k^\mu}{m})|k\rangle \simeq \frac{1}{2}\frac{m^2}{\vec{k}^2}|k\rangle \ , \tag{6.38}$$

which vanishes for $m^2 \ll \vec{k}^2$.

### 6.6.2 Helicity Decomposition for a massive Spin-2 Particle

In this section, we analyse the theory of dRGT massive gravity [72] in terms of a helicity decomposition following our work in [76]. For our analysis to be valid, the following set of conditions must hold for any theory subject to it:

- The theory has to be Poincaré invariant describing a massive spin-2 particle on a Minkowski background. The field must live in an irreducible representation of the Poincaré group labelled by its Casimir operators, its spin $s = 2$ and its mass $m \neq 0$.
- The theory must be local which means it can be expressed in terms of polynomial interactions of the massive spin-2 field.
- The theory must be weakly coupled in at least a finite energy interval $m \ll E \ll \Lambda$, where $\Lambda$ denotes the effective field theory cutoff.

We have seen in section 6.6.1 that a helicity decomposition for a massive spin-1 particle diagonalizes the kinetic term at high energies. The same is true for a massive spin-2 particle. For $k \gg m$ its Poincaré representation decomposes into the two helicity-2, two helicity-1 and one helicity-0 degree of freedom. A massive spin-2 particle can be written in terms of its helicities[4]

$$h_{\mu\nu} = \tilde{h}_{\mu\nu} + \frac{\partial_{(\mu}A_{\nu)}}{m} + \frac{1}{3}\left(\frac{\partial_\mu\partial_\nu\chi}{m^2} + \frac{1}{2}\eta_{\mu\nu}\chi\right) , \tag{6.39}$$

where $\tilde{h}_{\mu\nu}$ describes the helicity-2, $A_\mu$ the helicity-1 and $\chi$ the helicity-0 part of the massive spin-2 Poincaré representation. When analyzing scattering amplitudes, (6.39) becomes particularly useful since for high energies the asymptotic states can be described by the individual helicities. Accordingly, the power of the decomposition (6.39) can be seen explicitly when inserted into the quadratic action (6.2),

$$\begin{aligned}\mathcal{L}_{\mathrm{PF}} &= \tilde{h}^{\mu\nu}\mathcal{E}^{\rho\sigma}_{\mu\nu}\tilde{h}_{\rho\sigma} - \frac{1}{8}F_{\mu\nu}F^{\mu\nu} + \frac{1}{12}\chi\Box\chi - \frac{1}{2}m^2\left(\tilde{h}^{\mu\nu}\tilde{h}_{\mu\nu} - \tilde{h}^2\right) + \frac{1}{6}m^2\chi^2 \\ &\quad + \frac{1}{2}m^2\chi\tilde{h} + m\left(\tilde{h}\partial_\mu A^\mu - \tilde{h}^{\mu\nu}\partial_\mu A_\nu\right) + \frac{m}{2}\chi\partial_\mu A^\mu \, ,\end{aligned} \tag{6.40}$$

where $\tilde{h}^{\mu\nu}\mathcal{E}^{\rho\sigma}_{\mu\nu}\tilde{h}_{\rho\sigma} = \partial_\mu\tilde{h}^{\mu\nu}\partial_\nu\tilde{h} - \partial_\mu\tilde{h}^{\rho\sigma}\partial_\rho\tilde{h}^\mu_\sigma + \frac{1}{2}\partial_\mu\tilde{h}^{\rho\sigma}\partial^\mu\tilde{h}_{\rho\sigma} - \frac{1}{2}\partial_\mu\tilde{h}\partial^\mu\tilde{h}$ describes the linear part of the Einstein action. For $k^2 \gg m^2$, the action becomes diagonal in field space. The individual kinetic terms for $\tilde{h}_{\mu\nu}$ and $A_\mu$ correspond to massless linearized Einstein and Maxwell theory, respectively. Thus, in the limit where the mixing of the individual fields can be neglected, $\tilde{h}_{\mu\nu}$ carries precisely the two helicity-2, $A_\mu$ the two helicity-1 and $\chi$ the single helicity-0 degrees of freedom.

Note that requiring the diagonalization of the kinetic term fixes the relative factor of $1/2$ between the $\chi$-terms in (6.39). Similarly, the factors of $m$ in (6.39) normalize the kinetic terms. They can be determined by the coupling of $h_{\mu\nu}$ to sources $\sim \int d^4x T^{\mu\nu}h_{\mu\nu}$. The propagator of a massive spin-2 field $h_{\mu\nu}$ between two conserved sources $T_{\mu\nu}$ and $\tau_{\mu\nu}$ is

[4] Note, that the fields $\tilde{h}_{\mu\nu}, A_\mu, \chi_\mu$ only represent the three helicities at high energies. Otherwise they correspond to an admixture of all of them. The decomposition is, of course, valid for all energies.

given by

$$\begin{aligned}
T^{\mu\nu} D_{\mu\nu,\rho\sigma} \tau^{\rho\sigma} &= T^{\mu\nu} \frac{\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \frac{2}{3}\eta_{\mu\nu}\eta_{\rho\sigma}\right)}{p^2 - m^2} \tau^{\rho\sigma} \\
&= T^{\mu\nu} \frac{\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma}\right)}{p^2 - m^2} \tau^{\rho\sigma} + T^{\mu\nu} \frac{1}{6} \frac{\eta_{\mu\nu}\eta_{\rho\sigma}}{p^2 - m^2} \tau^{\rho\sigma} \; .
\end{aligned} \tag{6.41}$$

The first term in the last line corresponds to the helicity-2 state $\tilde{h}_{\mu\nu}$. The second term is an additional interaction from the extra scalar degree of freedom $\chi$ and fixes the overall normalization of it in the helicity decomposition. By considering non-conserved sources one can accordingly fix the normalization of $A_\mu$ in (6.39).

In order for (6.39) to describe the correct number of degrees of freedom it is mandatory that there are additional redundancies in the components since on the left-hand side of (6.39) the tensor has ten components and on the right-hand side there are, a priori, 15 components. These redundancies are reflected in the transformations

$$\begin{aligned}
\tilde{h}_{\mu\nu} &\rightarrow \tilde{h}_{\mu\nu} + \partial_{(\mu}\xi_{\nu)} + \frac{1}{2}\eta_{\mu\nu} m \Sigma \; , \\
A_\mu &\rightarrow A_\mu + \partial_\mu \Sigma - m\xi_\mu \; , \\
\chi &\rightarrow \chi - 3m\Sigma \; ,
\end{aligned} \tag{6.42}$$

which leave (6.39) and hence the Lagrangian (6.40) invariant. Here, $\Sigma$ denotes a scalar and $\xi_\mu$ a four vector and both are arbitrary functions of spacetime. Together they remove five redundant components of the decomposition (6.39). Also note here again that by construction the validity of the decomposition is limited to a theory of a weakly coupled massive spin-2 particle. If $h_{\mu\nu}$ is used to describe different degrees of freedom, (6.39) is no longer guaranteed to capture the correct physics. This is consistent with the group theoretical arguments outlined above.

One can thus argue that finding inconsistencies in the analysis of the helicity components, as for example higher derivatives, leads to the conclusions that one of the above assumptions is violated and hence

- The theory contains ghosts.
- There is no weak coupling regime for $k^2 \gg m^2$
- The weakly coupled degrees of freedom cannot be grouped to form a massive spin-2 particle. This happens explicitly for example in Lorentz violating theories.
- Additional degrees of freedom are required to enter the theory at some scale or the theory is shielded otherwise [245].
- The theory is nonlocal.

It is important to bear in mind that these conclusions can only be drawn from the analysis within its realm of applicability, i.e. within the limits of the effective field theory.

## 6.7 Einsteinian Interactions

### 6.7.1 Cubic Vertex

In this section, we apply the helicity decomposition (6.39) to analyze the theory of (6.29) up to and including cubic order which we have carried out in [76]. This analysis complements the one of [72, 73] from a different point of view.

Two advantages of the approach using the helicity decomposition are that due to the assumption of weak coupling, one can concentrate on the cubic order interaction term at first. Furthermore, as we have argued in section 6.4 the helicity basis constitutes an appropriate choice of degrees of freedom for a stability analysis concerning ghosts since one expects the possible instability to arise on energy scales much larger than $m$. We will consider only couplings to sources which are linear in the helicity fields, i.e. which arise from $h_{\mu\nu}T^{\mu\nu}$. We will comment on this choice later on. With this choice of source-coupling, we will find that higher-derivative terms on the helicities persist in the theory. However, the ghostly mode never drops below the cutoff on backgrounds accessible within the effective field theory.

The cubic order interaction Lagrangian can be obtained by expanding (6.23) and adding all possible cubic non-derivative interaction terms:

$$
\begin{aligned}
\mathcal{L}^{(3)} \;=\; & \frac{1}{M_P}\left[\frac{1}{4}h^{\alpha\beta}\partial_\alpha h^{\mu\nu}\partial_\beta h_{\mu\nu} - \frac{1}{4}h^{\alpha\beta}\partial_\alpha h\partial_\beta h + h^{\alpha\beta}\partial_\beta h\partial_\mu h^\mu_\alpha - \frac{1}{2}h^{\mu\nu}\partial_\alpha h\partial^\alpha h_{\mu\nu}\right.\\
& +\frac{1}{8}h\partial_\mu h\partial^\mu h - h^{\mu\nu}\partial_\alpha h^\alpha_\mu\partial_\beta h^\beta_\nu - h^{\mu\nu}\partial_\nu h^\alpha_\mu\partial_\beta h^\beta_\alpha + \frac{1}{2}h\partial_\mu h^{\mu\nu}\partial_\alpha h^\alpha_\nu\\
& +\frac{1}{2}h^{\mu\nu}\partial^\alpha h_{\mu\nu}\partial_\beta h^\beta_\alpha - \frac{1}{4}h\partial_\alpha h\partial_\beta h^{\alpha\beta} + \frac{1}{2}h^{\mu\nu}\partial_\alpha h_{\nu\beta}\partial^\beta h^\alpha_\mu + \frac{1}{2}h^{\mu\nu}\partial_\beta h_{\nu\alpha}\partial^\beta h^\alpha_\mu\\
& -\frac{1}{4}h\partial_\alpha h_{\mu\nu}\partial^\nu h^{\mu\alpha} - \frac{1}{8}h\partial_\alpha h^{\mu\nu}\partial^\alpha h_{\mu\nu}\\
& \left.+m^2(k_1 h^\mu_\nu h^\nu_\rho h^\rho_\mu + k_2 h h_{\mu\nu}h^{\mu\nu} + k_3 h^3)\right], \qquad (6.43)
\end{aligned}
$$

where $k_1, k_2, k_3$ are free parameters. We will at first set external sources to zero as the coupling $h_{\mu\nu}T^{\mu\nu}$ does not introduce higher derivatives. Inserting the decomposition (6.39), one immediately encounters higher derivatives on $A_\mu$ and $\chi$. Those operators with seven or eight derivatives are boundary terms and disappear on the equations of motion and are hence irrelevant. Terms with the largest number of derivatives contributing to the equation of motion are suppressed by the scale $\Lambda_5 = \sqrt[5]{m^4 M_P}$. This is the lowest scale of the theory and constitutes the effective field theory cutoff. Interactions at this scale are the first ones to become important for the stability analysis. Taking the decoupling limit $M_P \to \infty$ and $m \to 0$ while keeping $\Lambda_5$ fixed annihilates all interactions apart from the ones suppressed by $\Lambda_5$. The resulting Lagrangian takes the form

$$\begin{aligned}\mathcal{L}_{\text{dec}} &= \mathcal{L}_{\text{kin}}(\tilde{h}_{\mu\nu}, A_\mu, \chi) - \frac{1}{18\Lambda_5^5}\Big(\tilde{h}^{\mu\nu}\partial_\mu\partial_\alpha\partial_\beta\chi\partial_\nu\partial^\alpha\partial^\beta\chi - \tilde{h}^{\mu\nu}\partial_\mu\partial_\alpha\partial_\nu\chi\partial^\alpha\Box\chi \\ &\quad -\frac{1}{2}\tilde{h}\partial_\mu\partial_\alpha\partial_\beta\chi\partial^\mu\partial^\alpha\partial^\beta\chi + \frac{1}{2}\tilde{h}\partial_\alpha\Box\chi\partial^\alpha\Box\chi\Big) \\ &\quad +\frac{1}{432\Lambda_5^5}\Big((2+8k_1+16k_2+32k_3)(\Box\chi)^3 + (2-24k_1-16k_2)\chi\Box\chi\Box^2\chi \\ &\quad +(1-12k_1-8k_2)\chi^2\Box^3\chi\Big)\,, \end{aligned} \tag{6.44}$$

where $\mathcal{L}_{\text{kin}}$ contains the kinetic terms of all helicities (6.40) with mass equal to zero due to the decoupling limit.

On the equations of motion, one can then use the freedom in the parameters $k_1, k_2, k_3$ to eliminate the higher derivative self-interactions of $\chi$. One can already infer from (6.44) that it is not possible to eliminate all higher derivative interactions. In order to cancel the $\chi$ self-interactions we choose

$$\begin{aligned} 1+4k_1+8k_2+16k_3 &= 0 \\ 1-12k_1-8k_2 &= 0\,. \end{aligned} \tag{6.45}$$

The remaining interactions at the scale $\Lambda_5$

$$\mathcal{L}_{\text{dec5}}^{(2+3)} = \mathcal{L}_{\text{kin}}(\tilde{h}_{\mu\nu}, A_\mu, \chi) - \frac{1}{36\Lambda_5^5}\tilde{h}_{\rho\sigma}\mathcal{E}^{\rho\sigma\mu\nu}\partial_\mu\partial_\alpha\chi\partial_\nu\partial^\alpha\chi \tag{6.46}$$

indicate the appearance of an additional degree of freedom on a background for $\tilde{h}_{\mu\nu}$ [76]. However, for perturbative backgrounds which are accessible within the effective description, the ghost mass always remains above the cutoff. Hence, the ghost does not appear in the physical spectrum. The stipulated UV completion must then take care of this possible instability. Within the framework of the gravitational Higgs mechanism, a similar instability for the Lagrangian proposed in [72] was found at fourth order in [267].

Without sources, the above conclusion, however, only remains true if the theory is truncated at the cubic level. As has later been studied in [74], adding the fourth order interaction in $h_{\mu\nu}$ shows that the interactions at $\Lambda_5$ are given by a redundant operator which thus can be removed by a field redefinition. The fourth order interactions are actually the only other interactions which come with the scale $\Lambda_5$ and are given by

$$\frac{1}{36^2\Lambda_5^{10}}\partial_\mu\partial^\rho\phi\partial_\rho\partial_\nu\phi\mathcal{E}^{\mu\nu\alpha\beta}\partial_\alpha\partial_\sigma\phi\partial_\beta\partial^\sigma\phi\,. \tag{6.47}$$

Now, one can perform a nonlinear field redefinition of the field $\tilde{h}_{\mu\nu}$ of the following form

$$\tilde{h}_{\mu\nu} = \bar{h}_{\mu\nu} + \frac{1}{36\Lambda_5^5}\partial_\mu\partial_\alpha\chi\partial_\nu\partial^\alpha\chi \tag{6.48}$$

which yields a free cubic Lagrangian for the scale $\Lambda_5$. Note that this field redefinition is canonical and invertible. The Lagrangian after the field redfinition is

$$\mathcal{L} = \frac{1}{2}\bar{h}^{\mu\nu}\mathcal{E}^{\alpha\beta}_{\mu\nu}\bar{h}_{\alpha\beta} - \frac{1}{8}F^{\mu\nu}F_{\mu\nu} + \frac{1}{12}\chi\Box\chi\ . \tag{6.49}$$

Thus, we recover a free theory at the scale $\Lambda_5$. As pointed out in [74], all scales lower than $\Lambda_3$ are redundant operators if one considers all higher order interaction terms appearing with the respective scale. In terms of helicities, they can be removed by appropriately defined field redefinitions which are always local and invertible suggesting that they cannot change the number of degrees of freedom.

Let us make a very important observation here. We have seen that all scales below $\Lambda_3$ are in fact redundant. Therefore, the true strong coupling scale of the theory is $\Lambda_3$. However, it is not possible to find an expansion in terms of $\Lambda_3$ of the operators $h_{\mu\nu}$ which can be truncated at a given order. To be more precise, let us not consider the theory in terms of helicities or Stückelberg fields but rather in terms of the spin-2 field $h_{\mu\nu}$ corresponding to what was called unitary gauge in section 6.5.1. For instance, the theory considered as an expansion in powers of $h_{\mu\nu}$ can only then have $\Lambda_5$ as a redundant coupling if all appropriate powers of $h_{\mu\nu}$ are included. Even though the theory is weakly coupled at $\Lambda_5$ when including both cubic and quartic interactions of $h_{\mu\nu}$, it is not possible to truncate the theory at cubic order in $h_{\mu\nu}$, because this would reintroduce the strong coupling scale $\Lambda_5$. The reason for this is that the helicity-0 polarizations, which are the part of $h_{\mu\nu} \propto \frac{k_\mu k_\nu}{m^2}$, are already strongly coupled at the scale $\Lambda_5$ and therefore one needs to consider all higher order interactions suppressed by this scale. If the coefficients are tuned appropriately, they combine to redundant operators.

### 6.7.2 Coupling to Sources

Let us now analyse how the coupling to sources can affect the above conclusions. In order to be as concrete as possible, first, consider the following example Lagrangian, also discussed in [74]

$$\mathcal{L} = \frac{1}{2}\phi\Box\phi + \frac{1}{2}\psi\Box\psi + \frac{\Box\psi\,(\partial_\mu\partial_\nu\phi)^2}{\Lambda^5} + \frac{(\partial_\mu\partial_\nu\phi)^2\,\Box\,(\partial_\alpha\partial_\beta\phi)^2}{\Lambda^{10}}\ . \tag{6.50}$$

In [74] it was demonstrated that though seemingly a higher derivative action, four initial conditions suffice to solve the corresponding equations of motion, indicating the absence of additional degrees of freedom.

Performing a Hamiltonian analysis of (6.50) leads to the same conclusion. Introducing two auxiliary fields $\mu = \ddot{\phi}$ and $\rho = \dot{\phi}$ to account for higher derivatives allows for a straightforward counting of constraints. While initially the phase space dimension is enlarged from four to eight, one first class and two second class constraints are found which removes four phase space dimensions, cf. [74]. Thus, only two of the four fields are propagating.

At the same time, however, introducing sources into the action can lead to very different conclusions. A linear coupling $J_\phi \phi + J_\psi \psi$ reintroduces the ghost problem into the action as was also discussed in [74]. In the Hamiltonian analysis, this is reflected by the fact that a linear coupling to sources converts the first class constraint into a second class constraint, while at the same time eliminating the tertiary second class constraint, see [74]. A system of two second class constraints can only reduce the phase space dimension by two in our case from eight to six, indicating the presence of the ghost. On the other hand, specifying a coupling of the form

$$J_\phi \phi + J_\psi \left( \psi + \frac{1}{\Lambda^5} (\partial_\mu \partial_\nu \phi)^2 \right) \tag{6.51}$$

avoids this problem. This can again be confirmed in a Hamiltonian analysis of the full action, where both the first class and the tertiary constraint are now preserved.

For a classical analysis, cf. section 6.4, to detect the possible appearance of ghost-like degrees of freedom on certain backgrounds, it is sufficient to prove the existence of very mild "$\epsilon$-backgrounds" that induce additional poles of the propagator. It is straightforward to see that a weak background of the form $\Box\psi = \epsilon$ results in the following kinetic action for $\phi$:

$$\frac{1}{2}\phi \left( \Box\phi + \frac{2\epsilon}{\Lambda^5} \Box^2 \phi \right) . \tag{6.52}$$

While the resulting ghost obviously has a mass which is parametrically larger than the cut-off $\Lambda$, this is of no immediate interest. Eq. (6.52) simply proves the existence of backgrounds with additional degrees of freedom.

How can this be reconciled with the previously cited Hamiltonian analysis for (6.50)? The answer is quite simple. Specifying the coupling to sources makes implicit statements about the physical degrees of freedom of the system. With a linear coupling, the fields $\phi$ and $\psi$ are the physical degrees of freedom. They can be excited independently. A background $\Box\psi = \epsilon$ is a valid physical configuration and leads to a higher derivative action for $\phi$. The additional pole seen in (6.52) corresponds to the additional modes found in a Hamiltonian analysis of (6.50) with a linear coupling to sources. On the other hand, specifying a coupling of the form (6.51) implies that $\phi$ and $\psi + \frac{1}{\Lambda^5}(\partial_\mu \partial_\nu \phi)^2$ are the physical degrees of freedom. This immediately signals that a nonlinear field redefinition should be performed to simplify an analysis of the properties of the theory. Furthermore, the nonlinear coupling to sources has to be explicitly taken into account for the ghost analysis. Choosing a background $\Box\psi = \epsilon$, and accordingly $J_\psi = -\epsilon$, leads to a contribution to the quadratic Lagrangian from the source coupling that exactly cancels the higher derivative term in (6.52).

We have now understood that a ghost analysis relies crucially on specifying the physical degrees of freedom and this is exactly where shortcomings of a Hamiltonian analysis become visible. Performing such an analysis without specifying the coupling to sources is not sufficient to exclude the appearance of ghosts on viable backgrounds. The Hamiltonian analysis of (6.50) allowed one to conclude that the system is free of ghosts, seemingly contradicting a straightforward stability analysis. However, we have seen that the latter includes additional information, which, when correctly translated into the Hamiltonian, leads to an agreement between both methods.

We shall now apply this reasoning to the scenario of massive Einstein gravity. While it is immediately clear that a Hamiltonian analysis without specifying a coupling to sources does not capture the correct physics on arbitrary backgrounds, the question whether one can restrict couplings to sources, as addressed in [74], is much more subtle. Restrictions in this case consist of explicit conditions on the sources, such as demanding covariant conservation. It is thus not as straightforward to identify the physical degrees of freedom as in the previous case, where only the linearly coupled fields deserved the name of propagating degrees of freedom.

In order for the coupling $\Lambda_5$ in [72, 74] to remain redundant when introducing sources $T_{\mu\nu}$, it is important also to include the nonlinear couplings to the sources. Assuming the source is covariantly conserved, $\sqrt{-g}\nabla_\mu T^{\mu\nu} \equiv \partial_\mu(\sqrt{-g}T^{\mu\nu}) + \sqrt{-g}\Gamma^\nu_{\mu\gamma}T^{\mu\gamma} = 0$, where $\nabla_\mu$ is the covariant derivative with respect to the full metric $g_{\mu\nu}$ and $\Gamma^\nu_{\mu\gamma} = \frac{1}{2}g^{\nu\delta}(\partial_\mu g_{\delta\gamma} + \partial_\gamma g_{\delta\mu} - \partial_\delta g_{\mu\gamma})$ are the Christoffel symbols of the metric. The fact that the sources are only covariantly conserved is a consequence of the backreaction of gravity on matter, which is is equivalent to considering an infinite power series of nonlinear couplings $h^n_{\mu\nu}T^{\mu\nu}$, where $n \in \mathbb{N}$. Once again, it is then important that one considers the higher order terms corresponding to the covariant conservation if $\Lambda_5$ is to be a redundant coupling. It has been shown in [74] that if one does so, one can indeed retain this redundancy. For this reason, in terms of the helicities, the field redefinition (6.48) does not introduce any higher derivatives in the coupling to sources.

Let us briefly note that on the classical level one can always choose sources such that the condition of covariant conservation is fulfilled. In contrast to gravity however in massive gravity there is no symmetry a priori which would protect the covariant conservation of sources. Therefore, for a full quantum analysis one has to make sure that radiative corrections leave the coupling $\Lambda_5$ redundant. In [74] it was suggested that this is indeed the case.

We want to end this section with a general comment on nonlinear field redefinitions. While well-defined, invertible field redefinitions obviously just correspond to a renaming of variables and cannot change the physical content of a self-contained theory, these properties have to be carefully checked. Furthermore, once the observable degrees of freedom of a theory are specified, a coupling to sources has to be taken into account, as we have seen in the example given above. Otherwise, nonlinear redefinitions, even if invertible, may change the notion of physical degrees of freedom and can thus give misleading results.

## 6.8 Cubic Interaction for a massive Spin-2 Particle

Interacting massless spin-2 theories which are Lorentz-invariant, weakly coupled and unitary have been shown by Weinberg and others, see e.g. [80–82], to be uniquely described by Einstein's theory in a low-momentum expansion of operators in four dimensions. This means that the interactions of a massless spin-2 particle described by operators with up to two derivatives have to be precisely those of General Relativity. This is quite a remarkable property.

The situation changes, however, when considering massive spin-2 particles. The above theorem no longer applies and, a priori, there are no restrictions on the interactions apart from Lorentz invariance and that they should propagate only the five helicities of the massive spin-2 representation. In section 6.2, it was argued that the Lagrangian for a free massive spin-2 particle propagating two helicity-2, two helicity-1 and one helicity-0 degree of freedom has to be of the Fierz-Pauli form (6.2) [67, 255].

The previous sections were concerned with the analysis of massive spin-2 theories which keep the full non-linear Einsteinian derivative structure but add potential interactions. Since these theories are expected to reduce to General Relativity, barring the vDVZ discontinuity, they are referred to as massive gravity theories.

Starting instead solely from the free Lagrangian of a massive spin-2 field, one can add interactions and restrict their structure such that they do not change the number of propagating degrees of freedom, i.e. that the number of constraints is conserved. Following our work in [76], let us first take the Fierz-Pauli Lagrangian (6.2) and add all possible Lorentz invariant cubic interaction terms with up to two derivatives,

$$\begin{aligned}
\mathcal{L}^{(3)} \;=\; & \frac{1}{\Lambda^7}\big(k_1 h^{\alpha\beta}\partial_\alpha h^{\mu\nu}\partial_\beta h_{\mu\nu} + k_2 h^{\alpha\beta}\partial_\alpha h\partial_\beta h + k_3 h^{\alpha\beta}\partial_\beta h\partial_\mu h^\mu_\alpha + k_4 h^{\mu\nu}\partial_\alpha h\partial^\alpha h_{\mu\nu} \\
& + k_5 h\partial_\mu h\partial^\mu h + k_6 h^{\mu\nu}\partial_\alpha h^\alpha_\mu\partial_\beta h^\beta_\nu + k_7 h^{\mu\nu}\partial_\nu h^\alpha_\mu\partial_\beta h^\beta_\alpha + k_8 h\partial_\mu h^{\mu\nu}\partial_\alpha h^\alpha_\nu \\
& + k_9 h^{\mu\nu}\partial^\alpha h_{\mu\nu}\partial_\beta h^\beta_\alpha + k_{10} h\partial_\alpha h\partial_\beta h^{\alpha\beta} + k_{11} h^{\mu\nu}\partial_\alpha h_{\nu\beta}\partial^\beta h^\alpha_\mu + k_{12} h^{\mu\nu}\partial_\beta h_{\nu\alpha}\partial^\beta h^\alpha_\mu \\
& + k_{13} h\partial_\alpha h_{\mu\nu}\partial^\nu h^{\mu\alpha} + k_{14} h\partial_\alpha h^{\mu\nu}\partial^\alpha h_{\mu\nu}\big) \\
& + \frac{1}{\Lambda^5}\big(k_{15} h^\mu_\nu h^\nu_\rho h^\rho_\mu + k_{16} h h_{\mu\nu} h^{\mu\nu} + k_{17} h^3\big)\,. 
\end{aligned} \tag{6.53}$$

Here $\Lambda$ constitutes the effective field theory cutoff. In order for the theory to be a useful description, the cutoff should be larger than $m$, i.e. $\sqrt[7]{k_{(1-13)}}^{\,-1}, \sqrt[5]{k_{(14-17)}}^{\,-1} \gg \frac{m}{\Lambda}$. It was argued in section 6.6 that the helicity decomposition (6.39) identifies a specific representation of the massive spin-2 field which is useful when analyzing particular instabilities as we have carried out in [76]. Higher derivative terms generally indicate the presence of additional degrees of freedom – at least on the classical level. Additionally, the helicity decomposition has an intriguing correspondence to the constraint structure of the Lagrangian in terms of the components of the tensor $h_{\mu\nu}$. We will find that the Lagrangian which is free of higher derivatives on the helicity components $\tilde{h}_{\mu\nu}$, $A_\mu$ and $\chi$ is simultaneously the Lagrangian for which $h_{00}$ is a Lagrange multiplier and $h_{0i}$ remains non-dynamical.

The idea is the following. Inserting the helicity decomposition (6.39) in (6.53), one can use the freedom in the parameter space for $k_i$ to eliminate higher derivatives on the equation of motion of the fields $\tilde{h}_{\mu\nu}$, $A_\mu$ and $\chi$. The advantage of working directly on the equation of motion is that all higher derivative terms appearing are relevant, as boundary terms do not contribute.

The detailed derivation can be found in appendix B. The resulting Lagrangian is [76]

$$\begin{aligned}\mathcal{L}^{(3)} \;=\; & \frac{k_1}{\Lambda^7}\big(h^{\alpha\beta}\partial_\alpha h^{\mu\nu}\partial_\beta h_{\mu\nu} - h^{\alpha\beta}\partial_\alpha h\partial_\beta h + 4h^{\alpha\beta}\partial_\beta h\partial_\mu h^\mu_\alpha - 2h^{\mu\nu}\partial_\alpha h\partial^\alpha h_{\mu\nu} + h\partial_\mu h\partial^\mu h \\ & -3h^{\mu\nu}\partial_\alpha h^\alpha_\mu\partial_\beta h^\beta_\nu - 4h^{\mu\nu}\partial_\nu h^\alpha_\mu\partial_\beta h^\beta_\alpha + 3h\partial_\mu h^{\mu\nu}\partial_\alpha h^\alpha_\nu + 2h^{\mu\nu}\partial^\alpha h_{\mu\nu}\partial_\beta h^\beta_\alpha \\ & -2h\partial_\alpha h\partial_\beta h^{\alpha\beta} + h^{\mu\nu}\partial_\alpha h_{\nu\beta}\partial^\beta h^\alpha_\mu + 2h^{\mu\nu}\partial_\beta h_{\nu\alpha}\partial^\beta h^\alpha_\mu - h\partial_\alpha h_{\mu\nu}\partial^\nu h^{\mu\alpha} \\ & -h\partial_\alpha h^{\mu\nu}\partial^\alpha h_{\mu\nu}\big) + \frac{1}{2}\frac{k_{15}}{\Lambda^5}\big(2h^\mu_\nu h^\nu_\rho h^\rho_\mu - 3hh_{\mu\nu}h^{\mu\nu} + h^3\big)\;. \end{aligned} \tag{6.54}$$

whose corresponding equations of motion are free of higher time derivatives.

That this theory still propagates the right number of degrees of freedom can be also easily seen by counting the number of constraints for $h_{\mu\nu}$. As explained in section 6.2, in the free theory, the five constraints on $h_{\mu\nu}$, (6.8) and (6.9), reduce the number of degrees of freedom to five. For the Lagrangian (6.54), these constraints are preserved. For example, the non-derivative part is given by

$$\mathcal{L} = \frac{1}{2}k_{15}(3h_{00}(h_{ii}^2 - h_{ij}^2) + 6(h_{0i}h_{0j}h_{ij} - h_{0i}^2 h_{ii}) + 3h_{ii}h_{ij}^2 - 2h_{ij}h_{jl}h_{li} - h_{ii}^3). \tag{6.55}$$

Hence, $h_{00}$ and $h_{0i}$ appear in the same way as in the free action. We do not display the explicit expression for the derivative part because the expression is rather lengthy. Still, one can easily check that also there $h_{0i}$ remains non-dynamical and can be solved for algebraically, yielding 3 constraints on $h_{\mu\nu}$. Furthermore, $h_{00}$ appears as a Lagrange multiplier in (6.54) and accordingly eliminates another two degrees of freedom [76].

It is easy to understand why the helicity decomposition corresponds to retaining the same constraints as at linear order. The helicity decomposition takes care of too many, i.e. greater than two, time-derivatives on the fields $\chi$ and $A_\mu$ on the equations of motion. The components $h_{00}$ and $h_{0i}$ are exactly those components of $h_{\mu\nu}$ which can introduce these higher derivatives as in terms of helicities these correspond to $\partial_0^2\chi$, $\partial_0 A_0$, $\partial_0\partial_i\chi$ and $\partial_0 A_i$.

Up to boundary terms, one can rewrite the above Lagrangian in a compact form [66] as follows

$$\mathcal{L}^{(3)} = k_1\epsilon^{\alpha_1\ldots\alpha_4}\epsilon^{\beta_1\ldots\beta_4}\partial_{\alpha_1}\partial_{\beta_1}h_{\alpha_2\beta_2}\ldots h_{\alpha_4\beta_4} + k_{15}\epsilon^{\alpha_1\ldots\alpha_3\sigma_4}\epsilon^{\beta_1\ldots\beta_3}_{\sigma_4}h_{\alpha 1\beta_1}\ldots h_{\alpha_3\beta_3}\,. \tag{6.56}$$

$\epsilon^{\alpha_1\ldots\alpha_4}$ denotes the totally anti-symmetric four-tensor in four dimension. From its anti-symmetry properties it is then simple to conclude that the constraint structure of the free Lagrangian is preserved. If there is one $h_{00}$ in (6.56), then there cannot be any other factor of $h_{00}$ in that term. Therefore, $h_{00}$ can only appear as a Lagrange multiplier. Terms with $h_{0i}$ can carry at most one time derivative and one power of $h_{0i}$ or only spatial derivatives and at most two powers of $h_{0i}$ and all other terms have spatial indices. Variation with respect to $h_{0i}$, thus, leads to a constraint equation for itself which defines it algebraically in terms of the components $h_{ij}$.

Of course we have only proven the absence of ghosts. As any cubic theory, (6.54) will still contain tachyonic instabilities. One may however easily extend our formalism to higher orders, see for example also [277]. Such a theory, however, will not reduce to General Relativity in the massless limit. Instead, one could consider it as an effective theory describing the interactions of a massless spin-2 meson.

## 6.9 Summary

In this chapter, we considered IR modifications of Einstein gravity. We saw that simply adding a mass term to General Relativity leads to inconsistencies and the appearance of a sixth propagating mode – the Boulware-Deser ghost. Adding additional non-derivative interactions de Rham, Gabadadze and Tolley showed that it is possible to construct a theory, the dRGT model for massive gravity, which only propagates five degrees of freedom; exactly the correct number for an on-shell massive spin-2 particle. In this chapter, we have employed an analysis in terms of helicity components of the spin-2 particle in order to investigate the dynamics of the dRGT model. We found that when truncating the theory at any given order higher derivative interactions appear on the equation of motion. Although they do not necessarily lead to ghost instabilities, their existence makes the counting of degrees of freedom to add up to more than five. However, it is possible to reconcile the findings of dRGT with ours by noticing that if the theory is allowed to include all orders in nonlinearities, there exists a redundant coupling. This redundancy can be removed by making a field redefinition which in turn eliminates the higher derivatives order by order.

In addition, we started from the free theory of a massive spin-2 particle and constructed possible cubic interaction terms which are free of higher derivatives for the helicity degrees of freedom at this order. We found that these coefficients do not reproduce the cubic Einsteinian derivative structure, but instead that the Lagrange multiplier $h_{00}$ of the free theory retains its role as such. This action could, for example, describe interactions of a massive spin-2 meson.

# Chapter 7

# Conclusions and Outlook

The advancement of effective field theories has largely shaped the way we understand theoretical physics today. We know that it is not necessary to understand the underlying physics up to arbitrarily high energies if we want to describe physical interactions at a given (low-energy) scale. Within the framework of effective field theories, it is possible to formulate gravity as a quantum theory similar to other fundamental forces, like, e.g., the Standard Model interactions.

In this thesis, we utilized methods of effective field theory in order to study various aspects of gravitational interactions. At first (in chapter 2), we followed the ideas of Dvali and Gomez [7] who argued that quantum gravity at high energy scales is described by light degrees of freedom which are already present in the IR theory. This so-called self-completeness of Einstein gravity is due to black hole formation and implies that short distances are shielded from observations and thus physically inaccessible. Black holes formed during an attempt to probe such short distances are described by weakly interacting gravitons – degrees of freedom which are present in the IR. Building on this idea, we argued that the same properties that make Einstein gravity self-complete in the deep UV are also responsible for the self-completeness of any UV modification of gravity. The requirement that only positive norm states appear in the spectrum of the weakly coupled theory leads to an even "earlier" encounter of black hole formation in such theories. We established that this has important consequences for attempts of a standard Wilsonian UV completion of gravity since any weakening of gravitational interactions can only take place in the strong coupling domain. However, it is precisely this regime which is shielded by black hole formation. We thus concluded that suggested UV completions of gravity which rely on an asymptotic weakening of the gravitational interactions cannot be realized in a physically meaningful way.

Furthermore, we considered the black hole condensate picture proposed by Dvali and Gomez in [18, 19]. In this picture black holes are described as self-bound condensates of many weakly interacting gravitons. This proposal points a way towards the resolution of such long-standing problems in black hole physics as the information paradox and the microscopic origin of the black hole entropy. Since the condensates are thought to be at the critical point of a quantum phase transition, quantum correlations are large and thus need to be taken into account in order to obtain a correct description of the condensate. In chapter 3, we studied a toy model system which shares certain characteristics with black hole systems. Specifically, we investigated a condensate which undergoes a collapse during which it loses particles due to incoherent scattering. We showed that it is possible to obtain solutions in which the system undergoes a self-similar collapse while staying at the critical point. In addition, by computing the excitation spectrum, we could qualitatively establish the appearance of a light mode.

To summarize, the toy model of a collapsing condensate with a mass given by its inverse size can reproduce some of the crucial features of the quantum N-portrait [18, 19] in which black holes are described by a condensate of weakly interacting gravitons. It would be interesting to analyse how this analogy extends to more involved toy models which, e.g., either feature long-range derivative interactions (see e.g. [278]) or a relativistic dispersion relation which may even be realized in tabletop experiments using ultracold atoms in optical lattices. For example, a relativistic dispersion relation for fermions has been realized in [279]. Furthermore, a straightforward extension of our work is to compute the depletion of the condensate. Our numerical method provides a straightforward implementation of this and we will investigate it in a future work. More insight could also be gained from finding a toy model which exhibits the properties expected for a star collapsing to form a black hole. In particular, with such a toy model one could study the dynamical evolution from a graviton condensate which is far from the critical point, which is the case for the gravitational field of a star, to the critical point at the end of the collapse. Another aspect to be studied in detail in connection with the proposed self-completeness of gravity is to find a toy model which starts out from a system of two scattering gravitons with a very high center of mass energy and subsequently forms a black hole of many gravitons.

However, we want to note that the origin of entropy in toy models with non-derivative contact interactions is still under debate. Usually too few light modes appear at the critical point (see e.g. [126, 127]). Derivative couplings, such as, for example, present in classicalization could, however, provide a larger number of light modes. There are many different directions the graviton condensate picture could be extended to. For example, there are efforts to understand de Sitter and Anti-de Sitter spaces in terms of graviton condensates.

In the second part of this thesis, we were concerned with the possible effects of quantum gravity corrections. Quantum gravity is believed to violate global symmetries. However, neither the exact form of the corresponding symmetry violating operators nor their suppression strength are known. Such corrections could have important consequences for the axion solution of the strong CP problem, see chapter 4. By working in the dual three-form description of the QCD axion solution, it is possible to parametrize these effects efficiently by introducing an additional three-form coupled to the axion. In this description, the axion solution corresponds to giving a mass to the QCD three-form gauge field and thereby screening the $\theta$-angle. If an additional gravitational three-form is induced which is coupled to the axion, the latter can no longer completely screen the QCD field. However, we showed that the neutrino lepton number $U(1)$ symmetry of the Standard Model provides an anomalous current for gravity which can resolve the tension with the axion solution. Effectively, there are now two independent mass terms generated for both the QCD and the gravitational three-form. Although for non-zero neutrino masses this screening mechanism starts to become inefficient. As long as the neutrino masses are small, it is still possible to obtain sufficient screening such that the $\theta$-parameter is within the experimental bounds.

The question whether and how quantum gravitational effects violate global symmetries is still not settled. Usually the arguments for such a violation rely on the fact that black holes in the standard semi-classical treatment have no hair. In the quantum N-portrait of black holes discussed in chapter 3, Dvali and Gomez argued that black holes could carry quantum hair which are only suppressed by $\sim 1/N$ (N is the number of constituent gravitons) instead of the usually assumed exponential suppression $e^{-N}$. These hair would indeed imply that black holes do not "destroy" global charges, but instead the information about global charges $Q$ inside the black could be stored in the $Q/N$ suppressed hair. In consequence, it could be subsequently released during the evaporation. This is certainly an interesting aspect of the black hole condensate picture which requires further studies.

Regarding the strong CP problem, the specific form of the QCD vacuum energy is important. This was another concern of our work. In the presence of massless quarks, the $\theta$-dependence of the QCD vacuum becomes unobservable. Indeed, for small quark masses the $\theta$-dependence is linear in the lightest quark mass as was shown in [44]. In chapter 4, we considered the screening of the $\theta$-term in terms of bubble nucleation of $\eta'$ vacua. This mechanism is analogous to Schwinger pair creation in a strong electric field. Using the potential for $\eta'$ derived from standard instanton calculations, we found that the linear dependence cannot be reproduced. In fact, taking into account Witten's arguments about the nonanalyticity of the QCD vacuum energy for $\theta$, this could have been expected. Thus, we conclude that indeed the stipulated non-analyticities are important when relying on the vacuum energy for values of $\theta$ away from zero.

Gravitational effects appear also on large scales in a cosmological context. An important question is what constitutes the dark matter which is thought to be responsible for the (gravitational) formation of structure. In chapter 5, we considered models of axion dark matter in light of the newest Planck data [187]. Generically, axion dark matter models in which the dark matter density originates from the so-called misalignment mechanism are subject to additional constraints from isocurvature perturbations. These are perturbations in the axion number density which are induced during inflation when the axion is essentially massless. Observations from Planck have put a tight bound on isocurvature perturbations. In order for axion dark matter to observe this bound, in general, the inflationary scale has to be low. On the other hand, theoretical models for low inflationary scenarios which are compatible with the observations from Planck are rare. In view of these problems, apart from considering a specific model which accommodates a low inflationary scale, we proposed a mechanism which suppresses the isocurvature perturbations of the axion independently of the inflationary scale. This was achieved by considering a non-minimal kinetic coupling for the axion field which is large on an inflating background. Furthermore, we considered the possibility of realizing a cosmological scenario in which inflation is successfully driven by the Standard Model

Higgs boson and dark matter produced by the axion. Postulating a generic non-minimal kinetic coupling for scalar fields on a de Sitter background, we found that this is possible without introducing a large hierarchy of scales in the coupling strengths.

In our approach, we used the fact that we could treat the non-minimal coupling operator as the covariant resummation of a series of operators of an effective field theory on FRW with strong coupling scale $\Lambda$. In this context, an interesting future analysis could be to understand how the effective operators change during the transition from the FRW background to Minkowski. Furthermore, we have hinted that such a coupling to the Higgs boson on Minkowski could lead to interesting signatures in the running of the effective Higgs self-coupling. Currently, we are considering whether the additional terms could improve the running at high energies and help to avoid the electroweak instability [244].

In the last part of this thesis, we considered theories of massive spin-2 particles. The introduction of a small mass term for General Relativity has profound consequences. On the non-linear level, such an addition leads to the appearance of a sixth mode, the so-called Boulware-Deser ghost. By adding an infinite series of non-derivative interactions, first put forward by de Rham, Gabadadze and Tolley, it has been shown that this mode disappears [72, 73, 75]. In chapter 6, we have analyzed the truncated cubic order theory of dRGT massive gravity. Utilizing a helicity decomposition for the massive spin-2 particle, we found that there appear higher derivative terms on the equation of motion, suppressed by the scale $\Lambda_5$ indicating the presence of additional degrees of freedom. However, considering also higher order interactions which enter with the scale $\Lambda_5$, it was argued in [74] that these operators are redundant and can thus be removed by a field redefinition. This automatically removes the terms containing higher derivatives. On the other hand, this implies that the theory of [72] cannot be truncated without reintroducing the scale $\Lambda_5$ and the accompanying higher derivative terms. However, since the analysis in terms of helicities is only applicable within the effective field theory, i.e. below the cutoff, the presence of higher derivative terms does not necessarily imply that there are negative-norm states in the spectrum.

A second concern of our work was to identify a cubic order self-consistent interaction for a massive spin-2 particle. Starting from the free theory, we considered all possible (Lorentz invariant) cubic interaction terms with up to two derivatives and performed a stability analysis in terms of helicities. By requiring that the theory propagates only the five helicity components of the massive spin-2 particles, which we ensured by eliminating higher derivative interactions, we constructed a viable cubic interaction term. One interesting aspect is that, in terms of the components of the two tensor $h_{\mu\nu}$, the constraint structure of the linear theory is preserved in the sense that $h_{00}$ remains a Lagrange multiplier and $h_{0i}$ non-dynamical. Such an interaction could, for example, describe an interacting spin-2 meson.

In the context of massive gravity an obvious next step is to covariantize the theory. For dRGT massive gravity, this has been considered, for example, in [280]. However, in the context of the cubic interaction for a massive spin-2 particle we derived in chapter 6, this has not yet been studied. In this case, a first step would be to put the massive spin-2 particle on a fixed curved background geometry. In the linear theory, the constraint structure is kept on the curved background but in order to have a well-defined Cauchy problem the addition of an extra curvature term is needed [281]. A naive attempt, by simply replacing the partial derivatives with covariant ones at cubic order, already does not conserve the constraint structure. In this respect, it would be of much interest to see in the future whether it is possible to put the Lagrangian (6.54) on a curved background without loosing any constraints.

# Appendix A

# Corrections to the Energy-Momentum Tensor of the Source

The energy-momentum tensor of colliding particles is modified during a scattering process due to their coupling to gravity. This modification is encoded in the conservation equation

$$\nabla_\alpha T^\alpha_\beta = 0 \tag{A.1}$$

which is valid at all orders in nonlinearities. The conservation equation (A.1) is a result of the diffeomorphism invariance of the action. At linear order, the conservation equation is obtained from the interaction

$$\int d^4x \frac{h^{\alpha\beta}}{M_p} T^{(0)}_{\alpha\beta} \ , \tag{A.2}$$

where $T^{(0)}$ is the energy momentum tensor calculated in absence of gravity, or in other words, by considering the energy-momentum tensor as an *external* source. In fact by considering the linear diffeomorphism group under which the perturbation of the metric transforms as $h_{\alpha\beta} = \partial_{(\alpha}\xi_{\beta)}$ we obtain

$$\partial^\alpha T^{(0)}_{\alpha\beta} = 0 \ , \tag{A.3}$$

which is the zeroth order in (A.1). Obviously one may consider the first order in (A.1). In this case the energy momentum tensor can no longer be consider as an external source. This is similar, for example, to radiative corrections in QED. However, this contribution is only important whenever the operator (A.2) is a large, i.e., after black hole formation (since the colliding particle masses are small with respect to the Planck scale). However this regime is hidden behind a black hole. Concluding, although it is true that the energy-momentum tensor is not an external source at full non-linear level, *it is*, however, at linearized level, which is the regime considered in this thesis.

The following computation is to show that the first order corrections to the stress-energy tensor of the "external" particle are indeed negligible. In this case, the particle can no longer be considered as a point-like $\delta$-function source. Instead we model the particle as a perfect fluid ball of radius of its Compton wavelength $R_c$ with constant density $\rho = \frac{M}{V} = const$ for $r < R_c = M^{-1}$, where $V$ is the volume of the ball and $M$ the mass of the particle. The stress-energy tensor of such a ball of fluid is given by

$$T^{(0)}_{\alpha\beta} = (\rho + p) u_\alpha u_\beta + p g_{\alpha\beta} \ . \tag{A.4}$$

We assume the matter to be non-relativistic to first approximation, i.e. $\rho \gg p$. In a static spacetime the fluid velocity 4-vector points in the same direction as the static Killing vector field $u_\alpha \propto (dt)_\alpha$, which in our coordinates means $u_\alpha \propto \delta^0_\alpha$. A timelike 4-velocity gives the constraint

$$u_\alpha u^\alpha = -1 \ , \tag{A.5}$$

and it follows that $u_\alpha = \frac{1}{\sqrt{-g^{00}}} \delta^0_\alpha$.

On a Minkowski background, (A.1) is satisfied by $\rho = const$, $p = 0$ and $u_\alpha$ being a solution to the geodesic equation

$$u_\alpha \nabla^\alpha u_\beta = 0 \ . \tag{A.6}$$

This source yields the first order perturbations in the metric, see (2.35). What is the effect of these perturbations on the source itself? The 4-velocity up to first order corrections is

$$u_\alpha = (1 - \frac{1}{2} h^{00}) \delta^0_\alpha \ . \tag{A.7}$$

From (2.35) we know that $h_{\mu\nu} = \frac{2Gm(r)}{r} \delta_{\mu\nu}$ with $m(r) = \int_0^{R_c} d^3x \rho$. Let us split eq. (A.1) in two orthogonal parts; one in the direction of $u_\alpha$ and the other orthogonal to it:

$$u^\alpha \nabla_\alpha \rho + (\rho + p) \nabla_\alpha u^\alpha = 0 \ , \tag{A.8}$$

$$(p + \rho) u^\alpha \nabla_\alpha u_\beta + (g_{\alpha\beta} + u_\alpha u_\beta) \nabla^\alpha p = 0 \ . \tag{A.9}$$

Equation (A.8) gives $\partial_t \rho = 0$ which is satisfied trivially. Equation (A.9) gives us the correction to the pressure due to the selfinteraction of the gravitational source. In a static spacetime the pressure cannot depend on $t$ and we find

$$-\frac{1}{2} \rho \, \partial_i h_{tt} + \partial_i p = 0 \ , \tag{A.10}$$

where $i$ denotes the three spatial coordinates. Together with the boundary condition that $p(R_c) = 0$ we find that $p^{(1)}(r) = \frac{1}{2} \rho (h^{(1)}_{00}(r) - h^{(1)}_{00}(R_c))$. So the first order correction to $T_{\alpha\beta}$ is given by

$$T^{(1)}_{\alpha\beta} = 2 \rho u^{(1)}_\alpha u^{(0)}_\beta + p^{(1)} \eta_{\alpha\beta} \ , \tag{A.11}$$

where $u^{(1)}_\alpha = -\frac{1}{2} h^{00} \delta^0_\alpha$ and $u^{(0)}_\beta = \delta^0_\beta$.

We see that the first order correction is always subleading as long as $h_{\mu\nu} \ll 1$. This is the point where a black hole starts forming and hence our approximation ceases to be valid. We conclude that we can safely neglect the back-reaction of the gravitational field on the source in the weak coupling regime.

# Appendix B

# Derivation of the cubic order Lagrangian

This appendix derives the Lagrangian (6.54). Starting from the interaction Lagrangian (6.53), we first derive the equations of motion for the helicity-0 component $\chi$ and subsequently eliminate higher time derivatives. Eradicating $\Box^2\chi\Box^2\chi$, $\Box\chi\Box^3\chi$, $\partial_\mu\Box\chi\partial^\mu\Box^2\chi$ and $\Box\chi\Box^2\chi$ fixes four coefficients:

$$
\begin{aligned}
&2k_{10} - k_2 - k_3 + k_4 + 2k_5 - k_6 + k_7 + 2k_8 + k_9 = 0\ ,\\
&k_{13} + k_{14} + k_5 + k_8 + \frac{1}{2}\left(k_2 + k_3 - k_4 - 2k_5 + k_6 - k_7 - 2k_8 - k_9\right) = 0\ ,\\
&k_1 + k_{11} + k_{12} + k_2 + k_3 + k_4 + k_6 + k_7 + k_9 = 0\ ,\\
&8k_{16} + 24k_{17} + \left(8k_1 - k_2 - 7k_3 + 3k_4 - 18k_5 - 13k_6 + 17k_7 + 18k_8 + 9k_9\right) m^2 = 0\ .
\end{aligned}
$$

We proceed with eliminating terms such as $\partial_\mu\Box\chi\partial^\mu\Box\chi$, $\chi\Box^2\chi$ yielding

$$
\begin{aligned}
6k_{15} + 4k_{16} - \left(2k_1 + 11k_2 + 5k_3 + 3k_4 - k_6 + 2k_7\right) m^2 &= 0\ ,\\
-2k_1 + k_3 + 2k_5 + 2k_6 - 3k_7 - 2k_8 - 2k_9 &= 0\ .
\end{aligned}
$$

Next, consider the equation of motion for the vector $A^\mu$. Cancelling terms of the form $\partial_\mu A^\mu\Box^2 A_\alpha$, $\partial_\alpha A^\mu\Box^2 A_\mu$, $\Box A_\alpha\Box\partial_\mu A^\mu$, $\Box A^\mu\Box\partial_\alpha A_\mu$ and $\Box A_\mu\Box\partial^\mu A_\alpha$ sets five coefficients:

$$
\begin{aligned}
2k_1 + 2k_{13} + 2k_2 + k_3 - 2k_4 - 2k_5 + k_7 &= 0\ ,\\
2k_2 + 3k_3 + 2k_4 + 2k_5 + 2k_6 - k_7 - 2k_8 &= 0\ ,\\
2k_1 - 2k_4 - 2k_5 + k_7 &= 0\ ,\\
2k_2 + k_3 + 2\left(k_4 + k_5\right) &= 0\ ,\\
k_{11} + 2k_2 + k_3 + k_6 &= 0\ .
\end{aligned}
$$

Reverting to mixed interactions, the EOM for $\chi$ contains terms such as $\Box\tilde{h}\Box^2\chi$, $\Box^2\tilde{h}\Box\chi$, $\partial^\mu\partial^\nu\chi\Box^2\tilde{h}_{\mu\nu}$ and $\partial_\mu\chi\partial^\mu\Box\tilde{h}$ requiring

$$\begin{aligned}
-k_1 + k_{11} + 4k_2 + k_3 &= 0 \, , \\
4k_2 + 2k_3 + 2k_4 &= 0 \, , \\
k_1 + k_2 &= 0 \, , \\
4k_1 - k_3 &= 0 \, .
\end{aligned}$$

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