# Towards an Arithmetic for Partial Computable Functionals

**Basil A. Karádais**

Munich 2013

# Towards an Arithmetic for Partial Computable Functionals

**Basil A. Karádais**

Dissertation
an der Fakultät für Mathematik, Informatik und Statistik
der Ludwig–Maximilians–Universität
München

vorgelegt von
Basil A. Karádais
aus Thessaloniki

München, den 24. Juni 2013

Erstgutachter: Prof. Dr. Helmut Schwichtenberg
Zweitgutachter: Prof. Giovanni Sambin
Tag der mündlichen Prüfung: 12. August 2013

# Eidesstattliche Versicherung

Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

Vasileios Karádais

München, den 21. Juni 2013

# Abstract

***In English.*** The thesis concerns itself with non-flat Scott information systems as an appropriate denotational semantics for the proposed theory $\mathrm{TCF}^+$, a constructive theory of higher-type partial computable functionals and approximations. We prove a definability theorem for type systems with at most unary constructors via atomic-coherent information systems, and give a simple proof for the density property for arbitrary finitary type systems using coherent information systems. We introduce the notions of token matrices and eigen-neighborhoods, and use them to locate normal forms of neighborhoods, as well as to demonstrate that even non-atomic information systems feature implicit atomicity. We then establish connections between coherent information systems and various point-free structures. Finally, we introduce a fragment of $\mathrm{TCF}^+$ and show that extensionality can be eliminated.

***Auf Deutsch.*** Diese Dissertation befasst sich mit nicht-flachen Scott-Informationssystemen als geeignete denotationelle Semantik für die vorgestellte Theorie $\mathrm{TCF}^+$, eine konstruktive Theorie von partiellen berechenbaren Funktionalen und Approximationen in höheren Typen. Auf Basis von atomisch-kohärenten Informationssystemen wird ein Definierbarkeitssatz für Typsysteme mit höchstens einstelligen Konstruktoren gegeben und ein einfacher Beweis des Dichtheitssatzes von beliebigen finitären Typsystemen auf kohärenten Informationssystemen erbracht. Token-Matrizen und Eigenumgebungen werden eingeführt und verwendet, um Normalformen von Umgebungen aufzufinden und um aufzuzeigen, dass auch nicht-atomische Informationssysteme über implizite Atomizität verfügen. Im Anschluss werden Verbindungen zwischen kohärenten Informationssystemen und verschiedenen punktfreien Strukturen geknüpft. Schlussendlich wird ein Fragment von $\mathrm{TCF}^+$ vorgestellt und gezeigt, dass Extensionalität umgangen werden kann.

# Dues

Mathematical research texts, according to the modern norm, bear to my mind a certain ironic resemblance to the sort of narratives you typically get when reading stories by Raymond Carver or watching films by Clint Eastwood: you're shown the waves, but it's the undercurrent that matters. It's an ironic resemblance, since the modern mathematician takes frantic care in leaving out any possible hint at the undercurrent; it's perceived as a taboo. Mathematics after all, as we all so dutifully agree, is no art; it's not historic, nor political, stormy love affairs, family tragedies, international financial trends or crises are all irrelevant, and so on and so forth.

Yet the undercurrent is strong, and usually manages to sneak in the text anyway. Like in the acknowledgments. So let me follow the norm.

It's not wise to let an endeavor like this last for so long, nor, rather equivalently, is it wise to allow yourself a truly social life in the meantime: people you become thankful to accumulate at an unwieldy rate. I'll try nevertheless.

Helmut Schwichtenberg is the one who really gave me the chance to live and work in Munich. He's been patient and trustful all these years, he gave me a lot of leeway, and yet he was always alert and ready to spring to aid when I would get lost in all this freedom. I guess I will have to let it sink in for a while before I can tell just how much I really owe him.

I am deeply thankful to my family, my mom, my sister, and my little niece Christina who always tried to help me with my mathematical problems over the phone—if I gathered her suggestions in a little book, I'd get a collection of mathy fairy tales reminiscent of both Edwin Abbott and E.E. Cummings. I'm especially thankful to my father, who nevertheless won't be reading this (as far as I know; my metaphysics is a bit rusty).

And I have to thank all people who I hold dear, like Giorgos, Thanos, Apostolos, Kostas, Brent (who also did a great job shaping up my englisch), Pavel, Ian 'n' Leah, Sarah, Spyros, Dirk, and certainly Rhea, who's been there when I'd otherwise be truly alone—let alone she taught me a couple of cool programming tricks. Most of these people have even had a traceable impact on the work itself, but to follow these traces would be to dive deep into the undercurrent. There are more I could mention here, but I kept it to the guys and girls that I burdened with the thesis the most, I guess.

Then the colleagues: Diana Ratiu, Lucchino Chiarabini, Bogomil Kovachev, Trifon Trifonov, Stefan Schimanski, Freiric Barral, Klaus Thiel; Florian Ranzi and Simon Huber; Parmenides Garcia Cornejo; Josef Berger; Daniel Bembé, Kathrin Bild, Sifis Petrakis, Kenji Miyamoto, Davide Rinaldi, and Fredrik Nordvall Forsberg; Andreas Abel, too. All of them, at one time or another, have helped, both as friends as well as scientifically.

And the teachers: Giovanni Sambin, Peter Schuster, Douglas Bridges, Detlef Dürr, Sofia Kalpazidou, Symeon Bozapalidis, Giorgos Rachonis, Thanasis Tzouvaras. I must

also mention Thierry Coquand, whose few and strictly scientific remarks guided this work to an unexpected degree.

The indirect teachers too: Thomas Kuhn [26], Imre Lakatos [27], and especially Ludwig Wittgenstein's [58], have been invaluable, instructive, and highly soothing companions whose influence ran deep throughout my research experience. It's sad how downplayed, at times even ridiculed (see Georg Kreisel's unfortunate [23]), such works are by the modern working mathematician, but this is I believe a topic not best suited here. These people are now dead, and aren't going to read this text either, but hopefully somebody else who reads this will read them.

Finally, a thanks to the places: *Alter Simpl* and *Zeitgeist* of Türkenstraße; *Landhaus* at Tal, a.k.a. "the tree bar"; *Cabane* of Theresienstraße; *Flaschenöffner* of Fraunhoferstraße; *Altes Kreuz* of Falkenstraße; *Gartensalon* of Amalienpassage; all of them, places fit to ponder over calculations, when the office or home felt too small and the streets too open. At least, of course, up to the time when Munich was still farther east than any U.S. city and you could still have a decent smoke indoors without feeling guilty like a teenager does after sex; it's a shame how this picturesque city's gotten too sterile to allow for an honest way of life, but again, I guess this is a topic for some other kind of text.

**Acknowledgments**

I was supported by a Marie Curie Early Stage Training fellowship (MEST-CT-2004-504029) for three years. Unfortunately, I don't know who to thank for that, other than the local MATHLOGAPS committee in Munich who trusted me with the allotted funds, and our dutiful secretary, Gerlinde Bach, who dealt with much of the dreary paperwork; for despite the outrageous bureaucratic disguise, Marie Curie funding is a blessing in most ways.

I also want to thank the people who've trusted me to help as a TA with their students, providing me with the means to pay my rent in my post-fellowship years: Schwichtenberg and Schuster again, and also Günther Krauss, Erwin Schörner, Rudolf Fritsch, Michael Prähofer, Max von Renesse, Martin Hofmann, and Franz Merkl.

At this point, I would like to give a big thanks to the students too, whose trust has often been more precious and motivating than that of peers or supervisors. Clichés are often based on truths.

That's about it.

Basil K., Summer 2013

# Contents

# Introduction

In computability theory one has to deal with algorithms which are not sure to terminate. These algorithms naturally give rise to functionals that are definable only on the arguments for which the algorithm does terminate, that is, to *partial functionals*. The classical way to deal with this notion of partiality, originating in Stephen Kleene's [21] and Georg Kreisel's [24], is to suppose that the arguments of these functionals are themselves *total*, in the sense that they are always defined; this approach does not prove so elegant though when one wants to develop a general theory of computation at *higher types*: one needs a unified and intuitively natural way to deal with functionals, which can accept partial arguments as well as total ones.

A more appropriate setting for this, where partiality is not introduced externally anymore, is provided by the theory of *domains*, which started with Dana Scott's [50]: here one handles functionals which are total, in that they respond to every given argument, but where the arguments themselves might be "partial", in a sense to be acknowledged on the formal level: the notion of partiality should come built-in with the corresponding logical theory. The notion of approximations would then be formulated quite naturally in terms of concrete elements, characterizing the arguments at hand, as it happens for example in computations on real numbers in terms of their rational approximations.

## Types and algebras

The *type system* that we consider in what follows builds upon "algebras". A *higher type* will be formed by already given types $\rho$ and $\sigma$ as the corresponding *function space* $\rho \to \sigma$, and every *base type* $\alpha$ will be given by an *algebra*, that is, by a finite set of *constructors*; every such constructor $C$ is given with a *constructor type*:

$$C : \vec{\rho}_0 \to (\vec{\rho}_1 \to \alpha) \to \cdots \to (\vec{\rho}_r \to \alpha) \to \alpha \ ,$$

where $\vec{\rho}_i$, for $i \geq 0$, are vectors of *type variables* which may not include $\alpha$—obviously, our type system is defined by mutual induction[1]. The *arity* $\mathrm{ar}(C)$ of the constructor is defined by

$$\mathrm{ar}(C) := (\vec{\rho}_0, \vec{\rho}_1 \to \alpha, \ \cdots, \vec{\rho}_r \to \alpha) \ .$$

The arguments of type $\vec{\rho}_0$ are *parametric arguments* and the arguments of type $\vec{\rho}_i \to \alpha$, for $i > 0$, are called *recursive arguments*.

The vectors $\vec{\rho}_i$, for all $i$'s, may be empty. When this is the case for $i \geq 0$, we call $\alpha$ a *finitary algebra*; when $\vec{\rho}_0$ may not be empty but $\vec{\rho}_i$ is, for all $i > 0$, we call it *structure-finitary*; in the cases where $\vec{\rho}_i$, for some $i > 0$, are not empty, we talk of an *infinitary*

[1] Also called *simultaneous induction*. The type system we employ in this thesis is a simplified version of the one defined in [49, Chapter 6].

*algebra*. In a finitary algebra, a constructor with $r$ recursive arguments is simply said to have arity $r$. In the case of a non-parametric algebra (where $\vec{\rho}_0$ is empty), to avoid it being empty we require that it comes with at least one *nullary* constructor, often written 0.

So, the algebra $\mathbb{N}$ of the *natural numbers* comes with a nullary constructor $0 : \mathbb{N}$ and a unary constructor $S : \mathbb{N} \to \mathbb{N}$. The algebra $\mathbb{B}$ of the *boolean numbers* comes with two nullary constructors $\mathsf{tt} : \mathbb{B}$ and $\mathsf{ff} : \mathbb{B}$. The algebra $\mathbb{O}$ of the *ordinal numbers* comes with a nullary constructor $0 : \mathbb{O}$, and two unary ones, $S : \mathbb{O} \to \mathbb{O}$ and $\cup : (\mathbb{N} \to \mathbb{O}) \to \mathbb{O}$. Algebras $\mathbb{N}$ and $\mathbb{B}$ are finitary, but $\mathbb{O}$ is infinitary.

As for parametric examples, the algebra $\mathbb{L}(\rho)$ of *lists of $\rho$-objects* comes with a nullary constructor $\mathsf{nil} : \mathbb{L}(\rho)$ and a binary constructor $\mathrm{cons} : \rho \to \mathbb{L}(\rho) \to \mathbb{L}(\rho)$. Another parametric example is the *product algebra $\rho \times \sigma$* of *ordered pairs of $\rho$- and $\sigma$-objects*, with a binary constructor $(\ ,\ ) : \rho \to \sigma \to \rho \times \sigma$ (observe here the absence of nullary constructors). Both of these parametric algebras are structure-finitary.

## Constructor expressions and partiality

A naive understanding of the structure that elements in such algebras must have comes from universal algebra (see for example in [56]). If we view the set $K$ of *all* constructors involved in the simultaneous consideration of given algebras as a *many-sorted signature*—one sort per algebra—then we can easily form the *free $K$-algebra* in the well-known way, namely, as the class of all *$K$-trees* (or *$K$-terms*).

In particular, supposing for example that we had to deal with the algebras $\mathbb{N}$ and $\mathbb{O}$, the aforementioned free $K$-algebra would be two-sorted, with $K = \{0^{\mathbb{N}}, S^{\mathbb{N}}, 0^{\mathbb{O}}, S^{\mathbb{O}}, \cup^{\mathbb{O}}\}$, and among its trees one would expect to find expressions like the following[2]:

$$\text{sort } \mathbb{N}: \quad 0, S0, SS0, SSS0, \ldots$$
$$\text{sort } \mathbb{O}: \quad 0, S0, SS0, SSS0, \ldots, \cup(0^{\mathbb{N}}, 0^{\mathbb{O}}), \ldots, \cup(SS0, \cup(SSS0, SS0)), \ldots .$$

Indeed, these expressions form the backbone of the carrier sets which we will use in practice; the differences will stem from the desire to allow for *partiality*.

If we think of the above expressions as denoting "completed", "total" entities, we want to allow for expressions denoting "incomplete", "partial" entities as well; in order for these to be completed more information would be needed. We achieve that by introducing yet another symbol $*_\alpha$ for each algebra $\alpha$ we consider, meaning "least information of sort $\alpha$" and behaving exactly like an extra nullary constructor, the *(partial) pseudo-constructor*, so that, in the previous setting, we would additionally obtain expressions like the following:

$$\text{sort } \mathbb{N}: \quad *, S*, SS*, SSS*, \ldots$$
$$\text{sort } \mathbb{O}: \quad *, S*, SS*, SSS*, \ldots, \cup(*^{\mathbb{N}}, *^{\mathbb{O}}), \ldots, \cup(SS*, \cup(SSS*, SS*)), \ldots .$$

However, since we really want to discuss computability—in other words, *information* on the construction and behavior of numbers, functions, and functionals that may not always be defined—the carrier sets have to be so devised, as to portray partial entities in a bit more intricate way than the straightforward free $K$-algebra above; namely,

[2]Note that the pairs $(a^{\mathbb{N}}, b^{\mathbb{O}})$ in expressions with the supremum constructor refer to elements of the *graph* of a sequence of ordinals, understood as a *mapping* of type $\mathbb{N} \to \mathbb{O}$, and not to elements of the corresponding product space $\mathbb{N} \times \mathbb{O}$.

in a way that will treat *consistency* and *entailment of computational information* in satisfactory technical detail. This calls for more structure upon our free algebras, which will be given by so called *Scott information systems*; for example, in algebra $\mathbb{N}$, the *information token* $S0$ is considered consistent with the token $S*$ but not to $SS0$, and, in algebra $\mathbb{O}$, the *neighborhood* (combined information) $\{\cup(0, S0), \cup(S0, S0)\}$ entails the token $\cup(S0, S*)$ but not $\cup(S0, 0)$. One may already see that in this way we obtain *non-flat domains* (see Figure 1.1 on page 30), as was already premised by Helmut Schwichtenberg in [47].

## Contributions to the semantics: acises, matrices, and point-free structures

It is known (see for example [49]) that every free algebra induces a non-flat Scott information system which, as it turns out, always falls into the subclass of "coherent" information systems, where consistency can be fully described by a binary predicate. Moreover, in the special case of algebras with at most unary constructors, like $\mathbb{N}$ or $\mathbb{B}$, and function spaces over them, an even simpler version of coherent information systems suffices, the ones called "atomic" in [47], where also entailment can be fully described by a binary predicate.

Non-superunary algebras like the latter, that is, algebras with at most unary constructors, represent data types that govern a reasonably essential part of known applications, and we focus on *atomic-coherent information systems* in Chapter 1. We introduce a version of them that we call *acis*, as in [47][3], but given directly as a structure of tokens with two binary relations. We investigate varying notions of functions over acises, in particular token- and neighborhood-mappings versus ideals. We isolate a maximal normal form for neighborhoods. We then prove a *definability theorem* for non-superunary type systems, that is, for type systems based on non-superunary algebras, extending previously known results. In the end, we outline limits to our proof of definability through a characterization of non-superunary algebras by *comparability properties*.

In Chapter 2 we lose the strict atomicity demand and engage in general *coherent information systems*. Still, the main feature of the chapter is actually the uncovering of atomicity that hides even in these more general structures, making acises important not just because of their simplicity but also because of their fundamental role in the model. We introduce the idea of forming a *matrix of tokens*, and then develop a *matrix theory over acises* for both finitary and infinitary algebras; we show that *entailment at base type is implicitly atomic* by characterizing it through matrix application, since matrices form an atomic information system. We isolate yet another normal form for base type neighborhoods, the *homogeneous normal form* and prove a *matrix representation theorem* for it. We then show that in the basic case of finitary algebras, base type neighborhoods attain an even simpler normal form, their *eigentoken*.

Then we move on to higher types, where we single out a crucial special case of a neighborhood, the *eigen-neighborhood*; we show that *higher type entailment is again implicitly atomic* by characterizing it through atomic entailment on the level of eigen-neighborhoods. We also find *canonical monotone forms for finitary neighborhoods*. Finally, in way of exemplifying the introduced notions, we point the way to an intrinsic approach to the well-known *density theorem* for finitary type systems: we first give a

[3]Pronounce \\ˈeɪsɪs\\ to avoid sounding vulgar. We will be using the term as a proper english noun, allowing for the forms *acises* for the plural and *acis's* for the possessive.

proof that considerably simplifies previously known proofs in settings close to ours, and then we give examples of how it may be applied.

In Chapter 3 we broaden our viewpoint to answer a simple and reasonable, yet up to this point lurking question: what kind of point-free structures correspond to the coherent information systems that we use here? Point-free topology and higher-type computability are intertwined to a considerable extent—the former providing the topological understanding for the type systems of the latter—and this is a connection that ought to be made. To this end, besides *domains*, we consider two well-known point-free structures, namely *precusls* and *formal topologies*, and impose further *coherence conditions* on them that achieve the correspondence that we seek.

## ... and a contribution to the syntax: towards an arithmetic for partial computable functionals

The motivation for delving so deeply into a *mathematical theory of coherent Scott information systems*, comes from implementation considerations. The overall project is a *logical theory of arithmetic with approximations* to be implemented in a *proof assistant*—the first steps in such a theory are described in [18]—but for such a goal one must firstly have a refined enough understanding of the model of the theory. Implementation guides one to (a) avoid using abstract domains as abstract higher-type computability theory would have it and rather turn to their tangible representation through information systems, (b) narrow one's focus on the relevant *coherent* information systems, and (c) try to find viewpoints within the model that present it in both intuitive and technically simple ways; the premise being that the latter should lead to a simpler logical theory and thus to a simpler implementation.

This necessary process of understanding the model, that is, the mathematical theory of coherent information systems, proved enough to fill up three chapters, and for the anticipated logical theory we can afford here merely one. In Chapter 4 we provide a version of the old argument of Robin Gandy's [12] and show how *extensionality can be eliminated* in such a theory, as one would like to have.

## Organization of the material

Every chapter starts with a brief preview, followed by the main sections and ending with notes, where we pay dues to colleagues and existing literature, discuss issues that digress from the main route, and give an outlook on future work. Particularly important results are labeled "theorems". Well-known results from domain theory which didn't fit in the main text were relegated to Appendix A. A selective index can be found at the end of the text.

# Chapter 1

# Atomic-coherent information systems

In this chapter we concentrate on data types as simple as the natural or the boolean numbers—in general, types of objects that constructors of at most unary arity[1] can build. Such objects are of well-known value application-wise, but also play a fundamental role in the mathematical theory, as we will see in Chapter 2. To model such types we use the *atomic-coherent information systems*, or *acises*, that were introduced in [47].

### Preview

The main plot of the chapter concerns *definability for non-superunary types*. In section 1.1 we go through basic facts concerning acises and their function spaces, and we linger a bit on a study of different notions of mappings between them. In section 1.2 we study ideals of acises from an elementary topological and category-theoretic viewpoint. In section 1.3 we show how given non-superunary algebras induce acises, state simple facts about them, and describe a normal form for their neighborhoods. In section 1.4 we prove the *definability theorem* 1.46, as well as the characterization of non-superunary algebras via comparability properties in Theorem 1.50.

## 1.1 Acises and function spaces

### Consistency and entailment as binary predicates

An *atomic-coherent information system*, being a special kind of a Scott information system, was first described in [47] as a triple $\rho = (T, \mathsf{Con}, \succ)$[2], whereby $T$ is a countable set, $\mathsf{Con}$ is a nonempty set of finite subsets of $T$, and $\succ$ is a binary relation, such that

1. $\succ$ is reflexive and transitive, that is, a preorder,

[1]For the arity of a constructor see page 1.

[2]A notational convention throughout the text is that $\iota$, $\alpha$, $\beta$, $\gamma\ldots$ denote *base types*, whereas $\rho$, $\sigma$, $\tau\ldots$ either denote *arbitrary types*, or, in absence of a type system (namely, in sections 1.1, 1.2, 2.1, as well as in Chapter 3), they denote *arbitrary Scott information systems*.

2. $\varnothing \in \mathsf{Con} \wedge \forall_{a\in T} \{a\} \in \mathsf{Con}$,

3. $U \in \mathsf{Con} \Leftrightarrow \forall_{a,b\in U} \{a,b\} \in \mathsf{Con}$,

4. $\{a,b\} \in \mathsf{Con} \wedge b \succ c \to \{a,c\} \in \mathsf{Con}$.

Call the elements of $T$ *tokens*, of $\mathsf{Con}$ *neighborhoods* (or *consistent sets*), and $\succ$ *entailment relation* of $\rho$, and write $U \succ a$ for $\exists_{b\in U}\, b \succ a$ and $U \succ V$ for $\forall_{a\in V}\, U \succ a$.

The *coherence property*, stated in axiom 3 above, makes it possible to describe this structure graph-theoretically, as a set with two binary relations. Call an *acis graph* a triple $\rho = (T, \asymp, \succ)$, whereby $T$ is again a countable set and $\asymp$ and $\succ$ are two binary relations on $T$ such that

1. $\asymp$ is reflexive and symmetric,

2. $\succ$ is reflexive and transitive,

3. if $a \asymp b$ and $b \succ c$ then $a \asymp c$, for all $a,b,c \in T$.

Call $\asymp$ a *consistency relation* and write $U \asymp a$ for $\forall_{b\in U}\, b \asymp a$ and $U \asymp V$ for $\forall_{a\in V}\, U \asymp a$. Let us also call the third axiom *propagation of consistency*. One can see that such a graph has all $\asymp$- and $\succ$-loops and is $\asymp$-undirected but $\succ$-directed.

The notion of an acis graph is equivalent to the notion of an atomic-coherent information system. First, it is easy to notice that in an acis graph $(T, \asymp, \succ)$ it is

$$\forall_{a,b\in A}\; a \succ b \to a \asymp b\ ,$$

by the reflexivity and propagation of consistency. Then we have the following.

**Proposition 1.1.** *Every acis graph corresponds to an atomic-coherent information system, and vice-versa.*

*Proof.* For the right direction, define the neighborhoods of an acis graph to be the finite sets which have the *coherence property*, that is

$$U \in \mathsf{Con} \coloneqq U \subseteq^f T \wedge \forall_{a,b\in U}\; a \asymp b$$

—in graph-theoretic terms, the neighborhoods of an acis graph are exactly its finite $\asymp$-clusters. For the left direction, define the consistency relation in an atomic-information system by

$$a \asymp b \coloneqq \{a,b\} \in \mathsf{Con}\ .$$

It is easy to check that the details hold. □

This justifies our use of the term *atomic-coherent information system* with the meaning "acis graph", and indeed in what follows we will not differentiate between the two.[3]

[3]For a more general discussion of atomicity and coherence in the context of Scott information systems as we know them see Chapter 3.

## Basic notions and facts

So an *atomic-coherent information system*, or simply an *acis*, is a triple $\rho = (T, \asymp, \succ)$, where $T$ is the *carrier*, a nonempty countable set, the elements of which are called *tokens* (or *atoms*), $\asymp$ is the *consistency*, a reflexive and symmetric binary relation on $T$ and $\succ$ is the *entailment*, a reflexive and transitive binary relation on $T$, such that *consistency propagates through entailment*, that is,

$$\forall_{a,b,c \in T} (a \asymp b \wedge b \succ c \to a \asymp c) \ .$$

For $U, V \subseteq T$, write $U \asymp V$ for $\forall_{a \in U} \forall_{b \in V} \, a \asymp b$ and $U \succ V$ for $\forall_{b \in V} \exists_{a \in U} \, a \succ b$.

The classes of *(formal) neighborhoods* (or *consistent sets*) and *ideals* (or *elements*) in $\rho$ are defined respectively by

$$U \in \mathsf{Con} := (U \subseteq^f T) \wedge (\forall_{a,b \in U} a \asymp b) \ ,$$

$$u \in \mathsf{Ide} := \forall_{a,b \in u} a \asymp b \wedge \forall_{a \in u} (a \succ b \to b \in u) \ .$$

Denote the *empty ideal* $\varnothing \in \mathsf{Ide}$ by $\bot$.

**Proposition 1.2.** *The following hold in any acis, for tokens $a, a', b, b'$, neighborhoods $U, U', V, V', W$ and ideals $u, v$:*

1. $a \succ b \to a \asymp b$.
2. $a \succ b \wedge a' \succ b' \wedge a \asymp a' \to b \asymp b'$.
3. $U \succ V \wedge U' \succ V' \wedge U \asymp U' \to V \asymp V'$.
4. $U \asymp V \wedge V \succ W \to U \asymp W$.
5. $U, V \in \mathsf{Con} \to U \cap V \in \mathsf{Con}$.

*Proof.* The first two statements follow from reflexivity and propagation of consistency. For the third statement: Suppose that $U_1 \succ V_1$ and $U_2 \succ V_2$; this unfolds to $\forall_{b_1 \in V_1} \exists_{a_1 \in U_1} a_1 \succ b_1$ and $\forall_{b_2 \in V_2} \exists_{a_2 \in U_2} a_2 \succ b_2$; since also $\forall_{a_1 \in U_1} \forall_{a_2 \in U_2} a_2 \asymp a_1$, by the second statement, we obtain $\forall_{b_1 \in V_1} \forall_{b_2 \in V_2} b_1 \asymp b_2$, that is, $V_1 \asymp V_2$.

For the fourth statement: Let $U \asymp V$ and $V \succ W$; we have $U \cup V \succ U$ and $U \cup V \succ W$, so by the previous statement we take $U \asymp W$. More concretely: let $a \in U$ and $c \in W$; then there is a $b \in V$ for which $a \asymp b$ and $b \succ c$; propagation for tokens yields $a \asymp c$.

The fifth statement is direct to show. □

For a set of tokens $X \subseteq T$, define its *(deductive) closure* and the *cone (of ideals)* above it by

$$\overline{X} := \{a \in T \mid X \succ a\} \ \text{ and } \ \nabla X := \{u \in \mathsf{Ide} \mid X \subseteq u\}$$

respectively. Denote by $\overline{\mathsf{Con}}$ the class of all closures of neighborhoods and by $\mathsf{Kgl}$ the class of all cones in the acis and write $\overline{a}$ for $\overline{\{a\}}$ and $\nabla a$ for $\nabla\{a\}$. Note that the closure of a neighborhood is finite—hence itself a neighborhood—only if the entailment relation is finitarily branching and well-founded; so, in general, $\overline{\mathsf{Con}} \not\subseteq \mathsf{Con}$. Note moreover, that the cone above a set of tokens is nonempty only when the set is consistent, that is, a neighborhood.

The following are straightforward to check by the previous proposition:

**Proposition 1.3.** *Let $X, Y \subseteq T$ and $U, V \in \mathsf{Con}$.*

1. *If $X$ is finite then $X \in \mathsf{Con}$ if and only if $\overline{X} \in \mathsf{Ide}$.*
2. $\overline{X} \cup \overline{Y} = \overline{X \cup Y}$.
3. $\overline{X} \cap \overline{Y} \supseteq \overline{X \cap Y}$.
4. $\overline{X} = \bigcup_{a \in X} \overline{a}$.
5. *$X \asymp Y$ if and only if $\overline{X} \asymp \overline{Y}$.*
6. *$X \succ Y$ if and only if $\overline{X} \supseteq \overline{Y}$.*
7. $\nabla \bot = \mathsf{Ide}$.
8. $\overline{U} \in \nabla U$.
9. $\nabla \overline{U} = \nabla U$.
10. $U \asymp V \to \nabla U \cap \nabla V = \nabla (U \cup V)$.
11. $\nabla U = \bigcap_{a \in U} \nabla a$.

### Simple constructs

Given an acis $\rho$, any acis $(T, \asymp, \succ)$ with $T \subseteq T_\rho$, $\asymp \subseteq \asymp_\rho$ and $\succ \subseteq \succ_\rho$ is a *sub-acis* of $\rho$. In particular, for any subset $\Omega \subseteq T_\rho$, we can define $\rho|_\Omega := (\Omega, \asymp_\rho|_{\Omega \times \Omega}, \succ_\rho|_{\Omega \times \Omega})$. Clearly, this is again an acis, and

$$\mathsf{Ide}_{\rho|_\Omega} = \mathsf{Ide}_\rho|_\Omega \ .$$

Let $\rho = (T_\rho, \asymp_\rho, \succ_\rho)$ and $\sigma = (T_\sigma, \asymp_\sigma, \succ_\sigma)$ be two acises. Define their *disjoint union* $\rho \cup \sigma$ by

$$\begin{aligned} T_{\rho \cup \sigma} &:= T_\rho \cup T_\sigma \ , \\ a \asymp_{\rho \cup \sigma} b &:= a \asymp_\rho b \vee a \asymp_\sigma b \ , \\ a \succ_{\rho \cup \sigma} b &:= a \succ_\rho b \vee a \succ_\sigma b \ , \end{aligned}$$

provided they have disjoint carriers, that is, $T_\rho \cap T_\sigma = \varnothing$. Define their *intersection* $\rho \cap \sigma$ by

$$\begin{aligned} T_{\rho \cap \sigma} &:= T_\rho \cap T_\sigma \ , \\ a \asymp_{\rho \cap \sigma} b &:= a \asymp_\rho b \wedge a \asymp_\sigma b \ , \\ a \succ_{\rho \cap \sigma} b &:= a \succ_\rho b \wedge a \succ_\sigma b \ . \end{aligned}$$

Define their *set-theoretic product* $\rho \otimes \sigma$ by

$$\begin{aligned} T_{\rho \otimes \sigma} &:= T_\rho \times T_\sigma \ , \\ (a_1, b_1) \asymp_{\rho \otimes \sigma} (a_2, b_2) &:= a_1 \asymp_\rho a_2 \wedge b_1 \asymp_\sigma b_2 \ , \\ (a_1, b_1) \succ_{\rho \otimes \sigma} (a_2, b_2) &:= a_1 \succ_\rho a_2 \wedge b_1 \succ_\sigma b_2 \ , \end{aligned}$$

and their *cartesian product* $\rho \times \sigma$ by

$$\begin{aligned}
T_{\rho\times\sigma} &:= T_\rho \cup T_\sigma \ , \\
a \asymp_{\rho\times\sigma} b &:= (a \in T_\rho \wedge b \in T_\sigma) \vee (a \in T_\sigma \wedge b \in T_\rho) \vee a \asymp_\rho b \vee a \asymp_\sigma b \ , \\
a \succ_{\rho\times\sigma} b &:= a \succ_\rho b \vee a \succ_\sigma b \ ,
\end{aligned}$$

provided again that $T_\rho \cap T_\sigma = \varnothing$. It is easy to check the following.

**Proposition 1.4.** *The disjoint union, intersection, set-theoretic product and cartesian product of two acises is again an acis. Furthermore, for the corresponding ideals, the following statements hold up to isomorphism:* $\mathsf{Ide}_{\rho\star\sigma} = \mathsf{Ide}_\rho \star \mathsf{Ide}_\sigma$, *for* $\star \in \{\uplus, \cap, \otimes\}$, *and* $\mathsf{Ide}_{\rho\times\sigma} \supseteq \mathsf{Ide}_\rho \times \mathsf{Ide}_\sigma$.

*Proof.* All of the cases are pretty much direct to show. We show the equality of ideals in the set-theoretic product case. Let $u \in \mathsf{Ide}_{\rho\times\sigma}$ and set $u^\rho := \{a \in T_\rho \mid \exists_b (a,b) \in u\}$ and $u^\sigma := \{b \in T_\sigma \mid \exists_b (a,b) \in u\}$. We show that $u^\rho \in \mathsf{Ide}_\rho$ (we work similarly for the $\sigma$ case): For consistency, let $a_1, a_2 \in u^\rho$, so there is a $b_i$ for $i = 1,2$ with $(a_i, b_i) \in u$; since $u$ is an ideal, it is $(a_1, b_1) \asymp_{\rho\times\sigma} (a_2, b_2)$; by the definition of $\asymp_{\rho\times\sigma}$, it is $a_1 \asymp_\rho a_2$. For closure under entailment, let $a \in u^\rho$ and $a \succ_\rho a'$; by the definition of $u^\rho$, there is a $b$ with $(a,b) \in u$; by the definition of $\succ_{\rho\times\sigma}$, we have $(a,b) \succ_{\rho\times\sigma} (a',b)$; since $u$ is an ideal, it is $(a',b) \in u$, that is, $a' \in u^\rho$.

Conversely, let $u^\rho \in \mathsf{Ide}_\rho$, $u^\sigma \in \mathsf{Ide}_\sigma$ and set $u := u^\rho \times u^\sigma$. We show that $u \in \mathsf{Ide}_{\rho\times\sigma}$. For consistency, let $(a_i, b_i) \in u$, $i = 1,2$; since $u^\rho$ and $u^\sigma$ are ideals, it is $a_1 \asymp_\rho a_2$ and $b_1 \asymp_\sigma b_2$, so, by the definition of $\asymp_{\rho\times\sigma}$, it is $(a_1, b_1) \asymp_{\rho\times\sigma} (a_2, b_2)$. For closure under entailment, let $(a,b) \in u$ and $(a,b) \succ_{\rho\times\sigma} (a',b')$; by the definition of $\succ_{\rho\times\sigma}$, we get $a \succ_\rho a'$ and $b \succ_\sigma b'$ and since $u^\rho$ and $u^\sigma$ are ideals, it is $a' \in u^\rho$ and $b' \in u^\sigma$, that is, $(a',b') \in u$.

For the cartesian product case: The properties of $\asymp_{\rho\times\sigma}$ and $\succ_{\rho\times\sigma}$ are direct. For the propagation of consistency, starting without loss of generality with the definition, we have:

$$\begin{aligned}
a \asymp_{\rho\times\sigma} b \wedge b \succ_{\rho\times\sigma} c \Leftrightarrow & \big((a \in T_\rho \wedge b \in T_\sigma) \vee a \asymp_\rho b \vee a \asymp_\sigma b\big) \\
& \wedge \big(b \succ_\rho c \vee b \succ_\sigma c\big) \\
\Leftrightarrow & \big((a \in T_\rho \wedge b \in T_\sigma \wedge b \succ_\rho c) \\
& \quad \vee (a \in T_\rho \wedge b \in T_\sigma \wedge b \succ_\sigma c)\big) \\
& \vee \big((a \asymp_\rho b \wedge b \succ_\rho c) \vee (a \asymp_\rho b \wedge b \succ_\sigma c)\big) \\
& \vee \big((a \asymp_\sigma b \wedge b \succ_\rho c) \vee (a \asymp_\sigma b \wedge b \succ_\sigma c)\big) \\
\Rightarrow & (\bot \vee (a \in T_\rho \wedge c \in T_\rho)) \vee (a \asymp_\rho c \vee \bot) \vee (\bot \vee a \asymp_\sigma c) \\
\overset{\text{def}}{\Leftrightarrow} & a \asymp_{\rho\times\sigma} c \ .
\end{aligned}$$

For the inclusion of the ideals, consider the correspondence defined by $(u,v) \mapsto u \cup v$, for $u \in \mathsf{Ide}_\rho$, $v \in \mathsf{Ide}_\sigma$, which is bijective. □

### Function Spaces

For our purposes, the most important construct between two acises $\rho$ and $\sigma$ is their *function space* $\rho \to \sigma = (T, \asymp, \succ)$, which is defined by

$$\begin{aligned} T &:= \mathsf{Con}_\rho \times T_\sigma \ , \\ (U,a) \asymp (V,b) &:= U \asymp_\rho V \to a \asymp_\sigma b \ , \\ (U,a) \succ (V,b) &:= V \succ_\rho U \wedge a \succ_\sigma b \ . \end{aligned}$$

**Proposition 1.5.** *The function space between two acises is again an acis.*

*Proof.* The axioms for $\asymp$ and $\succ$ are easy to check. For the axiom of propagation: Suppose that $(U,a) \asymp (V,b)$ and $(V,b) \succ (W,c)$; by the definition of consistency and entailment in the function space we have $U \asymp_\rho V \to a \asymp_\sigma b$ and $W \succ_\rho V \wedge b \succ_\sigma c$; we want to show that $(U,a) \asymp (W,c)$, or equivalently that $U \asymp_\rho W \to a \asymp_\sigma c$; let $U \asymp_\rho W$; by the second statement of Proposition 1.2, since $U \asymp_\rho W \wedge W \succ_\rho V$, we have $U \asymp_\rho V$, which by the assumption of consistency in $\rho \to \sigma$ yields $a \asymp_\sigma b$; propagation of consistency in $\sigma$ gives $a \asymp_\sigma c$. □

The following are direct consequences of the definition of the function space.

**Proposition 1.6.** *For a function space $\rho \to \sigma$ the following hold:*

1. $U \not\asymp_\rho V \to \forall_{a,b \in T_\sigma} (U,a) \asymp_{\rho \to \sigma} (V,b)$.
2. $(U,a) \asymp_{\rho \to \sigma} (V,b) \to (U,b) \asymp_{\rho \to \sigma} (V,a)$.
3. $a \succ_\sigma b \to \forall_{U \in \mathsf{Con}_\rho} (U,a) \succ_{\rho \to \sigma} (U,b)$.
4. $V \succ_\rho U \to \forall_{a \in T_\sigma} (U,a) \succ_{\rho \to \sigma} (V,a)$.

## Morphisms of acises

### Token-mappings

A *token-mapping* $f$ from $\rho$ to $\sigma$ is a total mapping $f : T_\rho \to T_\sigma$. It is *monotone* when

$$a \succ_\rho b \to f(a) \succ_\sigma f(b) \ ,$$

*consistency-preserving* when

$$a \asymp_\rho b \to f(a) \asymp_\sigma f(b) \ ,$$

and a *homomorphism* when it is both monotone and consistency-preserving. A homomorphism is furthermore a *monomorphism*, *epimorphism* or *isomorphism* when the token-mapping is injective, surjective or bijective respectively.

For an arbitrary token-mapping $f : T_\rho \to T_\sigma$ define the *idealization* of $f$ by the class $\mathrm{i\!i} f \subseteq T_{\rho \to \sigma}$ by

$$\mathrm{i\!i} f := \{(U,b) \mid \underset{a \in T_\rho}{\exists} \big(U \succ_\rho a \wedge f(a) \succ_\sigma b\big)\} \ .$$

For example, consider the *identity token-mapping* $\mathrm{id} : T_\rho \to T_\rho$, defined by $\mathrm{id}(a) := a$; then

$$\mathrm{i\!i}\,\mathrm{id} = \{(U,a) \mid U \succ_\rho a\} \ .$$

Another example is the *constant token-mapping* $\mathrm{cnst}_{b_0} : T_\rho \to T_\sigma$, defined by $\mathrm{cnst}_{b_0}(a) := b_0$, for a fixed $b_0 \in T_\sigma$; then

$$\ddot{\imath}\mathrm{cnst}_{b_0} = \{(U,b) \mid b_0 \succ_\sigma b\} \ .$$

The choice of the name stems from the following observation, due to Helmut Schwichtenberg.

**Proposition 1.7.** *A token-mapping $f : T_\rho \to T_\sigma$ is consistency-preserving if and only if $\ddot{\imath}f \in \mathsf{Ide}_{\rho\to\sigma}$.*

*Proof.* For the right direction: To show consistency, let $(U_1,b_1),(U_2,b_2) \in \ddot{\imath}f$; we want to show that $(U_1,b_1) \asymp_{\rho\to\sigma} (U_2,b_2)$, so let $U_1 \asymp_\rho U_2$; by the definition of $\ddot{\imath}f$ we get $a_i$'s such that $U_i \succ_\rho a_i \wedge f(a_i) \succ_\sigma b_i$, for $i = 1,2$; by the assumption we have $U_1 \cup U_2 \succ_\rho a_i$, for $i = 1,2$, hence $a_1 \asymp_\rho a_2$; since $f$ preserves consistency, it is $f(a_1) \asymp_\sigma f(a_2)$, so Proposition 1.2(2) yields $b_1 \asymp_\sigma b_2$.

To show closure under entailment, let $(U_1,b_1) \in \ddot{\imath}f$ and $(U_1,b_1) \succ_{\rho\to\sigma} (U_2,b_2)$, or, equivalently, $U_2 \succ_\rho U_1 \wedge b_1 \succ_\sigma b_2$; by the definition of $\ddot{\imath}f$ we have an $a$ with $U_1 \succ_\rho a \wedge f(a) \succ_\sigma b_1$; by the transitivity we have $U_2 \succ_\rho a \wedge f(a) \succ_\sigma b_2$, which is by definition $(U_2,b_2) \in \ddot{\imath}f$.

For the left direction: Let $\ddot{\imath}f \in \mathsf{Ide}_{\rho\to\sigma}$ and $a \asymp_\rho b$. Since $(\{a\}, f(a)),(\{b\}, f(b)) \in \ddot{\imath}f$ and $\ddot{\imath}f$ is an ideal, it follows that $f(a) \asymp_\sigma f(b)$. □

Clearly, $\ddot{\imath}\mathrm{id}$ and $\ddot{\imath}\mathrm{cnst}_{b_0}$, as defined above, are ideals of $\rho \to \rho$ and $\rho \to \sigma$ respectively.

### Neighborhood-mappings

It is tempting to carry the idea of idealization from the case of mappings between tokens to the case of mappings between *sets* of tokens: given a mapping from $\mathscr{P}(T_\rho)$ to $\mathscr{P}(T_\sigma)$, induce an ideal of the corresponding function space $\rho \to \sigma$ by collecting all $(U,b)$'s that have an "intermediary" $X \subseteq T_\rho$, that is, a set entailed by $U$ and having an image that entails $b$. Clearly, such an intermediary $X$ has to be consistent, otherwise it couldn't possibly be entailed by the neighborhood $U$.

A *neighborhood-mapping* $f$ from $\rho$ to $\sigma$ is a total mapping $f : \mathsf{Con}_\rho \to \mathsf{Con}_\sigma$. It is *monotone* when

$$U \succ_\rho V \to f(U) \succ_\sigma f(V) \ ,$$

*consistency-preserving* when

$$U \asymp_\rho V \to f(U) \asymp_\sigma f(V) \ ,$$

and a *homomorphism* when it is both monotone and consistency-preserving. A homomorphism is furthermore a *monomorphism*, *epimorphism* or *isomorphism* when the neighborhood-mapping is injective, surjective or bijective respectively.[4]

**Proposition 1.8.** *If a neighborhood-mapping $f : \mathsf{Con}_{\rho\times\sigma} \to \mathsf{Con}_\tau$ is consistency-preserving then it is consistency-preserving in each component.*

*Proof.* Let $U_1 \asymp_\rho U_2$ and $V_1 \asymp_\sigma V_2$; then $f(U,V_1) \asymp_\tau f(U,V_2)$ for each $U \in \mathsf{Con}_\rho$ and similarly $f(U_1,V) \asymp_\tau f(U_2,V)$ for each $V \in \mathsf{Con}_\sigma$. □

[4] A neighborhood-mapping from $\rho$ to $\sigma$ is nothing but a *token-mapping* from $N\rho$ to $N\sigma$, where by $N\rho$ we denote the corresponding *neighborhood information system* of an acis $\rho$ (see page 108).

For an arbitrary neighborhood-mapping $f : \mathsf{Con}_\rho \to \mathsf{Con}_\sigma$ define the *idealization* of $f$ by the class $\mathrm{ii}f \subseteq T_{\rho\to\sigma}$ by

$$\mathrm{ii}f := \{(U,b) \mid \underset{V\in\mathsf{Con}_\rho}{\exists} \big(U \succ_\rho V \wedge f(V) \succ_\sigma b\big)\} \ .$$

For example, consider the *identity neighborhood-mapping* $\mathrm{id} : \mathsf{Con}_\rho \to \mathsf{Con}_\rho$, defined by $\mathrm{id}(U) := U$; then

$$\mathrm{iiid} = \{(U,a) \mid U \succ_\rho a\} \ .$$

Another example is the *constant neighborhood-mapping* $\mathrm{cnst}_{V_0} : Con_\rho \to \mathsf{Con}_\sigma$, defined by $\mathrm{cnst}_{V_0}(U) := V_0$, for a fixed $V_0 \in \mathsf{Con}_\sigma$; then

$$\mathrm{iicnst}_{V_0} = \{(U,b) \mid V_0 \succ_\sigma b\} \ .$$

**Proposition 1.9.** *A neighborhood-mapping $f : \mathsf{Con}_\rho \to \mathsf{Con}_\sigma$ is consistency-preserving if and only if $\mathrm{ii}f \in \mathsf{Ide}_{\rho\to\sigma}$.*

*Proof.* We proceed similarly as in the proof of Proposition 1.7. For the right direction: To show consistency, let $(U_1,b_1),(U_2,b_2) \in \mathrm{ii}f$; we want to show that $(U_1,b_1) \asymp_{\rho\to\sigma} (U_2,b_2)$, so let $U_1 \asymp_\rho U_2$; by the definition of $\mathrm{ii}f$ we get $V_i$'s such that $U_i \succ_\rho V_i \wedge f(V_i) \succ_\sigma b_i$ for $i = 1,2$; by the assumption and Proposition 1.2(3) we have $V_1 \asymp_\rho V_2$; since $f$ preserves consistency, it is $f(V_1) \asymp_\sigma f(V_2)$, so, again by Proposition 1.2(3), it is $b_1 \asymp_\sigma b_2$.

To show closure under entailment, let $(U_1,b_1) \in \mathrm{ii}f$ and $(U_1,b_1) \succ_{\rho\to\sigma} (U_2,b_2)$, or, equivalently, $U_2 \succ_\rho U_1 \wedge b_1 \succ_\sigma b_2$; by the definition of $\mathrm{ii}f$ we have a $V$ with $U_1 \succ_\rho V \wedge f(V) \succ_\sigma b_1$; by transitivity and assumption we have $U_2 \succ_\rho V \wedge f(V) \succ_\sigma b_2$, which is by definition $(U_2,b_2) \in \mathrm{ii}f$.

For the other direction: Let $\mathrm{ii}f \in \mathsf{Ide}_{\rho\to\sigma}$ and $U \asymp_\rho V$. Since, for any $a \in f(U)$ and $b \in f(V)$, it is $(U,a),(V,b) \in \mathrm{ii}f$ and $\mathrm{ii}f$ is an ideal, it follows that $a \asymp_\sigma b$, so $f(U) \asymp_\sigma f(V)$. □

Clearly again, $\mathrm{iiid}$ and $\mathrm{iicnst}_{V_0}$, as defined above, are ideals of $\rho \to \rho$ and $\rho \to \sigma$ respectively.

We now link neighborhood-mappings to token-mappings. Let $f : T_\rho \to T_\sigma$ be a token-mapping. Define a mapping $\mathrm{m}f : \mathsf{Con}_\rho \to \mathscr{P}_f(T_\sigma)$ by

$$\mathrm{m}f(U) := \{f(a) \mid a \in U\} \ .$$

**Proposition 1.10.** *Let $\rho$ and $\sigma$ be acises.*

1. *The mapping $\mathrm{m}f$ is a well-defined neighborhood-mapping from $\rho$ to $\sigma$ when $f$ is consistency-preserving. In this case, $\mathrm{m}f$ is also consistency-preserving.*

2. *The mapping $\mathrm{m}f$ is monotone when $f$ is monotone.*

3. *If $f$ is a consistency-preserving token-mapping then $\mathrm{ii}f = \mathrm{iim}f$.*

*Proof.* For the first statement: Let $U \in \mathsf{Con}_\rho$; the set $\mathrm{m}f(U)$ is finite by definition, since $U$ is finite; furthermore, if $b,b' \in \mathrm{m}f(U)$, then there must exist $a,a' \in U$ for which $f(a) = b$ and $f(a') = b'$; but $U$ is a neighborhood, so $a \asymp_\rho a'$; since $f$ is consistency-preserving we get $b \asymp_\sigma b'$. Now, let $U \asymp_\rho V$ and $b \in \mathrm{m}f(U)$, $b' \in \mathrm{m}f(V)$; then there are $a \in U$, $a' \in V$ with $f(a) = b$, $f(a') = b'$; by the assumption and by the preservation of consistency of $f$, we get $b \asymp_\sigma b'$.

For the second statement: Let $U \succ_\rho V$ and $b' \in \mathbb{im} f(V)$; by definition there exists an $a' \in V$ such that $f(a') = b'$; by the assumption, there must be some $a \in U$ such that $a \succ_\rho a'$; set $b := f(a) \in \mathbb{im} f(U)$; by monotonicity of $f$ we get $b \succ_\sigma b'$.

For the third statement: Let $f : T_\rho \to T_\sigma$ be a consistency-preserving token-mapping; then we have

$$\begin{aligned}
(U,b) \in \mathbb{i}\mathbb{im} f &\overset{\text{def}}{\Leftrightarrow} \underset{V \in \mathsf{Con}_\rho}{\exists} \big(U \succ_\rho V \wedge \mathbb{im} f(V) \succ_\sigma b\big) \\
&\overset{\text{def}}{\Leftrightarrow} \underset{V \in \mathsf{Con}_\rho}{\exists} \left(U \succ_\rho V \wedge \underset{a \in T_\rho}{\exists} (a \in V \wedge f(a) \succ_\sigma b)\right) \\
&\Leftrightarrow \underset{a \in T_\rho}{\exists} \underset{V \in \mathsf{Con}_\rho}{\exists} \big(U \succ_\rho V \wedge a \in V \wedge f(a) \succ_\sigma b\big) \\
&\overset{(\star)}{\Leftrightarrow} \underset{a \in T_\rho}{\exists} \big(U \succ_\rho a \wedge f(a) \succ_\sigma b\big) \\
&\overset{\text{def}}{\Leftrightarrow} (U,b) \in \mathbb{i} f \ ,
\end{aligned}$$

where $(\star)$ holds leftwards for $V := \{a\}$. □

### Closure-mappings

Given the well-foundedness of entailment in the source acis, we can move a step further and consider mappings between *closures* $\overline{U}$ of neighborhoods. The primary reason for this is that we can achieve a decent converse route from ideals to mappings between sets of tokens, which we cannot have in the case of token-mappings. In particular, we will establish a bijective correspondence between closure-homomorphisms from $\rho$ to $\sigma$ and a class of ideals of $\rho \to \sigma$, when $\rho$ has a well-founded entailment relation.

A *closure-mapping* $f$ from $\rho$ to $\sigma$ is a total mapping $f : \overline{\mathsf{Con}_\rho} \to \overline{\mathsf{Con}_\sigma}$. It is *monotone* when

$$\overline{U} \succ_\rho \overline{V} \to f(\overline{U}) \succ_\sigma f(\overline{V}) \ ,$$

or, equivalently by Proposition 1.2(5),

$$\overline{U} \supseteq_\rho \overline{V} \to f(\overline{U}) \supseteq_\sigma f(\overline{V}) \ ,$$

*consistency-preserving* when

$$\overline{U} \asymp_\rho \overline{V} \to f(\overline{U}) \asymp_\sigma f(\overline{V}) \ ,$$

and a *homomorphism* when it is both monotone and consistency-preserving. A homomorphism is furthermore a *monomorphism*, *epimorphism* or *isomorphism* when the closure-mapping is injective, surjective or bijective respectively.

For an arbitrary closure-mapping $f : \overline{\mathsf{Con}_\rho} \to \overline{\mathsf{Con}_\sigma}$ define the *idealization* of $f$ by the class $\mathbb{i} f \subseteq T_{\rho\to\sigma}$ by

$$\mathbb{i} f := \{(U,b) \mid \underset{V \in \mathsf{Con}_\rho}{\exists} \big(U \succ_\rho V \wedge f(\overline{V}) \succ_\sigma b\big)\} \ .$$

For example, consider the *identity closure-mapping* $\mathrm{id} : \overline{\mathsf{Con}_\rho} \to \overline{\mathsf{Con}_\rho}$, defined by $\mathrm{id}(\overline{U}) := \overline{U}$; then

$$\mathbb{i}\mathrm{id} = \{(U,a) \mid U \succ_\rho a\} \ .$$

Another example is the *constant closure-mapping* $\mathrm{cnst}_{V_0} : \overline{\mathsf{Con}_\rho} \to \overline{\mathsf{Con}_\sigma}$, defined by $\mathrm{cnst}_{V_0}(\overline{U}) := \overline{V_0}$, for a fixed $V_0 \in \mathsf{Con}_\sigma$; then

$$\mathbb{i}\mathrm{cnst}_{V_0} = \{(U,b) \mid V_0 \succ_\sigma b\} \ .$$

**Proposition 1.11.** *If* $f : \overline{\mathsf{Con}_\rho} \to \overline{\mathsf{Con}_\sigma}$ *is a consistency-preserving closure-mapping then* $\mathbb{i}f \in \mathsf{Ide}_{\rho\to\sigma}$.

*Proof.* Again, we proceed similarly as in the proof of Proposition 1.7. To show consistency, let $(U_1,b_1),(U_2,b_2) \in \mathbb{i}f$; we want to show that $(U_1,b_1) \asymp_{\rho\to\sigma} (U_2,b_2)$, so let $U_1 \asymp_\rho U_2$; by the definition of $\mathbb{i}f$ we get $V_i$'s with $U_i \succ_\rho V_i \wedge f(\overline{V_i}) \succ_\sigma b_i$ for $i = 1,2$; by the assumption and Proposition 1.2(3) we have $V_1 \asymp_\rho V_2$; since $f$ preserves consistency, it is $f(\overline{V_1}) \asymp_\sigma f(\overline{V_2})$, so, again by Proposition 1.2(3), we have $b_1 \asymp_\sigma b_2$.

To show closure under entailment, let $(U_1,b_1) \in \mathbb{i}f$ and $(U_1,b_1) \succ_{\rho\to\sigma} (U_2,b_2)$, or, equivalently, $U_2 \succ_\rho U_1 \wedge b_1 \succ_\sigma b_2$; by the definition of $\mathbb{i}f$ we have a $V$ with $U_1 \succ_\rho V \wedge f(\overline{V}) \succ_\sigma b_1$; by transitivity and assumption we have $U_2 \succ_\rho V \wedge f(\overline{V}) \succ_\sigma b_2$, which is by definition $(U_2,b_2) \in \mathbb{i}f$. □

Clearly again, $\mathbb{i}\mathrm{id}$ and $\mathbb{i}\mathrm{cnst}_{V_0}$, as defined above, are ideals of $\rho \to \rho$ and $\rho \to \sigma$ respectively.

Now let $u \in \mathsf{Ide}_{\rho\to\sigma}$. Call $u$ a *finitely valued ideal*, if for all $U \in \mathsf{Con}_\rho$ the set

$$\mathrm{mxl}\,u(U) := \mathrm{mxl}\{b \in T_\sigma \mid (U,b) \in u\}$$

is finite. Denote the class of all finitely valued ideals of $\rho \to \sigma$ by $\mathsf{FVIde}_{\rho\to\sigma}$. In general

$$\mathsf{FVIde}_{\rho\to\sigma} \subseteq \mathsf{Ide}_{\rho\to\sigma}\ .$$

It is easy to see that $\mathbb{i}\mathrm{id} \in \mathsf{FVIde}_{\rho\to\rho}$ and $\mathbb{i}\mathrm{cnst}_{V_0} \in \mathsf{FVIde}_{\rho\to\sigma}$.

*Remark.* For the sake of a counterexample, let us anticipate the arithmetical acis $\mathbb{N} \to \mathbb{N}$ (see page 30); it is easy to see that $\{0, S^n * \mid n = 0,1,\ldots\} \in \mathsf{Ide}_{\mathbb{N}\to\mathbb{N}} \smallsetminus \mathsf{FVIde}_{\mathbb{N}\to\mathbb{N}}$. □

For a finitely valued ideal $u \in \mathsf{Ide}_{\rho\to\sigma}$, define a mapping $\mathbb{h}u : \overline{\mathsf{Con}_\rho} \to \overline{\mathsf{Con}_\sigma}$ by

$$\mathbb{h}u(\overline{U}) := \overline{\mathrm{mxl}\,u(U)}\ .$$

**Proposition 1.12.** *If* $u \in \mathsf{FVIde}_{\rho\to\sigma}$ *then the mapping* $\mathbb{h}u$ *is a well-defined closure-homomorphism from* $\rho$ *to* $\sigma$.

*Proof.* For the well-definedness: It is easy to see that $\mathbb{h}u$ is indeed single-valued. Furthermore, the class $\mathrm{mxl}\,u(U)$ is finite, since $u$ is finitely valued. Now let $U \in \mathsf{Con}_\rho$ and $b,b' \in \mathrm{mxl}\,u(U)$; we have $(U,b),(U,b') \in u$; since $u$ is an ideal, $(U,b) \asymp_{\rho\to\sigma} (U,b')$, and since $U \asymp_\rho U$, it is $b \asymp_\sigma b'$, so $\mathrm{mxl}\,u(U) \in \mathsf{Con}_\sigma$, and then $\overline{\mathrm{mxl}\,u(U)} \in \overline{\mathsf{Con}_\sigma}$.

For the preservation of consistency: Let $U,V \in \mathsf{Con}_\rho$ with $U \asymp_\rho V$ and arbitrary $b \in \mathbb{h}u(\overline{U})$ and $c \in \mathbb{h}u(\overline{V})$; by the definition of $\mathbb{h}u$ we have $(U,b),(V,c) \in u$; by the definition of an ideal, $(U,b) \asymp_{\rho\to\sigma} (V,c)$; by the consistency in $\rho \to \sigma$ and by the assumption, we get $b \asymp_\sigma c$, that is, $\mathbb{h}u(\overline{U}) \asymp_\sigma \mathbb{h}u(\overline{V})$.

For the monotonicity: Let $U,V \in \mathsf{Con}_\rho$ with $U \succ_\rho V$ and let $c \in \mathbb{h}u(\overline{V})$; by the definition of $\mathbb{h}u$ we have $(V,c) \in u$; by the assumption we get $(V,c) \succ_{\rho\to\sigma} (U,c)$ and since $u$ is an ideal, $(U,c) \in u$, that is, $c \in \mathbb{h}u(\overline{U})$. □

**Proposition 1.13.** *If* $\succ_\rho$ *is well-founded and* $f : \overline{\mathsf{Con}_\rho} \to \overline{\mathsf{Con}_\sigma}$ *is a consistency-preserving closure-mapping then* $\mathbb{i}f \in \mathsf{FVIde}_{\rho\to\sigma}$.

*Proof.* We have proved that $\mathbb{i}f$ is indeed an ideal in Proposition 1.9. It remains to prove that it is moreover finitely valued. So let $U \in \mathsf{Con}_\rho$ and consider the set $M_U \coloneqq \operatorname{mxl} \mathbb{i}f(U)$; by definition it is

$$M_U = \operatorname{mxl}\{b \in T_\sigma \mid (U,b) \in \mathbb{i}f\};$$

by the definition of $\mathbb{i}f$, it is

$$M_U = \operatorname{mxl}\{b \in T_\sigma \mid \underset{V \in \mathsf{Con}_\rho}{\exists} \big(U \succ_\rho V \wedge f(\overline{V}) \succ_\sigma b\big)\} ,$$

or, since $f(\overline{V}) \in \overline{\mathsf{Con}_\sigma}$, by the definition of deductive closure we have

$$M_U = \operatorname{mxl}\{b \in T_\sigma \mid \underset{V \in \mathsf{Con}_\rho}{\exists} \big(U \succ_\rho V \wedge b \in f(\overline{V})\big)\} ;$$

in particular, since $f(\overline{V})$ is the deductive closure of a neighborhood, namely, there is a $W_V \in \mathsf{Con}_\sigma$ such that $f(\overline{V}) = \overline{W_V}$, we can write

$$M_U = \operatorname{mxl}\{b \in T_\sigma \mid \underset{V \in \mathsf{Con}_\rho}{\exists} \big(U \succ_\rho V \wedge b \in \overline{W_V}\big)\} ;$$

by the definition of deductive closure again, it is

$$M_U = \operatorname{mxl}\{b \in T_\sigma \mid \underset{V \in \mathsf{Con}_\rho}{\exists} \big(U \succ_\rho V \wedge W_V \succ_\sigma b\big)\} .$$

Now, since $\succ_\rho$ is well-founded, the index set $I_U \coloneqq \{V \in \mathsf{Con}_\rho \mid U \succ_\rho V\}$ is finite, hence

$$M_U = \bigcup_{V \in I_U} \operatorname{mxl}\{b \in T_\sigma \mid W_V \succ_\sigma b\} = \bigcup_{V \in I_U} \operatorname{mxl} W_V$$

is also finite, because every $W_V$ is. □

**Theorem 1.14** (Finitely valued ideals)**.** *Let $\rho$, $\sigma$ be acises with $\succ_\rho$ being well-founded. The closure-homomorphisms $f : \overline{\mathsf{Con}_\rho} \to \overline{\mathsf{Con}_\sigma}$ and the finitely valued ideals $u \in \mathsf{FVIde}_{\rho\to\sigma}$ are in a bijective correspondence, that is,* $\mathsf{Hom}(\overline{\mathsf{Con}_\rho}, \overline{\mathsf{Con}_\sigma}) \cong \mathsf{FVIde}_{\rho\to\sigma}$.

*Proof.* We have to show that $\mathbb{h}$ and $\mathbb{i}$ are mutually inverse, that is, that $\mathbb{h}\mathbb{i}f = f$ as well as $\mathbb{i}\mathbb{h}u = u$. For the first one we have

$$\begin{aligned}
b \in \mathbb{h}\mathbb{i}f(\overline{U}) &\overset{\text{def}}{\Leftrightarrow} b \in \overline{\operatorname{mxl} \mathbb{i}f(U)} \\
&\overset{\text{def}}{\Leftrightarrow} b \in \overline{\operatorname{mxl}\{b' \in T_\sigma \mid (U,b') \in \mathbb{i}f\}} \\
&\overset{\text{def}}{\Leftrightarrow} \operatorname{mxl}\{b' \in T_\sigma \mid (U,b') \in \mathbb{i}f\} \succ_\sigma b \\
&\Leftrightarrow \{b' \in T_\sigma \mid (U,b') \in \mathbb{i}f\} \succ_\sigma b \\
&\overset{\text{def}}{\Leftrightarrow} \underset{b' \in T_\sigma}{\exists} \big((U,b') \in \mathbb{i}f \wedge b' \succ_\sigma b\big) \\
&\Leftrightarrow (U,b) \in \mathbb{i}f \\
&\overset{\text{def}}{\Leftrightarrow} \underset{V \in \mathsf{Con}_\rho}{\exists} \big(U \succ_\rho V \wedge f(\overline{V}) \succ_\sigma b\big) \\
&\overset{\text{mon}}{\Leftrightarrow} \underset{V \in \mathsf{Con}_\rho}{\exists} \big(f(\overline{U}) \succ_\sigma f(\overline{V}) \wedge f(\overline{V}) \succ_\sigma b\big) \\
&\Leftrightarrow f(\overline{U}) \succ_\sigma b \\
&\Leftrightarrow b \in f(\overline{U}) ,
\end{aligned}$$

and for the second one

$$
\begin{aligned}
(U,b) \in \mathrm{ii}\,\mathrm{Ih}u &\overset{\text{def}}{\Leftrightarrow} \underset{V \in \mathsf{Con}_\rho}{\exists} \big(U \succ_\rho V \wedge \mathrm{Ih}u(\overline{V}) \succ_\sigma b\big) \\
&\overset{\text{def}}{\Leftrightarrow} \underset{V \in \mathsf{Con}_\rho}{\exists} \Big(U \succ_\rho V \wedge \overline{\mathrm{mxl}\, u(V)} \succ_\sigma b\Big) \\
&\overset{\text{def}}{\Leftrightarrow} \underset{V \in \mathsf{Con}_\rho}{\exists} \Big(U \succ_\rho V \wedge \overline{\mathrm{mxl}\{b' \in T_\sigma \mid (V,b') \in u\}} \succ_\sigma b\Big) \\
&\Leftrightarrow \underset{V \in \mathsf{Con}_\rho}{\exists} \big(U \succ_\rho V \wedge \mathrm{mxl}\{b' \in T_\sigma \mid (V,b') \in u\} \succ_\sigma b\big) \\
&\Leftrightarrow \underset{V \in \mathsf{Con}_\rho}{\exists} \big(U \succ_\rho V \wedge \{b' \in T_\sigma \mid (V,b') \in u\} \succ_\sigma b\big) \\
&\overset{\text{def}}{\Leftrightarrow} \underset{V \in \mathsf{Con}_\rho}{\exists} \Big(U \succ_\rho V \wedge \underset{b' \in T_\sigma}{\exists} \big((V,b') \in u \wedge b' \succ_\sigma b\big)\Big) \\
&\Leftrightarrow \underset{V \in \mathsf{Con}_\rho}{\exists} \big(U \succ_\rho V \wedge (V,b) \in u\big) \\
&\Leftrightarrow (U,b) \in u \ ,
\end{aligned}
$$

as we wanted. □

Finally, we link closure-mappings to token-mappings. Let $f : T_\rho \to T_\sigma$ be a token-mapping. Define a mapping $\overline{f} : \overline{\mathsf{Con}}_\rho \to \mathscr{P}(T_\sigma)$ by

$$\overline{f}(\overline{U}) \coloneqq \{f(a) \mid U \succ_\rho a\} \ .$$

**Proposition 1.15.** *Let $\rho$ and $\sigma$ be acises.*

1. *The mapping $\overline{f}$ is a well-defined closure-mapping from $\rho$ to $\sigma$ when $f$ is consistency-preserving. In this case, $\overline{f}$ is also consistency-preserving.*
2. *The mapping $\overline{f}$ is monotone when $f$ is monotone.*
3. *If $f$ is a consistency-preserving token-mapping then $\mathrm{ii} f = \mathrm{ii}\overline{f}$.*

*Proof.* We prove the third statement, merely using the definitions. Let $f : T_\rho \to T_\sigma$ be a consistency-preserving token-mapping; then its closure is well-defined, and we have:

$$
\begin{aligned}
(U,b) \in \mathrm{ii}\overline{f} &\overset{\text{def}}{\Leftrightarrow} \underset{V \in \mathsf{Con}_\rho}{\exists} \big(U \succ_\rho V \wedge \overline{f}(\overline{V}) \succ_\sigma b\big) \\
&\overset{\text{def}}{\Leftrightarrow} \underset{V \in \mathsf{Con}_\rho}{\exists} \Big(U \succ_\rho V \wedge \underset{b' \in T_\sigma}{\exists} \big(b' \in \overline{f}(\overline{V}) \wedge b' \succ_\sigma b\big)\Big) \\
&\overset{\text{def}}{\Leftrightarrow} \underset{V \in \mathsf{Con}_\rho}{\exists} \Big(U \succ_\rho V \wedge \underset{a \in T_\rho}{\exists} \big(f(a) \in \overline{f}(\overline{V}) \wedge f(a) \succ_\sigma b\big)\Big) \\
&\overset{\text{def}}{\Leftrightarrow} \underset{V \in \mathsf{Con}_\rho}{\exists} \Big(U \succ_\rho V \wedge \underset{a \in T_\rho}{\exists} \big(V \succ_\rho a \wedge f(a) \succ_\sigma b\big)\Big) \\
&\Leftrightarrow \underset{a \in T_\rho}{\exists} \underset{V \in \mathsf{Con}_\rho}{\exists} \big(U \succ_\rho V \wedge V \succ_\rho a \wedge f(a) \succ_\sigma b\big) \\
&\Leftrightarrow \underset{a \in T_\rho}{\exists} \big(U \succ_\rho a \wedge f(a) \succ_\sigma b\big) \\
&\overset{\text{def}}{\Leftrightarrow} (U,b) \in \mathrm{ii} f \ ,
\end{aligned}
$$

as we wanted. □

## Approximable maps

We now turn to a more traditional path. A relation $r \subseteq \mathsf{Con}_\rho \times T_\sigma$ between two acises $\rho$ and $\sigma$ is called a *(unary) approximable map* from $\rho$ to $\sigma$, and we write $r \in \mathsf{Apx}_{\rho\to\sigma}$, if it is *consistently defined*, that is,

$$r(U,a) \wedge r(U,b) \to a \asymp_\sigma b \ ,$$

and furthermore,

$$V \succ_\rho U \wedge r(U,a) \wedge a \succ_\sigma b \to r(V,b) \ ,$$

which expresses that $r$ is closed under entailment (that is, deductively closed). Write $r(U) := \{b \in T_\sigma \mid r(U,b)\}$. The intuition is that $r(U,a)$, where $r$ behaves like a black-box, means "input $U$ suffices for the output $a$".

**Proposition 1.16.** *The ideals of $\rho \to \sigma$ are exactly the approximable maps from $\rho$ to $\sigma$, that is, $\mathsf{Ide}_{\rho\to\sigma} = \mathsf{Apx}_{\rho\to\sigma}$.*

*Proof.* For the right direction: Let $u \in \mathsf{Ide}_{\rho\to\sigma}$. Suppose that $(U,a) \in u \wedge (U,b) \in u$; since ideals are consistent, we have $(U,a) \asymp (U,b)$; by definition this is $U \asymp_\rho U \to a \asymp_\sigma b$, that is, $a \asymp_\sigma b$. Suppose furthermore that $V \succ_\rho U \wedge (U,a) \in u \wedge a \succ_\sigma b$; by the definition of entailment in a function space we get $(U,a) \in u \wedge (U,a) \succ (V,b)$, which by closure under propagation yields $(V,b) \in u$; so $u \in \mathsf{Apx}_{\rho\to\sigma}$.

For the other direction: Let $f \in \mathsf{Apx}_{\rho\to\sigma}$. Suppose that $f(U,a) \wedge f(V,b)$. We want to show that $(U,a) \asymp (V,b)$; suppose that $U \asymp_\rho V$; we can then write $U \cup V \succ_\rho U \wedge U \cup V \succ_\rho V$; by the second property of approximable maps we get $f(U\cup V,a) \wedge f(U\cup V,b)$; since the first property of approximable maps yields $a \asymp_\sigma b$, we have proved that $U \asymp_\rho V \to a \asymp_\sigma b$, that is $(U,a) \asymp (V,b)$. Suppose furthermore that $f(U,a) \wedge (U,a) \succ (V,b)$; by the definition of entailment in function spaces we have $f(U,a) \wedge V \succ_\rho U \wedge a \succ_\sigma b$, which by the second property of approximable maps gives $f(V,b)$; so $f \in \mathsf{Ide}_{\rho\to\sigma}$. □

## Application

In the following we will be largely concerned with the "application of ideals". In general, define *(set) application* $\cdot : \mathscr{P}(T_{\rho\to\sigma}) \times \mathscr{P}(T_\rho) \to \mathscr{P}(T_\sigma)$, by

$$\{(X_i,a_i)\}_{i\in I} \cdot Y :=_\sigma \{a_i \mid Y \succ_\rho X_i\} \ .$$

**Proposition 1.17.** *For the application operation the following hold.*

1. *It is consistency-preserving, that is, if $\{(U_i,a_i)\}_{i\in I} \in \mathsf{Con}_{\rho\to\sigma}$ and $U \in \mathsf{Con}_\rho$, then $\{(U_i,a_i)\}_i U \in \mathsf{Con}_\sigma$, and so it is a well-defined operation on $\mathsf{Con}_{\rho\to\sigma} \times \mathsf{Con}_\rho \to \mathsf{Con}_\sigma$. In particular, it is consistency-preserving as a neighborhood mapping, that is, if $\{(U_i,b_i)\}_i \asymp_{\rho\to\sigma} \{(V_j,c_j)\}_j$ and $U \asymp_\rho V$ then $\{(U_i,b_i)\}_i \cdot U \asymp_\sigma \{(V_i,c_i)\}_i \cdot V$. Consequently, the idealization of application is an ideal, that is, $\ddot{\imath}\cdot \in \mathsf{Ide}_{(\rho\to\sigma)\times\rho\to\sigma}$.*

2. *It is $\{(U_i,a_i)\}_i \succ_{\rho\to\sigma} \{(V_j,b_j)\}_j$ if and only if, for all $U \in \mathsf{Con}_\rho$, $\{(U_i,a_i)\}_i \cdot U \succ_\sigma \{(V_j,b_j)\}_j \cdot U$.*

3. *For all $\{(U_i,a_i)\}_i \in \mathsf{Con}_{\rho\to\sigma}$, if $U \succ_\rho V$ then $\{(U_i,a_i)\}_i \cdot U \succ_\sigma \{(U_i,a_i)\}_i \cdot V$.*

4. *It commutes with deductive closure, that is, $\overline{\{(U_i,a_i)\}_{i\in I}} \cdot \overline{U} = \overline{\{(U_i,a_i)\}_{i\in I} \cdot U}$.*

5. *Fix $\rho$ and $\sigma$. For $X \subseteq^f T_{\rho\to\sigma}$, $Y \subseteq^f T_\rho$ and $Z \subseteq^f T_\sigma$, the relation $Z = X \cdot Y$ is $\Sigma_1^0$-definable.*

*Proof.* For the first statement: It is easy to see that set application is single-valued. Furthermore, let $\{(U_i,a_i)\}_i U = \{a_i \mid U \succ_\rho U_i\}$; we want to show that for all $i_1,i_2 \in I$ it is $a_{i_1} \asymp_\sigma a_{i_2}$, so let $i_1,i_2 \in I$; since $\{(U_i,a_i)\}_{i\in I} \in \mathsf{Con}_{\rho\to\sigma}$, it is $(U_{i_1},a_{i_1}) \asymp_{\rho\to\sigma} (U_{i_2},a_{i_2})$, or, equivalently, $(U_{i_1} \asymp_\rho U_{i_2} \to a_{i_1} \asymp_\sigma a_{i_2})$; by Proposition 1.2 we have what we wanted.

Furthermore, let $U \succ_\rho U_i$ and $V \succ_\rho V_j$ for some $i$ and $j$; since $U \asymp_\rho V$, Proposition 1.2(3) gives us $U_i \asymp_\rho V_j$; by the definition of consistency in function spaces we get $b_i \asymp_\sigma c_j$. That the idealization of application is an ideal follows from Proposition 1.9.

For the second statement: For the right direction, let $\{(U_i,a_i)\}_{i\in I} \succ_{\rho\to\sigma} \{(V_j,b_j)\}_{j\in J}$, which by definition is $\forall_{j\in J}\exists_{i\in I}(V_j \succ_\rho U_i \wedge a_i \succ_\sigma b_j)$; we want to show that $\{(U_i,a_i)\}_i \cdot U \succ_\sigma \{(V_j,b_j)\}_j \cdot U$, which by definition is $\{a_i \mid U \succ_\rho U_i\} \succ_\sigma \{b_j \mid U \succ_\rho V_j\}$, which is provided by the assumption. For the other way around, let $\{(U_i,a_i)\}_i \cdot U \succ_\sigma \{(V_j,b_j)\}_j \cdot U$, or $\{a_i \mid U \succ_\rho U_i\} \succ_\sigma \{b_j \mid U \succ_\rho V_j\}$; we have to show that $\{(U_i,a_i)\}_{i\in I} \succ_{\rho\to\sigma} \{(V_j,b_j)\}_{j\in J}$, which by definition is $\forall_{j\in J}\exists_{i\in I}(V_j \succ_\rho U_i \wedge a_i \succ_\sigma b_j)$; for every $l \in J$ we may put $U := V_l$ and the assumption then yields $\{a_i \mid V_l \succ_\rho U_i\} \succ_\sigma \{b_j \mid V_l \succ_\rho V_j\}$; since $V_l \succ_\rho V_i$, there is a $k \in I$ such that $V_l \succ_\rho U_k$ and $a_k \succ_\sigma b_l$.

For the third statement: Let $U \succ_\rho V$; due to transitivity of entailment we have $\forall_i\big(V \succ_\rho U_i \to U \succ_\rho U_i\big)$, which proves what we need.

For the fourth statement, we have

$$\begin{aligned}\overline{\{(U_i,a_i)\}_{i\in I}}\cdot\overline{U} &\stackrel{\text{def}}{=} \{a \mid \underset{V\in\mathsf{Con}_\rho}{\exists}\ \underset{i\in I}{\exists}\big(U \succ_\rho V \wedge V \succ_\rho U_i \wedge a_i \succ_\sigma a\big)\}\\ &= \{a \mid \underset{i\in I}{\exists}\big(U \succ_\rho U_i \wedge a_i \succ_\sigma a\big)\}\\ &\stackrel{\text{def}}{=} \overline{\{a_i \mid U \succ_\rho U_i\}}\\ &\stackrel{\text{def}}{=} \overline{\{(U_i,a_i)\}_{i\in I}\cdot U}\ .\end{aligned}$$

For the last statement, we write

$$\begin{aligned}Z =_\sigma \{(X_i,a_i)\}_{i\in I}\cdot Y &\Leftrightarrow Z =_\sigma \{a_i \mid Y \succ_\rho X_i\}\\ &\Leftrightarrow a \in Z \leftrightarrow \underset{i\in I}{\exists}\left(a = a_i \wedge \underset{b\in X_i}{\forall}\ \underset{c\in Y}{\exists}\ c \succ_\rho b\right)\ .\end{aligned}$$

Since $X$, $Y$ and $Z$ are finite, this is a $\Sigma_1^0$-expression. □

## 1.2 Ideals

In this section we make a minimal exposition of topological as well as category-theoretic aspects of the collection of ideals of a given acis.

### Topological spaces

We recall basic notions and facts that we will use later. Let $P$ be a (nonempty) set of *points* and $\mathscr{T}$ a collection of subsets of $P$. The couple $(P,\mathscr{T})$ is a *topological space* with *open sets* the elements of $\mathscr{T}$, if the following are fulfilled:

- the empty subset as well as the universal set is in $\mathscr{T}$, that is, $\varnothing, P \in \mathscr{T}$.
- the collection $\mathscr{T}$ is closed under finite intersection, that is, if $X_1, \ldots, X_n \in \mathscr{T}$ then $\bigcap_{1 \le i \le n} X_i \in \mathscr{T}$.
- the collection $\mathscr{T}$ is closed under arbitrary union, that is, if $X_1, \ldots, X_n, \ldots \in \mathscr{T}$ then $\bigcup_i X_i \in \mathscr{T}$.

An open set $X$ is a *neighborhood* of a point $p$ if $p \in X$.

A topological space $(P, \mathscr{T})$ is a *Kolmogorov space* if it satisfies the $T_0$-*separation axiom*:

$$p \not\equiv q \to \underset{X \in \mathscr{T}}{\exists} (p \in X \wedge q \notin X) \vee (q \in X \wedge p \notin X)) \ ,$$

and a *Hausdorff space* if it satisfies the $T_2$-*separation axiom*:

$$p \not\equiv q \to \underset{X,Y \in \mathscr{T}}{\exists} p \in X \wedge q \in Y \wedge X \cap Y = \varnothing \ .$$

A *basis* for $\mathscr{T}$ is a family $\{U_i\}_{i \in I} \subseteq \mathscr{T}$ of *basic* open sets that can provide a union decomposition for every nonempty open set:

$$\underset{X \in \mathscr{T}}{\forall} X = \bigcup \{U \mid U \in \{U_i\} \wedge U \subseteq X\} \ .$$

**Fact 1.18.** *Let $(P, \mathscr{T})$ be a topological space and $\mathscr{B} \subseteq \mathscr{T}$. The following are equivalent:*

1. *The family $\mathscr{U}$ is a topological basis.*
2. *For every point of the space and every neighborhood of the point there is a set in $\mathscr{U}$ which contains the point and is a subset of its neighborhood:*

$$\underset{p \in P, X \in \mathscr{T}}{\forall} (p \in X \to \underset{U \in \mathscr{U}}{\exists} (p \in U \wedge U \subseteq X)) \ .$$

3. *Every point of the space belongs to some set in $\mathscr{U}$:*

$$\underset{p \in P}{\forall} \underset{U \in \mathscr{U}}{\exists} p \in U \ ,$$

*and whenever a point belongs to two sets in $\mathscr{U}$, there is a third set in $\mathscr{U}$ which contains it, which is a subset of the other two:*

$$\underset{U,V \in \mathscr{U}}{\exists} p \in U \cap V \to \underset{W \in \mathscr{U}}{\exists} (p \in W \wedge W \subseteq U \cap V) \ .$$

Let $(T, \geq)$ be an ordered set. The *Alexandrov topology* $(T, \mathscr{T}_{\geq})$ on $(T, \geq)$ is defined by the *upward closed* subsets of $T$:

$$X \in \mathscr{T}_{\geq} := \underset{a,b \in T}{\forall} (a \in X \wedge b \geq a \to b \in X) \ .$$

Let $(P, \mathscr{T})$ and $(P', \mathscr{T}')$ be two topological spaces. A *continuous mapping* $f$ from $(P, \mathscr{T})$ to $(P', \mathscr{T}')$ is a mapping $f : P \to P'$ whose inverse preserves openness, that is, such that if $X \in \mathscr{T}'$ then $f^{-1}(X) \in \mathscr{T}$.

**Fact 1.19.** *A mapping $f : (P, \mathscr{T}) \to (P', \mathscr{T}')$ is continuous if and only if its inverse preserves openness on basic sets, that is, if and only if, for a base $\mathscr{B}'$ in $P'$, $f^{-1}(U) \in \mathscr{T}$ for all $U \in \mathscr{B}'$.*

## The Scott topology

The most commonly used, natural topology over ideals of information systems, the so called *Scott topology*, turns out to form a Kolmogorov space.[5] Call a collection of ideals $\mathscr{U} \subseteq \mathsf{Ide}$ a *Scott-open set* if it is closed under supersets (*Alexandrov condition*),

$$\forall_{u \in \mathscr{U}} (u \subseteq v \to v \in \mathscr{U}) ,$$

and is "finitely representable" (*Scott condition*), in the sense that

$$\forall_{u \in \mathscr{U}} \exists_{U \subseteq^f u} \overline{U} \in \mathscr{U} .$$

Denote the set of Scott-open sets by $\mathscr{S}$. Recall (page 7) that the cone of ideals over $U \in \mathsf{Con}$ is given by $\nabla X = \{u \in \mathsf{Ide} \mid X \subseteq u\}$ and that $\mathsf{Kgl}_\rho$ denotes the collection of cones of $\rho$.

**Proposition 1.20.** *For every acis $\rho$, the collection of its cones $\mathsf{Kgl}_\rho$ provides a base for its Scott topology. Furthermore, $(\mathsf{Ide}_\rho, \mathscr{S}_\rho)$ constitutes a Kolmogorov space.*

*Proof.* For the base: Every $u \in \mathsf{Ide}_\rho$ satisfies $u \in \nabla\bot$. Furthermore, let $u \in \nabla U \cap \nabla V$; it is $U \asymp_\rho V$, since otherwise the intersection would be empty; by Proposition 1.3, we have $u \in \nabla(U \cup V)$.

For the Kolmogorov separation: Let $u \neq v$; then, by choice, there is a token $a \in T_\rho$ such that either $a \in u \wedge a \notin v$ or $a \in v \wedge a \notin u$, which yields either $u \in \nabla a \wedge v \notin \nabla a$ or $v \in \nabla a \wedge u \notin \nabla a$. □

**Proposition 1.21.** *The following hold.*

1. *A collection of ideals $\mathscr{U} \subseteq \mathsf{Ide}$ is a Scott-open set if and only if $\mathscr{U} = \bigcup_{\overline{U} \in \mathscr{U}} \nabla U$.*
2. *Let $\mathscr{U} \subseteq \mathsf{Ide}$ satisfy the* strong Scott condition

$$\forall_{u \in \mathscr{U}} \exists_{a \in u} \overline{a} \in \mathscr{U} .$$

*Then $\mathscr{U}$ is a Scott-open set if and only if $\mathscr{U} = \bigcup_{\overline{a} \in \mathscr{U}} \nabla a$.*

*Proof.* For the first statement, let $\mathscr{U}$ be a Scott-open set and let $u \in \mathscr{U}$; by the Scott condition, there exists a $U \subseteq^f u$ such that $\overline{U} \in \mathscr{U}$, that is, $u \in \nabla U$; conversely, let $u \in \nabla U$ for some $U$ with $\overline{U} \in \mathscr{U}$; then $\overline{U} \subseteq u$; since $\overline{U} \in \mathscr{U}$, the Alexandrov condition gives $u \in \mathscr{U}$. For the other direction, let $\mathscr{U} = \bigcup_{\overline{U} \in \mathscr{U}} \nabla U$; then it is a Scott-open set because the cones make up a topological basis.

For the second statement proceed similarly. □

## Continuous mappings

Traditionally, an ideal-mapping (or just *mapping*) $f : \mathsf{Ide}_\rho \to \mathsf{Ide}_\sigma$ will be called *Scott-continuous* if it preserves Scott-openness on basic sets, that is, on cones:

$$\nabla V \in \mathsf{Kgl}_\sigma \to f^{-1}[\nabla V] \in \mathscr{S}_\rho ,$$

[5] For recent thoughts on using a more manageable Hausdorff topology, the so called *liminf topology* in [13, p. 232], see [34].

where $f^{-1}[\nabla V] := \{u \mid V \subseteq^f f(u)\}$. Call an ideal-mapping $f : \mathsf{Ide}_\rho \to \mathsf{Ide}_\sigma$ *monotone*, if it preserves inclusion, that is,

$$u \subseteq v \to f(u) \subseteq f(v) \ .$$

Furthermore, say that it satisfies the *principle of finite support* if

$$b \in f(u) \to \underset{U \subseteq^f u}{\exists} b \in f(\overline{U}) \ .$$

Finally, say that it *commutes with directed unions* if

$$f(\bigcup_{u \in \mathscr{D}} u) = \bigcup_{u \in \mathscr{D}} f(u) \ ,$$

where $\mathscr{D}$ is a directed set of ideals in $\rho$.

**Proposition 1.22.** *Let $\rho$ and $\sigma$ be two acises and $f : \mathsf{Ide}_\rho \to \mathsf{Ide}_\sigma$ an ideal-mapping. The following are equivalent.*

1. *The mapping $f$ is Scott-continuous.*
2. *The mapping is monotone and satisfies the principle of finite support.*
3. *The mapping is monotone and commutes with directed unions.*

*Proof.* For the equivalence of (1) and (2): Let $f$ be a Scott-continuous mapping; for monotonicity, let $u \subseteq v$ and let $b \in f(u)$, that is, $\{b\} \subseteq^f f(u)$; the Scott-open set $f^{-1}[\nabla b] = \{w \mid \{b\} \subseteq^f f(w)\}$ satisfies the Alexandrov condition, so, since $u \subseteq v$, we have $\{b\} \subseteq^f f(v)$, that is, $b \in f(v)$; for the principle of finite support, let $b \in f(u)$; the Scott-open set $f^{-1}[\nabla b]$ satisfies the Scott condition, so for $U \subseteq^f u$ we have $\{b\} \subseteq^f f(\overline{U})$.

Conversely, let $f$ be monotone and satisfy the principle of finite support and let $V \in \mathsf{Con}_\sigma$; we have to show that the set $f^{-1}[\nabla V] = \{u \mid V \subseteq^f f(u)\}$ is Scott-open; we show that

$$\{u \mid V \subseteq^f f(u)\} = \cup\{\nabla U \mid U \in \mathsf{Con}_\rho \wedge V \subseteq^f f(\overline{U})\};$$

for the right direction, let $V \subseteq^f f(u)$; by finite support there exists a $U \in \mathsf{Con}_\rho$ for which $U \subseteq^f u$ and $V \subseteq^f f(\overline{U})$, that is, $u \in \nabla U$; for the left direction, let $u \in \nabla U$ for some $U \in \mathsf{Con}_\rho$ for which $V \subseteq^f f(\overline{U})$; then $\overline{U} \subseteq u$, and monotonicity gives $V \subseteq^f f(u)$.

For the equivalence of (2) and (3): Let $f$ be monotone and satisfy the principle of finite support and let $\mathscr{D} \subseteq \mathsf{Ide}_\rho$ be a directed set of ideals; by monotonicity we immediately get $f(\bigcup_{u \in \mathscr{U}} u) \supseteq \bigcup_{u \in \mathscr{D}} f(u)$; for the converse inclusion, let $b \in f(\bigcup_{u \in \mathscr{D}} u)$; finite support gives a $U \subseteq^f \bigcup_{u \in \mathscr{D}} u$; directedness and finiteness of $U$ gives a $w$ for which $U \subseteq^f w$; since $b \in f(\overline{U})$ and $f$ is monotone, we have $b \in f(w)$.

Conversely, let $f$ commute with directed unions and let $b \in f(u)$; then

$$f(u) = f(\bigcup_{U \subseteq^f u} \overline{U}) = \bigcup_{U \subseteq^f u} f(\overline{U}) \ ,$$

and $b \in f(\overline{U})$, for some $U \subseteq^f u$. □

A direct consequence of Propositions 1.7 and 1.22 is that consistency-preserving token-mappings induce monotone ideal-mappings. Moreover, we have the following.

**Proposition 1.23.** *Let $f : \mathsf{Ide}_\rho \to \mathsf{Ide}_\sigma$ be monotone and $U_1, U_2 \in \mathsf{Con}_\rho$. Then $U_1 \asymp_\rho U_2$ implies $f(\overline{U_1}) \asymp_\sigma f(\overline{U_2})$, and $U_1 \succ_\rho U_2$ implies $f(\overline{U_2}) \subseteq f(\overline{U_1})$.*

*Proof.* For the preservation of consistency, let $b_i \in f(\overline{U_i})$, $i = 1,2$. It is $U_i \subseteq U_1 \cup U_2$ for both $i = 1,2$, and monotonicity of $f$ yields $b_i \in f(\overline{U_i}) \subseteq f(\overline{U_1 \cup U_2})$, so $b_1 \asymp_\sigma b_2$.

For the preservation of entailment, we have

$$U_1 \succ_\rho U_2 \Rightarrow \overline{U_2} \subseteq \overline{U_1} \Rightarrow f(\overline{U_2}) \subseteq f(\overline{U_1}) \ . \qquad \square$$

Say that an ideal-mapping $f : \mathsf{Ide}_\rho \to \mathsf{Ide}_\sigma$ satisfies the *principle of atomic support* if

$$b \in f(u) \to \underset{a \in u}{\exists}\, b \in f(\overline{a}) \ .$$

**Proposition 1.24.** *Let $\rho$, $\sigma$ be acises where for every $U \in \mathsf{Con}_\rho$, $\{\overline{a} \mid a \in U\}$ is a directed set. An ideal-mapping $f : \mathsf{Ide}_\rho \to \mathsf{Ide}_\sigma$ is Scott-continuous if and only if it is monotone and it satisfies the principle of atomic support.*

*Proof.* That atomic support implies finite support is direct. Conversely, let $f$ satisfy the principle of finite support and let $b \in f(u)$ for some $u \in \mathsf{Ide}_\rho$; by finite support we get $U \subseteq^f u$ with $b \in f(\overline{U})$, or, by Proposition 1.3(4), with $b \in f(\bigcup_{a \in U} \overline{a})$; therefore

$$\underset{U \subseteq^f u}{\exists}\, b \in \bigcup_{a \in U} f(\overline{a}) \Rightarrow \underset{U \subseteq^f u}{\exists}\ \underset{a \in U}{\exists}\, b \in f(\overline{a}) \Rightarrow \underset{a \in u}{\exists}\, b \in f(\overline{a}) \ .$$

The commutativity of $f$ with the union of $\{\overline{a} \mid a \in U\}$ follows from the assumption. $\square$

**Proposition 1.25.** *Let $\rho$, $\sigma$ be acises. The continuous ideal-mappings $f : \mathsf{Ide}_\rho \to \mathsf{Ide}_\sigma$ and the ideals $r \in \mathsf{Ide}_{\rho \to \sigma}$ are in a bijective correspondence, that is, $\mathsf{Ide}_\rho \to \mathsf{Ide}_\sigma \cong \mathsf{Ide}_{\rho \to \sigma}$.*

*Proof.* With an ideal $r \in \mathsf{Ide}_{\rho \to \sigma}$, associate a mapping $\mathrm{cm}(r) : \mathsf{Ide}_\rho \to \mathsf{Ide}_\sigma$ by

$$\mathrm{cm}(r)(u) := \{b \in T_\sigma \mid \underset{U \in \mathsf{Con}_\rho}{\exists} (U \subseteq u \wedge (U,b) \in r)\} \ .$$

This is well-defined: Let $b, b' \in \mathrm{cm}(r)(u)$; there are $U, U' \subseteq^f u$ such that $(U,b), (U',b') \in r$; but $r$ is an ideal, so $(U,b) \asymp_{\rho \to \sigma} (U',b')$, that is $U \asymp_\rho U' \to b \asymp_\sigma b'$; since $U, U' \subseteq u$ and $u$ is an ideal, $U \asymp_\rho U'$, so $b \asymp_\sigma b'$ and $\mathrm{cm}(r)(u)$ is consistent. Furthermore, let $b \in \mathrm{cm}(r)(u)$ and $b \succ_\sigma b'$; there is a $U \subseteq^f u$ such that $(U,b) \in r$; but $U \succ_\rho U \wedge b \succ_\sigma b'$, we get $(U,b) \succ_{\rho \to \sigma} (U,b')$ and since $r$ is an ideal, we have $(U,b') \in r$, that is, $b' \in \mathrm{cm}(r)(u)$ and $\mathrm{cm}(r)(u)$ is closed under entailment.

It is also continuous: Let $V \in \mathsf{Con}_\sigma$; we shall prove that $\mathrm{cm}(r)^{-1}(\nabla V)$ is a Scott-open set. For the Alexandrov condition, let $u \in \mathrm{cm}(r)^{-1}(\nabla V)$ and $u \subseteq v$; we have

$$\begin{aligned}
u \in \mathrm{cm}(r)^{-1}(\nabla V) &\Rightarrow \mathrm{cm}(r)(u) \in \nabla V \\
&\Rightarrow V \subseteq^f \mathrm{cm}(r)(u) \\
&\Rightarrow V \subseteq^f \{b \in T_\sigma \mid \underset{U \in \mathsf{Con}_\rho}{\exists} (U \subseteq u \wedge (U,b) \in r)\} \\
&\Rightarrow V \subseteq^f \{b \in T_\sigma \mid \underset{U \in \mathsf{Con}_\rho}{\exists} (U \subseteq u \subseteq v \wedge (U,b) \in r)\} \\
&\Rightarrow V \subseteq^f \mathrm{cm}(r)(v) \\
&\Rightarrow \mathrm{cm}(r)(v) \in \nabla V \\
&\Rightarrow v \in \mathrm{cm}(r)^{-1}(\nabla V) \ ,
\end{aligned}$$

and for the Scott condition, let $u \in \mathrm{cm}(r)^{-1}(\nabla V)$; we have

$$\begin{aligned} u \in \mathrm{cm}(r)^{-1}(\nabla V) &\Rightarrow \mathrm{cm}(r)(u) \in \nabla V \\ &\Rightarrow V \subseteq^f \mathrm{cm}(r)(u) \\ &\Rightarrow V \subseteq^f \{b \in T_\sigma \mid \exists_{U \in \mathsf{Con}_\rho} (U \subseteq u \wedge (U,b) \in r)\} \\ &\Rightarrow V \subseteq^f \{b \in T_\sigma \mid \exists_{U \in \mathsf{Con}_\rho} \big(U \subseteq \overline{U} \subseteq u \wedge (U,b) \in r\big)\} \\ &\Rightarrow V \subseteq^f \mathrm{cm}(r)(\overline{U}) \\ &\Rightarrow \mathrm{cm}(r)(\overline{U}) \in \nabla V \\ &\Rightarrow \overline{U} \in \mathrm{cm}(r)^{-1}(\nabla V) \ . \end{aligned}$$

Conversely, with a continuous ideal-mapping $f : \mathsf{Ide}_\rho \to \mathsf{Ide}_\sigma$, associate a set $\mathrm{is}(f) \in \mathsf{Ide}_{\rho\to\sigma}$ by

$$(U,b) \in \mathrm{is}(f) :\!= b \in f(\overline{U}) \ .$$

It is well-defined: For consistency, let $(U_i,b_i) \in \mathrm{is}(f)$, $i = 1,2$, with $U_1 \asymp_\rho U_1$; by definition, $b_i \in f(\overline{U_i})$ and so, by Proposition 1.23, $b_i \in f(\overline{U_1 \cup U_2})$, which is an ideal, so $b_1 \asymp_\sigma b_2$. For closure under entailment, let $(U,b) \in \mathrm{is}(f)$ and $(U,b) \succ_{\rho\to\sigma} (U',b')$; by the definition of entailment in function spaces, $U' \succ_\rho U \wedge b \succ_\sigma b'$; by definition, $b \in f(\overline{U})$; by Proposition 1.23 again, $f(\overline{U'}) \succ_\sigma f(\overline{U})$; since both of them are ideals, by Proposition 1.2 we have $b' \in f(\overline{U'})$, that is, $(U',b') \in \mathrm{is}(f)$.

Finally, the associations cm and is are inverse to each other, that is,

$$\mathrm{cm}(\mathrm{is}(f)) = f \text{ and } \mathrm{is}(\mathrm{cm}(r)) = r \ .$$

For the left one

$$\begin{aligned} b \in \mathrm{cm}(\mathrm{is}(f))(u) &\overset{\text{def}}{\Leftrightarrow} \exists_{U \in \mathsf{Con}_\rho} \big(U \subseteq^f u \wedge (U,b) \in \mathrm{is}(f)\big) \\ &\overset{\text{def}}{\Leftrightarrow} \exists_{U \in \mathsf{Con}_\rho} \big(U \subseteq^f u \wedge b \in f(\overline{U})\big) \\ &\Leftrightarrow b \in f(u) \ , \end{aligned}$$

and for the right one

$$\begin{aligned} (U,b) \in \mathrm{is}(\mathrm{cm}(r)) &\overset{\text{def}}{\Leftrightarrow} b \in \mathrm{cm}(r)(\overline{U}) \\ &\overset{\text{def}}{\Leftrightarrow} \exists_{U' \in \mathsf{Con}_\rho} \big(U' \subseteq^f \overline{U} \wedge (U',b) \in r\big) \\ &\overset{(\star)}{\Leftrightarrow} (U,b) \in r \ , \end{aligned}$$

where at $(\star)$ we let $U :\!= U'$. □

Let $r \in \mathsf{Ide}_{\rho\to\sigma}$ and $u \in \mathsf{Ide}_\rho$; the ideal $\mathrm{cm}(r)(u)$, written $r(u)$, is called the *application* of $r$ to $u$. By the proposition above, the application of an ideal to an ideal is a continuous operation.

The following are easy observations that we will need in section 1.4.

**Proposition 1.26.** *The following hold for all ideals of proper types:*

1. $\bot(u) = \bot$.
2. $r_1 \cup r_2(u) = r_1(u) \cup r_2(u)$.

## Cartesian products

We turn our attention now to cartesian products. Let $\rho$ and $\sigma$ be two acises with $T_\rho \cap T_\sigma = \varnothing$. Define the *projections* $\pi_\rho : \rho \times \sigma \to \rho$ and $\pi_\sigma : \rho \times \sigma \to \sigma$ by

$$\pi_\rho(u,v) := u \text{ and } \pi_\sigma(u,v) := v \ .$$

**Proposition 1.27.** *The projections from a cartesian product to its components are continuous mappings.*

*Proof.* For monotonicity: Let $(u,v) \subseteq (u',v')$, that is, $u \subseteq u'$ and $v \subseteq v'$; then immediately by definition $\pi(u,v) \subseteq \pi(u',v')$, for both projections.

For the principle of finite support: Without no loss of generality, let $b \in \pi_\rho(u,v)$, that is, $b \in u$; then $b \in \pi_\rho(\overline{b}, \overline{\varnothing})$. ◻

**Proposition 1.28** (Universal property of the cartesian product)**.** *Let $\rho$, $\sigma$ and $\tau$ be acises with $T_\rho \cap T_\sigma = \varnothing$. For every pair $f : \tau \to \rho$, $g : \tau \to \sigma$ of continuous mappings, there exists a unique continuous mapping $h : \tau \to \rho \times \sigma$ such that $f = \pi_\rho \circ h$ and $g = \pi_\sigma \circ h$.*

*Proof.* For all $u \in \mathsf{Ide}_\tau$ let $h(u) := (f(u), g(u))$. Monotonicity of $h$ follows directly from the motonicity of $f$ and $g$. For the principle of finite support, let $b \in h(u)$; since the carriers are disjoint, suppose with no loss of generality that $b \in T_\rho$, so it will be $b \in f(u)$; but $f$ is continuous, so it satisfies the principle of finite support, that is, there is a $U \subseteq^f u$ such that $b \in f(\overline{U})$; hence $b \in h(\overline{U})$. The uniqueness of $h$ follows directly from its definition. ◻

By the previous result we can define the *cartesian product* $f \times g : \rho \times \sigma \to \tau \times \upsilon$ of two continuous mappings $f : \rho \to \tau$ and $g : \sigma \to \upsilon$, where $T_\rho \cap T_\sigma = \varnothing$, by

$$f \times g(u,v) := (f(u), g(v)) \ .$$

Finally, we have the following.

**Proposition 1.29.** *Let $\rho$, $\sigma$ and $\tau$ be acises with $T_\rho \cap T_\sigma = \varnothing$. A mapping $f : \rho \times \sigma \to \tau$ is continuous if and only if it is continuous in each component separately, that is, if and only if all sections $f^v_\rho : \rho \to \tau$, $v$ fixed and all sections $f^u_\sigma : \sigma \to \tau$, $u$ fixed, defined by $f^v_\rho(u) := f(u,v)$ and $f^u_\sigma(v) := f(u,v)$, are continuous.*

*Proof.* The mapping $u \mapsto (u,v)$ for a fixed $v$ is obviously continuous. Since composition preserves continuity, $f^v_\rho$ is also continuous. For $f^u_\sigma$ the argument is similar.

Conversely, let all sections $f^v_\rho$, $f^u_\sigma$ be continuous. For monotonicity: Let $u \subseteq u'$ and $v \subseteq v'$, where $u, u' \in \mathsf{Ide}_\rho$, $v, v' \in \mathsf{Ide}_\sigma$; by monotonicity of the sections we immediately have

$$f(u,v) \subseteq f(u',v) \subseteq f(u',v') \ .$$

For the principle of finite support: Let $b \in f(u,v)$; by the principle of finite support for $f^u_\sigma$ we have

$$b \in f^u_\sigma(\overline{V}) = f(u, \overline{V}) = f^{\overline{V}}_\rho(u) \ ,$$

for some $V \subseteq^f v$; by the principle of finite support for $f^v_\rho$ and Proposition 1.3(2) we have

$$b \in f^{\overline{V}}_\rho(\overline{U}) = f(\overline{U}, \overline{V}) = f(\overline{U \cup V}) \ ,$$

for some $U \subseteq^f u$. ◻

## Evaluation and currying

Let $\rho$, $\sigma$ and $\tau$ be acises. Define the *evaluation* mapping $\mathrm{eval} : (\rho \to \sigma) \times \rho \to \sigma$ by

$$\mathrm{eval}(f,u) := f(u) \ ,$$

and the *currying* mapping $\mathrm{curry} : (\rho \times \sigma \to \tau) \to (\rho \to (\sigma \to \tau))$ by

$$\mathrm{curry}(f)(u,v) := f(u,v) \ .$$

**Proposition 1.30.** *The evaluation and currying mappings are well-defined and continuous.*

*Proof.* By Proposition 1.29 it suffices to show continuity in separate components. *For evaluation*. For the second argument: For monotonicity, let $u \subseteq v$; then by monotonicity of the fixed $f$ we get

$$\mathrm{eval}(f,u) := f(u) \subseteq f(v) =: \mathrm{eval}(f,v) \ .$$

For the principle of finite support, let $b \in \mathrm{eval}(f,u)$, that is, $b \in f(u)$; the fixed $f$ satisfies the principle of finite support, so there is a $Y \subseteq^f u$ such that $b \in f(\overline{U})$, hence $b \in \mathrm{eval}(f,\overline{U})$.

For the first argument: For monotonicity, let $f \subseteq g$; by the definition of the associated continuous mapping to an ideal, for a fixed $u$ we have:

$$\begin{aligned}
b \in \mathrm{eval}(f,u) &\overset{\text{def}}{\Leftrightarrow} b \in f(u) = \mathrm{cm}(f)(u) \\
&\overset{\text{def}}{\Leftrightarrow} \exists_{U \in \mathsf{Con}_\rho} (U \subseteq u \wedge (U,b) \in f) \\
&\Rightarrow \exists_{U \in \mathsf{Con}_\rho} (U \subseteq u \wedge (U,b) \in g) \\
&\overset{\text{def}}{\Leftrightarrow} b \in \mathrm{cm}(g)(u) = g(u) \\
&\overset{\text{def}}{\Leftrightarrow} b \in \mathrm{eval}(g,u) \ ,
\end{aligned}$$

so $\mathrm{eval}(f,u) \subseteq \mathrm{eval}(g,u)$. For the principle of finite support, let $b \in \mathrm{eval}(f,u)$, that is, $b \in \mathrm{cm}(f)(u)$; by definition, there is a $U \subseteq^f u$ such that $(U,b) \in f$; then $b \in \mathrm{eval}(\overline{\{U,b\}},u)$.

*For currying*. Fix $f \in \mathsf{Ide}_{\rho\times\sigma\to\tau}$. For a fixed $u \in \mathsf{Ide}_\rho$, the mapping $\mathrm{cm}(f)^u_\sigma$ (that is, $f^u_\sigma$ viewed as a continuous mapping) is continuous as a section of the continuous $\mathrm{cm}(f)$.

We show that the mapping $h : u \mapsto \mathrm{is}(\mathrm{cm}(f)^u_\sigma)$ (where now $f^u_\sigma$ is viewed as an ideal) is continuous. For monotonicity, let $u \subseteq u'$; since $\mathrm{cm}(f)$ is monotone we have

$$\begin{aligned}
(V,c) \in h(u) &\overset{\text{def}}{\Leftrightarrow} (V,c) \in \mathrm{is}(\mathrm{cm}(f)^u_\sigma) \\
&\overset{\text{def}}{\Leftrightarrow} c \in \mathrm{cm}(f)^u_\sigma(\overline{V}) \\
&\overset{\text{def}}{\Leftrightarrow} c \in \mathrm{cm}(f)(u,\overline{V}) \\
&\Rightarrow c \in \mathrm{cm}(f)(u',\overline{V}) \\
&\overset{\text{def}}{\Leftrightarrow} c \in \mathrm{cm}(f)^{u'}_\sigma(\overline{V}) \\
&\overset{\text{def}}{\Leftrightarrow} (V,c) \in \mathrm{is}(\mathrm{cm}(f)^u_\sigma) \\
&\overset{\text{def}}{\Leftrightarrow} (V,c) \in h(u') \ .
\end{aligned}$$

For the principle of finite support, by finite support for $\mathrm{cm}(f)^{v}_{\rho}$, we have

$$
\begin{aligned}
(V,c) \in h(u) &\overset{\text{def}}{\Leftrightarrow} (V,c) \in \mathrm{is}(\mathrm{cm}(f)^{u}_{\sigma}) \\
&\overset{\text{def}}{\Leftrightarrow} c \in \mathrm{cm}(f)^{u}_{\sigma}(\overline{V}) \\
&\overset{\text{def}}{\Leftrightarrow} c \in \mathrm{cm}(f)(u,\overline{V}) \\
&\overset{\text{def}}{\Leftrightarrow} c \in \mathrm{cm}(f)^{\overline{V}}_{\rho}(u) \\
&\Rightarrow \exists_{U \subseteq^f u} c \in \mathrm{cm}(f)^{\overline{V}}_{\rho}(\overline{U}) \\
&\overset{\text{def}}{\Leftrightarrow} \exists_{U \subseteq^f u} c \in \mathrm{cm}(f)(\overline{U},\overline{V}) \\
&\overset{\text{def}}{\Leftrightarrow} \exists_{U \subseteq^f u} c \in \mathrm{cm}(f)^{\overline{U}}_{\sigma}(\overline{V}) \\
&\overset{\text{def}}{\Leftrightarrow} \exists_{U \subseteq^f u} (V,c) \in \mathrm{is}(\mathrm{cm}(f)^{\overline{U}}_{\sigma}) \\
&\overset{\text{def}}{\Leftrightarrow} \exists_{U \subseteq^f u} (V,c) \in h(\overline{U}) \; .
\end{aligned}
$$

We show finally that the mapping $g : f \mapsto \mathrm{is}(h)$ is continuous. Monotonicity follows from the definition of $\mathrm{cm}(f)$. For the principle of finite support, let $(U,V,c) \in g(f)$; then $(U,V,c) \in g(\overline{\{U \cup V, c\}})$, with $\{U \cup V, c\} \subseteq^f f$. □

## Category theoretic characterization of ideals

Let $\rho$ be an acis. Define the *identity* $\mathrm{id}_\rho \in \mathsf{Ide}_{\rho\to\rho}$ by

$$(U,a) \in \mathrm{id}_\rho := U \succ_\rho a \; .$$

Furthermore, define the *composition* of $u \in \mathsf{Ide}_{\rho\to\sigma}$ and $v \in \mathsf{Ide}_{\sigma\to\tau}$ to be the set $v \circ u \subset T_{\rho\to\tau}$ where

$$(U,c) \in v \circ u := \exists_{V \in \mathsf{Con}_\sigma} \left( \forall_{b \in V} (U,b) \in u \wedge (V,c) \in v \right) \; .$$

**Proposition 1.31.** *The sets of ideals of acises together with the ideals of their function spaces form a category. Namely:*

1. *The identity is an ideal.*
2. *The composition of two ideals is again an ideal.*
3. *The identity is neutral with respect to composition.*
4. *The composition of ideals is associative.*

*Proof.* For the first statement, let $\rho$ be an acis. For consistency, let $(U_i,a_i) \in \mathrm{id}_\rho$, $i = 1,2$, and $U_1 \asymp_\rho U_2$; by the definition of the identity we have $U_i \succ_\rho a_i$, which, by Proposition 1.2, gives $a_1 \asymp_\rho a_2$. Closure under entailment follows by the definition of entailment, identity and by transitivity of entailment.

For the second statement, let $u \in \mathsf{Ide}_{\rho\to\sigma}$ and $v \in \mathsf{Ide}_{\sigma\to\tau}$. For consistency: Let $(U_i,c_i) \in v \circ u$, $i = 1,2$, and $U_1 \asymp_\rho U_2$; by the definition of composition, there are $V_1, V_2 \in \mathsf{Con}$ such that

$$\forall_{b_i \in V_i} (U_i,b_i) \in u \wedge (V_i,c_i) \in v \; ,$$

for $i = 1,2$; since $U_1 \asymp_\rho U_2$, we have $b_1 \asymp_\sigma b_2$ for all $b_i \in V_i$, that is, $V_1 \asymp_\sigma V_2$; then, by the consistency of $v$, we get $c_1 \asymp_\tau c_2$. For closure under entailment: Let $(U,c) \in v \circ u$ and $(U,c) \succ_{\rho\to\tau} (U',c')$; by the definition of entailment in function spaces, we have $U' \succ_\rho U \wedge c \succ_\tau c'$; by the definition of composition, there is a $V \in \mathsf{Con}_\sigma$ for which

$$\forall_{b\in V} (U,b) \in u \wedge (V,c) \in v \; ;$$

for all $b \in V$ it is $(U',b) \in u$ as well as $(V,c') \in v$, hence $(U',c') \in v \circ u$.

For the third statement, let $u \in \mathsf{Ide}_{\rho\to\sigma}$ and $v \in \mathsf{Ide}_{\sigma\to\rho}$. We prove that the identity ideal is left-neutral, that is, that $\mathrm{id}_\rho \circ v = v$. For the right direction we have

$$\begin{aligned}
(U,a) \in \mathrm{id}_\rho \circ v &\overset{\text{def}}{\Leftrightarrow} \exists_{V\in\mathsf{Con}_\rho} \left( \forall_{b\in V} (U,b) \in v \wedge (V,a) \in \mathrm{id}_\rho \right) \\
&\overset{\text{def}}{\Leftrightarrow} \exists_{V\in\mathsf{Con}_\rho} \left( \forall_{b\in V} (U,b) \in v \wedge V \succ_\rho a \right) \\
&\overset{\text{def}}{\Leftrightarrow} \exists_{V\in\mathsf{Con}_\rho} \left( \forall_{b\in V} (U,b) \in v \wedge \exists_{b\in V} b \succ_\rho a \right) \\
&\Rightarrow (U,a) \in v \; ,
\end{aligned}$$

and for the left direction

$$\begin{aligned}
(U,a) \in v &\Rightarrow (U,a) \in v \wedge \{a\} \succ_\rho a \\
&\overset{(\star)}{\Rightarrow} \exists_{V\in\mathsf{Con}_\rho} \left( \forall_{b\in V} (U,b) \in v \wedge V \succ_\rho a \right) \\
&\overset{\text{def}}{\Leftrightarrow} \exists_{V\in\mathsf{Con}_\rho} \left( \forall_{b\in V} (U,b) \in v \wedge (V,a) \in \mathrm{id}_\rho \right) \\
&\overset{\text{def}}{\Leftrightarrow} (U,a) \in \mathrm{id}_\rho \circ v \; ,
\end{aligned}$$

where at $(\star)$ we let $V := \{a\}$. Now we prove that the identity ideal is right-neutral, that is, that $u \circ \mathrm{id}_\rho = u$. For the right direction we have

$$\begin{aligned}
(U,b) \in u \circ \mathrm{id}_\rho &\overset{\text{def}}{\Leftrightarrow} \exists_{V\in\mathsf{Con}_\rho} \left( \forall_{a\in V} (U,a) \in \mathrm{id}_\rho \wedge (V,b) \in u \right) \\
&\overset{\text{def}}{\Leftrightarrow} \exists_{V\in\mathsf{Con}_\rho} \left( \forall_{a\in V} U \succ_\rho a \wedge (V,b) \in u \right) \\
&\overset{\text{def}}{\Leftrightarrow} \exists_{V\in\mathsf{Con}_\rho} \left( U \succ_\rho V \wedge (V,b) \in u \right) \\
&\Rightarrow (U,b) \in u \; ,
\end{aligned}$$

and for the left direction

$$\begin{aligned}
(U,b) \in u &\Rightarrow \forall_{a\in U} U \succ_\rho a \wedge (U,b) \in u \\
&\overset{(\star)}{\Rightarrow} \exists_{V\in\mathsf{Con}_\rho} \left( \forall_{a\in V} (U,a) \in \mathrm{id}_\rho \wedge (V,b) \in u \right) \\
&\overset{\text{def}}{\Leftrightarrow} (U,b) \in u \circ \mathrm{id}_\rho \; ,
\end{aligned}$$

where at $(\star)$ we let $V := U$.

For the fourth statement, let $u \in \mathsf{Ide}_{\rho\to\sigma}$, $v \in \mathsf{Ide}_{\sigma\to\tau}$, and $w \in \mathsf{Ide}_{\tau\to\upsilon}$. We prove the associativity, that is, that $(w \circ v) \circ u = w \circ (v \circ u)$. We have, from left to right

$$\begin{aligned}
&(U,d) \in (w \circ v) \circ u \\
&\quad \overset{\text{def}}{\Leftrightarrow} \underset{V \in \mathsf{Con}_\sigma}{\exists} \left( \underset{b \in V}{\forall} (U,b) \in u \wedge (V,d) \in w \circ v \right) \\
&\quad \overset{\text{def}}{\Leftrightarrow} \underset{V \in \mathsf{Con}_\sigma}{\exists} \left( \underset{b \in V}{\forall} (U,b) \in u \wedge \underset{W \in \mathsf{Con}_\tau}{\exists} \left( \underset{c \in W}{\forall} (V,c) \in v \wedge (W,d) \in w \right) \right) \\
&\quad \Leftrightarrow \underset{V \in \mathsf{Con}_\sigma}{\exists} \underset{W \in \mathsf{Con}_\tau}{\exists} \left( \underset{b \in V}{\forall} (U,b) \in u \wedge \underset{c \in W}{\forall} (V,c) \in v \wedge (W,d) \in w \right) ,
\end{aligned}$$

and similarly, from right to left

$$\begin{aligned}
&(U,d) \in w \circ (v \circ u) \\
&\quad \overset{\text{def}}{\Leftrightarrow} \underset{W \in \mathsf{Con}_\tau}{\exists} \left( \underset{c \in W}{\forall} (U,c) \in v \circ u \wedge (W,d) \in w \right) \\
&\quad \overset{\text{def}}{\Leftrightarrow} \underset{W \in \mathsf{Con}_\tau}{\exists} \left( \underset{c \in W}{\forall} \underset{V \in \mathsf{Con}_\sigma}{\exists} \left( \underset{b \in V}{\forall} (U,b) \in u \wedge (V,c) \in v \right) \wedge (W,d) \in w \right) \\
&\quad \Leftrightarrow \underset{W \in \mathsf{Con}_\tau}{\exists} \underset{V \in \mathsf{Con}_\sigma}{\exists} \left( \underset{b \in V}{\forall} (U,b) \in u \wedge \underset{c \in W}{\forall} (V,c) \in v \wedge (W,d) \in w \right) ,
\end{aligned}$$

as we needed. □

From the above it follows that the category of ideals is cartesian closed.

## 1.3 Algebraic acises

We introduced the notion of an algebra given by constructors on page 1. For the purposes of this chapter it suffices for $\alpha$ to be finitary (we will allow for more generality in Chapter 2).

Let $\alpha$ be an algebra given by constructors $C_1, \ldots, C_k$, with at least one nullary constructor among them. To $\alpha$ we further attach a nullary *partiality pseudo-constructor* $*_\alpha$. We may drop subscripts when the context suffices.

For each constructor $C$ of arity $r$ define inductively the following:

- if $a_1, \ldots, a_r \in T_\alpha$ then $Ca_1 \cdots a_r \in T_\alpha$; moreover, $*_\alpha \in T_\alpha$;
- if $a_1 \asymp_\alpha a'_1, \ldots, a_r \asymp_\alpha a'_r$ then $Ca_1 \cdots a_r \asymp_\alpha Ca'_1 \cdots a'_r$; moreover, $*_\alpha \asymp_\alpha a$ and $a \asymp_\alpha *_\alpha$ for all $a \in T_\alpha$;
- if $a_1 \succ_\alpha a'_1, \ldots, a_r \succ_\alpha a'_r$ then $Ca_1 \cdots a_r \succ_\alpha Ca'_1 \cdots a'_r$; moreover, $a \succ_\alpha *_\alpha$, for all $a \in T_\alpha$.

These inductive clauses define the predicates $T_\alpha$, $\asymp_\alpha$ and $\succ_\alpha$.

*Remark.* Notice that *equality of tokens in* $\alpha$, $=_\alpha$, is defined by

$$a =_\alpha b := a = b = * \vee \left( \underset{i}{\exists} \left( a = C_i\vec{a} \wedge b = C_i\vec{b} \right) \wedge \underset{j}{\forall} a_j =_\alpha b_j \right) .$$

Consequently, equality for neighborhoods of tokens should be understood as *set equality* (though in Chapter 2 it will be *list equality*). For simplicity's sake though, we keep this implicit in what follows. □

**Proposition 1.32.** *For an algebra $\alpha$ given by constructors, the triple $(T_\alpha, \asymp_\alpha, \succ_\alpha)$ is an acis.*

*Proof by induction on the formation of tokens.* We show that propagation holds, while the rest of the properties are shown similarly. Let $Ca_1 \cdots a_r \asymp_\alpha Ca'_1 \cdots a'_r$ and $Ca'_1 \cdots a'_r \succ_\alpha Ca''_1 \cdots a''_r$. By the definition of consistency and entailment, we have $a_i \asymp_\alpha a'_i$ and $a'_i \succ_\alpha a''_i$, for each $i = 1, \ldots, r$. The induction hypothesis gives $a_i \asymp_\alpha a''_i$, for each $i$, and the definition of consistency yields $Ca_1 \cdots a_r \asymp_\alpha Ca''_1 \cdots a''_r$. □

This is the *acis induced by* $\alpha$. Call an acis *algebraic* if it is either induced by an algebra or it is a function space composed by algebraic acises. When in need of distinguishing between the two, use *basic algebraic* and *composite algebraic* respectively; we denote basic algebraic acises by $\alpha, \beta, \gamma \ldots$ and composite ones by $\rho, \sigma, \tau \ldots$.

It is easy to see that every basic algebraic acis has a well-founded entailment relation. Furthermore, the homomorphisms between basic algebraic acises are exactly their monotone token-mappings, as the following proposition establishes.

**Proposition 1.33.** *Let $\alpha$ and $\beta$ be basic algebraic acises.*

1. *If two tokens in $\alpha$ are consistent then they have a least common entailer, that is, a* least upper bound*: for any $a, b \in T_\alpha$, if $a \asymp_\alpha b$ then there exists a token $c \in T_\alpha$ such that $c \succ_\alpha a$, $c \succ_\alpha b$, and $c' \succ_\alpha c$, for any $c' \in T_\alpha$ with $c' \succ_\alpha a$ and $c' \succ_\alpha b$.*
2. *If $f : T_\alpha \to T_\beta$ is a monotone token-mapping then it also preserves consistency.*

*Proof.* For statement 1. By induction on the formation of the tokens, if, without loss of generality, $b = *$, then $a \succ_\alpha \{a, b\}$ and obviously it is the least such token; if $a = C\vec{a}$ and $b = C\vec{b}$, with $a_i \asymp_\alpha b_i$, for every $i$, then by the induction hypothesis, it is

$$\forall_i \exists_{c_i} \left( c_i \succ_\alpha \{a_i, b_i\} \wedge \forall_{c'_i \in T_\alpha} \left( c'_i \succ_\alpha \{a_i, b_i\} \to c'_i \succ_\alpha c_i \right) \right) ,$$

and then,

$$C\vec{c} \succ_\alpha \{a, b\} \wedge \forall_{c' \in T_\alpha} \left( c' \succ_\alpha \{a, b\} \to c' \succ_\alpha C\vec{c} \right) .$$

For statement 2: Let $a, b \in T_\alpha$ be such that $a \asymp_\alpha b$; by 1, they will have a least common entailer $c := \mathrm{lub}(a, b)$; by monotonicity we get $f(c) \succ_\beta \{f(a), f(b)\}$, and by Proposition 1.2, $f(a) \asymp_\beta f(b)$. □

We now turn our attention to constructors. First, it is easy to see that, by the definition of entailment in an algebraic acis, every constructor defines a monotone token-mapping. Furthermore, every constructor $C$ of arity $n$ generates a subset of $\mathsf{Con}_\alpha{}^n \times T_\alpha$ by

$$r_C^* := \{(\vec{U}, *)\} \cup \{(\vec{U}, C\vec{a}) \mid \forall_{1 \le j \le n} U_j \succ a_j)\} ;$$

of particular importance is also the set $r_C := r_C^* \setminus \{(\vec{U}, *)\}$. If $\rho$ and $\sigma$ are algebraic acises, then a relation $r \subseteq \mathsf{Con}_\alpha^n \times T_\sigma$ is an *n-ary approximable map* from $\rho$ to $\sigma$—write $r : \rho^n \to \sigma$—if

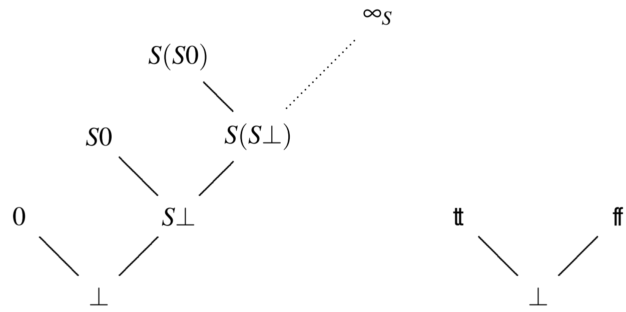


Figure 1.1: The ideals of $\mathbb{N}$ and $\mathbb{B}$, and their inclusion.

- $r(\vec{U},a) \wedge r(\vec{U},b) \to a \asymp_\sigma b$, and
- $\forall_{1\leq j\leq n} V_j \succ_\rho U_j \wedge r(\vec{U},a) \wedge a \succ_\sigma b \to r(\vec{V},b)$,

and the corresponding *n-ary continuous mapping* is

$$\mathrm{cm}(r)(\vec{u}) := \{b \in T_\sigma \mid \underset{U_1\subseteq u_1}{\exists} \cdots \underset{U_n\subseteq u_n}{\exists} r(\vec{U},b)\} \ .$$

It is easy to check the following.

**Proposition 1.34.** *Let $C$ and $C'$ be two distinguished constructors of arities $n$ and $n'$ respectively.*

1. *The sets $r_C^*, r_C : \alpha^n \to \alpha$ are n-ary approximable maps.*
2. *The continuous map*

$$\mathrm{cm}(r_C)(\vec{u}) = \{C\vec{a} \in T_\alpha \mid \underset{U_1\subseteq u_1,\dots,U_n\subseteq u_n}{\exists} \ \underset{1\leq j\leq n}{\forall} U_j \succ_\alpha a_j\}$$

*is injective. Furthermore, it is*

$$\mathrm{cm}(r_C)(\mathsf{Con}_\alpha{}^n) \cap \mathrm{cm}(r_{C'})(\mathsf{Con}_\alpha{}^{n'}) = \varnothing \ .$$

## Natural and boolean numbers

The *algebra of natural numbers* $\mathbb{N}$ is given by the nullary constructor 0 for zero and the unary $S$ for the successor. For brevity, we may write $S^n$ for $S\cdots S$ ($n$ times) and $n$ for $S^n0$. For all $n = 0,1,\dots$, the pair $(\{S^{n+m}0 \mid m \geq 0\} \cup \{S^{n+m}* \mid m \geq 0\}, \asymp_\mathbb{N})$ is a (graph-theoretic) star centered at $S^n*$. The ideals of $\mathbb{N}$ take one of the following forms:

$$\{n, S^n*,\dots,S*\}, \{S^n*,\dots,S*\}, \{\dots,S^n*,\dots,S*\} \ ,$$

while the neighborhoods are all the possible finite subsets of an ideal. Notice that the empty set $\bot$ is also an ideal. Call the ideals of $\mathbb{N}$ *partial natural numbers* and the ideals of function spaces above $\mathbb{N}$ *partial arithmetical functionals*. The set $r_S$ induced by the successor constructor is such an arithmetical function, as an ideal of $\mathbb{N} \to \mathbb{N}$.

Other important examples of algebras are the *algebra of boolean numbers* $\mathbb{B}$, given by two nullary constructors, tt and ff, as well as the parametric *algebra of lists of $\rho$-tokens* $\mathbb{L}(\rho)$, given by a nullary constructor $\mathrm{Nil}_\rho$ for the empty list, and a unary constructor $\mathrm{Cons}_\rho$ for the concatenator.

Call algebraic acises based on $\mathbb{N}$ and $\mathbb{B}$ *arithmetical acises*, and use $\iota$ to denote either $\mathbb{N}$ or $\mathbb{B}$. The equivalence between ideals, approximable maps and Scott-continuous maps justifies the definition of a *model $\mathscr{C}^\omega$ of partial continuous functionals (of finite type)*, based on arithmetical acises, by

$$\mathrm{C}_0 := \mathsf{Ide}_\iota \ , \quad \mathrm{C}_{\rho\Rightarrow\sigma} := \mathsf{Ide}_{\rho\to\sigma} \ , \quad \mathscr{C}^\omega := \bigcup_{\rho\in\mathbf{T}} \mathrm{C}_\rho \ .$$

Here are two simple but useful observations.

**Proposition 1.35.** *Let $\iota$ be a basic arithmetical acis and $\rho$ an arithmetical acis.*

1. *For all $a,b \in T_\iota$, we have the following* comparability property*: if $a \asymp_\iota b$ then either $a \succ_\iota b$ or $b \succ_\iota a$.*
2. *The application of a finitely generated ideal of type $\vec{\rho} \to \mathbb{N}$ to an ideal of type $\vec{\rho}$ is finite: if $W \in \mathsf{Con}_{\vec{\rho}\to\mathbb{N}}$ and $\vec{u} \in \mathsf{Ide}_{\vec{\rho}}$, then $\overline{W}(\vec{u}) \in \mathsf{Con}_{\mathbb{N}}$.*

Define the *total ideals $G_\rho \subseteq \mathsf{Ide}_\rho$*, in an arithmetical acis $\rho$, inductively on the types:

- It is $\mathtt{tt} \in G_{\mathbb{B}}$ and $\mathtt{ff} \in G_{\mathbb{B}}$. It is $0 \in G_{\mathbb{N}}$ and if $u \in G_{\mathbb{N}}$ then $Su \in G_{\mathbb{N}}$.
- If for every $v \in G_\rho$ it is $u(v) \in G_\sigma$, then $u \in G_{\rho\to\sigma}$.

In the remainder of the chapter we will not elaborate on totality, but merely use it to formalize indices, namely, we will use $G_{\mathbb{N}}$ as our standard denumerable set of indices when needed (see section 1.4). In section 2.4, we will discuss totality in the more general setting of not necessarily atomic information systems.

## Partial height of partial numbers

We need a continuous means to compare partial numbers. A reasonable idea is to use their height, that is, the number of constructors they are built upon, but this quickly leads to discontinuity, since this number is total. Indeed, if we define $l : T_{\mathbb{N}} \to T_{\mathbb{N}}$ by $l(*) := 0$ and $l(Ca) := S\,(l(a))$, for $C = 0, S$, then $\mathrm{ii}l \notin \mathsf{Ide}_{\mathbb{N}\to\mathbb{N}}$, since $l$ is not consistency-preserving (see Proposition 1.7): it is

$$S^m * \asymp_{\mathbb{N}} S^n * \quad \wedge \quad l(S^m *) = m \not\asymp_{\mathbb{N}} n = l(S^n *) \ ,$$

for any $m \neq n$ in $G_{\mathbb{N}}$.

We use instead a notion of "partial height", which intuitively stands for the projection of an ideal on the partial axis of the constructor $S$. Define a token-mapping $\mathrm{plength} : T_{\mathbb{N}} \to T_{\mathbb{N}}$ by

$$\begin{aligned} \mathrm{plength}(*) &:= * \ , \\ \mathrm{plength}(Ca) &:= S\,\mathrm{plength}(a) \ , \end{aligned}$$

where $C$ is either 0 or $S$. This token-mapping is trivially consistency-preserving, so its idealization is indeed an ideal of $\mathbb{N} \to \mathbb{N}$, and we have for example $\mathrm{ii}\mathrm{plength}(2) = \mathrm{ii}\mathrm{plength}(S^3\bot) = S^3\bot$, or $\mathrm{ii}\mathrm{plength}(\infty) = \infty$.

Let $u, v \in \mathsf{Ide}_{\mathbb{N}}$. Say that *$u$ is above $v$*, and write $u \rhd v$, if

$$\mathrm{ii}\mathrm{plength}(u) \succ_{\mathbb{N}} \mathrm{ii}\mathrm{plength}(v) \ .$$

Note that aboveness is not antisymmetric, since $S^n 0 \rhd S^{n+1}\bot$ and $S^{n+1}\bot \rhd S^n 0$ for all $n$. It is also obvious that aboveness between ideals is a total preorder with single maximum element $\infty$ and single minimum element $\bot$, as well as that, for total ideals, it reduces to the standard $\geq$ relation. Aboveness will prove crucial in section 1.4. Indeed, it is a sufficient step, beyond the techniques used by Plotkin, towards proving definability in our non-flat setting.

**Proposition 1.36.** *Concerning aboveness $\rhd \subseteq \mathsf{Ide}_{\mathbb{N}} \times \mathsf{Ide}_{\mathbb{N}}$ the following hold:*

1. *An ideal is above everything it entails, that is, if $u \succ_{\mathbb{N}} v$ then $u \rhd v$.*
2. *Aboveness is antisymmetrical for consistent ideals; more generally, for $u \in \mathsf{Ide}_{\mathbb{N}}$, if $v \rhd w$ and $w \rhd v$ for all $v, w \subseteq u$, then $v = w$.*
3. *For consistent ideals, aboveness reduces to entailment, that is, if $u \asymp_{\mathbb{N}} v$ and $u \rhd v$ then $u \succ_{\mathbb{N}} v$.*
4. *Ideals that are inconsistent with ideals above them are total, that is, if $u \rhd v$ and $u \not\asymp_{\mathbb{N}} v$ then $v \in G_{\mathbb{N}}$.*

*Proof.* The first statement derives from transitivity of entailment. For the second statement we have

$$\begin{aligned} v \asymp_{\mathbb{N}} w \wedge v \rhd w \wedge w \rhd v &\overset{\text{def}}{\Leftrightarrow} v \asymp_{\mathbb{N}} w \wedge \forall_n (v \rhd n \leftrightarrow w \rhd n) \\ &\Rightarrow \forall_n \left(v = w = S^{n-1}0 \vee (v \succ_{\mathbb{N}} S^n * \leftrightarrow w \succ_{\mathbb{N}} S^n *)\right) \\ &\Rightarrow v = w \ . \end{aligned}$$

The third one derives from the comparability property and the previous statement:

$$\begin{aligned} u \asymp_{\mathbb{N}} v \wedge u \rhd v &\overset{\text{P. 1.35}}{\Leftrightarrow} (u \succ_{\mathbb{N}} v \vee v \succ_{\mathbb{N}} u) \wedge u \rhd v \\ &\Rightarrow (u \succ_{\mathbb{N}} v \wedge u \rhd v) \vee (v \succ_{\mathbb{N}} u \wedge u \rhd v) \\ &\overset{(2)}{\Leftrightarrow} u \succ_{\mathbb{N}} v \vee (v \rhd u \wedge u \rhd v) \\ &\overset{(3)}{\Leftrightarrow} u \succ_{\mathbb{N}} v \vee u = v \\ &\Leftrightarrow u \succ_{\mathbb{N}} v \ . \end{aligned}$$

The last statement is obvious. □

## Maximal form of algebraic neighborhoods

Now we address the issue of making neighborhoods as parsimonious as possible, that is, with no redundant information: the subset $\{S^3*, S*\}$ informs us that the successor has been applied three times *and* that the successor has been applied one time—clearly, the second piece of information is redundant.

By the very definition of entailment in an abstract acis, we have non-antisymmetricity, that is, we can have two different tokens entailing one another. Call an acis *antisymmetric* when antisymmetricity for entailment holds. By induction on the formation of tokens one can directly prove the following.

**Proposition 1.37.** *All basic algebraic acises are antisymmetric.*[6]

[6] A parametric basic algebraic acis, like $\mathbb{L}(\rho)$, is antisymmetric if the parameter acis $\rho$ is antisymmetric. For simplicity's sake, we focus here on non-parametric algebraic acises.

Even in the case of an antisymmetric acis though, non-antisymmetricity may appear in its neighborhoods as well as in tokens and neighborhoods of its function spaces. For any acis $\rho$, recall the equivalence relation of mutual entailment on $\mathsf{Con}_\rho$:

$$U \sim_\rho V :\!= U \succ_\rho V \wedge V \succ_\rho U \ .$$

Nontrivial examples of equivalent neighborhoods in arithmetical acises are the following:

$$\begin{aligned} \{S^2*\} &\sim_\iota \{S^2*, S*\} \ , \\ \{(\{S^2*\}, S^2*)\} &\sim_{\iota\to\iota} \{(\{S^2*\}, S^2*), (\{2\}, S*)\} \ , \\ \{(\{(\{0\}, S*)\}, 0)\} &\sim_{(\iota\to\iota)\to\iota} \{(\{(\{0\}, S*)\}, 0), (\{(\{0\}, S*), (\{0\}, 1)\}, 0)\} \ , \\ \{(\{S^2*\}, \{S^2*\}, S*)\} &\sim_{\iota\to(\iota\to\iota)} \{(\{S^2*, S*\}, \{S^2*, S*\}, S*)\} \ . \end{aligned}$$

We would like to have a notion of "normal form" for neighborhoods, so that every neighborhood would have a normal form and two neighborhoods in normal form would be equivalent if and only if they were equal. This turns out to be easily feasible for algebraic acises, as we now show.

The definition of the set $\mathsf{NF}_\alpha$ of neighborhoods in *(atomic) maximal form*, for an algebraic acis $\rho$, is inductive on the formation of the acis:

- for a basic algebraic acis $\alpha$, a neighborhood $U \in \mathsf{Con}_\alpha$ is in maximal form if it *contains no entailments*, that is, if none of its elements entails some other:

$$\{a_i\}_i \in \mathsf{NF}_\alpha :\!= \forall_i \forall_{j\neq i} a_i \not\succ_\alpha a_j \ ;$$

- for a composite algebraic acis $\rho \to \sigma$, a neighborhood $\{(U_i, b_i)\}_i \in \mathsf{Con}_{\rho\to\sigma}$ is in maximal form if all its lower-type objects are either already in maximal form or else tokens and if it contains no entailments:

$$\{(U_i, a_i)\}_i \in \mathsf{NF}_{\rho\to\sigma} :\!= \forall_i \left( U_i \in \mathsf{NF}_\rho \wedge a_i \in \mathsf{NF}_\sigma \cup T_\sigma \wedge \forall_{j\neq i} (U_i, a_i) \not\succ_{\rho\to\sigma} (U_j, a_j) \right) \ .$$

**Theorem 1.38** (Atomic maximal form)**.** *For all algebraic acises $\rho$ the following hold:*

1. *For all $U \in \mathsf{Con}_\rho$ there is a $U' \in \mathsf{NF}_\rho$ such that $U \sim_\rho U'$.*
2. *For all $U, V \in \mathsf{NF}_\rho$ it is $U \sim_\rho V \leftrightarrow U = V$.*

*Proof.* We prove the more general step cases. For the first statement: Let $\{(U_i, a_i)\}_{i\in I} \in \mathsf{Con}_{\rho\to\sigma}$ with $U_i \in \mathsf{NF}_\rho$, $a_i \in \mathsf{NF}_\sigma \cup T_\sigma$, for every $i$; suppose that there are $k, l \in I$ such that $(U_k, a_k) \succ_{\rho\to\sigma} (U_l, a_l)$; set $I' :\!= I - \{l\}$; it is easy to see that $\{(U_i, a_i)\}_{i\in I} \sim_{\rho\to\sigma} \{(U'_i, a'_i)\}_{i'\in I'}$.

The left direction of the second statement is obvious. For the right direction let $\{(U_i, a_i)\}_{i\in I}, \{(V_j, b_j)\}_{j\in J} \in \mathsf{NF}_{\rho\to\sigma}$ be such that $\{(U_i, a_i)\}_{i\in I} \sim_{\rho\to\sigma} \{(V_j, b_j)\}_{j\in J}$; this unfolds to

$$\forall_j \exists_{i(j)} (U_{i(j)}, a_{i(j)}) \succ_{\rho\to\sigma} (V_j, b_j) \ \wedge \ \forall_i \exists_{j(i)} (V_{j(i)}, b_{j(i)}) \succ_{\rho\to\sigma} (U_i, a_i) \ ,$$

which is equivalent to $\forall_j \exists_i (V_j, b_j) \sim_{\rho\to\sigma} (U_i, a_i)$; by definition we get $\forall_j \exists_i (V_j \sim_\rho U_i \wedge b_j \sim_\sigma a_i)$, which, by the assumption and the induction hypothesis, yields $\forall_j \exists_i (V_j, b_j) = (U_i, a_i)$; similarly we have $\forall_i \exists_j (U_i, a_i) = (V_j, b_j)$, which concludes the proof. ◻

Notice that, in the special case of arithmetical acises, neighborhoods have fairly simple maximal forms, since they are built on singletons of $T_\iota$: $a$ at $\iota$, $\{(a_i^1,a_i^2)\}_i$ at $\iota \to \iota$, $\{(\{(a_{j_i}^1,a_{j_i}^2)\}_{j_i},a_i^3)\}_i$ at $(\iota \to \iota) \to \iota$, $\{(a_i^1,(a_i^2,a_i^3))\}_i$ at $\iota \to (\iota \to \iota)$, and so on, where curly brackets of singletons have been omitted.

It is also important to notice that $\mathsf{NF}_\alpha$ is *not* closed under application; for example, $\{(\{S*\},S*),(\{S^2*\},S^2*)\} \in \mathsf{NF}_{\mathbb{N}\to\mathbb{N}}$ and $\{2\} \in \mathsf{NF}_{\mathbb{N}}$ but

$$\{(\{S*\},S*),(\{S^2*\},S^2*)\} \cdot \{2\} = \{S^2*,S*\} \notin \mathsf{NF}_\iota \ .$$

Nevertheless, we have the following *monotonicity property*.

**Proposition 1.39.** *If $\rho$ is an arithmetical acis and $\{(U_i,b_i)\}_{i<n} \in \mathsf{NF}_{\rho\to\mathbb{N}}$, then*

$$U_i \succ_\rho U_j \to b_i \succ_{\mathbb{N}} b_j \ ,$$

*for all $i,j<n$.*

*Proof.* Let $U_i \succ_\rho U_j$ for some $i,j<n$ with $i \neq j$ (no loss of generality); then $U_i \asymp_\rho U_j$, hence, by the consistency of the neighborhood, $b_i \asymp_{\mathbb{N}} b_j$; the property of comparability gives $b_i \succ_{\mathbb{N}} b_j$, since $b_j \succ_{\mathbb{N}} b_i$ would contradict normality. □

*Remark.* Note that it can be $U_i \asymp_\rho U_j \wedge b_i \succ_{\mathbb{N}} b_j$, with $U_i \not\succ_\rho U_j$ and $U_j \not\succ_\rho U_i$; take for example

$$\{(\{(S0,S0)\},S0),(\{(S^2 0,S^2 0)\},S*)\} \in \mathsf{NF}_{(\mathbb{N}\to\mathbb{N})\to\mathbb{N}} \ .$$

We will revisit monotone neighborhoods in not necessarily atomic information systems on page 96. □

## 1.4 Computability over arithmetical functionals

In this section we have the main result of this chapter, a *definability result* (Theorem 1.46): we show that a functional is computable exactly when it is defined in terms of certain basic arithmetical functionals. Results of this kind for given PCF-like languages have been considered numerous times, for example in [35, 11, 46, 31]. We prove this result here directly for arithmetical acises, in a way that is directly applicable to any non-superunary algebra. Yet, in the case of algebras with superunary constructors, one cannot hope to apply the same methods, as we show in the end of the chapter.

For the remainder of the section we consider exclusively arithmetical acises, unless otherwise stated.

### Recursion on parallel conditionals and existentials

Call a partial continuous functional *computable* if it is $\Sigma_1^0$-definable as a set of tokens. It is direct to check that evaluation and currying functionals are computable, that composition, application and cartesian products of computable functionals are computable, as well as that projections are computable.

Let $\rho$ be an acis and $f : \rho \to \rho$ a continuous mapping. An ideal $u \in \mathsf{Ide}_\rho$ is said to be the *least fixed point* of $f$ if

$$f(u) = u \wedge \underset{v\in\mathsf{Ide}_\rho}{\forall} (f(v) = v \to u \subseteq v) \ .$$

**Proposition 1.40.** *Let $\rho$ be an acis and $f : \rho \to \rho$ a continuous mapping. The mapping $f$ has a least fixed point given by the equation*

$$\mathsf{Y}(f) = \bigcup_{n \in G_{\mathbb{N}}} f^n(\bot) .$$

*Proof.* Since $f$ is continuous, it is monotone and commutes with directed unions. By monotonicity,

$$\bot \subseteq f(\bot) \subseteq \cdots \subseteq f^n(\bot) \subseteq \cdots ,$$

which yields $u := \bigcup_{n \in G_{\mathbb{N}}} f^n(\bot) \in \mathsf{Ide}_\rho$; by commutativity with directed unions,

$$f(u) = f(\bigcup_{n \in G_{\mathbb{N}}} f^n(\bot)) = \bigcup_{n \in G_{\mathbb{N}}} f(f^n(\bot)) = \bigcup_{n \in G_{\mathbb{N}}} f^{n+1}(\bot) = u .$$

So $u$ is a fixed point. Let $v$ be another fixed point of $f$; we have

$$\bot \subseteq v \Rightarrow f(\bot) \subseteq f(v) = v \Rightarrow \cdots \Rightarrow f^n(\bot) \subseteq v \Rightarrow \cdots ,$$

which yields $u = \bigcup_{n \in G_{\mathbb{N}}} f^n(\bot) \subseteq v$, so $u$ is the least fixed point. □

**Proposition 1.41.** *The* least fixed point functional $\mathsf{Y} : (\rho \to \rho) \to \rho$ *is continuous and computable for any* $\rho$.

*Proof.* For monotonicity, if $f \subseteq g$, then trivially $\bigcup_{n \in G_{\mathbb{N}}} f^n(\bot) \subseteq \bigcup_{n \in G_{\mathbb{N}}} g^n(\bot)$. For the principle of finite support, let $b \in \mathsf{Y}(f)$, that is, $b \in \bigcup_{n \in G_{\mathbb{N}}} f^n(\bot)$; equivalently,

$$\exists_{n \in G_{\mathbb{N}}} b \in f^n(\bot) \overset{\text{PFS}}{\Rightarrow} \exists_{n \in G_{\mathbb{N}}} \exists_{W \subseteq^f f} b \in \overline{W}^n(\bot) \overset{\text{def}}{\Leftrightarrow} \exists_{W \subseteq^f f} b \in \bigcup_{n \in G_{\mathbb{N}}} \overline{W}^n(\bot) ,$$

so $\exists_{W \subseteq^f f} b \in \mathsf{Y}(\overline{W})$. For $\Sigma^0_1$-definability, let $(U, a) \in \mathsf{Y}_\rho$, that is, $a \in \mathsf{Y}_\rho(\overline{U})$; then equivalently

$$a \in \bigcup_{n \in G_{\mathbb{N}}} \overline{U}^n(\bot) \overset{\text{def}}{\Leftrightarrow} \exists_{n \in G_{\mathbb{N}}} a \in \overline{U}^n(\bot) \overset{\text{P. 1.17(4)}}{\Leftrightarrow} \exists_{n \in G_{\mathbb{N}}} a \in \overline{U^n \varnothing} \overset{\text{def}}{\Leftrightarrow} \exists_{n \in G_{\mathbb{N}}} U^n \varnothing \succ_\rho a ;$$

but this is equivalent to the formula

$$\exists_{n \in G_{\mathbb{N}}} \exists_{V_0} \cdots \exists_{V_n} \left( V_0 = \varnothing \wedge \forall_{i<n} (V_{i+1} = UV_i) \wedge V_n \succ_\rho a \right) ,$$

which is a $\Sigma^0_1$-expression by Proposition 1.17(5). □

Define the *parallel conditional functional* $\mathsf{pcond} : \mathbb{B} \to \mathbb{N} \to \mathbb{N} \to \mathbb{N}$ by

$$\mathsf{pcond}(p, u, v) := \begin{cases} u , & p = \mathtt{tt} , \\ v , & p = \mathtt{ff} , \\ u \cap v , & p = \bot . \end{cases}$$

**Proposition 1.42.** *The parallel conditional functional is continuous and computable.*

*Proof.* For monotonicity: Let $u \subseteq u'$ and $v \subseteq v'$; we distinguish cases according to the boolean values:

$$\begin{aligned}
\mathsf{pcond}(\mathtt{tt}, u, v) &= u \subseteq u' = \mathsf{pcond}(\mathtt{tt}, u', v') \ , \\
\mathsf{pcond}(\mathtt{ff}, u, v) &= v \subseteq v' = \mathsf{pcond}(\mathtt{ff}, u', v') \ , \\
\mathsf{pcond}(\bot, u, v) &= u \cap v \subseteq u' \cap v' = \mathsf{pcond}(\bot, u', v') \ , \\
\mathsf{pcond}(\bot, u, v) &= u \cap v \subseteq u \subseteq u' = \mathsf{pcond}(\mathtt{tt}, u', v') \ , \\
\mathsf{pcond}(\bot, u, v) &= u \cap v \subseteq v \subseteq v' = \mathsf{pcond}(\mathtt{ff}, u', v') \ .
\end{aligned}$$

For the principle of finite support: if $a \in \mathsf{pcond}(\mathtt{tt}, u, v)$ then $a \in \mathsf{pcond}(\overline{\mathtt{tt}}, \overline{a}, \overline{\varnothing})$; if $a \in \mathsf{pcond}(\mathtt{ff}, u, v)$ then $a \in \mathsf{pcond}(\overline{\mathtt{ff}}, \overline{\varnothing}, \overline{a})$; finally, if $a \in \mathsf{pcond}(\bot, u, v)$ then $a \in \mathsf{pcond}(\overline{\varnothing}, \overline{a}, \overline{a})$. For $\Sigma^0_1$-definability, let $(P, U, V, a) \in \mathsf{pcond}$, that is, $a \in \mathsf{pcond}(\overline{P}, \overline{U}, \overline{V})$; this is equivalent to the formula

$$(P \succ_{\mathbb{B}} \mathtt{tt} \wedge U \succ_{\mathbb{N}} a) \vee (P \succ_{\mathbb{B}} \mathtt{ff} \wedge V \succ_{\mathbb{N}} a) \vee (P = \varnothing \wedge U \succ_{\mathbb{N}} a \wedge V \succ_{\mathbb{N}} a) \ ,$$

which is a $\Sigma^0_1$-expression. □

Define the *(parallel) existential functional* $\mathsf{exist} : (\mathbb{N} \to \mathbb{B}) \to \mathbb{B}$ by

$$\mathsf{exist}(f) = \begin{cases} \mathtt{ff} \ , & \exists_{n \in G_{\mathbb{N}}} \left( f(S^n \bot) = \mathtt{ff} \wedge \forall_{k \le n} f(k) = \mathtt{ff} \right) \ , \\ \mathtt{tt} \ , & \exists_{n \in G_{\mathbb{N}}} f(n) = \mathtt{tt} \ , \\ \bot \ , & \text{otherwise} \ . \end{cases}$$

**Proposition 1.43.** *The existential functional is continuous and computable.*

*Proof.* For monotonicity: Let $f, f' \in \mathsf{Ide}_{\mathbb{N} \to \mathbb{B}}$ be such that $f \subseteq f'$. We distinguish three cases: In case $\exists_{n \in G_{\mathbb{N}}} \left( f(S^n \bot) = \mathtt{ff} \wedge \forall_{k \le n} f(k) = \mathtt{ff} \right)$, then, by definition,

$$\underset{n \in G_{\mathbb{N}}}{\exists} \left( \underset{U \in \mathsf{Con}_{\mathbb{N}}}{\exists} (S^n \bot \succ_{\mathbb{N}} U \wedge (U, \mathtt{ff}) \in f) \wedge \underset{k \le n}{\forall} \underset{V_k \in \mathsf{Con}_{\mathbb{N}}}{\exists} (k \succ_{\mathbb{N}} V_k \wedge (V_k, \mathtt{ff}) \in f) \right) \ ;$$

since $f \subseteq f'$, we trivially get

$$\underset{n \in G_{\mathbb{N}}}{\exists} \left( \underset{U \in \mathsf{Con}_{\mathbb{N}}}{\exists} \left(S^n \bot \succ_{\mathbb{N}} U \wedge (U, \mathtt{ff}) \in f'\right) \wedge \underset{k \le n}{\forall} \underset{V_k \in \mathsf{Con}_{\mathbb{N}}}{\exists} \left(k \succ_{\mathbb{N}} V_k \wedge (V_k, \mathtt{ff}) \in f'\right) \right) \ ,$$

which in turn means that $\exists_{n \in G_{\mathbb{N}}} \left( f'(S^n \bot) = \mathtt{ff} \wedge \forall_{k \le n} f'(k) = \mathtt{ff} \right)$, so $\mathsf{exist}(f) = \mathsf{exist}(f') = \mathtt{ff}$. In case $\exists_{n \in G_{\mathbb{N}}} f(n) = \mathtt{tt}$, we have

$$\underset{n \in G_{\mathbb{N}}}{\exists} \underset{U \in \mathsf{Con}_{\mathbb{N}}}{\exists} \left( n \succ_{\mathbb{N}} U \wedge (U, \mathtt{tt}) \in f \subseteq f' \right) \ ,$$

which gives $\exists_{n \in G_{\mathbb{N}}} f'(n) = \mathtt{tt}$, so $\mathsf{exist}(f) = \mathsf{exist}(f') = \mathtt{tt}$. Finally, in case $\mathsf{exist}(f) = \bot$, it is obviously $\mathsf{exist}(f) \subseteq \mathsf{exist}(f')$ for any possible value of the latter.

For the principle of finite support: Let $b \in \underline{\mathsf{exist}(f)}$; if there exists an $n \in G_{\mathbb{N}}$ such that $f(S^n \bot) = \mathtt{ff} \wedge \forall_{k \le n} f(k) = \mathtt{ff}$ then $b \in \mathsf{exist}(\overline{(S^n \varnothing, \mathtt{ff})})$; if there exists an $n \in G_{\mathbb{N}}$ such that $f(n) = \mathtt{tt}$ then $b \in \mathsf{exist}(\overline{(n, \mathtt{tt})})$ (the case $b \in \bot$ is absurd).

For $\Sigma^0_1$-definability: Let $(\{(U_i, b_i)\}_{i \le m}, b) \in \mathsf{exist}$, that is, $b \in \mathsf{exist}(\overline{\{(U_i, b_i)\}_{i \le m}})$; this is equivalent to the formula

$$\begin{aligned}
&\left( \mathtt{tt} \succ_{\mathbb{B}} b \wedge \underset{n \in G_{\mathbb{N}}}{\exists} \{(U_i, b_i)\}_{i \le m} \succ_{\mathbb{N} \to \mathbb{B}} (S^n 0, \mathtt{tt}) \right) \\
\vee &\left( \mathtt{ff} \succ_{\mathbb{B}} b \wedge \underset{n \in G_{\mathbb{N}}}{\exists} \left( \{(U_i, b_i)\}_{i \le m} \succ_{\mathbb{N} \to \mathbb{B}} (S^n \varnothing, \mathtt{ff}) \wedge \underset{k \le n}{\forall} \{(U_i, b_i)\}_{i \le m} \succ_{\mathbb{N} \to \mathbb{B}} (S^k 0, \mathtt{ff}) \right) \right) ,
\end{aligned}$$

which is a $\Sigma^0_1$-expression. □

In terms of pcond and exist we can easily define the following:

1. The *conditional functional* $\mathsf{cond} : \mathbb{B} \to \mathbb{N} \to \mathbb{N} \to \mathbb{N}$ defined by

$$\mathsf{cond}(p,u,v) := \mathsf{pcond}(p, \mathsf{pcond}(p,u,\bot), \mathsf{pcond}(p,\bot,v)) \ ,$$

which can be unfolded to

$$\mathsf{cond}(p,u,v) = \begin{cases} u \ , & p = \mathsf{tt} \ , \\ v \ , & p = \mathsf{ff} \ , \\ \bot \ , & p = \bot \ . \end{cases}$$

Note that $\mathsf{pcond}(p,u,v) = \mathsf{cond}(p,u,v)$ if and only if $u \cap v = \bot$.

2. The *disjunction functional* $\mathsf{or} : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$ defined by

$$\mathsf{or}(p,q) := \mathsf{pcond}(p, \mathsf{tt}, q) \ ,$$

which unfolds to

$$\mathsf{or}(p,q) := \begin{cases} \mathsf{tt} \ , & p = \mathsf{tt} \vee q = \mathsf{tt} \ , \\ \mathsf{ff} \ , & p = q = \mathsf{ff} \ , \\ \bot \ , & \text{otherwise} \ . \end{cases}$$

3. The *conjunction functional* $\mathsf{and} : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$ defined by

$$\mathsf{and}(p,q) := \mathsf{pcond}(p, q, \mathsf{ff}) \ ,$$

which unfolds to

$$\mathsf{and}(p,q) := \begin{cases} \mathsf{tt} \ , & p = q = \mathsf{tt} \ , \\ \mathsf{ff} \ , & p = \mathsf{ff} \vee q = \mathsf{ff} \ , \\ \bot \ , & \text{otherwise} \ . \end{cases}$$

4. The *negation functional* $\mathsf{not} : \mathbb{B} \to \mathbb{B}$ defined by

$$\mathsf{not}(p) := \mathsf{pcond}(p, \mathsf{ff}, \mathsf{tt}) \ ,$$

which unfolds to

$$\mathsf{not}(p) := \begin{cases} \mathsf{tt} \ , & p = \mathsf{ff} \ , \\ \mathsf{ff} \ , & p = \mathsf{tt} \ , \\ \bot \ , & \text{otherwise} \ . \end{cases}$$

5. The *implication functional* $\mathsf{implies} : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$ defined by

$$\mathsf{implies}(p,q) := \mathsf{or}(\mathsf{not}(p), q) \ ,$$

which unfolds to

$$\mathsf{implies}(p,q) := \begin{cases} \mathsf{tt} \ , & p = \mathsf{ff} \vee q = \mathsf{tt} \ , \\ \mathsf{ff} \ , & p = \mathsf{tt} \wedge q = \mathsf{ff} \ , \\ \bot \ , & \text{otherwise} \ . \end{cases}$$

All of these are continuous and computable functionals since they are defined by pcond. The $p$-ary generalization of or we will denote by $\mathsf{OR}_{i=1}^{p}$.

## Enumeration and inconsistency functionals

To make case distinctions in subsequent proofs less branching, we hereafter adopt the following convention: let $\rho$, $\sigma$ and $\tau$ be arithmetical acises; a functional $f : \rho \to \mathbb{N} \to \sigma \to \tau$ that satisfies $f(u,x,v) = \bot$ for $x \notin G_\mathbb{N}$, will be informally typed by $\rho \to G_\mathbb{N} \to \sigma \to \tau$, and will be treated only on its arguments where $x$ is total.

For every arithmetical acis we fix an enumeration of neighborhoods $\{U_n\}_{n\in G_\mathbb{N}}$, such that (i) $U_0 = \varnothing$ holds, and (ii) the following are primitive recursive relations: $U_n \asymp_\rho U_m$, $U_n \succ_\rho U_m$, $U_n \cdot U_m = U_k$ (for appropriate types), and $U_n \cup U_m = U_k$, with $k = 0$ if $U_n \not\asymp_\rho U_m$. In the following we may write $b$ for *singleton neighborhoods* $\{b\}$, whenever we want to stress that they behave as *tokens*.

Define the *conditional extension functional* (also *continuous union*, see note on page 47) $\mathsf{condext} : \mathbb{N} \to G_\mathbb{N} \to \mathbb{N}$ by

$$\mathsf{condext}(v,n) := \begin{cases} v\ , & b_n \in v\ , \\ \overline{b_n}\ , & \text{otherwise}\ . \end{cases}$$

The condext operator extends $\overline{b_n}$ to $v$ whenever this is allowed.

**Proposition 1.44.** *The conditional extension functional is continuous and computable.*

*Proof.* For monotonicity: Let $v, v' \in \mathsf{Ide}_\mathbb{N}$, with $v \subseteq v'$. In case $b_n \in v$, then $b_n \in v'$ as well, so $\mathsf{condext}(v,n) = v \subseteq v' = \mathsf{condext}(v',n)$. In case $b_n \notin v$, then either $b_n \in v'$ and so $\mathsf{condext}(v,n) = \overline{b_n} \subseteq v' = \mathsf{condext}(v',n)$, or $b_n \notin v'$ and so $\mathsf{condext}(v,n) = \overline{b_n} = \mathsf{condext}(v',n)$.

For the principle of finite support, let $b \in \mathsf{condext}(v,n)$. If $b_n \in v$ then $b \in v$ as well, and $b \in \mathsf{condext}(\overline{b} \cup \overline{b_n}, n)$. If $b_n \notin v$ then $b \in \overline{b_n}$, and $b \in \mathsf{condext}(\overline{b_n}, n)$.

For $\Sigma^0_1$-definability, consider a token $(V,N,b) \in \mathsf{condext}$. This means that $b \in \mathsf{condext}(\overline{V}, \overline{N})$, which is equivalent to the formula

$$\exists_{n \in G_\mathbb{N}} (N \succ_\mathbb{N} n \wedge (V \not\asymp_\mathbb{N} b_n \vee V \succ_\mathbb{N} b \vee b_n \succ_\mathbb{N} b))\ ;$$

this is a $\Sigma^0_1$-expression. □

**Proposition 1.45.** *Let $\rho$ be an arithmetical acis.*

1. *There exists an* enumeration functional $\mathsf{en}_\rho : G_\mathbb{N} \to \mathbb{N} \to \rho$*, with the properties*

$$\mathsf{en}_\rho(m,x) = \overline{U_m},\ \textit{when}\ x \notin G_\mathbb{N}\ ,$$
$$\mathsf{en}_\rho(m,n) = \overline{U_n},\ \textit{when}\ U_n \succ_\rho U_m\ .$$

2. *There exists an* inconsistency functional $\mathsf{incns}_\rho : \rho \to G_\mathbb{N} \to \mathbb{B}$*, with the property*

$$\mathsf{incns}_\rho(u,n) = \begin{cases} \mathtt{tt}\ , & u \not\asymp_\rho U_n\ , \\ \mathtt{ff}\ , & u \succ_\rho U_n\ . \end{cases}$$

The inconsistency functionals concern application: let $\sigma$ be an arithmetical acis and $(U_n, b_n)$ a token of a partial continuous functional $v$ of type $\rho \to \sigma$; the only case where the token may contribute the information $b_n$ to the value $v(u)$ is when $\mathsf{incns}_\rho(u,n)$ is ff. In order to define the functionals $\mathsf{incns}_\rho$, we will need the *enumeration functionals*, that enumerate all finitely generated extensions of $\overline{U_m}$, for $U_m \in \mathsf{Con}_\rho$.

*Proof by induction on types.* First we deal with the *enumeration functionals*. Let $\rho = \rho_1 \to \cdots \to \rho_p \to \mathbb{N}$, $U_m \in \mathsf{Con}_{\rho_1\to\cdots\to\rho_p\to\mathbb{N}}$ and $f_1,\ldots,f_p$, $g$, and $h$ be primitive recursive functions such that

$$U_m = \{(U_{f_1(m,l)},\ldots,U_{f_p(m,l)},b_{g(m,l)})\}_{l<h(m)} ,$$

with

$$l > l' \to \overline{b_{g(m,l)}} \triangleright \overline{b_{g(m,l')}} . \tag{1.1}$$

In this representation we have $U_{f_i(m,l)} \in \mathsf{Con}_{\rho_i}$ and $b_{g(m,l)} \in \mathsf{Con}_{\mathbb{N}}$, for all $l < h(m)$ and all $1 \le i \le p$, while $h(m)$ denotes the number of elements of $U_m$. Consider the collection $\{U_{m,l}\}_l$ of *progressive approximations* of $U_m$; in particular, for $l \in G_{\mathbb{N}}$, define:

$$\begin{aligned} U_{x,l} &\coloneqq \varnothing \text{ if } x \notin G_{\mathbb{N}} , \\ U_{m,0} &\coloneqq \varnothing , \\ U_{m,l+1} &\coloneqq U_{m,l} \cup \{(U_{f_1(m,l)},\ldots,U_{f_p(m,l)},b_{g(m,l)})\} . \end{aligned}$$

Observe that $U_{m,h(m)} = U_m$.

For an arbitrary argument $\vec{u} \in \mathsf{Ide}_{\rho_1\to\cdots\to\rho_p}$, define an *argument test* $q_{\vec{u},m,l}$ expressing whether the application $\overline{U_{m,l+1}}(\vec{u})$ does *not* contribute information to the value of $\overline{U_{m,l}}(\vec{u})$:

$$q_{\vec{u},m,l} \coloneqq \operatorname*{OR}_{i=1}^{p} \mathsf{incns}_{\rho_i}(u_i, f_i(m,l)) = \begin{cases} \mathtt{tt} , & \exists_{i=1}^{p} u_i \not\asymp_{\rho_i} U_{f_i(m,l)} , \\ \mathtt{ff} , & \forall_{i=1}^{p} u_i \succ_{\rho_i} U_{f_i(m,l)} , \\ \bot , & \text{otherwise} . \end{cases}$$

Observe that in the last case we have that $\exists_{i=1}^{p} u_i \not\succ_{\rho_i} U_{f_i(m,l)}$ while we still have that $\forall_{i=1}^{p} u_i \asymp_{\rho_i} U_{f_i(m,l)}$. Similarly, for an arbitrary partial number $v \in \mathsf{Ide}_{\mathbb{N}}$, we will use $\mathsf{condext}(v, g(m,l))$—written $\mathsf{condext}(v,m,l)$—as a *value test*.

Define now an auxiliary functional $\Psi : \rho_1 \to \cdots \to \rho_p \to \mathbb{N} \to \mathbb{N} \to G_{\mathbb{N}} \to \mathbb{N}$ by

$$\begin{aligned} \Psi_{\vec{u},x}(v,l) &\coloneqq v \text{ if } x \notin G_{\mathbb{N}} , \\ \Psi_{\vec{u},m}(v,0) &\coloneqq v , \\ \Psi_{\vec{u},m}(v,l+1) &\coloneqq \mathsf{pcond}\left(q_{\vec{u},m,l}, \Psi_{\vec{u},m}(v,l), \mathsf{condext}(\Psi_{\vec{u},m}(v,l),m,l)\right) . \end{aligned}$$

The following two claims show that this functional is designed to yield extensions of $\overline{U_m}$. In particular, $\Psi_{\vec{u},m}(v,l)$ is meant to add $v$ to the value of the application of the $l$-th progressive approximation of $\overline{U_m}$, to $\vec{u}$, whenever possible.

**Claim 1.** *It is* $\Psi_{\vec{u},x}(\bot,l) = \overline{U_{x,l}}(\vec{u})$.

*Proof.* For $x \notin G_{\mathbb{N}}$ it is obvious. For $m \in G_{\mathbb{N}}$, we proceed by induction on $l$. For $l = 0$, it is obvious. For $l+1$, we reason by cases on $q_{\vec{u},m,l}$:

- If $q_{\vec{u},m,l} = \mathtt{tt}$ then $\exists_{i=1}^{p} u_i \not\asymp_{\rho_i} U_{f_i(m,l)}$, so

  $$\overline{(U_{f_1(m,l)},\ldots,U_{f_p(m,l)},b_{g(m,l)})}(\vec{u}) = \bot ,$$

  which yields $\overline{U_{m,l+1}}(\vec{u}) = \overline{U_{m,l}}(\vec{u})$. On the other hand,

  $$\Psi_{\vec{u},m}(\bot,l+1) = \Psi_{\vec{u},m}(\bot,l) \overset{\text{IH}}{=} \overline{U_{m,l}}(\vec{u}) .$$

- If $q_{\vec{u},m,l} = \mathtt{ff}$ then $\forall_{i=1}^{p} u_i \succ_{\rho_i} U_{f_i(m,l)}$, so

$$\overline{(U_{f_1(m,l)}, \ldots, U_{f_p(m,l)}, b_{g(m,l)})}(\vec{u}) = \overline{b_{g(m,l)}} \ ,$$

  which gives $\overline{U_{m,l+1}}(\vec{u}) = \overline{b_{g(m,l)}}$, due to (1.1) and Proposition 1.36(3). On the other hand,

$$\begin{aligned}\Psi_{\vec{u},m}(\bot, l+1) &= \mathsf{condext}(\Psi_{\vec{u},m}(\bot, l), m, l)\\ &\overset{\text{IH}}{=} \mathsf{condext}(\overline{U_{m,l}}(\vec{u}), m, l)\\ &= \overline{b_{k(m,l)}} \ .\end{aligned}$$

  For the last step: for $b_{g(m,l)} \in \overline{U_{m,l}}(\vec{u})$ it is $\overline{U_{m,l}}(\vec{u}) = \overline{b_{g(m,l)}}$ by (1.1), while for $b_{g(m,l)} \notin \overline{U_{m,l}}(\vec{u})$ it is immediate.

- Finally, if $q_{\vec{u},m,l} = \bot$ then $\exists_{i=1}^{p} u_i \not\succ_{\rho_i} U_{f_i(m,l)}$ and $\forall_{i=1}^{p} u_i \asymp_{\rho_i} U_{f_i(m,l)}$, so

$$\overline{(U_{f_1(m,l)}, \ldots, U_{f_p(m,l)}, b_{g(m,l)})}(\vec{u}) = \bot \ ,$$

  which yields $\overline{U_{m,l+1}}(\vec{u}) = \overline{U_{m,l}}(\vec{u})$. On the other hand,

$$\begin{aligned}\Psi_{\vec{u},m}(\bot, l+1) &= \Psi_{\vec{u},m}(\bot, l) \cap \mathsf{condext}(\Psi_{\vec{u},m}(\bot, l), m, l)\\ &\overset{\text{IH}}{=} \overline{U_{m,l}}(\vec{u}) \cap \mathsf{condext}(\overline{U_{m,l}}(\vec{u}), m, l)\\ &= \overline{U_{m,l}}(\vec{u}) \ .\end{aligned}$$

  For the last step: If $b_{g(m,l)} \in \overline{U_{m,l}}(\vec{u})$, the result is immediate. If $b_{g(m,l)} \notin \overline{U_{m,l}}(\vec{u})$, then let $\overline{U_{m,l}}(\vec{u}) = \overline{b_{g(m,l')}}$, for some $l' < l$ (if $\overline{U_{m,l}}(\vec{u}) = \bot$ there is nothing to show). By (1.1) it is $\overline{b_{g(m,l)}} \rhd \overline{b_{g(m,l')}}$. Since $\forall_{i=1}^{p} u_i \asymp_{\rho_i} U_{f_i(m,l)}$ and $\forall_{i=1}^{p} u_i \succ_{\rho_i} U_{f_i(m,l')}$, by the propagation of consistency in each $\rho_i$ we have $\forall_{i=1}^{p} U_{f_i(m,l)} \asymp_{\rho_i} U_{f_i(m,l')}$; this, by the definition of consistency in function spaces, yields $b_{g(m,l)} \asymp_{\mathbb{N}} b_{g(m,l')}$; by Proposition 1.36(3) we get $b_{g(m,l)} \succ_{\mathbb{N}} b_{g(m,l')}$, and so, $\overline{b_{k(m,l')}} \cap \overline{b_{k(m,l)}} = \overline{b_{k(m,l')}} = \overline{U_{m,l}}(\vec{u})$. □

**Claim 2.** *For a given $v \in \mathsf{Ide}_{\mathbb{N}}$, suppose that $q_{\vec{u},m,l} = \mathtt{ff} \to v \succ_{\mathbb{N}} \overline{b_{g(m,l)}}$ and $q_{\vec{u},m,l} = \bot \to \overline{b_{g(m,l)}} \asymp_{\mathbb{N}} v$. Then $\forall_{l<h(m)} \Psi_{\vec{u},m}(v, l) = v$.*

*Proof.* We proceed again by induction on $l$. For $l = 0$ it is trivial. For $l+1$, we reason by cases on $q_{\vec{u},m,l}$.

- If $q_{\vec{u},m,l} = \mathtt{tt}$ then $\Psi_{\vec{u},m}(v, l+1) = \Psi_{\vec{u},m}(v, l) \overset{\text{IH}}{=} v$.

- If $q_{\vec{u},m,l} = \mathtt{ff}$ then $\Psi_{\vec{u},m}(v, l+1) = \mathsf{condext}(\Psi_{\vec{u},m}(v, l), m, l) \overset{\text{IH}}{=} \mathsf{condext}(v, m, l) \overset{\text{H}}{=} v$.

- If $q_{\vec{u},m,l} = \bot$ then $\Psi_{\vec{u},m}(v, l+1) = \Psi_{\vec{u},m}(v, l) \cap \mathsf{condext}(\Psi_{\vec{u},m}(v, l), m, l) \overset{\text{IH}}{=} v \cap \mathsf{condext}(v, m, l)$. In case $b_{g(m,l)} \in v$ we're done; in case $b_{g(m,l)} \notin v$, by hypothesis and comparability, $\overline{b_{g(m,l)}} \succ_{\mathbb{N}} v$, so $\overline{b_{g(m,l)}} \cap v = v$.

So in all cases the claim holds. □

Now let

$$\mathsf{en}(m,x)(\vec{u}) :\!= \Psi_{\vec{u},m}(\Psi_{\vec{u},x}(\bot,\overline{h}(x)),h(m)) .$$

We show that the desired properties of $\mathsf{en}$ hold: For the first one, suppose that $x \notin G_{\mathbb{N}}$; we have

$$\begin{aligned} \mathsf{en}(m,x)(\vec{u}) &= \Psi_{\vec{u},m}(\Psi_{\vec{u},x}(\bot,\overline{h}(x)),h(m)) \\ &\overset{\text{def}}{=} \Psi_{\vec{u},m}(\bot,h(m)) \\ &\overset{\text{Cl. 1}}{=} \overline{U_{m,h(m)}}(\vec{u}) \\ &= \overline{U_m}(\vec{u}) . \end{aligned}$$

For the second one, suppose that $U_n \succ_{\rho_1\to\cdots\to\rho_p\to\mathbb{N}} U_m$, for $n \in G_{\mathbb{N}}$; we have:

$$\begin{aligned} \mathsf{en}(m,n)(\vec{u}) &= \Psi_{\vec{u},m}(\Psi_{\vec{u},n}(\bot,h(n)),h(m)) \\ &\overset{\text{Cl. 1}}{=} \Psi_{\vec{u},m}(\overline{U_{n,h(n)}}(\vec{u}),h(m)) \\ &= \Psi_{\vec{u},m}(\overline{U_n}(\vec{u}),h(m)) \\ &\overset{\text{Cl. 2}}{=} \overline{U_n}(\vec{u}) . \end{aligned}$$

Claim 2 that we appealed to in the last step indeed holds: it is $\overline{U_n}(\vec{u}) \succ_{\mathbb{N}} \overline{U_m}(\vec{u})$ by Proposition 1.17; let $l < h(m)$; if $q_{\vec{u},m,l} = \mathsf{ff}$, then

$$\bigvee_{i=1}^{p} u_i \succ_{\rho_i} U_{f_i(m,l)} \Rightarrow \overline{U_{m,l}}(\vec{u}) = \overline{b_{g(m,l)}} \Rightarrow \overline{U_m}(\vec{u}) \succ_{\mathbb{N}} b_{g(m,l)} ,$$

while if $q_{\vec{u},m,l} = \bot$, then, by hypothesis,

$$\begin{aligned} &\bigvee_{i=1}^{p} u_i \asymp_{\rho_i} U_{f_i(m,l)} \wedge \exists_{l'<h(n)} \left( \bigvee_{i=1}^{p} U_{f_i(m,l)} \succ_{\rho_i} U_{f_i(n,l')} \wedge b_{g(n,l')} \succ_{\mathbb{N}} b_{g(m,l)} \right) \\ &\overset{\text{prpg}}{\Rightarrow} \exists_{l'<h(n)} \left( \bigvee_{i=1}^{p} u_i \asymp_{\rho_i} U_{f_i(n,l')} \wedge b_{g(n,l')} \succ_{\mathbb{N}} b_{g(m,l)} \right) \\ &\overset{\text{def}}{\Rightarrow} \exists_{l'<h(n)} \left( \overline{U_n}(\vec{u}) \asymp_{\mathbb{N}} b_{g(n,l')} \wedge b_{g(n,l')} \succ_{\mathbb{N}} b_{g(m,l)} \right) \\ &\overset{\text{prpg}}{\Rightarrow} \overline{U_n}(\vec{u}) \asymp_{\mathbb{N}} b_{g(m,l)} . \end{aligned}$$

Now we deal with the *inconsistency functionals*. Since we are working with atomic-coherent structures, we can start by expressing inconsistency between an ideal and a token (again, expressed as a singleton neighborhood). For all types $\rho$, $\sigma$, fix an enumeration $\{(U_{f(i)},b_{g(i)})\}_{i\in G_{\mathbb{N}}} \subseteq \mathsf{Con}_{\rho\to\sigma}$ of $T_{\rho\to\sigma}$, through primitive recursive functions $f$ and $g$. We need a term for a functional with the following behavior:

$$\mathsf{ic}_{\rho\to\sigma}(u,i) = \begin{cases} \mathsf{tt} & u \not\asymp_{\rho\to\sigma} (U_{f(i)},b_{g(i)}) , \\ \mathsf{ff} & u \succ_{\rho\to\sigma} (U_{f(i)},b_{g(i)}) . \end{cases}$$

We have

$$\begin{aligned} \mathsf{ic}_{\rho\to\sigma}(u,i) = \mathsf{tt} &\overset{\text{def}}{\Leftrightarrow} u \not\asymp_{\rho\to\sigma} (U_{f(i)},b_{g(i)}) \\ &\overset{(*)}{\Leftrightarrow} \exists_{n'} \left( U_{f(n')} \succ_{\rho} U_{f(i)} \wedge u(\overline{U_{f(n')}}) \not\asymp_{\sigma} b_{g(i)} \right) \\ &\overset{(*)}{\Leftrightarrow} \exists_{n'} \left( u(\mathsf{en}_{\rho}(f(i),f(n'))) \not\asymp_{\sigma} b_{g(i)} \right) \\ &\overset{\text{def}}{\Leftrightarrow} \exists_{n'} \left( \mathsf{ic}_{\sigma}(u(\mathsf{en}_{\rho}(f(i),f(n'))),i) = \mathsf{tt} \right) , \end{aligned}$$

where for $(*)$'s we let $U_{f(n')} := U \cup U_{f(i)}$, for $U \asymp_\rho U_{f(i)}$; furthermore

$$\begin{aligned}\mathsf{ic}_{\rho\to\sigma}(u,i) = \mathsf{ff} &\overset{\text{def}}{\Leftrightarrow} u \succ_{\rho\to\sigma} (U_{f(i)}, b_{g(i)}) \\ &\Leftrightarrow u(\overline{U_{f(i)}}) \succ_\sigma b_{g(i)} \\ &\overset{\text{def}}{\Leftrightarrow} u(\mathsf{en}_\rho(f(i),\bot)) \succ_\sigma b_{g(i)} \\ &\overset{\text{def}}{\Leftrightarrow} \mathsf{ic}_\sigma(u(\mathsf{en}_\rho(f(i),\bot)),i) = \mathsf{ff}\ ,\end{aligned}$$

so, writing $n$ for $f(n')$, define

$$\mathsf{ic}_{\rho\to\sigma}(u,i) := \mathsf{exist}(\lambda_n\ \mathsf{ic}_\sigma(u(\mathsf{en}_\rho(f(i),n)),i))\ .$$

We are now able to express inconsistency between an ideal and a neighborhood $U_n = \{b_{j(n,l)}\}_{l<h(n)}$, by letting

$$\mathsf{incns}_\rho(u,n) := \underset{l<h(n)}{\mathsf{OR}}\ \mathsf{ic}_\rho(u,j(n,l)) = \begin{cases}\mathsf{tt}\ , & \exists_{l<h(n)}\,\mathsf{ic}_\rho(u,j(n,l)) = \mathsf{tt}\ , \\ \mathsf{ff}\ , & \forall_{l<h(n)}\,\mathsf{ic}_\rho(u,j(n,l)) = \mathsf{ff}\ ,\end{cases}$$

which is exactly what we were after. □

## Definability

Call a partial continuous functional $u \in \mathsf{Ide}_{\rho_1\to\cdots\to\rho_p\to\mathbb{N}}$ *recursive in* pcond*,* exist *and* condext if it can be defined explicitly for all arguments $v_1,\ldots,v_p$ by an equation

$$u(v_1,\ldots,v_p) = t(v_1,\ldots,v_p)\ ,$$

where $t$ is a simply-typed lambda term built up from variables $v_1,\ldots,v_p$, $\lambda$-abstraction, application, algebra constructors, fixed point functionals, parallel conditional functionals, existential functionals, and conditional extension functionals.

**Theorem 1.46** (Definability)**.** *A partial continuous functional of type $\rho\to\mathbb{N}$ over $\mathbb{N}$ and $\mathbb{B}$ is computable if and only if it is recursive in* pcond*,* exist*, and* condext*.*

*Proof.* Let $\Omega : \rho_1 \to \cdots \to \rho_p \to \mathbb{N}$ be a computable functional. It will be represented as a primitive recursively enumerable set of tokens, that is,

$$\Omega = \{(U_{f_1(n)},\ldots,U_{f_p(n)},b_{g(n)})\}_{n\in G_\mathbb{N}}\ ,$$

where, for each $i=1,\ldots,p$, $U_{f_i(n)}$ follows an enumeration of $\mathsf{Con}_{\rho_i}$, $b_{g(n)}$ follows an enumeration of $\mathsf{Con}_\mathbb{N}$, and $f_1,\ldots,f_p,g$ are fixed primitive recursive functions.

For arbitrary $\vec{u} \in \mathsf{Ide}_{\rho_1\to\cdots\to\rho_p}$ and $v \in \mathsf{Ide}_\mathbb{N}$, define an *argument test* by

$$q_{\vec{u},n} := \underset{i=1}{\overset{p}{\mathsf{OR}}}\,\mathsf{incns}_{\rho_i}(u_i,f_i(n)) = \begin{cases}\mathsf{tt}\ , & \exists_{i=1}^p\, u_i \not\succ_{\rho_i} U_{f_i(n)}\ , \\ \mathsf{ff}\ , & \forall_{i=1}^p\, u_i \succ_{\rho_i} U_{f_i(n)}\ , \\ \bot\ , & \text{otherwise}\ ,\end{cases}$$

and use $\mathsf{condext}(v,n)$ as a *value test*. Define a functional $\omega : \rho_1 \to \cdots \to \rho_p \to (\mathbb{N}\to\mathbb{N}) \to G_\mathbb{N} \to \mathbb{N}$ by

$$\omega_{\vec{u}}(\psi)(n) := \mathsf{pcond}\left(q_{\vec{u},n},\psi(n+1),\mathsf{condext}(\psi(n+1),n)\right)\ .$$

We show that $\Omega(\vec{u}) = \mathsf{Y}(\omega_{\vec{u}})(0)$. In particular, we show that both recursion on $\omega_{\vec{u}}$ at 0 and $\Omega(\vec{u})$ entail the very same information, that is,

$$\forall_{b \in T_{\mathbb{N}}} (\Omega(\vec{u}) \succ_{\mathbb{N}} b \leftrightarrow \mathsf{Y}(\omega_{\vec{u}})(0) \succ_{\mathbb{N}} b) \ .$$

For the right direction, suppose that there exists a token $b \in T_{\mathbb{N}}$ such that $\Omega(\vec{u}) \succ_{\mathbb{N}} b$. This means that

$$\exists_{n \in G_{\mathbb{N}}} \left( \forall_{i=1}^{p} u_i \succ_{\rho_i} U_{f_i(n)} \wedge b_{g(n)} \succ_{\mathbb{N}} b \right) \ . \tag{1.2}$$

We claim that

$$\forall_{k \leq n} \omega_{\vec{u}}^{k+1}(\lambda_x \perp)(n-k) \succ_{\mathbb{N}} b_{g(n)} \tag{1.3}$$

and we prove it by induction on $k$. For $k = 0$:

$$\omega_{\vec{u}}(\lambda_x \perp)(n) \overset{(1.2)}{=} \mathsf{condext}((\lambda_x \perp)(n+1), n) = \mathsf{condext}(\perp, n) = \overline{b_{g(n)}} \succ_{\mathbb{N}} b_{g(n)} \ .$$

For brevity, let $v := \omega_{\vec{u}}^{k+2}(\lambda_x \perp)(n-k-1)$ and $v_0 := \omega_{\vec{u}}^{k+1}(\lambda_x \perp)(n-k)$. The induction hypothesis is that $v_0 \succ_{\mathbb{N}} b_{g(n)}$. For $k+1$ we have:

$$v = \omega_{\vec{u}}(\omega_{\vec{u}}^{k+1}(\lambda_x \perp))(n-k-1) = \mathsf{pcond}\left(q_{\vec{u},n-k-1}, v_0, \mathsf{condext}(v_0, n-k-1)\right) \ .$$

We argue by cases on the argument test:

- If $q_{\vec{u},n-k-1} = \mathsf{tt}$, then $v = v_0 \succ_{\mathbb{N}} b_{g(n)}$ by the induction hypothesis.
- If $q_{\vec{u},n-k-1} = \mathsf{ff}$, then, for $b_{g(n-k-1)} \in v_0$, it is $v = \mathsf{condext}(v_0, n-k-1) = v_0 \succ_{\mathbb{N}} b_{g(n)}$, by the induction hypothesis. For $b_{g(n-k-1)} \notin v_0$, it is $v = \overline{b_{g(n-k-1)}}$. By the definition of $\mathsf{incns}$ we have

$$\begin{aligned} &\forall_{i=1}^{p} u_i \succ_{\rho_i} U_{f_i(n-k-1)} \\ &\overset{(\star)}{\Rightarrow} \forall_{i=1}^{p} U_{f_i(n)} \asymp_{\rho_i} U_{f_i(n-k-1)} \\ &\overset{\text{def}}{\Rightarrow} b_{g(n)} \asymp_{\mathbb{N}} b_{g(n-k-1)} \\ &\overset{(\star\star)}{\Rightarrow} b_{g(n)} \succ_{\mathbb{N}} b_{g(n-k-1)} \vee b_{g(n-k-1)} \succ_{\mathbb{N}} b_{g(n)} \\ &\overset{(\star\star\star)}{\Rightarrow} b_{g(n-k-1)} \succ_{\mathbb{N}} b_{g(n)} \ , \end{aligned}$$

  where: $(\star)$ holds by (1.2) and Proposition 1.2; $(\star\star)$ holds by Proposition 1.35; and $(\star\star\star)$ holds since if it was $b_{g(n)} \succ_{\mathbb{N}} b_{g(n-k-1)}$, then the induction hypothesis would yield $v_0 \succ_{\mathbb{N}} b_{g(n-k-1)}$, which contradicts the running hypothesis. So $v \succ_{\mathbb{N}} b_{g(n)}$.
- If it is $q_{\vec{u},n-k-1} = \perp$, then, for $b_{g(n-k-1)} \in v_0$, it is $v = v_0 \cap v_0 = v_0 \succ_{\mathbb{N}} b_{g(n)}$, by the induction hypothesis. For $b_{g(n-k-1)} \notin v_0$, it is $v = v_0 \cap \overline{b_{g(n-k-1)}}$. By the definition of $\mathsf{incns}$ we have

$$\begin{aligned} &\forall_{i=1}^{p} u_i \asymp_{\rho_i} U_{f_i(n-k-1)} \\ &\overset{(\star)}{\Rightarrow} U_{f_i(n-k-1)} \asymp_{\rho_i} U_{f_i(n)} \\ &\overset{\text{def}}{\Rightarrow} b_{g(n-k-1)} \asymp_{\mathbb{N}} b_{g(n)} \\ &\overset{(\star\star)}{\Rightarrow} b_{g(n)} \succ_{\mathbb{N}} b_{g(n-k-1)} \vee b_{g(n-k-1)} \succ_{\mathbb{N}} b_{g(n)} \\ &\overset{(\star\star\star)}{\Rightarrow} b_{g(n-k-1)} \succ_{\mathbb{N}} b_{g(n)} \ , \end{aligned}$$

where: ($\star$) holds by (1.2) and propagation of consistency; ($\star\star$) holds by Proposition 1.35; and ($\star\star\star$) holds since if it was $b_{g(n)} \succ_{\mathbb{N}} b_{g(n-k-1)}$, then the induction hypothesis would yield $v_0 \succ_{\mathbb{N}} b_{g(n-k-1)}$, which contradicts the running hypothesis. So $v \succ_{\mathbb{N}} b_{g(n)}$.

We proved that $v \succ_{\mathbb{N}} b_{g(n)}$ in all cases. Letting $k = n$ in statement (1.3), we get

$$\omega_{\vec{u}}^{n+1}(\lambda_x \bot)(0) \succ_{\mathbb{N}} b_{g(n)} \Rightarrow \mathsf{Y}(\omega_{\vec{u}})(0) \succ_{\mathbb{N}} b \ ,$$

by (1.2) and the definition of the fixed point functional. So, $\mathsf{Y}(\omega_{\vec{u}})(0)$ entails all information of $\Omega(\vec{u})$.

Conversely, suppose that

$$\mathsf{Y}(\omega_{\vec{u}})(0) \succ_{\mathbb{N}} b \ . \tag{1.4}$$

We claim that

$$\underset{m \in G_{\mathbb{N}}}{\forall} \left( \omega_{\vec{u}}^{n+1}(\lambda_x \bot)(m) \succ_{\mathbb{N}} b \to \overset{m+n}{\underset{l=m}{\exists}} \left( \overset{p}{\underset{i=1}{\forall}} u_i \succ_{\rho_i} U_{f_i(l)} \wedge b_{g(l)} \succ_{\mathbb{N}} b \right) \right) \ , \tag{1.5}$$

and we prove it by induction on $n$. For $n = 0$:

$$\begin{aligned}
&\omega(\lambda_x \bot)(m) \succ_{\mathbb{N}} b \\
&\quad \overset{\text{def.}}{\Leftrightarrow} \mathsf{pcond}(q_{\vec{u},m}, (\lambda_x \bot)(m+1), \mathsf{condext}((\lambda_x \bot)(m+1), m)) \succ_{\mathbb{N}} b \\
&\quad \overset{\text{def.}}{\Leftrightarrow} \mathsf{pcond}(q_{\vec{u},m}, \bot, \mathsf{condext}(\bot, m)) \succ_{\mathbb{N}} b \\
&\quad \Rightarrow q_{\vec{u},m} = \mathsf{ff} \wedge \mathsf{condext}(\bot, m) = b_{g(m)} \succ_{\mathbb{N}} b \\
&\quad \overset{\text{def.}}{\Leftrightarrow} \overset{p}{\underset{i=1}{\forall}} u_i \succ_{\rho_i} U_{f_i(m)} \wedge b_{g(m)} \succ_{\mathbb{N}} b \ ,
\end{aligned}$$

so $l := m$ does the job. For $n+1$ we have

$$\begin{aligned}
&\omega(\omega^{n+1}(\lambda_x \bot))(m) \succ_{\mathbb{N}} b \\
&\quad \overset{\text{def.}}{\Leftrightarrow} \mathsf{pcond}(q_{\vec{u},m}, \omega^{n+1}(\lambda_x \bot)(m+1), \mathsf{condext}(\omega^{n+1}(\lambda_x \bot)(m+1), m)) \succ_{\mathbb{N}} b \\
&\quad \overset{\text{def.}}{\Leftrightarrow} \left( q_{\vec{u},m} = \mathsf{tt} \wedge \omega^{n+1}(\lambda_x \bot)(m+1) \succ_{\mathbb{N}} b \right) \vee \left( q_{\vec{u},m} = \mathsf{ff} \wedge b_{g(m)} \succ_{\mathbb{N}} b \right) \ ,
\end{aligned}$$

which is granted by the induction hypothesis. So, $\Omega(\vec{u})$ entails all information of $\mathsf{Y}(\omega_{\vec{u}})(0)$. □

## Comparability properties

Entailment in general—indeed, in most natural instances as well—does nothing more than *pre*order the carrier set. In the proof of Theorem 1.46 however we saw that comparability of tokens (stated in Proposition 1.35) proves of crucial importance. Is it possible to extend the definability result using the same techniques in order to cover algebras with constructors of arbitrary arity?

The answer is no. In this section we explore principles of comparability for an acis in general and then clarify the connection to algebraic coherent information systems. We show that the demand that constructors be at most unary is essential to our proof of Theorem 1.46.

Given an arbitrary acis $\rho$, introduce a *principle of comparability* (or *conditional dichotomy*) for its tokens, its neighborhoods, and its ideals, as follows:

$$\forall_{a,b\in T} (a \asymp b \to a \succ b \vee b \succ a) \ , \tag{PC-T}$$

$$\forall_{U,V\in\mathsf{Con}} (U \asymp V \to U \succ V \vee V \succ U) \ , \tag{PC-N}$$

$$\forall_{u,v\in\mathsf{Ide}} (u \asymp v \to v \subseteq u \vee u \subseteq v) \ . \tag{PC-I}$$

We will readily show that all three principles are equivalent.

For an arbitrary neighborhood $U$, call a *maximum of* $U$, denoted by $\max U$, an element $a \in U$ that entails all elements of $U$, that is,$\forall_{b\in U}\, a \succ b$. If a maximum exists at all it doesn't have to be unique, as $(T,\succ)$ is in general just a preorder. Assuming PC-T though, entailment would order any nonempty neighborhood totally, so a maximum element would exist and be unique. So we have the following.

**Proposition 1.47.** *If principle PC-T holds, then for any nonempty* $U \in \mathsf{Con}$*,* $\max U$ *exists and is unique.*

**Proposition 1.48.** *Principles PC-T and PC-N are equivalent.*

*Proof.* Assume PC-T and let $U,V \in \mathsf{Con}$. It is

$$\begin{aligned} U \asymp V &\overset{\text{def}}{\Leftrightarrow} \forall_{a\in U, b\in V}\ a \asymp b \\ &\overset{\text{PC-T}}{\Rightarrow} \forall_{a\in U, b\in V} (a \succ b \vee b \succ a) \\ &\overset{\text{P. 1.47}}{\Rightarrow} \max U \succ \max V \vee \max V \succ \max U \\ &\overset{\text{Def.}}{\Rightarrow} U \succ V \vee V \succ U \ , \end{aligned}$$

which proves PC-N. Conversely, assume PC-N and let $a,b \in T$. It is

$$a \asymp b \Leftrightarrow \{a\} \asymp \{b\} \overset{\text{PC-N}}{\Rightarrow} \{a\} \succ \{b\} \vee \{b\} \succ \{a\} \Leftrightarrow a \succ b \vee b \succ a \ ,$$

which proves PC-T. □

**Proposition 1.49.** *Principles PC-T and PC-I must be equivalent.*

*Proof by contradiction.* Assume PC-T and let $u,v \in \mathsf{Ide}$, for which $u \asymp v$. With no loss of generality, assume $u$ and $v$ to be distinct, so there is either an $a_0 \in T$ such that $u \ni a_0 \notin v$ or a $b_0 \in T$ such that $u \not\ni b_0 \in v$. In the first case we have

$$\begin{aligned} u \asymp v &\overset{\text{def}}{\Leftrightarrow} \forall_{a\in u, b\in v}\ a \asymp b \\ &\Rightarrow \forall_{b\in v}\ a_0 \asymp b \\ &\overset{\text{PC-T}}{\Rightarrow} \forall_{b\in v} (a_0 \succ b \vee b \succ a_0) \\ &\overset{(\star)}{\Rightarrow} \forall_{b\in v}\ a_0 \succ b \ , \end{aligned}$$

so $v \subseteq u$. For step $(\star)$, should there exist some $b \in v$ such that $b \succ a_0$, then, by the definition of an ideal, we would have $a_0 \in v$, against our supposition. In the second case we similarly have $u \subseteq v$. This proves PC-I.

Conversely, assume PC-I and let $a,b \in T$. Then

$$a \asymp b \Rightarrow \overline{a} \asymp \overline{b} \overset{\text{PC-I}}{\Rightarrow} \overline{b} \subseteq \overline{a} \vee \overline{a} \subseteq \overline{b} \Rightarrow a \succ b \vee b \succ a \ ,$$

which proves PC-T. □

The sole presence of at most unary constructors in an algebra yields comparability. In fact, comparability *characterizes* algebras with at most unary constructors within our type system.

**Theorem 1.50** (Comparability)**.** *An acis induced by an algebra has at most unary constructors if and only if it satisfies PC-T.*

*Proof.* Let $\alpha$ have at most unary constructors; we perform induction on the length of the tokens. Let $C_i$ and $C_j$ be two constructors of $\alpha$, both nullary. Then

$$C_i \asymp C_j \Rightarrow i = j \ .$$

Let $Ca$ and $Cb$ be two tokens. We have

$$Ca \asymp Cb \overset{\text{def.}}{\Leftrightarrow} a \asymp b \overset{\text{IH}}{\Rightarrow} a \succ b \vee b \succ a \overset{\text{def.}}{\Leftrightarrow} Ca \succ Cb \vee Cb \succ Ca \ .$$

This proves PC-T for all tokens of $T_\alpha$.

Conversely, let $C$ be a constructor of arity $r > 1$ in $\alpha$; for a nullary constructor 0, we have tokens of the form $a = C\vec{a}0\vec{b}{*}\vec{c}$ and $b = C\vec{a}{*}\vec{b}0\vec{c}$, for which:

$$a \asymp b \wedge a \not\succ b \wedge b \not\succ a \ ,$$

so comparability fails. □

Based on this observation we may think of non-superunary algebras as *comparability algebras*.

## 1.5 Notes

### On binary entailment

The notion of *atomic* information systems was introduced by Helmut Schwichtenberg in [47], after a suggestion by Ulrich Berger, as a particularly simple though far-reaching special case of Scott information systems. As far as we know, apart from this thesis, the structure has not been particularly studied in its own right, though an interesting exception is its use by Bucciarelli, Carraro, Ehrhard and Salibra as a *model for intuitionistic linear logic* in [8].

### Normal forms of neighborhoods

Normal forms for neighborhoods in *flat* information systems were already treated in [44]. The maximal form that we describe in section 1.3 (page 32) actually generalizes the normal form that is described there. We will return to the subject of normal and canonical forms of neighborhoods more than once in Chapter 2.

### The conditional extension functional

In page 38 we define the conditional extension functional condext, which also features in [18] as *continuous union* and denoted by $\cup_\sharp$. This functional is actually decidable. Algorithms for this conditional extension are

$$\mathsf{condext}(v, \sharp b) = \mathsf{cond}\left(\mathsf{implies}(v \triangleright \overline{b}, \overline{b} \in G_{\mathbb{N}}), \overline{b}, v\right)$$

and

$$\mathsf{condext}(v, \sharp S^n a) = \mathsf{cond}\left(\mathsf{or}(\overline{a} \in G_{\mathbb{N}}, v \dot{-} (n \dot{-} 1) = 0), \overline{S^n a}, v\right) ,$$

where $\sharp b$ denotes the code of token $b$, $a$ is either $\varnothing$ or 0, and $\dot{-}$ is the standard modified subtraction. Notice that "$\in G_{\mathbb{N}}$" expresses a *totality test* in the first case and a *zero test* in the second—the latter is due to Simon Huber and Florian Ranzi.

### Plotkin's definability theorem

Our Theorem 1.46 originates in Gordon Plotkin's original definability theorem for the language PCF in his seminal paper [35] (the result there is listed as Theorem 5.1). Helmut Schwichtenberg in [46] recounts Plotkin's result using *flat* information systems for the denotational semantics. Here we adapt the former to the setting of *non-flat information systems*, which seems to be more appropriate for a development of a *theory of higher-type computability with approximations*.

Our proof departs from the one in [46] in two main points, both due to the non-flatness of the setting that we adopted: (a) it makes heavy and nontrivial use of comparability of base-type tokens (see Proposition 1.35 and section 1.4)—recall that comparability for non-partial (that is, non-bottom) base-type tokens in flat systems reduces to identity; (b) it uses an *extra* "parallel" functional, namely, *conditional extension*.

Our approach was presented for the first time at the CiE 2008 conference. For a formalization of the result in the theory $\mathrm{TCF}^+$, see [18].

### The Coquand counterexample

The structure of acises begins to show fatigue with Theorem 1.50: comparability, a basic tool for our argument of definability, fails in acises induced by algebras with constructors of superunary arities.

There is another problem with atomic systems, not so much of a technical but rather of a conceptual nature. In an algebra $\alpha$ with a binary constructor $B : \alpha \to \alpha \to \alpha$ and a nullary $0 : \alpha$, together with the partiality pseudo-constructor $*$, it is natural to expect that the *combined* information $B0*$ and $B{*}0$ should entail the information $B00$; this can not be explained atomically in a direct manner, since neither of $B0*$ and $B{*}0$ can afford the information $B00$ on its own:

$$\{B0*, B{*}0\} \not\vdash B00 . \qquad (\star)$$

We refer to this as the *Coquand counterexample to atomicity*, as it was initially pointed out by Thierry Coquand in a session of the MAP 2006 summer school, where acises were presented. We will invoke it again and again—see in particular pages 106 and 113.

The observation, simple as it is, proves fundamentally crucial. It points to the limits of the applicability of acises, since it shows that the problem $(\star)$ propagates to the case of functional information, that is, information on the graph of a function: this information comes as a pair of *combined information on the argument* of the intended function and *simple information on the respective value* of the intended function; consequently, the problem propagates further to the case of infinitary algebras, since they involve constructors with nontrivial recursive arguments, as in $\cup : (\mathbb{N} \to \mathbb{O}) \to \mathbb{O}$ (for the algebra of ordinal numbers see page 2). The counterexample really marks the transition from the realm of atomicity to the realm of non-atomicity.

In fact, we embark on non-atomic information systems in the next chapter based on a counter-observation to Coquand's counterexample: the reason that we would rather have the pair $\{B0*, B*0\}$ *entail* $B00$, is that the information *in every position* of the latter is atomically entailed by the former; that is, despite the apparent non-atomicity, the entailment in $(\star)$ features *implicit* atomicity, and this lends itself to elaboration.

### Outlook

In view of the previous note, maybe the most reasonable question to ask is how to approach the definability requirement for types over algebras where superunary constructors are present. As already noted, this calls for different and more advanced techniques. We move in this direction in the context of the more general coherent information systems in the next chapter.

# Chapter 2

# Matrices and coherent information systems

In Chapter 1 we already alluded to the inadequacy of acises. In order to embrace more complicated—but nonetheless indispensable—kinds of algebras, namely, algebras with superunary constructors and infinitary algebras, we need to move to a non-atomic setting. But how?

A way to do it would be top-down: to consider Scott information systems and impose the property of coherence upon them, as we do in Chapter 3. This is a simple way—it is actually the way that has governed the relevant research for the most part up to now—but it is also a rather naive way: in doing so we miss the close and intricate relation between algebraic atomicity and algebraic nonatomicity. This relation can be best revealed in a bottom-up fashion, that is, starting from acises and arriving at coherent information systems.

## Non-atomic coherent information systems

To justify better the bottom-up course of this chapter, let us first see what a direct definition of an algebraic coherent system looks like, and what it might be hiding. The standard definition of a *coherent Scott information system* $(T, \mathsf{Con}, \vdash)$, for $T$ a countable set, $\mathsf{Con}$ a collection of finite subsets of $T$, and $\vdash$ a relation of the sort $\mathsf{Con} \times T$, demands that the following axioms be fulfilled (in section 3.1 we go into more detail concerning information systems in general, but for now the definitions suffice).

1. consistency is reflexive, that is, $\{a\} \in \mathsf{Con}$, for all $a \in T$,
2. consistency is closed under subsets, that is, if $U \in \mathsf{Con}$ and $V \subseteq U$ then $V \in \mathsf{Con}$,
3. consistency is coherent, that is, if $\{a, a'\} \in \mathsf{Con}$, for all $a, a' \in U$, then $U \in \mathsf{Con}$,
4. entailment is reflexive, that is, if $a \in U$ then $U \vdash a$,
5. entailment is transitive, that is, if $U \vdash V$ and $V \vdash c$ then $U \vdash c$, and
6. consistency propagates through entailment, that is, if $U \in \mathsf{Con}$ and $U \vdash b$ then $U \cup \{b\} \in \mathsf{Con}$,

where $U \vdash V$ is as usual a shorthand for $\forall_{b \in V} U \vdash b$.

To each algebra $\alpha$ given by constructors (see page 1), we assign a nullary *partiality pseudo-constructor* $*_\alpha$; we may also say *proper constructor* when we want to stress that we don't mean $*_\alpha$. Let $\alpha$ be such an algebra (for simplicity, we assume it is non-parametric). For any constructor $C$ of arity $(\vec{\rho}_1 \to \alpha, \ldots, \vec{\rho}_r \to \alpha)$ we define the following:

- if $a_1 \in T_{\vec{\rho}_1 \to \alpha}, \ldots, a_r \in T_{\vec{\rho}_r \to \alpha}$, then
$$Ca_1 \cdots a_r \in T_\alpha \ ;$$
moreover, $*_\alpha \in T_\alpha$;
- if $\{a_{11}, \ldots, a_{1l}\} \in \mathsf{Con}_{\vec{\rho}_1 \to \alpha}, \ldots, \{a_{r1}, \ldots, a_{rl}\} \in \mathsf{Con}_{\vec{\rho}_r \to \alpha}$, then
$$\{Ca_{11} \cdots a_{r1}, \ldots, Ca_{1l} \cdots a_{rl}\} \in \mathsf{Con}_\alpha \ ;$$
moreover, if $U \in \mathsf{Con}_\alpha$ then $U \cup \{*_\alpha\} \in \mathsf{Con}$;
- if $\{a_{11}, \ldots, a_{1l}\} \vdash_{\vec{\rho}_1 \to \alpha} a_1, \ \ldots, \ \{a_{r1}, \ldots, a_{rl}\} \vdash_{\vec{\rho}_r \to \alpha} a_r$, with $\{a_{11}, \ldots, a_{1l}\} \in \mathsf{Con}_{\vec{\rho}_1 \to \alpha}, \ldots, \{a_{r1}, \ldots, a_{rl}\} \in \mathsf{Con}_{\vec{\rho}_r \to \alpha}$, then
$$\{Ca_{11} \cdots a_{r1}, \ldots, Ca_{1l} \cdots a_{rl}\} \vdash_\alpha Ca_1 \cdots a_r \ ;$$
moreover, if $U \in \mathsf{Con}_\alpha$, then $U \vdash_\alpha *_\alpha$.

One can directly show that these definitions qualify for $(T_\alpha, \mathsf{Con}_\alpha, \vdash_\alpha)$ to form a coherent information system; this will be the *coherent information system induced by* $\alpha$. Note that if $\alpha$ had no proper nullary constructors it would be empty, hence the induced information system would consist solely of partial tokens—here we do not allow this situation.

As for the *function space* $\rho \to \sigma$ of two given information systems $\rho$ and $\sigma$, which models the corresponding higher type, one defines the following[1]:

- if $U \in \mathsf{Con}_\rho$ and $b \in T_\sigma$ then
$$\langle U, b \rangle \in T_{\rho \to \sigma} \ ;$$
- let $U_1, \ldots, U_l \in \mathsf{Con}_\rho$, $b_1, \ldots, b_l \in T_\sigma$, and $J := \{1, \ldots, l\}$; if for all $I \subseteq J$, $\cup_{i \in I} U_i \in \mathsf{Con}_\rho$ implies $\cup_{i \in I} \{b_i\} \in \mathsf{Con}_\sigma$, then
$$\{\langle U_j, b_j \rangle \mid j \in J\} \in \mathsf{Con}_{\rho \to \sigma} \ ;$$
- let $U_1, \ldots, U_l, U \in \mathsf{Con}_\rho$, $b_1, \ldots, b_l, b \in T_\sigma$, and $J := \{1, \ldots, l\}$; if for some $I \subseteq J$, it is both $U \vdash_\rho U_i$ for all $i \in I$ and $\{b_i \mid i \in I\} \vdash_\sigma b$, then
$$\{\langle U_j, b_j \rangle \mid j \in J\} \vdash_{\rho \to \sigma} \langle U, b \rangle \ .$$

The definition of entailment here can be formulated in terms of a *(list) application* between formal neighborhoods: $\{\langle U_1, b_1 \rangle, \ldots, \langle U_l, b_l \rangle\} U := \{b_i \mid U \vdash_\rho U_i, i \in J\}$; so

- if $\{\langle U_1, b_1 \rangle, \ldots, \langle U_l, b_l \rangle\} U \vdash_\sigma b$, then $\{\langle U_1, b_1 \rangle, \ldots, \langle U_l, b_l \rangle\} \vdash_{\rho \to \sigma} \langle U, b \rangle$.

Again, one can directly show that the triple $(T_{\rho \to \sigma}, \mathsf{Con}_{\rho \to \sigma}, \vdash_{\rho \to \sigma})$ is a coherent information system every time $\rho$ and $\sigma$ are coherent information systems (it actually suffices for $\sigma$ to be coherent, see [49, § 6.1.6]).

[1] Due to legibility, in this chapter we choose to write $\langle U, b \rangle$ instead of $(U, b)$ for tokens in the function space.

## Some examples of coherent information systems

***Conjunction-implication information system.*** Call a set of propositional formulas $\{F_1,\dots,F_l\}$ *consistent* if $\vdash F_j$, $j=1,\dots,l$, and say that $\{F_1,\dots,F_l\}$ *entails* $F$ if $\vdash F_1 \wedge \cdots \wedge F_l \to F$. It is easy to see that these definitions build a coherent information system with tokens all valid propositional formulas, which we will call here the *conjunction-implication information system* for propositional calculus.

***Binary trees.*** Let $\mathbb{D}$ be the algebra given by a nullary constructor $0:\mathbb{D}$ and a binary $B:\mathbb{D}\to\mathbb{D}\to\mathbb{D}$. In the coherent information system induced by $\mathbb{D}$ as explained above, one has (we write "$*$" for "$*_{\mathbb{D}}$")

$$\begin{aligned}
&*, 0, B{*}0, B0{*}, B(B({*}{*}))(B0(B00)) \in T_{\mathbb{D}}\ ,\\
&\{B0{*}, B{*}0\}, \{*, B(B0{*}){*}, B{*}(B0{*})\} \in \mathsf{Con}_{\mathbb{D}}\ ,\\
&\{B(B0{*}){*}, B(B{*}0)0\} \vdash_{\mathbb{D}} *, B{*}{*}, B(B00)0\ .
\end{aligned}$$

***Ordinal numbers.*** Having defined the algebra of natural numbers $\mathbb{N}$, the algebra $\mathbb{O}$ of ordinal numbers is given by the constructors $0:\mathbb{O}$, $S:\mathbb{O}\to\mathbb{O}$, and $\cup:(\mathbb{N}\to\mathbb{O})\to\mathbb{O}$. In its coherent information system one has

$$\begin{aligned}
&*_{\mathbb{O}}, 0, SS0, SSS{*_{\mathbb{O}}}, S\cup\langle\{S{*_{\mathbb{N}}}\}, S{*_{\mathbb{O}}}\rangle \in T_{\mathbb{O}}\ ,\\
&\{\cup\langle\{S*\},\cup\langle\{SS0,*\},SS0\rangle\rangle, \cup\langle\{S0\},\cup\langle\{*\},SSS*\rangle\rangle, \cup\langle\{0\},SS0\rangle\} \in \mathsf{Con}_{\mathbb{O}}\ ,\\
&\{\cup\langle\{SS*\},\cup\langle\{S0\},S0\rangle\rangle, \cup\langle\{S*\},\cup\langle\{SS*\},SS0\rangle\rangle\} \vdash_{\mathbb{O}} \cup\langle\{S0\},\cup\langle\{SS0\},SS*\rangle\rangle\ .
\end{aligned}$$

## Implicit atomicity

The claim of this chapter is that for any algebra that we may consider, entailment in its induced information system will implicitly feature atomicity—which is already explicit in the case of non-superunary algebras.

To see that, consider firstly a finitary algebra $\alpha$ with an $r$-ary constructor $C$. By the above definition, for *nullary* tokens (that is, either 0's or $*_\alpha$'s) $a_{ij}, a_i \in T_\alpha$, $i=1,\dots,r$, $j=1,\dots,l$, we have

$$\begin{aligned}
&\{Ca_{11}\cdots a_{r1},\dots,Ca_{1l},\cdots a_{rl}\} \vdash_\alpha Ca_1\cdots a_r\\
&\Leftrightarrow \forall_{i=1}^{r} \{a_{i1},\dots,a_{il}\} \vdash_\alpha a_i\\
&\Leftrightarrow \forall_{i=1}^{r} \exists_{j=1}^{l} \{a_{ij}\} \vdash_\alpha a_i\\
&\Leftrightarrow \forall_{i=1}^{r} \{a_{i1},\dots,a_{il}\} \succ_\alpha a_i\ ,
\end{aligned}$$

where $\succ_\alpha$ is the atomic entailment as we defined it in the previous chapter. We can phrase this situation as follows: nonatomic entailment translates to point-wise atomic entailment.

It seems fair to anticipate that this implicit atomicity carries over to more complex tokens so that it will permeate the whole entailment relation of $\alpha$; in other words, that $\vdash_\alpha$ may be completely explained in terms of $\succ_\alpha$. But even supposing that this turns out to be true, what about the function spaces of algebras, or, to take it a step further, what about infinitary algebras?

To this, consider a simple-looking but nasty enough infinitary pseudo-algebra $\omega$ with a nullary constructor $0:\omega$ and a unary constructor with a functional argument

$\Omega : (\omega \to \omega) \to \omega$. The following example, for $a_1, a_2, a, b_1, b_2, b \in T_\omega$ suffices to get a feeling of how entailment behaves here:

$$\begin{aligned} &\{\Omega\langle\{a_1\}, b_1\rangle, \Omega\langle\{a_2\}, b_2\rangle\} \vdash_\omega \Omega\langle\{a\}, b\rangle \\ &\Leftrightarrow \{\langle\{a_1\}, b_1\rangle, \langle\{a_2\}, b_2\rangle\} \vdash_{\omega\to\omega} \langle\{a\}, b\rangle \\ &\Leftrightarrow \{\langle\{a_1\}, b_1\rangle, \langle\{a_2\}, b_2\rangle\}\{a\} \vdash_\omega b\ . \end{aligned}$$

The last formula depends on lower-order entailment again, and it looks plausible that the procedure eventually stops, and one has to compare sets of nullary tokens on the left side and a nullary token on the right side (in fact, the presence of proper nullary constructors ensures the termination); so again, entailment translates to point-wise atomic entailment.

In the following we carry out this idea in a rigorous way.

*Remark.* Structures like the above $\omega$, where constructors may have "negative" recursive argument types, can not be tolerated in a formal higher-type computability theory, as they can't afford a natural semantic meaning; they are not *algebras* as we want them. Nevertheless, the matrix theory to be developed in the following, deals with such formal structures as well. □

## Preview

In section 2.1 we introduce the notion of a *matrix consisting of tokens* from a given acis, which generalizes the notion of a formal neighborhood, and develop an elementary formal matrix theory on arbitrary acises. In section 2.2 we consider *algebraic matrices*, that is, matrices over basic algebraic acises. We define the *application of a constructor to a matrix*, which basically makes sense of the intuitive equation

$$\{Ca_{11}\cdots a_{r1}, \ldots, Ca_{1l}\cdots a_{rl}\} \text{ “}=\text{” } C \begin{bmatrix} a_{11} & \cdots & a_{1l} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rl} \end{bmatrix} ,$$

and use it to show that the non-atomic entailment is characterized by the entailment between matrices, which is essentially atomic. Then we capture the *homogeneous normal form* of a neighborhood and prove that it has a unique *matrix representation* (Theorem 2.17). The homogeneous form leads to *eigentokens* in finitary algebras, the simplest possible normal forms for neighborhoods that one could expect. Finally we generalize the matrix representation theorem for the infinitary case.

In the rest of the chapter we deal with higher types. In section 2.3 we define non-atomic function spaces. We investigate maximal neighborhoods of lists and establish necessary and sufficient conditions for a sublist of a given list to be a maximal neighborhood. Then we introduce the important notion of the *set of eigen-neighborhoods* of a neighborhood, and, among several applications, we show that *an eigen-neighborhood behaves as a generalized token in an atomic setting*: non-atomic entailment is characterized by a binary preorder on the level of eigen-neighborhoods, thus establishing that *there is an implicit atomicity at higher types as well*. Finally, in section 2.4 we give a proof of the *density theorem* (Theorem 2.37) that improves previous known arguments for our setting, and additionally give a couple of applications that aim to shed some light on the structure of higher types and the role of eigen-neighborhoods.

## 2.1 A formal matrix theory

Let $\rho = (T_\rho, \asymp_\rho, \succ_\rho)$ be an arbitrary acis (for the definition of an acis see page 7). An $r \times l$ *matrix* over $\rho$ is any two-dimensional array

$$\begin{bmatrix} a_{11} & \cdots & a_{1l} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rl} \end{bmatrix} ,$$

where all $a_{ij}$'s are in $T_\rho$. Call such a matrix *coherently consistent* if

$$\overset{r}{\underset{i=1}{\forall}} \; \overset{l}{\underset{j,j'=1}{\forall}} \; a_{ij} \asymp_\rho a_{ij'} ,$$

and write $\mathsf{Mat}_\rho$ for the class of all coherently consistent matrices over $\rho$. When the need appears, write $\mathsf{Mat}_\rho(r,l)$, and $\mathsf{Mat}_\rho(r)$ for $r \times l$ matrices, and matrices with $r$ rows (and arbitrarily many columns) respectively. We also allow ourselves to say *lists* or *vectors*, write $\mathsf{Lst}_\rho(l)$ or $\mathsf{Vec}_\rho(r)$, and mean row matrices with $l$ entries or column matrices with $r$ entries respectively. Denote the $(i,j)$-th element of a matrix $A$ by $A(i,j)$, and the $i$-th row by $A_i$. Finally, write $\mathsf{Arr}_\rho$ instead of $\mathsf{Mat}_\rho$, for the class of arbitrary, that is, not necessarily coherently consistent matrices over $\rho$.

It should now be obvious how matrices generalize formal neighborhoods (the latter regarded as *lists*[2]): a coherently consistent $(r,l)$-matrix is nothing but a vertical appending of $r$ formal neighborhoods of length $l$. The motivation, as we pictured before, is that this is a correct entity to use as an argument to an $r$-ary constructor (see section 2.2).

Let $A, B \in \mathsf{Mat}_\rho(r)$. Call them *(mutually) consistent* and write $A \asymp_\rho B$ when

$$\overset{r}{\underset{i=1}{\forall}} \; \overset{l}{\underset{j=1}{\forall}} \; \overset{l'}{\underset{j'=1}{\forall}} \; A(i,j) \asymp_\rho B(i,j') .$$

Say that $A$ *entails* $B$ and write $A \succ_\rho B$ when

$$\overset{r}{\underset{i=1}{\forall}} \; \overset{l'}{\underset{j'=1}{\forall}} \; \overset{l}{\underset{j=1}{\exists}} \; A(i,j) \succ_\rho B(i,j') .$$

**Proposition 2.1.** *Given an acis $\rho$, the triple $M(\rho) = (\mathsf{Mat}_\rho, \asymp_\rho, \succ_\rho)$ itself forms an acis, the* matrix acis *of $\rho$.*

*Proof.* We show that $M(\rho)$ satisfies the properties for an acis. For reflexivity of consistency: For any $A \in \mathsf{Mat}_\rho(r,l)$ it is by definition

$$\overset{r}{\underset{i=1}{\forall}} \; \overset{l}{\underset{j=1}{\forall}} \; \overset{l}{\underset{j'=1}{\forall}} \; A(i,j) \asymp_\rho A(i,j') \Leftrightarrow A \asymp_\rho A .$$

For symmetry of consistency: Let $A, B \in \mathsf{Mat}_\rho(r)$; it is

$$A \asymp B \Leftrightarrow \overset{r}{\underset{i=1}{\forall}} \; \overset{l}{\underset{j=1}{\forall}} \; \overset{l'}{\underset{j'=1}{\forall}} \; A(i,j) \asymp_\rho B(i,j') \Leftrightarrow \overset{r}{\underset{i=1}{\forall}} \; \overset{l'}{\underset{j'=1}{\forall}} \; \overset{l}{\underset{j=1}{\forall}} \; B(i,j') \asymp_\rho A(i,j) \Leftrightarrow B \asymp_\rho A .$$

[2] It should be noted that in this chapter we move to an ordered environment, that is, we view formal neighborhoods as *lists*, rather than *sets*, and at times we allow ourselves to be sloppy about that: unless otherwise mentioned, a formal neighborhood will be identified with any list consisting of the same tokens.

For reflexivity of entailment: Let $A \in \mathsf{Mat}_\rho(r,l)$; then

$$\operatorname*{\forall}_{i=1}^{r}\operatorname*{\forall}_{j=1}^{l} A(i,j) \succ_\rho A(i,j) \overset{j':=j}{\Rightarrow} \operatorname*{\forall}_{i=1}^{r}\operatorname*{\forall}_{j=1}^{l}\operatorname*{\exists}_{j'=1}^{l} A(i,j') \succ A(i,j) \Leftrightarrow A \succ_\rho A \ .$$

For transitivity of entailment: Let $A,B,C \in \mathsf{Mat}_\rho(r)$; it is

$$\begin{aligned} & A \succ_\rho B \wedge B \succ_\rho C \\ & \Leftrightarrow \operatorname*{\forall}_{i=1}^{r}\left(\operatorname*{\forall}_{j'=1}^{l'}\operatorname*{\exists}_{j=1}^{l} A(i,j) \succ_\rho B(i,j') \wedge \operatorname*{\forall}_{j''=1}^{l''}\operatorname*{\exists}_{j'=1}^{l'} B(i,j') \succ_\rho C(i,j'')\right) \\ & \overset{\text{trns}}{\Rightarrow} \operatorname*{\forall}_{i=1}^{r}\operatorname*{\forall}_{j''=1}^{l''}\operatorname*{\exists}_{j=1}^{l} A(i,j) \succ_\rho C(i,j'') \\ & \Leftrightarrow A \succ_\rho C \ . \end{aligned}$$

For propagation of consistency through entailment: Let $A,B,C \in \mathsf{Mat}_\rho(r)$; it is

$$\begin{aligned} & A \asymp_\rho B \wedge B \succ_\rho C \\ & \Leftrightarrow \operatorname*{\forall}_{i=1}^{r}\left(\operatorname*{\forall}_{j=1}^{l}\operatorname*{\forall}_{j'=1}^{l'} A(i,j) \asymp_\rho B(i,j') \wedge \operatorname*{\forall}_{j''=1}^{l''}\operatorname*{\exists}_{j'=1}^{l'} B(i,j') \succ_\rho C(i,j'')\right) \\ & \overset{\text{prpg}}{\Rightarrow} \operatorname*{\forall}_{i=1}^{r}\operatorname*{\forall}_{j=1}^{l}\operatorname*{\forall}_{j''=1}^{l''} A(i,j) \asymp_\rho C(i,j'') \\ & \Leftrightarrow A \asymp_\rho C \ , \end{aligned}$$

as we wanted. □

We now discuss some basic operations on matrices. Let $A \in \mathsf{Mat}_\rho(r,l)$ and $B \in \mathsf{Mat}_\rho(r',l')$. The *transpose* of $A$ is defined as usual by

$$A^t(i,j) := A(j,i) \ ,$$

but notice that it is not necessary that the transpose of a coherently consistent matrix is itself coherently consistent. In case $r = r'$, define the *horizontal append* or *sum* $A+B \in \mathsf{Arr}_\rho(r,l+l')$ by

$$(A+B)(m,n) := \begin{cases} A(m,n) \ , & n \leq l \ , \\ B(m,n-l) \ , & n > l \ , \end{cases}$$

and in case $l = l'$, define the *vertical append* or *product* $A \cdot B \in \mathsf{Arr}_\rho(r+r',l)$ by

$$(A \cdot B)(m,n) := \begin{cases} A(m,n) \ , & m \leq r \ , \\ B(m-r,n) \ , & m > r \ . \end{cases}$$

In addition, we consider *empty* $0 \times l$- and $r \times 0$-matrices, which we collectively denote by $\varnothing_\rho$, and depend on the context to determine their exact dimensions. It is $\varnothing_\rho \in \mathsf{Mat}_\rho$, for every such matrix. Empty matrices will be particularly helpful when discussing infinitary algebras and non-atomic function spaces (pages 85 and 87 respectively).

**Proposition 2.2.** *Let $A,B,C,D \in \mathsf{Arr}_\rho$. The following hold whenever well-defined.*

1. *If* $A,B \in \mathsf{Mat}_\rho$ *then* $A \cdot B \in \mathsf{Mat}_\rho$*, and if furthermore* $A \asymp_\rho B$ *it is also* $A + B \in \mathsf{Mat}_\rho$.
2. *Both append operations are monotone, that is,*

$$A \succ_\rho A' \wedge B \succ B' \to A \circ A' \succ_\rho B \circ B' \ ,$$

*where* $\circ$ *is either* $+$ *or* $\cdot$.

3. *It is* $A \asymp_\rho \varnothing$ *and* $A \succ_\rho \varnothing$*, for every* $A \in \mathsf{Mat}_\rho$.
4. *The following equations hold:*

$$\begin{aligned} (A+B)+C &= A+(B+C) \ , \\ A+\varnothing &= A = \varnothing + A \ , \\ (A \cdot B) \cdot C &= A \cdot (B \cdot C) \ , \\ A \cdot \varnothing &= A = \varnothing \cdot A \ , \\ (A+B) \cdot (C+D) &= (A \cdot C) + (B \cdot D) \ , \\ A \cdot B &= \left(A^t + B^t\right)^t \ . \end{aligned}$$

With the convention of writing $a$ for a one-element matrix $[a]$, Proposition 2.2 yields two alternative notations for an arbitrary matrix $A \in \mathsf{Arr}_\rho(r,l)$, an *additive-multiplicative* and a *multiplicative-additive* (or *sigma-pi* and *pi-sigma* respectively):

$$A = \sum_{j=1}^{l} \prod_{i=1}^{r} A(i,j) = \prod_{i=1}^{r} \sum_{j=1}^{l} A(i,j) \ ,$$

both of which will later prove useful.

## Mixed matrices

In anticipation of both infinitary algebras (where functional recursive arguments appear) and parametric algebras, we also introduce a kind of generalized matrices, where each row might draw from a different acis. By a *mixed matrix* $A$ of dimensions $(\rho_1, \dots, \rho_r) \times l$ we will mean a two-dimensional array

$$\begin{bmatrix} a_{11} & \cdots & a_{1l} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rl} \end{bmatrix} ,$$

where, for each $i = 1, \dots, r$, all $a_{ij}$'s are in $T_{\rho_i}$; we write $A \in \mathsf{Arr}(\rho_1, \dots, \rho_r, l)$.

All notions pertaining to simple matrices, like coherent consistency, mutual consistency, and entailment, as well as the structural operations and all relevant facts, carry over to mixed matrices without any problems—to convince oneself, one should notice how all definitions work row-wise, as the following observation ensures.

**Proposition 2.3.** *Let* $\rho_1, \dots, \rho_r$ *be acises. It is*

$$\mathsf{Arr}(\rho_1, \dots, \rho_r, l) \simeq \mathsf{Arr}_{\rho_1}(1,l) \times \cdots \times \mathsf{Arr}_{\rho_r}(1,l) \ ,$$

*for all* $l > 0$.

*Proof.* A matrix $A \in \mathsf{Arr}(\rho_1,\dots,\rho_r,l)$ can be mapped to the $r$-tuple $(A_1,\dots,A_r)$ of its rows; conversely, given an $r$-tuple $(A_1,\dots,A_r)$, with $A_i \in \mathsf{Arr}_{\rho_i}(1,l)$ for every $i = 1,\dots,r$, we have $\prod_{i=1}^{r} A_i \in \mathsf{Arr}(\rho_1,\dots,\rho_r,l)$. The mappings are clearly an isomorphism pair. □

## Atomic function spaces I

We already have enough to go on with the main course, which is, matrices over algebras, or *algebraic matrices* (section 2.2). Still, we linger on the formal level for the remainder of this section, and examine the case of *atomic* function spaces. This, apart from making the formal exposition more complete, will also help obtain a better understanding of the behavior of matrices, and also, allow to take a glimpse at interesting structures that arise on the way (see *nbr-acises*, page 100), and try out ideas that will play a crucial role in the case of the algebraic matrices, and their non-atomic function spaces (see *test matrices*, page 58).

We will readily see that the generalization of the above to atomic function spaces is not so direct; here logic plays a dominant role, which takes some effort to be translated into matrix terms. Let us begin in the obvious and natural way. An *$r \times l$ matrix* over $\rho \to \sigma$ is any two-dimensional array

$$\begin{bmatrix} \langle U_{11}, a_{11}\rangle & \cdots & \langle U_{1l}, a_{1l}\rangle \\ \vdots & \ddots & \vdots \\ \langle U_{r1}, a_{r1}\rangle & \cdots & \langle U_{rl}, a_{rl}\rangle \end{bmatrix},$$

where all $U_{ij}$'s are in $\mathsf{Con}_\rho$ and $a_{ij}$'s are in $T_\sigma$. We denote the class of all such arrays by $\mathsf{Arr}_{\rho\to\sigma}$. Obstacles appear already when one considers the subclass of coherently consistent matrices over $\rho \to \sigma$, and further the notions of mutual consistency and entailment. For the latter two we have the following. Let

$$A := \begin{bmatrix} \langle U_{11}, a_{11}\rangle & \cdots & \langle U_{1l}, a_{1l}\rangle \\ \vdots & \ddots & \vdots \\ \langle U_{r1}, a_{r1}\rangle & \cdots & \langle U_{rl}, a_{rl}\rangle \end{bmatrix} \text{ and } B := \begin{bmatrix} \langle V_{11}, b_{11}\rangle & \cdots & \langle V_{1l'}, b_{1l'}\rangle \\ \vdots & \ddots & \vdots \\ \langle V_{r1}, b_{r1}\rangle & \cdots & \langle V_{rl'}, b_{rl'}\rangle \end{bmatrix}.$$

The matrices $A, B \in \mathsf{Mat}_\rho(r)$ should be *(mutually) consistent* when

$$\begin{aligned} A \asymp_{\rho\to\sigma} B &\Leftrightarrow \underset{i=1}{\overset{r}{\forall}}\, \underset{j=1}{\overset{l}{\forall}}\, \underset{j'=1}{\overset{l'}{\forall}}\, A(i,j) \asymp_{\rho\to\sigma} B(i,j') \\ &\Leftrightarrow \underset{i=1}{\overset{r}{\forall}}\, \underset{j=1}{\overset{l}{\forall}}\, \underset{j'=1}{\overset{l'}{\forall}}\, \langle U_{ij}, a_{ij}\rangle \asymp_{\rho\to\sigma} \langle V_{ij'}, b_{ij'}\rangle \\ &\Leftrightarrow \underset{i=1}{\overset{r}{\forall}}\, \underset{j=1}{\overset{l}{\forall}}\, \underset{j'=1}{\overset{l'}{\forall}}\, \left(U_{ij} \asymp_\rho V_{ij'} \to a_{ij} \asymp_\sigma b_{ij'}\right) \end{aligned}$$

—note that coherent consistency of a matrix is a special case of mutual consistency of two matrices: it is $A \in \mathsf{Mat}$ if and only if $A \asymp A$. Further, $A$ should *entail* $B$ when

$$\begin{aligned} A \succ_{\rho\to\sigma} B &\Leftrightarrow \mathop{\forall}_{i=1}^{r} \mathop{\forall}_{j'=1}^{l'} \mathop{\exists}_{j=1}^{l} A(i,j) \succ_{\rho\to\sigma} B(i,j') \\ &\Leftrightarrow \mathop{\forall}_{i=1}^{r} \mathop{\forall}_{j'=1}^{l'} \mathop{\exists}_{j=1}^{l} \langle U_{ij}, a_{ij}\rangle \succ_{\rho\to\sigma} \langle V_{ij'}, b_{ij'}\rangle \\ &\Leftrightarrow \mathop{\forall}_{i=1}^{r} \mathop{\forall}_{j'=1}^{l'} \mathop{\exists}_{j=1}^{l} \left(V_{ij'} \succ_\rho U_{ij} \wedge a_{ij} \succ_\sigma b_{ij'}\right) \ . \end{aligned}$$

Clearly, it is not trivial to formulate the above formulas in matrix terms. In order to do that we will use some extra tools, in particular, appropriate versions of "adjacency matrices" for consistency and entailment.

## Test matrices

Let $I = \{1,\dots,r\}$, $J = \{1,\dots,l\}$, $J' = \{1,\dots,l'\}$, and $I' = \{1,\dots,rl\}$. Consider the *blocking* map $b : I \times J \times J' \to I' \times J'$, and the *unblocking* maps $u_0 : I' \times J' \to I$, $u_1 : I' \times J' \to J$, and $u_0 : I' \times J' \to J'$, that are defined by

$$\begin{aligned} b(i,j,j') &:= \left((i-1)l + j, j'\right) \ , \\ u_0(i',j') &:= (i'-1) \setminus l + 1 \ , \\ u_1(i',j') &:= ((i'-1) \bmod l) + 1 \ , \\ u_2(i',j') &:= j' \ , \end{aligned}$$

where $m \setminus n$ and $m \bmod n$ are the *integer quotient* and the *remainder* of the division of $m$ by $n$, and $i \in I$, $j \in J$, $j' \in J'$, $i' \in I'$.

**Proposition 2.4.** *The mappings $b : I \times J \times J' \to I' \times J'$ and $u := (u_0, u_1, u_2) : I' \times J' \to I \times J \times J'$ form a bijection pair, called* block coding (of indices).

*Proof.* One has to show that

$$\begin{aligned} \mathop{\forall}_{i\in I, j\in J, j'\in J'} u(b(i,j,j')) &= \left(i,j,j'\right) \ , \\ \mathop{\forall}_{i'\in I', j'\in J'} b(u(i',j')) &= \left(i',j'\right) \ . \end{aligned}$$

For the first property, let $i \in I$, $j \in J$, and $j' \in J'$. It is

$$\begin{aligned} u(b(i,j,j')) &= u\left((i-1)l + j, j'\right) \\ &= \left(u_0\left((i-1)l + j, j'\right), u_1\left((i-1)l + j, j'\right), u_2\left((i-1)l + j, j'\right)\right) \ , \end{aligned}$$

with

$$\begin{aligned} u_0\left((i-1)l + j, j'\right) &= ((i-1)l + j - 1) \setminus l + 1 = i \ , \\ u_1\left((i-1)l + j, j'\right) &= ((i-1)l + j - 1) \bmod l + 1 = j \ , \\ u_2\left((i-1)l + j, j'\right) &= j' \ , \end{aligned}$$

so $u(b(i,j,j')) = (i,j,j')$.

For the converse property, let $i' \in I'$ and $j' \in J'$. Then

$$\begin{aligned} b(u(i',j')) &= b\big(u_0(i',j'),u_1(i',j'),u_2(i',j')\big) \\ &= \big((u_0(i',j')-1)l+u_1(i',j'),u_2(i',j')\big) \\ &= \big(((i'-1)\setminus l)l+(i'-1) \bmod l+1,j'\big) \\ &= (i',j') \ , \end{aligned}$$

where for the last step we used the identity

$$(m \setminus n)n + m \bmod n = m \ ,$$

for $m,n \in \mathbb{N}$. □

If $A \in \mathsf{Arr}_\rho(r,l)$ and $B \in \mathsf{Arr}_\rho(r,l')$ are two arbitrary matrices, we can use this block coding to form a *block matrix* with $r$ blocks; each block will have $l$ rows and $l'$ columns, so that it may serve as a "comparison table" between (positions of) elements of $A$ and $B$ in their respective row; we understand $b(i,j,j')$ as "the position $(j,j')$ in the $i$-th block".

In particular we need such comparison tables to test consistency and entailment between elements of the two matrices in the same row. For an arbitrary acis $\rho$ define the mappings $\overset{?}{\asymp}_\rho, \overset{?}{\succ}_\rho \colon \mathsf{Arr}_\rho(r,l) \times \mathsf{Arr}_\rho(r,l') \to \mathsf{Arr}_\mathbb{B}(rl,l')$ by

$$A \overset{?}{\asymp}_\rho B(i',j') \coloneqq \begin{cases} \mathtt{tt} \ , & A(u_{01}(i',j')) \asymp_\rho B(u_{02}(i',j')) \ , \\ *_\mathbb{B} \ , & \text{otherwise} \ , \end{cases}$$

$$A \overset{?}{\succ}_\rho B(i',j') \coloneqq \begin{cases} \mathtt{tt} \ , & A(u_{01}(i',j')) \succ_\rho B(u_{02}(i',j')) \ , \\ *_\mathbb{B} \ , & \text{otherwise} \ , \end{cases}$$

where $u_{01} \coloneqq (u_0,u_1)$ and $u_{02} \coloneqq (u_0,u_2)$. We call $A \overset{?}{\asymp}_\rho B$ and $A \overset{?}{\succ}_\rho B$ the *consistency* and *entailment test matrix* respectively of $A$ and $B$.

**Proposition 2.5.** *Let $A \in \mathsf{Arr}_\rho(r,l)$, $B \in \mathsf{Arr}_\rho(r,l')$, $C \in \mathsf{Arr}_\rho(r,l'')$, and let $i = 1,\dots,r$, $j = 1,\dots,l$, $j' = 1,\dots,l'$, $j'' = 1,\dots,l''$.*

1. *It is $A \asymp_\rho B$ if and only if their consistency test matrix is the* true matrix*, that is,*

$$\forall_i \forall_j \forall_{j'} A \overset{?}{\asymp}_\rho B(b(i,j,j')) = \mathtt{tt} \ .$$

2. *It is $A \overset{?}{\asymp}_\rho A(b(i,j,j)) = \mathtt{tt}$, for all $i$, $j$.*

3. *It is $A \overset{?}{\asymp}_\rho B(b(i,j,j')) = B \overset{?}{\asymp}_\rho A(b(i,j',j))$, for all $i$, $j$, $j'$.*

4. *It is $A \succ_\rho B$ if and only if their entailment test matrix* consists of blocks with non-blank columns*, that is,*

$$\forall_i \forall_{j'} \exists_j A \overset{?}{\succ}_\rho B(b(i,j,j')) = \mathtt{tt} \ .$$

5. *It is $A \overset{?}{\succ}_\rho A(b(i,j,j)) = \mathtt{tt}$, for all $i$, $j$.*

6. *If* $A \overset{?}{\succ}_\rho B(b(i,j,j')) = \mathtt{tt}$ *and* $B \overset{?}{\succ}_\rho C(b(i,j',j'')) = \mathtt{tt}$*, then* $A \overset{?}{\succ}_\rho C(b(i,j,j'')) = \mathtt{tt}$*, for all* $i,\ j,\ j',\ j''$.
7. *If* $A \overset{?}{\asymp}_\rho B(b(i,j,j')) = \mathtt{tt}$ *and* $B \overset{?}{\succ}_\rho C(b(i,j',j'')) = \mathtt{tt}$*, then* $A \overset{?}{\asymp}_\rho C(b(i,j,j'')) = \mathtt{tt}$*, for all* $i,\ j,\ j',\ j''$.

*Proof.* For 1:

$$A \asymp_\rho B \Leftrightarrow \forall_i \forall_j \forall_{j'} A(i,j) \asymp_\rho B(i,j') \Leftrightarrow \forall_i \forall_j \forall_{j'} A \overset{?}{\asymp}_\rho B(b(i,j,j')) = \mathtt{tt}\ .$$

Statement 2 is easy to see. For 3:

$$\begin{aligned} A \overset{?}{\asymp}_\rho B(b(i,j,j')) = \mathtt{tt} &\Leftrightarrow A(i,j) \asymp_\rho B(i,j') \\ &\Leftrightarrow B(i,j') \asymp_\rho A(i,j) \\ &\Leftrightarrow B \overset{?}{\asymp}_\rho A\left((i-1)l + j', j\right) = \mathtt{tt} \\ &\Leftrightarrow A \overset{?}{\asymp}_\rho B(b(i,j,j')) = \mathtt{tt}\ . \end{aligned}$$

For 4:

$$A \succ_\rho B \Leftrightarrow \forall_i \forall_{j'} \exists_j A(i,j) \succ_\rho B(i,j') \Leftrightarrow \forall_i \forall_{j'} \exists_j A \overset{?}{\succ}_\rho B(b(i,j,j')) = \mathtt{tt}\ .$$

Statement 5 is again easy to see, while for 6 we have

$$\begin{aligned} & A \overset{?}{\succ}_\rho B(b(i,j,j')) = \mathtt{tt} \wedge B \overset{?}{\succ}_\rho C(b(i,j',j'')) = \mathtt{tt} \\ &\quad \Leftrightarrow A(i,j) \succ_\rho B(i,j') \wedge B(i,j') \succ_\rho C(i,j'') \\ &\quad \Leftrightarrow A(i,j) \succ_\rho C(i,j'') \\ &\quad \Leftrightarrow A \overset{?}{\succ}_\rho C(b(i,j,j'')) = \mathtt{tt}\ . \end{aligned}$$

Finally, for the propagation statement 7 we have

$$\begin{aligned} & A \overset{?}{\asymp}_\rho B(b(i,j,j')) = \mathtt{tt} \wedge B \overset{?}{\succ}_\rho C(b(i,j',j'')) = \mathtt{tt} \\ &\quad \Leftrightarrow A(i,j) \asymp_\rho B(i,j') \wedge B(i,j') \succ_\rho C(i,j'') \\ &\quad \Leftrightarrow A(i,j) \asymp_\rho C(i,j'') \\ &\quad \Leftrightarrow A \overset{?}{\asymp}_\rho C(b(i,j,j'')) = \mathtt{tt}\ . \qquad \square \end{aligned}$$

## Overlapping and inclusion

We introduce two relations between matrices of the same dimension. Let $A, B \in \mathsf{Arr}_\rho(r,l)$. Define

$$\begin{aligned} A \between B &:= \overset{r}{\underset{i=1}{\forall}} \overset{l}{\underset{j=1}{\exists}} \left(A(i,j) = B(i,j) \neq *_\rho\right)\ , \\ A \subset B &:= \overset{r}{\underset{i=1}{\forall}} \overset{l}{\underset{j=1}{\forall}} \left(A(i,j) = *_\rho \vee A(i,j) = B(i,j)\right)\ . \end{aligned}$$

In the first case *A and B overlap*; in the second case *A is included in B*.

**Proposition 2.6** (nbr-acis)**.** *The following hold.*

1. *Matrix overlapping is reflexive on matrices with non-blank rows, and symmetric.*
2. *Matrix inclusion is reflexive and transitive.*
3. *If $B$ is a matrix with non-blank rows, then*

$$A \between B \wedge B \subset C \to A \between C \ .$$

*That is, overlapping (with a non-blank-row-matrix) propagates through inclusion.*

*Proof.* For reflexivity of overlapping, if $A$ is a non-blank row matrix, that is, if $\forall_i \exists_j A(i,j) \neq *_\rho$, then the statement follows immediately. For the symmetry of overlapping:

$$A \between B \Leftrightarrow \underset{i}{\forall}\underset{j}{\exists} A(i,j) = B(i,j) \neq *_\rho \Leftrightarrow \underset{i}{\forall}\underset{j}{\exists} B(i,j) = A(i,j) \neq *_\rho \Leftrightarrow B \between A \ .$$

Reflexivity of inclusion again is immediate, since for any $A$ and all its $i$'s, $j$'s, it is $A(i,j) = *_\rho$ or $A(i,j) = A(i,j)$, so $A \subset A$. We show transitivity of inclusion: let $A \subset B$ and $B \subset C$, that is,

$$\underset{i}{\forall}\underset{j}{\forall} \left( \left(A(i,j) = *_\rho \vee A(i,j) = B(i,j)\right) \wedge \left(B(i,j) = *_\rho \vee B(i,j) = C(i,j)\right) \right) \ ;$$

in case $A(i,j) = *_\rho$, then $A(i,j) = *_\rho \vee A(i,j) = C(i,j)$ holds; on the other hand, in the case that $A(i,j) \neq *_\rho$ and $A(i,j) = B(i,j)$ and $B(i,j) = *_\rho \vee B(i,j) = C(i,j)$, then $A(i,j) = B(i,j) = C(i,j) \neq *_\rho$ holds; in both cases we get $A \subset C$.

Finally, for the propagation of overlapping through inclusion, let $B$ be a non-blank row matrix; then

$$\begin{aligned}
&A \between B \wedge B \subset C \\
&\qquad \Leftrightarrow \underset{i}{\forall}\underset{j}{\exists} A(i,j) = B(i,j) \neq *_\rho \\
&\qquad\qquad\qquad \wedge \underset{i}{\forall}\underset{j}{\forall} \left(B(i,j) = *_\rho \vee B(i,j) = C(i,j)\right) \\
&\qquad \Rightarrow \underset{i}{\forall}\underset{j}{\exists} \left(A(i,j) = B(i,j) \neq *_\rho \wedge \left(B(i,j) = *_\rho \vee B(i,j) = C(i,j)\right)\right) \\
&\qquad \Rightarrow \underset{i}{\forall}\underset{j}{\exists} A(i,j) = B(i,j) = C(i,j) \neq *_\rho \\
&\qquad \Leftrightarrow A \between C \ ,
\end{aligned}$$

as we wanted. ◻

## Atomic function spaces II

We will make use of a generalization of the transpose operation on block matrices, where all blocks have the same dimension. Let $A \in \mathsf{Arr}(r,l)$, with $r = Mr_0$, $l = Nl_0$. Define its *(inner)* $(M,N)$*-transpose* by

$$A^{t(M,N)} := \sum_{n=1}^{N} \prod_{m=1}^{M} (A^{(m,n)})^t \ ,$$

where $A^{(m,n)}$ is the $(m,n)$*-th block* of $A$:

$$A^{(m,n)}(i,j) := A((m-1)r_0+i,(n-1)l_0+j) \ ,$$

for $m=1,\ldots,M$, $n=1,\ldots,N$, $i=1,\ldots,r_0$, $j=1,\ldots,l_0$.

**Proposition 2.7.** *Let* $A \in \mathsf{Arr}(r,l)$*. The following hold:*

1. $A^t = A^{t(1,1)} = A^{t(r,l)}$.
2. *If* $r = Mr_0$ *and* $l = Nl_0$*, then* $(A^{t(M,N)})^{t(M,N)} = A$.

*Proof.* The properties are direct to show. For the first one we have

$$A^{t(1,1)} = \sum_{n=1}^{1}\prod_{m=1}^{1}(A^{(m,n)})^t = (A^{(1,1)})^t = A^t \ ,$$

as well as

$$\begin{aligned} A^{t(r,l)} &= \sum_{n=1}^{l}\prod_{m=1}^{r}(A^{(m,n)})^t = \sum_{n=1}^{l}\prod_{m=1}^{r}\left(\sum_{j=1}^{1}\prod_{i=1}^{1}A((m-1)+i,(n-1)+j)\right)^t \\ &= \sum_{n=1}^{l}\prod_{m=1}^{r}A^t(m,n) = \sum_{n=1}^{l}\prod_{m=1}^{r}A(n,m) = A^t \ . \end{aligned}$$

For the second statement, let $r = Mr_0$ and $l = Nl_0$. Then:

$$\begin{aligned} (A^{t(M,N)})^{t(M,N)} &= \left(\left(\sum_{n=1}^{N}\prod_{m=1}^{M}A^{(m,n)}\right)^{t(M,N)}\right)^{t(M,N)} = \left(\sum_{n=1}^{N}\prod_{m=1}^{M}(A^{(m,n)})^t\right)^{t(M,N)} \\ &= \sum_{n=1}^{N}\prod_{m=1}^{M}\left((A^{(m,n)})^t\right)^t = \sum_{n=1}^{N}\prod_{m=1}^{M}A^{(m,n)} = A \ . \end{aligned}$$

We're done. □

Let us now return to the situation where we had to figure out consistency and entailment between functional matrices. Let $\rho$ and $\sigma$ be acises. In section 1.1 we defined what $\rho \to \sigma$ is. Here we will define $M(\rho \to \sigma)$.

For a functional matrix $A \in \mathsf{Mat}_{\rho\to\sigma}(r,l)$ with

$$A = \begin{bmatrix} \langle U_{11},a_{11}\rangle & \cdots & \langle U_{1l},a_{1l}\rangle \\ \vdots & \ddots & \vdots \\ \langle U_{r1},a_{r1}\rangle & \cdots & \langle U_{rl},a_{rl}\rangle \end{bmatrix} ,$$

define its *argument part* $\arg A \in \mathsf{Arr}_{N\rho}(r,l)$ and *value part* $\mathrm{val} A \in \mathsf{Arr}_{\sigma}(r,l)$ by

$$\arg A := \begin{bmatrix} U_{11} & \cdots & U_{1l} \\ \vdots & \ddots & \vdots \\ U_{r1} & \cdots & U_{rl} \end{bmatrix} \text{ and } \mathrm{val} A := \begin{bmatrix} a_{11} & \cdots & a_{1l} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rl} \end{bmatrix}$$

respectively.

**Theorem 2.8** (Matrix characterization of functional consistency and entailment)**.** *Let $A, B \in \mathsf{Mat}_{\rho\to\sigma}(r)$. Consistency and entailment between A and B are characterized by inclusion and overlapping respectively of the test matrices of their parts:*

1. $A \asymp_{\rho\to\sigma} B$ *if and only if* $\arg A \overset{?}{\asymp}_{N\rho} \arg B \subset \operatorname{val} A \overset{?}{\asymp}_{\sigma} \operatorname{val} B$.

2. $A \succ_{\rho\to\sigma} B$ *if and only if* $\arg B \overset{?}{\succ}_{N\rho} \arg A \between \left(\operatorname{val} A \overset{?}{\succ}_{\sigma} \operatorname{val} B\right)^{t(r,1)}$.

*Proof.* For consistency:

$$\begin{aligned}
&\arg A \overset{?}{\asymp}_{N\rho} \arg B \subset \operatorname{val} A \overset{?}{\asymp}_{\sigma} \operatorname{val} B \\
\Leftrightarrow &\mathop{\forall}_{i'=1}^{rl} \mathop{\forall}_{j'=1}^{l'} \Big(\arg A \overset{?}{\asymp}_{\rho} \arg B(i',j') = * \\
&\qquad \vee \arg A \overset{?}{\asymp}_{\rho} \arg B(i',j') = \operatorname{val} A \overset{?}{\asymp}_{\rho} \operatorname{val} B(i',j') = \mathtt{tt}\Big) \\
\Leftrightarrow &\mathop{\forall}_{i'=1}^{rl} \mathop{\forall}_{j'=1}^{l'} \left(U_{u_{01}(i',j')} \not\asymp_{\rho} V_{u_{02}(i',j')} \vee a_{u_{01}(i',j')} \asymp_{\sigma} b_{u_{02}(i',j')}\right) \\
\Leftrightarrow &\mathop{\forall}_{i'=1}^{rl} \mathop{\forall}_{j'=1}^{l'} \left(U_{u_{01}(i',j')} \asymp_{\rho} V_{u_{02}(i',j')} \to a_{u_{01}(i',j')} \asymp_{\sigma} b_{u_{02}(i',j')}\right) \\
\Leftrightarrow &\mathop{\forall}_{i=1}^{r} \mathop{\forall}_{j=1}^{l} \mathop{\forall}_{j'=1}^{l'} \left(U_{(i,j)} \asymp_{\rho} V_{(i,j')} \to a_{(i,j)} \asymp_{\sigma} b_{(i,j')}\right) \\
\Leftrightarrow &A \asymp_{\rho\to\sigma} B\ .
\end{aligned}$$

For entailment:

$$\begin{aligned}
&\arg B \overset{?}{\succ}_{N\rho} \arg A \between \left(\operatorname{val} A \overset{?}{\succ}_{\sigma} \operatorname{val} B\right)^{t(r,1)} \\
\Leftrightarrow &\mathop{\forall}_{i'=1}^{rl'} \mathop{\exists}_{j=1}^{l} \left(\arg B \overset{?}{\succ}_{\rho} \arg A(i',j) = \left(\operatorname{val} A \overset{?}{\succ}_{\sigma} \operatorname{val} B\right)^{t(r,1)}(i',j) = \mathtt{tt}\right) \\
\Leftrightarrow &\mathop{\forall}_{i'=1}^{rl'} \mathop{\exists}_{j=1}^{l} \left(V_{u_{02}(i',j)} \succ_{\rho} U_{u_{01}(i',j)} \wedge a_{u_{01}(i',j)} \succ_{\sigma} b_{u_{02}(i',j)}\right) \\
\Leftrightarrow &\mathop{\forall}_{i=1}^{r} \mathop{\forall}_{j'=1}^{l'} \mathop{\exists}_{j=1}^{l} \left(V_{(i,j')} \succ_{\rho} U_{(i,j)} \wedge a_{(i,j)} \succ_{\sigma} b_{(i,j')}\right) \\
\Leftrightarrow &A \succ_{\rho\to\sigma} B\ . \qquad \square
\end{aligned}$$

## 2.2 Algebraic matrices

From now on we abandon the generality of abstract information systems and focus on matrices over *basic algebraic acises* (see page 29). We have already used such matrices, namely over $\mathbb{B}$ on page 58. A further example of a coherently consistent $4\times 3$-matrix over a basic algebraic acis $\alpha$ with constructors $0 : \alpha$, $S : \alpha \to \alpha$, $B : \alpha \to \alpha \to \alpha$,

$C : \alpha \to \alpha \to \alpha \to \alpha$, and $\Omega : (\mathbb{B} \to \alpha) \to \alpha$ is the following:

$$\begin{bmatrix} C(B00)*_\alpha(S*_\alpha) & *_\alpha & C*_\alpha*_\alpha(S0) \\ SB*_\alpha 0 & SB*_\alpha*_\alpha & SB(\Omega\langle[\mathtt{tt}],\Omega\langle[\mathtt{ff}],B*_\alpha*_\alpha\rangle\rangle)*_\alpha \\ B*_\alpha*_\alpha & B(C(S0)*_\alpha*_\alpha)*_\alpha & B*_\alpha(B*_\alpha 0) \\ \Omega\langle[*_\mathbb{B}],SB*_\alpha 0\rangle & \Omega\langle[\mathtt{tt}],SB0*_\alpha\rangle & \Omega\langle[\mathtt{ff}],SB(S0)*_\alpha\rangle \end{bmatrix} .$$

A matrix over a basic algebraic acis is called *basic* if it consists solely of $*_\alpha$'s or nullary constructors; it is called *blank* if it is basic without any nullary constructors (so it consists solely of $*$'s); write $*^{r,l}$ for the blank $r \times l$-matrix.

*Remark.* Note that a blank matrix expresses "least information", and is definitely not an *empty* matrix. However, we have $*^{r,l}_\alpha \succ \varnothing_\alpha$ and $\varnothing_\alpha \succ *^{r,l}_\alpha$ for all matching dimensions. These are redundancies that can prove quite helpful, as for example in the convention of page 86. □

## Non-atomic entailment

As we saw in the beginning of the chapter, the concept of a matrix is motivated by the way entailment behaves in algebraic coherent information systems. In order to make this behavior explicit, we introduce the concept of the matrix operators induced by the constructors of an algebra.

For the most part we will be interested in finitary algebras, unless we clearly state otherwise. Every constructor $C$ of $\alpha$ with arity $r > 0$ (so, not a nullary one) induces a *constructor operator* $\dot{C} : \mathsf{Mat}_\alpha(r) \to \mathsf{Mat}_\alpha(1)$ by

$$\dot{C} \begin{bmatrix} a_{11} & \cdots & a_{1l} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rl} \end{bmatrix} := \begin{bmatrix} Ca_{11}\cdots a_{r1} & \cdots & Ca_{1l}\cdots a_{rl} \end{bmatrix} ,$$

or, in sigma-pi notation:

$$\dot{C}\left(\sum_{j=1}^{l}\prod_{i=1}^{r} a_{ij}\right) := \sum_{j=1}^{l} Ca_{1j}\cdots a_{rj} .$$

**Proposition 2.9.** *The respective constructor operator of any constructor is well-defined.*

*Proof.* That the operation is single-valued is clear. One has to show further that the result of an application of a constructor operator to a coherently consistent matrix yields a coherently consistent matrix as well.

Let $[a_{ij}]_{i,j} \in \mathsf{Mat}_\alpha(r,l)$ and $C$ an $r$-ary constructor of $\alpha$; then $a_{ij} \asymp_\alpha a_{ij'}$ for all $j, j' = 1, \ldots, l$, so, by definition, $Ca_{1j}\cdots a_{rj} \asymp_\alpha Ca_{1j'}\cdots a_{rj'}$, that is,

$$\sum_{j=1}^{l} Ca_{1j}\cdots a_{rj} = \dot{C}\left(\sum_{j=1}^{l}\prod_{i=1}^{r} a_{ij}\right) = [a_{ij}]_{i,j}$$

is indeed an element of $\mathsf{Mat}_\alpha(1,l)$. □

Define the *blank inclusion* of a list $U$ to a list $V$, and write $U \subseteq^* V$ if $a \in U \to a \in V$ and $a \in V \wedge a \notin U \to a = *$; so a list is blankly included in another list if they either contain exactly the same tokens or else the second contains a surplus of $*$'s. For an algebra $\alpha$, we define entailment as follows.

- It is $U \vdash_\alpha *_\alpha$, for all $U \in \mathsf{Con}_\alpha$.
- If 0 is a nullary constructor, then $0 + \cdots + 0 \vdash 0$; if $U_1 \vdash_\alpha a_1, \ldots, U_r \vdash_\alpha a_r$ then $\dot{C}(U_1 \cdots U_r) \vdash_\alpha Ca_1 \cdots a_r$, for every $r$-ary constructor $C$, $r > 0$.
- If $U \supseteq^* U'$ and $U' \vdash_\alpha a$ then $U \vdash_\alpha a$.

We will write $U \vdash a_1 + \cdots + a_l$ for $\forall_{j=1}^{l} U \vdash a_j$, as usual, and also

$$\prod_{i=1}^{r} U_i \vdash \prod_{i=1}^{r} V_i := \mathop{\forall}_{i=1}^{r} U_i \vdash V_i$$

for matrices.

We show that we have indeed defined a proper entailment relation.

**Theorem 2.10** (Entailment)**.** *Entailment is reflexive, transitive, and propagates consistency, that is,*

1. $a \in U \to U \vdash a$,
2. $U \vdash V \to V \vdash a \to U \vdash a$,
3. $U \vdash a \to U \asymp a$.

*Proof by induction.* For $a = *$ all statements follow directly from the definition, so we assume that $a \neq *$.

*Reflexivity.* In case $* \notin U$, we have

$$\begin{aligned} a \in U &\Rightarrow Ca_1 \cdots a_r \in \dot{C}(U_1 \cdots U_r) \\ &\Rightarrow a_1 \in U_1 \wedge \cdots \wedge a_r \in U_r \\ &\overset{\text{IH}}{\Rightarrow} U_1 \vdash a_1 \wedge \cdots \wedge U_r \vdash a_r \\ &\Rightarrow \dot{C}(U_1 \cdots U_r) \vdash Ca_1 \cdots a_r \\ &\Rightarrow U \vdash a\ . \end{aligned}$$

In case $* \in U$, let $U' \subseteq^* U$ be such that $* \notin U'$; since $a \neq *$, $a \in U$ yields $a \in U'$; by the induction hypothesis, $U' \vdash a$, so $U \vdash a$.

*Transitivity.* In case $U$ and $V$ have no $*$'s, we have

$$\begin{aligned} U \vdash V \wedge V \vdash a &\Rightarrow \dot{C}(U_1 \cdots U_r) \vdash \dot{C}(V_1 \cdots V_r) \wedge \dot{C}(V_1 \cdots V_r) \vdash Ca_1 \cdots a_r \\ &\Rightarrow (U_1 \vdash V_1 \wedge V_1 \vdash a_1) \wedge \ldots \wedge (U_r \vdash V_r \wedge V_r \vdash a_r) \\ &\overset{\text{IH}}{\Rightarrow} U_1 \vdash a_1 \wedge \cdots \wedge U_r \vdash a_r \\ &\Rightarrow \dot{C}(U_1 \cdots U_r) \vdash Ca_1 \cdots a_r \\ &\Rightarrow U \vdash a\ . \end{aligned}$$

In case $* \in U$, let $U' \subseteq^* U$ and $V' \subseteq^* V$ such that $* \notin U', V'$; since $a \neq *$, $U'$ and $V'$ are nonempty, and $U \vdash V \wedge V \vdash a$ yields $U' \vdash V' \wedge V' \vdash a$; by the induction hypothesis, $U' \vdash a$, so $U \vdash a$.

*Propagation.* In case $* \notin U$, we have

$$\begin{aligned} U \vdash a &\Rightarrow \dot{C}(U_1 \cdots U_r) \vdash Ca_1 \cdots a_r \\ &\Rightarrow U_1 \vdash a_1 \wedge \cdots \wedge U_r \vdash a_r \\ &\overset{\text{IH}}{\Rightarrow} U_1 \asymp a_1 \wedge \cdots \wedge U_r \asymp a_r \\ &\Rightarrow \dot{C}(U_1 \cdots U_r) \asymp Ca_1 \cdots a_r \\ &\Rightarrow U \asymp a\ . \end{aligned}$$

In case $* \in U$, let $U' \subseteq^* U$ be such that $* \notin U'$; since $a \neq *$, $U \vdash a$ yields $U' \vdash a$; by the induction hypothesis, $U' \asymp a$, so $U \asymp a$, since $* \asymp a$. □

The two entailments have indeed the relationship that we would expect.

**Proposition 2.11.** *The following hold.*

1. $U \succ b \leftrightarrow \exists_{a \in U} [a] \vdash b$,
2. $U \succ b \rightarrow U \vdash b$.

*Proof.* Let $U = a_1 + \cdots + a_l$. For the statement 1: If $U \succ b$, that is, $a_1 + \cdots + a_l \succ b$, then by the definition of entailment in $M\alpha$ it is $\exists_{j=1}^{l} a_j \succ b$; by the definition of atomic entailment in $\alpha$ it is

$$\exists_{j=1}^{l} \left( b = * \vee (a_j = 0 \wedge b = 0) \vee (a_j = C a_{1j} \cdots a_{rj} \wedge b = C b_1 \cdots b_r \wedge \forall_{i=1}^{r} a_{ij} \succ b_i) \right) ;$$

but this, by the definition of (non-atomic) entailment, is $\exists_{j=1}^{l} [a_j] \vdash b$.

For the statement 2: If $U \succ b$, that is, $a_1 + \cdots + a_l \succ b$, then by 1 it is $\exists_{j=1}^{l} [a_j] \vdash b$; by Proposition 2.10, this implies that $\exists_{j=1}^{l} ([a_j] \vdash b \wedge U \vdash a_j)$, which in turn implies that $U \vdash b$. □

## Constructor contexts

As we saw, the application of a single constructor operator to a matrix results in a consistent list, that is, a neighborhood. Put conversely, a neighborhood is nothing but *a matrix with a coefficient consisting of one constructor*[3]. As matrices are generalizations of neighborhoods, we may accordingly generalize this application to the case where the coefficient is a whole *vector* of constructors $(C_1, \ldots, C_k)$, with respective arities $r_1, \ldots, r_k$. This vector induces a constructor operator vector $(\dot{C}_1, \ldots, \dot{C}_k) : \mathsf{Mat}_\alpha(r_1 + \cdots + r_k) \rightarrow \mathsf{Mat}_\alpha(k)$ in the following way:

$$(\dot{C}_1, \ldots, \dot{C}_k) \begin{bmatrix} a_{11} & \cdots & a_{1l} \\ \vdots & \ddots & \vdots \\ a_{r_1 1} & \cdots & a_{r_1 l} \\ & \vdots & \\ a_{r_1+\cdots+r_{k-1}+1,1} & \cdots & a_{r_1+\cdots+r_{k-1}+1,l} \\ \vdots & \ddots & \vdots \\ a_{r_1+\cdots+r_k,1} & \cdots & a_{r_1+\cdots+r_k,l} \end{bmatrix}$$

$$:= \begin{bmatrix} C_1 a_{11} \cdots a_{r_1 1} & \cdots & C_1 a_{1l} \cdots a_{r_1 l} \\ \vdots & \ddots & \vdots \\ C_k a_{r_1+\cdots+r_{k-1}+1,1} \cdots a_{r_1+\cdots+r_k,1} & \cdots & C_k a_{r_1+\cdots+r_{k-1}+1,l} \cdots a_{r_1+\cdots+r_k,l} \end{bmatrix}$$

—in sigma-pi notation:

$$(\dot{C}_1, \ldots, \dot{C}_k) \left( \sum_{j=1}^{l} \prod_{i=1}^{r_1} a_{ij} \cdots \prod_{i=r_1+\cdots+r_{k-1}+1}^{r_1+\cdots+r_k} a_{ij} \right) = \sum_{j=1}^{l} \prod_{i=1}^{k} C_i a_{r_1+\cdots+r_{i-1}+1,j} \cdots a_{r_1+\cdots+r_i,j}.$$

[3] This is not strictly true when we consider lists where some of their elements are $*$; but we will later see that this is not an essential problem, as every list can be "homogenized" without altering its information content (see from page 69 on).

Finally in practice we will also need *blank* or *identity operators* $\dot{*}_\alpha : \mathsf{Mat}_\alpha(1,l) \to \mathsf{Mat}_\alpha(1,l)$, defined by

$$\dot{*}_\alpha(U) := U ,$$

for every list $U \in \mathsf{Mat}_\alpha(1,l)$.

Actually, in view of the following factorization properties and the resulting normal form of matrices, we will have to deal with even more general "coefficients" than simple operators or vectors of operators. We generalize and formalize the above by introducing the notion of a *constructor context* $K$; these, in contrast to the token terms, can be visualized as trees that may in general be "leafless", since nullary constructors are kept out of the factorization we have in mind, as well as "rootless", as we allow for vectors of operators.

We define the terms $K$ mutually with their *left arity* $\mathrm{lar}(K)$ and *right arity* $\mathrm{rar}(K)$, and we write $K \in \mathsf{Kon}_\alpha(\mathrm{lar}(K), \mathrm{rar}(K))$, as follows:

- if $C$ is a supernullary constructor of $\alpha$, then $\dot{C} \in \mathsf{Kon}_\alpha(1, \mathrm{ar}(C))$; moreover, $\dot{*}_\alpha \in \mathsf{Kon}_\alpha(1,1)$;
- if $K_1 \in \mathsf{Kon}_\alpha(s_1, s'_1), \dots, K_r \in \mathsf{Kon}_\alpha(s_r, s'_r)$, then $(K_1, \dots, K_r) \in \mathsf{Kon}_\alpha(r, s'_1 + \cdots + s'_r)$;
- if $K_1 \in \mathsf{Kon}_\alpha(s_1, s)$, $K_2 \in \mathsf{Kon}_\alpha(s, s_2)$, then $K_1 K_2 \in \mathsf{Kon}_\alpha(s_1, s_2)$.

Notice that it is $\mathrm{lar}(K) \leq \mathrm{rar}(K)$ for every constructor context $K$. Write $(K_1, \dots, K_m, K'_1, \dots, K'_n)$ for $((K_1, \dots, K_m), (K'_1, \dots, K'_n))$.

Define the application of a constructor context $K$ with right arity $\mathrm{rar}(K) = r$ to a matrix $A \in \mathsf{Mat}_\alpha(r)$ inductively, as expected.

- If $K = \dot{C}$ then $K(A) := \dot{C}(A)$ as above; if $K = \dot{*}$, then $\dot{*}(A) := A$.
- If $K = (K_1, \dots, K_m)$ and $A = A_1 \cdots A_m$, with $A_i \in \mathsf{Mat}_\alpha(\mathrm{rar}(K_i))$, then $K(A) := K_1(A_1) \cdots K_m(A_m)$.
- If $K = K_1 K_2$, then $K(A) = K_1(K_2(A))$.

**Proposition 2.12.** *Application of constructor contexts preserves matrix consistency and entailment, that is, for $A, B \in \mathsf{Mat}_\alpha(r)$ and $K \in \mathsf{Kon}_\alpha$, with $\mathrm{rar}(K) = r$, the following hold.*

1. $A \asymp B \to K(A) \asymp K(B)$,
2. $A \vdash B \to K(A) \vdash K(B)$.

*Proof by induction on K. For* (1). Let $A \asymp B$. If $K = (\dot{*}, \dots, \dot{*})$ ($r$ times), then there is nothing to show. If $K = \dot{C}$ with $\mathrm{ar}(C) = r$, then

$$\dot{C}(A) = \dot{C}(A_1 \cdots A_r) \overset{\asymp_\alpha}{\asymp} \dot{C}(B_1 \cdots B_r) = \dot{C}(B) ,$$

where $A_i, B_i \in \mathsf{Mat}_\alpha(1)$, for all $i = 1, \dots, r$. If $K = (K_1, \dots, K_s)$, $s \leq r$, then

$$(K_1, \dots, K_s)(A) = K_1(A_1) \cdots K_s(A_s) \overset{\text{IH}}{\asymp} K_1(B_1) \cdots K_s(B_s) = (K_1, \dots, K_s)(B) ,$$

where $A_i, B_i \in \mathsf{Mat}_\alpha(\mathrm{rar}(K_i))$, for all $i = 1, \dots, s$. If $K = K_1 K_2$, then

$$(K_1 K_2)(A) = K_1(K_2(A)) \overset{\text{IH}}{\asymp} K_1(K_2(B)) = (K_1 K_2)(B) .$$

*For* (2). Let $A \vdash B$. If $K = (*, \dots, *)$ ($r$ times), then there is nothing to show. If $K = \dot{C}$ with $\mathrm{ar}(C) = r$, then

$$\dot{C}(A) = \dot{C}(A_1 \cdots A_r) \overset{\vdash_\alpha}{\vdash} \dot{C}(B_1 \cdots B_r) = \dot{C}(B) \ ,$$

where $A_i, B_i \in \mathsf{Mat}_\alpha(1)$, for all $i = 1, \dots, r$. If $K = (K_1, \dots, K_s)$, $s \le r$, then

$$(K_1, \dots, K_s)(A) = K_1(A_1) \cdots K_s(A_s) \overset{\text{IH}}{\vdash} K_1(B_1) \cdots K_s(B_s) = (K_1, \dots, K_s)(B) \ ,$$

where $A_i, B_i \in \mathsf{Mat}_\alpha(\mathrm{rar}(K_i))$, for all $i = 1, \dots, s$. If $K = K_1 K_2$, then

$$(K_1 K_2)(A) = K_1(K_2(A)) \overset{\text{IH}}{\vdash} K_1(K_2(B)) = (K_1 K_2)(B) \ . \qquad \square$$

*Remark.* Notice that application of a constructor context to a matrix is *not* compatible with matrix entailment $\succ_\alpha$ as it is with matrix consistency:

$$A \succ_\alpha B \not\Rightarrow K(A) \succ_\alpha K(B) \ .$$

A Coquand counterexample (page 47) helps here again:

$$\begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix} \succ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \wedge \dot{B} \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix} \not\succ \dot{B} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \ .$$

This comes as no surprise, since we used application exactly in order to define the more intricate non-atomic entailment (see page 63).

On the other hand of course, it is

$$A \succ_\alpha B \overset{\text{Pr. 2.11 (2)}}{\Rightarrow} A \vdash_\alpha B \Rightarrow K(A) \vdash_\alpha K(B) \ . \qquad \square$$

The next proposition gathers some simple but quite important properties concerning the application of a constructor context to a matrix.

**Proposition 2.13** (Factorization). *Let $K$, $K'$, $K_1, \dots, K_r$ be constructor contexts of $\alpha$, with $\mathrm{rar}(K) = r$, and $A, A', A_1, \dots A_r \in \mathsf{Mat}_\alpha$. The following hold whenever well-defined:*

1. $K(A) + K(A') = K(A + A')$.
2. $K(A) \cdot K'(A') = (K, K')(A \cdot A')$.
3. $K(K_1(A_1) \cdot \cdots \cdot K_r(A_r)) = K(K_1, \dots, K_r)(A_1 \cdot \cdots \cdot A_r)$.
4. $(\dots, K, \dots)(\dots, K_1, \dots, K_r, \dots)(A) = (\dots, K(K_1, \dots, K_r), \dots)(A)$.

An easy consequence of this proposition (in particular, of the fourth statement) is that any constructor context attains a *normal form*, which can be stated as follows.

- Every $\dot{C}$ is in normal form; $(*, \dots, *)$ is also in normal form.
- If $K, K_1, \dots, K_r$ are in normal form, with $K$ and at least one $K_i$ proper (that is, not $*$), then $K(K_1, \dots, K_r)$ is in normal form.
- If $K_1, \dots, K_r$ are in normal form, with at least one $K_i$ proper, then $(K_1, \dots, K_r)$ is in normal form.

So, $(\dot{B},\dot{S})(*,\dot{B},*)$ is not in normal form, but $(\dot{B}(*,\dot{B}),\dot{S})$ is. Note also, that the only case where we allow for a blank constructor context in a normal form, is when it is the only constructor context present.

It turns out that constructor contexts of an acis may be organized themselves into an acis when regarded as tokens. Write $\mathsf{Kon}_\alpha$ for the set of all constructor contexts of $\alpha$; define consistency between them by the following inductive clauses (we drop the subscripts):

- for all $K \in \mathsf{Kon}$, it is $K \asymp K$; moreover, it is $K \asymp *$ and $* \asymp K$;
- if $K_1 \asymp K'_1, \dots, K_r \asymp K'_r$, then $(K_1,\dots,K_r) \asymp (K_1,\dots,K_r)$;
- if $K_1 \asymp K'_1$ and $K_2 \asymp K'_2$, then $K_1K_2 \asymp K'_1K'_2$;
- if $K_1 \asymp K'_1$, then $K_1K \asymp K'_1$ and $K_1 \asymp K'_1K$, for all $K$ with appropriate left arity, that is, $\mathrm{lar}(K) = \mathrm{rar}(K_1) = \mathrm{rar}(K'_1)$;

define their entailment by the following ones:

- for all $K \in \mathsf{Kon}$, it is $K \succ K$; moreover, it is $K \succ *$;
- if $K_1 \succ K'_1, \dots, K_r \succ K'_r$, then $(K_1,\dots,K_r) \succ (K_1,\dots,K_r)$;
- if $K_1 \succ K'_1$ and $K_2 \succ K'_2$, then $K_1K_2 \succ K'_1K'_2$;
- if $K_1 \succ K'_1$, then $K_1K \succ K'_1$, for all $K$ with appropriate left arity, that is, $\mathrm{lar}(K) = \mathrm{rar}(K_1)$.

**Proposition 2.14.** *The triple $K(\alpha) = (\mathsf{Kon}_\alpha, \asymp, \succ)$ is an acis.*

*Proof by induction. Reflexivity of consistency.* It is $* \asymp *$ and $\dot{C} \asymp \dot{C}$ for every constructor $C$. If $K_1 \asymp K_1, \dots, K_r \asymp K_r$ then $(K_1,\dots,K_r) \asymp (K_1,\dots,K_r)$. If $K_1 \asymp K_1$ and $K_2 \asymp K_2$ then $K_1K_2 \asymp K_1K_2$.

*Symmetry of consistency.* It is $* \asymp K$ and $K \asymp *$ for every $K$, and if $\dot{C}_1 \asymp \dot{C}_2$ then $C_1 = C_2$, so $\dot{C}_2 \asymp \dot{C}_1$ as well. Let $(K_1,\dots,K_r) \asymp (K'_1,\dots,K'_r)$; then $K_1 \asymp K'_1, \dots, K_r \asymp K'_r$; by the induction hypothesis, $K'_1 \asymp K_1, \dots, K'_r \asymp K_r$; then $(K'_1,\dots,K'_r) \asymp (K_1,\dots,K_r)$. Let $K_1K_2 \asymp K'_1K'_2$; then $K_1 \asymp K'_1$ and $K_2 \asymp K'_2$; by the induction hypothesis, $K'_1 \asymp K_1$ and $K'_2 \asymp K_2$, so $K'_1K'_2 \asymp K_1K_2$. For the unequal length cases, let $K_1K \asymp K'_1$; then $K_1 \asymp K'_1$; by the induction hypothesis, $K'_1 \asymp K_1$, so $K'_1 \asymp K_1K$ (similarly for the other case).

*Reflexivity of entailment* is shown just like the reflexivity of consistency.

*Transitivity of entailment.* The cases involving $*$'s are easy; if $\dot{C}_1 \succ \dot{C}_2$ and $\dot{C}_2 \succ \dot{C}_3$, then $C_1 = C_2 = C_3$, so $\dot{C}_1 \succ \dot{C}_3$. Let $(K_1,\dots,K_r) \succ (K'_1,\dots,K'_r)$ and $(K'_1,\dots,K'_r) \succ (K''_1,\dots,K''_r)$; then $K_1 \succ K'_1 \wedge K'_1 \succ K''_1, \dots, K_r \succ K'_r \wedge K'_r \succ K''_r$; by the induction hypothesis, $K_1 \succ K''_1 \dots, K_r \succ K''_r$, so $(K_1,\dots,K_r) \succ (K''_1,\dots,K''_r)$. Let $K_1K_2 \succ K'_1K'_2$ and $K'_1K'_2 \succ K''_1K''_2$; then $K_1 \succ K'_1 \wedge K'_1 \succ K''_1$ and $K_2 \succ K'_2 \wedge K'_2 \succ K''_2$; by the induction hypothesis, $K_1 \succ K''_1$ and $K_2 \succ K''_2$, so $K_1K_2 \succ K''_1K''_2$. The only unequal length case that makes sense is $K_1K \succ K'_1$ and $K'_1 \succ K''_1$; then $K_1 \succ K'_1$, and by the induction hypothesis $K_1 \succ K''_1$, so $K_1K \succ K''_1$.

*Propagation of consistency.* Again, the $*$ cases are easy; if $\dot{C}_1 \asymp \dot{C}_2$ and $\dot{C}_2 \succ \dot{C}_3$, then $C_1 = C_2 = C_3$, so $\dot{C}_1 \asymp \dot{C}_3$. Let $(K_1,\dots,K_r) \asymp (K'_1,\dots,K'_r)$ and $(K'_1,\dots,K'_r) \succ (K''_1,\dots,K''_r)$; then $K_1 \asymp K'_1 \wedge K'_1 \succ K''_1, \dots, K_r \asymp K'_r \wedge K'_r \succ K''_r$; by the induction hypothesis, $K_1 \asymp K''_1 \dots, K_r \asymp K''_r$, so $(K_1,\dots,K_r) \asymp (K''_1,\dots,K''_r)$. Let $K_1K_2 \asymp K'_1K'_2$ and $K'_1K'_2 \succ K''_1K''_2$; then $K_1 \asymp K'_1 \wedge K'_1 \succ K''_1$ and $K_2 \asymp K'_2 \wedge K'_2 \succ K''_2$; by the induction hypothesis, $K_1 \doteq\!\!\asymp K_1''$ and $K_2 \doteq\!\!\asymp K_2''$, so $K_1K_2 \doteq\!\!\asymp K_1''K_2''$. The only unequal length case that makes sense here is $K_1 \doteq\!\!\asymp K_1'K$ and $K_1'K \dot{\succ} K_1''$; then $K_1 \doteq\!\!\asymp K_1'$ and $K_1' \dot{\succ} K_1''$; by the induction hypothesis $K_1 \doteq\!\!\asymp K_1''$. □

## Homogeneous form of algebraic neighborhoods

We will employ a notion of "homogenization" of a token $a$ with respect to a token $b$, the intuitive meaning being that $a$ may rise up to the "structure" of $b$ without acquiring more information content than the latter has; as we will eventually understand, the "structure" of a token is essentially determined by the constructor context serving as its "coefficient".

Define the *homogenization* $h_b(a)$ *of* $a$ *with respect to* $b$ by the following:

- if $b$ is a *nullary token*, that is, either $*$ or some nullary constructor, then $h_b(a) := a$;
- if $b = C_b b_1 \cdots b_{r_b}$ and $a = C_a a_1 \cdots a_{r_a}$, $r_b, r_a > 0$, with $C_b \neq C_a$, then $h_b(a) := a$;
- if $b = Cb_1 \cdots b_r$, $r > 0$, and $a = *$, then $h_b(*) := Ch_{b_1}(*) \cdots h_{b_r}(*)$;
- if $b = Cb_1 \cdots b_r$ and $a = Ca_1 \cdots a_r$, $r > 0$, then $h_b(Ca_1 \cdots a_r) := Ch_{b_1}(a_1) \cdots h_{b_r}(a_r)$.

The crucial clauses of the definition are the middle two. The second one expresses that at each step of the procedure we check the *head* of the tokens in question, and proceed only if consistency still holds. Call $a$ and $b$ *head-consistent* if they start with the same constructor; two consistent tokens that are different than $*$, are always head-consistent, while the converse doesn't hold for supernullary constructors:

$$B{*}0 \not\asymp B(S*)(S*) \; , \;\; S(S0) \not\asymp S0 \; .$$

The third clause is the drastic one: if $a$ happens to be uninformative, we "lift" it to the "structure" of $b$ in the *least informative way* and we proceed; notice also that the "lifting" takes place only when we encounter supernullary constructors.

**Proposition 2.15.** *For the token-mapping* $h : T_\alpha \times T_\alpha \to T_\alpha$ *the following hold.*

$$h_b(a) \succ a \tag{2.1}$$
$$b \asymp a \to b \asymp h_b(a) \; , \tag{2.2}$$
$$b \succ a \to b \succ h_b(a) \; , \tag{2.3}$$
$$b \asymp b' \to h_b(h_{b'}(a)) = h_{b'}(h_b(a)) \; , \tag{2.4}$$
$$b \succ b' \to h_b(a) = h_b(h_{b'}(a)) \; , \tag{2.5}$$
$$a \asymp b \to a' \asymp b \to a \asymp a' \to h_b(a) \asymp h_b(a') \; , \tag{2.6}$$
$$a \asymp b \to a \asymp b' \to b \asymp b' \to h_b(a) \asymp h_{b'}(a) \; , \tag{2.7}$$
$$a \asymp b \to a \succ a' \to h_b(a) \succ h_b(a') \; , \tag{2.8}$$
$$a \asymp b \to b \succ b' \to h_b(a) \succ h_{b'}(a) \; , \tag{2.9}$$
$$a \succ b \to a \succ c \to a \succ h_b(c) \; . \tag{2.10}$$

*Proof. For* (2.1). For $b$ and $a$ head-inconsistent or $b$ nullary, there is nothing to show. Let $b = Cb_1 \cdots b_r$; for $a = *$ it is again immediate; for $a = Ca_1 \cdots a_r$, it is:

$$h_b(a) = h_{Cb_1 \cdots b_r}(Ca_1 \cdots a_r) = Ch_{b_1}(a_1) \cdots h_{b_r}(a_r) \overset{\text{IH}}{\succ} Ca_1 \cdots a_r = a \; .$$

*For* (2.2). Let $b \asymp a$. If $b$ is nullary, then it is immediate. Let $b = Cb_1 \cdots b_r$; for $a = *$ it is

$$h_b(a) = h_{Cb_1 \cdots b_r}(*) = Ch_{b_1}(*) \cdots h_{b_r}(*) \overset{\text{IH}}{\asymp} Cb_1 \cdots b_r = b \ ;$$

for $a = Ca_1 \cdots a_r$ it is

$$h_b(a) = h_{Cb_1 \cdots b_r}(Ca_1 \cdots a_r) = Ch_{b_1}(a_1) \cdots h_{b_r}(a_r) \overset{\text{IH}}{\asymp} Cb_1 \cdots b_r = b \ .$$

*For* (2.3). Let $b \succ a$; then, by the propagation, $b \asymp a$. If $b$ is nullary, then it is immediate. Let $b = Cb_1 \cdots b_r$; for $a = *$ it is

$$h_b(a) = h_{Cb_1 \cdots b_r}(*) = Ch_{b_1}(*) \cdots h_{b_r}(*) \overset{\text{IH}}{\prec} Cb_1 \cdots b_r = b \ ;$$

for $a = Ca_1 \cdots a_r$ it is

$$h_b(a) = h_{Cb_1 \cdots b_r}(Ca_1 \cdots a_r) = Ch_{b_1}(a_1) \cdots h_{b_r}(a_r) \overset{\text{IH}}{\prec} Cb_1 \cdots b_r = b \ .$$

*For* (2.4). Let $b \asymp b'$. If one of them has no information, say $b' = *$, then

$$h_b(h_{b'}(a)) = h_b(a) = h_{b'}(h_b(a)) \ .$$

Let $b = Cb_1 \cdots b_r$ and $b' = Cb'_1 \cdots b'_r$. If $b'$ and $a$ are head-inconsistent, then $b$ and $a$ are also head-inconsistent, and it is

$$h_b(h_{b'}(a)) = h_b(a) = a = h_{b'}(a) = h_{b'}(h_b(a)) \ .$$

For $a = *$ it is

$$\begin{aligned} h_b(h_{b'}(a)) &= h_b(h_{Cb'_1 \cdots b'_r}(*)) \\ &= h_{Cb_1 \cdots b_r}(Ch_{b'_1}(*) \cdots h_{b'_r}(*)) \\ &= Ch_{b_1}(h_{b'_1})(*) \cdots h_{b_r}(h_{b'_r})(*) \\ &\overset{\text{IH}}{=} Ch_{b'_1}(h_{b_1}(*)) \cdots h_{b'_r}(h_{b_r}(*)) \\ &= h_{b'}(h_b(a)) \ , \end{aligned}$$

and for $a = Ca_1 \cdots a_r$ it is

$$\begin{aligned} h_b(h_{b'}(a)) &= h_b(h_{Cb'_1 \cdots b'_r}(Ca_1 \cdots a_r)) \\ &= h_{Cb_1 \cdots b_r}(Ch_{b'_1}(a_1) \cdots h_{b'_r}(a_r)) \\ &= Ch_{b_1}(h_{b'_1})(a_1) \cdots h_{b_r}(h_{b'_r})(a_r) \\ &\overset{\text{IH}}{=} Ch_{b'_1}(h_{b_1})(a_1) \cdots h_{b'_r}(h_{b_r})(a_r) \\ &= h_{b'}(h_b(a)) \ . \end{aligned}$$

*For* (2.5). For $b'$ and $a$ head-inconsistent or $b'$ nullary, it is $h_b(h_{b'}(a)) = h_b(a)$. Let $b' = Cb'_1 \cdots b'_r$. For $a = *$ it is

$$\begin{aligned} h_b(h_{b'}(a)) &= h_b(h_{Cb'_1 \cdots b'_r}(*)) \\ &= h_{Cb_1 \cdots b_r}(Ch_{b'_1}(*) \cdots h_{b'_r}(*)) \\ &= Ch_{b_1}(h_{b'_1})(*) \cdots h_{b_r}(h_{b'_r})(*) \\ &\overset{\text{IH}}{=} Ch_{b_1}(*) \cdots h_{b_r}(*) \\ &= h_b(a) \ ; \end{aligned}$$

for $a = Ca_1 \cdots a_r$ it is

$$\begin{aligned} h_b(h_{b'}(a)) &= h_b(h_{Cb'_1 \cdots b'_r}(Ca_1 \cdots a_r)) \\ &= h_{Cb_1 \cdots b_r}(Ch_{b'_1}(a_1) \cdots h_{b'_r}(a_r)) \\ &= Ch_{b_1}(h_{b'_1})(a_1) \cdots h_{b_r}(h_{b'_r})(a_r) \\ &\overset{\text{IH}}{=} Ch_{b_1}(a_1) \cdots h_{b_r}(a_r) \\ &= h_b(a) \ . \end{aligned}$$

*For* (2.6). If $b$ is nullary then it is immediately $h_b(a) = a \asymp a' = h_b(a')$. Let $b = Cb_1 \cdots b_r$; for $a = a' = *$, there is nothing to show; let $a = Ca_1 \cdots a_r$; if $a' = *$, then

$$\begin{aligned} h_b(a) &= h_{Cb_1 \cdots b_r}(Ca_1 \cdots a_r) \\ &= Ch_{b_1}(a_1) \cdots h_{b_r}(a_r) \\ &\overset{\text{IH}}{\asymp} Ch_{b_1}(*) \cdots h_{b_r}(*) \\ &= h_{Cb_1 \cdots b_r}(*) \\ &= h_b(a') \ ; \end{aligned}$$

if $a' = Ca'_1 \cdots a'_r$, then

$$\begin{aligned} h_b(a) &= h_{Cb_1 \cdots b_r}(Ca_1 \cdots a_r) \\ &= Ch_{b_1}(a_1) \cdots h_{b_r}(a_r) \\ &\overset{\text{IH}}{\asymp} Ch_{b_1}(a'_1) \cdots h_{b_r}(a'_r) \\ &= h_{Cb_1 \cdots b_r}(Ca'_1 \cdots a'_r) \\ &= h_b(a') \ . \end{aligned}$$

*For* (2.7). If $b$ and $b'$ are nullary, then $h_b(a) = a = h_{b'}(a)$. Let $b' = Cb'_1 \cdots b'_r$. If $b$ is nullary, then for $a = *$ it is immediately $h_b(a) = * \asymp h_{b'}(a)$, and for $a = Ca_1 \cdots a_r$, it is

$$h_b(a) = a = Ca_1 \cdots a_r \overset{(2.1)}{\asymp} Ch_{b'_1}(a_1) \cdots h_{b'_r}(a_r) = h_{Cb'_1 \cdots b'_r}(Ca_1 \cdots a_r) = h_{b'}(a) \ .$$

Further, let $b = Cb_1 \cdots b_r$; if $a = *$, then

$$\begin{aligned} h_b(a) &= h_{Cb_1 \cdots b_r}(*) \\ &= Ch_{b_1}(*) \cdots h_{b_r}(*) \\ &\overset{\text{IH}}{\asymp} Ch_{b'_1}(*) \cdots h_{b'_r}(*) \\ &= h_{Cb'_1 \cdots b'_r}(*) \\ &= h_{b'}(a) \ , \end{aligned}$$

whereas, if $a = Ca_1 \cdots a_r$, it is

$$\begin{aligned} h_b(a) &= h_{Cb_1 \cdots b_r}(a_1 \cdots a_r) \\ &= Ch_{b_1}(a_1) \cdots h_{b_r}(a_r) \\ &\overset{\text{IH}}{\asymp} Ch_{b'_1}(a_1) \cdots h_{b'_r}(a_r) \\ &= h_{Cb'_1 \cdots b'_r}(Ca_1 \cdots a_r) \\ &= h_{b'}(a) \ . \end{aligned}$$

*For* (2.8). Let $a \succ a'$ and $b \asymp a$; by the propagation, it is also $b \asymp a'$. Let $b = Cb_1 \cdots Cb_r$. If $a = *$, then necessarily $a' = *$, and $h_b(a) = h_b(a')$. Let further $a = Ca_1 \cdots a_r$; for $a' = *$, it is

$$\begin{aligned} h_b(a) &= h_{Cb_1\cdots b_r}(Ca_1 \cdots a_r) \\ &= Ch_{b_1}(a_1) \cdots h_{b_r}(a_r) \\ &\overset{\text{IH}}{\succ} Ch_{b_1}(*) \cdots h_{b_r}(*) \\ &= h_{Cb_1\cdots b_r}(*) \\ &= h_b(a') , \end{aligned}$$

whereas, if $a' = Ca'_1 \cdots a'_r$, it is

$$\begin{aligned} h_b(a) &= h_{Cb_1\cdots b_r}(Ca_1 \cdots a_r) \\ &= Ch_{b_1}(a_1) \cdots h_{b_r}(a_r) \\ &\overset{\text{IH}}{\succ} Ch_{b_1}(a'_1) \cdots h_{b_r}(a'_r) \\ &= h_{Cb_1\cdots b_r}(Ca'_1 \cdots a'_r) \\ &= h_b(a') . \end{aligned}$$

*For* (2.9). Let $b \succ b'$ and $b \asymp a$; by the propagation, it is also $b' \asymp a$. If $b$ is nullary, then $b'$ must also be nullary, and $h_b(a) = a = h_{b'}(a)$. Let $b = Cb_1 \cdots b_r$. For $b'$ nullary, it is $h_b(a) \overset{(2.1)}{\succ} a = h_{b'}(a)$. Let then $b' = Cb'_1 \cdots b'_r$ as well. If $a = *$ then

$$\begin{aligned} h_b(a) &= h_{Cb_1\cdots b_r}(*) \\ &= Ch_{b_1}(*) \cdots h_{b_r}(*) \\ &\overset{\text{IH}}{\succ} Ch_{b'_1}(*) \cdots h_{b'_r}(*) \\ &= h_{Cb'_1\cdots b'_r}(*) \\ &= h_{b'}(a) ; \end{aligned}$$

if $a = Ca_1 \cdots a_r$, it is

$$\begin{aligned} h_b(a) &= h_{Cb_1\cdots b_r}(Ca_1 \cdots a_r) \\ &= Ch_{b_1}(a_1) \cdots h_{b_r}(a_r) \\ &\overset{\text{IH}}{\succ} Ch_{b'_1}(a_1) \cdots h_{b'_r}(a_r) \\ &= h_{Cb'_1\cdots b'_r}(Ca_1 \cdots a_r) \\ &= h_{b'}(a) . \end{aligned}$$

*For* (2.10). Let $a \succ b$ and $a \succ c$; then it is $b \asymp c$. If $a$ is nullary then $b$, $c$ must also be nullary, so we don't have anything to do. Let $a = Ca_1 \cdots a_r$. For $b$ nullary, there is again nothing to show, so let also $b = Cb_1 \cdots b_r$. Now if $c$ is nullary we have

$$h_b(c) = h_{Cb_1\cdots b_r}(*) = Ch_{b_1}(*) \cdots h_{b_r}(*) \overset{\text{IH}}{\prec} Ca_1 \cdots a_r = a ,$$

where if $c = Cc_1 \cdots c_r$, we have

$$h_b(c) = h_{Cb_1\cdots b_r}(Cc_1 \cdots c_r) = Ch_{b_1}(c_1) \cdots h_{b_r}(c_r) \overset{\text{IH}}{\prec} Ca_1 \cdots a_r = a . \qquad \square$$

Now we gradually extend the notion of homogenization to matrices, in the following way:

$$\begin{aligned} h_{b_1+\cdots+b_{l'}}(a) &:= h_{b_{l'}}\cdots h_{b_1}(a) \ , \\ h_V(a_1+\cdots+a_l) &:= h_V(a_1)+\cdots+h_V(a_l) \ , \\ h_{V_1\cdots V_r}(U_1\cdots U_r) &:= h_{V_1}(U_1)\cdots h_{V_r}(U_r) \ , \end{aligned}$$

where $U_i$'s and $V_i$'s, $i=1,\dots,r$, are lists of equal length, and we write $h_1\cdots h_l$ for $h_1\circ\cdots\circ h_l$ to save space. Put more bluntly, the *homogenization* $h_B(A)$ *of* $A\in\mathsf{Arr}_\alpha(r,l)$ *with respect to* $B\in\mathsf{Arr}_\alpha(r,l')$ is defined by

$$h_{\prod_{i=1}^r\sum_{j'=1}^{l'}b_{ij'}}\left(\prod_{i=1}^r\sum_{j=1}^l a_{ij}\right) := \prod_{i=1}^r\sum_{j=1}^l h_{b_{il'}}\cdots h_{b_{i1}}(a_{ij}) \ ;$$

it is obviously well-defined.[4] We will just write $h(A)$ for $h_A(A)$; call $A$ *homogeneous* if it is already $A=h(A)$.

**Proposition 2.16.** *The mapping* $h:\mathsf{Arr}_\alpha(r,l')\times\mathsf{Arr}_\alpha(r,l)\to\mathsf{Arr}_\alpha(r,l)$ *is well-defined and satisfies the following.*

$$\begin{aligned}
&h_B(A)\succ A \ , && (2.11)\\
&B\asymp B'\to h_B(h_{B'}(A))=h_{B'}(h_B(A)) \ , && (2.12)\\
&B\succ B'\to h_B(A)=h_B(h_{B'}(A)) \ , && (2.13)\\
&A\asymp B\to(A'\asymp B\vee A\asymp B')\to A'\asymp B' && \\
&\qquad\to A\asymp A'\to B\asymp B'\to h_B(A)\asymp h_{B'}(A') \ , && (2.14)\\
&A\asymp B\to A\succ A'\to B\succ B'\to h_B(A)\succ h_{B'}(A') \ , && (2.15)\\
&A\asymp B\to A\vdash A'\to h_B(A)\vdash h_B(A') \ , && (2.16)\\
&A\asymp B\to B\vdash B'\to h_B(A)\vdash h_{B'}(A) \ , && (2.17)\\
&A\vdash h(A) \ , && (2.18)\\
&h(K(A))=K(h(A)) \ , && (2.19)\\
&A=K(B)\to(A=h(A)\leftrightarrow B=h(B)) \ , && (2.20)\\
& && (2.21)
\end{aligned}$$

*Proof.* That the mapping is well-defined is clear. For the following, let $A\in\mathsf{Mat}(r,l_A)$, $B\in\mathsf{Mat}(r,l_B)$, $A'\in\mathsf{Mat}(r,l_{A'})$, and $B'\in\mathsf{Mat}(r,l_{B'})$.

*For* (2.11). It is

$$h_B(A)=\prod_{i=1}^r\sum_{j_A=1}^{l_A}h_{b_{il_B}}\cdots h_{b_{i1}}(a_{ij_A})\overset{(2.1)}{\succ}\prod_{i=1}^r\sum_{j_A=1}^{l_A}a_{ij_A}=A \ .$$

[4]Notice though that the value in general depends on the order of the elements in the list; for example,

$$h_{S*+B**}(*)=S*\neq B**=h_{B**+S*}(*) \ .$$

This is not the case though when the list is consistent, that is, a neighborhood, as we will readily see.

*For* (2.12). It is

$$\begin{aligned} h_B(h_{B'}(A)) &= \prod_{i=1}^{r} \sum_{j_A=1}^{l_A} h_{b_{il_B}} \cdots h_{b_{i1}} h_{b'_{il_{B'}}} \cdots h_{b'_{i1}}(a_{ij_A}) \\ &\overset{(2.4)}{=} \prod_{i=1}^{r} \sum_{j_A=1}^{l_A} h_{b'_{il_{B'}}} \cdots h_{b'_{i1}} h_{b_{il_B}} \cdots h_{b_{i1}}(a_{ij_A}) \\ &= h_{B'}(h_B(A)) \ . \end{aligned}$$

*For* (2.13). It is

$$\begin{aligned} h_B(h_{B'}(A)) &= \prod_{i=1}^{r} \sum_{j_A=1}^{l_A} h_{b_{il_B}} \cdots h_{b_{i1}} h_{b'_{il_{B'}}} \cdots h_{b'_{i1}}(a_{ij_A}) \\ &\overset{(2.4),\,(2.5)}{=} \prod_{i=1}^{r} \sum_{j_A=1}^{l_A} h_{b_{il_B}} \cdots h_{b_{i1}}(a_{ij_A}) \\ &= h_B(A) \ . \end{aligned}$$

*For* (2.14). Let $A \asymp B$, $A' \asymp B$, and then $A \asymp A'$, and $B \asymp B'$. It is

$$\begin{aligned} h_B(A) &= \prod_{i=1}^{r} \sum_{j_A=1}^{l_A} h_{b_{il_B}} \cdots h_{b_{i1}}(a_{ij_A}) \\ &\overset{(2.6)}{\asymp} \prod_{i=1}^{r} \sum_{j_{A'}=1}^{l_{A'}} h_{b_{il_B}} \cdots h_{b_{i1}}(a'_{ij_{A'}}) \\ &\overset{(2.7),\,(2.4),\,(2.5)}{\asymp} \prod_{i=1}^{r} \sum_{j_{A'}=1}^{l_{A'}} h_{b'_{il_{B'}}} \cdots h_{b'_{i1}}(a'_{ij_{A'}}) \\ &= h_{B'}(A') \ . \end{aligned}$$

The assumption that $A' \asymp B$ enabled us to take the second step; we would need the alternative assumption $A \asymp B'$ if we wanted to use (2.7) first, and then (2.6).

*For* (2.15). Let $A \asymp B$, $A \succ A'$, and $B \succ B'$. It is

$$\begin{aligned} h_B(A) &= \prod_{i=1}^{r} \sum_{j_A=1}^{l_A} h_{b_{il_B}} \cdots h_{b_{i1}}(a_{ij_A}) \\ &\overset{(2.8)}{\succ} \prod_{i=1}^{r} \sum_{j_{A'}=1}^{l_{A'}} h_{b_{il_B}} \cdots h_{b_{i1}}(a'_{ij_{A'}}) \\ &\overset{(2.9),\,(2.4),\,(2.5)}{\succ} \prod_{i=1}^{r} \sum_{j_{A'}=1}^{l_{A'}} h_{b'_{il_{B'}}} \cdots h_{b'_{i1}}(a'_{ij_{A'}}) \\ &= h_{B'}(A') \ . \end{aligned}$$

For the following statements, since entailment between matrices works row-wise, we restrict our arguments to lists with no loss of generality. Furthermore, because of (2.5), we may safely *assume that the lists $V$, $V'$, with respect to which we homogenize, will consist of tokens where none entails some other* (we make this assumption in the proofs of (2.16) and (2.17)). In this case, for a non-blank list $V$ of length $l'$ and a list $U$ of length $l$, such that $U \asymp V$, we either have $U$ non-blank as well, in which case

$$\begin{aligned}
h_V(U) &= h_{b_1+\cdots+b_{l'}}(a_1+\cdots+a_l)\\
&= h_{b_1+\cdots+b_{l'}}(a_1)+\cdots+h_{b_1+\cdots+b_{l'}}(a_l)\\
&= h_{b_1}\cdots h_{b_{l'}}(a_1)+\cdots+h_{b_1}\cdots h_{b_{l'}}(a_l)\\
&= h_{Cb_{11}\cdots b_{r1}}\cdots h_{Cb_{1l'}\cdots b_{rl'}}(Ca_{11}\cdots a_{r1})\\
&\qquad +\cdots+h_{Cb_{11}\cdots b_{r1}}\cdots h_{Cb_{1l'}\cdots b_{rl'}}(Ca_{1l}\cdots a_{rl})\\
&= C(h_{b_{11}}\cdots h_{b_{1l'}}(a_{11}))\cdots(h_{b_{r1}}\cdots h_{b_{rl'}}(a_{r1}))\\
&\qquad +\cdots+C(h_{b_{11}}\cdots h_{b_{1l'}}(a_{1l}))\cdots(h_{b_{r1}}\cdots h_{b_{rl'}}(a_{rl}))\\
&= \dot{C}\begin{bmatrix} h_{b_{11}}\cdots h_{b_{1l'}}(a_{11}) & \cdots & h_{b_{11}}\cdots h_{b_{1l'}}(a_{1l})\\ \vdots & \ddots & \vdots\\ h_{b_{r1}}\cdots h_{b_{rl'}}(a_{r1}) & \cdots & h_{b_{r1}}\cdots h_{b_{rl'}}(a_{rl})\end{bmatrix},
\end{aligned}$$

that is,

$$h_{\dot{C}(V_1\cdots V_r)}(\dot{C}(U_1\cdots U_r)) = \dot{C}(h_{V_1}(U_1)\cdots h_{V_r}(U_r)) , \tag{2.22}$$

or $U$ is a blank list, $U = *^{1,l}$, in which case we similarly obtain

$$h_{\dot{C}(V_1\cdots V_r)}(*^{1,l}) = \dot{C}(h_{V_1}(*^{1,l})\cdots h_{V_r}(*^{1,l})) . \tag{2.23}$$

*For* (2.16). We show that

$$U \asymp V \to U \vdash U' \to h_V(U) \vdash h_V(U') ,$$

for $U, U', V$ lists. Let $U \asymp V$ and $U \vdash U'$. We proceed by induction on $V$ (under our assumption above). If $V$ is a nullary (more precisely, it consists of a single nullary token), then $h_V(U) = U \vdash U' = h_V(U')$, so let $V = b_1+\cdots+b_{l_V} = \dot{C}(V_1\cdots V_r)$. For $U = *^{1,l_U}$, it must also be that $U' = *^{1,l_{U'}}$, so

$$h_V(U) = \underbrace{h_V(*)+\cdots+h_V(*)}_{l_U \text{ times}} \succ \underbrace{h_V(*)+\cdots+h_V(*)}_{l_{U'} \text{ times}} = h_V(U') .$$

For $U = \dot{C}(U_1\cdots U_r)$, if $U' = *^{1,l_{U'}}$, it is

$$\begin{aligned}
h_V(U) &= h_{\dot{C}(V_1\cdots V_r)}(\dot{C}(U_1\cdots U_r))\\
&\overset{(2.22)}{=} \dot{C}(h_{V_1}(U_1)\cdots h_{V_r}(U_r))\\
&\overset{\text{IH}}{\vdash} \dot{C}(h_{V_1}(*^{1,l_{U'}})\cdots h_{V_r}(*^{1,l_{U'}}))\\
&\overset{(2.23)}{=} h_{\dot{C}(V_1\cdots V_r)}(*^{1,l_{U'}})\\
&= h_V(U') ,
\end{aligned}$$

whereas if $U' = \dot{C}(U'_1\cdots U'_r)$ it is

$$\begin{aligned}
h_V(U) &= h_{\dot{C}(V_1\cdots V_r)}(\dot{C}(U_1\cdots U_r))\\
&\overset{(2.22)}{=} \dot{C}(h_{V_1}(U_1)\cdots h_{V_r}(U_r))\\
&\overset{\text{IH}}{\vdash} \dot{C}(h_{V_1}(U'_1)\cdots h_{V_r}(U'_r))\\
&\overset{(2.22)}{=} h_{\dot{C}(V_1\cdots V_r)}(U')\\
&= h_V(U') .
\end{aligned}$$

Finally, if $U^* \supseteq^* U$, then

$$\begin{aligned} h_V(U^*) &= h_V(\vec{*} + a_1 + \vec{*} + \cdots + \vec{*} + a_{l_U} + \vec{*}) \\ &= h_V(\vec{*}) + h_V(a_1) + h_V(\vec{*}) + \cdots + h_V(\vec{*}) + h_V(a_{l_U}) + h_V(\vec{*}) \\ &\succ h_V(a_1) + \cdots + h_V(a_{l_U}) \\ &= h_V(a_1 + \cdots + a_{l_U}) \\ &= h_V(U) \\ &\overset{\text{IH}}{\vdash} h_V(U') \ . \end{aligned}$$

*For* (2.17). We show that

$$U \asymp V \to V \vdash V' \to h_V(U) \vdash h_{V'}(U) \ ,$$

for $U, V, V'$ appropriate lists. Let $U \asymp V$ and $V \vdash V'$. According to our assumption, let first $V$ be nullary, $V'$ must also be nullary; then $h_V(U) = U = h_{V'}(U)$. Then let $V = \dot{C}(V_1 \cdots V_r)$; if $V'$ is still a nullary, then, by (2.11), $h_V(U) \succ U = h_{V'}(U)$; if $V' = \dot{C}(V'_1 \cdots V'_r)$, then for $U = *^{1,l_U}$ it is

$$\begin{aligned} h_V(U) &= h_{\dot{C}(V_1 \cdots V_r)}(*^{1,l_U}) \\ &\overset{(2.23)}{=} \dot{C}(h_{V_1}(*^{1,l_U}) \cdots h_{V_r}(*^{1,l_U})) \\ &\overset{\text{IH}}{\vdash} \dot{C}(h_{V'_1}(*^{1,l_U}) \cdots h_{V'_r}(*^{1,l_U})) \\ &\overset{(2.23)}{=} h_{\dot{C}(V'_1 \cdots V'_r)}(*^{1,l_U}) \\ &= h_{V'}(U) \ , \end{aligned}$$

for $U = \dot{C}(U_1 \cdots U_r)$ it is

$$\begin{aligned} h_V(U) &= h_{\dot{C}(V_1 \cdots V_r)}(\dot{C}(U_1 \cdots U_r)) \\ &\overset{(2.22)}{=} \dot{C}(h_{V_1}(U_1) \cdots h_{V_r}(U_r)) \\ &\overset{\text{IH}}{\vdash} \dot{C}(h_{V'_1}(U_1) \cdots h_{V'_r}(U_r)) \\ &\overset{(2.22)}{=} h_{\dot{C}(V'_1 \cdots V'_r)}(\dot{C}(U_1 \cdots U_r)) \\ &= h_{V'}(U) \ , \end{aligned}$$

and for $U^* \supseteq^* U$ it is

$$h_V(U^*) \overset{(2.16)}{\vdash} h_V(U) \overset{\text{IH}}{\vdash} h_{V'}(U) \overset{(2.16)}{\vdash} h_{V'}(U^*) \ .$$

*For* (2.18). We show that

$$U \vdash h(U) \ ,$$

for any list $U$. If $U$ is a nullary, then immediately $h(U) = U$. If $U = \dot{C}(U_1 \cdots U_r)$; then

$$h(U) = h(\dot{C}(U_1 \cdots U_r)) \overset{(2.22)}{=} \dot{C}(h(U_1) \cdots h(U_r)) \overset{\text{IH}}{\dashv} \dot{C}(U_1 \cdots U_r) = U \ .$$

Finally, if $U^* \supseteq^* U$, then

$$U^* \succ U \overset{\text{IH}}{\vdash} h(U) = h_U(U) \overset{(2.17)}{\vdash} h_{U^*}(U) \overset{(2.16)}{\vdash} h_{U^*}(U^*) = h(U^*) \ .$$

*For* (2.19). By induction on the constructor context $K$. For $K = *$, it is immediately $h(*(A)) = h(A) = *(A)$. For $K = \dot{C}$, it is

$$h(\dot{C}(A)) = h(\dot{C}(U_1 \cdots U_r)) \overset{(2.22)}{=} \dot{C}(h(U_1) \cdots h(U_r)) = \dot{C}(h(A)) \ .$$

For $K = (K_1, \ldots, K_m)$, it is

$$\begin{aligned} h(K(A)) &= h((K_1, \ldots, K_m)(A_1 \cdots A_m)) \\ &= h(K_1(A_1) \cdots K_m(A_m)) \\ &= h(K_1(A_1)) \cdots h(K_m(A_m)) \\ &\overset{\text{IH}}{=} K_1(h(A_1)) \cdots K_m(h(A_m)) \\ &= (K_1, \ldots, K_m)(h(A_1) \cdots h(A_m)) \\ &= K(h(A)) \ . \end{aligned}$$

Finally, for $K = K_1 K_2$, it is

$$\begin{aligned} h(K(A)) &= h(K_1 K_2(A)) \\ &= h(K_1(K_2(A_1))) \\ &\overset{\text{IH}}{=} K_1(h(K_2(A_1))) \\ &\overset{\text{IH}}{=} K_1(K_2(h(A_1))) \\ &= K_1 K_2(h(A)) \\ &= K(h(A)) \ . \end{aligned}$$

*For* (2.20). Let $A = K(B)$. We have

$$A = h(A) \Leftrightarrow K(B) = h(K(B)) \overset{(2.19)}{\Leftrightarrow} K(B) = K(h(B)) \Leftrightarrow B = h(B) \ . \qquad \square$$

Now we have the necessary means to state and prove the matrix form theorem.

**Theorem 2.17** (Finitary matrix form)**.** *Let $\alpha$ be a finitary algebra. For every homogeneous matrix $A \in \mathsf{Mat}_\alpha(r,l)$, there exist a unique constructor context $K_A \in \mathsf{Kon}_\alpha(r,r')$ in normal form, $r' \geq r$, and a unique basic matrix $M_A \in \mathsf{Mat}_\alpha(r',l)$, such that*

$$A = K_A(M_A) \ .$$

Call $K_A(M_A)$ the *matrix form* of $A$, $K_A$ the *basic coefficient* of $A$, and $M_A$ the *basis* of $A$.

*Proof by induction on the complexity of $A$.* Let $A \in \mathsf{Mat}_\alpha(r,l)$. If $A$ is already basic, then simply $K_A = (*, \ldots, *)$ ($r$ times) and $M_A = A$.

If $A = K(A')$, for some $K \in \mathsf{Kon}_\alpha(r,r')$ and $A' \in \mathsf{Mat}_\alpha(r',l)$, then, by (2.20), $A'$ is also homogeneous, so by the induction hypothesis it has a matrix form $K_{A'}(M_{A'})$, with $K_{A'} \in \mathsf{Kon}_\alpha(r',r'')$ and $M_{A'} \in \mathsf{Mat}_\alpha(r'',l)$ (it is $r \leq r' \leq r''$); then

$$A = K(A') = K(K_{A'}(M_{A'})) = (KK_{A'})(M_{A'}) \ ;$$

now if $K_{A'} = (*, \ldots, *)$ then $r' = r''$ and $K_A = K \in \mathsf{Kon}_\alpha(r,r'')$, otherwise $K_A = KK_{A'} \in \mathsf{Kon}_\alpha(r,r'')$; for the basis, it is $M_A = M_{A'} \in \mathsf{Mat}_\alpha(r'',l)$. $\square$

The *matrix form*, the *basic coefficient*, and the *basis* of an arbitrary matrix $A$ are defined to be the matrix form, the basic coefficient, and the basis of its homogenization $h(A)$ respectively, in symbols

$$K_A :\equiv K_{h(A)} \wedge M_A :\equiv M_{h(A)} \ ;$$

so the theorem in effect says that *every* matrix has a matrix form, which is unique up to homogenization.

*Example.* Let $\alpha$ be given by a nullary constructor 0, a unary $S$, a binary $B$, and a ternary $C$; we write $*$ for $*_\alpha$. Consider the list

$$U = C(B0*)** + C(B*0)(B*0)* + C**(SS0) + * \ .$$

Its homogenization $h(U)$ is given by

$$C(B0*)(B**)(SS*) + C(B*0)(B*0)(SS*) + C(B**)(B**)(SS0) + C(B**)(B**)(SS*) \ .$$

Exhaustive factorization proceeds as follows:

$$\begin{aligned}
&C(B0*)(B**)(SS*) + C(B*0)(B*0)(SS*) + C(B**)(B**)(SS0) + C(B**)(B**)(SS*) \\
&= \dot{C}\big((B0*)(B**)(SS*) + (B*0)(B*0)(SS*) + (B**)(B**)(SS0) + (B**)(B**)(SS*)\big) \\
&= \dot{C}(\dot{B},\dot{*},\dot{*})\big(0*(B**)(SS*) + *0(B*0)(SS*) + **(B**)(SS0) + **(B**)(SS*)\big) \\
&= \dot{C}(\dot{B},\dot{*},\dot{*})(\dot{*},\dot{*},\dot{B},\dot{*})\big(0***(SS*) + *0*0(SS*) + ****(SS0) + ****(SS*)\big) \\
&= \dot{C}(\dot{B},\dot{*},\dot{*})(\dot{*},\dot{*},\dot{B},\dot{*})(\dot{*},\dot{*},\dot{*},\dot{*},\dot{S})\big(0***(S*) + *0*0(S*) + ****(S0) + ****(S*)\big) \\
&= \dot{C}(\dot{B},\dot{*},\dot{*})(\dot{*},\dot{*},\dot{B},\dot{*})(\dot{*},\dot{*},\dot{*},\dot{*},\dot{S})(\dot{*},\dot{*},\dot{*},\dot{*},\dot{S})\big(0**** + *0*0* + ****0 + *****\big) \\
&= \dot{C}(\dot{B},\dot{B},\dot{S}\dot{S})\big(0**** + *0*0* + ****0 + *****\big)
\end{aligned}$$

so the matrix form of $U$ is

$$U = \dot{C}(\dot{B},\dot{B},\dot{S}\dot{S}) \begin{bmatrix} 0 & * & * & * \\ * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ * & * & 0 & * \end{bmatrix} \ . \qquad \square$$

## Using normal forms

Here we take a break and demonstrate how useful the matrix form can be in practice. Namely, we show that it readily provides a very clear and insightful characterization of entailment, which in turn leads rather automatically to a simple *atomic* characterization.

We will use two natural notions of equivalence for matrices. Call $A, B \in \mathsf{Mat}_\alpha$ *atomically equivalent* (or *atomically equientailing*), and write $A \sim^A B$, when $A \succ B \wedge B \succ A$; call them just *equivalent* (or *equientailing*), and write $A \sim B$, if $A \vdash B \wedge B \vdash A$. For a consistent list $U \in \mathrm{Lst}_\alpha(l)$, define its *induced set* $\mathrm{set}(U)$ by

$$a \in \mathrm{set}(U) :\equiv \mathop{\exists}_{j=1}^{l} U(j) = a \ .$$

**Proposition 2.18.** *The following hold whenever well-defined.*

1. *If $A \sim^A B$ then $A \sim B$.*
2. *It is $A \sim^A B$ if and only if* $\mathrm{set}(A_i) = \mathrm{set}(B_i)$*, for all $i = 1, \dots, r$.*
3. *It is*

$$A \sim^A B \leftrightarrow \underset{M}{\forall}(A \succ M \leftrightarrow B \succ M) \ ,$$
$$A \sim B \leftrightarrow \underset{M}{\forall}(A \vdash M \leftrightarrow B \vdash M) \ .$$

4. *It is*

$$A \sim A' \rightarrow A \asymp B \rightarrow A' \asymp B \ ,$$
$$A \sim^A A' \rightarrow B \sim^A B' \rightarrow \left(A + B \sim^A A' + B' \wedge A \cdot B \sim^A A' \cdot B'\right) \ ,$$
$$A \sim A' \rightarrow B \sim B' \rightarrow \left(A + B \sim A' + B' \wedge A \cdot B \sim A' \cdot B'\right) \ .$$

*Proof.* The first two statements are easy to see. As for 3, it is an immediate consequence of transitivity of entailment, atomic as well as not.

*For 4.* The first clause is an easy consequence of propagation of consistency through entailment. For the second clause, let $A \sim^A A'$ and $B \sim^A B'$; then for any matrix $M$ it is

$$\begin{aligned} A + B \succ M &\Leftrightarrow \overset{r}{\underset{i=1}{\forall}} \overset{l_M}{\underset{j_M=1}{\forall}} \left( \overset{l_A}{\underset{j_A=1}{\exists}} a_{ij_A} \succ c_{ij_M} \vee \overset{l_B}{\underset{j_B=1}{\exists}} b_{ij_B} \succ c_{ij_M} \right) \\ &\Leftrightarrow \overset{r}{\underset{i=1}{\forall}} \overset{l_M}{\underset{j_M=1}{\forall}} \left( \overset{l_{A'}}{\underset{j_{A'}=1}{\exists}} a'_{ij_{A'}} \succ c_{ij_M} \vee \overset{l_{B'}}{\underset{j_{B'}=1}{\exists}} b'_{ij_{B'}} \succ c_{ij_M} \right) \\ &\Leftrightarrow A' + B' \succ M \ , \end{aligned}$$

so $A + B \sim^A A' + B'$; similarly we have

$$A \cdot B \succ M \Leftrightarrow \overset{r}{\underset{i=1}{\forall}} \overset{l_M}{\underset{j_M=1}{\forall}} \overset{l}{\underset{j=1}{\exists}} a_{ij} \succ c_{ij_M} \Leftrightarrow \overset{r}{\underset{i=1}{\forall}} \overset{l_M}{\underset{j_M=1}{\forall}} \overset{l'}{\underset{j'=1}{\exists}} a'_{ij'} \succ c_{ij_M} \Leftrightarrow A' \cdot B' \succ M \ ,$$

that is, $A \cdot B \sim^A A' \cdot B'$. For the third clause, more swiftly, from $A \sim A'$ and $B \sim B'$ we obtain

$$A \vdash A' \wedge B \vdash B' \overset{\text{trns}}{\Rightarrow} A + B \vdash A' \wedge A + B \vdash B' \Rightarrow A + B \vdash A' + B' \ ,$$

and similarly $A' + B' \vdash A + B$; moreover, if $A \sim A' \wedge B \sim B'$, then $AB \sim A'B'$ by the definition of entailment for matrices. □

### Characterization of consistency and entailment through homogenization

Some inspection of examples leads us to acknowledge three different cases when checking if $U \vdash a$, with respective normal forms $K_U(M_U)$ and $K_a(M_a)$. For simplicity (which does not harm generality, as we will show) we assume that both basic matrices have the same row dimension.

- It is $U \asymp a$ and $U \vdash a$; in this case it is necessarily both $K_U \dot{\succ} K_a$ and $M_U \succ M_a$.

- It is $U \asymp a$ but $U \nvdash a$; in this case $a$ holds more information in some positions, either concerning some supernullary constructors or nullary ones, hence it is either $K_U \not\dot{\succ} K_a$ or $M_U \not\succ M_a$ (or both, of course).

- It is $U \not\asymp a$; in this tricky case it is possible to have both $K_U \dot{\succ} K_a$ and $M_U \succ M_a$, as the following example illustrates: if $U = B(C00*)(S0) + B(C00(S0))*$ and $a = B(C000)(S*)$, then the normal form of $U$ is

$$\dot{B}(\dot{C}(\dot{*},\dot{*},\dot{S}),\dot{S}) \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ * & 0 \\ 0 & * \end{bmatrix}$$

and the normal form of $a$ is

$$\dot{B}(\dot{C},\dot{S}) \begin{bmatrix} 0 \\ 0 \\ 0 \\ * \end{bmatrix} ;$$

it is $K_U \dot{\succ} K_a$ and $M_U \succ M_a$, but $U \not\asymp a$, which we can see easily if we factorize just $\dot{B}(\dot{C},\dot{S})$ out of $U$:

$$U = \dot{B}(\dot{C},\dot{S}) \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ S* & S0 \\ 0 & * \end{bmatrix} \not\asymp \dot{B}(\dot{C},\dot{S}) \begin{bmatrix} 0 \\ 0 \\ 0 \\ * \end{bmatrix} .$$

This case indicates that we should be interested in the consistency of the *augmented matrix* $U + a$.

**Proposition 2.19.** *The following hold.*

1. *It is $A \sim h(A)$. Consequently, it is $A \sim B$ if and only if $h(A) \sim h(B)$.*

2. *If $A = A_1 \cdots A_s \in \mathsf{Mat}_\alpha$, then $K_A = (K_{A_1}, \dots, K_{A_s})$ and $M_A = M_{A_1} \cdots M_{A_s}$.*

*Proof. For 1.* By (2.11) and (2.18). The homogeneous characterization follows from transitivity of entailment.

*For 2.* It is

$$A_1 \cdots A_s = K_{A_1}(M_{A_1}) \cdots K_{A_s}(M_{A_s}) \overset{\text{P.2.13 (3)}}{=} (K_{A_1}, \dots, K_{A_s})(M_{A_1} \cdots M_{A_s}) . \qquad \square$$

**Theorem 2.20** (Calculus of homogeneous forms)**.** *Let $U, V \in \mathsf{Con}_\alpha$. The following hold.*

1. *It is $U \asymp V$ if and only if*

$$K_{h_{U+V}(U)} = K_{h_{U+V}(V)} \wedge M_{h_{U+V}(U)} \asymp M_{h_{U+V}(V)} .$$

2. *It is $U \vdash V$ if and only if*

$$K_U = K_{h_U(V)} \wedge M_U \succ M_{h_U(V)} .$$

*Proof.* By Proposition 2.19. 1, it is $U \asymp V$ and $U \vdash V$ if and only if $h(U) \asymp h(V)$ and $h(U) \vdash h(V)$ respectively. Based on this observation, we may assume that the lists we consider are homogeneous, with no loss of generality.

*For 1.* If both lists are nullary, then, for $W := U + V = *^{1,l+l'}$,

$$K_{h_W(U)} = K_U = \dot{*} = K_V = K_{h_W(V)}$$

and

$$U \asymp V \Leftrightarrow M_{h_W(U)} = M_U = U \asymp V = M_V = M_{h_W(V)} \ .$$

If $U = \dot{C}(U_1 \cdots U_r)$ and $V = *^{1,l'}$, for $W := U + V$ we have

$$\begin{aligned} &h_W(U) \overset{(2.13)}{=} h(U) \\ &h_W(V) \overset{(2.13)}{=} h_U(V) = h_{\dot{C}(U_1 \cdots U_r)}(*^{1,l'}) \overset{(2.23)}{=} \dot{C}\left(h_{U_1}(*^{1,l'}) \cdots h_{U_r}(*^{1,l'})\right) \ ; \end{aligned}$$

by the induction hypothesis, for $W_i := U_i + *^{1,l'}$, $i = 1, \ldots, r$, it is

$$\begin{aligned} \overset{r}{\underset{i=1}{\forall}} U_i \asymp *^{1,l'} \Leftrightarrow &\overset{r}{\underset{i=1}{\forall}} \left(K_{h_{W_i}(U_i)} = K_{h_{W_i}(*^{1,l'})} \wedge M_{h_{W_i}(U_i)} \asymp M_{h_{W_i}(*^{1,l'})}\right) \\ \Leftrightarrow &\left(K_{h_{W_1}(U_1)}, \ldots, K_{h_{W_r}(U_r)}\right) = \left(K_{h_{W_1}(*^{1,l'})}, \ldots, K_{h_{W_r}(*^{1,l'})}\right) \\ &\wedge M_{h_{W_1}(U_1)} \cdots M_{h_{W_r}(U_r)} \asymp M_{h_{W_1}(*^{1,l'})} \cdots M_{h_{W_r}(*^{1,l'})} \\ \Leftrightarrow &\dot{C}\left(K_{h_{W_1}(U_1)}, \ldots, K_{h_{W_r}(U_r)}\right) = \dot{C}\left(K_{h_{W_1}(*^{1,l'})}, \ldots, K_{h_{W_r}(*^{1,l'})}\right) \\ &\wedge M_{h_{W_1}(U_1)} \cdots M_{h_{W_r}(U_r)} \asymp M_{h_{W_1}(*^{1,l'})} \cdots M_{h_{W_r}(*^{1,l'})} \\ \overset{\text{P.2.19}}{\Leftrightarrow} &K_{h_W(U)} = K_{h_W(V)} \wedge M_{h_W(U)} \asymp M_{h_W(V)} \ . \end{aligned}$$

If $U = \dot{C}(U_1 \cdots U_r)$ and $V = \dot{C}(V_1 \cdots V_r)$, for $W := U + V$ and $W_i := U_i + V_i$, $i = 1, \ldots, r$, we have

$$\begin{aligned} U \asymp V \Leftrightarrow &\overset{r}{\underset{i=1}{\forall}} U_i \asymp V_i \\ \overset{\text{IH}}{\Leftrightarrow} &\overset{r}{\underset{i=1}{\forall}} \left(K_{h_{W_i}(U_i)} = K_{h_{W_i}(V_i)} \wedge M_{h_{W_i}(U_i)} \asymp M_{h_{W_i}(V_i)}\right) \\ \Leftrightarrow &\left(K_{h_{W_1}(U_1)}, \ldots, K_{h_{W_r}(U_r)}\right) = \left(K_{h_{W_1}(V_1)}, \ldots, K_{h_{W_r}(V_r)}\right) \\ &\wedge M_{h_{W_1}(U_1)} \cdots M_{h_{W_r}(U_r)} \asymp M_{h_{W_1}(V_1)} \cdots M_{h_{W_r}(V_r)} \\ \Leftrightarrow &\dot{C}\left(K_{h_{W_1}(U_1)}, \ldots, K_{h_{W_r}(U_r)}\right) = \dot{C}\left(K_{h_{W_1}(V_1)}, \ldots, K_{h_{W_r}(V_r)}\right) \\ &\wedge M_{h_{W_1}(U_1)} \cdots M_{h_{W_r}(U_r)} \asymp M_{h_{W_1}(V_1)} \cdots M_{h_{W_r}(V_r)} \\ \overset{\text{P.2.19}}{\Leftrightarrow} &K_{h_W(U)} = K_{h_W(V)} \wedge M_{h_W(U)} \asymp M_{h_W(V)} \ . \end{aligned}$$

*For 2.* If both lists are nullary, then

$$K_U = \dot{*} = K_V = K_{h_U(V)}$$

and

$$U \vdash V \Leftrightarrow M_U = U \succ V = M_V = M_{h_U(V)} \ .$$

If $U = \dot{C}(U_1 \cdots U_r)$ and $V = *^{1,l'}$, we have

$$h_U(V) = h_{\dot{C}(U_1 \cdots U_r)}(*^{1,l'}) \overset{(2.23)}{=} \dot{C}\left(h_{U_1}(*^{1,l'}) \cdots h_{U_r}(*^{1,l'})\right) \ ;$$

by the induction hypothesis, it is

$$\begin{aligned}
\overset{r}{\underset{i=1}{\forall}} U_i \vdash *^{1,l'} \Leftrightarrow & \overset{r}{\underset{i=1}{\forall}} \left(K_{U_i} = K_{h_{U_i}(*^{1,l'})} \wedge M_{U_i} \succ M_{h_{U_i}(*^{1,l'})}\right) \\
\Leftrightarrow & (K_{U_1}, \dots, K_{U_r}) = \left(K_{h_{U_1}(*^{1,l'})}, \dots, K_{h_{U_r}(*^{1,l'})}\right) \\
& \wedge M_{U_1} \cdots M_{U_r} \succ M_{h_{U_1}(*^{1,l'})} \cdots M_{h_{U_r}(*^{1,l'})} \\
\Leftrightarrow & \dot{C}(K_{U_1}, \dots, K_{U_r}) = \dot{C}\left(K_{h_{U_1}(*^{1,l'})}, \dots, K_{h_{U_r}(*^{1,l'})}\right) \\
& \wedge M_{U_1} \cdots M_{U_r} \succ M_{h_{U_1}(*^{1,l'})} \cdots M_{h_{U_r}(*^{1,l'})} \\
\overset{\text{P.2.19}}{\Leftrightarrow} & K_U = K_{h_U(V)} \wedge M_U \succ M_{h_U(V)} \ .
\end{aligned}$$

If $U = \dot{C}(U_1 \cdots U_r)$ and $V = \dot{C}(V_1 \cdots V_r)$, we have

$$\begin{aligned}
U \vdash V \Leftrightarrow & \overset{r}{\underset{i=1}{\forall}} U_i \vdash V_i \\
\overset{\text{IH}}{\Leftrightarrow} & \overset{r}{\underset{i=1}{\forall}} \left(K_{U_i} = K_{h_{U_i}(V_i)} \wedge M_{U_i} \succ M_{h_{U_i}(V_i)}\right) \\
\Leftrightarrow & (K_{U_1}, \dots, K_{U_r}) = \left(K_{h_{U_1}(V_1)}, \dots, K_{h_{U_r}(V_r)}\right) \\
& \wedge M_{U_1} \cdots M_{U_r} \succ M_{h_{U_1}(V_1)} \cdots M_{h_{U_r}(V_r)} \\
\Leftrightarrow & \dot{C}(K_{U_1}, \dots, K_{U_r}) = \dot{C}\left(K_{h_{U_1}(V_1)}, \dots, K_{h_{U_r}(V_r)}\right) \\
& \wedge M_{U_1} \cdots M_{U_r} \succ M_{h_{U_1}(V_1)} \cdots M_{h_{U_r}(V_r)} \\
\overset{\text{P.2.19}}{\Leftrightarrow} & K_U = K_{h_U(V)} \wedge M_U \succ M_{h_U(V)} \ . \qquad \square
\end{aligned}$$

**Characterization of consistency and entailment through eigentokens**

Let $U \in \mathsf{Lst}_\alpha(l)$; a token $e \in T_\alpha$ is an *eigentoken* (or a *characteristic token*) of $U$, if it is equivalent to $U$, that is, if

$$U \sim e \Leftrightarrow \underset{b \in T_\alpha}{\forall} (U \vdash b \leftrightarrow e \vdash b) \ .$$

An *eigenvector* (or a *characteristic vector*) of a matrix $A = U_1 \cdots U_r \in \mathsf{Mat}_\alpha(r,l)$ is a vector $E = e_1 \cdots e_r \in \mathsf{Vec}_\alpha(r,1)$ where $e_i$ is an eigenvector of $U_i$, for $i = 1, \dots, r$.

**Proposition 2.21.** *Let $\alpha$ be a finitary algebra.*

1. *For every $A \in \mathsf{Mat}_\alpha(r)$ there exists a unique eigenvector $e(A) \in \mathsf{Vec}_\alpha(r)$.*
2. *If $A = K(B)$ then $e(A) = K(e(B))$.*
3. *The eigenvector of any $A \in \mathsf{Mat}_\alpha$ is given by $K_A(e(M_A))$.*

*Proof. For 1.* We show it first for a list $U \in \mathsf{Lst}(l)$: if $U$ is blank or $U = 0 + \cdots + 0$, for some nullary 0, then obviously $e(U) = *$ or $e(U) = 0$ respectively; if $U = \dot{C}(U_1 \cdots U_r)$, then

$$\begin{aligned} U \vdash b \Leftrightarrow & \dot{C}(U_1 \cdots U_r) \vdash Cb_1 \cdots b_r \\ \Leftrightarrow & \overset{r}{\underset{i=1}{\forall}} U_i \vdash b_i \\ \overset{\text{IH}}{\Leftrightarrow} & \overset{r}{\underset{i=1}{\forall}} e(U_i) \vdash b_i \\ \Leftrightarrow & Ce(U_1) \cdots e(U_r) \vdash Cb_1 \cdots b_r \ , \end{aligned}$$

so $e(U) = Ce(U_1) \cdots e(U_r)$; finally, if $U \supseteq^* U'$, then clearly $e(U') \vdash U$, so $e(U) = e(U')$.

For a matrix $A = U_1 \cdots U_r$, it is

$$\begin{aligned} A \vdash B \Leftrightarrow & U_1 \cdots U_r \vdash V_1 \cdots V_r \\ \Leftrightarrow & \overset{r}{\underset{i=1}{\forall}} U_i \vdash V_i \\ \Leftrightarrow & \overset{r}{\underset{i=1}{\forall}} e(U_i) \vdash V_i \\ \Leftrightarrow & e(U_1) \cdots e(U_r) \vdash V_1 \cdots V_r \ , \end{aligned}$$

for all $B = V_1 \cdots V_r \in \mathsf{Mat}(r)$, so $e(A) = e(U_1) \cdots e(U_r)$.

*For 2.* Let $A = K(B)$. By (1), $A$ and $B$ have eigenvectors, say $e_A$ and $e_B$ respectively. Then

$$B \sim e_B \overset{\text{P.2.12 (2)}}{\Rightarrow} K(B) \sim K(e_B) \overset{\text{H}}{\Rightarrow} e_A \sim K(e_B) \overset{\text{L.1.37}}{\Rightarrow} e_A = K(e_B) \ .$$

The statement 3 is a direct consequence of 2 and the homogeneous form theorem 2.17. □

*Example (continued).* Consider the $3 \times 4$ matrix

$$A = (B0*) ** + (B*0)(B*0) * + **(SS0) + *** \ ,$$

given in sigma-pi form, which is obviously coherently consistent. Its rows are the lists

$$\begin{aligned} A_1 &= B0* + B*0 + * + * \ , \\ A_2 &= * + B*0 + * + * \ , \\ A_3 &= * + * + SS0 + * \ , \end{aligned}$$

with respective eigentokens $B00$, $B*0$, and $SS0$. It is

$$e(A) = \begin{bmatrix} B00 \\ B*0 \\ SS0 \end{bmatrix} \ .$$ □

A further use of the eigentokens is the following characterization of consistency and entailment, which is a simple yet important application, since, in the second case, it turns non-atomic comparisons of entailment to atomic ones.

**Theorem 2.22** (Implicit atomicity at base types)**.** *Let $\alpha$ be a finitary algebra, and $A, B \in \mathsf{Mat}_\alpha$. The following hold.*

1. *It is $A \asymp B$ if and only if $e(A) \asymp e(B)$.*

2. *It is $A \vdash B$ if and only if $e(A) \succ e(B)$.*

3. *It is $A \sim B$ if and only if $e(A) = e(B)$.*

*Proof.* For 1, one uses propagation twice in each direction. For 2, similarly, one uses transitivity of entailment twice in each direction, as well as Proposition 2.11. Finally, the eigentoken characterization 3 further needs Proposition 1.37. □

**Normal forms combined**

In section 1.3 we saw that neighborhoods in basic *atomic* coherent information systems afford a normal form which consists of the maximal elements of the neighborhood: if $U = a_1 + \cdots + a_l$, then $U$ is in *(atomic) maximal form* if

$$\underset{1 \leq j, j' \leq l}{\forall} \left( a_j \succ a_{j'} \to j = j' \right) \ .$$

We write $m(U)$ for the maximal form of $U \in \mathsf{Con}$; $U$ is in maximal normal form if $U = m(U)$. A natural question is how the maximal and the homogeneous form, and also the eigentokens combine. Notice that, in general, they may all differ: for the list

$$U = B{*}(S{*}) + B(S0){*} + B(S{*}){*} \ ,$$

we have

$$\begin{aligned} h(U) &= B(S{*})(S{*}) + B(S0)(S{*}) + B(S{*})(S{*}) \ , \\ m(U) &= B{*}(S{*}) + B(S0){*} \ , \\ e(U) &= B(S0)(S{*}) \ . \end{aligned}$$

Call a matrix $A = A_1 + \cdots + A_l \in \mathsf{Mat}_\alpha(r,l)$ *atomically maximal* if no column is (atomically) entailed by some other, that is, if

$$\underset{1 \leq j, j' \leq l}{\forall} \left( A_j \succ A_{j'} \to j = j' \right) \ .$$

**Proposition 2.23.** *Let $U$ be a neighborhood. The following hold.*

1. *If $M_U$ is atomically maximal then $U$ is in maximal normal form.*

2. *If $m(U)$ is homogeneous, then $M_{m(U)}$ is atomically maximal.*

*Proof. For 1.* Let $U = a_1 + \cdots + a_l$, with $M_U$ atomically maximal. Suppose that, for $j, j' = 1, \ldots, l$, it is $a_j \succ a_{j'}$; by 2.15, it is $h_U(a_j) \succ h_U(a_{j'})$, which is the same as $K_U(M_j) \succ K_U(M_{j'})$, for $M_j$ the $j$-th column of $M_U$; this in turn yields $M_j \succ M_{j'}$, which by hypothesis gives $j = j'$.

*For 2.* Let $m(U) = a_1 + \cdots + a_l$ be homogeneous, with homogeneous form $K(M)$. Suppose that, in its basis $M$, it is $M_j \succ M_{j'}$ for two columns; then $K(M_j) \succ K(M_{j'})$, which means $a_j \succ a_{j'}$; this, by hypothesis, yields $j = j'$. □

*Remark.* A further natural question would be to examine *non-atomic* maximal normal forms, and their connection to the homogeneous ones—in this chapter, after all, we *are* concerned with non-atomic coherent systems in general. But here a problem of well-definedness arises which muddles thing up, and seems to disallow for an intuitively clear course of action.

The natural notion of a non-atomic maximal form, would require a neighborhood $U$ to satisfy the following:

$$\underset{V\subseteq U}{\forall}\ \underset{a\in U}{\forall}\,(V\vdash a \to a\in V)\ .$$

This does not always define a single neighborhood though: consider the list

$$U = C{*}00 + C0{*}0 + C00{*}\ ;$$

all three of the following neighborhoods satisfy the above condition, being equivalent to $U$:

$$U_1 = C{*}00 + C0{*}0\ ,\ U_2 = C{*}00 + C00{*}\ ,\ U_3 = C0{*}0 + C00{*}\ .$$

Notice that, at the same time, $U$'s *atomic* maximal form exists uniquely (and here coincides with $U$). □

The lesson to be learned from this short study seems to be that the best working normal form of a list in a basic algebraic coherent information system that we can have, is really its *eigentoken*, which, as Theorem 2.22 portrays, is determined uniquely up to equality. As it turns out, this situation can not hold for more general cases of algebras, as we will see in the following section; still, in type systems that build on finitary algebras—as is the case with Chapter 4, and plenty of other natural models—this should certainly prove instrumental when breaking higher-type statements down to base-type ones.

## Infinitary algebras

To cover algebras $\alpha$ that feature constructors with functional recursive arguments, we need to employ mixed matrices (see page 55). A constructor $C$ of type $(\vec{\rho}_1 \to \alpha) \to \cdots \to (\vec{\rho}_r \to \alpha) \to \alpha$ induces an operator $\dot{C} : \mathsf{Mat}(\vec{\rho}_1 \to \alpha, \ldots, \vec{\rho}_r \to \alpha) \to \mathsf{Mat}_\alpha(1)$, where $r > 1$, exactly as in the finitary case (page 63). The definition is also well-defined, as is the generalization to *vectors* of constructor operators and *constructor contexts*, with the difference that arities are now taken on the type level (compare to page 66):

- $\dot{*}_\alpha \in \mathsf{Kon}_\alpha(\alpha;\alpha)$ and if $C$ is a supernullary constructor of $\alpha$, then $\dot{C} \in \mathsf{Kon}_\alpha(\alpha, \mathrm{ar}(C))$;
- if $K_1 \in \mathsf{Kon}_\alpha(\vec{\sigma}_1;\vec{\sigma}_1')$, $\ldots$, $K_r \in \mathsf{Kon}_\alpha(\vec{\sigma}_r;\vec{\sigma}_r')$, then $(K_1,\ldots,K_r) \in \mathsf{Kon}_\alpha(\vec{\sigma}_1,\ldots,\vec{\sigma}_r;\vec{\sigma}_1',\ldots,\vec{\sigma}_r')$;
- if $K_1 \in \mathsf{Kon}_\alpha(\vec{\sigma}_1;\vec{\sigma})$, $K_2 \in \mathsf{Kon}_\alpha(\vec{\sigma};\vec{\sigma}_2)$, then $K_1K_2 \in \mathsf{Kon}_\alpha(\vec{\sigma}_1;\vec{\sigma}_2)$;

notice that the vectors $\vec{\sigma}$ cannot be empty. It is also direct to see that the factorization rules of Proposition 2.13 hold.

Essential difficulties appear when we ponder the existence of homogeneous forms and eigenvectors of mixed algebraic matrices. We will readily see that we cannot hope for the latter, though we can have a reasonably straightforward notion of the former.

We will make use of the following convention. Let $C$ be a constructor of arity $(\vec{\rho}_1 \to \alpha, \dots, \vec{\rho}_r \to \alpha)$. We will write $[*_{\vec{\rho}_i \to \alpha}] := \varnothing_{\vec{\rho}_i \to \alpha} \in \mathsf{Mat}_{\vec{\rho}_i \to \alpha}(1,0)$, for every $i = 1, \dots, r$, and consequently (in multiplicative notation)

$$C(\varnothing_{\vec{\rho}_1 \to \alpha} \cdots \varnothing_{\vec{\rho}_r \to \alpha}) = C([*_{\vec{\rho}_1 \to \alpha}] \cdots [*_{\vec{\rho}_r \to \alpha}]) = [C *_{\vec{\rho}_1 \to \alpha} \cdots *_{\vec{\rho}_r \to \alpha}] := [*_\alpha] \ .$$

Using this trick we can stretch the concept of homogenization between tokens (see page 69) as follows:

- if $b$ is either a *nullary token*, or an *alien* token, that is, a token from another algebra, then $h_{[b]}([a]) = [a]$;
- if $b = C_b b_1 \cdots b_{r_b}$ and $a = C_a a_1 \cdots a_{r_a}$, $r_b, r_a > 0$, with $C_b \neq C_a$, then $h_{[b]}([a]) = [a]$;
- if $b = Cb_1 \cdots b_r$, $r > 0$, and $a = *$, then
$$h_{[b]}([*_\alpha]) = \left[C h_{[b_1]}(*_{\vec{\rho}_1 \to \alpha}) \cdots h_{[b_r]}(*_{\vec{\rho}_r \to \alpha})\right] \ ;$$
- if $b = Cb_1 \cdots b_r$ and $a = Ca_1 \cdots a_r$, $r > 0$, then
$$h_{[b]}([Ca_1 \cdots a_r]) = C h_{[b_1]}([a_1]) \cdots h_{[b_r]}([a_r]) \ ,$$

where $C$ is a constructor of $\alpha$ with the above arity. It is direct to see that, for finitary algebras, we get
$$h_{[b]}([a]) = [h_b(a)] \ .$$

We extend the notion to matrices by

$$h_{\prod_{i=1}^r \sum_{j'=1}^{l'} b_{ij'}} \left( \prod_{i=1}^r \sum_{j=1}^l a_{ij} \right) := \prod_{i=1}^r \sum_{j=1}^l h_{b_{il'}} \cdots h_{b_{i1}}(a_{ij}) \ .$$

The matrix $A$ is *homogeneous* if it is already $A = h(A)$. Adopting the above generalized notions, it is direct to see that the normal form theorem 2.17 holds for matrices over infinitary algebras as well, with one crucial difference: the basis is no longer necessarily a *basic* matrix, but a *pseudo-basic* one: a matrix is called *pseudo-basic (for $\alpha$)* if it consists solely of $*_\alpha$'s, nullary constructors, or *alien tokens* (in which case, it is a mixed matrix).

**Theorem 2.24** (Infinitary matrix form)**.** *Let $\alpha$ be an infinitary algebra. For every homogeneous matrix $A \in \mathsf{Mat}(\vec{\sigma}, l)$, there exist a unique constructor context $K_A \in \mathsf{Kon}_\alpha(\vec{\sigma}; \vec{\sigma}')$ in normal form, and a unique pseudo-basic matrix $M_A \in \mathsf{Mat}(\vec{\sigma}', l)$, such that*
$$A = K_A(M_A) \ .$$

Again, call $K_A(M_A)$ the *matrix form* of $A$, $K_A$ the *basic coefficient* of $A$, and $M_A$ the *basis* of $A$; the *normal form*, the *basic coefficient*, and the *basis* of an arbitrary matrix $A$ over an infinitary algebra are defined to be the normal form, the basic coefficient, and the basis of its homogenization $h(A)$ respectively.

*Example.* Recall the matrix on page 63 (we write $*$ for $*_\alpha$, but keep the subscript for $*_{\mathbb{B}\to\alpha}$):

$$\begin{bmatrix} C(B00)*(S*) & * & C**(S0) \\ SB*0 & SB** & SB(\Omega\langle[\mathtt{tt}],\Omega\langle[\mathtt{ff}],B**\rangle\rangle)* \\ B** & B(C(S0)**)* & B*(B*0) \\ \Omega\langle[*_{\mathbb{B}}],SB*0\rangle & \Omega\langle[\mathtt{tt}],SB0*\rangle & \Omega\langle[\mathtt{ff}],SB(S0)*\rangle \end{bmatrix}.$$

Its homogenization is given by

$$\begin{bmatrix} C(B00)*(S*) & C(B**)*(S*) & C(B**)*(S0) \\ SB(\Omega*_{\mathbb{B}\to\alpha})0 & SB(\Omega*_{\mathbb{B}\to\alpha})* & SB(\Omega\langle[\mathtt{tt}],\Omega\langle[\mathtt{ff}],B**\rangle\rangle)* \\ B(C(S*)**)(B**) & B(C(S0)**)(B**) & B(C(S*)**)(B*0) \\ \Omega\langle[*_{\mathbb{B}}],SB*0\rangle & \Omega\langle[\mathtt{tt}],SB0*\rangle & \Omega\langle[\mathtt{ff}],SB(S0)*\rangle \end{bmatrix};$$

exhaustive factorization yields the basic coefficient

$$(\dot{C}(\dot{B},\dot{*},\dot{S}),\dot{S}\dot{B}(\dot{\Omega},\dot{*}),\dot{B}(\dot{C}(\dot{S},\dot{*},\dot{*}),\dot{B}),\dot{\Omega})$$

and the basis

$$\begin{bmatrix} 0 & * & * \\ 0 & * & * \\ * & * & * \\ * & * & 0 \\ *_{\mathbb{B}\to\alpha} & *_{\mathbb{B}\to\alpha} & \langle[\mathtt{tt}],\Omega\langle[\mathtt{ff}],B**\rangle\rangle \\ 0 & * & * \\ * & 0 & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & 0 \\ \langle[*_{\mathbb{B}}],SB*0\rangle & \langle[\mathtt{tt}],SB0*\rangle & \langle[\mathtt{ff}],SB(S0)*\rangle \end{bmatrix}. \qquad \square$$

As the matrix form theorem claims and the example illustrates, due to the deliberate relativization of homogenization, the procedure halts whenever it encounters alien tokens, tokens that do not belong to the carrier of $\alpha$—in particular, functional tokens that serve as arguments of the constructor $\Omega$. Consequently, the normal form may (as in this example) contain alien tokens; homogenization, so to speak, is a procedure restricted to the algebra at hand.

## 2.3 Algebraic function spaces

We examined earlier atomic function spaces (see page 56) which do not suffice for the the study of the (not necessarily atomic) algebraic higher-type entailment. We further need a notion of non-atomic "application", tailored-made for lists, as we already saw in the beginning of the chapter. Here we will express application through boolean tests, in a pretty similar way as with the test matrices for the atomic function spaces.

We first define the *boolean application* or *(boolean) switch* $\odot : \mathsf{Mat}_{\mathbb{B}}(1,l) \to \mathsf{Mat}_\rho(1,1) \to \mathsf{Mat}_\rho(1,l)$, $l \le 1$, by

$$p \odot a := \begin{cases} a & p = [\mathtt{tt}] \\ \varnothing_\rho & \text{otherwise}, \end{cases}$$

for any type $\rho$, basic or otherwise. Then we define the *(non-atomic) entailment test matrix* $\vdash^{?}\colon \mathsf{Mat}_\rho(1,l) \to \mathsf{Mat}_\rho(1,1) \to \mathsf{Mat}_{\mathbb{B}}(1,1)$, inductively on the height of $\rho$:

- for a basic $\alpha$ define

$$U \vdash^{?}_\alpha [a] \coloneqq \begin{cases} [\mathtt{tt}] & U \vdash_\alpha a \\ \varnothing_{\mathbb{B}} & \text{otherwise;} \end{cases}$$

- for a function space $\rho \to \sigma$ define

$$\sum_j \langle U_j, b_j \rangle \vdash^{?}_{\rho\to\sigma} \langle U, b \rangle \coloneqq \sum_j (U \vdash^{?}_\rho U_j) \odot b_j \vdash^{?}_\sigma b \ ,$$

  where the entailment test between two *neighborhoods* is understood as follows:

$$U \vdash^{?}_\rho \sum_{j=1}^{l} a_j \coloneqq \bigodot_{j=1}^{l} (U \vdash^{?}_\rho a_j) = (U \vdash^{?}_\rho a_1) \odot \cdots \odot (U \vdash^{?}_\rho a_l) \ ,$$

  associated to the left.

With these, we may now simply define the *higher-type non-atomic entailment* by

$$U \vdash_{\rho\to\sigma} b \coloneqq U \vdash^{?}_{\rho\to\sigma} b = [\mathtt{tt}] \ .$$

It is easy to see that the list application we appealed to in the beginning of the chapter can be expressed in our terms as follows:

$$\left( \sum_j \langle U_j, b_j \rangle \right) U = \sum_j (U \vdash^{?} U_j) \odot b_j \ .$$

This concludes what we set out to demonstrate: all entailments eventually break down to entailments in basic acises, which, in turn, are determined by *atomic* entailments on the level of the corresponding matrices.

**Proposition 2.25.** *Let $\rho$ and $\sigma$ be types built upon finitary algebras.*

1. *The relation $\vdash_{\rho\to\sigma}$ is a proper entailment relation, that is, it is reflexive, transitive, and propagates consistency at $\rho \to \sigma$:*

$$\begin{aligned} &\langle U, b \rangle \in W \to W \vdash_{\rho\to\sigma} \langle U, b \rangle \ , \\ &W_1 \vdash_{\rho\to\sigma} W_2 \wedge W_2 \vdash_{\rho\to\sigma} \langle U, b \rangle \to W_1 \vdash_{\rho\to\sigma} \langle U, b \rangle \ , \\ &W \in \mathsf{Con}_{\rho\to\sigma} \wedge W \vdash_{\rho\to\sigma} \langle U, b \rangle \to W + \langle U, b \rangle \in \mathsf{Con}_{\rho\to\sigma} \ . \end{aligned}$$

2. *Application of neighborhoods is monotonic in both arguments:*

$$\begin{aligned} &U \vdash_\rho U' \to WU \vdash_\sigma WU' \ , \\ &W \vdash_{\rho\to\sigma} W' \to WU \vdash_\sigma W'U \ , \end{aligned}$$

   *as well as consistency-preserving in both arguments:*

$$\begin{aligned} &U \asymp_\rho U' \to WU \asymp_\sigma WU' \ , \\ &W \asymp_{\rho\to\sigma} W' \to WU \asymp_\sigma W'U \ . \end{aligned}$$

3. *Application distributes over left appending, that is,*

$$(W+W')U = WU + W'U \ .$$

*Proof.* We show the third statement. Let $W_1 = \sum_{j_1} \langle U^1_{j_1}, b^1_{j_1}\rangle$ and $W_2 = \sum_{j_2} \langle U^2_{j_2}, b^2_{j_2}\rangle$, with $W_1 \asymp_{\rho\to\sigma} W_2$, and $U \in \mathsf{Con}_\rho$; it is

$$\begin{aligned}
(W_1+W_2)U &= \left(\sum_{j_1=1}^{l_1} \langle U^1_{j_1}, b^1_{j_1}\rangle + \sum_{j_2=1}^{l_2} \langle U^2_{j_2}, b^2_{j_2}\rangle\right) U \\
&\overset{(\star)}{=} \left(\sum_{j=1}^{l_1+l_2} \langle U_j, b_j\rangle\right) U \\
&= \sum_{j=1}^{l_1+l_2} (U \vdash^2 U_j) \odot b_j \\
&\overset{(\star)}{=} \sum_{j_1=1}^{l_1} \left(U \vdash^2 U^1_{j_1}\right) \odot b^1_{j_1} + \sum_{j_2=1}^{l_2} \left(U \vdash^2 U^2_{j_2}\right) \odot b^2_{j_2} \\
&= \left(\sum_{j_1=1}^{l_1} \langle U^1_{j_1}, b^1_{j_1}\rangle\right) U + \left(\sum_{j_2=1}^{l_2} \langle U^2_{j_2}, b^2_{j_2}\rangle\right) U \\
&= W_1U + W_2U \ ,
\end{aligned}$$

where for $(\star)$ we naturally let $U_j \coloneqq U^1_j$ for $j \le l_1$ and $U_j \coloneqq U^2_{j-l_1}$ for $j > l_1$. □

*Remark.* Note that in general we don't have right distributivity, that is,

$$\neg \underset{W\in\mathsf{Con}_{\rho\to\sigma}}{\forall} \ \underset{U,U'\in\mathsf{Con}_\rho}{\forall} \left(U \asymp_\rho U' \to W(U+U') = WU + WU'\right) \ .$$

A Coquand counterexample (page 47) can show this already for type level 1: it is $\langle B00, 0\rangle(B0{*} + B{*}0) = 0$ but $\langle B00,0\rangle B0{*} + \langle B00,0\rangle B{*}0 = \varnothing$. This is a situation that easily extends to higher types: at type level 2 for example, we have

$$\langle\langle S0, B00\rangle, 0\rangle(\langle S{*}, B0{*}\rangle + \langle S{*}, B{*}0\rangle) = 0$$

but

$$\langle\langle S0, B00\rangle, 0\rangle\langle S{*}, B0{*}\rangle + \langle\langle S0, B00\rangle, 0\rangle\langle S{*}, B{*}0\rangle = \varnothing \ .$$

Consider though the following situation where right distributivity works:

$$\big(\langle\langle S0, {*}\rangle, B0{*}\rangle + \langle\langle B00, {*}\rangle, B{*}0\rangle\big)\big(\langle S{*}, 0\rangle + \langle B{*}{*}, 1\rangle\big) = B0{*} + B{*}0$$

and

$$\begin{aligned}
&\big(\langle\langle S0, {*}\rangle, B0{*}\rangle + \langle\langle B00, {*}\rangle, B{*}0\rangle\big)\langle S{*}, 0\rangle = B0{*} \ , \\
&\big(\langle\langle S0, {*}\rangle, B0{*}\rangle + \langle\langle B00, {*}\rangle, B{*}0\rangle\big)\langle B{*}{*}, 1\rangle = B{*}0 \ .
\end{aligned}$$

More generally, this indicates that we can hope for right distributivity when the two appendices $U$ and $U'$ are *trivially consistent*, meaning, when their respective arguments are inconsistent; apparently, this is something to discuss only at type levels higher than 1. □

*Example.* For clarity's sake, let's see in some detail how a higher type non-atomic entailment unfolds, and eventually breaks down to basic entailments. We consider four finitary algebras (basic acises) $\alpha$, $\beta$, $\gamma$, and $\delta$, and form the function space $((\alpha \to \beta) \to \gamma) \to \delta$; in order to save space, we also write $\alpha\beta$ for $\alpha \to \beta$, so the higher space is written $((\alpha\beta)\gamma)\delta$. We consider the general case of an entailment in $((\alpha\beta)\gamma)\delta$:

$$\sum_i \langle \sum_j \langle \sum_k \langle U_{ijk}, b_{ijk} \rangle, c_{ij} \rangle, d_i \rangle \vdash_{((\alpha\beta)\gamma)\delta} \langle \sum_{j'} \langle \sum_{k'} \langle U'_{j'k'}, b'_{j'k'} \rangle, c'_{j'} \rangle, d' \rangle \ ,$$

which, by definition, is equivalent to

$$\sum_i \langle \sum_j \langle \sum_k \langle U_{ijk}, b_{ijk} \rangle, c_{ij} \rangle, d_i \rangle \vDash_{((\alpha\beta)\gamma)\delta} \langle \sum_{j'} \langle \sum_{k'} \langle U'_{j'k'}, b'_{j'k'} \rangle, c'_{j'} \rangle, d' \rangle = [\mathrm{tt}] \ .$$

The term on the left unfolds as follows:

$$\begin{aligned}
&\sum_i \langle \sum_j \langle \sum_k \langle U_{ijk}, b_{ijk} \rangle, c_{ij} \rangle, d_i \rangle \vDash_{((\alpha\beta)\gamma)\delta} \langle \sum_{j'} \langle \sum_{k'} \langle U'_{j'k'}, b'_{j'k'} \rangle, c'_{j'} \rangle, d' \rangle \\
&= \sum_i \left( \sum_{j'} \langle \sum_{k'} \langle U'_{j'k'}, b'_{j'k'} \rangle, c'_{j'} \rangle \vDash_{(\alpha\beta)\gamma} \sum_j \langle \sum_k \langle U_{ijk}, b_{ijk} \rangle, c_{ij} \rangle \right) \odot d_i \vDash_\delta d' \\
&= \sum_i \left( \bigodot_j \sum_{j'} \langle \sum_{k'} \langle U'_{j'k'}, b'_{j'k'} \rangle, c'_{j'} \rangle \vDash_{(\alpha\beta)\gamma} \langle \sum_k \langle U_{ijk}, b_{ijk} \rangle, c_{ij} \rangle \right) \odot d_i \vDash_\delta d' \\
&= \sum_i \left( \bigodot_j \sum_{j'} \left( \sum_k \langle U_{ijk}, b_{ijk} \rangle \vDash_{\alpha\beta} \sum_{k'} \langle U'_{j'k'}, b'_{j'k'} \rangle \right) \odot c'_{j'} \vDash_\gamma c_{ij} \right) \odot d_i \vDash_\delta d' \\
&= \sum_i \left( \bigodot_j \sum_{j'} \left( \bigodot_{k'} \sum_k \langle U_{ijk}, b_{ijk} \rangle \vDash_{\alpha\beta} \langle U'_{j'k'}, b'_{j'k'} \rangle \right) \odot c'_{j'} \vDash_\gamma c_{ij} \right) \odot d_i \vDash_\delta d' \\
&= \sum_i \left( \bigodot_j \sum_{j'} \left( \bigodot_{k'} \sum_k \left( U'_{j'k'} \vDash_\alpha U_{ijk} \right) \odot b_{ijk} \vDash_\beta b'_{j'k'} \right) \odot c'_{j'} \vDash_\gamma c_{ij} \right) \odot d_i \vDash_\delta d'
\end{aligned}$$

One can see that, in order to decide the entailment in the space $((\alpha\beta)\gamma)\delta$, one has to successively decide entailments in $\alpha$, then $\beta$, then $\gamma$, and finally $\delta$ (one should also notice how contravariance of entailment expresses itself in the zig-zag succession of non-primed and primed indices); since all four are basic finitary acises, these entailments are conducted by means of *atomic* entailments, as we previously saw. □

## Lists and maximal neighborhoods

It is very natural that higher-type considerations may heavily rely on considerations about *lists of tokens which are not necessarily consistent*: such lists may appear every time we regard the arguments in a given neighborhood. This necessitates a careful examination of $\mathsf{Arr}_\rho$ in its own right—in the following we just write $\mathsf{Arr}_\rho$ for $\mathsf{Arr}_\rho(1,l)$. As we will see, this examination unavoidably leads us to purely combinatorial grounds.

Let $\Gamma \in \mathsf{Arr}_\rho$ be a list, not necessarily consistent, and denote its consistent sublists by $\mathsf{Con}_\Gamma$; for example, it is $a \in \mathsf{Con}_\Gamma$ (seen as a neighborhood), for every $a \in \Gamma$, as well as $\varnothing \in \mathsf{Con}_\Gamma$ for every $\Gamma \in \mathsf{Arr}_\rho$. Clearly, if $\Gamma \in \mathsf{Con}_\rho$ already, then $\mathsf{Con}_\Gamma = \mathscr{P}(\Gamma)$, while in general it is $\mathsf{Con}_\Gamma \subseteq \mathscr{P}(\Gamma)$.

Call $M \in \mathsf{Con}_\Gamma$ a *maximal neighborhood* in $\Gamma$, and write $M \in \operatorname{Max}\Gamma$, if

$$\underset{a\in\Gamma}{\forall}\left(a \asymp_\rho M \to a \in M\right) .$$

An easy observation is that for all $V \in \mathsf{Con}_\rho$, it is $U \vdash_\rho V$ for some $U \in \mathsf{Con}_\Gamma$ if and only if $M \vdash_\rho V$ for some $M \in \operatorname{Max}\Gamma$ (leftwards let $U := M$ and rightwards use transitivity of entailment at type $\rho$).

Moreover, the consistency of a list is characterized easily through maximal neighborhoods: at base types, clearly, a list is consistent if and only if it is its own sole maximal neighborhood, while for higher types we have the following (recall the definitions of $\arg A$ and $\operatorname{val} A$ on page 61).

**Proposition 2.26.** *A list $\Theta \in \mathsf{Arr}_{\rho\to\sigma}$ is a neighborhood if and only if for each left maximal $M \in \operatorname{Max}(\arg\Theta)$ there is a right maximal $N \in \operatorname{Max}(\operatorname{val}\Theta)$ with $\Theta M \subseteq N$.*

*Proof.* Write $\Gamma$ and $\Delta$ for $\arg\Theta$ and $\operatorname{val}\Theta$ respectively.

From left to right, let $\Theta \in \mathsf{Con}_{\rho\to\sigma}$ and $M \in \operatorname{Max}\Gamma$. If $U, U' \in M$, it will be $U \asymp_\rho U'$, and then $WU \asymp_\sigma WU'$; so there must be a maximal $N \in \operatorname{Max}\Delta$ with $WU \subseteq N$ for every $U \in \operatorname{Max}$.

From right to left, let $\Theta \in \mathsf{Arr}_{\rho\to\sigma}$ with the property that for each left maximal $M \in \operatorname{Max}\Gamma$ there is a right maximal $N \in \operatorname{Max}\Delta$ such that $\Theta M \subseteq N$; let $\langle U, b\rangle, \langle U', b'\rangle \in \Theta$ with $U \asymp_\rho U'$; there will be a maximal $M \in \operatorname{Max}\Gamma$ with $U, U' \in \Gamma$; by hypothesis, there will be an $N \in \operatorname{Max}\Delta$ such that

$$b + b' \subseteq WU + WU' \subseteq \Theta W \subseteq N ,$$

so $\Theta$ is consistent. □

Write $\mathrm{c}_j\Gamma$ ("converging" in $\Gamma$) and $\mathrm{d}_j\Gamma$ ("diverging" in $\Gamma$) for the tokens in $\Gamma$ which are consistent and inconsistent with its $j$'th element respectively, that is,

$$\mathrm{c}_j\Gamma := \{a_{j'} \in \Gamma \mid a_j \asymp_\rho a_{j'}\} ,$$
$$\mathrm{d}_j\Gamma := \{a_{j'} \in \Gamma \mid a_j \not\asymp_\rho a_{j'}\} .$$

It is obviously $\Gamma = \mathrm{c}_j\Gamma + \mathrm{d}_j\Gamma$, for all $j$'s.

**Proposition 2.27.** *Let $\Gamma \in \mathsf{Arr}_\rho$ and $M \subseteq \Gamma$. It is $M \in \operatorname{Max}\Gamma$ if and only if*

$$M = \bigcap_{j\in I(M)} \mathrm{c}_j\Gamma .$$

*Proof.* From left to right, let $M \in \operatorname{Max}\Gamma$; it is

$$a_i \in \bigcap_{j\in I(M)} \mathrm{c}_j\Gamma \Leftrightarrow \underset{j\in I(M)}{\forall} a_i \in \mathrm{c}_j\Gamma \Leftrightarrow \underset{j\in I(M)}{\forall} a_i \asymp_\rho a_j \overset{(\star)}{\Leftrightarrow} a_i \in M ,$$

where $(\star)$ holds by the maximality of $M$.

For the other direction, let $M = \bigcap_{j\in I(M)} \mathrm{c}_j\Gamma$; for the consistency, let $a_i, a_j \in M$, that is, $a_i \asymp_\rho a_k$ and $a_j \asymp_\rho a_k$, for all $k \in I(M)$; since $i, j \in I(M)$ already, we have $a_i \asymp_\rho a_j$; for maximality, let $a_i \asymp_\rho M$, for some $i \in I(\Gamma)$; then $a_i \asymp_\rho a_j$, for all $j \in I(M)$, that is, $a_i \in \mathrm{c}_j\Gamma$, for all $j \in I(M)$, so $a_i \in \bigcap_{j\in I(M)} \mathrm{c}_j\Gamma$, hence $a_i \in M$ by the hypothesis. □

By the graph-theoretic intuition we appealed to in the previous remark, Proposition 2.27 expresses that a maximal clique is characterized by the intersection of the *stars* induced by its nodes.

We are interested in the way maximal neighborhoods form as we move to a higher type, so that we may in principle argue about lists in a type-inductive fashion. Introduce the following shorthands for the sake of readability. Given a list $\Theta \in \mathsf{Arr}_{\rho\to\sigma}$, and $M \in \mathsf{Con}_\rho$, $N \in \mathsf{Con}_\sigma$, write $\langle M,N\rangle_\Theta$ for all $\langle U,b\rangle \in \Theta$ with $M \vdash_\rho U$ and $N \vdash_\sigma b$. If $\Gamma \in \mathsf{Arr}_\rho$, and $\Delta \in \mathsf{Arr}_\sigma$, then write $\langle \Gamma,\Delta\rangle$ for $\sum_{M\in\mathrm{Max}\,\Gamma}\sum_{b\in\Delta}\langle M,b\rangle$.

**Proposition 2.28.** *Let $\Theta \in \mathsf{Arr}_{\rho\to\sigma}$. The following hold.*

1. $\mathrm{c}_j\Theta = \langle \mathrm{d}_j\arg\Theta, \mathrm{val}\,\Theta\rangle_\Theta \cup \langle \arg\Theta, \mathrm{c}_j\mathrm{val}\,\Theta\rangle_\Theta$ ,
2. $\mathrm{d}_j\Theta = \langle \mathrm{c}_j\arg\Theta, \mathrm{val}\,\Theta\rangle_\Theta \cap \langle \arg\Theta, \mathrm{d}_j\mathrm{val}\,\Theta\rangle_\Theta$ .

*Proof.* For 1:

$$\begin{aligned}\langle U_{j'},b_{j'}\rangle \in \mathrm{c}_j\Theta &\Leftrightarrow U_j \not\asymp_\rho U_{j'} \vee b_j \asymp_\sigma b_{j'}\\ &\Leftrightarrow U_{j'} \in \mathrm{d}_j\arg\Theta \vee b_{j'} \in \mathrm{c}_j\mathrm{val}\,\Theta\\ &\Leftrightarrow \langle U_j,b_j\rangle \in \langle \mathrm{d}_j\arg\Theta, \mathrm{val}\,\Theta\rangle_\Theta \cup \langle \arg\Theta, \mathrm{c}_j\mathrm{val}\,\Theta\rangle_\Theta\ .\end{aligned}$$

Statement 2 is shown similarly:

$$\begin{aligned}\langle U_{j'},b_{j'}\rangle \in \mathrm{d}_j\Theta &\Leftrightarrow U_j \asymp_\rho U_{j'} \wedge b_j \not\asymp_\sigma b_{j'}\\ &\Leftrightarrow U_{j'} \in \mathrm{c}_j\arg\Theta \wedge b_{j'} \in \mathrm{d}_j\mathrm{val}\,\Theta\\ &\Leftrightarrow \langle U_j,b_j\rangle \in \langle \mathrm{c}_j\arg\Theta, \mathrm{val}\,\Theta\rangle_\Theta \cap \langle \arg\Theta, \mathrm{d}_j\mathrm{val}\,\Theta\rangle_\Theta\ . \ \square\end{aligned}$$

The following is a characterization of higher-type maximal neighborhoods.

**Proposition 2.29.** *Let $\Theta \in \mathsf{Arr}_{\rho\to\sigma}$ and $P \subseteq \Theta$. It is $P \in \mathrm{Max}\,\Theta$ if and only if*

$$P = \bigcup_{K\subseteq I(P)}\left(\bigcap_{k\in K}\langle \mathrm{d}_k\arg\Theta, \mathrm{val}\,\Theta\rangle_\Theta \cap \bigcap_{k\in I(P)\smallsetminus K}\langle \arg\Theta, \mathrm{c}_k\mathrm{val}\,\Theta\rangle_\Theta\right)\ .$$

*Proof.* Let $\Theta \in \mathsf{Arr}_{\rho\to\sigma}$, and write $\Gamma := \arg\Theta$, $\Delta := \mathrm{val}\,\Theta$. By Proposition 2.27 we know that $P \in \mathrm{Max}\,\Theta$ if and only if

$$P = \bigcap_{j\in I(P)} c_j\Theta \overset{\text{P. 2.28}}{=} \bigcap_{j\in I(P)}\left(\langle \mathrm{d}_j\Gamma,\Delta\rangle_\Theta \cup \langle \Gamma,\mathrm{c}_j\Delta\rangle_\Theta\right)\ ,$$

so we want to show that the following equation holds:

$$\begin{aligned}&\bigcap_{j\in I(P)}\left(\langle \mathrm{d}_j\Gamma,\Delta\rangle_\Theta \cup \langle \Gamma,\mathrm{c}_j\Delta\rangle_\Theta\right)\\ &\quad= \bigcup_{K\subseteq I(P)}\left(\bigcap_{k\in K}\langle \mathrm{d}_k\Gamma,\Delta\rangle_\Theta \cap \bigcap_{k\in I(P)\smallsetminus K}\langle \Gamma,\mathrm{c}_k\Delta\rangle_\Theta\right)\ .\end{aligned}\tag{2.24}$$

We relax the notation even more by writing $d_j$ for $\langle \mathrm{d}_j\Gamma,\Delta\rangle_\Theta$ and $c_j$ for $\langle \Gamma,\mathrm{c}_j\Delta\rangle_\Theta$, so the equation (2.24) becomes

$$\bigcap_{j\in I(P)}(d_j \cup c_j) = \bigcup_{K\subseteq I(P)}\left(\bigcap_{k\in K} d_k \cap \bigcap_{k\in I(P)\smallsetminus K} c_k\right)\ .$$

We show that the equation holds by induction on $n = |I(P)| > 0$. The base case is direct to see. Let $I(P) = \{1, \dots, n, n+1\}$; then

$$\begin{aligned}
\bigcap_{j=1}^{n+1} (d_j \cup c_j) &= \bigcap_{j=1}^{n} (d_j \cup c_j) \cap (d_{n+1} \cup c_{n+1}) \\
&\overset{\text{(IH)}}{=} \bigcup_{K \subseteq \{1,\dots,n\}} \left( \bigcap_{k \in K} d_k \cap \bigcap_{k \notin K} c_k \right) \cap (d_{n+1} \cup c_{n+1}) \\
&= \bigcup_{K \subseteq \{1,\dots,n\}} \left( \bigcap_{k \in K} d_k \cap \bigcap_{k \notin K} c_k \cap d_{n+1} \right) \cup \bigcup_{K \subseteq \{1,\dots,n\}} \left( \bigcap_{k \in K} d_k \cap \bigcap_{k \notin K} c_k \cap c_{n+1} \right) \\
&= \bigcup_{K \subseteq \{1,\dots,n+1\}} \left( \bigcap_{k \in K} d_k \cap \bigcap_{k \notin K} c_k \right) ,
\end{aligned}$$

where the last steps require careful but elementary set theory. □

*Remark.* Note that in the proof we have used the following convention: if $\Theta \in \mathsf{Arr}_{\rho \to \sigma}$ and $Q \subseteq \Theta$, indexed by $I(Q)$, then $\bigcap_{k \in \varnothing} Q(k) = \Theta$ ("an intersection over the empty index set yields the universe"). □

We need to be a bit careful with the way we use the lists of arguments in given neighborhoods: do we mean them as lists of neighborhoods, that is, lists of type $N\rho$, or as lists of the neighborhoods' tokens, that is, lists of type $\rho$? For our purposes, it turns out that, given a list in $N\rho$, we can work with its underlying "flat" list in $\rho$, and then draw safe conclusions about it in $N\rho$ again.

Define a *flattening* mapping $\mathrm{fl} : \mathsf{Arr}_{N\rho} \to \mathsf{Arr}_{\rho}$ in the usual way:

$$\mathrm{fl}(\Gamma) := \sum_{U \in \Gamma} \sum_{a \in U} a \, ;$$

in set-theoretical notation we may as well write $\mathrm{fl}(\Gamma) = \cup \Gamma$ (we return to this in Chapter 3, on page 108).

**Proposition 2.30.** *Let $\Gamma \in \mathsf{Arr}_{N\rho}$. Then*

$$\underset{M \in \mathrm{Max}\,\Gamma}{\forall} \; \underset{M_f \in \mathrm{Max}\,\mathrm{fl}(\Gamma)}{\exists!} \; M_f \vdash_{N\rho} M \;\; \wedge \;\; \underset{M_f \in \mathrm{Max}\,\mathrm{fl}(\Gamma)}{\forall} \; \underset{M \in \mathrm{Max}\,\Gamma}{\exists!} \; M_f \vdash_{N\rho} M \, .$$

*Proof.* Let $\Gamma \in \mathsf{Arr}_{N\rho}$. For the first conjunct, let $M \in \mathrm{Max}\,\Gamma$; it is consistent, so $U \asymp_{N\rho} U'$, for all $U \in M$, which means that $a \asymp_{\rho} a'$, for all $a \in U$, $a' \in U'$; then there must exist a maximal neighborhood $M_f$ in $\mathrm{fl}(\Gamma)$, which will contain all $a \in U$, for any $U \in M$, so $M_f \vdash_{N\rho} M$. Suppose that $M_f'$ is yet another maximal neighborhood in $\mathrm{fl}(\Gamma)$, with $M_f' \vdash_{N\rho} M$; then, since entailment preserves consistency, for all $a \in \mathrm{fl}(\Gamma)$ it is

$$[a] \asymp_{N\rho} M \to a \asymp_{\rho} M_f \wedge a \asymp_{\rho} M_f' \, ,$$

which yields $M_f = M_f'$ due to their maximality with respect to consistency.

For the second conjunct, let $M_f \in \mathrm{Max}\,\mathrm{fl}(\Gamma)$; since the situation is finite, we may argue indirectly; let $M_1, \dots, M_T$ be all maximals in $\Gamma$, and suppose that $M_f \nvdash_{N\rho} M_t$ for any $t = 1, \dots, T$; by the first conjunct, there is an $M_f^t \in \mathrm{Max}\,\mathrm{fl}(\Gamma)$, with $M_f^t \vdash M_t$, for every $t$, and since $M_t$'s together cover $\Gamma$, their corresponding $M_f^t$'s together must cover $\mathrm{fl}(\Gamma)$; on the other hand, the supposition yields $M_f \not\asymp_{\rho} M_f^t$, for all $t$, which would mean that there are $a \in M_f \smallsetminus \mathrm{fl}(\Gamma)$, a contradiction. Assume now that $M_f \vdash_{N\rho} M$ and $M_f \vdash_{N\rho} M'$, for $M, M' \in \mathrm{Max}\,\Gamma$; then $M \asymp_{N\rho} M'$, so $M = M'$ by maximality. □

## Eigen-neighborhoods

Let $\Gamma \in \mathsf{Arr}_\rho$ be an arbitrary list. In the following we drop the maximality requirement of the previous section and consider arbitrary neighborhoods $U \subseteq \Gamma$ which are still "closed under entailment relatively to $\Gamma$", in a way that we describe below.

Let $W \in \mathsf{Con}_{\rho\to\sigma}$ be some neighborhood, and consider a sublist $E \subseteq W$ which has the property

$$U_1 \asymp_\rho U_2 \wedge b_1 \asymp_\sigma b_2 \ ,$$

for all $\langle U_i, b_i\rangle \in E$, $i = 1,2$. Suppose further that all these arguments $U$ in $E$ are part of an ideal $x \in \mathsf{Ide}_\rho$; then $b \in W(x)$, for all the corresponding values $b$ in $E$. Conversely, if such an ideal $x$ entails *exactly* $\arg E$ out of all arguments of $W$—in which case, due to closure of $x$, $\arg E$ would already contain all of $\arg W$ that it entails—then the elicited value should be *exactly* $\mathrm{val}\,E$; in other words, this sub-neighborhood would serve as a pointer to all ideals like $x$ extending its argument *exclusively* in $W$, while simultaneously providing the exact value that $W$ would give out as a result by application. Let's try and flesh out this intuition.

Call a sublist $E \subseteq W$ an *eigen-neighborhood* of $W$, and write $E \in \mathrm{Eig}\,W$, if it is *left-consistent*, that is,

$$\forall_{U_1,U_2 \in \arg E} U_1 \asymp_\rho U_2$$

(so consequently $b_1 \asymp_\sigma b_2$ for the corresponding arguments as well), and *left-closed under entailment relatively to* $W$, that is,

$$\forall_{\langle U,b\rangle \in W} \left(\arg E \vdash_{N\rho} U \to \langle U,b\rangle \in E\right) \ ,$$

where $N\rho$ is the corresponding *information system of the neighborhoods of* $\rho$[5]. Every $U \in \arg W$ generates an eigen-neighborhood $E_U$ of $W$, by

$$\langle U', b\rangle \in E_U :\!= \langle U', b\rangle \in W \wedge U \vdash_\rho U' \ .$$

Furthermore, it is clear that $W \sim_{\rho\to\sigma} \sum_{E \in \mathrm{Eig}\,W} E$.

This is not an inductively defined concept, but merely focuses on the arguments of the higher-type neighborhood at hand. Still, we can conventionally define the eigen-neighborhoods of a base-type neighborhood $U \in \mathsf{Con}_\alpha$ to be $U$ itself and $\varnothing_\alpha$.

*Remark.* For some intuition on this convention one can think of every base type $\alpha$ as being isomorphic to the function space $\mathbb{U} \to \alpha$, where $\mathbb{U}$ is the unit type. Otherwise, one can quite easily see that the *set* of eigen-neighborhoods of a given neighborhood can indeed be defined inductively. This leads to a method of inductive proof that we implicitly use in section 2.4. □

The eigen-neighborhoods behave as generalized tokens to some extent, enough to reveal a quite unexpected atomic behavior that underlies the otherwise non-atomic algebraic entailment.

---

[5]The *neighborhood information system* of $\rho$ is the triple $N\rho = (\mathsf{Con}_\rho, \mathsf{Con}_{N\rho}, \vdash_{N\rho})$, where $\sum_{j=1}^{l} U_j \in \mathsf{Con}_{N\rho}$ if and only if $\cup_{j=1}^{l} U_j \in \mathsf{Con}_\rho$ and $\sum_{j=1}^{l} U_j \vdash_{N\rho} U$ if and only if $\cup_{j=1}^{l} U_j \vdash_\rho U$. This is an information system which is coherent if $\rho$ is coherent and moreover the two have isomorphic domains of ideals (see Chapter 3, page 108).

**Theorem 2.31** (Implicit atomicity at higher types)**.** *Let $W_1, W_2 \in \mathsf{Con}_{\rho\to\sigma}$. The following hold:*

$$W_1 \asymp_{\rho\to\sigma} W_2 \leftrightarrow \underset{E_1\in\operatorname{Eig}W_1}{\forall}\ \underset{E_2\in\operatorname{Eig}W_2}{\forall} \left(\arg E_1 \asymp_{N\rho} \arg E_2 \to \operatorname{val} E_1 \asymp_\sigma \operatorname{val} E_2\right) ,$$

$$W_1 \vdash_{\rho\to\sigma} W_2 \leftrightarrow \underset{E_2\in\operatorname{Eig}W_2}{\forall}\ \underset{E_1\in\operatorname{Eig}W_1}{\exists} \left(\arg E_2 \vdash_{N\rho} \arg E_1 \wedge \operatorname{val} E_1 \vdash_\sigma \operatorname{val} E_2\right) .$$

*Proof.* Let $W_1, W_2 \in \mathsf{Con}_{\rho\to\sigma}$. For consistency, let first $W_1 \asymp_{\rho\to\sigma} W_2$ and $E_i \in \operatorname{Eig} W_i$, $i = 1,2$, with $\arg E_1 \asymp_{N\rho} \arg E_2$; the latter means $U_1 \asymp_\rho U_2$, for any $U_i \in \arg E_i$, which by the consistency of $W$ yields $b_1 \asymp_\sigma b_2$, for any $b_i \in \operatorname{val} E_i$. Conversely, let

$$\arg E_1 \asymp_{N\rho} \arg E_2 \to \operatorname{val} E_1 \asymp_\sigma \operatorname{val} E_2 ,$$

for all $E_i \in \operatorname{Eig} W_i$, $i = 1,2$, and $\langle U_i, b_i\rangle \in W_i$ with $U_1 \asymp_\rho U_2$; let $E_{U_i}$ be the eigen-neighborhood in $W_i$ generated by $U_i$; by the propagation of consistency, it is $\arg E_{U_1} \asymp_{N\rho} \arg E_{U_2}$, so the assumption yields $b_1 \in \operatorname{val} E_{U_1} \asymp_\sigma \operatorname{val} E_{U_1} \ni b_2$.

For entailment, it is:

$$\begin{aligned} W_1 \vdash_{\rho\to\sigma} W_2 &\Leftrightarrow \underset{E_2\in\operatorname{Eig}W_2}{\forall} W_1 \vdash_{\rho\to\sigma} E_2 \\ &\Leftrightarrow \underset{E_2\in\operatorname{Eig}W_2}{\forall} W_1 \arg E_2 \vdash_\sigma \operatorname{val} E_2 \\ &\Leftrightarrow \underset{E_2\in\operatorname{Eig}W_2}{\forall}\ \underset{\langle U_1,b_1\rangle,\dots,\langle U_l,b_l\rangle\in W_1}{\exists} \left(\overset{l}{\underset{j=1}{\forall}} \arg E_2 \vdash_\rho U_l \wedge \sum_{j=1}^{l} b_j \vdash_\sigma \operatorname{val} E_2\right) \\ &\overset{(\star)}{\Leftrightarrow} \underset{E_2\in\operatorname{Eig}W_2}{\forall}\ \underset{E_1\in\operatorname{Eig}W_1}{\exists} \left(\arg E_2 \vdash_{N\rho} \arg E_1 \wedge \operatorname{val} E_1 \vdash_\sigma \operatorname{val} E_2\right) , \end{aligned}$$

where $(\star)$ holds by setting $\langle U, b\rangle \in E_1$ for some $\langle U, b\rangle \in W_1$, whenever $\sum_{j=1}^{l} U_j \vdash_\rho U$; that this is indeed an eigen-neighborhood of $W_1$ follows directly by the propagation of consistency and transitivity of entailment at $\rho$. ☐

Write $\langle U, V\rangle$ for $\sum_{b\in V}\langle U, b\rangle$ and $U \sim_\rho U'$ for $U \vdash_\rho U' \wedge U' \vdash_\rho U$ (equientailment is clearly an equivalence on neighborhoods). With the use of eigen-neighborhoods we can achieve manageable *conservative extensions* of a neighborhood.

**Proposition 2.32.** *Let $W \in \mathsf{Con}_{\rho\to\sigma}$, and $E_1, \dots, E_m \in \operatorname{Eig} W$. For any choice of $U_1, \dots, U_m \in \mathsf{Con}_\rho$ and $V_1, \dots, V_m \in \mathsf{Con}_\sigma$ with the property that $U_i \vdash_\rho \arg E_i$ and $\operatorname{val} E_i \vdash_\sigma V_i$, for $i = 1, \dots, m$, it is*

$$W \sim_{\rho\to\sigma} W + \sum_{i=1}^{m} \langle U_i, V_i\rangle .$$

*Proof.* For the consistency of the extension $W + \sum_{i=1}^{m}\langle U_i, V_i\rangle$, let $i, j = 1, \dots, m$; then

$$U_i \asymp_\rho U_j \Rightarrow \arg E_i \asymp_\rho \arg E_j \Rightarrow \operatorname{val} E_i \asymp_\sigma \operatorname{val} E_j \Rightarrow V_i \asymp_\sigma V_j ,$$

by the propagation of consistency and consistency of $W$; this suffices.

For the equientailment, let $i = 1, \dots, m$; it is

$$U_i \vdash_{N\rho} \arg E_i \wedge \operatorname{val} E_i \vdash_\sigma V_i \Rightarrow E_i \vdash_{\rho\to\sigma} \langle U_i, V_i\rangle ,$$

so $W \vdash_{\rho\to\sigma} W + \sum_{i=1}^{m}\langle U_i, V_i\rangle$. The converse is trivial. ☐

For every $E \in \operatorname{Eig} W$ there is exactly one $U^E \in \mathsf{Con}_\rho$ (up to equientailment) and exactly one $V^E \in \mathsf{Con}_\sigma$ (up to equientailment), such that $E \sim_{\rho\to\sigma} \langle U^E, V^E \rangle$; just set

$$U^E := \mathrm{fl}(\arg E) \quad \text{and} \quad V^E := \operatorname{val} E$$

(for the definition of the flattening mapping see page 93). Say that $W$ is in *eigenform*, if for every $E \in \operatorname{Eig} W$ it is $\langle U^E, V^E \rangle \subseteq W$. Furthermore call $W$ *monotone*, if for all $\langle U_1, V_1 \rangle, \langle U_2, V_2 \rangle \subseteq W$ it is

$$U_1 \vdash_\rho U_2 \to V_1 \vdash_\sigma V_2 \ .$$

For example, $\langle S0, B01 \rangle + \langle S*, B{*}1 \rangle$ is monotone whereas $\langle S0, B{*}1 \rangle + \langle S*, B01 \rangle$ isn't.

**Proposition 2.33.** *Let $\rho$, $\sigma$ be types. For all $W \in \mathsf{Con}_{\rho\to\sigma}$, there is a monotone $W' \in \mathsf{Con}_{\rho\to\sigma}$ in eigenform, such that $W \sim_{\rho\to\sigma} W'$.*

*Proof.* Let $W \in \mathsf{Con}_{\rho\to\sigma}$. Set $\Gamma := \mathrm{fl}(\arg W)$, and

$$W' := \sum_{U \in \mathsf{Con}_\Gamma} \langle U, WU \rangle \ .$$

It is easy to see that this list is finite and consistent; that it is monotone follows from the monotonicity of application; that it is in eigenform is obvious by construction.

We show the equientailment. Let $\langle U, b \rangle \in W$; since obviously $U \in \mathsf{Con}_\Gamma$ and $b \in WU$, it is immediate that $W' \vdash_{\rho\to\sigma} W$. For the other direction, let $\langle U, V \rangle \subseteq W'$, for some $U \in \mathsf{Con}_\Gamma$; since obviously $WU \vdash_\sigma WU$, the definition of higher-type entailment immediately yields $W \vdash_{\rho\to\sigma} \langle U, WU \rangle$. □

So the non-monotone neighborhood $\langle S0, B{*}1 \rangle + \langle S*, B01 \rangle$ has the equivalent monotone eigenform

$$\langle \varnothing, \varnothing \rangle + \langle S0, B{*}1 + B01 \rangle + \langle S*, B01 \rangle + \langle S0 + S*, B{*}1 + B01 \rangle \ ,$$

or, written in tokens,

$$\langle S0, B{*}1 \rangle + \langle S0, B01 \rangle + \langle S*, B01 \rangle + \langle S0 + S*, B{*}1 \rangle + \langle S0 + S*, B01 \rangle \ .$$

Now let $\Theta \in \mathsf{Arr}_{\rho\to\sigma}$. We can use eigen-neighborhoods to define an *eigensplitting* mapping $\mathrm{sp} : \mathsf{Arr}_{\rho\to\sigma} \to \mathsf{Arr}_{\rho\to\sigma}$ by

$$\mathrm{sp}\Theta := \sum_{\langle U, b \rangle \in \Theta} \sum_{E \in \operatorname{Eig} U} \langle E, b \rangle \ ,$$

such that it commutes with the flattening mapping.

**Proposition 2.34.** *Let $W_1 +_N \cdots +_N W_J \in \mathsf{Arr}_{N(\rho\to\sigma)}$. Then*

$$\mathrm{sp}(\mathrm{fl}(W_1 +_N \cdots +_N W_J)) = \mathrm{fl}(\mathrm{sp}W_1 +_N \cdots +_N \mathrm{sp}W_J) \ .$$

*Proof.* It is

$$\begin{aligned} \mathrm{sp}(\mathrm{fl}(W_1 +_N \cdots +_N W_J)) &= \mathrm{sp}(W_1 + \cdots + W_J) \\ &\overset{(\star)}{=} \mathrm{sp}(W_1) + \cdots + \mathrm{sp}(W_J) \\ &= \mathrm{fl}(\mathrm{sp}(W_1) +_N \cdots +_N \mathrm{sp}(W_J)) \ , \end{aligned}$$

where $(\star)$ is direct to see from the definition of eigensplitting. □

## 2.4 Totality and density

In a finitary basic algebraic coherent information system a token is total when it involves no stars, and a neighborhood is total when it *entails* a total token; write $GT$ and $G\mathsf{Con}$ for total tokens and total neighborhoods respectively. Define the *total ideals* at type $\rho$, and write $G_\rho$, by the following: at base type $\alpha$, $x$ is total if it contains a total token; at type $\rho \to \sigma$, $f$ is total when, applied to a total ideal, it yields a total ideal, that is, when

$$\underset{x \in \rho}{\forall} \left( x \in G_\rho \to f(x) \in G_\sigma \right) ,$$

where $b \in f(x)$ if and only if $\langle U, b \rangle \in f$, for some $U \subseteq x$. A type $\rho$ is called *dense* when every neighborhood $U \in \mathsf{Con}_\rho$ can be extended to a total ideal $x \in G_\rho$. The last main result of this chapter is the *density theorem* (Theorem 2.37), which states that every type is dense.

At every base type $\alpha$ we demand the existence of a distinguished nullary constructor $0_\alpha$. Noting that every type $\rho$ has the form $\rho_1 \to \cdots \to \rho_p \to \alpha$, for some base type $\alpha$ and some non-negative number $p$, write $\rho^-$ for the (possibly empty) finite sequence $(\rho_1, \ldots, \rho_p)$ of the argument types, and $\mathsf{Con}_{\rho^-}$ for $\mathsf{Con}_{\rho_1} \times \cdots \times \mathsf{Con}_{\rho_p}$; elements of $\mathsf{Con}_{\rho^-}$ are (possibly empty) finite sequences of neighborhoods of the corresponding types.

**Lemma 2.35** (Separation)**.** *Let $\rho$ be a type. For every $U_1, U_2 \in \mathsf{Con}_\rho$, if $U_1 \not\asymp_\rho U_2$ then there exists a* separator *$U^- \in \mathsf{Con}_{\rho^-}$ such that $U_1 U^- \not\asymp_\alpha U_2 U^-$.*

*Proof by induction on types.* At a base type there is nothing to show. Let $W_1, W_2 \in \mathsf{Con}_{\rho\to\sigma}$, with $W_1 \not\asymp_{\rho\to\sigma} W_2$. There will be $\langle U_i, b_i \rangle \in W_i$, $i = 1, 2$, with $\langle U_1, b_1 \rangle \not\asymp_{\rho\to\sigma} \langle U_2, b_2 \rangle$, that is, with $U_1 \asymp_\rho U_2 \wedge b_1 \not\asymp_\sigma b_2$. By the induction hypothesis, there is a separator of $b_1$, $b_2$, that is, a finite sequence $V^- \in \mathsf{Con}_{\sigma^-}$ such that $b_1 V^- \not\asymp_\alpha b_2 V^-$. Set $U^- := (U_1 + U_2, V^-)$; this is obviously a finite sequence in $\mathsf{Con}_{(\rho\to\sigma)^-}$ and satisfies what we need. □

*Remark.* Notice that the choice of $U^-$ in the proof is not unique, since there may be more than one inconsistency between $W_1$ and $W_2$. □

Now we state another useful lemma without proof (see [49, §6.5]).

**Lemma 2.36** (Extension)**.** *Let $\rho$ be a type and $f, g \in \mathsf{Ide}_\rho$. If $f \in G_\rho$ and $f \subseteq g$ then $g \in G_\rho$.*

**Theorem 2.37** (Density)**.** *Let $\rho$ be a type. For every $U \in \mathsf{Con}_\rho$ there is an $x \in \mathsf{Ide}_\rho$, such that $U \subseteq x$ and $x \in G_\rho$.*

*Proof by induction on types.* At base type $\alpha$, let $U \in \mathsf{Con}_\alpha$, and $b := e(U)$ be its eigentoken. Clearly, it suffices to find a total token $a \in GT_\alpha$ with $a \vdash_\alpha b$, and then set $x := \overline{a}$. If $b = 0$, for some nullary constructor 0, then set $a := b$; if $b = *_\alpha$, set $a := 0_\alpha$. If $b = Cb_1 \cdots b_r$, then the induction hypothesis gives $a_i \in GT_\alpha$, with $a_i \vdash_\alpha b_i$, for $i = 1, \ldots, r$, so setting $a := Ca_1 \cdots a_r \in GT_\alpha$ gives $a \vdash_\alpha b$.

At type $\rho \to \sigma$, assume that $\sigma$ and all types in $\rho^-$ are dense. Let $W = \sum_{k=1}^{n} \langle U_k, b_k \rangle$ be a neighborhood. Observe first that, if there are indices $i, j = 1, \ldots, n$ such that $b_i \not\asymp_\sigma b_j$, then it is also $U_i \not\asymp_\rho U_j$, by the consistency of $W$. Then by Lemma 2.35 there is a separator $U_{ij}^- \in \mathsf{Con}_{\rho^-}$ of $U_i$, $U_j$. In the following we assume that $U_{ij}^- = U_{ji}^-$.

For an arbitrary neighborhood $U \in \mathsf{Con}_\rho$ gather its fitting values in $W$, if any, by

$$V_U := \{b_k \mid \underset{i=1}{\overset{n}{\forall}} \left(b_i \not\asymp_\sigma b_k \to UU^-_{ik} \vdash_\alpha U_k U^-_{ik}\right)\} ;$$

notice that $b_k \in V_{U_k}$. This is a neighborhood: let $b_i, b_j \in V_U$, and suppose that $b_i \not\asymp_\sigma b_j$; by the definition of $V_U$ it is

$$UU^-_{ij} \vdash_\alpha U_i U^-_{ij} \ \wedge \ UU^-_{ij} \vdash_\alpha U_j U^-_{ij} \Rightarrow U_i U^-_{ij} \asymp_\alpha U_j U^-_{ij} \ ,$$

which contradicts that $U^-_{ij}$ is a separator. Moreover, it is monotone in $U$: let $U, U' \in \mathsf{Con}_\rho$, with $U \vdash_\rho U'$; by monotonicity of application it is $UU^-_{ik} \vdash_\alpha U' U^-_{ik}$; by the definition of $V_U$, it is direct to check that, if $b_k \in V_{U'}$, then $b_k \in V_U$ as well, so the desired monotonicity follows trivially by inclusion.

By the induction hypothesis we have density at type $\sigma$, so $V_U$ will have a total extension $v_U$. Set

$$f := \{\langle U, b\rangle \mid \left( \underset{1 \le i,j \le n}{\forall} \left(b_i \not\asymp_\sigma b_j \to UU^-_{ij} \in G\mathsf{Con}_\alpha\right) \wedge b \in v_U\right) \vee V_U \vdash_\sigma b\} \ . \quad (2.25)$$

Since $b_k \in V_{U_k}$, it is clear that $f$ contains $W$. We show that $f$ is an ideal, by showing that it is an approximable mapping.

First, it is consistently defined: Let $\langle U, V\rangle \subseteq f$; we show that $V \in \mathsf{Con}_\sigma$. For arbitrary $b, b' \in V$, by the definition of $f$, it is either $b, b' \in v_U$, or $b \in v_U \wedge V_U \vdash b'$, or $V_U \vdash b, b'$, so $b \asymp_\sigma b'$ in all three cases.

Then, it is deductively closed on the right: Let $\langle U, V\rangle \subseteq f$ and $V \vdash_\sigma b$; we show that $\langle U, b\rangle \in f$. There are two cases: either $b' \in v_U$, for some $b' \in V$, or $V_U \vdash_\sigma b'$, for all $b' \in V$. In the first case, we have $V \subseteq v_U$, and since $v_U$ is deductively closed, $b \in v_U$. In the second case, we have $V_U \vdash_\sigma b$ by transitivity at $\sigma$.

Finally, it is deductively closed on the left: Let $U \vdash_\rho U'$ and $\langle U', b\rangle \in f$; we show that $\langle U, b\rangle \in f$. In case $U' U^-_{ij} \in G\mathsf{Con}_\alpha$, for $b_i \not\asymp_\sigma b_j$, by monotonicity of application it is also $UU^-_{ij} \in G\mathsf{Con}_\alpha$. Moreover, since $U \vdash_\rho U'$, both $U' U^-_{ij}$ and $U' U^-_{ij}$ will entail the same total token, so $V_U = V_{U'}$, and then $b \in v_U$. In case $V_{U'} \vdash_\sigma b$, monotonicity of $V_U$ in $U$ yields $V_U \vdash b$.

It remains to show totality. Let $x \in G_\rho$. We show that $f(x) \in G_\sigma$. By the definition of application we have

$$b \in f(x) \Leftrightarrow \underset{U \subseteq x}{\exists} \langle U, b\rangle \in f \ .$$

For all $i, j = 1, \dots, n$ with $b_i \not\asymp_\sigma b_j$ we have separators $U^-_{ij} \in \mathsf{Con}_{\rho^-}$; by the induction hypotheses at types in $\rho^-$, for every such separator there exists a finite sequence $u^-_{ij}$ of corresponding total extensions. By totality of $x$ we have $x(u^-_{ij}) \in G_\alpha$. Let $U$ be the union of all $U_{ij} \subseteq x$ with $U_{ij} u^-_{ij} \in G\mathsf{Con}_\alpha$; it is also $U u^-_{ij} \in G\mathsf{Con}_\alpha$. Then, for all $b \in v_U$, it is $\langle U, b\rangle \in f$, which means that $v_U \subseteq f(x)$. By Lemma 2.36 we have that $f(x) \in G_\sigma$. □

Now let us examine what it means for our setting that the density statement holds. Write $x \asymp_\rho U$ if $U^x \asymp_\rho U$ for all $U^x \subseteq x$.

**Proposition 2.38.** *For all $x \in G_\rho$ and $U \in \mathsf{Con}_\rho$ it is effectively either $x \asymp_\rho U$ or $x \not\asymp_\rho U$.*

*Proof by induction on types.* At base type $\alpha$, let $x \in G_\alpha$ and $U \in \mathsf{Con}_\alpha$; let $a \in x$ be the total token of $x$; then either $a \asymp_\alpha U$ or $a \not\asymp_\alpha U$, so either $x \asymp_\alpha U$ or $x \not\asymp_\alpha U$.

At type $\rho \to \sigma$, let $f \in G_{\rho\to\sigma}$ and $W \in \mathsf{Con}_{\rho\to\sigma}$; let $\langle U,V\rangle \in \mathrm{Eig}(W)$; by Theorem 2.37 there exists an $x \in G_\rho$ with $U \subseteq x$; $f$ is total, so $f(x) \in G_\sigma$, and it is either $f(x) \asymp_\sigma V$ or $f(x) \not\asymp_\sigma V$ by the induction hypothesis at $\sigma$; so we can decide if $f \asymp_{\rho\to\sigma} \langle U,V\rangle$ holds for all eigen-neighborhoods of $W$ or if $f \not\asymp_{\rho\to\sigma} \langle U,V\rangle$ for some of them; in the first case we have $f \asymp_{\rho\to\sigma} W$, and in the second we have $f \not\asymp_{\rho\to\sigma} W$. □

Note that there is a neighborhood at each type $\rho$, which is consistent with *all* total ideals, namely $\varnothing_\rho$, the empty one. Now suppose that we are given the list of arguments of some higher-type neighborhood, and that it contains two inconsistent neighborhoods; the following implies that we can witness their inconsistency in a minimal way.

**Proposition 2.39.** *For all $x \in G_\rho$ and $U_1, U_2 \in \mathsf{Con}_\rho$ in eigenform, if $x \asymp_\rho U_1$ and $U_1 \not\asymp_\rho U_2$ then*

1. *$x \not\asymp_\rho U_2$, and furthermore,*
2. *there is a $U^{x,1} \in \mathsf{Con}_\rho$, such that $U_1 \vdash_\rho U^{x,1} \subseteq x$ and $U^{x,1} \not\asymp_\rho U_2$.*

*Proof by induction on types.* At type $\alpha$, let $x \in G_\alpha$ and $U_1, U_2 \in \mathsf{Con}_\alpha$, with $x \asymp_\alpha U_1$ and $U_1 \not\asymp_\alpha U_2$; then $U_1 \subseteq x$, and the second assumption yields $x \not\asymp_\alpha U_2$, so the property 1 holds. The property 2 holds as well, for $U^{x,1} := U_1$.

At type $\rho \to \sigma$, let $f \in G_{\rho\to\sigma}$ and $W_1, W_2 \in \mathsf{Con}_{\rho\to\sigma}$, with $f \asymp_{\rho\to\sigma} W_1$ and $W_1 \not\asymp_{\rho\to\sigma} W_2$; by Theorem 2.31 and the fact that the neighborhoods are in eigenform, there will be $\langle U_i, V_i\rangle \subseteq W_i$, $i = 1,2$, such that

$$\langle U_1, V_1\rangle \not\asymp_{\rho\to\sigma} \langle U_2, V_2\rangle \ ,$$

which means

$$U_1 \asymp_\rho U_2 \wedge V_1 \not\asymp_\sigma V_2 \ ;$$

by Theorem 2.37 there is an $x \in G_\rho$ with $U_1 + U_2 \subseteq x$, and by the totality of $f$ we have $f(x) \in G_\sigma$, for which

$$f(x) \asymp_\sigma V_1 \wedge V_1 \not\asymp_\sigma V_2 \ .$$

The induction hypothesis that 1 holds at $\sigma$ yields $f(x) \not\asymp_\sigma V_2$, so $f \not\asymp_{\rho\to\sigma} \langle U_2, V_2\rangle$, and consequently $f \not\asymp_{\rho\to\sigma} W_2$, so the property 1 holds at $\rho \to \sigma$ as well.

The induction hypothesis that 2 holds at $\sigma$ yields a $V^{f(x),1} \in \mathsf{Con}_\sigma$, with

$$V_1 \vdash_\sigma V^{f(x),1} \subseteq f(x) \ \wedge \ V^{f(x),1} \not\asymp_\sigma V_2 \ ;$$

by the definition of application of ideals, let $U^{f(x),1} \in \mathsf{Con}_\rho$ be an argument for $V^{f(x),1}$ in $f$, that is, be such that

$$\langle U^{f(x),1}, V^{f(x),1}\rangle \subseteq f \ \wedge \ U^{f(x),1} \subseteq x \ ;$$

then $\langle U^{f(x),1} + U_1 + U_2, V^{f(x),1}\rangle \subseteq f$, by the deductive closure of $f$; it is

$$\langle U_1, V_1\rangle \vdash_{\rho\to\sigma} \langle U^{f(x),1} + U_1 + U_2, V^{f(x),1}\rangle \subseteq f \ ,$$

and

$$\langle U^{f(x),1} + U_1 + U_2, V^{f(x),1}\rangle \not\asymp_{\rho\to\sigma} \langle U_2, V_2\rangle \ ;$$

so it suffices to set

$$W^{f,1} := \sum_{U_1 \in \mathsf{Con}_{\mathrm{fl}(\arg W_1)}} \langle U^{f(x),1} + U_1 + U_2, V^{f(x),1} \rangle \ ,$$

and the property 2 holds at $\rho \to \sigma$ as well. □

## 2.5 Notes

### Non-blank-row matrices

Proposition 2.6 interestingly implies that matrices with non-blank rows together with matrix overlapping and matrix inclusion form an acis; let us call this *nbr-acis* and denote it by $(\mathsf{Arr}^{nbr}, \between, \subset)$. This acis is different from the $(\mathsf{Mat}, \asymp, \succ)$, since the carriers are clearly distinct: blank arrays belong to $\mathsf{Mat}$ but not to $\mathsf{Arr}^{nbr}$ and the former contains only coherently consistent matrices. However, one can easily show that if $A \subset B$ then $B \succ A$, though no similar remark can be made for $\between$ and $\asymp$.

This last remark is nevertheless misleading as far as the nature of this structure is concerned. The nbr-acis essentially mimics the conjunction-implication systems of page 51: given two (decidable) formulas, suppose we induce corresponding matrices (like the test matrices we used for the function spaces); then overlapping of the latter mimics the conjunction of the former while inclusion of the latter mimics implication of the former, as is portrayed by Theorem 2.8.

### On normal forms

The notion of a "normal form", as is well known, is a prime player in the context of rewriting systems (see for example [54]). A *rewriting system* is just a set $A$ with a binary relation $\to$, its *rewrite relation*. We say that an element $a \in A$ is in *normal form* if it is not further rewritable, that is, if there is no $b \in A$ for which $a \to b$. The rewriting system is *uniquely normalizing* if for every $a \in A$ there is a unique normal form $a' \in A$ such that $a \to^\star a'$ ($\to^\star$ being the reflexive-transitive closure of $\to$); we say that $a'$ is *the normal form of $a$*.

Under this terminology, there are *three* different notions of normal forms of a given neighborhood $U$ that we have encountered: the maximal form $m(U)$ (pages 33 and 84), the homogeneous form $h(U)$ (page 69) and the eigentoken, or *eigenform*, $e(U)$ (page 82). As we have seen, the only ubiquitous one is the homogeneous form, as it makes sense for all finitary and infinitary algebras alike. On the other hand, the eigenform exists only for finitary algebras, while the maximal form is unique only for finitary algebras with at most unary constructors.

In the preceding sections we have studied these normal forms rather algebraically, as *functions* $h$, $e$, and $m$—even, the "matrix form" $K_U(M_U)$ of $U$ is strictly speaking not a *normal* form, since the analysis to an application of a constructor context to a basic matrix is just a (unique) *representation* of the homogeneous form. An approach to these notions as normal forms of explicit rewrite relations $\to^h$, $\to^e$, and $\to^m$, should shed appropriate light on the equational logic of any logical system that would attempt to formalize our model for higher-type computability, like the system $\mathrm{TCF}^+$ of [18].

### On general formal matrix theories

Except of course for the fully developed linear algebraic matrix theory and its wide applications and generalizations in mathematics and science in general, it seems that little or no work has been done on matrices over non-field-like structures. Given their combined intuitiveness and practicability as mathematical objects, it could be expected that a treatment of matrices over an arbitrary mathematical structure (in the sense of mathematical logic), or maybe over graphs and hypergraphs (in the sense of graph theory and theoretical computer science) should be direct and easily attainable to a considerable extent; still, in view of the lack of relevant literature, it seems that something like that has hardly ever been considered worthwhile to pursue until now.

To the best knowledge of the author, this is the first time that a non-numerical, relational structure (here, information systems) motivates a study of matrices; one can wonder what a more general approach to a formal matrix theory could bring about.

### Implicit atomicity formally

It is interesting to notice that there is a flavor of *uniformity* about atomic entailment. Let $U = K(A)$ and $b = K(B)$ be the matrix forms of a neighborhood and a token at a finitary base type, where $A \in \mathsf{Mat}(r,l)$ and $B \in \mathsf{Mat}(r,1)$; then

$$U \vdash b \leftrightarrow \mathop{\forall}_{i=1}^{r} \mathop{\exists}_{j=1}^{l} A(i,j) \succ B(i) ,$$

whereas

$$U \succ b \leftrightarrow \mathop{\exists}_{j=1}^{l} \mathop{\forall}_{i=1}^{r} A(i,j) \succ B(i) .$$

Furthermore, in the light of our study of eigen-neighborhoods, we may state the implicit atomicity of non-atomic systems in a strict formal way. Say that an information system $\rho$ features *implicit atomicity* when for each neighborhood, there is an equivalent *atomic* one, that is,

$$\mathop{\forall}_{U \in \mathsf{Con}_\rho} \mathop{\exists}_{U^E \in \mathsf{Con}_\rho} \left( U^E \sim U \wedge \mathop{\forall}_{b \in T_\rho} \left( U^E \vdash b \rightarrow \mathop{\exists}_{a \in U^E} \{a\} \vdash b \right) \right) .$$

Acises are implicitly atomic in a trivial way, since *every* neighborhood there is defined to be atomic. Using Theorems 2.22 and 2.31 (in conjunction with Proposition 2.33) one may show that *all* algebraic coherent information systems are implicitly atomic: at each type, each $U$ has an atomic eigenform $U^E$.

### The maximal clique problem

In page 90 we start a short study of maximal neighborhoods in arbitrary lists. We should note that the notion of a maximal neighborhood has a clear graph-theoretic nature, which can give us a good intuition about its behavior. A list $\Gamma \in \mathsf{Arr}_\rho(n)$ induces an undirected graph $G_\Gamma$ with all loops, where the nodes are (the indices of) its elements and an edge appears whenever two nodes are consistent. A maximal neighborhood in $\Gamma$ corresponds then simply to a *maximal clique* in $G_\Gamma$.

The problem of determining all maximal cliques in a given graph is a well-known subject in the related literature (see for example in [19]). An algorithm which solves the problem is the following:

1. set $\mathrm{Max}\,\Gamma := \{\Gamma\}$;
2. for all pairs $(i,j) \notin G_\Gamma$ do the following:
   (a) let INC be the set of the current $M \in \mathrm{Max}\,\Gamma$, for which $i,j \in M$; reset

   $$\mathrm{Max}\,\Gamma := \bigcup_{M \in \mathrm{INC}} (\mathrm{Max}\,\Gamma \smallsetminus M) \cup \{M \smallsetminus i, M \smallsetminus j\}) \ ,$$

   so that the inconsistency between $a_i$ and $a_j$ is lifted by separating $M$ into two subsets, one including the former, the other the latter;

   (b) reset

   $$\mathrm{Max}\,\Gamma := \{M \in \mathrm{Max}\,\Gamma \mid \forall_{M' \in \mathrm{Max}\,\Gamma} \big(M \subseteq M' \to M = M'\big)\} \ ,$$

   so that only the (set-theoretically) maximal sets are kept.

It is not hard to see that the algorithm in the end yields the maximal cliques: the for-loop (2a) ensures that all inconsistencies are raised from each possible maximal neighborhoods, while maximality is ensured by step (2b); that no element is forgotten in the process is ensured by the initial value at step (1).

As an example, consider a list $\Gamma \in \mathsf{Arr}_\rho(8)$, with inconsistency pairs $(1,4)$, $(2,6)$, $(4,6)$, and $(5,6)$. The algorithm proceeds as follows:

$$\begin{aligned}
\Gamma &= (1+2+3+4+5+6+7+8) \\
&\overset{(1,4)}{=} (\mathbf{1}+2+3+5+6+7+8) + (2+3+\mathbf{4}+5+6+7+8) \\
&\overset{(2,6)}{=} (1+\mathbf{2}+3+5+7+8) + (1+3+5+\mathbf{6}+7+8) \\
&\qquad + (\mathbf{2}+3+4+5+7+8) + (3+4+5+\mathbf{6}+7+8) \\
&\overset{(4,6)}{=} (1+2+3+5+7+8) + (1+3+5+6+7+8) + (2+3+4+5+7+8) \\
&\qquad + \cancel{(3+\mathbf{4}+5+7+8)} + \cancel{(3+5+\mathbf{6}+7+8)} \\
&\overset{(5,6)}{=} (1+2+3+5+7+8) + \cancel{(1+3+\mathbf{5}+7+8)} + \\
&\qquad + (1+3+\mathbf{6}+7+8) + (2+3+4+5+7+8) \ ,
\end{aligned}$$

where the canceled lists are proper subsets of preexisting lists each time; the resulting maximal cliques are the remaining three.

Complexity lies beyond our current scope, but we should notice that in the worst-case scenario, where $\Gamma$ consists of pairwise inconsistent elements, thus inducing a so-called *totally disconnected* graph—to be precise, "totally" up to the loops—the algorithm terminates in $2^{\frac{1}{2}n(n-1)}$ steps (taking both symmetry and reflexivity into account)—in which case, naturally, the maximal neighborhoods would consist of one element of $\Gamma$ each.[6] What is particular in the above algorithm to our approach, is that it employs a recurring idea: "an inconsistency may serve as a pivot".

## On density

The density theorem was first proved by Stephen Kleene in [21] and Georg Kreisel in [24], and since the development of domain theory, and after Yuri Ershov's [10], it

[6] Thanks to Mihalis Yannakakis for the enlightening private correspondence on this issue.

has been studied a number of times, notably in Ulrich Berger's [4] and [5]—see [52], [45], [25], and also [32].

The proof of our density theorem (page 97) follows closely the corresponding proof in [18] (where it was also formalized in $\mathrm{TCF}^+$). It differs in the following points: (a) it is "linear", in that it does not employ a mutual induction anymore, where each type was shown to be both dense and separating, but breaks down to a lemma for separation and a single induction for density; (b) the separators are not *infinite* objects (total ideals) any more, but mere *finite* lists (neighborhoods). A third difference is that we restrict here attention to types over *finitary* algebras, but it is easy to see that this does not really harm generality.

### Outlook

There are at least two directions to pursue. On the one hand, the matrix theory which we developed in the first half of the chapter could afford a lot of streamlining, in particular on such a proof assistant as MINLOG (see `http://www.math.lmu.de/~logik/minlog/index.php` as well as [3], [48], and [6]). On the other hand, the higher-type notion of an eigen-neighborhood suggests novel techniques that we could use to obtain native proofs of definability, density, and other deep results which up to now could have been available only as adaptations of much more general arguments.

# Chapter 3

# Connections to point-free structures

In this chapter we seek to establish direct connections between information systems and well-known point-free topological structures. This should be viewed as a step in the spirit of Giovanni Sambin's abstract of [38]:

> *[...] how much of domain theory can be generalized to formal topology? My impression is that some open problems in one of the two fields could already have a solution in the other, and that is why an intensification of contact should be rewarding.*

### Preview

In contrast to the previous chapters, we adopt here a traditional and general view, where no algebras are considered. In section 3.1 we state basic facts and observations concerning Scott information systems, and in section 3.2 we introduce the notions of atomicity and coherence in the traditional top-down way, where among other things we point to the fact that atomic and coherent versions of Scott information systems feature more ideals than the generic version (Theorem 3.7).

In section 3.3 we consider well-known point-free structures, like domains and formal topologies, and impose appropriate *coherence properties* on them to show that they correspond to coherent information systems.

## 3.1 Scott information systems

A *(Scott) information system* is a triple $\rho = (T, \mathsf{Con}, \vdash)$ where $T$ is a countable set of *tokens*, $\mathsf{Con} \subseteq \mathscr{P}_f(T)$ is a collection of *(formal) neighborhoods* and $\vdash \subseteq \mathsf{Con} \times T$ is an *entailment* relation, which obey the following:

1. consistency is reflexive, in the sense that a token suffices to form a formal neighborhood:
$$\{a\} \in \mathsf{Con} \ ;$$
2. consistency is closed under subsets:
$$U \in \mathsf{Con} \wedge V \subseteq U \to V \in \mathsf{Con} \ ;$$

3. entailment is reflexive, in the sense that a neighborhood entails its elements:

$$a \in U \to U \vdash a \ ;$$

4. entailment is transitive:

$$U \vdash V \land V \vdash c \to U \vdash c \ ;$$

5. consistency propagates through entailment:

$$U \in \mathsf{Con} \land U \vdash b \to U \cup \{b\} \in \mathsf{Con} \ ,$$

where $U \vdash V$ is a shorthand for $\forall_{b \in V} U \vdash b$.

The following follow directly from the definition of an information system.

**Proposition 3.1.** *In every information system the following hold:*

1. $\varnothing \in \mathsf{Con}$,
2. $U \vdash V \to U \cup V \in \mathsf{Con}$,
3. $U \vdash U$,
4. $U \vdash V \land V \vdash W \to U \vdash W$,
5. $U' \supseteq U \land U \vdash V \land V \supseteq V' \to U' \vdash V'$,
6. $U \vdash V \land U \vdash V' \to U \vdash V \cup V'$.

An *ideal* in $\rho$ is a set $u \subseteq T$ which is *consistent* and *closed under entailment* in the following sense:

$$\forall_{U \subseteq^f u} U \in \mathsf{Con} \land \forall_{U \subseteq^f u} \left( U \vdash a \to a \in u \right) \ .$$

Denote the empty ideal by $\perp$ and the collection of all ideals of $\rho$ by $\mathsf{Ide}_\rho$. Define the *deductive closure* of a neighborhood $U \in \mathsf{Con}$ by

$$\mathrm{cl}_\rho(U) := \{a \in T \mid U \vdash a\} \ .$$

When $\rho$ is clear from the context we just write $\overline{U}$. Write $\overline{\mathsf{Con}}_\rho$ for the collection of all such closures.

**Proposition 3.2.** *If* $U \in \mathsf{Con}$ *then* $\overline{U} \in \mathsf{Ide}_\rho$.

*Proof.* Let $U \in \mathsf{Con}$, and $V \subseteq^f \overline{U}$; it is $U \vdash V$, so $U \cup V \in \mathsf{Con}$ and so $V \in \mathsf{Con}$. If, further, $V \subseteq^f \overline{U} \land V \vdash a$, then $U \vdash V \vdash a$, so $a \in \overline{U}$ by definition. □

## Two examples

We give two examples of finite information systems that we will use later. Consider the alphabet $\Omega = \{l, r, m\}$, and denote the empty word of $\Omega^\star$ by $\varepsilon$. The first one, the *Coquand information system* $\mathscr{C}$, offers a formal version of Coquand's counterexample to atomicity (see page 47), and is defined by

$$T_\mathscr{C} := \{\varepsilon, l, r\} \ ,$$
$$\mathsf{Con}_\mathscr{C} := \mathscr{P}_f(T_\mathscr{C}) \ ,$$

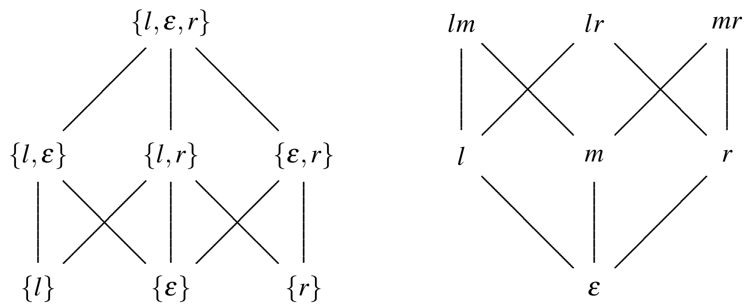


Figure 3.1: Entailments in $\mathsf{Con}_{\mathscr{C}} \setminus \varnothing$ and $T_{\mathscr{L}}$.

and its entailment by (see Figure 3.1)

$$\begin{array}{ccc} \{l\} \vdash_{\mathscr{C}} l\,, & \{r\} \vdash_{\mathscr{C}} r\,, & \{\varepsilon\} \vdash_{\mathscr{C}} \varepsilon\,, \\ \{l,r\} \vdash_{\mathscr{C}} l,r,\varepsilon\,, & \{r,\varepsilon\} \vdash_{\mathscr{C}} r,\varepsilon\,, & \{l,\varepsilon\} \vdash_{\mathscr{C}} l,\varepsilon\,, \\ & \{l,r,\varepsilon\} \vdash_{\mathscr{C}} l,r,\varepsilon\,. & \end{array}$$

It is easy to check that $\mathscr{C}$ indeed satisfies the axioms of an information system. Furthermore, understanding that $\{l,r\} \vdash_{\mathscr{C}} \varepsilon$ is the only nontrivial entailment at hand, it is easy to see that

$$\mathsf{Ide}_{\mathscr{C}} = \mathscr{P}_f(T_{\mathscr{C}}) \setminus \{l,r\}\,.$$

The *Plotkin information system* $\mathscr{L}$ is defined by

$$\begin{aligned} T_{\mathscr{L}} &:= \{\varepsilon, l, m, r, lm, lr, mr\}\,, \\ U \in \mathsf{Con}_{\mathscr{L}} &:= \exists_{a \in T_{\mathscr{L}}} \forall_{b \in U}\, a \text{ superword of } b\,, \\ U \vdash_{\mathscr{L}} b &:= \exists_{a \in U}\, a \text{ superword of } b\,. \end{aligned}$$

Again, it is easy to check that $\mathscr{L}$ satisfies the axioms of an information system. Its ideals, $\mathsf{Ide}_{\mathscr{L}}$, are the following:

$$\begin{aligned} &\bot\,,\ \{\varepsilon\}\,, \\ &\{l,\varepsilon\}\,,\ \{m,\varepsilon\}\,,\ \{r,\varepsilon\}\,, \\ &\{l,m,\varepsilon\}\,,\ \{l,r,\varepsilon\}\,,\ \{m,r,\varepsilon\}\,, \\ &\{lm,l,m,\varepsilon\}\,,\ \{lr,l,r,\varepsilon\}\,,\ \{mr,m,r,\varepsilon\}\,. \end{aligned}$$

## Approximable maps

Let $\rho$ and $\sigma$ be two information systems. A relation $r \subseteq \mathsf{Con}_\rho \times T_\sigma$ is called an *approximable map* from $\rho$ to $\sigma$ if

- it is *consistently defined*, that is,

$$\forall_{b \in V} (U,b) \in r \to V \in \mathsf{Con}_\sigma\,,$$

  for $U \in \mathsf{Con}_\rho$, $V \subseteq^f T_\sigma$, and

- it is *deductively closed*, that is,

$$U' \vdash_\rho U \wedge \underset{b \in V}{\forall} (U,b) \in r \wedge V \vdash_\sigma b' \to (U',b') \in r ,$$

  for $U, U' \in \mathsf{Con}_\alpha$, $V \in \mathsf{Con}_\sigma$ and $b' \in T_\sigma$.

Let us stress the fact that an approximable map need not be a single-valued mapping in general. Write $\mathsf{Apx}_{\rho\to\sigma}$ for all approximable maps from $\rho$ to $\sigma$. One can easily generalize Proposition 1.16 and show that the ideals of $\rho \to \sigma$ are exactly the approximable maps from $\rho$ to $\sigma$, that is, that $\mathsf{Ide}_{\rho\to\sigma} = \mathsf{Apx}_{\rho\to\sigma}$.

As already remarked in Scott's seminal paper [51, §5] (where actually the converse route was taken), any approximable map $r$ from $\rho$ to $\sigma$ induces a relation $\hat{r} \subseteq \mathsf{Con}_\rho \times \mathsf{Con}_\sigma$ by letting

$$(U,V) \in \hat{r} :\simeq \underset{b \in V}{\forall} (U,b) \in r .$$

**Proposition 3.3.** *Let $r$ be an approximable map from $\rho$ to $\sigma$. For the relation $\hat{r}$ the following hold:*

1. $(\varnothing,\varnothing) \in \hat{r}$,
2. $(U,V) \in \hat{r} \wedge (U,V') \in \hat{r} \to (U, V \cup V') \in \hat{r}$,
3. $U' \vdash_\rho U \wedge (U,V) \in \hat{r} \wedge V \vdash_\sigma V' \to (U',V') \in \hat{r}$.

*Conversely, if $R \subseteq \mathsf{Con}_\rho \times \mathsf{Con}_\sigma$ satisfies the above, then the relation $\check{R} \subseteq \mathsf{Con}_\rho \times T_\sigma$ defined by*

$$(U,b) \in \check{R} :\simeq \underset{V \in \mathsf{Con}_\sigma}{\exists} (b \in V \wedge (U,V) \in R)$$

*is an approximable map from $\rho$ to $\sigma$.*

In what follows we will identify $r$ with $\hat{r}$ and $R$ with $\check{R}$.

## Ideal-wise equality

Call two information systems $\rho$ and $\sigma$ *ideal-wise equal* or just *equal*, and write $\rho \simeq \sigma$, if they give rise to the same ideals up to isomorphism, that is, if $\mathsf{Ide}_\rho \simeq \mathsf{Ide}_\sigma$. Ideal-wise equal information systems define isomorphic domains (see section 3.3), hence the overload on the isomorphism symbol "$\simeq$". Call $\rho$ *ideal-wise smaller* or just *smaller* than $\sigma$, and write $\rho \preceq \sigma$, if the ideals of the former can be embedded into the ideals of the latter, that is, if $\mathsf{Ide}_\rho \hookrightarrow \mathsf{Ide}_\sigma$. Notice that, in general, it is $\rho \preceq \sigma \wedge \sigma \preceq \rho \not\to \rho \simeq \sigma$.

Let $\rho = (T, \mathsf{Con}, \vdash)$ be an information system. Its *neighborhood information system* $N\rho = (NT, N\mathsf{Con}, \vdash^N)$ is defined by

$$NT :\simeq \mathsf{Con} ,$$
$$\mathscr{U} \in N\mathsf{Con} :\simeq \bigcup \mathscr{U} \in \mathsf{Con} ,$$
$$\mathscr{U} \vdash^N V :\simeq \bigcup \mathscr{U} \vdash V .$$

**Proposition 3.4.** *If $\rho$ is an information system then $N\rho$ is an information system, and it is $\rho \simeq N\rho$.*

*Proof.* For reflexivity of consistency: Let $U \in \mathsf{Con}$; then obviously $\{U\} \in N\mathsf{Con}$ by definition. For closure under subsets: Let $\mathscr{U}, \mathscr{V} \in \mathscr{P}_f(\mathsf{Con})$; then

$$\begin{aligned}
&\mathscr{U} \in N\mathsf{Con} \wedge \mathscr{V} \subseteq \mathscr{U} \\
&\quad \Rightarrow \bigcup \mathscr{U} \in \mathsf{Con} \wedge \bigcup \mathscr{V} \subseteq \bigcup \mathscr{U} \\
&\quad \overset{\text{clsr}}{\Rightarrow} \bigcup \mathscr{V} \in \mathsf{Con} \\
&\quad \Leftrightarrow \mathscr{V} \in N\mathsf{Con} .
\end{aligned}$$

For reflexivity of entailment: Let $\mathscr{U} \in \mathscr{P}_f(\mathsf{Con})$ and $U \in \mathsf{Con}$; then

$$U \in \mathscr{U} \Rightarrow U \subseteq \bigcup \mathscr{U} \overset{\text{refl. in } \rho}{\Rightarrow} \bigcup \mathscr{U} \vdash U \Leftrightarrow \mathscr{U} \vdash^N U .$$

For transitivity of entailment: Let $\mathscr{U}, \mathscr{V} \in \mathscr{P}_f(\mathsf{Con})$ and $W \in \mathsf{Con}$; then

$$\begin{aligned}
&\mathscr{U} \vdash^N \mathscr{V} \wedge \mathscr{V} \vdash^N W \\
&\quad \Leftrightarrow \underset{V \in \mathscr{V}}{\forall} \mathscr{U} \vdash^N V \wedge \mathscr{V} \vdash^N W \\
&\quad \Leftrightarrow \underset{V \in \mathscr{V}}{\forall} \bigcup \mathscr{U} \vdash V \wedge \bigcup \mathscr{V} \vdash W \\
&\quad \Leftrightarrow \bigcup \mathscr{U} \vdash \bigcup \mathscr{V} \wedge \bigcup \mathscr{V} \vdash W \\
&\quad \overset{\text{trns}}{\Rightarrow} \bigcup \mathscr{U} \vdash W \\
&\quad \Leftrightarrow \mathscr{U} \vdash^N W .
\end{aligned}$$

For propagation of consistency through entailment: Let $\mathscr{U} \in \mathscr{P}_f(\mathsf{Con})$ and $V \in \mathsf{Con}$; then

$$\begin{aligned}
&\mathscr{U} \in N\mathsf{Con} \wedge \mathscr{U} \vdash^N V \\
&\quad \Leftrightarrow \bigcup \mathscr{U} \in \mathsf{Con} \wedge \bigcup \mathscr{U} \vdash V \\
&\quad \overset{\text{prpg}}{\Rightarrow} \bigcup \mathscr{U} \cup V \in \mathsf{Con} \\
&\quad \Leftrightarrow \mathscr{U} \cup \{V\} \in N\mathsf{Con} .
\end{aligned}$$

To show that the classes of ideals in the two information systems are isomorphic, take the correspondence defined by the two mappings $\mathscr{P}_f : \mathsf{Ide}_\rho \to \mathsf{Ide}_{N\rho}$ and $\bigcup : \mathsf{Ide}_{N\rho} \to \mathsf{Ide}_\rho$. It is direct to see that both of them are well-defined injections, as well as that they are mutually inverse. $\square$

Consider now the *quotient information system* $Q\rho = \left(QT, Q\mathsf{Con}, \vdash^Q\right)$, defined by

$$\begin{aligned}
QT &:= \mathsf{Con}/\sim , \\
\{[U_i]\}_{i<n} \in Q\mathsf{Con} &:= \bigcup_{i<n} U_i \in \mathsf{Con} , \\
\{[U_i]\}_{i<n} \vdash^Q [V] &:= \bigcup_{i<n} U_i \vdash V ,
\end{aligned}$$

where

$$U \sim V := U \vdash V \wedge V \vdash U$$

is the *equientailment* or *equivalence* of formal neighborhoods. An easy characterization of equientailment in terms of closures is given by the following.

**Proposition 3.5.** *It is $U \sim V$ if and only if $\overline{U} = \overline{V}$.*

*Proof.* The right direction follows from transitivity and the left one from reflexivity and transitivity of entailment. □

**Proposition 3.6.** *Equientailment is compatible with $Q\mathsf{Con}$ and $\vdash^Q$. Furthermore, if $\rho$ is an information system then $Q\rho$ is an information system, and it is $\rho \simeq Q\rho$.*

*Proof.* For compatibility with $Q\mathsf{Con}$: Let $\{U_1, \dots, U_n\} \in Q\mathsf{Con}$ and $U_i \sim V_i$, for all $i = 1, \dots, n$; by definition $\bigcup_{i=1}^n U_i \in \mathsf{Con}$ and $U_i \vdash V_i \wedge V_i \vdash U_i$, for all $i$'s; then $\bigcup_{i=1}^n U_i \vdash V_i$, for all $i$'s, so $\bigcup_{i=1}^n V_i \in \mathsf{Con}$, and by definition $\{V_1, \dots, V_n\} \in Q\mathsf{Con}$.

For compatibility with $\vdash^Q$: Let $\{U_1, \dots, U_n\} \vdash^Q U$ and $U_i \sim V_i$, for $i = 1, \dots, n$, as well as $U \sim V$; by definition $\bigcup_{i=1}^n U_i \vdash U$ and $U_i \vdash V_i \wedge V_i \vdash U_i$, for all $i$'s, as well as $U \vdash V \wedge V \vdash U$; then $\bigcup_{i=1}^n V_i \vdash \bigcup_{i=1}^n U_i \vdash U \vdash V$, so by transitivity and definition $\{V_1, \dots, V_n\} \vdash^Q V$.

For reflexivity of consistency: Let $[U] \in QT$, that is, $[U] \in \mathsf{Con}/\sim$; from $U \in \mathsf{Con}$, we immediately have $\{[U]\} \in Q\mathsf{Con}$ by definition. For closure under subsets: Let $\{[U_i]\}_{i<n}, \{[V_j]\}_{j<m} \in \mathscr{P}_f(QT)$; then

$$\begin{aligned} &\{[U_i]\}_{i<n} \in Q\mathsf{Con} \wedge \{[V_j]\}_{j<m} \subseteq \{[U_i]\}_{i<n} \\ &\quad \Leftrightarrow \bigcup_{i<n} U_i \in \mathsf{Con} \wedge \bigcup_{j<m} V_j \subseteq \bigcup_{i<n} U_i \\ &\quad \overset{\text{clsr}}{\Rightarrow} \bigcup_{j<m} V_j \in \mathsf{Con} \\ &\quad \Leftrightarrow \{[V_j]\}_{j<m} \in Q\mathsf{Con} \;. \end{aligned}$$

For reflexivity of entailment: Let $\{[U_i]\}_{i<n} \in \mathscr{P}_f(QT)$ and $[U] \in QT$; then

$$\begin{aligned} [U] \in \{[U_i]\}_{i<n} &\Leftrightarrow \underset{i<n}{\exists} [U] = [U_i] \\ &\Rightarrow \underset{i<n}{\exists} U_i \vdash U \\ &\overset{(\star)}{\Rightarrow} \bigcup_{i<n} U_i \vdash U \\ &\Leftrightarrow \{[U_i]\}_{i<n} \vdash^Q [U] \;, \end{aligned}$$

where at $(\star)$ we used reflexivity and transitivity at $\rho$. For transitivity of entailment: Let $\{[U_i]\}_{i<n}, \{[V_j]\}_{j<m} \in \mathscr{P}_f(QT)$ and $[W] \in QT$; then

$$\begin{aligned} &\{[U_i]\}_{i<n} \vdash^Q \{[V_j]\}_{j<m} \wedge \{[V_j]\}_{j<m} \vdash^Q [W] \\ &\quad \Leftrightarrow \underset{j<m}{\forall} \{[U_i]\}_{i<n} \vdash^Q [V_j] \wedge \{[V_j]\}_{j<m} \vdash^Q [W] \\ &\quad \Leftrightarrow \underset{j<m}{\forall} \bigcup_{i<n} U_i \vdash^Q V_j \wedge \bigcup_{j<m} V_j \vdash^Q W \\ &\quad \overset{(\star)}{\Rightarrow} \bigcup_{i<n} U_i \vdash^Q \bigcup_{j<m} V_j \wedge \bigcup_{j<m} V_j \vdash^Q W \\ &\quad \overset{\text{trns}}{\Rightarrow} \{U_i\}_{i<n} \vdash W \\ &\quad \Leftrightarrow \{[U_i]\}_{i<n} \vdash^Q [W] \;, \end{aligned}$$

where the step $(\star)$ we got by Proposition 3.1(6). For propagation of consistency through entailment: Let $\{[U_i]\}_{i<n} \in \mathscr{P}_f(QT)$ and $[V] \in QT$; then

$$\begin{aligned} &\{[U_i]\}_{i<n} \in Q\mathsf{Con} \wedge \{[U_i]\}_{i<n} \vdash^Q [V] \\ &\Leftrightarrow \bigcup_{i<n} U_i \in \mathsf{Con} \wedge \bigcup_{i<n} U_i \vdash V \\ &\overset{\text{prpg}}{\Rightarrow} \bigcup_{i<n} U_i \cup V \in \mathsf{Con} \\ &\Leftrightarrow \{[U_i]\}_{i<n} \cup \{[V]\} \in Q\mathsf{Con} \ . \end{aligned}$$

We show that $Q\rho \simeq N\rho$, so that $Q\rho \simeq \rho$ will follow from Proposition 3.4. For $\mathsf{Ide}_{Q\rho} \hookrightarrow \mathsf{Ide}_{N\rho}$ take $u \mapsto \{V \in NT \mid \exists_{U \in \mathsf{Con}}([U] \in u \wedge V \in [U])\}$, and for $\mathsf{Ide}_{N\rho} \hookrightarrow \mathsf{Ide}_{Q\rho}$ take $u \mapsto \{[U] \in QT \mid U \subseteq u\}$. It is direct to see that these mappings are well-defined injections, as well as that they are mutually inverse. □

## 3.2 Atomicity and coherence in information systems

For an arbitrary information system $\rho = (T, \mathsf{Con}, \vdash)$, define *atomic entailment* and *coherent neighborhoods* respectively by

$$\begin{aligned} U \vdash^A b &:= \exists_{a \in U} \{a\} \vdash b \ , \\ U \in H\mathsf{Con} &:= \forall_{a,a' \in U} \{a, a'\} \in \mathsf{Con} \ , \end{aligned}$$

where $U \in \mathscr{P}_f(T)$.

**Theorem 3.7.** *The following hold:*

1. *It is* $\vdash^A \subseteq \vdash$ *and* $\mathsf{Con} \subseteq H\mathsf{Con}$.
2. *The triples* $A\rho := (T, \mathsf{Con}, \vdash^A)$ *and* $H\rho := (T, H\mathsf{Con}, \vdash^H)$ *are both information systems, where, in* $H\rho$*, entailment is extended to coherent neighborhoods* $U \in H\mathsf{Con}$ *trivially, that is, by inductively adding all reflections and transitions of* $H\mathsf{Con}$ *on* $\vdash$*:*
$$U \vdash^H b := U \vdash b \vee b \in U \vee \exists_{V \in H\mathsf{Con}} \left(U \vdash^H V \wedge V \vdash^H b\right) \ .$$
3. *It is* $\mathrm{cl}_{A\rho}(U) \subseteq \mathrm{cl}_\rho(U)$ *and* $\forall_{U \in \mathsf{Con}} \mathrm{cl}_{H\rho}(U) = \mathrm{cl}_\rho(U)$.
4. *It is* $\rho \preceq A\rho$ *and* $\rho \preceq H\rho$.
5. *Atomicity and coherence are idempotent, in the sense that* $A(A\rho) \simeq A\rho$ *and* $H(H\rho) \simeq H\rho$.

*Proof. For the first statement*: we have $\vdash^A \subseteq \vdash$ directly by definition, and reflexivity and transitivity of $\vdash$, while for every $U \in \mathsf{Con}$, by closure under subsets, we have $\forall_{a,a' \in U} \{a, a'\} \in \mathsf{Con}$, so $U \in H\mathsf{Con}$.

*For the second statement*: For $A\rho$ we have to check the laws of the definition concerning entailment. For reflexivity, if $U \in \mathsf{Con}$ then $\{a\} \vdash a$ for all $a \in U$, so

$U \vdash^A a$ for all $a \in U$. For transitivity:

$$\begin{aligned} U \vdash^A V \wedge V \vdash^A c &\overset{\text{def}}{\Leftrightarrow} \underset{b \in V}{\forall}\ \underset{a_b \in U}{\exists} \{a_b\} \vdash b \wedge \underset{b \in V}{\exists} \{b\} \vdash c \\ &\overset{a:=a_b}{\Rightarrow} \underset{a \in U}{\exists} \{a\} \vdash c \\ &\overset{\text{def}}{\Leftrightarrow} U \vdash^A c \end{aligned}$$

Finally, consistency propagates through atomic entailment, since it does so through entailment in general (see previous statement).

For $H\rho$: Reflexivity for coherent consistency follows from the previous statement, since all singletons are already in $\mathsf{Con}$. For closure of coherent consistency under subsets, let $U \in H\mathsf{Con}$, $V \subseteq U$; by closure under subsets for neighborhoods, it is $V \in \mathsf{Con}$, so it is a finite set; for $a, a' \in V$, since $V \subseteq U$, it is $\{a, a'\} \in \mathsf{Con}$, so $V \in H\mathsf{Con}$. Reflexivity of coherent entailment follows directly from the definition. For the transitivity of coherent entailment let's consider the most complicated case, where $U, V \in H\mathsf{Con} \setminus \mathsf{Con}$ with

$$\underset{b \in V}{\forall} \left( \underset{W_U \in H\mathsf{Con}}{\exists} \left( U \vdash^H W_U \wedge W_U \vdash^H b \right) \right) \wedge \underset{W_V \in H\mathsf{Con}}{\exists} \left( V \vdash^H W_V \wedge W_V \vdash^H c \right) .$$

By applying the induction hypothesis first for $W_U$ via $V$ to $W_V$ and then for $W_U$ via $W_V$ to $c$, we get $W_U \vdash^H c$, so by definition it is $U \vdash^H c$. For propagation of coherent consistency through coherent entailment, let $U \vdash^H b$ for some $U \in H\mathsf{Con}$: if $U \vdash b$ then $U \cup \{b\} \in \mathsf{Con} \subseteq H\mathsf{Con}$; if $b \in U$, then $U \cup \{b\} = U \in H\mathsf{Con}$; if $U \vdash^H V$ and $V \vdash^H b$ for some $V \in H\mathsf{Con}$, then, by the induction hypothesis, it is $U \cup V \in H\mathsf{Con}$ and $U \cup V \vdash^H b$; by the induction hypothesis again, $U \cup V \cup \{b\} \in H\mathsf{Con}$; by closure under subsets, $U \cup \{b\} \in H\mathsf{Con}$.

The *third statement* is direct to show.

*For the fourth statement*: We show that $\mathsf{Ide}_\rho \subseteq \mathsf{Ide}_{A\rho}$; let $u \in \mathsf{Ide}_\rho$; if $U \subseteq^f u$, then $U \in \mathsf{Con}$ by the consistency in $\rho$; if further $U \vdash^A b$, then $U \vdash b$ since $\vdash^A \subseteq \vdash$, so $b \in u$ by deductive closure in $\rho$.

We now show that $\mathsf{Ide}_\rho \subseteq \mathsf{Ide}_{H\rho}$: let $u \in \mathsf{Ide}_\rho$; if $U \subseteq^f u$, then $U \in \mathsf{Con} \subseteq H\mathsf{Con}$ by the consistency in $\rho$; if further $U \vdash^A b$, then $U \vdash b$ since $U \in \mathsf{Con}$ again, so $b \in u$ by deductive closure in $\rho$.

*For the fifth statement*: For the idempotence of $A$, the only thing we have to show is that $\vdash^{A^A} = \vdash^A$:

$$\begin{aligned} U \vdash^{A^A} b &\Leftrightarrow \underset{a \in U}{\exists} \{a\} \vdash^A b \\ &\Leftrightarrow \underset{a \in U}{\exists}\ \underset{a' \in \{a\}}{\exists} \{a'\} \vdash b \\ &\Leftrightarrow \underset{a \in U}{\exists} \{a\} \vdash b . \end{aligned}$$

So it is $A(A\rho) \simeq A\rho$, actually with the trivial isomorphism. For the idempotence of $H$, we have $\mathsf{Con}_{H\rho} = H\mathsf{Con}_\rho$, so:

$$\begin{aligned} U \in H(H\mathsf{Con}_\rho) &\Leftrightarrow \underset{a, a' \in U}{\forall} \{a, a'\} \in H\mathsf{Con}_\rho \\ &\overset{(*)}{\Leftrightarrow} \underset{a, a' \in U}{\forall} \{a, a'\} \in \mathsf{Con}_\rho \\ &\Leftrightarrow U \in H\mathsf{Con}_\rho , \end{aligned}$$

and $U \in \mathsf{Con}_{H\rho}$; step $(*)$ holds since two-element sets that are consistent are also coherently consistent and vice versa. Finally, let $U \vdash^{H^H} b$; if it is $U \vdash^H b$ or $b \in U$, then we're done; if there is a $V \in H(H\mathsf{Con})$ such that $U \vdash^{H^H} V$ and $V \vdash^{H^H} b$, then, as we just saw, it is $V \in H\mathsf{Con}$, which means that $U \vdash^H V$ and $V \vdash^H b$, so by transitivity, $U \vdash^H b$. □

Notice that we cannot do better than Theorem 3.7(4) in general. Indeed, both atomicity and coherence may provide "extra" ideals that are absent in the original information system. For example, consider the information systems $\mathscr{C}$ and $\mathscr{L}$ (page 106); it is direct to check that

$$\mathsf{Ide}_{\mathscr{C}} \not\ni \{l, r\} \in \mathsf{Ide}_{A\mathscr{C}}$$

(follows by the Coquand counterexample) and also that

$$\mathsf{Ide}_{\mathscr{L}} \not\ni \{l, m, r, \varepsilon\} \in \mathsf{Ide}_{H\mathscr{L}} \,.$$

So atomic and coherent information systems are generally ideal-wise richer. Notice, moreover, that they are the richest we can get in this manner, a fact that is expressed by Theorem 3.7(5).

## 3.3 Coherent point-free structures

*Formal topology* is point-free topology from a predicative and constructive viewpoint ([40], [39], [42], [9]). Links between domain theory and formal topology have been noticed and studied by several people already (see [41], [38], [30], [33], [52]). The main objective of this section is to match coherent information systems to respective "coherent" point-free structures, domains included.

The basic problem that one faces in such an endeavor lies, not surprisingly, in the very nature of coherence. By its definition (page 111), coherence involves comparisons between tokens, but tokens are unobservable in any point-free setting; everything there starts with neighborhoods. Luckily, we have the following easy characterization.

**Proposition 3.8.** *A finite set of tokens is a coherent neighborhood if and only if every two of its subsets have a coherent union: it is $U \in H\mathsf{Con}$ if and only if $U \in \mathscr{P}_f(T)$ and $U_1 \cup U_2 \in H\mathsf{Con}$, for any $U_1, U_2 \subseteq U$.*

Notice that we could equivalently say "$U_0 \in H\mathsf{Con}$ for any $U_0 \subseteq U$" instead of "$U_1 \cup U_2 \in H\mathsf{Con}$ for any $U_1, U_2 \subseteq U$"; we choose the latter to stress that the issue of validity of a neighborhood in a coherent system is raised from *comparisons of its tokens*, to *comparisons of its subsets*. The coherence conditions (3.1), (3.2), and (3.4), that we pose in the following are all modeled after Proposition 3.8.

In this section we restrict ourselves to the case where we have countable carrier sets.

### Domains

We start with the known correspondence of arbitrary Scott information systems and domains (of countable base), which we quickly recount here without proofs, to set the mood for what comes next (see Appendix A for relevant basic notions and facts that we may refer to but omit here).

Let $D = (D, \sqsubseteq, \bot, \mathrm{lub})$ be a domain and define $I(D) \coloneqq (T, \mathsf{Con}, \vdash)$ by

$$\begin{aligned} T &\coloneqq D_c \ , \\ \{u_i\}_{i\in I} \in \mathsf{Con} &\coloneqq \{u_i\}_{i\in I} \subseteq^f D_c \wedge \mathrm{lub}\{u_i\}_{i\in I} \in D \ , \\ \{u_i\}_{i\in I} \vdash u &\coloneqq u \sqsubseteq \mathrm{lub}\{u_i\}_{i\in I} \ . \end{aligned}$$

Notice that, by Fact A.8.1, if $\{u_i\}_{i\in I} \in \mathsf{Con}$ then $\mathrm{lub}\{u_i\}_{i\in I} \in D_c$. Conversely, for an information system $\rho = (T, \mathsf{Con}, \vdash)$, define $D(\rho) \coloneqq (\mathsf{Ide}_\rho, \subseteq, \bot, \bigcup)$.

The following comprises Propositions 6.1.6, 6.1.8, and Theorem 6.1.9[1] of [52].

**Proposition 3.9.** *If $D$ is a domain and $\rho$ an information system, then $I(D)$ is an information system and $D(\rho)$ a domain, where compact elements are given by deductive closures, that is, where $D(\rho)_c = \overline{\mathsf{Con}_\rho}$.*

*Furthermore, if $D$ is a domain then $\mathsf{Ide}_{I(D)} \simeq D$, through the isomorphism pair $u \mapsto \mathrm{lub}\, u$ and $u \mapsto \mathrm{apx}(u)$, where $\mathrm{apx}(u)$ is the set of the compact approximations of $u$.*

Let now $r$ be an approximable map from $\rho$ to $\sigma$. Define a mapping $D(r) : D(\rho) \to D(\sigma)$ by

$$D(r)(u) \coloneqq \bigcup \{V \in \mathsf{Con}_\sigma \mid U \subseteq^f u \wedge (U,V) \in r\} \ .$$

Conversely, let $f$ be a continuous mapping from a domain $D$ to a domain $E$. Define a relation $I(f) \subseteq \mathsf{Con}_{I(D)} \times \mathsf{Con}_{I(E)}$ by

$$(U,V) \in I(f) \coloneqq \mathrm{lub}\, V \leq f(\mathrm{lub}\, U) \ .$$

These establish a bijective correspondence, as the following statement expresses (Theorem 6.1.12 of [52]).

**Proposition 3.10.** *If $r$ is an approximable map from $\rho$ to $\sigma$ then $D(r) : D(\rho) \to D(\sigma)$ is a continuous mapping. Conversely, if $f : D \to E$ is a continuous mapping then $I(f)$ is an approximable map from $I(D)$ to $I(E)$. Furthermore, the collection of continuous mappings from $D$ to $E$ is in a bijective correspondence with the collection of approximable maps between $I(D)$ and $I(E)$.*

### Coherent domains

Let $D = (D, \sqsubseteq, \bot)$ be a domain and $\{u_i\}_{i\in I} \subseteq^f D_c$ an arbitrary *finite* set of compact elements. Call $D$ a *coherent domain* if

$$\mathrm{lub}\{u_i\}_{i\in I} \in D_c \leftrightarrow \underset{i,j\in I}{\forall} \mathrm{lub}\{u_i, u_j\} \in D_c \ . \tag{3.1}$$

Note that the choice of $D_c$ instead of the more modest $D$ is justified by Fact A.8(1).

**Theorem 3.11.** *Let $D$ be a coherent domain and $\rho$ a coherent information system. Then $I(D)$ is a coherent information system and $D(\rho)$ is a coherent domain.*

*Proof.* Let $D$ be a coherent domain and $\{u_i\}_{i\in I} \in \mathsf{Con}_{I(D)}$. By the definition of $I(D)$, $\{u_i\}_{i\in I} \subseteq^f D_c$ and $\mathrm{lub}\{u_i\}_{i\in I} \in D$. By Fact A.8(1), $\mathrm{lub}\{u_i\}_{i\in I} \in D_c$. By the coherence, $\forall_{i,j\in I}\, \mathrm{lub}\{u_i, u_j\} \in D_c$ and we're done.

[1] This is called *second representation theorem* in [52], besides the *first* representation theorem ("a domain is represented by its compact elements", see Fact A.9) and the *third* representation theorem ("a domain is represented by its induced Scott space").

Let now $\rho$ be a coherent information system and $\{U_i\}_{i\in I} \subseteq^f \mathsf{Con}_\rho$. By the coherence and Proposition 3.8, it is

$$\bigcup_{i\in I} U_i \in \mathsf{Con}_\rho \leftrightarrow \underset{i,j\in I}{\forall} U_i \cup U_j \in \mathsf{Con}_\rho \ ,$$

so also

$$\bigcup_{i\in I} \overline{U_i} \in \overline{\mathsf{Con}}_\rho \leftrightarrow \underset{i,j\in I}{\forall} \overline{U_i} \cup \overline{U_j} \in \overline{\mathsf{Con}}_\rho \ . \qquad \square$$

## Precusl's

A *preordered conditional upper semilattice with a distinguished least element*, or just *precusl*, is a 'consistently complete preordered set with a distinguished least element', that is, a quadruple $P = (N, \sqsubseteq, \sqcup, \perp)$ where $\sqsubseteq$ is a *pre*order on $N$, $\perp$ is a (distinguished) least element and $\sqcup$ is a partial binary operation on $N$ which is defined only on *consistent* pairs, that is, on pairs having an upper bound, and then yields a (distinguished) least upper bound:

$$U \sqcup V \in N :\eqq \underset{W\in N}{\exists} (U \sqsubseteq W \wedge V \sqsubseteq W) \ ,$$

$$U \sqcup V \in N \Rightarrow U \sqsubseteq U \sqcup V \wedge V \sqsubseteq U \sqcup V$$
$$\wedge \underset{W\in N}{\forall} (U \sqsubseteq W \wedge V \sqsubseteq W \to U \sqcup V \sqsubseteq W) \ .$$

We think of $N$ as "a set of formal basic opens", $\sqsubseteq$ as "formal inclusion", $\perp$ as "a formal empty set" and $\sqcup$ as "a formal union". Call a subset $u \subseteq N$ a *precusl ideal* when it satisfies

$$\perp \in u \ \wedge \underset{U,V\in u}{\forall} U \sqcup V \in u \ \wedge \underset{U\in u}{\forall} (V \sqsubseteq U \to V \in u) \ .$$

Write $\mathsf{Ide}_P$ for the class of all precusl ideas of $P$. Observe that the second of the three requirements for a precusl ideal expresses the property of being upward directed, so it follows that *any* finite subset in a precusl ideal will have a least upper bound in the ideal (for details on the relation of precusl's and information systems that are not given here, one should consult [52, § 6.3]).

Let now $P = (N, \sqsubseteq, \sqcup, \perp)$ be a precusl and define $I(P) = (T, \mathsf{Con}, \vdash)$ by

$$T :\eqq N \ ,$$
$$\mathscr{U} \in \mathsf{Con} :\eqq \mathscr{U} \subseteq^f N \wedge \bigsqcup \mathscr{U} \in N \ ,$$
$$\mathscr{U} \vdash U :\eqq U \sqsubseteq \bigsqcup \mathscr{U} \ .$$

Conversely, let $\rho = (T, \mathsf{Con}, \vdash)$ be an information system and define $P(\rho) = (N, \sqsubseteq, \perp, \sqcup)$ by

$$N :\eqq \mathsf{Con} \ ,$$
$$U \sqsubseteq V :\eqq V \vdash U \ ,$$
$$\perp :\eqq \varnothing \ ,$$
$$U \sqcup V :\eqq U \cup V \text{ if } U \cup V \in \mathsf{Con} \ .$$

The following is Theorem 6.3.4 of [52].

**Proposition 3.12.** *If $P$ is a precusl and $\rho$ an information system, then $I(P)$ is an information system and $P(\rho)$ is a precusl. Furthermore, it is $\mathsf{Ide}_P = \mathsf{Ide}_{I(P)}$ and $\mathsf{Ide}_\rho \simeq \mathsf{Ide}_{P(\rho)}$.*

*Proof.* We just show the existence of isomorphisms between corresponding *information system ideals* and *precusl ideals*, as the proof is omitted in the above reference.

For $\mathsf{Ide}_P \subseteq \mathsf{Ide}_{I(P)}$, let $u \in \mathsf{Ide}_P$ and $\mathscr{U} \subseteq^f u$. Since $u$ is an ideal in $P$, $\bigsqcup \mathscr{U} \in u \subseteq N$, so $\mathscr{U} \in \mathsf{Con}_{I(P)}$ by definition. If further $\mathscr{U} \vdash_{I(P)} U$, it is $U \sqsubseteq \bigsqcup \mathscr{U}$ by definition, so $U \in u$, again because $u$ is an ideal in $P$.

For $\mathsf{Ide}_{I(P)} \subseteq \mathsf{Ide}_P$, let $u \in \mathsf{Ide}_{I(P)}$. That $\bot := \varnothing \in u$, follows from downward closure in $I(P)$. Let $U,V \in u$; since $u$ is an ideal in $I(P)$, it is $\{U,V\} \in \mathsf{Con}_{I(P)}$, and then $U \sqcup V \in N$ by definition; since $\{U,V\} \vdash_{I(P)} U \sqcup V$ and $u$ is an ideal in $I(P)$, it is $U \sqcup V \in u$. If now $U \in u$ and $V \sqsubseteq U$, then $\{U\} \vdash_{I(P)} V$ by definition, and $V \in u$ follows again because $u$ is an ideal in $I(P)$.

For $\mathsf{Ide}_\rho \simeq \mathsf{Ide}_{P(\rho)}$ take the following isomorphism pair:

$$\mathsf{Ide}_\rho \ni u \mapsto \mathscr{P}_f(u) \in \mathsf{Ide}_{P(\rho)} \ ,$$
$$\mathsf{Ide}_{P(\rho)} \ni u \mapsto \bigcup u \in \mathsf{Ide}_\rho \ .$$

These mappings are well-defined. For the right embedding, it is obviously $\bot := \varnothing \in \mathscr{P}_f(u)$; for every $U,V \in \mathscr{P}_f(u)$ it is also $U \cup V \subseteq^f u$, so it is $U \sqcup V \in \mathscr{P}_f(u)$ by definition; if $U \in \mathscr{P}_f(u)$ and $V \sqsubseteq U$, then $\bigcup\{U\} = U \vdash V$ by definition, so since $u$ is an ideal in $\rho$, it is $V \subseteq^f u$, that is, $V \in \mathscr{P}_f(u)$. For the left embedding, if $\{a_i\}_{i<n} \subseteq^f \bigcup u$, then $\forall_{i<n} \exists_{U_i \in u} a_i \in U_i$; since $u$ is an ideal in $P(\rho)$, it is $\bigsqcup_{i<n} U_i \in u \subseteq^f N_{P(\rho)}$, so, by definition, $\bigcup_{i<n} U_i \in \mathsf{Con}$; by closure of consistency under subsets in $\rho$, it is $\{a_i\}_{i<n} \in \mathsf{Con}$. If now $U \subseteq^f \bigcup u$ and $U \vdash a$, then $\{a\} \sqsubseteq U$ by definition; since $u$ is an ideal in $P(\rho)$, we have $\{a\} \in u$, so $a \in \bigcup u$.

It remains to show that the two embeddings are indeed mutually inverse. Let $u \in \mathsf{Ide}_\rho$; we have

$$a \in \bigcup \mathscr{P}_f(u) \Leftrightarrow \underset{U \subseteq^f u}{\exists} a \in U \overset{(\star)}{\Leftrightarrow} a \in u \ ,$$

where $(\star)$ holds leftwards for $U := \{a\}$, and

$$\begin{aligned}
\{a_i\}_{i<n} \in \mathscr{P}_f\left(\bigcup u\right) &\Leftrightarrow \{a_i\}_{i<n} \subseteq^f \bigcup u \\
&\Leftrightarrow \underset{i<n}{\forall} \underset{U_i \in u}{\exists} a_i \in U_i \\
&\overset{(\star)}{\Leftrightarrow} \underset{i<n}{\forall} \underset{U_i \in u}{\exists} \{a_i\} \sqsubseteq U_i \\
&\overset{(\star)}{\Leftrightarrow} \underset{i<n}{\forall} \underset{U_i \in u}{\exists} \{a_i\} \sqsubseteq \bigsqcup_{i<n} U_i \\
&\overset{(\star)}{\Leftrightarrow} \{a_i\}_{i<n} \in u \ ,
\end{aligned}$$

where $(\star)$ hold leftwards for $U_i := \{a_i\}$, $i < n$. □

A *precusl approximable map* from $P$ to $P'$ is a relation $\mathscr{R} \subseteq N \times N'$ which satisfies the following:

- $(\bot, \bot') \in \mathscr{R}$,
- $(U,V) \in \mathscr{R} \wedge (U,V') \in \mathscr{R} \rightarrow (U, V \sqcup V') \in \mathscr{R}$,

- $U \sqsubseteq U' \wedge (U,V) \in \mathscr{R} \wedge V' \sqsubseteq V \to (U',V') \in \mathscr{R}$,

where $(U, V \sqcup V') \in \mathscr{R}$ naturally implies that $V \sqcup V'$ is defined. Write $\mathsf{Apx}_{P \to P'}$ for all precusl approximable maps from $P$ to $P'$. For every $\mathscr{R} \in \mathsf{Apx}_{P \to P'}$ define a relation $I(\mathscr{R}) \subseteq \mathsf{Con}_{I(P)} \times \mathsf{Con}_{I(P')}$ by

$$(\mathscr{U}, \mathscr{V}) \in I(\mathscr{R}) :\!= \left(\bigsqcup \mathscr{U}, \bigsqcup \mathscr{V}\right) \in \mathscr{R} .$$

Conversely, let $r$ be an approximable map from $\rho$ to $\sigma$. Define a relation $P(r) \subseteq N_{P(\rho)} \times N_{P(\sigma)}$ by

$$(U,V) \in P(r) :\!= (U,V) \in r .$$

One can show (see [52, pp. 151–2]) that these establish a bijective correspondence.

**Proposition 3.13.** *If $r$ is an approximable map from $\rho$ to $\sigma$ then $P(r)$ is a precusl approximable map from $P(\rho)$ to $P(\sigma)$. Conversely, if $\mathscr{R}$ is a precusl approximable map from $P$ to $P'$ then $I(\mathscr{R})$ is an approximable map from $I(P)$ to $I(P')$. Furthermore, it is $\mathsf{Apx}_{\rho \to \sigma} \simeq \mathsf{Apx}_{P(\rho) \to P(\sigma)}$ and $\mathsf{Apx}_{P \to P'} \simeq \mathsf{Apx}_{I(P) \to I(P')}$.*

### Coherent precusl's

Call a precusl *coherent* if it satisfies the following property for a finite collection $\mathscr{U} \subseteq^f N$:

$$\bigsqcup \mathscr{U} \in N \leftrightarrow \underset{U,V \in \mathscr{U}}{\forall} U \sqcup V \in N . \tag{3.2}$$

**Theorem 3.14.** *If $P$ is a coherent precusl then $I(P)$ is a coherent information system. Conversely, if $\rho$ is a coherent information system then $P(\rho)$ is a coherent precusl.*

*Proof.* Suppose first that $P$ is coherent, that is, such that (3.2) holds for all $\mathscr{U} \subseteq^f N$. Let $\mathscr{U} \in \mathsf{Con}_{I(P)}$; by the definition,

$$\begin{aligned}\mathscr{U} \subseteq^f N \wedge \bigsqcup \mathscr{U} \in N &\overset{(3.2)}{\Leftrightarrow} \mathscr{U} \subseteq^f N \wedge \underset{U,V \in \mathscr{U}}{\forall} U \sqcup V \in N \\ &\overset{\text{def}}{\Leftrightarrow} \mathscr{U} \subseteq^f N \wedge \underset{U,V \in \mathscr{U}}{\forall} \{U,V\} \in \mathsf{Con}_{I(P)} ,\end{aligned}$$

so $I(P)$ is a coherent information system.

Now suppose that $\rho$ is a coherent information system, that is, such that

$$U \in \mathsf{Con} \leftrightarrow \underset{a,b \in U}{\forall} \{a,b\} \in \mathsf{Con} , \tag{3.3}$$

for all $U \subseteq^f T$. Let $\mathscr{U} \subseteq^f N_{P(\rho)}$, that is, $\mathscr{U} \subseteq^f \mathsf{Con}$; we have

$$\begin{aligned}\bigsqcup \mathscr{U} \in N_{P(\rho)} &\overset{\text{def}}{\Leftrightarrow} \bigcup \mathscr{U} \in \mathsf{Con} \\ &\overset{(*)}{\Leftrightarrow} \bigcup \mathscr{U} \subseteq^f T \wedge \underset{U,V \subseteq \bigcup \mathscr{U}}{\forall} U \cup V \in \mathsf{Con} \\ &\overset{\text{def}}{\Leftrightarrow} \bigcup \mathscr{U} \subseteq^f T \wedge \underset{U,V \subseteq \bigcup \mathscr{U}}{\forall} U \sqcup V \in N_{P(\rho)} \\ &\Rightarrow \underset{U,V \in \mathscr{U}}{\forall} U \sqcup V \in N_{P(\rho)} ,\end{aligned}$$

where at ($\star$) we used (3.3) and Proposition 3.8. Conversely, we have

$$
\begin{aligned}
\mathscr{U} \subseteq^f N_{P(\rho)} \wedge \forall_{U,V\in\mathscr{U}} U \sqcup V \in N_{P(\rho)} &\overset{\text{def}}{\Leftrightarrow} \mathscr{U} \subseteq^f \mathsf{Con} \wedge \forall_{U,V\in\mathscr{U}} U \sqcup V \in \mathsf{Con} \\
&\overset{(3.3)}{\Leftrightarrow} \mathscr{U} \subseteq^f \mathsf{Con} \wedge \forall_{U,V\in\mathscr{U}} \forall_{a,b\in U\cup V} \{a,b\} \in \mathsf{Con} \\
&\Rightarrow \mathscr{U} \subseteq^f \mathsf{Con} \wedge \forall_{a,b\in\bigcup\mathscr{U}} \{a,b\} \in \mathsf{Con} \\
&\overset{(3.3)}{\Leftrightarrow} \bigcup\mathscr{U} \in \mathsf{Con} \\
&\overset{\text{def}}{\Leftrightarrow} \bigsqcup\mathscr{U} \in N_{P(\rho)} \ ,
\end{aligned}
$$

so $P(\rho)$ is indeed a coherent precusl. □

## Scott–Ershov formal topologies

The *structure* of a "formal topology" was defined by Giovanni Sambin as early as 1987 in [37], and as the area has developed a number of alternative definitions has appeared. Suitable for our purposes is a version of the definition in [30], whose main difference from Sambin's original is the disposal of the "positivity predicate"—see [30, §2.4 ] or [39, Footnote 13] for a justification of this. In fact, we depart a bit from this definition as well, in that we require the presence of a top element among the formal basic opens, this is however an inessential difference (see Exercise 6.5.21 of [52]). For a list of nomenclature discrepancies between our exposition and the literature, see notes in 3.4.

We will use order-theoretic notions which are *dual* to notions appearing before, namely a *greatest* or *top element* and *greatest lower bounds* of sets of elements; all these are to be understood in the usual order-theoretic way.

Define a *formal topology* as a triple $\mathscr{T} = (N, \sqsubseteq, \prec)$ where $N$ is the collection of *formal basic opens*, $\sqsubseteq \subseteq N \times N$ is a preorder with a top element $\top$, which formalizes inclusion between basic opens, and $\prec \subseteq N \times \mathscr{P}(N)$, called *covering*, formalizes inclusion between arbitrary opens, and satisfies the following:

1. it is reflexive,
$$U \in \mathscr{U} \to U \prec \mathscr{U} \ ,$$
2. it is transitive,
$$U \prec \mathscr{U} \wedge \mathscr{U} \prec \mathscr{V} \to U \prec \mathscr{V} \ ,$$
3. it is *localizing*,
$$U \prec \mathscr{U} \wedge U \prec \mathscr{V} \to U \prec \mathscr{U} \downarrow \mathscr{V} \ ,$$
and
4. it extends formal inclusion between formal basic opens,
$$V \sqsubseteq U \wedge U \prec \mathscr{U} \to V \prec \mathscr{U} \ ,$$

where $\mathscr{U} \prec \mathscr{V} := \forall_{U\in\mathscr{U}} U \prec \mathscr{V}$ and

$$\mathscr{U} \downarrow \mathscr{V} := \{W \in N \mid \exists_{U\in\mathscr{U}} \exists_{V\in\mathscr{V}} (W \sqsubseteq U \wedge W \sqsubseteq V)\} \ .$$

A *formal point* in $\mathscr{T}$ is a subset $u \subseteq N$ such that

1. $\top \in u$,
2. $\forall_{U,V\in u}\exists_{W\in u}(W \sqsubseteq U \wedge W \sqsubseteq V)$,
3. $\forall_{U\in u}(U \prec \mathscr{U} \to \exists_{V\in\mathscr{U}} V \in u)$.

Dually to the case of precusl ideals, the second of the three requirements for a formal point expresses the peperty of being downward directed, so it follows that *any* finite subset in a formal point will have a greatest lower bound in the ideal. Denote the collection of formal points of $\mathscr{T}$ by $\mathrm{Pt}_{\mathscr{T}}$.

Call a formal topology $\mathscr{T}$ *unary* if

$$U \prec \mathscr{U} \to \underset{V\in\mathscr{U}}{\exists} U \prec V \ ,$$

where we write $U \prec V$ for $U \prec \{V\}$, and *consistently complete* if

$$\underset{U,V\in N}{\forall}\left(\underset{W\in N}{\exists}(W \sqsubseteq U \wedge W \sqsubseteq V) \to \underset{W\in N}{\exists} W = U \sqcap V\right) ,$$

where $\bigsqcap\mathscr{U}$ denotes the greatest lower bound of $\mathscr{U}$. Finally, call $\mathscr{T}$ a *Scott–Ershov formal topology* if it is both unary and consistently complete.

One can prove that every domain can be represented by the collection of formal points of a certain Scott–Ershov formal topology (see Theorem 4.35 of [30] and Theorem 6.2.15 of [52]). Here we proceed to link formal topologies *directly* to information systems.

Let $\mathscr{T} = (N, \sqsubseteq, \prec)$ be a Scott–Ershov formal topology. Define $I(\mathscr{T}) = (T, \mathsf{Con}, \vdash)$ by

$$\begin{aligned} T &:= N \ , \\ \mathscr{U} \in \mathsf{Con} &:= \mathscr{U} \subseteq^f N \wedge \bigsqcap \mathscr{U} \in N \ , \\ \mathscr{U} \vdash U &:= \bigsqcap \mathscr{U} \sqsubseteq U \ . \end{aligned}$$

Conversely, let $\rho = (T, \mathsf{Con}, \vdash)$ be an information system. Define $F(\rho) = (N, \sqsubseteq, \prec)$ by

$$\begin{aligned} N &:= \overline{\mathsf{Con}} \ , \\ \overline{U} \sqsubseteq \overline{V} &:= U \vdash V \ , \\ \overline{U} \prec \mathscr{U} &:= \underset{\overline{V}\in\mathscr{U}}{\exists} U \vdash V \ . \end{aligned}$$

Note that the definition is independent from the choice of representatives—see Proposition 3.5.

**Proposition 3.15.** *If $\mathscr{T}$ is a Scott–Ershov formal topology and $\rho$ an information system, then $I(\mathscr{T})$ is an information system and $F(\rho)$ is a Scott–Ershov formal topology. Furthermore, it is $\mathrm{Pt}_{\mathscr{T}} = \mathsf{Ide}_{I(\mathscr{T})}$ and $\mathsf{Ide}_\rho \simeq \mathrm{Pt}_{F(\rho)}$.*

*Proof.* First let $\mathscr{T}$ be a Scott–Ershov formal topology. We check the defining properties of an information system for $I(\mathscr{T})$. For reflexivity of consistency, let $U \in N$; it is $U \sqsubseteq U$, so $\bigsqcap\{U\} \in N$ and $\{U\} \in \mathsf{Con}$ by definition. For closure under subsets, let $\mathscr{U} \in \mathsf{Con}$ and $\mathscr{V} \subseteq \mathscr{U}$; then $\bigsqcap\mathscr{U} \in N$ and $\forall_{V\in\mathscr{V}} \bigsqcap\mathscr{U} \sqsubseteq V$, so $\bigsqcap\mathscr{V} \in N$ and $\mathscr{V} \in \mathsf{Con}$

by definition. For reflexivity of entailment, let $\mathscr{U} \in \mathsf{Con}$ and $U \in \mathscr{U}$; then $\bigsqcap \mathscr{U} \sqsubseteq U$, so $\mathscr{U} \vdash U$ by definition. For transitivity of entailment, let $\mathscr{U} \vdash \mathscr{V}$ and $\mathscr{V} \vdash W$; then $\bigsqcap \mathscr{U} \sqsubseteq \bigsqcap \mathscr{V}$ and $\bigsqcap \mathscr{V} \sqsubseteq W$; by transitivity we get $\bigsqcap \mathscr{U} \sqsubseteq W$, so $\mathscr{U} \vdash W$ by definition. Finally, for propagation of consistency through entailment, let $\mathscr{U} \in \mathsf{Con}$ and $\mathscr{U} \vdash V$; by definition, $\bigsqcap \mathscr{U} \in N$ and $\bigsqcap \mathscr{U} \sqsubseteq V$, so $\bigsqcap(\mathscr{U} \cup \{V\}) \in N$ and $\mathscr{U} \cup \{V\} \in \mathsf{Con}$ by definition.

Now let $\rho$ be an information system. We check the defining properties of a Scott–Ershov formal topology for $F(\rho)$. That $\sqsubseteq$ is a preorder with $\top := \overline{\varnothing}$ is direct to see. For reflexivity of covering, let $\overline{U} \in \mathscr{U}$; since $U \vdash U$, it is $\overline{U} \prec \mathscr{U}$ by definition. For transitivity of covering, we have

$$\begin{aligned} \overline{W} \prec \mathscr{U} \wedge \mathscr{U} \prec \mathscr{V} &\Leftrightarrow \underset{\overline{U} \in \mathscr{U}}{\exists} \left( W \vdash U \wedge \underset{\overline{V} \in \mathscr{V}}{\exists} U \vdash V \right) \\ &\overset{\text{trns}}{\Rightarrow} \underset{\overline{V} \in \mathscr{V}}{\exists} W \vdash V \\ &\Leftrightarrow \overline{W} \prec \mathscr{V} \,. \end{aligned}$$

For localization, we have

$$\begin{aligned} \overline{W} \prec \mathscr{U} \wedge \overline{W} \prec \mathscr{V} &\Leftrightarrow \underset{\overline{U} \in \mathscr{U}}{\exists} \left( W \vdash U \wedge \underset{\overline{V} \in \mathscr{V}}{\exists} W \vdash V \right) \\ &\Leftrightarrow \underset{\overline{U} \in \mathscr{U}}{\exists} \left( \overline{W} \sqsubseteq \overline{U} \wedge \underset{\overline{V} \in \mathscr{V}}{\exists} \overline{W} \sqsubseteq \overline{V} \right) \\ &\Leftrightarrow \overline{W} \in \mathscr{U} \downarrow \mathscr{V} \\ &\overset{\text{refl}}{\Rightarrow} \overline{W} \prec \mathscr{U} \downarrow \mathscr{V} \,. \end{aligned}$$

To show that the covering extends formal inclusion between formal basic opens, we have:

$$\begin{aligned} \overline{W} \sqsubseteq \overline{U} \wedge \overline{U} \prec \mathscr{V} &\Leftrightarrow W \vdash U \wedge \underset{\overline{V} \in \mathscr{V}}{\exists} U \vdash V \\ &\overset{\text{trns}}{\Rightarrow} \underset{\overline{V} \in \mathscr{V}}{\exists} W \vdash V \\ &\Leftrightarrow \overline{W} \prec \mathscr{V} \,. \end{aligned}$$

So $F(\rho)$ is indeed a formal topology. To show that it is unary is easy: let $\overline{U} \prec \mathscr{U}$; by definition there is a $\overline{V} \in \mathscr{U}$, for which $U \vdash V$, that is, $\overline{U} \sqsubseteq \overline{V}$; by reflexivity and extension, we get $\overline{U} \prec \{\overline{V}\}$. To show, finally, that it is consistently complete, let $\overline{U}, \overline{V}, \overline{W} \in N$, with $\overline{W} \sqsubseteq \overline{U}$ and $\overline{W} \sqsubseteq \overline{U}$, that is, $W \vdash U$ and $W \vdash V$; by Proposition 3.1(6), we get $W \vdash U \cup V$, and so, $\overline{W} \sqsubseteq \overline{U \cup V}$; let $\overline{U} \sqcap \overline{V} := \overline{U \cup V}$; that this does the job is direct to see.

We now show the bijective correspondence between *information system ideals* and *formal points*. For $\mathrm{Pt}_{\mathscr{T}} \subseteq \mathsf{Ide}_{I(\mathscr{T})}$, let $u \in \mathrm{Pt}_{\mathscr{T}}$ and $\mathscr{U} \subseteq^f u$. Since $u$ is downward directed in $\mathscr{T}$, it is $\bigsqcap \mathscr{U} \in$, and so $\mathscr{U} \in \mathsf{Con}_{I(\mathscr{T})}$ by definition. If further $\mathscr{U} \vdash_{I(\mathscr{T})} U$, it is $\bigsqcap \mathscr{U} \sqsubseteq U$ by definition, and $\bigsqcap \mathscr{U} \prec \{U\}$; hence $U \in u$ by the third formal point property.

For $\mathsf{Ide}_{I(\mathscr{T})} \subseteq \mathrm{Pt}_{\mathscr{T}}$, let $u \in \mathsf{Ide}_{I(\mathscr{T})}$. That $\top = \overline{\varnothing} \in u$, follows from downward closure in $I(\mathscr{T})$. Let $U, V \in u$; by the consistency in $I(\mathscr{T})$, $\{U, V\} \in \mathsf{Con}_{I(\mathscr{T})}$, and then $U \sqcap V \in N$ by definition; since $\{U, V\} \vdash_{I(\mathscr{T})} U \sqcap V$, it is $U \sqcap V \in u$ by the deductive closure in $I(\mathscr{T})$. If now $U \in u$ and $U \prec \mathscr{U}$, then, since $\mathscr{T}$ is unary, we have $\exists_{V \in \mathscr{U}} U \sqsubseteq V$, and $\exists_{V \in \mathscr{U}} \{U\} \vdash_{I(\mathscr{T})} V$ by the definition, so that $\exists_{V \in \mathscr{U}} V \in u$ follows from the deductive closure in $I(P)$.

For $\mathsf{Ide}_\rho \simeq \mathrm{Pt}_{F(\rho)}$ take the following isomorphism pair:

$$\mathsf{Ide}_\rho \ni u \mapsto \mathscr{P}_c(u) \in \mathrm{Pt}_{F(\rho)}$$
$$\mathrm{Pt}_{F(\rho)} \ni u \mapsto \bigcup u \in \mathsf{Ide}_\rho$$

where $\mathscr{P}_c(u) := \{\overline{U}\}_{U \subseteq^f u}$ contains the closures of subsets of $u$.

Indeed, for the right embedding, since $\varnothing \subseteq^f u$, it is $\top \in \mathscr{P}_c(u)$; for every $U, V \subseteq^f u$, since $U \cup V \subseteq^f u$, it is also $\overline{U} \sqcap \overline{V} \in \mathscr{P}_c(u)$; if $U \subseteq^f u$ and $\overline{U} \prec \mathscr{U}$, then $\exists_{\overline{V} \in \mathscr{U}} \overline{U} \sqsubseteq \overline{V}$, that is, $\exists_{\overline{V} \in \mathscr{U}} U \vdash V$ by definition; then $\exists_{\overline{V} \in \mathscr{U}} V \subseteq^f u$ by deductive closure in $\rho$.

For the left embedding, if $\{a_i\}_{i<n} \subseteq^f \bigcup u$, then $\forall_{i<n} \exists_{\overline{U_i} \in u} U_i \vdash a_i$; since $u$ is downward directed in $F(\rho)$ we have that $\exists_{\overline{W} \in u} \forall_{i<n} \overline{W} \sqsubseteq \overline{U_i}$; by definition, $\exists_{\overline{W} \in u} \forall_{i<n} W \vdash U_i \vdash a_i$, so $\{a_i\}_{i<n} \in \mathsf{Con}$ by transitivity of entailment and Proposition 3.1(6). If now $U \subseteq^f \bigcup u$ and $U \vdash a$, then, by definition, $\overline{U} \sqsubseteq \overline{a}$, that is $\overline{U} \prec \{\overline{a}\}$; by the third formal point property in $F(\rho)$, we have $\overline{a} \in u$, so $a \in \bigcup u$.

That the two embeddings are mutually inverse is also quite direct. Indeed, let $u \in \mathsf{Ide}_\rho$. We have

$$a \in \bigcup \mathscr{P}_c(u) \Leftrightarrow \exists_{U \subseteq^f u} U \vdash a \overset{(\star)}{\Leftrightarrow} a \in u$$

where $(\star)$ holds leftwards for $U := \{a\}$, and

$$\begin{aligned}
\overline{\{a_i\}_{i<n}} \in \mathscr{P}_c\left(\bigcup u\right) &\Leftrightarrow \{a_i\}_{i<n} \subseteq^f \bigcup u \\
&\Leftrightarrow \forall_{i<n} \exists_{\overline{U_i} \in u} U_i \vdash a_i \\
&\overset{(\star)}{\Leftrightarrow} \forall_{i<n} \exists_{\overline{U_i} \in u} \overline{U_i} \sqsubseteq \overline{a_i} \\
&\overset{(\star)}{\Leftrightarrow} \forall_{i<n} \exists_{\overline{U_i} \in u} \bigsqcap_{i<n} \overline{U_i} \sqsubseteq \overline{a_i} \\
&\overset{(\star)}{\Leftrightarrow} \forall_{i<n} \exists_{\overline{U_i} \in u} \bigsqcap_{i<n} \overline{U_i} \prec \{\overline{a_i}\} \\
&\overset{(\star)}{\Leftrightarrow} \forall_{i<n} \overline{a_i} \in u \\
&\Leftrightarrow \{\overline{a_i}\}_{i<n} \in u \\
&\Leftrightarrow \overline{\{a_i\}_{i<n}} \in u
\end{aligned}$$

where $(\star)$ hold leftwards for $U_i := \{a_i\}$, $i < n$. □

An *approximable map of Scott–Ershov formal topologies* from $\mathscr{T}$ to $\mathscr{T}'$ is a relation $\mathscr{R} \subseteq N \times N'$ which satisfies the following:

- $(\top, \top') \in \mathscr{R}$,
- $(U, V) \in \mathscr{R} \wedge (U, V') \in \mathscr{R} \to (U, V \sqcap V') \in \mathscr{R}$,
- $U' \sqsubseteq U \wedge (U, V) \in \mathscr{R} \wedge V \sqsubseteq V' \to (U', V') \in \mathscr{R}$.

Write $\mathsf{Apx}_{\mathscr{T}\to\mathscr{T}'}$ for all approximable maps of Scott–Ershov formal topologies from $\mathscr{T}$ to $\mathscr{T}'$. For every $\mathscr{R} \in \mathsf{Apx}_{\mathscr{T}\to\mathscr{T}'}$ define a relation $I(\mathscr{R}) \subseteq \mathsf{Con}_{I(\mathscr{T})} \times \mathsf{Con}_{I(\mathscr{T}')}$ by

$$(\mathscr{U},\mathscr{V}) \in I(\mathscr{R}) :\equiv \left(\bigsqcap \mathscr{U}, \bigsqcap \mathscr{V}\right) \in \mathscr{R} .$$

Conversely, let $r$ be an approximable map from $\rho$ to $\sigma$. Define a relation $F(r) \subseteq N_{F(\rho)} \times N_{F(\sigma)}$ by

$$(\overline{U},\overline{V}) \in F(r) :\equiv (U,V) \in r .$$

Again, it is easy to see that the definition does not rely on the choice of the representatives, due to deductive closure of $r$.

We show that these establish a bijective correspondence.

**Proposition 3.16.** *If $r$ is an approximable map from $\rho$ to $\sigma$ then $F(r)$ is an approximable map of Scott–Ershov formal topologies from $F(\rho)$ to $F(\sigma)$. Conversely, if $\mathscr{R}$ is an approximable map of Scott–Ershov formal topologies from $\mathscr{T}$ to $\mathscr{T}'$ then $I(\mathscr{R})$ is an approximable map from $I(\mathscr{T})$ to $I(\mathscr{T}')$. Furthermore, it is $\mathsf{Apx}_{\rho\to\sigma} \simeq \mathsf{Apx}_{F(\rho)\to F(\sigma)}$ and $\mathsf{Apx}_{\mathscr{T}\to\mathscr{T}'} \simeq \mathsf{Apx}_{I(\mathscr{T})\to I(\mathscr{T}')}$.*

*Proof.* Let $r$ be an approximable map from $\rho$ to $\sigma$. Since, by Proposition 3.3, $(\varnothing,\varnothing) \in r$, it is $(\top,\top') \in F(r)$. If $(\overline{U},\overline{V}),(\overline{U},\overline{V'}) \in F(r)$, then, by definitions, $(U,V\cup V') \in r$, so $(\overline{U},\overline{V\cup V'}) \in F(r)$, and $(\overline{U},\overline{V}\sqcap\overline{V'}) \in F(r)$. If $\overline{U'} \sqsubseteq \overline{U}$, $(\overline{U},\overline{V}) \in F(r)$ and $\overline{V} \sqsubseteq \overline{V'}$, then, by definitions, $U' \vdash U$, $(U,V) \in r$ and $V \vdash V'$ respectively, so, $(U',V') \in r$ and $(\overline{U'},\overline{V'}) \in F(r)$.

Conversely, let $\mathscr{R}$ be an approximable map of Scott–Ershov formal topologies from $\mathscr{T}$ to $\mathscr{T}'$. Since $(\top,\top') \in \mathscr{R}$, it is $(\varnothing,\varnothing) \in I(\mathscr{R})$. If $(\mathscr{U},\mathscr{V}),(\mathscr{U},\mathscr{V}') \in I(\mathscr{R})$, then, by definition, $(\bigsqcap\mathscr{U},\bigsqcap\mathscr{V}),(\bigsqcap\mathscr{U},\bigsqcap\mathscr{V}') \in \mathscr{R}$; since $\mathscr{R}$ is an approximable map of Scott–Ershov formal topologies, $(\bigsqcap\mathscr{U},(\bigsqcap\mathscr{V})\sqcap(\bigsqcap\mathscr{V}')) \in \mathscr{R}$, or, $(\bigsqcap\mathscr{U},\bigsqcap(\mathscr{V}\sqcap\mathscr{V}')) \in \mathscr{R}$, that is, $(\mathscr{U},\mathscr{V}\sqcap\mathscr{V}') \in I(\mathscr{R})$. If now $\mathscr{U}' \vdash \mathscr{U}$, $(\mathscr{U},\mathscr{V}) \in I(\mathscr{R})$ and $\mathscr{V} \vdash \mathscr{V}'$, by definition we obtain $\bigsqcap\mathscr{U}' \sqsubseteq \bigsqcap\mathscr{U}$, $(\bigsqcap\mathscr{U},\bigsqcap\mathscr{V}) \in \mathscr{R}$ and $\bigsqcap\mathscr{V} \sqsubseteq \bigsqcap\mathscr{V}'$; then $(\bigsqcap\mathscr{U}',\bigsqcap\mathscr{V}') \in \mathscr{R}$, so $(\mathscr{U}',\mathscr{V}') \in I(\mathscr{R})$.

We show that $F : \mathsf{Apx}_{\rho\to\sigma} \to \mathsf{Apx}_{F(\rho)\to F(\sigma)}$ is bijective. To show injectivity, let $F(r) = F(r')$; then

$$(U,V) \in r \overset{\text{def.}F}{\Leftrightarrow} (\overline{U},\overline{V}) \in F(r) \Leftrightarrow (\overline{U},\overline{V}) \in F(r') \overset{\text{def.}F}{\Leftrightarrow} (U,V) \in r' ,$$

so $r = r'$. To show surjectivity, let $\mathscr{R} \in \mathsf{Apx}_{F(\rho)\to F(\sigma)}$; set

$$(U,V) \in r :\equiv (\overline{U},\overline{V}) \in \mathscr{R} ;$$

it is straightforward to check that $r \in \mathsf{Apx}_{\rho\to\sigma}$ and $F(r) = \mathscr{R}$.

We show finally that $I : \mathsf{Apx}_{\mathscr{T}\to\mathscr{T}'} \to \mathsf{Apx}_{I(\mathscr{T})\to I(\mathscr{T}')}$ is bijective. To show injectivity, let $I(\mathscr{R}) = I(\mathscr{R}')$ and $(U,V) \in \mathscr{R}$; then, by the definition of $I$, there are $\mathscr{U} \in \mathsf{Con}_{I(\mathscr{T})}$ and $\mathscr{V} \in \mathsf{Con}_{I(\mathscr{T}')}$, such that $U = \bigsqcap\mathscr{U}$, $V = \bigsqcap\mathscr{V}$ and $(\mathscr{U},\mathscr{V}) \in I(\mathscr{R})$; by the assumption we get equivalently that $(\mathscr{U},\mathscr{V}) \in I(\mathscr{R})$, so $(U,V) \in \mathscr{R}$, and $\mathscr{R} = \mathscr{R}'$. To show surjectivity, let $r \in \mathsf{Apx}_{I(\mathscr{T})\to I(\mathscr{T}')}$; set

$$(U,V) \in \mathscr{R} :\equiv \underset{\mathscr{U}\in\mathsf{Con}_{I(\mathscr{T})}}{\exists}\ \underset{\mathscr{V}\in\mathsf{Con}_{I(\mathscr{T}')}}{\exists} \left(U = \bigsqcap\mathscr{U} \wedge V = \bigsqcap\mathscr{V} \wedge (\mathscr{U},\mathscr{V}) \in r\right) ;$$

it is $\mathscr{R} \in \mathsf{Apx}_{\mathscr{T}\to\mathscr{T}'}$, since (i) by $r \in \mathsf{Apx}_{I(\mathscr{T})\to I(\mathscr{T}')}$ we get $(\varnothing,\varnothing) \in r$, which yields $(\top,\top') \in \mathscr{R}$, (ii) it is

$$\begin{aligned}
&(U,V_1) \in \mathscr{R} \wedge (U,V_2) \in \mathscr{R} \\
&\Leftrightarrow \underset{\mathscr{U}_1,\mathscr{V}_1}{\exists} \Big(U = \bigsqcap \mathscr{U}_1 \wedge V_1 = \bigsqcap \mathscr{V}_1 \wedge (\mathscr{U}_1,\mathscr{V}_1) \in r\Big) \\
&\qquad \wedge \underset{\mathscr{U}_2,\mathscr{V}_2}{\exists} \Big(U = \bigsqcap \mathscr{U}_2 \wedge V_2 = \bigsqcap \mathscr{V}_2 \wedge (\mathscr{U}_2,\mathscr{V}_2) \in r\Big) \\
&\overset{(*)}{\Rightarrow} \underset{\mathscr{U},\mathscr{V}_1,\mathscr{V}_2}{\exists} \big(U = \bigsqcap \mathscr{U} \wedge V_1 = \bigsqcap \mathscr{V}_1 \wedge V_2 = \bigsqcap \mathscr{V}_2 \\
&\qquad\qquad \wedge (\mathscr{U},\mathscr{V}_1) \in r \wedge (\mathscr{U},\mathscr{V}_2) \in r\big) \\
&\Rightarrow \underset{\mathscr{U},\mathscr{V}_1,\mathscr{V}_2}{\exists} \Big(U = \bigsqcap \mathscr{U} \wedge V_1 = \bigsqcap \mathscr{V}_1 \wedge V_2 = \bigsqcap \mathscr{V}_2 \wedge (\mathscr{U},\mathscr{V}_1 \cup \mathscr{V}_2) \in r\Big) \\
&\overset{(**)}{\Rightarrow} \underset{\mathscr{U},\mathscr{V}}{\exists} \Big(U = \bigsqcap \mathscr{U} \wedge V_1 \sqcap V_2 = \bigsqcap \mathscr{V} \wedge (\mathscr{U},\mathscr{V}) \in r\Big) \\
&\Leftrightarrow (U, V_1 \sqcap V_2) \in \mathscr{R}\ ,
\end{aligned}$$

where $(*)$ holds for $\mathscr{U} := \{U\}$, and $(**)$ for $\mathscr{V} := \mathscr{V}_1 \cup \mathscr{V}_2$, and (iii) it is

$$\begin{aligned}
&U' \sqsubseteq U \wedge (U,V) \in \mathscr{R} \wedge V \sqsubseteq V' \\
&\Leftrightarrow U' \sqsubseteq U \wedge \underset{\mathscr{U},\mathscr{V}}{\exists} \Big(U = \bigsqcap \mathscr{U} \wedge V = \bigsqcap \mathscr{V} \wedge (\mathscr{U},\mathscr{V}) \in r\Big) \wedge V \sqsubseteq V' \\
&\overset{(*)}{\Rightarrow} \underset{\mathscr{U}'}{\exists} \Big(U' = \bigsqcap \mathscr{U}' \wedge \underset{U \in \mathscr{U}}{\forall} \mathscr{U}' \vdash_{I(\mathscr{T})} U\Big) \\
&\qquad \wedge \underset{\mathscr{U},\mathscr{V}}{\exists} \Big(U = \bigsqcap \mathscr{U} \wedge V = \bigsqcap \mathscr{V} \wedge (\mathscr{U},\mathscr{V}) \in r\Big) \\
&\qquad \wedge \mathscr{V} \vdash_{I(\mathscr{T}')} V' \\
&\overset{(**)}{\Rightarrow} \underset{\mathscr{U}',\mathscr{V}'}{\exists} \Big(U' = \bigsqcap \mathscr{U}' \wedge V' = \bigsqcap \mathscr{V}' \wedge (\mathscr{U}',\mathscr{V}') \in r\Big) \\
&\Leftrightarrow (U',V') \in \mathscr{R}\ ,
\end{aligned}$$

where $(*)$ holds for $\mathscr{U}' := \{U\}$ and because

$$V \sqsubseteq V' \Rightarrow \bigsqcap \mathscr{V} \sqsubseteq V' \Rightarrow \mathscr{V} \vdash_{I(\mathscr{T}')} V'\ ,$$

and $(**)$ holds for $\mathscr{V}' := \{V\}$ and because $r$ is an approximable map; finally, direct application of the definitions gives

$$(\mathscr{U},\mathscr{V}) \in I(\mathscr{R}) \Leftrightarrow \Big(\bigsqcap \mathscr{U}, \bigsqcap \mathscr{V}\Big) \in \mathscr{R} \Leftrightarrow (\mathscr{U},\mathscr{V}) \in r\ ,$$

which means that $I(\mathscr{R}) = r$. □

### Coherent Scott–Ershov formal topologies

Call a Scott–Ershov formal topology *coherent* if it satisfies the following property for a finite collection $\mathscr{U} \subseteq^f N$:

$$\bigsqcap \mathscr{U} \in N \leftrightarrow \underset{U,V \in \mathscr{U}}{\forall} U \sqcap V \in N\ . \tag{3.4}$$

**Theorem 3.17.** *If $\mathscr{T}$ is a coherent Scott–Ershov formal topology then $I(\mathscr{T})$ is a coherent information system. Conversely, if $\rho$ is a coherent information system then $F(\rho)$ is a coherent Scott–Ershov formal topology.*

*Proof.* Suppose first that $\mathscr{T}$ is coherent, that is, such that (3.4) holds for all $\mathscr{U} \subseteq^f N$. Let $\mathscr{U} \in \mathsf{Con}_{I(\mathscr{T})}$; by definition,

$$\mathscr{U} \subseteq^f N \wedge \bigsqcap \mathscr{U} \in N \overset{(3.4)}{\Leftrightarrow} \mathscr{U} \subseteq^f N \wedge \underset{U,V \in \mathscr{U}}{\forall} U \sqcap V \in N$$
$$\overset{\text{def}}{\Leftrightarrow} \mathscr{U} \subseteq^f N \wedge \underset{U,V \in \mathscr{U}}{\forall} \{U,V\} \in \mathsf{Con}_{I(\mathscr{T})} ,$$

so $I(\mathscr{T})$ is a coherent information system.

Now suppose that $\rho$ is a coherent information system, that is, such that

$$U \in \mathsf{Con} \leftrightarrow \underset{a,b \in U}{\forall} \{a,b\} \in \mathsf{Con} , \tag{3.3}$$

for all $U \subseteq^f T$. Let $\mathscr{U} \subseteq^f N_{F(\rho)}$, that is, $\mathscr{U} \subseteq^f \mathsf{Con}$; we have

$$\bigsqcap \mathscr{U} \in N_{F(\rho)} \overset{\text{def}}{\Leftrightarrow} \bigcup \mathscr{U} \in \overline{\mathsf{Con}}$$
$$\overset{(\star)}{\Leftrightarrow} \underset{U,V \subseteq^f \bigcup \mathscr{U}}{\forall} \overline{U \cup V} \in \overline{\mathsf{Con}}$$
$$\overset{\text{def}}{\Leftrightarrow} \underset{U,V \subseteq^f \bigcup \mathscr{U}}{\forall} \overline{U} \sqcap \overline{V} \in N_{F(\rho)}$$
$$\Rightarrow \underset{\overline{U},\overline{V} \in \mathscr{U}}{\forall} \overline{U} \sqcap \overline{V} \in N_{F(\rho)} ,$$

where at $(\star)$ we used (3.3) and Proposition 3.8. Conversely, we have

$$\mathscr{U} \subseteq^f N_{F(\rho)} \wedge \underset{\overline{U},\overline{V} \in \mathscr{U}}{\forall} \overline{U} \sqcap \overline{V} \in N_{F(\rho)} \overset{\text{def}}{\Leftrightarrow} \mathscr{U} \subseteq^f \overline{\mathsf{Con}} \wedge \underset{\overline{U},\overline{V} \in \mathscr{U}}{\forall} \overline{U} \sqcap \overline{V} \in \overline{\mathsf{Con}}$$
$$\overset{(3.3)}{\Leftrightarrow} \mathscr{U} \subseteq^f \overline{\mathsf{Con}} \wedge \underset{\overline{U},\overline{V} \in \mathscr{U}}{\forall} \underset{a,b \in \overline{U \cup V}}{\forall} \{a,b\} \in \mathsf{Con}$$
$$\Rightarrow \mathscr{U} \subseteq^f \overline{\mathsf{Con}} \wedge \underset{a,b \in \bigcup \mathscr{U}}{\forall} \overline{\{a,b\}} \in \overline{\mathsf{Con}}$$
$$\overset{(3.3)}{\Leftrightarrow} \bigcup \mathscr{U} \in \overline{\mathsf{Con}}$$
$$\overset{\text{def}}{\Leftrightarrow} \bigsqcap \mathscr{U} \in N_{F(\rho)} ,$$

so $F(\rho)$ is indeed a coherent Scott–Ershov formal topology. □

## 3.4 Notes

### On the notion of atomicity

The defining property of a *unary* formal topology (page 119) looks similar to the atomicity property for an information system (page 111)—in fact, unary formal topologies are called "atomic" by Erik Palmgren in a preliminary version of [33]—but the two are not essentially related from our viewpoint.

The property of being unary for a formal topology expresses *atomicity of compact covering*, whereas in information systems we have *atomicity of information flow*: in the first case, an "atom" would be a formal basic open while in the second case, an atom (that is, a token) represents a simple piece of data.

In order to avoid confusions, one should notice how the transition from an information system to a point-free structure—domains included—involves jumping from the level of atomic pieces of data to (finitely determined) *sets* of atomic pieces of data: atomicity of information appears in the presence of atomic pieces of data, which become indiscernible when one moves to a point-free setting (see however the last note).

### On the notion of coherence

Coherence in domain theory is in no way considered here for the first time. *Coherent cpo's* appear already in Gordon Plotkin's [36], where he attributes the notion to George Markowsky and Barry Rosen [29]. In the handbook chapter of Samson Abramsky and Achim Jung [1], coherence is studied in the more general setting of *continuous domains*. Viggo Stoltenberg-Hansen et al [52] introduce the notion too in an exercise. We should also mention Jean-Yves Girard's *coherence spaces* [14], which he uses alternatively to Scott–Ershov domains. On the other hand, coherence has been considered in point-free topology as well, at least since Peter Johnstone's [20], where *coherent locales* are discussed.

### Featuring Coquand and Plotkin

Both of the finite Scott information systems $\mathscr{C}$ and $\mathscr{L}$ of section 3.1 are elaborations of existing counterexamples.

The first one, due to Thierry Coquand, was first given as a *counterexample to atomicity* in *algebraic* information systems (see page 47). In Chapter 2 however, we have shown that atomicity remains a concept worth exploring, since it lies in the very fundaments of the more general algebraic entailment, and at the same time gives rise to surprising and utilizable notions, namely, matrices on the base-type case, and eigen-neighborhoods in the higher-type case. And of course, it is a perfectly sufficient property for algebras like natural numbers $\mathbb{N}$ or boolean numbers $\mathbb{B}$, which have at most unary constructors (see Chapter 1).

The second Scott information system stems from Plotkin's [36], where he uses the entailment graph of $\mathscr{L}$ as an example of a "consistently complete" but not "coherent" complete partial order. It is indeed *Plotkin's counterexample to coherence*: in the entailment diagramme of $\mathscr{L}$ in Figure 3.1, one has $\{l,m\},\{l,r\},\{m,r\} \in \mathsf{Con}_{\mathscr{L}}$ but $\{l,m,r\} \notin \mathsf{Con}_{\mathscr{L}}$.

Notice also that $\mathscr{C}$ is non-atomic but coherent, and that $\mathscr{L}$ is incoherent but atomic.

### Dues and nomenclature discrepancies

The questions answered in sections 3.2 and 3.3 occurred to the author at the 3WFTop workshop; the results were presented for the first time in a *Forschungstutorium* held at LMU during the winter semester of 2007–8 and led by Peter Schuster.

Giovanni Sambin et al [41] call consistently complete ordered sets *coherent*. Sara Negri [30] says *Scott formal topology* for a unary formal topology and *consistently complete Scott formal topology* for a Scott–Ershov formal topology (modulo the presence of a top formal basic open). Viggo Stoltenberg-Hansen et al [52] say *formal space* for a consistently complete formal topology and *Scott space* for a Scott–Ershov formal topology.

### Outlook

The issue of linking the theory of information systems and formal topology has many facets, at least as many as the various point-free structures that are currently studied by the community. Apart from the ones that we have covered in this chapter, further links should be attainable in various other settings, from the *event structures* of [57] to the *apartness spaces* of [7], by imposing an appropriate coherence property on the structure every time, one that would reflect Proposition 3.8; this suggests a rather straightforward cartographic endeavor, but still quite important, as Sambin described (see page 105).

Moreover, as we now know (see the note on page 101) that atomicity is a notion that implicitly permeates much more than non-superunary algebras, particularly one that may leave traces in terms of *eigen-neighborhoods*, one may expect that it could be feasible to describe it in point-free topological settings after all. The question would be to understand if and how it may manifest in point-free structures, and what would its presence ensue for the latter.

# Chapter 4

# Elimination of extensionality

The previous chapters focus on *coherent information systems as a model for higher-type computability*; the main motivation for this is a desire to devise an appropriate *constructive logical theory of higher-type computability*, one that will lend itself as painlessly as possible to *implementation on a proof assistant*. The leading idea in such a theory should be to provide the necessary means to talk not only about *objects* (ideals, that is, numbers, functions, and functionals) but also about their *finite approximations* (tokens, formal neighborhoods). First steps in this direction were presented in [18] under the name *Theory of Partial Computable Functionals*, or $\mathrm{TCF}^+$, extending the Theory of Computable Functionals TCF of [49], which covers just objects.

Pertaining to the objects in TCF and $\mathrm{TCF}^+$, partial and total alike, is the notion of *extensionality*, which roughly posits that *two equal arguments draw equal values from the same function*—a version of Leibniz' *indiscernibility of identicals* (for objects rather than predicates). This is a natural demand that nevertheless presents well known proof-theoretical problems (see William Howard's counterexample to Dialectica realizability in [17]). Dealing with the axiom of extensionality has since become a reasonable first challenge to pose to a proposed logical theory.

### Preview

In this chapter we concentrate on the part of $\mathrm{TCF}^+$ that will encompass *arithmetic*. In the style of [55], we present the generic *Heyting arithmetic* and its extensional version, and then show how *extensionality can be eliminated*.

## 4.1 Heyting arithmetic in all finite types

We introduce the theories $\mathrm{HA}^\omega$ and $\mathrm{E\text{-}HA}^\omega$, of *Heyting arithmetic* and *extensional Heyting arithmetic* in finite types respectively. Denote by $\mathbb{N}$ the base type, prototypically denoting natural numbers; if $\rho$ and $\sigma$ are types then $\rho \to \sigma$ is also a type.

### Language of $\mathrm{E\text{-}HA}^\omega$

We have the logical symbols $\wedge$, $\vee$, $\to$, as well as $\forall^\rho$, $\exists^\rho$ quantifier symbols for every type (but we just write $\forall$, $\exists$); the *lambda operator* $\lambda . \cdot$ and *application parentheses* $(\cdot\cdot)$; object variables $x, y, z, \ldots$, $f, g, h, \ldots$, for any type (write $x^\rho$ for an object $x$ of type

$\rho$); an object constant $0^{\mathbb{N}}$ for *zero*; the function constants $S^{\mathbb{N}\to\mathbb{N}}$ for the *successor*, and $R_\rho^{(\rho\to\mathbb{N}\to\rho)\to\rho\to\mathbb{N}\to\rho}$ for *recursors*, for all types; relation variables; a nullary relation constant $\bot$ for *falsum*; a relation constant $=_\rho$ for a generic *equality* for all types $\rho$.

Every object variable $x^\rho$ is a term; if $t^{\rho\to\sigma}$ and $s^\rho$ are terms, then the *application* $(ts)^\sigma$ is a term; if $x^\rho$ is a variable and $t^\sigma$ is a term, then the $\lambda$*-abstraction* $(\lambda_x\, t)^{\rho\to\sigma}$ is a term; especially for the constants, if $t$, $s$, $r$ are terms of appropriate types, then $St$ and $Rtsr$ are terms as well.

If $t^\rho$ and $s^\rho$ are terms then $t^\rho =_\rho s^\rho$ is a (prime) formula; if $A$ and $B$ are formulas then so are $A\wedge B$, $A\vee B$, $A\to B$; if $x^\rho$ is an object variable and $A$ is a formula then $\forall_{x^\rho} A$ and $\exists_{x^\rho} A$ are formulas.

For *negation*, *classical disjunction* and *classical existence* write $\neg A$ for $A\to\bot$, $A\tilde{\vee}B$ for $\neg A\to\neg B\to\bot$, and $\tilde{\exists}_x A$ for $\neg\forall_x\neg A$. For reasons of readability we write $f,g,h$ for objects that we use as functions and $x,y,z$ for objects that we use as arguments or values.

## Calculus of E-HA$^\omega$

We require the following inference rules: arbitrary *assumptions*

$$u : A\ ,$$

*arrow introduction* and *arrow elimination* (or *modus ponens*) rules

$$\dfrac{\begin{matrix}[u:A]\\ \mid \\ B\end{matrix}}{A\to B}\to^+_u\ ,\qquad\qquad \dfrac{\begin{matrix}\mid\\ A\to B\end{matrix}\qquad \begin{matrix}\mid\\ A\end{matrix}}{B}\to^-\ ,$$

as well as *forall introduction* and *forall elimination* rules

$$\dfrac{\begin{matrix}\mid\\ A\end{matrix}}{\forall_x A}\forall^+_x\ ,\qquad\qquad \dfrac{\begin{matrix}\mid\\ \forall_x A\end{matrix}\qquad r}{A[x:=r]}\forall^-\ ,$$

where for $\forall^+_x$, $x$ should be fresh.

Further, we require the following axioms. *Disjunction introduction* and *disjunction elimination* axioms:

$$\begin{aligned}&\vee^+_0 : A\to A\vee B\ ,\\ &\vee^+_1 : B\to A\vee B\ ,\\ &\vee^- : A\vee B\to(A\to P)\to(B\to P)\to P\ .\end{aligned}$$

*Conjunction introduction* and *conjunction elimination* axioms:

$$\begin{aligned}&\wedge^+ : A\to B\to A\wedge B\ ,\\ &\wedge^- : A\wedge B\to(A\to B\to P)\to P\ .\end{aligned}$$

*Exists introduction* and *exists elimination* axioms:

$$\begin{aligned}&\exists^+ : A\to \exists_x A\ ,\\ &\exists^- : \exists_x A\to(\forall_x A\to P)\to P\ ,\end{aligned}$$

for $x \notin \mathrm{FV}(P)$. The *falsum elimination* axiom:

$$\bot^- : \bot \to A \ .$$

Concerning equality, we require the following well-known (generic) *axioms of equality*, restricted to the base type:

$$x =_{\mathbb{N}} x \ ,$$
$$x =_{\mathbb{N}} y \to y =_{\mathbb{N}} x \ ,$$
$$x =_{\mathbb{N}} y \to y =_{\mathbb{N}} z \to x =_{\mathbb{N}} z \ ,$$

and define *(point-wise) equality* for higher types by

$$f =_{\rho \to \sigma} g :\equiv \forall_x fx =_{\sigma} gx \ ,$$

for $x$ of type $\rho$ and $f$, $g$ of type $\rho \to \sigma$. Further, take the following defining axioms for the constants:

$$Sx \neq_{\mathbb{N}} 0 \ ,$$
$$Sx =_{\mathbb{N}} Sy \leftrightarrow x =_{\mathbb{N}} y \ ,$$

$$R_\rho fz0 =_\rho z \ ,$$
$$R_\rho fz(Sx) =_\rho f(R_\rho fzx)x \ ,$$

where $z$ is of type $\rho$, $f$ is of type $\rho \to \mathbb{N} \to \rho$, and $x$ is of type $\mathbb{N}$. We also require $\beta$*-reduction* conversion rules:

$$(\lambda_x\, t)s^\rho =_\sigma t[x := s] \ ,$$

where $t[x := s]$, or just $t(s)$, means "$t$, with $s$ substituted for $x$"; to anticipate a substitution of $x$ by some other term in $t$, we also write $t(x)$ (not to be confused with the application parentheses).

Finally, we have an *induction axiom scheme*:

$$A(0) \wedge \forall_x (A(x) \to A(Sx)) \to \forall_x A(x) \ ,$$

for arbitrary formulas $A$ and $x$ of type $\mathbb{N}$, and the *extensionality axioms*:

$$x_1 =_\rho x_2 \to fx_1 =_\sigma fx_2 \ ,$$

for $f$ of type $\rho \to \sigma$ and both $x_i$'s of type $\rho$.

The resulting system is denoted here by E-HA$^{\omega}$. If we drop the extensionality axioms, we denote it by HA$^{\omega}$.

## 4.2 Elimination of extensionality in E-HA$^{\omega}$

Extensionality is quite a natural property to have in a system designed for mathematics, so intuitively there is plenty of reason to demand it. As it turns out, it is also an axiom that doesn't really cumber the theory: with an appropriate translation, one can make without the axioms of extensionality and just work within HA$^{\omega}$.

Define the *extensionality predicate*, or just *extensionality*, and the *extensional equality* by mutual induction:

$$E_{\mathbb{N}}x := x =_{\mathbb{N}} x \ ,$$
$$x_1 \,{}^e{=}_{\mathbb{N}}\, x_2 := x_1 =_{\mathbb{N}} x_2 \ ,$$
$$E_{\rho\to\sigma}f := \forall_{x_1,x_2} \left(x_1 \,{}^e{=}_\rho\, x_2 \to f x_1 \,{}^e{=}_\sigma\, f x_2\right) \ ,$$
$$f_1 \,{}^e{=}_{\rho\to\sigma}\, f_2 := E_{\rho\to\sigma}f_1 \wedge E_{\rho\to\sigma}f_2 \wedge \forall_x \left(E_\rho x \to f_1 x \,{}^e{=}_\sigma\, f_2 x\right) \ .$$

When $Ef$ holds, call $f$ *extensional*. Intuitively, an element is "extensional" when it is, so to speak, right-compatible, that is, stable under application. Furthermore, to talk about "extensionality of equality", is to stress equality's left-compatibility on stable elements. It should also be clear that extensional equality is in fact point-wise equality restricted to extensional elements.

**Proposition 4.1.** *Extensional equality is symmetric and transitive. Furthermore, it is reflexive on extensional elements, i.e.,* $E_\rho x \to x \,{}^e{=}_\rho\, x$.

*Proof.* Symmetry and transitivity are direct. For reflexivity on extensional elements, by induction on the type. The base case is direct. For the step case, let $E_{\rho\to\sigma}f$ and $E_\rho x$; by the induction hypothesis we have $E_{\rho\to\sigma}f$ and $x \,{}^e{=}_\rho\, x$; by the definition of $E_{\rho\to\sigma}$ we have $fx \,{}^e{=}_\sigma\, fx$; since $Ef$ and $x$ is arbitrary, by the definition of ${}^e{=}_{\rho\to\sigma}$ we have $f \,{}^e{=}_{\rho\to\sigma}\, f$. □

**Proposition 4.2.** *Extensional equality can is characterized as follows:*

$$f_1 \,{}^e{=}_{\rho\to\sigma}\, f_2 \leftrightarrow \forall_{x_1,x_2} \left(x_1 \,{}^e{=}_\rho\, x_2 \to f_1 x_1 \,{}^e{=}_\sigma\, f_2 x_2\right) \ .$$

*Proof.* For the right direction: Let $f_1 \,{}^e{=}_{\rho\to\sigma}\, f_2$ and $x_1 \,{}^e{=}_\rho\, x_2$; by the definition of ${}^e{=}_{\rho\to\sigma}$ we have $E_{\rho\to\sigma}f_1 \wedge E_{\rho\to\sigma}f_2 \wedge \forall_x \left(E_\rho x \to f_1 x \,{}^e{=}_\sigma\, f_2 x\right)$ and $x_1 \,{}^e{=}_\rho\, x_2$; by the definition of $E_{\rho\to\sigma}$ we get $\forall_{x_1,x_2} \left(x_1 \,{}^e{=}_\rho\, x_2 \to f_i x_1 \,{}^e{=}_\sigma\, f_i x_2\right)$ for $i = 1,2$ and we still have $\forall_x \left(E_\rho x \to f_1 x \,{}^e{=}_\sigma\, f_2 x\right)$ and $x_1 \,{}^e{=}_\rho\, x_2$; by the definition of ${}^e{=}_\rho$ and modus ponens we have $\forall_{x_1,x_2} \left(x_1 \,{}^e{=}_\rho\, x_2 \to f_i x_1 \,{}^e{=}_\sigma\, f_i x_2\right)$ and $f_1 x_i \,{}^e{=}_\sigma\, f_2 x_i$ for $i = 1,2$ and we still have $x_1 \,{}^e{=}_\rho\, x_2$; by modus ponens now we get $f_1 x_1 \,{}^e{=}_\sigma\, f_1 x_2$ and we still have $f_1 x_2 \,{}^e{=}_\sigma\, f_2 x_2$; by transitivity of ${}^e{=}_\sigma$ (Proposition 4.1) we finally get $f_1 x_1 \,{}^e{=}_\sigma\, f_2 x_2$.

For the left direction: Let $\forall_{x_1,x_2} \left(x_1 \,{}^e{=}_\rho\, x_2 \to f_1 x_1 \,{}^e{=}_\sigma\, f_2 x_2\right)$; we have to show that (i) $E_{\rho\to\sigma}f_i$ for $i = 1,2$ and that (ii) $\forall_x \left(E_\rho x \to f_1 x \,{}^e{=}_\sigma\, f_2 x\right)$. For (i): Let $x_1 \,{}^e{=}_\rho\, x_2$ and we still have $\forall_{x_1,x_2} \left(x_1 \,{}^e{=}_\rho\, x_2 \to f_1 x_1 \,{}^e{=}_\sigma\, f_2 x_2\right)$; by the definition of ${}^e{=}_\rho$ we have $E_\rho x_1$ and $E_\rho x_2$, which by Proposition 4.1 yield $x_1 \,{}^e{=}_\rho\, x_1$ and $x_2 \,{}^e{=}_\rho\, x_2$, while we still have $x_1 \,{}^e{=}_\rho\, x_2$ and $\forall_{x_1,x_2} \left(x_1 \,{}^e{=}_\rho\, x_2 \to f_1 x_1 \,{}^e{=}_\sigma\, f_2 x_2\right)$; by three different applications of modus ponens we get $f_1 x_1 \,{}^e{=}_\sigma\, f_2 x_2$, $f_1 x_1 \,{}^e{=}_\sigma\, f_2 x_1$, $f_1 x_2 \,{}^e{=}_\sigma\, f_2 x_2$ and we still have $x_1 \,{}^e{=}_\rho\, x_2$; by transitivity and reflexivity of ${}^e{=}_\sigma$ we get $f_1 x_1 \,{}^e{=}_\sigma\, f_1 x_2$ and $f_2 x_1 \,{}^e{=}_\sigma\, f_2 x_2$ and we still have $x_1 \,{}^e{=}_\rho\, x_2$; by the definition of $E_{\rho\to\sigma}$ we get $E_{\rho\to\sigma}f_i$, for $i = 1,2$. For (ii): Let $E_\rho x$, which by Proposition 4.1 yields $x \,{}^e{=}_\rho\, x$ and we still have $\forall_{x_1,x_2} \left(x_1 \,{}^e{=}_\rho\, x_2 \to f_1 x_1 \,{}^e{=}_\sigma\, f_2 x_2\right)$; by modus ponens we immediately get $f_1 x \,{}^e{=}_\sigma\, f_2 x$. □

**Proposition 4.3.** *Term formation preserves both extensional equality and extensionality, that is, if $r(\vec{x})$ is a well-formed term, with free variables among $\vec{x}$, it is*

$$\vec{x}_1 \,{}^e{=}\, \vec{x}_2 \to r(\vec{x}_1) \,{}^e{=}\, r(\vec{x}_2)$$

*and*

$$E\vec{x} \to Er(\vec{x})$$

*respectively.*

*Proof by mutual induction on the term. Base for $E$.* For variables it is clear. For constant 0 we have it by the definition of $E_{\mathbb{N}}$ (and forall elimination). For constant $S$, we want to know that $\forall_{x_1,x_2}(x_1 \,{}^e{=}_{\mathbb{N}}\, x_2 \to Sx_1 \,{}^e{=}_{\mathbb{N}}\, Sx_2)$, which is provided by the axioms for the successor. For constant $R$: we need to show that

$$\forall_{f_1,f_2}\forall_{z_1,z_2}\forall_{x_1,x_2}\left(f_1 \,{}^e{=}_{\rho\to\mathbb{N}\to\rho}\, f_2 \to z_1 \,{}^e{=}_\rho\, z_2 \to x_1 \,{}^e{=}_{\mathbb{N}}\, x_2 \to Rf_1z_1x_1 \,{}^e{=}_{\mathbb{N}}\, Rf_2z_2x_2\right) ,$$

so let $f_1 \,{}^e{=}\, f_2$, $z_1 \,{}^e{=}\, z_2$, $x_1 \,{}^e{=}\, x_2$; by the definition of $E_{\mathbb{N}}$, the latter is $x_1 =_{\mathbb{N}} x_2 =_{\mathbb{N}} x$; we use the axioms for the recursor (that is, we perform a side induction on $x$); firstly, since $Rf_iz_i0 = z_i$ for $i = 1,2$, and $z_1 \,{}^e{=}\, z_2$ by the induction hypothesis, it is $Rf_1z_10 \,{}^e{=}\, Rf_2z_20$; secondly, since $Rf_iz_i(Sx) = f_i(Rf_iz_ix)x$ and $f_1(Rf_1z_1x)x \,{}^e{=}\, f_2(Rf_2z_2x)x$ by the induction and the side induction hypotheses, we obtain $Rf_1z_1(Sx) \,{}^e{=}\, Rf_2z_2(Sx)$.

*Base for ${}^e{=}$.* For variables, it is clear. For constant 0, we have it by the definition of ${}^e{=}_{\mathbb{N}}$ and the fact that $E_{\mathbb{N}}0$ from above. For constant $S$, it is clear by the definition of ${}^e{=}_{\mathbb{N}\to\mathbb{N}}$ that we just need that $E_{\mathbb{N}\to\mathbb{N}}S$, which we showed above. For constant $R$, again we just need $ER$ which we have from above.

*Step for ${}^e{=}$.* For application: let $\vec{x}_1 \,{}^e{=}\, \vec{x}_2$; by the induction hypothesis we have $r(\vec{x}_1) \,{}^e{=}\, r(\vec{x}_2)$ and $s(\vec{x}_1) \,{}^e{=}\, s(\vec{x}_2)$; by Proposition 4.2 we immediately get $r(\vec{x}_1)s(\vec{x}_1) \,{}^e{=}\, r(\vec{x}_2)s(\vec{x}_2)$, that is, $(rs)(\vec{x}_1) \,{}^e{=}\, (rs)(\vec{x}_2)$. For $\lambda$-abstraction: we have to show that $\vec{x}_1 \,{}^e{=}\, \vec{x}_2 \to (\lambda_x\, r)(x,\vec{x}_1) \,{}^e{=}\, (\lambda_x\, r)(x,\vec{x}_2)$; let $\vec{x}_1 \,{}^e{=}\, \vec{x}_2$; by the characterization of Proposition 4.2, if $x$ is of type $\rho$ and $r$ is of type $\sigma$, it is enough to show that $\forall_{x_1,x_2}\left(x_1 \,{}^e{=}_\rho\, x_2 \to (\lambda_x\, r)(x,\vec{x}_1)x_1 \,{}^e{=}_\sigma\, (\lambda_x\, r)(x,\vec{x}_2)x_2\right)$; by $\beta$-reduction this is equivalent to $\forall_{x_1,x_2}\left(x_1 \,{}^e{=}_\rho\, x_2 \to r(x_1,\vec{x}_1) \,{}^e{=}_\sigma\, r(x_2,\vec{x}_2)\right)$; this is immediately provided by the induction hypothesis.

*Step for $E$.* For application: let $E_{\rho\to\sigma}f$ and $E_\rho x$; by Proposition 4.1 we get $x \,{}^e{=}_\rho\, x$ and we still have $E_{\rho\to\sigma}f$; by the definition of $E_{\rho\to\sigma}$ we have $fx \,{}^e{=}_\sigma\, fx$; by the definition of ${}^e{=}_\sigma$ we have $E_\sigma fx$. For $\lambda$-abstraction: let $E_\rho x$ and $r$ be of type $\sigma$; we want to show that $E_{\rho\to\sigma}\lambda_x\, r$, which by the definition of $E_{\rho\to\sigma}$ is $\forall_{x_1,x_2}\left(x_1 \,{}^e{=}_\rho\, x_2 \to (\lambda_x\, r)x_1 \,{}^e{=}_\sigma\, (\lambda_x\, r)x_2\right)$; by $\beta$-reduction this becomes $\forall_{x_1,x_2}\left(x_1 \,{}^e{=}_\rho\, x_2 \to r(x_1) \,{}^e{=}_\sigma\, r(x_2)\right)$, which is immediate by preservation of ${}^e{=}$ for $r$. □

Now let $A$ be a formula. Write $A^E$ for the *extensional translation* of the formula $A$, that is, the formula which results from $A$ after relativizing each of its quantifiers to $E$:

$$\begin{aligned} A^E &:= A \text{ , for } A \text{ prime,} \\ (A \diamond B)^E &:= A^E \diamond B^E \text{ , for } \diamond \in \{\vee, \wedge, \to\}, \\ \left(\exists_x A\right)^E &:= \exists_x \left(Ex \wedge A^E\right) , \\ \left(\forall_x A\right)^E &:= \forall_x \left(Ex \to A^E\right) . \end{aligned}$$

**Proposition 4.4.** *It is $(E_\rho x)^E \leftrightarrow E_\rho x$ and $(x_1 \,{}^e{=}_\rho\, x_2)^E \leftrightarrow x_1 \,{}^e{=}_\rho\, x_2$.*

*Proof by mutual induction on the type.* Base: Immediate from the definition of $E_{\mathbb{N}}$ and ${}^{e}{=}_{\mathbb{N}}$. Step for $E$: Let $(E_{\rho\to\sigma}f)^E$, which by definition means $\left(\forall_{x_1,x_2}\left(x_1 \,{}^{e}{=}_{\rho}\, x_2 \to fx_1 \,{}^{e}{=}_{\sigma}\, fx_2\right)\right)^E$; by the definition of the extensional translation, this is equivalent to $\forall_{x_1,x_2}\left(E_\rho x_1 \wedge E_\rho x_2 \to (x_1 \,{}^{e}{=}_{\rho}\, x_2)^E \to (fx_1 \,{}^{e}{=}_{\sigma}\, fx_2)^E\right)$, where the first two clauses are redundant; the induction hypothesis for ${}^{e}{=}$ yields equivalently $\forall_{x_1,x_2}\left(x_1 \,{}^{e}{=}_{\rho}\, x_2 \to fx_1 \,{}^{e}{=}_{\sigma}\, fx_2\right)$, which, by the definition of $E_{\rho\to\sigma}$, is $E_{\rho\to\sigma}f$. Step for ${}^{e}{=}$: Let $(f_1 \,{}^{e}{=}_{\rho\to\sigma}\, f_2)^E$; by the definition of ${}^{e}{=}_{\rho\to\sigma}$, this is equivalent to $\left(E_{\rho\to\sigma}f_1 \wedge E_{\rho\to\sigma}f_2 \wedge \forall_x\left(E_\rho x \to f_1x \,{}^{e}{=}_{\sigma}\, f_2x\right)\right)^E$; by the definition of the extensional translation, this is equivalent to $(E_{\rho\to\sigma}f_1)^E \wedge (E_{\rho\to\sigma}f_2)^E \wedge \forall_x\left(E_\rho x \to f_1x \,{}^{e}{=}_{\sigma}\, f_2x\right)$; by the step for $E$, we get the equivalent $E_{\rho\to\sigma}f_1 \wedge E_{\rho\to\sigma}f_2 \wedge \forall_x\left(E_\rho x \to f_1x \,{}^{e}{=}_{\sigma}\, f_2x\right)$, which is by definition equivalent to $f_1 \,{}^{e}{=}_{\rho\to\sigma}\, f_2$. □

**Proposition 4.5.** *Point-wise equality is equivalent to extensional equality up to extensional translation, that is,*

$$\left(E_\rho x_1 \to E_\rho x_2 \to (x_1 =_\rho x_2)^E\right) \leftrightarrow x_1 \,{}^{e}{=}_{\rho}\, x_2 \; .$$

*Proof by induction on the type.* Base: Immediate. Step: Let $E_{\rho\to\sigma}f_1 \wedge E_{\rho\to\sigma}f_2 \wedge (f_1 =_{\rho\to\sigma} f_2)^E$; by the definition of $=_{\rho\to\sigma}$ we have $E_{\rho\to\sigma}f_1 \wedge E_{\rho\to\sigma}f_2 \wedge (\forall_x f_1x =_\sigma f_2x)^E$; by the definition of the extensional translation we have $E_{\rho\to\sigma}f_1 \wedge E_{\rho\to\sigma}f_2 \wedge \forall_x\left(E_\rho x \to (f_1x =_\sigma f_2x)^E\right)$; by preservation of extensionality (Proposition 4.3) we can write $E_{\rho\to\sigma}f_1 \wedge E_{\rho\to\sigma}f_2 \wedge \forall_x\left(E_\rho x \to E_\sigma f_1x \wedge E_\sigma f_2x \wedge (f_1x =_\sigma f_2x)^E\right)$; by the induction hypothesis for type $\sigma$ we get $E_{\rho\to\sigma}f_1 \wedge E_{\rho\to\sigma}f_2 \wedge \forall_x\left(E_\rho x \to f_1x \,{}^{e}{=}_{\sigma}\, f_2x\right)$, which by the definition of ${}^{e}{=}_{\rho\to\sigma}$ is precisely $f_1 \,{}^{e}{=}_{\rho\to\sigma}\, f_2$. □

**Theorem 4.6** (Elimination of extensionality)**.** *The formula $A(\vec{x})$ is derivable within* E-HA$^\omega$ *from assumptions $u_i : A_i(\vec{x}_i)$ if and only if the formula $E(\vec{x}) \to A^E(\vec{x})$ is derivable within* HA$^\omega$ *from assumptions $u_i^E : E\vec{x}_i \to A_i^E(\vec{x}_i)$.*

*Proof by induction on the calculus.* We argue informally for the right ("only-if") direction, which is the most important.

*Leaf cases*. For assumptions it is trivial. Moreover, since extensional translation affects only those formulas where quantification appears, it is trivial to prove the axioms $\vee^+$, $\vee^-$, $\wedge^+$, $\wedge^-$, $\bot^-$, as well as the ones for the recursor and $\beta$-reduction. Then, the axioms of equality and the successor deal with prime formulas and are also trivial.

For exists introduction: For a formula $A(x,\vec{x})$, we have to show that $Ex \wedge E\vec{x} \to (A \to \exists_x A)^E$; use $\exists^+$ for the formula $Ex \wedge A^E$.

For exists elimination: For formulas $A(x,\vec{x})$ and $B(\vec{y})$ with $x$ not among $\vec{y}$, we have to show that $Ex \wedge E\vec{x} \wedge E\vec{y} \to ((\forall_x A \to B) \to \exists_x A \to B)^E$; use $\exists^-$ for the formulas $Ex \wedge A^E$, $B^E$.

For induction axioms: For a formula $A(x,\vec{x})$, we have to show that $Ex \wedge E\vec{x} \to (A(0,\vec{x}) \to \forall_x (A(x,\vec{x}) \to A(Sx,\vec{x})) \to \forall_x A(x,\vec{x}))^E$; use $\mathrm{Ind}_{\mathbb{N}}$ for the formula $Ex \to A(x,\vec{x})$.

For extensionality axioms: We have to show that $E_\rho x_1 \wedge E_\rho x_2 \wedge E_{\rho\to\sigma}f_1 \wedge E_{\rho\to\sigma}f_2 \to (x_1 =_\rho x_2 \to f_1x_1 =_\sigma f_2x_2)^E$, that is, $E_\rho x_1 \wedge E_\rho x_2 \wedge E_{\rho\to\sigma}f_1 \wedge E_{\rho\to\sigma}f_2 \to (x_1 =_\rho x_2)^E \to (f_1x_1 =_\sigma f_2x_2)^E$; by preservation of extensionality (Proposition 4.3) we get $Ex_1 \wedge Ex_2 \wedge E_\sigma f_1x_1 \wedge E_\sigma f_2x_2 \to (x_1 =_\rho x_2)^E \to (f_1x_1 =_\sigma f_2x_2)^E$; by Proposition 4.5, we get $x_1 \,{}^{e}{=}_{\rho}\, x_2 \to f_1x_1 \,{}^{e}{=}_{\sigma}\, f_2x_2$, or, equivalently by Proposition 4.2, $f_1 \,{}^{e}{=}_{\rho\to\sigma}\, f_2$, which is an atomic formula within HA$^\omega$.

*Step cases.* For arrow introduction: Let $E\vec{y} \to B^E(\vec{y})$, derived from assumption $E\vec{x} \to A^E(\vec{x})$, be the premise; we want to derive $E\vec{x} \wedge E\vec{y} \to A^E(\vec{x}) \to B^E(\vec{y})$; so suppose $E\vec{x}$, $E\vec{y}$, $A^E$ and show $B^E$; use $\to^+$ for the formulas $A^E$ and $B^E$.

For arrow elimination: Let $E\vec{x} \wedge E\vec{y} \to A^E(\vec{x}) \to B^E(\vec{y})$ and $E\vec{x} \to A^E(\vec{x})$ be the premise; we want to derive $E\vec{y} \to B^E(\vec{y})$; so suppose $E\vec{y}$ and show $B^E(\vec{y})$; use $\to^-$ for the formulas $A^E \to B^E$ and $A^E$.

For forall introduction: Let $Ex \wedge E\vec{x} \to A^E(x,\vec{x})$ be the premise; we want to derive $E\vec{x} \to \forall_x \left(Ex \to A^E\right)$; so suppose $E\vec{x}$ and show $\forall_x \left(Ex \to A^E\right)$; use $\forall^+$ for the formula $Ex \to A^E(x,\vec{x})$.

For forall elimination: Let $E\vec{x} \to \forall_x \left(Ex \to A^E(x,\vec{x})\right)$ and $r(\vec{y})$ be the premise, where $r$ is a term of the same type as $x$; we want to derive $E\vec{x} \wedge E\vec{y} \to A^E[x := r]$; so suppose $E\vec{x}$ and $E\vec{y}$ and show $A^E[x := r]$; use $\forall^-$ for the formula $A^E(x,\vec{x})$ and the variable $x$, where extensionality is preserved by Proposition 4.3. □

## 4.3 Notes

### Dues

The arguments of this chapter were devised under the guidance of Helmut Schwichtenberg in 2006, and carried through in juxtaposition to the exposition of Horst Luckhardt in [28] and its simplification by Ulrich Kohlenbach in an earlier draft of [22]; both of these are based on Robin Gandy's [12], while Luckhardt further reports that elimination of extensionality by relativization is to be found in Gaisi Takeuti's [53] (earlier than Gandy's seemingly independent effort) as well as in Kurt Schütte's [43]. In another direction, elimination of extensionality in Martin-Löf type theory is the subject of Martin Hofmann's [15] (see also [16]).

### Outlook

Elimination of extensionality as we presented it will find its place within TCF$^+$ as soon as the latter matures enough to formally encompass Heyting arithmetic; an embedding of Heyting arithmetic into TCF$^+$ is a straightforward and early question to pursue while developing TCF$^+$.

# Appendix A

# Some domain theory

We collect here notions and statements from elementary domain theory which are used in the text as known facts. For details one can look into [52, 1, 2].

## Preordered and ordered sets

Let $T$ be a set with equality[1] and $\geq$ a binary relation on $T$. The relation $\geq$ is *well-founded* if there is no infinite sequence of the sort $x_0 \geq x_1 \geq \cdots$ in $T$. It is *finitely branching* if the class $\{y \in T \mid x \geq y\}$ is finite for every $x \in T$. Say that $y \in T$ is an *immediate successor* of $x \in T$ if it is

$$x \geq y \wedge \underset{w\in T}{\forall} ((x \geq w \rightarrow y \geq w) \wedge (w \geq y \rightarrow w \geq x)) \ ,$$

and write $x \geq^1 y$ (note that it is also $x \geq^1 x$). The relation $\geq$ is *finitarily branching* (or *locally finitely branching*) if the set of immediate successors $x_{\geq^1} := \{y \in T \mid x \geq^1 y\}$ is finite for every $x \in T$.

The couple $(T, \geq)$ is a *preordered set* if $\geq$ is reflexive and transitive and a *(partially) ordered set* (or *poset*) if $\geq$ is reflexive, transitive and antisymmetric.

**Fact A.1.** *A preordered set $(T, \geq)$ induces an ordered set $(T/\sim, \geq)$, where*

$$a \sim b \Leftrightarrow a \geq b \wedge b \geq a$$

A *maximal* element $a$ in a preordered set $(T, \geq)$ is such that $\forall_{b\in T} (b \geq a \rightarrow b \equiv a)$; dually, a *minimal* element $a$ is such that $\forall_{b\in T} (a \geq b \rightarrow a \equiv b)$. Denote the set of maximal elements of $T$ by $\mathrm{mxl}\, T$. A *least element* (or *minimum*) $\bot$ is such that $\forall_{a\in T}\, a \geq \bot$. An *upper bound* of a subset $\{a_i\}$ is an element $a \in T$ such that $\forall_i\, a \geq a_i$; $a$ is a *least upper bound* (or *supremum*) if

$$\underset{i}{\forall} a \geq a_i \ \wedge \ \underset{i}{\forall} b \geq a_i \rightarrow b \geq a \ ;$$

write $a = \mathrm{lub}_i\{a_i\}$. Call $a, b \in T$ *consistent* if they have a common upper bound. We remark the following:

1. Least elements of $T$ or least upper bounds of an arbitrary subset of $T$, don't have to exist, but, in case $T$ is an *ordered* set, if they do, they are unique.

[1] Say that a set $T$ is a *set with equality* if it is equipped with an equivalence relation $\equiv_T$.

2. Let $S \subseteq T$, where $T$ is an ordered set. Concerning the cardinality of the set $\operatorname{mxl} S$, we have the following possible cases:
   (a) It may be empty; for example, consider $S$ being an infinite ascending chain.
   (b) It may be finite; this is the case with $S$ being a singleton, or a discrete subset (that is, consisting of mutually incomparable elements), or, say, a finite (therefore finitely branching) tree.
   (c) It may be infinite; for example, take $S$ to be an infinitely branching flat tree.

**Fact A.2.** *Let $(T, \geq)$ be an ordered set, $U \subseteq T$ a subset and $U' \subseteq U$ such that*

$$\underset{a \in U}{\forall} \underset{a' \in U'}{\exists} a' \geq a \ .$$

*The least upper bound of $U$ exists if and only if the least upper bound of $U'$ exists and, if it does,* $\operatorname{lub} U = \operatorname{lub} U'$.

Let $(T, \geq)$ be an ordered set. A subset $U \subseteq T$ is *(upwards) directed* when every two of its elements have a common upper bound in $U$, that is,

$$\underset{a_1, a_2 \in U}{\forall} \underset{a \in U}{\exists} (a \geq a_1 \wedge a \geq a_2) \ .$$

Write $U \in \mathscr{P}_d(T)$ and $U \subseteq^d T$. Further, $U$ is *closed (under the order)* when

$$a \in U \wedge a \geq a' \rightarrow a' \in U \ .$$

**Fact A.3.** *Let $(T, \geq)$ be an ordered set.*

1. *Let $U \subseteq T$. The* closure $\overline{U} := \{a \in T \mid \exists_{a' \in U} a' \geq a\}$ *is the smallest closed set in $(T, \geq)$ that contains $U$.*
2. *For every $a \in T$, the closure $\overline{a}$ is a directed set.*

A pre-ordered set $(T, \geq)$ is *complete* when every subset has a least upper bound, $\delta$*-complete* if every directed subset has a least upper bound, $\omega$*-complete* when it has a least element and every denumerable directed subset has a least upper bound, and *consistently complete* when every consistent pair has a least upper bound; note that, in the case of an ordered set, all of these least upper bounds are unique. Call the relation $\geq$ *well-ordered* if it yields no infinite *descending chains* in $T$, that is,

$$\underset{\{a_n\}_{n \in \mathbb{N}} \subseteq T}{\forall} \left( \underset{n \in \mathbb{N}}{\forall} a_n \geq a_{n+1} \rightarrow \underset{n_0 \in \mathbb{N}}{\exists} \underset{n \geq_{\mathbb{N}} n_0}{\forall} a_n \equiv a_{n_0} \right) \ .$$

**Fact A.4.** *An ordered set $(T, \geq)$ with a least element is $\omega$-complete if and only if every increasing sequence in it has a least upper bound.*

An *order-preserving mapping* $f$ from an ordered set $(T, \geq)$ to an ordered set $(T', \geq')$, is a mapping $f : T \rightarrow T'$ for which

$$a \geq b \rightarrow f(a) \geq' f(b)$$

If both $(T, \geq)$, $(T', \geq')$ are complete, $\delta$-complete or $\omega$-complete then an order-preserving mapping from one to the other is *continuous*, $\delta$*-continuous* or $\omega$*-continuous* if it commutes with supremums of subsets, directed subsets or denumerable directed subsets respectively:

$$f(\operatorname{lub} U) = \operatorname{lub} f(U)$$

where $f(U) := \{f(a) \in T' \mid a \in U\}$.

**Fact A.5.** *Let $(T, \geq)$, $(T', \geq')$ be $\omega$-complete ordered sets. An order-preserving mapping $f : T \to T'$ is $\omega$-complete if and only if it commutes with supremums of increasing sequences in $T$.*

## Domains

We restrict our attention to $\delta$-complete ordered sets with least element, which we just call *complete ordered sets*, or *cpo's*, and we interpret $a \geq b$ as "$a$ is approximated by $b$". For a detailed exposition we refer to [52, §§3.1–2].

Let $D = (T, \geq, \bot)$ be a cpo. Define its *compact* (or *finite*) elements by

$$a \in D_c :\equiv \underset{U \subseteq^d T}{\forall} \left( \operatorname{lub} U \geq a \to \underset{b \in U}{\exists} b \geq a \right) .$$

So compact elements are those which, whenever they approximate the least upper bound of a directed set $U$, they do so with a witness $b \in U$. Let $\operatorname{apx}(a)$ denote the set of compact approximations of $a$, that is, $\operatorname{apx}(a) :\equiv \{b \in D_c \mid a \geq b\}$. The cpo $D$ is called *algebraic* if every element $a \in T$ can be characterized by its compact approximations, that is, if

$$\underset{a \in T}{\forall} (\operatorname{apx}(a) \in \mathscr{P}_d(T) \wedge a = \operatorname{lub} \operatorname{apx}(a)) .$$

Finally, an algebraic cpo $D = (T, \geq, \bot)$ is called a *(Scott-Ershov) domain* when every pair of consistent compact elements has a least upper bound (not necessarily compact itself), that is, when

$$\underset{a,b \in D_c}{\forall} \left( \underset{c \in T}{\exists} (c \geq a \wedge c \geq b) \to \operatorname{lub}\{a, b\} \in T \right) .$$

**Fact A.6.** *Let $D = (T, \geq, \bot)$ be an algebraic cpo. The following hold.*

1. *For all $a, b \in T$ it is $a \geq b$ if and only if* $\operatorname{apx}(a) \geq \operatorname{apx}(b)$.
2. *If $U \subseteq^d T$ then* $\operatorname{apx}(\operatorname{lub} U) = \bigcup_{a \in U} \operatorname{apx}(a)$.

**Fact A.7.** *Let $D = (T, \geq, \bot)$ be an algebraic cpo and $D' = (T', \geq', \bot')$ a cpo. The following hold.*

1. *A mapping $f : D \to D'$ is continuous if and only if $f(a) = \operatorname{lub}\{f(b)\}_{b \in \operatorname{apx}(a)}$, for all $a \in T$. Moreover, every monotone function $f : D_c \to D'$ has a unique continuous extension $\hat{f} : D \to D'$ given by $\hat{f}(a) :\equiv \operatorname{lub}\{f(b)\}_{b \in \operatorname{apx}(a)}$.*
2. *Let $D'$ be algebraic as well and $f : D \to D'$. Then $f$ is continuous if and only if it is monotone and it satisfies the* principle of finite support*, that is,*

$$\underset{a \in D}{\forall} \left( b \in \operatorname{apx}(f(a)) \to \underset{c \in \operatorname{apx}(a)}{\exists} f(c) \geq b \right) .$$

**Fact A.8.** *Let $D$ be a domain.*

1. *If $U \subseteq^f D_c$ is* consistent*, that is, if it has an upper bound, then it has a compact least upper bound* $\operatorname{lub} U \in D_c$.

2. *Every domain is* consistently complete*, that is, every upper bounded set has a least upper bound.*

An *ideal* $u$ in a domain $D = (T, \geq, \perp)$ is a set $u \subseteq D_c$ which contains the least element, is downwards closed and consistently complete, that is, it satisfies the following:

- $\perp \in u$
- $a \in u \wedge a \geq b \to b \in u$
- $\forall_{a,b \in u} \operatorname{lub}\{a, b\} \in u$

Let $\overline{a} := \{b \in D_c \mid a \geq b\}$ and write $\mathsf{Ide}_D$ to denote the set of ideals in $D$.

**Fact A.9** (First Representation Theorem)**.** *The triple* $\overline{D_c} := (\mathsf{Ide}_D, \supseteq, \overline{\perp})$ *constitutes a domain. Moreover, it is* $\overline{D_c} \cong D$*, through the mapping* $u \mapsto \operatorname{lub} u$*.*

The domain $\overline{D_c}$ is called the *ideal completion* of $D_c$.

# Bibliography


[1] Samson Abramsky and Achim Jung. Domain theory. In *Handbook of logic in computer science, Vol. 3*, volume 3 of *Handb. Log. Comput. Sci.*, pages 1–168. Oxford Univ. Press, New York, 1994. (Cited on pages 125 and 135.)

[2] Roberto M. Amadio and Pierre-Louis Curien. *Domains and lambda-calculi*, volume 46 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, Cambridge, 1998. (Cited on page 135.)

[3] Holger Benl, Ulrich Berger, Helmut Schwichtenberg, Monika Seisenberger, and Wolfgang Zuber. Proof theory at work: Program development in the MINLOG system. In W. Bibel and P.H. Schmitt, editors, *Automated Deduction — A Basis for Applications, Volume II Systems and Implementation*. Kluwer, 1998. (Cited on page 103.)

[4] Ulrich Berger. *Totale Objekte und Mengen in der Bereichstheorie*. PhD thesis, Mathematisches Institut der Universität München, 1990. (Cited on page 103.)

[5] Ulrich Berger. Total sets and objects in domain theory. *Ann. Pure Appl. Logic*, 60(2):91–117, 1993. (Cited on page 103.)

[6] Ulrich Berger, Kenji Miyamoto, Helmut Schwichtenberg, and Monika Seisenberger. MINLOG — a tool for program extraction supporting algebras and coalgebras. In Andrea Corradini, Bartek Klin, and Corina Cîrstea, editors, *CALCO*, volume 6859 of *Lecture Notes in Computer Science*, pages 393–399. Springer, 2011. (Cited on page 103.)

[7] Douglas S. Bridges and Luminita S. Vîta. *Apartness and Uniformity*. Theory and Applications of Computability. Springer, 2011. (Cited on page 126.)

[8] Antonio Bucciarelli, Alberto Carraro, Thomas Ehrhard, and Antonino Salibra. On linear information systems. In *LINEARITY*, pages 38–48, 2009. (Cited on page 46.)

[9] Thierry Coquand, Giovanni Sambin, Jan Smith, and Silvio Valentini. Inductively generated formal topologies. *Ann. Pure Appl. Logic*, 124(1-3):71–106, 2003. (Cited on page 113.)

[10] Yuri L. Ershov. Maximal and everywhere-defined functionals. *Algebra and Logic*, 13:210–225, 1974. (Cited on page 102.)

[11] Martín Hötzel Escardó. *PCF extended with real numbers: a domain-theoretic approach to higher-order exact real number computation*. PhD thesis, University of London, Imperial College of Science, Technology and Medicine, Department of Computing, 1997. (Cited on page 34.)

[12] Robin O. Gandy. On the axiom of extensionality—Part I. *J. Symb. Log.*, 21(1):36–48, 1956. (Cited on pages 4 and 133.)

[13] Gerhard Gierz, Karl Heinrich Hofmann, Klaus Keimel, Jimmie D. Lawson, Michael Mislove, and Dana S. Scott. *Continuous Lattices and Domains*, volume 93 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 2003. (Cited on page 20.)

[14] Jean-Yves Girard, Paul Taylor, and Yves Lafont. *Proofs and types*, volume 7 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, Cambridge, 1989. (Cited on page 125.)

[15] Martin Hofmann. Elimination of extensionality in Martin-Löf type theory. In *Types for proofs and programs (Nijmegen, 1993)*, volume 806 of *Lecture Notes in Comput. Sci.*, pages 166–190. Springer, Berlin, 1994. (Cited on page 133.)

[16] Martin Hofmann. *Extensional concepts in intensional type theory*. PhD thesis, University of Edinburgh, 1995. (Cited on page 133.)

[17] William A. Howard. Appendix: Hereditarily majorizable functionals of finite type. In Troelstra [55], pages 454–461. Lecture Notes in Mathematics, Vol. 344. (Cited on page 127.)

[18] Simon Huber, Basil A. Karádais, and Helmut Schwichtenberg. Towards a formal theory of computability. In *Ways of Proof Theory (Pohler's Festschrift)*, pages 257–282. Ontos Verlag, Frankfurt, 2010. (Cited on pages 4, 47, 100, 103, and 127.)

[19] David S. Johnson, Christos H. Papadimitriou, and Mihalis Yannakakis. On generating all maximal independent sets. *Inf. Process. Lett.*, 27(3):119–123, 1988. (Cited on page 101.)

[20] Peter T. Johnstone. *Stone spaces*, volume 3 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1982. (Cited on page 125.)

[21] Stephen C. Kleene. Countable functionals. In *Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting)*, Studies in Logic and the Foundations of Mathematics, pages 81–100, Amsterdam, 1959. North-Holland Publishing Co. (Cited on pages 1 and 102.)

[22] Ulrich Kohlenbach. *Applied proof theory: proof interpretations and their use in mathematics*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2008. (Cited on page 133.)

[23] Georg Kreisel. Wittgenstein's remarks on the foundations of mathematics. *British Journal for the Philosophy of Science*, 9(34):135–158, 1958. (Cited on page x.)

[24] Georg Kreisel. Interpretation of analysis by means of constructive functionals of finite types. In *Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting)*, Studies in Logic and the Foundations of Mathematics, pages 101–128, Amsterdam, 1959. North-Holland Publishing Co. (Cited on pages 1 and 102.)

[25] Lill Kristiansen and Dag Normann. Total objects in inductively defined types. *Archive for Mathematical Logic*, 36:405–436, 1997. (Cited on page 103.)

[26] Thomas S. Kuhn. *The structure of scientific revolutions*. University of Chicago Press, Chicago, 1970. (Cited on page x.)

[27] Imre Lakatos. *Proofs and refutations: the logic of mathematical discovery*. Cambridge University Press, 12th edition, 1976. (Cited on page x.)

[28] Horst Luckhardt. *Extensional Gödel functional interpretation. A consistency proof of classical analysis.* Lecture Notes in Mathematics. 306. Berlin-Heidelberg-New York: Springer-Verlag. VI, 161 p., 1973. (Cited on page 133.)

[29] George Markowsky and Barry K. Rosen. Bases for chain-complete posets. *IBM J. Res. Develop.*, 20(2):138–147, 1976. (Cited on page 125.)

[30] Sara Negri. Continuous domains as formal spaces. *Math. Structures Comput. Sci.*, 12(1):19–52, 2002. (Cited on pages 113, 118, 119, and 125.)

[31] Dag Normann. Computability over the partial continuous functionals. *J. Symbolic Logic*, 65(3):1133–1142, 2000. (Cited on page 34.)

[32] Dag Normann. Applications of the Kleene–Kreisel Density Theorem to Theoretical Computer Science. In S. Barry Cooper, Benedikt Lwe, and Andrea Sorbi, editors, *New Computational Paradigms*, pages 119–138. Springer New York, 2008. (Cited on page 103.)

[33] Erik Palmgren. A note on domain representability and formal topology. Revised version of U.U.D.M. Report 2007:28, Department of Mathematics, University of Uppsala, 2007, 2007. (Cited on pages 113 and 124.)

[34] Iosif Petrakis. The topology of the strong Scott condition. Preprint, 2012. (Cited on page 20.)

[35] Gordon D. Plotkin. LCF considered as a programming language. *Theoret. Comput. Sci.*, 5(3):223–255, 1977/78. (Cited on pages 34 and 47.)

[36] Gordon D. Plotkin. $T^{\omega}$ as a universal domain. *J. Comput. System Sci.*, 17(2):209–236, 1978. (Cited on page 125.)

[37] Giovanni Sambin. Intuitionistic formal spaces — a first communication. In *Mathematical logic and its applications (Druzhba, 1986)*, pages 187–204. Plenum, New York, 1987. (Cited on page 118.)

[38] Giovanni Sambin. Formal topology and domains. In *Electronic notes in theoretical computer science, 35 (Remagen-Rolandseck, 1998)*, volume 35 of *Electron. Notes Theor. Comput. Sci.*, page 14 pp. (electronic). Elsevier, Amsterdam, 2000. (Cited on pages 105 and 113.)

[39] Giovanni Sambin. Some points in formal topology. *Theoret. Comput. Sci.*, 305(1-3):347–408, 2003. Topology in computer science (Schloß Dagstuhl, 2000). (Cited on pages 113 and 118.)

[40] Giovanni Sambin and Silvia Gebellato. A preview of the basic picture: a new perspective on formal topology. In *Types for proofs and programs (Irsee, 1998)*, volume 1657 of *Lecture Notes in Comput. Sci.*, pages 194–207. Springer, Berlin, 1999. (Cited on page 113.)

[41] Giovanni Sambin, Silvio Valentini, and Paolo Virgili. Constructive domain theory as a branch of intuitionistic pointfree topology. *Theoret. Comput. Sci.*, 159(2):319–341, 1996. (Cited on pages 113 and 125.)

[42] Peter Schuster. Formal Zariski topology: positivity and points. *Ann. Pure Appl. Logic*, 137(1-3):317–359, 2006. (Cited on page 113.)

[43] Kurt Schütte. *Grundlagen der Analysis im Rahmen einer einfachen Typenlogik: Ausz. (mit einigen Erg.) aus d. Vorlesung: Grundlagen d. Math., München, W.S. 1966/67*. 1967. (Cited on page 133.)

[44] Helmut Schwichtenberg. Eine Normalform für endliche Approximationen von partiellen stetigen Funktionalen. In J. Diller, editor, *Logik und Grundlagenforschung, Festkolloquium zum 100. Geburtstag von Heinrich Scholz*, pages 89–95, 1986. (Cited on page 46.)

[45] Helmut Schwichtenberg. Density and choice for total continuous functionals. In *Kreiseliana. About and Around Georg Kreisel*, pages 335–362, 1996. (Cited on page 103.)

[46] Helmut Schwichtenberg. Classifying recursive functions. In E. Griffor, editor, *Handbook of computability theory*, volume 140 of *Stud. Logic Found. Math.*, pages 533–586. North-Holland, Amsterdam, 1999. (Cited on pages 34 and 47.)

[47] Helmut Schwichtenberg. Recursion on the partial continuous functionals. In C. Dimitracopoulos, L. Newelski, D. Normann, and J. Steel, editors, *Logic Colloquium '05*, volume 28 of *Lecture Notes in Logic*, pages 173–201. Association for Symbolic Logic, 2006. (Cited on pages 3, 5, and 46.)

[48] Helmut Schwichtenberg. MINLOG. In Freek Wiedijk, editor, *The Seventeen Provers of the World*, volume 3600 of *Lecture Notes in Computer Science*, pages 151–157. Springer, 2006. (Cited on page 103.)

[49] Helmut Schwichtenberg and Stanley S. Wainer. *Proofs and computations*. Perspectives in Logic. Cambridge University Press, Cambridge, 2012. (Cited on pages 1, 3, 50, 97, and 127.)

[50] Dana S. Scott. A type-theoretical alternative to ISWIM, CUCH, OWHY. *Manuscript*, 1969. (Cited on page 1.)

[51] Dana S. Scott. Domains for denotational semantics. In *Automata, languages and programming (Aarhus, 1982)*, volume 140 of *Lecture Notes in Comput. Sci.*, pages 577–613. Springer, Berlin, 1982. (Cited on page 108.)

[52] Viggo Stoltenberg-Hansen, Ingrid Lindström, and Edward R. Griffor. *Mathematical theory of domains*, volume 22 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 1994. (Cited on pages 103, 113, 114, 115, 117, 118, 119, 125, 135, and 137.)

[53] Gaisi Takeuti. On a generalized logic calculus. *Japanese Journal of Mathematics*, 23:39–96, 1953. Errata: ibid, vol. 24 (1954), 149–156. (Cited on page 133.)

[54] Terese. *Term rewriting systems*, volume 55 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, Cambridge, 2003. (Cited on page 100.)

[55] Anne S. Troelstra, editor. *Metamathematical investigation of intuitionistic arithmetic and analysis*. Springer-Verlag, Berlin, 1973. Lecture Notes in Mathematics, Vol. 344. (Cited on pages 127 and 139.)

[56] Wolfgang Wechler. *Universal algebra for computer scientists*, volume 25 of *EATCS Monographs on Theoretical Computer Science*. Springer-Verlag, Berlin, 1992. (Cited on page 2.)

[57] Glynn Winskel. Event structures. In *Advances in Petri nets 1986, Part II (Bad Honnef, 1986)*, volume 255 of *Lecture Notes in Comput. Sci.*, pages 325–392. Springer, Berlin, 1987. (Cited on page 126.)

[58] Ludwig Wittgenstein. *Bemerkungen über die Grundlagen der Mathematik*. Suhrkamp, Frankfurt am Main, 1984. (Cited on page x.)

# Index