Differentiability of Loeb Measures and Applications

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Für

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Abstract

In this thesis a concept for differentiability especially of Loeb measures will be developed. A theory of differentiability of standard measures had been already introduced by S.V. Fomin in 1966 and has been further developed by a number of mostly Russian mathematicians. Some basic results of this extensive theory that are essential for the understanding of our concept and its applications will be stated at the beginning of this work, mostly modified in a way that makes them suitable for the application of the principles of nonstandard analysis.

Internal measures of a nonstandard model of mathematics build the basis for Loeb measures which are defined on specially rich σ-fields. We introduce different forms of differentiability for internal measures and discuss the resulting questions. The achieved insights about internal measures will be used later to prove a number of results concerning Loeb measures. The main result of this thesis is theorem 11.3, which shows the very general assumptions that lead to one of the strongest forms of differentiability - the so-called Fomin-differentiability - for Loeb measures.

Studies of the differentiability properties of the Loeb measure produced by an internal Gaussian measure on an internal Euclidian space lead to a number of new results in context with the corresponding measure space. Thereby the operators resulting from the differentiability of this measure - a kind of Malliavin derivative as well as a form of the Skorokhod integral - will be discussed.

Hereby the consequences for an image measure of Γ_L defined on a standard σ-field enable - among other things - a new and obvious proof for the fact that the classical Wiener space is in particular an abstract Wiener space.

Zusammenfassung

In der vorliegenden Arbeit wird ein Konzept der Differenzierbarkeit speziell von Loeb Maßen entwickelt. Eine Theorie der Differenzierbarkeit von standard Maßen wurde bereits 1966 von S.V. Fomin eingeführt und seither von einer großen Anzahl vorwiegend russischer Mathematiker weiterentwickelt. Einige grundlegende Inhalte dieser sehr umfangreichen Theorie, die für das Verständnis unseres Konzepts und seiner Anwendungen unerlässlich sind, werden zu Beginn der Arbeit aufgezeigt, größtenteils so modifiziert, dass sie sich für die Anwendung der Grundprinzipien der Nonstandard Analysis eignen.

Interne Maße eines Nonstandard Modells der Mathematik sind die Grundlage für die auf besonders reichhaltigen σ-Algebren definierten Loeb Maße. Wir führen verschiedene Formen der Differenzierbarkeit für interne Maße ein und diskutieren die sich daran anknüpfenden Fragestellungen. Die dabei gewonnenen Erkenntnisse über interne Maße werden im folgenden genutzt, um eine Reihe von Ergebnissen über Loeb Maße zu beweisen. Das entscheidende Resultat dieser Arbeit ist Theorem 11.3, in dem gezeigt wird, unter welchen sehr allgemeinen Voraussetzungen eine der stärksten Formen der Differenzierbarkeit, die sogenannte Fomin-Differenzierbarkeit, für Loeb Maße folgt.

Untersuchungen über die Differenzierbarkeitseigenschaften des Loeb Maßes Γ_L, das durch ein internes Gauß Maß Γ auf einem internen euklidischen Raum erzeugt wird, führen zu einer Reihe von neuen Ergebnissen im Zusammenhang mit dem entsprechenden Maßraum. Dabei werden auch die sich aus der Differenzierbarkeit dieses Maßes ergebenden Operatoren - eine Form der Malliavin Ableitung sowie eine Form des Skorokhod Integrals - diskutiert.

Die sich daraus ergebenden Folgerungen für ein Bildmaß von Γ_L, das auf einer standard σ-Algebra definiert ist, führen unter anderem zu einem neuen eleganten Beweis für die Tatsache, dass der klassische Wiener Raum die Bedingungen für einen abstrakten Wiener Raum erfüllt.

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Introduction

The theory of differentiable measures on infinite dimensional spaces, introduced by Fomin [15] in 1966 (see Bogachev [9] for more details), has become the foundation for many applications in different fields such as quantum field theory (see e.g. Kirillov [22] or Smolyanov and Weizsäcker [34]) or stochastic analysis (see e.g. Bogachev [9], [7], [8] and Smolyanov and Weizsäcker [35]).

Differentiability (in the sense mentioned above) for Loeb measures has not been studied previously as far as we are aware. In this thesis we present its foundation and some basic results. As application we investigate differentiability properties of a special Loeb measure Γ_L induced by an internal Gaussian measure Γ on $\mathbb{F}^H$, where $\mathbb{F}$ is an internal Euclidian space and H an infinite natural number. This Loeb measure is introduced by H. Osswald in his approach to Malliavin calculus in abstract Wiener spaces [30], generalizing Cutland's and Ng's construction of Brownian motion [12].

The thesis is organized as follows: In the first sections we try to give an impression of the richness of the (standard) theory of differentiable measures by presenting some of the basic ideas, referring mainly to the works of Weizsäcker, Smolyanov and Bogachev. We pay particular attention to Gaussian measures as they are differentiable and the space of differentiability coincides with the characteristic Cameron-Martin space.

Since we want to obtain results for Loeb measures, in Chapter 7 to 10 we provide and discuss natural and very general assumptions for the underlying internal measures, starting up from the questions dealt with by the standard literature. The arising results for the Loeb measures - in particular a powerful theorem for the case of Fomin-differentiability - are presented and discussed in Chapter 11. Most of the ideas of these sections have already been published in [1].

Chapter 12 contains several lifting results we need for our studies of the Loeb measure Γ_L introduced in Chapter 13. This section also presents an image measure W of Γ_L defined on the Borel subsets of the space $C_{\mathbb{B}}$ of continuous functions $f : [0;1] \to \mathbb{B}$, $f(0) = 0$, where $\mathbb{B}$ denotes a separable Banach space. The construction of both measure spaces is derived from Osswald's approach. In the following sections we present nonstandard and standard representations of the dual $C'_{\mathbb{B}}$ and of a subspace $C_{\mathbb{H}}$ of $C_{\mathbb{B}}$ that later turns out to be the Cameron-Martin subspace $H(W)$. We show that the measure W is a Gaussian measure and generalize Cutland's nonstandard representation of Wiener integrals in [11].

Chapter 21 takes up again our main theme of measure differentiability. We show the Fomin-differentiability of the Loeb measure Γ_L. This yields the Fomin-differentiability of the image measure W along the elements of $C_{\mathbb{H}}$. At the end of our thesis we present the well known - but not trivial - standard result that the pair $(C_{\mathbb{B}}; C_{\mathbb{H}})$, and therefore in particular the classical Wiener space, is an abstract Wiener space. This follows from our previous sections.

Describing the standard theory of differentiable measures and sketching Osswald's approach many nice results of excelent works will be cited. Although we tried to present a well comprehensible text in these sections we occasionally omitted proofs, especially if one can find them in important books like those of Ash [5], Bogachev [10], Diestel and Uhl [13], Heuser [19], Kuo [24] or Osswald [30].

1 Some Definitions, Notations and Basics

We indicate here some definitions, notations and basic facts that will frequently be used throughout this work. For sets A and B we write $A \subset B$ if A is a subset of B, where the inclusion needs not to be strict. $A \cup B$ denotes the union of A and B, $A \cap B$ the intersection. $A \setminus B$ denotes the set of all elements of A, that do not belong to B. Let $\mathbb{N} := \{1, 2, \ldots\}$. The set of real numbers is denoted by $\mathbb{R}$, $\mathbb{R}^n$ denotes n-dimensional Euclidian space and $\mathbb{R}^\infty$ the set of all sequences $(a_n)_{n \in \mathbb{N}}$ with $a_n \in \mathbb{R}$. For $s, t \in \mathbb{R}$, $s < t$, we define the intervals $[s;t] := \{x \in \mathbb{R} : s \leq x \leq t\}$ and $]s;t[:= \{x \in \mathbb{R} : s < x < t\}$.

Let Ω be any set and $\mathcal{F}$ a σ-field on Ω, then $(\Omega, \mathcal{F})$ is called a **measurable space**. If Ω is a topological space we denote by $\mathfrak{b}_\Omega$ the Borel σ-field on Ω. Given any measurable space $(\Omega, \mathcal{F})$, a **measure** ν on Ω denotes in this work always a real-valued, countably additive function on $\mathcal{F}$. The triple $(\Omega, \mathcal{F}, \nu)$ is called **measure space**. Note that a measure may take on negative values. For $B \in \mathcal{F}$ set

$$\nu^+ (B) := \sup \{\nu (A) : A \in \mathcal{F}, A \subset B\},$$

$$\nu^- (B) := -\inf \{\nu (A) : A \in \mathcal{F}, A \subset B\}.$$

By the Jordan-Hahn decomposition theorem (see Ash [5]) ν^+ and ν^- are measures on $\mathcal{F}$ and $\nu = \nu^+ - \nu^-$. We define the **norm of total variation** of the measure ν as

$$\|\nu\| := \nu^+ (\Omega) + \nu^- (\Omega).$$

Let $(\Omega, \mathcal{F})$ be a measurable space, μ and ν two measures on $\mathcal{F}$ such that ν is nonnegative. We say that μ is **absolutely continuous with respect to** ν if $\nu (B) = 0$ implies $\mu (B) = 0$ for all $B \in \mathcal{F}$. In this case we write $\mu \ll \nu$. If $\mu \ll \nu$, by the Radon-Nikodym theorem (see Ash [5]) there exists the Radon-Nikodym derivative, i.e. a ν-integrable function $\xi : \Omega \to \mathbb{R}$ so that for all $B \in \mathcal{F}$

$$\mu (B) = \int_B \xi (\omega) \, d\nu (\omega).$$

We will often use the term $\frac{d\mu}{d\nu}$ instead of ξ.

Let $(\Omega, \mathcal{F}, \nu)$ and $(\Omega', \mathcal{F}', \nu')$ be two measure spaces. We denote by $\mathcal{F} \otimes \mathcal{F}'$ the product σ-field of $\mathcal{F}$ and $\mathcal{F}'$ and by $\nu \otimes \nu'$ the product measure of ν and ν'.

Let $(\Omega, \mathcal{F}, \nu)$ be a measure space such that ν is nonnegative. If a condition holds outside of a set $B \in \mathcal{F}$ with $\nu (B) = 0$ we say that the condition holds

ν**-almost surely** or for ν**-almost all** $\omega \in \Omega$. Instead of ν-almost surely and ν-almost all we simply write ν**-a.s.** and ν**-a.a**.

We will use the following theorem due to Nikodym (see Dunford and Schwartz [14], Section III.7.4, Corollary 4) :

1.1 Proposition

Let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of measures on a σ-field $\mathcal{F}$. If the limit

$$\nu(B) := \lim_{n\to\infty} \nu_n(B)$$

exists for each $B \in \mathcal{F}$, then ν is a measure and the σ-additivity of the ν_n is uniform in n.

Given any locally convex space E, let E' denote the **topological dual** of E, i.e. the space of all real-valued, linear and continuous functions on E. Sometimes we say **dual space** instead of topological dual. Let $E^\star$ denote the **algebraic dual** of E, i.e. the space of all real-valued, linear functions on E. If $\mathbb{B}$ is a Banach space, we denote the norm by $|\cdot|$ or $|\cdot|_{\mathbb{B}}$ or $|\cdot|$ additionally provided with another characteristic index. The topological dual $\mathbb{B}'$ equipped with the norm

$$|\varphi|_{\mathbb{B}'} := \sup\{\varphi(x) : x \in \mathbb{B}, |x|_{\mathbb{B}} \le 1\}$$

is itself a Banach space. Given any Hilbert space $\mathbb{H}$ we denote the inner product by $<;>_{\mathbb{H}}$ and the corresponding norm by $\|\cdot\|_{\mathbb{H}}$. If $\mathbb{H}$ is understood, we simply write $<;>$ and $\|\cdot\|$. By the Riesz representation theorem (see Heuser [18]), we can identify the dual space $\mathbb{H}'$ with $\mathbb{H}$. Let $x, y \in \mathbb{H}$. Then x is called **orthogonal** to y if $\langle x; y\rangle = 0$. In this case we write $x \perp y$. Fix $F, G \subset \mathbb{H}$. We define

$$x \perp F :\Leftrightarrow \text{ if } x \perp z \text{ for all } z \in F,$$

$$G \perp F :\Leftrightarrow \text{ if } x \perp F \text{ for all } x \in G,$$

$$F^\perp := \{x \in \mathbb{H} : x \perp F\}.$$

We further denote by **span** F the linear subspace of $\mathbb{H}$ generated by F. Let F be a finite dimensional subspace of $\mathbb{H}$. By the projection theorem (see Heuser [18], 22.1), each $x \in \mathbb{H}$ can be uniquely composed into a sum $x = y_x + z_x$, where $y_x \in F$

and $z_x \in F^{\perp}$. The mapping

$$pr_F^{\mathbb{H}} : \mathbb{H} \to F,\, x \mapsto y_x$$

is called **orthogonal projection** from $\mathbb{H}$ onto F. For a subset $B \subset F$ we define

$$B + F^{\perp} := \left\{ x \in \mathbb{H} : x = y_x + z_x \text{ where } \ y_x \in B \text{ and } z_x \in F^{\perp} \right\}.$$

Let $(\Omega, \mathcal{F}, \nu)$ be a measure space. For $B \in \mathcal{F}$ we denote by 1_B the indicator function. Finite linear combinations of indicator functions are called **simple functions**. Let $p \in \mathbb{N}$. Then $L^p(\Omega, \nu)$ denotes the Hilbert space of all Borel measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\int_{\Omega} |f(\omega)|^p \, d\nu(\omega) < \infty,$$

where we identify two Borel measurable functions f and g with $f = g$ $\nu-$a.s. A family $\mathfrak{F} \subset L^1(\Omega, \nu)$ is said to be **uniformly integrable** if

$$\lim_{k \to \infty} \sup_{\{f \in \mathfrak{F}\}} \int_{\{|f| \geq k\}} |f| \, d\nu = 0.$$

We will use the following applications of uniform integrability which are modifications of Theorem 7.5.2 and Theorem 7.5.3 in Ash [5], but they can be proved similarly.

1.2 Proposition

Let $\varepsilon > 0$, $(f_t)_{-\varepsilon < t < \varepsilon}$ be a uniformly integrable family of Borel measurable functions on Ω, such that for all $\omega \in \Omega$ the limit

$$\lim_{t \to 0} f_t(\omega) =: f(\omega)$$

exists. Then f is integrable and

$$\lim_{t \to 0} \int_{\Omega} f_t(\omega) \, d\nu(\omega) = \int_{\Omega} f(\omega) \, d\nu(\omega).$$

1.3 Proposition

A family $(f_t)_{-\varepsilon<t<\varepsilon}$ of Borel measurable functions on Ω is uniformly integrable if and only if the integrals $\int_\Omega f_t d\nu$ are uniformly bounded and

$$\sup_{t\in]-\varepsilon;\varepsilon[} \int\limits_B f_t(\omega)\, d\nu(\omega) \to 0,$$

as $\nu(B) \to 0$.

Given any vector space Ω, a function $g : \Omega \to \mathbb{R}$ is called **Gateaux differentiable** in the direction of a vector $h \in \Omega$, if for each $\omega \in \Omega$ the limit

$$\lim_{t\to 0} \frac{g(\omega + t \cdot h) - g(\omega)}{t} =: g'(\omega)(h)$$

exists.

Fix any Banach space $(\mathbb{B}, |\cdot|_{\mathbb{B}})$. For a function $f : [0,1] \to \mathbb{B}$ and a partition $P : 0 = t_0 < t_1 < \ldots < t_n = 1$ of $[0;1]$ we define

$$V(P) := \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|_{\mathbb{B}}\,.$$

Let $V(f)$ denote the supremum of $V(P)$ over all partitions of $[0;1]$. $V(f)$ is called the **variation** of f. If $V(f) < \infty$, we say that f is of **bounded variation**.

Let $(\Omega, \mathcal{F}, \nu)$ be a measure space. The Banach space $\mathbb{B}$ shall be equipped with its Borel σ-field. Functions of the form

$$\sum_{i=1}^{k} \alpha_i \cdot 1_{B_i},$$

with $B_i \in \mathcal{F}$ and $\alpha_i \in \mathbb{B}$, are also called **simple functions**. For a simple function the **Bochner integral** is defined by

$$\int\limits_B \left(\sum_{i=1}^{k} \alpha_i \cdot 1_{B_i} \right) d\nu := \sum_{i=1}^{k} \alpha_i \cdot \nu(B_i \cap B)\,.$$

In general, a measurable function $g : \Omega \to \mathbb{B}$ is called **Bochner integrable** if there exists a sequence $(g_n)_{n\in\mathbb{N}}$ of simple functions $g_n : \Omega \to \mathbb{B}$ such that

$$\lim_{n\to\infty} \int\limits_{\Omega} |g_n - g|_{\mathbb{B}} \, d\nu = 0.$$

In this case the **Bochner integral** $\int_B g d\nu$ is defined for each $B \in \mathcal{F}$ by

$$\int\limits_B g d\nu := \lim_{n\to\infty} \int\limits_B g_n d\nu,$$

where the convergence is in $(\mathbb{B}, |\cdot|_{\mathbb{B}})$.

We will need the following important propositions about Bochner integrable functions.

1.4 Proposition

(Diestel [13], II.2. Theorem 2.) *A measurable function $g : \Omega \to \mathbb{B}$ is Bochner integrable if and only if $|g|_{\mathbb{B}}$ is element of $L^1(\Omega, \nu)$.*

1.5 Proposition

(Diestel [13], II.2. Theorem 9.) *Let $a, b \in \mathbb{R}, a < b$, $g : [a; b] \to \mathbb{B}$ be Bochner integrable with respect to Lebesgue measure λ. Then*

$$g(t) = \lim_{r\to 0} \frac{1}{r} \int\limits_t^{t+r} g(s) ds \quad \textit{for } \lambda - a.a.\ t \in [a; b],$$

where the convergence is in $(\mathbb{B}, |\cdot|_{\mathbb{B}})$.

A Bochner integrable function $g : \Omega \to \mathbb{B}$ is called **square Bochner integrable** if

$$\int\limits_{\Omega} |g|_{\mathbb{B}}^2 \, d\nu < \infty.$$

The set of all square Bochner integrable functions $g : \Omega \to \mathbb{B}$ is denoted by $L^2(\nu, \mathbb{B})$. Let $\mathcal{J}$ be a sub-σ-field of $\mathcal{F}$. The functions $f, g : \Omega \to \mathbb{B}$ shall be Bochner integrable. g is said to be the **conditional expectation** of f relative to

$\mathcal{J}$ if g is $\mathcal{J}$-measurable and

$$\int_B g d\nu = \int_B f d\nu$$

for all $B \in \mathcal{J}$. The next proposition is an application of Jensen's inequality.

1.6 Proposition

(Diestel [13], V.1.Theorem 4.) *Let $f : \Omega \to \mathbb{B}$ be square Bochner integrable, g the conditional expectation of f relative to a sub-σ-field $\mathcal{J}$ of $\mathcal{F}$. Then*

$$\left(\int_\Omega |g|_{\mathbb{B}}^2 \, d\nu\right)^{\frac{1}{2}} \leq \left(\int_\Omega |f|_{\mathbb{B}}^2 \, d\nu\right)^{\frac{1}{2}}.$$

The reader should be familiar with the concept of nonstandard analysis, in particular the Loeb measure construction, presented e.g. by Albeverio et al. [2], Osswald [30] or Loeb [27]. We will denote by * an elementary embedding from the standard model of mathematics into an extended polysaturated model. Let Ω be an internal set. An internal function $F : \Omega \to {}^*\mathbb{R}$ is called **S-bounded** if there exists a number $k \in \mathbb{N}$ such that $|F(x)| \leq k$ for all $x \in \Omega$. For $a, b \in {}^*\mathbb{R}$ we write $a \approx b$ if for all standard $n \in \mathbb{N}$

$$|b - a| < \frac{1}{n}.$$

An element $a \in {}^*\mathbb{R}$ is called **limited** if there exists an $n \in \mathbb{N}$ such that $|a| < n$, otherwise a is called **unlimited**. An element $a \in {}^*\mathbb{R}$ is called **nearstandard** if there is a real number $b \in \mathbb{R}$ with $a \approx b$.

1.7 Proposition

(Osswald [30], Proposition 8.7.1.) *An element $a \in {}^*\mathbb{R}$ is limited if and only if there is a uniquely determined $b \in \mathbb{R}$ with $a \approx b$.*

We denote by *Lim* the set of all limited or - equivalently - all nearstandard numbers. For any $a \in {}^*\mathbb{R}$ we define the **standard part** ${}^\circ a$ of a by

$$
{}^\circ a := \begin{cases} b & \text{if } a \text{ is limited and } b \in \mathbb{R} \text{ with } a \approx b, \\ \infty & \text{if } a \text{ is unlimited and } a > 0, \\ -\infty & \text{if } a \text{ is unlimited and } a < 0. \end{cases}
$$

The **standard part map** st is given by

$$
st : Lim \to \mathbb{R},\ a \mapsto {}^\circ a.
$$

Now we generalize these notations. Fix an infinite integer $H \in {}^*\mathbb{N}$ and define $T := \{1, \ldots, H\}$. In this context st denotes the surjective mapping

$$
st : T \to [0;1]\,,\ n \mapsto {}^\circ\left(\frac{n}{H}\right).
$$

Let ν be the internal counting measure on the internal set ${}^*\mathcal{P}(T)$ of all internal subsets of T, i.e. $\nu(A) = \frac{|A|}{H}$ for all $A \in {}^*\mathcal{P}(T)$, where $|A|$ denotes the *finite number of elements of A. We denote the Loeb-space over $(T, {}^*\mathcal{P}(T), \nu)$ by $(T, L_\nu({}^*\mathcal{P}(T)), \nu_L)$.

1.8 Proposition

(Albeverio et al. [2] or Osswald [30], Lemma 10.5.1.) *A subset $B \subset [0;1]$ is Lebesgue measurable if and only if $st^{-1}[B] \in L_\nu({}^*\mathcal{P}(T))$. In this case*

$$
\lambda(B) = \nu_L\left(st^{-1}[B]\right).
$$

Let $(M;d)$ be a metric space. For $x, y \in {}^*M$ we write $x \approx_M y$ if ${}^*d(x;y) \approx 0$. If M is a normed space, we write also $x \approx_{|\cdot|_M} y$ or simply $x \approx_{|\cdot|} y$. A vector $x \in {}^*M$ is called **nearstandard** if there exists an element $y \in M$ such that ${}^*d(x;y) \approx 0$. Note, that in this case the vector y is uniquely determined. We call y the **standard part** of x and write $y = {}^\circ x$. An internal function $F : {}^*M \to {}^*\mathbb{R}$ is called S**-continuous** if $F(x) \in Lim$ for all $x \in {}^*M$ and $F(x) \approx F(y)$ whenever $x \approx_M y$.

Given any internal measure space $(\Omega, \mathcal{A}, \mu)$ we denote by $(\Omega, L_\mu(\mathcal{A}), \mu_L)$ the associated Loeb space. Let $(\mathbb{B}, |\cdot|_{\mathbb{B}})$ be a Banach space.

An internal, $\mathcal{A}$-measurable function $F : \Omega \to {}^*\mathbb{B}$ is called S_μ**-integrable** if for all unlimited $K \in {}^*\mathbb{N}$

$$\int\limits_{\{|F|_{^*\mathbb{B}} \geq K\}} |F|_{^*\mathbb{B}} \, d\mu \approx 0.$$

This property is equivalent to the condition, that $\int_\Omega |F|_{^*\mathbb{B}} \, d\mu \in Lim$ and for each $\varepsilon \in \mathbb{R}^+$ there exists a $\delta \in \mathbb{R}^+$ such that

$$\int\limits_A |F|_{^*\mathbb{B}} \, d\mu < \varepsilon,$$

for all $A \in \mathcal{A}$ with $\mu(A) < \delta$.

1.9 Proposition

(Osswald [30], Corollary 10.8.2.) *Let $F : \Omega \to {}^*\mathbb{R}_0^+$ be an internal, $\mathcal{A}$-measurable function. F is S_μ-integrable if and only if*

$$^\circ\int\limits_\Omega F d\mu = \int\limits_\Omega {}^\circ F d\mu_L < \infty.$$

We will very often use the following application of the so called "witness criterion" (see Osswald [30], Section 10.11).

1.10 Proposition

Let $F : \Omega \to {}^*\mathbb{R}$ be an internal, $\mathcal{A}$-measurable function. If

$$\int\limits_\Omega F^2 d\mu \in Lim,$$

then F is S_μ-integrable.

For $p \in \mathbb{N}$ set

$$SL^p(\mu) := \left\{F : \Omega \to {}^*\mathbb{B} : |F|^p_{^*\mathbb{B}} \text{ is } S_\mu\text{-integrable}\right\}.$$

The next proposition is a slight modification of a lifting theorem of Anderson [4] and Loeb [27].

1.11 Proposition

Let $F : \Omega \to {}^*\mathbb{B}$ *be an internal,* $\mathcal{A}$*-measurable function.*
(a) If $\int_\Omega |F|^p_{{}^*\mathbb{B}} \, d\mu$ *is limited, then* $|F|$ *is limited* μ_L*-a.s.*
(b) F *belongs to* $SL^p(\mu)$ *if and only if there exists a sequence* $(G_n)_{n\in\mathbb{N}}$ *in* $SL^p(\mu)$ *such that*

$$\int\limits_\Omega |F - G_n|_{{}^*\mathbb{B}} \, d\mu < \frac{1}{n}$$

for each $n \in \mathbb{N}$.

2 Definitions of Measure Differentiability

There are many different notions of measure differentiability. The following two are most common (see e.g. Bogachev [9], Fomin [15] or Skorohod [33]). Let E be a locally convex space equipped with its Borel σ-field $\mathfrak{b}_E$ and y an element of E.
1) The measure ν on $\mathfrak{b}_E$ is called Fomin-differentiable along y if for all $B \in \mathfrak{b}_E$ the limit

$$\lim_{t\to 0} \frac{\nu(B + t \cdot y) - \nu(B)}{t}$$

exists. By Proposition 1.1 the limit is a measure on $\mathfrak{b}_E$.
2) The measure ν on $\mathfrak{b}_E$ is called Skorokhod-differentiable along y, if there exists another measure ν' on $\mathfrak{b}_E$, such that for all continuous real-valued bounded functions g on E

$$\lim_{t\to 0} \frac{\int_E g(x - t \cdot y) d\nu(x) - \int_E g(x) d\nu(x)}{t} = \int_E g(x) d\nu'(x).$$

The relationships between these two approaches were studied amongst others by Averbukh, Smolyanov and Fomin [6] and Bogachev [9]. A very general definition of differentiability of a curve of measures is given by Smolyanov and Weizsäcker in [35].

In this work we use the following modification of Smolyanov's and Weizsäcker's definition that is more suitable to serve as a basis for the definition of differentiability of internal measures and Loeb measures. Let $(\Omega, \mathcal{F})$ be a measurable space, $(\nu_t)_{-\varepsilon<t<\varepsilon}$, $\varepsilon \in \mathbb{R}^+$, a curve of measures on $\mathcal{F}$ and $\mathcal{C}$ a set of $\mathcal{F}$-measurable real-valued bounded functions on Ω. We say that the measure $\nu := \nu_0$ is ***differentiable in*** $]-\varepsilon;\varepsilon[$ ***with respect to the set*** $\mathcal{C}$ if there exists a measure ν' on $\mathcal{F}$, such that for all functions g of $\mathcal{C}$

$$\lim_{t\to 0} \frac{\int_\Omega g d\nu_t - \int_\Omega g d\nu}{t} = \int_\Omega g d\nu'.$$

The measure ν' is called a **derivative measure** or simply a **derivative** of ν. Generally, derivative measures are not uniquely determined.

Note that this definition covers the cases mentioned above, since for a locally convex space Ω and a fixed vector $y \in \Omega$ a curve $(\nu_t)_{-\varepsilon<t<\varepsilon}$ can be defined by $\nu_t(B) = \nu(B + t \cdot y)$ for all $B \in \mathfrak{b}_\Omega$. When choosing $\mathcal{C}$ as the set of all continu-

ous, real-valued, bounded functions we obtain Skorohod-differentiability along y. When choosing $\mathcal{C} = \{1_B : B \in \mathcal{F}\}$, we obtain Fomin-differentiability along y.
We will use the term **Fomin-differentiable** if the differentiability is with respect to $\mathcal{C} = \{1_B : B \in \mathcal{F}\}$ and the term **Skorokhod-differentiable** if the differentiability is with respect to the set of all continuous, real-valued, bounded functions.

Let $(\Omega, \mathcal{F}, \nu)$ be a measure space. If Ω is a vector space, the idea of differentiating along a vector can be extended as follows. Let h be a **vector field**, i.e. a measurable mapping $h : \Omega \to \Omega$. Define transformations T_t for $-\varepsilon < t < \varepsilon$ by

$$T_t : \Omega \to \Omega,\ x \mapsto T_t\,(x) := x - t \cdot h(x).$$

Then a curve $(\nu_t)_{-\varepsilon<t<\varepsilon}$ is given by the image measures $\nu_t := \nu \circ T_t^{-1}$. If ν is differentiable in $]-\varepsilon;\varepsilon[$ with respect to a set $\mathcal{C}$, then ν is also called differentiable **along the vector field** h or simply **along** h**.**

The following integration by parts formula is a modification of a part of Proposition 3 in the article [35] of Smolyanov and Weizsäcker. We adapted the assumptions there to our terminology. Furthermore, we don't need boundedness of the vector field, instead we have different demands on the set $\mathcal{C}$.

2.1 Proposition

Let Ω be a locally convex space, ν a measure on $\mathfrak{b}_\Omega$ and h a vector field on Ω. Suppose $\mathcal{C}$ to be a set of measurable, real-valued, bounded functions on Ω that are Gateaux differentiable in all directions $h(x)$ with $x \in \Omega$. If $g \in \mathcal{C}$ the functions g_s, $s \in \mathbb{R} \setminus \{0\}$, defined by

$$g_s\,(x) := \frac{g(x + s \cdot h(x)) - g(x)}{s},$$

shall be uniformly integrable and $g'(x)(h(x)) := \lim_{s\to 0} g_s\,(x)$. Let $g'(x)(h(x)) := \lim_{s\to 0} g_s\,(x)$. Then the measure ν is differentiable along the vector field h if and only if there is a measure ν' such that for all $g \in \mathcal{C}$

$$\int_\Omega g'(x)(h(x))d\nu\,(x) = -\int_\Omega g(x)d\nu'\,(x)\,.$$

In this case ν' is a derivative of ν.

Proof: Let $g \in \mathcal{C}$. At first, note that for any $t, s \in \mathbb{R} \setminus \{0\}$, $s = -t$, we have

$$\frac{\int_\Omega g d\nu_t - \int_\Omega g d\nu}{t} = \frac{\int_\Omega g \circ T_t d\nu - \int_\Omega g d\nu}{t} =$$

$$\int_\Omega \frac{g(x - t \cdot h(x)) - g(x)}{t} d\nu\,(x) = -\int_\Omega \frac{g(x + s \cdot h(x)) - g(x)}{s} d\nu\,(x) = -\int_\Omega g_s d\nu.$$

Since the functions g_s are uniformly integrable and $\lim_{s\to 0} g_s\,(x) = g'(x)(h(x))$ for each $x \in \Omega$, we can apply Proposition 1.2 to obtain that $x \mapsto g'(x)(h(x))$ is integrable and

$$\lim_{s\to 0} \int_\Omega g_s d\nu = \int_\Omega g'(x)(h(x)) d\nu\,(x)\,.$$

Hence

$$\lim_{t\to 0} \frac{\int_\Omega g d\nu_t - \int_\Omega g d\nu}{t} = -\int_\Omega g'(x)(h(x)) d\nu\,(x)\,.$$

Now if ν is differentiable with derivative ν', then

$$\int_\Omega g d\nu' = \lim_{t\to 0} \frac{\int_\Omega g d\nu_t - \int_\Omega g d\nu}{t} = -\int_\Omega g'(x)(h(x)) d\nu\,(x)\,.$$

On the other side, if the integration by parts formula is fulfilled, then there is a measure ν' with

$$\int_\Omega g d\nu' = -\int_\Omega g'(x)(h(x)) d\nu\,(x) = \lim_{t\to 0} \frac{\int_\Omega g d\nu_t - \int_\Omega g d\nu}{t}. \quad \square$$

If a measure ν is differentiable and a derivative ν' is absolutely continuous with respect to ν, then the Radon-Nikodym derivative $\frac{d\nu'}{d\nu}$ is also called **logarithmic derivative**. The next proposition and its proof stem from Section 2 in the article [35] of Smolyanov and Weizsäcker.

2.2 Proposition

Let $(\Omega, \mathcal{F})$ be a measurable space, $(\nu_t)_{-\varepsilon<t<\varepsilon}$ a curve of nonnegative measures on $\mathcal{F}$. If $\nu = \nu_0$ is Fomin-differentiable in $]-\varepsilon;\varepsilon[$, then the derivative ν' is absolutely continuous with respect to ν.

Proof: Let $\nu(N) = 0$. Consider the function

$$f :]-\varepsilon;\varepsilon[\longrightarrow \mathbb{R}_0^+,\ t \mapsto \nu_t\,(N)$$

Then f is differentiable at $t = 0$ and $f'(0) = \nu'\,(N)$. Since f is nonnegative and $f(0) = 0$, the first derivative of f in 0 must be 0. Hence $\nu'\,(N) = 0$. $\square$

3 Relationships between Different Forms of Differentiability

The relationships between different forms of differentiability are described in great detail in the standard literature. In this section we give a very short account, confining ourselves on results we either need in the following or which gave the inspiration for our contributions to the theory of differentiable Loeb measures. Later in this thesis we will introduce several forms of S-differentiability of internal measures and deal with their relationships. In this section, our aim is to scetch a clear, tightly structured outline as basis for our following explanations.

Throughout this chapter, E is a locally convex space, equipped with its Borel σ-field $\mathfrak{b}_E$, and the differentiability is always along a vector $y \in E$. Hence, for any measure ν, the measure ν_t is given by the shift

$$\nu_t(A) = \nu(A + t \cdot y).$$

The following helpful lemma is a special case of Proposition 2. of Smolyanov and Weizsäcker [35].

3.1 Lemma

Let ν be a measure in $\mathfrak{b}_E$. Suppose ν to be differentiable in $]-\varepsilon;\varepsilon[$ along a vector $y \in E$ with respect to a set $\mathcal{C}$. Let ν' be a derivative of ν. If for all $t \in]-\varepsilon;\varepsilon[$ the set $\mathcal{C}$ coincides with

$$\{g : E \to \mathbb{R} : \text{there is a function } f \in \mathcal{C} \text{ with } g(x) = f(x + t \cdot y) \text{ for all } x \in E\},$$

then all measures ν_t are differentiable along y with respect to $\mathcal{C}$ and the measures $(\nu')_t$ are derivatives.

Proof: Fix $t \in]-\varepsilon;\varepsilon[$. Choose $g, f \in \mathcal{C}$ with $g(x) = f(x + t \cdot y)$. Then

$$\lim_{s\to 0} \frac{\int_E g(x) d\,(\nu_t)_s\,(x) - \int_E g(x) d\nu_t(x)}{s} =$$

$$\lim_{s\to 0} \frac{\int_E g(x - s \cdot y) d\nu_t(x) - \int_E g(x) d\nu_t(x)}{s} =$$

$$\lim_{s\to 0}\frac{\int_E g(x-s\cdot y-t\cdot y)d\nu(x)-\int_E g(x-t\cdot y)d\nu(x)}{s} =$$

$$\lim_{s\to 0}\frac{\int_E f(x-s\cdot y)d\nu(x)-\int_E f(x)d\nu(x)}{s} = \int_E f(x)d\nu'(x) =$$

$$\int_E g(x-t\cdot y)d\nu'(x) = \int_E g(x)d\left(\nu'\right)_t(x). \qquad \square$$

Note that the assumptions of Lemma 3.1 are in particular fulfilled, if ν is Skorokhod-differentiable or Fomin-differentiable along y. The following proposition shows the strength of Fomin-differentiability. It is a special case of a part of Proposition 3 in Smolyanov and Weizsäcker [35]. In Section 8 we state a related result for internal measures.

3.2 Proposition

Let ν be a Borel measure on E. If ν is Fomin-differentiable in $]-\varepsilon;\varepsilon[$ along a vector $y\in E$, then ν is differentiable along y with respect to the set $\mathcal{C}$ of all bounded Borel functions on E.

Proof: (See [35].) We have to show that

$$\lim_{t\to 0}\frac{\int_E gd\nu_t-\int_E gd\nu}{t} = \int_E gd\nu'$$

for all $g\in\mathcal{C}$. Obviously this is true for any simple function g. Now for arbitrary $g\in\mathcal{C}$, there exists a sequence $(g_n)_{n\in\mathbb{N}}$ of simple functions g_n, which converges uniformly to g (see e.g. Ash [5], Section 1.5). Note that for $t\in\,]-\varepsilon;\varepsilon[$ also the sequence $(g_n(x-ty))_{n\in\mathbb{N}}$ converges uniformly in x to $g(x-ty)$. Moreover, since the transformations $T_t: x\mapsto x-ty$ are bijective, for every $\delta>0$, there is an $n_0\in\mathbb{N}$, such that for all $n\geq n_0$, for all $x\in E$ and all $t\in\,]-\varepsilon;\varepsilon[$ the following inequality (+) holds:

$$(+)\qquad |g_n(x-ty)-g(x-ty)|<\delta.$$

Now define for all $n\in\mathbb{N}$

$$f_n: \,]-\varepsilon;\varepsilon[\,\to\mathbb{R},\ t\mapsto\int_E g_n d\nu_t.$$

By Lemma 3.1, f_n is differentiable on $]-\varepsilon;\varepsilon[$ with

$$f_n'(t) = \int_E g_n d\,(\nu_t)' = \int_E g_n(x-ty)d\nu'(x).$$

By the inequality (+), the sequence $(f_n')_{n\in\mathbb{N}}$ converges uniformly on $]-\varepsilon;\varepsilon[$ to a function defined by $]-\varepsilon;\varepsilon[\longrightarrow \mathbb{R}$, $t\mapsto \int_E g(x-ty)d\nu'(x)$. Since

$$\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty}\int_E g_n(x)d\nu(x) = \int_E g(x)d\nu(x),$$

by elementary analysis, we can exchange the limits as follows

$$\lim_{t\to 0}\frac{\int_E g d\nu_t - \int_E g d\nu}{t} = \lim_{t\to 0}\lim_{n\to\infty}\frac{\int_E g_n d\nu_t - \int_E g_n d\nu}{t} =$$

$$\lim_{n\to\infty}\lim_{t\to 0}\frac{\int_E g_n d\nu_t - \int_E g_n d\nu}{t} = \lim_{n\to\infty}\int_E g_n d\nu' = \int_E g d\nu'. \qquad \square$$

We will now show under which assumptions differentiability with respect to a set $\mathcal{C}$ implies Fomin-differentiability. For this purpose we need the following two lemmas, where the first one is obvious. Recall that $\|\nu\| = \nu^+(E)+\nu^-(E)$ is the norm of total variation of a measure ν.

3.3 Lemma

For any measure ν on $\mathfrak{b}_E$ and any $t\in\mathbb{R}$ we have $\|\nu_t\| = \|\nu\|$.

We call a space $\mathcal{C}$ of bounded Borel functions **norm-defining** if for each measure ν on $\mathfrak{b}_E$

$$\|\nu\| = \sup\left\{\int f d\nu \;:\; f\in\mathcal{C} \text{ and } \|f\|_{\sup}\le 1\right\}.$$

Note that for any measure ν being differentiable with respect to a norm-defining set $\mathcal{C}$ the derivative ν' is uniquely determined. Furthermore, we have the following application of Lemma 1.5 in Weizsäcker [37].

3.4 Lemma

Let ν be differentiable in $]-\varepsilon;\varepsilon[$ along a vector $y \in E$ with respect to a norm-defining space $\mathcal{C}$ that coincides with

$$\{g : E \to \mathbb{R} : \textit{there is a function } f \in \mathcal{C} \textit{ with } g(x) = f(x + t \cdot y) \textit{ for all } x \in E\}$$

for all $t \in\,]-\varepsilon;\varepsilon[$. Then $]-\varepsilon;\varepsilon[\, \ni t \mapsto \nu_t$ is continuous with respect to $\|\cdot\|$.

Proof: (In the proof we follow the argumentation in [37].) Choose $g \in \mathcal{C}$ and define $f_g(t) := \int g d\nu_t$. By the assumptions and Lemma 3.1 the function f_g is differentiable on $]-\varepsilon;\varepsilon[$ with

$$\left|f_g'(t)\right| = \left|\int_E g d\left(\nu_t\right)'\right| \leq \|g\|_{\sup} \cdot \left\|\left(\nu_t\right)'\right\| = \|g\|_{\sup} \cdot \|\nu'\|.$$

For $t, s \in\,]-\varepsilon;\varepsilon[$ the mean value theorem yields

$$|f_g(t) - f_g(s)| \leq \|g\|_{\sup} \cdot \|\nu'\| \cdot |t - s|.$$

Since the choice of g was arbitrary and $\mathcal{C}$ is norm-defining we obtain

$$\|\nu_t - \nu_s\| \leq \|\nu'\| \cdot |t - s|,$$

and therefore the desired continuity. □

The next proposition can be seen as application either of Section 4 in Norin [28] or of Proposition 1.6 in Weizsäcker [37].

3.5 Proposition

Let ν be a measure in $\mathfrak{b}_E$, $\varepsilon > 0$, $\mathcal{C}$ be a norm-defining space that coincides with

$$\{g : E \to \mathbb{R} : \textit{there is a function } f \in \mathcal{C} \textit{ with } g(x) = f(x + t \cdot y) \textit{ for all } x \in E\}$$

for all $t \in\,]-\varepsilon;\varepsilon[$. Let ν be differentiable in $]-\varepsilon;\varepsilon[$ with respect to $\mathcal{C}$ along a vector $y \in E$ with derivative ν'. If for each $B \in \mathfrak{b}_E$ the mapping $]-\varepsilon;\varepsilon[\, \ni t \mapsto (\nu')_t\,(B)$ is Lebesgue measurable and if ν' is absolutely continuous with respect to ν then ν is Fomin-differentiable in $]-\varepsilon;\varepsilon[$ along y.

Proof: (The proof is very close to Weizsäcker.) By Lemma 3.1 with ν all measures $\nu_t, t \in \left]-\varepsilon;\varepsilon\right[$, are differentiable with respect to $\mathcal{C}$ with derivatives $(\nu_t)' = (\nu')_t$ and with the assumption each measure $(\nu_t)'$ is absolutely continuous with respect to ν_t. In the first part of the proof we establish a further measure μ on $\mathfrak{b}_E$ and two product measurable functions f and f' on $\left]-\varepsilon;\varepsilon\right[\times E$ such that $\nu_t \ll \mu$ with Radon-Nikodym derivative $f(t,\cdot)$ and $\nu_t' \ll \mu$ with Radon-Nikodym derivative $f'(t,\cdot)$. Furthermore, there shall be the following relationship (++) between f and f':

$$(++) \quad f(b,x) - f(a,x) = \int_a^b f'(t,x)dt \quad \text{for all } a,b \in \left]-\varepsilon;\varepsilon\right[\text{ and all } x \in E.$$

In the second part of the proof we will work with these functions to show Fomin-differentiability.
To define μ, choose an enumeration $(t_n)_{n\in\mathbb{N}}$ of $\mathbb{Q}$, and set

$$\mu := \sum_{n\in\mathbb{N}} \frac{1}{2^n} \cdot \frac{|\nu_{t_n}|}{\|\nu_{t_n}\|} = \sum_{n\in\mathbb{N}} \frac{1}{2^n} \cdot \frac{|\nu_{t_n}|}{\|\nu\|}.$$

Then μ is a measure and $\nu_{t_n} \ll \mu$ for all n. With Lemma 3.4 it is not hard to see that even $\nu_t \ll \mu$ for all t. Now we define a measure n' on the product $\sigma-$field on $\left]-\varepsilon;\varepsilon\right[\times E$ by

$$n'(A,B) := \int_A \nu_t'(B)dt.$$

We show that $n' \ll \lambda \otimes \mu$, where λ denotes Lebesgue measure. Let $\lambda \otimes \mu(N) = 0$, then, by Fubini, $\mu(N_t) = 0$ for $\lambda-$a.a. $t \in \left]-\varepsilon;\varepsilon\right[$, hence also $\nu'(N_t) = 0$ for $\lambda-$a.a. $t \in \left]-\varepsilon;\varepsilon\right[$ and therefore $n'(N) = 0$. Let $\tilde{f}'$ be a Radon-Nikodym derivative $\frac{dn'}{d(\lambda\otimes\mu)}$ of n' with respect to $\lambda \otimes \mu$. Since $\mu \geq 0$, we have $\|\nu_t'\| = \int_E \left|\tilde{f}'(t,x)\right| d\mu$ λ-a.s. By Lemma 3.3, $\|\nu_t'\| = \|\nu'\|$ for all $t \in \left]-\varepsilon;\varepsilon\right[$. Therefore, $\int_a^b \left(\int_E \left|\tilde{f}'(t,\omega)\right| d\mu\right) dt < \infty$ for all $a,b \in \left]-\varepsilon;\varepsilon\right[$. Hence we can define the functions f and f' as follows:

$$f'(t,x) := \begin{cases} 0 & \text{if} \quad x \in \bigcup_{n\in\mathbb{N}\cap]-\varepsilon;\varepsilon[} \left\{\tilde{x} : \int_{-n}^{n} \left|\tilde{f}'(t,\tilde{x})\right| dt = \infty\right\}, \\ \tilde{f}'(x,t) & \text{otherwise.} \end{cases}$$

Of course, also f' is a Radon-Nikodym derivative $\frac{dn'}{d(\lambda\otimes\mu)}$ and for each $t \in \left]-\varepsilon;\varepsilon\right[$ the function $f'(t,\cdot)$ is a Radon-Nikodym derivative $\frac{d\nu'_t}{d\mu}$. Set

$$f(0,x) := \frac{d\nu_0}{d\mu}(x) \quad \text{and} \quad f(t,x) := f(0,x) + \int_0^t f'(s,x)ds.$$

To see that $f(t,\cdot) = \frac{d\nu_t}{d\mu}$ choose an arbitrary function $g \in \mathcal{C}$. Then

$$\int_E \left(g(\omega) \int_0^t f'(s,\omega)ds \right) d\mu(\omega) = \int_{[0,t]\times E} g(\omega)\cdot f'(s,\omega)d\lambda\otimes\mu(s,\omega) =$$

$$\int_0^t \int_E g(\omega)dn'(s,\omega) = \int_0^t \left(\int_E g(\omega)d(\nu')_s(\omega) \right) ds.$$

The differentiability of ν_t and the boundedness of $(\|(\nu')_s\|)_{s\in]-\varepsilon;\varepsilon[}$ imply that $s \mapsto \int_E g(\omega)d(\nu')_s$ is integrable on $\left]-\varepsilon;\varepsilon\right[$. Hence

$$\int_0^t \left(\int_E g d(\nu')_s \right) ds = \int_E g d(\nu_t - \nu_0).$$

Since the choice of g was arbitrary and since $\mathcal{C}$ is norm-defining, we obtain that $\int_0^t f'(s,x)ds = \frac{d(\nu_t-\nu_0)}{d\mu}(x)$, and therefore $\frac{d\nu_t}{d\mu}(x) = \int_0^t f'(s,x)ds + \frac{d\nu_0}{d\mu}(x) = f(t,x)$. Hence the relationship (++) is proved.

Now let us regard the functions $f$and f' as mappings from $\left]-\varepsilon;\varepsilon\right[$ to $L^1(E,\mu)$. Then f' is Bochner integrable and

$$\int_t^{t+r} f'(s)ds = f(t+r) - f(t).$$

By Proposition 1.5,

$$\lim_{r\to 0} \left\| \frac{f(t+r)-f(t)}{r} - f'(t) \right\|_{L^1(E,\mu)} = 0 \quad \lambda\text{-a.s.}$$

Hence

$$\lim_{r\to 0} \left\| \frac{\nu_{t+r}-\nu_t}{r} - \nu'_t \right\| = 0 \quad \lambda\text{-a.s.}$$

Since for each $B \in \mathfrak{b}_E$

$$\left| \frac{\nu_{t+r}(B) - \nu_t(B)}{r} - \nu_t'(B) \right| \leq \left\| \frac{\nu_{t+r} - \nu_t}{r} - \nu_t' \right\|$$

we obtain Fomin-differentiabilty of ν_t for λ-a.a. $t \in]-\varepsilon;\varepsilon[$. By Lemma 3.1 each ν_t is Fomin-differentiable, in particular the measure ν. $\square$

4 Gaussian Measures and their Subspaces of Differentiability

Gaussian measures have a nice differentiability property we will need later. Hence we will describe it in this section. To this end we have to give a short insight into the theory of Gaussian measures on locally convex spaces and their corresponding Cameron-Martin spaces. It is based on the works of Bogachev [10] and Norin [28]. We regard only centered measures, but the results are also true for the general case.

For $n \in \mathbb{N}$ and $\sigma \in \mathbb{R}^+$ we define a Borel measure $\gamma^{n,\sigma}$ on $\mathbb{R}^n$ by

$$\gamma^{n,\sigma}(B) := \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \int_B \exp\left(-\frac{1}{2\sigma^2}\left(x_1^2 + \ldots + x_n^2\right)\right) dx_1 \ldots dx_n.$$

$\gamma^{n,\sigma}$ is called **centered Gaussian measure with variance** σ^2 **on** $\mathbb{R}^n$. The following lemma can be easily proved by induction and integration by parts.

4.1 Lemma

Let $m \in \mathbb{N}$ and $\sigma \in \mathbb{R}^+$. *Then*

$$\int\limits_{\mathbb{R}} x^m d\gamma^{1,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} x^m e^{-\frac{x^2}{2\sigma^2}} dx = \begin{cases} 0 & if\ m\ is\ odd \\ 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (m-1) \cdot \sigma^m & if\ m\ is\ even. \end{cases}$$

The next lemma is a special case of Lemma 2.4.4. in Bogachev [10]. For $t \in \mathbb{R}$ we define the shifted measure $\gamma_t^{1,\sigma}$ by $\gamma_t^{1,\sigma}(B) := \gamma^{1,\sigma}(B+t)$.

4.2 Lemma

Let $\gamma^{1,\sigma}$ be a centered Gaussian measure on $\mathbb{R}$. Then, for any real number $t \in \mathbb{R}$, we have

$$\left\|\gamma^{1,\sigma} - \gamma_t^{1,\sigma}\right\| \geq 2 - 2\exp\left(-\frac{1}{8}\sigma^2 t^2\right).$$

Throughout this chapter, let E be a locally convex space and $\mathfrak{b}_E$ the σ-field of Borel subsets. A probability measure ν in $\mathfrak{b}_E$ is called a **centered Gaussian measure** if for each $\varphi \in E'$ the measure $\nu \circ \varphi^{-1}$ is a centered Gaussian measure on $\mathbb{R}$. Obviously, each measure $\gamma^{n,\sigma}$ on $\mathbb{R}^n$ is a centered Gaussian measure. The following example shows that this also holds on $\mathbb{R}^\infty$.

4.3 Example

(Bogachev [10], 2.3.5.) Regard $\mathbb{R}^\infty$ together with the family $(p_n)_{n\in\mathbb{N}}$ of seminorms where

$$p_n(x) = p_n\left((x_1, x_2, x_3, \ldots, x_n, \ldots)\right) := |x_n| \,.$$

Let γ^∞ be the countable product of $\gamma^{1,1}$. Choose $\varphi \in (\mathbb{R}^\infty)'$. By Proposition 64.10 in Heuser [18] there exists a real $r \in \mathbb{R}^+$ and finitely many $p_{n_1}, \ldots, p_{n_k}$ such that for all $x \in \mathbb{R}^\infty$

$$|\varphi(x)| \leq r \cdot \max_{i\in\{1,\ldots,k\}} p_{n_i}(x) = r \cdot \max_{i\in\{1,\ldots,k\}} |x_{n_i}| \,.$$

Hence $\varphi(x) = 0$ if $x_{n_i} = 0$ for all $i \in \{1.\ldots, k\}$. This and the linearity of φ yield that φ is a linear combination $\sum_{i=1}^k \alpha_i \cdot \varphi_{n_i}$ of the coordinate functions $\varphi_{n_i} : x \mapsto x_{n_i}$. To see that $\gamma^\infty \circ \varphi^{-1}$ is a centered Gaussian measure on $\mathbb{R}$ choose $B \in \mathfrak{b}_{\mathbb{R}}$. Then

$$\gamma^\infty \circ \varphi^{-1}[B] = \gamma^\infty\left(\left\{x \in \mathbb{R}^\infty : \sum_{i=1}^k \alpha_i \cdot x_{n_i} \in B\right\}\right) =$$

$$\gamma^{k,1}\left(\left\{(x_1, \ldots, x_k) \in \mathbb{R}^k : \sum_{i=1}^k \alpha_i \cdot x_i \in B\right\}\right) = \gamma^{k,1} \circ \tilde{\varphi}^{-1}[B,]$$

where $\tilde{\varphi} \in \left(\mathbb{R}^k\right)'$.

To describe the subspaces of differentiability of Gaussian measures we need some properties of these measures, which we present in the following.

4.4 Lemma

If ν is a centered Gaussian measure in $\mathfrak{b}_E$, then $E' \subset L^2(E, \nu)$.

Proof: For $l \in E'$ the measure $\nu \circ l^{-1}$ is a centered Gaussian measure $\gamma^{1,\sigma}$

on $\mathbb{R}$. Hence

$$\int\limits_E l^2 d\nu = \int\limits_{\mathbb{R}} x^2 d\nu \circ l^{-1}(x) = \int\limits_{\mathbb{R}} x^2 d\gamma^{1,\sigma}(x) = \sigma^2,$$

where we have used Lemma 4.1. $\square$

4.5 Proposition

If ν is a centered Gaussian measure in $\mathfrak{b}_E$ and $l \in E'$ then $\exp(|l|)$ is an element of $L^p(E,\nu)$ for all $p \in \mathbb{N}$.

Proof: Firstly, we show that $\exp(l)$ is $\nu-$integrable. Let again $\gamma^{1,\sigma} = \nu \circ l^{-1}$.

$$\int\limits_E \exp(l)\, d\nu = \int\limits_{\mathbb{R}} \exp(x)\, d\gamma^{1,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} \exp(x) \exp\left(-\frac{x^2}{2\sigma^2}\right) dx =$$

$$\frac{1}{\sigma\sqrt{2\pi}} \int\limits_{\mathbb{R}} \exp\left(-\frac{1}{2\sigma^2}(x^2 - 2\sigma^2 x)\right) dx =$$

$$\frac{1}{\sigma\sqrt{2\pi}} \int\limits_{\mathbb{R}} \exp\left(-\frac{1}{2\sigma^2}(x-\sigma^2)^2\right) \exp\left(\frac{\sigma^2}{2}\right) dx = \exp\left(\frac{\sigma^2}{2}\right).$$

Therefore, $\exp(l)$ is ν-integrable. Obviously, also $\exp(-l)$ is ν-integrable. Define

$$A := \{x \in E : l(x) > 0\}.$$

Then $\int_A \exp(l(x))\, d\nu(x) < \infty$ and $\int_{E\setminus A} \exp(-l(x))\, d\nu(x) < \infty$. Thus

$$\int\limits_E \exp(|l(x)|)\, d\nu(x) =$$

$$\int\limits_A \exp(l(x))\, d\nu(x) + \int\limits_{E\setminus A} \exp(-l(x)\, d\nu(x) < \infty.$$

Let $p \in \mathbb{N}$. Then $(\exp(|l|))^p = \exp(p \cdot |l|) = \exp(|p \cdot l|)$. Since $p \cdot l$ is also an element of E', $\exp(|p \cdot l|) \in L^1(E,\nu)$. $\square$

The following two lemmas are due to Bogachev [10]. Let ν be a centered Gaussian measure on E. For $h \in E$ we define the shifted measure ν_h by $\nu_h(A) := \nu(A+h)$.

4.6 Lemma

If $\varphi \in E'$, then

$$\|\nu_h - \nu\| \geq \left\| \left(\nu \circ \varphi^{-1}\right)_{\varphi(h)} - \nu \circ \varphi^{-1} \right\|$$

for all $h \in E$.

Proof: Choose $\varphi \in E'$ and $h \in E$. For any $A \in \mathfrak{b}_{\mathbb{R}}$ set $B_A = \varphi^{-1}(A)$. It's easily verified that $\varphi^{-1}(A + \varphi(h)) = B_A + h$. Hence,

$$(\nu_h - \nu)(B_A) = \left(\left(\nu \circ \varphi^{-1}\right)_{\varphi(h)} - \nu \circ \varphi^{-1} \right)(A).$$

Since the choice of A was arbitrary we obtain that

$$\sup\left\{(\nu_h - \nu)(B) : B \in \mathfrak{b}_E\right\} \geq \sup\left\{ \left(\left(\nu \circ \varphi^{-1}\right)_{\varphi(h)} - \nu \circ \varphi^{-1} \right)(A) : A \in \mathfrak{b}_{\mathbb{R}} \right\}$$

and

$$\inf\left\{(\nu_h - \nu)(B) : B \in \mathfrak{b}_E\right\} \leq \inf\left\{ \left(\left(\nu \circ \varphi^{-1}\right)_{\varphi(h)} - \nu \circ \varphi^{-1} \right)(A) : A \in \mathfrak{b}_{\mathbb{R}} \right\}.$$

Thus the result follows. □

Two measures ν and μ in $\mathfrak{b}_E$ are called ***singular*** if there exists a set $B \in \mathfrak{b}_E$ such that $\nu(B) = 0$ and $\mu(E \setminus B) = 0$.

4.7 Lemma

Let $h \in E$. If $\|\nu_h - \nu\| = 2$, then the measures ν and ν_h are singular.

Proof: The proof uses only elementary measure theory (see for example Ash [5], Section 2). □

Note that for any arbitrary measure ν on E one has the **Fourier transform** $\tilde{\nu}$ defined by

$$\tilde{\nu}(l) = \int_E \exp\left(i \cdot l(x)\right) d\nu(x) \quad \text{for all} \quad l \in E'.$$

The following proposition is well known. It is a special case of Theorem 2.2.4. in Bogachev [10].

4.8 Proposition

A measure ν on E is a centered Gaussian measure if and only if its Fourier transform has the form

$$\tilde{\nu}(l) = \exp\left(-\frac{1}{2}\int\limits_E l^2 d\nu\right) \quad \textit{for all} \quad l \in E'.$$

In the following we assume ν to be a centered Gaussian measure in E. Let us denote by E'_ν the closure of E' in $L^2(E,\nu)$. The next lemma and the idea of its proof are due to Norin [28].

4.9 Lemma

For all $l \in E'$ and $g \in E'_\nu$ we have

$$\int\limits_E i \cdot g(x) \exp\left(i \cdot l(x)\right) d\nu = -\int\limits_E l(x)g(x)d\nu \cdot \exp\left(-\frac{1}{2}\int\limits_E l^2(x)d\nu\right).$$

Proof: Choose arbitrary elements $l, g \in E'$ and $t \in \mathbb{R}$. Then also $l + t \cdot g \in E'$. By Proposition 4.8 the following equation (+) holds:

$$(+) \qquad \int\limits_E \exp\left(i \cdot (l(x) + tg(x))\right) d\nu\,(x) = \exp\left(-\frac{1}{2}\int\limits_E (l(x) + tg(x))^2 d\nu\,(x)\right).$$

We will differentiate both sides of equation (+) with respect to t at $t = 0$. To differentiate the left-hand side we use the following estimation, induced from the Taylor development:

$$\left|\frac{\exp\left(i \cdot (l(x) + tg(x))\right) - \exp\left(i \cdot l(x)\right)}{t}\right| = \left|\exp\left(i \cdot l(x)\right) \cdot \frac{\exp\left(i \cdot t \cdot g(x)\right) - 1}{t}\right| \leq$$

$$\exp\left(|l(x)|\right) \cdot \left|\frac{\exp\left(i \cdot t \cdot g(x)\right) - 1}{t}\right| \leq \exp\left(|l(x)|\right) \cdot |g(x)| \cdot \exp\left(|t \cdot g(x)|\right).$$

By Proposition 4.5 and the Hölder inequality (see Ash [5])

$$x \mapsto \exp\left(|l(x)|\right) \cdot |g(x)| \cdot \exp\left(|t \cdot g(x)|\right)$$

is integrable. Hence, as application of the theorem of dominated convergence (see Ash [5] and Heuser [19], 44.7) we obtain

$$\frac{d}{dt}\left(\int_E \exp\left(i \cdot (l + tg)\right) d\nu\right)_{t=0} = \lim_{t\to 0}\int_E \frac{\exp\left(i\cdot(l + t\cdot g)\right) - \exp\left(i \cdot l\right)}{t} d\nu =$$

$$\int_E \lim_{t\to 0} \frac{\exp\left(i\cdot(l+tg)\right) - \exp\left(i\cdot l\right)}{t} d\nu = \int_E i \cdot g \cdot \exp\left(i \cdot l\right) d\nu.$$

Now differentiating also the right-hand side of (+) yields:

$$\int_E i \cdot g(x) \exp\left(i \cdot l(x)\right) d\nu\left(x\right) = -\int_E l(x) g(x) d\nu\left(x\right) \cdot \exp\left(-\frac{1}{2}\int_E l^2(x) d\nu\left(x\right)\right).$$

For fixed l this relation is true for all $g \in E'$, hence also for all $g \in E'_\nu$. □

We will now introduce a Hilbert subspace of E, depending on the measure ν. We will do this according to Bogachev [10]. Recall that E'_ν denotes the closure of E' in $L^2(E,\nu)$. Together with the inner product of $L^2(E,\nu)$, E'_ν becomes a Hilbert space, usually called **reproducing kernel Hilbert space** of the measure ν. Now for $h \in E$ define

$$\|h\| := sup\left\{l(h) : l \in E' \text{ and } \|l\|_{L^2(E,\nu)} \leq 1\right\}$$

and set

$$H(\nu) := \{h \in E : \|h\| < \infty\}.$$

$H\left(\nu\right)$ is called **Cameron-Martin space of ν on E** or **Cameron-Martin subspace of E.**

4.10 Proposition

(Bogachev [10], Lemma 2.4.1.) *Any given $h \in E$ lies in $H(\nu)$ if and only if there exists a function $g \in E'_\nu$ such that $l(h) = \int_E g \cdot l d\nu$ for all $l \in E'$. In this case we* *obtain* $\|h\| = \|g\|_{L^2(E,\nu)}$.

Proof: Take $h \in E$ and regard the mapping $\tilde{h} : E' \to \mathbb{R}$, $l \mapsto l(h)$. If $h \in H(\nu)$ this mapping is linear and bounded, hence continuous. By the theorem of Hahn-Banach (see e.g. Heuser [18]) it can be extended to a linear and continuous functional on E'_ν. Since E'_ν is a Hilbert space, we can apply the theorem of Riesz (see e.g. Heuser [18]) to obtain an element $g \in E'_\nu$ such that for all $l \in E'$

$$l(h) = \tilde{h}\,(l) = \langle g; l\rangle_{L^2(E,\nu)} = \int\limits_E g \cdot l d\nu.$$

On the other side, if there exists a function $g \in E'_\nu$ with $l(h) = \int_E g \cdot l d\nu$ for all $l \in E'$, the mapping $\tilde{h} : E' \to \mathbb{R}$, $l \mapsto l(h)$ is continuous on E', hence bounded. This implies $\|h\| < \infty$.
With elementary analysis it is easily shown that

$$sup\left\{l(h) : l \in E' \text{ and } \|l\|_{L^2(E,\nu)} \leq 1\right\} = \|g\|_{L^2(E,\nu)}. \quad \Box$$

4.11 Example

Let γ^∞ be the Gaussian measure on $\mathbb{R}^\infty$, defined in Example 4.3. Let $\varphi, \psi \in (\mathbb{R}^\infty)'$. In Example 4.3 we have shown that φ and ψ are finite linear combinations of coordinate functions. Hence,

$$\int \varphi(x)\psi(x)\mathrm{d}\gamma^\infty = \int_E \left(\sum_{i=1}^k \alpha_i \cdot x_{n_i}\right) \cdot \left(\sum_{i=1}^l \beta_i \cdot x_{m_i}\right)\mathrm{d}\gamma^\infty = \sum_{i=1}^{min(k,l)} \alpha_i\beta_i.$$

So, it is easy to verify that the Cameron-Martin space $H(\gamma^\infty)$ coincides with the Hilbert space

$$l^2 = \left\{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^\infty : \sum_{n\in\mathbb{N}} x_n^2 < \infty\right\}$$

where the norm is given by

$$\|x\|_{l^2} = \sqrt{\sum_{n\in\mathbb{N}} x_n^2}.$$

A measure μ in $\mathfrak{b}_E$ is called **Radon measure** if for every $B \in \mathfrak{b}_E$ and every

$\varepsilon \in \mathbb{R}^+$ there is a compact set $K_\varepsilon \subset B$ with $\mu(B \setminus K_\varepsilon) < \varepsilon$. The following three propositions supply important information about Radon Gaussian measures, we will need also in the next section. The first one follows from Section 2.4 and Section 3.2. in Bogachev [10].

4.12 Proposition

Let ν be a Radon Gaussian measure on E. Then there exists an isomorphism $F_\nu : H(\nu) \to E'_\nu$, such that for any $h \in H(\nu)$,

$$l(h) = \int_E F_\nu(h) \cdot l d\nu$$

for all $l \in E'$. $H(\nu)$ together with its norm

$$\|h\| = \sqrt{\int_E F_\nu^2(h)\, d\nu}$$

turns out to be a separable Hilbert space.

The other two propositions are parts of Theorem 3.3.6 and Theorem 3.5.1, respectively, in Bogachev [10].

4.13 Proposition

Let μ and ν be two Radon Gaussian measures on E. Then the following conditions are equivalent:
(1) $\int_E \varphi^2 d\mu \leq \int_E \varphi^2 d\nu$ for all $\varphi \in E'$.
(2) $\mu(A) \geq \nu(A)$ for all convex Borel sets A with $\{-x : x \in A\} = A$.

4.14 Proposition

Let ν be a Radon Gaussian measure on E, $(h_n)_{n\in\mathbb{N}}$ an orthonormal base of the Cameron Martin subspace $H(\nu)$. Then

$$x = \sum_{n=1}^{\infty} F_\nu(h_n)(x) \cdot h_n \quad \nu\text{-a.s.},$$

where the convergence is with respect to the topology of E and F_ν is the isomorphism, introduced in Proposition 4.12.

With the following main result of this section we return to measure differentiability.

4.15 Theorem

(Norin [28], Theorem 4.15 (i).) *Let ν be a centered Gaussian measure on a separable Banach space E. A vector $h \in E$ lies in the Cameron-Martin space $H(\nu)$ if and only if ν is Fomin-differentiable along h.*

Proof: The first part of the proof is due to Norin [28], Chapter 4. Let $h \in H(\nu)$. By Proposition 4.10 there exists a function $g \in E'_\nu$ such that $l(h) = \int_E g \cdot l d\nu$ for all $l \in E'$. Let $l \in E'$. By Lemma 4.9 we have

$$\int\limits_E i \cdot g(x) \exp\left(i \cdot l(x)\right) d\nu\left(x\right) =$$

$$-\int\limits_E l(x) g(x) d\nu\left(x\right) \cdot \exp\left(-\frac{1}{2}\int\limits_E l^2(x) d\nu\left(x\right)\right) = -l(h) \cdot \exp\left(-\frac{1}{2}\int\limits_E l^2(x) d\nu\left(x\right)\right).$$

Define a measure ν' by $\nu'(A) := -\int_A g d\nu$ for each $A \in \mathfrak{b}_E$. Then we obtain the following relationship for the Fourier transforms $\tilde{\nu}$ and $\tilde{\nu'}$:

$$\tilde{\nu'}(l) = -i \cdot l(h) \cdot \tilde{\nu}(l).$$

We will show that ν is Skorokhod-differentiable along h with derivative ν'. Choose $l \in E'$ and $t \in \mathbb{R}$. According to Section 3 we define $\nu_t\left(B\right) := \nu\left(B + t \cdot h\right)$. Then

$$\int\limits_E \exp\left(i \cdot l(x)\right) d(\nu_t - \nu)\left(x\right) =$$

$$\exp\left(-i \cdot t \cdot l(h)\right) \cdot \int\limits_E \exp\left(i \cdot l(x)\right) d\nu\left(x\right) - \int\limits_E \exp\left(i \cdot l(x)\right) d\nu\left(x\right) =$$

$$\int\limits_0^t \left(\int\limits_E \exp\left(i \cdot l(x) - i \cdot s \cdot l(h)\right) d\nu'\left(x\right)\right) ds = \int\limits_E \exp\left(i \cdot l(x)\right) d(\nu' \star \mu^{\lambda,t})(x),$$

where $\mu^{\lambda,t}$ is the image measure of Lebesgue measure λ on $[0,t]$ under the mapping

$$[0;t] \ni s \mapsto -s \cdot h,$$

and $\nu' \star \mu^{\lambda,t}$ denotes the convolution. Since the choice of $l \in E'$ was arbitrary, the Fourier transforms of $\nu_t - \nu$ and $\nu' \star \mu^{\lambda,t}$ coincide on E'. Since E is a Banach space, the measures $\nu_t - \nu$ and $\nu' \star \mu^{\lambda,t}$ coincide on $\mathfrak{b}_E$. Therefore, we obtain for all bounded, Borel measurable functions f:

$$\int_E (f(x - t \cdot h) - f(x))\, d\nu\,(x) = \int_E f(x) d(\nu_t - \nu)\,(x) =$$

$$\int_E f(x) d(\nu' \star \mu^{\lambda,t}) = \int_0^t \left(\int_E f(x - sh) d\nu'\,(x) \right) ds.$$

If we suppose f to be continuous, we can apply Fubini's theorem (see Ash [5]) to obtain

$$\int_0^t \left(\int_E f(x - sh) d\nu'\,(x) \right) ds = \int_E \left(\int_0^t f(x - sh) ds \right) d\nu'\,(x)\,.$$

Since f is bounded there is a $k \in \mathbb{N}$ such that $f\,(x) \leq k$ for all $x \in E$, hence

$$\frac{\int_0^t f\,(x - sh)\, ds}{t} \leq \frac{t \cdot k}{t} = k.$$

By the theorem of dominated convergence (see Ash [5]) and the fundamental theorem of calculus we have

$$\lim_{t \mapsto 0} \frac{\int_E (f(x - t \cdot h) - f(x))\, d\nu\,(x)}{t} = \int_E f(x) d\nu'\,(x)\,.$$

Hence, for any $\varepsilon \in \mathbb{R}^+$, ν is Skorokhod-differentiable in $]-\varepsilon; \varepsilon[$ along h. It is easily verified that the space $\mathcal{C}$ of all real-valued, bounded and continuous functions on E is norm-defining and that the mappings $t \mapsto (\nu_t)'\,(B)$ are Lebesgue measurable for each $B \in \mathfrak{b}_E$. Since $\nu' := -g \cdot \nu$ we have $\nu' \ll \nu$. Hence we can apply Proposition 3.5 to obtain that ν is Fomin-differentiable along h.

Now let $h \in E \setminus H\,(\nu)$. We show that then the measures ν and ν_h are singular.

This part of the proof is due to Bogachev [10], Section 2.4. By definition,

$$sup \left\{ l(h) \mid l \in E' \text{ and } \|l\|_{L^2(E,\nu)} \leq 1 \right\} = \infty.$$

Hence, for any $n \in \mathbb{N}$ there exists a functional $l \in E'$, with $\|l\|_{L^2(E,\nu)} = 1$ and $l(h) > n$. Let us regard the Gaussian measure $\nu \circ l^{-1}$. Since $\int_{\mathbb{R}} x^2 d\left(\nu \circ l^{-1}\right)(x) = \int_E l^2 d\nu = 1$, the measure $\nu \circ l^{-1}$ has variance $\sigma^2 = 1$. Applying Lemma 4.2 and Lemma 4.6 we get

$$\|\nu_h - \nu\| \geq 2 - 2\exp\left(-\frac{1}{8}n^2\right)$$

for all $n \in \mathbb{N}$. Hence, $\|\nu_h - \nu\| = 2$ and therefore, by Lemma 4.7, the measures ν_h and ν are singular. Note that $h \in E \setminus H(\nu)$ implies $\frac{1}{n} \cdot h \in E \setminus H(\nu)$ for all $n \in \mathbb{N}$. Let $A_n \in \mathfrak{b}_E$ such that $\nu(A_n) = 0$ and $\nu_{\frac{1}{n}\cdot h}(E \setminus A_n) = \nu\left(E \setminus A_n + \frac{1}{n} \cdot h\right) = 0$. Since ν is a probability measure, $\nu_{\frac{1}{n}\cdot h}(A_n) = 1$ for all $n \in \mathbb{N}$. Define $A := \bigcup_{n\in\mathbb{N}} A_n$. Then

$$\frac{\nu\left(A + \frac{1}{n} \cdot h\right) - \nu(A)}{\frac{1}{n}} = n.$$

Thus ν is not Fomin-differentiable along h. $\square$

In Section 21 we will prove a related result for Loeb measures, which yields the differentiability of a special Gaussian measure, the Wiener measure.

5 Abstract Wiener Spaces

We will now apply Theorem 4.15 to abstract Wiener spaces. The concept of abstract Wiener spaces was developed by L. Gross in [16] to obtain Gaussian measures in infinite dimensional spaces. Since we need some facts about abstract Wiener spaces also later, the essential ideas and results of Gross's theory are sketched in this section. For the proofs of most of the presented facts and more details consult Osswald [30], Kuo [24] or Gross [17]. Our starting point is an arbitrary infinite dimensional and separable Hilbert space $(\mathbb{H}; \|\cdot\|)$. At first we establish a finitely additive set function in $\mathbb{H}$. To this end we start from Borel subsets of finite dimensional subspaces of $\mathbb{H}$. Let

$$\mathcal{E}(\mathbb{H}) = \{E \subset \mathbb{H} : E \text{ is a finite dimensional subspace of } \mathbb{H}\}.$$

Fix $\sigma \in \mathbb{R}^+$. For any given subspace $E \in \mathcal{E}(\mathbb{H})$ with orthonormal basis ($\mathfrak{e}_1$, ..., $\mathfrak{e}_n$) a centered Gaussian measure $\gamma^{E,\sigma}$ on E can be defined by

$$\gamma^{E,\sigma}(B) := \gamma^{n,\sigma}\left(\left\{(\alpha_1, ..., \alpha_n) \in \mathbb{R}^n : \sum_{i=1}^{n} \alpha_i \mathfrak{e}_i \in B\right\}\right) =$$

$$\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \int_{\left\{(\alpha_1,...,\alpha_n)\in\mathbb{R}^n | \sum_{i=1}^{n} \alpha_i \mathfrak{e}_i \in B\right\}} \exp\left(-\frac{1}{2\sigma^2}\left(x_1^2 + \ldots + x_n^2\right)\right) dx_1 \ldots dx_n,$$

where $B \in \mathfrak{b}_E$.

5.1 Lemma

(Osswald [30], Lemma 4.2.1.) *The measure $\gamma^{E,\sigma}$ on E does not depend on the choice of the orthonormal basis of E.*

A subset $Z \subset \mathbb{H}$ is called a **cylinder set in** $\mathbb{H}$ if there are elements $E \in \mathcal{E}(\mathbb{H})$ and $B \in \mathfrak{b}_E$ such that $Z = B + E^\perp$. In this case we define $\gamma^{\mathbb{H},\sigma}(Z) := \gamma^{E,\sigma}(B)$. By Lemma 4.1.3 in Osswald [30], $\gamma^{\mathbb{H},\sigma}(Z)$ is well defined. We will also need the following characterisation of cylinder sets.

5.2 Lemma

(Osswald [30], Proposion 4.1.4.) *A set* $Z \in \mathfrak{b}_{\mathbb{H}}$ *is a cylinder set if and only if there exist* $\varphi_i \in \mathbb{H}'$, $n \in \mathbb{N}$ *and* $A \in \mathfrak{b}_{\mathbb{R}^n}$ *such that*

$$Z = (\varphi_1, \dots, \varphi_n)^{-1}(A).$$

with $n \in \mathbb{N}$, $\varphi_i \in \mathbb{H}'$ *and* $A \in \mathfrak{b}_{\mathbb{R}^n}$.

Let $\mathcal{Z}_{\mathbb{H}}$ denote the set of all cylinder sets in $\mathbb{H}$.

5.3 Proposition

(Osswald [30], Proposition 4.1.5, Proposition 4.2.4 and Kuo [24], Chapter I. Proposition 4.1.)
(1) $\mathcal{Z}_{\mathbb{H}}$ *is a field.*
(2) $\gamma^{\mathbb{H},\sigma}$ *is a finitely additive mapping on* $\mathcal{Z}_{\mathbb{H}}$.
(3) $\gamma^{\mathbb{H},\sigma}$ *is not countably additive on* $\mathcal{Z}_{\mathbb{H}}$.

To obtain σ-additivity, it was L. Gross's idea to find a suitable extension of $\mathbb{H}$. If $(\mathbb{B}; |\cdot|)$ is a Banach space such that $\mathbb{H} \subset \mathbb{B}$ we can define **cylinder sets in** $\mathbb{B}$ according to Lemma 5.2 by

$$Z = (\varphi_1, \dots, \varphi_n)^{-1}(A)$$

where $\varphi_i \in \mathbb{B}'$, $n \in \mathbb{N}$ and $A \in \mathfrak{b}_{\mathbb{R}^n}$. Denote by $\mathcal{Z}_{\mathbb{B}}$ the collection of all cylinder sets in $\mathbb{B}$. The following Lemma is obvious.

5.4 Lemma

If for all functionals of $\mathbb{B}'$ *the restrictions on* $\mathbb{H}$ *are elements of* $\mathbb{H}'$*, we obtain for each* $Z \in \mathcal{Z}_{\mathbb{B}}$ *that* $Z \cap \mathbb{H} \in \mathcal{Z}_{\mathbb{H}}$.

If the assumption of Lemma 5.4 holds we may define a mapping $\gamma^{\mathbb{B},\sigma}$ on $\mathcal{Z}_{\mathbb{B}}$ by

$$\gamma^{\mathbb{B},\sigma}(Z) := \gamma^{\mathbb{H},\sigma}(Z \cap \mathbb{H}).$$

L. Gross introduced the concept of measurable norms. A norm $|\cdot|$ on $\mathbb{H}$ is called **measurable (with respect to σ)** if for each $\varepsilon > 0$ there is an $E_\varepsilon \in \mathcal{E}(\mathbb{H})$ such that for all $E \in \mathcal{E}(\mathbb{H})$ with $E \perp E_\varepsilon$

$$\gamma^{E,\sigma}(\{x \in E :| x |> \varepsilon\}) \leq \varepsilon.$$

5.5 Lemma

(Kuo [24], Chapter I. Lemma 4.2) *If $|\cdot|$ is a measurable norm on $\mathbb{H}$ then there exists a constant $c \in \mathbb{R}^+$ such that $| h |\leq c \cdot \|h\|$ for all $h \in \mathbb{H}$.*

Now let $\mathbb{H}$ and $\gamma^{\mathbb{H},\sigma}$ be as above. Assume $|\cdot|$ to be a measurable norm (with respect to σ) on $\mathbb{H}$ and $(\mathbb{B}, |\cdot|)$ the Banach space completion of $(\mathbb{H}, |\cdot|)$. Then $(\mathbb{H}, |\cdot|)$ is a dense subspace of $(\mathbb{B}, |\cdot|)$. Therefore and by Lemma 5.5, $\mathbb{B}'$ can be regarded as a subset of $\mathbb{H}'$ and therefore - since we can identify $\mathbb{H}'$ with $\mathbb{H}$ - as a subset of $\mathbb{H}$. Furthermore we have

5.6 Lemma

The space $\mathbb{B}'$ is dense in $(\mathbb{H}; \|\cdot\|)$.

Proof: Let $h \in \mathbb{H}$ such that $\langle\varphi; h\rangle_{\mathbb{H}} = 0$ for all $\varphi \in \mathbb{B}'$. Therefore, $\varphi(h) = 0$ for all $\varphi \in \mathbb{B}'$. Hence, $h = 0$. $\square$

By Lemma 5.4 we can establish the mapping $\gamma^{\mathbb{B},\sigma}$ on $\mathcal{Z}_{\mathbb{B}}$. The measurability of the norm is essential for the proof of the following important proposition, that is sometimes called theorem of Gross.

5.7 Proposition

(Kuo [24], Chapter I. Theorem 4.1 und Theorem 4.2.) *$\gamma^{\mathbb{B},\sigma}$ can be extended uniquely to a countably additive measure in the σ-field generated by $Z_{\mathbb{B}}$. This σ-field is the Borel field $\mathfrak{b}_{\mathbb{B}}$ of $\mathbb{B}$.*

Let us denote the countably additive extension by γ^σ. Then γ^σ is called **Wiener measure** and the pair $(\mathbb{H}, \mathbb{B})$ is called an **abstract Wiener space**.

5.8 Lemma

The Wiener measure γ^σ is a centered Gaussian measure on $\mathbb{B}$.

Proof: The idea of the proof is due to Osswald [30]. Let $\varphi \in \mathbb{B}'$ and $A \in \mathfrak{b}_{\mathbb{R}}$. Since φ is also an element of $\mathbb{H}' = \mathbb{H}$, the space $E = \{\lambda \cdot \varphi : \lambda \in \mathbb{R}\}$ is a subspace of $\mathbb{H}$ and we get

$$\gamma^\sigma \circ \varphi^{-1}(A) = \gamma^{\mathbb{H},\sigma}(\{x \in \mathbb{H} : \varphi(x) \in A\}) =$$

$$\gamma^{\mathbb{H},\sigma}\left(\{x \in E : \varphi(x) \in A\} + E^\perp\right) =$$

$$\frac{1}{\sigma \cdot \|\varphi\|_{\mathbb{H}} \sqrt{2\pi}} \cdot \int_A \exp\left(-\frac{x^2}{2\left(\sigma \cdot \|\varphi\|_{\mathbb{H}}\right)^2}\right) dx. \quad \square$$

In Section 13, we will construct an internal probability space with the aid of an abstract Wiener space.

To describe the differentiability of a Wiener measure we will show a relationship between the underlying Hilbert space $\mathbb{H}$ and the Cameron-Martin space $H(\gamma^\sigma)$. For this purpose we need the following lemma.

5.9 Lemma

If ν is a centered Gaussian measure on a separable Banach space $(\mathbb{B}; |\cdot|)$, such that the Cameron-Martin space $H(\nu)$ is dense in $\mathbb{B}$, then for all functionals φ of $\mathbb{B}'$ the restrictions on $H(\nu)$ are elements of $H(\nu)'$. If we identify the restriction $\varphi \upharpoonright H(\nu)$ with an element of $H(\nu)$, then $\mathbb{B}'$ is a dense subset of $H(\nu)$. Furthermore,

$$\|\varphi \upharpoonright H(\nu)\|^2_{H(\nu)} = \int_{\mathbb{B}} \varphi^2(x)\, d\nu(x).$$

Proof: Let $\varphi \in \mathbb{B}'$ and $\varepsilon \in \mathbb{R}^+$. Set

$$\delta := \frac{\varepsilon}{\sqrt{\int_{\mathbb{B}} \varphi^2(x)\, d\nu(x)}}.$$

Now choose $h \in H(\nu)$, satisfying $\|h\|_{H(\nu)} < \delta$. Let F_ν be the isomorphism between the Cameron-Martin subspace $H(\nu)$ and the Hilbert space $\mathbb{B}'_\nu$, introduced in

Proposition 4.12. Then

$$|\varphi(h)| = \left| \int_{\mathbb{B}} \varphi(x) F_\nu(h)(x) d\nu(x) \right| \leq$$

$$\left(\int_{\mathbb{B}} \varphi^2(x) d\nu(x) \cdot \int_{\mathbb{B}} (F_\nu(h))^2 (x) d\nu(x) \right)^{\frac{1}{2}} =$$

$$\left(\int_{\mathbb{B}} \varphi^2(x) d\nu(x) \right)^{\frac{1}{2}} \cdot \|h\|_{H(\nu)} < \varepsilon.$$

Hence, $\varphi \restriction H(\nu) \in H(\nu)'$. Since $\varphi \restriction H(\nu)$ can be identified with an element of $H(\nu)$, $\mathbb{B}'$ can be regarded as subset of $H(\nu)$. With the same arguments as in the proof of Lemma 5.6 follows that $\mathbb{B}'$ is a dense subset of $H(\nu)$. The last part of the Lemma follows from the definition of $\|\cdot\|_{H(\nu)}$. $\square$

5.10 Proposition

(Bogachev [10], Theorems 3.9.5. and 3.9.6.) *Let $(\mathbb{H}, \mathbb{B})$ be an abstract Wiener space with Wiener measure γ^1. Then $\mathbb{H}$ coincides with the Cameron Martin space $H(\gamma^1)$ of γ^1 of $\mathbb{B}$. Conversely, if ν is a centered Gaussian measure on a separable Banach space $\mathbb{B}$ with norm $|\cdot|$ such that the Cameron Martin space $H(\nu)$ is dense in $\mathbb{B}$, then the norm $|\cdot|$ restricted to $H(\nu)$ is measurable with respect to $\sigma = 1$ and $(H(\nu), \mathbb{B})$ is an abstract Wiener space with $\nu = \gamma^1$ as Wiener measure.*

Proof: The proof is very close to the proofs in [10]. Let $(\mathbb{H}, \mathbb{B})$ be an abstract Wiener space with Wiener measure γ^1. Choose $\varphi, \psi \in \mathbb{B}'$. By Lemma 5.5 φ and ψ can also be regarded as elements of $\mathbb{H}$. To avoid confusion, we denote these elements of $\mathbb{H}$ by h_φ and h_ψ. First note that $\int_{\mathbb{B}} \varphi \cdot \psi d\gamma^1 = \langle h_\varphi; h_\psi \rangle$, since by Lemma 5.8,

$$\int_{\mathbb{B}} \varphi^2 d\gamma^1 = \int_{\mathbb{R}} x^2 d\left(\gamma^1 \circ \varphi^{-1}\right)(x) = \|h_\varphi\|^2.$$

Let F_{γ^1} be the isomorphism between the Cameron Martin subspace $H(\gamma^1)$ and the Hilbert space $\mathbb{B}'_{\gamma^1}$ introduced in Proposition 4.12. Set $h = F_{\gamma^1}^{-1}(\varphi)$. Then

$$\psi(h) = \int_{\mathbb{B}} F_{\gamma^1}(h)(x) \cdot \psi(x)\, d\gamma^1(x) = \int_{\mathbb{B}} \varphi(x) \cdot \psi(x)\, d\gamma^1(x) = \langle h_\varphi; h_\psi \rangle = \psi(h_\varphi).$$

Hence $F_{\gamma^1}^{-1}(\varphi) = h_\varphi$. Since $\|h_\varphi\|^2 = \int_{\mathbb{B}} \varphi^2(x)\, d\gamma^1(x) = \|h\|_{H(\gamma^1)}$ and since $\mathbb{B}'$ is dense in $H(\gamma^1)$ as well as in $\mathbb{H}$, $H(\gamma^1)$ coincides with $\mathbb{H}$.

To prove the converse, note that - again by Lemma 5.9 - each element of $\mathbb{B}'$ can be regarded as an element of $H(\nu)'$. Hence, we can define the measure $\gamma^{\mathbb{B},1}$ on $\mathcal{Z}_{\mathbb{B}}$. Let $\varphi \in \mathbb{B}'$. By Lemma 5.9,

$$\|\varphi \upharpoonright H(\nu)\|_{H(\nu)} = \int_{\mathbb{B}} \varphi^2(x)\, d\nu(x).$$

This implies that ν coincides with $\gamma^{\mathbb{B},1}$ on $\mathcal{Z}_{\mathbb{B}}$. It remains to prove the measurability with respect to $\sigma = 1$ of the norm $|\cdot|$ on $H(\nu)$. Fix $\varepsilon > 0$. Let $(\varphi_n)_{n\in\mathbb{N}}$ be an orthonormal base of $\mathbb{B}'_\nu$ satisfying $(\varphi_n)_{n\in\mathbb{N}} \subset \mathbb{B}'$. Define

$$P_n : \mathbb{B} \to H(\nu),\ x \mapsto \sum_{i=1}^{n} \varphi_i(x) \cdot F_\nu^{-1}(\varphi_i).$$

By Proposition 4.14, we have

$$\lim_{n\to\infty} |x - P_n(x)| = 0 \quad \nu\text{-a.s.}$$

By Egoroff's theorem (see Ash [5]) the convergence is in measure, hence there exists a number $N \in \mathbb{N}$ such that for all $n, m \geq N$

$$\nu(\{x \in \mathbb{B} : |P_n(x) - P_m(x)| > \varepsilon\}) < \varepsilon.$$

To verify that $\{x \in \mathbb{B} : |P_n(x) - P_m(x)| > \varepsilon\} \in \mathcal{Z}_{\mathbb{B}}$, note that

$$\{x \in \mathbb{B} : |P_n(x) - P_m(x)| > \varepsilon\} = \left\{x \in \mathbb{B} : \left|\sum_{i=m+1}^{n} \varphi_i(x) \cdot F_\gamma^{-1}(\varphi_i)\right| > \varepsilon\right\} =$$

$$\{x \in \mathbb{B} : (\varphi_{m+1}(x), \ldots, \varphi_n(x)) \in A\},$$

where $A = \left\{(a_{m+1}, \ldots, a_n) \in \mathbb{R}^{m-n} : \left|\sum_{i=m+1}^{n} a_i \cdot F_\nu^{-1}(\varphi_i)\right| > \varepsilon\right\} \in \mathfrak{b}_{\mathbb{R}^{\text{m-n}}}$. Since also

$$\left\{x \in H(\nu) : \left|\sum_{i=m+1}^{n} \varphi_i(x) \cdot F_\nu^{-1}(\varphi_i)\right| > \varepsilon\right\} =$$

$$\{x \in H(\nu) : (\varphi_{m+1}(x), \ldots, \varphi_n(x)) \in A\},$$

we obtain

$$\nu(\{x \in \mathbb{B} : |P_n(x) - P_m(x)| > \varepsilon\}) = \gamma^{H(\nu),1}(\{x \in H(\nu) : |P_n(x) - P_m(x)| > \varepsilon\}).$$

Let $E_\varepsilon = span\{F_\nu^{-1}(\varphi_1), \ldots, F_\nu^{-1}(\varphi_N)\}$. Choose any $E \in \mathcal{E}(H(\nu))$ with $E \perp E_\varepsilon$. By the projection theorem (see Section 1), each $x \in \mathbb{H}$ can be composed uniquely into a sum $x = a + y$ where $a \in E$ and $y \in E^\perp$. Let $pr_E^{H(\nu)} : H(\nu) \to E$, $x \mapsto a$ be the orthogonal projection. Then $P_N\left(pr_E^{H(\nu)}(x)\right) = 0$ and

$$\lim_{k\to\infty} P_{N+k}\left(pr_E^{H(\nu)}(x)\right) = pr_E^{H(\nu)}(x).$$

Again by the theorem of Egoroff (see Ash [5]) and by the definition of $\gamma^{H(\nu),1}$ it is easily proved that

$$\gamma^{E,1}(\{x \in E : |x| > \varepsilon\}) = \lim_{k\to\infty} \gamma^{E,1}(\{x \in E : |P_{N+k}(x)| > \varepsilon\}) =$$

$$\lim_{k\to\infty} \gamma^{H(\nu),1}\left(\left\{x \in H(\nu) : \left|P_{N+k}\left(pr_E^{H(\nu)}(x)\right)\right| > \varepsilon\right\}\right) =$$

$$\lim_{k\to\infty} \gamma^{H(\nu),1}\left(\left\{x \in H(\nu) : \left|(P_{N+k} - P_N)\left(pr_E^{H(\nu)}(x)\right)\right| > \varepsilon\right\}\right).$$

For the last step of the proof we use Proposition 4.13. Set

$$D := \{x \in H(\nu) : |P_{N+k}(x) - P_N(x)| \le \varepsilon\}.$$

Obviously $D \in \mathcal{Z}_{H(\nu)}$, hence $D = \tilde{B} + \tilde{E}^\perp$ where $\tilde{E} \in \mathcal{E}(H(\nu))$ and $\tilde{B} \subset \tilde{E}$. We regard the smallest finite dimensional space $\bar{E} \in \mathcal{E}(H(\nu))$ containing $\tilde{E}$ and E and the measures $\gamma^{\bar{E},1}$ and $\gamma^{\bar{E},1} \circ \left(pr_E^{\bar{E}}\right)^{-1}$. For $y \in \bar{E}$ define $\varphi(x) := \langle y; x\rangle_{\overline{E}}$. Note that

$$\varphi\left(pr_E^{\bar{E}}(x)\right) = \left\langle y; pr_E^{\bar{E}}(x)\right\rangle_{\bar{E}} = \left\langle pr_E^{\bar{E}}(y); x\right\rangle_{\bar{E}}.$$

By Lemma 5.9,

$$\int\limits_{\bar{E}} \varphi^2(x)\, d\left(\gamma^{\bar{E},1} \circ \left(pr_E^{\bar{E}}\right)^{-1}\right)(x) = \int\limits_{\bar{E}} \left(\varphi\left(pr_E^{\bar{E}}(x)\right)\right)^2_{\bar{E}} d\gamma^{\bar{E},1}(x) =$$

$$\left\|pr_E^{\bar{E}}(y)\right\|^2_{\bar{E}} \leq \|y\|^2_{\bar{E}} = \int\limits_{\bar{E}} \varphi^2(x)\, d\gamma^{\bar{E},1}(x).$$

Hence Proposition 4.13 implies

$$\gamma^{H(\nu),1}\left(\left(pr_E^{H(\nu)}\right)^{-1}(D)\right) = \gamma^{\bar{E},1} \circ \left(pr_E^{\bar{E}}\right)^{-1}\left(\tilde{B} + \left\{x \in \bar{E} : x \perp \tilde{E}\right\}\right) \geq$$

$$\gamma^{\bar{E},1}\left(\tilde{B} + \left\{x \in \bar{E} : x \perp \tilde{E}\right\}\right) = \gamma^{H(\nu),1}(D).$$

Therefore,

$$\gamma^{H(\nu),1}\left(\left\{x \in H(\nu) : \left|(P_{N+k} - P_N)\, pr_E^{H(\nu)}(x)\right| > \varepsilon\right\}\right) \leq$$

$$\gamma^{H(\nu),1}\left(\left\{x \in H(\nu) : |P_{N+k}(x) - P_N(x)| > \varepsilon\right\}\right).$$

Finally we obtain $\gamma^{E,1}(\{x \in E : |x| > \varepsilon\}) \leq \varepsilon$. □

We close this section with the announced application of Theorem 4.15 to abstract Wiener spaces.

5.11 Corollary

The Wiener measure $\gamma^{\mathbb{H},1}$ *of an abstract Wiener space* $(\mathbb{H}, \mathbb{B})$ *is Fomin-differentiable along a vector* $h \in \mathbb{B}$ *if and only if* $h \in \mathbb{H}$ *.*

6 An Application of Measure Differentiability

One of the applications of measure differentiabilty is the construction of operators comparable to those of Malliavin and Skorokhod in the Gaussian case. We will describe this now, following closely the presentation of Weizsäcker in [37]. Let ν be a nonnegative measure on a locally convex space E and $(\mathbb{H}; \|\cdot\|_{\mathbb{H}})$ a Hilbert subspace of E, continuously embedded in E. Recall that in this case each element $\varphi \in E'$ can be regarded as an element of $\mathbb{H}$. Let $C_c^{\infty}(\mathbb{R}^n)$ be the set of all infinitely continuously differentiable functions $g : \mathbb{R}^n \to \mathbb{R}$ with compact support and denote by $\partial^i g$ the partial derivatives. Define

$$C := \{f : E \to \mathbb{R}, f = g(\varphi_1, \ldots, \varphi_n) : \varphi_i \in E', g \in C_c^{\infty}(\mathbb{R}^n)\}.$$

Then C is dense in $L^2(E, \nu)$ (see e.g. Smolyanov and Weizsäcker [35]). Let $f \in C$ and $y \in E$. Then f is Gateaux differentiable in the direction of y and for $x \in E$ we obtain $f'(x)(y) = \sum_{i=1}^{n} \partial^i g(\varphi_1(x), \ldots, \varphi_n(x)) \cdot \varphi_i(y)$. If $y \in \mathbb{H}$, then the following inequality (+) holds:

$$(+) \quad |f'(x)(y)| \leq n \cdot sup_{\tilde{x} \in \mathbb{R}^n} \left(\sum_{i=1}^{n} \partial^i g(\tilde{x}) \cdot \tilde{x}_i \right) \cdot max_{1 \leq i \leq n} \|\varphi_i\|_{\mathbb{H}} \cdot \|y\|_{\mathbb{H}} =: c_f \cdot \|y\|_{\mathbb{H}}.$$

Hence, the linear mapping $f'(x) : \mathbb{H} \to \mathbb{R}, y \mapsto f'(x)(y)$ is bounded, and therefore an element of $\mathbb{H}'$. Note that $\mathbb{H}'$ can be identified with $\mathbb{H}$. Since the constant c_f depends only on f, the function

$$f' : E \to \mathbb{H}, x \mapsto f'(x)$$

is a bounded vector field from E to $\mathbb{H}$. Denote by $L^2_{\mathbb{H}}(\nu)$ the set of all vector fields $h : E \to \mathbb{H}$, such that

$$\|h\|_{L^2_{\mathbb{H}}(\nu)} = \left(\int_E \|h(x)\|^2_{\mathbb{H}} \, d\nu \right)^{\frac{1}{2}} < \infty.$$

Now we are ready to define a derivative operator $D_0 : C \to L^2_{\mathbb{H}}(\nu)$ by $D_0(f) := f'$. Up to now we have not claimed nor used measure differentiability in this chapter. But for the following, the measure ν shall be Fomin-differentiable along all vectors $h \in \mathbb{H}$. By Proposition 3.2, the derivatives are absolutely continuous with respect to ν. We denote by ξ_y the logarithmic derivatives. Moreover, we assume that

$\xi_y \in L^2(E,\nu)$ for all $y \in \mathbb{H}$.

6.1 Proposition

(Smolyanov and Weizsäcker [35], Proposition 6.) *The operator D_0 is closable in $L^2(E,\nu)$, i.e. if $C \ni f_n \underset{L^2(E,\nu)}{\rightarrow} f$ and $Df_n \underset{L^2_{\mathbb{H}}(\nu)}{\rightarrow} g$ then $Df = g$.*

Proof: Let $(f_n)_{n\in\mathbb{N}} \subset C$, $f_n \underset{L^2(E,\nu)}{\rightarrow} 0$ and $D_0 f_n \underset{L^2_{\mathbb{H}}(\nu)}{\rightarrow} g$. We will show that $g = 0$. To this end choose arbitrary elements $u \in C$ and $y \in \mathbb{H}$. Then

$$\int_E u(x)\cdot\langle g(x);y\rangle_{\mathbb{H}}\,d\nu(x) = \lim_{n\to\infty}\int_E u(x)\cdot\langle D_0 f_n(x);y\rangle_{\mathbb{H}}\,d\nu(x) =$$

$$\lim_{n\to\infty}\int_E u(x)\cdot f_n'(x)(y)d\nu(x).$$

Since the functions u and f_n are Gateaux differentiable in the direction of y, the product rule yields that also each $u\cdot f_n$ is Gateaux differentiable in the direction of y and

$$(u\cdot f_n)'(x)(y) = u'(x)(y)\cdot f_n(x) + u(x)\cdot f_n'(x)(y).$$

Hence,

$$\lim_{n\to\infty}\int_E u(x)\cdot f_n'(x)(y)d\nu(x) = \lim_{n\to\infty}\int_E \left((u\cdot f_n)'(x)(y) - u'(x)(hy)\cdot f_n(x)\right)d\nu(x),$$

and by (+),

$$|(u\cdot f_n)'(x)(y)| \le c_u\cdot\|y\|_{\mathbb{H}}\cdot\|f_n\|_{sup} + c_f\cdot\|y\|_{\mathbb{H}}\cdot\|u\|_{sup} =: a.$$

We like to apply Proposition 2.1 with respect to the functions $u\cdot f_n$ and the constant vector field y. Let $t\in\mathbb{R}\setminus\{0\}$. By the mean value theorem there exists a $t_x\in\mathbb{R}$ such that

$$\frac{(u\cdot f_n)(x+ty) - (u\cdot f_n)(x)}{t} = (u\cdot f_n)'(x+t_x y)(y).$$

Hence we obtain for all $t\in\mathbb{R}\setminus\{0\}$,

$$\int\limits_E \frac{(u\cdot f_n)(x+ty)-(u\cdot f_n)(x)}{t}d\nu(x) \le \int\limits_E a\,d\nu(x).$$

By Proposition 1.3, the functions

$$t \mapsto \frac{(u \cdot f_n)(x + ty) - (u \cdot f_n)(x)}{t}$$

are uniformly integrable. Thus we can apply Proposition 2.1 to obtain

$$\int_E (u \cdot f_n)'(x)(y) d\nu(x) = \int_E (u \cdot f_n)(x) d\nu'(x).$$

Now

$$\lim_{n\to\infty} \int_E u(x) \cdot f_n'(x)(y) d\nu(x) = \lim_{n\to\infty} \int_E ((u \cdot f_n)'(x)(y) - u'(x)(hy) \cdot f_n(x)) d\nu(x),$$

$$= \lim_{n\to\infty} \int_E (u \cdot f_n)(x) d\nu'(x) - \lim_{n\to\infty} \int_E u'(x)(y) \cdot f_n(x) d\nu(x) =$$

$$\lim_{n\to\infty} \int_E ((u \cdot f_n)(x) \cdot \xi_y(x) - u'(x)(h) \cdot f_n(x)) d\nu(x) = 0,$$

since u, u' are bounded and $\xi_y \in L^2(E, \nu)$. Hence, $g = 0$. $\square$

Let us denote by D the closure of D_0. Then D is a closed operator defined on a dense subset of $L^2(E, \nu)$. The definition of D is analogical with the common definition of the Malliavin derivative presented for example in Nualart [29]. Just like there we introduce also the adjoint operator. Since D is densely defined, by Reed and Simon [31], Section VIII, there exists the adjoint operator δ_ν (from $L^2_{\mathbb{H}}(\nu)$ to $L^2(E,\nu)$), and the domain of δ_ν can be characterised as follows. A vector field $h \in L^2_{\mathbb{H}}(\nu)$ is in the domain of δ_ν if and only if there exists a constant K, such that for all $g \in \mathcal{C}$

$$\left| \int_E g'(x)(h(x)) d\nu(x) \right| \leq K \cdot \sqrt{\int_E g^2(x) d\nu(x)}.$$

In this case there exists a uniquely determined element $\delta_\nu(h) \in L^2(E, \nu)$, such that for all $g \in \mathcal{C}$

$$\int_E g'(x)(h(x)) d\nu(x) = \int_E g(x) \cdot \delta_\nu(h)(x) d\nu(x).$$

The definition of δ_ν is in accordance with the introduction of the Skorokhod operator by Nualart [29]. The next proposition summarizes several comments of Weizsäcker's [37] Section 4.

6.2 Proposition

Let E, $\mathbb{H}$, $\mathcal{C}$ and ν defined as above. Fix a vector field $h \in L^2_{\mathbb{H}}(\nu)$. Then h lies in the domain of δ_ν if and only if ν is differentiable along h with respect to $\mathcal{C}$, the derivative ν' is absolutely continuous with respect to ν and the logarithmic derivative $\frac{d\nu'}{d\nu}$ is an element of $L^2(E,\nu)$. In this case we have $\frac{d\nu'}{d\nu} = -\delta_\nu(h)$.

Proof: Firstly, we show that the assumptions of Proposition 2.1 are fulfilled. Let $g \in \mathcal{C}$. Then g is Gateaux differentiable along all elements of $\mathbb{H}$, in particular along all vectors $h(x)$. By the same arguments as in the proof of Proposition 6.1, we obtain for all $t \in \mathbb{R} \setminus \{0\}$

$$\int_E \frac{g(x+th(x)) - g(x)}{t} d\nu(x) = \int_E g'(x + t_x \cdot h(x))(h(x))\, d\nu(x) \leq$$

$$c_g \cdot \int_E \|h(x)\|_{\mathbb{H}}\, d\nu(x),$$

and therefore uniform integrability.
Now let h be an element of the domain of δ_ν. Since for all $g \in \mathcal{C}$

$$\int_E g'(x)(h(x))d\nu(x) = \int_E g(x) \cdot \delta_\nu(h)(x)\, d\nu(x),$$

by Proposition 2.1, ν is differentiable along h with derivative

$$\nu'(A) = \int_A (-\delta_\nu(h)(x))\, d\nu(x)$$

and the logarithmic derivative $-\delta_\nu(h)$ lies in $L^2(E,\nu)$.

For the other direction we apply again Proposition 2.1. Then for all $g \in \mathcal{C}$

$$\int_E g'(x)(h(x))d\nu(x) = -\int_E g(x)d\nu'(x) = -\int_E g(x) \cdot \xi_h(x)\, d\nu(x),$$

with $\xi_h \in L^2(E,\nu)$. Hence, by the Hölder inequality (see e.g. Ash [5]),

$$\left| \int_E g'(x)(h(x)) d\nu(x) \right| \leq \int_E |g(x) \cdot \xi_h(x)| \, d\nu(x) \leq$$

$$\sqrt{\int_E g^2(x) \, d\nu(x)} \cdot \sqrt{\int_E \xi_h^2(x) \, d\nu(x)}.$$

Hence, h is in the domain of δ_ν and $\delta_\nu(h) = -\xi_\nu$. $\square$

7 S-Differentiability of Internal Measures[1]

In this chapter we start dealing with internal measures. Throughout Section 7 to Section 11 we have the following standing assumption. Let Ω be an internal set, $\mathcal{A}$ an internal ${}^*\sigma$-field on Ω and $\mu \geq 0$ an internal S-bounded (${}^*\sigma$-additive) measure on $\mathcal{A}$. Let $(\mu_t)_{t\in J}$ be an internal curve of nonnegative S-bounded measures on $\mathcal{A}$. Since we want to obtain an external curve of Loeb measures, we assume that for some $\varepsilon \in \mathbb{R}^+$ the internal parameter set J is either an interval of ${}^*\mathbb{R}$ containing the standard interval $I =]-\varepsilon, \varepsilon[$ or J is a discrete interval $\{\frac{-k}{H}, \frac{-k+1}{H}, \ldots, \frac{k-1}{H}, \frac{k}{H}\}$ with $H \in {}^*\mathbb{N} \setminus \mathbb{N}$, $k \in \{1, \ldots, H\}$ and $\varepsilon \leq \frac{k}{H}$. Finally we assume that $\mu = \mu_0$.

The internal curve $(\mu_t)_{t\in J}$ is called **S-continuous** if $\mu_t(A) \approx \mu_s(A)$ for each $A \in \mathcal{A}$ whenever $t \approx s$. We now introduce S-differentiability for internal measures. The choice of the examples and the questions dealt with are based on the standard literature especially those written by Smolyanov, Weizsäcker and Bogachev. Suppose we have a (not necessarily internal) set $\mathcal{C}$ of internal ${}^*\mathbb{R}$-valued functions on Ω, each being $\mathcal{A}$-measurable and S-bounded. We say that the internal measure μ is **S-differentiable with respect to the set $\mathcal{C}$** if there exists an internal S-bounded measure μ' on $\mathcal{A}$ so that for all $f \in \mathcal{C}$ and for all infinitesimals $t \in J$, $t \neq 0$

$$\frac{\int_\Omega f(\omega)d\mu_t(\omega) - \int_\Omega f(\omega)d\mu(\omega)}{t} \approx \int_\Omega f(\omega)d\mu'(\omega).$$

We call μ' an (internal) **derivative (measure)** of μ. If there exists an internal μ-integrable function β so that for all $A \in \mathcal{A}$

$$\mu'(A) \approx \int_A \beta(\omega)\, d\mu(\omega),$$

then also β is called (internal) **logarithmic derivative**.

Note that a derivative measure is not uniquely determined by the above definition. If μ is S-differentiable with respect to a set $\mathcal{C}$ and if for some infinitesimal $t \in J$ the internal measure $\frac{\mu_t - \mu}{t}$ has limited values, then $\frac{\mu_t - \mu}{t}$ is a derivative of μ.

In the next section we will regard and compare S-differentiability for different

[1] The main results and ideas of Section 7 to Section 11 have already been published in [1].

sets $\mathcal{C}$ of functions.

8 Different Forms of S-Differentiability and their Relationships

Following the standard literature we define S-Fomin-differentiability. The internal measure μ is called S**-Fomin-differentiable** if the differentiability is with respect to $\mathcal{C} = \{1_A : A \in \mathcal{A}\}$. This definition yields the first lemma.

8.1 Lemma

If μ is S-Fomin-differentiable each measure $\frac{\mu_t - \mu}{t}$ with $t \approx 0$, $t \in J \setminus \{0\}$, is a derivative of μ.

The next proposition shows the power of S-Fomin-differentiability.

8.2 Proposition

If μ is S-Fomin-differentiable and μ' is a derivative of μ, then μ is S-differentiable with repect to the set $\mathcal{C}$ of all (internal) S-bounded ${}^*\mathbb{R}$-valued $\mathcal{A}$-measurable functions. The Fomin-derivative μ' is also a derivative with respect to $\mathcal{C}$.

Proof: Let μ be S-Fomin-differentiable and let μ' be a derivative of μ. Let $t \neq 0$ be an infinitesimal of J and set $\tilde{\mu} := \frac{\mu_t - \mu}{t}$. Since $\mu' \approx \tilde{\mu}$ on $\mathcal{A}$ we obtain $\int_\Omega f(\omega) d\mu'(\omega) \approx \int_\Omega f(\omega) d\tilde{\mu}(\omega)$ for all S-bounded $\mathcal{A}$-measurable functions $f : \Omega \to {}^*\mathbb{R}$. $\square$

Now let Ω be a subset of ${}^*M$ where M is a metric space and let $\mathcal{A}$ be an internal ${}^*\sigma$-field on Ω. The measure μ is called S**-Skorokhod-differentiable** if it is S-differentiable with respect to the set of all S-bounded, $\mathcal{A}$-measurable functions $f : \Omega \to {}^*\mathbb{R}$ that are S-continuous.

As a consequence of Proposition 8.2, S-Fomin-differentiability implies S-Skorokhod-differentiability. The following example shows that the converse is not true.

8.3 Example

Fix a natural number $H \in {}^*\mathbb{N} \setminus \mathbb{N}$ and let $\Omega \subset {}^*\mathbb{R}$, $\Omega = \{\frac{1}{H} \cdot z : z \in {}^*\mathbb{Z}\}$. The measure μ is the counting measure defined on the field of internal subsets of Ω, i.e.

$$\mu(A) \;=\; \frac{\mid A\cap[0,\frac{H-1}{H}]\mid}{H}.$$

Here $[0,\frac{H-1}{H}] \;=\; \{0,\frac{1}{H},\ldots,\frac{H-1}{H}\}$ and $\mid A\mid$ is the internal number of elements. Now let $J\subset\Omega$, $J=[-\frac{l}{H},\frac{l}{H}]$ with $l\in{}^*\mathbb{N}$, $\frac{l}{H}\approx\frac{1}{2}$ and let $(\mu_t)_{t\in J}$ be defined by $\mu_t(A)=\mu(A+t)$. Note that for any $k\in{}^*\mathbb{N}$ with $\frac{k}{H}\in J$ we obtain:

$$(+)\;\; \frac{\mu_{\frac{k}{H}}-\mu}{\frac{k}{H}}(A) \;=\; \frac{1}{k}\Big(\mid A\cap\left[\frac{-k}{H},\frac{-1}{H}\right]\mid - \mid A\cap\left[\frac{H-k}{H},\frac{H-1}{H}\right]\mid\Big)$$

and

$$(++)\;\; \frac{\mu_{\frac{-k}{H}}-\mu}{\frac{-k}{H}}(A) \;=\; \frac{1}{k}\Big(\mid A\cap\left[0,\frac{k-1}{H}\right]\mid - \mid A\cap\left[\frac{H}{H},\frac{H+k-1}{H}\right]\mid\Big).$$

Now choose $k\in{}^*\mathbb{N}$ with $\frac{k}{H}\in J$ and $\frac{k}{H}\approx 0$ and set $A=[0,\frac{k-1}{H}]$. Then

$$\frac{\mu_{\frac{k}{H}}-\mu}{\frac{k}{H}}(A) \;=\; 0 \;\text{ and }\; \frac{\mu_{\frac{-k}{H}}-\mu}{\frac{-k}{H}}(A) \;=\; 1.$$

Hence μ is not S-Fomin-differentiable. Now we show that μ is S-Skorokhod-differentiable with internal derivatives $\frac{\mu_t-\mu}{t}$ for all infinitesimals $t\in J\setminus\{0\}$. Assume $f:\Omega\to{}^*\mathbb{R}$ is an internal S-bounded, $\mathcal{A}$-measurable and S-continuous function and $k\in{}^*\mathbb{N}$ with $\frac{k}{H}\in J$ and $\frac{k}{H}\approx 0$. Then, using (+) and the S-continuity of f

$$\frac{\int_\Omega f(\omega)d\mu_{\frac{k}{H}}(\omega)-\int_\Omega f(\omega)d\mu(\omega)}{\frac{k}{H}} = \frac{1}{k}\sum_{i=-k}^{-1} f\left(\frac{i}{H}\right) - \frac{1}{k}\sum_{i=H-k}^{H-1} f\left(\frac{i}{H}\right) \approx f(0)-f(1).$$

Of course, (++) yields the same result for $-k$, which completes the proof. We will give a - well known - standard application of this example in Example 11.2.

Finally we consider S-differentiability with respect to *continuous functions. We will see that this is equivalent to S-Fomin-differentiability. Let Ω be an internal *normal space, $\mathcal{A}$ the internal field of Borel subsets. Note that an internal nonnegative measure μ is ***regular** if for all $A\in\mathcal{A}$ we have

$$\mu\left(A\right)=\sup\left\{\mu\left(C\right): C\subseteq A\,\text{and}\,C\,\text{is}\;{}^*\text{closed}\right\}$$

and

$$\mu(A) = \inf\{\mu(O) : A \subseteq O \,\text{and}\, O \,\text{is}\, {}^*\text{open}\}.$$

A measure μ with Jordan-Hahn-decomposition $\mu = \mu^+ + \mu^-$ is called *regular if μ^+ and μ^- are both *regular.

8.4 Proposition

*Let Ω be an internal *normal space, $\mathcal{A}$ the internal field of Borel subsets. If $(\mu_t)_{t\in J}$ is a curve of *regular measures, then μ is S-differentiable with respect to the set $\mathcal{C}$ of all internal *continuous S-bounded functions if and only if μ is S-Fomin-differentiable.*

Proof: By Proposition 8.2 we only have to prove " $\Rightarrow$ ". Let $A \in \mathcal{A}$. We have to show that for all infinitesimals $t, s \in J$,

$$\frac{\mu_t - \mu}{t}(A) \approx \frac{\mu_s - \mu}{s} \quad \in \quad Lim.$$

Fix $t, s \approx 0$. It is easy to verify that with μ also $\frac{\mu_t-\mu}{t}$ and $\frac{\mu_s-\mu}{s}$ are *regular measures. By the definition of *regular there exists an open set O and a closed set C with $C \subset A \subset O$ and so that we get both:

$$(\frac{\mu_t - \mu}{t})^+(O \setminus C) \;\approx\; 0 \text{ and } (\frac{\mu_t - \mu}{t})^-(O \setminus C) \;\approx\; 0$$

and

$$(\frac{\mu_s - \mu}{s})^+(O \setminus C) \;\approx\; 0 \text{ and } (\frac{\mu_s - \mu}{s})^-(O \setminus C) \;\approx\; 0.$$

By transfer of Urysohn's Lemma (see Ash [5]) there exists a *continuous function $f : \Omega \to {}^*[0,1]$ so that $f \equiv 1$ on C and $f \equiv 0$ on $\Omega \setminus O$. Hence

$$\int_\Omega f(\omega)\, d\frac{\mu_t - \mu}{t}(\omega) \;=\; \int_\Omega f(\omega)\, d\left(\frac{\mu_t - \mu}{t}\right)^+(\omega) - \int_\Omega f(\omega)\, d\left(\frac{\mu_t - \mu}{t}\right)^-(\omega) \;\approx$$

$$\left(\frac{\mu_t - \mu}{t}\right)^+(A) - \left(\frac{\mu_t - \mu}{t}\right)^-(A) \;=\; \frac{\mu_t - \mu}{t}(A).$$

Similarly we have

$$\int f(\omega)\, d\frac{\mu_s - \mu}{s}(\omega) \;\approx\; \frac{\mu_s - \mu}{s}(A).$$

Now the Skorokhod-differentiability yields

$$\frac{\mu_t - \mu}{t}(A) \approx \frac{\mu_s - \mu}{s}(A) \in Lim,$$

as required. $\square$

9 S-Differentiability on Interal Vector Spaces

Throughout this section let Ω be an internal vector space. We change the standing assumption of Section 7 in so far as we do not start with a given family of measures, but only with one measure μ.

Due to the standard definitions, presented in Chapter 2, we use an internal vector field $h : \Omega \to \Omega$ to define a transformation T_t for each $t \in J$ by

$$T_t : \Omega \to \Omega,\ t \mapsto T_t(\omega) := \omega - t \cdot h(\omega).$$

Then the curve $(\mu_t)_{t\in J}$ is given by the internal image measures $\mu_t := \mu \circ T_t^{-1}$. If μ is S-differentiable with respect to a set $\mathcal{C}$, then μ is called S-differentiable **along the vector field** h. For a constant vector field h, with $h(\omega) = y$ for all ω, we say that μ is S-differentiable **along the vector** y.

9.1 Proposition

Let $(\mu_t)_{t\in J}$ be defined as above, h an internal vector field. Let $\mathcal{C}$ be a set of internal ${}^*\mathbb{R}$-valued, $\mathcal{A}$-measurable and S-bounded functions on Ω that have the following property: for all $\omega \in \Omega$ the functions

$$J \setminus \{0\} \to {}^*\mathbb{R},\ t \mapsto \frac{f(\omega + t \cdot h(\omega)) - f(\omega)}{t}$$

*are S-continuous. Then μ is S-differentiable with respect to $\mathcal{C}$ along h if and only if there exists an internal S-*bounded *measure μ' such that for all $f \in \mathcal{C}$ and $t \in J \setminus \{0\}$ with $t \approx 0$*

$$\int_\Omega \frac{f(\omega + t \cdot h(\omega)) - f(\omega)}{t} d\mu(\omega) \approx -\int_\Omega f(\omega) d\mu'(\omega).$$

In this case the measure μ' is a derivative of μ.

Proof: " $\Rightarrow$ " Let μ be S-differentiable with respect to $\mathcal{C}$ along h with derivative μ'. Choose $f \in \mathcal{C}$ and $t \in J \setminus \{0\}$ with $t \approx 0$. Set $s := -t$. Then

$$\int_\Omega f(\omega) d\mu'(\omega) \approx \frac{\int_\Omega f(\omega) d\mu_s(\omega) - \int_\Omega f(\omega) d\mu(\omega)}{s} =$$

$$\frac{\int_\Omega f(T_s(\omega)) d\mu(\omega) - \int_\Omega f(\omega) d\mu(\omega)}{s} = \frac{\int_\Omega f(\omega - s \cdot h(\omega)) d\mu(\omega) - \int_\Omega f(\omega) d\mu(\omega)}{s} =$$

$$= -\frac{\int_\Omega f(\omega + t \cdot h(\omega))d\mu(\omega) - \int_\Omega f(\omega)d\mu(\omega)}{t}.$$

" $\Leftarrow$ " This direction follows the same argumentation. □

The following lemma is an internal version of Lemma 3.1. It is easily verified.

9.2 Lemma

Let μ be S-differentiable along a vector $y \in \Omega$ with respect to a set $\mathcal{C}$. Let ν' be a derivative of ν. If for all $t \in J$ the set $\mathcal{C}$ coincides with

$$\{g : \Omega \to {}^*\mathbb{R} : \text{there is an element } f \in \mathcal{C} \text{ with } g(x) = f(x + t \cdot y) \text{ for all } x \in \Omega\},$$

then all measures ν_t are S-differentiable along y with respect to $\mathcal{C}$ and the measures $(\nu')_t$ are derivatives.

9.3 Lemma

Let μ be S-Fomin-differentiable along a vector y, then the curve $(\mu_t)_{t \in J}$ is S-continuous.

Proof: Choose $s, t \in J$ with $s \approx t$ and $A \in \mathcal{A}$. Set $B := \{\omega \in \Omega : \omega - t \cdot y \in A\}$. Since $s - t \approx 0$, the S-Fomin-differentiability and Lemma 9.2 yield

$$\frac{\mu_s(B) - \mu_t(B)}{s - t} \in Lim,$$

and therefore $\mu_s(B) - \mu_t(B) \approx 0$. □

10 S-Differentiability on the Euclidian Space ${}^*\mathbb{R}^M$

In this section we regard measures with nonnegative Lebesgue densities on ${}^*\mathbb{R}^M$ with $M \in {}^*\mathbb{N}$.

10.1 Theorem

Let $\Omega = {}^*\mathbb{R}^M$ with $M \in {}^*\mathbb{N}$, $\mathcal{A}$ the internal field of Borel subsets, μ an internal S-bounded measure on $\mathcal{A}$ and $h : {}^*\mathbb{R}^M \to {}^*\mathbb{R}^M$ an internal vector field. Take $J = {}^*\mathbb{R}$ and define $T_t : {}^*\mathbb{R}^M \to {}^*\mathbb{R}^M$, $t \mapsto T_t(x) := x - t \cdot h(x)$ for all $t \in J$. Finally set $\mu_t := \mu \circ T_t^{-1}$. *Assume that μ has an internal Lebesgue density f satisfying the following three conditions:*
(1) $f \geq 0$, $f = 0$ only on a set of internal Lebesgue measure zero.
(2) f is *(Gateaux-)differentiable in the direction of $h(x)$ for all $x \in {}^*\mathbb{R}^M$ (with derivative $f'_{h(x)}(x)(h(x)) := \langle f'(x); h(x)\rangle_{{}^*\mathbb{R}^M}$).
(3) If $t \approx 0$, then there is a μ-integrable internal function $g : {}^*\mathbb{R}^M \to {}^*\mathbb{R}$ with $\int_{{}^*\mathbb{R}^M} g(x)d\mu(x) \approx 0$ and for all $x \in {}^*\mathbb{R}^M$ with $f(x) \neq 0$

$$\left| \frac{1}{t}\Big(\frac{f(x + t \cdot h(x))}{f(x)} - 1\Big) \; - \; \frac{f'_{h(x)}(x)}{f(x)} \right| \leq g(x).$$

Now if the internal function $\beta^h_\mu : {}^*\mathbb{R}^M \to {}^*\mathbb{R}$, defined by

$$\beta^h_\mu(x) = \begin{cases} \frac{f'_{h(x)}(x)}{f(x)} & \textit{if } f(x) \neq 0 \\ 0 & \textit{if } f(x) = 0, \end{cases}$$

is S_μ-integrable, then μ is S-Fomin-differentiable along the vector field h and if μ' is a derivative then β^h_μ is a logarithmic derivative, hence for all $A \in \mathcal{A}$

$$\mu'(A) \approx \int_A \beta^h_\mu(x)d\mu(x).$$

Proof: Since β^h_μ is S_μ-integrable, $A \mapsto \int_A \beta^h_\mu(x)d\mu(x)$ defines an S-bounded measure. Now let $N = \{x \in {}^*\mathbb{R}^M : f(x) = 0\}$. Then $\lambda^M(N) = 0$, where λ is the internal Lebesgue measure. If $t \approx 0$, $t \neq 0$, then

$$\left| \frac{\mu_t(A) - \mu(A)}{t} - \int_A \beta_\mu^h(x) d\mu(x) \right| \leq$$

$$\int_A \left| \frac{f(x + t \cdot h(x)) - f(x)}{t} - \beta_\mu^h(x) \cdot f(x) \right| d\lambda^M(x) \; =$$

$$\int_{A \setminus N} \left| \frac{1}{t} \Big(\frac{f(x + t \cdot h(x))}{f(x)} - 1 \Big) - \frac{f'_{h(x)}(x)}{f(x)} \right| d\mu(x) \; \leq \; \int_{{}^*\mathbb{R}^M} g(x) d\mu(x) \; \approx 0.$$

Here we have used condition (3). $\square$

The following example is the nonstandard version of a typical standard measure.

10.2 Example

Let $\Omega = {}^*\mathbb{R}$, $\mathcal{A}$ be the internal field of Borel subsets and μ defined by $\mu(A) = \int_A \frac{1}{1+x^2} \, d\lambda(x)$. Fix an element $y \in {}^*\mathbb{R}$ and define $h(x) \equiv y$. This yields

$$\mu_t(A) = \int_{A+ty} \frac{1}{1+x^2} \, d\lambda(x), \quad t \in {}^*\mathbb{R}.$$

If y is limited, then it is easy to see that the assumptions of Theorem 10.1 are satisfied. Hence the measure μ is S-Fomin-differentiable and if μ' is a derivative, then

$$\mu'(A) \approx \int_A \frac{-2xy}{1+x^2} \, d\mu(x)$$

for each $A \in \mathcal{A}$. If y is an unlimited element of ${}^*\mathbb{R}$, then μ is not S-Fomin-differentiable, because for $t = \frac{1}{y}$ and for the internal interval $A = {}^*[0,1] \subset {}^*\mathbb{R}$ the value $\frac{\mu_t(A) - \mu(A)}{t}$ is unlimited.

In Section 21 we will use Theorem 10.1 to show S-Fomin-differentiability of a nonstandard representation of the Wiener measure.

11 Differentiability of Loeb Measures

In this section we show how S-differentiability of an internal measure yields differentiability of the corresponding Loeb measure. Recall the standing assumption of Section 7. In addition we claim that the curve $(\mu_t)_{t\in J}$ is S-continuous. Then we can define a curve of Loeb measures in a unique way. Let $\varepsilon \in \mathbb{R}^+$ as described in the standing assumption, $I =]-\varepsilon, \varepsilon[$. For each $r \in I$ choose $t \in J$ such that $t \approx r$ and set $\mu_r := \mu_t$. Let us denote the associated Loeb spaces by $(\Omega, L_{\mu_r}(\mathcal{A}), \mu_L)$. Since the Loeb σ-fields $L_{\mu_r}(\mathcal{A})$ are not necessarily identical we choose a joint σ-field $\mathcal{F} \subset \bigcap_{r\in I} L_{\mu_r}(\mathcal{A})$. We now define the curve $((\mu_L)_r)_{r\in I}$ of measures on $\mathcal{F}$ by $(\mu_L)_r := (\mu_r)_L$ restricted to $\mathcal{F}$. First let us mention some obvious connections between S-differentiability and differentiability.

11.1 Lemma

Let μ be S-differentiable with respect to a set $\mathcal{C}$ of internal ${}^*\mathbb{R}$-valued, $\mathcal{A}$-measurable and S-bounded functions on Ω and let μ' be an internal derivative of μ.
(1) Then for all $f \in \mathcal{C}$

$$\lim_{r\to 0} \frac{{}^\circ\Big(\int_\Omega f(\omega) d\mu_r(\omega)\Big) - {}^\circ\Big(\int_\Omega f(\omega) d\mu(\omega)\Big)}{r} = {}^\circ\Big(\int_\Omega f(\omega) d\mu'(\omega)\Big).$$

The convergence is uniform, if $\mathcal{C}$ is internal.
(2) Suppose

$$\mathcal{C}_L := \{\, g : \Omega \to \mathbb{R} : \textit{there is a function } f \in \mathcal{C} \textit{ with}$$

$$ {}^\circ(f(\omega)) = g(\omega) \textit{ for all } \omega \in \Omega \,\}$$

and

$$\mathcal{F}_{\mu'} := (\bigcap_{r\in I} L_{\mu_r}(\mathcal{A})) \cap L_{\mu'}(\mathcal{A}).$$

Then the Loeb measure μ_L, restricted to $\mathcal{F}_{\mu'}$, is differentiable with respect to the set $\mathcal{C}_L$ and the Loeb extension $(\mu')_L$ of ${}^\circ(\mu')$, restricted to $\mathcal{F}_{\mu'}$, is a derivative $(\mu_L)'$ of μ_L. This means that for all $g \in \mathcal{C}_L$:

$$\lim_{r\to 0} \frac{\int_\Omega g(\omega) d\,(\mu_L)_r\,(\omega) - \int_\Omega g(\omega) d\mu_L(\omega)}{r} = \int_\Omega g(\omega) d\,(\mu')_L\,(\omega).$$

Note that if μ' and γ' are two internal derivatives of μ then $\int_\Omega g(\omega)d\left(\mu'\right)_L(\omega) = \int_\Omega g(\omega)d\left(\gamma'\right)_L$ for all $g \in \mathcal{C}_L$, but the Loeb measures $\left(\mu'\right)_L$ and $\left(\gamma'\right)_L$ on $\mathcal{A}$ and hence also the σ-fields $L_{\mu'}(\mathcal{A})$ and $L_{\gamma'}(\mathcal{A})$ may be different.

11.2 Example

In Example 8.3 we defined an internal counting measure μ. It is easily verified that the curve $(\mu_t)_{t\in J}$ is S-continuous. Since μ is S-Skorokhod-differentiable with internal derivatives $\frac{\mu_t-\mu}{t}$, $t \in J \setminus \{0\}$, $t \approx 0$, we can apply Lemma 11.1. But as we have seen in Example 8.3, for $k \in {}^*\mathbb{N}$ with $\frac{k}{H} \in J$ and $\frac{k}{H} \approx 0$ and $A = [0, \frac{k-1}{H}]$

$$\left(\frac{\mu_{\frac{k}{H}} - \mu}{\frac{k}{H}}\right)_L (A) \;=\; 0 \quad\text{and}\quad \left(\frac{\mu_{-\frac{k}{H}} - \mu}{-\frac{k}{H}}\right)_L (A) \;=\; 1.$$

Nevertheless, there exists a σ-field, on which the Loeb measures $\left(\frac{\mu_t-\mu}{t}\right)_L$ coincide for all infinitesimals $t \in J \setminus \{0\}$. Let $\mathfrak{b}_{\mathbb{R}}$ be the (standard) σ-field of all Borel subsets of $\mathbb{R}$. For $B \in \mathfrak{b}_{\mathbb{R}}$ let $st^{-1}[B] = \{\omega \in \Omega : {}^\circ\omega \in B\}$. By Proposition 1.8, the set $st^{-1}[B]$ is an element of $L_{\mu_r}(\mathcal{A})$ for all $r \in I$ and $(\mu_r)_L(st^{-1}[B]) = \nu_r(B)$, where $\nu_r(B) = \int_{B+r} 1_{[0,1]}(x)d\lambda(x)$ with standard Lebesgue measure λ. But $st^{-1}[B]$ is also an element of $L_{\frac{\mu_t-\mu}{t}}(\mathcal{A})$ for all infinitesimals $t \in J \setminus \{0\}$ and

$$\left(\frac{\mu_t - \mu}{t}\right)_L (st^{-1}[B]) \;=\; 1_B(0) - 1_B(1).$$

Hence the Loeb measures $\left(\frac{\mu_t-\mu}{t}\right)_L$, restricted to the σ-field $\{st^{-1}[B] : B \in \mathfrak{b}_{\mathbb{R}})\}$, coincide for all infinitesimals $t \in J \setminus \{0\}$.
If we define a measure ν' on $\mathfrak{b}_{\mathbb{R}}$ by $\nu'(B) = 1_B(0) - 1_B(1)$, then the S-differentiability of the internal counting measure μ yields the Skorohod-differentiability (along the vector $1 \in \mathbb{R}$) of the standard measure ν with derivative ν', a well known standard result.

We will now see that in the case of S-Fomin-differentiability the σ-field $\mathcal{F}$ doesn't depend on the chosen internal derivative and the derivative $(\mu_L)'$ of μ_L is uniquely determined. Moreover, the differentiability of μ_L is true not only with respect to standard parts of internal functions, but also with respect to all $\mathcal{F}$-measurable bounded real-valued functions on Ω.

We gather these facts together into the following theorem, which is the main result of this thesis.

11.3 Theorem

Let μ be S-Fomin-differentiable and $\mathcal{F} = \bigcap_{r\in I} L_{\mu_r}(\mathcal{A})$. Then μ_L is differentiable on $\mathcal{F}$ with respect to the set $\mathcal{C}_L = \{1_B : B \in \mathcal{F}\}$ and the differentiability is uniform on $\mathcal{C}_L$. The derivative $(\mu_L)'$ is uniquely determined and is absolutely continuous with respect to μ_L. If μ' is an internal derivative of μ, then the Loeb extension $(\mu')_L$ is defined on $\mathcal{F}$ and coincides with $(\mu_L)'$. In particular this is true for all internal measures $\frac{\mu_t-\mu}{t}$, where $t \in J \setminus \{0\}$ is infinitesimal.

Proof: Let μ' be a derivative of μ. We will show the following statements:
(A) For all internal sets $A \in \mathcal{A}$ the limit

$$\lim_{r\to 0} \frac{(\mu_L)_r(A) - \mu_L(A)}{r}$$

exists and is equal to $(\mu')_L(A)$. The convergence is uniform on the internal field $\mathcal{A}$.
(B) If $N \in \mathcal{F}$ is a μ_L-nullset, then

$$\lim_{r\to 0} \frac{(\mu_L)_r(N) - \mu_L(N)}{r} = 0.$$

The convergence is uniform for all μ_L-nullsets of $\mathcal{F}$. Any μ_L-nullset of $\mathcal{F}$ is also a $(\mu')_L$-nullset of $\mathcal{F}$.
(C) If $B \in \mathcal{F}$ and if $A \in \mathcal{A}$ is μ_L-equivalent to B, then

$$\lim_{r\to 0} \frac{(\mu_L)_r(B) - \mu_L(B)}{r} = \lim_{r\to 0} \frac{(\mu_L)_r(A) - \mu_L(A)}{r},$$

in particular the left limit exists. The convergence is uniform on $\mathcal{F}$.
(D) The Loeb extension $(\mu')_L$ is defined on $\mathcal{F}$ and for all $B \in \mathcal{F}$

$$(\mu')_L(B) = \lim_{r\to 0} \frac{(\mu_L)_r(B) - \mu_L(B)}{r}.$$

(A) follows from Lemma 11.1.

To prove *(B)* let N be a μ_L-nullset of $\mathcal{F}$, i.e. there exists a sequence $(N_{\frac{1}{n}})_{n\in\mathbf{N}} \subset$

$\mathcal{A}$ so that for all $n \in \mathbb{N}$ we have $N \subseteq N_{\frac{1}{n}}$, $N_{\frac{1}{n+1}} \subseteq N_{\frac{1}{n}}$ and $\mu(N_{\frac{1}{n}}) \leq \frac{1}{n}$. Let $\tilde{N} := \bigcap_{n=1}^{\infty} N_{\frac{1}{n}}$. Then $\tilde{N} \in \mathcal{F}$, $\mu_L(\tilde{N}) = 0$ and we obtain for all $r \in I$

$$(\mu_L)_r(N) \leq (\mu_L)_r(\tilde{N}).$$

Since $\mu_L(N) = 0$ we get

$$\left| \frac{(\mu_L)_r(N) - \mu_L(N)}{r} \right| = \left| \frac{(\mu_L)_r(N)}{r} \right| \leq \left| \frac{(\mu_L)_r(\tilde{N})}{r} \right| .$$

Therefore it is sufficient to show that $\lim_{r\to 0} \frac{(\mu_L)_r(\tilde{N})}{r} = 0$.
Now since $\mu_L(\tilde{N}) = 0$,

$$\frac{(\mu_L)_r(\tilde{N})}{r} = \frac{(\mu_L)_r(\tilde{N}) - \mu_L(\tilde{N})}{r} = \frac{\lim_{n\to\infty}(\mu_L)_r(N_{\frac{1}{n}}) - \lim_{n\to\infty}\mu_L(N_{\frac{1}{n}})}{r} =$$

$$\lim_{n\to\infty} \frac{(\mu_L)_r(N_{\frac{1}{n}}) - \mu_L(N_{\frac{1}{n}})}{r}.$$

It follows from *(A)*, that for each $n \in \mathbb{N}$ the limit $\lim_{r\to 0} \frac{(\mu_L)_r(N_{\frac{1}{n}}) - \mu_L(N_{\frac{1}{n}})}{r}$ exists and is equal to $(\mu')_L(N_{\frac{1}{n}})$. Since $(\mu')_L$ is defined on the smallest σ-field containing $\mathcal{A}$, the limit $\lim_{n\to\infty}(\mu')_L(N_{\frac{1}{n}}) = (\mu')_L(\tilde{N})$ also exists. Because of the uniform convergence stated in *(A)* and since $(N_{\frac{1}{n}})_{n\in\mathbf{N}} \subset \mathcal{A}$ we can exchange the limits as follows:

$$\lim_{r\to 0} \frac{(\mu_L)_r(\tilde{N})}{r} = \lim_{r\to 0} \lim_{n\to\infty} \frac{(\mu_L)_r(N_{\frac{1}{n}}) - \mu_L(N_{\frac{1}{n}})}{r} =$$

$$\lim_{n\to\infty} \lim_{r\to 0} \frac{(\mu_L)_r(N_{\frac{1}{n}}) - \mu_L(N_{\frac{1}{n}})}{r} = \lim_{n\to\infty} (\mu')_L(N_{\frac{1}{n}}) = (\mu')_L(\tilde{N}).$$

So the measure μ_L is differentiable at $\tilde{N}$ and the value of the derivative is $(\mu')_L(\tilde{N})$.
By Proposition 2.2 $(\mu')_L$ is absolutely continuous with respect to μ_L.
The uniformity of the convergence can be seen by using again the uniform convergence on $\mathcal{A}$.
Of course $(\mu')_L(N) = 0$ since $N \subset \tilde{N}$. Hence the μ_L-nullsets of $\mathcal{F}$ are also $(\mu')_L$-nullsets of $\mathcal{F}$.

(C) Here we show the differentiability for an arbitrary element of $\mathcal{F}$. So let $B \in \mathcal{F}$, $A \in \mathcal{A}$ μ_L-equivalent to B and $r \in I$. Since $\mu_L(B) = \mu_L(A)$, it is sufficient to show that

$$\lim_{r\to 0} \frac{(\mu_L)_r(B) - (\mu_L)_r(A)}{r} = 0.$$

Since μ_L is nonnegative the following estimate is easily verified.

$$| (\mu_L)_r(B) - (\mu_L)_r(A) | \leq (\mu_L)_r(A \triangle B),$$

where $A \triangle B$ denotes the symmetric difference $(A \setminus B) \cup (B \setminus A)$. Now *(B)* yields

$$\lim_{r\to 0} | \frac{(\mu_L)_r(B) - (\mu_L)_r(A)}{r} | \leq \lim_{r\to 0} \frac{(\mu_L)_r(A \triangle B)}{| r |} = 0.$$

The uniform convergence follows from *(A)* and *(B)*. Hence *(C)* is proved.

(D) follows from *(A)*, *(B)* and *(C)*. □

11.4 Lemma

Let μ be a measure with internal Lebesgue density satisfying the assumptions of Theorem 10.1 for a fixed $y \in {}^*\mathbb{R}^H$. Recall that the internal curve is given by $\mu_t(A) = \mu(A + ty)$ for all Borel subsets $A \subset {}^*\mathbb{R}^H$. Let $\varepsilon \in \mathbb{R}^+$, $I =]-\varepsilon, \varepsilon[$. Since μ is S-Fomin-differentiable along y, we can apply Lemma 9.3 and Theorem 11.3. Hence μ_L is differentiable on $\mathcal{F} = \bigcap_{r\in I} L_{\mu_r}(\mathcal{A})$ with respect to $\mathcal{C}_L = \{1_B : B \in \mathcal{F}\}$. Moreover, for the uniquely determined derivative $(\mu_L)'$ we obtain:

$$(\mu_L)'(B) = \int_B {}^\circ\beta_\mu^y(x) d\mu_L(x)$$

for all $B \in \mathcal{F}$, where β_μ^y is defined in Lemma 10.1.

The power of Fomin-differentiability is also shown in the last result. It can be proved directly using routine integration theory or it can be derived from a more general version of Proposition 3.2 in Smolyanov and Weizsäcker [36].

11.5 Corollary

If μ is S-Fomin-differentiable with an internal derivative μ' and $\mathcal{F}$ is defined as in Theorem 11.3, then the Loeb measure μ_L, restricted to $\mathcal{F}$, is differentiable with respect to the set $\mathcal{C}_L$ of all $\mathcal{F}$-measurable real-valued bounded functions on Ω. The Loeb measure $(\mu')_L$, restricted to $\mathcal{F}$, is the derivative of $(\mu)_L$.

12 Lifting and Integrability Results

Now we provide several lifting results we will need for the nonstandard approach to abstract Wiener spaces in the following sections.

We fix an infinite integer $H \in {}^*\mathbb{N}$. Recall that $T = \{1, \ldots, H\}$ and st denotes the standard part map

$$st : T \to [0;1]\,,\; n \mapsto {}^\circ\left(\frac{n}{H}\right).$$

Let ν be the internal counting measure on the internal set ${}^*\mathcal{P}(T)$ of all internal subsets of T. Let $(\mathbb{B}, |\cdot|)$ be a normed space. An internal function $F : T \to {}^*\mathbb{B}$ is called a **lifting** of $f : [0,1] \to \mathbb{B}$ if for ν_L-a.a. $n \in T$

$$F(n) \approx_{|\cdot|} (f({}^\circ(\frac{n}{H}))).$$

We say that a function $F : T \to {}^*\mathbb{B}$ is S-**continuous** if

(1) $F(n)$ is nearstandard in $\mathbb{B}$ for all $n \in T$ and

(2) for all $n, m \in T$ with $\frac{n}{H} \approx \frac{m}{H}$ we have $F(n) \approx_{|\cdot|} F(m)$.

12.1 Proposition

(Osswald [30], Proposition 9.7.1.) *If $F : T \to {}^*\mathbb{B}$ is S-continuous, then*

$$f : [0,1] \to \mathbb{B}, t \mapsto {}^\circ F(n) \quad \textit{with} \quad \frac{n}{H} \approx t,$$

is well defined and continuous.

Let $\mathbb{H}$ be a separable Hilbert space. Recall that $\mathcal{E}(\mathbb{H})$ is the set of all finite dimensional subspaces of $\mathbb{H}$. The next lemma is a direct application of saturation.

12.2 Lemma

There exists a set $\mathbb{F} \in {}^*\mathcal{E}(\mathbb{H})$ such that each orthonormal basis $(\mathfrak{e}_i)_{i\in\mathbb{N}}$ of $\mathbb{H}$ can be extended to an internal orthonormal basis $(\mathfrak{f}_i)_{i\le\omega}$ of $\mathbb{F}$, i.e. $(\mathfrak{f}_i)_{i\le\omega}$ is an orthonormal basis of $\mathbb{F}$ and $\mathfrak{f}_i = {}^*\mathfrak{e}_i$ for all $i \in \mathbb{N}$.

Fix any $\mathbb{F}$ with the property of Lemma 12.2. If $h \in \mathbb{H}$ we define

$$\|{}^*h\|_{\mathbb{F}} := \left\|pr_{\mathbb{F}}^{{}^*\mathbb{H}}({}^*h)\right\|_{\mathbb{F}}.$$

Then it is easy to see that

$$\|{}^*h\|_{\mathbb{F}} \approx \|h\|_{\mathbb{H}}.$$

We call an element $x \in \mathbb{F}$ **nearstandard in** $\mathbb{H}$, if there exists an $h \in \mathbb{H}$ such that $\left\|pr_{\mathbb{F}}^{{}^*\mathbb{H}}({}^*h) - x\right\|_{\mathbb{F}} \approx 0$. Then we write ${}^*h \approx_{\mathbb{F}} x$ and call ${}^\circ x := h$ the **standard part** of x. Note that we use the symbol ${}^\circ x$ also for the standard part of $x \in {}^*\mathbb{H}$ in $\mathbb{H}$.

12.3 Corollary

(1) Assume that $x \in \mathbb{F}$ is nearstandard in $\mathbb{H}$. Then $\|x\|_{\mathbb{F}} \approx \|{}^\circ x\|_{\mathbb{H}}$.
(2) Assume that $x \in {}^*\mathbb{H}$ is nearstandard in $\mathbb{H}$ with standardpart $h \in \mathbb{H}$. Then also $pr_{\mathbb{F}}^{{}^*\mathbb{H}}(x)$ is nearstandard in $\mathbb{H}$ with standardpart h.

Now we can introduce $\mathbb{F}$-valued liftings. An internal function $F : T \to \mathbb{F}$ is called a **lifting** of $f : [0,1] \to \mathbb{H}$ if for ν_L-a.a. $n \in T$

$$F(n) \approx_{\mathbb{F}} {}^*(f({}^\circ(\frac{n}{H}))).$$

We call a function $f : [0,1] \to \mathbb{H}$ **continuous λ-a.e.**, if the set

$$\{t \in [0;1] : f \text{ is continuous in } t\}$$

is a Lebesgue set of measure 1.

12.4 Proposition

Let $f : [0,1] \to \mathbb{H}$ be Lebesgue measurable and continuous λ-a.e. Then

$$F : T \to \mathbb{F}, \;\; n \mapsto pr_{\mathbb{F}}^{{}^*\mathbb{H}}\left({}^*f(\frac{n}{H})\right)$$

is a lifting of f.

Proof: Define $T_f := st^{-1}\left[\{t \in [0;1] : f \text{ is continuous in } t\}\right]$. By Proposition 1.8,

$\nu_L(T_f) = 1$. Now let $n \in T_f$. Since f is continuous in $^\circ\left(\frac{n}{H}\right)$ we obtain

$$^*f(s) \approx_{^*\mathbb{H}} f\left(^\circ(\frac{n}{H})\right)$$

for all $s \in {}^*[0;1]$ with $s \approx \frac{n}{H}$. Hence $^*f\left(\frac{n}{H}\right) \approx_{^*\mathbb{H}} f\left(^\circ(\frac{n}{H})\right)$. By Corollary 12.3 (2) follows

$$F(n) = pr_{\mathbb{F}}^{^*\mathbb{H}}\left(^*f(\frac{n}{H})\right) \approx_{\mathbb{F}} {}^*(f(^\circ(\frac{n}{H})). \qquad \square$$

The next proposition is a slight modification of the Loeb-Anderson lifting theorem (see Loeb and Osswald [26]).

12.5 Proposition

(a) A function $f : [0;1] \to \mathbb{H}$ is Lebesgue measurable if and only if f has a lifting $F : T \to \mathbb{F}$.
(b) A function $f : [0;1] \to \mathbb{H}$ is Bochner integrable with respect to Lebesgue measure and $\left(\int_0^1 \|f\|_{\mathbb{H}}^p \, d\lambda\right)^{\frac{1}{p}} < \infty$ if and only if f has a lifting $F : T \to \mathbb{F}$ with $F \in SL^p(\nu)$. In this case

$$\int_0^1 f(t) d\lambda\,(t) \approx_{\mathbb{F}} \int_T F(n) d\nu\,(n)$$

and

$$\int_0^1 \|f(t)\|_{\mathbb{H}}^p \, d\lambda\,(t) \approx \int_T \|F(n)\|_{\mathbb{F}}^p \, d\nu\,(n)\,.$$

Let $f : [0;1] \to \mathbb{H}$ be a mapping. We define $pr_{\mathbb{F}}^{^*\mathbb{H}}\,(^*f) : {}^*[0;1] \to \mathbb{F}$ by

$$pr_{\mathbb{F}}^{^*\mathbb{H}}\,(^*f)\,(t) := pr_{\mathbb{F}}^{^*\mathbb{H}}\,(^*f\,(t))$$

for all $t \in {}^*[0;1]$.

12.6 Lemma

(a) Let $f : [0;1] \to \mathbb{H}$ be Bochner integrable with respect to the Lebesgue measure. Then $pr_{\mathbb{F}}^{^*\mathbb{H}}\,(^*f)$ is $S_{^*\lambda}$-integrable.
(b) Let $f : [0;1] \to \mathbb{H}$ be Bochner integrable with respect to the Lebesgue measure

where $\left(\int_0^1 \|g\|_{\mathbb{H}}\, d\lambda\right)^{\frac{1}{p}} < \infty$. *Then* $\left\|pr_{\mathbb{F}}^{*\mathbb{H}}\left({}^*f\right)\right\|_{\mathbb{F}}^p$ *is* $S_{^*\lambda}$*-integrable.*

Proof: (a) First note that by transfer $\int_0^1 \|{}^*f(s)\|_{^*\mathbb{H}}\, d^*\lambda(s)$ is limited. Hence also $\int_0^1 \left\|pr_{\mathbb{F}}^{*\mathbb{H}}\left({}^*f\right)(s)\right\|_{\mathbb{F}} d\,{}^*\lambda(s)$ is limited. It remains to show that for each $\varepsilon > 0$ there is a $\delta > 0$ such that for all internal Lebesgue measurable sets B

$$ {}^*\lambda(B) < \delta \Rightarrow \int_B \left\|pr_{\mathbb{F}}^{*\mathbb{H}}\left({}^*f\right)(s)\right\|_{\mathbb{F}} d\,{}^*\lambda(s) < \varepsilon. $$

Fix $\varepsilon > 0$. By Proposition 12.5(b) f has an S_ν-integrable lifting F. Hence there exists $\delta > 0$ so that for all internal $\tilde{A} \subset T$

$$ \nu(\tilde{A}) < \delta \Rightarrow \int_{\tilde{A}} \|F(n)\|_{\mathbb{F}}\, d\nu(n) < \varepsilon. $$

Now let A be a Lebesgue set with $\lambda(A) < \delta$. Then $\lambda(A) = \nu_L\left(st^{-1}(A)\right) = {}^\circ\left(\nu(B)\right)$ where B is a ν_L-approximation of $st^{-1}(A)$. Hence

$$ \int_A \|f(s)\|_{\mathbb{H}}\, d\lambda(s) = \int_{st^{-1}(A)} {}^\circ\|F(n)\|_{\mathbb{F}}\, d\nu_L(n) = $$

$$ \int_B {}^\circ\|F(n)\|_{\mathbb{F}}\, d\nu_L(n) \approx \int_B \|F(n)\|_{\mathbb{F}}\, d\nu(n) < \varepsilon. $$

By transfer, we obtain for all internal Lebesgue sets B with ${}^*\lambda(B) < \delta$

$$ \int_B \left\|pr_{\mathbb{F}}^{*\mathbb{H}}\left({}^*f\right)(s)\right\|_{\mathbb{F}} d\,{}^*\lambda(s) \leq \int_B \|{}^*f(s)\|_{^*\mathbb{H}}\, d\,{}^*\lambda(s) < \varepsilon. $$

(b) can be proved similarly. □

The next useful lifting result is a special case of Anderson's Luzin theorem [4]:

12.7 Proposition

Let $f : [0;1] \to \mathbb{H}$ *be measurable. Then for* ${}^*\lambda$*-a.a.* $t \in {}^*[0;1]$

$$ {}^*f(t) \approx_{^*\mathbb{H}} f\left({}^\circ t\right). $$

In [32] Rodenhausen gives a lifting construction for elements of $L^1(\mathbb{R}, \lambda)$. Cutland presents in [11] a new proof for the same statement. We extend now this construction to $\mathbb{H}$-valued functions. Our proof is due to Cutland.

12.8 Lemma

Let $f : [0,1] \to \mathbb{H}$ be Bochner integrable and define $F : T \to \mathbb{F}$ by

$$F(n) := H \cdot pr_{\mathbb{F}}^{*\mathbb{H}} \left(\int_{\frac{n}{H}}^{\frac{n+1}{H}} {}^*f(s) d\,{}^*\lambda(s) \right),$$

then F is an S_ν-integrable lifting of f. If in addition, $\left(\int_0^1 \|f\|_{\mathbb{H}}\, d\lambda\right)^{\frac{1}{p}} < \infty$, for $p \in \mathbb{N}$, then $F \in SL^p(\nu)$.

Proof: Let us regard ${}^*f : {}^*[0;1] \to \mathbb{F}$. By Proposition 12.7 and Corollary 12.3 we have

$${}^*f(s) \approx_{\mathbb{F}} f({}^\circ s)$$

for ${}^*\lambda$- a.a. $s \in {}^*[0;1]$. By Lemma 12.6(a) $pr_{\mathbb{F}}^{*\mathbb{H}}({}^*f)$ is $S_{^*\lambda}$-integrable. The crucial idea is to regard the bijective, measurable mapping

$$l : T \times {}^*[0;\frac{1}{H}[\to {}^*[0;1[,\ (n;\tau) \mapsto \frac{n-1}{H} + \tau.$$

On T we have again the counting measure ν, on $[0;\frac{1}{H}[$ we take the internal measure $\mu = H \cdot {}^*\lambda$. Then we have ${}^*\lambda(A) = \nu \otimes \mu(l^{-1}(A))$. Hence we can apply Keisler's Fubini theorem in the version for Bochner integrals (see Osswald [30], 10.12.4). Thus for ν_L-a.a $n \in T$ the function

$${}^*f_{\frac{n}{H}} : [0;\frac{1}{H}[\to \mathbb{F},\ s \mapsto pr_{\mathbb{F}}^{*\mathbb{H}}({}^*f)\left(\frac{n}{H} + s\right)$$

is S_μ-integrable and the function $n \mapsto \int_0^{\frac{1}{H}} pr_{\mathbb{F}}^{*\mathbb{H}}({}^*f)\left(\frac{n}{H} + s\right) d\mu(s)$ is S_ν-integrable. Since

$$\int_0^{\frac{1}{H}} pr_{\mathbb{F}}^{*\mathbb{H}}({}^*f)\left(\frac{n}{H} + s\right) d\mu(s) = H \cdot \int_0^{\frac{1}{H}} pr_{\mathbb{F}}^{*\mathbb{H}}({}^*f)\left(\frac{n}{H} + s\right) d\,{}^*\lambda(s) =$$

$$H \cdot \int_{\frac{n}{H}}^{\frac{n+1}{H}} pr_{\mathbb{F}}^{*\mathbb{H}}({}^*f)(s)\, d\,{}^*\lambda(s) = F(n),$$

F is S_ν-integrable and for ν_L-a.a. $n \in T$

$$f\left(^\circ\left(\frac{n}{H}\right)\right) = \int_0^{\frac{1}{H}} f\left(^\circ\left(\frac{n}{H}\right)\right) d\mu_L(s) =$$

$$^\circ \int_0^{\frac{1}{H}} pr_{\mathbb{F}}^{*\mathbb{H}}\left(^* f\right)\left(\frac{n}{H} + s\right) d\mu(s) = {}^\circ\left(F(n)\right).$$

Hence F is an S_ν-integrable lifting of f.

Now assume that $\left(\int_0^1 \|f\|_{\mathbb{H}}\, d\lambda\right)^{\frac{1}{p}} < \infty$ for some $p \in \mathbb{N}$. By Lemma 12.6(b), $\left\|pr_{\mathbb{F}}^{*\mathbb{H}}\left(^* f\right)\right\|_{\mathbb{F}}^p$ is $S_{^*\lambda}$-integrable. Let A be an internal subset of T and $\tilde{A} = \bigcup_{n\in A}[\frac{n}{H}; \frac{n+1}{H}[$. Note that $^*\lambda\left(\tilde{A}\right) = \nu\left(A\right)$. Thus

$$\frac{1}{H}\sum_{n\in A}\|F(n)\|_{\mathbb{F}}^p = \frac{1}{H}\sum_{n\in A}\left\|H\cdot\int_{\frac{n}{H}}^{\frac{n+1}{H}} pr_{\mathbb{F}}^{*\mathbb{H}}\left(^* f\right)(s)\, d^*\lambda(s)\right\|_{\mathbb{F}}^p \leq$$

$$\sum_{n\in A}\int_{\frac{n}{H}}^{\frac{n+1}{H}} \left\|pr_{\mathbb{F}}^{*\mathbb{H}}\left(^* f\right)(s)\right\|_{\mathbb{F}}^p d^*\lambda(s) =$$

$$\int_{\tilde{A}} \left\|^* f(s)\right\|_{\mathbb{F}}^p d^*\lambda(s) \begin{cases} \in Lim & \text{for all internal } A \subseteq T, \\ \approx 0 & \text{if } \nu(A) \approx 0. \end{cases} \qquad \square$$

In the remain of this section we regard lifting conditions for functions of bounded variation. Note that any function of bounded variation is in particular bounded.

12.9 Proposition

Let $f : [0,1] \to \mathbb{H}$ be of bounded variation. Then f is measurable, Bochner integrable and $\left(\int_0^1 \|f\|_{\mathbb{H}}\, d\lambda\right)^{\frac{1}{p}} < \infty$ for all $p \in \mathbb{N}$. Furthermore, f is continuous a.e.

Proof: f is measurable if and only if for each $a \in \mathbb{H}$ the function $f_a(t) = \langle a; f(t)\rangle$ is measurable. Let $a \in \mathbb{H}$ and $0 = t_0 < t_1 < \ldots < t_n = 1$ be a partition of $[0,1]$. Then

$$\sum_{i=1}^n |f_a(t_i) - f_a(t_{i-1})| = \sum_{i=1}^n |\langle f(t_i) - f(t_{i-1}); a\rangle_{\mathbb{H}}| \leq \sum_{i=1}^n \|f(t_i) - f(t_{i-1})\|_{\mathbb{H}} \cdot \|a\|_{\mathbb{H}},$$

hence f_a is also of bounded variation. With elementary analysis it is easy to prove that f_a is then Riemann integrable and therefore Lebesgue measurable. So also f is measurable. This and the boundedness of f yield the Bochner integrability and that $\left(\int_0^1 \|f\|_{\mathbb{H}}\, d\lambda\right)^{\frac{1}{p}} < \infty$ for all $p \in \mathbb{N}$.
Now we show that the set of all $t \in [0;1]$ so that f is not continuous in t is countable. We denote the variation $V(f)$ by k. For each $n \in \mathbb{N}$ define

$$A_{\frac{1}{n}} := \{t \in [0;1] : \text{ for all } \ \delta > 0 \,\text{there is a } \ t_\delta \in [0;1] \ \text{ with } |t - t_\delta| < \delta$$

$$\text{and } \ \|f(t) - f(t_\delta)\|_{\mathbb{H}} > \frac{1}{n}\}.$$

Assume $A_{\frac{1}{n}}$ to be infinite. Then we can choose $n \cdot k$ elements $t_1, \ldots, t_{n \cdot k}$ of $A_{\frac{1}{n}}$. Let $\delta > 0$ so that

$$\delta < \min \{|t_i - t_j| : i, j \in \{1, \ldots, n \cdot k\}, i \neq j\},$$

and choose for each $i \in \{1, \ldots, n \cdot k\}$ a $t_{i_\delta} \in [0;1]$ with $|t_i - t_{i_\delta}| < \delta$ and

$$\|f(t_i) - f(t_{i_\delta})\|_{\mathbb{H}} > \frac{1}{n}.$$

The set $\{t_i, t_{i_\delta} : i \in \{1, \ldots, n \cdot k\}\}$ can be reordered to an increasing sequence and extended to a partition $0 = t_0 < t_1 < \ldots < t_l = 1$ of $[0, 1]$. Then

$$\sum_{j=1}^{l} \| \, f(t_j) - f(t_{j-1}) \, \|_{\mathbb{H}} > k \cdot n \cdot \frac{1}{n} = k,$$

in contrary to our preliminary. Hence $A_{\frac{1}{n}}$ is finite and the set $A = \bigcup_{n \in \mathbb{N}} A_{\frac{1}{n}}$ is countable. Thus f is continuous a.e. $\square$

12.10 Lemma

If $f : [0, 1] \to \mathbb{H}$ is of bounded variation then $F : T \to \mathbb{F}$, $n \mapsto pr_{\mathbb{F}}^{*\mathbb{H}}\left({}^*f(\frac{n}{H})\right)$ is a lifting of f and

$$\sum_{i=2}^{H} \| \, F(i) - F(i-1) \, \|_{\mathbb{F}} \in Lim.$$

Furthermore, $\|F(n)\|_{\mathbb{F}}$ is limited for all $n \in T$ and therefore, $F \in SL^1(\nu)$.

Proof: The lifting result follows immediately from Proposition 12.9 and Proposition 12.4.

Since f is of bounded variation we obtain for all $n \in \mathbb{N}$

$$\sum_{i=2}^{n} \left\| f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right) \right\|_{\mathbb{H}} \leq V(f).$$

By transfer, we get

$$\sum_{i=2}^{H} \left\| {}^*f\left(\frac{i}{H}\right) - {}^*f\left(\frac{i-1}{H}\right) \right\|_{{}^*\mathbb{H}} \leq V(f).$$

Since

$$\|F(i) - F(i-1)\|_{\mathbb{F}} = \left\| pr_{\mathbb{F}}^{{}^*\mathbb{H}} ({}^*f) \left(\frac{i}{H}\right) - pr_{\mathbb{F}}^{{}^*\mathbb{H}} ({}^*f) \left(\frac{i-1}{H}\right) \right\|_{\mathbb{F}} =$$

$$\left\| pr_{\mathbb{F}}^{{}^*\mathbb{H}} \left({}^*f\left(\frac{i}{H}\right) - {}^*f\left(\frac{i-1}{n}\right) \right) \right\|_{\mathbb{F}} \leq \left\| {}^*f\left(\frac{i}{n}\right) - {}^*f\left(\frac{i-1}{n}\right) \right\|_{{}^*\mathbb{H}}$$

the first part of the Lemma is proved. Since F is a lifting, there is of course one $n \in T$ so that $\|F(n)\|_{\mathbb{F}}$ is limited. Now take any other $m \in T$. Without loss of generality assume $m < n$. Then

$$|\|F(n)\|_{\mathbb{F}} - \|F(m)\|_{\mathbb{F}}| \leq \|F(n) - F(m)\|_{\mathbb{F}} \leq \sum_{i=m+1}^{n} \|F(i) - F(i-1)\|_{\mathbb{F}} \leq V(f).$$

Hence also $\|F(m)\|_{\mathbb{F}}$ is limited. Thus this is true for all $n \in T$. Obviously F is then an element of $SL^1(\nu)$.

13 The Internal Probability Space

From now on we work with abstract Wiener spaces and with a nonstandard model of these, developed by Osswald (see for example [30]). In this section we will sketch Osswald's construction by presenting the essential steps and those parts of the proofs we will need. For the detailed proofs see [30], Section11. We use the notation in Section 4 and 5. Let us start from an arbitrary separable Hilbert space $(\mathbb{H}; \|\cdot\|)$. Choose any $\mathbb{F} \in {}^*\mathcal{E}(\mathbb{H})$ with the property of Lemma 12.2 and denote its dimension by ω. Fix an infinite integer $H \in {}^*\mathbb{N}$ to define the space $\mathbb{F}^H$. Let $\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}$ be the internal Gaussian measure on ${}^*\mathbb{R}^{\omega\cdot H}$, i.e.

$$\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}(B) = \left(\sqrt{\frac{H}{2\pi}}\right)^{\omega\cdot H} \int_B \exp\left(-\frac{H}{2}\sum_{i\leq\omega,j\leq H} x_i^2(j)\right) dx_i(j)_{i\leq\omega,j\leq H}.$$

Fix an orthonormal basis $(\mathfrak{e}_i)_{i\leq\omega}$ of $\mathbb{F}$. According to the definition of $\gamma^{E,\sigma}$ in Section 5 we define an internal Gaussian measure Γ on $\mathbb{F}^H$ by

$$\Gamma(A) := \gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}\left(\left\{(x_i(j))_{i\leq\omega,j\leq H} : \left(\sum_{i=1}^{\omega} x_i(j)\cdot\mathfrak{e}_{\mathrm{i}}\right)_{j\leq H} \in A\right\}\right).$$

for each internal $A \in \mathfrak{b}_{\mathbb{F}^{\mathbb{H}}}$. By Lemma 5.1 the measure Γ does not depend on the choice of the ONB of $\mathbb{F}$.

Choose a norm $|\cdot|$ on $\mathbb{H}$, which is measurable with respect to $\sigma = 1$ and let $(\mathbb{B}; |\cdot|)$ be the Banach completion. By Lemma 5.5, for each $\varphi \in \mathbb{B}'$ the restriction $\varphi \upharpoonright \mathbb{H}$ is continuous with respect to $\|\cdot\|$. Therefore we can consider $\mathbb{B}'$ as a subspace of $\mathbb{H}$. By Lemma 5.6 $\mathbb{B}'$ is dense in $(\mathbb{H}; \|\cdot\|)$. Denote by $(C_{\mathbb{B}}; |\cdot|_{\sup})$ the Banach space of all continuous functions $f : [0;1] \to \mathbb{B}$, $f(0) = 0$, together with the supremums norm $|f|_{\sup} = \sup\{|f(t)| : t \in [0;1]\}$. Now we describe the relationship between $\mathbb{F}^{\mathrm{H}}$ and the Banach space $C_{\mathbb{B}}$, given by

the internal mapping[2]:

$$B : \mathbb{F}^{\mathrm{H}} \times T \to \mathbb{F},\ (X, n) \mapsto \sum_{i=1}^{n} X_i.$$

[2]Note that B is an internal Brownian motion (see [30], Chapter 11 for the definition and details). Since - apart from Lemma 13.1 - we don't use the typical properties of Brownian motions we go on without this term.

We use the following abbreviations:

$$B_n : \mathbb{F}^{\mathrm{H}} \to \mathbb{F},\ X \mapsto \sum_{i=1}^{n} X_i$$

$$B(X) : T \to \mathbb{F},\ n \mapsto \sum_{i=1}^{n} X_i.$$

13.1 Lemma

(a) Let $a \in \mathbb{F}$ with $\|a\|_{\mathbb{F}} = 1$. Then $\langle a; B_n \rangle$ is normally distributed with variance $\frac{n}{H}$.
(b) Let $a \in \mathbb{F}$ with $\|a\|_{\mathbb{F}} = 1$, $n, m \in T$ with $m > n$. Then $\langle a; B_n - B_m \rangle$ is normally distributed with variance $\frac{n-m}{H}$.

The aim is to establish a relationship between the internal functions $B(X)$ and the elements of $C_{\mathbb{B}}$. Since $|\cdot|$ is measurable with respect to $\sigma = 1$, there exists a function $g : \mathbb{N} \to \mathcal{E}(\mathbb{H})$ with $g(n) \subseteq g(n+1)$ for each $n \in \mathbb{N}$ and such that for each $m \in \mathbb{N}$ and for each $E \in \mathcal{E}(\mathbb{H})$ with $E \perp g(m)$

$$\gamma^{E,1}\left(\left\{x \in E : |x| \geq \frac{1}{2^m}\right\}\right) < \frac{1}{2^{m+1}}.$$

By transfer, the inequality is also true for each $m \in {}^*\mathbb{N}$ and for each $E \in {}^*\mathcal{E}(\mathbb{H})$ with $E \perp {}^*g(m)$.

13.2 Lemma

Let $m \in {}^*\mathbb{N}$ and $E \in {}^*\mathcal{E}(\mathbb{H})$ with $E \perp {}^*g(m)$ and $E \subset \mathbb{F}$. Then

1. $\gamma^{E,1}\left(\left\{x \in E : |x| \geq \frac{1}{2^m}\right\}\right) = \Gamma\left(\left\{X \in \mathbb{F}^{\mathrm{H}} : \left|pr_E^{\mathbb{F}}\left(B_H(X)\right)\right| \geq \frac{1}{2^m}\right\}\right)$ *and*
2. $\Gamma\left(\left\{X \in \mathbb{F}^{\mathrm{H}} : \max_{n \in T} \left|pr_E^{\mathbb{F}}\left(B_n(X)\right) \geq \frac{1}{2^m}\right|\right\}\right) \leq$
   $2\Gamma\left(\left\{X \in \mathbb{F}^{\mathrm{H}} : \left|pr_E^{\mathbb{F}}\left(B_H(X)\right)\right| \geq \frac{1}{2^m}\right\}\right)$.[3]

Hence we obtain for each $m \in {}^*\mathbb{N}$ and each $E \in {}^*\mathcal{E}(\mathbb{H})$ with $E \perp {}^*g(m)$ and $E \subset \mathbb{F}$

$$\Gamma\left(\left\{X \in \mathbb{F}^{\mathrm{H}} : \max_{n \in T} \left|Pr_E^{\mathbb{F}}\left(B_n(X)\right) \geq \frac{1}{2^m}\right|\right\}\right) \leq 2 \cdot \frac{1}{2^{m+1}} = \frac{1}{2^m}.$$

[3] Note that **2.** is a version of Levy's inequality [25].

The next step is the construction of an ONB $(\mathfrak{h}_i)_{i\in\mathbb{N}}$ of $\mathbb{H}$ by recursion. Fix any ONB $(\mathfrak{e}_i)_{i\in\mathbb{N}}$ of $\mathbb{H}$ and start with the space $span\,\{g(1)\cup\{\mathfrak{e}_1\}\}$. Let $(\mathfrak{h}_i)_{i\le l_1}$ be an ONB of this space. Assume that $(\mathfrak{h}_i)_{i\le l_n}$ is already defined and is an ONB of $span\,\{g(n)\cup\{\mathfrak{e}_1,\ldots,\mathfrak{e}_n\}\}$. Then let $(\mathfrak{h}_i)_{i\le l_{n+1}}$ be an extension of $(\mathfrak{h}_i)_{i\le l_n}$ to an ONB of $span\,\{g(n+1)\cup\{\mathfrak{e}_1,\ldots,\mathfrak{e}_{n+1}\}\}$. Then $(\mathfrak{h}_i)_{i\in\mathbb{N}}$ is an ONB of $\mathbb{H}$ and, by Lemma 12.2, it can be extended to an internal ONB $(\mathfrak{h}_i)_{i\le\omega}$ of $\mathbb{F}$. The next Lemma follows from the inequality above.

13.3 Lemma

For all $m\in\mathbb{N}$ and all $k\in{}^*\mathbb{N}$ with $l_m\le k<\omega$

$$\Gamma\left(\left\{X\in\mathbb{F}^{\mathrm{H}}:\max_{n\in T}\left|\sum_{i=k+1}^{\omega}\langle B_n(X);\mathfrak{h}_i\rangle\cdot\mathfrak{h}_i\right|\ge\frac{1}{2^m}\right\}\right)<\frac{1}{2^m}.$$

Now set for each $m\in\mathbb{N}$

$$U_m:=\left\{X\in\mathbb{F}^{\mathrm{H}}:\max_{n\in T}\left|\sum_{i=l_m}^{\omega}\langle B_n(X);\mathfrak{h}_i\rangle\cdot\mathfrak{h}_i\right|\ge\frac{1}{2^m}\right\}$$

and define

$$U_0:=\bigcup_{p\in\mathbb{N}}\bigcap_{m\ge p}U_m.$$

It is easy to see that $\Gamma_L(U_0)=1$. Finally let $U:=$

$$U_0\cap\bigcap_{i\in\mathbb{N}}\{X\in\mathbb{F}^{\mathrm{H}}:\text{the function }\langle B(X);\mathfrak{h}_i\rangle:T\to{}^*\mathbb{R},\ n\mapsto\sum_{j=1}^{n}\langle X_j;\mathfrak{h}_i\rangle$$

$$is\ S-continuous\}.$$

13.4 Lemma

(1) $\Gamma_L(U)=1$.
(2) Let $X\in U$. For all $n,k\in T$ with $\frac{n}{H}\approx\frac{k}{H}$ and all $m\in\mathbb{N}$ we have

$$B_n(X)\approx_{|\cdot|}B_k(X).$$

In addition, there exists a function $h_X : \mathbb{N} \to \mathbb{N}$, *such that for each* $k \in \mathbb{N}$ *the sequence* $\left(\sum_{i=1}^{h_X(m)} {}^\circ \langle B_k(X); \mathfrak{h}_i \rangle \cdot \mathfrak{h}_i\right)_{m \in \mathbb{N}}$ *is a Cauchy sequence in* $\mathbb{B}$ *and*

$$B_k(X) \approx_{|\cdot|} \lim_{m \to \infty} \sum_{i=1}^{h_X(m)} {}^\circ \langle B_k(X); \mathfrak{h}_i \rangle \cdot \mathfrak{h}_i \in \mathbb{B}.$$

Hence $B_k(X)$ *is nearstandard in* $\mathbb{B}$.

The next result follows from the previous lemma and Proposition 12.1.

13.5 Proposition

For Γ_L*-a.a.* $X \in \mathbb{F}^{\mathrm{H}}$ *the internal function* $B(X)$ *is* S*-continuous. Therefore the function*

$$b_X : [0;1] \to \mathbb{R},\ t \mapsto {}^\circ B_n(X)$$

with $\frac{n}{H} \approx t$, *is well defined and an element of* $C_{\mathbb{B}}$.

According to the construction of H. Osswald in [30] we define $b : \mathbb{F}^{\mathrm{H}} \times [0;1] \to \mathbb{B}$,

$$b\left(X, {}^\circ\left(\frac{n}{H}\right)\right) := \begin{cases} b_X\left({}^\circ\left(\frac{n}{H}\right)\right) = {}^\circ B_n(X) & \text{if } B(X) \text{ is } S\text{-continuous} \\ 0 & \text{otherwise.} \end{cases}$$

Before we define a measure on $C_{\mathbb{B}}$ using b, we state a kind of reverse of Proposition 13.5.

13.6 Proposition

For each $f \in C_{\mathbb{B}}$ *there exists an* $X \in \mathbb{F}^{\mathrm{H}}$ *such that* $B(X)$ *is* S*-continuous and* $b_X = f$.

The last step in the construction of Osswald is the definition of a probability measure W on $\mathfrak{b}_{\mathcal{C}_{\mathbb{B}}}$. To this end set

$$\kappa : \mathbb{F}^{\mathrm{H}} \to C_{\mathbb{B}},\ X \mapsto b_X.$$

Then κ is surjective and Borel measurable. Note that $\kappa^{-1}[A] = \left\{X \in \mathbb{F}^{\mathrm{H}} : b_X \in A\right\}$ for all $A \in \mathfrak{b}_{\mathcal{C}_{\mathbb{B}}}$. Set $\mathcal{F}_0 = \{\kappa^{-1}[A] : A \in \mathfrak{b}_{\mathcal{C}_{\mathbb{B}}}\}$ and denote by $\mathcal{F}$ the σ-field generated by $\mathcal{F}_0$ and all Γ_L-nullsets. Now let us define the probability measure W for each $A \in \mathfrak{b}_{C_{\mathbb{B}}}$ by

$$W(A) = \Gamma_L\left(\kappa^{-1}[A]\right).$$

In the following sections we will see that W is a Gaussian measure and we will specify the corresponding Cameron Martin subspace of $C_{\mathbb{B}}$. In contrast to Osswald in [30] we will do this on the basis of measure differentiability.

14 A Nonstandard Representation of $C_{\mathbb{H}}$

We work again with the abstract Wiener space $(\mathbb{H}, \mathbb{B})$ and the Banach space $\left(C_{\mathbb{B}}; |\cdot|_{\text{sup}}\right)$ introduced in Chapter 13. In this section we present a nonstandard characterisation of a subspace $C_{\mathbb{H}}$ of $C_{\mathbb{B}}$ that in Section 22 turns out to be the Cameron Martin subspace $H(W)$. Define

$$C_{\mathbb{H}} := \{f : [0;1] \to \mathbb{H} : \text{there exists a square Bochner integrable function}$$

$$\dot{f} \in L^2(\lambda, \mathbb{H}) \text{ so that } f(t) = \int_0^t \dot{f}(s)ds \text{ for all } t \in [0;1]\}.$$

Since $\mathbb{H} \subset \mathbb{B}$, $C_{\mathbb{H}}$ is a subspace of $C_{\mathbb{B}}$ containing all absolutely continuous, $\mathbb{H}$-valued functions with square Bochner integrable derivative.

14.1 Lemma

$C_{\mathbb{H}}$ is dense in $\left(C_{\mathbb{B}}; |\cdot|_{sup}\right)$.

Proof: Denote by A_n the set of all functions $f \in C_{\mathbb{B}}$ that take values of $\mathbb{H}$ on $\left\{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\right\}$ and that are linear between these points. Set $A = \bigcup_{n\in\mathbb{N}} A_n$. Clearly, $A \subset C_{\mathbb{H}}$. Since $\mathbb{H}$ is dense in $\mathbb{B}$, and since all functions of $C_{\mathbb{B}}$ are absolutely continuous, A is dense in $\left(C_{\mathbb{B}}; |\cdot|_{sup}\right)$. Hence, this is also true for $C_{\mathbb{H}}$. $\square$

An inner product on $C_{\mathbb{H}}$ is given by

$$\langle f; g\rangle_{C_{\mathbb{H}}} := \int_0^1 \dot{f}(s) \cdot \dot{g}(s)ds$$

for $f, g \in C_{\mathbb{H}}$. It is easy to check that $\left(C_{\mathbb{H}}; \|\cdot\|_{C_{\mathbb{H}}}\right)$ is a separable Hilbert space. As shown in Section 13, there is a relationship between $C_{\mathbb{B}}$ and $\mathbb{F}^{\mathrm{H}}$ given by the mapping $B : \mathbb{F}^H \times T \to \mathbb{F}$, $(X, n) \mapsto \sum_{i=1}^n X_i$. We will show now which elements of $\mathbb{F}^{\mathrm{H}}$ are mapped by B onto functions of $C_{\mathbb{H}}$. Define

$$H_{\mathbb{F}} := \{Y \in \mathbb{F}^H : H \cdot \sum_{i=1}^{H} \|Y_i\|_{\mathbb{F}}^2 \in \mathit{Lim} \text{ and } \sum_{i=1}^{n} Y_i$$

$$\text{is nearstandard in } \mathbb{H} \text{ for all } n \in T\}.$$

14.2 Remark

Note the following obvious relationship between elements of $C_{\mathbb{H}}$ and $H_{\mathbb{F}}$: For $Y \in H_{\mathbb{F}}$ define the derivative $\frac{\Delta B(Y)}{\triangle t}$ of $B(Y)$ by

$$\frac{\Delta B(Y)}{\triangle t}:\ n \mapsto \frac{\sum_{i=1}^{n} Y_i - \sum_{i=1}^{n-1} Y_i}{\frac{1}{H}} = H \cdot Y_n.$$

Therefore, $H \cdot \sum_{i=1}^{H} \|Y_i\|_{\mathbb{F}}^2$ is the integral $\int_T \left\| \frac{\triangle B(Y)(i)}{\triangle t} \right\|_{\mathbb{F}}^2 d\nu\,(i) = \frac{1}{H} \cdot \sum_{i=1}^{H} \|H \cdot Y_i\|_{\mathbb{F}}^2$.

14.3 Proposition

(1) For $Y \in H_{\mathbb{F}}$ the internal function $B(Y)$ is S-continuous and $b_Y \in C_{\mathbb{H}}$.
(2) For $f \in C_{\mathbb{H}}$ the vector $Y_f = \left(pr_{\mathbb{F}}^{*\mathbb{H}}\left({}^*f\left(\frac{n}{H}\right) - {}^*f\left(\frac{n-1}{H}\right)\right)\right)_{n \in T}$ lies in $H_{\mathbb{F}}$ and $b_Y = f$.

Proof: (1) Let $Y \in H_{\mathbb{F}}$. Define the internal function $G : T \to {}^*\mathbb{R}$, $i \mapsto H \cdot Y_i$. Since

$$\int_T \|G(i)\|_{\mathbb{F}}^2\, d\nu(i) = \frac{1}{H} \sum_{i=1}^{H} \left(H^2 \cdot \|Y_i\|_{\mathbb{F}}^2\right) = H \cdot \sum_{i=1}^{H} \|Y_i\|_{\mathbb{F}}^2 \in Lim,$$

by Proposition 1.10, G is S_ν-integrable. Hence, for $n, m \in T$ with $\frac{n}{H} \approx \frac{m}{H}$ and $m < n$

$$0 \approx \int_m^n \|G(i)\|_{\mathbb{F}}\, d\nu(i) = \frac{1}{H} \sum_{i=m}^{n} H \cdot \|Y_i\|_{\mathbb{F}} = \sum_{i=m}^{n} \|Y_i\|_{\mathbb{F}}.$$

Therefore,

$$\|B_m(Y) - B_n(Y)\|_{\mathbb{F}} = \left\| \sum_{i=m+1}^{n} Y_i \right\|_{\mathbb{F}} \leq \sum_{i=m+1}^{n} \|Y_i\|_{\mathbb{F}} \approx 0.$$

Since, by the definition of $H_{\mathbb{F}}$, $B_n(Y)$ is nearstandard in $\mathbb{H}$ for all $n \in T$, $B(Y)$ is S-continuous. Hence b_Y is well defined and an element of $C_{\mathbb{B}}$. Obviously b_Y is $\mathbb{H}$-valued.
We prove now that $f := b_Y$ is absolutely continuous. Let $\varepsilon > 0$. Since G is S_ν-integrable there is a $\widetilde{\delta} > 0$ such that $\int_A \|G(i)\|_{\mathbb{F}}\, d\nu(i) < \varepsilon$ if $A \in \mathcal{C}$ and $\nu(A) < \tilde{\delta}$. Set $\delta := \frac{\tilde{\delta}}{2}$ and let $\{]a_1; b_1[, \ldots]a_k; b_k[\}$ be a family of pairwise disjunct open subintervals of $[0; 1]$ with total length at most δ. Choose $n_{a_i}, n_{b_i} \in T$ such that $a_i \approx \frac{n_{a_i}}{H}, b_i \approx \frac{n_{b_i}}{H}$ for all $i \in \{1, \ldots, k\}$ and such that $n_{b_i} \neq n_{a_{i+1}}$ for all

$i \in \{1, \ldots, k-1\}$. Set $A := \bigcup_{i=1}^{k}]n_{a_i}; n_{b_i}]$ where $]n_{a_i}; n_{b_i}]$ is an interval in T. Then

$$\nu(A) = \sum_{i=1}^{k} \frac{n_{b_i} - n_{a_i}}{H} = \sum_{i=1}^{k} \left(\frac{n_{b_i}}{H} - \frac{n_{a_i}}{H} \right) \leq \tilde{\delta} < \delta.$$

Hence

$$\sum_{i=1}^{k} \|f(b_i) - f(a_i)\|_{\mathbb{H}} \approx \sum_{i=1}^{k} \left\| \sum_{j=n_{a_i}+1}^{n_{b_i}} Y_j \right\|_{\mathbb{F}} \leq \sum_{j \in A} \|Y_j\|_{\mathbb{F}} = \int_A \|G(j)\|_{\mathbb{F}} \, d\nu(j) < \varepsilon.$$

So f is absolutely continuous. It remains to prove that $\dot{f}$ is square Bochner integrable. We will do this due to N. Cutland's ideas to the real valued case (see Theorem 5.1 with proof in [11]). Since G is S_ν-integrable, $\|G(n)\|_{\mathbb{F}}$ is nearstandard for the elements n of a Loeb-set T_ν with $\nu_L(T_\nu) = 1$. Choose a sequence $(A_m)_{m \in \mathbb{N}}$ of increasing internal subsets of T_ν such that $\nu(A_m) \geq 1 - \frac{1}{m}$. Define $A = \bigcup_{m \in \mathbb{N}} A_m$. Then $A \subset T_\nu$ and $\nu_L(A) = 1$. Define $g : T \to \mathbb{H}$ by

$$g(n) := \begin{cases} {}^\circ G(n) & \text{if } n \in A \\ 0 & \text{if } n \notin A. \end{cases}$$

G is an S_ν-integrable lifting of g, hence g is $L_\nu(\mathcal{C})$-measurable and Bochner ν_L-integrable. Now we use the fact that $H \cdot \sum_{i=1}^{H} \|Y_i\|_{\mathbb{F}}^2 \in Lim$ to show first that g and as a result of this that $\dot{f}$ is square Bochner-integrable. Set $M := H \cdot \sum_{i=1}^{H} \|Y_i\|_{\mathbb{F}}^2$. Since $\int_T \|G^2(j)\|_{\mathbb{F}} \, d\nu(j) = M$ it is easy to check that $\|G \cdot 1_{A_m}\|_{\mathbb{F}}^2$ is S_ν-integrable for all $m \in \mathbb{N}$. Since $G \cdot 1_{A_m}$ is a lifting of $g \cdot 1_{A_m}$, Proposition 12.5(b) implies

$$\int_T \|g(j)\|_{\mathbb{H}}^2 \cdot 1_{A_m}(j) d\nu_L(j) = {}^\circ \left(\int_T \|G(j)\|_{\mathbb{F}}^2 \cdot 1_{A_m}(j) d\nu(j) \right) \leq {}^\circ M.$$

By the theorem of monotone convergence (see Ash [5], 1.6.2) we obtain

$$\int_T \|g(j)\|_{\mathbb{H}}^2 \, d\nu_L(j) \leq {}^\circ M.$$

For $a, b \in [0; 1]$, $n_a, n_b \in T$ with $\frac{n_a}{H} \approx a$, $\frac{n_b}{H} \approx b$ we have

$$\int_a^b \dot{f}(t) dt = f(b) - f(a) = {}^\circ \left(\int_{n_a}^{n_b} G(j) d\nu(j) \right) =$$

$$\int_{n_a}^{n_b} g(j) d\nu_L(j) = \int_{st^{-1}([a;b])} g(j) d\nu_L(j).$$

Since $\dot{f} \circ st : T \to \mathbb{H}$, $n \mapsto \dot{f}(^\circ \left(\frac{n}{H}\right)$ is measurable with respect to the σ-field $\mathcal{J} = st^{-1}\left(\mathfrak{b}_{[0;1]}\right)$ and since the Lebesgue measure is the image measure of ν_L with respect to st we obtain for all $B \in \mathcal{J}$

$$\int_B \dot{f}\left(^\circ\left(\frac{n}{H}\right)\right) d\nu_L(n) = \int_B g(n) d\nu_L(n).$$

So $\dot{f} \circ st$ is the conditional expectation of g relative to $\mathcal{J}$. Since g is square integrable Proposition 1.6 says

$$\left(\int_T \left\|\left(\dot{f} \circ st\right)(n)\right\|_{\mathbb{H}}^2 d\nu_L(n)\right)^{0,5} \leq \left(\int_T \|g(n)\|_{\mathbb{H}}^2 d\nu_L(n)\right)^{0,5}$$

and thus

$$\int_0^1 \left\|\dot{f}(t)\right\|_{\mathbb{H}}^2 dt = \int_T \left\|\left(\dot{f} \circ st\right)(n)\right\|_{\mathbb{H}}^2 d\nu_L(n) \leq \int_T \|g(n)\|_{\mathbb{H}}^2 d\nu_L(n) \leq {^\circ}M.$$

This proves (1).
(2) (See also Cutland [11], Theorem 3.5.) For $f \in C_{\mathbb{H}}$ let

$$Y^f = \left(Ppr_{\mathbb{F}}^{*\mathbb{H}}\left({^*f}\left(\frac{n}{H}\right) - {^*f}\left(\frac{n-1}{H}\right)\right)\right)_{n\in T}.$$

So $Y_n^f = pr_{\mathbb{F}}^{*\mathbb{H}}\left(\int_{\frac{n-1}{H}}^{\frac{n}{H}} {^*\dot{f}}(s) ds\right)$, and therefore, by Lemma 12.8, $n \mapsto H \cdot Y_n^f$ is an element of $SL^2(\nu)$ and it is a lifting of $\dot{f}$. Hence

$$H \cdot \sum_{n=1}^{H} \|Y_i\|_{\mathbb{F}}^2 \approx \int_0^1 \left\|\dot{f}(s)\right\|_{\mathbb{H}}^2 ds.$$

Since $\sum_{i=1}^n Y_i^f = pr_{\mathbb{F}}^{*\mathbb{H}}\left({^*f}\left(\frac{n}{H}\right)\right)$, Lemma 12.5 implies that $\sum_{i=1}^n Y_i^f$ is nearstandard for all $n \in T$. Hence $Y^f \in H_{^*\mathbb{R}}$ and obviously $b_{Y^f} = f$. $\square$

In the next section the special case $\mathbb{B} = \mathbb{H} = \mathbb{R}$ is required. Then

$C_{\mathbb{H}} = \{f : [0;1] \to \mathbb{R}$: there exists a square integrable function $\dot{f} \in L^2([0;1], \lambda)$

$$\text{so that } \; f(t) = \int_0^t \dot{f}(s)ds \text{ for all } t \in [0;1]\}$$

and

$$H_{\mathbb{F}} = H_{^*\mathbb{R}} = \{Y \in {}^*\mathbb{R}^H \mid H \cdot \sum_{i=1}^{H} Y_i^2 \in \; Lim \; \}.$$

The nearstandard condition follows here directly from the fact that $H \cdot \sum_{i=1}^{H} Y^2 \in Lim$.

15 A Standard Characterisation of $C_{\mathbb{H}}$

For the sake of completeness we will also state a natural standard characterisation of $C_{\mathbb{H}}$. It has been already mentioned by Kuelbs and Lepage [23], but we give here our own proof.

15.1 Proposition

Let $f : [0;1] \to \mathbb{H}$ be measurable. The following statements are equivalent.
(1) $f \in C_{\mathbb{H}}$.
(2) For each ONB $(\mathfrak{e}_i)_{i\in\mathbb{N}}$ of $\mathbb{H}$ there exists a sequence $\left(\dot{f}_i\right)_{i\in\mathbb{N}} \subset L^2([0;1],\lambda)$ with $\sum_{i\in\mathbb{N}} \int_0^1 \dot{f}_i^2(s)ds < \infty$ and such that

$$f(t) = \sum_{i\in\mathbb{N}} \left(\int_0^t \dot{f}_i(s)ds \cdot \mathfrak{e}_i \right)$$

for all $t \in [0;1]$.
If (1) and (2) hold, we have $\|f\|^2_{C_{\mathbb{H}}} = \sum_{i\in\mathbb{N}} \int_0^1 f_i^2(s)ds$, where $f_i(t) := \int_0^t \dot{f}_i(s)ds$.

Proof: "(1) $\Rightarrow$ (2)" Assume $f \in C_{\mathbb{H}}$. Let $Y \in H_{\mathbb{F}}$ with $b_Y = f$. Choose an arbitrary ONB $(\mathfrak{e}_i)_{i\in\mathbb{N}}$ of $\mathbb{H}$ and let $(\mathfrak{e}_i)_{i\le\omega}$ be the extension to $\mathbb{F}$. We denote by Y_{ij} the scalar $\langle Y_i; \mathfrak{e}_j\rangle_{\mathbb{F}}$. Since

$$H \cdot \sum_{i=1}^{H} \sum_{j=1}^{\omega} Y_{ij}^2 = H \cdot \sum_{i=1}^{H} \|Y_i\|^2_{\mathbb{F}} \in Lim,$$

we have in particular that $H \cdot \sum_{i=1}^{H} Y_{ij}^2$ is limited for all $j \in \{1,\dots,\omega\}$. Hence, $(Y_{ij})_{1\le i\le H} \in H_{*\mathbb{R}}$. By Proposition 14.3, for each $j \in \{1,\dots,\omega\}$ there exists a function $\dot{f}_j \in L^2([0;1],\lambda)$ with $\int_0^t \dot{f}_j(s)ds = {}^\circ\left(\sum_{i=1}^{n} Y_{ij}\right)$, where $\frac{n}{H} \approx t$, and - as shown in the proof of Proposition 14.3 - such that $\int_0^1 \dot{f}_j^2(s)ds \le {}^\circ\left(\sum_{i=1}^{H} Y_{ij}^2\right)$. Hence, for all $m \in \mathbb{N}$

$$\sum_{j=1}^{m} \int_0^1 \dot{f}_j^2(s)ds \le \sum_{j=1}^{m} {}^\circ\left(H \cdot \sum_{i=1}^{H} Y_{ij}^2\right) =$$

$${}^\circ\left(H \cdot \sum_{j=1}^{m} \sum_{i=1}^{H} Y_{ij}^2\right) \le {}^\circ\left(H \cdot \sum_{j=1}^{\omega} \sum_{i=1}^{H} Y_{ij}^2\right).$$

Thus $\sum_{j=1}^{\infty} \int_0^1 \dot{f}_j^2(s)ds < \infty$.
Now fix any $t \in [0;1]$. Choose $n \in T$ with $\frac{n}{H} \approx t$. Then

$$f(t) = {}^\circ\left(\sum_{i=1}^{n} Y_i\right) = {}^\circ\left(\sum_{j=1}^{\omega}\left(\sum_{i=1}^{n} Y_{ij}\right)\cdot \mathfrak{e}_j\right)$$

and

$$\sum_{j=1}^{\infty}\int_0^t \dot{f}_j(s)ds \cdot \mathfrak{e}_j = \sum_{j=1}^{\infty} {}^\circ\left(\sum_{i=1}^{n} Y_{ij}\right)\cdot \mathfrak{e}_j.$$

Let $\varepsilon > 0$. Since $\sum_{i=1}^{n} Y_i$ is nearstandard in $\mathbb{H}$ it is easily checked that there is an $m_0 \in \mathbb{N}$ such that for all $m \in \{m_0, \ldots, \omega\}$

$$\sum_{j=m+1}^{\omega}\left(\sum_{i=1}^{n} Y_{ij}\right)^2 < \varepsilon^2.$$

Now for each $m \in \mathbb{N}, m \geq m_0$

$$\left\| f(t) - \sum_{j=1}^{m}\int_0^t \dot{f}_j(s)ds \cdot \mathfrak{e}_j \right\|_{\mathbb{H}} =$$

$$\left\| {}^\circ\left(\sum_{j=1}^{\omega}\left(\sum_{i=1}^{n} Y_{ij}\right)\cdot \mathfrak{e}_j\right) - \sum_{j=1}^{m} {}^\circ\left(\sum_{i=1}^{n} Y_{ij}\right)\cdot \mathfrak{e}_j \right\|_{\mathbb{H}} =$$

$$\left\| {}^\circ\left(\sum_{j=1}^{m}\left(\sum_{i=1}^{n} Y_{ij}\right)\cdot \mathfrak{e}_j\right) + {}^\circ\left(\sum_{j=m+1}^{\omega}\left(\sum_{i=1}^{n} Y_{ij}\right)\cdot \mathfrak{e}_j\right) - \sum_{j=1}^{m} {}^\circ\left(\sum_{i=1}^{n} Y_{ij}\right)\cdot \mathfrak{e}_j \right\|_{\mathbb{H}} =$$

$$\left\| {}^\circ\left(\sum_{j=m+1}^{\omega}\left(\sum_{i=1}^{n} Y_{ij}\right)\cdot \mathfrak{e}_j\right)\right\|_{\mathbb{H}} \approx \left\| \sum_{j=m+1}^{\omega}\left(\sum_{i=1}^{n} Y_{ij}\right)\cdot \mathfrak{e}_j \right\|_{\mathbb{F}} =$$

$$\sqrt{\sum_{j=m+1}^{\omega}\left(\sum_{i=1}^{n} Y_{ij}\right)^2} < \varepsilon.$$

Hence

$$\sum_{j=1}^{\infty}\int \dot{f}_j(s)ds \cdot \mathfrak{e}_j = \lim_{m\to\infty}\sum_{j=1}^{m}\int \dot{f}_j(s)ds \cdot \mathfrak{e}_j = f(t)$$

as required.
"(2) $\Rightarrow$ (1)" Assume (2) holds for f. Fix any ONB $(\mathfrak{e}_i)_{i\in\mathbb{N}}$ of $\mathbb{H}$ and a corresponding sequence $\left(\dot{f}_i\right)_{i\in\mathbb{N}} \subset L^2([0;1], \lambda)$. For each $i \in \mathbb{N}$ the product $\dot{f}_i \cdot e_i$ lies in $L^2(\lambda, \mathbb{H})$,

because $\int_0^1 \left\| \dot{f}_i(s) \cdot e_i \right\|_{\mathbb{H}} ds = \int_0^1 \dot{f}_i^2(s)ds$. Hence $\sum_{i=1}^n \dot{f}_i \cdot e_i \in L^2(\lambda, \mathbb{H})$ for each $n \in \mathbb{N}$. We show now that $\left(\sum_{i=1}^n \dot{f}_i \cdot e_i\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\lambda, \mathbb{H})$. Let $\varepsilon \in \mathbb{R}^+$. Choose $n_0 \in \mathbb{N}$ such that $\sum_{i=n}^m \int_0^1 \dot{f}_i^2(s)ds < \varepsilon^2$ for all $n, m > n_0$. Hence

$$\left\| \sum_{i=n}^m \dot{f} \cdot e_i \right\|_{L^2(\lambda,\mathbb{H})} = \sqrt{\int_0^1 \left\| \sum_{i=n}^m \dot{f}_i(s) \cdot e_i \right\|_{\mathbb{H}}^2 ds} =$$

$$\sqrt{\int_0^1 \left(\sum_{i=n}^m \dot{f}_i(s)^2 \right) ds} = \sqrt{\sum_{i=n}^m \left(\int_0^1 \dot{f}_i(s)^2 \right) ds} < \varepsilon.$$

Since $L^2(\lambda, \mathbb{H})$ is complete (see e.g. Diestel [13]) there exists a limit function $\tilde{f} \in L^2(\lambda, \mathbb{H})$. To see that $f(t) = \int_0^t \tilde{f}(s)ds$ for any $t \in [0,1]$ we show that the sequence

$$\sum_{i=1}^n \left(\int_0^t \dot{f}_i(s)\, ds \right) \cdot e_i$$

converges to $\int_0^t \tilde{f}(s)ds$ for $n \to \infty$. Let $\varepsilon \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\left\| \sum_{i=1}^n \dot{f}_i \cdot e_i - \tilde{f} \right\|_{L^2(\lambda,\mathbb{H})} < \varepsilon.$$

Then

$$\left\| \sum_{i=1}^n \left(\int_0^t \dot{f}(s)ds \right) \cdot e_i - \int_0^t \tilde{f}(s)ds \right\|_{\mathbb{H}} = \left\| \sum_{i=1}^n \left(\int_0^t \dot{f}(s) \cdot e_i ds \right) - \int_0^t \tilde{f}(s)ds \right\|_{\mathbb{H}} =$$

$$\left\| \int_0^t \left(\sum_{i=1}^n \dot{f}(s) \cdot e_i - \tilde{f}(s) \right) ds \right\|_{\mathbb{H}} \leq \int_0^t \left\| \sum_{i=1}^n \dot{f}_i(s) \cdot e_i - \tilde{f}(s) \right\|_{\mathbb{H}} ds \leq$$

$$\sqrt{\int_0^t \left\| \sum_{i=1}^n \dot{f}_i(s) \cdot e_i - \tilde{f}(s) \right\|_{\mathbb{H}}^2 ds} \leq \sqrt{\int_0^1 \left\| \sum_{i=1}^n \dot{f}_i(s) \cdot e_i - \tilde{f}(s) \right\|_{\mathbb{H}}^2 ds} < \varepsilon.$$

The first inequality follows from Diestel [13], II. Theorem 4. Hence $\tilde{f}$ is a derivative of f and therefore $f \in C_{\mathbb{H}}$.

Now (2) yields

$$\|f\|^2_{C_{\mathbb{H}}} = \int\limits_0^1 \|f(s)\|^2_{\mathbb{H}}\, ds = \int\limits_0^1 \left\| \sum_{i\in\mathbb{N}} \left(\int_0^t \dot{f}_i(s)ds \cdot \mathfrak{e}_i \right) \right\|^2_{\mathbb{H}} ds =$$

$$\int\limits_0^1 \sum_{i\in\mathbb{N}} \left(\int_0^t \dot{f}_i(s)ds \right)^2 ds = \sum_{i\in\mathbb{N}} \int_0^1 f_i^2(s)ds. \quad \square$$

16 From $\mathbb{B}'$ to $C'_{\mathbb{B}}$

Recall that $C'_{\mathbb{B}}$ is the Banach space of all continuous and linear real-valued functions on $C_{\mathbb{B}}$. We will represent the elements of $C'_{\mathbb{B}}$ by integrals following the way of proceeding in the real valued case (see e.g. Heuser [18], Section 56). To this end we use $\mathbb{B}'$-valued functions. Let $f \in C_{\mathbb{B}}$ and

$$\varphi : [0;1] \to \mathbb{B}',\ t \mapsto \varphi_t.$$

Define for each partition $P_n : 0 = t_0 < t_1 < \ldots < t_n = 1$ of $[0;1]$

$$\int f(P_n) d\varphi(P_n) := \sum_{i=1}^{n} \left(\varphi_{t_i} - \varphi_{t_{i-1}}\right) \left(f(t_{i-1})\right).$$

We say that **the integral of f with respect to φ exists** if for each sequence $(P_n)_{n\in\mathbb{N}}$ of partitions with $\max_{i\in\{0,1,\ldots,n\}} |t_i - t_{i-1}| \to 0$ for $n \to \infty$ the limit $\lim_{n\to\infty} \int f(P_n) d\varphi(P_n)$ exists and it does not depend on the choice of the sequence. We denote this limit then by $\int f(t) d\varphi(t)$. The following proposition is easily verified.

16.1 Proposition

Let $\varphi : [0;1] \to \mathbb{B}'$ be of bounded variation with variation $V(\varphi) =: k$. Then for each $f \in C_{\mathbb{B}}$ the integral of f with respect to φ exists. Furthermore,

$$\int_0^1 f(t) d\varphi(t) \leq k \cdot |f|_{sup}.$$

In the following example we construct a special class of functions of bounded variation we need later.

16.2 Example

Let $\Phi \in C'_{\mathbb{B}}$. According to the theorem of Hahn-Banach (see e.g. Heuser [18], 36.2) there exists a linear and continuous extension $\tilde{\Phi}$ of Φ to the space $B_{\mathbb{B}}$ of all bounded $\mathbb{B}$-valued functions on $[0;1]$ such that

$$\sup\left\{\left|\tilde{\Phi}\left(x\right)\right| : x \in B_{\mathbb{B}} \text{ and } \sup_{t\in[0;1]} |x(t)|_{\mathbb{B}} \leq 1\right\} = |\Phi|_{C'_{\mathbb{B}}}.$$

Now set for each $t \in [0;1]$ and $b \in \mathbb{B}$

$$\varphi_t(b) := \tilde{\Phi}\left(1_{[0;t[} \cdot b\right).$$

Then $\varphi_t \in \mathbb{B}'$. We will prove now that the function

$$\varphi : [0;1] \to \mathbb{B}',\ t \mapsto \varphi_t$$

is of bounded variation. To this end define $k := |\Phi|_{C'_{\mathbb{B}}}$ and let $\varepsilon > 0$. Take a partition $P_n : 0 = t_0 < t_1 < \ldots < t_n = 1$ of $[0;1]$. We will show that

$$\sum_{i=1}^{n} \left|\varphi_{t_i} - \varphi_{t_{i-1}}\right|_{\mathbb{B}'} < k + \varepsilon.$$

Let us choose $b_i \in \mathbb{B}$ for each $i \in \{1,\ldots,n\}$ such that $|b_i|_{\mathbb{B}} \leq 1$ and

$$\big|\,|\ \varphi - \varphi_{t_{i-1}}\ |_{\mathbb{B}'} - |\ \varphi(b_i) - \varphi_{t_{i-1}}(b_{i-1})\ |\,\big| < \frac{\varepsilon}{n}.$$

For $\alpha \in \mathbb{R}$ we use the abbreviation

$$sgn\left\{\alpha\right\} := \begin{cases} 0 & \text{if } \alpha = 0, \\ \frac{\alpha}{|\alpha|} & \text{if } \alpha \neq 0. \end{cases}$$

Then

$$\sum_{i=1}^{n} \left|\varphi_{t_i} - \varphi_{t_{i-1}}\right|_{\mathbb{B}'} < \sum_{i=1}^{n} \left|\varphi_{t_i}(b_i) - \varphi_{t_{i-1}}(b_i)\right| + \varepsilon =$$

$$\sum_{i=1}^{n} \left|\tilde{\Phi}\left(b_i \cdot 1_{[t_{i-1};t_i[}\right)\right| + \varepsilon = \sum_{i=1}^{n} sgn\left\{\tilde{\Phi}\left(b_i \cdot 1_{[t_{i-1};t_i[}\right)\right\} \cdot \tilde{\Phi}\left(b_i \cdot 1_{[t_{i-1};t_i[}\right) + \varepsilon =$$

$$\tilde{\Phi}\left(\sum_{i=1}^{n} sgn\left\{\Phi\left(b_i \cdot 1_{[t_{i-1};t_i[}\right)\right\} \cdot b_i \cdot 1_{[t_{i-1};t_i[}\right) + \varepsilon.$$

Since

$$\sup_{t\in[0;1]} \left|\sum_{i=1}^{n} sgn\left\{\Phi\left(b_i \cdot 1_{[t_{i-1};t_i[}\right)\right\} \cdot b_i \cdot 1_{[t_{i-1};t_i[}(t)\right|_{\mathbb{B}} \leq 1$$

we have

$$\tilde{\Phi}\left(\sum_{i=1}^{n} sgn\left\{\Phi\left(b_i \cdot 1_{[t_{i-1};t_i[}\right)\right\} \cdot b_i \cdot 1_{[t_{i-1};t_i[}\right) \leq k$$

and hence $\tilde{\Phi}\left(\sum_{i=1}^{n} sgn\left\{\Phi\left(b_i \cdot 1_{[t_{i-1};t_i[}\right)\right\} \cdot b_i \cdot 1_{[t_{i-1};t_i[}\right) + \varepsilon \leq k + \varepsilon$, as required.

Now we state an integral representation of the elements of $C'_{\mathbb{B}}$.

16.3 Proposition

(1) Let $\varphi : [0;1] \to \mathbb{B}'$ be of bounded variation. Then the function

$$\Phi : C_{\mathbb{B}} \to \mathbb{R},\ f \mapsto \int_0^1 f(t)d\varphi(t)$$

is well defined and an element of $C'_{\mathbb{B}}$.
(2) Let $\Phi \in C'_{\mathbb{B}}$. Then there exists a function $\varphi : [0;1] \to \mathbb{B}'$ of bounded variation such that for all $f \in C_{\mathbb{B}}$

$$\Phi(f) = \int_0^1 f(t)d\varphi(t).$$

Proof: (1) Let $\varphi : [0;1] \to \mathbb{B}'$ be of bounded variation. By Proposition 16.1 the integral is well defined. The linearity of φ_t yields the linearity of the integral. The continuity of the integral is also a consequence of Proposition 16.1.
(2) Let $\Phi \in C'_{\mathbb{B}}$ and φ as in Example 16.2. For $f \in C_{\mathbb{B}}$ define

$$f_n := \sum_{i=1}^{n} f(t_{i-1}) \cdot 1_{]t_{i-1};t_i]}.$$

Then $f_n \in B_{\mathbb{B}}$ and $\lim_{n\to\infty} \sup_{t\in[0;1]} |f(t) - f_n(t)| = 0$. Thus

$$\lim_{n\to\infty} \tilde{\Phi}(f_n) = \tilde{\Phi}(f) = \Phi(f).$$

But

$$\tilde{\Phi}(f_n) = \sum_{i=1}^{n} \tilde{\Phi}\left(f(t_{i-1}) \cdot \left(1_{[0;t_i]} - 1_{[0;t_{i-1}]}\right)\right) =$$

$$\sum_{i=1}^{n} \left(\varphi_{t_i} - \varphi_{t_{i-1}}\right)\left(f(t_{i-1})\right) = f(P_n)d\varphi(P_n).$$

Therefore

$$\Phi(f) = \lim_{n\to\infty} f(P_n)d\varphi(P_n) = \int_0^1 f(t)d\varphi(t). \quad \square$$

In the next section we use Proposition 16.3 for a nonstandard characterisation of $C'_{\mathbb{B}}$.

17 A Nonstandard Characterisation of $C'_{\mathbb{B}}$

Remember that

$$H_{\mathbb{F}} := \{Y \in \mathbb{F}^H : H \cdot \sum_{i=1}^{H} \|Y_i\|_{\mathbb{F}}^2 \in Lim \text{ and } \sum_{i=1}^{n} Y_i \text{ is nearstandard in } \mathbb{H} \text{ for all } n \in T\}$$

is a nonstandard characterisation of the space

$$C_{\mathbb{H}} := \{f : [0;1] \to \mathbb{H} : \text{there exists a square Bochner integrable function}$$

$$\dot{f} \in L^2(\lambda, \mathbb{H}) \text{ so that } f(t) = \int_0^t \dot{f}(s) ds \text{ for all } t \in [0;1]\}.$$

To give a nonstandard characterisation of $C'_{\mathbb{B}}$ we need the following operator norm on $\mathbb{F}$

$$\mid y \mid_{\mathbb{B}'} = \sup\{|< y, x >_{\mathbb{F}}| : x \in \mathbb{F} \text{ and } \mid x \mid_{\mathbb{B}} \leq 1\}.$$

Recall that there is a constant $c > 0$ such that $|x|_{\mathbb{B}} \leq c \cdot \|x\|_{\mathbb{F}}$ for all $x \in \mathbb{F}$. So for $y \in \mathbb{F}$ we have $\|y\|_{\mathbb{F}} \leq c \cdot |y|_{\mathbb{B}'}$. Now let us define

$$H_{\mathbf{A},\mathbb{F}} := \Big\{Y \in \mathbb{F}^H : Y_H = 0,\ H \cdot \sum_{i=2}^{H} \mid Y_i - Y_{i-1} \mid_{\mathbb{B}'} \in Lim$$

$$\text{and } \sum_{i=1}^{n} Y_i \text{ is nearstandard in } \mathbb{H} \text{ for all } n \in T\Big\}.$$

At first we list some properties of $H_{\mathbf{A},\mathbb{F}}$.

17.1 Lemma

$H_{\mathbf{A},\mathbb{F}} \subset H_{\mathbb{F}}$.

Proof: It is enough to show that $H \cdot \sum_{i=1}^{H} \|Y_i\|_{\mathbb{F}}^2 \in Lim$ for all $Y \in H_{\mathbf{A},\mathbb{F}}$. Let $Y \in H_{A,\mathbb{F}}$. Then

$$H \cdot \sum_{i=2}^{H} \mid Y_i - Y_{i-1} \mid_{\mathbb{B}'} \in Lim$$

and therefore

$$H \cdot \sum_{i=2}^{H} \| Y_i - Y_{i-1} \|_{\mathbb{F}} \in Lim.$$

Since $Y_H = 0$ we obtain for all $n \in T$ that $H\cdot \parallel Y_n \parallel_{\mathbb{F}} \in Lim$. Choose $m \in T$ such that $\parallel Y_m \parallel_{\mathbb{F}} = \max_{i \in T} \parallel Y_i \parallel_{\mathbb{F}}$. Let $r := H\cdot \parallel Y_m \parallel_{\mathbb{F}}$. Since $r \in Lim$ we get

$$H \cdot \sum_{i=1}^{H} \parallel Y_i \parallel^2 \leq H \cdot H \cdot \parallel Y_m \parallel^2 = r^2 \in Lim. \quad \square$$

In particular $b(Y)$ is well defined and an element of $H_{\mathbb{F}}$ for all $Y \in H_{\mathbf{A},\mathbb{F}}$. The following lemma is very useful. It can be easily proved by induction.

17.2 Lemma

For all vectors X and Y in $\mathbb{F}^H$ there is the following partition of the scalar product

$$\sum_{i=1}^{H-1} < Y_i, X_i >_{\mathbb{F}} = \sum_{i=1}^{H-1} < Y_H, X_i >_{\mathbb{F}} + \sum_{i=2}^{H} < Y_{i-1} - Y_i, \sum_{j=1}^{i-1} X_j >_{\mathbb{F}} .$$

Note that for arbitrary $Y, \tilde{Y} \in H_{\mathbb{F}}$ with $b_Y = b_{\tilde{Y}}$ we cannot generally conclude that

$$H \cdot \sum_{i=1}^{H} \parallel Y_i \parallel_{\mathbb{F}}^2 \approx H \cdot \sum_{i=1}^{H} \parallel \tilde{Y}_i \parallel_{\mathbb{F}}^2 .$$

But this is true if $Y, \tilde{Y} \in H_{\mathbf{A},\mathbb{F}}$.

17.3 Lemma

Let $Y \in H_{\mathbf{A},\mathbb{F}}$. Then for all $X, \tilde{X} \in U$ such that $b_X = b_{\tilde{X}}$ we have

$$H \cdot \sum_{i=1}^{H} < Y_i, X_i >_{\mathbb{F}} \approx H \cdot \sum_{i=2}^{H} < Y_i, \tilde{X}_i >_{\mathbb{F}} \in Lim.$$

In particular if $Y, \tilde{Y} \in H_{\mathbf{A},\mathbb{F}}$ such that $b_Y = b_{\tilde{Y}}$ then

$$H \cdot \sum_{i=1}^{H} \parallel Y_i \parallel^2 \approx H \cdot \sum_{i=1}^{H} \parallel \tilde{Y}_i \parallel^2 .$$

Proof: If $Y_H = 0$ we obtain by Lemma 17.2 that

$$\sum_{i=1}^{H} < Y_i, X_i > \; = \; \sum_{i=1}^{H} < Y_{i-1} - Y_i, \sum_{j=1}^{i-1} X_j > .$$

Now let $Y \in H_{\mathbf{A},\mathbb{F}}$, $X, \tilde{X} \in U$ such that $b_X = b_{\tilde{X}}$. Then

$$\left| H \cdot \sum_{i=1}^{H} \langle Y_i; X_i \rangle_{\mathbb{F}} - H \cdot \sum_{i=1}^{H} \left\langle Y_i; \tilde{X}_i \right\rangle_{\mathbb{F}} \right| = \left| H \cdot \sum_{i=1}^{H} \left\langle Y_i; X_i - \tilde{X}_i \right\rangle_{\mathbb{F}} \right| =$$

$$\left| H \cdot \sum_{i=2}^{H} \left\langle Y_{i-1} - Y_i; \sum_{j=1}^{i-1} \left(X_j - \tilde{X}_j \right) \right\rangle_{\mathbb{F}} \right| =$$

$$\left| H \cdot \sum_{\substack{i \in \{2, \ldots, H\} \text{ and} \\ \sum_{j=1}^{i-1} \left(X_j - \tilde{X}_j \right) \neq 0}} \left| \sum_{j=1}^{i-1} \left(X_j - \tilde{X}_j \right) \right|_{\mathbb{B}} \cdot \left\langle Y_i - Y_{i-1}; \frac{\sum_{j=1}^{i-1} \left(X_j - \tilde{X}_j \right)}{\left| \sum_{j=1}^{i-1} \left(X_j - \tilde{X}_j \right) \right|_{\mathbb{B}}} \right\rangle_{\mathbb{F}} \right| \leq$$

$$\max_{i \in \{2, \ldots, H\}} \left| \sum_{j=1}^{i-1} \left(X_j - \tilde{X}_j \right) \right|_{\mathbb{B}} \cdot H \cdot \left(\sum_{i=1}^{H} |Y_i - Y_{i-1}|_{\mathbb{B}'} \right) \approx 0,$$

by the definition of $H_{A,\mathbb{F}}$ and since $\max_{i \in \{2, \ldots, H\}} \left| \sum_{j=1}^{i-1} \left(X_j - \tilde{X}_j \right) \right|_{\mathbb{B}} \approx 0$. It remains to show that the sums are limited.

$$\left| H \cdot \sum_{i=1}^{H} \langle Y_i; X_i \rangle_{\mathbb{F}} \right| = \left| H \cdot \sum_{i=2}^{H} \left\langle Y_i - Y_{i-1}; \sum_{j=1}^{i-1} X_j \right\rangle_{\mathbb{F}} \right| \leq$$

$$\max_{i \in \{2, \ldots, H\}} \left| \sum_{j=1}^{i-1} X_j \right|_{\mathbb{B}} \cdot \left(H \sum_{i=1}^{H} |Y_i - Y_{i-1}|_{\mathbb{B}'} \right) \; \in \; Lim,$$

since both factors are limited. □

Now we state the main theorem of this section.

17.4 Theorem

(1) Let $Y \in H_{\mathbf{A},\mathbb{F}}$*. Then the function*

$$\Phi_Y : C_{\mathbb{B}} \to \mathbb{R},\ f \mapsto {}^\circ\left(H \cdot \sum_{i=1}^{H} \langle X_i; Y_i\rangle_{\mathbb{F}}\right),$$

where X *is any element of* U *with* $b_X = f$*, is well defined and an element of* $C'_{\mathbb{B}}$*.*
(2) Conversely, let $\Phi \in C'_{\mathbb{B}}$*. Then there exists a vector* $Y^{\Phi} \in H_{\mathbf{A},\mathbb{F}}$ *such that for all* $f \in C_{\mathbb{B}}$ *and* $X \in U$ *with* $b_X = f$

$$\Phi(f) \approx H \cdot \sum_{i=1}^{H} \left\langle X_i; Y_i^{\Phi}\right\rangle_{\mathbb{F}}.$$

Proof: (1) Let $Y \in H_{\mathbf{A},\mathbb{F}}$, Φ_Y as in (1) and $r_Y := H \cdot \sum_{i=2}^{H} \mid Y_i - Y_{i-1} \mid_{\mathbb{B}'}$. By Lemma 17.3 Φ_Y is well defined. The linearity of Φ_Y follows immediately from the definition. To see that Φ_Y is continuous let $\varepsilon > 0$ and $\delta := \frac{\varepsilon}{{}^\circ r_Y}$. Since $\Phi_Y \equiv 0$ if $r_Y \approx 0$ and therefore an element of $C'_{\mathbb{B}}$, we may assume that ${}^\circ r_Y > 0$. Now let $f \in C_{\mathbb{B}}$ with $|f|_{C_{\mathbb{B}}} < \delta$ and $X \in U$ so that $b_X = f$. Then $\left|\sum_{j=1}^{i} X_j\right| \leq \delta$ for all $i \in T$. By Lemma 17.2

$$|\Phi_Y(f)| \approx \left|H \cdot \sum_{i=1}^{H} \langle Y_i; X_i\rangle_{\mathbb{F}}\right| = \left|H \cdot \sum_{i=2}^{H} \left\langle Y_i - Y_{i-1}; \sum_{j=1}^{i-1} X_j \right\rangle_{\mathbb{F}}\right| \leq \delta \cdot r_Y.$$

Thus $|\Phi_Y(f)| \leq \varepsilon$, as required.
(2) Let $\Phi \in C'_{\mathbb{B}}$. Define $\tilde{\Phi}$ and φ as in Example 16.2. Since we can identify $\mathbb{B}'$ with a subspace of $\mathbb{H}$, φ can be regarded as $\mathbb{H}$-valued, and for all $h \in \mathbb{H}$

$$\tilde{\Phi}\left(h \cdot 1_{[0;t[}\right) = \varphi_t(h) = \langle \varphi(t); h\rangle_{\mathbb{H}}.$$

By Lemma 5.5 φ is of bounded variation also with respect to $\|\cdot\|_{\mathbb{H}}$. Hence, by Lemma 12.10,

$$A : T \to \mathbb{F},\ n \mapsto A_n = pr_{\mathbb{F}}^{{}^*\mathbb{H}}\left({}^*\varphi\left(\frac{n}{H}\right)\right)$$

is an S_ν-integrable lifting of φ and

$$\sum_{n=2}^{H} \| F(n) - F(n-1) \|_{\mathbb{F}} \in Lim.$$

Thus $A = (A_n)_{n\in T} \in \mathbb{F}^H$ and for all $x \in \mathbb{F}$ and $n \in T$ we have ${}^*\varphi_{\frac{n}{H}}(x) = \langle A_n; x\rangle_{\mathbb{F}}$.
Now let us define $Y^\Phi \in \mathbb{F}^H$ by

$$Y_n^\Phi := \frac{A_H - A_n}{H} \quad \text{for all } n \in T.$$

First, we show that for all $f \in C_{\mathbb{B}}$ and $X \in U$ with $b_X = f$

$$\Phi(f) \approx H \cdot \sum_{i=1}^{H} \left\langle X_i; Y_i^\Phi \right\rangle_{\mathbb{F}}.$$

Let $f \in C_{\mathbb{B}}$ and $X \in U$ with $b_X = f$. Define

$$\tilde{f} : {}^*[0;1] \to {}^*\mathbb{B},\ t \mapsto \sum_{i=2}^{H} \left(\sum_{j=1}^{i-1} X_j \right) \cdot 1_{[\frac{i-1}{H};\frac{i}{H}[}(t).$$

Then $f \in {}^*B_{\mathbb{B}}$ and $f \approx_{|\cdot|_{B_{\mathbb{B}}}} \tilde{f}$ and therefore ${}^*\tilde{\Phi}\left(\tilde{f}\right) \approx \tilde{\Phi}(f) = \Phi(f)$. Thus

$$\Phi(f) \approx {}^*\tilde{\Phi}\left(\sum_{i=2}^{H} \left(\sum_{j=1}^{i-1} X_j \right) \cdot 1_{[\frac{i-1}{H};\frac{i}{H}[} \right) = \sum_{i=2}^{H} {}^*\tilde{\Phi}\left(\left(\sum_{j=1}^{i-1} X_j \right) \cdot 1_{[\frac{i-1}{H};\frac{i}{H}[} \right) =$$

$$\sum_{i=2}^{H} \left({}^*\varphi_{\frac{i}{H}}\left(\sum_{j=1}^{i-1} X_j \right) - {}^*\varphi_{\frac{i-1}{H}}\left(\sum_{j=1}^{i-1} X_j \right) \right) =$$

$$\sum_{i=2}^{H} \left(\left\langle A_i; \sum_{j=1}^{i-1} X_j \right\rangle_{\mathbb{F}} - \left\langle A_{i-1}; \sum_{j=1}^{i-1} X_j \right\rangle_{\mathbb{F}} \right) =$$

$$\sum_{i=1}^{H-1} \langle A_H; X_i\rangle_{\mathbb{F}} - \sum_{i=1}^{H-1} \langle A_i; X_i\rangle_{\mathbb{F}} = \sum_{i=1}^{H} \langle A_H - A_i; X_i\rangle_{\mathbb{F}} = H \cdot \sum_{i=1}^{H} \left\langle Y_i^\Phi; X_i \right\rangle,$$

where we again have used Lemma 17.2.
It remains to show that $Y^\Phi \in H_{A,\mathbb{F}}$. Since A is an S_ν-integrable lifting of φ we have for all $n \in T$

$$\frac{1}{H} \cdot \sum_{i=1}^{n} A_i \approx_{\mathbb{F}} \int_0^{\circ\left(\frac{n}{H}\right)} \varphi(t)dt,$$

hence $\frac{1}{H} \cdot \sum_{i=1}^{n} A_i$ is nearstandard in $\mathbb{H}$. Now

$$\textstyle\sum_{i=1}^{n} Y_i^\Phi = \displaystyle\sum_{i=1}^{n} \frac{A_H - A_i}{H} = A_H - \frac{1}{H} \cdot \sum_{i=1}^{n} A_i.$$

Thus $\sum_{i=1}^{n} Y_i^{\Phi}$ is also nearstandard in $\mathbb{H}$.
To prove that

$$H \cdot \sum_{i=2}^{H} \mid Y_i^{\Phi} - Y_{i-1}^{\Phi} \mid_{\mathbb{B}'} \in Lim,$$

we regard φ again as function with values in $\mathbb{B}'$. Since φ is of bounded variation, we can define $M := V(\varphi) \in \mathbb{R}^+$ such that for all $n \in \mathbb{N}$

$$\sum_{i=1}^{n} \left| \varphi\left(\frac{i}{n}\right) - \varphi\left(\frac{i-1}{n}\right) \right|_{\mathbb{B}'} \leq M.$$

By transfer we have for all $n \in {}^*\mathbb{N}$

$$\sum_{i=1}^{n} \left| {}^*\varphi\left(\frac{i}{n}\right) - {}^*\varphi\left(\frac{i-1}{n}\right) \right|_{{}^*(\mathbb{B}')} \leq M.$$

Fix $i \in \{2, \ldots, H\}$. Then

$$H \cdot \mid Y_i^{\Phi} - Y_{i-1}^{\Phi} \mid_{\mathbb{B}'} = \mid A_i - A_{i-1} \mid_{\mathbb{B}'} =$$

$$\sup\{ |< A_i - A_{i-1}, x >_{\mathbb{F}}| : x \in \mathbb{F} \text{ and } \mid x \mid_{\mathbb{B}} \leq 1\} =$$

$$\sup\left\{\left| < {}^*\varphi\left(\frac{i}{H}\right) - {}^*\varphi\left(\frac{i-1}{H}\right), x >_{\mathbb{F}} \right| : x \in \mathbb{F} \text{ and } \mid x \mid_{\mathbb{B}} \leq 1\right\} \leq$$

$$\sup\left\{\left| {}^*\varphi\left(\frac{i}{H}\right)(b) - {}^*\varphi\left(\frac{i-1}{H}\right)(b) \right| : b \in {}^*\mathbb{B} \text{ and } \mid b \mid_{\mathbb{B}} \leq 1\right\} =$$

$$\left| {}^*\varphi\left(\frac{i}{H}\right) - {}^*\varphi\left(\frac{i-1}{H}\right) \right|_{{}^*(\mathbb{B}')}.$$

Thus

$$H \cdot \sum_{i=2}^{H} \mid Y_i^{\Phi} - Y_{i-1}^{\Phi} \mid_{\mathbb{B}'} \leq \sum_{i=1}^{H} \left| {}^*\varphi\left(\frac{i}{n}\right) - {}^*\varphi\left(\frac{i-1}{n}\right) \right|_{{}^*(\mathbb{B}')} \leq M,$$

as required. □

18 The Dual Space $C'_{\mathbb{B}}$ as Subspace of $C_{\mathbb{H}}$

The theorem of Riesz (see e.g. Heuser [18]) presents analytic representations of linear functionals. So for each element Ψ of $C'_{\mathbb{H}}$ there exists a vector $h \in C_{\mathbb{H}}$ so that $\left\langle h; \tilde{h} \right\rangle_{C_{\mathbb{H}}} = \Psi\left(\tilde{h}\right)$ for all $\tilde{h} \in C_{\mathbb{H}}$. Conversely, each vector $h \in C_{\mathbb{H}}$ determines an element Ψ of $C'_{\mathbb{H}}$ by this equality. It is also well known that an arbitrary element of $C'_{\mathbb{H}}$ needs not to have a continuous extension to an element of $C'_{\mathbb{B}}$. Functionals of $C'_{\mathbb{H}}$, that have this property, will be characterised in this section. At first, we show that $C'_{\mathbb{B}}$ can be regarded as subset of $C'_{\mathbb{H}}$, in the sense that for each $\varphi \in C'_{\mathbb{B}}$ the restriction $\varphi \upharpoonright \mathbb{H}$ belongs to $C'_{\mathbb{H}}$. This follows immediately from the first Lemma:

18.1 Lemma

There exists a constant $c \in \mathbb{R}^+$ such that $|h|_{\sup} \leq c \cdot \|h\|_{C_{\mathbb{H}}}$ for all $h \in C_{\mathbb{H}}$.

Proof: By Lemma 5.5, there is a constant $c \in \mathbb{R}^+$ such that $|\ x\ |_{\mathbb{B}} \leq c \cdot \|x\|_{\mathbb{H}}$ for all $x \in \mathbb{H}$. Let $h \in C_{\mathbb{H}}$ and $t \in [0;1]$. Then by Diestel [13], II.2. Theorem 4, and the Hölder inequality

$$\left|\int_0^t \dot{h}(s)\, ds\right|_{\mathbb{B}} \leq c \cdot \left\|\int_0^t \dot{h}(s)\, ds\right\|_{\mathbb{H}} \leq c \cdot \int_0^t \left\|\dot{h}(s)\right\|_{\mathbb{H}} ds \leq$$

$$c \cdot \sqrt{\int_0^t \left\|\dot{h}(s)\right\|_{\mathbb{H}}^2 ds} \leq c \cdot \|h\|_{C_{\mathbb{H}}}.$$

Since t was arbitrary, $|h|_{\sup} \leq c \cdot \|h\|_{C_{\mathbb{H}}}$. $\square$

18.2 Proposition

Let $h \in C_{\mathbb{H}}$ so that the corresponding functional can be extended to an element Φ of $C'_{\mathbb{B}}$. Define φ as in Example 16.2. Then for all $t \in [0;1]$

$$h(t) = \int_0^t (\varphi(1) - \varphi(s))\, ds.$$

Proof: Let $u := \varphi(1) - \varphi$. Then u is of bounded variation and therefore, by

Lemma 12.9, square Bochner integrable. Thus

$$h_u : [0;1] \to \mathbb{H},\ t \mapsto \int_0^t u(s)ds$$

is an element of $C_{\mathbb{H}}$. It remains to prove that $h_u = h$. This can be done by showing

$$\Phi\left(\tilde{h}\right) = \int_0^1 \left\langle u(s); \dot{\tilde{h}}(s) \right\rangle_{\mathbb{H}} ds$$

for all $\tilde{h} \in C_{\mathbb{H}}$. Let Y^{Φ} be defined as in the proof of Theorem 17.4. Then the internal function

$$U : T \to \mathbb{F},\ \ n \mapsto H \cdot Y_n^{\Phi}$$

is an element of $SL^2(\nu)$ and a lifting of u. To obtain a suitable lifting of $\tilde{h}$, let us define $G(n) := H \cdot pr_{\mathbb{F}}^{*\mathbb{H}} \left(\int_{\frac{n-1}{H}}^{\frac{n}{H}} {}^*\dot{\tilde{h}}(s)ds \right)$. By Lemma 12.8, $G \in SL^2(\nu)$ and it is a lifting of $\dot{\tilde{h}}$. Furthermore, as in the proof of Theorem 14.3(2) shown, $\frac{G(\cdot)}{H}$ is S-continuous and $b_{\left(\frac{G(n)}{H}\right)_{n \in T}} = \tilde{h}$. It is easily checked that $\langle U(\cdot); G(\cdot) \rangle_{\mathbb{F}}$ is an S_ν-integrable lifting of $\left\langle u(\cdot); \dot{\tilde{h}}(\cdot) \right\rangle_{\mathbb{H}}$. Hence

$$\int_0^1 \left\langle u(s); \dot{\tilde{h}}(s) \right\rangle_{\mathbb{H}} ds \approx \frac{1}{H} \cdot \sum_{i=1}^{H} \langle U(i); G(i) \rangle_{\mathbb{F}} = H \cdot \sum_{i=1}^{H} \left\langle Y_i^{\Phi}; \frac{G(i)}{H} \right\rangle_{\mathbb{F}} \approx \Phi\left(\tilde{h}\right). \quad \square$$

Now we can characterise $C'_{\mathbb{B}}$ as subset of $C_{\mathbb{H}}$.

18.3 Theorem

The corresponding functional of an element $h \in C_{\mathbb{H}}$ can be extended to an element of $C'_{\mathbb{B}}$ if and only if h has a derivative $\dot{h} : [0;1] \to \mathbb{B}'$, that is of bounded variation.

Proof: By Proposition 18.2 it is enough to prove " $\Leftarrow$ ". Assume $h \in C_{\mathbb{H}}$ has a derivative $\dot{h} : [0;1] \to \mathbb{B}'$, that is of bounded variation. Then it is easily seen that $(Y_n)_{n \in T}$ with $Y_n := \frac{1}{H} \cdot pr_{\mathbb{F}}^{*\mathbb{H}} \left({}^*\dot{h}\left(\frac{n}{H}\right) \right)$, for $n \in \{1, 2, \ldots, H-1\}$ and $Y_H = 0$, where we again identify elements of $\mathbb{B}'$ with elements of $\mathbb{H}$, lies in $H_{A,\mathbb{F}}$. It remains to show that $\Phi_Y\left(\tilde{h}\right) = \left\langle h; \tilde{h} \right\rangle_{C_{\mathbb{H}}}$. Choose again the lifting G with $G(n) := H \cdot pr_{\mathbb{F}}^{*\mathbb{H}} \left(\int_{\frac{n-1}{H}}^{\frac{n}{H}} {}^*\dot{\tilde{h}}(s)ds \right)$ of $\dot{\tilde{h}}$. Then, with the same arguments as in the proof of Proposition 18.2

$$\Phi_Y\left(\tilde{h}\right) = H \cdot \sum_{i=1}^{H} \left\langle Y_i; \frac{G(i)}{H} \right\rangle_{\mathbb{F}} = \frac{1}{H} \cdot \sum_{i=1}^{H} \left\langle pr_{\mathbb{F}}^{*\mathbb{H}} \left({}^*\dot{h}\left(\frac{i}{H}\right)\right); G(i) \right\rangle_{\mathbb{F}}$$

$$\approx \int_0^1 \left\langle \dot{h}(s); \dot{\tilde{h}}(s) \right\rangle_{\mathbb{H}} ds. \quad \square$$

It has to be mentioned that the results presented in Proposition 18.2 and Theorem 18.3 are already well known.

18.4 Corollary

$\left(C'_{\mathbb{B}}; \|\cdot\|_{C_{\mathbb{H}}}\right)$ is a dense subspace of $\left(C_{\mathbb{H}}; \|\cdot\|_{C_{\mathbb{H}}}\right)$.

Proof: Note that each simple function is of bounded variation. By Diestel [13], Section IV.1, the simple functions $\psi : [0; 1] \to \mathbb{H}$ are dense in $L^2(\lambda, \mathbb{H})$. Since also $\mathbb{B}'$ is dense in $(\mathbb{H}; \|\cdot\|)$ the result follows from Theorem 18.3. $\square$

19 W is a Gaussian Measure

The fact that the measure W on $(C_{\mathbb{B}}, \mathcal{F}_0)$, introduced in Section 13 is a Gaussian measure can be derived from the literature (see e.g. Osswald [30]) but we will present here our own proof. We will use the following lemma, that we also need in the next sections.

19.1 Lemma

Let $X, Y \in \mathbb{F}^H$. Then

$$\int_{\mathbb{F}^H} \left(\sum_{i=1}^{H} \langle X_i; Y_i \rangle_{\mathbb{F}} \right)^2 d\Gamma(X) = \frac{1}{H} \sum_{i=1}^{H} \|Y_i\|_{\mathbb{F}}^2$$

and

$$\int_{\mathbb{F}^H} \left(\sum_{i=1}^{H} \langle X_i; Y_i \rangle_{\mathbb{F}} \right)^4 d\Gamma(X) = 3 \cdot \left(\frac{1}{H} \sum_{i=1}^{H} \|Y_i\|_{\mathbb{F}}^2 \right)^2 .$$

Proof: We prove only the second equality. The proof of the first one is then straightforward. Choose any ONB $(\mathfrak{e}_n)_{n \leq \omega}$ of $\mathbb{F}$. We again denote the scalar $\langle Y_i; \mathfrak{e}_n \rangle_{\mathbb{F}}$ by Y_{in}. Applying Lemma 4.1 it is easily verified that

$$\int_{\mathbb{F}^H} \left(\sum_{i=1}^{H} \langle X_i; Y_i \rangle_{\mathbb{F}} \right)^4 d\Gamma(X) =$$

$$\frac{3}{H^2} \cdot \sum_{(i;n) \in T \times \{1,\ldots,\omega\}} Y_{in}^4 + 3 \cdot \frac{1}{H^2} \cdot \sum_{\substack{(i;n) \in T \times \{1,\ldots,\omega\} \\ (j;m) \in T \times \{1,\ldots,\omega\} \\ (i;n) \neq (j;m)}} Y_{in}^2 Y_{jm}^2 =$$

$$3 \cdot \left(\left(\frac{1}{H} \cdot \sum_{i=1}^{H} \|Y_i\|_{\mathbb{F}}^2 \right) \cdot \left(\frac{1}{H} \cdot \sum_{i=1}^{H} \|Y_i\|_{\mathbb{F}}^2 \right) \right). \quad \square$$

19.2 Proposition

The probability measure W on $\mathfrak{b}_{C_{\mathbb{B}}}$ is a centered Gaussian measure.

Proof: By Proposition 4.8 it is enough to show that the Fourier transform $\tilde{W}$ of W has the representation

$$\tilde{W}(\Phi) = \exp\left(-\frac{1}{2}\int\limits_{C_{\mathbb{B}}} \Phi^2(f)\, dW(f)\right)$$

for all $\Phi \in C'_{\mathbb{B}}$. Fix $\Phi \in C'_{\mathbb{B}}$. We have to show that

$$\int\limits_{C_{\mathbb{B}}} \exp(i \cdot \Phi(f))\, dW(f) = \exp\left(-\frac{1}{2}\int\limits_{C_{\mathbb{B}}} \Phi^2(f)\, dW(f)\right).$$

We will do this with our nonstandard representations. Let $Y^{\Phi} \in H_{A,\mathbb{F}}$ be the corresponding vector, introduced in Theorem 17.4. First we use Proposition 1.10 to see that $X \mapsto \sum_{i=1}^{H} \left\langle X_i; Y_i^{\Phi}\right\rangle_{\mathbb{F}} \in SL^2(\Gamma)$. By the lemma above and since $Y^{\Phi} \in H_{A,\mathbb{F}}$

$$\int\limits_{\mathbb{F}^H} \left(H \cdot \sum_{i=1}^{H} \left\langle X_i; Y_i^{\Phi}\right\rangle_{\mathbb{F}}\right)^4 d\Gamma(X) \;=\; 3 \cdot \left(H \cdot \sum_{i=1}^{H} \left\| Y_i^{\Phi}\right\|_{\mathbb{F}}^2\right)^2 \;\in\; Lim.$$

By Proposition 1.9

$$^{\circ}\int\limits_{\mathbb{F}^H} \left(H \cdot \sum_{i=1}^{H} \left\langle X_i; Y_i^{\Phi}\right\rangle_{\mathbb{F}}\right)^2 d\Gamma(X) = \int\limits_{\mathbb{F}^H} {}^{\circ}\left(H \cdot \sum_{i=1}^{H} \left\langle X_i; Y_i^{\Phi}\right\rangle_{\mathbb{F}}\right)^2 d\Gamma_L(X) =$$

$$\int\limits_{\mathbb{F}^H} \Phi^2(\kappa(X))\, d\Gamma_L(X) = \int\limits_{C_{\mathbb{B}}} \Phi^2(f)\, dW(f).$$

With similar arguments one can prove that

$$^{\circ}\int\limits_{\mathbb{F}^H} \exp\left(i \cdot \left(H \cdot \sum_{i=1}^{H} \left\langle X_i; Y_i^{\Phi}\right\rangle_{\mathbb{F}}\right)\right) d\Gamma = \int\limits_{C_{\mathbb{B}}} \exp(i \cdot \Phi(f))\, dW(f).$$

Now we apply transfer of Proposition 4.8 to the internal Gaussian measure Γ and obtain for the internal Fourier transform $\tilde{\Gamma}$ and for all elements Y of $\left(\mathbb{F}^H\right)' = \mathbb{F}^H$

$$\int\limits_{\mathbb{F}^H} \exp\left(i \cdot \left(\sum_{i=1}^{H} \langle X_i; Y_i\rangle_{\mathbb{F}}\right)\right) d\Gamma(X) = \exp\left(-\frac{1}{2}\int\limits_{\mathbb{F}^H} \left(\sum_{i=1}^{H} \langle X_i; Y_i\rangle_{\mathbb{F}}\right)^2 d\Gamma(X)\right).$$

Since $H \cdot Y^{\Phi} \in \mathbb{F}^{H}$, we get

$$\int\limits_{C_{\mathbb{B}}} \exp\left(i \cdot \Phi\left(f\right)\right) dW\left(f\right) = {}^{\circ}\int\limits_{\mathbb{F}^{H}} \exp\left(i \cdot \left(\sum_{i=1}^{H} \left\langle X_i; H \cdot Y_i^{\Phi} \right\rangle_{\mathbb{F}}\right)\right) d\Gamma\left(X\right) =$$

$$^{\circ}\left(\exp\left(-\frac{1}{2}\int\limits_{\mathbb{F}^{H}} \left(\sum_{i=1}^{H} \left\langle X_i; H \cdot Y_i^{\Phi} \right\rangle_{\mathbb{F}}\right)^{2} d\Gamma\left(X\right)\right)\right) =$$

$$\exp\left(-\frac{1}{2}\int\limits_{C_{\mathbb{B}}} \Phi^{2}\left(f\right) dW\left(f\right)\right). \quad \square$$

20 The Wiener Integral

A further class of functionals on $C_{\mathbb{B}}$ is given by the Wiener integrals. We will introduce these integrals now by generalizing the usual definition for the classical Wiener space represented for example by Kuo [24] or Norin [28]. Let $g \in L^2(\lambda, \mathbb{H})$ and choose simple functions $g_n = \sum_{i=1}^{k_n-1} \alpha_i \cdot 1_{[t_i;t_{i+1}[}$ with $g^n \underset{L^2(\lambda,\mathbb{H})}{\longrightarrow} g$. By Lemma 5.6 $(\mathbb{B}', \|\cdot\|)$ is dense in $(\mathbb{H}, \|\cdot\|)$, hence all α_i can be chosen in $\mathbb{B}'$. Define the Wiener integral I_{g^n} on $C_{\mathbb{B}}$ by

$$I_{g^n}(f) = \int_0^1 g^n(t)df_t := \sum_{i=1}^{k_n-1} \alpha_i(f(t_{i+1}) - f(t_i))$$

for each $f \in C_{\mathbb{B}}$. Since W is a Gaussian measure, by Proposition 4.5 $I_{g^n} \in L^2(C_{\mathbb{B}}, W)$. We show, that the sequence $(I_{g^n})_{n\in\mathbb{N}}$ is Cauchy in $L^2(C_{\mathbb{B}}, W)$. Choose $n, m \in \mathbb{N}$. Let $g^n - g^m = \sum_{j=1}^{k} \beta_j \cdot 1_{[t_j;t_{j+1}[}$ with $\beta_j \in \mathbb{B}'$. Then $(I_{g^n} - I_{g^m})(f) = \sum_{j=1}^{k} \beta_j (f(t_{j+1}) - f(t_j))$. Choose $n_1, \ldots, n_k \in T$, such that $\frac{n_j}{H} \approx t_j$. If $X \in \mathbb{F}^H$ with $b(X) = f$, then

$$\sum_{j=1}^{k} < {}^*\beta_j; X_{n_j+1} + X_{n_j+2} \ldots + X_{n_{j+1}} >_{\mathbb{F}} \approx \sum_{j=1}^{k} \beta_j (f(t_{j+1}) - f(t_j)).$$

By Lemma 19.1,

$$\int_{\mathbb{F}^H} \left(\sum_{j=1}^{k} < {}^*\beta_j; X_{n_j+1} + X_{n_j+2} \ldots + X_{n_{j+1}} > \right)^4 d\Gamma(X) =$$

$$3 \cdot \left(\frac{1}{H} \sum_{j=1}^{k} \|{}^*\beta_j\|_{\mathbb{F}}^2 \cdot (n_{j+1} - n_j) \right)^2 =$$

$$3 \cdot \left(\sum_{j=1}^{k} \|{}^*\beta_j\|_{\mathbb{F}}^2 \cdot \left(\frac{n_{j+1}}{H} - \frac{n_j}{H} \right) \right)^2 \approx 3 \cdot \left(\sum_{j=1}^{k} \|\beta_j\|_{\mathbb{H}}^2 \cdot (t_{j+1} - t_j) \right)^2.$$

By Proposition 1.10 the internal function

$$X \to \sum_{j=1}^{k} < {}^*\beta_j; X_{n_j+1} + X_{n_j+2} \ldots + X_{n_{j+1}} >_{\mathbb{F}}$$

is an element of $SL^2(\Gamma)$. The definition of the measure W, Proposition 1.9 and Lemma 19.1 yield

$$\int\limits_{C_{\mathbb{B}}} \left(\sum_{j=1}^{k} \beta_j \left(f(t_{j+1}) - f(t_j) \right) \right)^2 dW(f) =$$

$$\int_{\mathbb{F}^H} \left(\sum_{j=1}^{k} \beta_j \left(\kappa(X)(t_{j+1}) - \kappa(X)(t_j) \right) \right)^2 d\Gamma_L(X) =$$

$$\int_{\mathbb{F}^H} {}^{\circ}\left(\sum_{j=1}^{k} < {}^*\beta_j; X_{n_j+1} + X_{n_j+2} \ldots + X_{n_{j+1}} > \right)^2 d\Gamma_L(X) =$$

$${}^{\circ}\left(\int_{\mathbb{F}^H} \left(\sum_{j=1}^{k} < {}^*\beta_j; X_{n_j+1} + X_{n_j+2} \ldots + X_{n_{j+1}} > \right)^2 d\Gamma(X) \right) =$$

$$\sum_{j=1}^{k} \|\beta_j\|_{\mathbb{H}}^2 \cdot (t_{j+1} - t_j).$$

Since

$$\|g^n - g^m\|_{L^2(\lambda,\mathbb{H})}^2 = \int_0^1 \left\| \sum_{j=1}^{k} \beta_j \cdot 1_{[t_j;t_{j+1}[}(t) \right\|_{\mathbb{H}}^2 d\lambda(t) = \sum_{j=1}^{k} \|\beta_j\|_{\mathbb{H}}^2 \cdot (t_{j+1} - t_j),$$

we obtain that $\|I_{g^n} - I_{g^m}\|_{L^2(C_{\mathbb{B}},W)} = \|g^n - g^m\|_{L^2(\lambda,\mathbb{H})}$. Since $(g^n)_{n\in\mathbb{N}}$ is Cauchy in $L^2(\lambda, \mathbb{H})$ the sequence of Wiener integrals $(I_{g^n})_{n\in\mathbb{N}}$ is Cauchy in $L^2(C_{\mathbb{B}}, W)$. Hence the Wiener integral I_g of g can be defined as $L^2(C_{\mathbb{B}}, W)$-limit of $(I_{g^n})_{n\in\mathbb{N}}$. Obviously I_g is independent from the choice of the step functions. Now we give a nonstandard characterisation of the Wiener integrals. This is again inspired from Cutland [11], who described the classical Wiener space, and from the introductions to the Ito-integral by Osswald in [30].

20.1 Theorem

Let $g \in L^2(\lambda, \mathbb{H})$ and $G : T \to \mathbb{F}$ be a lifting of g with $G \in SL^2(\nu)$. Then for Γ_L-a.a. $X \in \mathbb{F}^H$

$${}^{\circ}\left(\sum_{i=1}^{H} \langle G(i); X_i \rangle_{\mathbb{F}} \right) = I_g(f), \text{ where } b(X) = f.$$

Proof: First we show that the representation is independent from the choice of the lifting. Let $G, G' \in SL^2(\nu)$ be liftings of g. Then by Lemma 19.1

$$\int_{\mathbb{F}^H} \left(\sum_{i=1}^{H} \langle G(i); X_i \rangle_{\mathbb{F}} - \sum_{i=1}^{H} \langle G'(i); X_i \rangle_{\mathbb{F}} \right)^2 d\Gamma =$$

$$\sum_{i=1}^{H} \frac{\|G(i) - G'(i)\|_{\mathbb{F}}^2}{H} = \int_T \|G(i) - G'(i)\|_{\mathbb{F}}^2 \, d\nu(i) \approx 0.$$

Thus $\sum_{i=1}^{H} \langle G(i); X_i \rangle_{\mathbb{F}} \approx \sum_{i=1}^{H} \langle G'(i); X_i \rangle_{\mathbb{F}}$ for Γ_L-a.a. $X \in \mathbb{F}^H$. Hence it is enough to prove the result for one particular lifting. As above, suppose $(g^m)_{m\in\mathbb{N}}$ to be a sequence of $\mathbb{B}'$-valued step-functions $g^m = \sum_{j=1}^{k_m-1} \alpha_j \cdot 1_{[t_j;t_{j+1}[}$ with $g^m \underset{L^2(\lambda,\mathbb{H})}{\longrightarrow} g$. Choose $n_1, \ldots, n_{k_m} \in T$, such that $\frac{n_j}{H} \approx t_j$ and define

$$G^m : T \to \mathbb{F}, i \mapsto \sum_{j=1}^{k_m-1} {}^*\alpha_j \cdot 1_{[n_j;n_{j+1}[}.$$

Notice that each G^m is an element of $SL^2(\nu)$ and a lifting of g^m, and

$$\|G^m - G^n\|_{L^2(\nu,\mathbb{F})} \approx \|g^m - g^n\|_{L^2(\lambda,\mathbb{H})}.$$

Furthermore, for $f \in C_{\mathbb{B}}$ and $X \in \mathbb{F}^H$ with $b(X) = f$ we obtain

$$\sum_{i=1}^{H} \langle G^m(i); X_i \rangle_{\mathbb{F}} = \sum_{j=1}^{k_m-1} \left\langle {}^*\alpha_j; \sum_{i=n_j}^{n_{j+1}} X_i \right\rangle_{\mathbb{F}} \approx \sum_{j=1}^{k_m-1} \alpha_j \left(f(t_{j+1}) - f(t_j) \right) = I_{g^m}(f)$$

We will obtain a suitable lifting by using the ideas of Osswald in his proof of Theorem 10.8.1 in [30]. Let again $G' \in SL^2(\nu)$ be a lifting of g. Since $g^m \underset{L^2(\lambda,\mathbb{H})}{\longrightarrow} g$, the convergence is also in measure. Hence for all $\varepsilon > 0$

$$\lim_{n\to\infty} {}^\circ\nu\left(\{i \in T : \|G^m(i) - G'(i)\|_{\mathbb{F}} > \varepsilon\}\right) \leq$$

$$\lim_{n\to\infty} \lambda\left(\left\{t \in [0;1] : \|g^m(t) - g(t)\|_{\mathbb{H}} > \frac{\varepsilon}{2}\right\}\right) = 0.$$

By saturation, $(G^m)_{m\in\mathbb{N}}$ can be extended to an internal sequence $(G^m)_{m\in{}^*\mathbb{N}}$ of measurable functions and there exists a $K \in {}^*\mathbb{N}\setminus\mathbb{N}$ such that

$$\nu\left(\left\{i \in T : \left\|G^M(i) - G'(i)\right\|_{\mathbb{F}} > \varepsilon\right\}\right) \approx 0$$

for all $M \in {}^*\mathbb{N}\setminus\mathbb{N}$ with $M \le K$ and for all $\varepsilon > 0$. Hence G^M is a lifting of g. Again by saturation, there is a strictly increasing function $k : \mathbb{N} \to \mathbb{N}$ and an unlimited $M \le K$, such that

$$\int\limits_T \left\|G^M(i) - G^{k(n)}\right\|^2_{\mathbb{F}} d\nu < \frac{1}{n}$$

for all $n \in \mathbb{N}$. By Proposition 1.11(b) it follows that $G^M \in SL^2(\nu)$. Let $m \in \mathbb{N}$ or $m = M$. Since $G^m \in SL^2(\nu)$ we obtain by Lemma 19.1

$$\int_{\mathbb{F}^H}\left(\sum_{i=1}^H \langle G^m(i); X\rangle_{\mathbb{F}}\right)^4 d\Gamma(X) =$$

$$3\cdot\left(\left(\frac{1}{H}\cdot\sum_{i=1}^H \|G^m(i)\|^2_{\mathbb{F}}\right)\cdot\left(\frac{1}{H}\cdot\sum_{i=1}^H \|G^m(i)\|^2_{\mathbb{F}}\right)\right) \in Lim.$$

By Proposition 1.7, $X \to \sum_{i=1}^H \langle G^m(i); X\rangle_{\mathbb{F}}$ is an element of $SL^2(\Gamma)$. Lemma 19.1 implies for all $n \in \mathbb{N}$

$$\int_{\mathbb{F}^H}\left(\sum_{i=1}^H \left\langle G^M(i); X\right\rangle_{\mathbb{F}} - \sum_{i=1}^H \left\langle G^{k(n)}(i); X\right\rangle_{\mathbb{F}}\right)^2 d\Gamma(X) =$$

$$\frac{1}{H}\cdot\sum_{i=1}^H \left\|G^M(i) - G^{k(n)}(i)\right\|^2_{\mathbb{F}} < \frac{1}{n}.$$

Note that by Proposition 1.9

$$\int_{\mathbb{F}^H}\left(\sum_{i=1}^H \left\langle G^M(i); X\right\rangle_{\mathbb{F}} - \sum_{i=1}^H \left\langle G^{k(n)}(i); X\right\rangle_{\mathbb{F}}\right)^2 d\Gamma(X) =$$

$$\int_{\mathbb{F}^H} {}^\circ\left(\sum_{i=1}^H \left\langle G^M(i); X\right\rangle_{\mathbb{F}} - \sum_{i=1}^H \left\langle G^{k(n)}(i); X\right\rangle_{\mathbb{F}}\right)^2 d\Gamma_L(X).$$

By Proposition 1.11(a) there exists a set $A \in L_\Gamma(\mathbb{F}^H)$ with $\Gamma_L(A) = 1$ und such that for all $X \in A$

$$\left|\sum_{i=1}^{H} \left\langle G^M(i); X \right\rangle_{\mathbb{F}}\right| \in Lim \quad \text{and} \quad \left|\sum_{i=1}^{H} \left\langle G^{k(n)}(i); X \right\rangle_{\mathbb{F}}\right| \in Lim.$$

We apply again Proposition 1.9 to obtain that

$$\int_A \left({}^\circ\left(\sum_{i=1}^{H} \left\langle G^M(i); X \right\rangle_{\mathbb{F}}\right) - {}^\circ\left(\sum_{i=1}^{H} \left\langle G^{k(n)}(i); X \right\rangle_{\mathbb{F}}\right)\right)^2 d\Gamma_L(X) =$$

$$\int_A {}^\circ\left(\sum_{i=1}^{H} \left\langle G^M(i); X \right\rangle_{\mathbb{F}} - \sum_{i=1}^{H} \left\langle G^{k(n)}(i); X \right\rangle_{\mathbb{F}}\right)^2 d\Gamma_L(X).$$

Therefore

$$\int_A \left({}^\circ\left(\sum_{i=1}^{H} \left\langle G^M(i); X \right\rangle_{\mathbb{F}}\right) - {}^\circ\left(\sum_{i=1}^{H} \left\langle G^{k(n)}(i); X \right\rangle_{\mathbb{F}}\right)\right)^2 d\Gamma_L(X) \leq \frac{1}{n}.$$

According to Ash [5], Section 2.5, there is a subsequence $\tilde{k}(n)$ such that for $\Gamma_L - a.a.\ X \in \mathbb{F}^H$

$$\lim_{n\to\infty} {}^\circ\left(\sum_{i=1}^{H} \left\langle G^{\tilde{k}(n)}(i); X_i \right\rangle_{\mathbb{F}}\right) = {}^\circ\left(\sum_{i=1}^{H} \left\langle G^M(i); X \right\rangle_{\mathbb{F}}\right).$$

Hence we obtain for Γ_L-a.a. X with $b(X) = f$ that

$$ {}^\circ\left(\sum_{i=1}^{H} \left\langle G^M(i); X \right\rangle_{\mathbb{F}}\right) = I_g(f). \quad \square$$

Now we want to introduce an internal Wiener integral. We will see in the next section that in the context with differentiability of Γ, the internal Wiener integral is the negative logarithmic derivative. For $Y \in \mathbb{F}^H$ let I_Y be the mapping

$$I_Y : \mathbb{F}^H \to {}^*\mathbb{R},\ X \mapsto H \cdot \sum_{i=1}^{H} \langle X_i; Y_i \rangle_{\mathbb{F}}.$$

We call I_Y the **internal Wiener integral**.

20.2 Proposition

For $Y \in \mathbb{F}^H$ the Wiener integral I_Y is an element of $SL^2(\Gamma)$ if and only if $H \cdot \sum_{i=1}^{H} \|Y_i\|_{\mathbb{F}}^2 \in Lim$.

Proof: " $\Rightarrow$ " If the Wiener integral I_Y is an element of $SL^2(\Gamma)$, then

$$\int_{\mathbb{F}^H} \Big(I_Y(X)\Big)^2 d\Gamma(X) \in Lim.$$

Since by Lemma 19.1

$$\int_{\mathbb{F}^H} \Big(I_Y(X)\Big)^2 d\Gamma(X) = \int_{\mathbb{F}^H} \Big(H \cdot \sum_{i=1}^{H} \langle X_i; Y_i\rangle_{\mathbb{F}}\Big)^2 d\Gamma(X) =$$

$$H \sum_{i=1}^{H} \|Y_i\|_{\mathbb{F}}^2 \cdot \frac{1}{H} = H \cdot \sum_{i=1}^{H} \|Y_i\|_{\mathbb{F}}^2,$$

this direction is proved.

" $\Leftarrow$ " To see the converse let $Y \in \mathbb{F}^H$ such that $H \cdot \sum_{i=1}^{H} \|Y_i\|_{\mathbb{F}}^2 \in Lim$. We will show that then $\int_{\mathbb{F}^H} \left[\Big(I_Y(X)\Big)^2\right]^2 d\Gamma(X) \in Lim$. Then, by Proposition 1.10, the Wiener integral I_Y is an element of $SL^2(\Gamma)$. Now by Lemma 19.1

$$\int_{\mathbb{F}^H} \left[\Big(I_Y(X)\Big)^2\right]^2 d\Gamma(X) =$$

$$H^4 \cdot \int_{\mathbb{F}^H} \left(\sum_{i=1}^{H} \langle X_i; Y_i\rangle_{\mathbb{F}}\right)^4 d\Gamma(X) = H^4 \cdot 3 \cdot \left(\frac{1}{H} \sum_{i=1}^{H} \|Y_i\|_{\mathbb{F}}^2\right)^2 =$$

$$3 \cdot \left(\Big(H \cdot \sum_{i=1}^{H} \|Y_i\|_{\mathbb{F}}^2\Big) \cdot \Big(H \cdot \sum_{i=1}^{H} \|Y_i\|_{\mathbb{F}}^2\Big)\right) \in \text{Lim}. \quad \square$$

20.3 Corollary

If $Y \in H_{\mathbb{F}}$, then the Wiener integral I_Y is an element of $SL^2(\Gamma)$. *In the special case $\mathbb{B} = \mathbb{H} = \mathbb{R}$, $Y \in H_{\mathbb{R}}$ if and only if I_Y is an element of $SL^2(\Gamma)$.*

20.4 Remark

Let $g \in C_{\mathbb{H}}$, with derivative $\dot{g}$. Let Y be defined as in the proof of Proposition 14.3B, i.e. $Y_n = pr_{\mathbb{F}}^{*\mathbb{H}}\left(\int_{\frac{n-1}{H}}^{\frac{n}{H}} {}^*\dot{g}(s)ds\right)$. We have seen that $Y \in H_{\mathbb{F}}$ and $b_Y = g$. In addition, we have proved that

$$G : \mathrm{T} \to \mathbb{F},\ i \mapsto H \cdot Y_i$$

is an element of $SL^2(\Gamma)$ and a lifting of $\dot{g}$. By the above, we obtain that the Wiener integral $I_{\dot{g}}$ of the derivative $\dot{g}$ of g is the standard part of the internal Wiener integral I_Y. So for Γ_L-a.a. $X \in \mathbb{F}^H$

$$I_{\dot{g}}(f) = \int_0^1 \dot{g}(t)df_t = {}^\circ\left(\sum_{i=1}^{H} \langle G(i); X_i\rangle_{\mathbb{F}}\right) = {}^\circ\left(H\sum_{i=1}^{H} \langle Y_i; X_i\rangle_{\mathbb{F}}\right) = {}^\circ\left(I_Y(X)\right),$$

where $b(X) = f$. Since two derivatives $\dot{g}_1$ and $\dot{g}_2$ of g differ only on a set of Lebesgue measure 0, the Wiener integral $I_{\dot{g}}$ is uniquely determined.

21 Fomin-Differentiabilitiy of Γ_L

This section takes up again our main theme of measure differentiability. Now we will show the Fomin-differentiability of the Loeb measure Γ_L. Recall that $\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}$ is the internal Gaussian measure on ${}^*\mathbb{R}^{\omega\cdot H}$, i.e.

$$\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}(B) = \left(\sqrt{\frac{H}{2\pi}}\right)^{\omega\cdot H} \int_B \exp\left(-\frac{H}{2}\sum_{i\leq\omega,j\leq H} x_i^2(j)\right) dx_i(j)_{i\leq\omega,j\leq H}$$

for all internal Borel subsets $B \subset {}^*\mathbb{R}^{\omega\cdot H}$ and Γ is the internal Gaussian measure on $\mathbb{F}^H$ defined by

$$\Gamma(A) = \gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}\left(\left\{(x_i(j))_{i\leq\omega,j\leq H} : \left(\sum_{i=1}^{\omega} x_i(j)\cdot\mathfrak{e}_{\mathrm{i}}\right)_{j\leq H} \in A\right\}\right).$$

for each internal $A \in \mathfrak{b}_{\mathbb{F}^H}$, where $(\mathfrak{e}_i)_{i\leq\omega}$ is an arbitrary orthonormal basis of $\mathbb{F}$. At first we prove the following standard lemma we will need below.

21.1 Lemma

For all $t \in [-1;1]\setminus\{0\}$, $r \in \mathbb{R}_0^+$ and $z \in \mathbb{R}$ the following inequality holds:

$$\left|\frac{\exp(-tz - t^2 r) - 1}{t} + z\right| \leq |t|\cdot\exp(r)\cdot(\exp(-z) + \exp(+z)).$$

Proof: We will use the Taylor series $\exp(x) = \sum_{k=0}^{\infty}\frac{x^k}{k!}$. So,

$$\left|\frac{\exp(-tz - t^2 r) - 1}{t} + z\right| = \left|\frac{1 + \sum_{k=1}^{\infty}\frac{\left(-tz-t^2r\right)^k}{k!} - 1}{t} + z\right| =$$

$$\left|\sum_{k=1}^{\infty}\frac{t^{k-1}\cdot(-z - t\cdot r)^k}{k!} + z\right| = \left|-z - t\cdot r + \sum_{k=2}^{\infty}\frac{t^{k-1}(-z - t\cdot r)^k}{k!} + z\right| \leq$$

$$|t|\cdot r + \sum_{k=2}^{\infty}\frac{|t|^{k-1}(|z| + |t|\cdot r)^k}{k!} = (+).$$

Since $|t| \leq 1$, we have

$$(+) \leq |t| \cdot r + \sum_{k=2}^{\infty} \frac{|t|\left(|z|+r\right)^k}{k!} \leq |t| \cdot (r+1) + |t| \cdot |z| + \sum_{k=2}^{\infty} \frac{|t|\left(|z|+r\right)^k}{k!} \leq$$

$$|t| \cdot \left(1 + |z| + r + \sum_{k=2}^{\infty} \frac{\left(|z|+r\right)^k}{k!}\right) = |t| \cdot \exp\left(|z|+r\right) = |t| \cdot \exp\left(|z|\right) \cdot \exp\left(r\right) \leq$$

$$|t| \cdot \exp\left(r\right) \cdot \left(\exp\left(-z\right) + \exp\left(+z\right)\right). \square$$

Now we show that $\gamma^{\omega \cdot H, \sqrt{\frac{1}{H}}}$ is S-Fomin-differentiable along all measurable vector fields $\tilde{h} : {}^*\mathbb{R}^{\omega \cdot H} \to {}^*\mathbb{R}^{\omega \cdot H}$, which fulfill the following conditions:

(A) There is a fixed $k \in \mathbb{N},$ such that $H \cdot \sum_{i=1}^{\omega \cdot H} \left(\tilde{h}(x)\right)_i^2 \leq k$ for all $X \in {}^*\mathbb{R}^{\omega \cdot H}$.

(B) $\int_{{}^*\mathbb{R}^{\omega \cdot H}} \exp\left(-H \cdot \sum_{i=1}^{\omega \cdot H} x_i \cdot \tilde{h}(x)_i\right) d\gamma^{\omega \cdot H, \sqrt{\frac{1}{H}}}(x)$ is limited.

(C) $\int_{{}^*\mathbb{R}^{\omega \cdot H}} \exp\left(+H \cdot \sum_{i=1}^{\omega \cdot H} x_i \cdot \tilde{h}(x)_i\right) d\gamma^{\omega \cdot H, \sqrt{\frac{1}{H}}}(x)$ is limited.

(D) $x \to -H \cdot \sum_{i=1}^{\omega \cdot H} x_i \cdot \tilde{h}(x)_i$ is $S_{\gamma^{\omega \cdot H, \sqrt{\frac{1}{H}}}}$-integrable.

21.2 Theorem

Let $\tilde{h} : {}^*\mathbb{R}^{\omega \cdot H} \to {}^*\mathbb{R}^{\omega \cdot H}$ be a measurable vector field, such that the conditions (A), (B), (C) and (D) hold. Define $T_t(x) = x - t \cdot \tilde{h}(x)$ and $\left(\gamma^{\omega \cdot H, \sqrt{\frac{1}{H}}}\right)_t (A) := \gamma^{\omega \cdot H, \sqrt{\frac{1}{H}}}(T_t^{-1}[A])$ for all internal Borel subsets $A \subseteq {}^*\mathbb{R}^{\omega \cdot H}$ and all $t \in J := {}^*\mathbb{R}$. Then $\gamma^{\omega \cdot H, \sqrt{\frac{1}{H}}}$ is S-Fomin-differentiable along $\tilde{h}$ and if $\left(\gamma^{\omega \cdot H, \sqrt{\frac{1}{H}}}\right)'$ is a derivative of $\gamma^{\omega \cdot H, \sqrt{\frac{1}{H}}}$, then for all internal Borel subsets $A \subset {}^*\mathbb{R}^{\omega \cdot H}$

$$\left(\gamma^{\omega \cdot H, \sqrt{\frac{1}{H}}}\right)'(A) \approx \int_A \beta^h_{\gamma^{\omega \cdot H, \sqrt{\frac{1}{H}}}}(x)\, d\gamma^{\omega \cdot H, \sqrt{\frac{1}{H}}}(x).$$

where $\beta^h_{\gamma^{\omega \cdot H, \sqrt{\frac{1}{H}}}}(X) := -H \cdot \sum_{i=1}^{\omega \cdot H} x_i \cdot \tilde{h}(x)_i$.

Proof: We show that $\gamma^{\omega \cdot H, \sqrt{\frac{1}{H}}}$ fulfills the conditions of Theorem 10.1. Since $\gamma^{\omega \cdot H, \sqrt{\frac{1}{H}}}$ is nonnegative and $\gamma^{\omega \cdot H, \sqrt{\frac{1}{H}}}\left({}^*\mathbb{R}^{\omega \cdot H}\right) = 1$, $\gamma^{\omega \cdot H, \sqrt{\frac{1}{H}}}$ is S-bounded. $\gamma^{\omega \cdot H, \sqrt{\frac{1}{H}}}$

has the positive Lebesgue density

$$f : {}^*\mathbb{R}^{\omega\cdot H} \to {}^*\mathbb{R},\ x \mapsto \left(\sqrt{\frac{H}{2\pi}}\right)^{\omega\cdot H} \exp\left(-\frac{H}{2}\sum_{i=1}^{\omega\cdot H} x_i^2\right)$$

that is *differentiable in the direction of all $x \in {}^*\mathbb{R}^{\omega\cdot H}$. To prove condition 3) let $t \approx 0, t \neq 0$. Then

$$\left|\frac{1}{t}\left(\frac{f(x+t\cdot\tilde{h}(x))}{f(x)} - 1\right) - \frac{f'(x)\left(\tilde{h}(x)\right)}{f(x)}\right| =$$

$$\left|\frac{1}{t}\left(\frac{\exp\left(-\frac{H}{2}\sum_{i=1}^{\omega\cdot H}\left(x_i + t\cdot\tilde{h}(x)_i\right)^2\right)}{\exp\left(-\frac{H}{2}\sum_{i=1}^{\omega\cdot H} x_i^2\right)} - 1\right) - \right.$$

$$\left.\frac{-H\cdot\left(\sum_{i=1}^{\omega\cdot H} x_i\cdot\tilde{h}(x)_i\right)\cdot\left(\sqrt{\frac{H}{2\pi}}\right)^{\omega\cdot H}\cdot\exp\left(-\frac{H}{2}\sum_{i=1}^{\omega\cdot H} x_i^2\right)}{\left(\sqrt{\frac{H}{2\pi}}\right)^{\omega\cdot H}\cdot\exp\left(-\frac{H}{2}\sum_{i=1}^{\omega\cdot H} x_i^2\right)}\right| =$$

$$\left|\frac{1}{t}\left(\exp\left(\sum_{i=1}^{\omega\cdot H}\left(-Htx_i\tilde{h}(x)_i - \frac{H}{2}t^2\left(\tilde{h}(x)_i\right)^2\right)\right) - 1\right) + \right.$$

$$\left. H\cdot\left(\sum_{i=1}^{\omega\cdot H} x_i\cdot\tilde{h}(x)_i\right)\right| \leq$$

$$|t|\cdot\exp\left(\sum_{i=1}^{\omega\cdot H}\frac{H}{2}\left(\tilde{h}(x)\right)_i^2\right)\cdot\left(\exp\left(-\sum_{i=1}^{\omega\cdot H} Hx_i\tilde{h}(x)_i\right) + \exp\left(+\sum_{i=1}^{\omega\cdot H} Hx_i\tilde{h}(x)_i\right)\right).$$

For the last inequality we used transfer of Lemma 21.1. Now let us define

$$g : {}^*\mathbb{R}^{\omega\cdot H} \to {}^*\mathbb{R}, \quad \text{by } g(x) :=$$

$$|t|\cdot\exp\left(\sum_{i=1}^{\omega\cdot H}\frac{H}{2}\left(\tilde{h}(x)\right)_i^2\right)\cdot\left(\exp\left(-\sum_{i=1}^{\omega\cdot H} Hx_i\tilde{h}(x)_i\right) + \exp\left(+\sum_{i=1}^{\omega\cdot H} Hx_i\tilde{h}(x)_i\right)\right).$$

Of course, g is *Borel-measurable, and by the conditions (A), (B) and (C) we get

$$\int_{{}^*\mathbb{R}^{\omega\cdot H}} g(x)\,d\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}(x) =$$

$$\int_{^*\mathbb{R}^{\omega\cdot H}} |t| \cdot \exp\left(\sum_{i=1}^{\omega\cdot H} \frac{H}{2}\left(\tilde{h}(x)\right)_i^2\right) \cdot \exp\left(-\sum_{i=1}^{\omega\cdot H} Hx_i\tilde{h}(x)_i\right) d\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}(x) +$$

$$\int_{^*\mathbb{R}^{\omega\cdot H}} |t| \cdot \exp\left(\sum_{i=1}^{\omega\cdot H} \frac{H}{2}\left(\tilde{h}(x)\right)_i^2\right) \cdot \exp\left(+\sum_{i=1}^{\omega\cdot H} Hx_i\tilde{h}(x)_i\right) d\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}(x) \leq$$

$$e^k \cdot |t| \int_{^*\mathbb{R}^{\omega\cdot H}} \exp\left(-\sum_{i=1}^{\omega\cdot H} Hx_i\tilde{h}(x)_i\right) d\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}(x) +$$

$$e^k \cdot |t| \int_{^*\mathbb{R}^{\omega\cdot H}} \exp\left(+\sum_{i=1}^{\omega\cdot H} Hx_i\tilde{h}(x)_i\right) d\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}(x) \approx 0$$

since $|t| \approx 0$. Hence condition (3) is verified. Now for all $x \in {}^*\mathbb{R}^{\omega\cdot H}$

$$\frac{f'(x)\left(\tilde{h}\left(x\right)\right)}{f(x)} = -H \cdot \left(\sum_{i=1}^{\omega\cdot H} x_i \cdot \tilde{h}(x)_i\right).$$

By condition the internal function $\beta^{\tilde{h}}_{\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}} : {}^*\mathbb{R}^{\omega\cdot H} \to {}^*\mathbb{R}$ with

$$\beta^{\tilde{h}}_{\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}}(x) = \frac{f'(x)\left(\tilde{h}\left(x\right)\right)}{f(x)} = -H \cdot \sum_{i=1}^{\omega\cdot H} x_i \cdot \tilde{h}(x)_i$$

is $S_{\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}}$-integrable. Thus all preliminaries of Theorem 10.1 are given, and therefore the theorem is proved. □

We will see now that Theorem 21.2 holds in particular for all constant vector fields $\tilde{h}\left(x\right) \equiv y \in {}^*\mathbb{R}^{\omega\cdot H}$, where $H \cdot \sum_{i=1}^{\omega\cdot H} y_i^2 \in \ Lim$.

Note that, as a consequence of the translation invariance of $\lambda^{\omega\cdot H}$, for all $y \in {}^*\mathbb{R}^{\omega\cdot H}$

$$\left(\sqrt{\frac{H}{2\pi}}\right)^{\omega\cdot H} \int_{^*\mathbb{R}^{\omega\cdot H}} \exp\left(-\frac{H}{2}\sum_{i=1}^{\omega\cdot H}\left(x_i - y_i\right)^2\right) d\lambda^{\omega\cdot H}(x) = \gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}\left({}^*\mathbb{R}^{\omega\cdot H}\right) = 1.$$

We need also the next lemma.

21.3 Lemma

For $y \in {}^*\mathbb{R}^{\omega\cdot H}$ *define the internal functions* f *and* g *on* ${}^*\mathbb{R}^{\omega\cdot H}$ by $f(x) := \exp\left(2\cdot H\cdot\sum_{i=1}^{\omega\cdot H} x_i y_i\right)$ *and* $g(x) := \exp\left(-2\cdot H\cdot\sum_{i=1}^{\omega\cdot H} x_i y_i\right)$. *Then*

$$\int_{{}^*\mathbb{R}^{\omega\cdot H}} f(x)d\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}(x) = \int_{{}^*\mathbb{R}^{\omega\cdot H}} g(x)d\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}(x) = \exp\left(2H\cdot\sum_{i=1}^{\omega\cdot H} y_i^2\right).$$

Proof:

$$\int_{{}^*\mathbb{R}^{\omega\cdot H}} f(x)d\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}(x) =$$

$$\left(\sqrt{\frac{H}{2\pi}}\right)^{\omega\cdot H}\cdot\int_{{}^*\mathbb{R}^{\omega\cdot H}}\exp\left(2\cdot H\cdot\sum_{i=1}^{\omega\cdot H} x_i y_i\right)\cdot\exp\left(-\frac{H}{2}\cdot\sum_{i=1}^{\omega\cdot H} x_i^2\right)d\lambda^{\omega\cdot H}(x) =$$

$$\left(\sqrt{\frac{H}{2\pi}}\right)^{\omega\cdot H}\cdot\int_{{}^*\mathbb{R}^{\omega\cdot H}}\exp\left(-\frac{H}{2}\cdot\sum_{i=1}^{\omega\cdot H}(x_i-2y_i)^2\right)\cdot\exp\left(2H\cdot\sum_{i=1}^{\omega\cdot H} y_i^2\right)d\lambda^{\omega\cdot H}(x) =$$

$$\exp\left(2H\cdot\sum_{i=1}^{\omega\cdot H} y_i^2\right)\cdot\left(\sqrt{\frac{H}{2\pi}}\right)^{\omega\cdot H}\cdot\int_{{}^*\mathbb{R}^{\omega\cdot H}}\exp\left(-\frac{H}{2}\cdot\sum_{i=1}^{\omega\cdot H}(x_i-2y_i)^2\right)d\lambda^{\omega\cdot H}(x) =$$

$$\exp\left(2H\cdot\sum_{i=1}^{\omega\cdot H} y_i^2\right).$$

The proof for g is similar. □

21.4 Proposition

Fix $y \in {}^*\mathbb{R}^{\omega\cdot H}$ *and define* $\left(\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}\right)(A) := \gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}(A+ty)$ *for all internal Borel subsets* $A \subset {}^*\mathbb{R}^{\omega\cdot H}$ *and all* $t \in J := {}^*\mathbb{R}$ *. If* $H\cdot\sum_{i=1}^{\omega\cdot H} y_i^2$ *is limited, then* $\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}$ *is S-Fomin differentiable along* y*. The function* $\beta^y_{\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}} : {}^*\mathbb{R}^{\omega\cdot H} \to {}^*\mathbb{R}$, defined by $\beta^y_{\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}}(x) := -H\cdot\sum_{i=1}^{\omega\cdot H} x_i y_i$ *is an element of* $SL^2\left(\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}\right)$ *and if* $\left(\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}\right)'$ *is a derivative of* $\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}$*, then for all internal Borel subsets* $A \subset {}^*\mathbb{R}^{\omega\cdot H}$

$$\left(\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}\right)'(A) \approx \int_A \beta^y_{\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}}(x)\,d\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}(x).$$

Proof: By Corollary 20.3 $\beta^y_{\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}}$ is an element of $SL^2\left(\gamma^{\omega\cdot H,\sqrt{\frac{1}{H}}}\right)$. We only have to check the conditions (B) and (C) of Theorem 21.2. But they follow immediately from Lemma 21.3. Hence, all preliminaries of Theorem 21.2 are given. □

With Theorem 21.2 and Proposition 21.4 the following main theorem is obvious.

21.5 Theorem

Let $h : \mathbb{F}^H \to \mathbb{F}^H$ be a measurable vector field which fulfills the following conditions:

(A) *There is a fixed $k \in \mathbb{N}$, such that $H \cdot \sum_{i=1}^{H} \|h(X)_i\|_{\mathbb{F}}^2 \leq k$ for all $X \in \mathbb{F}^H$.*

(B) $\int_{\mathbb{F}^H} \exp\left(-H \cdot \sum_{i=1}^{H} \langle X_i; h(X)_i\rangle_{\mathbb{F}}\right) d\Gamma(X)$ is limited.

(C) $\int_{\mathbb{F}^H} \exp\left(+H \cdot \sum_{i=1}^{H} \langle X_i; h(X)_i\rangle_{\mathbb{F}}\right) d\Gamma(X)$ is limited.

(D) $X \mapsto -H \cdot \sum_{i=1}^{H} \langle X_i; h(X)_i\rangle_{\mathbb{F}}$ is S_Γ-integrable.

Then Γ is S-Fomin-differentiable along the vector field h and if Γ' is a derivative of Γ then for all internal Borel subsets $A \in \mathfrak{b}_{\mathbb{F}^H}$

$$\Gamma'(A) \approx \int_A \beta_\Gamma^h(X)\, d\Gamma(X).$$

where $\beta_\Gamma^h(X) = -H \cdot \sum_{i=1}^{H} \langle X_i; h(X)_i\rangle_{\mathbb{F}}$. This holds in particular for constant vector fields $h \equiv Y \in \mathbb{F}^H$, when $H \cdot \sum_{i=1}^{H} \|Y_i\|_{\mathbb{F}}^2 \in Lim$, in which case we obtain that the logarithmic derivative $\frac{d\Gamma'}{d\Gamma}$ is the negative internal Wiener integral I_Y.

Now we apply Theorem 11.3 and Lemma 11.4 to obtain Fomin-differentiability of the Loeb measure Γ_L. Recall Lemma 9.3, that for S- Fomin-differentiable measures the curve of measures is S-continuous.

21.6 Proposition

We take the assumptions of Theorem 21.5. For some $\varepsilon \in \mathbb{R}^+$ let J be an interval of ${}^*\mathbb{R}$ containing the standard interval $I =]-\varepsilon, \varepsilon[$. For $r \in I$ choose $t \in J$ such that $t \approx r$ and set $\mu_r := \mu_t$ and $\mathcal{F} := \bigcap_{r\in\mathbb{R}} L_{\Gamma_r}(\mathfrak{b}_{\mathbb{F}^H})$. Then we can define the curve $((\mu_L)_r)_{r\in I}$ of Loeb measures on $\mathcal{F}$ by $(\mu_L)_r := (\mu_r)_L$ restricted to $\mathcal{F}$. Since Γ is S-Fomin-differentiable along the vector field h, the Loeb measure Γ_L is Fomin-differentiable along h with derivative measure $\Gamma_L' : \mathcal{F} \to \mathbb{R}$,

$$B \mapsto \Gamma_L'(B) = \int_B {}^\circ\beta_\Gamma^h(X)\, d\Gamma_L(X),$$

where $\beta_\Gamma^h(X) := -H \cdot \sum_{i=1}^H \langle X_i; h(X)_i\rangle_{\mathbb{F}}$. *If* $Y \in \mathbb{F}^H$ *with* $H \cdot \sum_{i=1}^H \|Y_i\|_{\mathbb{F}}^2 \in Lim$, Γ_L *is Fomin-differentiable along* Y *with* $\frac{d\Gamma'}{d\Gamma}(X) = {}^\circ I_Y(X)$. *This holds in particular when* $Y \in H_{\mathbb{F}}$.

At the end of this chapter we apply Proposition 9.1 to get an integration by part formula. Note that by Corollary 11.5 S-Fomin-differentiability of Γ implies the differentiability of Γ_L with respect to the set $\mathcal{C}_L$ of all $\mathcal{F}$-measurable real-valued bounded functions on $\mathbb{F}^H$. This yields the next proposition.

21.7 Proposition

We take the assumptions of Proposition 21.6. Let $\mathcal{C}$ *be a set of internal* ${}^*\mathbb{R}$*-valued,* $\mathfrak{b}_{\mathbb{F}^H}$*-measurable and* S*-bounded functions on* $\mathbb{F}^H$ *that have the following property: for each* $X \in \mathbb{F}^H$ *the function*

$$J \setminus \{0\} \to {}^*\mathbb{R},\ t \mapsto \frac{f(X + t \cdot h(X)) - f(X)}{t}$$

is S*-continuous. Let*

$$\mathcal{C}_L := \{ g : \mathbb{F}^H \to \mathbb{R} : g \text{ is Gateaux differentiable in all directions } h(X)$$

$$\text{and there is an element } f \in \mathcal{C} \text{ with } {}^\circ(f(X)) = g(X) \text{ for all } X \in \mathbb{F}^H\}.$$

Then Γ_L *is differentiable with respect to* $\mathcal{C}_L$ *along* h *and for all* $g \in \mathcal{C}_L$

$$\int_{\mathbb{F}^H} g'(X)(h(X))d\Gamma_L(X) = -\int_{\mathbb{F}^H} g(X)d\Gamma'_L(X) = -\int_{\mathbb{F}^H} g(X) \cdot {}^\circ\beta_\Gamma^h(X)\, d\Gamma_L(X).$$

In Section 6 we introduced generalizations of Malliavin derivative and Skorokhod operator. In the situation of Proposition 21.7 g' is a kind of Malliavin derivative and $-\beta_\Gamma^h$ is a kind of internal Skorokhod integral of the vector field h. If $h(X) \equiv Y \in \mathbb{F}^H$, with $H \cdot \sum_{i=1}^H \|Y_i\|_{\mathbb{F}}^2 \in Lim$, this Skorokhod integral is the internal Wiener integral.

22 Fomin-Differentiability of the Wiener Measure

Recall the construction of the Wiener measure W on $\mathfrak{b}_{C_{\mathbb{B}}}$ in Section 13. There κ is the surjective and Borel measurable mapping $\kappa : \mathbb{F}^H \to C_{\mathbb{B}}$, $X \mapsto b_X$, $\mathcal{F}_0 = \{\kappa^{-1}[A] \mid A \in \mathfrak{b}_{C_{\mathbb{B}}}\}$ and

$$W(A) := \Gamma_L\left(\kappa^{-1}[A]\right).$$

for each $A \in \mathfrak{b}_{C_{\mathbb{B}}}$. Denote by $\bar{\mathcal{F}}$ the σ-field generated by $\mathcal{F}_0$. We shall see in this section that W is Fomin-differentiable along all elements of $C_{\mathbb{H}}$. Note that for $f \in C_{\mathbb{H}}$ and $A \in \mathfrak{b}_{C_{\mathbb{B}}}$ also $A + r \cdot f \in \mathfrak{b}_{C_{\mathbb{B}}}$ for all $r \in \mathbb{R}$.

22.1 Theorem

Let $f \in C_{\mathbb{H}}$. Then there is a measure W' on $\mathfrak{b}_{C_{\mathbb{B}}}$ such that for $A \in \mathfrak{b}_{C_{\mathbb{B}}}$

$$\lim_{r\to 0} \frac{W(A + r\cdot f) - W(A)}{r} = W'(A).$$

Furthermore, $W'(A) = \int_A \xi(\omega)\, dW(\omega)$, and the logarithmic derivative ξ is the Wiener integral I_f.

Proof: Let $f \in C_{\mathbb{H}}$. By Proposition 14.3(2) the vector

$$Y_f = \left(Pr_{\mathbb{F}}^{{}^*\mathbb{H}}\left({}^*f\left(\frac{n}{H}\right) - {}^*f\left(\frac{n-1}{H}\right)\right)\right)_{n\in T}$$

lies in $H_{\mathbb{F}}$ and $b_Y = f$. Hence for all $r \in \mathbb{R}$ and $A \in \mathfrak{b}_{C_{\mathbb{B}}}$

$$\kappa^{-1}[A + r\cdot f] = \kappa^{-1}[A] + r \cdot Y_f.$$

Of course, $\mathcal{F}_{\mathsf{o}} \subset \mathcal{F} = \bigcap_{r\in\mathbb{R}} L_{\Gamma_r}(\mathcal{A})$. Now

$$\lim_{r\to 0} \frac{W(A + r\cdot f) - W(A)}{r} =$$

$$\lim_{r\to 0} \frac{\Gamma_L(\kappa^{-1}[A] + rY) - \Gamma_L(\kappa^{-1}[A])}{r} = \Gamma_L'(\kappa^{-1}[A]) =$$

$$\int_{\kappa^{-1}[A]} {}^{\circ}\beta_{\Gamma}^{Y}(X)\, d\Gamma_L = \int_B \xi(\omega)\, dW(\omega),$$

where we have used Proposition 21.6. □

23 $(C_{\mathbb{B}}, C_{\mathbb{H}})$ as Abstract Wiener Space

In the last chapter we list some well established standard results that follow from the previous sections.

23.1 Theorem

The space $\left(C_{\mathbb{H}}; \|\cdot\|_{C_{\mathbb{H}}}\right)$ *is the Cameron-Martin subspace* $H(W)$ *of* $\left(C_{\mathbb{B}}; |\cdot|_{\sup}\right)$.

Proof: According to Theorem 22.1 the measure W is Fomin-differentiable along all elements of $C_{\mathbb{H}}$. Theorem 4.15 shows that $C_{\mathbb{H}} \subset H(W)$. By Lemma 14.1 $C_{\mathbb{H}}$ is dense in $\left(C_{\mathbb{B}}; |\cdot|_{\sup}\right)$. Hence, also $H(W)$ is dense in $\left(C_{\mathbb{B}}; |\cdot|_{\sup}\right)$. Therefore, we can apply Lemma 5.9 to obtain that $C'_{\mathbb{B}}$ is a dense subspace of $\left(H(W); \|\cdot\|_{H(W)}\right)$. Finally, as proved in Section 18, $C'_{\mathbb{B}}$ is a dense subspace of $\left(C_{\mathbb{H}}; \|\cdot\|_{C_{\mathbb{H}}}\right)$. By Theorem 18.3 it is enough to show that

$$\|f\|_{H(W)} = \|f\|_{C_{\mathbb{H}}}$$

for all $f \in C_{\mathbb{H}}$ which have a derivative $\dot{f}$ of bounded variation.

Let $f \in C_{\mathbb{H}}$ with a derivative $\dot{f} : [0;1] \to \mathbb{B}'$, that is of bounded variation. By Theorem 18.3 the corresponding functional on $C_{\mathbb{H}}$ can be extended to an element $\Phi_f \in C'_{\mathbb{B}}$. At first we show that $\Phi_f = R_W(f)$. As in the proof of 18.3 we define $\left(Y_n^f\right)_{n \in T}$ by $Y_n^f := \frac{1}{H} \cdot pr_{\mathbb{F}}^{*\mathbb{H}}\left({}^*\dot{f}\left(\frac{n}{H}\right)\right)$ for $n \in \{1, 2, \ldots, H-1\}$ and $Y_H^f = 0$, and use that $\left(Y_n^f\right)_{n \in T}$ lies in $H_{\mathbf{A},\mathbb{F}}$, $b_{Y^f} = f$ and $n \mapsto H \cdot Y_n^f$ is an element of $SL^2(\nu)$ and a lifting of $\dot{f}$. Choose an arbitrary functional $\Phi \in C'_{\mathbb{B}}$ and let $Y \in H_{\mathbf{A},\mathbb{F}}$ the nonstandard representation, introduced in Section 17. By Lemma 19.1 the functions $X \to H \cdot \sum_{i=1}^H \langle Y_i; X_i\rangle_{\mathbb{F}}$ and $X \to H \cdot \sum_{i=1}^H \left\langle Y_i^f; X_i\right\rangle_{\mathbb{F}}$ are both elements of $SL^2(\Gamma)$. Hence we obtain, again using that W is the image measure of Γ_L and using the properties of Gaussian measures:

$$\int\limits_{C_{\mathbb{B}}} \Phi_f(\omega) \cdot \Phi(\omega) dW(\omega) =$$

$$\int\limits_{\mathbb{F}^H} {}^\circ\left(H \cdot \sum_{i=1}^H \left\langle Y_i^f; X_i\right\rangle_{\mathbb{F}}\right) \cdot {}^\circ\left(H \cdot \sum_{i=1}^H \langle Y_i; X_i\rangle_{\mathbb{F}}\right) d\Gamma_L(X) =$$

$$^\circ\left(\sum_{i=1}^{H}\int_{\mathbb{F}^H} H^2\cdot\left\langle Y_i^f;X_i\right\rangle\cdot\langle Y_i;X_i\rangle_{\mathbb{F}}\,d\Gamma(X)\right)={}^\circ\left(\sum_{i=1}^{H} H\cdot\left\langle Y_i^f;Y_i\right\rangle\right)=\Phi(f).$$

The fact that $n\to H\cdot Y_n^f$ is an element of $SL^2(\nu)$ and a lifting of $\dot f$ yields that $n\to\left\langle H\cdot Y_n^f;H\cdot Y_n^f\right\rangle_{\mathbb{F}}$ is an S_ν-integrable lifting of $\dot f^2$. Hence,

$$\|f\|^2_{H(W)}=\int_{C_{\mathbb{B}}}\Phi_f^2(\omega)dW(\omega)=\int_{\mathbb{F}^H}{}^\circ\left(H\cdot\sum_{i=1}^{H}\left\langle Y_i^f;X_i\right\rangle_{\mathbb{F}}\right)^2 d\Gamma_L(X)=$$

$$^\circ\left(H\cdot\sum_{i=1}^{H}\left\langle Y_i^f;Y_i^f\right\rangle\right)={}^\circ\left(\frac{1}{H}\cdot\sum_{i=1}^{H}\left\langle H\cdot Y_i^f;H\cdot Y_i^f\right\rangle\right)=$$

$$^\circ\int_T\left\langle H\cdot Y_n^f;H\cdot Y_n^f\right\rangle d\nu=\int_0^1\dot f^2(s)\,ds=\|f\|^2_{C_{\mathbb{H}}}.\quad\square$$

Now we show that the set of Wiener integrals coincides with the reproducing kernel Hilbert space.

23.2 Proposition

Let $l\in L^2(C_{\mathbb{B}},W)$. Then there exists a $g\in L^2(\lambda,\mathbb{H})$ such that

$$l(f)=I_g(f)$$

for W-a.a. $f\in C_{\mathbb{B}}$ if and only if $l\in C'_{\mathbb{B},W}$.

Proof:" $\Rightarrow$ " is obvious. " $\Leftarrow$ " Let $l\in C'_{\mathbb{B},W}$. Then there exists an element $h\in C_{\mathbb{H}}$ with $\Phi(h)=\int_{C_{\mathbb{B}}}l(\omega)\cdot\Phi(\omega)dW(\omega)$ for all $\Phi\in C'_{\mathbb{B}}$. With the same arguments as in the proof of Theorem 23.1 one can show that $\Phi(h)=\int_{C_{\mathbb{B}}}I_{\dot h}(\omega)\cdot\Phi(\omega)dW(\omega)$. Proposition 4.12 yields that $l(f)=I_{\dot h}(f)$ W-a.s. $\square$

23.3 Theorem

(Osswald [30]) $(C_{\mathbb{B}},C_{\mathbb{H}})$ *together with the measure W is an abstract Wiener space.*

Proof: By Lemma 14.1 the space $C_{\mathbb{H}}$ is dense in $C_{\mathbb{B}}$. Therefore, the result follows from Proposition 5.10. $\square$

We close the thesis with a view at the classical Wiener space. It arises from the special case $\mathbb{B} = \mathbb{H} = \mathbb{R}$. Recall that then

$$B : {}^*\mathbb{R}^H \times T \to {}^*\mathbb{R},\ (X, n) \mapsto \sum_{i=1}^{n} X_i,$$

$$C_{\mathbb{H}} = \{f : [0;1] \to \mathbb{R} : \text{there exists a square integrable function } \dot{f} \in L^2\,[0;1]$$

$$\text{so that } \ f(t) = \int_0^t \dot{f}(s)ds \text{ for all } t \in [0;1]\}$$

and

$$H_{\mathbb{F}} = H_{{}^*\mathbb{R}} = \{Y \in {}^*\mathbb{R}^H : H \cdot \sum_{i=1}^{H} Y^2 \in \ Lim\ \}.$$

The next lemma follows from Proposition 13.1. The independence is easily verified.

23.4 Lemma

Let $1 \le n < m \le k < l \in T$. Then $B_m - B_n$ and $B_l - B_k$ are independent, normally distributed random variables with variances $\frac{m-n}{H}$ and $\frac{l-k}{H}$.

Note that the classical Wiener measure P is a probability Borel measure on $C_{\mathbb{R}}$ such that for any partition $0 = t_0 < t_1 < \ldots < t_n \le 1$ and $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$

$$P\left(\{f \in C_{\mathbb{R}} : f(t_i) \le \alpha_i \,\text{for all}\, 1 \le i \le n\}\right) =$$

$$\int_{-\infty}^{\alpha_1} \ldots \int_{-\infty}^{\alpha_n} \prod_{i=0}^{n-1} \frac{1}{\sqrt{2\pi\,(t_{i+1} - t_i)}} \cdot \exp\left(-\frac{(y_{i+1} - y_i)^2}{2\,(t_{i+1} - t_i)}\right) dy_1 \ldots dy_n,$$

where $y_0 = 0$. (See e.g. Cutland [11] or Kuo [24].)

23.5 Proposition

(Cutland [11], Theorem 2.2.) *The Wiener measure W on $C_{\mathbb{R}}$ is the classical Wiener measure.*

Proof: (Cutland [11]) Let $0 = t_0 < t_1 < \ldots < t_n \le 1$ and $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$.

Pick $n_i \in T$, $\frac{n_i}{H} \approx t_i$ for each i. Then, by Lemma 23.4

$$\Gamma_L \left(st^{-1} \{ f \in C_{\mathbb{R}} : f(t_i) \leq \alpha_i \text{ for all } 1 \leq i \leq n \} \right) =$$

$$\Gamma_L \left(\left\{ X \in {}^*\mathbb{R}^H : {}^\circ \left(B_{n_i}(X) \right) \leq \alpha_i \text{ for all } 1 \leq i \leq n \right\} \right) =$$

$$\lim_{m \to \infty} \Gamma_L \left(\left\{ X \in {}^*\mathbb{R}^H : B_{n_i}(X) \leq \alpha_i + \frac{1}{m} \text{ for all } 1 \leq i \leq n \right\} \right) =$$

$$\lim_{m \to \infty} \Gamma_L(\{ X \in {}^*\mathbb{R}^H : B_{n_1}(X) \leq \alpha_1 + \frac{1}{m},$$

$$B_{n_i}(X) - B_{n_{i-1}}(X) \leq \alpha_i + \frac{1}{m} - B_{n_{i-1}}(X) \text{ for all } 2 \leq i \leq n \}) =$$

$$\lim_{m \to \infty} \int_{-\infty}^{\alpha_1} \int_{-\infty}^{\alpha_2 - x_1} \int_{-\infty}^{\alpha_3 - x_2 - x_1} \cdots \int_{-\infty}^{\alpha_n - x_{n-1} - x_{n-2} - \ldots - x_1} \prod_{i=0}^{n-1} \frac{\exp\left(-\frac{x_{i+1}^2}{2(t_{i+1} - t_i)} \right)}{\sqrt{2\pi \left(t_{i+1} - t_i \right)}} dx_1 \ldots dx_n =$$

$$\int_{-\infty}^{\alpha_1} \cdots \int_{-\infty}^{\alpha_n} \prod_{i=0}^{n-1} \frac{1}{\sqrt{2\pi \left(t_{i+1} - t_i \right)}} \cdot \exp\left(-\frac{\left(y_{i+1} - y_i \right)^2}{2 \left(t_{i+1} - t_i \right)} \right) dy_1 \ldots dy_n,$$

where $y_0 = 0, y_1 = x_1, y_i = \sum_{j=1}^{i} x_j$. Hence the image measure W of Γ_L coincides with the classical Wiener measure P. $\square$

By proving this last proposition we completed a new proof for the old - but not trivial - result that the classical Wiener space is an abstract Wiener space.

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Eidesstattliche Versicherung

(Siehe Promotionsordnung vom 12.07.11, § 8, Abs. 2 Pkt. .5.)

Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.