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Regularity for Degenerate Elliptic and Parabolic Systems

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Introduction

Three different types of systems will be studied in this work. The three model cases are as follows: The model case for Chapter 1 is the inhomogeneous p-Laplace equation

$$-\Delta_p u = -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) = -\mathrm{div} f. \tag{0.1}$$

In Chapter 2 it is the incompressible p-Stokes equation

$$\begin{aligned} -\mathrm{div}\big(|\varepsilon u|^{p-2}\varepsilon u\big) + \nabla\pi &= -\mathrm{div} f \\ \mathrm{div} u &= 0 \end{aligned} \tag{0.2}$$

where $\varepsilon u = \frac{1}{2}(\nabla + \nabla^T)u$ is the symmetric gradient. In Chapter 3 it is the parabolic p-Laplace equation

$$\partial_t u - \Delta_p u = -\mathrm{div} f. \tag{0.3}$$

The basic question of the inhomogeneous regularity theory is what impact do the qualities of f have on u. We will demonstrate the technique on Poisson's equation which is the natural starting point for all partial differential equations studied in this work.

$$-\Delta u = -\mathrm{div}\nabla u = -\mathrm{div} f. \tag{0.4}$$

Although our estimates will be stated in local form (and for local solutions), we will discuss the case of the entire space in the introduction, which is easier to state and therefore better to get insights.

If $f \in L^2(\mathbb{R}^n;\mathbb{R}^N)$, then there exists a unique $u \in W_0^{1,2}(\mathbb{R}^n;\mathbb{R}^N)$ which is a minimizer of the following functional

$$u = \arg \min_{W_0^{1,2}(\mathbb{R}^n)^N} \frac{1}{2}\int |\nabla v|^2 dx - \int f\cdot\nabla v dx$$

The first regularity statement is therefore $f \in L^2(\mathbb{R}^n)$ implies $\nabla u \in L^2(\mathbb{R}^n)$. But in fact many more qualities of f can be transferred. Indeed, the mapping $f \mapsto \nabla u$ can be characterized by a singular integral operator and the classical Calderón Zygmund theory implies the following regularity.

(1) $f \in L^q(\mathbb{R}^n)$ implies $\nabla u \in L^q(\mathbb{R}^n)$ for $1 < q < \infty$
(2) $f \in C^{k,\alpha}(\mathbb{R}^n)$ implies $\nabla u \in C^{k,\alpha}(\mathbb{R}^n)$ for $k \in \mathbb{N}$ and $\alpha \in (0,1)$.
(3) $f \in \mathrm{BMO}(\mathbb{R}^n)$ implies $\nabla u \in \mathrm{BMO}(\mathbb{R}^n)$ but $f \in L^\infty(\mathbb{R}^n)$ does not imply $\nabla u \in L^\infty(\mathbb{R}^n)$.

The function space $\mathrm{BMO}(\mathbb{R}^n)$ is the space of bounded mean oscillation which is of special interest. It is the right substitute of L^∞ in the regularity theory of equations in divergence form. We want to provide a different insight in this theory which is closer to the non-linear setting.

However, the non-linear Calderón Zygmund theory found a way of interpreting the matter above. It was founded by Iwaniec [**27, 28**]. By refining his technique we were able to show the following for Poisson's equations.

Theorem 0.1. *Let $u \in W_0^{1,2}(\mathbb{R}^n;\mathbb{R}^N)$ be a solution to (0.4). Then for almost every $x \in \mathbb{R}^n$*

$$M_2^\sharp(\nabla u)(x) \le c M_2^\sharp(f)(x).$$

$M_2^\sharp$ is the Fefferman Stein maximal operator defined in (0.10) and c only depends on the dimensions.

The beauty of the proof provided here, is that it is done purely by tools of the non-linear Calderón-Zygmund theory. Theorem 0.1 implies (1) immediately by the bounds of $M_2^\sharp$ in L^q for $q > 2$ (see [54]). As $f \in \mathrm{BMO}(\mathbb{R}^n)$ if and only if $M_2^\sharp(f) \in L^\infty(\mathbb{R}^n)$, we gain (3). By refining $M_2^\sharp$ by additional powers of the radii, we gain (2) for $k = 0$. These are precisely the regularity properties that can be shown in the non-linear case of the p-Laplacian. Analogous to the case $p = 2$, we have that if $f \in L^{p'}(\mathbb{R}^n; \mathbb{R}^{nN})$, there exists a unique solution of (0.1). For these solutions Iwaniec [28, 29] proved that $f \in L^q(\mathbb{R}^n)$ implies $|\nabla u|^{p-2}\nabla u \in L^q(\mathbb{R}^n)$ for $q \geq p'$. The case $1 < q < p'$ can not be treated by this technique. However, in [29, 36] the authors, using different techniques, were able to treat the case $p' - \delta < q \leq p'$ for a small $\delta > 0$. The case $1 < q < p' - \delta$ is an important open problem up to now. As a consequence to Chapter 1 $f \in \mathrm{BMO}(\mathbb{R}^n)$ implies $|\nabla u|^{p-2}\nabla u \in \mathrm{BMO}(\mathbb{R}^n)$ and $f \in C^\alpha(\mathbb{R}^n)$ implies $|\nabla u|^{p-2}\nabla u \in C^\alpha(\mathbb{R}^n)$ for (0.1) and α small. Therefore the conjecture, which we believe to be true, but are unable to prove is that

$$M_{p'}^\sharp(|\nabla u|^{p-2}\nabla u)(x) \leq c M_{p'}^\sharp(f)(x)$$

for almost every $x \in \mathbb{R}^n$ and $u \in W_0^{1,p}(\mathbb{R}^n; \mathbb{R}^N)$ a solution to (0.1). See Remark 1.21 for a further discussion on that matter.

In Chapter 1 we present the result of [15]. The difference to this article is, that we allow systems with coefficients. We can use that to show BMO results up to the boundary (see Section 1.5). It is part of a collaboration with Dominic Breit, Lars Diening and Andrea Cianchi. The two following chapters are two extensions of the techniques presented in Chapter 1. First we will suit it such that we can prove BMO and Campanato estimates for local solutions of (0.2), this is a work together with Lars Diening and Petr Kaplický [16]. Although our techniques are independent of the dimension, we have to restrict to the 2-dimensional case. In Chapter 3 we discuss the borderline case $q \to \infty$ for the parabolic p-Laplace, which is still to be published in a scientific journal.

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General Notation

Within this work we will use $\cdot$ as the standard scalar product on $\mathbb{R}^n$ or $\mathbb{R}^{N\times n}$ and $|\cdot|$ as the induced norm on $\mathbb{R}^n$ or $\mathbb{R}^{N\times n}$ We use c as a generic constant which may change from line to line, but does not depend on the crucial quantities. Moreover we write $f \sim g$ if and only if there exist constants $c, C > 0$ such that $c\,f \leq g \leq C\,f$. Note that we do not point out the dependencies of the constants on the fixed dimensions n and N. For $v \in L^1_{\text{loc}}(\mathbb{R}^n)$ and a measurable set $E \subset \mathbb{R}^n$ we define

$$\langle v\rangle_E := -\!\!\!\!\!\!\int_E v(x)dx := \frac{1}{|E|}\int_E v(x)dx, \tag{0.5}$$

where $|E|$ is the n-dimensional Lebesgue measure of E. For $\lambda > 0$ we denote by λB the ball with the same center as B but λ-times the radius. By r_B we mean the radius of B. By B_r we mean a ball with radius r. For a set $M \subset \mathbb{R}^n$ we denote χ_M as the characteristic function of the set M, i.e. $\chi(x) = 1$ if $x \in M$ otherwise it equals zero. We write $\mathbb{R}^{\geq 0} = [0, +\infty)$ and $\mathbb{R}^{>0} = (0, +\infty)$. We denote by

$$\operatorname{osc}_E(f) := \sup_{x,y\in E} |f(x) - f(y)|$$

the oscillations of f on E. We say that a function $\rho : [0,\infty) \to [0,\infty)$ is *almost increasing* if there is $c > 0$ such that for all $0 \leq s \leq t$ the inequality $\rho(s) \leq c\,\rho(t)$ is valid. We say that ρ is *almost decreasing* if there is $c > 0$ such that for all $0 \leq s \leq t$ the inequality $\rho(s) \geq c\,\rho(t)$ is valid. We say that ρ is *almost monotone* if it is almost increasing or almost decreasing.

We now will discuss N–functions.

Definition 0.2. *A real function $\varphi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is said to be an N-function if it satisfies the following conditions: There exists the derivative φ' of φ. This derivative is right continuous, non-decreasing and satisfies $\varphi'(0) = 0$ and $\varphi'(t) > 0$ for $t > 0$. Especially, φ is convex.*

The complementary function φ^* is given by

$$\varphi^*(u) := \sup_{t\geq 0}\big(ut - \varphi(t)\big)$$

and satisfies $(\varphi^*)'(t) = (\varphi')^{-1}(t)$. For any $t \geq 0$ we have

$$\varphi(t) \leq \varphi'(t)\,t \leq \varphi(2t), \qquad \varphi^*(\varphi'(t)) \leq \varphi(2t). \tag{0.6}$$

Moreover, $(\varphi^*)^* = \varphi$.

Definition 0.3. *We say that the N-function φ satisfies the Δ_2–condition, if there exists $c_1 > 0$ such that for all $t \geq 0$ it holds $\varphi(2t) \leq c_1 \varphi(t)$. By $\Delta_2(\varphi)$ we denote the smallest constant c_1. For a family Φ of N-functions we define $\Delta_2(\Phi) := \sup_{\varphi\in\Phi} \Delta_2(\varphi)$.*

For all $\delta > 0$ there exists c_δ (only depending on $\Delta_2(\varphi^*)$) such that for all $t, u \geq 0$

$$t\,u \leq \delta\,\varphi(t) + c_\delta\,\varphi^*(u). \tag{0.7}$$

This inequality is called Young's inequality. For all $t \geq 0$

$$\frac{t}{2}\varphi'\Big(\frac{t}{2}\Big) \leq \varphi(t) \leq t\,\varphi'(t), \qquad \varphi\bigg(\frac{\varphi^*(t)}{t}\bigg) \leq \varphi^*(t) \leq \varphi\bigg(\frac{2\,\varphi^*(t)}{t}\bigg). \tag{0.8}$$

Therefore, uniformly in $t \geq 0$

$$\varphi(t) \sim \varphi'(t)\,t, \qquad \varphi^*\big(\varphi'(t)\big) \sim \varphi(t), \tag{0.9}$$

where the constants only depend on $\Delta_2(\varphi, \varphi^*)$.

For an N-function φ with $\Delta_2(\varphi) < \infty$, we denote by L^φ and $W^{1,\varphi}$ the classical Orlicz and Sobolev-Orlicz spaces, i. e. $u \in L^\varphi$ if and only if $\int \varphi(|u|)\,dx < \infty$ and $u \in W^{1,\varphi}$ if and only if $u, \nabla u \in L^\varphi$. By $W_0^{1,\varphi}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,\varphi}(\Omega)$.

We define for B a ball and $g \in L^1_{\text{loc}}(\mathbb{R}^n)$

$$\begin{aligned} M_B^{\sharp,q} f &= \bigg(\fint_B |g - \langle g\rangle_B|^q dx\bigg)^{\frac{1}{q}}, \\ (M^{\sharp,q} g)(x) &= \sup_{B\ni x} M_B^{\sharp,q} g. \end{aligned} \tag{0.10}$$

We define $M_B^\sharp = M_B^{\sharp,1}$ and $M^\sharp = M^{\sharp,1}$. Finally we define the Hardy Littlewood maximal operator by

$$M_q(g)(x) = \sup_{x\ni B} \langle |g|^q\rangle_B^{\frac{1}{q}}.$$

The space BMO of bounded mean oscillations is defined via the following semi norm (for Ω open)

$$\|g\|_{\text{BMO}(\Omega)} := \sup_{B\subset\Omega} \fint_B |g - \langle g\rangle_B|\,dx = \sup_{B\subset\Omega} M_B^\sharp g;$$

saying that $g \in$ BMO(B), whenever its semi norm is bounded. Therefore $g \in$ BMO($\mathbb{R}^n$) if and only if $M^\sharp g \in L^\infty(\mathbb{R}^n)$.

Throughout the work we will need the following typical estimate for mean oscillations, which we will refer to as *best constant property*. For $f \in L^p(Q)$, $p \in [1, \infty)$ we have that

$$M_B^{\sharp,q} f \leq 2\bigg(\fint |f - c|^q dx\bigg)^{\frac{1}{q}} \text{ for all } c \in \mathbb{R}.$$

We will also need the famous *John-Nierenberg estimate* [30], see also [20, Corollary 6.12],

$$M_B^{\sharp,q} f \leq c_q\,\|f\|_{\text{BMO(B)}}$$

for $1 \leq q < \infty$.

We introduce the refined BMO spaces, see [53]. For a non-decreasing function $\omega : (0,\infty) \to (0,\infty)$ we define

$$M^\sharp_{\omega,B} g = \frac{1}{\omega(R_B)} \fint_B |g - \langle g \rangle_B| dx,$$

where R_B is the radius of B. We define the semi norm

$$\|g\|_{\mathrm{BMO}_\omega(\Omega)} := \sup_{B \subset \Omega} M^\sharp_{\omega,B} g.$$

The choice $\omega(r) = 1$ gives the usual BMO semi norm. When $\omega(r) = r^\beta$ with $0 < \beta \leq 1$, we gain by Campanato's characterization that $\mathrm{BMO}_\beta := \mathrm{BMO}_{\mathrm{r}^\beta} \equiv \mathrm{C}^{0,\beta}$.

Contents

CHAPTER 1

Elliptic Systems

We study solutions of an inhomogeneous elliptic system

$$-\mathrm{div}(A(\nabla u)) = -\mathrm{div} f \tag{1.1}$$

on a domain $\Omega \subset \mathbb{R}^n$, where $u : \Omega \to \mathbb{R}^N$ and $f : \Omega \to \mathbb{R}^{N\times n}$. We assume that $f \in \mathrm{BMO}$, where BMO is the space of functions with bounded mean oscillation, and A is given by

$$A(\nabla u) = \varphi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}$$

for a suitable N-function φ. Throughout the chapter we will assume φ satisfies the following assumption.

Assumption 1.1. *Let φ be a convex function on $[0,\infty)$ such that φ is C^1 on $[0,\infty)$ and C^2 on $(0,\infty)$. Moreover, let $\varphi'(0) = 0$, $\lim_{t\to\infty}\varphi'(t) = \infty$ and*

$$\begin{aligned} \varphi'(t) &\sim t\,\varphi''(t) \\ |\varphi''(s+t) - \varphi''(t)| &\le c\varphi''(t)\Big(\frac{|s|}{t}\Big)^{\sigma} \end{aligned} \tag{1.2}$$

uniformly in $t > 0$ with $|s| \le \frac{1}{2}t$ and $\sigma \in (0,1]$. The constants in (1.2) and σ are called the characteristics of *φ.*

The assumptions on φ are such that the induced operator $-\mathrm{div}(A(\nabla u))$ is strictly monotone. If we define the energy

$$\mathcal{J}(v) := \int \varphi(|\nabla v|)\,dx - \int f\cdot\nabla v\,dx,$$

then the system (1.1) is its Euler-Lagrange system and solutions of (1.1) are local minimizers of $\mathcal{J}$.

A significant example of the considered model is the p-Laplacian system, for which $p \in (1,\infty)$, $\varphi(t) = \frac{1}{p}t^p$, $A(\nabla u) = |\nabla u|^{p-2}\nabla u$, and the system (1.1) has the form

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = -\mathrm{div} f.$$

Note that $\varphi(t) = \frac{1}{p}t^p$ satisfies[1] Assumption 1.1.

We know from the linear theory of Poisson's equation (corresponding to $p = 2$) that $f \in L^\infty$ cannot imply $\nabla u \in L^\infty$. The natural question is, does $f \in \mathrm{BMO}$ imply $A(\nabla u) \in \mathrm{BMO}$? The first BMO result was done by DiBenedetto and Manfredi in [12]. Their result, however, only treated the super-quadratic case $p \ge 2$. Our inequalities are more precise and therefore valid for all $p \in (1,\infty)$ and even for more general growth.

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[1] Also $\varphi(t) = \frac{1}{p}\int_0^t(\mu+s)^{p-2}s\,ds$ and $\varphi(t) = \frac{1}{p}\int_0^t(\mu^2+s^2)^{\frac{p-2}{2}}s\,ds$ with $\mu \ge 0$ satisfy Assumption 1.1.

Theorem 1.2. *Let $B \subset \mathbb{R}^n$ be a ball. Let u be a solution of (1.1) on $2B$, with φ satisfying Assumption 1.1.*

If $f \in \mathrm{BMO(2B)}$, then $A(\nabla u) \in \mathrm{BMO(B)}$. Moreover,

$$\|A(\nabla u)\|_{\mathrm{BMO(B)}} \leq c \fint_{2B} |(A(\nabla u)) - \langle A(\nabla u)\rangle_{2B}|\, dx + c\|f\|_{\mathrm{BMO(2B)}}.$$

The constant c depends only on the characteristics of φ.

This theorem is a special case of our main result in Theorem 1.23.

Additionally to Theorem 1.2, we are able to transfer any modulus of continuity of the mean oscillation from f to $A(\nabla u)$. This includes the case of VMO, see Corollary 1.25. Moreover, $f \in C^{0,\beta}(2B)$ implies $A(\nabla u) \in C^{0,\beta}(B)$ with corresponding local estimates, see Corollary 1.26. The β is restricted by the regularity of the p-harmonic functions.

Our results also hold in the context of differential forms on $\Omega \subset \mathbb{R}^n$, where we get the corresponding estimates, see Remark 1.30. By conjugation we can also treat solutions of systems of the form $d^*(A(dv + g)) = 0$.

The special case $f = 0$ in Corollary 1.26 allows us to derive new decay estimates for φ-harmonic functions. On one hand we get decay estimates for $A(\nabla u)$, see Remark 1.27. On the other hand by conjugation, see Remark 1.30 we also get decay estimates for ∇u, see (1.25).

We study systems, where the right-hand side is given in divergence form, since it simplifies the presentation. The results can also be applied to the situation, where the right-hand side $\mathrm{div} f$ of (1.1) is replaced by a function g. Note that any functional from $(W_0^{1,\varphi}(\Omega))^*$ can be represented in such divergence form. Whenever, such g can be represented as $g = \mathrm{div} f$ with $f \in \mathrm{BMO}_\omega$ (a refinement of BMO, see Section 1.4), then our results immediately provide corresponding inequalities. For example we show in Remark 1.28 that $g \in L^n$ implies locally $A(\nabla u) \in \mathrm{VMO}$. This complements the results of [**8, 22**], who proved $A(\nabla u) \in L^\infty$ for $g \in L^{n,1}$ (Lorentz space; subspace of L^n), where the result of [**8**] is for equations only but up to the boundary; just recently the same authors extend their result to systems: "Global boundedness of the gradient for a class of nonlinear elliptic systems, Arch.Rat.Mech.Anal."

All these above results where first published in [**15**]. In this chapter we allow an additional perturbation by a Hölder continuous matrix. For that we denote $T^2 : \mathbb{R}^n \to R^{nN\times nN}$ uniformly elliptic

$$\frac{|x|^2}{\lambda} \leq x^t T^2 x \leq \lambda |x|^2.$$

In Theorem 1.23 we show BMO_ω-regularity for solutions of

$$-\mathrm{div}\Big(\varphi'(\sqrt{T^2\nabla u\cdot\nabla u})\frac{T^2\nabla u}{\sqrt{T^2\nabla u\cdot\nabla u}}\Big) = -\mathrm{div}(f).$$

We can write $T^2 = M^T\Lambda^2 M$, where M is orthonormal and Λ a diagonal matrix. We define $T := \Lambda M$, then $T^2 = T^t T$, then the system above can be written as

$$-\mathrm{div}(A_T(\nabla u)) = -\mathrm{div}(f), \quad \text{for} \quad A_T(\nabla u) = T^t A(T\nabla u). \tag{1.3}$$

We will be able to show BMO estimates also for these equations, as long as T is "close" to a rank one matrix.

Assumption 1.3. *We require*

(a) $T = (T_{ij,kl}) = (t^{ik} t_{jl}) : \Omega \to \mathbb{R}^{nN \times nN}$, *where* $T^* = (t^{ik}) : \Omega \to \mathbb{R}^{N \times N}$ *and* $T_* = (t_{jl}) : \Omega \to \mathbb{R}^{n \times n}$ *with full rank.*

(b) $T = \Lambda M$, *where* M *is orthonormal and* Λ *a diagonal matrix with strict positive entries* $\frac{1}{\lambda} \leq \Lambda_{ii} \leq \lambda$.

(c) $T \in C^{0,\gamma}(\Omega)$ *for a* $\gamma \in (0,1)$. *I.e.*$|T(x) - T(y)| \leq c|x-y|^{\gamma}$.

The quantities γ, λ *and* c *are called the characteristics of* T.

To include these perturbation a refined decay for homogeneous solutions of (1.3) with constant matrix T was shown; Corollary 1.19 which might be interesting on its own. One major advantage of these estimates, is that can be used to prove regularity up to the boundary. In Section 1.5 we proof local BMO_ω estimates up to the boundary for systems. It can be regarded as non-linear Schauder theory. The BMO_ω case has not been studied before. Higher integrability results have been studied before. Kinnunen and Zhou [**37**] studied perturbed equations ($N = 1$) for the p-Laplacien in divergence form. They prove higher integrability for $T \in \mathrm{VMO}(\Omega)$. In [**38**] they where able to show higher integrability for equations up to the boundary; the authors neither covered systems nor the BMO-case.

1.1. Preliminary Results

Assumption 1.1 (see for example [**3**]) implies that φ and φ^* are N-function and satisfy the Δ_2-condition i.e. $\varphi(2t) \leq c\,\varphi(t)$ and $\varphi^*(2t) \leq c\,\varphi^*(t)$ uniformly in $t \geq 0$, where the constants only depend on the characteristics of φ.

As a further consequence of Assumption 1.1 there exists $1 < p \leq q < \infty$ and $K_1 > 0$ such that

$$\varphi(st) \leq K_1 \max\{s^p, s^q\}\varphi(t) \tag{1.4}$$

for all $s, t \geq 0$. The exponents p and q are called the lower and upper index of φ. We say that φ is of type $T(p,q,K_1)$ if it satisfies (1.4), where we allow $1 \leq p \leq q < \infty$ in this definition. Note that (1.4) implies

$$\min\left\{s^p, s^q\right\}\varphi(t) \leq K_1 \varphi(st) \tag{1.5}$$

for all $a, t \geq 0$. Every $\varphi \in T(p,q,K_1)$ satisfies the Δ_2-condition; indeed $\varphi(2t) \leq K_1\, 2^q \varphi(t)$.

Lemma 1.4. *Let* φ *be of type* $T(p,q,K_1)$, *then* $\varphi^* \in T(q', p', K_2)$ *for some* $K_2 = K_2(p,q,K_1)$.

This lemma is well know. However, for the sake of completeness the proof is found in the Appendix. In particular, if $\varphi \in T(p,q,K)$ with $1 < p \leq q < \infty$, then also φ^* satisfies the Δ_2-condition. Under the assumtion of Lemma 1.4 we also get the following versions of *Young's inequality*. For all $\delta \in (0,1]$ and all $t, s \geq 0$ it holds

$$\begin{aligned} ts &\leq K_1\, K_2^{q-1}\, \delta^{1-q}\, \varphi(t) + \delta\, \varphi^*(s), \\ ts &\leq \delta\, \varphi(t) + K_2 K_1^{p'-1}\, \delta^{1-p'}\, \varphi^*(s). \end{aligned} \tag{1.6}$$

For an N-function φ we introduce the family of shifted N-functions $\{\varphi_a\}_{a \geq 0}$ by $\varphi_a'(t)/t := \varphi'(a+t)/(a+t)$. If φ satisfies Assumption 1.1 then $\varphi_a''(t) \sim \varphi''(a+t)$ uniformly in $a, t \geq 0$. The following lemmas show important invariants in terms of shifts.

Lemma 1.5 (Lemma 22, [17]). *Let φ hold Assumption 1.1. Then $(\varphi_{|P|})^*(t) \sim (\varphi^*)_{|A(P)|}(t)$ holds uniformly in $t \geq 0$ and $P \in \mathbb{R}^{N\times n}$. The implicit constants depend on p, q and K only.*

We define

$$\overline{p} := \min\{p, 2\} \text{ and } \overline{q} := \max\{q, 2\}. \tag{1.7}$$

Lemma 1.6. *Let φ be of type $T(p, q, K_1)$ and $P \in \mathbb{R}^{N\times n}$, then $\varphi_{|P|}$ is of type $T(\overline{p}, \overline{q}, \overline{K})$ and $(\varphi_{|P|})^*$ and $(\varphi^*)_{|A(P)|}$ are of type $T(\overline{q}', \overline{p}', K)$.*

The proof of this lemma is postponed to the Appendix. We define $V : \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ by

$$|V(Q)|^2 = A(Q)\cdot Q \qquad \text{and} \qquad \frac{V(Q)}{|V(Q)|} = \frac{A(Q)}{|A(Q)|} = \frac{Q}{|Q|},$$

in particular we have

$$V(Q) = \sqrt{\varphi'(|Q|)|Q|}\,\frac{Q}{|Q|} =: \psi(|Q|)\frac{Q}{|Q|}.$$

In the case of the p-Laplacian, we have $\varphi(t) = \frac{1}{p}t^p$, $A(Q) = |Q|^{p-2}Q$ and $V(Q) = |Q|^{\frac{p-2}{2}}Q$.

The connection between A, V, and the shifted N-functions is best reflected in the following lemma, which is a summary of Lemmas 3, 21, and 26 of [13].

Lemma 1.7. *Let φ satisfy Assumption 1.1. Then*

$$\big(A(P) - A(Q)\big)\cdot\big(P - Q\big) \sim \big|V(P) - V(Q)\big|^2 \tag{1.8a}$$

$$\sim \varphi_{|Q|}\big(|P - Q|\big) \tag{1.8b}$$

$$\sim \big(\varphi^*\big)_{|A(Q)|}\big(|A(P) - A(Q)|\big) \tag{1.8c}$$

uniformly in $P, Q \in \mathbb{R}^{N\times n}$. Moreover,

$$\begin{aligned} A(Q)\cdot Q = |V(Q)|^2 &\sim \varphi(|Q|), \ \textit{and} \\ |A(P) - A(Q)| &\sim \big(\varphi_{|Q|}\big)'\big(|P - Q|\big), \end{aligned} \tag{1.8d}$$

uniformly in $P, Q \in \mathbb{R}^{N\times n}$.

The following lemma is a simple modification of Lemma 35 and Corollary 26 of [17] by use of Young's inequality in the form (1.6) and Lemma 1.5.

Lemma 1.8 (Shift change). *For every $\varepsilon \in (0, 1]$, it holds*

$$\begin{aligned} \varphi_{|P|}(t) &\leq c\,\varepsilon^{1-\overline{p}'}\varphi_{|Q|}(t) + \varepsilon|V(P) - V(Q)|^2, \\ (\varphi_{|P|})^*(t) &\leq c\,\varepsilon^{1-\overline{q}}(\varphi_{|Q|})^*(t) + \varepsilon|V(P) - V(Q)|^2, \\ (\varphi^*)_{|A(P)|}(t) &\leq c\,\varepsilon^{1-\overline{q}}(\varphi^*)_{|A(Q)|}(t) + \varepsilon|V(P) - V(Q)|^2, \end{aligned}$$

for all $P, Q \in \mathbb{R}^{N\times n}$ and all $t \geq 0$. The constants only depend on the characteristics of φ.

By L^φ and $W^{1,\varphi}$ we denote the classical Orlicz and Sobolev-Orlicz spaces, i.e. $f \in L^\varphi$ if and only if $\int \varphi(|f|)\,dx < \infty$ and $f \in W^{1,\varphi}$ if and only if $f, \nabla f \in L^\varphi$. By $W_0^{1,\varphi}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,\varphi}(\Omega)$.

We can prove the following substitute for Lemma 1.7 for perturbated systems.

Lemma 1.9. *For all $Q, P \in \mathbb{R}^{nN}$ and $T_i = \Lambda_i M_i$ we find*

$$\Big(A_T(Q) - A_T(P)\Big) \cdot \Big(Q - P\Big) \sim |V(Q) - V(P)|^2$$

and

$$|A_{T_1}(Q) - A_{T_2}(P)| \le c|T_1 - T_2|\varphi'(|T_1 Q|) + c\Big(\varphi_{|T_1 Q|}\Big)'(|T_1 Q - T_2 P|),$$

especially for $|T_1 - T_2| \le 1$

$$|A_{T_1}(Q) - A_{T_2}(Q)| \le c|T_1 - T_2|^{\overline{p}-1}\varphi'(|Q|).$$

The constants depend only on the characteristics of φ and λ.

Proof. The first inequality is proved by Lemma 1.7. We find that

$$\begin{aligned}
&\Big(A_T(Q) - A_T(P)\Big) \cdot \Big(Q - P\Big) \\
&= \Big(T^t\Big(\varphi'(|TQ|)\frac{TQ}{|TQ|} - \varphi'(|TP|)\frac{TP}{|TP|}\Big)\Big) \cdot \Big(Q - P\Big) \\
&= \Big(\Big(\varphi'(|TQ|)\frac{TQ}{|TQ|} - \varphi'(|TP|)\frac{TP}{|TP|}\Big)\Big) \cdot \Big(TQ - TP\Big) \\
&\sim \varphi_{|TQ|}(|T(Q - P)|) \\
&\sim |V(Q) - V(P)|^2.
\end{aligned} \tag{1.9}$$

For the second statement we use (1.8d)

$$\begin{aligned}
|A_{T_1}(Q) - A_{T_2}(P)| &= \Big|\varphi'(|T_1 Q|)\frac{T_1^2 Q}{|T_1 Q|} - \varphi'(|T_2 P|)\frac{T_2^2 P}{|T_2 P|}\Big| \\
&\le \Big|(T_1^t - T_2^t)\varphi'(|T_1 Q|)\frac{T_1 Q}{|T_1 Q|}\Big| + |T_2|\Big|\varphi'(|T_1 Q|)\frac{T_1 Q}{|T_1 Q|} - \varphi'(|T_2 P|)\frac{T_2 P}{|T_2 P|}\Big| \\
&\le c|T_1 - T_2|\varphi'(|T_1 Q|) + c\Big(\varphi_{|T_1 Q|}\Big)'(|T_1 Q - T_2 P|).
\end{aligned}$$

The last statement follows by Lemma 1.6, as

$$\Big(\varphi_{|Q|}\Big)'(|T_1 - T_2||Q|) \le c|T_1 - T_2|^{\overline{p}-1}\varphi'(|Q|),$$

whenever $|T_1 - T_2| \le 1$. □

Let us introduce the right condition for the perturbation matrix such that regularity is preserved. If one shows Hölder regularity, one can only assume Hölder perturbations. In elliptic systems this is the classical Schauder theory.

A function T is a BMO_ω*-multiplier*, if $Tf \in \mathrm{BMO}(\Omega)$ for all $f \in \mathrm{BMO}(\Omega)$. We introduce BMO-multipliers with following lemma. Its proof can be found in the appendix of this chapter.

Lemma 1.10. *If $T \in L^\infty(\Omega)$ and holds*

$$\|T - T(y)\|_{L^\infty(B_r(y))}\frac{1}{\omega(r)} \fint_{B_r(y)} f\,dx \le c\|f\|_{\mathrm{BMO}_\omega(\Omega)} + c\|f\|_{L^1(\Omega)},$$

for all $B_r \subset \Omega$, then T is a $\mathrm{BMO}_\omega(\Omega)$ multiplier.

We say, that $T \in L^\infty(\Omega)$ satisfies the *vanishing* BMO_ω*-multiplier condition* on Ω if there is a function $\delta(r)$ positive continuous quasi increasing, such that $\delta(r) \to 0$ for $r \to 0$ and

$$c\|T - T(y)\|_{L^\infty(B_r)} \frac{1}{\omega(r)} \fint_{B_r(y)} f\, dx \le \delta(r)\|f\|_{\mathrm{BMO}_\omega(\Omega)} + c\|f\|_{L^1(\Omega)}.$$

We need the following calculation:

$$|\langle g\rangle_{\frac{1}{2}B} - \langle g\rangle_B| \le \fint_{\frac{1}{2}B} |g - \langle g\rangle_B| dx \le 2^n M^\sharp_B g.$$

By m iterations of the previous we find

$$|\langle g\rangle_{2^{-m}B} - \langle g\rangle_B| \le 2^n \sum_{i=0}^{m-1} M^\sharp_{2^{-i}B} g \le m2^n \max_{0\le i\le m-1} M^\sharp_{2^{-i}B} g. \tag{1.10}$$

Note also, that the best constant property implies

$$\fint_B ||g| - \langle |g|_B\rangle|\, dx \le 2\fint_B ||g| - |\langle g\rangle_B||\, dx \le 2\fint_B |g - \langle g\rangle_B|\, dx. \tag{1.11}$$

This can be used to show the following refined BMO-multiplier lemma

Lemma 1.11. *Let $\omega : (0,\infty) \to (0,\infty)$ be non decreasing, such that $\omega(r)r^{-\beta}$ is almost decreasing, then for $\gamma > \beta$ we find that if $T \in C^{0,\gamma}(\Omega)$, then T holds the vanishing BMO_ω-multiplier condition on Ω. Moreover, for $B(x) \subset \Omega$ and $B_i := 2^{-i}B(x)$ it holds*

$$\|T - T(x)\|_{L^\infty(B_m)} \frac{1}{\omega(R_{B_m})} \langle |g|\rangle_{B_m} \le cm2^{-m(\gamma-\beta)} \max_{0\le i\le m-1} M^\sharp_{\omega,2^{-i}B} g + \frac{c}{\omega(R_B)}\langle |g|\rangle_B,$$

the constant c only depends on $\gamma - \beta$ and on the Hölder continuity constant of T.

Proof. Without loss of generality we assume the radius of B to be one. We use the above iteration (1.10), (1.11), the assumptions on T and the assumption on ω to estimate

$$\begin{aligned}\|T - T(x)\|_{L^\infty(B_m)} \frac{1}{\omega(2^{-m})} \langle |g|\rangle_{B_m} &\le c2^{-m\gamma} \frac{1}{\omega(2^{-m})} (|\langle |g|\rangle_{B_m} - \langle |g|\rangle_B| + \langle |g|\rangle_B) \\ &\le cm2^{-m\gamma} \frac{1}{\omega(2^{-m})} \max_{0\le i\le m-1} M^\sharp_{2^{-i}B} g + m\frac{2^{m(\beta-\gamma)}}{\omega(1)} \langle |g|\rangle_B \\ &\le cm2^{-m(\gamma-\beta)} \max_{0\le i\le m-1} M^\sharp_{\omega,2^{-i}B} g + \frac{c}{\omega(1)} \langle |g|\rangle_B,\end{aligned}$$

as $\beta < \gamma$. □

1.2. Reverse Hölder estimate

In this section we refine the reverse Hölder estimate of Lemma 3.4 [19], where the case $f = 0$ was considered. For this we need the following version of Sobolev-Poincaré from [13, Lemma 7].

Theorem 1.12 (Sobolev-Poincaré). *Let φ be an N-function such that φ and φ^* satisfies the Δ_2-condition. Then there exists $0 < \theta_0 < 1$ and $c > 0$ such that the following holds. If $B \subset \mathbb{R}^n$ is some ball with radius R and $v \in W^{1,\varphi}(B, \mathbb{R}^N)$, then*

$$\fint_B \varphi\bigg(\frac{|v - \langle v\rangle_B|}{R}\bigg)\,dx \le c\bigg(\fint_B \varphi^{\theta_0}(|\nabla v|)\,dx\bigg)^{\frac{1}{\theta_0}}. \tag{1.12}$$

For gradients of solutions of (1.1) and (1.3) we can deduce the following reverse Hölder inequality.

Lemma 1.13. *Let u be a solution of (1.3). There exists $\theta \in (0,1)$ such that for all $P, f_0 \in \mathbb{R}^{N\times n}$ and all balls B satisfying $2B \subset \Omega$*

$$\begin{aligned}\fint_B |V(\nabla u) - V(P)|^2\,dx &\le c\bigg(\fint_{2B} |V(\nabla u) - V(P)|^{2\theta}\,dx\bigg)^{\frac{1}{\theta}}\\ &\quad + c\fint_{2B} (\varphi^*)_{|A(P)|}(|f - f_0|)\,dx + c\|T - T(z)\|_{L^\infty(2B)}^{(\overline{p}-1)\overline{q}'}\fint_{2B} \varphi(|\nabla u|) + \varphi(|P|)\,dx\end{aligned}$$

for $z \in B$. The constants c and θ only depend on λ and the characteristics of φ.

Proof. Let $\eta \in C_0^\infty(2B)$ with $\chi_B \le \eta \le \chi_{2B}$ and $|\nabla\eta| \le c/R$, where R is the radius of B. Let $\alpha \ge \overline{q}$, then $(\alpha - 1)\overline{p}' \ge \alpha$. We define $\xi := \eta^\alpha(u - z)$, where z is a linear function such that $\langle u - z\rangle_{2B} = 0$ and $\nabla z = P$. Using ξ as a test function in the weak formulation of (1.1) we get for all $f_0 \in \mathbb{R}^{N\times n}$

$$\begin{aligned}(Ia) &:= |B|^{-1}\langle A_T(\nabla u) - A_{T(z)}(P), \eta^\alpha(\nabla u - P)\rangle\\ &= |B|^{-1}\langle f - f_0, \eta^\alpha(\nabla u - P)\rangle + |B|^{-1}\langle f - f_0, \alpha\eta^{\alpha-1}(u - z)\otimes\nabla\eta\rangle\\ &\quad - |B|^{-1}\langle A_T(\nabla u) - A_{T(z)}(P), \alpha\eta^{\alpha-1}(u - z)\otimes\nabla\eta\rangle\\ &=: (II) + (III) + (IV).\end{aligned}$$

With the help of Lemma 1.7 and Lemma 1.9 we get

$$(Ia) \ge c\fint_{2B} \eta^\alpha|V(\nabla u) - V(P)|^2\,dx - c\fint_{2B} |T - T(z)|^{\overline{p}-1}\varphi'(|\nabla u|)|\nabla u - P|\eta^\alpha dx.$$

By (1.6) for $\varphi_{|\nabla u|}$ and Lemma 1.6 we find that

$$(Ia) \ge (c - \varepsilon)\fint_{2B} \eta^\alpha|V(\nabla u) - V(P)|^2\,dx - c\fint_{2B} (\varphi_{|\nabla u|})^*\Big(\|T - T(z)\|_{L^\infty(2B)}^{\overline{p}-1}\varphi(|\nabla u|)\Big)\,dx.$$

By Lemma 1.6 we have $(\varphi_{|\nabla u|})^* \in T(\overline{q}', \overline{p}', K)$. Consequently

$$\begin{aligned}(I) &:= \fint_{2B} \eta^\alpha |V(\nabla u) - V(P)|^2\,dx \\ &\leq c\|T - T(z)\|_{L^\infty(2B)}^{(\overline{p}-1)\overline{q}'} \fint_{2B} \varphi(|\nabla u|)dx + c(II) + c(III) + c(IV).\end{aligned}$$

We used $(\varphi_{|\nabla u|})^*(|\nabla u|) \sim \varphi(|\nabla u|)$, which is a consequence of Lemma 1.7. By (1.6) for $\varphi_{|P|}$ and $\delta \in (0,1)$, by $(\varphi_{|P|})^* \sim (\varphi^*)_{|A(P)|}$ due to Lemma 1.5, $(\alpha-1)\overline{p}' \geq \alpha$ and by Lemma 1.7 we estimate

$$\begin{aligned}(II) &\leq c\,\delta^{1-\overline{p}'} \fint_{2B} (\varphi^*)_{|A(P)|}(|f - f_0|)\,dx + \delta \fint_{2B} \eta^\alpha \varphi_{|P|}(|\nabla u - P|)\,dx \\ &\leq c\,\delta^{1-\overline{p}'} \fint_{2B} (\varphi^*)_{|A(P)|}(|f - f_0|)\,dx + \delta c \fint_{2B} \eta^\alpha |V(\nabla u) - V(P)|^2\,dx.\end{aligned}$$

Similarly, we estimate with Lemma 1.7

$$(III) \leq c \fint_{2B} (\varphi^*)_{|A(P)|}(|f - f_0|)\,dx + c \fint_{2B} \varphi_{|P|}\left(\frac{|u-z|}{R}\right)dx.$$

To estimate (IV) we take

$$\begin{aligned}(IV) \leq c &\fint_{2B} \alpha\eta^{\alpha-1}|A_T(\nabla u) - A_T(P)|\left|\frac{u-z}{R}\right|dx \\ &+ c\fint_{2B} |A_T(P) - A_{T(z)}(P)|\left|\frac{u-z}{R}\right|dx = (V) + (VI).\end{aligned}$$

With Lemma 1.9, Young's inequality with $\varphi_{|P|}$, $(\alpha-1)\overline{q} \geq \alpha$ and (0.6) (second part) in combination with Lemma 1.7 we deduce analogously

$$\begin{aligned}(V) &\leq c \fint_{2B} \varphi'_{|P|}(|A(\nabla u) - P|)\,\eta^{\alpha-1}\frac{|u-z|}{R}\,dx \\ &\leq \delta \fint_{2B} \eta^\alpha (\varphi_{|P|})^*\big(\varphi'_{|P|}(|\nabla u - P|)\big)\,dx + c\,\delta^{1-\overline{q}} \fint_{2B} \varphi_{|P|}\left(\frac{|u-z|}{R}\right)dx. \\ &\leq \delta \fint_{2B} \eta^\alpha |V(\nabla u) - V(P)|^2\,dx + c\,\delta^{1-\overline{q}} \fint_{2B} \varphi_{|P|}\left(\frac{|u-z|}{R}\right)dx.\end{aligned}$$

As before we find by Young's inequality for $\varphi_{|P|}$ and Lemma 1.6,

$$(VI) \leq c\|T - T(z)\|_{L^\infty(2B)}^{(\overline{p}-1)\overline{q}'} \fint_{2B} \varphi(|P|)\,dx + c\fint_{2B} \varphi_{|P|}\left(\frac{|u-z|}{R}\right)dx.$$

Moreover, it follows from Theorem 1.12 for $\varphi_{|P|}$ for some $\theta \in (0,1)$, Lemma 1.7 and the facts that $\langle u - z\rangle_{2B} = 0$ and $\nabla z = P$ that

$$\fint_{2B} \varphi_{|P|}\left(\frac{|u-z|}{R}\right) dx \le c\left(\fint_{2B} \left(\varphi_{|P|}(|\nabla u - P|)\right)^{\theta} dx\right)^{\frac{1}{\theta}}$$

$$\le c\left(\fint_{2B} |V(\nabla u) - V(P)|^{2\theta}\, dx\right)^{\frac{1}{\theta}}.$$

For small δ we can absorb corresponding terms into (I) such that the claim follows. □

Our aim is to give estimates in terms of $A(\nabla u)$. We will give estimates exploiting reverse Hölder inequalities as well as BMO properties. These will enable us to replace the right hand side of Lemma 1.13 with adequate quantities. At first we need the following lemma for improving reverse Hölder estimates. It follows from [24, Remark 6.12] and [21, Lemma 3.2].

Lemma 1.14. *Let $B \subset \mathbb{R}^n$ be a ball, let $g, h : \Omega \to \mathbb{R}$ be an integrable functions and $\theta \in (0,1)$ such that*

$$\fint_{B} |g|\, dx \le c_0 \left(\fint_{2B} |g|^{\theta}\, dx\right)^{\frac{1}{\theta}} + \fint_{2B} |h|\, dx$$

for all balls B with $2B \subset \Omega$. Then for every $\gamma \in (0,1)$ there exists $c_1 = c_1(c_0, \gamma)$ such that

$$\fint_{B} |g|\, dx \le c_1 \left(\fint_{2B} |g|^{\gamma}\, dx\right)^{\frac{1}{\gamma}} + c_1 \fint_{2B} |h|\, dx.$$

We will use this result to prove the following inverse Jensen inequality.

Corollary 1.15. *Let $\Omega \subset \mathbb{R}^n$ and ψ be an N-function of type $T(1,q,K)$, $g \in L^{\psi}(\Omega)$ and $h \in L^1_{\mathrm{loc}}(\Omega)$. If there exists $\theta \in (0,1)$ such that*

$$\fint_{B} \psi(|g|)\, dx \le c_0 \left(\fint_{2B} \psi(|g|)^{\theta}\, dx\right)^{\frac{1}{\theta}} + \fint_{2B} |h|\, dx,$$

for all balls B with $2B \subset \Omega$, then there exists $c_1 = c_1(c_0, K, q)$ such that

$$\fint_{B} \psi(|g|)\, dx \le c_1\, \psi\left(\fint_{2B} |g|\, dx\right) + c_1 \fint_{2B} |h|\, dx.$$

Proof. By Lemma 1.14 we gain for a fixed $\gamma < \frac{1}{q}$

$$\fint_{B} \psi(|g|)\, dx \le c_1 \left(\fint_{2B} \psi(|g|)^{\gamma}\, dx\right)^{\frac{1}{\gamma}} + c_1 \fint_{2B} |h|\, dx$$

Due to Lemma 1.34, which can be found in the appendix, the function $(\psi(t))^{\gamma})^{-1}$ is quasi-convex; i.e. it is uniformly proportional to a convex function. Therefore the result follows by Jensen's inequality. □

The estimate of Lemma 1.13 can be improved in the following way.

Corollary 1.16. *Let u be a solution of (1.1). For all $P \in \mathbb{R}^{N\times n}$ and all balls B such that $2B \subset \Omega$*

$$\fint_B |V(\nabla u) - V(P)|^2\,dx \le c\,(\varphi^*)_{|A(P)|}\bigg(\fint_{2B} |A(\nabla u) - A(P)|\,dx\bigg) + c\,(\varphi^*)_{|A(P)|}(\|f\|_{\mathrm{BMO(2B)}}) + c\|T - T(z)\|^{(\overline{p}-1)\overline{q}'}_{L^\infty(2B)}(\langle\varphi(|\nabla u|)\rangle_{2B} + \varphi(|P|))$$

for $z \in B$. The constants only depend on the characteristics of φ and λ.

Proof. If follows from Lemma 1.7 that

$$|V(\nabla u) - V(P)|^2 \sim (\varphi^*)_{|A(P)|}(|A(\nabla u) - A(P)|).$$

Therefore we can apply Corollary 1.15 on the inequality proven in Lemma 1.13 to gain

$$\fint_B |V(\nabla u) - V(P)|^2\,dx \le c\,(\varphi^*)_{|A(P)|}\bigg(\fint_{2B} |A(\nabla u) - A(P)|\,dx\bigg) + c\fint_{2B}(\varphi^*)_{|A(P)|}(|f - f_0|)\,dx + c\|T - T(z)\|^{(\overline{p}-1)\overline{q}'}_{L^\infty(2B)}(\langle\varphi(|\nabla u|)\rangle_{2B} + \varphi(|P|))$$

for any $f_0 \in \mathbb{R}^{N\times n}$. The result follows by using Lemma 1.32 to the last integral

$$\fint_{2B}(\varphi^*)_{|A(P)|}(|f - f_0|)\,dx \le c\,(\varphi^*)_{|A(P)|}(\|f\|_{\mathrm{BMO(2B)}}).$$

This inequality reflects the reverse Jensen property of the BMO norm. □

Lemma 1.17. *Let u be a solution of (1.3). We find*

$$\fint_B \varphi(|\nabla u|)\,dx \le c(\varphi^*)_{|A(Q)|}\big(\langle A(\nabla u)\rangle_{2B}\big) + (\varphi^*)_{|A(Q)|}(\|f\|_{\mathrm{BMO(2B)}})$$

for $A(Q) = \langle A(\nabla u)\rangle_B$. The constants c only depend on λ and the characteristics of φ.

Proof. The proof goes analogously (but simpler) as was done for the oscillation integrals. We only give the important details. One uses $\xi = (u - \langle u\rangle_{2B})\eta^\alpha$ as a test function and find for $f_0 = \langle f\rangle_{2B}$

$$\langle A_T(\nabla u), \nabla u\eta^\alpha\rangle = \langle A_T(\nabla u), \alpha\eta^{\alpha-1}(u - \langle u\rangle_{2B}) \otimes \nabla\eta\rangle + \langle f - f_0, \nabla\xi\rangle.$$

The difference to Lemma 1.13 is that all terms that include ∇u can be absorbed. One uses Young's inequality 1.6 on $\varphi_{|Q|}$ and the fact that Lemma 1.7 and Lemma 1.33 imply $\langle\varphi(|\nabla u|)\rangle_B \sim \langle\varphi_{|Q|}(|\nabla u|)\rangle_B \sim \langle(\varphi)^*_{|A(Q)|}(A(|\nabla u|))\rangle_B$. This leads to

$$\fint_B \varphi(|\nabla u|)dx \le c(\varphi^*)_{|A(Q)|}(\|f\|_{\mathrm{BMO(2B)}}) + c\fint_{2B}\varphi_{|Q|}\Big(\Big|\frac{u - \langle u\rangle_{2B}}{R_B}\Big|\Big)\,dx.$$

Now the result follows analogous to the oscillation case by Poincaré's inequality and Corollary 1.15. □

1.3. Comparison

The key idea of the proof of our main result is to compare the solution u with a suitable φ-harmonic function h. Later we transfer the good properties of h to u. Regularity of φ-harmonic functions is well known in the case of p-Laplace system with $\varphi(t) = t^p$ for $p \in (1, \infty)$. Recently, the result was extended in [19, Theorem 6.4] for general φ satisfying Assumption 1.1:

Theorem 1.18 (Decay estimate for φ-harmonic maps). *Let $\Omega \subset \mathbb{R}^n$ be an open set, let φ satisfy Assumption 1.1, and let $h \in W^{1,\varphi}(\Omega, \mathbb{R}^N)$ be φ-harmonic on Ω. Then there exists $\alpha > 0$ and $c > 0$ such that for every ball $B \subset \Omega$ and every $\theta \in (0,1)$ holds*

$$\fint_{\theta B} |V(\nabla h) - \langle V(\nabla h)\rangle_{\theta B}|^2\, dx \le c\, \theta^{2\alpha} \fint_{B} |V(\nabla h) - \langle V(\nabla h)\rangle_{B}|^2\, dx.$$

Note that c and α depend only on the characteristics of φ.

The last Theorem can be extended. We take $T \in \mathbb{R}^{nN\times nN}$ elliptic. Let us look at local minimizers of functionals of the type

$$J(v) = \int_\Omega \varphi(|T^2\nabla v \cdot \nabla v|^{\frac{1}{2}}) = \int_\Omega \varphi(|T\nabla v|), \tag{1.13}$$

for $v \in W^{1,\varphi}(\Omega; \mathbb{R}^N)$.

Again $T^2 = T^t T$ and $T = \Lambda M$, with M being orthonormal and Λ being diagonal with all values strictly positive. We want to regain a $\tilde{\varphi}-$minimizer, on which we can apply Theorem 1.18. We define $\tilde{v}(x) := T^* v(T_* x)$, where $T^* \in \mathbb{R}^{N\times N}$ and $T_* \in \mathbb{R}^{n\times n}$ with full rank.

Now $\partial_{x_i}(v^k(T_*x)) = \sum_{j=1}^n (\partial_j v^k)(T_*x) t_{ji}$. This implies, that

$$\partial_i \tilde{v}^l = \sum_{k=1}^N \sum_{j=1}^n t^{lk} t_{ji} \partial_i v^k(T_* x).$$

Therefore whenever $T \in \mathbb{R}^{nN\times nN}$ has the form $T_{lj,ki} = t^{lk} t_{ji}$ we find that

$$\int_\Omega \varphi(|T\nabla v|)dx = |\det(T_*)| \int_{T_*^{-1}\Omega} \varphi(|T\nabla v(T_* x)|) =: \int_{\tilde{\Omega}} \tilde{\varphi}(|\nabla \tilde{v}|) =: \tilde{J}(\tilde{v}). \tag{1.14}$$

Corollary 1.19. *Let h be a minimizer of (1.13) with T is of the form as stated above and $B \subset \Omega$. Then*

$$\fint_{\theta B} |V(\nabla h) - \langle V(\nabla h)\rangle_{\theta B}|^2\, dx \le c\, \theta^{2\alpha} \fint_{B} |V(\nabla h) - \langle V(\nabla h)\rangle_{B}|^2\, dx.$$

Here α, c only depend on the constants of Theorem 1.18 and $|\det T_*|$.

PROOF. By (1.14) we find that every local minimizer h of J can be represented by a local minimizer $\tilde{h}$ of $\tilde{J}$. Now $V(\nabla\tilde{h})(x) = V(TDh)(T_* x)$. This implies by Lemma 1.9 , (1.9) and Theorem 1.18

$$\begin{aligned}
\fint_{\theta B} |V(\nabla h) - \langle V(\nabla h)\rangle_{\theta B}|^2\,dx &\sim \fint_{\theta B} |V(T\nabla h) - \langle V(T\nabla h)\rangle_{\theta B}|^2\,dx \\
&\leq c \fint_{\theta T_*^{-1} B} |V(\nabla\tilde{h}) - \langle V(\nabla\tilde{h})\rangle_{\theta B}|^2\,dx \\
&\leq c\,\theta^{2\alpha} \fint_{T_*^{-1} B} |V(\nabla\tilde{h}) - \langle V(\nabla\tilde{h})\rangle_{B}|^2\,dx \\
&\sim c\theta^{2\alpha} \fint_{B} |V(\nabla h) - \langle V(\nabla h)\rangle_{B}|^2\,dx.
\end{aligned}$$

□

For a given solution u of (1.3) let $h \in W^{1,\varphi}(B)$ be the unique solution

$$\begin{aligned}
-\mathrm{div} A_{T(z)}(\nabla h) &= 0 \quad \text{in } B, \\
h &= u \quad \text{on } \partial B
\end{aligned} \tag{1.15}$$

where z is the center of the ball. The next lemma estimates the distance of h to u.

Lemma 1.20. *Let u be a solution of (1.1). Further let h solve (1.15). Then for every $\delta > 0$ there exists $c_\delta \geq 1$ such that*

$$\begin{aligned}
\fint_B |V(\nabla u) - V(\nabla h)|^2\,dx &\leq \delta\,(\varphi^*)_{|\langle A(\nabla u)\rangle_{2B}|}\Big(\fint_{2B} |A(\nabla u) - \langle A(\nabla u)\rangle_{2B}|\,dx\Big) \\
&\quad + c\,\delta^{1-\overline{q}}\,(\varphi^*)_{|\langle A(\nabla u)\rangle_{2B}|}(\|f\|_{\mathrm{BMO(2B)}}) + c\|T - T(z)\|_{L^\infty(2B)}^{(\overline{p}-1)\overline{q}'}\langle\varphi(|\nabla u|)\rangle_{2B}
\end{aligned}$$

holds.

PROOF. We have for any $f_0 \in \mathbb{R}^{N\times n}$

$$|B|^{-1}\langle A_T(\nabla u) - A_{T(z)}(\nabla h), \nabla u - \nabla h\rangle = |B|^{-1}\langle f - f_0, \nabla u - \nabla h\rangle$$

We find by Lemma 1.9

$$\begin{aligned}
&\fint_B \big(A_{T(z)}(\nabla u) - A_{T(z)}(\nabla h)\big)\big(\nabla u - \nabla h\big) + \big(A_T(\nabla u) - A_{T(z)}(\nabla u)\big)\big(\nabla u - \nabla h\big) \\
&\sim \fint_B |V(\nabla u) - V(\nabla h)|^2\,dx + \fint_B \big(A_T(\nabla u) - A_{T(z)}(\nabla u)\big)\cdot\big(\nabla u - \nabla h\big).
\end{aligned}$$

Consequently

$$\begin{aligned}
(I) = \fint_B |V(\nabla u) - V(\nabla h)|^2\,dx &\leq c\fint_B |f - f_0||\nabla u - \nabla h| \\
&\quad + c\fint_B |A_T(\nabla u) - A_{T(z)}(\nabla u)||\nabla u - \nabla h| = (II) + (III).
\end{aligned} \tag{1.16}$$

We estimate (III) using Lemma 1.9 and Young's inequality (1.6) with $\varphi_{|\nabla u|}$.

$$\begin{aligned}(III) &\leq c \fint_B |T - T(z)|^{\overline{p}-1} \varphi'(|\nabla u|)|\nabla u - \nabla h| dx \\ &\leq \varepsilon(I) + c\varepsilon^{1-\overline{p}'} \|T - T(z)\|_{L^\infty(B)}^{(\overline{p}-1)\overline{q}'} \langle \varphi(|\nabla u|)\rangle_B.\end{aligned}$$

We estimate (II) by Young's inequality (1.6) with $\varphi_{|\nabla u|}$, Lemma 1.7 and Lemma 1.5

$$(II) \leq \varepsilon(I) + c \fint_B (\varphi^*)_{|A(\nabla u)|}(|f - f_0|)\, dx.$$

With the shift change of Lemma 1.8 with $A(Q) := \langle A(\nabla u)\rangle_{2B}$ we get for $\delta > 0$

$$(II) \leq \varepsilon(I) + c\, \delta^{1-\overline{q}} \fint_B (\varphi^*)_{|A(Q)|}(|f - f_0|)\, dx + \delta \fint_B |V(\nabla u) - V(Q)|^2\, dx. \tag{1.17}$$

We set $f_0 = \langle f\rangle_{2B}$ and estimate the first integral by Lemma 1.32. The second integral is estimated by Corollary 1.16 with $P := Q$. Then $\varphi(|Q|) \leq c\langle \varphi(|\nabla u|)\rangle_{2B}$, such that the claim follows by choosing $\delta, \varepsilon > 0$ conveniently. □

Remark 1.21. *Here we consider $u \in W_0^{1,\varphi}(\mathbb{R}^n; \mathbb{R}^N)$ a global solution of (1.1). We gain by (1.17) and Theorem 1.18 and Lemma 1.33,*

$$\begin{aligned}&\fint_{\theta B} |V(\nabla u) - \langle V(\nabla u)\rangle_{\theta B}|^2 dx \\ &\quad\leq c\theta^n \fint_B |V(\nabla u) - V(\nabla h)|^2 dx + c \fint_B |V(\nabla h) - \langle V(\nabla h)\rangle_{\theta B}|^2 dx \\ &\quad\leq c\, \delta^{1-\overline{q}} \fint_B (\varphi^*)_{|\langle A(\nabla u)\rangle_B|}(|f - f_0|)\, dx + \delta \fint_B |V(\nabla u) - \langle V(\nabla u)\rangle_B|^2\, dx.\end{aligned}$$

This estimate is very much in the spirit of Iwaniec [28]. We can deduce some global estimates from this inequality. First we discuss the case $\varphi(t) = t^p$. In the case of $p = 2$ we find that $(\varphi^*)_{|\langle A(\nabla u)\rangle_B|}(t) \sim t^2$. Therefore the last estimate implies Theorem 0.1 by taking the supremum over all radii and absorption (which is possible for almost every x).

If $p \geq 2$ we find $(\varphi^*)_{|\langle A(\nabla u)\rangle_B|} \leq (\varphi^*)$. Now the estimate implies (after taking the suprema over all radii and absorbation) for almost every x

$$M^{\sharp,2}(V(\nabla u)(x) \leq cM_2(|f|^{\frac{p'}{2}})(x).$$

For general φ we (only) find by Lemma 1.8 that there is a uniform $\delta > 0$ such that

$$M^{\sharp,2}(V(\nabla u)(x) \leq cM_2(\varphi^*(|f|)^{\frac{1}{2}})(x) + \delta M_2(V(\nabla u)).$$

By the maximal theorem's we find for $2 < q < \infty$ and general φ

$$\|V(\nabla u)\|_q \leq c\|\varphi^*(|f|)^{\frac{1}{2}}\|_q$$

especially for $p \geq 2$

$$\|V(\nabla u)\|_{\mathrm{BMO}}^2 \leq c\|\varphi^*(f)\|_\infty.$$

Theorem 1.23 will later imply proper global BMO*-estimates for general φ.*

1.4. BMO estimates for $A(\nabla u)$

Proposition 1.22. *Let $B \subset \mathbb{R}^n$ be a ball. Let α be the decay exponent for φ-harmonic functions as in Theorem 1.18. Then for every $m \in \mathbb{N}$ there exists $c_m \geq 1$ such that*

$$M^\sharp_{2^{-m}B}(A(\nabla u)) \leq c\, 2^{-m\frac{2\alpha}{\overline{p}'}} m \max_{0 \leq i \leq m} M^\sharp_{2^{1-i}B}(A(\nabla u)) + c_m \, \|f\|_{\mathrm{BMO(2B)}}$$
$$+ c_m \|T - T(z)\|^{(\overline{p}-1)\frac{\overline{q}'}{\overline{p}'}}_{L^\infty(2B)} \langle |A\nabla u| \rangle_{2B}.$$

The constant c_m is depending on α and the characteristics of φ and T. The constant c is independent of m and α.

Proof. Define $A(Q) := \langle A(\nabla u)\rangle_{2B}$ and $A(Q_m) := \langle A(\nabla u)\rangle_{2^{-m}B}$. With Lemma 1.6 we find $(\varphi^*)_{|A(P)|}$ is of type $T(\overline{q}', \overline{p}', K)$ for some K independent of P.

Let h be the φ-harmonic function on B with $u = h$ on the boundary ∂B as defined by (1.15). Then $V(\nabla h)$ satisfies the decay estimate of Theorem 1.19

(1.18)
$$\begin{aligned}
(I) &:= \fint_{2^{-m}B} |V(\nabla u) - \langle V(\nabla u)\rangle_{2^{-m}B}|^2 \, dx \\
&\leq c \fint_{2^{-m}B} |V(\nabla h) - \langle V(\nabla h)\rangle_{2^{-m}B}|^2 \, dx + c \fint_{2^{-m}B} |V(\nabla u) - V(\nabla h)|^2 \, dx \\
&\leq c\, 2^{-m2\alpha} \fint_B |V(\nabla h) - \langle V(\nabla h)\rangle_B|^2 \, dx + c 2^{mn} \fint_B |V(\nabla u) - V(\nabla h)|^2 \, dx \\
&\leq c\, 2^{-m2\alpha} \fint_B |V(\nabla u) - \langle V(\nabla u)\rangle_B|^2 \, dx + c\, 2^{mn} \fint_B |V(\nabla u) - V(\nabla h)|^2 \, dx. \\
&\leq c\, 2^{-m2\alpha} \fint_B |V(\nabla u) - V(Q)|^2 \, dx + c\, 2^{mn} \fint_B |V(\nabla u) - V(\nabla h)|^2 \, dx.
\end{aligned}$$

Now using Corollary 1.16, Lemma 1.20 and Lemma 1.17 we get

(1.19)
$$\begin{aligned}
(I) &\leq c\,(2^{-m2\alpha} + \delta 2^{mn})(\varphi^*)_{|A(Q)|}\bigg(\fint_{2B} |A(\nabla u) - A(Q)| \, dx\bigg) \\
&\quad + c\, 2^{mn} \delta^{1-\overline{q}} \left((\varphi^*)_{|A(Q)|}(\|f\|_{\mathrm{BMO(2B)}}) + \|T - T(z)\|^{(\overline{p}-1)\overline{q}'}_{L^\infty(2B)} \langle \varphi(|\nabla u|)\rangle_{2B}\right) \\
&\leq c\,(2^{-m2\alpha} + \delta 2^{mn})(\varphi^*)_{|A(Q)|}\bigg(\fint_{2B} |A(\nabla u) - A(Q)| \, dx\bigg) \\
&\quad + c\, 2^{mn} \delta^{1-\overline{q}} \, (\varphi^*)_{|A(Q)|}(\|f\|_{\mathrm{BMO(2B)}}) \\
&\quad + c\, 2^{mn} \delta^{1-\overline{q}} \|T - T(z)\|^{(\overline{p}-1)\overline{q}'}_{L^\infty(2B)} (\varphi^*)_{|A(Q)|}(\langle |A(\nabla u)|\rangle_{2B}).
\end{aligned}$$

We use Lemma 1.8 to change the shift $A(Q)$ to $A(Q_m)$ (for the first integral with $\varepsilon = 1$ and for the second and third integral with $\varepsilon = \frac{\tau}{2}$).

$$\begin{aligned}(I) \leq c\,(2^{-m2\alpha} + \delta 2^{mn})(\varphi^*)_{|A(Q_m)|}\big(M^\sharp_{2B}(A(\nabla u))\big)\\ + c\,2^{mn}\,\delta^{1-\overline{q}}\,\tau^{1-\overline{q}}(\varphi^*)_{|A(Q_m)|}(\|f\|_{\mathrm{BMO(2B)}})\\ + c\,2^{mn}\,\delta^{1-\overline{q}}\,\tau^{1-\overline{q}}\|T-T(z)\|^{(\overline{p}-1)\overline{q}'}_{L^\infty(2B)}(\varphi^*)_{|A(Q_m)|}(\langle|A(\nabla u)|\rangle_{2B}).\\ + c\,(2^{-m2\alpha} + \delta 2^{mn} + \tau)\,|V(Q) - V(Q_m)|^2.\end{aligned}$$

From Lemma 1.7 we know that

$$|V(Q) - V(Q_m)|^2 \leq c\,(\varphi^*)_{|A(Q_m)|}(|A(Q) - A(Q_m)|)$$

and from (1.10) that

$$|A(Q) - A(Q_m)| \leq 2^n \sum_{0\leq i\leq m-1} M^\sharp_{2^{-i}B}(A(\nabla u)).$$

The previous two estimates and $(\varphi^*)_{|A(Q_m)|} \in T(\overline{q}', \overline{p}', K)$ imply

$$|V(Q) - V(Q_m)|^2 \leq c\,(\varphi^*)_{|A(Q_m)|}\bigg(\sum_{0\leq i\leq m-1} M^\sharp_{2^{-i}B}(A(\nabla u))\bigg).$$

Overall, we get

$$\begin{aligned}(I) \leq c(2^{-m2\alpha} + \delta 2^{mn} + \tau)(\varphi^*)_{|A(Q_m)|}\Big(\sum_{0\leq i\leq m} M^\sharp_{2^{1-i}B}(A(\nabla u))\Big)\\ + c\,2^{mn}\,\delta^{1-\overline{q}}\,\tau^{1-\overline{q}}(\varphi^*)_{|A(Q_m)|}(\|f\|_{\mathrm{BMO(2B)}})\\ + c\,2^{mn}\,\delta^{1-\overline{q}}\,\tau^{1-\overline{q}}\|T-T(z)\|^{(\overline{p}-1)\overline{q}'}_{L^\infty(2B)}(\varphi^*)_{|A(Q_m)|}(\langle|A(\nabla u)|\rangle_{2B}).\end{aligned}$$

We fix $\tau := 2^{-m2\alpha}$ and $\delta := 2^{-m2\alpha - mn}$ to get

$$\begin{aligned}(I) \leq c\,2^{-m2\alpha}(\varphi^*)_{|A(Q_m)|}\Big(\sum_{0\leq i\leq m} M^\sharp_{2^{1-i}B}(A(\nabla u))\Big)\\ + c\,2^{mn+(m4\alpha+mn)(\overline{q}-1)}\,(\varphi^*)_{|A(Q_m)|}(\|f\|_{\mathrm{BMO(2B)}})\\ + c\,2^{mn+(m4\alpha+mn)(\overline{q}-1)}\|T-T(z)\|^{(\overline{p}-1)\overline{q}'}_{L^\infty(2B)}(\varphi^*)_{|A(Q_m)|}(\langle|A(\nabla u)|\rangle_{2B}).\end{aligned}$$

Note that for all $b \in [0, 1/K]$ and $t \geq 0$ we have by (1.5)

$$b\,(\varphi^*)_{|A(Q_m)|}(t) = \frac{1}{K}\,(bK)\,(\varphi^*)_{|A(Q_m)|}(t) \leq (\varphi^*)_{|A(Q_m)|}((bK)^{\frac{1}{\overline{p}'}}\,t).$$

Without loss of generality we can assume in the following that m is sufficiently large so $c2^{-m2\alpha} \leq 1/K$. Therefore

$$\begin{aligned}
(I) &\leq (\varphi^*)_{|A(Q_m)|}\Big(c\,2^{-m\frac{2\alpha}{\overline{p}'}} \sum_{0\leq i\leq m} M^\sharp_{2^{1-i}B}(A(\nabla u))\Big) \\
&\quad + (\varphi^*)_{|A(Q_m)|}(c_m\,\|f\|_{\mathrm{BMO(2B)}}) \\
&\quad + (\varphi^*)_{|A(Q_m)|}(c_m \|T - T(z)_{2B}\|^{(\overline{p}-1)\frac{\overline{q}'}{\overline{p}'}}_{L^\infty(2B)} \langle |A(\nabla u)|\rangle_{2B}) \qquad (1.20)\\
&\leq (\varphi^*)_{|A(Q_m)|}\Big(c\,2^{-m\frac{2\alpha}{\overline{p}'}} \sum_{0\leq i\leq m} M^\sharp_{2^{1-i}B}(A(\nabla u)) + c_m\|f\|_{\mathrm{BMO(2B)}} \\
&\quad + c_m \|T - T(z)\|^{(\overline{p}-1)\frac{\overline{q}'}{\overline{p}'}}_{L^\infty(2B)} \langle |A(\nabla u)|\rangle_{2B}\Big).
\end{aligned}$$

On one hand

$$\begin{aligned}
&\fint_{2^{-m}B} (\varphi^*)_{|A(Q_m)|}(|A(\nabla u) - A(Q_m)|)\,dx \\
&\qquad \leq c \fint_{2^{-m}B} (A(\nabla u) - A(Q_m)) \cdot (\nabla u - Q_m)\,dx \\
&\qquad \leq c \fint_{2^{-m}B} |V(\nabla u) - V(\langle \nabla u\rangle_{2^{-m}B})|^2\,dx
\end{aligned}$$

by Lemma 1.7 and $\langle A(\nabla u) - A(Q_m)\rangle_{2^{-m}B} = \langle \nabla u - \langle \nabla u\rangle_{2^{-m}B}\rangle_{2^{-m}B} = 0$.

Consequently we get using Lemma 1.4, Jensen's inequality and Lemma 1.33

$$\begin{aligned}
&(\varphi^*)_{|A(Q_m)|}\big(c\,M^\sharp_{2^{-m}B}(A(\nabla u))\big) \\
&\qquad \leq c\,(\varphi^*)_{|A(Q_m)|}\bigg(\fint_{2^{-m}B} |A(\nabla u) - A(Q_m)|)\,dx\bigg) \qquad (1.21)\\
&\qquad \leq c \fint_{2^{-m}B} (\varphi^*)_{|A(Q_m)|}(|A(\nabla u) - A(Q_m)|)\,dx \leq (I).
\end{aligned}$$

If we apply the inverse of $(\varphi^*)_{|A(Q_m)|}$ to the combination of (1.20) and (1.21) we obtain the claim. □

We can now prove our main result. It shows that the BMO_ω-regularity of f transfers to $A(\nabla u)$. Note that the case $\omega = 1$ is just Theorem 1.2.

Theorem 1.23. *Let $B \subset \mathbb{R}^n$ be a ball. Let u be a solution of (1.3) on $2B$, with φ satisfying Assumption 1.1 and T satisfying Assumption 1.3. Let $\omega : (0,\infty) \to (0,\infty)$ be non-decreasing such that for some $\beta \in (0, \min\{\frac{2\alpha}{\overline{p}'}, \gamma(\overline{p}-1)\frac{\overline{q}'}{\overline{p}'}\})$ the function $\omega(r)r^{-\beta}$ is almost decreasing. Then*

$$\max_{i\geq 0} M^\sharp_{\omega,2^{-m}B}(A(\nabla u)) \leq c\frac{\langle |A(\nabla u)|\rangle_{2B}}{\omega(2R)} + c\|f\|_{\mathrm{BMO}\omega(2\mathrm{B})}.$$

Moreover,

$$\|A(\nabla u)\|_{\mathrm{BMO}\omega(\mathrm{B})} \leq c\frac{\langle |A(\nabla u)|\rangle_{2B}}{\omega(2R)} + c\|f\|_{\mathrm{BMO}\omega(2\mathrm{B})}.$$

The constants depend on the characteristics of φ *and* T*,* β *and* c_0*.*

PROOF. Let $\sigma := \frac{2\alpha}{\overline{p}'}$, then $0 \leq \beta < \sigma$. We divide the estimate of Proposition 1.22 by $\omega(2^{-m}R)$, where R is the radius of B.

$$\begin{aligned}
M^\sharp_{\omega,2^{-m}B}(A(\nabla u)) &\leq c\, 2^{-m\sigma} m \max_{0\leq i\leq m} \frac{\omega(2^{1-i}R)}{\omega(2^{-m}R)} M^\sharp_{\omega,2^{1-i}B}(A(\nabla u)) \\
&\quad + c_m \frac{1}{\omega(2^{-m}R)} \Big(\|f\|_{\mathrm{BMO(2B)}} + \|T - T(z)\|^{(\overline{p}-1)\frac{\overline{q}'}{\overline{p}'}}_{L^\infty(2B)} \langle |A(\nabla u)| \rangle_{2B} \Big) \\
&\leq c\, 2^{-m\sigma} m \max_{0\leq i\leq m} \frac{(2^{1-i}R)^\beta}{(2^{-m}R)^\beta} M^\sharp_{\omega,2^{1-i}B}(A(\nabla u)) \\
&\quad + c_m \frac{\omega(2R)}{\omega(2^{-m}R)} \Big(\|f\|_{\mathrm{BMO}\omega\mathrm{(2B)}} + \|T - T(z)\|^{(\overline{p}-1)\frac{\overline{q}'}{\overline{p}'}}_{L^\infty(2B)} \frac{1}{\omega(2R)} \langle |A(\nabla u)| \rangle_{2B} \Big) \\
&\leq c\, 2^{-m(\sigma-\beta)} m \max_{0\leq i\leq m} M^\sharp_{\omega,2^{1-i}B}(A(\nabla u)) \\
&\quad + c_m\, 2^{(1+m)\beta} \Big(\|f\|_{\mathrm{BMO}\omega\mathrm{(2B)}} + \|T - T(z)\|^{(\overline{p}-1)\frac{\overline{q}'}{\overline{p}'}}_{L^\infty(2B)} \frac{1}{\omega(2R)} \langle |A(\nabla u)| \rangle_{2B} \Big).
\end{aligned}$$

Since $\sigma > \beta$, we find m_0 such that $c2^{-m(\sigma-\beta)}m \leq \frac{1}{4}$ for all $m \geq m_0$. This implies

$$\begin{aligned}
M^\sharp_{\omega,2^{-m}B}(A(\nabla u)) &\leq \frac{1}{4} \max_{0\leq i\leq m} M^\sharp_{\omega,2^{1-i}B}(A(\nabla u)) + c_0 \|f\|_{\mathrm{BMO}\omega\mathrm{(2B)}} \\
&\quad + c_0 \|T - T(z)\|^{(\overline{p}-1)\frac{\overline{q}'}{\overline{p}'}}_{L^\infty(2B)} \frac{\langle |A(\nabla u)| \rangle_{2B}}{\omega(2R)}.
\end{aligned}$$

Since the above estimate is independent of the ball, we find for $j \in \mathbb{N}$

$$\begin{aligned}
\max_{m_0\leq m\leq j} M^\sharp_{\omega,2^{-m}B}(A(\nabla u)) &\leq \frac{1}{4} \max_{0\leq i\leq j} M^\sharp_{\omega,2^{1-i}B}(A(\nabla u)) + c_0 \|f\|_{\mathrm{BMO}\omega\mathrm{(2B)}} \\
&\quad + c_0 \max_{0\leq i\leq j} \|T - T(z)\|^{(\overline{p}-1)\frac{\overline{q}'}{\overline{p}'}}_{L^\infty(2^{-i}B)} \frac{\langle |A(\nabla u)| \rangle_{2^{-i}B}}{\omega(2^{-i}R)}.
\end{aligned}$$

We want to remind the reader, that z is the center of 2^iB for all i. By our assumption on β we find for every $\delta \in (0,1)$ a $k_0 \in \mathbb{N}$ such that

$$k2^{-k((\overline{p}-1)\frac{\overline{q}'}{\overline{p}'}\gamma-\beta)}k \leq \delta$$

for all $k_0 \leq k$. We therefore can choose k_0 such that for $k_0 \leq k \leq j$ Lemma 1.11 implies

$$\|T - T(z)\|^{(\overline{p}-1)\frac{\overline{q}'}{\overline{p}'}}_{L^\infty(2^{-k}B)} \frac{\langle |A(\nabla u)| \rangle_{2^{-k}B}}{\omega(2^{-k}R)} \leq \frac{1}{4} \max_{0\leq i\leq j} M^\sharp_{\omega,2^{1-i}B}(A(\nabla u)) + c\frac{\langle |A(\nabla u)| \rangle_{2B}}{\omega(2R)}. \tag{1.22}$$

Using this estimate we find after absorbation for all $j \in \mathbb{N}$

$$\begin{aligned}
\max_{m_0\leq m\leq j} M^\sharp_{\omega,2^{-m}B}(A(\nabla u)) &\leq c \max_{0\leq i\leq m_0} M^\sharp_{\omega,2^{1-i}B}(A(\nabla u)) + c \max_{0\leq i\leq k_0} \frac{\langle |A(\nabla u)| \rangle_{2^{-i}B}}{\omega(2^{-i}R)} \\
&\quad + c_{m_0} \|f\|_{\mathrm{BMO}\omega\mathrm{(2B)}}.
\end{aligned}$$

The estimate

$$\max_{0\leq i\leq m_0} M^\sharp_{\omega,2^{1-i}B}(A(\nabla u)) \leq c \max_{0\leq i\leq k_0} \frac{\langle |A(\nabla u)| \rangle_{2^{-i}B}}{\omega(2^{-i}R)} \leq c\frac{\langle |A(\nabla u)| \rangle_{2B}}{\omega(2R)}$$

proves the first claim of the theorem. A standard covering argument proves the second claim. □

If T is not dependent on x, then the estimate can be sharpened.

Corollary 1.24. *If T is a constant matrix, then we find*

$$\max_{i\geq 0} M^{\sharp}_{\omega,2^{-m}B}(A(\nabla u)) \leq c\, M^{\sharp}_{\omega,2B}(A(\nabla u)) + c\|f\|_{\mathrm{BMO}\omega(2\mathrm{B})}.$$

Moreover,

$$\|A(\nabla u)\|_{\mathrm{BMO}\omega(\mathrm{B})} \leq c\, M^{\sharp}_{\omega,2B}(A(\nabla u)) + c\|f\|_{\mathrm{BMO}\omega(2\mathrm{B})}.$$

Corollary 1.25. *Let B be a ball in $\mathbb{R}^n$, u be a solution of (1.3) on $2B$, φ satisfy Assumption 1.1 and T Assumption 1.3. If $f \in \mathrm{VMO}(2B)$, then $A(\nabla u) \in \mathrm{VMO}(B)$.*

Proof. Since $f \in \mathrm{VMO}(2B)$, there exists a non-decreasing function $\tilde{\omega} : (0,\infty) \to (0,\infty)$ with $\lim_{r\to 0}\tilde{\omega}(r) = 0$, such that $\|f\|_{\mathrm{BMO}(\mathrm{B}_r)} \leq \tilde{\omega}(r)$, for all $B_r \subset 2B$. The result follows by Theorem 1.23 by defining $\omega(r) = \min\{\tilde{\omega}(r), r^{\beta}\}$. For $\beta \in (0, \min\{\frac{2\alpha}{\overline{p}'}, \gamma(\overline{p}-1)\frac{\overline{q}'}{\overline{p}'}\})$. □

The next result is a direct consequence of Theorem 1.23 with the choice of $\omega(r) = r^{\beta}$ and the equivalence of $\mathrm{BMO}_{\beta} := \mathrm{BMO}_{\mathrm{t}^{\beta}}$ and $C^{0,\beta}$.

Corollary 1.26. *Let φ hold Assumption 1.1 and T hold Assumption 1.3. Let u be a solution of (1.3) on a ball $2B \subset \mathbb{R}^n$. Let α be the Hölder coefficient (defined in Theorem 1.18) for φ-harmonic gradients.*

If $f \in C^{0,\beta}(2B)$ for $\beta < (0, \min\{\frac{2\alpha}{\overline{p}'}, \gamma(\overline{p}-1)\frac{\overline{q}'}{\overline{p}'}\})$, then $A(\nabla u) \in C^{0,\beta}(B)$. Moreover,

$$\|A(\nabla u)\|_{\mathrm{BMO}_{\beta}(\mathrm{B})} \leq c\|f\|_{\mathrm{BMO}_{\beta}(2\mathrm{B})} + c\frac{\langle |A(\nabla u)|\rangle_{2B}}{R^{\beta}}.$$

The constant depends on β, γ, the characteristics of φ and T.

Let us remark that the result in the Corollary 1.26 is optimal in the sense that any improvement of α in the decay estimate Theorem 1.18 transfers directly to the inhomogeneous case in the best possible way.

Remark 1.27. *If h is φ-harmonic on the open set $\Omega \subset \mathbb{R}^n$, then for any ball $B \subset \Omega$ we have the following decay estimate for $A(\nabla h)$. For any $\beta < \frac{2\alpha}{\overline{p}'}$ (where α is from Theorem 1.18) and any $\lambda \in (0,1]$ holds*

$$\begin{aligned} \fint_{\theta B} |A(\nabla h) - \langle A(\nabla h)\rangle_{\lambda B}| &\leq c_{\beta}\,(\theta R)^{\beta}\|A(\nabla h)\|_{\mathrm{BMO}_{\beta}(\mathrm{B})} \\ &\leq c_{\beta}\theta^{\beta}\fint_{B} |A(\nabla h) - \langle A(\nabla h)\rangle_{B}|. \end{aligned}$$

Remark 1.28. *Let us consider the system*

$$-\operatorname{div}(A(\nabla u)) = g \qquad \text{with } A(\nabla u) = \varphi'(|\nabla u|)\frac{\nabla u}{|\nabla u|},$$

where the right-hand side function g is not in divergence form. If $g \in L^n$, then there exists locally $f \in W^{1,n}$ with $\operatorname{div} f = g$ by solving the Laplace equation. Since $W^{1,n}$ embeds to VMO*, it follows by Corollary 1.25 that $A(\nabla u) \in \mathrm{VMO}$ locally.*

Let us compare this to the situation of [8] and [22], who studied the case $g \in L^{n,1}$ (Lorentz space) and proved $A(\nabla u) \in L^\infty$. Since $L^{n,1}$ embeds to L^n, we conclude that for such g additionally holds $A(\nabla u) \in \mathrm{VMO}$ locally.

Certainly, if $g \in L^s$ with $s > n$, then we find $f \in W^{1,s}$ and therefore $f \in C^{0,\sigma}$ with $\sigma = 1 - \frac{n}{s}$. Hence, by Corollary 1.26 we get Hölder continuity of $A(\nabla u)$.

Remark 1.29. *Let us explain that our result includes the estimates of [12] in the super-quadratic case $p \geq 2$ with $\varphi(t) = t^p$. Let $A(Q) := \langle A(\nabla u)\rangle_B$. Then $p \geq 2$ implies $\varphi(t) = t^p \leq \varphi_{|Q|}(t)$ and $(\varphi^*)_{|A(Q)|}(t) \leq \varphi^*(t) = c_p t^{p'}$. Hence, with Lemma 1.5, Lemma 1.32, Theorem 1.2 we estimate*

$$\begin{aligned}
\fint_B |\nabla u - Q|^p\,dx &\leq \fint_B \varphi_{|Q|}(|\nabla u - Q|)\,dx \\
&\leq c \fint_B (\varphi^*)_{|A(Q)|}(|A(\nabla u) - A(Q)|)\,dx \\
&\leq c \fint_B (\varphi^*)(|A(\nabla u) - A(Q)|)\,dx \\
&\leq c\,\|A(\nabla u)\|^{p'}_{\mathrm{BMO(B)}} \\
&\leq c\,\|f\|^{p'}_{\mathrm{BMO(2B)}} + c\left(M^\sharp_{2B}(A(\nabla u))\right)^{p'}.
\end{aligned}$$

Now, the estimate

$$\left(\fint_B |\nabla u - \langle \nabla u\rangle_B|\,dx\right)^p \leq \left(2\fint_B |\nabla u - Q|\,dx\right)^p$$

implies

$$\fint_B |\nabla u - Q|\,dx \leq c\,\|f\|^{\frac{1}{p-1}}_{\mathrm{BMO}} + c\left(M^\sharp_{2B}(A(\nabla u))\right)^{\frac{1}{p-1}}.$$

This is the same result as of Manfredi DiBenedetto [12].

Remark 1.30. *Our result also generalizes to the case of differential forms on $\Omega \subset \mathbb{R}^n$. In this Euclidean setting, we have the isometry $\Lambda^k \cong \mathbb{R}^{\binom{n}{k}}$, so the case of differential forms is just a special case of the vectorial situation. In particular, if $g \in \mathrm{BMO}(\Omega; \Lambda^k)$ and $d^*A(du) = d^*g$, with $u \in W^{1,\varphi}(\Omega; \Lambda^{k-1})$, then Theorem 1.23 (same ω) provides*

$$\|A(du)\|_{\mathrm{BMO}_\omega(\mathrm{B})} \leq c\|g\|_{\mathrm{BMO}_\omega(2\mathrm{B})} + cM^\sharp_{\omega,2B}(A(du)). \tag{1.23}$$

Let us show that a simple conjugation argument (see also [29, 26]*) provides another interesting result: We start with a solution* $v \in W^{1,\varphi}(\Omega; \Lambda^{k-1})$ *of*

$$d^*(A(dv + g)) = 0$$

which is a local minimizer of $\int \varphi(|dv + g|)\,dx$*. By Hodge theory we find* $w \in W^{1,\varphi^*}(\Omega, \Lambda^{k+1})$ *such that*

$$A(dv + g) = d^* w.$$

Applying A^{-1} *and then* d *we get the dual equation*

$$dg = d(A^{-1}(d^* w)).$$

If we define $A^* := (-1)^{k(n-k)} * A^{-1} *$*, then we can rewrite this equation as*

$$d^*(A^*(dw)) = \pm d^*(*g).$$

Moreover, we have (see [26]*) that* $A^*(dw) = (\varphi^*)'(|dw|)\frac{dw}{|dw|}$*. In particular, we are in the same situation as with* u *if we replace* φ *by* φ^* *and* dw *by* du*. Therefore, by* (1.23)

$$\|A^*(dw)\|_{\mathrm{BMO}_\omega(\mathrm{B})} \le c\|g\|_{\mathrm{BMO}_\omega(2\mathrm{B})} + cM^\sharp_{\omega,2B}(A^*(dw)).$$

This and $A(dv + g) = d^* w$ *implies*

$$\|dv + g\|_{\mathrm{BMO}_\omega(\mathrm{B})} \le c\|g\|_{\mathrm{BMO}_\omega(2\mathrm{B})} + cM^\sharp_{\omega,2B}(dv + g).$$

The triangle inequality gives

$$\|dv\|_{\mathrm{BMO}_\omega(\mathrm{B})} \le c\|g\|_{\mathrm{BMO}_\omega(2\mathrm{B})} + cM^\sharp_{\omega,2B}(dv). \tag{1.24}$$

In particular, we can apply this argument to φ*-harmonic function* h*. Then* (1.24) *(with* $g = 0$*) implies the decay estimate*

$$\fint_{\theta B} |\nabla h - \langle \nabla h\rangle_{\theta B}| \le c\theta^\beta \fint_{2B} |\nabla h - \langle \nabla h\rangle_B| \tag{1.25}$$

for all $\theta \in (0, 1]$ *with* $\beta = \frac{2\alpha}{\overline{q}}$*.*

1.5. A boundary result

Let us consider zero boundary values. We take $\Omega \subset \mathbb{R}^n$ with $C^{1,\sigma}$-boundary. Now we consider the following system with boundary values

$$\begin{aligned} -\mathrm{div}(A_T(\nabla u)) &= -\mathrm{div}(f) \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{1.26}$$

Higher integrability up to the boundary was shown for equations by Kinunnen and Zhou [38]. They used a boundary decay by Lieberman [45]. This does not exist in the case of systems, so we will proceed differently. At first we will follow the calculations of [38] to transfer the boundary problem to a half space problem. We take a boundary point; since solutions are translation invariant we can take it to be 0 and the outer normal to be $(0, ..., 0, -1)$. We will now imply a coordinate transform $\Psi : \overline{\Omega} \cap B_R(0) \to \{x_n \ge 0\}$, a $C^{1,\sigma}$–diffeomorphism, such that $\Psi(\partial\Omega \cap$

$B_R(0)) \subset \{x_n = 0\}$ and $\Psi(0) = 0$. We fix $(J_{ij}) = \partial_i \Psi^j$. We define $y = \Psi(x)$ and $\tilde{g}(y) = g \circ \Psi^{-1}(y)$. We will use the following calculations (also found in [38])

$$\nabla_x g(x) = J \circ \Psi^{-1}(y) \nabla_y \tilde{g}(y) \text{ and } \mathrm{div}_x g(x) = \mathrm{div}_y (J^t \circ \Psi^{-1} \tilde{g})(y),$$

for a differentiable function g. This implies that $|\nabla g| \sim |\nabla \tilde{g}|$, where the constants only depend on $|J|, |J^{-1}|$.

Now we define $v^i(x) = (T\nabla u)^i$. This implies that $\tilde{u}$ holds for $y \in B_R(0)^+$. We take $\nabla u = (\partial_1 u^1, ..., \partial_n u^1, \partial_1 u^2, ..., \partial_n u^N)^t$. We write the matrix $T = (T^1, ..., T^N)$, $T^i \in \mathbb{R}^{n \times nN}$ then we write $T_i = (T^i_1, ..., T^i_N)^t$, $T^i_j \in \mathbb{R}^{n\times n}$ such that $T\nabla u = \Big(\sum_{i=1}^N T^i_1 \nabla u^i, ..., \sum_{i=1}^N T^i_N \nabla u^i \Big)^t$. Then we find

$$T\nabla_x u(x) = \Big(\sum_{i=1}^N T^i_1 J \nabla_y u^i(y), ..., \sum_{i=1}^N T^i_N J \nabla_y u^i(y) \Big)^t =: \tilde{T}(y) \nabla_y \tilde{u}(y).$$

For the vector field $A^j_T(T\nabla u) = \sum_i (T^i_j)^t \frac{\varphi'(|T\nabla_x u|)}{|T\nabla_x u|} T^i_j \nabla_x u^i$ we have

$$\begin{aligned} -\mathrm{div}_x (A^j_T(\nabla_x u)) &= -\sum_{i=1}^N \mathrm{div}_y \Big(J^t (T^i_j)^t \Big(\frac{\varphi'(|T\nabla_x u|)}{|T\nabla_x u|} T^i_j \nabla_x u^i \Big) (\Psi^{-1}(y)) \\ &= -\sum_{i=1}^N \mathrm{div}_y J^t (T^i_j)^t \frac{\varphi'(|\tilde{T}\nabla_y u|)}{|T^2_j J \nabla_y u|} (T^i_j J \nabla_y u^i)(y). \end{aligned}$$

Therefore we gain the following system

$$(1.27) \quad \begin{aligned} -\mathrm{div}_y (A_{\tilde{T}} \nabla_y \tilde{u}(y)) = -\mathrm{div}_y \tilde{T}^t \varphi'(|\tilde{T}\nabla_y \tilde{u}|) \frac{\tilde{T}\nabla_y \tilde{u}}{|\tilde{T}\nabla \tilde{u}|} &= -\mathrm{div}_y (\tilde{f}) \text{ in } B_R(0)^+ \\ \tilde{u} = 0 \text{ on } \{x_n = 0\} \cap B_R(0), & \end{aligned}$$

where $\tilde{f}^j(y) = J^t f(\Psi^{-1}(y))$. Now we have a system on $B^+_R(0)$. We define

$$v = \tilde{u} \text{ if } y_n \geq 0 \text{ and } v(y) = -\tilde{u}(R_n y) \text{ if } y_n < 0.$$

Here R_n is the reflection on the y_n-axes. Consequently all v^j are odd with respect to y_n. This implies that $\partial_n v$ (and therefore ∇v) is well defined. Indeed, on the critical line $\{y_n = 0\}$ we find that $v \equiv 0$ and

$$\frac{v^j(y_1, ..., h)}{h} = \frac{v^j(y_1, ..., -h)}{-h}.$$

Then we find $(\nabla v^i)(R_n(y)) = -R_n \nabla \tilde{u}(y)$, as $R_n^{-1} = R_n$. We reflect $\tilde{T}$ as well such that v is a solution on $B_R(0)$. For h positive we define $(\mathcal{T}^i_j)(y_1, ..., -h) = -R_n(\tilde{T}^i_j)(y_1, ..., h)$ else $\mathcal{T} \equiv \tilde{T}$. Then we find for $y_n < 0$ that $\nabla v^i(y) = -R_n(\nabla \tilde{u}^i)(R_n y)$, as $R_n^{-1} = R_n$. With the same calculations as before we have

$$(1.28) \quad -\mathrm{div}_y (A_{\mathcal{T}} \nabla v) = -\mathrm{div} \Big(\mathcal{T}^t \varphi'(|\mathcal{T}\nabla v|) \frac{\mathcal{T}\nabla v}{|\mathcal{T}\nabla v|} \Big) = -\mathrm{div}_y (\overline{f}) \text{ in } B_R(0)$$

now $\overline{f}^j(y_1, ..., -h) = -R_n \tilde{f}^j(y_1, ..., h)$ and $\overline{f}^j(y_1, ..., h) \equiv \tilde{f}^j(y_1, ..., h)$ for h positive $1 \leq j \leq N$. On $B_R(0)$ we can apply the local theory which provides the following Theorem.

Theorem 1.31. *Let u be a solution of (1.26). Let φ satisfying Assumption 1.1 and T satisfying Assumption 1.3. Let $\omega:(0,\infty)\to(0,\infty)$ be non-decreasing such that for some $\beta\in(0,\min\{\frac{2\alpha}{\overline{p}'},\gamma\frac{\overline{q}'}{\overline{p}'}(\overline{p}-1),\sigma\frac{\overline{q}'}{\overline{p}'}(\overline{p}-1)\})$ the function $\omega(r)r^{-\beta}$ is almost decreasing in the sense that there is $c_0>0$ that $\omega(r)r^{-\beta}\leq c_0\,\omega(s)s^{-\beta}$ for all $r>s$. If $f\in \mathrm{BMO}(\overline{\Omega})$, then $A_T(\nabla u)\in(\overline{\Omega})$. Moreover, we find for $x\in\overline{\Omega}$ an $R>0$ such that*

$$\|A(\nabla u)\|_{\mathrm{BMO}\omega(\mathrm{B_R(x)}\cap\overline{\Omega})}\leq c\frac{\langle|A(\nabla u)|\rangle_{2B_R(x)\cap\overline{\Omega}}}{\omega(2R)}+c\|f\|_{\mathrm{BMO}\omega(2\mathrm{B_R(x)}\cap\overline{\Omega})}.$$

The constant depends on: the characteristics of φ, the properties of T, the the $C^{1,\sigma}-$properties of $\partial\Omega$ and $|B_R\cap\Omega|$.

Proof. We can assume that $x=0$ and that we have a $C^{1,\sigma}$–diffeomorphism $\Psi:\overline{\Omega}\cap 2B(0)\to 2B^+(0)$ with the desired properties. We define v to be the solution of (1.28) on $2B(0)$. As $\mathcal{T}\in C^{\min\{\sigma,\gamma\}}(2B)$ we can apply Theorem 1.23 on v.

We find by Lemma 1.7, the definition of $\tilde{u}(y)=u\circ\Psi^{-1}(y)$, the fact that consequently $|\nabla\tilde{u}(y)|\sim|\nabla u\circ\Psi^{-1}(y)|$ and the best constant property

$$\begin{aligned}
&M^\sharp_{\omega,2^{-m}B(0)\cap\overline{\Omega}}(A(\nabla u))\leq\frac{c}{\omega(2^{-m})}\fint_{\omega,2^{-m}B^+(0)}|A(\nabla u)\circ\Psi^{-1}-\langle A(\nabla u)\circ\Psi^{-1}\rangle_{B^+(0)}|dy\\
&\leq\frac{c}{\omega(2^{-m})}\fint_{\omega,2^{-m}B^+(0)}\sum_{i=1}^N\Big|\frac{\varphi'(|\nabla\tilde{u}|)}{|\nabla\tilde{u}|}J\nabla\tilde{u}^i-\langle\frac{\varphi'(|\nabla\tilde{u}|)}{|\nabla\tilde{u}|}J\nabla\tilde{u}^i\rangle_{B^+(0)}\Big|dy\\
&\leq\frac{c}{\omega(2^{-m})}\fint_{\omega,2^{-m}B^+(0)}\sum_{i=1}^N\Big|\frac{\varphi'(|\nabla\tilde{u}|)}{|\nabla\tilde{u}|}J\nabla\tilde{u}^i-J(0)\langle\frac{\varphi'(|\nabla\tilde{u}|)}{|\nabla\tilde{u}|}\nabla\tilde{u}^i\rangle_{B^+(0)}\Big|dy\\
&\leq\frac{c}{\omega(2^{-m})}\fint_{\omega,2^{-m}B^+(0)}|J|\Big|A(\nabla\tilde{u})-\langle A(\nabla\tilde{u})\rangle_{B^+(0)}\Big|+|J-J(0)|\big|\langle A(\nabla\tilde{u})\rangle_{B^+(0)}\big|dy\\
&\leq cM^\sharp_{\omega,2^{-m}B(0)^+}(A(\nabla v))+c\|J-J(0)\|_{L^\infty(2^{-m}B(0)^+)}\big|\langle A(\nabla v)\rangle_{2^{-m}B(0)^+}\big|\\
&\leq cM^\sharp_{\omega,2^{-m}B(0)}(A(\nabla v))+c\|J-J(0)\|_{L^\infty(2^{-m}B(0)^+)}\big|\langle A(\nabla v)\rangle_{2^{-m}B(0)}\big|\\
&\quad=I+II.
\end{aligned}$$

I can be estimated by Theorem 1.23. On II we can apply Lemma 1.11 just like in (1.22). This implies

$$\sup_{m\in\mathbb{N}}M^\sharp_{\omega,2^{-m}B(0)\cap\overline{\Omega}}(A(\nabla u))\leq c\frac{\langle|A(\nabla v)|\rangle_{2B}}{\omega(2R_B)}+c\|\overline{f}\|_{\mathrm{BMO}\omega(2\mathrm{B})}.$$

The left hand side is now immediately estimated by the wanted. Let us fix $B=B_R$. We find for every $x\subset B\cap\overline{\Omega}$ and $B_R(x)\subset 2B$. Consequently, the last estimate implies

$$\|A(\nabla u)\|_{\mathrm{BMO}_\omega(\mathrm{B}\cap\overline{\Omega})}\leq c\frac{\langle|A(\nabla u)|\rangle_{2B\cap\overline{\Omega}}}{\omega(2R)}+c\|f\|_{\mathrm{BMO}_\omega(2\mathrm{B}\cap\overline{\Omega})}.$$

□

1.6. Appendix

The classical John Nirenberg estimate [30] proves the following lemma in the case $\psi(t) = t^p$. We give an extension to N-functions ψ.

Lemma 1.32. *If ψ is an N-function, which satisfies the Δ_2 condition, $B \subset \mathbb{R}^n$ a ball and $g \in \mathrm{BMO(B)}$, then*

$$\fint_B \psi(|g - \langle g\rangle_B|)\,dx \le c\,\psi(\|g\|_{\mathrm{BMO(B)}}),$$

where c only depends on $\Delta_2(\psi)$.

Proof of Lemma 1.32. Because $\psi \in \Delta_2$, there exists $q < \infty$ only depending on $\Delta_2(\varphi)$ such that

$$\psi'(st) \le c_1 \max\{1, s^{q-1}\}\psi'(t),$$

where c_1 only depends on $\Delta_2(\psi)$.

Since $g \in \mathrm{BMO(B)}$ we find by the classical John-Nirenberg estimate which can be found in [30]:

$$\frac{|\{x \in B : |g(x) - \langle g\rangle| > \lambda\}|}{|B|} \le \exp\left(\frac{-c_2\lambda}{\|g\|_{\mathrm{BMO(B)}}}\right),$$

where $c_2 \in (0,1]$ only depends on the dimension. This implies

$$\begin{aligned}
\fint_B \psi(|g - \langle g\rangle|))\,dx &= \int_0^\infty \frac{|\{x \in B : |g(x) - \langle g\rangle| > \lambda\}|}{|B|}\psi'(\lambda)\,d\lambda \\
&\le \int_0^\infty \exp\left(\frac{-c_2\lambda}{\|g\|_{\mathrm{BMO(B)}}}\right)\psi'(\lambda)\,d\lambda \\
&= \frac{\|g\|_{\mathrm{BMO(B)}}}{c_2}\int_0^\infty \exp(-s)\psi'\left(\frac{s\|g\|_{\mathrm{BMO(B)}}}{c_2}\right)ds \\
&\le \frac{\|g\|_{\mathrm{BMO(B)}}}{c_2}\psi'\left(\frac{\|g\|_{\mathrm{BMO(B)}}}{c_2}\right)\int_0^\infty \exp(-s)\max\{1, s^{q-1}\}\,ds \\
&\le \frac{\|g\|_{\mathrm{BMO(B)}}}{c_2}\psi'\left(\frac{\|g\|_{\mathrm{BMO}}}{c_2}\right)(1 + \Gamma(q)). \\
&\le (1 + \Gamma(q))\psi\left(\frac{2\,\|g\|_{\mathrm{BMO(B)}}}{c_2}\right) \\
&\le (1 + \Gamma(q))\left(\frac{2}{c_0}\right)^q \psi(\|g\|_{\mathrm{BMO(B)}}).
\end{aligned}$$

□

Proof of Lemma 1.4. It has been shown in [25] that if $\varphi \in T(p,q,K)$, then $\varphi^{-1} \in T(1/q, 1/p, K_1)$, where K_1 only depends on p, q and K. From this, (1.5) and

$$t \le \varphi^{-1}(t)(\varphi^*)^{-1}(t) \le 2t$$

it follows, that $(\varphi^*)^{-1} \in T(1-1/p, 1-1/q, 2K_1)$ and as a consequence $\varphi^* \in T(q', p', K_2)$ with $K_2 = K_2(p,q,K)$. □

Proof of Lemma 1.6. Let $\varphi \in T(p,q,K)$. Then φ_a is of type $T(\overline{p}, \overline{q}, K_5)$, where K_5 only depends on K, p, q. Recall that every N-function ψ satisfies $\psi(t) \le \psi'(t)\, t \le \psi(2t)$, see for example [51]. This and $\varphi \in T(p,q,K)$ implies

$$\varphi'(st) \le \frac{\varphi(2st)}{st} \le K2^q \max\{s^p, s^q\} \frac{\varphi(t)}{st} \le K2^q \max\{s^{p-1}, s^{q-1}\} \varphi'(t).$$

We define $\tau = \frac{a+st}{a+t}$. This implies

$$\begin{aligned}\varphi_a'(st) &= \frac{\varphi'(\tau(a+t))}{a+st} st \le K2^q \max\{\tau^{p-1}, \tau^{q-1}\} \varphi'(a+t) \frac{st}{a+st}\\ &= K2^q s \max\{\tau^{p-2}, \tau^{q-2}\} \varphi_a'(t)\\ &\le K2^q s \max\{\tau^{\overline{p}-2}, \tau^{\overline{q}-2}\} \varphi_a'(t)\end{aligned}$$

for all $s, t \ge 0$. Now we split the cases $s \ge 1$ and $s \in (0,1)$ and apply $\overline{p} \le 2 \le \overline{q}$. It follows

$$\max\{\tau^{\overline{p}-2}, \tau^{\overline{q}-2}\} \le \max\{s^{\overline{p}-2}, s^{\overline{q}-2}\}.$$

This and the previous estimate proves the claim for $\varphi_{|P|}$. Since $\varphi \in T(p,q,K)$, we have $\varphi^*(q', p', K_2)$ by Lemma 1.4. This proves the claim for $(\varphi^*)_{|A(P)|}$. Now, the equivalence $(\varphi_{|P|})^*(t) \sim (\varphi^*)_{|A(P)|}(t)$ of Lemma 1.5 concludes the proof. □

Proof of Lemma 1.10. Let $g \in \mathrm{BMO}_\omega(\Omega)$ and $B_r \subset \Omega$.

$$\begin{aligned}&\frac{1}{\omega(r)} \fint_{B_r} |Tg - \langle Tg\rangle_{B_r}| dx \le c \frac{1}{\omega(r)} \fint_{B_r} |Tg - T(z)_{B_r} \langle g\rangle_{B_r}| dx\\ &\le c\|T\|_{L^\infty(B_r)} \frac{1}{\omega(r)} \fint_{B_r} |g - \langle g\rangle_{B_r}| dx + c\|T - T(z)_{B_r}\|_{L^\infty(\Omega)} \frac{1}{\omega(r)} \fint_{B_r} |g| dx.\end{aligned}$$

By the assumption we find that the right hand side is uniformly bounded. □

In the following equivalence Lemma is used in the proof of Proposition 1.22. It allows to express the mean oscillation of $V(\nabla u)$ in terms of different mean values.

Lemma 1.33. *Let φ satisfy Assumption 1.1. Let $B \subset \mathbb{R}^n$ be a ball and $g \in L^\varphi(B; \mathbb{R}^{N\times n})$. Define $g_A \in \mathbb{R}^{N\times n}$ by $A(g_A) := \langle A(g)\rangle_B$. Then*

$$\fint_B |V(g) - \langle V(g)\rangle_B|^2\, dx \sim \fint_B |V(g) - V(\langle g\rangle_B)|^2\, dx \sim \fint_B |V(g) - V(g_A)|^2\, dx$$

holds. The constants are independent of B and g; they only depend on the characteristics of φ.

Proof. Define $g_V \in \mathbb{R}^{N\times n}$ by $V(g_V) := \langle V(g)\rangle_B$. We denote the three terms by (I), (II) and (III). Note that

$$(I) = \inf_{P\in\mathbb{R}^{N\times n}} \fint_B |V(g) - P|^2\,dx,$$

which proves $(I) \le (II)$ and $(I) \le (III)$.

We calculate with Lemma 1.7 and $\langle A(g) - A(g_A)\rangle_B = 0$

$$(II) \sim \fint_B \big(A(g) - A(g_A)\big)\cdot(g - g_A)\,dx = \fint_B \big(A(g) - A(g_A)\big)\cdot(g - g_V)\,dx.$$

Again, by Lemma 1.7, Young's inequality with $\varphi_{|g|}$ in combination with (0.6) (second part) and again Lemma 1.7 we estimate

$$\begin{aligned}
(II) &\le c \fint_B \varphi'_{|g|}(|g - g_A|))|g - g_V|\,dx \\
&\le \delta \fint_B \varphi_{|g|}(|g - g_A|))\,dx + c_\delta \fint_B \varphi_{|g|}(|g - g_V|)\,dx \\
&\le \delta\, c \fint_B |V(g) - V(g_A)|^2\,dx + c_\delta \fint_B |V(g) - V(g_V)|^2\,dx \\
&\le \delta\, c\,(II) + c_\delta(I).
\end{aligned}$$

It follows that $(II) \le c\,(I)$.

On the other hand with Lemma 1.7 and $\langle g - \langle g\rangle_B\rangle_B = 0$ follows

$$(III) \sim \fint_B \big(A(g) - A(\langle g\rangle_B)\big)\cdot(g - \langle g\rangle_B)\,dx = \fint_B \big(A(g) - A(g_V)\big)\cdot(g - \langle g\rangle_B)\,dx.$$

By Young's inequality with $\varphi_{|g|}$ follows analogously to the estimates of (II) that $(III) \le c_\delta(I) + \delta\, c\,(III)$. Now, $(III) \le c\,(I)$ follows. □

Lemma 1.34. *Let ψ be of type $T(p, q, K)$ and let $\gamma \in (0, 1)$ such that $\gamma q \le 1$. Then the function $(\psi^\gamma)^{-1}$ is quasi-convex, i.e. there exists a convex function $\kappa : [0,\infty) \to [0,\infty)$ such that $(\psi^\gamma)^{-1}(t) \sim \kappa(t)$. The implicit constant only depends on q and K.*

Proof. Define $\rho(t) := \psi^\gamma(t)$. Since ψ is of type $T(p, q, K)$, there holds $\psi(st) \le K s^q \psi(t)$ for all $t \ge 0$ and $s \ge 1$. This implies $s\psi^{-1}(u) \le \psi^{-1}(K s^q u)$ for all $u \ge 0$ and $s \ge 1$. From $\rho^{-1}(u) = \psi^{-1}(u^{1/\gamma})$ and $\psi^{-1}(t) = \rho^{-1}(t^\gamma)$ we get $s\rho^{-1}(u) \le \rho^{-1}(K^\gamma s^{\gamma q} u)$. In particular, with $\gamma q \le 1$ follows

$$\frac{\rho^{-1}(u)}{u} \le \frac{\rho^{-1}(K^\gamma s^{\gamma q - 1}\, su)}{su} \le \frac{\rho^{-1}(K^\gamma\, su)}{su}$$

for all $u \ge 0$ and $s \ge 1$. Therefore Lemma 1.1.1 of [39] implies that ρ^{-1} is quasi-convex. □

CHAPTER 2

Degenerate Stokes

Let $\Omega \subset \mathbb{R}^2$ be a domain. In this chapter we study properties of the local weak solution $u \in W^{1,\varphi}(\Omega)$ and $\pi \in L^{\varphi^*}(\Omega)$ of the generalized Stokes problem

$$\begin{aligned} -\mathrm{div} A(\varepsilon u) + \nabla \pi &= -\mathrm{div} f \quad \text{in } \Omega, \\ \mathrm{div} u &= 0 \qquad\quad \text{in } \Omega \end{aligned} \tag{2.1}$$

for given $f : \Omega \to \mathbb{R}^{2\times 2}_{\text{sym}}$. Here u stands for the velocity of a fluid, εu the symmetric part of the gradient of u, i.e. $\varepsilon u = (\nabla u + (\nabla u)^T)/2$ and π for its pressure. We do not need boundary conditions, since our results are local. The model case is $A(Q) = \nu(\kappa + |Q|)^{p-2}Q$ corresponding to power law fluids with $\nu > 0$, $\kappa \geq 0$, $1 < p < \infty$ and Q symmetric. But we also allow more general growth conditions, which include for example Carreau type fluids $A(Q) = \mu_\infty Q + \nu(\kappa + |Q|)^{p-2}Q$ with $\mu_\infty \geq 0$ (see Section 2.1). In this chapter we are interested in the qualitative properties of $A(\varepsilon u)$ and π in terms of f. The divergence form of the right-hand side is only for convenience of the formulation of the result, since every g can be written as $-\mathrm{div} f$ with f symmetric, see Remark 2.12.

System (2.1) originates in fluid mechanics. It is a simplified stationary variant of the system

$$u_t - \mathrm{div} A(\varepsilon u) + [\nabla u]u + \nabla \pi = -\mathrm{div} f, \quad \mathrm{div} u = 0, \tag{2.2}$$

where u stands for a velocity of a fluid and π for its pressure. The extra stress tensor A determines properties of the fluid and must be given by a constitutive law. If $A(Q) = 2\nu Q$ with constant viscosity $\nu > 0$, then (2.2) is the famous Navier-Stokes system, which describes the flow of a Newtonian fluids. In the case of Non-Newtonian fluids however, the viscosity is not constant but may depend non-linearly on εu. The power law fluids and the Carreau type fluids are such examples, which are widely used among engineers. For a more detailed discussion on the connection with mathematical modeling see e.g. [**47, 50**]. The existence theory for such type of fluids was initiated by Ladyzhenskaya [**43, 44**] and Lions [**46**].

The main result of the chapter are the following Campanato type estimates for the local weak solutions of (2.1).

Theorem 2.1. *There is an $\alpha > 0$ such that for all $\beta \in [0, \alpha)$ there exists a constant $C > 0$ such that for every ball B with $2B \subset \Omega$*

$$\|A(\varepsilon u)\|_{\mathrm{BMO}_\beta(\mathrm{B})} + \|\pi\|_{\mathrm{BMO}_\beta(\mathrm{B})} \leq C \left(\|f\|_{\mathrm{BMO}_\beta(2\mathrm{B})} + R^{-\beta} \fint_{2B} |A(\varepsilon u) - \langle A(\varepsilon u)\rangle_{2B}| dx \right).$$

In particular, $f \in \mathrm{BMO}_\beta(2\mathrm{B})$ implies $A(\varepsilon u), \pi \in \mathrm{BMO}_\beta(\mathrm{B})$.

The spaces $\mathrm{BMO}_\beta(\mathrm{B})$ are the Campanato spaces, see Section 2.1. Our main theorem in particular includes the BMO-case (bounded mean oscillation), since

BMO $= \mathcal{L}^{1,2}$. Theorem 2.1 is a consequence of the refined BMO$_\omega$-estimates of Theorem 2.9, which also includes the case VMO (vanishing mean oscillation). The upper bound α is given by the maximal (local) regularity of the homogeneous generalized Stokes system. Our estimates hold up to this regularity exponent. Due to the Campanato characterization of Hölder spaces $C^{0,\alpha}$ our results can also be expressed in terms of Hölder spaces.

Theorem 2.1 is the limit case of the nonlinear Calderón-Zygmund theory, which was initiated by [**27, 28**]. The reduced regularity for (2.1) with $f = 0$ is the reason, why we can only treat the planar case $n = 2$ in this chapter. The crucial ingredient for Theorem 2.1 are the decay estimates for the homogeneous case $f = 0$ in terms of the gradients. In this chapter we are able to prove such decay estimates in the planar case $n = 2$, see Theorem 2.8. If such estimates can be proven for $n \geq 3$, then Theorem 2.1 would directly generalize to this situation. Unfortunately, this is an open problem, even in the absence of the pressure.

Theorem 2.1 can be used to improve the known regularity results for the stationary problem with convective term $[\nabla u]u$, see Section 2.4, and for the instationary problem (2.2), see Section 2.4. The first $C^{1,\alpha}$-regularity results for planar flows were obtained in the series of the articles [**32, 33, 34**] under various boundary conditions under the restriction $\kappa > 0$. See also [**52, 2**]. The stationary degenerate case $\kappa \geq 0$ was treated in [**56**] for $1 < p \leq 2$. To our knowledge the only result for $n \geq 2$ is the one obtained in [**9**] with $\kappa > 0$ and $1 < p \leq 2$ and small data and zero boundary values. Because of the zero boundary values (combined with the small data), we are not able to use this result for the higher regularity of the case $f = 0$.

Note that our result is optimal with respect to the regularity of f. All other planar results mentioned above need much stronger assumptions on the regularity of f. This is one of the advantages of the non-linear Calderón-Zygmund theory. This is the basis for our improved results in Section 2.3 and Section 2.4 for the system including the convective term. It is based on the fact, that the convective term can be written as $\operatorname{div}(u \otimes u)$ using $\operatorname{div} u = 0$ and therefore can be treated as a force term $\operatorname{div} f$.

2.1. Preliminary results and notation

For a mapping $u : \Omega \to \mathbb{R}^2$ we define $\varepsilon u = (\nabla u + (\nabla u)^T)/2$, $Wu = (\nabla u - (\nabla u)^T)/2$ and $([\nabla u]u)_j = \sum_{k=1}^2 u_k \partial_k u_j$. In the parts of the chapter dealing with evolutionary problems we will assume that $u : \Omega \times (0, T) \to \mathbb{R}^2$. In this case all operators ∇, ε, W and div are understood only with respect to the variable $x \in \Omega$.

For $P, Q \in \mathbb{R}^n$ with $n \geq 1$ we define $P \cdot Q = \sum_{j=1}^n P_j Q_j$. The symbol $\mathbb{R}^{2\times 2}_{\text{sym}}$ denotes the set of symmetric 2×2 matrices.

Throughout the chapter we will assume that φ satisfies Assumption 1.1. We remark that if φ satisfies Assumption 1.1 below, then $\Delta_2(\{\varphi, \varphi^*\}) < \infty$ will be automatically satisfied, where $\Delta_2(\{\varphi, \varphi^*\})$ depends only on the characteristics of φ, see for example [**3**] for a proof. Most steps in our proof do not require that φ'' is almost monotone. It is only needed in Theorem 2.7 for the derivation of the decay estimates of Theorem 2.8.

Let us now state the assumptions on A.

Assumption 2.2. *Let φ satisfy Assumption 1.1. The vector field $A : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$, $A \in C^{0,1}(\mathbb{R}^{2\times 2} \setminus \{0\}) \cap C^0(\mathbb{R}^{2\times 2})$ satisfies the non-standard φ-growth condition, i. e. there are $c, C > 0$ such that for all $P, Q \in \mathbb{R}^{2\times 2}_{\text{sym}}$ with $P \neq 0$*

$$\begin{aligned} \big(A(P) - A(Q)\big) \cdot \big(P - Q\big) &\geq c\,\varphi''(|P| + |Q|)\,|P - Q|^2, \\ |A(P) - A(Q)| &\leq C\,\varphi''(|P| + |Q|)\,|P - Q| \end{aligned} \tag{2.3}$$

holds. We also require that $A(\varepsilon)$ is symmetric for all $\varepsilon \in \mathbb{R}^{2\times 2}_{\text{sym}}$ and $A(0) = 0$.

Let us provide a few typical examples. If φ satisfies Assumption 1.1, then both $A(Q) := \varphi'(|Q|)\frac{Q}{|Q|}$ and $A(Q) := \varphi'(|Q^{\text{sym}}|)\frac{Q^{\text{sym}}}{|Q^{\text{sym}}|}$ satisfy Assumption 2.2. See [13] for a proof of this result. In this case, (2.1) is just the Euler-Lagrange equation of the local $W^{1,\varphi}_{\text{div}}$-minimizer of the energy $\mathcal{J}(w) := \int_\Omega \varphi(|\varepsilon w|)\,dx + \langle f, \nabla w\rangle$. Here $W^{1,\varphi}_{\text{div}}$ is the subspace of functions $w \in W^{1,\varphi}$ with $\text{div} w = 0$. The pressure acts as a Lagrange multiplier. This includes in particular the case of power law and Carreau type fluids:

(a) Power law fluids with $1 < p < \infty$, $\kappa \geq 0$ and $\nu > 0$

$$A(Q) = \nu(\kappa + |Q|)^{p-2} Q \qquad \text{and} \qquad \varphi(t) = \int_0^t \nu(\kappa + s)^{p-2}\, s\, ds$$

or

$$A(Q) = \nu(\kappa^2 + |Q|^2)^{\frac{p-2}{2}} Q \qquad \text{and} \qquad \varphi(t) = \int_0^t \nu(\kappa^2 + s^2)^{\frac{p-2}{2}}\, s\, ds.$$

(b) Carreau type fluids with $1 < p < \infty$, $\kappa, \mu_\infty \geq 0$ and $\nu > 0$

$$A(Q) = \mu_\infty Q + \nu(\kappa + |Q|)^{p-2} Q \qquad \text{and} \qquad \varphi(t) = \int_0^t \mu_\infty s + \nu(\kappa + s)^{p-2}\, s\, ds.$$

(c) For $1 < p < \infty$, $\mu_\infty > 0$, and $\nu \geq 0$

$$A(Q) = \mu_\infty Q + \nu \operatorname{arcsinh}(|Q|)\frac{Q}{|Q|} \qquad \text{and} \qquad \varphi(t) = \int_0^t \mu_\infty s + \nu \operatorname{arcsinh}(s)\, ds.$$

2.2. A BMO result for p-Stokes

Let u, π be the local weak solution of (2.1), in the sense that $u \in W^{1,\varphi}_{\text{div}}(\Omega)$, $\pi \in L^{\varphi^*}(\Omega)$, and

$$\forall \xi \in W^{1,\varphi}_0(\Omega) : \langle A(\varepsilon u), \varepsilon\xi\rangle - \langle \pi, \text{div}\xi\rangle = \langle f, \varepsilon\xi\rangle, \tag{2.4}$$

where we used that $A(\varepsilon u)$ and f are symmetric. To omit the pressure, we will use divergence free test function, i.e.

$$\forall \xi \in W^{1,\varphi}_{0,\text{div}}(\Omega) : \langle A(\varepsilon u), \varepsilon\xi\rangle = \langle f, \varepsilon\xi\rangle. \tag{2.5}$$

The method of the proof of Theorem 2.1 is like it was for the elliptic case in Chapter 1. It is based on a reverse Hölder inequality, a local comparison to a solution with zero right hand side and a decay estimate for this homogenous solution. These three properties are discussed in the subsequent subsections. Note that the restriction to the planar case and φ'' almost monotone is only needed for the decay estimate of Subsection 2.2.3. The first two subsections are valid independently of these extra assumptions.

2.2.1. Reverse Hölder inequality. In this section we show the reverse Hölder estimate for solutions of (2.1). To prove the result we need a Sobolev-Poincaré inequality in the Orlicz setting from [13, Lemma 7]. See Theorem 1.12 Remark, that it is not possible to replace the full gradient on the right hand side with the symmetric one only. Consider $v = (x_2, -x_1)$ on the unit ball.

We also need the following version of the Korn's inequality for Orlicz spaces, which is a minor modification of the one in [18, Theorem 6.13]. See [6] for sharp conditions for Korn's inequality on Orlicz spaces.

Lemma 2.3. *Let $B \subset \mathbb{R}^n$ be a ball. Let ψ be an N-function such that ψ and ψ^* satisfy the Δ_2-condition (for example let ψ satisfy Assumption 1.1). Then for all $v \in W^{1,\psi}(B)$ with $\langle Wv\rangle_B = 0$ the inequality*

$$\int_B \psi(|\nabla v|)\,dx \le C \int_B \psi(|\varepsilon v|)\,dx$$

holds. The constant $C > 0$ depends only on $\Delta_2(\{\psi, \psi^*\}) < \infty$.

Proof. From [18, Theorem 6.13] we know that

$$\int_B \psi(|\nabla v - \langle \nabla v\rangle_B|)\,dx \le C \int_B \psi(|\varepsilon v - \langle \varepsilon v\rangle_B|)\,dx. \tag{2.6}$$

Using $\langle Wv\rangle_B = 0$ we have $\nabla v = (\nabla v - \langle \nabla v\rangle_B) + \langle \varepsilon v\rangle_B$. Thus, by triangle inequality and (2.6) we get

$$\int_B \psi(|\nabla v|)\,dx \le c \int_B \psi(|\varepsilon v - \langle \varepsilon v\rangle_B|)\,dx + c \int_B \psi(|\langle \varepsilon v\rangle_B|)\,dx,$$

where we also used $\Delta_2(\psi) < \infty$. Now, the claim follows by triangle inequality and Jensen's inequality. ◻

As in the elliptic case we need a reverse Hölder estimate for the oscillation of the gradients. Additional difficulties arise due to the symmetric gradient and the hidden pressure (so that the test functions must be divergence free).

Lemma 2.4. *Let u be a local weak solution of (2.1) and B be a ball satisfying $2B \subset \Omega$. There exists $\theta \in (0,1)$ and $c > 0$ only depending on the characteristics of φ, such that for all $P, f_0 \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$,*

$$\fint_B |V(\varepsilon u) - V(P)|^2 dx \le c \left(\fint_{2B} |V(\varepsilon u) - V(P)|^{2\theta} dx \right)^{\frac{1}{\theta}} + c \fint_{2B} (\varphi^*)_{|A(P)|}(|f - f_0|)dx$$

holds. The constant $c > 0$ depends only on the characteristics of $\varphi \in T(p, q, K)$ and the constants in Assumption 2.2.

PROOF. Let $\eta \in C_0^\infty(2B)$ with $\chi_B \leq \eta \leq \chi_{3B/2}$ and $|\nabla\eta| \leq c/R$, where R is the radius of B. We define $\psi = \eta^{\overline{q}}(u - z)$, where z is a linear function such that $\langle u - z\rangle_{2B} = 0$, $\varepsilon z = P$, and $Wz = \langle Wu\rangle_{2B}$. We cannot use ψ as test function in the pressure free formulation (2.5), since its divergence does not vanish. Therefore we correct ψ by help of the Bogovskiĭ operator Bog from [4]. In particular, $w = \text{Bog}(\text{div}\psi)$ is a special solution of the auxiliary problem

$$\begin{aligned} \text{div} w &= \text{div}\psi \quad &&\text{in } \tfrac{3}{2}B \\ w &= 0 &&\text{in } \partial(\tfrac{3}{2}B). \end{aligned}$$

We extend w by zero outside of $\frac{3}{2}B$. It has been shown in [18, Theorem 6.6] that ∇w can be estimated by $\text{div}\psi$ in any suitable Orlicz spaces. In our case we use the following estimate in terms of $\varphi_{|P|}$.

$$\fint_{2B} \varphi_{|P|}(|\varepsilon w|)dx \leq C \fint_{2B} \varphi_{|P|}(|\text{div}\psi|)\, dx.$$

The constant $C > 0$ depends only on the characteristics of φ.

Using $\text{div}u = 0$, we have

$$\text{div}\psi = \nabla(\eta^{\overline{q}})\,(u - z) + \eta^{\overline{q}}\text{div}(u - z) = \overline{q}\eta^{\overline{q}-1}\nabla\eta\,(u - z) - \eta^{\overline{q}}\text{tr}P.$$

This implies

$$\fint_{2B} \varphi_{|P|}(|\varepsilon w|)dx \leq C \fint_{2B} \varphi_{|P|}\bigg(\frac{|u - z|}{R}\bigg)\, dx + C \fint_{2B} \varphi_{|P|}(|\text{tr}P|)\, dx. \tag{2.7}$$

We define $\xi := \psi - w = \eta^{\overline{q}}(u - z) - w$, then $\text{div}\xi = 0$, which ensures that ξ is a valid test function for (2.1). We get

$$\begin{aligned} \langle A(\varepsilon u) - A(P), \eta^{\overline{q}}(\varepsilon u - P)\rangle = \langle f - f_0, \eta^{\overline{q}}(\varepsilon u - P)\rangle + \langle f - f_0, (u - z) \otimes_{sym} \nabla(\eta^{\overline{q}})\rangle \\ - \langle A(\varepsilon u) - A(P), (u - z) \otimes_{sym} \nabla(\eta^{\overline{q}})\rangle \\ - \langle f - f_0, \varepsilon w\rangle + \langle A(\varepsilon u) - A(P), \varepsilon w\rangle. \end{aligned} \tag{2.8}$$

The symbol $\otimes_{sym}$ denotes the symmetric part of $\otimes$, i.e.$(g \otimes_{sym} g)_{ij} := (g_i g_j + g_j g_i)/2$ for $g, g \in \mathbb{R}^2$. We divide (2.8) by $|2B|$ and estimate the two sides. Concerning the left hand side we find by Lemma 1.7

$$|2B|^{-1}\langle A(\varepsilon u) - A(P), \eta^{\overline{q}}(\varepsilon u - P)\rangle \sim \fint_{2B} \eta^{\overline{q}}|V(\varepsilon u) - V(P)|^2 dx =: (I).$$

We estimate the right hand side of (2.8) by Young's inequality (1.6) for $\varphi_{|P|}$ with $\delta \in (0,1)$ using also $(\varphi_{|P|})^* \sim (\varphi^*)_{|A(P)|}$ (see Lemma 1.5).

$$\begin{aligned}(I) \leq & c_\delta \fint_{2B} (\varphi^*)_{|A(P)|}(|f-f_0|)dx + \delta \fint_{2B} \eta^{\overline{pq}} \varphi_{|P|}(|\varepsilon u - P|)dx \\ & + c_\delta \fint_{2B} \varphi_{|P|}\left(\frac{|u-z|}{R}\right)dx + c_\delta \fint_{2B} \varphi_{|P|}(|\varepsilon w|)dx \\ & + \delta \fint_{2B} \eta^{(\overline{q}-1)\overline{q}'} (\varphi^*)_{|A(P)|}\big(|A(\varepsilon u) - A(P)|\big)\, dx \\ & =: (II) + (III) + (IV) + (V) + (VI).\end{aligned}$$

Now we use Lemma 1.7 to estimate $(III) + (VI) \leq \delta\, c(I)$, so these terms can be absorbed. Moreover, by (2.7)

$$(IV) + (V) \leq c\,(IV) + c \fint_{2B} \varphi_{|P|}(|\mathrm{tr} P|)\, dx.$$

Since P is constant, $\mathrm{tr}P = \mathrm{div} z$ and $\mathrm{div} u = 0$, we can estimate

$$\fint_{2B} \varphi_{|P|}(|\mathrm{tr}P|)\, dx = \left(\fint_{2B} (\varphi_{|P|})^\theta(|\mathrm{div}(u-z)|)\, dx\right)^{\frac{1}{\theta}} \leq \left(\fint_{2B} (\varphi_{|P|})^\theta(|\varepsilon u - \varepsilon z)|)\, dx\right)^{\frac{1}{\theta}}. \tag{2.9}$$

It remains to estimate (IV). We use Sobolev-Poincaré inequality of Theorem 1.12 with $\psi = \varphi_{|P|}$ such that $(\varphi_{|P|})^\theta$ is almost convex and

$$(IV) = c \fint_{2B} \varphi_{|P|}\left(\frac{|u-z|}{R}\right)dx \leq c \left(\fint_{2B} \varphi^\theta_{|P|}(|\nabla u - \nabla z|)dx\right)^{\frac{1}{\theta}}$$

with $\theta \in (0,1)$. The constants and θ are independent of $|P|$, since the $\Delta_2(\{\varphi_a\}_{a\geq 0})$ is bounded in terms of the characteristics of φ.

As $\langle W(u-z)\rangle_{2B} = 0$ we find by Korn's inequality (Lemma 2.3) with $\psi = \varphi^\theta_{|P|}$ (almost convex) and $\varepsilon z = P$ that

$$(IV) \leq c \left(\fint_{2B} \varphi^\theta_{|P|}(|\varepsilon u - \varepsilon z|)dx\right)^{\frac{1}{\theta}}.$$

The above estimates and Lemma 1.7 show that

$$(IV) + (V) \leq c \left(\fint_{2B} \varphi^\theta_{|P|}(|\varepsilon u - \varepsilon z|)dx\right)^{\frac{1}{\theta}} \leq c \left(\fint_{2B} |V(\varepsilon u) - V(P)|^{2\theta} dx\right)^{\frac{1}{\theta}}.$$

The lemma is proved. □

Lemma 2.4 allows to obtain the next corollary, in the same way Lemma 1.13 implied Corollary 1.16

Corollary 2.5. *Let the assumptions of Lemma 2.4 be satisfied. Then for all* $P \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$

$$\fint_B |V(\varepsilon u) - V(P)|^2 dx \le c\,(\varphi^*)_{|A(P)|}\bigg(\fint_{2B} |A(\varepsilon u) - A(P)|\, dx\bigg) + c(\varphi^*)_{|A(P)|}(\|f\|_{BMO(2B)}).$$

The constants only depend on the characteristics of φ *and the constants in Assumption 2.2.*

2.2.2. Comparison. Let u be a local weak solution of (2.1) and B be a ball satisfying $2B \subset \Omega$. We consider a solution h, ρ of the homogeneous problem

$$\begin{aligned} -\mathrm{div} A(\varepsilon h) + \nabla \rho &= 0 \quad \text{in } \Omega,\\ \mathrm{div} u &= 0 \quad \text{in } \Omega,\\ h &= u \quad \text{on } \partial\Omega. \end{aligned} \tag{2.10}$$

The next lemma estimates the natural distance between u and its approximation h.

Lemma 2.6. *For every* $\delta > 0$ *there exists* $c_\delta \ge 1$ *such that*

$$\fint_B |V(\varepsilon u) - V(\varepsilon h)|^2\, dx \le \delta\,(\varphi^*)_{|\langle A(\varepsilon u)\rangle_{2B}|}\bigg(\fint_{2B} |A(\varepsilon u) - \langle A(\varepsilon u)\rangle_{2B}| dx\bigg) + c_\delta(\varphi^*)_{|\langle A(\varepsilon u)\rangle_{2B}|}(\|f\|_{BMO(2B)})$$

holds. The constants depend only on the characteristics of φ *and the constants in Assumption 2.2.*

Proof. The estimate is obtained by testing the difference of the equations for u and h by $u - h$. The proof is exactly as for Lemma 1.20. One just needs to replace the gradient by the symmetric gradient. □

2.2.3. Decay estimate. In this section we derive decay estimates for our approximation h. The main ingredient is the following theorem which can be found in [14, Theorem 3.6]. It is valid in any dimension but needs φ'' to be almost monotone. This is the only place in the chapter, where we need this assumption on φ''.

Theorem 2.7. *Let* φ'' *be almost monotone. If* h *is a weak solution of* (2.10)*, then there is an* $r > 2$ *such that for every ball* $Q \subset B$ *with radius* $R > 0$

$$R^2\bigg(\fint_{\frac{1}{2}Q} |\nabla V(\varepsilon h)|^r dx\bigg)^{\frac{2}{r}} \le C \fint_Q |V(\varepsilon h) - \langle V(\varepsilon h)\rangle_Q|^2 dx.$$

The constants C *and* r *depend only on the characteristics of* φ *and the constants in Assumption 2.2.*

The regularity $V \in W^{1,r}$ with $r > 2$ ensures in two space dimensions that V is Hölder continuous. This is the reason, why our estimates can only be applied to planar flows. It is an open question if $V(\nabla u)$ is Hölder continuous in higher dimensions.

This provides the following decay estimates in the plane:

Theorem 2.8. *There exists $\gamma > 0$ such that for every $\lambda \in (0,1]$*

$$\fint_{\theta B} |V(\varepsilon h) - \langle V(\varepsilon h)\rangle_{\theta B}|^2 dx \le C\lambda^{2\gamma} \fint_{B} |V(\varepsilon h) - \langle V(\varepsilon h)\rangle_{B}|^2 dx.$$

The constant C and γ depend only on the characteristics of φ and the constants in Assumption 2.2.

Proof. The result is clear if $\lambda \ge \frac{1}{2}$, so we can assume $\lambda \in (0, \frac{1}{2})$. Let R denote the radius of B. We compute by Poincaré inequality on λB, Jensen's inequality with $r > 2$, enlarging the domain of integration and Theorem 2.7

$$\begin{aligned}
&\fint_{\theta B} |V(\varepsilon h) - \langle V(\varepsilon h)\rangle_{\theta B}|^2 dx \le C(\lambda R)^2 \fint_{\lambda B} |\nabla V(\varepsilon h)|^2 dx \\
&\le C(\lambda R)^2 \bigg(\fint_{\theta B} |\nabla V(\varepsilon h)|^r dx\bigg)^{\frac{2}{r}} \le C R^2 \lambda^{2(1-\frac{2}{r})} \bigg(\fint_{\frac{1}{2}B} |\nabla V(\varepsilon h)|^r dx\bigg)^{\frac{2}{r}} \\
&\le C\lambda^{2(1-\frac{2}{r})} \fint_{B} |V(\varepsilon h) - \langle V(\varepsilon h)\rangle_{B}|^2 dx.
\end{aligned}$$

As $r > 2$ the proof is completed. □

2.2.4. BMO-Estimates. Theorem 2.1 is a corollary of the following more general theorem.

Theorem 2.9. *Let $B \subset \mathbb{R}$ be a ball. Let u, π be a local weak solution of (2.1) on $2B$, with φ and A satisfying Assumption 2.2. Let $\omega : (0,\infty) \to (0,+\infty)$ be non-decreasing such that for some $\beta \in (0, \frac{2\gamma}{\overline{p}'})$ the function $\omega(r)r^{-\beta}$ is almost decreasing, where γ is defined in Theorem 2.8 and $\overline{p}$ in (1.7). Then*

$$\|\pi\|_{\mathrm{BMO}_\omega(\mathrm{B})} + \|A(\varepsilon u)\|_{\mathrm{BMO}_\omega(\mathrm{B})} \le c\, M^{\sharp}_{\omega,2B}(A(\varepsilon u)) + c\|f\|_{\mathrm{BMO}_\omega(2\mathrm{B})}. \tag{2.11}$$

The constants depend only on the characteristics of φ and the constants in Assumption 2.2.

Proof. The proof of the estimate of $A(\varepsilon u)$ follows line by line the proof of Theorem 1.23 as we do not consider a perturbation T here we get then the result by Corollary 1.24. It is based on Corollary 2.5, Lemma 2.6 and Theorem 2.8.

To estimate the pressure we define $H = A(\varepsilon u) - f$. It holds $H \in \mathrm{BMO}_\omega(\mathrm{B}) \subset \mathrm{BMO}(\mathrm{B})$. We fix a ball $Q \subset B$. Then equation (2.1) implies that

$$\forall \xi \in W^{1,2}_0(\Omega) : \langle \pi - \langle \pi\rangle_Q, \mathrm{div}\xi\rangle = \langle H - \langle H\rangle_Q, \nabla \xi\rangle. \tag{2.12}$$

Let $\xi \in W^{1,2}_0(Q)$ be the solution of the auxiliary problem

$$\mathrm{div}\xi = \pi - \langle \pi\rangle_Q \quad \text{in } Q, \qquad \xi = 0 \quad \text{on } \partial Q.$$

The existence of such a solution is ensured by the Bogovskiĭ operator [5] and we have $\|\nabla\xi\|_{L^2(Q)} \leq C\|\pi - \langle\pi\rangle_Q\|_{L^2(Q)}$. The constant $C > 0$ is independent of Q. Inserting such ξ into (2.12) we get

$$\|\pi - \langle\pi\rangle_Q\|^2_{L^2(Q)} = \langle\pi - \langle\pi\rangle_Q, \operatorname{div}\xi\rangle = \langle H - \langle H\rangle_Q, \nabla\xi\rangle.$$

This and $\|\nabla\xi\|_{L^2(Q)} \leq C\|\pi - \langle\pi\rangle_Q\|_{L^2(Q)}$ implies $\|\pi - \langle\pi\rangle_Q\|_{L^2(Q)} \leq c\,\|H - \langle H\rangle_Q\|_{L^2(Q)}$. We find by Jensen's inequality

$$\big(M^\sharp_Q\pi\big)^2 \leq \fint_Q |\pi - \langle\pi\rangle_Q|^2 dx \leq C\fint_Q |H - \langle H\rangle_Q|^2 dx \leq C\|H\|^2_{\mathrm{BMO(Q)}}.$$

In the last inequality we used the John-Nirenberg estimate. It follows that $\pi \in \mathrm{BMO(B)}$ and $\|\pi\|_{\mathrm{BMO(Q)}} \leq C\|H\|_{\mathrm{BMO(Q)}}$. This implies that

$$M^\sharp_{\omega,Q}(\pi) \leq C\frac{1}{\omega(R_Q)}\|H\|_{\mathrm{BMO(Q)}} \leq C\|H\|_{\mathrm{BMO}_\omega\mathrm{(B)}}$$

using the monotonicity of ω. Since Q is arbitrary, we have $\|\pi\|_{\mathrm{BMO}_\omega\mathrm{(B)}} \leq \|H\|_{\mathrm{BMO}_\omega\mathrm{(B)}}$. Now $H = A(\varepsilon u) - f$ and the estimate for $A(\varepsilon u)$ concludes the proof. □

The choice $\omega(t) = 1$ in Theorem 2.9 gives the BMO estimate. However, the choice $\omega(t) = t^\beta$, $\beta \in (0, 2\gamma/\overline{p}')$ Theorem 2.9 gives the estimates in Campanato space BMO_β, compare Corollary 1.26.

Remark 2.10. *It is possible to transfer the Hölder continuity of $A(\varepsilon u)$ to εu and ∇u. Let us discuss the case of power-law and Carreau type fluids. This follows from the fact that $A^{-1} \in C^{0,\sigma}_{\mathrm{loc}}$ for some $\sigma > 0$. If $\kappa = 0$, then $\sigma = \min\{1, p' - 1\}$. If $\kappa > 0$, then $\sigma = 1$. Now, $A(\varepsilon u) \in C^{0,\beta}$ implies $\varepsilon u \in C^{0,\beta\sigma}$. Due to Korn's inequality we get $\nabla u \in C^{0,\beta\sigma}$ as well.*

Remark 2.11. *Note that if $f \in \mathrm{VMO}(2B)$ in Theorem 2.9 we get that $A(\varepsilon u) \in \mathrm{VMO}(B)$. Indeed, since $f \in \mathrm{VMO}(2B)$ there exists a nondecreasing function $\tilde\omega : (0,\infty) \to (0,\infty)$ with $\lim_{r\to 0}\tilde\omega(r) = 0$, such that $\|f\|_{\mathrm{BMO(B_r)}} \leq \tilde\omega(r)$, for all $B_r \subset 2B$. Defining $\omega(r) = \min\{\tilde\omega(r), r^{\frac{\alpha}{\overline{p}'}}\}$ we obtain by Theorem 2.9 the BMO_ω-estimate for $A(\varepsilon u)$ and π, which implies that both are in VMO (compare to Corollary 1.25).*

Remark 2.12. *Let us now assume that the right hand side of (2.1) is not given in divergence form $-\operatorname{div} f$ with f symmetric, but rather as $g \in L^s$ with $s \geq 2$.*

Let $w \in W^{2,s}(2B) \cap W^{1,s}_0(2B)$ and $\sigma \in W^{1,s}(2B)$ with $\langle\sigma\rangle_{2B} = 0$ be the unique solution of the Stokes problem $-\operatorname{div}\varepsilon w + \nabla\sigma = g$ and $\operatorname{div} w = 0$ in $2B$ with $w = 0$ on $\partial(2B)$. Then $g = -\operatorname{div} f$ for $f := \varepsilon w - \sigma\mathrm{Id}$ and f is symmetric. If $s = 2$, then $f \in W^{1,2}(2B) \hookrightarrow \mathrm{VMO}(2B)$. If $s > 2$, then $f \in W^{1,s}(2B) \hookrightarrow \mathcal{L}^{1,2+(1-\frac{2}{s})}(2B) = C^{0,1-\frac{2}{s}}(2B)$. In particular, Theorem 2.9 is applicable and for all $s \geq 2$

$$\|\pi\|_{\mathcal{L}^{1,2+\beta}(B)} + \|A(\varepsilon u)\|_{\mathcal{L}^{1,2+\beta}(B)} \leq c\,R^{-\beta}M^\sharp_{2B}(A(\varepsilon u)) + c\|g\|_{L^s(2B)}$$

for $s \geq 2$ and $\beta \in (0, 1 - \frac{2}{s}] \cap (0, \frac{2\gamma}{\overline{p}'})$. We additionally get VMO estimates if $s = 2$.

The case $s = 2$ is obviously the limiting one in this setting. In the case of the p-Laplacian, i.e. no symmetric gradient and no pressure, it has been proven in [8, 22] that $g \in L^{n,1}(\mathbb{R}^n)$ (Lorentz space; subspace of L^n) implies $A(\nabla u) \in L^\infty$.

It is an interesting open problem, if this also holds for the system with pressure and symmetric gradients (at least in the plane). Note that our results imply in this situation $A(\varepsilon u), \pi \in \mathrm{VMO}$ *for* $n = 2$.

2.3. An application to the stationary Navier-Stokes problem

In this section we present an application of the previous results to the generalized Navier-Stokes problem. We assume that $u \in W^{1,\varphi}(\Omega)$, $\mathrm{div} u = 0$ and $\pi \in L^{\varphi^*}(\Omega)$ are local weak solutions of the generalized Navier-Stokes problem, in the sense that

$$\forall \xi \in W_0^{1,\varphi}(\Omega) : \langle A(\varepsilon u), \varepsilon\xi\rangle - \langle \pi, \mathrm{div}\xi\rangle = \langle f + u \otimes u, \varepsilon\xi\rangle \tag{2.13}$$

for a given mapping $f : \Omega \to \mathbb{R}^{2\times 2}_{\mathrm{sym}}$.

In order to handle the convective term we need the condition

$$\liminf_{s\to+\infty} \frac{\varphi(s)}{s^r} > 0 \quad \text{for some } r > \tfrac{3}{2}. \tag{2.14}$$

We have the following result

Theorem 2.13. *Let* φ *and* A *satisfy Assumption 2.2 and (2.14). Let* u *be a local weak solution of (2.13) on* Ω*. Let* $\beta \in (0, \frac{2\alpha}{\overline{p}'})$ *(*α *is defined in Theorem 2.8 and* $\overline{p}$ *in Lemma 1.6). If* B *is a ball with* $2B \subset \Omega$ *and* $f \in \mathrm{BMO}_\beta(2\mathrm{B})$*, then* $A(\varepsilon u), \pi \in \mathrm{BMO}_\beta(\mathrm{B})$.

Proof. According to [14, Remark 5.3] we get that $\varepsilon u \in L^q(3B/2)$ for all $q > 1$. Consequently by the Korn inequality and the Sobolev embedding we get that $u \otimes u \in \mathcal{L}^{1,n+\beta}(3B/2)$. Applying Theorem 2.1 we get the result. □

Exactly as in Remark 2.10 it is possible to transfer the Hölder continuity of $A(\varepsilon u)$ to εu and ∇u.

Remark 2.14. *A similar result has also been proved in [33], provided* $\kappa > 0$*, by a completely different method, which requires the stronger assumption* $\mathrm{div} f \in L^q(2B)$ *for some* $q > 2$.

The same result was also proved in [56] for power law fluids with $p \in (3/2, 2]$ *and* $\kappa \geq 0$*, again under the stronger assumption* $\mathrm{div} f \in L^q(2B)$ *for some* $q > 2$.

By our method we reprove these known results and improve them by weakening the assumption on the data of the problem.

2.4. An application to the parabolic Stokes problem

Now we apply the previous results to the evolutionary variant of the problem (2.1). We set $T > 0$ and $I = (0, T)$, $\Omega_T = \Omega \times I$ and assume that $u \in L^\infty(I, L^2(\Omega))$ with $\varepsilon u \in L^\varphi(\Omega_T)$ is a local weak solution of the problem

$$\begin{aligned} \partial_t u - \mathrm{div}(A(\varepsilon u)) + \nabla\pi &= g \quad \text{in } \Omega_T, \\ \mathrm{div} u &= 0 \quad \text{in } \Omega_T. \end{aligned} \tag{2.15}$$

If the system of equations (2.15) is complemented by a suitable boundary and initial condition and if the data of the problem are sufficiently smooth it is possible to show existence of a solution that moreover satisfies

$$\partial_t u \in L^\infty(I, L^2(\Omega)), \tag{2.16}$$

see for example [35, 31, 7]. If we know such regularity of $\partial_t u$ and g is smooth, it is easy to reconstruct the pressure π in such a way that $\pi \in L^q(\Omega_T)$ with some $q > 1$ and

$$\forall \xi \in C_0^\infty(\Omega_T) : \int_0^T -\langle \partial_t u, \xi \rangle + \langle A(\varepsilon u) - \pi I, \nabla \xi \rangle dt = \int_0^T \langle g, \xi \rangle \, dt. \tag{2.17}$$

The constant q is determined by the requirement $A(\varepsilon u) \in L^q(\Omega_T)$.

Applying the results from the previous sections of this chapter we obtain the next simple corollary.

Corollary 2.15. *Let A and φ satisfy Assumption 2.2. Let $u \in L^\infty(I, L^2(\Omega))$ with $\varepsilon u \in L^\varphi(\Omega_T)$ and $\operatorname{div} u = 0$ in Ω_T solve the problem (2.15) and satisfy (2.16). Let B be a ball with $2B \subset \Omega$ and $g \in L^\infty(I, L^2(\Omega))$. Then $A(\varepsilon u), \pi \in L^\infty(I, \mathrm{VMO}(B))$.*

PROOF. The result is immediate consequence of $\partial_t u \in L^\infty(I, L^2(\Omega))$ and Remark 2.12. □

Remark 2.16. *Certainly, we can obtain a similar result for the problem (2.15) with convection, as soon as $u \otimes u \in L^\infty(I, \mathrm{VMO}(\Omega))$. This follows for example from the fact that $V(\varepsilon u) \in W^{1,2}(I, L^2(\Omega)) \cap L^2(I, W^{1,2}(\Omega))$. Such kind of regularity is obtained, if it is possible to test with $\partial_t^2 u$ and Δu.*

In [35] a method was developed to construct regular solutions of (2.15). The essential assumption was that the growth of A is sufficiently fast. It was necessary to assume that

$$\liminf_{s \to +\infty} \frac{\varphi(s)}{s^r} > 0 \quad \text{for some } r > \tfrac{4}{3}. \tag{2.18}$$

This assumption was not due to the presence of the convective term in the analysis of [35]. It was necessary to overcome problems connected with the anisotropy of the evolutionary problem (2.15). The previous corollary is a first step to improve these results. If it is possible to show $\partial_t u \in L^\infty(I, L^s(\Omega))$ for some $s > 2$. Then for $g \in L^\infty(I, L^s(\Omega))$, we find by Remark 2.12 that $A(\varepsilon u) \in L^\infty(I, C^{0,\beta}(\Omega))$ for $\beta \in (0, 1 - \frac{2}{s}] \cap (0, \frac{2\gamma}{p'})$. This implies (locally) bounded gradients ∇u. So far the results of this chapter are of local nature. An extension of this technique up to the boundary would imply globally bounded gradients ∇u and we could reconstruct the result of [35] for the generalized Stokes problem without the restriction (2.18).

CHAPTER 3

Parabolic p-Laplace

We study local behavior of solutions $u : Q_T \to \mathbb{R}^N$ to the inhomogeneous parabolic p-Laplace system.

$$\partial_t u - \Delta_p u = \partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = -\operatorname{div} g. \tag{3.1}$$

If $g \in L^{p'}(Q_T)$, then this problem is well-posed and local solutions exist; here Q_T is a space time cylinder. Solutions with this type of term on the right hand side are called energy solutions. In [1] it was proven, that if $g \in L^{p'q}$ for $1 \leq q < \infty$, then $\nabla u \in L^{pq}$ for solution of (3.1) including local estimates. On the other hand side Misawa [48] proved that if g is Hölder continuous, then ∇u is Hölder continuous for conveniently small Hölder exponents. Later this result was refined and extended by Kuusi and Mingione [40] (see also [49]). It is the concern of this chapter to close the gap between higher integrability and Hölder continuity, especially the limit case $q = \infty$. In Chapter 1, we showed that $g \in \text{BMO}$ implies $|\nabla u|^{p-2}\nabla u \in \text{BMO}$ (locally). The task to find a satisfactory limit space in the parabolic setting turns out to be difficult. We introduce this matter by looking at the inhomogeneous heat equation. For the linear theory we have the natural space of parabolic bounded mean oscillation. We say that $f \in \text{BMO}_{\text{par}}(\Omega)$, if $f \in L^1(\Omega)$ and

$$\|f\|_{\text{BMO}_{\text{par}}(\Omega)} := \sup_{Q_{r^2,r} \subset \Omega} \fint_{Q_{r^2,r}} |f - \langle f \rangle_{Q_{r^2,r}}| dz < \infty.$$

If $p = 2$, then we find that $g \in \text{BMO}_{\text{par}}(\mathrm{Q_T})$ implies $\nabla u \in \text{BMO}_{\text{par}}(\mathrm{Q_T})$.

The non-linear version of this result is the boundedness over mean oscillation of the so called natural scaled cylinders: $Q_{\lambda^{2-p}r^2,r} =: Q_r^\lambda$, where

$$\lambda^p \geq \fint_{Q_r^\lambda} |\nabla u|^p dz. \tag{3.2}$$

We carefully construct cubes of the above type and are able to bound the mean oscillations of ∇u over these natural scaled cylinders for $p \geq 2$: see Proposition 3.11. However, these oscillation estimates are not very satisfactory. They depend very strongly on the solution itself. We will overcome this by proving some Bochner estimates. To motivate this result, we want to mention a result on which we worked simultaneously. There we will prove that $|g|^{p'} \in L^\infty(I, L^q(B))$ implies $|\nabla u|^p \in L^\infty(I, L^q(B))$ (locally). If $q \to \infty$ on this quantity we realize that the right borderline space should be a Bochner space of type $L^\infty(I, X)$. The first guess is of course $X = \text{BMO(B)}$. It turns out that this space is too small. Instead we obtained the following main theorem.

Theorem 3.1. *Let u be a solution on $I \times B$, for $p \geq 2$. If $g \in L^\infty(I, \mathrm{BMO(B)})$, then $u \in L^\infty_{\mathrm{loc}}(I, \mathcal{C}^1_{\mathrm{loc}}(B))$. Moreover, for every parabolic cylinder $Q_{2r} \subset I \times B$*

$$\|u\|_{L^\infty(I_{r^2}, \mathcal{C}^1(B_r))} \leq c\|g\|^{\frac{1}{p-1}}_{L^\infty(I,\mathrm{BMO(B)})} + c\|\nabla u\|_{L^p(Q_{2r})} + c,$$

where the constant c only depends on n, N, p.

Here $\mathcal{C}^1$ is the 1-Hölder-Zygmund space (see [57] and Section 3.1 for the exact definition). It is a known substitute for C^1 in the setting of PDE's. To fortify this we mention the following order of spaces on a bounded set $B \subset \mathbb{R}^n$

$$C^1(B) \subset W^{1,\mathrm{BMO}}(B) \subset \mathcal{C}^1(B) \subset \bigcap_{1 \leq q < \infty} W^{1,q}(B).$$

All estimates can be found in Triebel's book [55]. The difference between these spaces and details will be discussed in Section 3.1 and interpolation estimates, that follow from our estimates can be found in Remark 3.14.

Theorem 3.1 is the limit case which has not been proven before. To the authors knowledge these estimates are new even for the linear case $p = 2$. Our estimates are general enough that we can go beyond. Indeed, all our estimates can be stated in the form of weighted BMO_ω (see Section 3.1 for details). Those imply, for example, that Hölder continuity can be transferred from g to ∇u (see Proposition 3.15). This was already proven for all $\frac{2n}{n+2} \leq p$ in [48] and more recently in [40] and [49]. However, for the case (3.1) and $p \geq 2$ considered here, all such estimates are regained by our technique. Moreover, we can weaken the condition on g. Indeed, if $g \in L^\infty(I, C^{\gamma(p-1)}(B))$, then this already implies that $\nabla u \in C^\gamma_{\mathrm{par}}(I \times B)$ locally for small γ; see Proposition 3.15 at the end of the chapter.

The sub-quadratic case requires more difficult analysis. This can be seen in the elliptic case, where the sub-quadratic case was much more problematic to treat (see [15] for details on that matter). In the parabolic case it is not a straightforward extension, but needs other sophisticated tools. We hope that we can present these in a future work. Some advances for the $\frac{2n}{n+2} < p < 2$ are achieved. The first important step to gain BMO estimates is a decay estimate for homogeneous solutions (called p-caloric). In Theorem 3.3 we prove a decay in the spirit of Giaquinta and Modica [23] for p-caloric solutions. This decay is a distinctively stronger estimate on the Hölder behavior for the gradients of p-caloric solutions than other estimates known before. It refines the famous result of DiBenedetto and Friedman [11].

Let us mention some results if the right hand side of (3.1) can be characterized by Radon measures. In the case of systems little is known. In the case where u is scalar valued, Kuusi and Mingione provided pointwise estimates, which allow a direct control of ∇u by the right hand side, such that many regularity properties can be carried over. See [41],[42].

The structure of this chapter is as follows: first we prove the decay for p-caloric solutions (for all $\frac{2n}{n+2} < p < \infty$). This is done in Section 3.2. In Section 3.3.3 we derive a comparison estimate on so-called intrinsic cylinders (see Lemma 3.10). This leads to the boundedness of the intrinsic mean oscillations, which implies the Hölder-Zygmund estimate.

3.1. Spaces and notation

Through the chapter we will denote by I a time interval and B to be a ball in space. We define I_r, B_r as a time interval or ball in space with radius r. A time space cylinder with "center point" (t,x) $Q_{s,r}(t,x) := Q_{s,r}(t,x) := (t,t-s) \times B_r(x)$ and its parabolic boundary as $\partial_{\rm par} Q_{s,r}(t,x) := [t,t-s] \times \partial B_r(x) \cup (t-s) \times B_r(x)$. We introduce the λ–scaled cylinders $Q_r^\lambda(t,x) := (t,t-\lambda^{p-2}r^2) \times B_r(x)$, where p is the exponent of (3.1). For $\theta \in \mathbb{R}^{>0}$ we define $\theta Q_r^\lambda(t,x) := (t,t-\lambda^{2-p}(\theta r)^2) \times B_{\theta r}(x)$. If $\lambda = 1$, then we have a standard parabolic cylinder and we write $Q_r^1(t,x) =: Q_r(t,x)$. As solutions are translation invariant and our estimates are local, the center (t,x) of the cube is mostly of no importance and will often be omitted, to shorten notation. Finally, we call a cylinder K-intrinsic with respect to f, when

$$\frac{\lambda}{K} \leq \langle |Df|^p \rangle_{Q_r^\lambda}^{\frac{1}{p}} \leq K\lambda \text{ and } K\text{-sub-intrinsic w.r.t } f, \text{ when} \qquad (3.3)$$

$$\langle |Df|^p \rangle_{Q_r^\lambda}^{\frac{1}{p}} \leq K\lambda.$$

We say (sub-)intrinsic if $K = 1$.

We have to introduce a few parabolic function spaces. Let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ almost increasing. This means, that there is a $c > 0$ fixed, such that $\omega(r) \leq c\omega(\rho)$ for all $r < \rho$. We say that $f \in \mathrm{BMO}_\omega^{\rm par}(\mathrm{Q})$ the weighted space of mean oscillations, if

$$\|f\|_{\mathrm{BMO}_\omega^{\rm par}(\mathrm{Q})} = \sup_{Q_{r^2,r} \subset Q} \frac{1}{\omega(r)} \fint_{Q_{r^2,r}} |f - \langle f \rangle_{Q_{r^2,r}}| dxdt < \infty.$$

For $\omega(r) = 1$, we get the space of parabolic bounded mean oscillation: $\mathrm{BMO}_{\rm par}(\mathrm{Q})$. By the Campanato characterization, of Hölder spaces we find for $\beta \in (0,1)$ and $\omega(r) = r^\beta$ the space of Hölder continuous function in the parabolic metric.

We look at the Bochner spaces of refined BMO. Let $\omega : \mathbb{R}_+^2 \to \mathbb{R}_+$. We say that $f \in \mathrm{BMO}_\omega(\mathrm{I} \times \mathrm{B})$ if

$$\|f\|_{\mathrm{BMO}_\omega(\mathrm{Q})} := \sup_{I_s \times B_r \subset Q} \frac{1}{\omega(s,r)} \fint_{I_s} \fint_{B_r} |f - \langle f(t) \rangle_{B_r}| dxdt < \infty.$$

If $\omega \equiv 1$, then we have the space $L^\infty(I, \mathrm{BMO(B)})$. More general, if ω only depends on r, then we have the $L^\infty(I, \mathrm{BMO}_\omega(\mathrm{B}))$ spaces.

Let us introduce the Hölder–Zygmund spaces. We say that $f \in \mathcal{C}^\gamma(\Omega)$ if

$$\|f\|_{\mathcal{C}^\gamma(\Omega)} := \sup_{x \in \Omega} \sup_{[x,x+2h] \subset \Omega} \frac{|f(x+2h) - 2f(x+h) + f(x)|}{|h|^\gamma} + \|f\|_\infty < \infty.$$

This is a Banach space. By [55, Sec. 1.2.2] we find that $C^\gamma(\Omega) = \mathcal{C}^\gamma(\Omega)$ if $\gamma \notin \mathbb{N}$ but $C^1(\Omega) \subsetneq \mathcal{C}^1(\Omega)$.

We find in [55, Section 1.7.2], that $\mathcal{C}^1$ has a Campanato space like interpretation. Analogous to the spaces of BMO_ω we define the space of weighted bounded linear oscillation BLO_ω by the semi-norm

$$\|f\|_{\mathrm{BLO}_\omega^q(\Omega)} := \sup_{B_r \subset \Omega} \inf_{\ell \in P^1(B_r)} \frac{1}{\omega(r)} \left(\fint_{B_r} \left| \frac{f - \ell}{r} \right|^q dx \right)^{\frac{1}{q}}, \; 1 < q < \infty.$$

Here P^1 is the set of all polynomials with degree 1. For $q = 2$ we define $\ell_r(f)$ as the best linear approximation of f on B_r in with respect to $\|\cdot\|_2$, which is well defined for all $r > 0$ and $f \in L^2_{\rm loc}$. We find by [55, Section: 1.7.2] that $\mathrm{BLO}(\Omega) := \mathrm{BLO}_1^1(\Omega) \equiv \mathrm{BLO}_1^q(\Omega) \equiv \mathcal{C}^1(\Omega)$ for all $1 \leq q < \infty$; more general, for $\gamma \in (0,1)$ and $\omega(r) = r^\gamma$ the space $\mathrm{BLO}_\omega^q(\Omega) = \mathcal{C}^{1+\gamma}(\Omega)$ for $1 \leq q < \infty$. We define that f is in the space of vanishing linear oscillations VLO if $\|f\|_{\mathrm{BLO}(B_r(x))} \to 0$ for $r \to 0$ uniform in x. Please note

$$\frac{1}{\omega(r)}\|f\|_{\mathrm{BMO^q(B_r)}} \le c\|f\|_{\mathrm{BMO^q_\omega(B_r)}} \text{ or } \frac{1}{\omega(r)}\|f\|_{\mathrm{BLO}^q(B_r)} \le \|f\|_{\mathrm{BLO}^q_\omega(B_r)},$$

because ω is almost increasing. We will use this in this chapter without further reference.

3.2. Decay for p-Caloric functions

In this section we consider $h : Q_T \to \mathbb{R}^N$ to be locally p-caloric on a space time domain Q_T. I.e. h is a solution to the following system

$$\partial_t h - \operatorname{div}(|\nabla h|^{p-2}\nabla h) = 0$$

locally in Q_T. In this section we provide a decay for the natural quantity $V(\nabla h) = |\nabla h|^{\frac{p-2}{2}}\nabla h$. It is an extension to the known result of DiBenedetto and Friedmann [11] providing finer estimates for the continuity behavior. Our results are very much in the spirit of Giaquinta and Modica [23, Proposition 3.1-3.3]. We will prove a parabolic version of their decay for the p-caloric setting.

The first theorem we will need is the well-known weak Harnack inequality first proved by DiBenedetto and Friedmann [11], see also [10, VIII]. We will use the K-sub-intrinsic version of [1, Lemma 1+2].

Theorem 3.2. *Let $p > \frac{2n}{n+2}$ and h be p-caloric on Q_T. If for $Q_r^\lambda \subset Q_T$*

$$\fint_{Q_r^\lambda} |\nabla h|^p dz \le K\lambda^p,$$

then

$$\sup_{\frac{1}{2}Q_r^\lambda} |\nabla h| \le c\lambda.$$

The constant only depends on K, p and the dimensions.

Proof. If $p \ge 2$ it is the same statement as in [1, Lemma 1]. But also in the case of $\frac{2n}{n+2} < p < 2$ the statement holds. In [1, Lemma 2] it is proved that if

$$\fint_{Q_{s^2,\lambda^{\frac{p-2}{2}}s}} |\nabla h|^p dz \le K\lambda^p,$$

it follows

$$\sup_{Q_{s^2,\lambda^{\frac{p-2}{2}}s}} |\nabla h| \le c\lambda.$$

Now we define $r = \lambda^{\frac{p-2}{2}}s$ which implies, that $s^2 = \lambda^{2-p}r^2$. Therefore the estimate holds for all $\frac{2n}{n+2} < p < \infty$. □

The main theorem of this section is the following.

Theorem 3.3. *Let $\partial_t h - \operatorname{div}(|\nabla h|^{p-2}\nabla h) = 0$ on Q_ρ^λ, such that*

$$\frac{\lambda}{K} \le \left(\fint_{Q_\rho^\lambda} |\nabla h|^p dz\right)^{\frac{1}{p}} \le K\lambda,$$

then there exists a $c > 0$ and $\alpha, \tau \in (0,1)$ depending only on n, N, p, K, such that for every $\theta \in (0, \tau]$

$$\sup_{z,w\in\theta Q_\rho^\lambda} |V(\nabla h(w)) - V(\nabla h(z))|^2 \le c\theta^\alpha \fint_{Q_\rho^\lambda} |V(\nabla h) - \langle V(\nabla h)\rangle_{Q_r^\lambda}|^2 dz.$$

We start with a K-intrinsic cube $Q_\rho^\lambda \subset Q_T$ fixed. To be able to state the result neatly we define for $r < \rho$

$$M(r) := \sup_{Q_r^\lambda} |Dh| \tag{3.4}$$

$$\Phi(r) := \left(\fint_{Q_r^\lambda} \left| V(Dh) - \langle V(\nabla h)\rangle_{Q_r^\lambda} \right|^2 dz \right)^{\frac{1}{2}}. \tag{3.5}$$

The classic elliptic result of Giaquinta and Modica [23] was that there is a uniform constant c and an $\alpha \in (0,1)$, such that $\Phi(\theta\rho) \le c\theta^\alpha \Phi(\rho)$. It is then a standard procedure to gain the estimate of the oscillations. It actually follows by Lemma 3.18 which can be found in the appendix.

Theorem 3.4. *Let h be p-caloric on Q_r^λ, such that*

$$\left(\fint_{Q_r^\lambda} |\nabla h|^p dz \right)^{\frac{1}{p}} \le K\lambda,$$

then there exists an $\alpha, c > 0$ depending only on n, N, p, K, such that for every $\theta \in (0, \frac{1}{4}]$

$$\sup_{z,w\in\theta Q_r^\lambda} |V(\nabla h(w)) - V(\nabla h(z))|^2 \le c\theta^\alpha \lambda^p.$$

The theorem is a consequence of [10, IX, Prop 1.1,1.2], resp. [40, Prop. 3.1-3.3]. We combine these statements in the following proposition, as we will use them.

Proposition 3.5. *Let h be p-caloric. Let*

$$M(\rho) \le K\lambda.$$

Then one of the two alternatives hold:

Case 1, non degenerate: There exist $\beta, \delta_0 \in (0,1)$ depending only on n, N, p, K such that

$$\frac{\lambda}{4} \le \inf_{2\delta_0 Q_r^\lambda} |\nabla h| \le \sup_{2\delta_0 Q_r^\lambda} |\nabla h| \le K\lambda$$

$$\text{and } \operatorname{osc}_{Q_{\delta_\rho}^\lambda} (V(\nabla h))^{\frac{1}{2}} \le c\delta^\beta \Phi(\rho) \text{ for all } \delta \in (0, \delta_0).$$

Case 2, degenerate: There exist $\sigma, \eta \in (0,1)$ depending only on n, N, p, K such that

$$M(\sigma\rho) \le \eta K\lambda.$$

PROOF. We only have to show that in Case 1, $\operatorname{osc}_{Q_{\delta_\rho}^\lambda} (V(\nabla h))^{\frac{1}{2}} \le c\delta^\beta \Phi(\rho)$ for $\delta \in (0, \delta_0)$. Anything else can be found in [40, Proposition 3.1-3.3].

By [40, Proposition 3.1] we know, that if Case 1 does not hold, there exists $\delta_1 \in (0,1)$ such that for every sub cube $Q_r^\lambda(z) \subset \delta_1 Q_\rho^\lambda$ we have

$$\frac{\lambda}{4} \le \inf_{Q_r^\lambda(z)} |\nabla h| \le \sup_{Q_r^\lambda(z)} |\nabla h| \le K\lambda.$$

Therefore we have for all these sub cubes

$$\fint_{\theta Q_r^\lambda(z)} \left|V(\nabla h) - \langle V(\nabla h)\rangle_{\theta Q_r^\lambda(z)}\right|^2 \le c\theta^{2\beta} \fint_{Q_r^\lambda(z)} \left|V(\nabla h) - \langle V(\nabla h)\rangle_{Q_r^\lambda(z)}\right|^2$$

because of [40, Proposition 3.2]. This implies the result by Lemma 3.18 with $\delta_0 = \frac{\delta_1}{2}$. □

Proof of Theorem 3.3. Before we can prove the decay we have to do some preliminary work. If for Q_ρ^λ Case 1 of Proposition 3.5 holds, we have the desired decay.

If Case 2 holds, we shall iterate. In this case the degenerate alternative of Proposition 3.5 holds for Q_ρ^λ. We will now construct another smaller cube on which we can apply Proposition 3.5 again.

We find for $\lambda_1 = \eta\lambda$,

$$Q^\lambda_{\sigma\eta^{\frac{2-p}{2}}\rho} \subset Q^{\lambda_1}_{\sigma\rho} \subset Q^\lambda_{\sigma\rho} \text{ if } p<2, \text{ and } Q^\lambda_{\sigma\eta^{\frac{p-2}{2}}\rho} \subset Q^{\lambda_1}_{\sigma\eta^{\frac{p-2}{2}}\rho} \subset Q^\lambda_{\sigma\rho} \text{ if } p \ge 2.$$

We define

$$\rho_1 = a\rho \text{ where } a = \sigma \text{ for } p<2 \text{ and } a = \sigma\eta^{\frac{p-2}{2}} \text{ for } p \ge 2$$
$$\text{and } r_1 = b\rho \text{ with } b = \eta^{\frac{2-p}{2}}\sigma \text{ if } p<2 \text{ and } b = \sigma \text{ if } p \ge 2.$$

We find

$$M(r_1) \le \sup_{Q^{\lambda_1}_{\rho_1}} |\nabla h| \le M(\sigma\rho) \le K\eta\lambda = K\lambda_1.$$

Thus $Q^{\lambda_1}_{\rho_1}$ satisfies the assumption of Proposition 3.5. If Case 2 holds for this cube we can iterate further with

$$\lambda_i = \eta^i\lambda;\ \rho_i = a\rho_{i-1} \text{ and } r_i = br_{i-1}, \tag{3.6}$$

and a, b defined above. If Case 2 holds also for $Q^{\lambda_j}_{\rho_j}$ and $1 \le j \le i-1$, then we find

$$Q^\lambda_{r_i} \subset Q^{\lambda_i}_{\rho_i} \subset Q^{\lambda_{i-1}}_{\rho_{i-1}} \text{ and } \sup_{Q^{\lambda_i}_{\rho_i}} |\nabla h| \le \sup_{Q^{\lambda_{i-1}}_{\sigma\rho_{i-1}}} |\nabla h| \le K\eta\lambda_{i-1} = K\eta^i\lambda.$$

Let us fix $m \in \mathbb{N}$, such that $\eta^m K^2 \le \frac{1}{2}$. This implies that if the degenerate alternative holds for all $i \le m$, then

$$\sup_{Q^\lambda_{rm}} |\nabla h| \le \sup_{Q^{\lambda m}_{\rho m}} |\nabla h| \le K\eta^m\lambda \le \frac{1}{2}\langle|\nabla h|^p\rangle^{\frac{1}{p}}_{Q^\lambda_\rho} \tag{3.7}$$

by the assumption that Q_ρ^λ is intrinsic.

Now we are able to prove the decay. Let us first assume, that for one $i \in \{0, ..., m\}$ the non-degenerate Case 1 of Proposition 3.5 holds. This implies for $\delta \in (0,\tau)$, where $\tau = \frac{\delta_0}{b^m}$, that

$$\operatorname{osc}_{\delta Q_\rho^\lambda}(V(\nabla h))^{\frac{1}{2}} \le \operatorname{osc}_{\delta b^m Q^\lambda_{rm}}(V(\nabla h))^{\frac{1}{2}} \le c\delta^\beta \Phi^{\lambda_i}(\rho_i) \le c\delta^\beta\Phi^\lambda(\rho),$$

as

$$Q_{\rho_i}^{\lambda_i} \subset Q_\rho^\lambda \text{ and } \frac{|Q_\rho^\lambda|}{|Q_{\rho_i}^{\lambda_i}|} \leq \frac{|Q_\rho^\lambda|}{|Q_{\rho_m}^{\lambda_m}|} \leq c \text{ depending only on } n, N, p, K.$$

This leaves the case, that if for all $i \in \{0, ..., m\}$ the degenerate alternative (Case 2) holds. In this case we know by (3.7)

$$\sup_{Q_{\rho_m}^{\lambda_m}} |\nabla h| \leq K\eta^m \lambda \leq \frac{1}{2} \langle |\nabla h|^p \rangle_{Q_\rho^\lambda}^{\frac{1}{p}}.$$

This implies that

$$|\langle V(\nabla h) \rangle_{Q_{\rho_m}^{\lambda_m}}| \leq \frac{1}{2^{\frac{p}{2}}} \langle |V(\nabla h)|^2 \rangle_{Q_\rho^\lambda}^{\frac{1}{2}}.$$

Therefore we gain by Lemma 3.17.

$$\lambda^p \leq K^p \langle |\nabla h|^p \rangle_{Q_\rho^\lambda} \leq c \fint_{Q_\rho^\lambda} |V(\nabla h) - \langle V(\nabla h) \rangle_{Q_r^\lambda}|^2 dz,$$

again, as

$$Q_{\rho_i}^{\lambda_i} \subset Q_\rho^\lambda \text{ and } \frac{|Q_\rho^\lambda|}{|Q_{\rho_i}^{\lambda_i}|} \leq \frac{|Q_\rho^\lambda|}{|Q_{\rho_m}^{\lambda_m}|} \leq c \text{ depending only on } n, N, p, K.$$

Finally, the last estimate combined with Theorem 3.4 implies the decay also in this case. □

3.3. A BMO result for $p \geq 2$

Theorem 3.1 is a consequence of a more general result. From this we will conclude other Campanto like estimates.

Before proving the main result we will have to prove some intermediate results. The key ingredient is to carefully choose a family of intrinsic cylinders.

3.3.1. Finding a scaled sequence of cubes. To treat the scaling behavior in a way to gain a BMO result for (3.1) is quit delicate. Our estimates are based on comparison principles: Whenever one knows that $\|g\|_{L^\infty(I, \mathrm{BMO(B_r)})}$ is small, then u is "close" to a p-caloric comparison solution.

In the following we will construct sub-intrinsic cubes with properties convenient for our needs.

Lemma 3.6. *Let $p \geq 2$. Let $Q_{S,R}(t, x) \subset Q_T$ and $b \in (0, 2)$. For every $0 < r \leq R$ there exists $s(r)$, λ_r and $Q_{s(r),r}(t, x)$ with the following properties. Let $r, \rho \in (0, R]$ and $r < \rho$, then*

(a) *$0 \leq s(r) \leq S$ and $s(r) = \lambda_r^{2-p} r^2$. Especially $Q_{s(r),r}(t, x) = Q_r^{\lambda_r} \subset Q_T$.*
(b) *$s(r) \leq \left(\frac{r}{\rho}\right)^b s(\rho)$, the function s is continuous and strictly increasing on $[0, R]$. Especially $Q_r^{\lambda_r} \subset Q_\rho^{\lambda_\rho}$.*
(c) *$\fint_{Q_r^{\lambda_r}} |\nabla u|^p dz \leq \lambda_r^p$, i.e. $Q_r^{\lambda_r}$ is sub-intrinsic.*
(d) *if $s(r) < \left(\frac{r}{\rho}\right)^b s(\rho)$, then there exists $r_1 \in [r, \rho)$ such that $Q_{r_1}^{\lambda_{r_1}}$ is intrinsic.*
(e) *if for all $r \in (r_1, \rho)$, $Q_r^{\lambda_r}$ is strictly sub-intrinsic, then $\lambda_r \leq \left(\frac{r}{\rho}\right)^\beta \lambda_\rho$ for all $r \in [r_1, \rho]$ and $\beta = \frac{2-b}{p-2} \in (0, \frac{2}{p-2})$.*
(f) *for $\theta \in (0, 1]$, $\theta^\beta \lambda_r \leq \lambda_{\theta r} \leq \frac{c\lambda_r}{\theta^{\frac{n+2}{2}}}$.*

(g) *for* $\theta \in (0,1], |Q_{\theta r}^{\lambda_{\theta r}}|^{-1} \le c\theta^{-(n+2)(1+\frac{p-2}{2})}|Q_r^{\lambda_r}|^{-1}$.
(h) *for* $\theta \in (0,1]$*, we find* $Q_{\sigma r}^{\lambda_{\sigma r}} \subset \theta Q_r^{\lambda_r}$ *for* $\sigma = \theta^{\frac{2}{b}}$.

The constant only depends on the dimensions and p.

Proof. Let $Q_{S,R}(t,x) \subset Q_T$. In the following we often omit the point (t,x). We start, by defining for every $r \in (0,R]$

$$\tilde{s}(r) = \max\Big\{ s \le S \Big| \Big(\int_{t-s}^{t} \int_{B_r(x)} |\nabla u|^p dz \Big)^{p-2} s^2 \le r^{2p}|B_r|^{p-2} \Big\}. \tag{3.8}$$

The function $\tilde{s}(r)$ is well defined and strictly positive for $r > 0$. We define $\tilde{\lambda}_r$ by the equation $r^2\tilde{\lambda}_r^{2-p} = \tilde{s}(r)$. We will first show, that $Q_r^{\tilde{\lambda}_r} := Q_{\tilde{s}(r),r}$ holds (c). By construction we find, that

$$\Big(\int_{Q_{r,\tilde{s}(r)}} |\nabla u|^p dz \Big)^{p-2} \tilde{s}(r)^2 \le r^{2p}|B_r|^{p-2}. \tag{3.9}$$

This implies that

$$\Big(\fint_{Q_{r,\tilde{s}(r)}} |\nabla u|^p dz \Big)^{p-2} \tilde{s}(r)^p \le r^{2p} = (\tilde{\lambda}^{(p-2)}\tilde{s}(r))^p$$

which implies

$$\fint_{Q_{r,\tilde{s}(r)}} |\nabla u|^p dz \le \tilde{\lambda}_r^p, \text{ and if } \fint_{Q_{\tilde{s}(r),r}} |\nabla u|^p dz < \tilde{\lambda}_r^p, \text{ then } \tilde{s}(r) = S. \tag{3.10}$$

Next we will show, that $\tilde{s}(r)$ is continuous for $r \in (0,R]$. For $\varepsilon \le \tilde{s}(r) \le S - \varepsilon$ and $r_0 > 0$, we find that $\big(\int_{t-\tilde{s}(r)}^t \int_{B_r} |\nabla u|^p dz\big)^{p-2} s^2$ is growing of order 2. Because the growth rate is explicitly bounded by

$$\frac{|B_R|^{p-2}R^{2p}}{\varepsilon^2} \ge \Big(\int_{t-\tilde{s}(r)}^{t} \int_{B_r(x)} |\nabla u|^p dz \Big)^{p-2} \ge \frac{r_0^{2p}|B_{r_0}|^{p-2}}{S^2},$$

for $r \in [r_0, R]$. This implies that there exists a $\delta_{\varepsilon,r_0} > 0$, such that for all $r, r_1 \in [r_0, R]$ with $|r - r_1| < \delta_{\varepsilon,r_0}$

$$\begin{aligned} \Big(\int_{t-\tilde{s}(r)}^{t} \int_{B_{r_1}(x)} |\nabla u|^p dz \Big)^{p-2} (\tilde{s}(r) - \varepsilon)^2 &< r_1^{2p}|B_{r_1}^{p-2}| \\ &< \Big(\int_{t-\tilde{s}(r)}^{t} \int_{B_{r_1}(x)} |\nabla u|^p dz \Big)^{p-2} (\tilde{s}(r) + \varepsilon)^2 \end{aligned}$$

as $\left(\int_{t-s}^{t}\int_{B_r(x)}|\nabla u|^p dz\right)^{p-2}$ and $r^{2p}|B_r|^{p-2}$ are both uniformly continuous in r. Now we gain immediately

$$\left(\int\limits_{t-\tilde{s}(r)+\varepsilon}^{t}\int\limits_{B_{r_1}(x)}|\nabla u|^p dz\right)^{p-2}(\tilde{s}(r)-\varepsilon)^2 < r_1^{2p}|B_{r_1}^{p-2}| < \left(\int\limits_{t-\tilde{s}(r)-\varepsilon}^{t}\int\limits_{B_{r_1}(x)}|\nabla u|^p dz\right)^{p-2}(\tilde{s}(r)+\varepsilon)^2,$$

which implies that $|\tilde{s}(r)-\tilde{s}(r_1)| < 2\varepsilon$.

Let us define $s_\varepsilon(r) = \max\{\varepsilon, \min\{\tilde{s}(r), S-\varepsilon\}\}$. By the previous calculations we find that s_ε is uniformly continuous, especially $|s_\varepsilon(r)-s_\varepsilon(r_1)| \leq 2\varepsilon$ for $r, r_1 \in [r_0, R]$ with $|r-r_1| < \delta_{\varepsilon,r_0}$. Therefore

$$|\tilde{s}(r_1)-\tilde{s}(r)| \leq |\tilde{s}(r_1)-s_\varepsilon(r_1)| + |s_\varepsilon(r_1)-s_\varepsilon(r)| + |s_\varepsilon(r)-\tilde{s}(r)| \leq 4\varepsilon.$$

As r_0 was arbitrary we find that $\tilde{s}(r)$ is continuous on $(0, R]$.

Now it might happen, that $r < \rho$ and $\tilde{s}(r) > \tilde{s}(r)$. To avoid that we define for $b \in (0, 2)$

$$s(r) = \min_{R \geq a \geq r}\left(\frac{r}{a}\right)^b \tilde{s}(a).$$

The minimum exists, as $\left(\frac{r}{a}\right)^b \tilde{s}(a)$ is continuous in a. As for $\rho \in (r, R]$

$$s(r) = \min\left\{\min_{\rho \geq a \geq r}\left(\frac{r}{a}\right)^b \tilde{s}(a), \left(\frac{r}{\rho}\right)^b s(\rho)\right\} \tag{3.11}$$

we find that $s(r) < s(\rho)$. Now we define $\lambda_r := \left(\frac{r^2}{s(r)}\right)^{\frac{1}{p-2}} \geq \tilde{\lambda}_r$ and $Q_r^{\lambda_r} := Q_{s(r),r}$. By this definition we find (a) and (b), as $\lim_{r\to 0} s(r) \leq \lim_{r \to 0}\left(\frac{r}{R}\right)^b S(R) = 0$.

We show (c), by (3.9)

$$\fint\limits_{Q_{s(r),r}} |\nabla u|^p \leq \frac{\tilde{s}(r)}{s(r)} \fint\limits_{Q_{\tilde{s}(r),r}} |\nabla u|^p = \left(\frac{\lambda_r}{\tilde{\lambda}_r}\right)^{p-2} \fint\limits_{Q_{\tilde{s}(r),r}} |\nabla u|^p \leq \tilde{\lambda}_r^2 \lambda_r^{p-2} \leq \lambda_r^p. \tag{3.12}$$

To prove (d) we assume that $s(r) < \left(\frac{r}{\rho}\right)^b s(\rho)$. Then there exist a $r_1 \in [r, \rho)$, such that

$$\left(\frac{r}{r_1}\right)^b \tilde{s}(r_1) = s(r) = \min_{R \geq a \geq r}\left(\frac{r}{a}\right)^b \tilde{s}(a) \leq \left(\frac{r}{r_1}\right)^b \min_{R \geq a \geq r_1}\left(\frac{r_1}{a}\right)^b \tilde{s}(a) = \left(\frac{r}{r_1}\right)^b s(r_1).$$

Now because $\tilde{s}(r_1) \geq s(r_1)$ we find $\tilde{s}r_1 = s(r_1)$. Since also $s(r) < \left(\frac{r_1}{\rho}\right)^b s(\rho) \leq \left(\frac{r_1}{R}\right)^b S$ we find by (3.9) that $Q_{s(r_1),r_1} = Q_{r_1}^{\lambda_{r_1}}$ is intrinsic. This implies (d).

To prove (e) we gain by (d) that if $Q_a^{\lambda_a}$ is strictly sub-intrinsic for all $a \in (r, \rho)$, then $s(a) = \left(\frac{a}{\rho}\right)^b s(\rho)$ for all $a \in (r, \rho)$. Now we calculate

$$\lambda_a^{p-2} = \frac{a^2}{s(a)} = \frac{a^2}{\left(\frac{a}{\rho}\right)^b s(\rho)} = \left(\frac{a}{\rho}\right)^{2-b} \lambda_\rho^{p-2},$$

this proves (e), with $\beta = \frac{2-b}{p-2}$.

To prove (f) we take $\theta \in (0,1)$. If $s(\theta r) = \theta^b s(r)$ we are finished. If $s(\theta r) < \theta^b s(r)$, we find by (d) that there is a $\sigma \in [\theta, 1)$ with $s(\theta r) = \left(\frac{\theta}{\sigma}\right)^b s(\sigma r)$ and $Q_{\sigma r}^{\lambda_{\sigma r}}$ is intrinsic. This implies using also (c)

$$\lambda_{\sigma r}^2 = \frac{c}{(\sigma r)^{n+2}} \int_{Q_{s(\sigma r),\sigma r}} |\nabla u|^p dz \le \frac{cs(r)}{r^2 \sigma^{n+2}} \fint_{s(r),r} |\nabla u|^p dz \le \frac{c\lambda_r^2}{\theta^{n+2}}.$$

By the definition of λ_r we find for $\beta = \frac{2-b}{p-2}$ and the previous that

$$\lambda_r \le \theta^{-\beta} \lambda_{\theta r} \text{ and } \lambda_{\theta_r} \le \frac{c}{\theta^{\frac{n+2}{2}}} \lambda_r,$$

which implies (f) and (g). To prove (h) we take $\theta Q_{s(r),r} = Q_{\theta^2 s(r),\theta r}$ we define $\sigma < \theta$, such that $\sigma^b = \theta^2$. Now we find by (3.11), that $s(\sigma r) \le \sigma^b s(r) = \theta^2 s(r)$. $\square$

3.3.2. Comparison. In this section we will derive a comparison estimate which will allow us to gain BMO estimates. Let u be a solution to (3.1) on $I \times B$. As we want to use Theorem 3.3, we will have to start with an intrinsic cylinder. We therefore take any intrinsic cylinder $Q_R^{\lambda_0}(z) \subset I \times B$, i.e.

$$\fint_{Q_R^{\lambda_0}(z)} |\nabla u|^p = \lambda_0^p.$$

In this section we define $Q_r^{\lambda_r}$ as the sub-intrinsic cylinders all sharing the same center, which are constructed by Lemma 3.6. For the next result we will use Lemma 1.7, which states in our case for $P, Q \in \mathbb{R}^{N \times n}$ and $1 < p < \infty$

$$\begin{aligned} (|Q|^{p-2}Q - |P|^{p-2}P) \cdot (Q - P) &\sim |V(Q) - V(P)|^2 \\ ||Q|^{p-2}Q - |P|^{p-2}P| &\sim (|Q| + |Q - P|)^{p-2}|P - Q|^2. \end{aligned} \tag{3.13}$$

This implies for $p \ge 2$

$$|P - Q|^p \le c|V(Q) - V(P)|^2. \tag{3.14}$$

By comparison we mean the local comparison to a p-caloric function. I.e. for $r \in (0, R)$ we will compare u to solutions of

$$\begin{aligned} \partial_t h - \operatorname{div}(|\nabla h|^{p-2}\nabla h) = 0 \text{ on } Q_r^{\lambda_r} \\ h = u \text{ on } \partial_{par} Q_r^{\lambda_r}. \end{aligned} \tag{3.15}$$

Lemma 3.7. *Let $p \ge 2$, $(t, t-\lambda_r^{2-p}) \times B_r(x) =: Q_r^{\lambda_r} \subset I \times B$ and $g \in L^\infty(I, \mathrm{BMO(B)})$. For h the solution of (3.15) and u the solution of (3.1) we have*

$$\lambda_r^{p-2} \fint_{B_r(x)} \frac{|u - h|^2(t)}{r^2} \, \mathrm{d}y + \fint_{Q_r^{\lambda_r}} |V(\nabla u) - V(\nabla h)|^2 dz \le c\|g\|_{L^\infty(I,\mathrm{BMO(B_r(x))})}^{p'}.$$

PROOF. We take $u-h$ as a test function for both systems (3.1) and (3.15). We take the difference and find

$$\fint_{Q_r^{\lambda_r}} \partial_t \frac{|u-h|^2}{2} dz + \fint_{Q_r^{\lambda_r}} (|\nabla u|^{p-2}\nabla u - |\nabla h|^{p-2}\nabla h)\cdot \nabla(u-h) dz$$
$$= \fint_{Q_r^{\lambda_r}} g\cdot \nabla(u-h)\,\mathrm{d}y d\tau = \fint_{Q_r^{\lambda_r}} (g-\langle g(\tau)\rangle_{B_r})\cdot\nabla(u-h)\,\mathrm{d}y d\tau.$$

We find by (3.13), (3.14), (1.6) and as $p'\leq 2$

$$\lambda_r^{p-2}\fint_{B_r}\frac{|u-h|^2(t)}{r^2}\,\mathrm{d}y + \fint_{Q_r^{\lambda_r}} |V(\nabla u)-V(\nabla h)|^2 dz$$
$$\leq c\fint_{Q_r^{\lambda_r}} \Big(|\nabla u| + |g-\langle g(\tau)\rangle_{B_r(x)}|\Big)^{p'-2} |g-\langle g(\tau)\rangle_{B_r(x)}|^2\,\mathrm{d}y d\tau$$
$$+\,\delta\fint_{Q_r^{\lambda_r}} |V(\nabla u)-V(\nabla h)|^2 dz$$
$$\leq c\fint_{t-\lambda_r^{2-p}r^2}^{t}\fint_{B_r(x)} |g-\langle g(\tau)\rangle_{B_r(x)}|^{p'}\,\mathrm{d}y d\tau + \delta\fint_{Q_r^{\lambda_r}} |V(\nabla u)-V(\nabla h)|^2 dz.$$

We absorb and use John-Nirenberg to find that

$$\fint_{B_r(x)} |g-\langle g(\tau)\rangle_{B_r}|^{p'} dx \leq c\|g(\tau)\|^{p'}_{\mathrm{BMO(B_r(x))}},$$

which leads to the result. □

Proposition 3.8. *Let $Q_R^{\lambda_0}$ be intrinsic and $r\in(0,R)$ and $g\in L^\infty(I,\mathrm{BMO(B)})$. Let $\beta\leq\frac{\alpha}{1+\alpha\frac{p-2}{2}}$, such that $\beta<\frac{2}{p-2}$, where α is defined by Theorem 3.4. Then there exist $K,c>1$ depending only on n,N,p,β, such that one of the following two alternatives holds:*

Case 1: $\lambda_r^p\leq K\|g\|^{p'}_{L^\infty(I,\mathrm{BMO(B_r)})}$

Case 2: For the p-caloric comparison function h of (3.15) there exist a $\rho\in[r,R]$ such that

$$osc_{Q_{\sigma r}^{\lambda_{\sigma r}}}(V(\nabla h))^2 \leq c\Big(\frac{\sigma r}{\rho}\Big)^\beta \fint_{Q_\rho^{\lambda_\rho}} |V(\nabla u)-\langle V(\nabla u)\rangle_{Q_\rho^{\lambda_\rho}}|^2 dz$$
$$+\,c\sigma^\beta\|g\|^{p'}_{L^\infty(I,\mathrm{BMO(B_r)})}$$

for every $\sigma\in(0,\delta]$ and $Q_r^{\lambda_r}$ defined by Lemma 3.6. The constant $\delta\in(0,1)$ only depends on n,N,p.

PROOF. Suppose Case 1 does not hold. We find for $\varepsilon=\frac{1}{K}$

$$\|g\|^{p'}_{L^\infty(I,\mathrm{BMO(B_r)})}\leq\varepsilon\lambda_r^p. \tag{3.16}$$

Now let h be the solution of (3.15) on $Q_r^{\lambda_r}$, then Lemma 3.7 implies

$$\fint_{Q_r^{\lambda_r}} |\nabla h|^p dz \leq 2^p \fint_{Q_r^{\lambda_r}} |\nabla u|^p dz + c\|g\|^{p'}_{L^\infty(I,\mathrm{BMO(B_r)})} \leq c\lambda_r^p. \tag{3.17}$$

We therefore can apply Theorem 3.4 and find for $\theta < \frac{1}{4}$

$$\operatorname{osc}_{\theta Q_r^{\lambda_r}} (V(\nabla h))^2 \leq c\theta^\alpha \lambda_r^p. \tag{3.18}$$

We define

$$\rho := \min\{a \geq r | Q_a^{\lambda_a} \text{ is intrinsic}\}. \tag{3.19}$$

By construction $\rho \leq R$ exists as $Q_R^{\lambda_0}$ is intrinsic. Moreover, (see Lemma 3.6,(e)), we find that $\lambda_a \leq (\frac{a}{\rho})^\beta \lambda_\rho$ for every $r \leq a \leq \rho$.

If $\frac{\rho}{2} > r$, we find

$$\langle |\nabla u|^p \rangle_{Q_{\frac{1}{2}\rho}^{\lambda_{\frac{1}{2}\rho}}} \leq \lambda_{\frac{1}{2}\rho}^p \leq \frac{1}{2^\beta} \langle |\nabla u|^p \rangle_{Q_\rho^{\lambda_\rho}}.$$

Therefore Lemma 3.17 implies

$$\lambda_r^p \leq c\Big(\frac{r}{\rho}\Big)^\beta \lambda_\rho^p \leq c\Big(\frac{r}{\rho}\Big)^\beta \fint_{Q_\rho^{\lambda_\rho}} |V(\nabla u) - \langle V(\nabla u)\rangle_{Q_\rho^{\lambda_\rho}}|^2 dz. \tag{3.20}$$

If $\frac{\rho}{2} \leq r \leq \rho$, we either find that

$$\langle |\nabla u|^p \rangle_{Q_r^{\lambda_r}} \leq \frac{1}{2} \langle |\nabla u|^p \rangle_{Q_\rho^{\lambda_\rho}}$$

in which case we have (3.20) again by Lemma 3.17. Otherwise we have

$$\langle |\nabla u|^p \rangle_{Q_r^{\lambda_r}} > \frac{1}{2} \langle |\nabla u|^p \rangle_{Q_\rho^{\lambda_\rho}} \geq c\lambda_r^p.$$

We find by Lemma 3.7 and as Case 1 does not hold

$$\begin{aligned} \lambda_r^p &\leq c \fint_{Q_r^{\lambda_r}} |\nabla u|^p \leq c \fint_{Q_r^{\lambda_r}} |\nabla h|^p + c\|g\|^{p'}_{L^\infty(I,\mathrm{BMO(B_r)})} \\ &\leq c \fint_{Q_r^{\lambda_r}} |\nabla h|^p + c\varepsilon\lambda_r^p. \end{aligned} \tag{3.21}$$

We gain (if ε is small enough) through the previous combined with (3.17)

$$\lambda_r^p \sim \fint_{Q_r^{\lambda_r}} |\nabla h|^p dz. \tag{3.22}$$

Now we can apply Theorem 3.3. This implies together with Lemma 3.7 for $\theta \in (0, \tau)$

$$\operatorname{osc}_{\theta Q_r^{\lambda_r}} (V(\nabla h))^2 \leq c\theta^\alpha \fint_{Q_r^{\lambda_r}} |V(\nabla u) - \langle V(\nabla u)\rangle_{Q_r^{\lambda_r}}|^2 dz + c\theta^\alpha \|g\|^{p'}_{L^\infty(I,\mathrm{BMO(B_r)})}.$$

Combining the last estimate with (3.18) and (3.20) we find

$$\operatorname{osc}_{\theta Q_r^{\lambda_r}}(V(\nabla h))^2 \leq c\theta^\alpha \Big(\frac{r}{\rho}\Big)^\beta \fint_{Q_\rho^{\lambda_\rho}} |V(\nabla u) - \langle V(\nabla u)\rangle_{Q_\rho^{\lambda_\rho}}|^2 dz + c\theta^\alpha \|g\|^{p'}_{L^\infty(I,\mathrm{BMO}(\mathrm{B_r}))}.$$

To conclude the proof we use Lemma 3.6, (h): For $\sigma^{\frac{b}{2}} = \theta$ we have $Q_{\sigma r}^{\lambda_{\sigma r}} \subset \theta Q_r^{\lambda_r}$, therefore

$$\begin{aligned}
&\operatorname{osc}_{Q_{\sigma r}^{\lambda_{\sigma r}}}(V(\nabla h))^2 \\
&\leq c\sigma^{\frac{b\alpha}{2}}\Big(\frac{r}{\rho}\Big)^\beta \fint_{Q_\rho^{\lambda_\rho}} |V(\nabla u) - \langle V(\nabla u)\rangle_{Q_\rho^{\lambda_\rho}}|^2 dz + c\sigma^{\frac{b\alpha}{2}} \|g\|^{p'}_{L^\infty(I,\mathrm{BMO}(\mathrm{B_r}))} \\
&\leq c\Big(\frac{\sigma r}{\rho}\Big)^\beta \fint_{Q_\rho^{\lambda_\rho}} |V(\nabla u) - \langle V(\nabla u)\rangle_{Q_\rho^{\lambda_\rho}}|^2 dz + c\sigma^\beta \|g\|^{p'}_{L^\infty(I,\mathrm{BMO}(\mathrm{B_r}))}
\end{aligned}$$

by the choice of $\beta = \frac{2-b}{p-2} \leq \frac{b\alpha}{2}$ which is a consequence of our assumptions on β. $\square$

3.3.3. An intrinsic BMO result. The next proposition gives an intrinsic BMO estimate. We will prove it for the refined spaces BMO_ω. In the following let $\omega : [0,\infty) \to [0,\infty)$ be almost increasing. Moreover,

$$\frac{\omega(r)}{\omega(\sigma r)} \leq c_1 \sigma^{\frac{-\gamma}{p}} \text{ for } \sigma \in (0,1] \text{ where } \gamma < \min\Big\{\frac{\alpha}{1+\alpha\frac{p-2}{2}}, \frac{2}{p-2}\Big\}. \tag{3.23}$$

Lemma 3.9. *Let $Q_R^{\lambda_0}$ be intrinsic, ω hold (3.23) and $g \in L^\infty(I, \mathrm{BMO}_{\omega'}(\mathrm{B}))$, with $\omega' \equiv \omega^{p-1}$. Then there exist constants c, β depending on γ, c_1, n, N, p such that*

$$\begin{aligned}
\sup_{0<r<R} \frac{1}{\omega^p(r)} \fint_{Q_r^{\lambda_r}} |V(\nabla u) - \langle V(\nabla u)\rangle_{Q_r^{\lambda_r}}|^2 &\leq c\|g\|^{p'}_{L^\infty(I,\mathrm{BMO}_{\omega'}(\mathrm{B_r}))} \\
&+ \frac{c}{\omega^p(R)} \fint_{Q_R^{\lambda_0}} |V(\nabla u) - \langle V(\nabla u)\rangle_{Q_R^{\lambda_0}}|^2,
\end{aligned}$$

where $Q_r^{\lambda_r}$ is defined by Lemma 3.6 for a $\beta > \gamma$ fixed.

Proof. We fix $\gamma < \beta < \min\Big\{\frac{\alpha}{1+\alpha\frac{p-2}{2}}, \frac{2}{p-2}\Big\}$. Now we take $\sigma \in (0,1)$. We will define the size of σ in the end of the proof. If $r \geq \sigma R$, we find by Lemma 3.6, (g)

$$\frac{1}{\omega^p(r)} \fint_{Q_r^{\lambda_r}} |V(\nabla u) - \langle V(\nabla u)\rangle_{Q_r^{\lambda_r}}|^2 \leq \frac{c(\sigma)}{\omega^p(R)} \fint_{Q_R^{\lambda_0}} |V(\nabla u) - \langle V(\nabla u)\rangle_{Q_R^{\lambda_0}}|^2. \tag{3.24}$$

Now we will prove the estimate for $\sigma r \in (0, \sigma R]$. We apply Proposition 3.8 on the cylinder $Q_r^{\lambda_r}$. If Case 1 holds, we find as $Q_{\sigma r}^{\lambda_{\sigma r}}$ is sub-intrinsic, that

$$\begin{aligned}\frac{1}{\omega^p(\sigma r)} \fint_{Q_{\sigma r}^{\lambda_{\sigma r}}} |V(\nabla u) - \langle V(\nabla u)\rangle_{Q_{\sigma r}^{\lambda_{\sigma r}}}|^2 dz &\leq \frac{c(\sigma)}{\omega^p(r)} \lambda_r^p \\ &\leq \frac{cK^p}{\omega^p(r)} \|g\|_{L^\infty(I,\mathrm{BMO(B_r)})}^{p'} \leq c\|g\|_{L^\infty(I,\mathrm{BMO}_{\omega'}(\mathrm{B_r}))}^{p'}\end{aligned} \tag{3.25}$$

where we used that ω is almost increasing and that $\omega' \equiv \omega^{\frac{p}{p'}}$.

If Case 2 of Proposition 3.8 holds, we find using the best constant property, Lemma 3.7, (3.23) and Lemma 3.6 (g)

$$\begin{aligned}&\frac{1}{\omega^p(\sigma r)} \fint_{Q_{\sigma r}^{\lambda_{\sigma r}}} |V(\nabla u) - \langle V(\nabla u)\rangle|_{Q_{\sigma r}^{\lambda_{\sigma r}}}^2 dz \\ &\leq \frac{c}{\omega^p(\sigma r)} \fint_{Q_{\sigma r}^{\lambda_{\sigma r}}} |V(\nabla h) - \langle V(\nabla h)\rangle_{Q_{\sigma r}^{\lambda_{\sigma r}}}|^2 dz + \frac{c(\sigma)}{\omega^p(r)} \fint_{Q_r^{\lambda_r}} |V(\nabla u) - V(\nabla h)|^2 dz \\ &\leq \frac{c}{\omega^p(\sigma r)} \operatorname{osc}_{Q_{\sigma r}^{\lambda_{\sigma r}}} (V(\nabla h))^2 + c\|g\|_{L^\infty(I,\mathrm{BMO}_{\omega'}(\mathrm{B_r}))}^{p'}.\end{aligned} \tag{3.26}$$

By Proposition 3.8 and (3.23) we find for $\sigma \in (0, \delta)$ and $\rho \geq r$

$$\begin{aligned}\frac{1}{\omega^p(\sigma r)} \operatorname{osc}_{Q_{\sigma r}^{\lambda_{\sigma r}}} (V(\nabla h)) &\leq \sigma^{\beta-\gamma} \frac{c}{\omega^p(\rho)} \fint_{Q_\rho^{\lambda_\rho}} |V(\nabla u) - \langle V(\nabla u)\rangle_{Q_\rho^{\lambda_\rho}}|^2 dz \\ &\quad + \sigma^{\beta-\gamma} \frac{c}{\omega^p(r)} \|g\|_{L^\infty(I,\mathrm{BMO(B_r)})}^{p'}.\end{aligned}$$

Combining the last estimate with (3.24),(3.25) and (3.26) leads to

$$\begin{aligned}&\sup_{a<r<R} \frac{1}{\omega(r)} \fint_{Q_r^{\lambda_r}} |V(\nabla u) - \langle V(\nabla u)\rangle_{Q_r^{\lambda_r}}|^2 \\ &\leq c\|g\|_{L^\infty(I,\mathrm{BMO}_{\omega'}(\mathrm{B_r}))}^{p'} + \frac{c}{\omega(R)} \fint_{Q_R^{\lambda_0}} |V(\nabla u) - \langle V(\nabla u)\rangle_{Q_R^{\lambda_0}}|^2 \\ &\quad + c\sigma^{\beta-\gamma} \sup_{\frac{a}{\sigma}<r<R} \frac{1}{\omega(r)} \fint_{Q_r^{\lambda_r}} |V(\nabla u) - \langle V(\nabla u)\rangle_{Q_r^{\lambda_r}}|^2.\end{aligned} \tag{3.27}$$

Now fix σ conveniently, such that we can absorb the last term. The result follows by $a \to 0$. $\square$

In Proposition 3.11 we show the intrinsic BMO estimate. Before we need another lemma on cylinders.

Lemma 3.10. *Let $Q_R^{\lambda_0}$ be sub-intrinsic. For every $z \in Q_{R/2}^{\lambda_0}$ there exist a sub-intrinsic cube $Q_{R/2}^{\lambda_{R/2}}(z) \subset Q_R^{\lambda_0}$ and $\lambda_{R/2} \sim \lambda_0$.*

Let $Q_R = (t, t-R^2) \times B_R(x)$. Then for every $z \in Q_{R/2}$ there exists a sub-intrinsic cube $Q_{R/2}^{\lambda_{R/2}}(z) \subset Q_R$ and $\lambda_{R/2} \sim \max\{(\fint_{Q_R} |\nabla u|^p)^{\frac{1}{2}}, 1\}$.

PROOF. We start with the first statement. Since $Q_R^{\lambda_0}$ is sub-intrinsic we find for fixed $z \in Q_{\frac{R}{2}}^{\lambda_0}$

$$\frac{1}{|Q_R^{\lambda_0}|} \int_{Q_{\frac{R}{2}}^{\lambda_0}(z)} |\nabla u|^p \leq \lambda_0^p.$$

Hence, for $2^{\frac{n+2}{p-2}}\lambda_0 = \lambda_{R/2} \geq \lambda_0$ we find

$$\Bigg(\fint_{Q_{\frac{R}{2}}^{\lambda_{\frac{R}{2}}}(z)} |\nabla u|^p \Bigg)^{\frac{1}{p}} \leq \lambda_{R/2} \leq 2^{\frac{n+2}{p-2}}\lambda_0.$$

To prove the second statement we define $\tilde{\lambda}_0$ by $\fint_{Q_R} |\nabla u|^p = \tilde{\lambda}_0^2$. If $\tilde{\lambda}_0 \leq 1$, then $\fint_{Q_R} |\nabla u|^p \leq 1^p$, in this case we define $\lambda_0 = 1$. If $\tilde{\lambda}_0 \geq 1$ (and $\tilde{\lambda}_0^{2-p} \leq 1$), we define $\lambda_0 = \tilde{\lambda}_0$ and find for any $Q_R^{\lambda_0}(t) := (t, t-\lambda_0^{2-p}R^2) \times B_R \subset Q_R$ that $\fint_{Q_R^{\lambda_0}(t)} |\nabla u|^p \leq \lambda_0^p$. Now we gain the result by proceeding as before. □

Proposition 3.11. *Let $Q_R^{\lambda_0}$ be sub-intrinsic, ω hold* (3.23) *and $g \in L^\infty(I, \mathrm{BMO}_{\omega'}(\mathrm{B}))$, with $\omega' \equiv \omega^{p-1}$. Then there exist a constant c, β depending on γ, c_1, n, p, N such that*

$$\sup_{z \in Q_{\frac{R}{2}}^{\lambda_0}} \sup_{r < \frac{R}{2}} \frac{1}{\omega(r)} \Bigg(\fint_{Q_r^{\lambda_r}(z)} |V(\nabla u) - \langle V(\nabla u)\rangle_{Q_r^{\lambda_r}(z)}|^2 \Bigg)^{\frac{1}{p}}$$
$$\leq c\|g\|_{L^\infty(I,\mathrm{BMO}_{\omega'}(\mathrm{B_R}))}^{\frac{1}{p-1}} + \frac{c\lambda_0}{\omega(R)}$$

where $Q_{R/2}^{\lambda_{R/2}}(z)$ is defined by Lemma 3.10 and $Q_r^{\lambda_r}(z) \subset Q_{R/2}^{\lambda_{R/2}}(z)$ is defined by Lemma 3.6 for $\beta > \gamma$ fixed.

PROOF. We fix $\rho := \sup\{a < \frac{R}{2} | Q_a^{\lambda_a}(z) \text{ is intrinsic}\}$. By (e) of Lemma 3.6, (3.23) and Lemma 3.10 we find for $\rho \leq r \leq \frac{R}{2}$

$$\frac{1}{\omega^p(r)} \fint_{Q_r^{\lambda_r}(z)} |V(\nabla u) - \langle V(\nabla u)\rangle_{Q_r^{\lambda_r}}(z)|^2 \leq \frac{c\lambda_r^p}{\omega^p(r)} \leq c(\sigma)\frac{r^\beta \omega^p(R)}{R^\beta \omega^p(r)} \frac{\lambda_{R/2}^p}{\omega^p(R)} \leq \frac{c\lambda_0^p}{\omega^p(R)}.$$

For $r \leq \rho$ we can apply Lemma 3.9 and find by the previous that

$$\frac{1}{\omega^p(r)} \fint_{Q_r^{\lambda_r}(z)} |V(\nabla u) - \langle V(\nabla u)\rangle_{Q_r^{\lambda_r}(z)}|^2$$
$$\leq c\|g\|_{L^\infty(I,\mathrm{BMO}_{\omega'}(\mathrm{B_R}))}^{p'} + \frac{c}{\omega^p(\rho)} \fint_{Q_\rho^{\lambda_\rho}(z)} |V(\nabla u) - \langle V(\nabla u)\rangle_{Q_\rho^{\lambda_\rho}(z)}|^2$$
$$\leq c\|g\|_{L^\infty(I,\mathrm{BMO}_{\omega'}(\mathrm{B_R}))}^{p'} + \frac{c\lambda_0^p}{\omega^p(R)}.$$

This finishes the proof. □

We can generalize this result by the following purely intrinsic result.

Corollary 3.12. *Let $Q_R^{\lambda_0}$ be sub-intrinsic, ω hold (3.23) and for every cube $Q_r^{\lambda_r}(z)$ constructed as in Proposition 3.11*

$$\sup_{z\in Q_R^{\lambda_0}}\sup_{r<R}\frac{1}{\omega^p(r)}\fint_{Q_r^{\lambda_r}(z)}|g-\langle g\rangle_{Q_r^{\lambda_r}(z)}|^{p'}=:\|\|g\|\|^{p'}<\infty,$$

then there exist a constant c,β depending on γ,c_1,n,p such that

$$\sup_{z\in Q_{\frac{R}{2}}^{\lambda_0}}\sup_{r<\frac{R}{2}}\frac{1}{\omega^p(r)}\fint_{Q_r^{\lambda_r}(z)}|V(\nabla u)-\langle V(\nabla u)\rangle_{Q_r^{\lambda_r}(z)}|^2 \leq c\|\|g\|\|^{p'}+\frac{c\lambda_0^p}{\omega^p(R)}.$$

Proof. One simply replaces $\|g\|_{L^\infty(I,\mathrm{BMO}(\mathrm{B_r}))}$ by $\|\|g\|\|$ in Lemma 3.7, Lemma 3.8 and Proposition 3.11. Anything else follows analogously. □

3.3.4. Main Results. We are now able to prove the main theorem on weighted BLO spaces.

Theorem 3.13. *Let $p\geq 2$ and the wight $\omega:\mathbb{R}^{>0}\to\mathbb{R}^{>0}$ be almost increasing and satisfy (3.23). Let u be a solution to (3.1) on $I\times B$ and $g\in L^\infty(I,\mathrm{BMO}_{\omega'}(\mathrm{B}))$, with $\omega'\equiv\omega^{p-1}$, then $u\in L^\infty(I,BLO_\omega(B))$ locally. Moreover, there exists c,δ depending on n,N,p,γ,c_1 such that for every sub-intrinsic cylinder $Q_R^{\lambda_0}\subset I\times B$*

$$\begin{aligned}&\|u\|_{L^\infty(I_{\lambda_0^{2-p}R^2/4},BLO_\omega(B_{\delta R/2}))}\\ &\leq \sup_{(t,x)\in Q_{\frac{R}{2}}^{\lambda_0}}\sup_{r\in(0,\frac{\delta R}{2}]}\frac{1}{\omega(r)}\bigg(\fint_{B_r(x)}\Big|\frac{u(t,y)-\ell_r(u)(t)}{r}\Big|^2\mathrm{d}y\bigg)^{\frac12}\\ &\leq c\|g\|_{L^\infty(I,\mathrm{BMO}_{\omega'}(\mathrm{B_R}))}^{\frac{1}{p-1}}+\frac{c\lambda_0}{\omega(R)}.\end{aligned}$$

Proof. We fix $(t,x)\in Q_{R/2}^{\lambda_0}$ and construct $Q_{R/2}^{\lambda_{R/2}}(t,x)$ by Lemma 3.10. Then we define $Q_r^{\lambda_r}:=(t,t-\lambda_r^{2-p}r^2)\times B_r(x)\subset Q_R^{\lambda_0}$ for $\frac{R}{2}>r>0$ by Lemma 3.6 with respect to $Q_{R/2}^{\lambda_{R/2}}(t,x)$ for a convenient β. In the following all balls in space are centered in (t,x). Our aim is to estimate

$$N_r^\omega(u)(t,x):=\frac{1}{\omega^2(r)}\fint_{B_r(x)}\Big|\frac{u(y,t)-\ell_r(u)(t)}{r}\Big|^2dy.$$

Here $\ell_r(u)(t)$ is the best linear approximation of u on $\{t\}\times B_r(x)$. We show the result for $\delta r\in(0,\frac{\delta R}{2})$. The constant δ is fixed by Proposition 3.8. We will divide the proof in the two cases of Proposition 3.8.

Case 1: $\lambda_r^p\leq K\|g\|_{L^\infty(I,\mathrm{BMO}(\mathrm{B_r}))}^{p'}$.

Case 2: $\|g\|_{L^\infty(I,\mathrm{BMO}(\mathrm{B_r}))}^{p'}\leq\frac{1}{K}\lambda_r^p$.

If Case 1 holds, we find $\langle|\nabla u|^p\rangle_{Q_r^{\lambda_r}}\leq\lambda_r^p\leq\mu_r^p=:K\|g\|_{L^\infty(I,\mathrm{BMO}(\mathrm{B_r}))}^{p'}$. As $\lambda_r\leq\mu_r$ we find that $Q_r^{\mu_r}\subset Q_r^{\lambda_r}$ and by (3.12) that $\langle|\nabla u|^p\rangle_{Q_r^{\mu_r}}\leq\mu_r^p$. We take h to be the solution of (3.15) on $Q_r^{\mu_r}$. Now Lemma 3.7 provides

$$\mu_r^{p-2} \fint_{\{t\}\times B_r} \Big|\frac{u-h}{r}\Big|^2 dx + \fint_{Q_r^{\mu_r}} |V(\nabla u) - V(\nabla h)|^2 dz \leq c\mu_r^p. \tag{3.28}$$

We use $\ell_{\delta r}(u)$ as the best linear approximation of u on $B_{\delta r} := \{t\} \times B_{\delta r}(x)$ and Poincaré's inequality to gain

(3.29)

$$\begin{aligned} N_{\delta r}^{\omega}(u) &\leq \frac{1}{\omega^2(\delta r)} \fint_{B_{\delta r}} \Big|\frac{u-\ell_{\delta r}(h)}{r}\Big|^2 dx \leq \frac{c}{\omega^2(\delta r)} \fint_{B_{\delta r}} \Big|\frac{u-h}{r}\Big|^2 + \Big|\frac{h-\ell_{\delta r}(h)}{r}\Big|^2 dx \\ &\leq \frac{c(\delta)}{\omega^2(r)} \fint_{B_r} \Big|\frac{u-h}{r}\Big|^2 dx + \frac{c}{\omega^2(\delta r)} \sup_{B_{\delta r}} |\nabla h|^2 = I + II. \end{aligned}$$

For I we find by (3.28)

$$I \leq \frac{c(\delta)}{\omega^2(r)} \mu_r^2 \leq c\|g\|_{L^\infty(I,\mathrm{BMO}_{\omega'}(\mathrm{B_r}))}^{\frac{2}{p-1}}.$$

To estimate II we find by (3.28)

$$\langle |\nabla h|^p \rangle_{Q_r^{\mu_r}} \leq \langle |\nabla u|^p \rangle_{Q_r^{\mu_r}} + \fint_{Q_r^{\mu_r}} |V(\nabla u) - V(\nabla h)|^2 dz \leq c\mu_r^p.$$

Now Theorem 3.2 implies

$$\frac{c}{\omega^2(\delta r)} \sup_{\{t\}\times B_{\delta r}(x)} |\nabla h|^2 \leq \frac{c}{\omega^2(\delta r)} \sup_{Q_{\delta r}^{\mu_r}} |\nabla h|^2 \leq \frac{c}{\omega^2(r)} \mu_r^2 \leq c\|g\|_{L^\infty(I,\mathrm{BMO}_{\omega'}(\mathrm{B_R}))}^{\frac{2}{p-1}}.$$

This closes Case 1.

In the following Case 2 holds. Remember, that $\delta r \in (0, \delta\frac{R}{2})$. We start similar to (3.29). we take h to be the solution of (3.15) on $Q_r^{\lambda_r}$. Now Lemma 3.7 gives

$$\lambda_r^{p-2} \fint_{\{t\}\times B_r} \Big|\frac{u-h}{r}\Big|^2 dx + \fint_{Q_r^{\lambda_r}} |V(\nabla u) - V(\nabla h)|^2 dz \leq c\|g\|_{L^\infty(I,\mathrm{BMO}(\mathrm{B_r}))}^{p'}. \tag{3.30}$$

Similar to Case 1 we find

$$\begin{aligned} N_{\delta r}^{\omega}(u) &\leq \frac{1}{\omega^2(\delta r)} \fint_{B_{\delta r}} \Big|\frac{u-\ell_{\delta r}(h)}{r}\Big|^2 dx \leq \frac{c}{\omega^2(\delta r)} \fint_{B_{\delta r}} \Big|\frac{u-h}{r}\Big|^2 + \Big|\frac{h-\ell_{\delta r}(h)}{r}\Big|^2 dx \\ &\leq \frac{c(\delta)}{\omega^2(r)} \fint_{B_r} \Big|\frac{u-h}{r}\Big|^2 dx + \frac{c}{\omega^2(\delta r)} \mathrm{osc}_{B_{\delta r}}(\nabla h)^2 = I + II. \end{aligned}$$

Here we used the Poincaré's inequality. We estimate I by Lemma 3.7; as Case 2 holds we deduce from (3.30)

$$\fint_{B_r} \Big|\frac{u-h}{r}\Big|^2 dx \leq \lambda_r^{2-p} \|g\|_{L^\infty(I,\mathrm{BMO}(\mathrm{B_r}))}^{p'} \leq \|g\|_{L^\infty(I,\mathrm{BMO}(\mathrm{B_r}))}^{\frac{2}{p-1}},$$

and consequently

$$I \leq c\|g\|_{L^\infty(I,\mathrm{BMO}_{\omega'}(\mathrm{B_r}))}^{\frac{2}{p-1}}. \tag{3.31}$$

We estimate II by using $p \geq 2$ and Proposition 3.8. As in the proof of Proposition 3.11 we fix $\rho := \sup\{a < \frac{R}{2} | Q_a^{\lambda_a}(t,x) \text{ is intrinsic}\}$. If $r \leq \rho$ Proposition 3.8 provides an $r_1 \leq \rho$ such that

$$\operatorname{osc}_{B_{\delta r}}(V(\nabla h))^2 \leq c\Big(\frac{\delta r}{r_1}\Big)^\beta \fint_{Q_{r_1}^{\lambda_{r_1}}} |V(\nabla u) - \langle V(\nabla u)\rangle_{Q_{r_1}^{\lambda_{r_1}}}|^2 dz + c\delta^\beta \|g\|^{p'}_{L^\infty(I,\mathrm{BMO}(\mathrm{B_r}))}.$$

This implies using (3.14)

$$\begin{aligned} II &= \frac{c}{\omega^2(\delta r)} \operatorname{osc}_{B_{\delta r}}(\nabla h)^2 \leq \frac{c}{\omega^2(\delta r)} \operatorname{osc}_{B_{\delta r}}(V(\nabla h))^{\frac{4}{p}} \\ &\leq c\frac{1}{\omega^2(\delta r)}\bigg(\Big(\frac{\delta r}{r_1}\Big)^\beta \fint_{Q_{r_1}^{\lambda_{r_1}}} |V(\nabla u) - \langle V(\nabla u)\rangle_{Q_{r_1}^{\lambda_{r_1}}}|^2 dz + \|g\|^{p'}_{L^\infty(I,\mathrm{BMO}(\mathrm{B_r}))}\bigg)^{\frac{2}{p}} \\ &\leq c\Big(\frac{\delta r}{r_1}\Big)^{\frac{2}{p}(\beta-\gamma)} \bigg(\frac{1}{\omega^p(r_1)} \fint_{Q_{r_1}^{\lambda_{r_1}}} |V(\nabla u) - \langle V(\nabla u)\rangle_{Q_{r_1}^{\lambda_{r_1}}}|^2 dz\bigg)^{\frac{2}{p}} \\ &\quad + c\|g\|^{\frac{2}{p-1}}_{L^\infty(I,\mathrm{BMO}_{\omega'}(\mathrm{B_r}))} \end{aligned}$$

as ω holds (3.23). On this we can apply Proposition 3.11 and find as $\gamma < \beta$

$$II \leq c\|g\|^{\frac{2}{p-1}}_{L^\infty(I,\mathrm{BMO}_{\omega'}(\mathrm{B_R}))} + \frac{c\lambda_0^2}{\omega^2(R)}. \tag{3.32}$$

If $\rho < r < \frac{R}{4}$ we have by (e) of Lemma 3.6 and the construction of $Q_{R/2}^{\lambda_{R/2}}(t,x)$, that $\lambda_r \leq \Big(\frac{r}{R/2}\Big)^\beta \lambda_0$ and therefore we find by (3.30) and it's consequences (3.17) and (3.18)

$$II \leq \frac{c}{\omega^2(\delta r)} \operatorname{osc}_{B_{\delta r}}(\nabla h)^2 \leq \frac{c}{\omega^2(\delta r)}\lambda_r^2 \leq \frac{c}{\omega^2(R)}\Big(\frac{r}{R}\Big)^{\beta-\gamma}\lambda_0^2.$$

Combining the last estimate with (3.31) and (3.32) closes case 2. As all estimates are independent of $(t,x) \in Q_{\frac{R}{2}}^{\lambda_0}$, the result is proved. □

Proof of Theorem 3.1. One fixes $\omega(r) \equiv 1$ and combines Lemma 3.10 with Theorem 3.13. Then the result follows by the Campanato characterization of $\mathcal{C}^1(B_{R/2}(x))$. □

Remark 3.14. *In* [55, Section: 1.7.2] *we find that* $BLO = \mathcal{C}^1 = F^1_{\infty,\infty}$*, here* $F^1_{\infty,\infty}$ *is the Triebel-Lizorkin space. The space* $W^{1,\mathrm{BMO}} = F^1_{\infty,2}$*. Consequently we find by our estimates that, if* g *is in* $L^\infty(2I,\mathrm{BMO}(2\mathrm{B}))$*, then* $u \in L^p(I,W^{1,p}(B)) \cap L^\infty(I,BLO(B)) = L^p(I,F^1_{p,2}(B)) \cap L^\infty(I,F^1_{\infty,\infty}(B))$*. By interpolation* $u \in L^q(I,W^{1,r}(B))$ *for every* $1 \leq q \leq \infty$, $1 \leq r < \infty$ *(see* [55, Section: 1.6.2]*); natural local estimates are available.*

Proposition 3.15. *Let* $\gamma p < \min\Big\{\frac{\alpha}{1+\alpha\frac{p-2}{2}}, \frac{2}{p-2}\Big\}$*. If* $g \in L^\infty(I,C^{\gamma(p-1)}(B))$*, then* $\nabla u \in C^\gamma_{par}(I \times B)$*. Moreover, for every sub-intrinsic cylinder* $Q_R^{\lambda_0}$ *we find*

$$\|\nabla u\|_{C^\gamma_{par}(Q^{\lambda_0}_{R/4})} \leq c\Big(\frac{1}{R^\gamma} + \frac{1}{(K^{2-p}R^2)^{\frac{\gamma}{2}}}\Big),$$

where $K = c\lambda_0 + cR^\gamma \|g\|^{\frac{1}{p-1}}_{L^\infty(I,C^{\gamma(p-1)}(B_R))}$ *and* c *depends on* γ, n, p, K.

Proof. We start by showing Hölder continuity in space. By Theorem 3.13 and $\omega(r) = r^\gamma$ we gain by the Campanato characterization that

$$\|\nabla u\|_{L^\infty(I_{\lambda_0^{2-p}R^2/4}, C^\gamma(B_{R/2}))} \leq c\|g\|^{\frac{1}{p-1}}_{L^\infty(I,C^{\gamma(p-1)}(B_R))} + \frac{c\lambda_0}{R^\gamma}. \tag{3.33}$$

This implies that ∇u is Hölder continuous in space. It implies also, that ∇u is bounded in $Q^{\lambda_0}_{R/2}$. Moreover, the previous implies

$$\max_{Q^{\lambda_0}_{R/2}} |\nabla u| \leq K < \infty$$

for $K = c\lambda_0 + cR^\gamma \|g\|^{\frac{1}{p-1}}_{L^\infty(I,C^{\gamma(p-1)}(B_R))}$.

In the following we prove Hölder continuity in time. I.e. we show for $(t,x) \in Q^{\lambda_0}_{R/4}$,

$$\Bigg(\fint_{t-s}^{t} \big|V(\nabla u)(\tau,x) - \langle V(\nabla u)(\tau,x)\rangle_{(t,t-s)}\big|^2 d\tau\Bigg)^{\frac{1}{p}} \leq K\Big(\frac{s}{S}\Big)^{\frac{\gamma}{2}} \tag{3.34}$$

for all $s \in (0,S)$, $S := K^{p-2}R^2$. From this estimate the Hölder continuety in time follows by (3.14) and the Campanato characterization of Hölder spaces.

In the following we prove (3.34). We take $(t,x) \in Q^{\lambda_0}_{R/4}$, fix $S(R) = K^{2-p}R$ and take $Q^K_{R/4}(t,x) \subset Q^{\lambda_0}_{R/2}$ as starting cylinder. Then for all $r < \frac{R}{4}$ we take $Q^{\lambda_r}_r(t,x)$ constructed by Lemma 3.6. We have that $\lambda_r \leq K$, (as $\tilde{\lambda} \leq K$ by (3.10)). Therefore Proposition 3.11 provides for all $r \in (0, \frac{R}{4}]$

$$\begin{aligned}\fint_{t-s(r)}^{t} \fint_{B_r(x)} &\big|V(\nabla u) - \langle V(\nabla u)\rangle_{Q^{\lambda_r}_r(z)}\big|^2 \\ &\leq c\Big(\frac{r}{R}\Big)^{p\gamma} K^p = c\Big(\frac{\lambda_r}{K}\Big)^{\frac{p-2}{2}}\Big(\frac{s(r)}{S(R)}\Big)^{\frac{\gamma p}{2}} K^p \leq c\Big(\frac{s(r)}{S(R)}\Big)^{\frac{\gamma p}{2}} K^p,\end{aligned} \tag{3.35}$$

as $\lambda_r \leq K$. Now we find by Lemma 3.6, (b), that $s(r) = \lambda_r^{2-p} r^2$ is continuous and $s(0) = 0$ and $s(R) = S(R)$. Therefore we can choose an $r(s)$ for every $0 < s \leq S$

such that $s = s(r) = \lambda_{r(s)}^{2-p} r^2(s)$. We estimate for $x \in B_{R/4}$ and s fixed

$$\begin{aligned}
&\fint_{t-s}^{t} \left|V(\nabla u)(\tau,x) - \langle V(\nabla u)(\tau,x)\rangle_{(t,t-s)}\right|^2 d\tau \\
&\quad \le c \fint_{t-s}^{t} \left|V(\nabla u)(\tau,x) - \langle V(\nabla u)\rangle_{Q_{r(s)}^{\lambda_{r(s)}}}\right|^2 d\tau \\
&\quad \le c \fint_{t-s}^{t} \left|V(\nabla u)(\tau,x) - \langle V(\nabla u)(\tau)\rangle_{B_{r(s)}(x)}\right|^2 d\tau \\
&\qquad + c \fint_{t-s}^{t} \left|\langle V(\nabla u)(\tau)\rangle_{B_{r(s)}(x)} - \langle V(\nabla u)\rangle_{Q_{r(s)}^{\lambda_{r(s)}}}\right|^2 d\tau = I + II.
\end{aligned}$$

I can be estimated by the $L^\infty(I_{R^2/4}, C^{1,\gamma}(B_{\frac{R}{2}}))$ estimate

$$I \le K^p \Big(\frac{r(s)}{R(S)}\Big)^{p\gamma} \le K^p \Big(\frac{s}{S}\Big)^{\frac{\gamma p}{2}}.$$

II can be estimated by (3.35)

$$II \le \fint_{t-s}^{t} \fint_{B_{r(s)}} \left|V(\nabla u) - \langle V(\nabla u)\rangle_{Q_{r(s)}^{\lambda_{r(s)}}}\right|^2 \le c\Big(\frac{s}{S}\Big)^{\frac{\gamma p}{2}} K^p,$$

where we used that $Q_{r(s)}^{\lambda_{r(s)}} = (t, t-s(r)) \times B_{r(s)}(x)$. This finishes the proof of (3.34). □

Remark 3.16. *The last result can be weakened. As long as the modulus of continuity is strong enough to imply the boundedness of $|\nabla u|$ we find the same natural estimates as in Proposition 3.15. We expect that the sharp bound would be the Dini continuity. I.e. f is Dini continuous on B_R if it's modulus of continuity ω holds $\sum_{i=1}^\infty \omega(2^{-i}R) < \infty$. We conjecture that in this case $BLO_\omega \equiv C^{1,\omega}$. If this would be true, then the Dini result of [40] could be gained similar to Proposition 3.15 with a weaker condition on g, i.e. $g \in L^\infty(I, C_{\omega(p-1)}(B))$, but restricted to (3.1) and $p \ge 2$.*

If we follow the estimates of [15, Corr. 5.4], we find directly, that $g \in L^\infty(I, VMO(B))$ implies that locally $u \in L^\infty(I, VLO(B))$.

3.4. Appendix

For $Q_1 \subset Q_2$ and $q \in [1,\infty)$ we find that

$$|\langle f\rangle_{Q_1} - \langle f\rangle_{Q_2}| \le \Big(\fint_{Q_1} |f - \langle f\rangle_{Q_2}|^q\Big)^{\frac{1}{q}} \le \Big(\frac{|Q_2|}{|Q_1|}\fint_{Q_2} |f - \langle f\rangle_{Q_2}|^q\Big)^{\frac{1}{q}}. \tag{3.36}$$

This estimate can be iterated for $i = \{0 \dots k\}$ and $Q_i \subset Q_{i-1}$ with $\frac{|Q_{i-1}|}{|Q_i|} \le c$

$$|\langle f\rangle_{Q_k} - \langle f\rangle_{Q_0}| \le \sum_{i=1}^{k} |\langle f\rangle_{Q_i} - \langle f\rangle_{Q_{1-i}}| \le c\sum_{i=1}^{k}\Big(\fint_{Q_{i-1}} |f - \langle f\rangle_{Q_{i-1}}|^q\Big)^{\frac{1}{q}}. \tag{3.37}$$

Lemma 3.17. *Let $Q_1 \subset Q$ be two Cylinders and $f \in L^q(Q)$ for $q \in [1,\infty)$. For $\varepsilon \in (0,1)$ we find:*

If $|\langle f\rangle_{Q_1}| \le \varepsilon\langle |f|^q\rangle_Q^{\frac{1}{q}}$, then

$$|\langle f\rangle_{Q_1}| \le \varepsilon\langle |f|^q\rangle_Q^{\frac{1}{q}} \le \frac{\varepsilon}{1-\varepsilon}\Big(1 + \Big(\frac{|Q|}{|Q_1|}\Big)^{\frac{1}{q}}\Big)\Big(\fint_Q |f - \langle f\rangle_Q|^q dx\Big)^{\frac{1}{q}}.$$

Proof. We find

$$\begin{aligned}\langle |f|^q\rangle_Q^{\frac{1}{q}} &\le \Big(\fint_Q |f - \langle f\rangle_{Q_1}|^q dx\Big)^{\frac{1}{q}} + |\langle f\rangle_{Q_1}| \\ &\le \Big(\fint_Q |f - \langle f\rangle_Q|^q dx\Big)^{\frac{1}{q}} + |\langle f\rangle_Q - \langle f\rangle_{Q_1}| + \varepsilon\langle |f|^q\rangle_Q^{\frac{1}{q}}.\end{aligned}$$

This implies that

$$\langle |f|^q\rangle_Q^{\frac{1}{q}} \le \frac{1}{1-\varepsilon}\Big(\fint_Q |f - \langle f\rangle_Q|^q dx\Big)^{\frac{1}{q}} + \frac{1}{1-\varepsilon}|\langle f\rangle_{Q_1} - \langle f\rangle_Q|.$$

We estimate the second integral by

$$\begin{aligned}|\langle f\rangle_{Q_1} - \langle f\rangle_Q| &\le \fint_{Q_1} |f - \langle f\rangle_Q| dx \le \Big(\fint_{Q_1} |f - \langle f\rangle_Q|^q dx\Big)^{\frac{1}{q}} \\ &\le \Big(\frac{|Q_1|}{|Q|}\fint_Q |f - \langle f\rangle_Q|^q dx\Big)^{\frac{1}{q}}.\end{aligned}$$

□

Lemma 3.18. *Let $f \in L^q(Q_R)$ with $q \in [1,\infty)$. Suppose that $\omega : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ is increasing and holds the following Dini condition: $\sum_i^\infty \omega(2^{-i}R) \le K$ (e.g. $\omega(r) = r^\gamma$). If*

$$\Big(\fint_{\theta B} |f - \langle f\rangle_{\theta B}|^q\Big)^{\frac{1}{q}} \le c_1\omega(\theta)\Big(\fint_B |f - \langle f\rangle_B|^q\Big)^{\frac{1}{q}},$$

then

$$osc_{\theta Q_\rho}(f) \le cK\omega(\theta)\Big(\fint_{Q_\rho} |f - \langle f\rangle_{Q_R}|^q\Big)^{\frac{1}{q}},$$

for all $\theta \in (0,\frac{1}{2})$, $\rho \le R$ and c depending only on q, n, c_1.

Proof. We only proof the first statement. For $k \in \mathbb{N}$ we define for $z \in \frac{1}{2}\theta Q_\rho$ we define $Q_i(z) := 2^{-i}\frac{1}{2}Q_{\theta\rho}(z)$ for $i = 1, ..., k$ and $Q_0(z) = \theta Q_\rho$. We estimate by (3.37)

$$|\langle f\rangle_{Q_k(z)} - \langle f\rangle_{\theta Q_\rho}| \leq \sum_{i=0}^{k-1} \Bigg(\fint_{Q_i(z)} |f - \langle f\rangle_{Q_i(z)}|^q \Bigg)^{\frac{1}{q}}$$

this can be estimated by assumption and because ω is increasing

$$\begin{aligned} |\langle f\rangle_{Q_k(z)} - \langle f\rangle_{\theta Q_\rho}| &\leq c\sum_{i=1}^{k-1} \omega(2^{-i}\theta\rho)\Bigg(\fint_{\theta Q_\rho} |f - \langle f\rangle_{Q_{\theta\rho}}|^q \Bigg)^{\frac{1}{q}} \\ &\leq cK\omega(\theta)\Bigg(\fint_{Q_\rho} |f - \langle f\rangle_{Q_\rho}|^q \Bigg)^{\frac{1}{q}}; \end{aligned}$$

the constant is independent of k; this implies that

$$|f(z) - \langle f\rangle_{\theta Q_\rho}| \leq cK\omega(\theta)\Bigg(\fint_{Q_\rho} |f - \langle f\rangle_{Q_\rho}|^q \Bigg)^{\frac{1}{q}}.$$

Consequently, we find for $z, w \in \frac{1}{2}\theta Q_\rho$

$$|f(z) - f(w)| \leq cK\omega(\theta)\Bigg(\fint_{Q_\rho} |f - \langle f\rangle_{Q_\rho}|^q \Bigg)^{\frac{1}{q}}.$$

$\square$

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