String Field Theory: Algebraic Structure, Deformation Properties and Superstrings

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Zusammenfassung

Diese Arbeit diskutiert verschiedene Aspekte der String-Feldtheorie. Zuerst befassen wir uns mit bosonischer String-Feldtheorie offener und geschlossener Strings und dessen algebraischer Struktur – die Quanten Homotopie-Algebra offener und geschlossener Strings. Die Quanten Homotopie-Algebra offener und geschlossener Strings lässt sich im Rahmen von involutiven Homotopie-Lie-Bialgebren als Morphismus von der Algebra der geschlossenen Strings zur involutiven Lie-Bialgebra auf dem Hochschild Komplex der offenen Strings formulieren. Die Beschreibung der klassischen/Quanten Homotopie-Algebra offener und geschlossener Strings als Morphismus offenbart Deformationseigenschaften von geschlossenen Strings auf offene String-Feldtheorie. Insbesondere zeigen wir, dass inäquivalente klassische offene String-Feldtheorien durch Lösungen der Feldgleichungen des geschlossenen Strings modulo Eichäquivalenz parametrisiert sind. Für die volle Quanten Theorie ist die Korrespondenz leer, jedoch existiert eine nicht triviale Korrespondenz für den topologischen String. Zudem beweisen wir das Zerlegungs-Theorem im Zusammenhang mit der Schleifen-Homotopie-Lie-Algebra geschlossener Strings, welches Eindeutigkeit der String-Feldtheorie geschlossener Strings auf einem vorgegebenen konformen Hintergrund impliziert.

Ein weiterer Inhaltspunkt ist die Erkenntnis, dass die Konstruktion einer String-Feldtheorie im allgemeinen durch Operaden beschrieben werden kann. Die Konstruktion einer String-Feldtheorie setzt sich aus zwei Teilen zusammen: Die Bedingung dass die pertubativen Streuamplituden korrekt reproduziert werden erfordert eine Zerlegung des Modulraumes der Weltflächen. Um nun die Vertices der String-Feldtheorie zu definieren benötigt man zusätzlich einen konformen Hintergrund. Jeder dieser Teile kann äquivalent als Morphismus zwischen geeigneten zyklischen/modularen Operaden interpretiert werden. Die algebraische Struktur der String-Feldtheorie wird durch die Komposition der beiden Morphismen beschrieben.

Zuletzt skizzieren wir die Konstruktion der Typ II Superstring-Feldtheorie. Spezifische Merkmale des Superstrings sind das Auftreten von Ramond Punktierungen und Darstellungswechsel Operatoren. Das zusammennähen von Ramond Punktierungen erfordert eine zusätzliche Einschränkung auf dem Zustandsraum der konformen Feldtheorie, so dass die zugehörige symplektische Form nicht entartet ist. Zudem formulieren wir ein geeignetes Extremalprobelm für Metriken auf Typ II Weltflächen, welches die Konstruktion einer kon-

sistenten Zerlegung des Modulraumes in Vertices und Graphen ermöglicht. Die algebraische Struktur der Typ II Superstring-Feldtheorie ist die einer $\mathcal{N} = 1$ Schleifen-Homotopie-Lie-Algebra im Quanten Fall, und die einer $\mathcal{N} = 1$ Homotopie-Lie-Algebra im klassischen Fall.

Abstract

This thesis discusses several aspects of string field theory. The first issue is bosonic open-closed string field theory and its associated algebraic structure – the quantum open-closed homotopy algebra. We describe the quantum open-closed homotopy algebra in the framework of homotopy involutive Lie bialgebras, as a morphism from the loop homotopy Lie algebra of closed string to the involutive Lie bialgebra on the Hochschild complex of open strings. The formulation of the classical/quantum open-closed homotopy algebra in terms of a morphism from the closed string algebra to the open string Hochschild complex reveals deformation properties of closed strings on open string field theory. In particular, we show that inequivalent classical open string field theories are parametrized by closed string backgrounds up to gauge transformations. At the quantum level the correspondence is obstructed, but for other realizations such as the topological string, a non-trivial correspondence persists. Furthermore, we proof the decomposition theorem for the loop homotopy Lie algebra of closed string field theory, which implies uniqueness of closed string field theory on a fixed conformal background.

Second, the construction of string field theory can be rephrased in terms of operads. In particular, we show that the formulation of string field theory splits into two parts: The first part is based solely on the moduli space of world sheets and ensures that the perturbative string amplitudes are recovered via Feynman rules. The second part requires a choice of background and determines the real string field theory vertices. Each of these parts can be described equivalently as a morphism between appropriate cyclic and modular operads, at the classical and quantum level respectively. The algebraic structure of string field theory is then encoded in the composition of these two morphisms.

Finally, we outline the construction of type II superstring field theory. Specific features of the superstring are the appearance of Ramond punctures and the picture changing operators. The sewing in the Ramond sector requires an additional constraint on the state space of the world sheet conformal field theory, such that the associated symplectic structure is non-degenerate, at least on-shell. Moreover, we formulate an appropriate minimal area metric problem for type II world sheets, which can be utilized to sketch the construction of a consistent set of geometric vertices. The algebraic structure of type II superstring field theory is that of a $\mathcal{N} = 1$ loop homotopy Lie algebra at the quantum level, and that of a $\mathcal{N} = 1$ homotopy Lie algebra at the classical level.

Chapter 1

Introduction

One of the most successful principles in fundamental physics is the gauge principle. In simplified terms, it can be stated as follows: Consider a physical system that admits a description in terms of fields, such that the physics is invariant under a continuous group of *local* transformations. The requirement of invariance under local transformations is then generically strong enough to determine the action for the corresponding fields almost uniquely. The most prominent example is electromagnetism, whose formulation in terms of gauge potentials rather than the electric and magnetic field itself reveals a local $U(1)$ symmetry. Moreover, the physics of elementary particles, as described by the standard model, is encoded in a unified theory based on a local $SU(3) \times SU(2) \times U(1)$ symmetry in the unbroken phase.

Despite the huge experimental evidence, including the recent discovery of the Higgs boson at the LHC, the standard model is believed to be inconsistent at very high energies. Open problems like the hierarchy problem, the mystery about dark matter and dark energy, the strong CP-problem and the matter-antimatter asymmetry in our universe demand for physics beyond the standard model. A plausible candidate that resolves the hierarchy problem and might also elucidate the nature of dark matter is supersymmetry – a symmetry between bosons and fermions. But from a conceptual point of view, there is another problem which is the apparent incompatibility of gravity and quantum mechanics. Thus, a theory that claims to be fundamental has to resolve the known problems of the standard model and provide a description for quantum gravity.

String theory is a promising candidate for the unification of gravity and particle physics consistent with the laws of quantum mechanics. At the classical level, strings are one dimensional objects – in contrast to the point particle approach of quantum field theory – which propagate in space-time. Topologically, there are two types of strings: The open string which has two endpoints and the closed string without endpoints. Upon quantization, the excitations of the string determine the particle spectrum. The massless particles are of major interest, since massive particles are too heavy to be experimentally accessible.

The open string spectrum contains a massless spin one particle – the gauge field – and the closed string spectrum contains a spin two particle – the graviton.

The bosonic string lacks of fermions, which are indispensable in particle physics, thus a generalization is required to treat this deficiency. Again, supersymmetry provides the answer to this shortcoming. Furthermore, in the supersymmetric extension of the bosonic string – the superstring – the tachyonic degree of freedom present in the bosonic string is absent. There turn out to be five realizations of the superstring, all of them requiring a 10-dimensional space-time. However, the various superstring theories are related by an intriguing web of dualities. Some dualities are of geometric nature, that is they relate a superstring theory defined on a given geometry to another superstring theory with the geometry of the former modified in a specific way. The simplest example is T-duality, which relates e.g. type IIA superstring theory compactified on a circle of radius R to type IIB superstring theory compactified on a circle of radius $1/R$. A more intricate and mathematically attractive duality is mirror symmetry, especially in the context of open strings and D-branes which is termed homological mirror symmetry [1]: It expresses a duality between string theories on Calabi-Yau manifolds and their mirror duals. On the other hand, some dualities relate a strongly coupled regime of one theory to a weakly coupled regime of another theory – the so called S-dualities. The most prominent example is the electro-magnetic duality of $\mathcal{N} = 2$ supersymmetric Yang-Mills theory, discovered in the seminal work of Seiberg and Witten [2, 3]. T- and S-dualities suggest that the various superstring theories describe different regimes of one and the same fundamental theory, but the picture does not complete before the introduction of a hypothetical 11-dimensional theory, called M-theory, whose low energy effective action is the unique 11-dimensional supergravity.

Finally, the holographic principle, an idea brought forth by t'Hooft [4], claims that the information inside some region of space-time can be represented as a hologram on the boundary of the region. Its physical implications are of particular interest for addressing conceptual problems of black hole physics. The most definite realization of the holographic principle has been formulated in the framework of superstring theory: Type IIB string theory on $AdS_5 \times S_5$ is conjectured to be dual to $\mathcal{N} = 4$ Yang-Mills gauge theory on the 4-dimensional boundary of AdS_5 [5]. Since $\mathcal{N} = 4$ Yang-Mills gauge theory is a conformal field theory, this duality is referred to as AdS/CFT duality, and its applications range from quantum chromodynamics to condensed matter physics.

Nevertheless, the current status of string theory as a fundamental theory of nature is quite unsatisfactory, for the following reasons: The formulation of string theory requires a choice of background, i.e. a choice of space-time on which the string propagates. Six out of the ten dimensions of space-time have to be compact and very tiny, in order to not contradict our everyday observation of four space-time dimensions. In contrast to general relativity, the background is not determined by the theory itself but is rather an auxiliary prescribed object. Certainly, there are consistency conditions for the background geometry like Ricci flatness, and the phenomenologically motivated constraint of $\mathcal{N} = 1$ supersymmetry in the residual four dimensional space-time, leading to Calabi-Yau manifolds. But still there is a huge number of geometries satisfying these requirements, which is even enhanced upon taking brane world scenarios into account, and so far string theory does not help to distinguish one of these. Since the explicit shape of the background determines the particle spectrum, the background independence issue is of outstanding importance. One way to address this problem is to seek for a background which accommodates the standard model by brute force, but a conceptually more appealing approach would be to distinguish a background by some physical principal. If there is no such physical principle, string theory does not seem to have predictive power at all. On the other hand, the theory of inflation assumes that our universe represents a bubble that nucleated in an ambient space-time which expands exponentially due to the presence of a vacuum energy density with negative pressure. Consequently, there might be several bubbles, each representing a universe on its own, and the assignment of a background to a bubble might be purely probabilistic. This multiverse scenario would strongly favour the anthropic principle, but anyhow, it does not help to determine the physical laws in our universe.

Furthermore, we do not yet sufficiently understand the underlying symmetries of string theory in order to determine an action principle. String theory merely provides a pattern to calculate scattering amplitudes. Due to the lack of a 'gauge principle' for string theory, one has to pursue a different strategy in order to construct a string field theory. The common approaches towards a second quantization of string theory are based on two fundamental requirements: First, the action of string field theory has to be designed such that one recovers the perturbative scattering amplitudes via Feynman rules. Second, background independence has to be incorporated. While the first requirement is manifestly satisfied due to the construction of string field theory as described in [6, 7], the issue of background independence is much more subtle.

The requirement of recovering the perturbative scattering amplitudes via Feynman rules amounts to a decomposition of the moduli space of Riemann surfaces into elementary vertices and graphs. Generically, we have to define an elementary vertex for every topological type of Riemann surfaces. For example, in closed string field theory the vertices are labeled by the genus and the number of punctures. Thus, every Riemann surface is either part of the subspace of the moduli space which represents an elementary vertex, or it can be constructed uniquely by sewing together Riemann surfaces of other elementary vertices along their punctures. We call the vertices of the moduli space the geometric vertices. The decomposition of the moduli space implies that the geometric vertices satisfy a certain Batalin-Vilkovisky (BV) master equation. The second ingredient of string field theory is a choice of background which determines a world sheet conformal field theory (CFT). The image of the geometric vertices under the CFT are called the algebraic vertices. The CFT preserves the BV structure and hence the algebraic vertices satisfy a BV master equation as well. Satisfying the BV master equation is just the requirement which guarantees that the theory can be quantized consistently. Besides that, the BV master equation also encodes the algebraic structure the theory entails. In section 2, we will review the construction of string field theory which originates in the moduli space of world sheets, and in section 3 we conclude with a discussion on background independence.

As alluded in the previous paragraph, the construction of string field theory naturally leads to a BV master equation on the moduli space of world sheets, and another BV master equation on the state space of the world sheet conformal field theory which represents the background. We will show in section 4, that satisfying the BV master equation on the state space of the CFT is equivalent to the axioms of some homotopy algebra. This correspondence is established in the framework of operads [8]. Furthermore, we give a short account on operads and the generic properties of homotopy algebras, with particular focus on the physical interpretation for not just string field theory but also for field theory in general.

In section 5, we discuss the homotopy algebra of open-cosed bosonic string field theory. There we describe the extension of the classical open-closed homotopy algebra of [9] to the full quantum level. The formulation of the classical open-closed algebra reveals a relation between closed string backgrounds and open string field theories. At the quantum level, this correspondence is in general obstructed, but in other realizations of the open-closed homotopy algebra, as e.g. in the topological string, there is still a non-trivial correspondence.

Finally, in section 6 we describe the adjustments which are necessary to apply the concepts developed in bosonic string field theory to type II superstring field theory. The main difficulty is the appearance of picture changing operators. The sewing of punctures in the Ramond sector inevitably generates a picture changing operator associated with the odd vector field that generates translations in the Ramond divisor. This fact requires a specific restriction of the state space of the world sheet superconformal field theory. The homotopy algebra of type II superstring field theory is the supersymmetric extension of a loop homotopy Lie algebra, and at the classical level it is the supersymmetric generalization of a homotopy Lie algebra.

Chapter 2

Geometric Approach to String Field Theory

This section is intended to illustrate the construction of string field theory on the basis of bosonic closed strings, following [6]. The primary objective of this approach is to guarantee that the vertices of the string field theory action produce the correct scattering amplitudes. Scattering amplitudes in string perturbation theory are defined by integrating an appropriate measure – which we will discuss below – over the space of inequivalent world sheets, i.e. the moduli space of closed Riemann surfaces. The requirement of reproducing the perturbative scattering amplitudes can be traced back to the moduli space itself: Vertices naturally represent subspaces of the full moduli space and propagators are defined by sewing together punctures along prescribed *coordinate curves*, such that the associated Feynman graphs constitute a single cover of the compactified moduli space.

Let $\mathcal{M}_{g,n}$ be the moduli space of genus g Riemann surfaces with n punctures. For a Riemann surface Σ, a coordinate curve is an embedded submanifold $S^1 \subset \Sigma$ homotopic to a puncture, i.e. a closed non-intersecting curve encircling a single puncture. The moduli space of Riemann surfaces together with a coordinate curve for every puncture is denoted by $\hat{\mathcal{P}}_{g,n}$. A coordinate curve determines a *local coordinate* system z, where the location of the puncture corresponds to $z = 0$ and the points of S^1 correspond to $|z| = 1$, up to rotations. Rotations are generated by $l_0 - \tilde{l}_0$, where $l_n = -z^{n+1}\partial_z$ denotes the Witt algebra. The moduli space of Riemann surfaces decorated with local coordinates is denoted by $\mathcal{P}_{g,n}$. Given two punctures p_1 and p_2 together with local coordinates z_1 and z_2, the sewing is described by identifying points according to

$$I(z_1) := -\frac{1}{z_1} = z_2 \, , \tag{2.1}$$

and similarly for the antiholomorphic sector. In figure 2.1, we illustrate the sewing operation of equation (2.1).

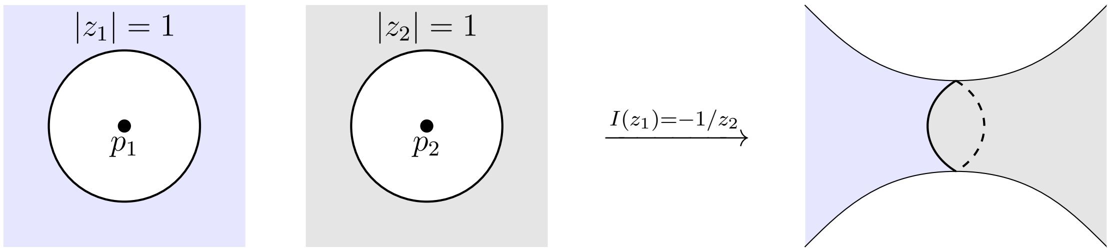


Figure 2.1: The sewing of two closed string punctures p_1 and p_2 with prescribed local coordinates.

It turns out that it is impossible to assign local coordinates around the punctures globally over moduli space in a continuous fashion, or in other words, $\mathcal{P}_{g,n}$ as a fibre bundle over $\mathcal{M}_{g,n}$ does not admit global sections [6]. In contrast, an appropriate minimal area problem for Riemann surfaces leads to the description of a global section for $\hat{\mathcal{P}}_{g,n}$ as a fibre bundle over $\mathcal{M}_{g,n}$, which we will briefly discuss below. A global section on moduli space is indispensable for the construction of string field theory, and thus we henceforth focus on $\hat{\mathcal{P}}_{g,n}$. The ambiguity of determining local coordinates from coordinate curves is parametrized by an angle $\vartheta \in [0, 2\pi]$, representing all possible rotations. Thus the sewing of punctures with prescribed coordinate curves naturally generates a 1-parameter family of Riemann surfaces associated with the twist angle ϑ. Explicitly, the sewing map reads

$$\Phi_\vartheta = (I \circ \varphi^{l_0}_{i\vartheta}, \tilde{I} \circ \tilde{\varphi}^{\tilde{l}_0}_{-i\vartheta}) \ , \tag{2.2}$$

where φ^{l_0} denotes the flow generated by l_0, and the tilde indicates the antiholomorphic sector.

Consider now the singular chain complex $C^\bullet(\hat{\mathcal{P}}_{g,n})$. The grading of the chains is defined by codimension, i.e.

$$\deg(\mathcal{A}_{g,n}) = \dim(\mathcal{M}_{g,n}) - \dim(\mathcal{A}_{g,n}) \ ,$$

where $\mathcal{A}_{g,n} \in C^{\deg(\mathcal{A}_{g,n})}(\hat{\mathcal{P}}_{g,n})$. Note that this choice of grading makes the boundary operator ∂ a degree one operator. Furthermore we endow the chains with an orientation. We denote the sewing operation induced on $C^\bullet(\hat{\mathcal{P}}_{g,n})$ by ${}_i\overset{\Phi}{\circ}_j$ and $\overset{\Phi}{\xi}_{ij}$ in the separating (the two punctures reside on two disconnected components) and non-separating (the two punctures reside on one connected component) case respectively, where i and j denote the corresponding punctures. The sewing operation in the separating and non-separating case is depicted in figure 2.2 and 2.3, respectively.

Due to the choice of grading, both ${}_i\overset{\Phi}{\circ}_j$ and $\overset{\Phi}{\xi}_{ij}$ are of degree one:

$${}_i\overset{\Phi}{\circ}_j : C^{k_1}(\hat{\mathcal{P}}_{g_1,n_1+1}) \times C^{k_2}(\hat{\mathcal{P}}_{g_2,n_2+1}) \to C^{k_1+k_2+1}(\hat{\mathcal{P}}_{g_1+g_2,n_1+n_2}) \ , \tag{2.3}$$

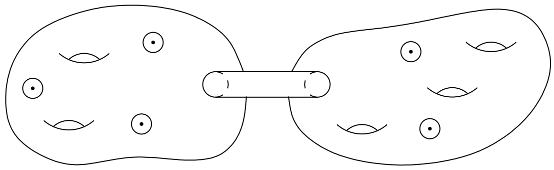


Figure 2.2: Sewing operation in the separating case.

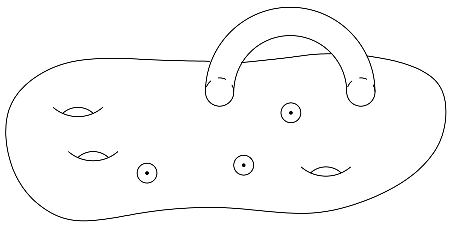


Figure 2.3: Sewing operation in the non-separating case.

$$\overset{\Phi}{\xi}_{ij} : C^k(\hat{\mathcal{P}}_{g,n+2}) \to C^{k+1}(\hat{\mathcal{P}}_{g+1,n}) \,. \tag{2.4}$$

As alluded in the beginning of this section, vertices represent subspaces of the moduli space, but we have to implement the indistinguishability of identical particles already at the geometric level by requiring invariance under permutations of punctures. The chain complex invariant under permutations of punctures is denoted by $C^\bullet_{\text{inv}}(\hat{\mathcal{P}}_{g,n})$. Now it can be shown that lifting ${}_i\overset{\Phi}{\circ}_j$ and $\overset{\Phi}{\xi}_{ij}$ to maps on $C^\bullet_{\text{inv}}(\hat{\mathcal{P}}_{g,n})$ induces the structure of a BV algebra [6]:

$$\Delta^{\text{geo}}\mathcal{B}_{g,n+2} := \overset{\Phi}{\xi}_{ij}\, \mathcal{B}_{g,n+2} \tag{2.5}$$

$$(\mathcal{B}_{g_1,n_1+1}, \mathcal{B}_{g_2,n_2+1})^{\text{geo}} := \sum_{\sigma\in \text{Sh}(n_1,n_2)} \sigma.\big(\mathcal{B}_{g_1,n_1+1}\ {}_i\overset{\Phi}{\circ}_j\, \mathcal{B}_{g_2,n_2+1}\big) \,, \tag{2.6}$$

where $\mathcal{B}_{g,n+2} \in C^\bullet_{\text{inv}}(\hat{\mathcal{P}}_{g,n+2})$ and $\mathcal{B}_{g_i,n_i} \in C^\bullet_{\text{inv}}(\hat{\mathcal{P}}_{g_i,n_i})$. In equation (2.6), $\text{Sh}(n_1, n_2) \subset \Sigma_{n_1+n_2}$ denotes the set of shuffles, i.e. the set of permutations constraint to $\sigma_1 < \cdots < \sigma_{n_1}$ and

$\sigma_{n_1+1} < \cdots < \sigma_{n_1+n_2}$. The axioms of a BV algebra read[1]

$$\begin{aligned} \partial^2 &= 0 \\ \Delta^2 &= 0 \\ \partial\Delta + \Delta\partial &= 0 \\ \partial \circ (\cdot,\cdot) &= (\partial,\cdot) - (\cdot,\partial) \\ \Delta \circ (\cdot,\cdot) &= (\Delta,\cdot) - (\cdot,\Delta) \\ (a,b) &= -(-1)^{(|a|+1)(|b|+1)}(b,a) \\ (-1)^{(|a|+1)(|c|+1)}((a,b),c)) + \text{cycl.} &= 0\,. \end{aligned} \tag{2.7}$$

For example, the property $(\Delta^{\text{geo}})^2 = 0$ follows from the fact that the sewing increases dimensionality by one due to the twist angle, and that the chains are endowed with an orientation.

The map of equation (2.2) describes the sewing of punctures with an intermediate cylinder of length zero. The propagator is defined by sewing in cylinders of arbitrary length $x \in [0,\infty)$, i.e. we identify points w.r.t.

$$P_{x,\vartheta} = (I \circ \varphi^{l_0}_{-x+i\vartheta}, \tilde{I} \circ \tilde{\varphi}^{\tilde{l}_0}_{-x-i\vartheta})\,, \tag{2.8}$$

and consequently generate a 2-parameter family of surfaces labeled by x and ϑ. Now the fundamental consistency condition for a collection of *geometric vertices* $\mathcal{V}_{g,n} \in C^0_{\text{inv}}(\hat{\mathcal{P}}_{g,n})$ reads,

$$\overline{\mathcal{M}}_{g,n} = \pi\big(\mathcal{V}_{g,n} \sqcup R^1_{g,n} \sqcup \cdots \sqcup R^{3g-3+n}_{g,n}\big)\,, \tag{2.9}$$

where $R^i_{g,n}$ denotes the collection of genus g graphs[2] with n legs, constructed from $\{\mathcal{V}_{g',n'}\}_{g',n'}$ and involving exactly i propagators. $3g-3+n$ is the maximal number of propagators, corresponding to the case where only $n=3$, $g=0$ vertices are involved, and $\pi : \hat{\mathcal{P}}_{g,n} \to \mathcal{M}_{g,n}$ denotes the projection map of the fibre bundle $\hat{\mathcal{P}}_{g,n}$. The right hand side comprises two types of boundaries: One which describes the boundary of the geometric vertices, and another which corresponds to the limit of infinitely short propagators $x \to 0$. Since the compactified moduli space on the the left hand side of equation (2.9) has no boundary, we conclude that these two boundary contributions have to cancel, or equivalently that the BV master equation

$$\partial\mathcal{V}_{g,n} + \sum_{\substack{n_1+n_2=n \\ g_1+g_2=g}} \Delta^{\text{geo}}\mathcal{V}_{g-1,n+2} + \frac{1}{2}(\mathcal{V}_{g_1,n_1+1}, \mathcal{V}_{g_2,n_2+1})^{\text{geo}} = 0 \tag{2.10}$$

[1] Indeed, the definition of a BV algebra includes a commutative multiplication, such that Δ is a second order derivation and ∂ is a first order derivation. Such a structure can be introduced on the chain complex of moduli spaces by disjoint union [6].

[2] The genus of a graph is the sum of the genera of the vertices plus the first Betti number of the graph.

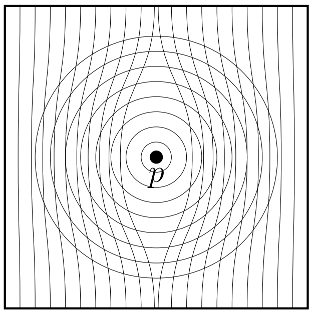


Figure 2.4: A torus together with a puncture p. The minimal area metric determines two bands of saturating geodesics which cover the surface completely. Opposite (thick) lines on the boundary of the square are identified.

is satisfied.

The main task now is to determine a set of geometric vertices. The appropriate tool is the concept of minimal area metrics: Given a Riemann surface Σ, we ask for the metric of least possible area under the constraint that there is no non-trivial closed curve which is shorter than 2π. Assuming the existence of minimal area metrics, one can easily show their uniqueness. A minimal area metric gives rise to *bands of saturating geodesics*: A saturating geodesic is a closed curve of length equal to 2π. Furthermore, saturating geodesics of the same homotopy type never intersect. The collection of all saturating geodesics of a certain homotopy type foliates a part of the surface, and is called a band of saturating geodesics. The unity of all bands of saturating geodesics covers the surface completely, but in general bands of saturating geodesics might intersect. We distinguish external and internal bands of saturating geodesics, by whether the geodesics are homotopic to a puncture or not. External bands of saturating geodesics have the topology of a semi-infinite cylinder, bounded by a geodesic from where the band extends infinitely towards the puncture. An internal band of saturating geodesics is topologically a finite cylinder and its height is defined to be the length of the shortest path between its two boundary components. The saturating geodesics can be interpreted as representing the closed string itself. We examplify the concept of bands of saturating geodesics in figure 2.4.

With the aid of saturating geodesics, there is a simple and intuitive way to define global sections of $\hat{\mathcal{P}}_{g,n}$ over $\mathcal{M}_{g,n}$: Consider a closed Riemann surface Σ equipped with its unique minimal area metric. For every external band of saturating geodesics, we define the coordinate curve to be the saturating geodesic a distance l separated from the bounding saturating geodesic. We denote the corresponding section by

$$\sigma^l : \mathcal{M}_{g,n} \to \hat{\mathcal{P}}_{g,n} \ ,$$

and the smallest possible value for l is π, in order to avoid the occurrence of closed curves

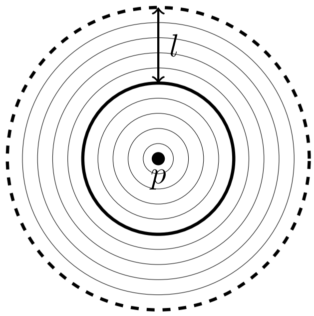


Figure 2.5: Saturating geodesics homotopic to a puncture p and definition of coordinate curves. The thick dashed curve represents the bounding saturating geodesic and the coordinate curve is defined to be the thick curve which is a distance l separated from the bounding saturating geodesic.

shorter than 2π through sewing. In figure 2.5, we visualize the construction of coordinate curves from minimal area metrics.

Finally, we define a one parameter family of geometric vertices $\mathcal{V}^l_{g,n}$ for $l \geq \pi$: A surface $\Sigma_{g,n} \in \mathcal{M}_{g,n}$ is part of $\mathcal{V}^l_{g,n}$, if there are no internal bands of saturating geodesics of height larger than $2l$. These vertices manifestly satisfy equation (2.9) and consequently also the BV master equation (2.10). The usual choice of geometric vertices used in closed string field theory corresponds to $l = \pi$, which represents the smallest possible subset of the moduli space satisfying the fundamental requirement of reproducing a single cover of moduli space. On the other hand, the limit $l \to \infty$ describes the Deligne-Mumford compactification.

The previous part of the construction of string field theory, formulated on the chain complex of moduli spaces, is manifestly background independent. A background refers to a choice of space-time. The Polyakov action on a given space-time defines a conformal field theory, and is invariant under Weyl transformations and world sheet reparametrizations. Quantization of the theory requires a gauge fixing procedure, which leads to Faddeev-Popov ghosts – the bc ghost system. The c and b ghost carries ghost number one and minus one, respectively. In general, after gauge-fixing a local symmetry, there remains an associated global symmetry – the BRST symmetry. This remnant symmetry manifests itself by the existence of a ghost number one operator Q that squares to zero – the BRST differential. In string theory, physical states correspond to cohomology classes of Q.

To every Riemann surface $\Sigma_{g,n} \in \mathcal{P}_{g,n}$ decorated with local coordinates, the combined CFT of the matter and the ghost sector assigns a multilinear map

$$Z(\Sigma_{g,n}) \in \mathrm{Hom}(\mathcal{H}^{\otimes n}, \mathbb{C}) ,$$

where $\mathcal{H}$ denotes the state space of the CFT. We denote the local operator corresponding to a state $\phi \in \mathcal{H}$ by $\mathcal{O}_\phi$. The bpz inner product of two states ϕ_1 and ϕ_2 is defined in terms of the sewing map of equation (2.1) by

$$\mathrm{bpz}(\phi_1, \phi_2) := \lim_{|z|\to 0} \Big\langle \big((I^*, \tilde{I}^*)\mathcal{O}_{\phi_1}\big)(z, \tilde{z})\, \mathcal{O}_{\phi_2}(z, \tilde{z})\Big\rangle \ . \tag{2.11}$$

Similar to the case of coordinate curves, we denote the map which sews puncture i with puncture j along prescribed local coordinates by ${}_i\overset{I}{\circ}_j$ and $\overset{I}{\xi}_{ij}$ in the separating and non-separating case, respectively:

$${}_i\overset{I}{\circ}_j : \mathcal{P}_{g_1,n_1+1} \times \mathcal{P}_{g_2,n_2+1} \to \mathcal{P}_{g_1+g_2,n_1+n_2} \ , \tag{2.12}$$

$$\overset{I}{\xi}_{ij} : \mathcal{P}_{g,n+2} \to \mathcal{P}_{g+1,n} \ . \tag{2.13}$$

Furthermore, we define

$${}_i\overset{\mathrm{bpz}}{\circ}_j : \mathrm{Hom}(\mathcal{H}^{\otimes n_1+1}, \mathbb{C}) \times \mathrm{Hom}(\mathcal{H}^{\otimes n_2+1}, \mathbb{C}) \to \mathrm{Hom}(\mathcal{H}^{\otimes n_1+n_2}, \mathbb{C})$$

and

$$\overset{\mathrm{bpz}}{\xi}_{ij} : \mathrm{Hom}(\mathcal{H}^{\otimes n+2}, \mathbb{C}) \to \mathrm{Hom}(\mathcal{H}^{\otimes n}, \mathbb{C})$$

to be the maps that contract inputs i and j w.r.t. the inverse of the bpz inner product.

The CFT satisfies the factorization properties

$$Z(\Sigma_{g_1,n_1+1}\, {}_i\overset{I}{\circ}_j \Sigma_{g_2,n_2+1}) = Z(\Sigma_{g_1,n_1+1})\, {}_i\overset{\mathrm{bpz}}{\circ}_j Z(\Sigma_{g_2,n_2+1}) \tag{2.14}$$

and

$$Z(\overset{I}{\xi}_{ij} \Sigma_{g,n+2}) = \overset{\mathrm{bpz}}{\xi}_{ij} Z(\Sigma_{g,n+2}) \ . \tag{2.15}$$

The concept of Schiffer variation allows to represent a tangent vector of $\mathcal{P}_{g,n}$ by a collection of n Witt vectors. The idea is to cut out a disc around a puncture, deforming it by the flow generated by the Witt vector and finally to sew it back in. The relation between a tangent vector $V \in T_{\Sigma_{g,n}}\mathcal{P}_{g,n}$ and the associated collection of Witt vectors $v^{(i)}$, $i \in \{1, \dots n\}$, is expressed by

$$V(Z) = Z \circ T(\vec{v}) \ ,$$

where $T(\vec{v}) = \sum_{i=1}^n T^{(i)}(v^{(i)})$, and $T(v)$ is determined by

$$T(l_n) := L_n$$

and linearity.

Finally, Z is BRST closed and defines a morphism of Lie algebras, i.e.

$$[V_1, V_2](Z) = Z \circ T([\vec{v}_1, \vec{v}_2])$$

and

$$Z(\Sigma) \circ \sum_{i=1}^{n} Q^{(i)} = 0 \, .$$

The presence of the b ghost enhances the CFT to what is called a TCFT in the mathematical literature: With the aid of the b ghost we can construct differential forms on moduli space with values in the space of multilinear maps of $\mathcal{H}$. Let $(V_1, \ldots, V_r)$ be a collection of tangent vectors to $\mathcal{P}_{g,n}$ at $\Sigma_{g,n}$. We define

$$\omega^{k}_{g,n}(V_1, \ldots, V_r) = N_{g,n} \cdot Z(\Sigma_{g,n}) \circ b(\vec{v}_1) \ldots b(\vec{v}_r) \tag{2.16}$$

In equation (2.16), k is related to r by $r = \dim(\mathcal{M}_{g,n}) - k$ in agreement with the grading introduced for the chain complex of moduli spaces, $b(\vec{v})$ is defined in analogy to $T(\vec{v})$ by $b(l_n) = b_n$ and $N_{g,n} = (2\pi i)^{-(3g-3+n)}$ is a normalization constant whose necessity will be elucidated below.

It can be shown that $Z(\Sigma)$ carries ghost number $6g - 6$, which implies that the ghost number of $\omega^{k}_{g,n}$ is $k - 2n$. Let Σ_n be the permutation group of n elements. The differential forms define Σ_n equivariant maps and satisfy the chain map property

$$d\omega^{k+1}_{g,n} = (-1)^{k} \omega^{k}_{g,n} \circ \sum_{i=1}^{n} Q^{(i)} \, . \tag{2.17}$$

As discussed above, we are forced to base the construction of string field theory on $\hat{\mathcal{P}}_{g,n}$ rather than $\mathcal{P}_{g,n}$, due to the absence of a global section on $\mathcal{P}_{g,n}$. The question now is, which modifications are necessary in order to pull the previously introduced structure defined for $\mathcal{P}_{g,n}$ back to $\hat{\mathcal{P}}_{g,n}$. It will turn out that a restriction of the state space is inevitable. The restrictions derive from requiring factorization properties analogous to those of equations (2.14) and (2.15):

$$\int\limits_{\mathcal{A}_{g_1,n_1+1} \; {}_i\overset{\Phi}{\circ}_j \; \mathcal{A}_{g_2,n_2+1}} \omega^{k_1+k_2+1}_{g_1+g_2,n_1+n_2} = \left(\int\limits_{\mathcal{A}_{g_1,n_1+1}} \omega^{k_1}_{g_1,n_1+1} \right) {}_i\overset{\omega}{\circ}_j \left(\int\limits_{\mathcal{A}_{g_2,n_2+1}} \omega^{k_2}_{g_2,n_2+1} \right) , \tag{2.18}$$

$$\int\limits_{\overset{\Phi}{\xi}_{ij} \, \mathcal{A}_{g-1,n+2}} \omega^{k+1}_{g,n} = \overset{\omega}{\xi}_{ij} \left(\int\limits_{\mathcal{A}_{g-1,n+2}} \omega^{k}_{g-1,n+2} \right) . \tag{2.19}$$

In equation (2.18) and (2.19), ${}_i\overset{\omega}{\circ}_j$ and $\overset{\omega}{\xi}_{ij}$ denote the contraction maps w.r.t. the inverse of the bpz inner product plus an additional insertion arising from the twist angle. Let us determine this insertion: Rotations are generated by $l_0 - \tilde{l}_0$, thus the rotation of an angle ϑ is described by $\exp(i\vartheta L_0^-)$. Furthermore the measure contributes an insertion $b(l_0^-) = b_0 - \tilde{b}_0 =: b_0^-$. Integrating out $\vartheta \in [0, 2\pi]$, we identify the contraction map to be

$$\omega^{-1} := 2\pi i b_0^- P_{L_0^-} \circ \mathrm{bpz}^{-1} \, , \tag{2.20}$$

where $P_{L_0^-}$ is the projection map onto states annihilated by L_0^- and bpz^{-1} is interpreted as a map from the dual space $\mathcal{H}^*$ to the state space $\mathcal{H}$. The restricted state space $\hat{\mathcal{H}}$ is now determined by requiring that $\omega^{-1} : \mathcal{H}^* \to \mathcal{H}$ is indeed the inverse of a map $\omega : \mathcal{H} \to \mathcal{H}^*$, the odd symplectic form relevant for BV quantization. This determines $\hat{\mathcal{H}}$ to be the space of states annihilated by L_0^- and b_0^-, and the symplectic structure reads

$$\omega = \mathrm{bpz}(\cdot, c_0^- \cdot) \,. \tag{2.21}$$

We absorbed a constant of $2\pi i$ in the definition (2.21) of the symplectic form, which is the origin for the necessity of the normalization constant for the differential forms of equation (2.16).

The *algebraic vertices* corresponding to a given background are now defined by integrating the geometric vertices $\mathcal{V}_{g,n}$ over the appropriate differential forms:

$$f_{g,n} = f_{g,n}(\mathcal{V}_{g,n}) = \int\limits_{\mathcal{V}_{g,n}} \omega^0_{g,n} \,. \tag{2.22}$$

Due to the equivariance property of the differential forms and the symmetry properties of the geometric vertices, the algebraic vertices are invariant under permutations of the inputs. We denote the space of multilinear maps invariant under permutations by $\mathrm{Hom}_{\mathrm{inv}}(\hat{\mathcal{H}}^{\otimes n}, \mathbb{C})$. What is still missing is the kinetic term. Since the symplectic form ω is the appropriate bilinear map for the restricted state space $\hat{\mathcal{H}}$ and since the cohomology of the BRST charge describes the on-shell spectrum, the kinetic term reads

$$\omega(Q\cdot, \cdot) \,.$$

The full master action S is given by weighing the vertices with symmetry factors and powers of $\hbar$:

$$S(\phi) = \omega(Q\phi, \phi) + \sum_{g,n} \frac{\hbar^g}{n!} f_{g,n}(\phi^{\wedge n}) \,.$$

Simliar to the BV structure introduced in equations (2.5) and (2.6), we define a BV structure on $\mathrm{Hom}_{\mathrm{inv}}(\hat{\mathcal{H}}^{\otimes n}, \mathbb{C})$, which is induced by the odd symplectic structure:

$$\Delta^{\mathrm{alg}} h_{g,n+2} := \overset{\omega}{\xi}_{ij}\, h_{g,n+2} \tag{2.23}$$

$$(h_{g_1,n_1+1}, h_{g_2,n_2+1})^{\mathrm{alg}} := \sum_{\sigma \in \mathrm{Sh}(n_1,n_2)} \sigma.\big(h_{g_1,n_1+1} \; {}_i\overset{\Phi}{\circ}_j \; h_{g_2,n_2+1}\big) \,. \tag{2.24}$$

Finally, the factorization and chain map properties of (2.18), (2.19) and (2.17) imply that the TCFT defines a morphism from the geometric BV algebra on the chain complex of moduli spaces to the algebraic BV algebra on the space of multilinear maps of the restricted state space. Hence, we infer that the algebraic vertices satisfy the BV master equation as well, i.e.

$$f_{g,n} \circ \sum_{i=1}^{n} Q^{(i)} + \Delta^{\mathrm{alg}} f_{g-1,n+2} + \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n}} (f_{g_1,n_1+1}, f_{g_2,n_2+1})^{\mathrm{alg}} = 0 \,. \tag{2.25}$$

The master equation (2.25) expresses a collection of algebraic constraints imposed on the vertices. It turns out that these constraints are the axioms of some homotopy algebra, which will be discussed in detail in section 4.

Although we described the construction of closed string field theory in this subsection, other realizations of string field theory can be described similarly [7, 10, 11]. The most successful realization of string field theory is definitely Witten's open string field theory [12]. It is special in the sense that besides the kinetic term it involves only a cubic vertex – the star product. Witten constructed this theory in a completely different manner than described above, by seeking for a Chern-Simons like action which possesses decent gauge symmetries. Anyhow, it has been realized later on that Witten's cubic string field theory arises indeed form the geometrical approach with appropriate minimal area metrics [10].

In conclusion, the construction of string field theory is performed in two steps: First, we have to find a decomposition of the moduli space of world sheets into elementary (geometric) vertices and graphs. The single cover requirement then implies that the corresponding geometric vertices satisfy a BV master equation. This part is based solely on the moduli space and does not refer to a background at all. With the additional input of a background, we can then determine the algebraic vertices. Thus the choice of a background is an essential ingredient in the construction of string field theory. Moreover, string field theory as described so far, does not help to distinguish a background. Up to consistency conditions like Ricci flatness, the choice of background is completely arbitrary. In the next subsection, we will discuss the background independence issue more thoroughly.

Chapter 3

Background Independence

In a complete formulation of string theory, background independence is required to be implemented manifestly. Unfortunately this is not the case in the current formulation of string field theory. Nevertheless, background independence might still be realized indirectly, at least to some extent. Let us formulate the problem more precisely: Consider closed string field theory[1] constructed on two distinct backgrounds x and y, each representing a world sheet conformal field theory with associated state spaces $\mathcal{H}_x$ and $\mathcal{H}_y$, respectively. Furthermore, the bpz inner product plus the c_0^- insertion furnishes the state space with an odd symplectic structure, ω_x and ω_y. Up to a constant, the state spaces carry a natural volume form vol_x and vol_y, and we denote the master actions by S_x and S_y respectively. Background independence means that string field theories constructed on distinct backgrounds indeed represent the same theory. More precisely, we require that we can map isomorphically observables in the theory constructed on x to observables in the theory constructed on y, such that their expectation values formally coincide [13,14]. This is guaranteed, if we find a map

$$F : \mathcal{H}_x \to \mathcal{H}_y \ , \tag{3.1}$$

such that it preserves the symplectic structure, i.e.

$$F^* \omega_y = \omega_x \ , \tag{3.2}$$

and establishes a relation between the corresponding master actions. At the classical level the requirement is [13]

$$F^* S_y = S_x \ , \tag{3.3}$$

while at the quantum level the appropriate condition reads [14]

$$F^* \left(\mathrm{vol}_y \, e^{2S_y/\hbar}\right) = \mathrm{vol}_x \, e^{2S_x/\hbar} \ . \tag{3.4}$$

[1] Similar considerations can be made for open string field theory.

$$\begin{tikzcd}
& \mathrm{TCFT}_x \arrow[dl, "\mathcal{V}"] \arrow[dr, "\varphi"] & \\
f(\mathcal{V})[0,0] \arrow[d, "\phi \to \phi_0 + \phi"] & & \mathrm{TCFT}_y \arrow[d, "\mathcal{V}"] \\
f(\mathcal{V})[0,\phi_0(\varphi)] & & f(\mathcal{V})[\varphi,0] \arrow[ll, "F"]
\end{tikzcd}$$

Figure 3.1: Background independence in string field theory.

Of course, the physics in different backgrounds is generically expected to be very different. The physical content of the background is indeed encoded in the constant shift of the map F [14], which does not enter in equations (3.1), (3.2), (3.3) and (3.4). In that sense, the equivalence is formal.

In the following, we investigate TCFTs which are related by an exactly marginal deformation φ, i.e. the TCFT y is given by the TCFT x plus $\int d^2z\, \varphi(z,\tilde{z})$. The central idea is now, that the string field theory constructed on x is related to the string field theory constructed on y by a shift in the string field $\phi \to \phi_0 + \phi$ [15]. Furthermore the shift $\phi_0 = \phi_0(\varphi)$ has to satisfy the equation of motion of closed string field theory, in order that the shifted BRST differential still squares to zero.

Unfortunately, a simple shift in the string field is in general not sufficient to map the action constructed on x to the action constructed on y. In addition to the shift, a field redefinition is required [15, 16], which reduces to the identity map on-shell [17]. Let us make that statement more precise: The collection of algebraic vertices in general depends on three ingredients: First, we have to choose a consistent set of geometric vertices $\mathcal{V}$. Second, we can deform the reference background x by an exactly marginal operator φ, and finally, we can shift the string field $\phi \to \phi_0 + \phi$. We denote the corresponding collection of algebraic vertices by $f(\mathcal{V})[\varphi, \phi]$. Background independence now amounts to the existence of a map F, satisfying equation (3.1), (3.2), (3.3)/(3.4), such that diagram 3.1 commutes.

In the case of infinitesimal deformations, the map F can be determined explicitly [13, 14]: The field redefinition is constructed in two steps. First, one utilizes a canonical connection Γ on the space of TCFTs to parallel transport the vertices $f(\mathcal{V})[\varphi, 0]$ from y to x. The resulting vertices on x almost coincide with the vertices $f(\mathcal{V})[0, \phi_0]$, where the deviation amounts to a different choice of geometric vertices $\mathcal{V}'$, i.e. the parallel transported

$$f(\mathcal{V})[0, \phi_0(\varphi)] \xleftarrow[\int_{\mathcal{B}} \omega^{-1}]{} f(\mathcal{V}')[0, \phi_0(\varphi)] \xleftarrow[\Gamma]{} f(\mathcal{V})[\varphi, 0]$$

Figure 3.2: Construction of field redefinition F for infinitesimal exactly marginal deformations. ω^{-1} represents the appropriate differential form (see equation (2.16)) to integrate the interpolating chain $\mathcal{B}$ of degree -1.

vertices read $f(\mathcal{V}')[0, \phi_0]$. The geometric vertices satisfy $\partial\mathcal{V} = \partial\mathcal{V}'$, such that one can define an interpolating chain $\mathcal{B}$ of degree -1 with the property $\partial\mathcal{B} = \mathcal{V}' - \mathcal{V}$. Integrating the appropriate forms over $\mathcal{B}$ then defines the second part of the field redefinition. The construction of F is schematically depicted in figure 3.2.

All together, background independence is realized in string field theory, although not manifestly. The physical content of two distinct backgrounds is described by the shift ϕ_0, whereas the field redefinition F does not change the physics at all. From the above discussion, we conclude that every exactly marginal deformation φ corresponds to a solution of string field theory $\phi_0(\varphi)$. Since it is generally very hard to find a solution to the e.o.m. of string field theory, it is of major interest to explicitly determine $\phi_0(\varphi)$ for a given φ. In the context of open string field theory, solutions have been constructed for a certain class of finite marginal deformations [18].

The relation between backgrounds and solutions to string field theory is of particular interest in the context of open string field theory, where a background refers to a boundary conformal field theory which encodes the D-brane configuration. Open string field theory constructed on the background describing a space filling D-brane contains a tachyon in the particle spectrum. The tachyon represents the instability of the D-brane. The effective potential of the tachyon has a maximum at zero, representing the D-brane instability. But furthermore, there is a local minimum for a non-vanishing vacuum expectation value of the tachyon, which describes the absence of the original D-brane. The physical consequences of this assertion culminate in the famous Sen conjectures [19]:

(i) The energy difference in the effective potential of the tachyon between the local maximum representing the D-brane and the local minimum corresponding to the absence of the D-brane has to be equal to the tension of the D-brane.

(ii) At the tachyon vacuum, which represents a background without any D-branes, there are no more open string, i.e. the cohomology $H(Q_{\psi_0})$ of the shifted BRST differential Q_{ψ_0} has to be empty, where ψ_0 represents the tachyon vacuum solution.

(iii) Lump solutions of open string field theory describe lower dimensional D-branes.

Progress in verifying these conjectures analytically has been initiated with the discovery of a solution which represents the tachyon vacuum [20]. For a comprehensive review of tachyon condensation in open string field theory, see reference [21]. This remarkable discovery raises the question, whether the correspondence holds in general, i.e. if the space of solutions of open string field theories modulo gauge transformations covers the space of open string backgrounds, or equivalently D-brane configurations, completely. One objection against a full correspondence is, that the state spaces for distinct D-brane configurations do not coincide in general. This cannot be implemented by a shift in the string field. Nevertheless, the correspondence might still hold on-shell, where the physical degrees of freedom are determined by the cohomology of the BRST charge, as it happens in the context of tachyon condensation.

Finally, there is an alternative approach to string field theory which tries to incorporate background independence manifestly [22, 23]. A key ingredient in this formulation is the RG-flow, which can be considered as a kind of evolution equation on the space of 2-dimensional quantum field theories. But still there are conceptual problems like a proper definition of the space of 2-dimensional quantum field theories, due to ultraviolet divergencies which arise upon including arbitrary local operators. Furthermore, to make sense out of a renormalizable theory, one has to choose a regulator and a renormalization scheme, but the choice is arbitrary which makes the whole approach indefinite. On the other hand, it has been proposed, that a *generalized* moduli space might lead to a more apparent background independence [24]. Moreover, such a formulation could also lead to a manifestation of S-duality already at the geometric level.

Chapter 4

Operadic Description and Homotopy Algebras

In subsection 2, we reviewed the construction of string field theory in the geometric approach, based on the moduli space of world sheets. The construction naturally leads to a BV master equation on the chain complex of moduli spaces, and a background defines a morphism of BV algebras such that the BV master equation is satisfied also on the restricted state space of the TCFT. The BV master equation on the restricted state space encodes the algebraic constraints the vertices have to satisfy. In the following, we will employ operads, in order to give a simple classification of the algebraic structure induced by the BV master equation. The main tool that we will utilize is a correspondence between algebras over the Feynman transform of a modular operad and solutions to an associated BV master equation [8]. We will start with a concise review of the relevant notions in operad theory, in particular we will introduce modular operads and the Feynman transform. This introductory part does not claim full mathematical rigor, but is rather intended to develop some intuition. We refer the interested reader to [8, 25, 26] for a thorough exposition.

[1]A stable Σ-module $\mathcal{P}$ is a collection of differential graded vector spaces $\mathcal{P}(g, n)$ endowed with a Σ_n action, for all $g \geq 0$ and $n \geq 0$ satisfying the stability condition $2g + n - 3 \geq 0$.

A graph G is a collection $(H(G), V(G), \pi, \sigma)$, where the half-edges $H(G)$ and the vertices $V(G)$ are finite sets, $\pi : H(G) \to V(G)$ and $\sigma : H(G) \to H(G)$ is an involution, i.e. $\sigma^2 = \mathrm{id}$.

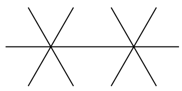
The preimage $\pi^{-1}(v) =: L(v)$ determines the half-edges attached to the vertex $v \in V(G)$. The cardinality of $L(v)$ is denoted by $n(v)$. The involution σ decomposes into 1-cycles and 2-cycles, where the 1-cycles define the legs (external lines) $L(G)$ and the 2-cycles define the edges (internal lines) $E(G)$ of the graph G.

A stable graph is a connected graph G together with a map $g : V(G) \to \mathbb{N}_0$, which assigns a genus to each vertex. For every vertex $v \in V(G)$ the stability condition $2g(v) +$

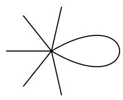
[1]This part, which introduces the theory of operads, is taken from [11].

$n(v) - 3 \geq 0$ has to hold. The genus of the graph G is defined by $g(G) = \sum_{v \in V(G)} g(v) + b_1(G)$, where $b_1(G)$ denotes the first Betti number. Furthermore we require a bijection between $L(G)$ and $\{1, \dots, n(G)\}$, where $n(G)$ denotes the cardinality of $L(G)$.

A morphism of graphs is a contraction of edges. Let G be a stable graph and $I \subset E(G)$ a subset of its edges. We denote the graph that arises from contracting the edges I of the graph G by G/I, and the corresponding morphism by $f_{G,I} : G \to G/I$. Every morphism can be decomposed into a collection of single edge contraction. There are two types of single edge contractions, corresponding to the separating and non-separating case, i.e. to the contraction of an edge connecting two vertices and the contraction of an edge forming a loop on one vertex respectively. In the following, we use a graphical representation for the single edge graphs



and



in the separating and non-separating case respectively. Stable graphs and morphism as described above define the category $\Gamma(g, n)$.

Let $\mathcal{P}$ be a stable Σ-module and G a stable graph. We define

$$\mathcal{P}(G) = \bigotimes_{v \in V(G)} \mathcal{P}(g(v), n(v)) \,.$$

A *modular operad* $\mathcal{P}$ is a stable Σ-module, which in addition defines a functor on the category of graphs. That is, for every morphism $f : G_1 \to G_2$ there is a morphism $\mathcal{P}(f) : \mathcal{P}(G_1) \to \mathcal{P}(G_2)$, and the associativity condition

$$\mathcal{P}(f \circ g) = \mathcal{P}(f) \circ \mathcal{P}(g)$$

has to hold. A *cyclic operad* is the tree level version of a modular operad, i.e. corresponds to $g = 0$.

Due to the functor property and the fact that every morphism of graphs can be decomposed into single edge contractions, a modular operad $\mathcal{P}$ is indeed determined by the underlying Σ-module together with the maps

$$\mathcal{P}\big(f_{\ast\!-\!\ast,\{e\}}\big) =: {}_i\circ_j$$

and

$$\mathcal{P}\big(f_{\ast\!\circ,\{e\}}\big) =: \xi_{ij} \,,$$

where i and j represent the half edges constituting the edge e.

Finally, there is the notion of twisted modular operads. The only twist we will need is the so called $\mathfrak{K}$-twist, which assigns degree one to the edges of a graph: For a stable graph G, $\mathfrak{K}(G)$ is defined to be the top exterior power of the vector space generated by the elements of $E(G) = \{e_1, \ldots, e_n\}$, suspended to degree n, i.e.

$$\mathfrak{K}(G) = \det(E(G)) := \uparrow^n \Lambda^n \big(\operatorname{span}(E(G))\big) .$$

The standard example of a modular operad is the endomorphism operad. Let (A, d) be a differential graded vector space endowed with a symmetric, bilinear and non-degenerate form $B : A^{\otimes 2} \to \mathbb{k}$ of degree zero, where $\mathbb{k}$ denotes some field or ring. The inverse B^{-1} of B is also symmetric and of degree zero. We define the Σ_n-modules

$$\mathcal{E}[A, d, B](g, n) = \operatorname{Hom}(A^{\otimes n}, \mathbb{k}) ,$$

where the action of Σ_n is defined by permutation of the inputs of the multilinear maps. Contractions w.r.t. B^{-1} make $\mathcal{E}[A, d, B]$ a modular operad. Similarly, consider a differential graded vector space (A, d) endowed with an odd symplectic structure of degree -1. The inverse ω^{-1} is then symmetric and of degree 1. Due to the degree of ω^{-1},

$$\mathcal{E}[A, d, \omega](g, n) = \operatorname{Hom}(A^{\otimes n}, \mathbb{k})$$

defines a $\mathfrak{K}$-twisted modular operad.

An algebra over a modular operad $\mathcal{P}$, called a $\mathcal{P}$-algebra, is a morphism α form $\mathcal{P}$ to some endomorphism operad.

The last ingredient we need is the Feynman transform of a modular operad. Let $\mathbb{M}$ be the functor from the category of stable Σ-modules to the category of modular operads, left adjoint to the forgetful functor. Consider a modular operad $\mathcal{P}$ and let $\mathcal{P}(g, n)^*$ be the dual space of $\mathcal{P}(g, n)$. For our purposes, it suffices to consider the case where the differential on $\mathcal{P}$ vanishes, i.e. $d_{\mathcal{P}} = 0$. The Feynman transform $\mathcal{FP}$ of $\mathcal{P}$ is defined to be the $\mathfrak{K}$-twisted modular operad freely generated from the dual spaces $\mathcal{P}(g, n)^*$, i.e.

$$\mathcal{FP} = \mathbb{M}_{\mathfrak{K}} \mathcal{P}^* := \bigoplus_{G \in [\Gamma(g,n)]} \big(\mathfrak{K}(G) \otimes \mathcal{P}(G)^*\big)_{\operatorname{Aut}(G)} ,$$

where $[\Gamma(g, n)]$ denotes the set of isomorphism classes of stable graphs. The main feature of the Feynman transform is that it endows $\mathcal{FP}$ with an additional differential: The *Feynman differential* $d_{\mathcal{FP}}$ is defined by

$$d_{\mathcal{FP}}\big|_{(\mathfrak{K}(G)\otimes\mathcal{P}(G)^*)_{\operatorname{Aut}(G)}} = \sum_{G'/\{e\} \simeq G} \uparrow e \otimes \mathcal{P}(f_{G',\{e\}})^* ,$$

i.e. for a given graph G it generates all graphs G' which are isomorphic to G upon contracting a single edge e.

Consider now a morphism α from the Feynman transform $\mathcal{FP}$ of a modular operad $\mathcal{P}$ to some $\mathfrak{K}$-twisted modular operad $\mathcal{Q}$. The morphism is Σ equivariant and defines a chain map, i.e.

$$d_{\mathcal{Q}} \circ \alpha = \alpha \circ d_{\mathcal{FP}} \,. \tag{4.1}$$

Furthermore, α is determined by

$$\alpha(g,n) : \mathcal{P}(g,n)^* \to \mathcal{Q}(g,n) \,, \tag{4.2}$$

and Σ_n equivariance implies that

$$\alpha(g,n) \in \big(\mathcal{Q}(g,n) \otimes \mathcal{P}(g,n)\big)^{\Sigma_n} \,.$$

Evaluating equation (4.2) on a graph consisting of a single vertex leads to [8]

$$\begin{aligned} d_{\mathcal{Q}} \circ \alpha(g,I) &= \mathcal{Q}\big(f_{\ast\!\!\circ,\{e\}}\big) \otimes \mathcal{P}\big(f_{\ast\!\!\circ,\{e\}}\big)\big(\uparrow\! e \otimes \alpha(g-1, I \sqcup \{i,j\})\big) \\ &+ \frac{1}{2} \sum_{\substack{I_1 \sqcup I_2 = I \\ g_1+g_2=g}} \mathcal{Q}\big(f_{\ast\!-\!\ast,\{e\}}\big) \otimes \mathcal{P}\big(f_{\ast\!-\!\ast,\{e\}}\big)\big(\uparrow\! e \otimes \alpha(g_1, I_1 \sqcup \{i\}) \otimes \alpha(g_2, I_2 \sqcup \{j\})\big) \,, \end{aligned} \tag{4.3}$$

where $I = \{1,\ldots,n\}$. Equation (4.3) can be interpreted as a BV master equation on $\big(\mathcal{Q}(g,n) \otimes \mathcal{P}(g,n)\big)^{\Sigma_n}$, by identifying the contractions w.r.t. $\mathcal{Q}$ and $\mathcal{P}$ together with the determinant of the edge as the antibracket $(\cdot,\cdot)$ in the separating, and the BV operator Δ in the non-separating case. $d^2_{\mathcal{FP}} = 0$ is then equivalent to the axioms of a BV algebra (without multiplication) listed in equation (2.7) [8]. Substituting $d_{\mathcal{Q}} \to -d_{\mathcal{Q}}$, equation (4.3) reads

$$d_{\mathcal{Q}} \circ \alpha(g,n) + \Delta\alpha(g-1,n+2) + \frac{1}{2} \sum_{\substack{n_1+n_2=n \\ g_1+g_2=g}} (\alpha(g_1,n_1+1), \alpha(g_2,n_2+1)) = 0 \,. \tag{4.4}$$

Theorem 1 ([8])**.** *Morphisms from the Feynman transform $\mathcal{FP}$ of a modular operad $\mathcal{P}$ to a $\mathfrak{K}$-twisted modular operad $\mathcal{Q}$ are in one-to-one correspondence with solutions to the BV master equation (4.4).*

Since, the geometric as well as the algebraic vertices satisfy a BV master equation, theorem 1 makes the usefulness of operads in the context of string field theory apparent. Let us again focus on closed string field theory: We define the $\mathfrak{K}$-twisted modular operad $C^\bullet(\hat{\mathcal{P}})$, whose underlying Σ_n modules are $C^\bullet(\hat{\mathcal{P}}_{g,n})$ with grading as defined in subsection 2. The single edge contractions are defined by

$$\begin{aligned} C^\bullet(\hat{\mathcal{P}})(f_{\ast\!-\!\ast,\{e\}})(\mathcal{A}_{g_1,n_1+1} \sqcup \mathcal{A}_{g_2,n_2+1}) &= \mathcal{A}_{g_1,n_1+1} \; {}_i\overset{\Phi}{\circ}_j \, \mathcal{A}_{g_2,n_2+1} \,, \\ C^\bullet(\hat{\mathcal{P}})(f_{\ast\!\!\circ,\{e\}})(\mathcal{A}_{g-1,n+2}) &= \overset{\Phi}{\xi}_{ij} \, \mathcal{A}_{g-1,n+2} \,, \end{aligned} \tag{4.5}$$

where $\overset{\Phi}{\xi}_{ij}$ and ${}_i\overset{\Phi}{\circ}_j$ are the sewing maps which have been introduced in equation (2.3) and (2.4). The closed string field theory vertices represent closed Riemann surfaces with punctures. Every permutation of punctures can be implemented by continuously moving the punctures on the surface. The indistinguishability of identical particles requires that this symmetry is respected by the vertices. The operad that describes this symmetry is the cyclic operad $\mathcal{C}om$ of commutative algebras for the classical vertices and the modular operad $\mathbf{Mod}(\mathcal{C}om)$ for the vertices to all orders in $\hbar$. Here $\mathbf{Mod}$ denotes the functor from the category of cyclic operads to the category of modular operads, left adjoint to the forgetful functor. $\mathcal{C}om(n)$ is a one dimensional vector space that carries the trivial representation of Σ_n. The single edge contraction reads

$$\mathcal{C}om(f_{\ast\!-\!\ast,\{e\}})(x_{n_1+1}\otimes x_{n_2+1}) = x_{n_1+n_2}\ ,$$

where x_n denotes the generator of $\mathcal{C}om(n)$. Similarly, $\mathbf{Mod}(\mathcal{C}om)(g,n)$ is a one dimensional vector space endowed with the trivial representation of Σ_n, and the single edge contractions are defined by

$$\begin{aligned}\mathbf{Mod}(\mathcal{C}om)(f_{\ast\!-\!\ast,\{e\}})(x_{g_1,n_1+1}\otimes x_{g_2,n_2+1}) &= x_{g_1+g_2,n_1+n_2}\ ,\\ \mathbf{Mod}(\mathcal{C}om)(f_{\ast\!\circlearrowright,\{e\}})(x_{g-1,n+2}) &= x_{g,n}\ .\end{aligned} \tag{4.6}$$

Note that $C^\bullet_{\mathrm{inv}}(\hat{\mathcal{P}}_{g,n}) = \big(C^\bullet(\hat{\mathcal{P}}_{g,n})\otimes\mathbf{Mod}(\mathcal{C}om)(g,n)\big)^{\Sigma_n}$. Thus, we infer from equation (2.10) and theorem 1, that the decomposition of the moduli space into elementary vertices and graphs implies the existence of a morphism

$$\alpha : \mathcal{F}\mathbf{Mod}(\mathcal{C}om)\to C^\bullet(\hat{\mathcal{P}})\ .$$

Second, the chain map (2.17) property and the factorization properties (2.18), (2.19) are equivalent to the statement that a TCFT defines a morphism

$$\beta : C^\bullet(\hat{\mathcal{P}})\to\mathcal{E}[A,Q,\omega]\ , \tag{4.7}$$

where $\mathcal{E}[A,Q,\omega]$ is the endomorphism operad of the double desuspended state space $A :=\downarrow^2 \hat{\mathcal{H}}$, with differential equal the BRST charge and ω the symplectic form defined in equation (2.21). We use the double desuspension of the state space, since with this choice of grading the symplectic form is of degree -1, and thus $\mathcal{E}[A,Q,\omega]$ indeed defines a $\mathfrak{K}$-twisted modular operad.

The algebraic BV master equation corresponds to the composition $\gamma := \beta\circ\alpha$ of these two morphisms, which defines an algebra over the Feynman transform of the modular operad $\mathbf{Mod}(\mathcal{C}om)$:

$$\gamma : \mathcal{F}\mathbf{Mod}(\mathcal{C}om)\to\mathcal{E}[A,Q,\omega]\ .$$

$$\begin{tikzcd}
\mathcal{F}\mathbf{Mod}(\mathcal{C}om) \arrow[r, "\text{decomposition}", "\text{of moduli space}"'] \arrow[dr, "\text{algebra}"'] & C^\bullet(\hat{\mathcal{P}}) \arrow[d, "\text{TCFT}"] \\
& \mathcal{E}[A,Q,\omega]
\end{tikzcd}$$

Figure 4.1: Construction of closed string field theory in terms of morphisms of modular operads.

Schematically, the construction of string field theory can be summarized as depicted in figure 4.1.

The following statements immediately reveal the algebraic constraints of string field theory as the axioms of some homotopy algebra:

Theorem 2 ([27]). *Let $\mathcal{P}$ be a Koszul cyclic operad. Algebras over the cobar transform (the tree level part of the Feynman transform) of the quadratic dual $\mathcal{P}^!$ of $\mathcal{P}$ are homotopy $\mathcal{P}$-algebras.*

Definition 1 ([28]). *Let $\mathcal{P}$ be a Koszul cyclic operad. Algebras over $\mathcal{F}\boldsymbol{Mod}(\mathcal{P}^!)$ are loop homotopy $\mathcal{P}$-algebras.*

The operad $\mathcal{C}om$ is a Koszul cyclic operad, and its quadratic dual is $\mathcal{L}ie$, the operad of Lie algebras [27]. Hence we conclude that closed string theory field vertices carry the structure of a homotopy Lie algebra (L_∞-algebra) at the classical level and that of a loop homotopy Lie algebra at the quantum level [6, 28].

Now we can readily apply this approach to other realizations of string field theory. In order to specify the algebraic structure of a certain type of string field theory, all we have to do is to determine the symmetry properties of the vertices and identify the (desuspended) restricted state space A together with the symplectic structure. Consider for example classical open string field theory. There the appropriate moduli space is the moduli space of discs with punctures on the boundary. The symmetries that can be implemented by a continuous translation of punctures, without collisions, is the group of cyclic permutations. The operad that reflects this symmetry is the cyclic operad $\mathcal{A}ss$ of associative algebras. Since the Koszul dual of $\mathcal{A}ss$ is $\mathcal{A}ss$ itself [27], we infer that the algebraic structure of classical open string field theory is that of a homotopy associative algebra (A_∞-algebra) [29]. In contrast to closed string field theory, a coordinate curve around a boundary puncture does not have rotational invariance, since the endpoints are fixed on the boundary. Thus no restriction of the state space is necessary and the symplectic structure is simply the bpz inner product.

A theory of only open strings is inconsistent at the quantum level, due to closed string poles that arise in loop amplitudes. On the other hand, combining closed strings and open strings yields a reasonable quantum theory, and the algebraic structure of open-closed string field theory bears interesting features which we will discuss in section 5. Furthermore, the geometric approach to string field theory applies even in the context of superstrings. In section 6, we outline the construction of type II superstring field theory. In particular, we derive the necessary restrictions of the state space and determine the operad which describes the algebraic structure.

In the remainder of this section, we state generic properties of homotopy algebras and comment on their physical significance. First, we would like to point out that the conclusions to follow do not just apply to string field theory, but to any field theory with gauge symmetry: In the most general case, quantization of a gauge theory requires the BV formalism [30, 31]. The main result of this approach is, that the BV action satisfies a BV master equation. Again, the algebraic constraints induced by the master equation are generically equivalent to the axioms of some homotopy algebra.

Let $\mathcal{P}$ be a Koszul cyclic operad. The cobar transform $\mathcal{BP}^!$ of the quadratic dual $\mathcal{P}^!$ of $\mathcal{P}$ is a resolution of $\mathcal{P}$ [27, 32]. This fact implies that the structure of a homotopy algebra is preserved under chain homotopy equivalences [32], which justifies the attribute homotopy. Let us explain what that means: A chain homotopy equivalence between two chain complexes (A, d_A) and (B, d_B) is a collection of chain maps $f : A \to B$ and $g : B \to A$, such that $f \circ g$ is chain homotopic to id_B and $g \circ f$ is chain homotopic to id_A. Thus, given a homotopy $\mathcal{P}$-algebra on (A, d_A), it induces the structure of a homotopy $\mathcal{P}$-algebra on (B, d_B). In particular, a chain complex (A, d_A) is chain homotopy equivalent to its cohomology $(H(A), d = 0)$ with vanishing differential. Thus, we conclude that we can associate to every homotopy $\mathcal{P}$-algebra on (A, d_A) a homotopy $\mathcal{P}$-algebra on $(H(A), d = 0)$. A homotopy algebra without differential is called minimal, and the fact that a homotopy algebra induces a minimal homotopy algebra on its cohomology is called the minimal model theorem. The explicit construction of the minimal model requires a hodge decomposition of the chain complex and involves graphs which are constructed from the hodge decomposition and the multilinear maps of the initial homotopy algebra [33, 34]. The physical application of the minimal model theorem has been elucidated in [35, 36]: The cohomology $H(A)$ of the state space A represents the physical on-shell states, i.e. the appropriate states for scattering processes. The hodge decomposition determines a gauge together with the corresponding propagator, and the graphs constructed from the Hodge decomposition and the multilinear maps are simply the Feynman graphs. We summarize this observation as follows: Assume that the vertices of some field theory satisfy the axioms of some homotopy algebra, then the S-matrix amplitudes satisfy the same axioms, but without a differential. The homotopy algebra axioms on the S-matrix amplitudes are just the Ward identities of the BRST symmetry.

There is a generalization of the minimal model theorem, which provides a decomposition of a given homotopy algebra into a minimal and a linear contractible part on the full state space [35–38]. The linear contractible part is just the differential, restricted to a subspace of the full state space such that its cohomology is trivial. This theorem can be utilized to proof uniqueness of string field theory on a fixed background [36, 37].

We conclude this section by listing several examples of field theories together with their algebraic structure: The topological string is based on a $\mathcal{N} = 2$ supersymmetric sigma model, where the target space is a Calabi-Yau 3-fold. In [39], Witten introduces a twist which renders the sigma model into TCFT, i.e. the field content is isomorphic to that of the BRST quantized bosonic string. There are two distinct ways to twist the sigma model, leading to the A- and B-model respectively. The A-model is sensitive to the symplectic structure, whereas the B-model is concerned with the complex structure. The field theories for open topological string theory have been constructed in [40]. The action has the form of a Chern-Simons theory, and the algebraic structure is that of a differential graded associative algebra, which is a special case of an A_∞-algebra. The closed string A-model is termed Kähler gravity [41], and the closed string B-model is referred to as Kodaira-Spencer theory of gravity [42]. Both the closed A- and B-model realized the structure of a differential graded Lie algebra, a special case of a L_∞-algebra. The A-model is conjectured to be related to the B-model by mirror symmetry, and thus the topological string provides a powerful guideline for a fascinating interplay between symplectic geometry and algebraic geometry. Of particular interest is the case of open strings and D-branes: Roughly speaking, the A-model studies Lagrangian submanifolds, whereas the B-model is concerned with holomorphic submanifolds. In the case of several D-branes, the notion of A_∞-algebras is replaced by that of an A_∞-category, where the D-branes are considered as objects and the morphisms are the open strings stretched between the D-branes.

The massless spectrum of open string field theory includes a gauge field. Thus, we expect that BV quantization of Yang Mills theory leads to the structure of an A_∞-algebra, just as open string field theory suggests. On the other hand a theory of gravity should naturally determine an L_∞-algebra. Indeed, the algebraic structure of Yang Mills theories has been analyzed in [43–46], and the algebraic content of certain supergravity theories reveals the structure of a L_∞-algebra [47]. The fact that Yang Mills theories carry the structure of an A_∞-algebra is interesting for the following reason: It has been shown that the amplitudes of Yang Mills theory satisfy a surprising recursion relation, which is called BFCW recursion relation according to the authors of the corresponding papers [48, 49]. Let us denote the collection of scattering amplitudes schematically by M. The scattering amplitudes satisfy the axioms of an A_∞-algebra due to the minimal model theorem. These axioms are quadratic in the maps, and we represent them schematically by $M^2 = 0$. On the other hand, the BCFW recursion relations express a scattering amplitude by a combination of two scattering amplitudes connected by a propagator, such that the total number of external legs is preserved. In our schematic language, we represent the BCFW recursion relations by $M = M \circ H \circ M$, where H denotes the propagator. The presence of both the A_∞-axioms and the BCFW recursion relations raises the following questions: Certainly, not every A_∞-algebra possesses the BCFW recursion relations. Hence, what is the extra structure on top of an A_∞-algebra that leads to BCFW recursion relations? On the other hand, given that the BCFW recursion relations hold, does it automatically imply that some homotopy algebra axioms are satisfied?

Chapter 5

Open-Closed String Field Theory and Related Algebraic Structure

Besides quantum closed string field theory, the only bosonic string field theory consistent at the quantum level is open-closed string field theory. Merging open and closed strings leads to interesting algebraic structures, which this section is concerned with. In the following, we focus on oriented strings. The construction of open-closed string field theory is described in [7]. The geometric input of open-closed string field theory is the moduli space of bounded Riemann surfaces [50]. The topological characteristics of a bounded Riemann surface are the genus g, the number of boundary components b, the number of bulk punctures n and the number of boundary punctures m_i for every boundary component i. We will abbreviate the collection $(m_i)_{i\in\{1,\dots,b\}}$ by m. The geometric vertices $\mathcal{V}^{b,g}_{n,m}$ represent a subspace of the full moduli space $\mathcal{M}^{b,g}_{n,m}$, together with an assignment of coordinate curves around the punctures. Moreover they satisfy a BV master equation, as a consequence of the condition that the vertices reproduce a single cover of the moduli space via Feynman rules. The symmetries of the vertices derive from the requirement that they can be implemented continuously: The vertices are invariant under

(i) cyclic permutations of open string punctures on a single boundary component,

(ii) arbitrary permutations of closed string punctures, and

(iii) arbitrary permutations of boundaries.

Similarly as in the closed string case, the algebraic vertices $f^{b,g}_{n,m}$ are defined by integrating the geometric vertices over appropriate differential forms, i.e. $f^{b,g}_{n,m} = \int_{\mathcal{V}^{b,g}_{n,m}} \omega^{b,g}_{n,m}$. The full quantum master action reads

$$S(\phi,\psi) = \sum_{b,g} \sum_{n,m} \frac{\hbar^{2g+b+n/2-1}}{b!\, n!\, m_1 \dots m_b} f^{b,g}_{n,m}(\phi^{\wedge n}; \psi^{\otimes m_1}, \dots, \psi^{\otimes m_b}) \,, \qquad (5.1)$$

where ϕ denotes the closed string field and ψ denotes the open string field. The action (5.1) satisfies a BV master equation, and the aim is to determine the corresponding homotopy algebra. Here we will not describe the algebraic structure as an algebra over the Feynman transform of some modular operad, which is the purpose of [51], but we rather express the axioms of quantum open-closed string field theory explicitly in the framework of IBL_∞-algebras [52]. The formulation to follow heavily relies on the concept of (higher order) coderivations on the symmetric algebra and the cyclic tensor algebra. We will not explain these notions here, but rather refer the interested reader to [53], [28] and also [54].

We denote the (desuspended) restricted state spaces by A_o and A_c, with corresponding odd symplectic forms ω_o and ω_c, where o and c refer to open and closed respectively. For the purely closed string vertices $f^{0,g}_{n,0} : A_c^{\wedge n} \to \mathbb{C}$, associated to surfaces without boundary, we define the maps $l^g_n : A_c^{\wedge n} \to A_c$ via

$$\omega_c(l^g_n, \cdot) := f^{0,g}_{n+1,0} \,.$$

We denote the lift of a multilinear map to a (higher order) coderivation on SA by a hat. The order of a coderivation is determined by the number of outputs of the multilinear map [28]. Note that ω_c^{-1} is a map with no inputs but two outputs, and hence we can lift it to a second order coderivation $\Omega_c^{-1} := \widehat{\omega_c^{-1}}$. The multilinear maps l^g_n have n inputs and one output, which determines a first order coderivation $L^g := \sum_n \widehat{l^g_n}$. The algebraic structure of closed string field theory can be summarized by the statement that

$$\mathfrak{L}_c := \sum_g \hbar^g L^g + \hbar \Omega_c^{-1} \tag{5.2}$$

squares to zero, i.e. $\mathfrak{L}_c^2 = 0$ [28, 54].

On the open string side, there are two types of sewing operations. Either we sew two punctures on a single boundary component, or we sew two punctures on distinct boundary components. Indeed, geometrically one also distinguishes the case where the two boundary components reside on a connected component of the surface or on two disconnected components, but algebraically the corresponding operations are equivalent.

Let $\mathcal{A}_o = \mathrm{Hom}^{cycl}(TA_o, \mathbb{k})$ be the cyclic Hochschild complex of open strings. An element of $\mathcal{A}_o$ physically represents a boundary component with an arbitrary number of punctures. We first consider the case of sewing two punctures on two boundary components, which is described by the Gerstenhaber bracket $[\cdot,\cdot] : \mathcal{A}_o^{\wedge 2} \to \mathcal{A}_o$: For $f, g \in \mathcal{A}_o$, we define

$$\begin{aligned}[f,g](a_1,\dots,a_{n+1}) =& (-1)^{f+g(i+1)} \sum_{i_1+i_2+i_3=n} (-1)^\epsilon f \circ (a_1,\dots,a_{i_1}, e_i, a_{i_1+i_2+1},\dots,a_n,a_{n+1}) \\ & \qquad\qquad\qquad\qquad \otimes g \circ (a_{i_1+1},\dots,a_{i_1+i_2}, e^i) \\ & + (-1)^{fg}(f \leftrightarrow g) \,,\end{aligned}$$

where $(-1)^\epsilon$ denotes the Koszul sign, $\{e_i\}$ is a basis of A_o and $\{e^i\}$ its corresponding dual basis satisfying $\omega_o({}_ie, e^j) = {}_i\delta^j$ (see [54, 55] for the sign conventions for left and right indices). The operation corresponding to the sewing of two open string punctures on a single boundary component is algebraically implemented by $\delta : \mathcal{A}_o \to \mathcal{A}_o^{\wedge 2}$:

$$\begin{aligned}
&(\delta f)(a_1, \dots, a_n)(b_1, \dots, b_m) \\
&:= (-1)^f \sum_{i=1}^{n} \sum_{j=1}^{m} (-1)^\epsilon f(e_k, a_i, \dots, a_n, a_1, \dots, a_{i-1}, e^k, b_j, \dots, b_m, b_1, \dots, b_{j-1}) \, .
\end{aligned}$$

The open string disc vertices without closed string punctures $m_o := \sum_n f^{1,0}_{0,n}$ are the vertices of classical open string field theory. Hence, they satisfy the axioms of an A_∞-algebra, which is equivalent to $[m_o, m_o] = 0$. The combination of m_o with the Gerstenhaber bracket $[\cdot, \cdot]$ defines the Hochschild differential $d_h = [m_o, \cdot]$. The collection $(\mathcal{A}_o, d_h, [\cdot, \cdot], \delta)$ defines an involutive Lie bialgebra [52, 56], which is equivalent to [54]

$$\mathfrak{L}_o^2 = 0 \, ,$$

where

$$\mathfrak{L}_o = \widehat{d_h} + \widehat{[\cdot, \cdot]} + \hbar \widehat{\delta} \, . \tag{5.3}$$

Finally, the open-closed vertices

$$\mathfrak{n} = \sum_{b=1}^{\infty} \sum_{g=0}^{\infty} \sum_{n,m} \hbar^{g+b-1} f^{b,g}_{n,m} - \delta_{b,1}\delta_{g,0} m_o$$

can be identified as an IBL_∞-morphism from the closed string loop homotopy Lie algebra to the involutive Lie bialgebra on the cyclic Hochschild complex [54], i.e.

$$e^{\mathfrak{n}} \circ \mathfrak{L}_c = \mathfrak{L}_o \circ e^{\mathfrak{n}} \, . \tag{5.4}$$

Equation (5.2), (5.3) and (5.4), together with $\mathfrak{L}_c^2 = 0$ and $\mathfrak{L}_o^2 = 0$, define the quantum open-closed homotopy algebra.

In the limit $\hbar \to 0$, one recovers the classical open-closed homotopy algebra of [9]. More precisely, the open-closed disc vertices $n = \sum_{n,m} f^{1,0}_{n,m} - m_o$ define a L_∞-morphism from the closed string L_∞-algebra $(A_c, L_c = L^0)$ to the differential graded Lie algebra on the Hochschild complex of open strings $(\mathcal{A}_o, d_h, [\cdot, \cdot])$, i.e.

$$e^n \circ L_c = L_o \circ e^n \, , \tag{5.5}$$

where $L_o = \widehat{d_h} + \widehat{[\cdot, \cdot]}$.

The benefit of reformulating the algebraic relations of open-closed string field theory in terms of homotopy algebra axioms is, that it makes deformation properties of closed strings on open string field theory apparent. Let us first discuss the classical case. A Maurer Cartan element of the L_∞-algebra (A_c, L_c) of classical closed string field theory is an element $\phi \in A_c$ satisfying $L_c(e^\phi) = 0$. Physically, the Maurer Cartan equation is the equation of motion of closed sting field theory. On the other hand, a Maurer Cartan element of $(\mathcal{A}_o, d_h, [\cdot,\cdot])$ describes a finite deformation of the initial open string A_∞-algebra (A_o, m_o), that is a finite deformation of the vertices of open string field theory. Since L_∞-morphisms preserve Maurer Cartan elements, we conclude that closed string backgrounds (solutions to the equations of motion of closed string field theory) induce a consistent deformation of open string field theory. The linear map n_1 of the morphism n of equation (5.5) is a chain map. Thus it relates closed string states to infinitesimal deformations of open string field theory. In [57], it has been argued that the map n_1 induces indeed an isomorphism on cohomology, and thus n defines a quasi-isomorphism. On the space of Maurer Cartan elements, there is the notion of gauge equivalence. Two Maurer Cartan elements are physically equivalent if they are related by a gauge transformation [35,36,38]. The space of Maurer Cartan elements modulo gauge transformations is called the moduli space of a L_∞-algebra. A theorem of Kontsevich [38] states, that the moduli spaces of quasi-isomorphic L_∞-algebras are isomorphic. Thus, we infer that closed string backgrounds modulo gauge transformations are in one-to-one correspondence with inequivalent classical open string field theories.

A particular realization of the classical open-closed homotopy algebra appears in the context of deformation quantization [38]. Consider a manifold M. The 'closed string side' is in this case represented by the space of poly-vectorfields, promoted to a Lie algebra by the Schouten-Nijenhuis bracket. Maurer Cartan elements of the space of poly-vectorfields are Poisson structures on M. On the 'open string side', one considers the space of poly-differential operators acting on $C^\infty(M)$, which is a subspace of the full Hochschild complex. The Gerstenhaber bracket is defined as before, and the Hochschild differential is induced by the pointwise multiplication on $C^\infty(M)$. The space of poly-differential operators is designed such that Maurer Cartan elements define star products. The main achievement of [38] is the construction of a L_∞-quasi-isomorphism from the space of poly-vectorfields to the space of poly-differential operator. The existence of such a quasi-isomorphism then guarantees that the respective moduli spaces are isomorphic, i.e. that there is a one-to-one correspondence between Poisson structures on M and star products on M, up to gauge transformations.

For open-closed strings at the quantum level, it is still true that the corresponding moduli spaces are isomorphic. The moduli space on the open string side represents consistent quantum open string field theories. The peculiar novelty is, that the closed string moduli space is empty [37]. This result is a consequence of the non-degeneracy of the symplectic form ω_c. Thus there are no consistent quantum open string field theories, which just confirms the common statement that open strings are inconsistent at the quantum level due to closed string poles arising in loop amplitudes. On the other hand, the situation is different for the topological string. The symplectic structure of topological closed string field theory is degenerate on-shell. This property is just the necessary modification of the assumptions, to admit a non-trivial open-closed correspondence at the quantum level. Indeed, inequivalent topological open string theories are parametrized by non-local closed string insertions [37].

We described the algebraic structure of quantum open-closed string field theory in the framework of IBL_∞-algebras. Certainly, a formulation via modular operads is possible, following the approach described in section 4. In [51], we determine the modular operad which has the property that algebras over its Feynman transform are quantum open-closed homotopy algebras.

Chapter 6

Type II Superstring Field Theory

In this section we adopt the concepts developed in the context of bosonic string field theory, to construct a covariant field theory for type II superstrings. We are brief where the construction is completely analogous to the bosonic case, but will discuss the specific peculiarities of the superstring more carefully. We follow the exposition of [11, 58–61]. First, we have to describe the appropriate moduli space, which is the space of inequivalent type II world sheets.

A super Riemann surface Σ is a complex $1|1$ dimensional supermanifold, endowed with a subbundle $\mathcal{D} \subset T\Sigma$ of rank $0|1$. We distinguish between Neveu-Schwarz (NS) and Ramond (R) punctures. A NS puncture is described by a point $(z, \theta) = (z_0, \theta_0)$, whereas a R puncture defines a divisor specified by $z = z_0$. The collection of all R punctures is called the Ramond divisor. The subbundle $\mathcal{D}$ has to satisfy a non-degenercy condition: For every local section D of $\mathcal{D}$, $[D, D]$ has to be linearly independent of D everywhere, except along the Ramond divisor where $[D, D] = 0$.

A type II world sheet $\mathbf{\Sigma}$ is a smooth $2|2$ dimensional submanifold of $\Sigma \times \tilde{\Sigma}$, where the reduced space of $\tilde{\Sigma}$ is the complex conjugate of the reduced space of Σ. Furthermore, the reduced space of $\mathbf{\Sigma}$ is the diagonal in the cartesian product of the reduced spaces of Σ and $\tilde{\Sigma}$. In general, there is no relation between the spin structures of Σ and $\tilde{\Sigma}$. Thus, we end up with four types of punctures, and we denote the number of punctures collectively by $\vec{n} = (n_\alpha)$ with $\alpha \in \{NS-NS, NS-R, R-NS, R-R\}$. Since the number of R punctures is always even, we conclude

$$\begin{aligned} n_{NS-NS} + n_{NS-R} &\in \mathbb{N}_0 \\ n_{NS-NS} + n_{R-NS} &\in \mathbb{N}_0 \\ n_{R-NS} + n_{R-R} &\in 2\mathbb{N}_0 \\ n_{NS-R} + n_{R-R} &\in 2\mathbb{N}_0 \end{aligned} \tag{6.1}$$

The moduli space $\mathfrak{M}^{II}_{g,\vec{n}}$ of type II world sheets has dimensionality

$$\dim\left(\mathfrak{M}^{II}_{g,\vec{n}}\right) = 6g-6+2n|4g-4+2n_{NS-NS}+\tfrac{3}{2}(n_{NS-R}+n_{R-NS})+n_{R-R}\ ,$$

where $n=\sum_\alpha n_\alpha$.

For an off-shell formulation of string theory, we need an explicit parametrization around each puncture to define the sewing operations. More precisely, we require a superconformal coordinate system $(z,\tilde{z},\theta,\tilde{\theta})$ around each puncture, where the puncture resides at

$$\begin{aligned} (z,\tilde{z},\theta,\tilde{\theta}) &= 0\ , &\quad& NS-NS \\ (z,\tilde{z},\theta) &= 0\ , && NS-R \\ (z,\tilde{z},\tilde{\theta}) &= 0\ , && R-NS \\ (z,\tilde{z}) &= 0\ , && R-R\ . \end{aligned}$$

We refer to such a coordinate system as *local coordinates*. The moduli space decorated with local coordinates is denoted by $\mathfrak{P}^{II}_{g,\vec{n}}$. As in the bosonic case, we require in addition that the extra structure necessary for the sewing operations can be assigned globally on moduli space in a continuous way. $\mathfrak{P}^{II}_{g,\vec{n}}$ does not satisfy this condition, i.e. it is not a trivial bundle over $\mathfrak{M}^{II}_{g,\vec{n}}$. The appropriate moduli space is indeed the moduli space of type II world sheets $\hat{\mathfrak{P}}^{II}_{g,\vec{n}}$ decorated with coordinate curves: Let $\boldsymbol{\Sigma}$ be a type II world sheet, and $S^{1|2}_\alpha$ the supercircle with two odd dimensions of type $\alpha\in\{NS-NS, NS-R, R-NS, R-R\}$. A *coordinate curve* is an embedding of $S^{1|2}_\alpha$ in Σ, homotopic to a puncture of type α.

A coordinate curve does not determine local coordinates uniquely. The ambiguity is characterized as follows: In any sector, local coordinates are determined by a coordinate curve up to a phase, i.e. rotations generated by $l_0-\tilde{l}_0$ and parametrized by $\vartheta\in[0,2\pi]$. Furthermore, in the Ramond sector (holomorphic and antiholomorphic separately), the odd coordinate is not fixed. This corresponds to translations in the Ramond divisor generated by g_0 and $\tilde{g}_0$, and parametrized by $\tau\in C^{0|1}$ and $\tilde{\tau}\in C^{0|1}$ respectively.

The sewing maps for prescribed local coordinates are the superconformal generalization of the sewing map (2.1) in the bosonic case. Corresponding to the four different types of punctures, we denote them by I_α. The GSO projection is already implemented at the geometric level, by taking the two additional spin structure in the non-separating case into account.

On the other hand, the sewing w.r.t. coordinate curves generates a whole family of world sheets, parametrized by the ambiguities of determining local coordinates. The dimensionality of the parameter families in the various sectors is displayed in table 6.1. We

sector	dimensionality
NS-NS	1|0
NS-R	1|1
R-NS	1|1
R-R	1|2

Table 6.1: Dimensionality of parameter family, generated by sewing of punctures decorated with coordinate curves.

denote the corresponding sewing maps by Φ_α, and explicitly they read

$$\begin{aligned}
(\Phi_{NS-NS})_\vartheta &= \big(I_{NS}\circ\varphi^{l_0}_{i\vartheta}\,,\ \tilde{I}_{NS}\circ\varphi^{\tilde{l}_0}_{-i\vartheta}\big) \\
(\Phi_{R-NS})_{\vartheta,\tau} &= \big(I_{R}\circ\varphi^{g_0}_{i\vartheta,\tau}\,,\ \tilde{I}_{NS}\circ\varphi^{\tilde{l}_0}_{-i\vartheta}\big) \\
(\Phi_{NS-R})_{\vartheta,\tilde\tau} &= \big(I_{NS}\circ\varphi^{l_0}_{i\vartheta}\,,\ \tilde{I}_{R}\circ\varphi^{\tilde{g}_0}_{-i\vartheta,\tilde\tau}\big) \\
(\Phi_{R-R})_{\vartheta,\tau,\tilde\tau} &= \big(I_{R}\circ\varphi^{g_0}_{i\vartheta,\tau}\,,\ \tilde{I}_{R}\circ\varphi^{\tilde{g}_0}_{-i\vartheta,\tilde\tau}\big)\ .
\end{aligned} \tag{6.2}$$

In equation (6.2), the maps φ^{l_0} and φ^{g_0} denote the flows generated by the vector fields l_0 and g_0, respectively.

Consider the chain complex $C^{\bullet|\bullet}(\hat{\mathfrak{P}}^{II}_{g,\vec{n}})$, where the chains are endowed with a $[+-]$ orientation[1] and the grading is defined by

$$k|l = \deg(\mathcal{A}_{g,\vec{n}}) = \dim(\mathfrak{M}^{II}_{g,\vec{n}}) - \dim(\mathcal{A}_{g,\vec{n}})\ ,$$

for $\mathcal{A}_{g,\vec{n}} \in C^{k|l}(\hat{\mathfrak{P}}^{II}_{g,\vec{n}})$. The maps ${}_i\overset{\Phi_\alpha}{\circ}{}_j$ and $\overset{\Phi_\alpha}{\xi}_{ij}$ are defined to be the sewing maps in the separating and non-separating case respectively, where i and j denote the punctures that are sewn together. ${}_i\overset{\Phi_\alpha}{\circ}{}_j$ and $\overset{\Phi_\alpha}{\xi}_{ij}$ are evaluated pointwise on a chain, and for every world sheet they generate a parameter family of world sheets corresponding to (6.2). Due to the choice of grading the boundary operator ∂, ${}_i\overset{\Phi_\alpha}{\circ}{}_j$ and $\overset{\Phi_\alpha}{\xi}_{ij}$ are of degree $1|0$.

We have to take the indistinguishability of identical particles into account, which amounts to requiring invariance under permutations of punctures of the same type. We define the *invariant chain complex*

$$C^{\bullet|\bullet}_{\text{inv}}(\hat{\mathfrak{P}}^{II}_{g,\vec{n}}) := \big(C^{\bullet|\bullet}(\hat{\mathfrak{P}}^{II}_{g,\vec{n}})] \otimes \mathbf{Mod}(\mathcal{C}om^{\mathcal{N}=1})\big)^{\Sigma_{\vec{n}}}\ ,$$

where $\Sigma_{\vec{n}} = \times_\alpha \Sigma_{n_\alpha}$. $\mathbf{Mod}(\mathcal{C}om^{\mathcal{N}=1})$ is the modular envelope of the colored[2] cyclic operad $\mathcal{C}om^{\mathcal{N}=1}$. The $\Sigma_{\vec{n}}$ modules $\mathcal{C}om^{\mathcal{N}=1}(g,\vec{n})$ are one dimensional vector spaces endowed with

[1] In supergeometry, there are four different notions of orientation. The relevant one for integrating differential forms is the $[+-]$ orientation [62].

[2] 'Colored' refers to the fact that we have different sectors labeled by α.

the trivial action of $\Sigma_{\vec{n}}$, and the single edge contractions are defined by

$$\mathcal{C}om^{\mathcal{N}=1}(f^{\alpha}_{\cancel{\ast\ast},\{e\}})(x_{g_1,\vec{n}_1+e_\alpha}\otimes x_{g_2,\vec{n}_2+e_\alpha}) = x_{g_1+g_2,\vec{n}_1+\vec{n}_2} \ ,$$

where $x_{g,n}$ denotes the generator of $\mathcal{C}om^{\mathcal{N}=1}(g,\vec{n})$ and e_α is the unit vector in direction α.

Lifting the maps ${}_i\overset{\Phi_\alpha}{\circ}_j$ and $\overset{\Phi_\alpha}{\xi_{ij}}$ to maps on the invariant chain complex, which involves a sum over all shuffles in the non-separating case, defines the geometric BV structure.

Now, a type II background is a superconformal field theory (SCFT), whose ghost content are the Grassmann odd ghosts b, c and the Grassmann even ghosts β, γ. The bpz inner product is defined by

$$\mathrm{bpz}_\alpha(\phi_1,\phi_2) = \langle I^*_\alpha(\phi_1)\phi_2\rangle .$$

To every world sheet $\mathbf{\Sigma}_{g,\vec{n}}\in\mathfrak{P}^{II}_{g,\vec{n}}$, the SCFT assign a multilinear map

$$Z(\mathbf{\Sigma}_{g,\vec{n}})\in\mathrm{Hom}(\mathcal{H}^{\vec{n}},\mathbb{C}^{1|1}) \ ,$$

where $\mathcal{H}_\alpha$ denotes the state space in sector α, and $\mathcal{H}^{\vec{n}} = \otimes_\alpha(\mathcal{H}_\alpha)^{n_\alpha}$. Furthermore, the SCFT satisfies the factorization properties

$$Z(\mathbf{\Sigma}_{g_1,\vec{n}_1+e_\alpha}\ {}_i\overset{I_\alpha}{\circ}_j\ \mathbf{\Sigma}_{g_2,\vec{n}_2+e_\alpha}) = Z(\mathbf{\Sigma}_{g_1,\vec{n}_1+e_\alpha})\ {}_i\overset{\mathrm{bpz}_\alpha}{\circ}_j\ Z(\mathbf{\Sigma}_{g_2,\vec{n}_2+e_\alpha}) \ ,$$

$$Z(\overset{I_\alpha}{\xi_{ij}}\mathbf{\Sigma}_{g,\vec{n}+2e_\alpha}) = \overset{\mathrm{bpz}_\alpha}{\xi_{ij}}\ Z(\mathbf{\Sigma}_{g,\vec{n}+2e_\alpha}) \ ,$$

where ${}_i\overset{\mathrm{bpz}_\alpha}{\circ}_j$ and $\overset{\mathrm{bpz}_\alpha}{\xi_{ij}}$ are the contraction maps w.r.t. the inverse of the bpz inner product.

A tangent vector V to $\hat{\mathfrak{P}}^{II}_{g,\vec{n}}$ can be represented by a collection of super Witt vectors $\vec{v} = \big((v^{(1)},\tilde{v}^{(1)}),\ldots,(v^{(n)},\tilde{v}^{(n)})\big)$, with $n=\sum_\alpha n_\alpha$. Let T be the superfield that comprises the energy momentum tensor and the super current, B the super field that contains the b and the β ghost, and (l_n), (g_n) the generators of the super Witt algebra. We define

$$\begin{aligned} T(l_n) &= L_n \ , \\ T(g_n) &= G_n \ , \\ b(l_n) &= b_n \ , \\ \beta(g_n) &= \beta_n \ , \end{aligned}$$

and the following relations hold:

$$\begin{aligned} V(Z) &= Z\circ T(\vec{v}) \ , \\ [V_1,V_2](Z) &= Z\circ T([\vec{v}_1,\vec{v}_2]) \ , \\ Z\circ\sum_{i=1}^{n}Q^{(i)} &= 0 \ . \end{aligned}$$

We define differential forms[3] $\omega^{k|l}_{g,\vec{n}}$ on $\mathfrak{P}^{II}_{g,\vec{n}}$ by,

$$\omega^{k|l}_{g,\vec{n}}(V_1,\ldots,V_r|\mathcal{V}_1,\ldots,\mathcal{V}_s) := N_{g,\vec{n}}\cdot Z(\mathbf{\Sigma}_{g,\vec{n}})\circ B(\vec{v}_1)\ldots B(\vec{v}_1)\delta(B(\vec{\nu}_1))\ldots\delta(B(\vec{\nu}_s))\ , \tag{6.3}$$

where $(V_1,\ldots,V_r|\mathcal{V}_1,\ldots,\mathcal{V}_s)$ is a collection of r even and s odd vectors in $T_{\mathbf{\Sigma}_{g,\vec{n}}}\mathfrak{P}^{II}_{g,\vec{n}}$. The normalization constant is given by $N_{g,\vec{n}} = (2\pi i)^{-(3g-3+n)}$, and $k|l$ is related to $r|s$ by $r|s = \dim(\mathfrak{M}^{II}_{g,\vec{n}}) - k|l$. The operator $\delta(B(\vec{\nu}))$ carries picture minus one[4]. The geometric counterpart of picture is the the odd dimensionality of a chain.

The differential forms (6.3) satisfy the chain map property

$$d\omega^{k+1|l}_{g,\vec{n}} = (-1)^k\omega_{g,\vec{n}^{k|l}}\circ\sum_{i=1}^{n}Q^{(i)}\ .$$

Furthermore, we require the factorization properties

$$\int\limits_{\mathcal{A}_{g_1,\vec{n}_1+e_\alpha}\ {}_i\overset{\Phi_\alpha}{\circ}_j\ \mathcal{A}_{g_1,\vec{n}_1+e_\alpha}}\omega^{k_1+k_2+1|l_1+l_2}_{g_1+g_2,\vec{n}_1+\vec{n}_2} = \Bigg(\int\limits_{\mathcal{A}_{g_1,\vec{n}_1+e_\alpha}}\omega^{k_1|l_1}_{g_1,\vec{n}_1+e_\alpha}\Bigg)\ {}_i\overset{\omega_\alpha}{\circ}_j\ \Bigg(\int\limits_{\mathcal{A}_{g_2,\vec{n}_2+e_\alpha}}\omega^{k_2|l_2}_{g_2,\vec{n}_2+e_\alpha}\Bigg)\ , \tag{6.4}$$

$$\int\limits_{\overset{\Phi_\alpha}{\xi_{ij}}\ \mathcal{A}_{g-1,\vec{n}+2e_\alpha}}\omega^{k+1|l}_{g,\vec{n}} = \overset{\omega_\alpha}{\xi_{ij}}\Bigg(\int\limits_{\mathcal{A}_{g-1,\vec{n}+2e_\alpha}}\omega^{k|l}_{g-1,\vec{n}+2e_\alpha}\Bigg)\ . \tag{6.5}$$

In equations (6.4), (6.5), the maps ${}_i\overset{\omega_\alpha}{\circ}_j$ and $\overset{\omega_\alpha}{\xi_{ij}}$ denote the contraction w.r.t. the inverse of the bpz inner product, plus additional insertions arising form the sewing operations (6.2). A SCFT, which in addition satisfies the factorization properties (6.4), (6.5) is referred to as superconformal topological field theory (STCFT). The sewing parameter τ, and similarly $\tilde{\tau}$, generates the picture changing operator $X_{g_0} = \frac{1}{2}(G_0\delta(\beta_0) - \delta(\beta_o)G_0)$. The picture changing operator X_{g_0} is not invertible. Non-degeneracy of the symplectic structure, at least on-shell, requires to restrict the state space to a subspace such that X_{g_0} is invertible. This leads to additional constraints besides the level matching condition and the $b_0^- = 0$ constraint in the Ramond sectors. The constraints defining the restricted state spaces $\hat{\mathcal{H}}_\alpha$ are listed in table 6.2.

The symplectic structure ω_α is defined to be the bpz inner product together with the inverse of the insertions arising from the sewing operations (6.2). Furthermore, ω_α induces a BV structure on the space of multilinear maps

$$\mathrm{Hom}_{\mathrm{inv}}(A^{\otimes\vec{n}},\mathbb{C}^{1|1}) := \big(\mathrm{Hom}(A^{\otimes\vec{n}},\mathbb{C}^{1|1})\otimes\mathbf{Mod}(\mathcal{C}om^{\mathcal{N}=1})(g,\vec{n})\big)^{\Sigma_{\vec{n}}}\ ,$$

[3]Here, we refer to differential forms in the sense of supergeometry. See e.g. [62] for a complete review of the subject.

[4]We do not use the conventional grading, but a grading which reflects the geometric origin of picture [11, 63, 64].

NS-NS	R-NS	NS-R	R-R
$L_0^- = 0$	$L_0^- = 0$	$L_0^- = 0$	$L_0^- = 0$
$b_0^- = 0$	$b_0^- = 0$	$b_0^- = 0$	$b_0^- = 0$
	$\beta_0^2 = 0$	$\tilde{\beta}_0^2 = 0$	$\beta_0^2 = \tilde{\beta}_0^2 = 0$
	$G_0\beta_0 - b_0 = 0$	$\tilde{G}_0\tilde{\beta}_0 - \tilde{b}_0 = 0$	$\tilde{G}_0\tilde{\beta}_0 - \tilde{b}_0 = G_0\beta_0 - b_0 = 0$

Table 6.2: Constraints defining restricted state spaces.

invariant under permutations of punctures of the same type[5]. The STCFT defines a morphism of BV algebras, or equivalently, it is a morphism between the $\mathfrak{K}$-twisted modular operads $C^{\bullet|\bullet}(\hat{\mathfrak{P}}^{II})$ and $\mathcal{E}[A_\alpha, Q_\alpha, \omega_\alpha]$.

With the aid of an appropriately formulated minimal area problem for type II world sheets, one can outline the construction of a consistent set of geometric vertices [11]. Consistent refers to the requirement that the geometric vertices provide a single cover of the full moduli space via Feynman graphs. This consistency condition implies that the geometric vertices satisfy the geometric BV master equation, which is due to theorem 1 equivalent to the existence of a morphism of $\mathfrak{K}$-twisted modular operads from $\mathcal{F}\mathbf{Mod}(\mathcal{C}om^{\mathcal{N}=1})$ to $C^{\bullet|\bullet}(\hat{\mathfrak{P}}^{II})$. Utilizing theorem 2 and definition 1, we conclude with the following theorem:

Theorem 3. *The vertices of the quantum/classical master action of type II superstring field theory satisfy the axioms of a $\mathcal{N}=1$ loop homotopy Lie-algebra/$\mathcal{N}=1$ homotopy Lie-algebra.*

[5] A_α is the restricted state space $\hat{\mathcal{H}}_\alpha$, desuspended in the grading such that the classical field carries degree $0|0$.

Chapter 7

Summary and Results

Covariant string field theory is constructed in two steps. First, the requirement of reproducing the perturbative string amplitudes via Feynman rules can be traced back to the moduli space of world sheets: Geometric vertices represent a subspace of the moduli space and propagators are defined by sewing punctures. The geometric vertices describe the background independent ingredient of string field theory. A background refers to a choice of a world sheet conformal field theory. Furthermore, the world sheet conformal field theory allows for the construction of differential forms on the moduli space of world sheets, and the algebraic vertices are defined by integrating the geometric vertices over appropriate forms.

The single cover condition of the moduli space implies that the geometric vertices satisfy a BV master equation. Since a world sheet conformal field theory satisfies certain factorization properties, the algebraic vertices satisfy a BV master equation as well. Moreover, the algebraic BV master equation encodes the algebraic constraints of the string field vertices.

In the first paper [54], we analyse the algebraic structure of quantum open-closed string field theory. We find an algebraic structure, the quantum open-closed homotopy algebra (QOCHA), which generalizes the open-closed homotopy algebra (OCHA) of Kajiura and Stasheff to the quantum level. The OCHA reveals deformation properties of closed strings on open string field theory. We conclude the paper with a discussion on the question, to what extend the correspondence between closed strings and open string field theories persists at the quantum level.

In a follow up paper [37], we discuss properties of the quantum open-closed homotopy algebra. In particular, we prove the decomposition theorem for loop homotopy Lie algebras, which describe the algebraic structure of the closed string sector of the QOCHA. On the basis of the decomposition theorem, we prove uniqueness of quantum closed string field theory on a fixed conformal background. Finally, we enlarge upon the correspondence between closed strings and open strings, and conclude that classical open string field theories are in one-to-one correspondence to closed string backgrounds. At the quantum level, the correspondence is obstructed, but in other realizations of string field theory such as the topological string, a non-trivial correspondence persists due to the fact that the symplectic structure degenerates on-shell.

The purpose of [65] is to review the algebraic structure of bosonic string field theory. Furthermore, we comment on the classification of inequivalent string field theories and include some thoughts about background independence.

Finally, the paper [11] is devoted to the construction of type II superstring field theory. We describe the BV structure on the moduli space of type II world sheets and review the operator formalism in the context of superstrings. Moreover, we outline the construction of a consistent choice of geometric vertices, by formulating an appropriate minimal area metric problem for type II world sheets. The algebraic vertices satisfy a BV master equation, and we utilize the theory of operads in order to readily interpret the induced algebraic structure in terms of the axioms of some homotopy algebra.

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Education

2010-present	ASC for Theoretical Physics Ludwig Maximilians Universität München PhD candidate (supervisor: Prof. Ivo Sachs)
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Teaching and Experience

2010-present	Tutorials in supersymmetry, differential geometry, superstring theory, string field theory, cosmology, statistical mechanics
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April, 2012	String Theory group at the Institute of Physics of the ASCR *"Homotopy algebras and string field theory"*
May, 2012	Bayrischzell Workshop 2012 Noncommutativity and Physics: Spacetime Quantum Geometry *"Homotopy algebras and string field theory"*
September, 2012	Algebra Geometry and Mathematical Physics - 8th workshop - Brno 2012 Operads, Feynman transform and string field theory *"Construction of string field theory revisited"*
October, 2012	String Field Theory and Related Aspects V, SFT 2012 *"Minimal model theorem of quantum closed string field theory"*

Enclosed Papers

This cumulative thesis consists of four papers, which are reprinted below with the permission of the corresponding journals.

1. K. Muenster, I. Sachs, "Quantum Open-Closed Homotopy Algebra and String Field Theory," Communications in Mathematical Physics (2012) 1-33[1].
2. K. Muenster, I. Sachs, "Homotopy Classification of Bosonic String Field Theory," *submitted to Communication in mathematical physics*, 1208.5626 [hep-th].
3. K. Muenster, I. Sachs, "On Homotopy Algebras and Quantum String Field Theory," *accepted in Miskolc Mathematical Notes*, 1303.3444 [math.QA].
4. B. Jurco, K. Muenster, "Type II Superstring Field Theory: Geometric Approach and Operadic Description," Journal of High Energy Physics, Volume 2013, Issue 04 (2013) 1-37.

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Communications in
**Mathematical
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Quantum Open-Closed Homotopy Algebra and String Field Theory

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Abstract: We reformulate the algebraic structure of Zwiebach's quantum open-closed string field theory in terms of homotopy algebras. We call it the quantum open-closed homotopy algebra (QOCHA) which is the generalization of the open-closed homotopy algebra (OCHA) of Kajiura and Stasheff. The homotopy formulation reveals new insights about deformations of open string field theory by closed string backgrounds. In particular, deformations by Maurer Cartan elements of the quantum closed homotopy algebra define consistent quantum open string field theories.

Contents

1. Introduction

String field theory is an off-shell formulation of string theory. Such a description is probably indispensable for a more fundamental understanding of string theory, in particular, its underlying symmetries and the relation between open and closed strings (see e.g. [1]). On the other hand, string field theory allows to address non-perturbative phenomena such as tachyon condensation for instance (see e.g. [2] for a review and references).

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The problem of constructing a string field theory is essentially that of the decomposition of moduli spaces $\mathcal{P}$ of bordered Riemann surfaces with closed string insertions in the bulk and open string insertions on the boundaries [13]. The most efficient way of doing so is based on the Batalin-Vilkovisky (BV) formalism, the simplest realization being Witten's cubic, bosonic, open string field theory [3]. This theory realizes a differential graded algebra (DGA), with the differential given by the open string BRST operator. More generally, the vertices of any consistent classical open string field theory satisfy the relations of an A_∞-algebra (strongly homotopy associative algebra) [14], which is a generalization of a DGA. In closed string theory there is no decomposition of the moduli space of Riemann surfaces compatible with Feynman rules obtained from a cubic action. Consequently one has to introduce higher string vertices and as a result the closed string field theory becomes non-polynomial. For the same reason the algebraic structure takes the form of a L_∞-algebra [12], that is, a differential graded Lie algebra up to homotopy.

Consider now classical open-closed string field theory.[1] This means that we include, in addition to the open string vertices with insertions on the boundary of the disc, and the closed string vertices with insertions on the sphere, disc vertices with an arbitrary number of open and closed string insertions. The set of these vertices satisfies the classical BV master equation of open and closed strings to 0^{th} order in $\hbar$. The point of the reformulation of this in terms of homotopy algebras is that it reveals a new structure that is not explicit in the BV formulation: The A_∞-structure of a consistent open string theory endows the space of multilinear maps, with the Hochschild differential, d_h. This differential, together with the Gerstenhaber bracket $[\cdot,\cdot]$ in turn, imply the structure of a differential graded Lie algebra (DGL). Now, an useful insight of Kajiura and Stasheff [17,18] was that the disc vertices with open and closed inputs can be interpreted as a L_∞-morphism from the L_∞-algebra of closed strings to the DGL on the cyclic Hochschild complex of open string multilinear maps. That defines the open-closed homotopy algebra (OCHA). An important property of L_∞-morphisms is that they preserve Maurer Cartan elements. Maurer Cartan elements on the closed string side represent solutions of the equations of motion of closed string field theory - classical closed string backgrounds. On the other hand, Maurer Cartan elements on the open string side define a consistent classical open string field theory. Thus the L_∞-morphism realized by the open-closed vertices on the disc associates to every classical closed string background a consistent classical open string field theory. This property is called the open-closed correspondence.

In this paper we generalize the OCHA to the quantum level. That is we do not restrict to vertices with genus zero and at most one boundary, but include all vertices with arbitrary genus and arbitrary number of boundaries. Consequently we have to consider the full quantum BV master equation, which involves besides the odd Poisson bracket (antibracket) also the BV operator. The algebraic structure of quantum closed string field theory reformulated in homotopy language is called loop algebra [20]. This is a special case of a more general algebraic structure, namely an involutive Lie bialgebra up to homotopy (IBL_∞-algebra) [22]. Furthermore, it has been realized recently that the cyclic Hochschild complex is equipped with a richer structure than just a Lie algebra, one can define an involutive Lie bialgebra (IBL-algebra) on it [21,22]. The main result of this paper is that the algebraic structure of quantum open-closed string field theory can be described by an IBL_∞-morphism from the loop algebra of closed strings to the

[1] This corresponds to taking the limit $\hbar \to 0$ after absorbing $\hbar^{\frac{1}{2}}$ in the closed string field. In this normalization the closed string anti-bracket is proportional to $\hbar$.

IBL-algebra defined on the cyclic Hochschild complex of open strings. This defines the quantum open-closed homotopy algebra (QOCHA).

The property that Maurer Cartan elements are mapped into Maurer Cartan elements holds also for IBL_∞-morphisms. The IBL_∞-morphism of QOCHA thus maps Maurer Cartan elements of the quantum closed string theory into consistent quantum theories with only open strings. This is the quantum version of the open-closed correspondence. On the other hand, we show that the quantum closed string Maurer Cartan equation implies that the closed string BRST operator on the corresponding classical closed string background has to have trivial cohomology. This is in agreement with what is known about the inconsistency of open string field theory due to the presence of the closed string tadpole.

The paper is organized as follows. In Sect. 2 we give a concise description of the concepts involved and summarize the main results. In Sect. 3 we introduce A_∞- and L_∞-algebras. The material in this section is standard. It is nevertheless included to make the paper self contained and accessible to mathematicians as well as physicists. In Sect. 4 IBL_∞-algebras are introduced as a generalization of L_∞-algebras. In Sect. 5 we explain how A_∞/L_∞- and, in particular, IBL_∞-algebras are realized in open-closed string theory. The main result of this section is the realization of quantum open-closed string field theory as an IBL_∞-algebra with the open-closed vertices realizing an IBL_∞-morphism. This is the advertised quantum generalization of the open-closed homotopy algebra (OCHA) of Kajiura and Stasheff. In Sect. 6 we analyze the closed string Maurer Cartan equation and its relation to consistent quantum open string field theories. In particular, we show that the closed string Maurer Cartan equation implies that the closed string BRST cohomology in the corresponding classical closed string background is trivial. Appendix A contains a short description of the symplectic structure in open-closed string field theory. The detailed proof of the equivalence of the quantum open-closed BV master equation and the quantum open-closed homotopy algebra is contained in Appendix B.

2. Summary

Since this paper is rather technical we will start with a summary of the main results leaving the technical details and most definitions to the later sections. Let A_o and A_c denote the space of open and closed string fields respectively. These spaces are equipped with a grading - the ghost number. The quantum BV action S, of Zwiebach's open-closed string field theory [13], is a collection of vertices with an arbitrary number of open and closed insertions, an arbitrary number of boundaries and arbitrary genus. Each vertex is invariant under the following transformations:

(i) cyclic permutation of open string inputs of one boundary,
(ii) arbitrary permutation of closed string inputs,
(iii) arbitrary permutation of boundaries.

Let $f^{b,g}_{n,m_1,\ldots,m_b}$ denote the vertex of genus g with n closed string insertions and b boundaries with m_i representing the number of insertions on the i^{th} boundary. This vertex comes with a certain power in $\hbar$ which is $2g+b+n/2-1$ [13]. The full BV action reads

$$S(c,a)=\sum_{b,g}\sum_{n}\sum_{m_1,\ldots,m_b}\hbar^{2g+b+n/2-1}\ f^{b,g}_{n,m_1,\ldots,m_b}(c,a),$$

where $c \in A_c$ is the closed string field and $a \in A_o$ is the open string field. A_o and A_c are modules over some ring R. In order to define a consistent quantum theory, the action S has to satisfy the quantum BV master equation

$$\hbar\Delta^{BV}S + \frac{1}{2}(S, S) = 0, \tag{1}$$

where Δ^{BV} denotes the BV operator and $(\cdot,\cdot)$ denotes the odd Poisson bracket also known as antibracket [13]. These operations are constructed with the aid of the odd symplectic structures ω_o and ω_c on A_o and A_c respectively. The BV equation (1) puts constraints on the collection of vertices $f^{b,g}_{n,m_1,\ldots,m_b}$ and our goal is to interpret these constraints in the language of homotopy algebras.

The idea is to split the set of all vertices into two disjoint sets. One contains all vertices corresponding to closed Riemann surfaces and the other contains the vertices associated to bounded Riemann surfaces. Let us focus on the set of vertices with only closed string insertions first. Since ω_c is non-degenerate and the vertices are invariant w.r.t. any permutation of the inputs there is a unique map $l^g \in \mathrm{Hom}^{cycl}(SA_c, A_c)$ such that

$$f^{0,g}_n(c) = \frac{1}{n!}\omega_c\left(l^g_{n-1}(c^{\wedge n-1}), c\right), \quad \forall g,$$

with $c^{\wedge n} \in SA_c$, where SA_c denotes the graded symmetric algebra of A_c. Upon summing over n we get

$$\sum_n \hbar^{2g+n/2-1}\ f^{0,g}_n(c) = \hbar^{2g-1}\omega_c(l^g, \cdot)(e^{\hbar^{1/2}c}).$$

Taking all symmetries of vertices with open and closed inputs into account we can write

$$\sum_n \sum_{m_1,\ldots,m_b} \hbar^{2g+b+n/2-1}\ f^{b,g}_{n,m_1,\ldots,m_b}(c, a) = \frac{1}{b!}\hbar^{2g+b-1} f^{b,g}(e^{\hbar^{1/2}c}; \underbrace{\bar{e}^a, \ldots, \bar{e}^a}_{b\text{ times}}), \tag{2}$$

where $f^{b,g} \in \mathrm{Hom}(SA_c, R) \otimes (\mathrm{Hom}^{cycl}(TA_o, R))^{\wedge b}$. Furthermore, $\bar{e}^a := \sum_{n=1}^{\infty} \frac{1}{n} a^{\otimes n}$ and TA_o denotes the tensor algebra of A_o. To summarize, the full BV quantum action of open-closed string field theory can be expressed as

$$S = \sum_{g=0}^{\infty} \hbar^{2g-1}\omega_c(l^g, \cdot)(e^{\hbar^{1/2}c}) + \sum_{b=1}^{\infty}\sum_{g=0}^{\infty} \frac{1}{b!}\hbar^{2g+b-1} f^{b,g}(e^{\hbar^{1/2}c}; \underbrace{\bar{e}^a, \ldots, \bar{e}^a}_{b\text{ times}}). \tag{3}$$

The classical open-closed homotopy algebra is then realized as follows: The genus zero maps $l_{cl} := l^0 : SA_c \to A_c$, parametrizing the classical closed string vertices, define a L_∞-algebra. That is there is a coderivation $L_{cl} : SA_c \to SA_c$ with $L_{cl}^2 = 0$. Similarly, the vertices of any consistent classical open string field theory realize an A_∞-algebra defined by a coderivation $M_{cl} : TA_o \to TA_o,\ M_{cl}^2 = 0$. This makes the space $\mathrm{Coder}^{cycl}(TA_o)$ of coderivations on TA_o a differential graded Lie algebra, where $[\cdot,\cdot]$ is simply defined by the graded commutator of coderivations and $d_h := [M_{cl}, \cdot]$. This a special case of a L_∞-algebra. Note that since $\mathrm{Coder}^{cycl}(TA_o) \cong \mathrm{Hom}^{cycl}(TA_o, R)$, there are induced maps on $\mathrm{Hom}^{cycl}(TA_o, R)$, which we will also denote by d_h and $[\cdot,\cdot]$. The set of open-closed disc vertices parametrized by $f^{1,0}$ is then identified as a

L_∞-morphism between the L_∞-algebra of closed strings and the DGL on the cyclic Hochschild complex $\mathrm{Hom}^{cycl}(TA_o, R)$ of open string multilinear maps

$$(A_c, L_{cl}) \xrightarrow{L_\infty-\text{morphism}} (\mathrm{Hom}^{cycl}(TA_o, R), d_h, [\cdot, \cdot]).$$

This is the open-closed homotopy algebra of Kajiura and Stasheff.

We shall be interested in the quantum version of this homotopy algebra. This works as follows: The closed string BV operator Δ^{BV} requires the inclusion of a so-called second order coderivation $D(\omega_c^{-1}) \in \mathrm{Coder}^2(SA_c)$ defined by

$$\pi_1 \circ D(\omega_c^{-1}) = 0 \qquad \text{and} \qquad \pi_2 \circ D(\omega_c^{-1}) = \omega_c^{-1},$$

that is $D(\omega_c^{-1})$ has no inputs but two outputs. On the other hand, the commutator of a first order coderivation with $D(\omega_c^{-1})$ gives again a coderivation where two inputs have been glued together. In this way one produces new objects, $L^g \in \mathrm{Coder}^{cycl}(SA_c)$, again equivalent to maps $l^g \in \mathrm{Hom}^{cycl}(SA_c, A_c)$, which in turn represent closed string vertices corresponding to Riemann surfaces of higher genus. The combination

$$\mathfrak{L}_c = \sum_{g=0}^{\infty} \hbar^g L^g + \hbar D(\omega_c^{-1})$$

together with the condition $\mathfrak{L}_c^2 = 0$ defines the homotopy loop algebra of closed string field theory [20]. This is a special case of an IBL_∞-algebra.

We have already mentioned that the space of open string multilinear maps $\mathrm{Hom}^{cycl}(TA_o, R)$ forms a Lie algebra. However, it turns out that we can make $\mathrm{Hom}^{cycl}(TA_o, R)$ even an involutive Lie bialgebra, i.e. there is a map

$$\delta : \mathrm{Hom}^{cycl}(TA_o, R) \to \mathrm{Hom}^{cycl}(TA_o, R)^{\wedge 2},$$

such that $[\cdot, \cdot]$ and δ satisfy the defining equations of an IBL-algebra. Concretely we define

$$\mathfrak{L}_o := \widehat{[\cdot, \cdot]} + \hbar\hat{\delta},$$

which satisfies $\mathfrak{L}_o^2 = 0$. This then allows us to define the quantum open-closed homotopy algebra (QOCHA) as an IBL_∞-morphisms from the IBL_∞-algebra of closed strings to the IBL-algebra of open strings

$$(A_c, \mathfrak{L}_c) \xrightarrow{IBL_\infty-\text{morphism}} (\mathrm{Hom}^{cycl}(TA_o, R), \mathfrak{L}_o),$$

with

$$\mathfrak{F} \circ \mathfrak{L}_c = \mathfrak{L}_o \circ \mathfrak{F}.$$

The IBL_∞-morphism $\mathfrak{F}$ is determined by the open-closed vertices $f^{b,g}$, $b \geq 1$, $g \geq 0$. This is the main mathematical result of this paper.

In order to get a grasp of the usefulness of the QOCHA we now focus on the Maurer Cartan elements in homotopy algebras. Consider a purely open string theory with vertices described by a collection of multilinear maps $\mathfrak{m}$. Quantum consistency of open string field theory then requires that $\mathfrak{L}_o(e^{\mathfrak{m}}) = 0$. This is just the BV equation (1) for a theory of only open strings. Since IBL_∞-morphisms preserve Maurer Cartan elements we can look for $\mathfrak{m}$ in the image of $\mathfrak{F}$. In this way we are guaranteed to find a consistent open string field theory via

$$e^{\mathfrak{m}} = \mathfrak{F}(e^{\mathfrak{c}}),$$

if $\mathfrak{c}$ defines a Maurer Cartan element of the closed string algebra, i.e. $\mathfrak{L}_c(e^{\mathfrak{c}}) = 0$. In order to see what this implies for the closed string background we have to understand the conditions implied by the closed string Maurer Cartan equation. It turns out that this equation is difficult to analyze in full generality. Therefore we make an ansatz of the form $\mathfrak{c} = c + \hbar c^{(1)}$, where $c \in A_c$ is a solution of the classical closed string equation of motion and $c^{(1)} \in A_c^{\wedge 2}$. We then find that to lowest order in $\hbar$ that the quantum Maurer Cartan equation implies that

$$Q_c[c] \circ h + h \circ Q_c[c] = -1,$$

where $Q_c[c]$ is the closed string BRST operator in the classical closed string background c and $h : A_c \to A_c$ is a map constructed out of $c^{(1)}$ and ω_c. In other words, the quantum closed string Maurer Cartan equation implies that c has to be a background where there are no perturbative closed string excitations. This is in agreement with standard argument that open string field theory is inconsistent due to closed string poles arising at the one loop level. Here, this result arises directly from analyzing the Maurer Cartan element for the closed string IBL_∞-algebra.

In the following two sections we define A_∞/ L_∞- and IBL_∞-algebras respectively. Readers familiar with these algebras may proceed directly to Sect. 4 or 5 respectively.

3. A_∞- and L_∞-Algebras

We start by reviewing the construction of A_∞- and L_∞-algebras. Here we establish the notation that will be used throughout the paper. Useful references in the context of A_∞-algebras include [24,25] and as a reference for L_∞-algebras we have chosen [23]. In the following $A = \bigoplus_{n\in\mathbb{Z}} A_n$ will denote a graded vector space over some field $\mathbb{F}$ (more generally we could consider a module A over some ring R). We will use the Koszul sign convention, that is we generate a sign $(-1)^{xy}$ whenever we permute two objects x and y. If we permute several objects we abbreviate the Koszul sign by $(-1)^{\epsilon}$.

3.1. A_∞-algebras. Following [24], we consider the tensor algebra of A

$$TA = \bigoplus_{n=0}^{\infty} A^{\otimes n},$$

and the comultiplication $\Delta : TA \to TA \otimes TA$ defined by

$$\Delta(a_1 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n} (a_1 \otimes \ldots \otimes a_i) \otimes (a_{i+1} \otimes \ldots \otimes a_n).$$

Δ makes TA a coassociative coalgebra, i.e.

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta.$$

In addition we define the canonical projection maps $\pi_n : TA \to A^{\otimes n}$ and inclusion maps $i_n : A^{\otimes n} \to TA$. A coderivation $D \in \mathrm{Coder}(TA)$ is defined by the property

$$(D \otimes \mathrm{id} + \mathrm{id} \otimes D) \circ \Delta = \Delta \circ D. \tag{4}$$

The defining property (4) implies that a coderivation $D \in \mathrm{Coder}(TA)$ is uniquely determined by a map $d \in \mathrm{Hom}(TA, A)$, i.e. $\mathrm{Coder}(TA) \cong \mathrm{Hom}(TA, A)$ [24]. Explicitly the correspondence reads

$$D \circ i_n = \sum_{i+j+k=n} 1^{\otimes i} \otimes d_j \otimes 1^{\otimes k},$$

where $d_n := d \circ i_n$, 1 denotes the identity map on A and $d = \pi_1 \circ D$. The space of coderivations $\mathrm{Coder}(TA)$ turns out to be a Lie algebra where the Lie bracket is defined by

$$[D_1, D_2] := D_1 \circ D_2 - (-1)^{D_1 D_2} D_2 \circ D_1.$$

Now an A_∞-algebra is defined by a coderivation $M \in \mathrm{Coder}(TA)$ of degree 1 (degree -1 is considered if m_1 is supposed to be a boundary operator rather than a coboundary operator) that squares to zero,

$$M^2 = \frac{1}{2}[M, M] = 0 \qquad \text{and} \qquad |M| = 1.$$

The corresponding homomorphism is defined by $m = \pi_1 \circ M$. In the case where only m_1 and m_2 are non-vanishing, we recover the definition of a differential graded associative algebra up to a shift: Take sA to be the space defined by $(sA)_i = A_{i-1}$. The map $s : A \to sA$ has the only effect of increasing the degree by 1. Likewise, the inverse map $s^{-1} : sA \to A$ decreases the degree by one. The maps corresponding to the shifted space sA are defined by

$$\tilde{m}_n := s \circ m_n \circ (s^{-1})^{\otimes n} : (sA)^{\otimes n} \to sA.$$

$\tilde{m}_1$ and $\tilde{m}_2$ then define a differential graded associative algebra, if $m_n = 0$ for $n \geq 3$.

Consider now two A_∞-algebras (A, M) and (A', M'). An A_∞-morphism $F \in \mathrm{Morph}(A, A')$ from (A, M) to (A', M') is defined by

$$\Delta \circ F = (F \otimes F) \circ \Delta, \qquad F \circ M = M' \circ F \qquad \text{and} \qquad |F| = 0. \tag{5}$$

The first equation in (5) implies that a morphism $F \in \mathrm{Morph}(A, A')$ is determined by a map $f \in \mathrm{Hom}(TA, A')$ [24]. The explicit relation reads

$$F = \sum_{n=0}^{\infty} f^{\otimes n} \circ \Delta_n, \tag{6}$$

where $\Delta_n : TA \to TA^{\otimes n}$ denotes the n-fold comultiplication and $f = \pi_1 \circ F$. We use the convention $\Delta_1 := \mathrm{id}$. An important property is that the composition of two A_∞-morphisms is again an A_∞-morphism, i.e. for $F \in \mathrm{Morph}(A, A')$ and $G \in \mathrm{Morph}(A', A'')$, $G \circ F \in \mathrm{Morph}(A, A'')$. This is a direct consequence of Eq. (5).

The concept of Maurer Cartan elements of A_∞-algebras is closely related to that of A_∞-morphisms. We define the exponential in TA as

$$e^a := \sum_{n=0}^{\infty} a^{\otimes n}.$$

A Maurer Cartan element $a \in A$ of an A_∞-algebra (A, M) is a degree zero element that satisfies

$$M(e^a) = 0 \qquad \Leftrightarrow \qquad \sum_{n=0}^{\infty} m_n(a^{\otimes n}) = 0.$$

Note that $\Delta(e^a) = e^a \otimes e^a$. Thus we can interpret the exponential e^a of a Maurer Cartan element $a \in A$ as a constant morphism $F \in \mathrm{Morph}(A, A)$, that is $f_0 = a$ and $f_n = 0$ for all $n \geq 1$.[2] Since we know that the composition of two A_∞-morphisms is again an A_∞-morphism and that a Maurer Cartan element can be interpreted as a constant A_∞-morphism, it follows that an A_∞-morphism maps Maurer Cartan elements into Maurer Cartan elements. The same statement is true for L_∞-algebras (see Sect. 3.2).

The language of coderivations is also very useful to describe deformations of A_∞-algebras. Deformations of an A_∞-algebra (A, M) are controlled by the differential graded Lie algebra $\mathrm{Coder}(TA)$ with differential $d_h := [M, \cdot]$ and bracket $[\cdot, \cdot]$. Since $\mathrm{Coder}(TA) \cong \mathrm{Hom}(TA, A)$, d_h and $[\cdot, \cdot]$ have their counterparts defined on $\mathrm{Hom}(TA, A)$, the Hochschild differential and the Gerstenhaber bracket. An infinitesimal deformation of an A_∞-algebra is characterized by the Hochschild cohomology $H^1(d_h, \mathrm{Coder}(TA))$, i.e. the cohomology of d_h at degree 1. A finite deformation of an A_∞ algebra is an element $D \in \mathrm{Coder}(TA)$ of degree 1 that satisfies the Maurer-Cartan equation

$$d_h(D) + \frac{1}{2}[D, D] = 0 \qquad \Leftrightarrow \qquad (M + D)^2 = 0.$$

We will need one more concept in the context of A_∞-algebras which is called cyclicity. Assume that A is an A_∞-algebra that is additionally endowed with an odd symplectic structure $\omega : A \otimes A \to \mathbb{F}$ of degree -1. We call $d \in \mathrm{Hom}(TA, A)$ cyclic, if the multilinear map

$$\omega(\, d\,,\ \cdot\,) : TA \to \mathbb{F}$$

is cyclically symmetric, i.e.

$$\omega(d_n(a_1, \ldots, a_n), a_{n+1}) = (-1)^{\epsilon} \omega(d_n(a_2, \ldots, a_{n+1}), a_1).$$

Since we have the notion of cyclicity for $\mathrm{Hom}(TA, A)$, we also have the notion of cyclicity for $\mathrm{Coder}(TA)$ due to the isomorphism $\mathrm{Coder}(TA) \cong \mathrm{Hom}(TA, A)$. We denote the space of cyclic coderivations by $\mathrm{Coder}^{cycl}(TA)$. An A_∞-algebra (A, M, ω) is called a cyclic A_∞-algebra if $M \in \mathrm{Coder}^{cycl}(TA)$. It is straightforward to prove that $\mathrm{Coder}^{cycl}(TA)$ is closed w.r.t. the Lie bracket $[\cdot, \cdot]$, and thus we can consider deformations of cyclic A_∞-algebras which are controlled by the differential graded Lie algebra $\mathrm{Coder}^{cycl}(TA)$. The cohomology $H(d_h, \mathrm{Coder}^{cycl}(TA))$ is called cyclic cohomology.

3.2. L_∞-algebras. Many of the constructions in the context of L_∞-algebras are analogous to that of A_∞-algebras. The main difference is that the formulation of L_∞-algebras is based on the graded symmetric algebra SA instead of the tensor algebra TA. The graded symmetric algebra SA is defined as the quotient TA/I, where I denotes the two sided ideal generated by the elements $c_1 \otimes c_2 - (-1)^{c_1 c_2} c_2 \otimes c_1$ with $c_1, c_2 \in A$. The

[2] Again we used the notation $f_n = f \circ i_n$ and $f \in \mathrm{Hom}(TA, A)$ denotes the homomorphism that corresponds to F (see Eq. (6)).

product $\otimes$ defined in TA induces the graded symmetric product $\wedge$ in SA. The symmetric algebra is the direct sum of the symmetric powers in A,

$$SA = \bigoplus_{n=0}^{\infty} A^{\wedge n}.$$

All that is simply saying that an element $c_1 \wedge \cdots \wedge c_n \in A^{\wedge n}$ is graded symmetric, that is $c_{\sigma_1} \wedge \cdots \wedge c_{\sigma_n} = (-1)^{\epsilon} c_1 \wedge \cdots \wedge c_n$ for any permutation $\sigma \in S_n$ (S_n denotes the permutation group of n elements).

The comultiplication $\Delta : SA \to SA \otimes SA$ is defined by

$$\Delta(c_1, \cdots, c_n) = \sum_{i=0}^{n} \sum_{\sigma}{}' (c_{\sigma_1} \wedge \cdots \wedge c_{\sigma_i}) \otimes (c_{\sigma_{i+1}} \wedge \cdots \wedge c_{\sigma_n}),$$

where $\sum_{\sigma}'$ indicates the sum over all permutations $\sigma \in S_n$ constrained to $\sigma_1 < \cdots < \sigma_i$ and $\sigma_{i+1} < \cdots < \sigma_n$, i.e. the sum over all inequivalent permutations.

A coderivation $D \in \text{Coder}(SA)$ is defined by

$$(D \otimes \text{id} + \text{id} \otimes D) \circ \Delta = \Delta \circ D. \tag{7}$$

Again the isomorphism $\text{Coder}(SA) \cong \text{Hom}(SA, A)$ holds, and the explicit correspondence between a coderivation $D \in \text{Coder}(SA)$ and its associated map $d = \pi_1 \circ D \in \text{Hom}(SA, A)$ is given by [23]

$$D \circ i_n = \sum_{i+j=n} \sum_{\sigma}{}' (d_i \wedge 1^{\wedge j}) \circ \sigma, \tag{8}$$

where on the right-hand side of Eq. (8) σ denotes the map that maps $c_1 \wedge \cdots \wedge c_n$ into $(-1)^{\epsilon} c_{\sigma_1} \wedge \cdots \wedge c_{\sigma_n}$ (again $d_n = d \circ i_n$ and 1 is the identity map on A).

A L_∞-algebra is defined by a coderivation $L \in \text{Coder}(SA)$ of degree 1 that squares to zero,

$$L^2 = 0 \qquad \text{and} \qquad |L| = 1.$$

A L_∞-morphism $F \in \text{Morph}(A, A')$ from a L_∞-algebra (A, L) to another L_∞-algebra (A', L') is defined by

$$\Delta \circ F = (F \otimes F) \circ \Delta, \qquad F \circ L = L' \circ F \qquad \text{and} \qquad |F| = 0. \tag{9}$$

Furthermore it is determined by the map $f = \pi_1 \circ F \in \text{Hom}(SA, A')$ through [23],

$$F = \sum_{n=0}^{\infty} \frac{1}{n!} f^{\wedge n} \circ \Delta_n, \tag{10}$$

where $\Delta_n : SA \to SA^{\otimes n}$ denotes the n-fold comultiplication.

Analogous to A_∞-algebras a Maurer Cartan element $c \in A$ of a L_∞-algebra (A, L) is essentially a constant morphism, that is

$$L(e^c) = 0 \qquad \text{and} \qquad |c| = 0,$$

where the exponential is defined by

$$e^c = \sum_{n=0}^{\infty} \frac{1}{n!} c^{\wedge n}$$

and satisfies $\Delta(e^c) = e^c \otimes e^c$.

Finally there is also the notion of cyclicity in the context of L_∞-algebras. Let (A, L) be a L_∞-algebra equipped with an odd symplectic structure ω of degree -1. We call a coderivation $D \in \mathrm{Coder}(SA)$ cyclic if the corresponding multilinear map $\omega(d\,,\,\cdot)$ is graded symmetric, i.e.

$$\omega(d_n(c_{\sigma_1}, \ldots, c_{\sigma_n}), c_{\sigma_{n+1}}) = (-1)^{\epsilon} \omega(d_n(c_1, \ldots, c_n), c_{n+1}).$$

We denote the space of cyclic coderivations by $\mathrm{Coder}^{cycl}(SA)$.

As a simple illustration of L_∞-morphisms, we consider a background shift in closed string field theory. Consider the classical action of closed string field theory, the theory with genus zero vertices l_{cl} only. The corresponding coderivation L_{cl} defines a L_∞-algebra and the action reads

$$S_{c,cl} = \omega_c(l_{cl}, \cdot)(e^c). \tag{11}$$

Shifting the background simply means that we expand the string field c around c' rather than zero. The action in the new background is $\omega_c(l_{cl}, \cdot)(e^{c'+c})$. Hence the vertices $l_{cl}[c']$ in the shifted background read

$$l_{cl}[c'] = l_{cl} \circ E(c'),$$

where $E(c')$ is the map defined by

$$E(c')(c_1 \wedge \ldots \wedge c_n) = e^{c'} \wedge c_1 \wedge \ldots \wedge c_n. \tag{12}$$

In the language of homotopy algebras this shift is implemented by

$$L_{cl}[c'] = E(-c') \circ L_{cl} \circ E(c'). \tag{13}$$

Obviously $E(-c')$ is the inverse map of $E(c')$. Furthermore $\Delta \circ E(c') = E(c') \otimes E(c')$ and therefore $L_{cl}[c']$ defines also a L_∞-algebra. Thus $E(c')$ is an L_∞-morphism. In fact, there is a subtlety if the new background does not satisfy the field equations. The initial L_∞-algebra is determined by the vertices $(l_{cl})_n$, where there is no vertex for $n = 0$, i.e. $(l_{cl})_0 = 0$. (A non-vanishing $(l_{cl})_0$ would correspond to a term in the action that depends linearly on the field.) Such an algebra is called a strong L_∞ -algebra [23]. In the new background we get

$$(l_{cl}[c'])_0 = \sum_{n=0}^{\infty} \frac{1}{n!} (l_{cl})_n (c'^{\wedge n}),$$

and thus the L_∞-algebra $L_{cl}[c']$ defines a strong L_∞-algebra only if c' satisfies the field equations [12]. If this is not the case the resulting algebra is called a weak L_∞-algebra. An odd property of a weak L_∞-algebra (A, L) is that l_1 is no longer a differential of A, the new relation reads $l_1 \circ l_1 + l_2 \circ (l_0 \wedge 1) = 0$.

4. Homotopy Involutive Lie Bialgebras

The homotopy algebras introduced in the preceding section are suitable for describing the algebraic structures of classical open-closed string field theory as defined in the Introduction [17]. If one tries to describe quantum open-closed string field theory - the set of vertices satisfying the full quantum BV master equation - in the framework of homotopy algebras, the appropriate language is that of homotopy involutive Lie bialgebras (IBL_∞-algebra). An IBL_∞-algebra is a generalization of a L_∞-algebra. It is formulated in terms of higher order coderivations - a concept that will be introduced in the next subsection - and requires an auxiliary parameter $x \in \mathbb{F}$ (later on we will identify that parameter with $\hbar$). We will also introduce the notion of morphisms and Maurer Cartan elements in the context of IBL_∞-algebras. Our exposition is based on the work of [22]. In the following we collect their results (in a slightly different notation) to make the paper self-contained.

4.1. Higher order coderivations. We already know what a coderivation (of order one) on SA is (see Eq. (7)). We defined it by an algebraic equation involving the comultiplication Δ. The essence of that equation was that a coderivation $D \in \mathrm{Coder}(SA)$ is uniquely determined by a homomorphism $d \in \mathrm{Hom}(SA, A)$. Explicitly we had

$$D \circ i_n = \sum_{i+j=n} \sum_{\sigma}{}' (d_i \wedge 1^{\wedge j}) \circ \sigma, \tag{14}$$

where $\pi_1 \circ D = d$.

There are two ways to define higher order coderivations. One is based on algebraic relations like that in Eq. (7) [20,26,27]. A coderivation of order two is for example characterized by

$$\Delta_3 \circ D - \sum_{\sigma}{}' \sigma \circ (\Delta \circ D \otimes \mathrm{id}) \circ \Delta + \sum_{\sigma}{}' \sigma \circ (D \otimes \mathrm{id}^{\otimes 2}) \circ \Delta_3 = 0,$$

where $\sum_\sigma'$ denotes the sum over inequivalent permutations in S_3 (the permutation group of three elements) and $\sigma : SA^{\otimes 3} \to SA^{\otimes 3}$ is the map that permutes the three outputs. For completeness we state the algebraic definition of a coderivation $D \in \mathrm{Coder}^n(SA)$ of order n [20],

$$\sum_{i=0}^{n} \sum_{\sigma}{}' (-1)^i \sigma \circ (\Delta_{n+1-i} \circ D \otimes \mathrm{id}^{\otimes i}) \circ \Delta_{i+1} = 0. \tag{15}$$

But similar to the case of a coderivation of order one, this algebraic relation is simply saying that - and this is the alternative definition of higher order coderivations - a coderivation $D \in \mathrm{Coder}^n(SA)$ of order n is uniquely determined by a map $d \in \mathrm{Hom}(SA, \Sigma^n A)$, where $\Sigma^n A = \oplus_{i=0}^n A^{\wedge i}$. Thus in contrast to a coderivation of order one a coderivation of order n is determined by a linear map on SA with n (and less) outputs rather than just one output. The explicit relation between $D \in \mathrm{Coder}^n(SA)$ and $d \in \mathrm{Hom}(SA, \Sigma^n A)$ is

$$D \circ i_n = \sum_{i+j=n} \sum_{\sigma}{}' (d_i \wedge 1^{\wedge j}) \circ \sigma, \tag{16}$$

which is the naive generalization of Eq. (14).

A trivial observation is that a coderivation of order $n-1$ is also a coderivation of order n (by simply defining the map with n outputs to be zero), that is

$$\mathrm{Coder}^{n-1}(SA) \subset \mathrm{Coder}^{n}(SA).$$

We call a coderivation $D \in \mathrm{Coder}^n(SA)$ of order n a strict coderivation of order n if the corresponding map d is in $\mathrm{Hom}(SA, A^{\wedge n})$, that is if the map d has exactly n outputs. In that case we can identify $d = \pi_n \circ D$.

To continue we define the graded commutator

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{D_1 D_2} D_2 \circ D_1,$$

where D_1, D_2 are arbitrary higher order coderivations. Using the defining equations (15) it can be shown that [20]

$$[\mathrm{Coder}^i(SA), \mathrm{Coder}^j(SA)] \subset \mathrm{Coder}^{i+j-1}(SA). \tag{17}$$

In the case $i = j = 1$ we recover that $[\cdot,\cdot]$ defines a Lie algebra on $\mathrm{Coder}^1(SA)$, but we see that $[\cdot,\cdot]$ does not define a Lie algebra at higher orders $n > 1$. Of course we can make the collection of all higher order coderivations a Lie algebra, but in the next subsection we will see that there is still a finer structure.

4.2. IBL_∞-algebra. Now we have the mathematical tools to define what an IBL_∞-algebra is. We will furthermore see that one recovers an involutive Lie bialgebra (IBL-algebra) as a special case of an IBL_∞-algebra. IBL_∞-algebras were introduced in [22] as well as the notion of IBL_∞-morphisms and Maurer Cartan elements.

Consider the space

$$\mathfrak{coder}(SA, x) := \bigoplus_{n=1}^{\infty} x^{n-1}\mathrm{Coder}^n(SA),$$

where $x \in \mathbb{F}$ is some auxiliary parameter. An element $\mathfrak{D} \in \mathfrak{coder}(SA, x)$ can be expanded

$$\mathfrak{D} = \sum_{n=1}^{\infty} x^{n-1} D^{(n)},$$

where $D^{(n)} \in \mathrm{Coder}^n(SA)$. In the following we will indicate coderivations of order n by a superscript (n) and strict coderivations of order n by a superscript n. We can decompose every coderivation of order n into strict coderivations of order smaller than or equal to n. Accordingly, we define the strict coderivation of order $n-g$ corresponding to a coderivation $D^{(n)}$ of order n by $D^{n-g,g}$, $g \in \{0, \dots, n-1\}$ (later one g will denote the genus). Thus we have

$$D^{(n)} = \sum_{g=0}^{n-1} D^{n-g,g},$$

and $\mathfrak{D}$ expressed in terms of strict coderivations reads

$$\mathfrak{D} = \sum_{n=1}^{\infty}\sum_{g=0}^{\infty} x^{n+g-1} D^{n,g}.$$

Due to Eq. (17) we have

$$[\mathfrak{D}_1, \mathfrak{D}_2] \in \mathfrak{coder}(SA, x),$$

that is, the commutator $[\cdot, \cdot]$ turns $\mathfrak{coder}(SA, x)$ into a graded Lie algebra. The space $\mathfrak{coder}(SA, x)$ is the Lie algebra on which the construction of IBL_∞-algebras is based. From a conceptual point of view nothing new happens in the construction of IBL_∞-algebras compared to the construction of L_∞- and A_∞-algebras. The difference is essentially that the underlying objects are more complicated. An IBL_∞-algebra is defined by an element $\mathfrak{L} \in \mathfrak{coder}(SA, x)$ of degree 1 that squares to zero [22]:

$$\mathfrak{L}^2 = 0 \qquad \text{and} \qquad |\mathfrak{L}| = 1.$$

For completeness we will now describe IBL-algebras as a special case of IBL_∞-algebras. Consider an element $\mathfrak{L} \in \mathfrak{coder}(SA, x)$ that consists of a strict coderivation of order one and a strict coderivation of order two only:

$$\mathfrak{L} = L^{1,0} + xL^{2,0}.$$

Furthermore we restrict to the case where the only non-vanishing components of $l^{1,0} := \pi_1 \circ L^{1,0} : SA \to A$ and $l^{2,0} := \pi_2 \circ L^{2,0} : SA \to A^{\wedge 2}$ are

$$d := l^{1,0} \circ i_1 : A \to A, \qquad [\cdot, \cdot] := l^{1,0} \circ i_2 : A^{\wedge 2} \to A,$$
$$\delta := l^{2,0} \circ i_1 : A \to A^{\wedge 2}.$$

To recover the definition of an involutive Lie bialgebra we have to shift the degree by one (see Sect. 3), i.e. we define maps on the shifted space sA by

$$\tilde{d} := s \circ d \circ s^{-1}, \qquad [\cdot \tilde{,} \cdot] := s \circ [\cdot, \cdot] \circ (s^{-1})^{\wedge 2},$$
$$\tilde{\delta} := s^{\wedge 2} \circ \delta \circ s^{-1}.$$

The requirement $\mathfrak{L}^2 = 0$ is then equivalent to the seven conditions

$$d^2 = 0 \Leftrightarrow \tilde{d} \text{ is a differential}, \tag{18}$$

$$d \circ [\cdot, \cdot] + [\cdot, \cdot] \circ (d \wedge 1 + 1 \wedge d) \Leftrightarrow \tilde{d} \text{ is a derivation over } [\cdot \tilde{,} \cdot],$$

$$(d \wedge 1 + 1 \wedge d) \circ \delta + \delta \circ d \Leftrightarrow \tilde{d} \text{ is a coderivation over } \tilde{\delta},$$

$$\sum_\sigma{}' [\cdot, \cdot] \circ ([\cdot, \cdot] \wedge 1) \circ \sigma = 0 \Leftrightarrow [\cdot \tilde{,} \cdot] \text{ satisfies the Jacobi identity},$$

$$(\delta \wedge 1 + 1 \wedge \delta) \circ \delta = 0 \Leftrightarrow \tilde{\delta} \text{ satisfies the co-Jacobi identity},$$

$$\sum_\sigma{}' ([\cdot, \cdot] \wedge 1) \circ \sigma \circ (\delta \wedge 1 + 1 \wedge \delta) + \delta \circ [\cdot, \cdot] = 0 \Leftrightarrow \text{compatibility of } \tilde{\delta} \text{ and } [\cdot \tilde{,} \cdot],$$

$$[\cdot, \cdot] \circ \delta = 0 \Leftrightarrow \text{involutivity of } \tilde{\delta} \text{ and } [\cdot \tilde{,} \cdot].$$

These are just the conditions defining a differential involutive Lie bialgebra.

4.3. IBL_∞-morphisms and Maurer Cartan elements. A L_∞-morphism was defined by two equations (9). The first involves the comultiplication and implies that a L_∞-morphism can be expressed by a homomorphism from SA to A (10), i.e. it determines its structure. We do not of know a suitable generalization of that equation to the case of IBL_∞-algebras, but instead one can easily generalize Eq. (10). The second equation is just saying that the morphism commutes with the differentials and looks identically in the case of IBL_∞-algebras.

Let $(A, \mathfrak{L})$ and $(A', \mathfrak{L}')$ be two IBL_∞-algebras. An IBL_∞-morphism $\mathfrak{F} \in \mathfrak{morph}(A, A')$ is defined by [22]

$$\mathfrak{F} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathfrak{f}^{\wedge n} \circ \Delta_n, \qquad \mathfrak{F} \circ \mathfrak{L} = \mathfrak{L}' \circ \mathfrak{F} \qquad \text{and} \qquad |\mathfrak{F}| = 0, \tag{19}$$

where

$$\mathfrak{f} = \sum_{n=0}^{\infty} x^{n-1} f^{(n)} \qquad \text{and} \qquad f^{(n)} : SA \to \Sigma^n A'.$$

Recall that $\Sigma^n A' = \oplus_{i=1}^n A'^{\wedge i}$. Thus we can decompose $f^{(n)}$ into a set of maps $f^{n-g,g} : SA \to A'^{\wedge n-g}$, $g \in \{0, \dots, n-1\}$ (in the same way we decomposed higher order coderivations). Expressed in terms of $f^{n,g}$ we have

$$\mathfrak{f} = \sum_{n=1}^{\infty} \sum_{g=0}^{\infty} x^{n+g-1} f^{n,g}. \tag{20}$$

Due to the lack of an algebraic relation governing the structure of an IBL_∞-morphism - an equation generalizing (9) - it is not obvious that the composition of two morphisms yields again a morphism. Nevertheless, in [22] this has been shown to be true.

To complete the section we finally state what a Maurer Cartan element of an IBL_∞-algebra $(A, \mathfrak{L})$ is. Let $c^{n,g} \in A^{\wedge n}$ be of degree zero. $\mathfrak{c} = \sum_{n=1}^{\infty} \sum_{g=0}^{\infty} x^{n+g-1} c^{n,g}$ is called a Maurer Cartan element of $(A, \mathfrak{L})$ if [22]

$$\mathfrak{L}(e^{\mathfrak{c}}) = 0,$$

that is we can interpret a Maurer Cartan element as a constant morphism on $(A, \mathfrak{L})$. (Here the exponential is the same as in the case of L_∞-algebras, i.e. $e^{\mathfrak{c}} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathfrak{c}^{\wedge n}$.)

5. Quantum Open-Closed Homotopy Algebra

After all the preliminary parts about homotopy algebras, we now turn to string field theory and show how these mathematical structures are realized therein. At the classical level the spaces of open- and closed strings, A_o and A_c are vector spaces over the field $\mathbb{C}$ but at the quantum level A_o and A_c become a module over the Grassmann numbers $\mathbb{C}_{\mathbb{Z}_2} = \mathbb{C}_0 \oplus \mathbb{C}_1$, where $\mathbb{C}_0$ resp. $\mathbb{C}_1$ represents the commuting resp. anticommuting numbers. That is at the quantum level we have to allow for both bosonic and fermionic component fields of the string field and the space of string fields becomes a bigraded space. The ghost number grading is denoted by $|\cdot|_{gh}$ whereas the Grassmann grading is denoted by $|\cdot|_{gr}$. We define a total $\mathbb{Z}_2$ grading by $|\cdot| = |\cdot|_{gh} + |\cdot|_{gr}\ mod\ 2$. The string fields c and a are of total degree zero, i.e. we pair ghost number even with Grassmann even and ghost number odd with Grassmann odd.

The full BV quantum action of open-closed string field theory reads (see Sect. 2)

$$S = \sum_{g=0}^{\infty} \hbar^{2g-1} \omega_c(l^g, \cdot)(e^{\hbar^{1/2}c}) + \sum_{b=1}^{\infty}\sum_{g=0}^{\infty} \hbar^{2g+b-1} f^{b,g}(e^{\hbar^{1/2}c}; \underbrace{\bar{e}^a, \ldots, \bar{e}^a}_{b \text{ times}}). \tag{21}$$

Note that $\bar{e}^a := \sum_{n=1}^{\infty} \frac{1}{n} a^{\otimes n}$ deviates from the definition of $e^a = \sum_{n=0}^{\infty} a^{\otimes n}$ in Sect. 3, but the symmetry factor of $1/n$ turns out to be convenient later on. The idea that the set of all vertices with open and closed inputs can be interpreted as a morphism between appropriate homotopy algebras came up in [17,18] in the context of classical open-closed string field theory, where one considers only vertices with genus zero and at most one boundary:

It is known that the vertices of classical closed string field theory define a L_∞-algebra (A_c, L_{cl}) [12]. On the other hand, the vertices of classical open string field theory define an A_∞-algebra (A_o, M_{cl}) [14,15,19], which makes the space $\text{Coder}^{cycl}(TA_o)$ a differential graded Lie algebra (see Sect. 3.1). Due to the isomorphism $\text{Coder}^{cycl}(TA_o) \overset{\pi_1}{\cong} \text{Hom}^{cycl}(TA_o, A_o) \overset{\omega}{\cong} \text{Hom}^{cycl}(TA_o \otimes A_o, \mathbb{F})$, we can transfer the differential graded Lie algebra structure from $\text{Coder}^{cycl}(TA_o)$ to the cyclic Hochschild complex $\text{Hom}^{cycl}(TA_o, \mathbb{F})$. The induced maps on $\text{Hom}^{cycl}(TA_o, \mathbb{F})$ are called the Hochschild differential and the Gerstenhaber bracket and will be denoted by d_h and $[\cdot,\cdot]$ as well. Since the isomorphism utilizes the odd symplectic structure, the induced maps d_h and $[\cdot,\cdot]$ define a DGL with the sign convention that naturally appears in the construction of L_∞-algebras (see Sect. 3.2). We will discuss the DGL structure on $\text{Hom}^{cycl}(TA_o, \mathbb{F})$ in more detail in Subsect. 5.2. A DGL is a special case of a L_∞-algebra and the set of open-closed vertices can be identified as a L_∞-morphism between the L_∞-algebra of closed strings and the DGL on the cyclic Hochschild complex of open strings,

$$(A_c, L_{cl}) \xrightarrow{L_\infty-\text{morphism}} (\text{Hom}^{cycl}(TA_o, \mathbb{F}), d_h, [\cdot,\cdot]). \tag{22}$$

This is the open-closed homotopy algebra of Kajiura and Stasheff [17,18].

In order to generalize this picture to the quantum level, we first have to identify the new structures on the closed string and on the open string side of (22), i.e. the algebraic structure on A_c and $\text{Hom}^{cycl}(TA_o, \mathbb{F})$. This is the content of the following two subsections. In the last part of this section we will connect the open and closed string part by an IBL_∞-morphism and finally define the quantum open-closed homotopy algebra - the algebraic structure of quantum open-closed string field theory.

5.1. Loop homotopy algebra of closed strings. The reformulation of the algebraic structures of closed string field theory in terms of homotopy algebras has been done in [20] and we will briefly review the results here. The corresponding homotopy algebra is called loop algebra.

The space of closed string fields A_c is endowed with an odd symplectic structure ω_c. Choose a homogeneous basis $\{e_i\}$ of A_c, where we denote the degree of e_i by $|e_i| = i$. We use DeWitt's sign convention [7], that is we introduce for every basis vector e_i the vector ${}_ie := (-1)^i e_i$. Einstein's sum convention is modified in that we sum over repeated indices whenever one of the indices is an upper left resp. right index and the other one is a lower right resp. left index. A vector $c \in A_c$ can be expanded in terms of the left or the right basis, i.e.

$$c = c^i\ {}_i e = e_i\ {}^i c, \tag{23}$$

and the expansion coefficients are related via ${}^i c = (-1)^{|c|i}\ c^i$. Let $\{e^i\}$ be its dual basis with respect to the symplectic structure ω_c, i.e.

$$\omega_c({}_i e, e^j) = {}_i\delta^j, \tag{24}$$

where ${}_i\delta^j$ denotes the Kronecker delta. Note that e^i has degree $1-i$ and hence ${}^i e = (-1)^{i+1} e^i$. These definitions ensure that $\omega_c({}^j e, e_i) = \omega_c({}_i e, e^j) = {}_i\delta^j$. ω_c regarded as a map from A_c to A_c^* is invertible and we denote its inverse by ω_c^{-1}. It follows that $\omega_c^{-1} = \frac{1}{2} e_i \wedge e^i \in A^{\wedge 2}$ and $|\omega_c^{-1}| = 1$. We can lift ω_c^{-1} to a strict coderivation $D(\omega_c^{-1}) \in \mathrm{Coder}^2(SA_c)$ of order two defined by

$$\pi_1 \circ D(\omega_c^{-1}) = 0 \qquad \text{and} \qquad \pi_2 \circ D(\omega_c^{-1}) = \omega_c^{-1}\ . \tag{25}$$

Utilizing the isomorphism $\mathrm{Hom}^{cycl}(SA_c, A_c) \cong \mathrm{Coder}^{cycl}(SA_c)$ we can lift the closed string vertices $l^g \in \mathrm{Hom}^{cycl}(SA_c, A_c)$ of the BV action (21) to a coderivation $L^g \in \mathrm{Coder}^{cycl}(SA_c)$, $g \in \mathbb{N}_0$. The combination

$$\mathfrak{L}_c = \sum_{g=0}^{\infty} \hbar^g L^g + \hbar D(\omega_c^{-1}) \tag{26}$$

defines an element in $\mathfrak{coder}(SA_c, \hbar)$ of degree 1. The homotopy loop algebra of closed string field theory is defined by [20]

$$\mathfrak{L}_c^2 = 0. \tag{27}$$

Thus the loop algebra is a special case of an IBL_∞-algebra. Furthermore Eq. (27) is equivalent to the following statements:

$$\sum_{\substack{g_1+g_2=g \\ i_1+i_2=n}} \sum_{\sigma}{}' l^{g_1}_{i_1+1} \circ (l^{g_2}_{i_2} \wedge 1^{\wedge i_1}) \circ \sigma + l^{g-1}_{n+2} \circ (\omega_c^{-1} \wedge 1^{\wedge n}) = 0, \tag{28}$$

$$e_i \wedge l^g_{n+1} \circ ({}^i e \wedge 1^{\wedge n}) = 0. \tag{29}$$

Equation (29) is merely saying that l^g has to be cyclic whereas Eq. (28) is called the main identity [12]. These are the algebraic relations of quantum closed string field theory expressed in terms of homotopy algebras, i.e. the algebraic relations of the loop algebra are equivalent to the BV equation with closed strings only.

5.2. IBL structure on cyclic Hochschild complex. In Sect. 3.1 we already saw that the space of cyclic coderivations $\mathrm{Coder}^{cycl}(TA)$ is a Lie algebra, with Lie bracket $[D_1, D_2] = D_1 \circ D_2 - (-1)^{D_1 D_2} D_2 \circ D_1$. If A is in addition a cyclic A_∞-algebra (A, M, ω), the space $\mathrm{Coder}^{cycl}(TA)$ becomes a DGL where the differential is defined by $d_h = [M, \cdot]$. First we will transfer the DGL structure from $\mathrm{Coder}^{cycl}(TA)$ to the cyclic Hochschild complex $\mathcal{A} := \mathrm{Hom}^{cycl}(TA, \mathbb{F})$. Let $f, g \in \mathcal{A}$, with both having at least one input. We define associated maps in $\mathrm{Hom}^{cycl}(TA, A)$ by

$$\omega(d_f, \cdot) := f, \qquad \omega(d_g, \cdot) := g,$$

and lift them to cyclic coderivations $D_f, D_g \in \mathrm{Coder}^{cycl}(TA)$. We define the Gerstenhaber bracket on the cyclic Hochschild complex $\mathcal{A}$ by

$$[f, g] := (-1)^{f+1}\omega\big(\pi_1 \circ [D_f, D_g], \cdot\big). \tag{30}$$

In the case where one of the maps $f, g \in \mathcal{A}$ has no inputs, we define the commutator to be identically zero. Note that the Gerstenhaber bracket as defined in (30) is graded symmetric and has degree one. Thus the structure induced on $\mathcal{A}$ is a Lie algebra up to a shift in degree, that is the actual Lie algebra lives on $s\mathcal{A}$. Furthermore the map that associates a cyclic coderivation to a element of the cyclic Hochschild complex defines a morphism of Lie algebras

$$[D_f, D_g] = (-1)^{f+1} D_{[f,g]}.$$

It turns out that we can make $\mathcal{A}$ even a differential involutive Lie bialgebra, i.e. there is a map $\delta : \mathcal{A} \to \mathcal{A}^{\wedge 2}$ such that d_h, $[\cdot, \cdot]$ and δ satisfy the defining equations (18) of an IBL-algebra. Following [21,22] we define $\delta : \mathcal{A} \to \mathcal{A}^{\wedge 2}$ by

$$\begin{aligned}&(\delta f)(a_1, \ldots, a_n)(b_1, \ldots, b_m)\\ &\quad:=(-1)^f \sum_{i=1}^{n}\sum_{j=1}^{m}(-1)^{\epsilon} f(e_k, a_i, \ldots, a_n, a_1, \ldots, a_{i-1}, e^k, b_j, \ldots, b_m, b_1, \ldots, b_{j-1}),\end{aligned} \tag{31}$$

where $\{e_k\}$ is a basis of A and $\{e^k\}$ denotes the corresponding dual basis w.r.t. the symplectic structure ω. This definition ensures that δf has the right symmetry properties. Furthermore, d_h, $[\cdot, \cdot]$ and δ satisfy all conditions of (18) [21,22]. Now let us put this into the language of IBL_∞-algebras. Lift the Hochschild differential, the Gerstenhaber bracket and the cobracket δ to coderivations on $S\mathcal{A}$:

$$\widehat{d_h} \in \mathrm{Coder}(S\mathcal{A}), \qquad \widehat{[\cdot, \cdot]} \in \mathrm{Coder}(S\mathcal{A}), \qquad \widehat{\delta} \in \mathrm{Coder}^2(S\mathcal{A}).$$

The statement that the maps d_h, $[\cdot, \cdot]$ and δ satisfy the defining relations of a differential IBL-algebra is then equivalent to

$$(\widehat{d_h} + \widehat{[\cdot, \cdot]} + x\widehat{\delta})^2 = 0.$$

If the algebra A is not endowed with the structure of a cyclic A_∞-algebra the differential d_h is absent, but still we have an IBL-algebra defined by

$$\mathfrak{L}_o^2 = 0,$$

where

$$\mathfrak{L}_o := \widehat{[\cdot, \cdot]} + x\widehat{\delta} \in \mathfrak{coder}(S\mathcal{A}, x) \qquad \text{and} \qquad |\mathfrak{L}_o| = 1. \tag{32}$$

This is the structure that will enter in the definition of the quantum open-closed homotopy algebra. That means that we do not anticipate that the vertices of classical open string field theory define an A_∞-algebra but rather derive it from the quantum open-closed homotopy algebra.

5.3. Quantum open-closed homotopy algebra. Now we can put the parts together and define the quantum open-closed homotopy algebra (QOCHA). The QOCHA is defined by an IBL_∞-morphisms from the IBL_∞-algebra of closed strings to the IBL-algebra of open strings,

$$(A_c, \mathfrak{L}_c) \xrightarrow{IBL_\infty-\text{morphism}} (\mathcal{A}_o, \mathfrak{L}_o), \tag{33}$$

where $\mathfrak{L}_c \in \mathfrak{coder}(SA_c, \hbar)$ is defined in Eq. (26) and $\mathfrak{L}_o \in \mathfrak{coder}(S\mathcal{A}_o, \hbar)$ is defined in Eq. (32). We use the abbreviation $\mathcal{A}_o = \mathrm{Hom}^{cycl}(TA_o, \mathbb{F})$. More precisely we have an IBL_∞-morphism $\mathfrak{F} \in \mathfrak{morph}(A_c, \mathcal{A}_o)$, that is

$$\mathfrak{F} \circ \mathfrak{L}_c = \mathfrak{L}_o \circ \mathfrak{F} \qquad \text{and} \qquad |\mathfrak{F}| = 0. \tag{34}$$

The morphism $\mathfrak{F}$ is determined by a map $\mathfrak{f}$ through (see Eqs. (19) and (20))

$$\mathfrak{F} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathfrak{f}^{\wedge n} \circ \Delta_n,$$

where

$$\mathfrak{f} = \sum_{b=1}^{\infty} \sum_{g=0}^{\infty} \hbar^{g+b-1} f^{b,g},$$

and

$$f^{b,g} : SA_c \to \mathcal{A}_o^{\wedge b}.$$

It turns out that (34) together with (26) and (32) is equivalent to the algebraic constraints imposed by the BV master equation (1) for the vertices in the action of open-closed string field theory provided we identify the maps l^g and $f^{b,g}$ with the closed and open-closed vertices of the BV action S in (21). The detailed proof of this equivalence is postponed to Appendix B. Schematically the equivalence goes as follows: The BV operator Δ^{BV} is a second order derivation on the space of functions (see e.g. [5,8]), whereas the odd Poisson bracket $(\cdot, \cdot)$ and the action S together define a derivation $(S, \cdot)$ on the space of functions. More precisely, the BV operator and the odd Poisson bracket split into open and closed parts:

$$\Delta^{BV} = \Delta_o^{BV} + \Delta_c^{BV} \qquad \text{and} \qquad (\cdot, \cdot) = (\cdot, \cdot)_o + (\cdot, \cdot)_c.$$

The counterpart of the open string BV operator Δ_o^{BV} is the second order coderivation δ and the derivation $(S, \cdot)_o$ translates into the coderivation $[\cdot, \cdot]$. In fact this is not quite correct since the BV operator Δ_o^{BV} will also partly play the role of $[\cdot, \cdot]$. The reason for this is that Δ_o^{BV} is not a strict second order derivation in contrast to δ. On the closed string side a similar identification holds. The counterpart of the closed string BV operator Δ_c^{BV} is $D(\omega_c^{-1})$ (see Eq. (25)) and that of the derivation $(S, \cdot)_c$ is the coderivation $\sum_g \hbar^g L^g$ of Eq. (26). Again this is just the naive identification since $(S, \cdot)_c$ partly translates into $D(\omega_c^{-1})$.

In order to gain a better geometric intuition of (34) it is useful to disentangle this equation. First consider the left hand side of Eq. (34). We have

$$\Delta_n \circ L^g = \sum_{i+j=n-1} (\mathrm{id}^{\otimes i} \otimes L^g \otimes \mathrm{id}^{\otimes j}) \circ \Delta_n \tag{35}$$

and

$$\Delta_n \circ D(\omega_c^{-1}) = \sum_{i+j=n-1} \left(\mathrm{id}^{\otimes i} \otimes D(\omega_c^{-1}) \otimes \mathrm{id}^{\otimes j}\right) \circ \Delta_n$$
$$+ \sum_{i+j+k=n-2} \left(\mathrm{id}^{\otimes i} \otimes D(e_i) \otimes \mathrm{id}^{\otimes j} \otimes D(e^i) \otimes \mathrm{id}^{\otimes k}\right) \circ \Delta_n, \qquad (36)$$

where $D(e_i)$ denotes the coderivation of order one defined by

$$\pi_1 \circ D(e_i) = e_i.$$

In the following we abbreviate $L_q = \sum_g \hbar^g L^g$. We get

$$\mathfrak{F} \circ \mathfrak{L}_c = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i+j=n-1} (\mathfrak{f}^{\wedge i} \wedge \mathfrak{f} \circ (L_q + \hbar D(\omega_c^{-1})) \wedge \mathfrak{f}^{\wedge j}) \circ \Delta_n$$
$$+ \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i+j+k=n-2} \hbar \left(\mathfrak{f}^{\wedge i} \wedge \mathfrak{f} \circ D(e_i) \wedge \mathfrak{f}^{\wedge j} \wedge \mathfrak{f} \circ D(e^i) \wedge \mathfrak{f}^{\wedge k}\right) \circ \Delta_n$$
$$= \left(\left(\mathfrak{f} \circ \mathfrak{L}_c + \frac{1}{2}\hbar(\mathfrak{f} \circ D(e_i) \wedge \mathfrak{f} \circ D(e^i)) \circ \Delta\right) \wedge \mathfrak{F}\right) \circ \Delta.$$

Now let us turn to the right hand side of Eq. (34). There we have the maps $\widehat{\delta}$ and $\widehat{[\cdot,\cdot]}$. The defining map $\delta = \pi_2 \circ \widehat{\delta}$ of $\widehat{\delta}$ has two outputs and one input. Recall that we defined the order of a coderivation by the number of outputs of the underlying defining map (see Sect. 4.1). Similarly we can define higher order derivations by the number of inputs of the underlying defining map [20]. So we can interpret $\widehat{\delta}$ either as a second order coderivation or a first order derivation and $\widehat{[\cdot,\cdot]}$ as a first order coderivation or a second order derivation. For our purpose here the second point of view will prove useful. Having these properties in mind one can show that

$$\widehat{\delta} \circ \mathfrak{F} = \left(\widehat{\delta} \circ \mathfrak{f} \wedge \mathfrak{F}\right) \circ \Delta$$

and[3]

$$\widehat{[\cdot,\cdot]} \circ \mathfrak{F} = \left(\left(\widehat{[\cdot,\cdot]} \circ \mathfrak{f} + \frac{1}{2}\widehat{[\cdot,\cdot]} \circ \left(\mathfrak{f} \wedge \mathfrak{f}\right) \circ \Delta - \left((\widehat{[\cdot,\cdot]} \circ \mathfrak{f}) \wedge \mathfrak{f}\right) \circ \Delta\right) \wedge \mathfrak{F}\right) \circ \Delta. \qquad (37)$$

Besides the properties of $\widehat{\delta}$ and $\widehat{[\cdot,\cdot]}$, we also used cocommutativity and coassociativity of Δ. Thus we can equivalently define the QOCHA by

$$\mathfrak{f} \circ \mathfrak{L}_c + \frac{\hbar}{2}\left(\mathfrak{f} \circ D(e_i) \wedge \mathfrak{f} \circ D(e^i)\right) \circ \Delta$$
$$= \mathfrak{L}_o \circ \mathfrak{f} + \frac{1}{2}\widehat{[\cdot,\cdot]} \circ \left(\mathfrak{f} \wedge \mathfrak{f}\right) \circ \Delta - \left((\widehat{[\cdot,\cdot]} \circ \mathfrak{f}) \wedge \mathfrak{f}\right) \circ \Delta. \qquad (38)$$

[3] Equation (37) can be derived in analogy to $\Delta^{BV} e^S = (\Delta^{BV} S + \frac{1}{2}(S,S))e^S$, where $(f,g) := (-1)^f (\Delta^{BV}(fg) - (\Delta^{BV} f)g - (-1)^f (\Delta^{BV} g))$, in the BV formalism (see [8] and Appendix B for more details).

This is the equation we will match with the quantum BV master equation in Appendix B. Furthermore, the individual terms in Eq. (38) can be identified with the five distinct sewing operations of bordered Riemann surfaces with closed string insertions (punctures in the bulk) and open string insertions (punctures on the boundaries) defined in [13]. The sewing either joins two open string insertions or two closed string insertions. In addition the sewing may involve a single surface or two surfaces.

(i) Take an open string insertion of one surface and sew it with another open string insertion on a second surface. The genus of the resulting surface is the sum of the genera of the indiviual surfaces, whereas the number of boundaries decreases by one. This operation is identified with

$$\frac{1}{2}\widehat{[\cdot,\cdot]}\circ\big(\mathfrak{f}\wedge\mathfrak{f}\big)\circ\Delta-\Big(\big(\widehat{[\cdot,\cdot]}\circ\mathfrak{f}\big)\wedge\mathfrak{f}\Big)\circ\Delta.$$

(ii) Sewing of two open string insertions living on the same boundary. This operation obviously increases the number of boundaries by one but leaves the genus unchanged. It is described by

$$\widehat{\delta}\circ\mathfrak{f},$$

in the homotopy language.

(iii) Consider a surface with more than one boundary. Take an open string insertion of one boundary and sew it with another open string insertion on a second boundary. This operation increases the genus by one and decreases the number of boundaries by one. It is identified with

$$\widehat{[\cdot,\cdot]}\circ\mathfrak{f}.$$

(iv) Sewing of two closed string insertion, both lying on the same surface. This attaches a handle to the surface and hence increases the genus by one, whereas the number of boundaries does not change. We identify it with

$$\mathfrak{f}\circ D(\omega_c^{-1}).$$

(v) Take a closed string insertion of one surface and sew it with another closed string insertion on a second surface. The genus and the number of boundaries of the resulting surface is the sum of the genera and the sum of the number of boundaries respectively of the input surfaces. The sewing where both surfaces have open and closed insertions is identified with

$$\big(\mathfrak{f}\circ D(e_i)\wedge\mathfrak{f}\circ D(e^i)\big)\circ\Delta,$$

whereas the sewing involving a surface with closed string insertions only and another surface with open and closed string insertions is identified with

$$\mathfrak{f}\circ L_q.$$

This provides the geometric interpretation of all individual terms in (38).

Let us now focus on the vertices with open string insertions only. These vertices are also comprised in the IBL_∞-morphism $\mathfrak{F}$ and defined by setting the closed string inputs to zero. More precisely, let $\mathfrak{m} = \mathfrak{f}|$ be the restriction of $\mathfrak{f}$ onto the subspace $A_c^{\wedge 0}$ without closed strings. The weighted sum of open string vertices is then given by

$$\mathfrak{m} = \sum_{b=1}^{\infty}\sum_{g=0}^{\infty} \hbar^{g+b-1} m^{b,g}, \qquad m^{b,g} \in \mathcal{A}_o^{\wedge b},$$

where $m^{b,g} = f^{b,g}|$. The complement of $\mathfrak{m}$ - the vertices with at least one closed string input - is denoted by $\mathfrak{g}$, so that

$$\mathfrak{f} = \mathfrak{m} + \mathfrak{g}. \tag{39}$$

In the classical limit $\hbar \to 0$ we expect to recover the OCHA defined by Kajiura and Stasheff [17, 18]. Indeed the IBL_∞-morphism $\mathfrak{F}$ reduces to a L_∞-morphism, the loop algebra $\mathfrak{L}_c$ of closed strings reduces to a L_∞-algebra $L_{cl} := L^0$ and the IBL-algebra on the space of cyclic coderivations becomes an ordinary Lie algebra. The defining equation (38) of the QOCHA simplifies to

$$f_{cl} \circ L_{cl} = \frac{1}{2}[f_{cl}, f_{cl}] \circ \Delta, \tag{40}$$

where $f_{cl} := f^{1,0}$ is the component of $\mathfrak{f}$ with one boundary and genus zero and the corresponding L_∞-morphism is given by $\sum_n \frac{1}{n!} f_{cl}{}^{\wedge n} \circ \Delta_n$ (see Sect. 3.2). Separating the purely open string vertices m_{cl} from f_{cl}, we see that those have to satisfy an A_∞-algebra (since $L_{cl}| = 0$), i.e. they define a classical open string field theory [14, 15]. Thus the space $\mathcal{A}_o$ turns into a DGL with differential $d_h = [m_{cl}, \cdot]$ and Eq. (40) finally reads

$$n_{cl} \circ L_{cl} = d_h \circ n_{cl} + \frac{1}{2}[n_{cl}, n_{cl}] \circ \Delta, \tag{41}$$

where $n_{cl} = f_{cl} - m_{cl} : SA_c \to \mathcal{A}_o$ denotes the vertices with at least one closed string input.

Similarly we define $\mathfrak{n} = \mathfrak{f} - m_{cl}$ and the QOCHA in terms of $\mathfrak{n}$ reads

$$\mathfrak{N} \circ \mathfrak{L}_c = \mathfrak{L}_o' \circ \mathfrak{N}, \tag{42}$$

where $\mathfrak{N} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathfrak{n}^{\wedge n} \circ \Delta_n$ and $\mathfrak{L}_o' = \widehat{d_h} + \mathfrak{L}_o$.

Equation (41) is precisely the OCHA defined in [17, 18]. The physical interpretation of n_{cl} is that it describes the deformation of open string field theory by turning on a closed string background. The vanishing of the r.h.s. is the condition for a consistent classical field theory while the l.h.s. vanishes if the closed string background solves the classical closed string field theory equations of motion. Equation (41) then implies that the open-closed vertices define a consistent classical open string field theory if the closed string background satisfies the classical closed string equations of motion. The inverse assertion does not follow from (41). However, it has been shown to be true for infinitesimal closed string deformations in [16]. More precisely, upon linearizing Eq. (41) in $c \in A_c$ we get

$$n_{cl}(L_{cl}(c)) = d_h(n_{cl}(c)). \tag{43}$$

$L_{cl} \in \mathrm{Coder}^{cycl}(SA_c)$ is determined by $l_{cl} = \pi_1 \circ L_{cl} \in \mathrm{Hom}^{cycl}(SA_c)$, the closed string vertices of genus zero (see Sect. 3.2). In string field theory the vertex with just one input $(l_{cl})_1$ is the closed string BRST operator Q_c. Thus Eq. (43) is equivalent to

$$n_{cl}(Q_c(c)) = d_h(n_{cl}(c)), \tag{44}$$

that is $n_{cl} \circ i_1$ induces a chain map from the BRST complex of closed strings to the cyclic Hochschild complex of open strings. The cohomology of Q_c (BRST cohomology) defines the space of physical states whereas the cohomology of d_h (cyclic cohomology) characterizes the infinitesimal deformations of the initial open string field theory m_{cl}. In [16] it has been shown that the BRST cohomology of closed strings is indeed isomorphic to the cyclic Hochschild cohomology of open strings.

6. Deformations and the Quantum Open-Closed Correspondence

The quantum open-closed homotopy algebra described in the last section is essentially a reformulation of the open-closed BV master equation in terms of homotopy algebras. However, we can also extract physical insight from this reformulation. The point is that we have the notion of Maurer Cartan elements in homotopy algebras, a concept that is not explicit in the BV formulation. An important property of IBL_∞-morphisms is that they map Maurer Cartan elements into Maurer Cartan elements. Thus a Maurer Cartan element of the closed string loop algebra will in turn define a Maurer Cartan element on the IBL-algebra of open string multilinear maps, or in other words, there is a correspondence between certain closed string backgrounds and consistent quantum open string field theories. To make this last statement more precise we will first give a definition of quantum open string field theory and then try to identify corresponding Maurer Cartan elements of the closed string algebra.

6.1. Quantum open string field theory. To start with we examine the QOCHA in the case where all closed string insertions are set to zero. In Eq. (39) we separated the vertices $\mathfrak{m}$ with open string inputs only from the vertices $\mathfrak{g}$ with both open and closed inputs. Similarly the IBL_∞-morphism separates into

$$\mathfrak{F} = e^{\mathfrak{m}} \wedge \sum_{n=0}^{\infty} \frac{1}{n!} \mathfrak{g}^{\wedge n} \circ \Delta_n.$$

Consider now the defining relation (34) of the QOCHA and set all closed string insertions to zero. We get

$$\hbar\, \mathfrak{F} \circ D(\omega_c^{-1}) = \mathfrak{L}_o(e^{\mathfrak{m}}). \tag{45}$$

On the other hand, if $\mathfrak{m}$ is supposed to define a consistent quantum open string field theory it has to satisfy the Maurer Cartan equation, that is

$$\mathfrak{L}_o(e^{\mathfrak{m}}) = 0. \tag{46}$$

This is just the quantum BV master equation for open string field theory. In the classical limit this definition reproduces the known result that the vertices of a classical open string field define an A_∞-algebra [14,15]. From (45) it is then clear that in a trivial closed string background we can have a consistent theory of open strings only if $D(\omega_c^{-1})$ is in the kernel of $\mathfrak{F}$. As an example we consider Witten's cubic string field theory [9,10]. Cubic string field theory is defined in terms of the BRST operator $Q_o : A_o \to A_o$ and the star product $* : A_o \otimes A_o \to A_o$. The BRST operator together with the star product define a DGA - a special case of an A_∞-algebra (see Sect. 3.1). The statement that Q_o

and $*$ form a DGA solves the constraint imposed by $[\cdot,\cdot]$ whereas the impact of δ can be summarized as

$$\omega_o(Q_o(e_i), e^i) = 0 \qquad \text{and} \qquad e_i * e^i = 0. \tag{47}$$

The first equation in (47) is equivalent to demanding that Q is traceless

$$\mathrm{Tr}(Q_o) = 0.$$

This is always guaranteed since Q_o is a cohomology operator - $Q_o^2 = 0$ - and thus the only eigenvalue is zero. On the other hand the constraint the star product $*$ has to satisfy is more delicate:

$$e_i * e^i \overset{!}{=} 0.$$

$e_i * e^i$ is precisely the term that arose in the attempt to quantize cubic string field theory [11]. This term is not zero but highly divergent and corresponds to the open string tadpole diagram. The open question is if this divergence can be cured by a suitable regulator, without introducing closed string degrees of freedom explicitly.

6.2. Quantum open-closed correspondence. In this section we consider generic closed string backgrounds demanding that they induce consistent quantum open string field theories. In the language of homotopy algebras this property manifests itself in the statement that an IBL_∞-morphism preserves Maurer Cartan elements. Let $\mathfrak{c}$ be a Maurer Cartan element of the closed string algebra

$$\mathfrak{c} = \sum_{n,g} \hbar^{g+n-1} c^{n,g}, \qquad c^{n,g} \in A_c^{\wedge n}, \qquad \mathfrak{L}_c(e^{\mathfrak{c}}) = 0. \tag{48}$$

Plugging this into Eq. (34) we get

$$\mathfrak{L}_o(\mathfrak{F}(e^{\mathfrak{c}})) = 0.$$

Since $e^{\mathfrak{c}}$ is a constant IBL_∞-morphism and the composition of two morphisms is again a morphism (see Sect. 4.3), we can conclude that $\mathfrak{F}(e^{\mathfrak{c}})$ is a constant morphism and thus

$$e^{\mathfrak{m}'} := \mathfrak{F}(e^{\mathfrak{c}}), \qquad \mathfrak{L}_o(e^{\mathfrak{m}'}) = 0, \tag{49}$$

where $\mathfrak{m}' = \sum_{g,b} \hbar^{g+b-1} m'^{\,b,g}$ and $m'^{\,b,g} \in \mathcal{A}_o^{\wedge b}$. $\mathfrak{m}'$ represents the open string vertices induced by the closed string Maurer Cartan element via the IBL_∞-morphism. Equation (49) states that these vertices satisfy the requirements of a quantum open string field theory (46). Thus every Maurer Cartan element of the closed string loop algebra defines a quantum open string field theory. We give that circumstance a name and call it the quantum open-closed correspondence. To call it a correspondence is maybe a bit misleading. We do not claim that the space of quantum open string field theories is isomorphic to the space of closed string Maurer Cartan elements since we cannot argue that $\mathfrak{F}$ is an isomorphism.

An interesting problem is then to find the closed string Maurer Cartan elements or at least to see if they exist. The general ansatz for a Maurer Cartan element of an IBL_∞-algebra is given in (48). However, the loop algebra of closed strings is a special case of an IBL_∞-algebra defined by a collection $\sum_g \hbar^g L^g$ of first order coderivations and a second order coderivation $\hbar D(\omega_c^{-1})$. In particular, it defines an IBL_∞-algebra without coderivations of order higher than two. Therefore we claim that a generic Maurer Cartan element of the loop algebra is defined by setting $c^{n,g} = 0$ for $n > 2$. Explicitly we make the ansatz

$$\mathfrak{c} = c + \hbar g^{-1},$$

where $c \in A_c$ and $g^{-1} \in A_c^{\wedge 2}$. Assume for a moment that g^{-1}, considered as a map from A_c^* to A_c, is invertible and denote its inverse by g. Then g defines a metric of degree zero on A_c. Let $\{u_i\}$ be a homogeneous basis of A_c and $\{u^i\}$ its dual basis w.r.t. g, that is

$$g({}_iu, u^j) = {}_i\delta^j = g({}^ju, u_i).$$

These two equations are compatible only if we use the sign convention

$$u_i = (-1)^i{}_iu \qquad \text{and} \qquad u^i = {}^iu.$$

Note that the sign convention for the dual basis of an odd symplectic form is different (see Sect. 5.1). With these conventions we can express g^{-1} as

$$g^{-1} = \frac{1}{2} u_i \wedge {}^iu. \tag{50}$$

In the following we relax the assumption that $g^{-1} : A_c^* \to A_c$ is invertible, but still we can express g^{-1} in the form (50) with the corresponding sign convention for u_i and u^i.

The Maurer Cartan equation for this particular ansatz reads

$$\big(L_q + \hbar D(\omega_c^{-1})\big)(e^{c+\hbar g^{-1}}) = 0, \tag{51}$$

where we abbreviated $L_q = \sum_g \hbar^g L^g$.

Let us now disentangle this equation and express it in terms of the vertices $l_q = \sum_g \hbar^g l^g = \pi_1 \circ L_q$. A straightforward calculation yields

$$\Delta\left(e^{c+\hbar g^{-1}}\right) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} e^{c+\hbar g^{-1}} \wedge d_{i_1} \wedge \ldots \wedge d_{i_n} \otimes e^{c+\hbar g^{-1}} \wedge {}^{i_n}d \wedge \ldots \wedge {}^{i_1}d.$$

Next we plug this into Eq. (51) using (8). Multiplying the resulting equation by $e^{-c-\hbar g^{-1}}$ one obtains

$$\sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \big(l_q[c] \circ E(\hbar g^{-1})\big)_n (d_{i_1} \wedge \ldots d_{i_n}) \wedge {}^{i_n}d \wedge \ldots \wedge {}^{i_1}d + \hbar\omega_c^{-1} = 0, \tag{52}$$

where $l_q[c]$ denotes the background shifted vertices. We proceed by decomposing Eq. (52) according to powers in A_c, i.e. we project with π_n onto $A_c^{\wedge n}$:

$$\big(l_q[c] \circ E(\hbar g^{-1})\big)_0 = 0, \tag{53}$$

$$\big(l_q[c] \circ E(\hbar g^{-1})\big)_1 (u_i) \wedge {}^iu + \omega_c^{-1} = 0, \tag{54}$$

$$\big(l_q[c] \circ E(\hbar g^{-1})\big)_n (d_{i_1} \wedge \ldots \wedge d_{i_n}) \wedge {}^{i_1}d \wedge \ldots \wedge {}^{i_n}d = 0, \qquad n \geq 3. \tag{55}$$

Furthermore we can split these equations by comparing coefficients in powers of $\hbar$. Equation (53) gives

$$(l_{cl}[c])_0 = \sum_{n=0}^{\infty} \frac{1}{n!}(l_{cl})_n(c^{\wedge n}) = 0. \tag{56}$$

at order $\hbar^0$, that is c satisfies the equations of motion of closed string field theory and hence defines a closed string background. This is what we already had in the classical case. The new feature is encoded in Eq. (54). The $\hbar^0$ component of this equation reads

$$(l_{cl}[c])_1(u_i) \wedge {}^i u + \omega_c^{-1} = 0. \tag{57}$$

Note that $(l_{cl}[c])_1$ defines the closed string BRST operator in the new background c. We write $(l_{cl}[c])_1 = Q_c[c]$. Equation (57) looks unfamiliar so far, but we can represent it in a more convenient form: First, we make use of the isomorphism $c_1 \wedge \ldots \wedge c_n \mapsto \frac{1}{n!}\sum_{\sigma \in S_n} c_{\sigma_1} \otimes \ldots \otimes c_{\sigma_n}$ that identifies elements of the symmetric algebra with symmetric tensors. Equation (57) then becomes an equation in $(A_c^{\otimes 2})_{sym}$, the set of second rank symmetric tensors. Now act on the second element of this equation with the isomorphism $\omega_c : A_c \to A_c^*$ and use that $(l_{cl}[c])_1$ is cyclic symmetric w.r.t. ω_c. Following these steps one obtains

$$Q_c[c] \circ h + h \circ Q_c[c] = -1, \tag{58}$$

where 1 denotes the identity map on A_c and $h = g^{-1} \circ \omega_c$. This equation implies that the cohomology of $Q_c[c]$ is trivial. In other words, Eq. (58) is saying that c has to be a background where there are no perturbative closed string excitations. This is in agreement with the standard argument that open string field theory is inconsistent due to closed string poles arising at the one loop level. Here, this result arises directly from analyzing the Maurer Cartan element for the closed string IBL_∞-algebra. We should stress, however, that the triviality of the closed string cohomology is neither necessary nor sufficient. It is not sufficient since we have only analyzed the lowest orders in the expansion of the Maurer Cartan Eq. (51) in u_i and $\hbar$. Furthermore, we have not shown that the IBL_∞ map $\mathfrak{F}$ is an isomorphism. Therefore we cannot exclude the existence of Maurer Cartan elements of $\mathfrak{L}_o$ which are not in the image of $\mathfrak{F}$.

To summarize, we found a class of Maurer Cartan elements of the closed string loop algebra involving a background $c \in A_c$ and a linear map $h : A_c \to A_c$ or equivalently an element $g^{-1} \in A_c^{\wedge 2}$ that can be interpreted, if it is non-degenerate, as the inverse of a metric g on the space of closed string fields. The Maurer Cartan equation implies that c has to be a background that does not admit any physical closed string excitations or, in other words the induced BRST charge $Q_c[c]$ has to have a trivial cohomology. This statement is deduced from Eq. (58), which involves the map h. There are further implications from the Maurer Cartan equation that are summarized by Eq. (51), but their physical meaning remains unclear and need further investigation. We chose a special ansatz for the Maurer Cartan elements by setting $c^{n,g} = 0$ for $n > 2$. However, we find that the conclusion that the closed string background has to have a trivial BRST cohomology persists for a general ansatz for the Maurer Cartan elements.

7. Outlook

We showed that Maurer Cartan elements of the closed string algebra induce consistent quantum open string field theories. Furthermore, we saw that this Maurer Cartan equation singles out closed string backgrounds whose associated BRST charges have a trivial cohomology. However, since we have not established that the IBL_∞-morphism between the closed string loop algebra and the IBL-algebra on the cyclic Hochschild complex of open strings is an isomorphism, the absence of perturbative closed string states is not proved to be necessary. It would be interesting if such an isomorphism could be established, possibly along the lines of [16]. On the other hand, triviality of the closed string cohomology is not sufficient either since there are further implications at higher orders in $\hbar$ whose physical interpretation is not clear yet. Progress in this direction should be useful to classify consistent open string field theories.

On another front it would be interesting to see how other versions of string field theory such as boundary string field theory [28–32] as well as toplogical strings [33,34] and refinements thereof [35] fit into the framework of homotopy algebra [36–38]. Finally, we expect that there should be a suitable generalization of the homotopy algebras described here, which encode the structure of superstring field theory [39].

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A. Symplectic Structures in String Field Theory

Here we review the basic ingredients in the formulation of bosonic string field theory [9,10,12,14,15]. Strings are described by a conformal field theory on the world sheet, where we denote the spatial resp. time coordinate by σ^1 resp. σ^2. This conformal field theory comprises matter and ghosts, where the ghosts arise from gauge fixing the Polyakov action. The space of states A corresponding to that conformal field theory (which is isomorphic to the space of local operators) is the space in which the string fields reside. Furthermore the ghosts endow the vector space A with a $\mathbb{Z}$-grading - the ghost number. In addition we can define an odd symplectic structure ω on A via the bpz conjugation. This symplectic structure is of outstanding importance for the formulation of string field theory, since the BV operator Δ and the odd Poisson bracket $(\cdot,\cdot)$ (the two operations that appear in the BV master equation for the string field action S) are constructed with the aid of ω.

A.1. Open strings. The world sheet of an open string is topologically the infinite strip $(0,\pi)\times\mathbb{R}$. By the conformal mapping $z=-e^{-iw}$ ($w=\sigma^1+i\sigma^2, (\sigma^1,\sigma^2)\in(0,\pi)\times\mathbb{R}$), the strip is mapped to the upper half plane H. The fields living on H can be separated into holomorphic and anti-holomorphic parts, but due to the boundary conditions these two parts combine to a single holomorphic field defined on the whole complex plane $\mathbb{C}$. We expand each field in a Laurant series (mode expansion)

$$i\partial X^\mu(z)=\sum_{n\in\mathbb{Z}}\frac{\alpha_n^\mu}{z^{n+1}},\quad c(z)=\sum_{n\in\mathbb{Z}}\frac{c_n}{z^{n-1}},\quad b(z)=\sum_{n\in\mathbb{Z}}\frac{b_n}{z^{n+2}}, \tag{59}$$

where the conformal weights are $h_{\partial X} = 1$, $h_c = -1$, $h_b = 2$, and the modes satisfy the commutation relations

$$[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}, \quad \{c_m, b_n\} = \delta_{m+n,0}. \tag{60}$$

The space of states $\tilde{A}_o$ is generated by acting with the creation operators on the $SL(2,\mathbb{R})$ invariant vacuum $|0,k\rangle$, where k denotes the momentum. The grading on $\tilde{A}_o$ is induced by assigning ghost number one to c, minus one to b and zero to X, i.e. every c mode increases the ghost number by one whereas the b modes decrease the ghost number by one. Utilizing the operator state correspondence, we can identify every state $\Psi \in \tilde{A}_o$ with a local operator $\mathcal{O}_\Psi$ and define the bpz inner product by [10]

$$(\Psi_1, \Psi_2)_{bpz} := \lim_{z\to 0} \left\langle (I^*\mathcal{O}_{\Psi_1})(z)\mathcal{O}_{\Psi_2}(z)\right\rangle_H, \tag{61}$$

where $I(z) = -1/z$, $\langle \dots \rangle_H$ is the correlator on the upper half plane and $I^*\mathcal{O}$ denotes the conformal transformation of $\mathcal{O}$ w.r.t. I. Since the correlator is $SL(2,\mathbb{R})$ invariant and $I \in SL(2,\mathbb{R})$, the bpz inner product is graded symmetric. Note that this correlator is non-vanishing only if it is saturated by three c ghost insertions, i.e. the correlator and consequently the pbz inner product carries ghost number -3. The classical string field is an element in $\tilde{A}_o$ of definite ghost number. From the kinetic term of the string field action $S_{kin} = \frac{1}{2}(\Psi, Q_o\Psi)_{bpz}$ [10], where Q_o is the open string BRST charge which carries ghost number one, we can conclude that the classical open string field Ψ must have ghost number one.

Now we would like to identify the bpz inner product with the odd symplectic structure ω, but at first sight this identification seems to fail since the bpz inner product is graded symmetric rather than graded anti-symmetric. To overcome that discrepancy we shift the degree by one (see Sect. 3.1) which turns an odd graded symmetric inner product into an odd symplectic structure

$$\omega_o := (\cdot,\cdot)_{bpz} \circ (s \otimes s) : A_o \otimes A_o \to \mathbb{C}, \tag{62}$$

where $A_o := s^{-1}\tilde{A}_o$.

To summarize we have an odd symplectic structure ω_o on A_o of degree -1 and the classical open string field is a degree zero element in A_o.

A.2. Closed strings. The topology of closed strings is that of an infinite cylinder. The conformal mapping $z = e^{-iw}$ maps the cylinder to the complex plane. Now we get twice as many modes as in the open string since the holomorphic modes are independent of the antiholomorphic ones.

$$i\partial X^\mu(z) = \sum_{n\in\mathbb{Z}} \frac{\alpha_n^\mu}{z^{n+1}}, \quad c(z) = \sum_{n\in\mathbb{Z}} \frac{c_n}{z^{n-1}}, \quad b(z) = \sum_{n\in\mathbb{Z}} \frac{b_n}{z^{n+2}}, \tag{63}$$

$$i\bar{\partial} X^\mu(\bar{z}) = \sum_{n\in\mathbb{Z}} \frac{\tilde{\alpha}_n^\mu}{\bar{z}^{n+1}}, \quad \tilde{c}(\bar{z}) = \sum_{n\in\mathbb{Z}} \frac{\tilde{c}_n}{\bar{z}^{n-1}}, \quad \tilde{b}(\bar{z}) = \sum_{n\in\mathbb{Z}} \frac{\tilde{b}_n}{\bar{z}^{n+2}}. \tag{64}$$

The construction of the vector space $\tilde{A}_c$ is equivalent to that of the open string, except that we constrain the space to the subset of states annihilated by $b_0 - \tilde{b}_0$ and furthermore impose the level matching condition [12]. We assign ghost number one to c and $\tilde{c}$, minus one to b and $\tilde{b}$ and zero to X. The correlator on the complex plane $\langle \ldots \rangle_{\mathbb{C}}$ is zero unless we saturate it with three c ghost and three $\tilde{c}$ ghost insertions, i.e. the correlator $\langle \ldots \rangle_{\mathbb{C}}$ has ghost number -6. The bpz inner product is defined by [12]

$$(\Phi_1, \Phi_2)_{bpz} := \lim_{|z|\to 0} \big\langle (I^* \mathcal{O}_{\Phi_1})(z, \bar{z}) \mathcal{O}_{\Phi_2}(z, \bar{z}) \big\rangle , \tag{65}$$

where $\mathcal{O}_\Phi$ is again the local operator corresponding to the state $\Phi \in \tilde{A}_c$ and $I(z, \bar{z}) = (1/z, 1/\bar{z})$. In contrast to open string field theory the kinetic term of closed string field theory is defined by an additional insertion of $c_0^- = \frac{1}{2}(c_0 - \tilde{c}_0)$, i.e. $S_{kin} = \frac{1}{2}(\Psi, c_0^- Q\Psi)_{bpz}$ [12]. This shows that the ghost number of the classical closed string field Φ has to be 2. To unify the presentation we shift the degree by two, such that the classical closed string field is a degree zero element in $A_c := s^{-2}\tilde{A}_c$. The odd symplectic structure of closed string field theory $\omega_c : A_c \otimes A_c \to \mathbb{C}$ is then identified as

$$\omega_c := \big(\cdot, c_0^- \cdot\big)_{bpz} \circ (s^2 \otimes s^2). \tag{66}$$

Due to the shift and the c_0^- insertion, ω_c is graded anti-symmetric and has degree -1.

B. BV Master Equation and QOCHA

In this section we show that the algebraic relations imposed by the BV master equation in open-closed string field theory are equivalent to the QOCHA. Preliminary we review the BV formalism of open-closed string field theory [13].

Let $A = \bigoplus_n A_n$ be a graded vector space over a field $\mathbb{F}$ endowed with an odd symplectic structure ω and $\{e_i\}$ be a homogeneous basis of A. The dual basis w.r.t. ω is denoted by $\{e^i\}$,

$$\omega({}_i e, e^j) = \omega({}^j e, e_i) = {}_i\delta^j,$$

where we use again the sign convention ${}_i e = (-1)^i e_i$ and ${}^i e = (-1)^{i+1} e^i$ (see Sect. 5.1). The corresponding bases of forms in A^* are denoted by $\{\sigma_i\}$ and $\{\sigma^i\}$, i.e.

$${}_i\sigma(e^j) = {}_i\delta^j = {}^j\sigma(e_i).$$

Consistency of these two equations requires the sign convention $\sigma^i = {}^i\sigma$ and $\sigma_i = {}_i\sigma$. We can consider the vector space A as a supermanifold. The points in this supermanifold are vectors $c \in A$ and expressed in components $c = c^i\,{}_i e = c_i\,{}^i e = e_i\,{}^i c = e^i\,{}_i c$. The tangent space of that manifold is spanned by the collection of derivatives w.r.t. the components of c. We distinguish between left and right derivatives. A left resp. right derivative acts from the left resp. right and is labelled by an arrow $\rightharpoonup$ resp. $\leftharpoonup$. We define

$$\begin{aligned} {}_i\partial &:= \overset{\rightharpoonup}{\partial}_{c^i}, \qquad {}^i\partial := \overset{\rightharpoonup}{\partial}_{c_i}, \\ \partial_i &:= \overset{\leftharpoonup}{\partial}_{{}^i c}, \qquad \partial^i := \overset{\leftharpoonup}{\partial}_{{}_i c}, \end{aligned}$$

and the differential of a function $f \in C^\infty(A)$ is defined by

$$df = \sigma^i\,{}_i\partial f = \sigma_i\,{}^i\partial f = f\partial_i\,{}^i\sigma = f\partial^i\,{}_i\sigma.$$

With this convention we have for example ${}_i\partial c = {}_i e$.

To every function $f \in C^\infty(A)$ we can assign a Hamiltonian vector field $X_f \in \mathrm{Vect}(A)$ by

$$df = -i_{X_f}\omega, \tag{67}$$

where i_X denotes the interior product, i.e. the contraction w.r.t. to the vector field X. The odd Poisson bracket (antibracket) $(\cdot, \cdot)$ is then defined by [5],

$$(f, g) = X_f(g), \tag{68}$$

for $f, g \in C^\infty(A)$. The BV operator Δ is defined by [5],

$$\Delta f = \frac{1}{2}\mathrm{div} X_f, \tag{69}$$

and squares to zero $\Delta^2 = 0$ since ω has degree -1. Here we suppress the superscript BV for the BV operator since it cannot be confused with the comultiplications in the present context. In components we get

$$(f, g) = (-1)^i f\partial_i{}^i\partial g \qquad \text{and} \qquad \Delta f = \frac{1}{2}{}_i\partial^i\partial f.$$

Equivalently the odd Poisson bracket can be defined via the BV operator

$$(f, g) = (-1)^f\Big(\Delta(fg) - \Delta(f)g - (-1)^f f\Delta(g)\Big), \tag{70}$$

i.e. the odd Poisson bracket is the deviation of Δ being a derivation. The BV operator is indeed a second order derivation [8], that is

$$\begin{aligned}\Delta(fgh) - \Delta(fg)h - (-1)^{f(g+h)}\Delta(gh)f - (-1)^{h(f+g)}\Delta(hf)g \\ + \Delta(f)gh + (-1)^{f(g+h)}\Delta(g)hf + (-1)^{h(f+g)}\Delta(h)fg = 0.\end{aligned}$$

Furthermore the following identities hold [8]:

$$\begin{aligned}\Delta(f, g) &= (\Delta f, g) + (-1)^{f+1}(f, \Delta g),\\ 0 &= (-1)^{(f+1)(g+1)}(f, (g, h)) + (-1)^{(g+1)(f+1)}(g, (h, f))\\ &\quad + (-1)^{(h+1)(g+1)}(h, (g, f)),\\ (f, gh) &= (f, g)h + (-1)^{(f+1)g}g(f, h).\end{aligned}$$

The first is saying that Δ is a derivation over $(\cdot, \cdot)$, the second is the Jacobi identity for $(\cdot, \cdot)$ and the third is saying that $(f, \cdot)$ is a derivation of degree $|f| + 1$ on the space of functions.

In open-closed string field theory the vector space is the direct sum $A = A_o \oplus A_c$ and the symplectic structure is $\omega = \omega_o \oplus \omega_c$. Hence the BV operator and the odd Poisson bracket also split into open and closed parts:

$$\Delta = \Delta_o + \Delta_c, \qquad (\cdot, \cdot) = (\cdot, \cdot)_o + (\cdot, \cdot)_c.$$

The quantum BV master equation reads

$$\hbar\Delta S + \frac{1}{2}(S, S) = 0,$$

where

$$S=\sum_{g=0}^{\infty}\hbar^{2g-1}\omega_c(l^g,\cdot)(e^{\hbar^{1/2}c})+\sum_{b=1}^{\infty}\sum_{g=0}^{\infty}\frac{1}{b!}\hbar^{2g+b-1}f^{b,g}(e^{\hbar^{1/2}c};\bar{e}^a,\dots,\bar{e}^a),$$

is the BV action of Eq. (21). Before we consider the general case, let us restrict to open string field theory in the classical limit. In this case the action reads

$$S_{o,cl}=\sum_{n=1}^{\infty}\frac{1}{n+1}\omega_o(m_n(a^{\otimes n}),a). \tag{71}$$

The classical BV equation of open string field theory is

$$\begin{aligned}
&(S_{o,cl},S_{o,cl})_o\\
&=(-1)^i\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\frac{1}{n_1+1}\frac{1}{n_2+1}\omega_o\big(m_{n_1}(a^{\otimes n_1}),a\big)\,\partial_i\,{}^i\partial\,\omega_o\big(m_{n_2}(a^{\otimes n_2}),a\big)\\
&=(-1)^i\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\omega_o\big(m_{n_1}(a^{\otimes n_1}),e_i\big)\,\omega_o\big(m_{n_2}(a^{\otimes n_2}),{}^ie\big)\\
&=\omega_o\big(m(e^a),e_i\big)\,\omega_o\big({}^ie,m(e^a)\big)=\omega_o\big(m(e^a),e_i\big)\,{}^i\sigma\big(m(e^a)\big)\\
&=\omega_o\big(m(e^a),m(e^a)\big)=\sum_{n=1}^{\infty}\frac{2}{n+2}\sum_{i+j+k=n}\omega_o\Big(m_{i+k+1}\big(a^{\otimes i}\otimes m_j(a^{\otimes j})\otimes a^{\otimes k}\big),a\Big)\\
&=\sum_{n=1}^{\infty}\frac{2}{n+2}\omega_o\big(\pi_1\circ M^2(a^{\otimes n}),a\big)=0.
\end{aligned} \tag{72}$$

All we had to use is cyclicity of m_n and $e_i\otimes{}^i\sigma=1$, where 1 denotes the identity map on A_o. $M\in\mathrm{Coder}^{cycl}(TA_o)$ is the coderivation corresponding to $m\in\mathrm{Hom}^{cylc}(TA_o,A_o)$ and Eq. (72) is equivalent to $M^2=0$, the well known statement that the vertices of a classical open string field theory define an A_∞-algebra [14]. Schematically we write

$$(S_{o,cl},S_{o,cl})_o\sim\omega_o\Big(m\big(e^a\otimes m(e^a)\otimes e^a\big),a\Big),$$

i.e. $\sim$ indicates that we will ignore the precise coefficients.

In order to keep the presentation clear, we will use this notation for the treatment of the quantum BV master equation of open and closed strings. Furthermore we abbreviate $c'=\hbar^{1/2}c$. We just collect the results here since the calculations are quite similar to that in (72). From the open string BV operator we get

$$\begin{aligned}
\hbar\Delta_oS\sim&\sum_{b,g}\hbar^{2g+b}\sum_{k=1}^{b}f^{b,g}(e^{c'};e^a,\dots,\underbrace{e_i\otimes e^a\otimes e^i\otimes e^a}_{k\text{-th boundary}},\dots,e^a)\\
&+\sum_{b,g}\hbar^{2g+b}\sum_{k\neq l}f^{b,g}(e^{c'};e^a,\dots,\underbrace{e_i\otimes e^a}_{k\text{-th bdry}},\dots,\underbrace{e^i\otimes e^a}_{l\text{-th bdry}},\dots,e^a).
\end{aligned} \tag{73}$$

The first term in (73) translated into homotopy language is equivalent to

$$\sum_{b,g} \hbar^{2g+b}\,\widehat{\delta} \circ f^{b,g},$$

whereas the second term is equivalent to

$$\sum_{b,g} \hbar^{2g+b}\,\widehat{[\cdot,\cdot]} \circ f^{b,g}.$$

Here we see that Δ_o partly translates into $\widehat{[\cdot,\cdot]}$ as anticipated in Sect. 5.3. The closed string BV operator contributes

$$\begin{aligned}\hbar\Delta_c S \sim &\sum_{g} \hbar^{2g+1}\omega_c(l^g,\cdot)(\omega_c^{-1}\wedge e^{c'})\\ &+\sum_{b,g}\hbar^{2g+b+1} f^{b,g}(\omega_c^{-1}\wedge e^{c'};e^a,\ldots,e^a).\end{aligned} \tag{74}$$

The second term in (74) is equivalent to

$$\sum_{b,g}\hbar^{2g+b+1} f^{b,g}\circ D(\omega_c^{-1}).$$

Next consider the open string Poisson bracket

$$\begin{aligned}(S,S)_o \sim &\sum_{\substack{g_1,g_2\\ b_1,b_2}} \hbar^{2(g_1+g_2)+b_1+b_2-2}\sum_{k,l=0}^{b} f^{b_1,g_1}(e^{c'};e^a,\ldots,\underbrace{e_i\otimes e^a}_{k\text{-th bdry}},\ldots,e^a)\\ &\cdot f^{b_2,g_2}(e^{c'};e^a,\ldots,\underbrace{e_i\otimes e^a}_{l\text{-th bdry}},\ldots,e^a).\end{aligned} \tag{75}$$

Equation (75) is equivalent to

$$\sum_{\substack{g_1,g_2\\ b_1,b_2}} \hbar^{2(g_1+g_2)+b_1+b_2-2}\,\frac{1}{2}\widehat{[\cdot,\cdot]}'\left(f^{b_1,g_1}\wedge f^{b_2,g_2}\right)\circ\Delta, \tag{76}$$

where the prime indicates that the first resp. second input must be out of f^{b_1,g_1} resp. f^{b_2,g_2}. If we express (76) in terms of the unrestricted $\widehat{[\cdot,\cdot]}$, we have to compensate by subtracting twice the part where $\widehat{[\cdot,\cdot]}$ acts on only one of the f's, i.e.

$$\begin{aligned}&\sum_{\substack{g_1,g_2\\ b_1,b_2}} \hbar^{2(g_1+g_2)+b_1+b_2-2}\,\frac{1}{2}\widehat{[\cdot,\cdot]}'\left(f^{b_1,g_1}\wedge f^{b_2,g_2}\right)\circ\Delta\\ &\quad=\sum_{\substack{g_1,g_2\\ b_1,b_2}} \hbar^{2(g_1+g_2)+b_1+b_2-2}\left(\frac{1}{2}\widehat{[\cdot,\cdot]}\left(f^{b_1,g_1}\wedge f^{b_2,g_2}\right)-\left(\widehat{[\cdot,\cdot]}f^{b_1,g_1}\right)\wedge f^{b_2,g_2}\right)\circ\Delta.\end{aligned}$$

Finally the closed string Poisson bracket yields

$$
\begin{aligned}
(S,S)_c \sim & \sum_{g_1,g_2} \hbar^{2g_1+2g_2-1}\ \omega_c\Big(l^{g_1}\big(l^{g_2}(e^{c'})\wedge e^{c'}\big),c'\Big) \\
& +2\sum_{g_1,g_2,b} \hbar^{2(g_1+g_2)+b-1}\ f^{b,g_1}\big(l^{g_2}(e^{c'})\wedge e^{c'};e^a,\dots,e^a\big) \\
& +\sum_{\substack{g_1,g_2\\ b_1,b_2}} \hbar^{2(g_1+g_2)+b_1+b_2-1}\ f^{b_1,g_1}\big(e_i\wedge e^{c'};e^a,\dots,e^a\big) \\
& \cdot f^{b_2,g_2}\big(e^i\wedge e^{c'};e^a,\dots,e^a\big).
\end{aligned} \tag{77}
$$

The second term in (77) is equivalent to

$$
\sum_{g_1,g_2,b} \hbar^{2(g_1+g_2)+b-1} f^{b,g_1}\circ L^{g_2},
$$

while the third term is equivalent to

$$
\sum_{\substack{g_1,g_2\\ b_1,b_2}} \hbar^{2(g_1+g_2)+b_1+b_2-1}\big(f^{b_1,g_1}\circ D(e_i)\wedge f^{b_2,g_2}\circ D(e^i)\big)\circ\Delta.
$$

We see that the third term is associated with $D(\omega_c^{-1})$, that is $(\cdot,\cdot)_c$ plays partly the role of $D(\omega_c^{-1})$ as we pointed out in 5.3. The fact that second order derivations in the BV formalism translate into not just second order but also first order coderivations in the homotopy language and vice versa, is actually the reason why the powers in $\hbar$ in the BV formalism ($\hbar^{2g+b+n/2-1}$) differ from that in the homotopy language ($\hbar^{g+b-1}$).

Let us collect the individual terms now. First consider the terms with closed strings only. By comparing coefficients in $\hbar$, we recover the loop algebra of closed strings:

$$
\sum_{\substack{g_1+g_2=g\\ i_1+i_2=n}} \sum_{\sigma}{}' \, l^{g_1}_{i_1+1}\circ(l^{g_2}_{i_2}\wedge 1^{\wedge i_1})\circ\sigma\ +\ l^{g-1}_{n+2}(\omega_c^{-1}\wedge 1^{\wedge n})=0.
$$

Finally turn to the parts with open and closed strings. First project onto $\mathcal{A}_o^{\wedge b}$, i.e. terms with a definite number of boundaries, and then compare coefficients in $\hbar$. Following that procedure we precisely obtain the QOCHA:

$$
\begin{aligned}
& \mathfrak{f}\circ\mathfrak{L}_c+\frac{\hbar}{2}\big(\mathfrak{f}\circ D(e_i)\wedge\mathfrak{f}\circ D(e^i)\big)\circ\Delta \\
& =\mathfrak{L}_o\circ\mathfrak{f}+\frac{1}{2}[\cdot,\cdot]\circ\big(\mathfrak{f}\wedge\mathfrak{f}\big)\circ\Delta-(([\cdot,\cdot]\circ\mathfrak{f})\wedge\mathfrak{f})\circ\Delta.
\end{aligned} \tag{78}
$$

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Homotopy Classification of Bosonic String Field Theory

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Abstract: We prove the decomposition theorem for the loop homotopy algebra of quantum closed string field theory and use it to show that closed string field theory is unique up to gauge transformations on a given string background and given S-matrix. For the theory of open and closed strings we use results in open-closed homotopy algebra to show that the space of inequivalent open string field theories is isomorphic to the space of classical closed string backgrounds. As a further application of the open-closed homotopy algebra we show that string field theory is background independent and locally unique in a very precise sense. Finally we discuss topological string theory in the framework of homotopy algebras and find a generalized correspondence between closed strings and open string field theories.

Contents

1. Introduction and Summary

Historically, the first consistent, interacting formulation of string field theory is Witten's open cubic string field theory [1–3]. Its algebraic structure is rather simple: The BRST differential Q and the star product $*$, which define the kinetic term and the cubic interaction respectively, satisfy the axioms of a differential graded associative algebra (DGA). More generally, it turns out [4] that any formulation of open string field theory realizes an A_∞-algebra, a generalization of a DGA where associativity holds only up to homotopy.

The general procedure of constructing covariant string field theory, as described by Zwiebach [5,6], requires a decomposition of the relevant moduli space of Riemann surfaces into elementary vertices and graphs. This decomposition guarantees a single cover of moduli space via Feynman rules and implies that the vertices satisfy a BV master equation. In a second step one employs the operator formalism of the world sheet conformal field theory to construct a morphism of BV algebras from the moduli space to the space of multilinear functions on the (restricted) state space of the conformal field theory. This is where background dependence enters in the construction.

At the classical level, the multilinear maps on the state space of the CFT satisfy the axioms of an A_∞- (open string) or L_∞- (closed string) algebra. The classification of physically inequivalent string field theories is then obtained with the help of the decomposition theorem [8,7]. This theorem establishes an isomorphism between a given homotopy algebra and the direct sum of a linear contractible algebra and a minimal model. In the context of string field theory, the structure maps of the minimal model are identical to the tree-level S-matrix elements of the perturbative string theory in the string background corresponding to the trivial Maurer-Cartan element of the homotopy algebra [7,9].

One purpose of this paper is to extend this classification to quantum closed SFT. To this end we proof the decomposition theorem for loop homotopy algebras, which are a special case of IBL_∞-algebras. We then utilize the decomposition theorem to show that string field theory is unique up to gauge transformations on a given string background. More precisely, two string field theories constructed on the same string background, in particular inducing the same S-matrix, are connected by a 1-parameter family of strong IBL_∞-isomorphisms. This is the algebraic counterpart of the statement that the string vertices at the geometric level define an unique element in the cohomology of the boundary operator plus BV operator on the moduli space of Riemann surfaces [10–12].

Given the above result one is naturally led to ask if changes in the closed string background are the only non-trivial deformations of closed string field theory compatible with the operator formalism. We will answer this question within the restriction to deformations which leave the state space of the CFT invariant. In this case we will first establish background independence which amounts to proving that shifts in the closed string background are equivalent to conjugation by Maurer-Cartan elements of the homotopy algebra. Since such transformations correspond to weak IBL_∞-isomorphisms we can define bigger equivalence classes where different closed string backgrounds are identified. We then establish uniqueness of closed string field theory in the sense that there is no non-trivial infinitesimal deformation of closed string field theory compatible with the operator formalism.

Next we turn to open-closed string field theory. The reformulation of open-closed SFT in terms of homotopy algebras (see [26,27] for the classical case and [21] for the quantum theory) relates (quantum) closed, open and open-closed vertices of the SFT to structure maps of $(IB)L_\infty$-, A_∞-algebras and $(IB)L_\infty$-morphism respectively. As we will explain, classical closed string Maurer-Cartan elements (closed string backgrounds) modulo closed string gauge transformations, are in one-to-one correspondence with classically consistent open string field theories modulo gauge transformations, which include open string background transformations as well as open string field redefinitions. Thus a closed string background not only determines a unique closed string field theory but also a unique classical open SFT, modulo gauge transformations.

We will show that the latter isomorphism persists at the quantum level although the complete quantum closed string Maurer-Cartan equation will generically have no solutions, which is a reflection of the fact that a SFT of just open strings is quantum mechanically incomplete. The exception to this is when the closed string symplectic structure is degenerate on shell, i.e. on the cohomology of the closed string BRST operator. This is one of the distinguishing features of the topological string. In the latter case the Maurer-Cartan equation decomposes into two irreducible parts: an equation for the background and linear equation for the propagator.

We should also emphasize the relevance of the open-closed correspondence in establishing background independence of closed string field theory described above. The isomorphism just described is instrumental in establishing background independence within the class of backgrounds that preserve the vector space of perturbative fluctuations. The details of this will be explained in the text.

2. The Homotopy Algebra of String Field Theory

String vertices represent subspaces, i.e. singular chains, of the moduli space of Riemann surfaces. The corresponding chain complex admits the structure of a BV algebra [5,6]. The basic requirement for any SFT, that it reproduces the S-matrix amplitudes of perturbative string theory, translates into the statement that the singular chains defining the string vertices satisfy the BV master equation. This is the background independent data of SFT [10,11]. A string background determines a world sheet conformal field theory where the state space A of this CFT (or a certain restriction thereof) is equipped with an odd symplectic structure ω. This in turn makes the space $C(A)$ of functions on A (the space of multilinear maps on A with suitable symmetry properties) a BV algebra. The world sheet CFT defines a morphism of BV algebras which implies that the BV master equation is also satisfied at the level of $C(A)$ [5,6].

The most general theory involves open and closed strings and we have to consider the moduli spaces $\mathcal{P}^{b,g}_{n,m}$ [6], where g is the genus, b is the number of boundaries, n is the number of closed string punctures and $m = (m_1, \ldots, m_b)$ where m_i is the number of open string punctures on the i-th boundary. Furthermore, the geometric vertices which we will denote by $\mathcal{V}^{b,g}_{n,m} \subset \mathcal{P}^{b,g}_{n,m}$, have to be invariant under the following transformations:

(i) cyclic permutation of open string punctures on one boundary
(ii) arbitrary permutation of closed string punctures
(iii) arbitrary permutation of boundaries

Consider now a fixed background, that defines a world sheet CFT. The corresponding state space of open strings is denoted by A_o and the restricted state space of closed strings (those states annihilated by b_0^- and L_0^-) by A_c. We use the conventions where the string fields have degree zero, both in the closed string and the open string sector [6,21]. The world sheet CFT preserves the above symmetry properties, that is

$$\mathcal{P}^{b,g}_{n,m} \supset \mathcal{V}^{b,g}_{n,m} \mapsto f^{b,g}_{n,m} \in \mathrm{Hom}\big(A_c^{\wedge n} \otimes (A_o^{\otimes m_1})^{cycl} \wedge \ldots \wedge (A_o^{\otimes m_b})^{cycl}, R\big) ,$$

where $\wedge$ denotes the graded symmetric product and R is the module of commuting and anti-commuting numbers. The maps $f^{b,g}_{n,m}$ are the algebraic string vertices. In the following we will usually not distinguish between algebraic and geometric vertices, whenever the meaning is clear from the context. The string field theory action for the open string field $a \in A_o$ and the closed string field $c \in A_c$ is then given by the sum of all string vertices, weighted with appropriate powers of $\hbar$ and symmetry factors [6]:

$$S(c, a) = \sum_{b,g} \sum_{n,m} \frac{1}{b!} \frac{1}{n!} \frac{1}{m_1 \ldots m_b} \, \hbar^{2g+b+n/2-1} \, f^{b,g}_{n,m}\big(c^{\wedge n}; a^{\otimes m_1}, \ldots, a^{\otimes m_b}\big) . \tag{1}$$

The quantum BV master equation reads

$$\hbar \Delta^{BV} S + \frac{1}{2}(S, S) = 0 , \tag{2}$$

where Δ^{BV} is the BV operator induced by the odd symplectic structure ω (bpz inner product) on the state space of the world sheet CFT, and $(\cdot, \cdot)$ is the associated odd Poisson bracket (antibracket) [21,25]. Since the odd symplectic structure splits into open and closed parts $\omega = \omega_o + \omega_c$, the BV operator and the odd Poisson bracket split as well:

$$\Delta^{BV} = \Delta^{BV}_o + \Delta^{BV}_c , \qquad (\cdot, \cdot) = (\cdot, \cdot)_o + (\cdot, \cdot)_c .$$

The geometric counterpart of Δ^{BV}_o and Δ^{BV}_c at the level of chain complexes of moduli spaces is the sewing of open and closed string punctures, respectively. The homotopy algebra corresponding to that full-blown theory is the quantum open-closed homotopy algebra (QOCHA) [21], but there are many sub-algebras corresponding to certain limits of this theory, which will be discussed in the following.

2.0.1. Classical Theory. Let us consider the limit where we restrict to those moduli spaces that are closed under sewing at tree level. For open SFT the relevant surfaces are discs with punctures on the boundary, whereas in closed SFT we have to consider punctured spheres.

Similarly such a theory satisfies the classical BV master equation

$$(S, S) = 0 .$$

In classical open SFT, the action thus reads (see equation (1))

$$S(a) = \sum_n \frac{1}{n} f^{1,0}_{0,n}(a^{\otimes n}) ,$$

Fig. 1. punctured disc/sphere

and the classical BV master equation implies that the multilinear maps $m_n : A_o^{\otimes n} \to A_o$ defined by

$$\omega_o(m_n\,,\,\cdot\,) := f^{1,0}_{0,n+1}$$

satisfy the relations of an A_∞-algebra[1] [4]. Similarly the multilinear maps $l_n : A_c^{\wedge n} \to A_c$ associated to the classical action $S(c)$ of closed SFT (after absorbing $\hbar^{1/2}$ in the closed string field)

$$S(c) = \sum_n \frac{1}{n!} f^{0,0}_{n,0}(c^{\wedge n})\,, \qquad \omega_c(l_n\,,\,\cdot\,) := f^{0,0}_{n+1,0}\,,$$

obey the relations of a L_∞-algebra [5].

Finally, there is also a sub-algebra corresponding to a theory of open and closed strings. We consider spheres with closed string punctures, discs with open string punctures and additionally discs with open and closed punctures.

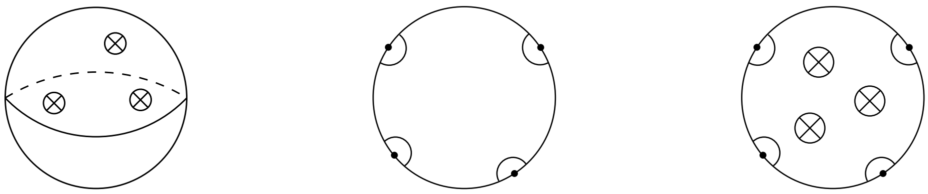

Fig. 2. surfaces of 'classical' open-closed SFT

In order to make this theory well defined, we have to exclude the operation of sewing a closed string puncture on one disc to another closed string puncture on a second disc. This would produce surfaces with more than one boundary, i.e. surfaces which are not part of the theory. Physically speaking, we treat the closed string as an external field. After absorbing $\hbar^{1/2}$ in the closed string field, the action reads

$$S(c,a) = \frac{1}{\hbar}\sum_n \frac{1}{n!} f^{0,0}_{n,0}(c^{\wedge n}) + \sum_n \frac{1}{n} f^{1,0}_{0,n}(a^{\otimes n}) + \sum_{n,m} \frac{1}{n!}\frac{1}{m} f^{1,0}_{n,m}(c^{\wedge n}; a^{\otimes m})\,,$$

and satisfies the classical BV master equation to order $\hbar^0$ (Note that the closed string Poisson bracket is proportional to $\hbar$ in this normalization.). Translated into the language of homotopy algebras we get the following: Let's define multilinear maps $n_{n,m} : A_c^{\wedge n} \otimes A_o^{\otimes m} \to A_o$ associated to discs with open and closed punctures by

$$\omega_o(n_{n,m}\,,\,\cdot\,) := f^{1,0}_{n,m+1}\,.$$

[1] An A_∞-algebra actually corresponds to the case of a single D-brane. For several D-branes, one obtains a Calabi-Yau A_∞ category (See for example [12,31]).

Furthermore we collect the individual maps to

$$\begin{aligned} l &:= \sum_n l_n \ : \ SA_c \to A_c \\ m &:= \sum_n m_n \ : \ TA_o \to A_o \\ n &:= \sum_{n,m} n_{n,m} \ : \ SA_c \otimes TA_o \to A_o \ , \end{aligned}$$

where TA and SA denote the tensor algebra and the graded symmetric tensor algebra respectively. To the first two maps we can associate a coderivation (see appendix A for details about coderivations and homotopy algebras). That is,

$$\begin{aligned} L &:= \widehat{l} \ \in \mathrm{Coder}^{cycl}(SA_c) \\ M &:= \widehat{m} \ \in \mathrm{Coder}^{cycl}(TA_o) \\ N &:= \widehat{n} \ : \ SA_c \to \mathrm{Coder}^{cycl}(TA_o) \ , \end{aligned}$$

where the map N, associated to discs with open and closed punctures, induces an L_∞-morphism from the L_∞-algebra (A_c, L) of closed strings to the differential graded Lie algebra $(\mathrm{Coder}^{cycl}(TA_o), d_h, [\cdot,\cdot])$ which controls deformations of the open string field theory (A_o, M) [26,27]:

$$(A_c, L) \xrightarrow{L_\infty-\text{morphism}} (\mathrm{Coder}^{cycl}(TA_o), d_h, [\cdot,\cdot]) \tag{3}$$

More precisely, we have

$$N \circ L = d_h \circ N + \frac{1}{2}[N,N] \circ \Delta \ , \tag{4}$$

where Δ denotes the comultiplication in SA_c. This algebra is called open-closed homotopy algebra (OCHA) [26,27] and will be essential in section 5.

2.0.2. Quantum Theory. At the quantum level there there is no consistent open SFT, since e.g. open string one-loop diagrams can be interpreted as closed string tree-level amplitudes. For a theory of closed strings we have to consider surfaces of arbitrary genus with an arbitrary number of punctures, and the action according to equation (1) reads (after absorbing appropriate powers of $\hbar$)

$$S(c) = \sum_g \sum_n \frac{\hbar^g}{n!} \, f^{0,g}_{n,0}(c^{\wedge n}) \ .$$

We define multilinear maps $l^g_n : A_c^{\wedge n} \to A_c$ via

$$\omega_c(l^g_n \, , \cdot) := f^{0,g}_{n+1,0} \ ,$$

and lift $l^g = \sum_n l^g_n$ to a coderivation

$$L^g := \widehat{l^g} \in \mathrm{Coder}^{cycl}(SA_c) \ .$$

The closed string BV operator Δ^{BV}_c requires the inclusion of a second order coderivation Ω_c^{-1}, which is defined to be the lift of the inverse of the odd symplectic structure ω_c:

$$\Omega_c^{-1} := \widehat{\omega_c^{-1}} \in \mathrm{Coder}^2(SA_c) \ .$$

The main identity of closed string field theory [5] together with the cyclicity condition is equivalent to the statement that $\mathfrak{L}_c \in \mathfrak{coder}(SA_c, \hbar)$ defined by

$$\mathfrak{L}_c := \sum_g \hbar^g L^g + \hbar \, \Omega_c^{-1} \ , \tag{5}$$

squares to zero [22]. This algebra is called loop homotopy algebra, which is obviously a special case of an IBL_∞-algebra (see appendix A).

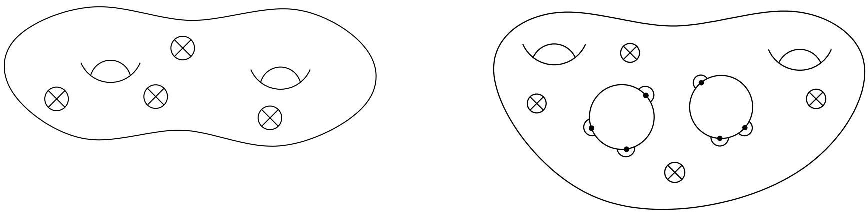

Fig. 3. surfaces in quantum SFT

The algebraic structure of quantum open-closed SFT can be described in a similar way as in 'classical' open-closed SFT. The surfaces with open and closed string punctures define a morphism, but in this case an IBL_∞-morphism rather than a L_∞-morphism. On the open string side of the OCHA (3) we had the differential graded Lie algebra $(\mathrm{Coder}^{cycl}(TA_o), d_h, [\cdot,\cdot])$, but note that due to the isomorphism $\mathrm{Coder}^{cycl}(TA_o) \overset{\pi_1}{\cong} \mathrm{Hom}^{cycl}(TA_o, A_o) \overset{\omega_o}{\cong} \mathrm{Hom}^{cycl}(TA_o, R)$ the Hochschild differential d_h and the Gerstenhaber bracket $[\cdot,\cdot]$ have their counterparts defined on $\mathcal{A}_o := \mathrm{Hom}^{cycl}(TA_o, R)$ (see e.g. [21] for more details), which we will also denote by

$$d_h : \mathcal{A}_o \to \mathcal{A}_o$$

and

$$[\cdot,\cdot] : \mathcal{A}_o^{\wedge 2} \to \mathcal{A}_o \ .$$

(The Gerstenhaber bracket is now symmetric in the inputs and has degree one, since ω_o has degree minus one.) In the following we will work with the space $\mathcal{A}_o$, which is called the cyclic Hochschild complex, rather than with $\mathrm{Coder}^{cycl}(TA_o)$. In order to take account of the open string BV operator Δ_o^{BV}, we have to supplement the differential graded Lie algebra $(\mathcal{A}_o, d_h, [\cdot,\cdot])$ by an additional operation

$$\delta : \mathcal{A}_o \to \mathcal{A}_o^{\wedge 2} \ ,$$

defined by

$$\begin{aligned}&(\delta f)(a_1,\dots,a_n)(b_1,\dots,b_m) \\ &:= (-1)^f \sum_{i=1}^{n}\sum_{j=1}^{m}(-1)^\epsilon f(e_k, a_i,\dots,a_n,a_1,\dots,a_{i-1},e^k,b_j,\dots,b_m,b_1,\dots,b_{j-1}) \ ,\end{aligned} \tag{6}$$

where $(-1)^\epsilon$ denotes the Koszul sign, $\{e_i\}$ is a basis of A_o and $\{e^i\}$ is the corresponding dual basis satisfying $\omega_o({}_ie, e^j) = {}_i\delta^j$ (see [30,21] for the sign conventions for left and right indices). In [31,20] it has been shown that $(\mathcal{A}_o, d_h, [\cdot,\cdot], \delta)$ defines an involutive Lie bialgebra, a special case of an IBL_∞-algebra. In the language of IBL_∞-algebras this is equivalent to the statement that

$$\mathfrak{L}_o := \widehat{d_h} + \widehat{[\cdot,\cdot]} + \hbar\,\widehat{\delta} \ \in \mathfrak{coder}(\mathcal{A}_o, \hbar)$$

squares to zero. Now for $b \geq 1$ and $g \geq 0$, we define maps $n^{b,g} \in \mathrm{Hom}(SA_c, \mathcal{A}_o^{\wedge b})$ by

$$n^{b,g} = \begin{cases} \sum_{n=1}^{\infty}\sum_m f_{n,m}^{1,0} & , b=1,\ g=0 \\ \sum_{n=0}^{\infty}\sum_m f_{n,m}^{b,g} & , \text{else} \end{cases} \ .$$

We exclude $f_{0,n}^{1,0}$ in the sum for $b=1$, $g=0$, since it is already taken into account via the Hochschild differential d_h. Finally, the algebraic structure of quantum open-closed SFT can be summarized in the following way: The open-closed vertices $n^{b,g}$ define an IBL_∞-morphism from the loop homotopy algebra of closed strings A_c to the involutive Lie bialgebra on the cyclic Hochschild complex of open strings $\mathcal{A}_o$

$$(A_c, \mathfrak{L}_c) \xrightarrow{IBL_\infty-\text{morphism}} (\mathcal{A}_o, \mathfrak{L}_o) \ .$$

That is we have

$$e^{\mathfrak{n}} \circ \mathfrak{L}_c = \mathfrak{L}_o \circ e^{\mathfrak{n}} \tag{7}$$

where

$$\mathfrak{n} = \sum_{b=1}^{\infty}\sum_{g=0}^{\infty} \hbar^{b+g-1}\, n^{b,g} \ .$$

This is the quantum open-closed homotopy algebra introduced in [21]. Equation (7) can be recast, such that the five distinct sewing operations in open-closed SFT become apparent [21]:

$$\begin{aligned} &\mathfrak{n}\circ\mathfrak{L}_c + \frac{\hbar}{2}\big(\mathfrak{n}\circ\widehat{e_i}\wedge\mathfrak{n}\circ\widehat{e^i}\big)\circ\Delta \\ &\quad = \mathfrak{L}_o\circ\mathfrak{n} + \frac{1}{2}\widehat{[\cdot,\cdot]}\circ\big(\mathfrak{n}\wedge\mathfrak{n}\big)\circ\Delta - \big((\widehat{[\cdot,\cdot]}\circ\mathfrak{n})\wedge\mathfrak{n}\big)\circ\Delta \ . \end{aligned} \tag{8}$$

In equation (8), e_i and e^i denote a basis and corresponding dual basis of A_c w.r.t. the symplectic structure ω_c. Obviously we recover the OCHA of equation (4) in the limit $\hbar \to 0$.

In [22] it has been shown that the closed string loop homotopy algebra (5) defines an algebra over the Feynman transform of $\mathbf{Mod}(\mathcal{C}om)$. Similarly, it is expected that the QOCHA of open-closed SFT actually describes an algebra over the Feynman transform of a (two colored) operad corresponding to the moduli spaces of [32]. For more information in this direction see [33,34].

3. Decomposition Theorem for closed String Loop Algebra

In the previous section we reformulated the BV master equation for the string vertices as axioms of some homotopy algebra. The connection between the S-matrix of SFT and the perturbative string amplitudes is then established via the minimal model theorem. Consider for example classical open SFT, and denote its corresponding A_∞-algebra by (A, M). The minimal model theorem states that the cohomology $H(A, d)$ of A with respect to the differential $d = \pi_1 \circ M \circ i_1$ admits the structure of an A_∞-algebra, denoted by $(H(A, d), \tilde{M})$, with vanishing differential $\pi_1 \circ \tilde{M} \circ i_1 = 0$. Furthermore, $(H(A, d), \tilde{M})$ and (A, M) are quasi-isomorphic, i.e. there is an A_∞-quasi-isomorphism $\tilde{F} : (H(A, d), \tilde{M}) \to (A, M)$. Note that in SFT the differential d is the BRST operator and the BRST cohomology $H(A, d)$ represents the physical states.

The construction of the minimal model is, in fact, identical to the construction of tree level S-matrix amplitudes via Feynman rules: First one chooses a certain gauge, such that we can define a propagator. With the aid of the propagator we construct all possible trees with vertices labeled $m_n := \pi_1 \circ M \circ i_n$ and internal lines labeled by the propagator. The collection of all these trees, restricted to the cohomology $H(A, d)$, then defines the multilinear maps $\tilde{m} = \pi_1 \circ \tilde{M}$. Thus $\tilde{m}$ represents the S-matrix amplitudes, and moreover the A_∞ relations for the S-matrix elements, $\tilde{M}^2 = 0$, can be identified as the Ward identities.

The relation between the minimal model and S-matrix amplitudes in classical open SFT is discussed in [7,9]. In classical closed SFT, the algebraic structure induced by the S-matrix elements on the BRST cohomology is accordingly that of an L_∞-algebra [35], and the minimal model in the context of L_∞-algebras is discussed in [36,8]. Furthermore there is a generalization of the minimal model theorem in the form of the decomposition theorem, which states that an A_∞/L_∞-algebra is isomorphic to the direct sum of a linear contractible part and a minimal part [8,7,9].

In this section we are concerned with analogous statements in quantum closed SFT. The Ward identities of quantum closed SFT can be interpreted as the loop homotopy algebra axioms [5,37]. In chapter 2, we pointed out that loop homotopy algebras are indeed algebras over the Feynman transform of a modular operad [22], and the minimal model theorem corresponding to such algebras has been established in [38,39]. The explicit construction of such minimal models resembles that in the case of A_∞-algebras, but where one has to consider graphs (allowing loops) instead of trees.

In the first subsection we will review what kind of extra structure is needed in order to define the minimal model/decomposition model, and the relation of these extra structures to the notion of gauge fixing in SFT. The second subsection is devoted to the proof of the decomposition theorem for loop homotopy algebras and finally we derive thereof the minimal model theorem. Indeed we will need the decomposition theorem, rather than the minimal model theorem, for the considerations in section 4.1. Besides an explicit construction of the decomposition model, we also give an explicit construction of the IBL_∞-isomorphism from the initial loop homotopy algebra to its decomposition model.

3.1. Hodge decomposition and gauge fixing. Let A be a graded module endowed with an odd symplectic structure ω of degree minus one and a compatible differential $d : A \to A$ of degree one, i.e.

$$d^2 = 0 \qquad \text{and} \qquad \omega(d, \cdot\,) + \omega(\,\cdot\,, d) = 0\ .$$

Definition 1. *A pre Hodge decomposition of A is a map $h : A \to A$ of degree minus one which is compatible with the symplectic structure and squares to zero.*

For a given pre Hodge decomposition of A, we define the map

$$P = 1 + dh + hd\ ,$$

which obviously satisfies $Pd = dP$ and $Ph = hP$.

Definition 2. *A Hodge decomposition of A is a pre Hodge decomposition which additionally satisfies $hdh = -h$.*

Let h be a Hodge decomposition of A and define $P_U = -hd$ and $P_T = -dh$. Then the following properties are satisfied:

$$P^2 = P\ , \qquad P_U^2 = P_U\ , \qquad P_T^2 = P_T\ .$$

That is P, P_U, P_T are projection maps and A decomposes into the corresponding projection subspaces $A_P \oplus A_U \oplus A_T$. Furthermore we have $Ph = hP = 0$.

Definition 3. *A Hodge decomposition of A is called harmonious if $dhd = -d$.*

For a harmonious Hodge decomposition the additional feature compared to a Hodge decomposition is $Pd = dP = 0$. Furthermore we have $A_P \perp A_U \oplus A_T$, $A_U \perp A_U$ and $A_T \perp A_T$. These definitions are borrowed from [39].

Let us now elucidate how the algebraic structures just described come into play in SFT. Let d be the BRST differential and A the space of string fields. Gauge fixing is required to obtain a well defined path integral, which amounts to fixing a representative for every element of the cohomology $H(A, d)$. More precisely, the gauge fixing determines a map

$$i : H(A, d) \to A\ ,$$

which maps an element of the cohomology to its corresponding representative. We will call i the inclusion map. We also have the projection map

$$\pi : A \to H(A, d)\ .$$

Obviously, the map $P := i \circ \pi : A \to A$ satisfies $P^2 = P$ and the image A_P of P is isomorphic to $H(A, d)$. That is A_P represents the physical states. Moreover P is a chain map, i.e. $Pd = dP = 0$, and its induced map on cohomology is the identity map. This implies that P is homotopic to 1, i.e. there is a map $h : A \to A$ of degree one such that

$$P - 1 = hd + dh\ .$$

Note that $P^2 = P$ implies $h^2 = 0$. Physically we can identify h as the propagator corresponding to the chosen gauge. We demand $hP = Ph = 0$, which means that we set the propagator to zero on the space of physical states. The subspace A_U corresponding to the projection map $P_U = -hd$ represents the unphysical states, i.e. the states not annihilated by d, and the subspace A_T represents the space of trivial states, i.e. d exact states. Thus we can summarize that choosing a gauge in SFT determines a harmonious Hodge decomposition, which decomposes the state space into physical, unphysical and trivial states [7,9]. When dealing with a pre Hodge decomposition, we will call the images of P, $-dh$, $-hd$ the physical space, trivial space, unphysical space as well.

In the next subsection we will see that the extra data required to construct a decomposition model is just a pre Hodge decomposition, whereas we need a harmonious Hodge decomposition to construct a minimal model.

3.2. Decomposition theorem of loop homotopy algebra. Let $(A, \mathfrak{L})$ be a loop homotopy algebra, i.e.

$$\mathfrak{L} = \sum \hbar^g L^g + \hbar \Omega^{-1} \ , \tag{9}$$

where $L^g = \widehat{l^g} \in \mathrm{Coder}^{cycl}(SA)$ and $\Omega^{-1} = \widehat{\omega^{-1}} \in \mathrm{Coder}^2(SA)$ is the lift of the inverse of the odd symplectic structure (see equation (5)). We define $l_q := \sum_g \hbar^g l^g$ and $l_{cl} := l^0$, where the subscripts indicate quantum and classical respectively. The differential on A is given by $d = l_{cl} \circ i_1$. Furthermore we abbreviate the collection of multilinear maps without the differential by $l_q^* := l_q - d$ and $l_{cl}^* := l_{cl} - d$.

In appendix A we introduced the lifting map, which lifts multilinear maps to a coderivations, but for notational convenience we will denote this map by D rather than a hat in the following. With these conventions equation (9) reads

$$\mathfrak{L} = D(d + l_q^* + \hbar\omega^{-1}) \ .$$

The loop homotopy algebra axioms are summarized by $\mathfrak{L}^2 = 0$ and can be recast to

$$d \circ l_q^* + l_q^* \circ D(l_q^*) + l_q^* \circ D(d) + l_q^* \circ D(\hbar\omega^{-1}) = 0 \ , \tag{10}$$

and

$$l_q^* \circ D(e_i) \wedge e^i = 0 \ , \tag{11}$$

where $\{e_i\}$ and $\{e^i\}$ denote a basis and corresponding dual basis of A w.r.t. the symplectic structure ω, that is $\omega^{-1} = \frac{1}{2} e_i \wedge e^i$. Equation (10) is called the main identity [5,22] whereas equation (11) states cyclicity of the maps l_q^*, i.e. that $\omega(l_n^g, \cdot)$ enjoys full symmetry in all $n+1$ inputs (see appendix A). The cyclicity condition (11) is essentially saying that there is actually no distinction between outputs and inputs.

To construct a decomposition model of the loop homotopy algebra (9), we additionally need the data of a pre Hodge decomposition $h : A \to A$. Again we define $P = 1 + dh + hd$, and in addition we introduce

$$g := -\omega \circ d \qquad \text{and} \qquad g^{-1} := h \circ \omega^{-1} \ , \tag{12}$$

where the symplectic structure ω and its inverse ω^{-1} are considered as a map from A to A^* and A^* to A, respectively. Since d and h are compatible with the symplectic structure, g is a symmetric map and $g^{-1} \in A^{\wedge 2}$, each of degree zero. Assume for a moment that h defines indeed a harmonious Hodge decomposition, then we saw that the full space A splits into $A_P \oplus A_U \oplus A_T$, where A_P, A_U, A_T represents the physical, unphysical, trivial space respectively. In this case g is non-vanishing only on the unphysical space A_U and g^{-1} defines its inverse upon restricting to A_U, that is g defines a metric on the unphysical space.

In the context of L_∞-algebras the decomposition theorem is proven by constructing trees from l_{cl}^* and h [36]. There is a nice way of generating these trees, by employing the tools developed in appendix A [7,9]: Consider trees where the root and the internal lines are labelled by the propagator h, the vertices by $l_n = l_{cl}^* \circ i_n$ and the leaves by the identity map 1. The collection of all these trees $\mathsf{T}_{cl} : SA \to A$, is defined recursively via

$$\mathsf{T}_{cl} = h \circ l_{cl}^* \circ e^{1 + \mathsf{T}_{cl}} \qquad \text{and} \qquad \mathsf{T}_{cl} \circ i_1 = 0 \ , \tag{13}$$

where e is the lifting map of multilinear maps to cohomomorphisms (see appendix A). In figure 4, we depict the first few terms of T_{cl} according to the number of inputs.

Likewise, for an arbitrary linear map $x : A \to A$ we define trees in the same way as in equation (13) but replacing the root by x, that is

$$\overset{(x)}{\mathsf{T}}_{cl} := x \circ l_{cl}^* \circ e^{1 + \mathsf{T}_{cl}} \ . \tag{14}$$

As anticipated in the beginning of this section, we have to consider graphs to prove the decomposition theorem for loop homotopy algebras. Graphs are essentially trees with loops attached. The strategy is thus to start with trees as in the L_∞ case. Attaching loops can then be implemented neatly by composing the trees with an appropriate cohomomorphism. So let us first define trees constructed recursively from l_q^* and h via

$$\mathsf{T}_q = h \circ l_q^* \circ e^{1 + \mathsf{T}_q} \qquad \text{and} \qquad \mathsf{T}_q \circ i_1 = 0 \ . \tag{15}$$

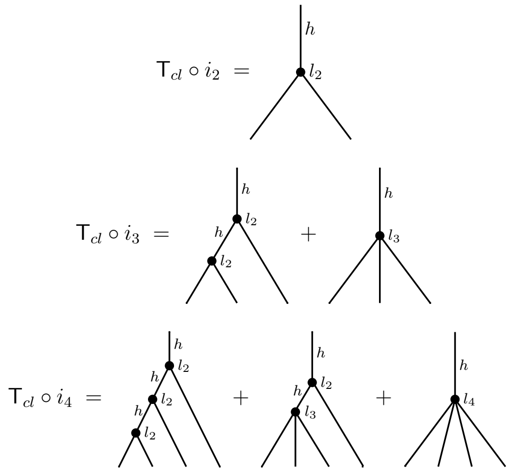

Fig. 4. Trees constructed from l^*_{cl} and h, where $l_n = l^*_{cl} \circ i_n$. Although not explicitly indicated, the inputs are understood to be symmetrized.

Consider now the cohomomorphism

$$E(\hbar g^{-1}) := e^{1+\hbar g^{-1}} \in \mathfrak{cohom}(SA, SA, \hbar) ,$$

where e is the lifting map and g^{-1} is the inverse metric on the unphysical space defined in equation (12). Let $\{u_i\}$ be a basis of the unphysical space and $\{u^i\}$ its dual basis w.r.t. g, i.e.

$$g({}_iu, u^j) = {}_i\delta^j ,$$

where we use the sign conventions of [21,30] that relate left indexed objects with right indexed objects. In terms of basis and dual basis, we can express the inverse metric as

$$g^{-1} = \frac{1}{2} u_i \wedge {}^iu .$$

Physically g^{-1} is interpreted as a loop, it connects two inputs by propagating the unphysical degrees of freedom. The cohomomorphism $E(\hbar g^{-1})$ is then the map that attaches loops in all possible ways. Thus $E(\hbar g^{-1})$ is the map that we have to compose with the trees T_q to obtain graphs. Since we are actually interested in graphs with many outputs (directed connected graphs), we define the collection of all these graphs $\mathbf{\Gamma}$ by

$$e^{1+\hbar g^{-1}+\mathbf{\Gamma}} = e^{1+\mathsf{T}_q} \circ E(\hbar g^{-1}) . \tag{16}$$

Note that since $E(\hbar g^{-1})$ acts on the collection of disconnected trees $e^{1+\mathsf{T}_q}$, every g^{-1} either generates a loop or increases the number of outputs by one. In figure 5 we depict a typical graph generated in that way. Upon amputating the loops g^{-1}, every graph reduces to a collection of connected trees.

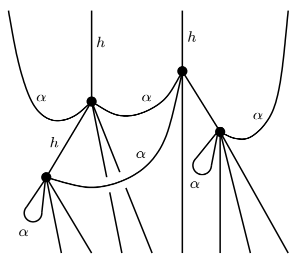

Fig. 5. Graph constructed from h, l^*_q and $\alpha = g^{-1}$. The curved lines represent the inverse metric g^{-1}.

A graph comes with a certain power of $\hbar$, which is the number of outputs minus one plus the number of loops (the first Betti number of the graph) plus the powers of $\hbar$ from the vertices. The number of loops plus the powers of $\hbar$ from the vertices define the genus of the graph. Thus we have $\mathbf{\Gamma} \in \bigoplus_{n=1}^{\infty} \hbar^{n-1}\mathrm{Hom}(SA, \Sigma^n A)$, with

$$\mathbf{\Gamma} = \sum_{n=1}^{\infty}\sum_{g=0}^{\infty} \hbar^{n+g-1}\mathbf{\Gamma}^{n,g} \ ,$$

where $\mathbf{\Gamma}^{n,g}$ represents the collection of graphs of genus g with n outputs and

$$\bar{\mathfrak{F}} := e^{1+\hbar g^{-1}+\mathbf{\Gamma}} \in \mathfrak{cohom}(SA, SA, \hbar) \ . \tag{17}$$

Now we are ready to state the decomposition theorem for loop homotopy algebras.

Theorem 1. *Let $(A, \mathfrak{L} = D(d + l_q^* + \hbar\omega^{-1}))$ be a loop homotopy algebra. For a given pre Hodge decomposition h, there is an associated loop homotopy algebra*

$$\bar{\mathfrak{L}} = D(d + \overset{(P)}{\mathsf{T}}_q \circ E(\hbar g^{-1}) + \hbar\bar{\omega}^{-1}) \ , \tag{18}$$

where $\bar{\omega}^{-1} = P^{\wedge 2}(\omega^{-1})$ is the restriction of ω^{-1} to the physical space and $\overset{(P)}{\mathsf{T}}_q \circ E(\hbar g^{-1})$ represents the graphs with a single output labeled by P. Furthermore $\bar{\mathfrak{F}} = e^{1+\hbar g^{-1}+\mathbf{\Gamma}}$ (see equation (17)) defines an IBL_∞-isomorphism from $(A, \bar{\mathfrak{L}})$ to $(A, \mathfrak{L})$. d is called the linear contractible part and $\overset{(P)}{\mathsf{T}}_q \circ E(\hbar g^{-1}) + \hbar\bar{\omega}^{-1}$ the minimal part.

Proof. Since we expressed graphs as the composition of trees with the cohomomorphism $E(\hbar g^{-1})$, the proof can be traced back to the level of trees. In the following we will leave out the subscript q, i.e. $l^* = l_q^*$ and $\mathsf{T}_q = \mathsf{T}$. In a first step we show

$$\overset{(hd)}{\mathsf{T}} + \mathsf{T} \circ D(d) + \mathsf{T} \circ D(\overset{(P)}{\mathsf{T}}) + \mathsf{T} \circ D^{con}(\hbar\omega^{-1}) = 0 \ , \tag{19}$$

where $\mathsf{T} \circ D^{con}(\hbar\omega^{-1})$ means that we consider only those terms where ω^{-1} acts on one vertex and not those where ω^{-1} connects two vertices. We prove equation (19) inductively: Using the main identity (10) and $P = 1 + dh + hd$, we get

$$\begin{aligned} \overset{(hd)}{\mathsf{T}} &= h \circ d \circ l^* \circ F = -h \circ l^* \circ D(d + l^* + \hbar\omega^{-1}) \circ F \\ &= -h \circ l^* \circ \Big(\big(d + d \circ \mathsf{T} + l^* \circ F + \hbar\omega^{-1}\big) \wedge F \Big) \circ \varDelta \\ &= -h \circ l^* \circ \Big(\big(d + \overset{(dh)}{\mathsf{T}} + \overset{(1)}{\mathsf{T}} + \hbar\omega^{-1}\big) \wedge F \Big) \circ \varDelta \\ &= -h \circ l^* \circ \Big(\big(d + \overset{(P)}{\mathsf{T}} + \hbar\omega^{-1} - \overset{(hd)}{\mathsf{T}}\big) \wedge F \Big) \circ \varDelta \ , \end{aligned} \tag{20}$$

where $F = e^{1+\mathsf{T}}$. In equation (20), we use identities like (see appendix A for more details)

$$D(l^*) \circ F = (l^* \wedge \mathrm{id}) \circ \varDelta \circ F = (l^* \circ F \wedge F) \circ \varDelta \ .$$

Upon iterating equation (20), we finally obtain equation (19).

Our strategy will be to first show $\mathfrak{L} \circ \bar{\mathfrak{F}} = \bar{\mathfrak{F}} \circ \bar{\mathfrak{L}}$. We start with calculating the left hand side:

$$\begin{aligned} \mathfrak{L} \circ \bar{\mathfrak{F}} &= D(d) \circ F \circ E(\hbar g^{-1}) + D(l^*) \circ F \circ E(\hbar g^{-1}) + D(\hbar\omega^{-1}) \circ F \circ E(\hbar g^{-1}) \\ &= \Big(\big(d + \overset{(dh)}{\mathsf{T}} + \overset{(1)}{\mathsf{T}} + \hbar\omega^{-1}\big) \wedge F \Big) \circ \varDelta \circ E(\hbar g^{-1}) \\ &= \Big(\big(d + \hbar\omega^{-1} + \overset{(P)}{\mathsf{T}} - \overset{(hd)}{\mathsf{T}}\big) \wedge F \Big) \circ \varDelta \circ E(\hbar g^{-1}) \end{aligned} \tag{21}$$

On the right hand side we have:

$$\bar{\mathfrak{F}} \circ \bar{\mathfrak{L}} = F \circ E(\hbar g^{-1}) \circ D(d + \overset{(P)}{\mathsf{T}} \circ E(\hbar g^{-1}) + \hbar\bar{\omega}^{-1}) \tag{22}$$

Let us consider the individual terms one by one. From the definition of g^{-1} it follows that

$$D(d)(g^{-1}) = d(d_i) \wedge {}^i d = \bar{\omega}^{-1} - \omega^{-1} ,$$

and thus

$$E(\hbar g^{-1}) \circ D(d) = D(d) \circ E(\hbar g^{-1}) + D(\hbar\omega^{-1}) \circ E(\hbar g^{-1}) - D(\hbar\bar{\omega}^{-1}) \circ E(\hbar g^{-1}) .$$

Therefore the first plus the third term of equation (22) yield

$$\begin{aligned} F \circ E(\hbar g^{-1}) \circ D(d + \hbar\bar{\omega}^{-1}) &= F \circ D(d + \hbar\omega^{-1}) \circ E(\hbar g^{-1}) \\ &= \Big(\big(d + \mathsf{T} \circ D(d) + \hbar\omega^{-1} + \hbar e_i \wedge \mathsf{T} \circ D(e^i) + \frac{\hbar}{2} \mathsf{T} \circ D(e_i) \wedge \mathsf{T} \circ D(e^i) \\ &\quad + \mathsf{T} \circ D(\hbar\omega^{-1}) \big) \wedge F \Big) \circ \Delta \circ E(\hbar g^{-1}) . \end{aligned}$$

Using the cyclicity property (11), we conclude that

$$\hbar e_i \wedge \mathsf{T} \circ D(e^i) = \frac{\hbar}{2} \mathsf{T} \circ D(e_i) \wedge \mathsf{T} \circ D(e^i) = 0 ,$$

and

$$\mathsf{T} \circ D(\hbar\omega^{-1}) = \mathsf{T} \circ D^{con}(\hbar\omega^{-1}) .$$

Similarly, cyclicity implies that the second term of equation (22) reduces to

$$\begin{aligned} F \circ E(\hbar g^{-1}) \circ D(\overset{(P)}{\mathsf{T}} \circ E(\hbar g^{-1})) &= F \circ D(\overset{(P)}{\mathsf{T}}) \circ E(\hbar g^{-1}) \\ &= \Big(\big(\overset{(P)}{\mathsf{T}} + \mathsf{T} \circ D(\overset{(P)}{\mathsf{T}}) \big) \wedge F \Big) \circ \Delta \circ E(\hbar g^{-1}) . \end{aligned}$$

Altogether we finally get

$$\bar{\mathfrak{F}} \circ \bar{\mathfrak{L}} = \Big(\big(d + \hbar\omega^{-1} + \overset{(P)}{\mathsf{T}} + \mathsf{T} \circ D(d) + \mathsf{T} \circ D(\overset{(P)}{\mathsf{T}}) + \mathsf{T} \circ D^{con}(\hbar\omega^{-1}) \big) \wedge F \Big) \circ \Delta \circ E(\hbar g^{-1}) ,$$

and $\mathfrak{L} \circ \bar{\mathfrak{F}} = \bar{\mathfrak{F}} \circ \bar{\mathfrak{L}}$ follows then directly form equation (19).

The second part of the proof, $\bar{\mathfrak{L}}^2 = 0$, follows directly from $\mathfrak{L} \circ \bar{\mathfrak{F}} = \bar{\mathfrak{F}} \circ \bar{\mathfrak{L}}$. Note that $\bar{\mathfrak{F}}$ is an IBL_∞-isomorphism, which implies that there is a unique inverse $\bar{\mathfrak{F}}^{-1}$. Thus we have

$$\bar{\mathfrak{L}}^2 = \bar{\mathfrak{F}}^{-1} \circ \mathfrak{L}^2 \circ \bar{\mathfrak{F}} = 0 .$$

3.3. Minimal model of loop homotopy algebra. The minimal model theorem follows readily from the decomposition theorem, but in contrast to the decomposition theorem we need a harmonious Hodge decomposition.

Theorem 2. *Let $(A, \mathfrak{L} = D(d + l_q^* + \hbar\omega^{-1}))$ be a loop homotopy algebra. For a given harmonious Hodge decomposition h, with corresponding inclusion map $i : H(A,d) \to A$ and projection map $\pi : A \to H(A,d)$, there is an associated loop homotopy algebra on the cohomology $H(A,d)$*

$$\tilde{\mathfrak{L}} = D(\overset{(\pi)}{\mathsf{T}}_q \circ E(\hbar g^{-1}) \circ I + \hbar\tilde{\omega}^{-1}) , \tag{23}$$

where $\tilde{\omega}^{-1} = \pi^{\wedge 2}(\omega^{-1})$ is the projection of ω^{-1} to the cohomology $H(A,d)$, $\overset{(\pi)}{\mathsf{T}}_q \circ E(\hbar g^{-1})$ represents the graphs with a single output labeled by π and $I = e^i$ is the lift of the inclusion map. Furthermore $\tilde{\bar{\mathfrak{F}}} = \bar{\mathfrak{F}} \circ I$ defines an IBL_∞-isomorphism from $(H(A,d), \tilde{\mathfrak{L}})$ to $(A, \mathfrak{L})$.

Proof. From the decomposition theorem we know

$$\mathfrak{L} \circ \bar{\mathfrak{F}} = \bar{\mathfrak{F}} \circ \bar{\mathfrak{L}} \qquad \text{and} \qquad \bar{\mathfrak{L}}^2 = 0 \ .$$

Furthermore the loop homotopy algebra of the decomposition model is related to the loop homotopy algebra of the minimal model by

$$\tilde{\mathfrak{L}} = \Pi \circ \bar{\mathfrak{L}} \circ I \ , \qquad I \circ \tilde{\mathfrak{L}} = \bar{\mathfrak{L}} \circ I \ , \tag{24}$$

where $\Pi = e^{\pi}$ is the lift of the projection map. Thus we have

$$\tilde{\mathfrak{F}} \circ \tilde{\mathfrak{L}} = \bar{\mathfrak{F}} \circ I \circ \tilde{\mathfrak{L}} = \bar{\mathfrak{F}} \circ \bar{\mathfrak{L}} \circ I = \mathfrak{L} \circ \bar{\mathfrak{F}} \circ I = \mathfrak{L} \circ \tilde{\mathfrak{F}} \ .$$

Let us denote $\mathcal{P} = e^{P}$. Using $\mathcal{P} \circ \bar{\mathfrak{L}} \circ I = \bar{\mathfrak{L}} \circ I$, we get

$$\tilde{\mathfrak{L}}^2 = \Pi \circ \bar{\mathfrak{L}} \circ \mathcal{P} \circ \bar{\mathfrak{L}} \circ I = \Pi \circ \bar{\mathfrak{L}}^2 \circ I = 0 \ .$$

Finally let us discuss the physical relevance of the minimal model. In the following we abbreviate

$$\tilde{l}^* = \overset{(\pi)}{\mathsf{T}}_q \circ E(\hbar g^{-1}) \circ I \ ,$$

and thus

$$\tilde{\mathfrak{L}} = D(\tilde{l}^* + \hbar\tilde{\omega}^{-1}) \ .$$

As for the initial loop homotopy algebra, the condition $\tilde{\mathfrak{L}}^2 = 0$ can be recast into two separate equations, one resembling the main identity (10)

$$\tilde{l}^* \circ D(\tilde{l}^*) + \tilde{l}^* \circ D(\hbar\tilde{\omega}^{-1}) = 0 \ , \tag{25}$$

and the other expressing cyclicity (11) with respect to $\tilde{\omega} = \omega \circ i^{\wedge 2}$, that is

$$\tilde{l}^* \circ D(p_i) \wedge p^i = 0 \ ,$$

where $\{p_i\}$ denotes a basis of $H(A,d)$ and $\{p^i\}$ denotes its dual basis w.r.t. the symplectic structure $\tilde{\omega}$.

Recall that $\tilde{l}^*$ represents the collection of graphs, whose tree lines and loop lines are labelled by h and $g^{-1} = h \circ \omega^{-1}$, respectively (see e.g. figure 5). Cyclicity tells us that there is actually no distinction between tree lines and loop lines, this separation is indeed a peculiarity of the formalism. The physical meaningful maps are

$$\tilde{\omega}(\tilde{l}^*_n, \cdot) : H(A,d)^{\wedge n+1} \to \mathbb{C} \ .$$

These are the full quantum S-matrix amplitudes, the sum over all possible Feynman graphs (amputated and restricted to the physical states $H(A,d)$) constructed from the vertices l^*_q and the propagator h. Finally the main identity (25) summarizes the Ward identities [5,37] for the S-matrix amplitudes.

4. Uniqueness of SFT

In the previous section we saw that the minimal model theorem is directly related to S-matrix amplitudes. In the following we exploit the more general decomposition theorem and explain its relevance in SFT. With the aid of the decomposition theorem we show in the first subsection that there is a unique SFT on a given background, compatible with the S-matrix of a given world sheet conformal field theory. In the second subsection we then consider the background independent deformation theory of closed string field theory. Concretely we restrict to deformations which preserve the CFT state space but not the BRST charge and the S-matrix. Such deformations are described by the Chevalley-Eilenberg cohomology. In particular, we argue that generic deformations of the closed string vertices are trivial. This is the closed string analogue of the uniqueness result for open string field theory in [44].

4.1. Fixed Background. For concretness we present the line of reasoning in the context of quantum closed SFT, but the same conclusion will hold for any bosonic SFT, all we need is actually the decomposition theorem and the concept of RG flow. Consider two string field theories on a fixed string background. The SFTs are determined by a choice of string vertices $\mathcal{V}$ at the geometric level. The world sheet conformal field theory then maps the geometric vertices to the algebraic vertices, preserving the BV structure. As pointed out in the previous sections, consistency requires first that the algebraic vertices define some homotopy algebra and second that the corresponding minimal model coincides with the S-matrix amplitudes of perturbative string theory. Denote the two string field theories by $(A, \mathfrak{L}_0)$ and $(A, \mathfrak{L}_1)$, where A is the (restricted) state space of the world sheet conformal field theory and $\mathfrak{L}_i$ is the loop homotopy algebra defining the SFT. That they are constructed on the same background implies that their BRST differentials and their symplectic structures coincide, i.e.

$$\mathfrak{L}_0 = D(d + l_0^* + \hbar\omega^{-1}) \qquad \text{and} \qquad \mathfrak{L}_1 = D(d + l_1^* + \hbar\omega^{-1}) \ .$$

Now choose a gauge, such that we can define a propagator h, and consider the minimal models corresponding to these two SFTs. Since both SFTs are constructed on the same background, their S-matrix amplitudes are identical and hence their minimal models coincide. Recall that the decomposition model is the sum of the linear contractible part (differential) plus the minimal part. Since both SFTs share the same BRST differential, we can finally conclude that their decomposition models coincide as well. Thus we have

$$\bar{\mathfrak{L}}_0 = \bar{\mathfrak{L}}_1 =: \bar{\mathfrak{L}} \ ,$$

where $\bar{\mathfrak{L}}_i$ denotes the decomposition model corresponding to $\mathfrak{L}_i$. In theorem 1 we proved that the decomposition model is IBL_∞-isomorphic to its corresponding initial loop homotopy algebra, and because an IBL_∞-isomorphism is invertible the first conclusion is that two SFTs defined on a given background are IBL_∞-isomorphic:

$$(A, \mathfrak{L}_0) \xleftarrow{IBL_\infty-\text{isomorphism}} (A, \mathfrak{L}) \xrightarrow{IBL_\infty-\text{isomorphism}} (A, \mathfrak{L}_1) \ .$$

This is precisely the argumentation of [7], that was used to show that classical open SFT on a fixed background is unique up to A_∞-isomorphisms. But we can go one step further by tracking the RG flows of the theories: Introduce a UV cut-off ξ for the propagator. The vertices of the action change upon varying the cut-off ξ. Geometrically, the variation of the vertices induced by the cut-off scale can be described as follows. To the initial vertices $\mathcal{V}$ we have to attach stubs of length ξ. Consistency requires that the string vertices generate a single cover of the full moduli space via Feynman graphs, where the propagator is the operation of sewing in stubs (cylinders) of arbitrary length. Upon attaching stubs to the initial vertices $\mathcal{V}$, we have to add those surfaces that can no longer be produced via Feynman graphs. These surfaces are exactly the ones that arise from graphs where we sew in stubs of length shorter than 2ξ [5,40,9]. Thus for every value of ξ we get a new set of vertices $\mathcal{V}_\xi$. At the algebraic level, the appropriate tool to describe the RG flow is the decomposition model for a certain choice of pre Hodge decomposition. For definiteness let us work in Siegel gauge, where the propagator takes the form

$$h = -b_0^+ \int_0^\infty d\tau e^{-\tau L_0^+}(1-P) \ ,$$

and P is the projection onto physical states, i.e. states annihilated by L_0^+. Using basic properties of the BRST charge $Q = d$, the energy momentum tensor and the b ghost, we find

$$dh + hd = P - 1 \ .$$

Now the operation of sewing in stubs of length shorter than 2ξ corresponds to the map

$$h_\xi = -b_0^+ \int_0^{2\xi} d\tau e^{-\tau L_0^+} \ .$$

Furthermore we have

$$d\, h_\xi + h_\xi\, d = e^{-2\xi L_0^+} - 1 \ ,$$

that is we can identify

$$P_\xi = e^{-2\xi L_0^+} \ .$$

The map P_ξ is the operation of attaching stubs of length 2ξ. Recall that the vertices of the decomposition model describe the collection of all graphs with internal lines labelled by the chosen pre Hodge decomposition h, the outputs labelled by the corresponding map P and the inputs labelled by the identity map 1. That is we attach stubs of length 2ξ to the outputs and no stubs to the inputs, which is equivalent to attaching stubs of length ξ to outputs and inputs (the two descriptions are IBL_∞-isomorphic). From the discussion above we can then conclude that the new vertices corresponding to a specific value of the cut-off ξ are given by the decomposition model with the choice of pre Hodge decomposition being h_ξ (see [9] for this discussion in the context of classical open SFT).

The limit $\xi \to \infty$ describes the decomposition model that corresponds to the S-matrix amplitudes whereas in the limit $\xi \to 0$ we recover the initial loop homotopy algebras. In other words we can interpolate continuously between the S-matrix theory and the initial SFT and thus two SFTs constructed on the same background are connected by a 1-parameter family of IBL_∞-isomorphisms. More precisely we have IBL_∞-isomorphisms parametrized by $t \in [0,1]$

$$\mathfrak{F}(t) : (A, \mathfrak{L}(t)) \to (A, \mathfrak{L}_0) \ ,$$

where $\mathfrak{F}(0) = \mathrm{id}$, $\mathfrak{L}(0) = \mathfrak{L}_0$ and $\mathfrak{L}(1) = \mathfrak{L}_1$.

Recall from appendix A that an IBL_∞-algebra on A is defined to be a Maurer-Cartan element of the Lie algebra $(\mathfrak{coder}(SA,\hbar), [\cdot,\cdot])$. The statement that the two loop homotopy algebras $(A,\mathfrak{L}_0)$ and $(A,\mathfrak{L}_1)$ are connected by a 1-parameter family of IBL_∞-isomorphisms implies that they are gauge equivalent Maurer-Cartan elements of $(\mathfrak{coder}(SA,\hbar), [\cdot,\cdot])$: The notion of gauge transformations of L_∞-algebras, and in particular Lie algebras, is reviewed in appendix A. Two Maurer-Cartan elements $\mathfrak{L}_0, \mathfrak{L}_1 \in \mathfrak{coder}(SA,\hbar)$ are gauge equivalent, if there is a $\Lambda(t) \in \mathfrak{coder}(SA,\hbar)$ of degree zero and a $\mathfrak{L}(t) \in \mathcal{MC}\big(\mathfrak{coder}(SA,\hbar), [\cdot,\cdot]\big)$, $t \in [0,1]$, such that

$$\frac{d}{dt}\mathfrak{L}(t) = -[\Lambda(t), \mathfrak{L}(t)] \qquad \text{and} \qquad \mathfrak{L}(0) = \mathfrak{L}_0, \quad \mathfrak{L}(1) = \mathfrak{L}_1 \ .$$

In our case we have a family of IBL_∞-isomorphisms $\mathfrak{F}(t)$, that is

$$\mathfrak{F}(t) \circ \mathfrak{L}(t) = \mathfrak{L}_0 \circ \mathfrak{F}(t) \ ,$$

and hence

$$\begin{aligned} \frac{d}{dt}\mathfrak{L}(t) &= \frac{d}{dt}\big(\mathfrak{F}(t)^{-1} \circ \mathfrak{L}_0 \circ \mathfrak{F}(t)\big) \\ &= \tfrac{d}{dt}\mathfrak{F}(t)^{-1} \circ \mathfrak{F}(t) \circ \mathfrak{L}(t) + \mathfrak{L}(t) \circ \mathfrak{F}(t)^{-1} \circ \tfrac{d}{dt}\mathfrak{F}(t) \\ &= -[\Lambda(t), \mathfrak{L}(t)] \ , \end{aligned}$$

where $\Lambda(t) = \mathfrak{F}(t)^{-1} \circ \frac{d}{dt}\mathfrak{F}(t)$. Thus, we showed that closed SFT on a given background defines a loop homotopy algebra on the (restricted) state space of the world sheet CFT which is unique up to gauge transformations, or in other words it defines a unique element in the moduli space $\mathcal{M}\big(\mathfrak{coder}(SA,\hbar), [\cdot,\cdot]\big)$.

4.2. Uniqueness of closed string field theory. In [44] it was shown that a closed string background defines a unique equivalence class of classically consistent open string field theories. The equivalence classes are defined w.r.t. to L_∞ gauge transformations. In this subsection we will describe the corresponding result for closed string field theory. For this we first need to understand the nature of generic gauge transformations and the geometry of the moduli space $\mathcal{M}\big(\mathrm{Coder}(SA), [\cdot,\cdot]\big)$. Clearly, L_∞ field redefinitions preserve the L_∞ structure and can be interpreted as gauge transformations if they are continuously connected to the identity. On the other hand, field redefinitions include shifts in the closed string background. These are easily seen to be L_∞-isomorphisms along the lines explained in appendix A for A_∞-algebras. This takes us right to he heart of the question about background independence in SFT: For a given homotopy algebra we can consider a non-vanishing Maurer-Cartan element c, $L(e^c) = 0$. We then obtain a new homotopy algebra $L[c] = E(-c) \circ L \circ E(c)$ upon conjugation. Background independence then would imply

$$\begin{tikzcd}
S_{CFT} \arrow[rrr, "\text{operator formalism}", "\mathcal{V}_n \mapsto l_n"'] \arrow[d, "S_{CFT} \to S_{CFT} + \int \phi"'] & & & \{l_n\}_{n\in\mathbb{N}} \arrow[d, "L \to E(-c)\circ L \circ E(c)"] \\
S'_{CFT} \arrow[rrr, "\text{operator formalism}", "\mathcal{V}_n \mapsto l'_n"'] & & & \{l'_n\}_{n\in\mathbb{N}}
\end{tikzcd}$$

Fig. 6. background independence

that the structure maps of the minimal model obtained from this homotopy algebra are equivalent to the perturbative S-matrix elements of the world-sheet CFT in the new background (see figure 6).

In [41,42] it was shown that exactly marginal deformation of the open string world-sheet theory correspond to classical solutions in open string field theory, that is Maurer-Cartan elements, thus establishing background independence in one direction for the open string, at least in a open neighborhood of a given open string background (see also [42] for some progress involving marginal deformations).

As explained above, closed string background shifts are $\big(\mathrm{Coder}(SA), [\cdot,\cdot]\big)$-gauge transformations and thus L_∞ algebras for different closed string backgrounds are within the same equivalence class[2]. We now want to argue that all infinitesimal deformations of a given closed string world sheet theory are trivial.

The proof of this assertion proceeds in close analogy with the corresponding open string result (section 4.2 of [44]). Let us denote the classical closed string vertices by $f_n \equiv f^{0,0}_{n,0}$ (see section 2). The bracket $[\cdot,\cdot]$ on Coder(SA) induces the Chevalley-Eilenberg differential $d_C = [L,\cdot]$ on the deformation complex. Any consistent infinitesimal deformation $\mathit{\Delta} f = \{\mathit{\Delta} f_n\}_{n\in\mathbb{N}}$ of the L_∞-structure $\{f_n\}_{n\in\mathbb{N}}$ is d_C-closed, $d_C(\mathit{\Delta} f) = 0$. Starting with $n = 2$ we conclude $(l_1\mathit{\Delta} f_2)(c_1,c_2) \equiv \mathit{\Delta} f_2(l_1c_1,c_2) + (-1)^{c_1}\mathit{\Delta} f_2(c_1,l_1c_2) = 0$ which implies that $\mathit{\Delta} f_2(c_1,c_2) = \omega_c(\mathit{\Delta} l_1c_1,c_2)$ with $[l_1,\mathit{\Delta} l_1] = 0$. For $n = 3$ we write

$$\mathit{\Delta} f_3(c_1,c_2,c_3) = \omega_c(\mathit{\Delta} l_2(c_1,c_2),c_3) \tag{26}$$

Then $\mathit{\Delta} f_2$ and $\mathit{\Delta} f_3$ are subject to the equation

$$(l_2\mathit{\Delta} f_2) + (l_1\mathit{\Delta} f_3) = 0. \tag{27}$$

It is not hard to see that this implies that $\mathit{\Delta} l_2 = [\mathcal{O},l_2] + g_2$ with $\mathcal{O}$ a linear operator and $[l_1,g_2] = 0$. To continue we can assume without restricting the generality that bpz$(\mathcal{O}) = \pm\mathcal{O}$. If $\mathcal{O}$ is BPZ-odd then

$$\mathit{\Delta} f_2(c_1,c_2) = \omega_c([\mathcal{O},l_1]c_1,c_2) + \omega_c(\mathcal{H}c_1,c_2) \tag{28}$$

where $[l_2,\mathcal{H}] = 0$. The latter condition together with $[l_1,\mathcal{H}] = 0$ from above is in contraction with the uniqueness of the world-sheet BRST charge Q. Thus $\mathcal{H} = 0$. Furthermore, for $\mathcal{O}$ BPZ-odd, $\mathit{\Delta} f_2$ and $\mathit{\Delta} f_3$ are exact. This leaves us with $\mathit{\Delta} l_1 = 0$ and bpz$(\mathcal{O}) = \mathcal{O}$. If $\mathcal{O}$ and g_2 are l_1-exact then $\mathit{\Delta} f_3$ is again trivial in the d_C-cohomology. To continue, we then assume that $\mathcal{O}$ and g_2 are in the cohomology of l_1 and consider $n = 4$ which gives the condition

$$(l_3\mathit{\Delta} f_2) + (l_2\mathit{\Delta} f_3) + (l_1\mathit{\Delta} f_4) = 0 \tag{29}$$

However, $\mathit{\Delta} f_2 = 0$ from the above and

$$(l_2\mathit{\Delta} f_3) = \omega_c([\mathit{\Delta} l_2,l_2](c_1,c_2,c_3),c_4) \tag{30}$$

[2] We should note, however, that field redefinitions do not preserve the decomposition of the homotopy algebra and, in particular, background shifts do not preserve the cohomology $H(A,Q)$

Now, since $\mathcal{O}$ and g_2 are in the cohomology of l_1 this term cannot be canceled by $(l_1 \Delta f_4)$ unless $g_2 = 0$ and $\mathcal{O}$ is a conformal invariant so that $\mathcal{O}$ can be pulled in the bulk. Indeed, since $\mathcal{O}$ is not l_1-exact, the only way the differential l_1 acting on Δf_4 can reproduce (30) is as a derivation on its moduli space. Since $\mathcal{O}$ is BPZ-even it cannot be a derivative. On the other hand if $\mathcal{O}$ can be pulled in the bulk then $[\mathcal{O}, l_1] = 0$ is equivalent to the closed string cohomology condition. Repeating these steps for $n > 4$ it then follows that the only non-trivial elements in the d_C-cohomology are given by a degree 0 insertion of the form $\mathcal{O}|0\rangle_c$. However, since the semi-relative closed string cohomology at degree -2 (ghost number zero)[3] and vanishing mass contains only the vacuum this completes the proof of our assertion.

In important consequence of this result is that the L_∞ structure $L_{CFT(\phi)}$ obtained from the world sheet theory in the new background ϕ corresponding to the MC-element c is L_∞ equivalent to the L_∞ stuture L_c obtained from L by conjugation, i.e.

$$L_{CFT(\phi)} = K^{-1} \circ L_c \circ K \tag{31}$$

where K is an L_∞ isomorphism continuously connected to the identity. Now, since the L_∞ equivalence classes identify all continuously connected closed backgrounds we cannot conclude from the above that $L_{CFT(\phi)}$ and L_c actually describe the same background. The necessary refinement for this is then provided by the open-closed homotopy algebra in the next section which implies that $K(1) = \mathbf{1}$. On the other hand, we should note that generic on-shell closed string backgrounds are not continuously connected to each other and furthermore do not preserve A_c. This puts a limitation on applicability of the proof of background independence given here.

5. Open-closed Correspondence

Let us turn to the theory of open and closed string. In the geometrical setting of bounded Riemann surfaces, it is generically impossible to distinguish whether a surface should be interpreted as the world sheet of a propagating open or closed string. From the point of view of open strings, a cylinder for example represents a one-loop diagram, whereas the alternative identification is the closed string propagator. There is an algebraic counterpart to this phenomenon which we will investigate. The main result of this section is then to describe an isomorphism between deformations of open string theory and closed string Maurer-Cartan elements.

5.1. Open-closed correspondence. Consider open-closed SFT in the 'classical' limit as described in section 2. That is we have vertices corresponding to discs with open, discs with open and closed and spheres with closed string punctures. The open-closed vertices define a L_∞-morphism from the L_∞-algebra of closed strings to the differential graded Lie algebra which controls deformations of the open string A_∞-algebra (see equation (3)). The OCHA (4) of [26,27] reads

$$N \circ L = d_h(N) + \frac{1}{2}[N, N] \circ \Delta \ ,$$

where N represents the open-closed vertices, L represents the closed vertices and $d_h = [M, \cdot]$ with M representing the open string vertices. L_∞-morphisms preserve Maurer-Cartan elements, thus let us identify the Maurer-Cartan elements on the closed and open side of the OCHA. The Maurer-Cartan elements of the closed string L_∞-algebra are solution of the equations of motion, whereas on the open string side a Maurer-Cartan element of the differential graded Lie algebra $(\mathrm{Coder}^{cycl}(TA), d_h, [\cdot,\cdot])$ defines a finite deformation of the A_∞-algebra M. Thus every solution of the closed string equations of motion defines a new open string field theory. This is the classical open-closed correspondence [26,27]. At the infinitesimal level, the open-closed vertex with just one closed input $N_1 := N \circ i_1$ defines a morphism of complexes, that is it maps physical closed string states to infinitesimal deformations of the open string field theory. Indeed we know more about this vertex. In [44] it has been shown that N_1 defines a quasi-isomorphism, that is it induces an isomorphism on cohomologies. A powerful theorem of Kontsevich

[3] The ghost number is related to the grading use here by a double shift (see e.g. [21]), i.e. the ground state $|0\rangle_c$ has degree minus two and ghost number zero, respectively.

[8] then guarantees isomorphism at the finite level, or more precisely that the moduli spaces of two L_∞-algebras connected by a L_∞-quasi-isomorphism are isomorphic. In our particular case, this means that the space of closed stings satisfying the equations of motion modulo gauge transformations is in one-to-one correspondence with the space of inequivalent deformations of the open string field theory M, i.e.

$$\mathcal{M}(A_c, L) \cong \mathcal{M}(\mathrm{Coder}^{cycl}(TA_o), d_h, [\cdot,\cdot]) \ .$$

In the previous section we argued that SFT is unique (up to gauge transformations) on a given background, thus we need not distinguish between SFT and world sheet conformal field theory. Let us then formulate the open-closed correspondence in terms of world sheet conformal field theories. We start with an open-closed world sheet conformal field theory. The restriction to open/closed strings induces an open/closed world sheet conformal field theory. The moduli space of the L_∞-algebra corresponding to the closed world sheet conformal field theory is isomorphic to the space of inequivalent open world sheet conformal field theories.

Since the open-closed correspondence relates Maurer-Cartan elements modulo gauge transformations, we give some examples of gauge transformations in order to develop some intuition. On the closed side we know what gauge transformations are, they leave the equations of motion invariant. Thus we will focus on the open side where we are in the context of A_∞-algebras, and represent three types of gauge transformations therein. In the following (A_o, M) is the A_∞-algebra describing the open SFT.

1. *1-parameter family of A_∞-isomorphisms:* For $t \in [0,1]$, let (A_o, M_t) be A_∞-algebras connected continuously to the initial A_∞-algebra $(A_o, M = M_{t=0})$ by A_∞-isomorphisms

$$F_t : (A_o, M_t) \to (A_o, M) \ .$$

In general, a gauge transformation of the differential graded Lie algebra $(\mathrm{Coder}^{cycl}(TA_o), [\cdot\cdot])$ is given by a a 1-parameter family of A_∞-algebras $M_t \in \mathrm{Coder}^{cycl}(TA_o)$, satisfying

$$\frac{d}{dt} M_t = -[\Lambda_t, M_t] \ , \tag{32}$$

for some $\Lambda_t \in \mathrm{Coder}^{cycl}(TA_o)$ of degree zero. Thus we conclude that (A, M) is gauge equivalent to (A, M_1), with $\Lambda_t = F_t^{-1} \circ \frac{d}{dt} F_t$.

2. *backgrounnd shifts:* A more concrete example is that of shifting the open string background. Let $a \in A_o$ be an open string state of degree zero. A background shift in the initial A_∞-algebra gives rise to a new A_∞-algebra $(A, M[a])$, defined by (see appendix A)

$$M[a] = E(-a) \circ M \circ E(a) \ ,$$

where $E(a) = e^{1+a}$ is the lift of the identity map 1 plus the background a to a cohomomorphism. $E(a)$ defines in fact an A_∞-isomorphism from $(A, M[a])$ to (A, M). We can easily construct a 1-parameter family of A_∞-isomorphisms by gradually scaling the background to zero, that is the A_∞-isomorphisms are $E(ta)$ and $\Lambda_t = \widehat{a}$, where $\widehat{a}$ denotes the lift of a to a coderivation.

3. *attaching strips - RG flow:* In section 4.1 we used the decomposition theorem to discuss the RG flow in closed SFT. Introducing a cut-off ξ for the propagator amounts to attaching stubs of length ξ to the vertices. In the case of open SFT and A_∞-algebras a similar discussion can be found in [9]: Attach strips of length ξ to the initial vertices and add those diagrams to the vertices that can no longer be produced by (tree-level) Feynman graphs. The new vertices M_ξ are given by the decomposition model for a suitable choice of pre Hodge decomposition, and the decomposition theorem provides an A_∞-isomorphism $F_\xi : (A, M_\xi) \to (A, M)$. Since we can vary the length of the stubs continuously, the initial SFT and the one with strips attached are related by a 1-parameter family of A_∞-isomorphisms and are thus gauge equivalent.

Relevance for Background Independence: Suppose that for a given non-trivial closed string Maurer-Cartan element c, $L(e^c) = 0$, we obtain the corresponding open string theory via the operator formalism of the world-sheet CFT on the background ϕ (see figure 6). The corresponding A_∞ structure is then necessarily given by an element $M_\phi \in \mathcal{M}(\mathrm{Coder}^{cycl}(TA_o), d_h, [\cdot,\cdot])$. Since this space is isomorphic to the moduli space of classical solutions of closed string field theory modulo gauge transformations, we can identify M_ϕ with the image of e^c under the open-closed L_∞-morphism

$$\begin{aligned} N : SA_c &\to \mathrm{Coder}(TA_o) \\ e^c &\mapsto N(e^c) = M_\phi \end{aligned} \tag{33}$$

On the other hand, the operator formalism defines a L_∞-morphism

$$N_\phi \ : \ (A_c, L_\phi) \to (\mathrm{Coder}(TA_o), [M_\phi, \cdot], [\cdot,\cdot]) \ .$$

Since the vector space A_c is assumed to be invariant we have

$$N_\phi(1) = N(e^c) = (N \circ E(c))(1) \ , \tag{34}$$

and thus $K(1) = \mathbf{1}$.

5.2. Quantum case. In the previous subsection we discussed the open-closed correspondence as it arises from 'classical' open-closed SFT. The correspondence is based on the property that L_∞-morphisms preserve Maurer-Cartan elements and the way the OCHA (3) is defined. The algebraic structure of quantum open-closed SFT is quite similar to that of the OCHA: The open-closed vertices define an IBL_∞-morphism from the loop algebra of closed strings to the involutive Lie bialgebra on the cyclic Hochschild complex of open strings (7). As in the classical case, IBL_∞-morphisms preserve Maurer-Cartan elements and thus we can look for closed string Maurer-Cartan elements which will in turn define consistent quantum SFTs of only open strings. The L_∞-morphism in the classical case was shown to be a L_∞-quasi-isomorphism [44], which implies that the IBL_∞-morphism is a quasi-isomorphism as well. Furthermore the moduli spaces of IBL_∞-quasi-isomorphic IBL_∞-algebras are isomorphic [20]. Thus quantum open SFTs, if there exists any, are in one-to-one correspondence with Maurer-Cartan elements of the closed string loop algebra (up to gauge transformations). Therefore let us investigate the Maurer-Cartan equation of the closed string loop algebra

$$\mathfrak{L}_c = D(d + l_q^* + \hbar\omega_c^{-1}) \in \mathfrak{coder}(SA_c, \hbar) \ ,$$

as described in section 3. A Maurer-Cartan element $\mathfrak{c} = \sum_{n,g} \hbar^{n+g-1} c^{n,g}$, $c^{n,g} \in A_c^{\wedge n}$, of $(A_c, \mathfrak{L}_c)$ satisfies

$$\mathfrak{L}_c(e^{\mathfrak{c}}) = 0 \ . \tag{35}$$

The corresponding quantum open string field theory $\mathfrak{m}[\mathfrak{c}]$ is defined by

$$e^{\mathfrak{m}[\mathfrak{c}]} = e^{\mathfrak{n}}(e^{\mathfrak{c}}) \ ,$$

and satisfies

$$\mathfrak{L}_o(e^{\mathfrak{m}[\mathfrak{c}]}) = e^{\mathfrak{n}} \circ \mathfrak{L}_c(e^{\mathfrak{c}}) = 0 \ , \tag{36}$$

due to equation (7). Similarly as in equation (8), equation (36) can be recast into

$$\widehat{d}_h(\mathfrak{m}[\mathfrak{c}]) + \widehat{[\cdot,\cdot]}(\mathfrak{m}[\mathfrak{c}]) + \frac{1}{2}\widehat{[\cdot,\cdot]}(\mathfrak{m}[\mathfrak{c}] \wedge \mathfrak{m}[\mathfrak{c}]) - \widehat{[\cdot,\cdot]}(\mathfrak{m}[\mathfrak{c}]) \wedge \mathfrak{m}[\mathfrak{c}] = 0 \ ,$$

which is the defining equation of a quantum A_∞-algebra [47]. The closed string Maurer-Cartan equation (35) was analyzed in [21] and implies the following:

(i) $c := c^{1,0}$ has to satisfy the classical Maurer-Cartan equation

$$\sum_{n=0}^{\infty} \frac{1}{n!} l_n^0(c^{\wedge n}) = 0 \ ,$$

that is c defines a closed string background.

(ii) Consider the part $g^{-1} := c^{2,0}$ of the Maurer-Cartan element $\mathfrak{c}$. Contracting one output of g^{-1} with the symplectic structure ω_c we obtain a linear map

$$h := g^{-1} \circ \omega_c : A_c \to A_c \ .$$

In leading order in $\hbar$, we found in [21] that the part of the Maurer-Cartan equation with two outputs implies

$$d[c] \circ h + h \circ d[c] = -1 \ . \tag{37}$$

Equation (37) states that the cohomology of $d[c]$ is trivial, i.e. that there are no physical states in the background c. Furthermore we can identify h as the propagator corresponding to this background and g^{-1} as the inverse metric on the unphysical states (see section 3). Note that in order to derive equation (37) we had to contract with the symplectic structure, and the identity map on the right hand side of equation (37) stems from the assumption that ω_c is non-degenerate. This observation will be crucial in the topological string, since there the symplectic structure degenerates on the physical states and BRST triviality does not follow from the Maurer-Cartan equation in that case.

From these two observations we will now prove by contradiction that the loop homotopy algebra of closed strings does not admit any Maurer-Cartan element (assuming that ω_c is non-degenerate). Assume that $\mathfrak{c}$ is a Maurer-Cartan element of $\mathfrak{L}_c$ and consider the background shifted loop algebra

$$\mathfrak{L}_c[c] = E(-c) \circ \mathfrak{L}_c \circ E(c) = D(d[c] + l_q^*[c] + \hbar\omega_c^{-1}) \ .$$

Again $c = c^{1,0}$ and $g^{-1} = c^{2,0}$. Next we construct the minimal model $(H(A_c, d[c]), \tilde{\mathfrak{L}}_c[c])$ of $(A_c, \mathfrak{L}_c[c])$. Since the cohomology of $d[c]$ is trivial, i.e. $H(A_c, d[c]) = \{0\}$, the only candidate Maurer-Cartan element of $(H(A_c, d[c]), \tilde{\mathfrak{L}}_c[c])$ is 0, but $\tilde{\mathfrak{L}}_c[c](e^0) = \tilde{\omega}_c^{-1} \neq 0$. Thus $(H(A_c, d[c]), \tilde{\mathfrak{L}}_c[c])$ has no Maurer-Cartan elements and likewise $\mathcal{M}(H(A_c, d[c]), \tilde{\mathfrak{L}}_c[c]) = \emptyset$. Since by construction, $(H(A_c, d[c]), \tilde{\mathfrak{L}}_c[c])$ is quasi-isomorphic to $(A_c, \mathfrak{L}_c[c])$ and quasi-isomorphic IBL_∞-algebras have isomorphic moduli spaces [20], we conclude

$$\emptyset = \mathcal{M}(H(A_c, d[c]), \tilde{\mathfrak{L}}_c[c]) \cong \mathcal{M}(A_c, \mathfrak{L}_c[c]) \cong \mathcal{M}(A_c, \mathfrak{L}_c) \ . \tag{38}$$

The second isomorphism in (38) follows from the observation that a Maurer-Cartan element $\mathfrak{c}$ of $\mathfrak{L}_c[c]$ corresponds to a Maurer-Cartan element $\mathfrak{c} + c$ of $\mathfrak{L}_c$.

Equation (38) states that there are no Maurer-Cartan elements of the closed string loop algebra, which in turn implies that there is no consistent quantum theory of only open strings, or in other words it is impossible to deform the classical open string field theory determined by an A_∞-algebra m into a quantum A_∞-algebra $\mathfrak{m}$.

5.3. Illustration. We will illustrate this last point with an simple toy example: We consider a differential Lie algebra $(A, d, [\cdot,\cdot])$. A harmonious Hodge decomposition is then defined by the triple d, h and[4] P. The classical Maurer-Cartan equation

$$dc + [c, c] = 0 \tag{39}$$

implies that the L_∞ structure of the corresponding minimal model follows from the equation $P[c, c] = 0$, where $c = c_P + c_U + c_T$ is recursively determined through

$$c_U = h[c, c] \ . \tag{40}$$

Thus, $\tilde{l}_2(c_P, c_P) = P[c_p, c_p]$, $\tilde{l}_3(c_P, c_P) = P[c_P, h[c_P, c_P]] + \cdots$ and so forth.

Let us now turn to the quantum Maurer-Cartan equation

$$D(d + [\cdot,\cdot] + \hbar\omega^{-1})(e^{c+\hbar g^{-1}}) = 0 \tag{41}$$

[4] For $A = \Omega^\bullet(M)$, the space of differential forms on M we have $h = -\frac{d^\dagger}{\Delta}$.

where $g^{-1} = \frac{1}{2} u_i \wedge {}^i u \in A \wedge A$. We can disentangle this equation by successive projections as in section VI.A of [21] onto A, $A \wedge A$ and $A \wedge A \wedge A$ respectively. This gives

$$0 = dc + [c, c] + \frac{\hbar}{2}[u_i, {}^i u] \tag{42}$$

$$0 = du_i \wedge {}^i u + [c, u_i] \wedge {}^i u + \omega^{-1} \tag{43}$$

$$0 = [u_i, u_j] \wedge {}^i u \wedge {}^j u \tag{44}$$

where (43) and (44) come with a global factor of $\hbar$ and $\hbar^2$ respectively. At order $\hbar^0$ we recover the classical Maurer-Cartan equation (39). The obstructions at the quantum level arise from (43). Upon composing (43) with ω to the right we recover the propagator equation in the background c,

$$d[c] \circ h + h \circ d[c] = P[c] - 1 \ , \tag{45}$$

provided ω is degenerate, i.e. vanishes on $H(A)$. If ω is non-degenerate then (43) has no solutions and consequently the quantum moduli space is the empty set. This is the case in bosonic string theory. Further obstructions can arise from (44). In Chern-Simons theory (44) is compatible with the propagator equation but we cannot exclude obstructions arising from (44) in general. The topological string, discussed in the next section, is another example where ω is degenerate and (43) and (44) are compatible.

6. Applications to topological strings

The world sheet description of the topological string is based on a supersymmetric sigma model whose target space is a Calabi-Yau 3-fold, X (see [53] for a good review). The world sheet theory admits 4 supercharges as well as an R-symmetry current and the energy momentum tensor. The latter can be twisted by the R-current in such a way that a linear combination of the supercharges defines a differential Q on the state space. Furthermore, the world sheet theory is topological on the cohomology of Q. There are two possible ways to twist the energy momentum tensor leading to two inequivalent theories, the A-model and the B-model. The algebra of the triplet consisting of the differential Q, together with the remaining supercharge and the stress tensor is isomorphic to that of Q, b-ghost and the energy momentum tensor of the BRST quantized bosonic string CFT. Thus we can apply the operator formalism as in bosonic string field theory to define vertices. The corresponding field theories have been constructed in [49] in the open case, and in the closed case in [51] and [52] for the B- and A-model, respectively.

In the A-model there is a natural chain map between the de Rham complex of X and the BRST complex of the twisted world sheet sigma model. If one restricts to local operators this map induces an isomorphism between the de Rham cohomology and the BRST cohomology. In particular, the degree $(1,1)$ elements of the BRST cohomology of the twisted world sheet sigma model are identified with the Kähler structure of X.

In the B-model, on the other hand, there is a chain map between the BRST cohomology and $\oplus_{p.q} H^p(X, \wedge^q T^{1,0} X)$. Again, this induces an isomorphism on the cohomology upon restriction to local operators. Consequently the degree $(1,1)$ elements of the BRST cohomology are identified with the changes of complex structure of X.

Although bosonic and topological string theory share some fundamental properties, there are many crucial differences which we summarize here:

(i) The action of topological open/closed string theory is cubic [49]/[51,52]. Furthermore these actions satisfy the quantum BV master equation. In particular, the closed string vertices define a loop homotopy algebra without including higher vertices.

(ii) The operator b_0^- generically does not have trivial cohomology. Thus it is impossible to define an operator c_0^-, such that $\{b_0^-, c_0^-\} = 1$. Such an operator exists only on all but the physical states, where the physical states are identified with the kernel of L_0^+ (i.e. we work in Siegel gauge). In other words, there is an operator c_0^-, such that

$$\{b_0^-, c_0^-\} = 1 - P \ ,$$

where P is the projection onto the physical states [51,52].

(iii) The minimal models corresponding to the off-shell L_∞-algebras of closed strings vanish identically [27].

(iv) Following the prescription of [5], the symplectic structure on the closed string side is defined by inserting the operator c_0^- into the inner product (bpz inner product in the context of bosonic string theory), that is

$$\omega_c = (\cdot, c_0^- \cdot) \ .$$

Since c_0^- is defined only on the trivial and unphysical states, the symplectic structure of closed strings degenerates on the physical states. Thus we conclude that the entire minimal model of the loop homotopy algebra of closed strings vanishes - the vertices and the symplectic structure.

Let us now turn to the open-closed correspondence in the context of topological string theory. At the classical level, the open-closed vertices again define an L_∞-morphism from the L_∞-algebra defined by the closed string vertices to the Hochschild complex of the open string A_∞-algebra. For the B-model, it has been shown in [50] that the L_∞-morphism is indeed a quasi-isomorphism. Furthermore they conjectured that this should be the case in any string field theory realization of the OCHA, which has been confirmed for the bosonic string [44] and also for Landau-Ginzburg models [54]. At a more abstract level, the results of [12] seem to support this conjecture as well. This implies that the moduli spaces of the L_∞-algebras connected by the quasi-isomorphism are isomorphic, that is

$$\mathcal{M}(A_c, L_c) \cong \mathcal{M}(\mathrm{Coder}^{cycl}(TA_o), d_h, [\cdot,\cdot]) \ ,$$

where L_c denotes the L_∞-algebra of closed strings and $d_h = [M, \cdot]$ is the Hochschild differential corresponding to the open string A_∞-algebra M. An L_∞-algebra is quasi-isomorphic to its minimal model which implies isomorphy of their respective moduli spaces. Recall that one of the distinguished properties of topological strings is that the closed string minimal model vanish identically, and thus we conclude

$$\mathcal{M}(\mathrm{Coder}^{cycl}(TA_o), d_h, [\cdot,\cdot]) \cong \mathcal{M}(A_c, L_c) \cong \mathcal{M}(H(A_c, d), \tilde{L_c}) = H_0(A_c, d_c) \ ,$$

where $\tilde{L_c} = 0$ denotes the minimal model corresponding to L_c and $H_0(A_c, d_c)$ represents the cohomology of d at degree zero. That is, inequivalent deformations of topological open string field theory are parametrized by physical closed string states.

On the other hand one can also ask for deformations of topological open string (tree-level) amplitudes induced by closed strings [54]. To attempt this question, it is useful to think of the OCHA as a single algebraic entity and take the minimal model of it [26]. The minimal model of an OCHA is described by the minimal model of its closed string L_∞-algebra linked to the deformation complex of the minimal model of its open string A_∞-algebra by an L_∞-morphism. If the L_∞-morphism of the initial OCHA is a quasi-isomorphism, then so is the L_∞-morphism of the corresponding minimal model. This implies

$$\mathcal{M}(\mathrm{Coder}^{cycl}(TH(A_o, d_o)), \tilde{d}_h, [\cdot,\cdot]) \cong \mathcal{M}(H(A_c, d), \tilde{L}_c) = H_0(A_c, d_c) \ ,$$

where $\tilde{d}_h = [\tilde{M}, \cdot]$ is the Hochschild differential induced by the minimal model of the open string A_∞-algebra and $H(A_o, d_o)$ represents the physical open string states. In other words, physical closed string states parametrize the space of inequivalent deformations of topological open string (tree-level) amplitudes[5].

Now we draw our attention to the quantum case. As state previously, the cubic closed string action satisfies the quantum BV master equation and thus defines a loop homotopy algebra. If the symplectic structure ω_c is non-degenerate, we showed in section 5 that the corresponding loop homotopy algebra does not admit any Maurer-Cartan element at all. In the topological string, the symplectic structure degenerates on the physical states. This implies that equation (37) is modified to

$$d_c[c] \circ h + h \circ d_c[c] = P[c] - 1 \ ,$$

where $P[c]$ is the projection onto the physical states in Siegel gauge. This is the propagator equation or in mathematical terms h is a harmonious Hodge decomposition of A_c (see section 3). Thus the Maurer-Cartan equation of the loop homotopy algebra of topological strings does not require a vanishing BRST

[5] This space includes deformations with a non-vanishing tadpole. Amplitudes without tadpole correspond to the moduli space of the full OCHA [26,27].

cohomology, and hence, in contrast to bosonic string theory, the conclusion that there cannot be any Maurer-Cartan elements does not persist here. Similar to bosonic string field theory, the full open-closed theory defines a QOCHA, where the open-closed vertices define an IBL_∞-morphism from the loop homotopy algebra of closed strings to the involutive Lie bialgebra on the cyclic Hochschild complex of open strings. The IBL_∞-morphism is a quasi-isomorphism, since the classical L_∞-morphism is, and thus the moduli spaces of the respective IBL_∞-algebras are isomorphic.

$$\begin{aligned}\mathcal{M}(\mathcal{A}_o, \mathfrak{L}_o) &\cong \mathcal{M}(A_c, \mathfrak{L}_c) \\ &\cong \mathcal{M}(H(A_c, d_c), \tilde{\mathfrak{L}}_c) \\ &= \big\{ \mathfrak{c} = \sum_{n,g} \hbar^{n+g-1} c^{n,g} \,\big|\, c^{n,g} \in H(A_c, d_c)^{\wedge n},\ |c^{n,g}| = 0 \big\}\end{aligned} \tag{46}$$

In equation (46) $\mathcal{A}_o = \mathrm{Hom}^{cycl}(TA_o, R)$ denotes the cyclic Hochschild complex, $\mathfrak{L}_o = \widehat{d_h} + \widehat{[\cdot,\cdot]} + \hbar\widehat{\delta}$, $\mathfrak{L}_c$ is the closed string loop homototpy algebra and $\tilde{\mathfrak{L}}_c = 0$ its corresponding minimal model (see appendix A and 3). Maurer-Cartan elements of $\mathfrak{L}_o$ represent deformations of the initial A_∞-algebra M to a quantum A_∞-algebra, or in other words, they represent consistent quantum theories of only open strings. Equation (46) states that the space of quantum open string theories is parametrized by symmetric tensors in $H(A_c, d_c)$ of degree zero, which generalizes the classical open-closed correspondence where we allowed just for vectors. As in the classical case, we can also ask for bulk induced deformations of open string amplitudes (including loops). Again the idea is to take the minimal model of the whole QOCHA, which is guaranteed to exist due to [38,39], and leads to the statement that

$$\mathcal{M}(\tilde{\mathcal{A}}_o, \tilde{\mathfrak{L}}_o) \cong \mathcal{M}(H(A_c, d_c), \tilde{\mathfrak{L}}_c) = \big\{ \mathfrak{c} = \sum_{n,g} \hbar^{n+g-1} c^{n,g} \,\big|\, c^{n,g} \in H(A_c, d_c)^{\wedge n},\ |c^{n,g}| = 0 \big\} ,$$

where $\tilde{\mathcal{A}}_o = \mathrm{Hom}^{cycl}(TH(A_o, d_o))$ and $\tilde{\mathfrak{L}}_o = \widehat{\tilde{d}_h} + \widehat{[\cdot,\cdot]} + \hbar\widehat{\delta}$. Thus we find that the topological open string amplitudes can be deformed by closed strings in a more general way then discussed in [47]. It is not just closed string backgrounds but also higher rank tensors that deform the topological open string amplitudes. In order to get a world-sheet interpretation of such deformations we recall the chain map from the de Rham complex (A-model) to the BRST complex reviewed at the beginning of this section. Correspondingly tensor deformations are implemented on the world-sheet by non-local CFT operators. It would be interesting to see if non-trivial deformations of this type exist.

7. Outlook

In this paper we discussed several properties of bosonic string field theory in terms of homotopy algebras. In particular, combining the open-closed homotopy algebra with the isomorphism between consistent infinitesimal deformations of classical open string field theory and physical closed string states, we established an isomorphism between closed string Maurer-Cartan elements and consistent finite deformations of open string field theory. The QOCHA also provides a simple algebraic description of the obstructions (notably absent in the topological string) to the existence of Maurer-Cartan elemnts at the quantum level.

We also proved a decomposition theorem for the loop algebra of quantum closed string field theory which, in turn, implies uniqueness of closed string field theory on a given background. Finally, we also addressed uniqueness and background independence of closed string field theory using OCHA.

In contrast, a complete formulation of super string field theory has not been developed yet [45,46]. Generalizing the prescription of [5,6] to the supersymmetric case, the first task would be to construct a BV algebra on the singular chains of super Riemann surfaces. The super conformal field theory of the super string is then expected to define a morphism of BV algebras and would lead to some novel algebraic structures on the corresponding state space.

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A. A_∞-, L_∞- and IBL_∞-algebras

A.1. A_∞- and L_∞-algebras. Let $A = \oplus_{n\in\mathbb{Z}} A_n$ be a graded module over some ring R and consider the tensor algebra

$$TA = \bigoplus_{n=0}^{\infty} A^{\otimes n} ,$$

with comultiplication $\Delta : TA \to TA \otimes TA$ defined by

$$\Delta(a_1 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n} (a_1 \otimes \ldots \otimes a_i) \otimes (a_{i+1} \otimes \ldots \otimes a_n) .$$

We have the canonical projection maps $\pi_n : TA \to A^{\otimes n}$ and inclusion maps $i_n : A^{\otimes n} \to TA$.

A coderivation $D \in \mathrm{Coder}(TA)$ is a linear map on TA that satisfies

$$(D \otimes \mathrm{id} + \mathrm{id} \otimes D) \circ \Delta = \Delta \circ D .$$

From this property it follows that there is an isomorphism $\mathrm{Coder}(TA) \cong \mathrm{Hom}(TA, A)$ induced by

$$D \mapsto \pi_1 \circ D ,$$

with inverse (lifting map)

$$d \mapsto \widehat{d} := (\mathrm{id} \otimes d \otimes \mathrm{id}) \circ \Delta_3 ,$$

where Δ_n denotes the n-fold comultiplication.

Similarly a cohomomorphisms $F \in \mathrm{Cohom}(TA, TA')$ is a linear map from TA to TA' satisfying

$$\Delta \circ F = (F \otimes F) \circ \Delta ,$$

which implies $\mathrm{Cohom}(TA, TA') \cong \mathrm{Hom}(TA, A')$, induced by

$$F \mapsto \pi_1 \circ F ,$$

with inverse (lifting map)

$$f \mapsto e^f := \sum_{n=0}^{\infty} f^{\otimes n} \circ \Delta_n .$$

The Gerstenhaber bracket $[\cdot,\cdot]$ defined by

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{D_1 D_2} D_2 \circ D_1$$

endows $\mathrm{Coder}(TA)$ with the structure of a graded Lie algebra. Now an A_∞-algebra is defined by a coderivation $M \in \mathrm{Coder}(TA)$ of degree 1 that squares to zero. This in turn makes $\mathrm{Coder}(TA)$ a differential graded Lie algebra (DGL) with Hochschild differential d_h defined by

$$d_h = [M, \cdot] ,$$

and deformations of M are controlled by this DGL. An A_∞-algebra M is denoted as strong or weak, corresponding to whether $\pi_1 \circ M \circ i_0$ is zero or non-zero respectively. Let (A, M) and (A', M') be A_∞-algebras, then an A_∞-morphism $F \in \mathrm{Morph}(A, A')$ is a cohomomorphism of degree zero which commutes with the differentials

$$F \circ M = M' \circ F .$$

Furthermore $F \in \mathrm{Morph}(A, A')$ is called an A_∞-quasi-isomorphism if the linear map $\pi_1 \circ F \circ i_1$ induces an isomorphism on cohomologies. Similarly it is called an A_∞-isomorphism if $\pi_1 \circ F \circ i_1$ defines an isomorphism. We also distinguish between strong and weak A_∞-morphisms, corresponding to whether $\pi_1 \circ f \circ i_0$ is zero or non-zero respectively.

A Maurer-Cartan element of an A_∞-algebra (A, M) is a degree zero element $a \in A$ that satisfies

$$M(e^a) = 0 ,$$

Table 1. A_∞- and L_∞-algebras in summary

	A_∞	L_∞
algebra	$M^2 = 0$, $\lvert M\rvert = 1$	$L^2 = 0$, $\lvert L\rvert = 1$
morphism	$F \circ M = M' \circ F$	$F \circ L = L' \circ F$
lift (coder)	$\widehat{d} = (\mathrm{id} \otimes d \otimes \mathrm{id}) \circ \Delta_3$	$\widehat{d} = (d \wedge \mathrm{id}) \circ \Delta$
lift (cohom)	$e^f = \sum_{n=0}^{\infty} f^{\otimes n} \circ \Delta_n$	$e^f = \sum_{n=0}^{\infty} \frac{1}{n!} f^{\wedge n} \circ \Delta_n$
background shift	$E(a)(a_1 \otimes \ldots \otimes a_n) =$ $e^a \otimes a_1 \otimes e^a \ldots e^a \otimes a_n \otimes e^a$	$E(c)(c_1 \wedge \ldots \wedge c_n) =$ $e^c \wedge c_1 \wedge \ldots \wedge c_n$
	$M[a] = E(-a) \circ M \circ E(a)$	$L[c] = E(-c) \circ L \circ E(c)$
Maurer-Cartan element	$M(e^a) = 0$, $\lvert a\rvert = 0$	$L(e^c) = 0$, $\lvert c\rvert = 0$
gauge transformation	$\frac{d}{dt} U_\lambda(t) = [M, \widehat{\lambda}(t)] \circ U_\lambda(t)$	$\frac{d}{dt} U_\lambda(t) = [L, \widehat{\lambda}(t)] \circ U_\lambda(t)$
cyclicity	$\omega(\pi_1 \circ D, \cdot)$ cyclic sym.	$\omega(\pi_1 \circ D, \cdot)$ full sym.

that is e^a is a constant (no inputs) A_∞-morphism on A. The space of all Maurer-Cartan elements is denoted by $\mathcal{MC}(A, M)$.

Furthermore we have the notion of gauge equivalence on the space of Maurer-Cartan elements: Gauge transformations are implemented by a family of A_∞-isomorphisms $U_\lambda(t)$, defined by

$$\frac{d}{dt} U_\lambda(t) = [M, \widehat{\lambda}(t)] \circ U_\lambda(t) \qquad \text{and} \qquad U_\lambda(0) = \mathrm{id}\ ,$$

where $\lambda(t) \in A$ is of degree minus 1 [7]. The moduli space of an A_∞-algebra is defined to be the Maurer-Cartan space modulo gauge transformations

$$\mathcal{M}(A, M) := \mathcal{MC}(A, M)/\sim\ ,$$

that is for $a, b \in \mathcal{MC}(A, M)$, $a \sim b$ if there is a gauge transformation $U_\lambda(t)$ such that $e^b = U_\lambda(1)(e^a)$.

A background shift by an element $a \in A$ of degree zero is implemented by the cohomomorphism $E(a)$ defined by

$$E(a)(a_1 \otimes \ldots \otimes a_n) = e^a \otimes a_1 \otimes e^a \otimes \ldots \otimes e^a \otimes a_n \otimes e^a\ .$$

For a given A_∞-algebra (A, M) the background shifted A_∞-algebra is defined by $M[a] = E(-a) \circ M \circ E(a)$, which makes $E(a)$ a weak A_∞-isomorphism.

Suppose the module A is endowed with an odd symplectic structure $\omega : A \otimes A \to R$. A coderivation $D \in \mathrm{Coder}(TA)$ is called cyclic, if the map

$$\omega(\pi_1 \circ D, \cdot) : TA \otimes A \to R$$

is cyclic symmetric in $TA \otimes A$. The space of cyclic coderivations is denoted by $\mathrm{Coder}^{cycl}(TA)$ and is closed under the Gerstenhaber bracket.

L_∞-algebras are constructed in a similar way, where instead of the tensor algebra TA one considers the symmetric algebra SA. The coalgebra structure on SA is given by

$$\Delta(c_1, \cdots, c_n) = \sum_{i=0}^{n} \sum_{\sigma}{}' (c_{\sigma_1} \wedge \cdots \wedge c_{\sigma_i}) \otimes (c_{\sigma_{i+1}} \wedge \cdots \wedge c_{\sigma_n})\ ,$$

where $\sum_\sigma'$ indicates the sum over all permutations $\sigma \in S_n$ constraint to $\sigma_1 < \cdots < \sigma_i$ and $\sigma_{i+1} < \cdots < \sigma_n$ (unshuffels). In table 1 we summarize the definitions from above, together with the corresponding counterparts in the L_∞ context.

A.2. IBL_∞-algebras. Homotopy involutive Lie bialgebras (IBL_∞-albegras) as presented in [20], are constructed similarly to L_∞-algebras. The definition includes an external parameter $\hbar$ and makes use of higher order coderivations [23,24,22]. In the previous section we saw that (first oder) coderivations on SA are in one-to-one correspondence with homomorphisms from SA to A, where the correspondence is established by the lifting map and the projection π_1. That is, a first order coderivations is the lift of a homomorphism with an arbitrary number of inputs and one output. Higher order coderivations are then introduced by allowing for several outputs of the homomorphism: The space $\mathrm{Coder}^n(SA)$ of coderivations of order n is isomorphic to $\mathrm{Hom}(SA, \Sigma^n A)$, where $\Sigma^n A := \bigoplus_{i=1}^n A^{\wedge i}$. The isomorphism is given by

$$\begin{aligned} \mathrm{Hom}(SA, \Sigma^n A) &\to \mathrm{Coder}^n(SA) \\ d &\mapsto \widehat{d} = (d \wedge \mathrm{id}) \circ \Delta \ , \end{aligned}$$

with inverse

$$\begin{aligned} \mathrm{Coder}^n(SA) &\to \mathrm{Hom}(SA, \Sigma^n A) \\ D &\mapsto \begin{cases} \pi_1 \circ D \\ +\Big(\pi_2 \circ D - (\pi_1 \circ D \wedge \pi_1) \circ \Delta\Big) \\ \vdots \\ +\Big(\pi_n \circ D - \sum_{i+j=n-1} (\pi_i \circ D \wedge \pi_{j+1}) \circ \Delta\Big) \end{cases} . \end{aligned}$$

The graded commutator

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{D_1 D_2} D_2 \circ D_1 \ ,$$

where D_1, D_2 are arbitrary higher order coderivations, satisfies the property

$$[\mathrm{Coder}^i(SA), \mathrm{Coder}^j(SA)] = \mathrm{Coder}^{i+j-1} \ . \tag{47}$$

Consider now the space

$$\mathfrak{coder}(SA, \hbar) := \bigoplus_{n=1}^{\infty} \hbar^{n-1} \mathrm{Coder}^n(SA) \ .$$

Equation (47) implies that the graded commutator raises $\mathfrak{coder}(SA, \hbar)$ to a graded Lie algebra. From the definition of higher order coderivations, we obtain the isomorphism

$$\mathfrak{coder}(SA, \hbar) \cong \bigoplus_{n=1}^{\infty} \hbar^{n-1} \mathrm{Hom}(SA, \Sigma^n A) \ .$$

For an element $\mathfrak{d} \in \bigoplus_{n=1}^{\infty} \hbar^{n-1} \mathrm{Hom}(SA, \Sigma^n A)$ we define associated maps $d^{n,g} \in \mathrm{Hom}(SA, A^n)$ by

$$\mathfrak{d} = \sum_{n=1}^{\infty} \sum_{g=0}^{\infty} \hbar^{n+g-1} \, d^{n,g} \ ,$$

that is we expand $\mathfrak{d}$ in the number of outputs.

The definition of IBL_∞-algebras, IBL_∞-morphisms, etc. resembles that of L_∞-algebras, except that we substitute $\mathrm{Hom}(SA, A)$ by $\bigoplus_{n=1}^{\infty} \hbar^{n-1} \mathrm{Hom}(SA, \Sigma^n A)$: An IBL_∞-algebra is defined by an element $\mathfrak{L} \in \mathfrak{coder}(SA, \hbar)$ of degree one that squares to zero. A cohomomorphism $\mathfrak{F} \in \mathfrak{cohom}(SA, SA', \hbar)$ is determined by a map $\mathfrak{f} \in \bigoplus_{n=1}^{\infty} \hbar^{n-1} \mathrm{Hom}(SA, \Sigma^n A)$ via the lifting map

$$\mathfrak{F} = e^{\mathfrak{f}} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathfrak{f}^{\wedge n} \ .$$

Let $(A, \mathfrak{L})$ and $(A', \mathfrak{L}')$ be IBL_∞-algebras. An IBL_∞-morphism $\mathfrak{F} \in \mathfrak{morph}(A, A')$ from $(A, \mathfrak{L})$ to $(A', \mathfrak{L}')$ is a cohomomorphism of degree zero, that commutes with the differentials

$$\mathfrak{F} \circ \mathfrak{L} = \mathfrak{L}' \circ \mathfrak{F} \ .$$

Similarly a Maurer-Cartan element of an IBL_∞-algebra $(A, \mathfrak{L})$ is an element $\mathfrak{c} \in \bigoplus_{n=1}^{\infty} \hbar^{n-1} \Sigma^n A$ of degree zero, satisfying

$$\mathfrak{L}(e^{\mathfrak{c}}) = 0 \ .$$

The space of Maurer-Cartan elements of an IBL_∞-algebra is denoted by $\mathcal{MC}(A, \mathfrak{L})$. In analogy to the L_∞ case we define gauge transformations by a family of IBL_∞-isomorphisms $\mathfrak{U}_\lambda(t)$ determined by

$$\frac{d}{dt}\mathfrak{U}_\lambda(t) = [\mathfrak{L}, \widehat{\lambda}(t)] \circ \mathfrak{U}_\lambda(t) \qquad \text{and} \qquad \mathfrak{U}_\lambda(0) = \text{id} \ ,$$

where $\lambda(t) \in \bigoplus_{n=1}^{\infty} \hbar^{n-1} \Sigma^n A$ is of degree minus one. Finally the moduli space of an IBL_∞-algebra is the space of Maurer-Cartan elements modulo gauge transformations, that is $\mathcal{M}(A, \mathfrak{L}) = \mathcal{MC}(A, \mathfrak{L})/\sim$. Obviously one recovers the L_∞ structures in the limit $\hbar \to 0$.

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ON HOMOTOPY ALGEBRAS AND QUANTUM STRING FIELD THEORY

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Abstract. We revisit the existence, background independence and uniqueness of closed, open and open-closed bosonic- and topological string field theory, using the machinery of homotopy algebra. In a theory of classical open- and closed strings, the space of inequivalent open string field theories is isomorphic to the space of classical closed string backgrounds. We then discuss obstructions of these moduli spaces at the quantum level. For the quantum theory of closed strings, uniqueness on a given background follows from the decomposition theorem for loop homotopy algebras. We also address the question of background independence of closed string field theory.

1. Introduction

The standard formulation of classical string theory consists of a set of rules to compute scattering amplitudes for a set of n (excited) strings typically propagating on a D-dimensional Minkowski space-time M_D. This prescription involves an integration over the moduli space of disks with n punctures for open strings (or spheres with n punctures for the closed strings). Comparing this with the approach taken for point particles the situation in string theory seems incomplete. Indeed, for point particles one starts with an action principle and then obtains the classical scattering amplitudes by solving the equations of motions deriving from this action. Since the various string excitations ought to be interpreted as particles one would hope to be able to apply the same procedure for the scattering of strings. The aim of string field theory is precisely to provide such an action principle so that the set of rules to compute scattering amplitudes for strings follow from this action. Since the string consists of a infinite linear superpositions of point particle excitations one would expect that such an action may be rather complicated. Yet the first construction of a consistent classical string field theory of interacting open strings [3] has a remarkably simple algebraic structure of a differential graded algebra (DGA) together with a non-degenerate odd symplectic form.

Key words and phrases. string field theory, homotopy algebra, topological string.

The geometric approach for the construction of string field theory [1, 2], starts with a decomposition of the relevant moduli space of Riemann surfaces into elementary vertices and graphs. The condition that the moduli space is covered exactly once, implies that the geometric vertices satisfy a classical Batalin-Vilkovisky master equation. From this one then anticipates that any string field theory action should realize some homotopy algebra. The subject of this talk is to investigate to what extend this algebraic structure is useful, and to determine certain additional properties that should be satisfied by any consistent string field theory. In particular, it is of interest to know in what sense string field theory is unique. Another related issue stems from the fact that the construction of string field theory assumes that the string propagates in a certain string background, whose geometry is that of Minkowski space. However, since string theory includes gravity, this background is dynamical. The question of background independence of this construction is thus relevant.

To set the stage, let us start with the well understood case of a single point particle propagating on a non-compact manifold M_D with a pseudo-Riemannian metric g. The world line of the particle is described by a curve $\phi : [a, b] \to M_D$ that extremizes the action

$$S[\phi, h] = \int\limits_{[a,b]} \frac{1}{\sqrt{h_{tt}}} g(\dot{\phi}, \dot{\phi}) dt$$

where h_{tt} is a non-dynamical "metric" on the world line that can be set to 1 by a suitable reparametrization of t. Similarly, for an open string we have a map $\phi : \Sigma = [a, b] \times [c, d] \to M_D$ that extremizes the action

$$S[\phi, h] = \int\limits_{\Sigma} \sqrt{h} h^{ij} g(\partial_i \phi, \partial_j \phi) \tag{1.1}$$

so that the area is minimal. If the Riemann curvature of M_D vanishes, then the action (1.1) is invariant under conformal mappings of the *world sheet* Σ. In particular, we can conformally map Σ to a disk with 2 punctures. Analogously, a world sheet describing $n-1$ strings joining into one can be mapped into a disk with n punctures. In order to specify which particles (or string excitations) are involved in the scattering amplitude we need to endow the puncture with additional structures. This is done by attaching conformal tensors $\{V_i[\phi]\}$ built out of the maps ϕ evaluated at the puncture and the coefficients of the Laurent polynomial of ϕ evaluated in local coordinates. The amplitude is then expressed in terms of the n-point correlator

$$< V_{i_1}(z_1), \cdots, V_{i_n}(z_n) > \ , \tag{1.2}$$

with respect to the (formal) Gaußian measure defined by $S[\phi]$. In fact the correlator (1.2) which is called a conformal field theory correlator in physics is not quite what one needs. In order to get the string scattering amplitude we need to integrate over the moduli space of the punctured disk. Now, since the action $S[\phi, h]$ is invariant under diffeomorphisms on the world sheet Σ as well as under Weyl re-scalings of the world sheet metric h we really want to integrate over the $(n-3)$-dimensional gauge-fixed moduli space M_{n-3} (for a review see e.g. [4] and references therein). Treating the gauge-fixed action using the standard BRST formalism we end up with an action $S[\phi, c, b]$ including odd world sheet tensors fields (BRST ghosts) together with an odd differential Q_o that generates the odd symmetry transformations of the gauge fixed action. Similarly, the insertions at the punctures of Σ contain added Laurent coefficients of the b and c ghosts. The string amplitude can be written schematically as in figure 1, where the $n-3$ meromorphic vector fields v_i are

FIGURE 1. Sketch of the CFT realization of the scattering amplitude of n open strings.

constant near the puncture P_i, and cannot be extended to the whole disk. These vector fields generate translations in the moduli space; they move the punctures. Concretely, this amplitude becomes

$$\int_{M_{n-3}} ds_1 \dots ds_{n-3} \, \langle b(v_1) \dots b(v_{n-3}) \, V_{i_1}[\phi, b, c](z_1) \dots V_{i_n}[\phi, b, c](z_n) \rangle, \quad (1.3)$$

where the correlator is evaluated with respect to the measure obtained from the world sheet action $S[\phi, c, b]$. What we have just described is what is usually referred to as the operator formalism of the world sheet conformal field theory (CFT), which dresses the geometric amplitudes (punctured disks) with the physical states (particles). The amplitudes (1.3) are well defined on the cohomology of Q_o.

The purpose of string field theory is two-fold. First to reproduce these amplitudes in terms of vertices and graphs built from them and second to generalize the amplitudes (1.3) on $coh(Q_o)$ to the module A_o of all conformal tensors with suitable regularity conditions. At the geometrical level, the simplest possible construction would be that of a single vertex of 3 joining strings which has no moduli, with all amplitudes recovered from graphs built from 3-vertices. This is indeed possible for the open bosonic string [3]. However, the decomposition of moduli space is not unique so that other realizations are possible where higher order vertices are needed to recover the amplitudes

(1.3). In any case the geometric vertices in any consistent decomposition form a BV algebra.

The world sheet CFT then defines a morphism of BV algebras between the set of geometric vertices $\{\mathcal{V}_n\}$, and the dressed "physical" vertices. It also provides us with an inner product on the graded module A_o generated by the conformal tensors $V_i[\phi, b, c]$ of the (ϕ, b, c) - CFT inserted at the origin in the local coordinate z around a puncture P on the disk. With the help of the latter we can interpret the set of physical vertices as multilinear maps $m_i : A_o^{\otimes i} \to A_o$, with some further symmetry properties implied by the cyclic symmetry of the vertices. We denote by $C(A_o)$, the space of such multilinear maps on A_o. It is then not hard to see that the BV-master equation implies that the maps m_i define an A_∞-structure. One way to see this is to define a coderivation M of degree 1 on the tensor algebra $TA_o = \oplus_n A_o^{\otimes n}$ with components

$$(M)_{n,u} = \sum_{\substack{r+s+t=n \\ r+1+t=u}} \mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}$$

Imposing vanishing of the graded commutator $[M, M]$, we obtain a characterization of all differentials compatible with the A_∞-structure.

The classical solutions of the string field theory action defined by the maps m_i together with $< \cdot, \cdot >$ are given by the Maurer-Cartan elements, $M(e^{\psi_0}) = 0$.

There is an analogous story for classical closed strings obtained from the above by replacing the punctured disk by a punctured sphere with world sheet conformal field theory $S[\Phi, c, \bar{c}, b, \bar{b}]$ and dressed by conformal tensors $V_i[\Phi, b, \bar{b}, c, \bar{c}]$ where $b, \bar{b}, c$ and $\bar{c}$ depend holomorphically and anti-holomorphically on the world sheet coordinates z and $\bar{z}$, respectively. The CFT then provides a morphism between the set of geometric vertices and the (dressed) physical vertices of closed string field theory. The latter can again be interpreted as maps, l_i on the garaded symmetric module $SA_c = \oplus_n A_c^{\wedge n}$. They accordingly realize an L_∞ algebra (A_c, L), with $[L, L] = 0$.

Finally, we let open and closed strings interact with each other. The open closed vertices consist of disks with punctures on the boundary as well as on the disk. These vertices realize an L_∞ morphism F, between the closed and open sector taken separately,

$$(A_c, L) \xrightarrow{F} (\mathrm{Coder}^{cycl}(TA_o), d_h, [\cdot, \cdot]) \ . \tag{1.4}$$

This is the open-closed homotopy algebra of Kajiura and Stasheff [5].

Remark 1. Note that, while the geometric decomposition of the moduli spaces appearing in the construction of string field theory, just reviewed, is independent of the details of M_D the operator formalism makes explicit use of the geometry of M_D as well as possible other background fields inserted at the punctures. In particular, the module A of conformal tensors typically depends on these data. This is where the background dependence enters in the construction of string field theory. This is in contrast to e.g. General Relativity where the action does not depend on any background metric on M_D.

A natural question that arises in the above context is whether for a given background (in the sense just described) the generalization of (1.3) as well as its closed string version is unique. For classical string field theory the answer to this question is affirmative, as follows form the decomposition theorem [5] for homotopy algebras. This theorem establishes an isomorphism between a given homotopy algebra and the direct sum of a linear contractible algebra and a minimal model. In the context of string field theory, the structure maps of the minimal model are given by (1.3).

In this talk we discuss the following generalizations of the results reviewed above:

- classification of inequivalent deformations of classical open string field theory.
- background independence of closed string field theory.
- decomposition theorem for quantum closed string field theory.
- quantization of the open closed homotopy algebra.

2. Results

Let us start with non-trivial deformations of open string field theory. That is we consider *continuous* deformations of the worldsheet CFT that do not preserve Q_o and (1.3) simultaneously . The usefulness of the homotopy formulation of SFT in this respect is that this problem can be formulated as a cohomology problem. Indeed, since any consistent open string field theory realizes an A_∞ algebra, i.e. defines a coderivation M of degree 1 on the tensor algebra TA_o with $[M, M] = 0$, any infinitesimal deformation $M + \delta M$ satisfies $d_H(\delta M) \equiv [M, \delta M] = 0$. For a given worldsheet CFT one would therefore like to determine $coh(d_H)$. The outcome of this analysis is contained in

Theorem 1 ([6])**.** *Let $S[\phi, c, b]$ be the open string world sheet CFT on M_D, A_o the corresponding module of conformal tensors, Q_o the BRST differential, and (1.3) the corresponding string amplitudes on $coh(Q_o)$. Then the only non-trivial infinitesimal deformations of $S[\phi, c, b]$ preserving A_o are infinitesimal deformations of the closed string background in the relative cohomology of Q_c,*

$$coh(d_H) \cong coh(b_0 - \bar{b}_0, Q_c) \ .$$

Remark 2. A particular class of deformations that do not preserve Q_o and (1.3) are shifts in the open string background $\phi_0 \to \phi_0 + \epsilon\delta\phi$ with $M(e^{\phi_0+\epsilon\delta\phi}) =$

$O(\epsilon^2)$. Such transformations are, however, d_H-exact as are all field redefinitions of ϕ. From a physics perspective, the interesting fact implied by theorem 1 is that open string theory already contains the complete information of the particle content of *closed* string theory.

Proof. The proof of this assertion proceeds via a detailed analysis of the deformations of the CFT correlator (1.3). □

Given the isomorphism between the cohomologies one may wonder whether this isomorphism holds for finite deformations. On the closed string side finite deformations correspond to classical solutions of the closed string field theory equation of motion, that is Maurer-Cartan elements $L(e^{\Phi}) = 0$, whereas finite deformations of open string field theory are Maurer-Cartan elements of $[\cdot,\cdot]$ on $\{M \in \mathrm{Coder}^{cycl}(TA_o)\}$, that is $[M, M] = 0$. A classic theorem of Kontsevich then guarantees isomorphism at the finite level, or more precisely that the moduli spaces of two L_∞-algebras connected by a L_∞-quasi-isomorphism are isomorphic. Thus, we have

Corollary 1. *Let $\mathcal{M}(A_c, L)$ and $\mathcal{M}(Coder^{cycl}(TA_o), [\cdot,\cdot])$ be the moduli space of Maurer-Cartan elements obtained by moding out L- and $[\cdot,\cdot]$ -gauge transformations respectively, then we have*

$$\mathcal{M}(A_c, L) \cong \mathcal{M}(Coder^{cycl}(TA_o), d_h, [\cdot,\cdot]) \ .$$

We will return to the question whether this isomorphism survives quantization below but first we would like to turn to background independence of closed string field theory. As mentioned above for a given background the operator formalisms realizes a certain L_∞ algebra. Furthermore, for a given classical solution Φ_0 in this field theory we then obtain a new homotopy algebra upon conjugation by this Maurer-Cartan element. Background independence then would imply that the structure maps of the minimal model obtained from this homotopy algebra are equivalent to the amplitudes (1.3) obtained with the measure of the world-sheet CFT $S[\Phi, c, \bar{c}, b, \bar{b}]$ in the new background (see figure 2).

We can answer this question by addressing the cohomology problem on $\{L \in \mathrm{Coder}^{sym}(SA_c)\}$. The bracket $[\cdot,\cdot]$ on $\mathrm{Coder}(SA)$ induces the Chevalley-Eilenberg differential $d_C = [L, \cdot]$ on the deformation complex. The analysis proceeds in close analogy with that for open string theory with the result,

Proposition 1. *Let $S[\Phi, c, \bar{c}, b, \bar{b}]$ be the closed string world sheet CFT on M_D, A_c the corresponding module of conformal tensors and Q_c the BRST differential. Then*

$$coh(d_c) = \emptyset \ .$$

$$\begin{tikzcd}
S_{CFT} \arrow[r, "\text{operator formalism}", "\mathcal{V}_n \mapsto l_n"'] \arrow[d, "S_{CFT} \to S_{CFT} + \int \Phi_0"'] & \{l_n\}_{n\in\mathbb{N}} \arrow[d, "L \to e^{-\Phi_0} \circ L \circ e^{\Phi_0}"] \\
S'_{CFT} \arrow[r, "\text{operator formalism}", "\mathcal{V}_n \mapsto l'_n"'] & \{l'_n\}_{n\in\mathbb{N}}
\end{tikzcd}$$

FIGURE 2. Background independence requires that the L_∞ maps $\{l'_n\}$ obtained upon conjugation by the MC-element e^{Φ_0} are equivalent to those obtained from the world sheet CFT in the background Φ_0.

An immediate consequence of this proposition is that the diagram in figure 2 commutes which, in turn, implies independence under shifts in the background that preserve A_c.

Remark 3. We should note that generic shifts in the background Φ will not preserve the module A_c.

Let us now return to the decomposition theorem which states that a homotopy algebra defined on a certain complex can be decomposed into the direct sum of a minimal and a linear contractible part. By definition, the linear contractible part is just a complex with vanishing cohomology, whereas the minimal part is a homotopy algebra of the same type as the initial one but without differential [7]. Furthermore, the initial an the decomposed algebra are isomorphic in the appropriate sense. Clearly, the minimal part can be extracted from the decomposed algebra by projection, and thus the decomposition theorem implies the minimal model theorem.

The relevance of the minimal model theorem in physics is as follows: Suppose that the vertices of some field theory satisfy the axioms of some homotopy algebra. Then the minimal model describes the corresponding S-matrix amplitudes [8, 9]. Furthermore, the S-matrix amplitudes and the field theory vertices are quasi-isomorphic, which implies that their respective moduli spaces are isomorphic (this follows in general from the minimal model theorem).

Now we conclude that string field theory is unique up to isomorphisms on a fixed conformal background (CFT): In string field theory, the differential is generically given by the BRST charge Q. Furthermore the CFT determines the S-matrix amplitudes. Thus a conformal background determines the minimal and the linear contractible part, which implies uniqueness up to isomorphisms.

An explicit construction of the decomposition model is known for the classical algebras (A_∞ and L_∞) [8, 9]. In the following we construct the decomposition model for quantum closed string field theory, formulated in the framework of IBL_∞-algebras (see e.g. [10, 11] for a definition).

Quantum closed string field theory has the algebraic structure of a loop homotopy Lie-algebra $(A, \mathfrak{L})$ [12], i.e.

$$\mathfrak{L} = \sum \hbar^g L^g + \hbar\Omega^{-1} , \quad \mathfrak{L}^2 = 0 , \tag{2.1}$$

where $L^g = D(l^g) \in \mathrm{Coder}^{cycl}(SA)$ and $\Omega^{-1} = D(\omega^{-1}) \in \mathrm{Coder}^2(SA)$ is the lift of the inverse of the odd symplectic structure (D denotes the lift from multilinear maps to coderivations). We define $l_q := \sum_g \hbar^g l^g$. The differential on A is given by $d = l_{cl} \circ i_1$. Furthermore we abbreviate the collection of multilinear maps without the differential by $l_q^* := l_q - d$.

Definition 1. A pre Hodge decomposition of A is a map $h : A \to A$ of degree minus one which is compatible with the symplectic structure and squares to zero.

For a given pre Hodge decomposition of A, we define the map

$$P = 1 + dh + hd , \tag{2.2}$$

and

$$g := -\omega \circ d \qquad \text{and} \qquad g^{-1} := h \circ \omega^{-1} \in A^{\wedge 2} , \tag{2.3}$$

where the symplectic structure ω and its inverse ω^{-1} are considered as a map from A to A^* and A^* to A, respectively. We define *trees* constructed recursively from l_q^* and h via

$$\mathsf{T}_q = h \circ l_q^* \circ e^{1+\mathsf{T}_q} \qquad \text{and} \qquad \mathsf{T}_q \circ i_1 = 0 . \tag{2.4}$$

Theorem 2 ([13]). *Let $(A, \mathfrak{L} = D(d + l_q^* + \hbar\omega^{-1}))$ be a loop homotopy Lie algebra. For a given pre Hodge decomposition h, there is an associated loop homotopy Lie algebra*

$$\bar{\mathfrak{L}} = D(d + \overset{(P)}{\mathsf{T}}_q \circ e^{\hbar g^{-1}} + \hbar\bar{\omega}^{-1}) , \tag{2.5}$$

where $\bar{\omega}^{-1} = P^{\wedge 2}(\omega^{-1})$ and $\overset{(P)}{\mathsf{T}}_q \circ e^{\hbar g^{-1}}$ represents the graphs with a single output labeled by P. Furthermore there is an IBL_∞-isomorphism from $(A, \bar{\mathfrak{L}})$

to $(A,\mathfrak{L})$. d *is called the linear contractible part and* $\overset{(P)}{\mathsf{T}}_q \circ E(\hbar g^{-1}) + \hbar\bar{\omega}^{-1}$ *the minimal part.*

Proof. The proof follows by explicit verification, using equation (2.1), (2.2) and (2.4). □

Finally, we describe the quantum generalization of the classical open-closed homotopy algebra (OCHA) of Kajiura and Stasheff. As already alluded in the introduction, the OCHA can be described by an L_∞-morphism, N, mapping from the closed string algebra (A_c, L) to the deformation complex of the open string algebra $(\mathrm{Coder}^{cycl}(TA_o), d_h, [\cdot,\cdot])$, i.e.

$$e^N \circ L = D(d_h + [\cdot,\cdot]) \circ e^N \ ,$$

or equivalently

$$N \circ L = d_h \circ N + \frac{1}{2}[N,N] \circ \Delta \ , \tag{2.6}$$

where N describes the open-closed vertices and the comultiplication $\Delta : TA \to TA \otimes TA$ is defined by

$$\Delta(a_1 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n} (a_1 \otimes \ldots \otimes a_i) \otimes (a_{i+1} \otimes \ldots \otimes a_n) \ .$$

In a similar way one can describe the QOCHA by an IBL_∞-morphism from the loop homotopy Lie algebra $(A_c, \mathfrak{L})$ of closed strings to the involutive Lie bialgebra $(\mathcal{A}_o, d_h, [\cdot,\cdot], \delta)$, where $\mathcal{A}_o := \mathrm{Hom}^{cycl}(TA_o, \mathbb{k})$ [1].

The operation

$$\delta : \mathcal{A}_o \to \mathcal{A}_o^{\wedge 2} \ ,$$

is defined by

$$\begin{aligned} &(\delta f)(a_1, \ldots, a_n)(b_1, \ldots, b_m) \\ &:= (-1)^f \sum_{i=1}^{n} \sum_{j=1}^{m} (-1)^\epsilon f(e_k, a_i, \ldots, a_n, a_1, \ldots, a_{i-1}, e^k, b_j, \ldots, b_m, b_1, \ldots, b_{j-1}) \ , \end{aligned} \tag{2.7}$$

where $(-1)^\epsilon$ denotes the Koszul sign, $\{e_i\}$ is a basis of A_o and $\{e^i\}$ is the corresponding dual basis satisfying $\omega_o({}_ie, e^j) = {}_i\delta^j$. This operation can be interpreted geometrically as the sewing of open strings on one boundary component. In [14, 10] it has been shown that $(\mathcal{A}_o, d_h, [\cdot,\cdot], \delta)$ defines an involutive Lie bialgebra, a special case of an IBL_∞-algebra. In the language of IBL_∞-algebras this is equivalent to the statement that

$$\mathfrak{L}_o := D(d_h + [\cdot,\cdot] + \hbar\, \delta)$$

[1] In the quantum case it is more convenient to work with $\mathrm{Hom}^{cycl}(TA_o, \mathbb{k})$ rather than with $\mathrm{Coder}^{cycl}(TA_o)$

squares to zero.

Definition 2 ([11])**.** The quantum open-closed homotopy algebra is defined by an IBL_∞-morphism from a loop homotopy Lie algebra $(A_c, \mathfrak{L}_c)$ to the involutive Lie bialgebra $(\mathcal{A}_o, \mathfrak{L}_o)$, i.e.

$$e^{\mathfrak{n}} \circ \mathfrak{L}_c = \mathfrak{L}_o \circ e^{\mathfrak{n}} \tag{2.8}$$

The maps $\mathfrak{n}$ describe the open-closed vertices to all orders in $\hbar$.

Equation (2.8) can be recast, such that the five distinct sewing operations in open-closed string field theory become apparent:

$$\begin{aligned} &\mathfrak{n} \circ \mathfrak{L}_c + \frac{\hbar}{2}\big(\mathfrak{n} \circ D(e)_i \wedge \mathfrak{n} \circ D(e)^i\big) \circ \Delta \\ &= \mathfrak{L}_o \circ \mathfrak{n} + \frac{1}{2} D([\cdot,\cdot]) \circ \big(\mathfrak{n} \wedge \mathfrak{n}\big) \circ \Delta - \big((D([\cdot,\cdot]) \circ \mathfrak{n}) \wedge \mathfrak{n}\big) \circ \Delta \ . \end{aligned} \tag{2.9}$$

In equation (2.9), e_i and e^i denote a basis and corresponding dual basis of A_c w.r.t. the symplectic structure ω_c. Obviously we recover the OCHA of equation (2.6) in the limit $\hbar \to 0$.

Similarly as in the classical case, the morphism $e^{\mathfrak{n}}$ is a quasi-isomorphism which implies isomorphism of the corresponding moduli spaces, i.e.

$$\mathcal{M}(A_c, \mathfrak{L}_c) \cong \mathcal{M}(A_o, \mathfrak{L}_o) \ .$$

Theorem 3 ([11])**.** *The moduli space of any loop homotopy Lie algebra is empty,*

$$\mathcal{M}(A_c, \mathfrak{L}_c) = \emptyset \ .$$

Proof. The proof follows from considering the order $\hbar$ term of the Maurer Cartan equation for a general ansatz. This equation, together with the non-degeneracy of the symplectic form implies triviality of the cohomology, which in turn implies that $\mathcal{M}(A_c, \mathfrak{L}_c) = \emptyset$. □

Remark 4. The story is different for the topological string, where the symplectic structure ω degenerates on-shell. Under this condition, theorem 3 does not hold anymore, which implies consistency of open topological string theory at the quantum level in contrast to bosonic string theory.

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Type II superstring field theory: geometric approach and operadic description

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Abstract: We outline the construction of type II superstring field theory leading to a geometric and algebraic BV master equation, analogous to Zwiebach's construction for the bosonic string. The construction uses the small Hilbert space. Elementary vertices of the non-polynomial action are described with the help of a properly formulated minimal area problem. They give rise to an infinite tower of superstring field products defining a $\mathcal{N} = 1$ generalization of a loop homotopy Lie algebra, the genus zero part generalizing a homotopy Lie algebra. Finally, we give an operadic interpretation of the construction.

Keywords: Superstrings and Heterotic Strings, String Field Theory

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Contents

1 Introduction and summary

The first attempt towards a field theory of superstrings was initiated by the work of Witten [1], by seeking a Chern-Simons like action for open superstrings similar to the one of open bosonic string field theory [2]. The major obstacle compared to the bosonic string is the necessity of picture changing operators. Indeed, the cubic superstring theory of [1] turns out to be inconsistent due to singularities arising form the collision of picture changing operators [3]. In order to circumvent this problem, another approach was pursued which sets the string field into a different picture [4, 5], but upon including the Ramond sector, the modified superstring field theory suffers from similar inconsistencies [6]. These two approaches are based on the small Hilbert space, the state space including the reparametrization ghosts and superghosts as they arise from gauge fixing. Upon bosonization of the superghosts, an additional zero mode arises which allows the formulation of a WZW like action for the NS sector of open superstring field theory [7]. In contrast to bosonic string field theory, BV quantization of this theory is more intricate than simply relaxing the ghost number constraint for the fields of the classical action [8, 9]. Finally, there is a formulation of open superstring field theory that differs from all other approaches in not fixing the picture of classical fields [10].

On the other hand, the construction of bosonic closed string field theory [11] takes its origin in the moduli space of closed Riemann surfaces. Vertices represent a subspace of the

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moduli space, such that the moduli space decomposes uniquely into vertices and graphs, and do not apriori require a background. Graphs are constructed from the vertices by sewing together punctures along prescribed local coordinates around the punctures. But an assignment of local coordinates around the punctures, globally on the moduli space, is possible only up to rotations. This fact implies the level matching condition and via gauge invariance also the $b_0^- = 0$ constraint.

In an almost unnoticed work [12], the geometric approach developed in bosonic closed string field theory, as described in the previous paragraph, has been generalized to the context of superstring field theory. Neveu-Schwarz punctures behave quite similar to punctures in the bosonic case, but a Ramond puncture describes a divisor on a super Riemann surface rather than a point. As a consequence, local coordinates around Ramond punctures, globally defined over super moduli space, can be fixed only up to rotations and translation in the Ramond divisor.

A given background provides forms on super moduli space [13, 14] in the sense of geometric integration theory on supermanifolds [15], and in particular the geometric meaning of picture changing operators has been clarified [16]: integrating along an odd direction in moduli space inevitably generates a picture changing operator. Thus, the ambiguity of defining local coordinates around Ramond punctures produces a picture changing operator associated with the vector field generating translations in the Ramond divisor. The bpz inner product plus the additional insertions originating from the sewing define the symplectic form relevant for BV quantization. As in the bosonic case, we require that the symplectic form has to be non-degenerate, but the fact that the picture changing operator present in the Ramond sector has a non-trivial kernel, forces to impose additional restrictions besides the level matching and $b_0^- = 0$ constraint on the state space.

The purpose of this paper is to describe the construction of type II superstring field theory in the geometric approach. We start in section 2 by defining a BV structure on the moduli space of type II world sheets decorated with coordinate curves. A coordinate curve determines local coordinates around the punctures up to rotations and translations in the Ramond divisors. The BV operator and the antibracket correspond to the sewing of punctures along coordinate curves in the non-separating (both punctures on a single connected world sheet) and separating (punctures located on two disconnected world sheets) case respectively.

In section 3, we then review the operator formalism in the context of superstrings and the construction of forms on super moduli space. We define the symplectic form in the various sectors and determine the corresponding restricted state spaces. The symplectic form induces a BV structure on the space of multilinear maps on the restricted state spaces, and the factorization and chain map properties of the forms make the combined superconformal field theory of the matter and ghost sector a morphism of BV algebras. Note that the relevant grading in the BV formalism is the ghost number but not the picture.

Finally, we propose a minimal area problem in section 4, which determines the geometric vertices of type II superstring field theory and furthermore induces a section from the super moduli space to the super moduli space decorated with coordinate curves. The requirement that Feynman graphs produce a single cover of moduli space implies that the

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geometric vertices satisfy the BV master equation. For a given background, the algebraic vertices are defined by integrating the geometric vertices w.r.t. the corresponding forms, and satisfy the BV master equation as well. The kinetic term of the theory is given by the symplectic form together with the BRST charge.

The construction of string field theory in the geometric approach manifestly leads to a BV master equation on the moduli space, which describes the background independent part of string field theory. The second ingredient is a background, which defines a morphism of BV algebras. In section 5, we elucidate the relevance of operads in the context of string field theory. The usefulness of operads in formulating string field theory derives from a theorem due to Barannikov [17], which establishes a one-to-one correspondence between morphisms over the Feynman transform of a modular operad and solutions to an associated BV master equation. We conclude that the decomposition of the moduli space into vertices and graphs defines a morphism from the Feynman transform of the modular operad encoding the symmetry properties of the vertices to the chain complex of moduli spaces. A background then corresponds to a morphism from the chain complex of moduli spaces to the endomorphism operad whose vector space is the state space, the differential is the BRST charge and the contraction maps are defined w.r.t. the symplectic form. Altogether, the composition of these two morphisms determines the algebraic structure of the vertices. In closed string field theory the vertices satisfy the axioms of a loop homotopy Lie-algebra [18], whose tree-level part is a homotopy Lie-algebra (L_∞-algebra). We introduce the relevant operad for type II superstring field theory and define algebras over its Feynman transform to be $\mathcal{N}=1$ loop homotopy Lie-algebras.

Appendix A includes a brief account of super Riemann surfaces, in order to make the paper self contained. In appendix B, we treat the superconformal field theory of type II superstring theory, with a particular focus on the ghost sector. We define ghost number and picture in an unconventional way, avoiding half integer picture number in the Ramond sector. Finally, appendix C, reviews the geometric integration theory on supermanifolds and its relation to superstring theory, following [13, 16].

2 Supermoduli space and geometric BV structure

The basic requirement of string field theory is, that its vertices reproduce the perturbative string amplitudes via Feynman rules. The fundamental object of interest is thus the appropriate moduli space of world sheets. Following [19, 20], a type II world sheet $\boldsymbol{\Sigma}$ is a smooth supermanifold embedded in $\Sigma \times \tilde{\Sigma}$, where Σ and $\tilde{\Sigma}$ are super Riemann surfaces s.t. the reduced space of $\tilde{\Sigma}$ is the complex conjugate of the reduced space of Σ. We refer to Σ as the holomorphic and $\tilde{\Sigma}$ as the antiholomorphic sector, in analogy to the bosonic case. We require that the total number of punctures on Σ and $\tilde{\Sigma}$ coincide, but not that the number of punctures for NS and R coincide separately. Furthermore there is no condition imposed on the spin structures. The dimension of $\boldsymbol{\Sigma}$ is $2|2$, whereas the dimension of $\Sigma\times\tilde{\Sigma}$ as a smooth supermanifold is $4|2$. Conversely, given reduced spaces Σ_{red} and $\tilde{\Sigma}_{\text{red}}$ which are complex conjugate to each other, $\boldsymbol{\Sigma}$ can be constructed by thickening the diagonal of $\Sigma_{\text{red}} \times \tilde{\Sigma}_{\text{red}}$ in the odd directions. The operation of thickening in the odd directions is unique up to homology, which is good enough since the world sheet action is defined by integrating $\mathbf{\Sigma}$ over a closed form.

The moduli space of super Riemann surfaces of genus g with n_{NS} NS punctures and n_R Ramond punctures is denoted by $\mathfrak{M}_{g,n_R,n_{NS}}$. Its complex dimension is

$$\dim(\mathfrak{M}_{g,n_R,n_{NS}}) = 3g-3+n_{NS}+n_R \mid 2g-2+n_{NS}+\frac{1}{2}n_R\,.$$

This is not quite the appropriate moduli space for type II strings. We need a moduli space that parametrizes inequivalent type II world sheets, and thus we proceed as in the previous paragraph: consider the reduced space $(\mathfrak{M}_{g,n_{NS},n_R})_{\text{red}}$ and its complex conjugate $\left(\tilde{\mathfrak{M}}_{g,\tilde{n}_{NS},\tilde{n}_R}\right)_{\text{red}}$. The moduli space of type II strings $\mathfrak{M}^{II}_{g,\vec{n}}$ is defined by thickening the diagonal of $(\mathfrak{M}_{g,n_{NS},n_R})_{\text{red}} \times \left(\tilde{\mathfrak{M}}_{g,\tilde{n}_{NS},\tilde{n}_R}\right)_{\text{red}}$ in the odd directions. Again this operation is unique up to homology, but since superstring amplitudes are defined by integrating $\mathfrak{M}^{II}_{g,\vec{n}}$ over a closed form, this ambiguity does not matter. We have four different kinds of punctures

$$\vec{n} = (n_{NS-NS}, n_{NS-R}, n_{R-NS}, n_{R-R})$$

satisfying

$$\begin{aligned} n_{NS-NS}+n_{NS-R} &= n_{NS} \in \mathbb{N}_0 \\ n_{NS-NS}+n_{R-NS} &= \tilde{n}_{NS} \in \mathbb{N}_0 \\ n_{R-R}+n_{R-NS} &= n_R \in 2\mathbb{N}_0 \\ n_{R-R}+n_{NS-R} &= \tilde{n}_R \in 2\mathbb{N}_0\,. \end{aligned} \tag{2.1}$$

Thus we conclude that the dimension of $\mathfrak{M}^{II}_{g,\vec{n}}$ as a smooth supermanifold is given by

$$\dim(\mathfrak{M}^{II}_{g,\vec{n}}) = 6g-6+2n \mid 4g-4+2n_{NS-NS}+\frac{3}{2}(n_{NS-R}+n_{R-NS})+n_{R-R}\,.$$

This describes the geometric data which is needed to define superstring perturbation theory. In a field theory formulation of string theory, however, we need additional structure. Vertices represent a subspace of the full moduli space, and Feynman graphs are constructed by sewing surfaces along punctures. To perform the sewing operation, we have to know which points in a neighborhood of one puncture to identify with which points in a neighborhood of the other puncture. The required extra structure is that of a *coordinate curve* around each puncture, which is an embedded submanifold $S^{1|2}_\alpha \subset \Sigma$ encircling a single puncture of type $\alpha \in \{NS-NS,\ NS-R,\ R-NS,\ R-R\}$, where $S^{1|2}_\alpha$ is the supercircle with two odd directions and boundary condition α. Such a coordinate curve determines a local superconformal coordinate system $(z,\tilde{z},\theta,\tilde{\theta})$, where the puncture is located at

$$\begin{aligned} (z,\tilde{z},\theta,\tilde{\theta}) &= 0\,, \qquad NS-NS \\ (z,\tilde{z},\tilde{\theta}) &= 0\,, \qquad R-NS \\ (z,\tilde{z},\theta) &= 0\,, \qquad NS-R \\ (z,\tilde{z}) &= 0\,, \qquad R-R\,, \end{aligned}$$

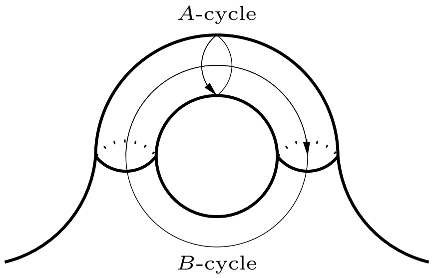

Figure 1. Sewing operation in the non-separating case.

up to rotations generated by $l_0^- := l_0 - \tilde{l}_0$ and translations in the Ramond divisors (if present) generated by g_0 and $\tilde{g}_0$.

We denote the moduli space of type II world sheets decorated with *coordinate curves* by $\hat{\mathfrak{P}}^{II}_{g,\vec{n}}$, whereas the moduli space decorated with *local coordinates* is denoted by $\mathfrak{P}^{II}_{g,\vec{n}}$. The decorated spaces are of course infinite dimensional and can be considered as a fibre bundle over $\mathfrak{M}^{II}_{g,\vec{n}}$ by discarding the information about the coordinate curves/local coordinates. In section 4, we propose that $\hat{\mathfrak{P}}^{II}_{g,\vec{n}}$ is indeed a trivial bundle, by outlining the construction of a global section. In contrast, the moduli space $\mathfrak{P}^{II}_{g,\vec{n}}$ does not admit global sections [11].

We will start by defining the sewing operations for given local coordinate systems: consider two punctures p and p' of the same type, together with local coordinates $(z, \tilde{z}, \theta, \tilde{\theta})$ and $(z', \tilde{z}', \theta', \tilde{\theta}')$. The punctures may either reside on a single connected surface or on two disconnected surfaces, which we call the non-separating and separating case respectively. First we will focus on the holomorphic sector. In the bosonic case, the sewing operation for two given coordinate systems z and z' is given by the identification

$$z' = I(z) := -\frac{1}{z}\,. \tag{2.2}$$

From equation (A.5), we can infer that the generalization of the sewing map (2.2) for the NS sector is given by

$$I_{(\pm,+)}(z,\theta) = \begin{pmatrix} -\frac{1}{z} \\ \pm\frac{\theta}{z} \end{pmatrix}\,. \tag{2.3}$$

In the separating case, there is no essential difference between $I_{(+,+)}$ and $I_{(-,+)}$, they are related by replacing $\theta \to -\theta$ on one surface globally. For the non-separating case the situation is different. Assume that a transition from (z,θ) to (z',θ') does not change the sign in the odd coordinate, i.e. that for a coordinate system (z'',θ'') covering (z,θ) and (z',θ'), the transition functions from (z'',θ'') to (z,θ) and (z'',θ'') to (z',θ') are both of the form (A.1) with the same sign in front of θ''. Under this assumption, the sewing with $I_{(\pm,+)}$ generates a handle with $\pm$ spin structure along the B-cycle, see figure 1.

In the R sector, the sewing map follows from generalizing (2.2) according to (A.6):

$$I_{(\pm,-)}(z,\theta) = \begin{pmatrix} -\frac{1}{z} \\ \pm i\theta \end{pmatrix}\,. \tag{2.4}$$

Similarly as in the NS sector, the sewing with $I_{(\pm,-)}$ in the non-separating case generates a handle with $\pm$ spin structure along the B-cycle. For the A-cycle, the $+$ and $-$ spin structure corresponds to NS and R respectively, which justifies the notation.

Modular invariance requires a sum over all spin structures. The modular invariant combination of spin structures is known to be

$$(+,+)-(-,+)-(+,-)\pm(--)\,.$$

Thus we can determine the sewing operations to be

$$I_{NS}=\frac{1}{2}\left(I_{(+,+)}-I_{(-,+)}\right)=\Pi^{GSO^-}\circ I_{(+,+)} \tag{2.5}$$

and

$$I_R^\pm=\frac{1}{2}\left(I_{(+,-)}\pm I_{(-,-)}\right)=\Pi^{GSO^\pm}\circ I_{(+,-)} \tag{2.6}$$

for the NS sand R sector, respectively. In equation (2.5) and (2.6), the sum has to be understood as generating two surfaces from a given one and taking their formal linear combination, which defines the GSO projection $\Pi^{GSO^\pm}$. These are the maps that determine the bpz conjugation in superconformal field theory (see appendix B). Combining the holomorphic and antiholomorphic sector, we end up with

$$I_\alpha(z,\tilde{z},\theta,\tilde{\theta})\,,\qquad \alpha\in\{NS-NS,\ NS-R,\ R-NS,\ R-R\}\,. \tag{2.7}$$

Now let us describe the sewing operation for given coordinate curves. As discussed previously, a coordinate curve does not uniquely determine a local coordinate system. This ambiguity naturally leads to a family of surfaces associated to the sewing of two punctures. We begin by restricting our considerations to the holomorphic sector. In the NS sector the local coordinate system is determined up to rotations generated by $l_0-\tilde{l}_0$. Let $\varphi_t^{l_0}$ be the flow generated by l_0,

$$\partial_t\varphi_t^{l_0}=l_0\circ\varphi_t^{l_0}\,,$$

which leads to

$$\varphi_t^{l_0}(z,\theta)=\begin{pmatrix}e^{-t}z\\ e^{-t/2}\theta\end{pmatrix}\,.$$

The family of local coordinate systems associated to a coordinate curve in the NS sector is parametrized by an angle $\vartheta\in[0,2\pi]$ and the corresponding sewing operation is given by

$$\phi_\vartheta=I_{NS}\circ\varphi_{i\vartheta}^{l_0}=\Pi^{GSO^-}\circ I_{(+,+)}\circ\varphi_{i\vartheta}^{l_0}\,, \tag{2.8}$$

which explicitly reads

$$I_{(+,+)}\circ\varphi_{i\vartheta}^{l_0}(z,\theta)=\begin{pmatrix}-\dfrac{e^{i\vartheta}}{z}\\ \theta\dfrac{e^{i\vartheta/2}}{z}\end{pmatrix}\,. \tag{2.9}$$

In the R sector the local coordinate system is determined up to rotations and translations in the Ramond divisor generated by g_0. Let $\varphi_{t,\tau}^{g_0}$ be the flow generated by g_0,

$$(\partial_\tau+\tau\partial_t)\varphi_{t,\tau}^{g_0}=g_0\circ\varphi_{t,\tau}^{g_0}$$

which leads to

$$\varphi^{g_0}_{t,\tau}(z,\theta) = \begin{pmatrix} e^{-t}z(1+\theta\tau) \\ \theta+\tau \end{pmatrix}.$$

We conclude, that in the R sector the family of local coordinate systems associated to a coordinate curve is parametrized by an angle $\vartheta \in [0, 2\pi]$ and an odd parameter $\tau \in \mathbb{C}^{0|1}$, and the corresponding sewing operation reads

$$\phi^{\pm}_{\vartheta,\tau} = I^{\pm}_R \circ \varphi^{g_0}_{i\vartheta,\tau} = \Pi^{GSO^{\pm}} \circ I_{(+,-)} \circ \varphi^{g_0}_{i\vartheta,\tau}\,. \tag{2.10}$$

Explicitly, we have

$$I_{(+,-)} \circ \varphi^{g_0}_{i\vartheta,\tau}(z,\theta) = \begin{pmatrix} -\dfrac{e^{i\vartheta}}{z}(1-\theta\tau) \\ i(\theta+\tau) \end{pmatrix}. \tag{2.11}$$

Combining holomorphic and antiholomorphic sectors, we identify the four sewing operations to be

$$\begin{aligned} (\Phi_{NS-NS})_{\vartheta} &= \left(I_{NS}\circ\varphi^{l_0}_{i\vartheta}\,,\ \tilde{I}_{NS}\circ\varphi^{\tilde{l}_0}_{-i\vartheta}\right) \\ (\Phi_{R-NS})_{\vartheta,\tau} &= \left(I_{R}\circ\varphi^{g_0}_{i\vartheta,\tau}\,,\ \tilde{I}_{NS}\circ\varphi^{\tilde{l}_0}_{-i\vartheta}\right) \\ (\Phi_{NS-R})_{\vartheta,\tilde{\tau}} &= \left(I_{NS}\circ\varphi^{l_0}_{i\vartheta}\,,\ \tilde{I}_{R}\circ\varphi^{\tilde{g}_0}_{-i\vartheta,\tilde{\tau}}\right) \\ (\Phi_{R-R})_{\vartheta,\tau,\tilde{\tau}} &= \left(I_{R}\circ\varphi^{g_0}_{i\vartheta,\tau}\,,\ \tilde{I}_{R}\circ\varphi^{\tilde{g}_0}_{-i\vartheta,\tilde{\tau}}\right). \end{aligned} \tag{2.12}$$

The geometric vertices of string field theory represent a subspace of the full moduli space. Thus the natural object to consider is the singular chain complex

$$C^{\bullet|\bullet}(\hat{\mathfrak{P}}^{II}_{g,\vec{n}})\,. \tag{2.13}$$

The grading for $\mathcal{A}_{g,\vec{n}} \in C^{k|l}(\hat{\mathfrak{P}}^{II}_{g,\vec{n}})$ is defined by codimension, i.e.

$$k|l = \deg(\mathcal{A}_{g,\vec{n}}) := \dim(\mathfrak{M}^{II}_{g,\vec{n}}) - \dim(\mathcal{A}_{g,\vec{n}})\,. \tag{2.14}$$

Furthermore we endow the chains with an orientation. In the context of supergeometry, there are different notions of orientation on a supermanifold $M^{m|n}$, corresponding to the four normal subgroups of the general linear group $\mathrm{GL}(m|n)$, described in appendix C. The relevant notion for integrating forms is that of a $[+-]$ orientation, see e.g. [15] or appendix C, which requires $\det(g_{00}) > 0$ for

$$\mathrm{GL}(m|n) \ni g = \begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix}.$$

Now we are going to describe the BV structure on the chain complex of moduli spaces. The final aim is of course to dress the punctures with vertex operators, which forces us to implement the indistinguishability of identical particles already at the geometric level. We proceed as follows: we define

$$\mathbf{Mod}(\mathcal{C}om^{\mathcal{N}=1})(g,\vec{n})$$

to be a one dimensional vector space.[1] Furthermore, the permutation group $\Sigma_{\vec{n}} := \times_\alpha \Sigma_{n_\alpha}$ acts on $\mathbf{Mod}(\mathcal{C}om^{\mathcal{N}=1})(g,\vec{n})$ by the trivial representation. According to the geometrical interpretation, we require $g \geq 0$ and the conditions of (2.1). Hence, the chains with appropriate symmetry properties can be described by the invariants

$$C^{\bullet|\bullet}_{\mathrm{inv}}(\hat{\mathfrak{P}}^{II}_{g,\vec{n}}) := \left(C^{\bullet|\bullet}(\hat{\mathfrak{P}}^{II}_{g,\vec{n}}) \otimes \mathbf{Mod}(\mathcal{C}om^{\mathcal{N}=1})(g,\vec{n}) \right)^{\Sigma_{\vec{n}}} , \tag{2.15}$$

where the permutation group $\Sigma_{\vec{n}}$ acts on $C^{\bullet|\bullet}(\hat{\mathfrak{P}}^{II}_{g,\vec{n}})$ by permutation of punctures. We call (2.15) the invariant chain complex. All that is just saying, that we restrict to chains which are invariant under permutations of punctures of the same type.

Let $\overset{\Phi_\alpha}{{}_i\circ_j}$ be the sewing operation in the separating case. The input of $\overset{\Phi_\alpha}{{}_i\circ_j}$ is a pair of surfaces decorated with coordinate curves, and its output is the family of surfaces generated by sewing together puncture i on the first surface with puncture j on the second surface according to (2.12), where both punctures i, j are of type α. Analogously, we define $\overset{\Phi_\alpha}{\xi_{ij}}$ to be the sewing operation in the non-separating case. For later use, we furthermore define maps $\overset{I_\alpha}{{}_i\circ_j}$ and $\overset{I_\alpha}{\xi_{ij}}$, involving the sewing (2.7) suitable for surfaces decorated with local coordinate around the punctures. The two former operations induce maps on the chain complex (2.13), which we also denote by $\overset{\Phi_\alpha}{{}_i\circ_j}$ and $\overset{\Phi_\alpha}{\xi_{ij}}$, by defining their action pointwise. From (2.12) and the definition of the grading (2.14), we conclude that for all α, $\overset{\Phi_\alpha}{{}_i\circ_j}$ and $\overset{\Phi_\alpha}{\xi_{ij}}$ are of degree $1|0$, that is

$$\overset{\Phi_\alpha}{{}_i\circ_j} : C^{k_1|l_1}(\hat{\mathfrak{P}}^{II}_{g_1,\vec{n}_1+e_\alpha}) \times C^{k_2|l_2}(\hat{\mathfrak{P}}^{II}_{g_2,\vec{n}_2+e_\alpha}) \to C^{k_1+k_2+1|l_1+l_2}(\hat{\mathfrak{P}}^{II}_{g_1+g_2,\vec{n}_1+\vec{n}_2}) \tag{2.16}$$

and

$$\overset{\Phi_\alpha}{\xi_{ij}} : C^{k|l}(\hat{\mathfrak{P}}^{II}_{g,\vec{n}+2e_\alpha}) \to C^{k+1|l}(\hat{\mathfrak{P}}^{II}_{g+1,\vec{n}}) , \tag{2.17}$$

where e_α denotes the unit vector in direction α and represents puncture i respectively j. Note also that the boundary operator

$$\partial : C^{k|l}(\hat{\mathfrak{P}}^{II}_{g,\vec{n}}) \to C^{k+1|l}(\hat{\mathfrak{P}}^{II}_{g,\vec{n}})$$

is of degree $1|0$ due to the choice of grading.

Finally, we want to lift $\overset{\Phi_\alpha}{{}_i\circ_j}$ and $\overset{\Phi_\alpha}{\xi_{ij}}$ to maps on the invariant chain complex (2.15), which will lead to the desired BV structure. Let $\mathcal{B}_{g_1,\vec{n}_1+e_\alpha} \in C^{k_1|l_1}_{\mathrm{inv}}(\hat{\mathfrak{P}}^{II}_{g_1,\vec{n}_1+e_\alpha})$ and $\mathcal{B}_{g_2,\vec{n}_2+e_\alpha} \in C^{k_2|l_2}_{\mathrm{inv}}(\hat{\mathfrak{P}}^{II}_{g_2,\vec{n}_2+e_\alpha})$ be invariant chains and consider the expression

$$\begin{aligned}(\mathcal{B}_{g_1,\vec{n}_1+e_\alpha}, \mathcal{B}_{g_2,\vec{n}_2+e_\alpha})^{\mathrm{geo}}_\alpha &:= \sum_{\sigma \in \mathrm{sh}(\vec{n}_1,\vec{n}_2)} \sigma . \left(\mathcal{B}_{g_1,\vec{n}_1+e_\alpha} \overset{\Phi_\alpha}{{}_i\circ_j} \mathcal{B}_{g_2,\vec{n}_2+e_\alpha} \right) , \\ (\cdot,\cdot)^{\mathrm{geo}} &:= \sum_\alpha (\cdot,\cdot)^{\mathrm{geo}}_\alpha .\end{aligned} \tag{2.18}$$

[1]The notation for this object will be justified in section 5, where we introduce operads and explain their applications to string field theory.

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First, note that since $\mathcal{B}_{g_1,\vec{n}_1+e_\alpha}$ and $\mathcal{B}_{g_2,\vec{n}_2+e_\alpha}$ are invariant under permutation of punctures of the same type, it does not matter which punctures i and j we choose for the sewing operation ${}_i\overset{\Phi_\alpha}{\circ}{}_j$. That is why i and j does not appear on the left hand side of (2.18). Second, $\mathrm{sh}\,(\vec{n}_1,\vec{n}_2)$ denotes the set of shuffles[2] of the punctures $\vec{n}_1$ and $\vec{n}_2$ that remain after sewing. In the non-separating case, we define

$$\begin{aligned}\Delta_\alpha^{\mathrm{geo}}\mathcal{B}_{g,\vec{n}+2e_\alpha} &:= \overset{\Phi_\alpha}{\xi}_{ij}\left(\mathcal{B}_{g,\vec{n}+2e_\alpha}\right)\,,\\ \Delta^{\mathrm{geo}} &:= \sum_\alpha \Delta_\alpha^{\mathrm{geo}}\,.\end{aligned}\tag{2.19}$$

for $\mathcal{B}_{g,\vec{n}+2e_\alpha} \in C^{k|l}_{\mathrm{inv}}(\hat{\mathfrak{P}}^{II}_{g,\vec{n}+2e_\alpha})$. Again the $\Sigma_{\vec{n}}$ invariance guarantees independence of the choice of punctures i and j.

Now one can show that ∂, $(\cdot,\cdot)^{\mathrm{geo}}$ and Δ^{geo} satisfy the axioms of a differential BV algebra, that is (leaving out the superscript geo)

$$\begin{aligned}\partial^2 &= 0\\ \Delta^2 &= 0\\ \partial\Delta+\Delta\partial &= 0\\ \partial\circ(\cdot,\cdot) &= (\partial,\cdot)-(\cdot,\partial)\\ \Delta\circ(\cdot,\cdot) &= (\Delta,\cdot)-(\cdot,\Delta)\\ (\mathcal{B}_{g_1,\vec{n}_1},\mathcal{B}_{g_2,\vec{n}_2}) &= -(-1)^{(k_1+1)(k_2+1)}(\mathcal{B}_{g_2,\vec{n}_2},\mathcal{B}_{g_1,\vec{n}_1})\\ (-1)^{(k_1+1)(k_3+1)}((\mathcal{B}_{g_1,\vec{n}_1},\mathcal{B}_{g_2,\vec{n}_2}),\mathcal{B}_{g_3,\vec{n}_3}) &+ \text{cycl.} = 0\,,\end{aligned}\tag{2.20}$$

where $\mathcal{B}_{g_i,\vec{n}_i} \in C^{k_i|l_i}_{\mathrm{inv}}(\hat{\mathfrak{P}}^{II}_{g_i,\vec{n}_i})$. Note that only the even part k of the grading $k|l$ enters in the expressions for the signs, thus the odd part l is not really a grading in the strict sense, it is merely an additional index representing the odd codimensionality of the chain. The reason for this resides in the fact that we chose the $[+-]$ orientation for the chains. The proof of the identities (2.20) follows directly from the proof in the bosonic case [11, 21], again due to the choice of orientation: the $[+-]$ orientation distinguishes an order for the even vectors but not for the odd vectors. For all α, the operations ${}_i\overset{\Phi_\alpha}{\circ}{}_j$ and $\overset{\Phi_\alpha}{\xi}_{ij}$ increase the even dimensionality by one due to the twist angle ϑ and thus the proof of (2.20) reduces to that in the bosonic case.

Indeed, a BV algebra also requires a graded commutative multiplication, such that Δ defines a second order derivation and ∂ a first order derivation. We do not describe this operation here, but definitely it can be defined similarly to the bosonic case by disjoint union [11, 21].

3 Operator formalism and algebraic BV structure

The geometric BV algebra discussed in the previous section describes the background independent ingredient of type II superstring field theory. A background refers to a super-

[2]The set of shuffles $\mathrm{sh}(n,m) \subset \Sigma_{n+m}$ contains all permutation $\sigma \in \Sigma_{n+m}$, satisfying $\sigma_1 < \cdots < \sigma_n$ and $\sigma_{n+1} < \cdots < \sigma_{n+m}$.

conformal field theory (SCFT) with additional structure provided by the superconformal ghosts and the BRST charge, which allows the construction of a measure on supermoduli space compatible with the sewing operations. Such a field theory is called a topological superconformal field theory (TSCFT) [13, 22].

We start by introducing differential forms on supermoduli space, following [13, 14]. Let $\mathcal{H}_\alpha$, $\alpha \in \{NS-NS,\ NS-R,\ R-NS,\ R-R\}$, denote the state spaces of a type II SCFT (see appendix B). For a given type II world sheet $\mathbf{\Sigma}_{g,\vec{n}} \in \mathfrak{P}^{II}_{g,\vec{n}}$ with local coordinates around the punctures, the SCFT assigns a multilinear map

$$Z(\mathbf{\Sigma}_{g,\vec{n}}) : \mathcal{H}^{\otimes\vec{n}} \to \mathbb{C}^{1|1}\,,$$

where

$$\mathcal{H}^{\otimes\vec{n}} := \bigotimes_\alpha (\mathcal{H}_\alpha)^{\otimes n_\alpha}\,.$$

Let ${}_i\overset{\mathrm{bpz}_\alpha}{\circ}_j$ be the map

$${}_i\overset{\mathrm{bpz}_\alpha}{\circ}_j : \mathrm{Hom}(\mathcal{H}^{\otimes\vec{n}_1+e_\alpha}, \mathbb{C}^{1|1}) \times \mathrm{Hom}(\mathcal{H}^{\otimes\vec{n}_2+e_\alpha}, \mathbb{C}^{1|1}) \to \mathrm{Hom}(\mathcal{H}^{\otimes\vec{n}_1+\vec{n}_2}, \mathbb{C}^{1|1})$$

that contracts input i of the first linear map with input j of the second linear map, both of type α, w.r.t. the inverse of the bpz inner product bpz_α^{-1}. Analogously, we define the map

$$\overset{\mathrm{bpz}_\alpha}{\xi_{ij}} : \mathrm{Hom}(\mathcal{H}^{\otimes\vec{n}+2e_\alpha}, \mathbb{C}^{1|1}) \to \mathrm{Hom}(\mathcal{H}^{\otimes\vec{n}}, \mathbb{C}^{1|1})\,.$$

The factorization properties

$$Z\left(\mathbf{\Sigma}_{g_1,\vec{n}_1+e_\alpha}\ {}_i\overset{I_\alpha}{\circ}_j\ \mathbf{\Sigma}_{g_2,\vec{n}_2+e_\alpha}\right) = Z\left(\mathbf{\Sigma}_{g_1,\vec{n}_1+e_\alpha}\right)\ {}_i\overset{\mathrm{bpz}_\alpha}{\circ}_j\ Z\left(\mathbf{\Sigma}_{g_2,\vec{n}_2+e_\alpha}\right) \tag{3.1}$$

and

$$Z\left(\overset{I_\alpha}{\xi_{ij}}\,\mathbf{\Sigma}_{g_1,\vec{n}+2e_\alpha}\right) = \overset{\mathrm{bpz}_\alpha}{\xi_{ij}}\ Z\left(\mathbf{\Sigma}_{g_1,\vec{n}+2e_\alpha}\right) \tag{3.2}$$

hold, with the sewing operations ${}_i\overset{I_\alpha}{\circ}_j$ and $\overset{I_\alpha}{\xi_{ij}}$ introduced in the previous section. Furthermore the tensor structure is preserved, i.e.

$$Z(\mathbf{\Sigma}_{g_1,\vec{n}_1} \sqcup \mathbf{\Sigma}_{g_2,\vec{n}_2}) = Z(\mathbf{\Sigma}_{g_1,\vec{n}_1}) \otimes Z(\mathbf{\Sigma}_{g_2,\vec{n}_2})\,.$$

A tangent vector $V \in T\mathfrak{P}^{II}_{g,\vec{n}}$ can be represented by a collection of pairs of holomorphic and antiholomorphic Virasoro vectors $\vec{v} = \left((v^{(1)}, \tilde{v}^{(1)}), \ldots, (v^{(n)}, \tilde{v}^{(n)})\right)$, $n = \sum_\alpha n_\alpha$, via Schiffer variation. We can think of Z as a function on $\mathfrak{P}^{II}_{g,\vec{n}}$ with values in $\mathrm{Hom}(\mathcal{H}^{\otimes\vec{n}}, \mathbb{C}^{1|1})$. The relation between the tangent vector V and its representation via Virasoro vectors is expressed by the relation

$$V(Z) = Z \circ T(\vec{v})\,, \tag{3.3}$$

where

$$T(\vec{v}) := \sum_{i=1}^{n} \left(T^{(i)}(v^{(i)}) + \tilde{T}^{(i)}(\tilde{v}^{(i)})\right)\,,$$

and

$$T(l_n) = L_n\,,$$
$$T(g_n) = G_n$$

defines $T(v)$ by linearity. Similarly, $B(v)$ is determined by

$$B(l_n) = b_n\,,$$
$$B(g_n) = \beta_n\,.$$

Furthermore, Z is BRST closed and a map of Lie algebras, that is

$$[V_1, V_2](Z) = Z \circ T([\vec{v}_1, \vec{v}_2])\,,$$
$$Z \circ \sum_{i=1}^{n} Q^{(i)} = 0\,.$$

Utilizing the B ghost, we can now define differential forms $\omega^{k|l}_{g,\vec{n}}$ on $\mathfrak{P}^{II}_{g,\vec{n}}$ with values in $\mathrm{Hom}(\mathcal{H}^{\otimes\vec{n}}, \mathbb{C}^{1|1})$ [13]: let $(V_1, \ldots, V_r|\mathcal{V}_1, \ldots, \mathcal{V}_s)$, be a collection of r even and s odd tangent vector to $\mathfrak{P}^{II}_{g,\vec{n}}$ at $\mathbf{\Sigma}_{g,\vec{n}}$, we define

$$\omega^{k|l}_{g,\vec{n}}(V_1, \ldots, V_r|\mathcal{V}_1, \ldots, \mathcal{V}_s) := N_{g,\vec{n}} \cdot Z(\mathbf{\Sigma}_{g,\vec{n}}) \circ B(\vec{v}_1) \ldots B(\vec{v}_r)\, \delta(B(\vec{\nu}_1)) \ldots \delta(B(\vec{\nu}_s))\,, \tag{3.4}$$

where $r|s = \dim(\mathfrak{M}^{II}_{g,\vec{n}}) - k|l$, in accordance with the grading (2.14) introduced for the chain complex of moduli spaces. The normalization constant $N_{g,\vec{n}} = (2\pi i)^{-(3g-3+n)}$ derives from the twist angle ϑ of the sewing operations (2.12) [11]. From (B.6) and (3.4), we conclude that $\omega^{k|l}_{g,\vec{n}}$ has ghost number and picture equal to

$$k - 2n|l - 2n_{NS-NS} - (n_{NS-R} + n_{R-NS})\,. \tag{3.5}$$

Moreover, the differential forms define chain maps in the sense that

$$d\omega^{k+1|l}_{g,\vec{n}} = (-1)^k \omega^{k|l}_{g,\vec{n}} \circ \sum_{i=1}^{n} Q^{(i)}\,. \tag{3.6}$$

Indeed, we would like to be able to pull this structure back to the finite dimensional moduli space $\mathfrak{M}^{II}_{g,\vec{n}}$. That is we need a natural way to assign local coordinates to type II world sheets, or in other words we require a global section of $\mathfrak{P}^{II}_{g,\vec{n}}$ as a fibre bundle over $\mathfrak{M}^{II}_{g,\vec{n}}$. As indicated in section 2, the topology of $\mathfrak{P}^{II}_{g,\vec{n}}$ does not admit global sections. The best we can get are global sections of $\hat{\mathfrak{P}}^{II}_{g,\vec{n}}$. In section 4, we outline the construction of a global section σ of $\hat{\mathfrak{P}}^{II}_{g,\vec{n}}$ as a fibre bundle over $\mathfrak{M}^{II}_{g,\vec{n}}$ via an analog of minimal area metrics in the superconformal setting. Therefore, in order to make use of the section σ, we first have to explain how to employ the forms (3.4) in the context of $\hat{\mathfrak{P}}^{II}_{g,\vec{n}}$. It turns out that using $\hat{\mathfrak{P}}^{II}_{g,\vec{n}}$ instead of $\mathfrak{P}^{II}_{g,\vec{n}}$ requires to restrict the state spaces $\mathcal{H}_\alpha$ in a certain way [11, 12], which

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we will denote by $\hat{\mathcal{H}}_\alpha$. The constraints leading to $\hat{\mathcal{H}}_\alpha$ follow from requiring factorization properties analogously to (3.1) and (3.2) [12]:

$$\int\limits_{\mathcal{A}_{g_1,\vec{n}_1+e_\alpha}\ {}_i\overset{\Phi_\alpha}{\circ}_j\ \mathcal{A}_{g_1,\vec{n}_1+e_\alpha}} \omega^{k_1+k_2+1|l_1+l_2}_{g_1+g_2,\vec{n}_1+\vec{n}_2} = \left(\ \int\limits_{\mathcal{A}^{k_1|l_1}_{g_1,\vec{n}_1+e_\alpha}} \omega^{k_1|l_1}_{g_1,\vec{n}_1+e_\alpha}\right)\ {}_i\overset{\omega_\alpha}{\circ}_j \left(\ \int\limits_{\mathcal{A}^{k_2|l_2}_{g_2,\vec{n}_2+e_\alpha}} \omega^{k_2|l_2}_{g_2,\vec{n}_2+e_\alpha}\right), \tag{3.7}$$

and

$$\int\limits_{\overset{\Phi_\alpha}{\xi}_{ij}\ \mathcal{A}^{k|l}_{g-1,\vec{n}+2e_\alpha}} \omega^{k+1|l}_{g,\vec{n}} = \overset{\omega_\alpha}{\xi}_{ij}\left(\ \int\limits_{\mathcal{A}^{k|l}_{g-1,\vec{n}+2e_\alpha}} \omega^{k|l}_{g-1,\vec{n}+2e_\alpha}\right). \tag{3.8}$$

The maps ${}_i\overset{\omega_\alpha}{\circ}_j$ and $\overset{\omega_\alpha}{\xi}_{ij}$ denote the contraction w.r.t. ω^{-1}_α, which is the inverse of the bpz inner product bpz^{-1}_α plus additional insertion originating from the sewing operations (2.12). In the following we determine these insertions.

In every sector α we have the twist angle ϑ, which leads to an insertion

$$\int_0^{2\pi} d\vartheta\, B(v_\vartheta)\exp(i\vartheta L_0^-)\,, \tag{3.9}$$

where $\exp(i\vartheta L_0^-)$ generates the twisting and $B(v_\vartheta)$ originates from the measure (3.4). The vector v_ϑ is determined by

$$\partial_\vartheta \exp(i\vartheta L_0^-) = iL_0^- \exp(i\vartheta L_0^-) \quad \Rightarrow \quad v_\vartheta = il_0^-\,.$$

In the case of R punctures in the holomorphic sector, we have the additional odd parameter τ. Consequently, the corresponding Virasoro vector is odd and the measure contributes a picture changing operator. The insertion associated to τ reads

$$\int d\tau\, \delta(B(v_\tau))\exp(\tau G_0)\,. \tag{3.10}$$

From

$$\partial_\tau \exp(\tau G_0) = (G_0 + \tau L_0)\exp(\tau G_0)$$

we conclude that

$$v_\tau = g_0 + \tau l_0\,.$$

Combining (3.9) and (3.10) and carrying out the integrals using some of the identities of appendix C, we end up with [12]

$$\begin{aligned}\omega^{-1}_{NS-NS} &= 2\pi i b_0^- P_{L_0^-} \circ \mathrm{bpz}^{-1}_{NS-NS} \\ \omega^{-1}_{R-NS} &= 2\pi i b_0^- P_{L_0^-} X_{g_0} \circ \mathrm{bpz}^{-1}_{R-NS} \\ \omega^{-1}_{NS-R} &= 2\pi i b_0^- P_{L_0^-} \tilde{X}_{\tilde{g}_0} \circ \mathrm{bpz}^{-1}_{R-NS} \\ \omega^{-1}_{R-R} &= 2\pi i b_0^- P_{L_0^-} X_{g_0}\tilde{X}_{\tilde{g}_0} \circ \mathrm{bpz}^{-1}_{R-R}\,.\end{aligned} \tag{3.11}$$

NS-NS	R-NS	NS-R	R-R
$L_0^- = 0$	$L_0^- = 0$	$L_0^- = 0$	$L_0^- = 0$
$b_0^- = 0$	$b_0^- = 0$	$b_0^- = 0$	$b_0^- = 0$
	$\beta_0^2 = 0$	$\tilde{\beta}_0^2 = 0$	$\beta_0^2 = \tilde{\beta}_0^2 = 0$
	$G_0\beta_0 - b_0 = 0$	$\tilde{G}_0\tilde{\beta}_0 - \tilde{b}_0 = 0$	$\tilde{G}_0\tilde{\beta}_0 - \tilde{b}_0 = G_0\beta_0 - b_0 = 0$

Table 1. Constraints defining restricted state spaces.

In equation (3.11), we think of bpz_α^{-1} as a map from the dual space $\mathcal{H}_\alpha^*$ to $\mathcal{H}_\alpha$, and $P_{L_0^-}$ denotes the projection onto states satisfying the level matching condition. Moreover, the operator

$$X_{g_0} = \frac{1}{2}\left(G_0\delta(\beta_0) - \delta(\beta_0)G_0\right)$$

is the picture changing operator associated to g_0 (see appendix C). In the following we will discard the factor of $2\pi i$ in (3.11), which has to be compensated by the normalization $N_{g,\vec{n}}$ introduced for the differential forms (3.4). The restricted state space $\hat{\mathcal{H}}_\alpha$ is now determined by demanding that ω_α^{-1} is indeed the inverse of a map $\omega_\alpha : \hat{\mathcal{H}} \to \hat{\mathcal{H}}^*$, the odd symplectic form relevant for the BV formalism. The constraint shared in all sectors is the level matching condition and $[Q, L_0^-] = b_0^- = 0$. Consider now the holomorphic R sector. Since $\beta_0^2 X_{g_0} = 0$, we conclude that states in the corresponding restricted state space have to satisfy

$$\beta_0^2 = 0\,.$$

Furthermore, gauge invariance requires also

$$\frac{1}{2}[Q, \beta_0^2] = G_0\beta_0 - b_0 = 0\,.$$

In table 1, we summarize the constraints defining the restricted state spaces in the various sectors.

An odd symplectic form is by definition an antisymmetric, closed, non-degenerate, bilinear map. But note that symmetry properties depend on the choice of grading. Consider for example a symmetric bilinear map $g : V^{\otimes 2} \to \mathbb{C}$ on a graded vector space $V = \oplus_n V_n$. The suspension map $\uparrow$ and the desuspension map $\downarrow$ are defined by $(\uparrow V)_n = V_{n-1}$ and $(\downarrow V)_n = V_{n+1}$. The map $g \circ (\downarrow \otimes \downarrow) : \uparrow V^{\otimes 2} \to \mathbb{C}$ induced on $\uparrow V$ defines then an antisymmetric map. From a mathematical point of view, the natural choice of grading in string field theory is determined by declaring the degree of a classical field to be zero. Of course this does not coincide with the ghost number, picture and Grassmann parity grading defined in appendices B and C: from the requirement that on-shell amplitudes are of degree $0|0|0$, which are defined by integrating $\omega_{g,\vec{n}}^{0|0}$ over the full moduli space $\mathfrak{M}_{g,\vec{n}}^{II}$, we infer from (3.5) that the degrees of classical fields are given by table 2.

For every sector separately, we define a new grading

$$g'\|l'\|\alpha' := g|p|\alpha - \text{degree of classical fields}\,, \tag{3.12}$$

NS-NS	R-NS	NS-R	R-R
2|2|0	2|1|1	2|1|1	2|0|0

Table 2. Degrees of classical fields.

NS-NS	R-NS	NS-R	R-R
$b_0^+ = 0$	$\beta_0 = 0$	$\tilde{\beta}_0 = 0$	$b_0^+ = 0\,, \qquad (G_0 \neq 0 \neq \tilde{G}_0)$

Table 3. Siegel gauge in the various sectors.

which sets the degree of classical fields to zero. We denote the corresponding desuspended space of $\hat{\mathcal{H}}_\alpha$ by A_α. On A_α the odd symplectic form, which we also denote by ω_α, reveals its natural properties, i.e. it is antisymmetric and of degree $-1\|0\|0$.[3] ω_α is the composite of the bpz inner product and an insertion which is inverse to the insertion of (3.11). The insertion has to be BRST closed and has to have the appropriate bpz parity.[4] The symplectic forms, expressed via the bpz inner product and the additional insertion read [12]

$$\begin{aligned}
\omega_{NS-NS} &= \mathrm{bpz}_\alpha(\cdot, c_0^- \cdot) \\
\omega_{R-NS} &= \mathrm{bpz}_\alpha\left(\cdot, -2c_0^- c_0^+ \delta'(\gamma_0)\cdot\right) \\
\omega_{NS-R} &= \mathrm{bpz}_\alpha\left(\cdot, -2c_0^- c_0^+ \delta'(\tilde{\gamma}_0)\cdot\right) \\
\omega_{R-R} &= \mathrm{bpz}_\alpha\left(\cdot, -c_0^- G_0^{-1} \tilde{G}_0^{-1} \delta(\gamma_0)\delta(\tilde{\gamma}_0)\cdot\right) .
\end{aligned} \tag{3.13}$$

In the restricted state space, we can impose the Siegel gauge conditions [4] as depicted in table 3. Surprisingly, the symplectic form in the R-R sector is non-local and degenerates on-shell, but in section 4 we will see that in combination with the BRST charge, this will reproduce the right expression for the propagator.

Finally, similarly to the geometric BV structure described in section 2, we can define a BV structure on

$$\mathrm{Hom}_{\mathrm{inv}}(A^{\otimes \vec{n}}, \mathbb{C}^{1|1}) := \left(\mathrm{Hom}(A^{\otimes \vec{n}}, \mathbb{C}^{1|1}) \otimes \mathbf{Mod}(\mathcal{C}om^{\mathcal{N}=1})(g, \vec{n})\right)^{\Sigma_{\vec{n}}} . \tag{3.14}$$

The antibracket is defined by

$$\begin{aligned}
(h_{g_1,\vec{n}_1+e_\alpha}, h_{g_2,\vec{n}_2+e_\alpha})_\alpha^{\mathrm{alg}} &:= \sum_{\sigma \in \mathrm{sh}(\vec{n}_1,\vec{n}_2)} \sigma.\left(h_{g_1,\vec{n}_1+e_\alpha}\ {}_i\overset{\omega_\alpha}{\circ}_j\ h_{g_2,\vec{n}_2+e_\alpha}\right) , \\
(\cdot,\cdot)^{\mathrm{alg}} &:= \sum_\alpha (\cdot,\cdot)_\alpha^{\mathrm{alg}} .
\end{aligned} \tag{3.15}$$

and the BV operator reads

$$\Delta_\alpha^{\mathrm{alg}} h_{g,\vec{n}+2e_\alpha} := \overset{\omega_\alpha}{\xi_{ij}}\left(h_{g,\vec{n}+2e_\alpha}\right) , \tag{3.16}$$

[3]Note that a suspension/desuspension in picture does not change the symmetry properties in contrast to ghost number and Grassmann parity.

[4]Since the bpz inner product is symmetric on $\mathcal{H}_\alpha$, ω_α being antisymmetric requires a bpz odd/even insertion and an even/odd number of desuspensions in ghost number and Grassmann parity.

$$\Delta^{\mathrm{alg}} := \sum_\alpha \Delta^{\mathrm{alg}}_\alpha \,.$$

for $h_{g,\vec{n}+2e_\alpha} \in \mathrm{Hom}_{\mathrm{inv}}(A^{\otimes\vec{n}+2e_\alpha}, \mathbb{C}^{1|1})$, $h_{g,\vec{n}_i+e_\alpha} \in \mathrm{Hom}_{\mathrm{inv}}(A^{\otimes\vec{n}_i+e_\alpha}, \mathbb{C}^{1|1})$. The permutation σ in equation (3.15) acts by permuting the inputs of the linear map.

From the factorization properties (3.7), (3.8) and the chain map property (3.6), we infer that the STCFT defines a morphism of BV algebras, i.e.

$$\mathrm{STCFT}: \left(C^{\bullet|\bullet}_{\mathrm{inv}}(\hat{\mathfrak{P}}^{II}_{g,\vec{n}}),\, \partial,\, \Delta^{\mathrm{geo}},\, (\cdot,\cdot)^{\mathrm{geo}}\right) \to \left(\mathrm{Hom}_{\mathrm{inv}}(A^{\otimes\vec{n}}, \mathbb{C}^{1|1}),\, Q,\, \Delta^{\mathrm{alg}},\, (\cdot,\cdot)^{\mathrm{alg}}\right) .$$

4 Vertices and BV master equation

In this part we construct the vertices for type II super string field theory. First, we discuss the kinetic term, and in particular its form in Siegel gauge. In a second step we treat the interactions and show that a consistent decomposition of the moduli space implies that the vertices satisfy a BV master equation. Finally, we outline an explicit construction of the vertices in close analogy to the bosonic case [11], by formulating a minimal area problem for type II world sheets.

4.1 Kinetic term

The kinetic term for a string field $\phi_\alpha \in A_\alpha$ of degree $0\|0\|0$ is defined by

$$\omega_\alpha(Q\phi_\alpha, \phi_\alpha)\,. \tag{4.1}$$

In Siegel gauge (see table 3) the kinetic term reduces to

$$\begin{aligned}
\omega_{NS-NS}\left(L_0^+ c_0^+\phi,\phi\right) &= \mathrm{bpz}_{NS-NS}\left(c_0^- c_0^+ L_0^+\phi,\phi\right) , && \phi\in A_{NS-NS} \\
\omega_{R-NS}\left(G_0\gamma_0\phi,\phi\right) &= \mathrm{bpz}_{R-NS}\left(-2c_0^- c_0^+\delta(\gamma_0)G_0\phi,\phi\right) , && \phi\in A_{R-NS} \\
\omega_{NS-R}\left(\tilde{G}_0\tilde{\gamma}_0\phi,\phi\right) &= \mathrm{bpz}_{NS-R}\left(-2c_0^- c_0^+\delta(\tilde{\gamma}_0)\tilde{G}_0\phi,\phi\right) , && \phi\in A_{NS-R} \\
\omega_{R-R}\left(L_0^+ c_0^+\phi,\phi\right) &= \mathrm{bpz}_{R-R}\left(-c_0^- c_0^+\delta(\gamma_0)\delta(\tilde{\gamma}_0)G_0^{-1}\tilde{G}_0^{-1}L_0^+\phi,\phi\right) , && \phi\in A_{R-R}\,.
\end{aligned} \tag{4.2}$$

The insertions in the bpz inner product of equation (4.2) lead precisely to the propagators known from perturbative string theory [23]. We conclude that the non-local form of the kinetic term in the R-R sector is probably related to the problem of finding an action principle for a self dual field strength.

4.2 Interactions

The covariant kinetic term defined in the previous subsection requires intrinsically a background. In contrast, the interactions represent a subspace of the moduli space. We call the corresponding vertices the *geometric vertices*. In order to be consistent with perturbative string theory, the geometric vertices have to reproduce a single cover of the full moduli space via Feynman rules. For a given background, which determines a TSCFT, the image of the geometric vertices under the TSCFT defines the corresponding *algebraic vertices*.

Thus the geometric vertices are background independent, whereas the algebraic vertices depend on the choice of background.

To formulate the consistency condition for the geometric vertices, we first have to define the notion of propagation on moduli space: the *geometric propagator* is defined by sewing of punctures w.r.t.

$$\begin{aligned}(P_{NS-NS})_{x,\vartheta} &= \left(I_{NS}\circ\varphi^{l_0}_{-x+i\vartheta}\,,\,\tilde{I}_{NS}\circ\varphi^{\tilde{l}_0}_{-x-i\vartheta}\right)\\ (P_{R-NS})_{x,\vartheta,\tau} &= \left(I_{R}\circ\varphi^{g_0}_{-x+i\vartheta,\tau}\,,\,\tilde{I}_{NS}\circ\varphi^{\tilde{l}_0}_{-x-i\vartheta}\right)\\ (P_{NS-R})_{x,\vartheta,\tilde{\tau}} &= \left(I_{NS}\circ\varphi^{l_0}_{-x+i\vartheta}\,,\,\tilde{I}_{R}\circ\varphi^{\tilde{g}_0}_{-x-i\vartheta,\tilde{\tau}}\right)\\ (P_{R-R})_{x,\vartheta,\tau,\tilde{\tau}} &= \left(I_{R}\circ\varphi^{g_0}_{-x+i\vartheta,\tau}\,,\,\tilde{I}_{R}\circ\varphi^{\tilde{g}_0}_{-x-i\vartheta,\tilde{\tau}}\right)\,,\end{aligned} \tag{4.3}$$

for $x\in[0,\infty)$, $\vartheta\in[0,2\pi]$ and $\tau,\tilde{\tau}\in\mathbb{C}^{0|1}$. The quantity x can be interpreted as the length of the cylinder sewn in between two puncture, and the sewing maps defined in equation (2.12) correspond to setting $x=0$. The induced maps on the invariant chain complex $C^{\bullet|\bullet}_{\text{inv}}(\hat{\mathfrak{P}}^{II}_{g,\vec{n}})$ carry degree $0|0$.

The geometric vertices $\mathcal{V}_{g,\vec{n}}$ represent a subspace of codimensionality $0|0$ of the moduli space decorated with coordinate curves, invariant under permutation of punctures of the same type. In other words, $\mathcal{V}_{g,\vec{n}}\in C^{0|0}_{\text{inv}}(\hat{\mathfrak{P}}^{II}_{g,\vec{n}})$. From the collection of geometric vertices, we can construct graphs with the aid of the propagator.[5] We denote the collection of genus g graphs with $\vec{n}$ punctures, constructed from $\mathcal{V}_{g,\vec{n}}$ and involving exactly i propagators, by $R^i_{g,\vec{n}}$. The requirement of a single cover reads [11]

$$\overline{\mathfrak{M}}^{II}_{g,\vec{n}} = \pi\left(\mathcal{V}_{g,\vec{n}}\sqcup R^1_{g,\vec{n}}\sqcup\cdots\sqcup R^{3g-3+n}_{g,\vec{n}}\right)\,, \tag{4.4}$$

where $3g-3+n$ is the maximal possible number of propagators, $\overline{\mathfrak{M}}^{II}_{g,\vec{n}}$ denotes the Deligne-Mumford compactification of $\mathfrak{M}^{II}_{g,\vec{n}}$ [20] and π denotes the projection map on $\hat{\mathfrak{P}}^{II}_{g,\vec{n}}$ as a fibre bundle over $\mathfrak{M}^{II}_{g,\vec{n}}$. The degenerations arise from infinitely long cylinders, i.e. correspond to $x\to\infty$.

The compactified moduli space on the left hand side of equation (4.4) has no boundary. On the other hand the right hand side of equation (4.4) involves two types of boundaries: one which describes the boundary of the geometric vertices itself and another which corresponds to a propagator collapse, i.e. $x\to 0$. Thus equation (4.4) implies that these two types of boundaries cancel each other. The required canellation of boundary terms is equivalent to the BV master equation [11]

$$\partial\mathcal{V}_{g,\vec{n}}+\sum_{\alpha}\Delta^{\text{geo}}_{\alpha}\mathcal{V}_{g-1,\vec{n}+2e_\alpha}+\frac{1}{2}\sum_{\alpha}\sum_{\substack{\vec{n}_1+\vec{n}_2=\vec{n}\\ g_1+g_2=g}}(\mathcal{V}_{g_1,\vec{n}_1+e_\alpha},\mathcal{V}_{g_2,\vec{n}_2+e_\alpha})^{\text{geo}}_{\alpha}=0\,. \tag{4.5}$$

To summarize, every consistent decomposition of the moduli space into vertices and graphs implies that the BV master equation (4.5) is satisfied.

[5] In section 5, while introducing operads, we will state more precisely what we mean by graphs.

In the rest of this section, we will introduce minimal area metrics on type II world sheets and outline their relevance for the construction of the geometric vertices. Following [20], a metric on a type II world sheet $\mathbf{\Sigma} \subset \Sigma \times \tilde{\Sigma}$ is determined by a collection of even local sections

$$\begin{aligned} E &\in \Gamma(U, \mathcal{D}^{-2})\,, \\ \tilde{E} &\in \Gamma(U, \tilde{\mathcal{D}}^{-2}) \end{aligned}$$

where $\mathcal{D}$ denotes the distinguished subbundle of $T\mathbf{\Sigma}$ (see appendix A). Overlapping sections are related by the gauge transformation

$$\begin{aligned} E' &= e^{iu} E\,, \\ \tilde{E}' &= e^{-iu} \tilde{E}\,, \end{aligned}$$

satisfying the reality condition $\overline{E} = \tilde{E}$ and $u \in \mathbb{R}$ for $\theta = \tilde{\theta} = 0$. The subbundle $\mathcal{D} \subset T\Sigma$ is locally spanned by $D_\theta = \partial_\theta + \theta \partial_z$, whereas $\mathcal{D}^{-2} \subset T^*\Sigma$ is locally spanned by $\Omega_z = dz + \theta d\theta$. Hence, (D_θ, ∂_z) describes a basis of $T\Sigma$ respecting the superconformal structure, with dual basis $(d\theta, \Omega_z)$. For a given coordinate system (z, θ), a local section $E/\tilde{E}$ determines $\varphi/\tilde{\varphi}$ via

$$\begin{aligned} E &= e^{\varphi} \Omega_z\,, \\ \tilde{E} &= e^{\tilde{\varphi}} \Omega_{\tilde{z}}\,. \end{aligned}$$

Furthermore, there is an odd one-form $F/\tilde{F}$ determined (up to a sign) by

$$\begin{aligned} \pi(dE) &= F \wedge F\,, \\ \tilde{\pi}(d\tilde{E}) &= \tilde{F} \wedge \tilde{F}\,, \end{aligned} \tag{4.6}$$

where $\pi/\tilde{\pi}$ denotes the projection maps onto $T^*\Sigma \otimes T^*\Sigma / T^*\tilde{\Sigma} \otimes T^*\tilde{\Sigma}$. From equation (4.6), we infer

$$\begin{aligned} F &= e^{\varphi/2} \left(d\theta + \frac{1}{2} D_\theta \varphi \Omega_z \right)\,, \\ \tilde{F} &= e^{\tilde{\varphi}/2} \left(d\tilde{\theta} + \frac{1}{2} D_{\tilde{\theta}} \tilde{\varphi} \Omega_{\tilde{z}} \right)\,. \end{aligned}$$

The full metric G, globally defined on $\mathbf{\Sigma}$, then reads

$$G = E \otimes \tilde{E} + \tilde{E} \otimes E + F \otimes \tilde{F} - \tilde{F} \otimes F\,.$$

The area of $\mathbf{\Sigma}$ measured w.r.t. the metric G is defined by

$$A(\mathbf{\Sigma}) = \int dz d\tilde{z} d\theta d\tilde{\theta}\, (\mathrm{sdet}({}_i G_j))^{1/2}\,. \tag{4.7}$$

Here we use the left and right index notion introduced in [24]. It can be shown [20], that the superdeterminant bundle $\mathrm{sdet}(\Sigma)$ is isomorphic to $\mathcal{D}^{-1}$. Thus a volume form for a type II world sheet naturally defines a section of $\mathcal{D}^{-1} \otimes \tilde{\mathcal{D}}^{-1}$. By a straightforward calculation, one can verify that

$$(\mathrm{sdet}({}_i G_j))^{1/2} = e^{(\varphi + \tilde{\varphi})/2}\,,$$

i.e. that it transforms as a section of $\mathcal{D}^{-1}\otimes\tilde{\mathcal{D}}^{-1}$. Finally, consider a supercircle $\gamma : S_\alpha^{1|2} \to \mathbf{\Sigma}$ embedded in $\mathbf{\Sigma}$. The length of γ measured with the induced metric reads

$$L(\gamma) = \int dtd\tau d\tilde{\tau}\,(\mathrm{sdet}({}_i(\gamma^* G)_j))\ .$$

Now we have all the necessary ingredients to formulate the appropriate minimal area problem: for a given type II world sheet $\mathbf{\Sigma}$, we ask for the metric of minimal length under the condition that there is no non-trivial supercircle which is shorter than 2π. We conjecture that this minimal area problem has a unique solution.

In analogy to the bosonic case [11], we claim that a minimal area metric on $\mathbf{\Sigma}$ gives rise to *bands of saturating geodesics*: a saturating geodesic is a supercircle whose length is exactly 2π. Furthermore, saturating geodesics of the same homotopy class are non-intersecting. The collection of all saturating geodesics of a certain homotopy class foliate a part of $\mathbf{\Sigma}$, which is called a band of saturating geodesics. Note that in general bands of saturating geodesics might intersect.

A band of saturating geodesics has the topology of a supercylinder. The height of a band of saturating geodesics is defined to be the shortest superpath between the two boundary components. We distinguish external bands from internal bands, by whether the saturating geodesics are homotopic to a puncture or not.

An external band describes a semi-infinite supercylinder, that is there is a bounding saturating geodesic from where the band extends infinitely towards the puncture. We can now define a section

$$\sigma^l : \mathfrak{M}^{II}_{g,\vec{n}} \to \hat{\mathfrak{P}}^{II}_{g,\vec{n}}\,, \tag{4.8}$$

by defining coordinate curves to be the saturating geodesic a distance l separated from the bounding saturating geodesic. The smallest possible choice for l is π, since for $l \leq \pi$ the sewing of two punctures would lead to supercircles shorter than 2π.

Finally, we describe a 1-parameter family of vertices satisfying condition (4.4) [11]: for given $l \geq \pi$, we define $\mathcal{U}^l_{g,\vec{n}}$ to be the collection of surfaces $\mathbf{\Sigma} \in \mathfrak{M}^{II}_{g,\vec{n}}$, which have no internal bands of saturating geodesics of height larger than l. The vertices together with coordinate curves are then defined by

$$\mathcal{V}^l_{g,\vec{n}} := \sigma^l\left(\mathcal{U}^l_{g,\vec{n}}\right) \in C^{0|0}_{\mathrm{inv}}(\hat{\mathfrak{P}}^{II}_{g,\vec{n}})\,. \tag{4.9}$$

According to (4.5), the BV master equation

$$\partial\mathcal{V}^l_{g,\vec{n}} + \sum_\alpha \Delta^{\mathrm{geo}}_\alpha \mathcal{V}^l_{g-1,\vec{n}+2e_\alpha} + \frac{1}{2}\sum_\alpha \sum_{\substack{\vec{n}_1+\vec{n}_2=\vec{n}\\ g_1+g_2=g}} (\mathcal{V}^l_{g_1,\vec{n}_1+e_\alpha}, \mathcal{V}^l_{g_2,\vec{n}_2+e_\alpha})^{\mathrm{geo}}_\alpha = 0\,, \tag{4.10}$$

is satisfied.

From a field theory point of view, the parameter $l-\pi$ can be interpreted as a cut-off. There are two interesting limits: the vertices corresponding to $l \to \pi$ describe the smallest possible subset of the moduli space consistent with (4.4). This is the natural choice of geometric vertices. On the other hand, in the limit $l \to \infty$ we have $\mathcal{U}^l_{g,\vec{n}} = \mathfrak{M}^{II}_{g,\vec{n}}$, and the corresponding master equation describes the Deligne-Mumford compactification. Indeed, in this singular limit the assignment of coordinate curves is obsolete [20], and thus the master equation describing the compactification can actually be formulated without a global section $\sigma : \mathfrak{M}^{II}_{g,\vec{n}} \to \hat{\mathfrak{P}}^{II}_{g,\vec{n}}$.

For a given TSCFT (background), the corresponding algebraic vertices $f_{g,\vec{n}} \in \mathrm{Hom}_{\mathrm{inv}}(A^{\otimes\vec{n}}, \mathbb{C}^{1|1})$ are now defined by

$$f_{g,\vec{n}} = \int_{\mathcal{V}_{g,\vec{n}}} \omega^{0|0}_{g,\vec{n}} \,. \tag{4.11}$$

Since the TSCFT defines a morphism of BV algebras (see section 3), the algebraic vertices satisfy the BV master equation

$$f_{g,\vec{n}} \circ \sum_{i=1}^{n} Q^{(i)} + \sum_{\alpha} \Delta^{\mathrm{alg}}_{\alpha} f_{g-1,\vec{n}+2e_\alpha} + \frac{1}{2} \sum_{\alpha} \sum_{\substack{\vec{n}_1+\vec{n}_2=\vec{n}\\ g_1+g_2=g}} (f_{g_1,\vec{n}_1+e_\alpha}, f_{g_1,\vec{n}_1+e_\alpha})^{\mathrm{alg}}_{\alpha} = 0 \,.$$

The relevant grading for the BV formalism is the ghost number and the Grassmann parity, but not the picture. That is, the picture number of fields and antifields coincides with the picture number of classical fields (see table 2). Fields have ghost number less then or equal to the ghost number of classical fields, and alternate in Grassmann parity. Similarly, antifields have ghost number greater then classical fields and alternate in Grassmann parity as well. In other words, we restrict the two outputs of the inverse of the symplectic structure appearing in the antibracket and the BV operator to the picture number of classical fields.

Finally, the full quantum action satisfying the BV master equation reads

$$S(\mathbf{c}) = \frac{1}{2} \sum_{\alpha} \omega_\alpha(Qc_\alpha, c_\alpha) + \sum_{g,\vec{n}} \frac{\hbar^g}{\prod_\alpha n_\alpha!} f_{g,\vec{n}}(\mathbf{c}^{\vec{n}}) \,, \tag{4.12}$$

where $\mathbf{c} = (c_\alpha)$ denotes the collection of fields and antifields in the various sectors.

5 Algebraic structure and operadic description

In this section, we employ operads in order to restate the result of the previous section in a uniform and concise way. It will turn out that the construction of string field theory can be formulated by two morphisms between appropriate modular operads, one which describes the decomposition of the moduli space and a second which represents the background.

We start with a brief introduction of modular operads and the Feynman transform. We then quote a result of [17], which establishes a relation between algebras over the Feynman transform of a modular operad and solutions to a corresponding BV master equation. This introductory part does not claim full mathematical rigor, but is rather intended to develop some intuition. We refer the interested reader to [17, 25, 26] for a thorough exposition.

A stable Σ-module $\mathcal{P}$ is a collection of differential graded vector spaces $\mathcal{P}(g, n)$ endowed with a Σ_n action, for all $g \geq 0$ and $n \geq 0$ satisfying the stability condition $2g + n - 3 \geq 0$.

A graph G is a collection $(H(G), V(G), \pi, \sigma)$, where the half-edges $H(G)$ and the vertices $V(G)$ are finite sets, $\pi : H(G) \to V(G)$ and $\sigma : H(G) \to H(G)$ is an involution, i.e. $\sigma^2 = \mathrm{id}$.

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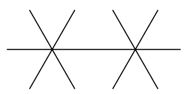
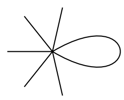
The preimage $\pi^{-1}(v) =: L(v)$ determines the half-edges attached to the vertex $v \in V(G)$. The cardinality of $L(v)$ is denoted by $n(v)$. The involution σ decomposes into 1-cycles and 2-cycles, where the 1-cycles define the legs (external lines) $L(G)$ and the 2-cycles define the edges (internal lines) $E(G)$ of the graph G.

A stable graph is a connected graph G together with a map $g : V(G) \to \mathbb{N}_0$, which assign a genus to each vertex. For every vertex $v \in V(G)$ the stability condition $2g(v) + n(v) - 3 \geq 0$ has to hold. The genus of the graph G is defined by $G(g) = \sum_{v \in V(G)} g(v) + b_1(G)$, where $b_1(G)$ denotes the first Betti number. Furthermore we require a bijection between $L(G)$ and $\{1, \ldots, n(G)\}$, where $n(G)$ denotes the cardinality of $L(G)$.

A morphism of graphs is a contraction of edges. Let G be a stable graph and $I \subset E(G)$ a subset of its edges. We denote the graph that arises from contracting the edges I of the graph G by G/I, and the corresponding morphism by $f_{G,I} : G \to G/I$. Every morphism can be decomposed into a collection of single edge contraction. There are two types of single edge contractions, corresponding to the separating and non-separating case, i.e. to the contraction of an edge connecting two vertices and the contraction of an edge forming a loop on one vertex respectively. In the following, we use a graphical representation for the single edge graphs



and



in the separating and non-separating case respectively. Stable graphs and morphism as described above define the category $\Gamma(g, n)$.

Let $\mathcal{P}$ be a stable Σ-module and G a stable graph. We define

$$\mathcal{P}(G) = \bigotimes_{v \in V(G)} \mathcal{P}(g(v), n(v)) \,.$$

A *modular operad* $\mathcal{P}$ is a stable Σ-module, which in addition defines a functor on the category of graphs. That is, for every morphism $f : G_1 \to G_2$ there is a morphism $\mathcal{P}(f) : \mathcal{P}(G_1) \to \mathcal{P}(G_2)$, and the associativity condition

$$\mathcal{P}(f \circ g) = \mathcal{P}(f) \circ \mathcal{P}(g)$$

has to hold. A *cyclic operad* is the tree level version of a modular operad, i.e. corresponds to $g = 0$.

Due to the functor property and the fact that every morphism of graphs can be decomposed into single edge contractions, a modular operad $\mathcal{P}$ is indeed determined by the underlying Σ-module together with the maps

$$\mathcal{P}\left(f_{\,\cdot\,,\{e\}}\right) =: {}_i\circ_j$$

and

$$\mathcal{P}\left(f_{\ast\!\!\circ,\{e\}}\right) =: \xi_{ij}\,,$$

where i and j represent the half edges constituting the edge e.

Finally, there is the notion of twisted modular operads. The only twist we will need is the so called $\mathfrak{K}$-twist, which assigns degree one to the edges of a graph: for a stable graph G, $\mathfrak{K}(G)$ is defined to be the top exterior power of the vector space generated by the elements of $E(G) = \{e_1, \ldots, e_n\}$, suspended to degree n, i.e.

$$\mathfrak{K}(G) = \det(E(G)) \,:= \uparrow^n \Lambda^n \left(\mathrm{span}(E(G))\right)\,.$$

The standard example of a modular operad is the endomorphism operad. Let (A, d) be a differential graded vector space endowed with a symmetric, bilinear and non-degenerate form $B : A^{\otimes 2} \to \mathbb{k}$ of degree zero, where $\mathbb{k}$ denotes some field or ring. The inverse B^{-1} of B is also symmetric and of degree zero. We define the Σ_n-modules

$$\mathcal{E}[A, d, B](g, n) = \mathrm{Hom}(A^{\otimes n}, \mathbb{k})\,,$$

where the action of Σ_n is defined by permutation of the inputs of the multilinear maps. Contractions w.r.t. B^{-1} make $\mathcal{E}[A, d, B]$ a modular operad. Similarly, consider a differential graded vector space (A, d) endowed with an odd symplectic structure of degree -1. The inverse ω^{-1} is then symmetric and of degree 1. Due to the degree of ω^{-1},

$$\mathcal{E}[A, d, \omega](g, n) = \mathrm{Hom}(A^{\otimes n}, \mathbb{k})$$

defines a $\mathfrak{K}$-twisted modular operad.

An algebra over a modular operad $\mathcal{P}$, called a $\mathcal{P}$-algebra, is a morphism α form $\mathcal{P}$ to some endomorphism operad.

The last ingredient we need is the Feynman transform of a modular operad. Let $\mathbb{M}$ be the functor from the category of stable Σ-modules to the category of modular operads, left adjoint to the forgetful functor. Consider a modular operad $\mathcal{P}$ and let $\mathcal{P}(g, n)^*$ be the dual space of $\mathcal{P}(g, n)$. For our purposes, it suffices to consider the case where the differential on $\mathcal{P}$ vanishes, i.e. $d_{\mathcal{P}} = 0$. The Feynman transform $\mathcal{FP}$ of $\mathcal{P}$ is defined to be the $\mathfrak{K}$-twisted modular operad freely generated from the dual spaces $\mathcal{P}(g, n)^*$, i.e.

$$\mathcal{FP} = \mathbb{M}_{\mathfrak{K}}\mathcal{P}^* \,:= \bigoplus_{G\in[\Gamma(g,n)]} \left(\mathfrak{K}(G) \otimes \mathcal{P}(G)^*\right)_{\mathrm{Aut}(G)}\,,$$

where $[\Gamma(g, n)]$ denotes the set of isomorphism classes of stable graphs. The main feature of the Feynman transform is that it endows $\mathcal{FP}$ with an additional differential: the *Feynman differential* $d_{\mathcal{FP}}$ is defined by

$$d_{\mathcal{FP}}\big|_{(\mathfrak{K}(G)\otimes\mathcal{P}(G)^*)_{\mathrm{Aut}(G)}} = \sum_{G'/\{e\}\simeq G} \uparrow e \otimes \mathcal{P}(f_{G',\{e\}})^*\,,$$

i.e. for a given graph G it generates all graphs G' which are isomorphic to G upon contracting a single edge e.

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Consider now a morphism α from the Feynman transform $\mathcal{FP}$ of a modular operad $\mathcal{P}$ to some $\mathfrak{K}$-twisted modular operad $\mathcal{Q}$. The morphism is Σ equivariant and defines a chain map, i.e.

$$d_{\mathcal{Q}} \circ \alpha = \alpha \circ d_{\mathcal{FP}} \,. \tag{5.1}$$

Furthermore, α is determined by

$$\alpha(g,n) : \mathcal{P}(g,n)^* \to \mathcal{Q}(g,n) \,, \tag{5.2}$$

and Σ_n equivariance implies that

$$\alpha(g,n) \in \left(\mathcal{Q}(g,n) \otimes \mathcal{P}(g,n)\right)^{\Sigma_n} \,.$$

Evaluating equation (5.2) on a graph consisting of a single vertex leads to [17]

$$\begin{aligned} d_{\mathcal{Q}} \circ \alpha(g,I) &= \mathcal{Q}\left(f_{\divideontimes\!\circ,\{e\}}\right) \otimes \mathcal{P}\left(f_{\divideontimes\!\circ,\{e\}}\right) (\uparrow e \otimes \alpha(g-1, I \sqcup \{i,j\})) \\ &+ \frac{1}{2} \sum_{\substack{I_1 \sqcup I_2 = I \\ g_1+g_2=g}} \mathcal{Q}\left(f_{\divideontimes\!\!-\!\!\divideontimes,\{e\}}\right) \otimes \mathcal{P}\left(f_{\divideontimes\!\!-\!\!\divideontimes,\{e\}}\right) (\uparrow e \otimes \alpha(g_1, I_1 \sqcup \{i\}) \otimes \alpha(g_2, I_2 \sqcup \{j\})) \,, \end{aligned} \tag{5.3}$$

where $I = \{1, \ldots, n\}$. Equation (5.3) can be interpreted as a BV master equation on $(\mathcal{Q}(g,n) \otimes \mathcal{P}(g,n))^{\Sigma_n}$, by identifying the contractions w.r.t. $\mathcal{Q}$ and $\mathcal{P}$ together with the determinant of the edge as the antibracket $(\cdot,\cdot)$ in the separating and the BV operator Δ in the non-separating case. $d^2_{\mathcal{FP}} = 0$ is then equivalent to the axioms of a BV algebra (without multiplication) listed in equation (2.20) [17]. Substituting $d_{\mathcal{Q}} \to -d_{\mathcal{Q}}$, equation (5.3) reads

$$d_{\mathcal{Q}} \circ \alpha(g,n) + \Delta\alpha(g-1,n+2) + \frac{1}{2} \sum_{\substack{n_1+n_2=n \\ g_1+g_2=g}} (\alpha(g_1,n_1+1), \alpha(g_2,n_2+1)) = 0 \,. \tag{5.4}$$

Theorem 1 (**[17]**) *Morphisms from the Feynman transform $\mathcal{FP}$ of a modular operad $\mathcal{P}$ to a $\mathfrak{K}$-twisted modular operad $\mathcal{Q}$ are in one-to-one correspondence with solutions to the BV master equation (5.4).*

In the previous sections we saw that the geometric approach to string field theory inevitably leads to a certain BV master equation that has to be satisfied. Thus, the link between the Feynman transform and soltions to an associated BV master equation immediately reveals the relevance of modular operads in the context of string field theory.

In type II superstring field theory, we have four different sectors $\alpha \in \{NS-NS, R-NS, NS-R, R-R\}$. Thus we need a slight generalization of a modular operad which allows for several sectors, i.e. a "colored" version of a modular operad. For our purposes, a "colored" modular operad $\mathcal{P}$ is a collection of differential graded vector spaces $\mathcal{P}(g,\vec{n})$, $\vec{n} = (n_\alpha)_{\alpha \in C}$, $n_\alpha \in \mathbb{N}_0$, satisfying the stability condition $2g + \sum_\alpha n_\alpha - 3 \geq 0$, where C denotes the set of colors. Half edges of a graph are labeled by a color, and only half edges of the same color can form an edge. Furthermore we are only allowed to permute half edges of the same color, i.e. $\mathcal{P}(g,\vec{n})$ is a $\Sigma_{\vec{n}}$-module, where $\Sigma_{\vec{n}} = \times_\alpha \Sigma_{n_\alpha}$.

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In the following we introduce the relevant operads for the formulation of type II superstring field theory in terms of morphisms of operads. The cyclic operad encoding the symmetries of the classical (genus zero) vertices is denoted by $\mathcal{C}om^{\mathcal{N}=1}$. It is a colored operad with $C = \{NS-NS, R-NS, NS-R, R-R\}$, and $\mathcal{C}om^{\mathcal{N}=1}(\vec{n})$ are one dimensional vector spaces of degree zero without differential. The permutation group $\Sigma_{\vec{n}}$ acts trivially on $\mathcal{C}om^{\mathcal{N}=1}(\vec{n})$. Furthermore, on top of the stability condition we impose the following constraints:

$$
\begin{aligned}
n_{NS-NS} + n_{NS-R} &\in \mathbb{N}_0 \\
n_{NS-NS} + n_{R-NS} &\in \mathbb{N}_0 \\
n_{R-R} + n_{R-NS} &\in 2\mathbb{N}_0 \\
n_{R-R} + n_{NS-R} &\in 2\mathbb{N}_0 \,.
\end{aligned}
\tag{5.5}
$$

Let $x_{\vec{n}}$ denote the element that generates the vector space $\mathcal{C}om^{\mathcal{N}=1}(\vec{n})$. The single edge contraction is defined by

$$
\mathcal{C}om^{\mathcal{N}=1}\left(f^{\alpha}_{\ast\!-\!\ast,\{e\}}\right)(x_{\vec{n}_1+e_\alpha}\otimes x_{\vec{n}_2+e_\alpha}) = x_{\vec{n}_1+\vec{n}_2}\,.
$$

It turns out that $\mathcal{C}om^{\mathcal{N}=1}$ is generated by the vector spaces with $\sum_\alpha n_\alpha = 3$. Such an operad is called a *quadratic* operad [26]. For $\sum_\alpha n_\alpha = 3$ there are five cases compatible with (5.5)

(i) $n_{NS-NS} = 3$

(ii) $n_{NS-NS} = 1$, $n_{R-NS} = 2$

(iii) $n_{NS-NS} = 1$, $n_{NS-R} = 2$

(iv) $n_{NS-NS} = 1$, $n_{R-R} = 2$

(iv) $n_{R-NS} = 1 = n_{NS-R}$, $n_{R-R} = 1$,

which correspond to the five possible types of genus zero surfaces with three punctures.

Let **Mod** be the functor from the category of cyclic operads to the category of modular operads, left adjoint to the forgetful functor. Consider now the modular operad $\mathbf{Mod}(\mathcal{C}om^{\mathcal{N}=1})$ associated to the cyclic operad $\mathcal{C}om^{\mathcal{N}=1}$, which encodes the symmetry properties of the vertices to all order in $\hbar$. Again $\mathbf{Mod}(\mathcal{C}om^{\mathcal{N}=1})(g,\vec{n})$ are one dimensional vector spaces endowed with the trivial action of $\Sigma_{\vec{n}}$, and the single edge contractions read

$$
\mathbf{Mod}(\mathcal{C}om^{\mathcal{N}=1})\left(f^{\alpha}_{\ast\!-\!\ast,\{e\}}\right)(x_{g_1,\vec{n}_1+e_\alpha}\otimes x_{g_2,\vec{n}_2+e_\alpha}) = x_{g_1+g_2,\vec{n}_1+\vec{n}_2}\,,
$$

$$
\mathbf{Mod}(\mathcal{C}om^{\mathcal{N}=1})\left(f^{\alpha}_{\ast\!\circlearrowright,\{e\}}\right)(x_{g-1,\vec{n}+2e_\alpha}) = x_{g,\vec{n}}\,,
$$

where $x_{g,\vec{n}}$ is the element that generates $\mathbf{Mod}(\mathcal{C}om^{\mathcal{N}=1})(g,\vec{n})$.

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$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathcal{F}\mathbf{Mod}(\mathcal{C}om^{\mathcal{N}=1}) \arrow[r, "\text{decomposition}", "\text{of moduli space}"'] \arrow[dr, "\text{algebra}"'] & C^{\bullet|\bullet}(\hat{\mathfrak{P}}^{II}) \arrow[d, "\text{TSCFT}"] \\
 & \mathcal{E}[A_\alpha, Q_\alpha, \omega_\alpha]
\end{tikzcd}
$$

Figure 2. Construction of type II superstring field theory in terms of morphisms of modular operads.

Next, we define the $\mathfrak{K}$-twisted modular operad $C^{\bullet|\bullet}(\hat{\mathfrak{P}}^{II})$. Its underlying $\Sigma_{\vec{n}}$-modules are $C^{\bullet|\bullet}(\hat{\mathfrak{P}}^{II}_{g,\vec{n}})$ with grading as defined in section 2, and the single edge contractions are defined by

$$
\begin{aligned}
C^{\bullet|\bullet}(\hat{\mathfrak{P}}^{II})\left(f^{\alpha}_{\ast\!-\!\ast,\{e\}}\right)(\mathcal{A}_{g_1,\vec{n}_1+e_\alpha}\sqcup\mathcal{A}_{g_2,\vec{n}_2+e_\alpha}) &= \mathcal{A}_{g_1,\vec{n}_1+e_\alpha}\;{}_i\overset{\Phi_\alpha}{\circ}_j\,\mathcal{A}_{g_2,\vec{n}_2+e_\alpha}\\
C^{\bullet|\bullet}(\hat{\mathfrak{P}}^{II})\left(f^{\alpha}_{\ast\!\circ,\{e\}}\right)(\mathcal{A}_{g-1,\vec{n}+2e_\alpha}) &= \overset{\Phi_\alpha}{\xi}_{ij}\mathcal{A}_{g-1,\vec{n}+2e_\alpha}\,,
\end{aligned}
$$

where ${}_i\overset{\Phi_\alpha}{\circ}_j$ and $\overset{\Phi_\alpha}{\xi}_{ij}$ are the sewing maps of equation (2.16) and equation (2.17) respectively.

Finally, consider a TSCFT which determines the endomorphism operad $\mathcal{E}[A_\alpha, Q_\alpha, \omega_\alpha]$, where A_α denotes the restricted state space with the grading of equation (3.12), Q_α is the BRST charge and ω_α is the odd symplectic structure as defined in (3.13).

As discussed in section 4, a consistent decomposition of the moduli space into vertices and graphs implies that the BV master equation (4.5) is satisfied, which is due to theorem 1 equivalent to a morphism α from $\mathcal{F}\mathbf{Mod}(\mathcal{C}om^{\mathcal{N}=1})$ to $C^{\bullet|\bullet}(\hat{\mathfrak{P}}^{II})$. Second, the factorization properties (3.7), (3.8) and the chain map property qualify a TCFT as a morphism β from $C^{\bullet|\bullet}(\hat{\mathfrak{P}}^{II})$ to $\mathcal{E}[A_\alpha, Q_\alpha, \omega_\alpha]$.

Schematically, the construction of string field theory can be summarized as depicted in figure 2.

The composition $\gamma := \beta\circ\alpha$ of the morphisms α and β then defines an algebra over $\mathcal{F}\mathbf{Mod}(\mathcal{C}om^{\mathcal{N}=1})$. Finally, we want to identify this algebraic structure as some homotopy algebra. We employ the following statements:

Theorem 2 ([27]) *Let $\mathcal{P}$ be a Koszul cyclic operad. Algebras over the cobar transform (the tree level part of the Feynman transform) of the quadratic dual $\mathcal{P}^!$ of $\mathcal{P}$ are homotopy $\mathcal{P}$-algebras.*

Definition 1 ([18]) *Let $\mathcal{P}$ be a Koszul cyclic operad. Algebras over $\mathcal{F}\boldsymbol{Mod}(\mathcal{P}^!)$ are loop homotopy $\mathcal{P}$-algebras.*

Let us first discuss the known results of bosonic string field theory. In closed string field theory, the cyclic operad encoding the symmetry properties of the classical vertices is the operad $\mathcal{C}om$, whose algebras are commutative algebras. $\mathcal{C}om$ is Koszul and its quadratic dual is $\mathcal{L}ie$, the operad whose algebras are Lie algebras. A consistent decomposition of the moduli space of closed Riemann surfaces $\mathcal{M}_{g,n}$ defines a morphism from $\mathcal{F}\mathbf{Mod}(\mathcal{C}om)$ to $C^\bullet(\hat{\mathcal{P}})$, and a background determines a topological conformal field theory which is a morphism from $C^\bullet(\hat{\mathcal{P}})$ to $\mathcal{E}[A, Q, \omega]$, where $\omega = \mathrm{bpz}(\cdot, c_0^- \cdot)$. Thus the algebraic structure of classical closed string field theory is that of a homotopy Lie-algebra (L_∞-algebra) [11], and quantum closed string field theory carries the structure of a loop homotopy Lie-algebra [18].

Inspired by that, we call an algebra over $\mathcal{F}\mathbf{Mod}(\mathcal{C}om^{\mathcal{N}=1})$ a $\mathcal{N}=1$ loop homotopy Lie-algebra and similarly an algebra over the cobar transform of $\mathcal{C}om^{\mathcal{N}=1}$ a $\mathcal{N}=1$ homotopy Lie-algebra.

We conclude this section with the following theorem:

Theorem 3 *The vertices of the quantum/classical master action of type II superstring field theory satisfy the axioms of a $\mathcal{N}=1$ loop homotopy Lie-algebra/$\mathcal{N}=1$ homotopy Lie-algebra.*

6 Outlook

In this paper we outline the construction of type II superstring field theory, leading to a geometric and an algebraic BV master equation analogous to the case in the bosonic string. The construction is based on the small Hilbert space, in contrast to other approaches to superstring field theory like [7, 28]. Picture changing operators arise as the consequence of the fact that we can not define local coordinates around punctures globally on moduli space but just coordinate curves. Pursuing the same idea for classical open superstring field theory would require a restriction of the state space in the Ramond sector, due to the translation invariance in the Ramond divisors. Such a theory might serve as an adequate description of classical open superstring field theory.

Recently, it has been shown that the moduli space of super Riemann surfaces is generically non-split [29]. An interesting question is whether the topology of the geometric vertices of type II superstring field theory is considerably simpler than that of the full moduli space, i.e. if the integrals defining the algebraic vertices can be reduced to integrals over the geometric vertices of bosonic closed string field theory.

A particular feature of type II superstring field theory is, that the symplectic form in the $R-R$ sector degenerates on-shell. On the other hand, this is a necessary condition for a non-trivial open-closed correspondence at the quantum level, as discussed in [30]. Thus, this indicates that in type I superstring field theory there might be backgrounds where closed strings decouple completely form open strings even at the quantum level, leading to a consistent theory of only open superstrings.

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Finally, we describe string field theory in terms of operads. For classical bosonic open strings on a single D-brane, the relevant operad is the operad $\mathcal{A}ss$ of associative algebras. First of all it would be interesting to generalize the operad $\mathcal{A}ss$ to several D-branes, such that algebras over its cobar transform are Calabi-Yau A_∞-categories [31]. Second, another project [32] is to specify the operad that describes the quantum open-closed homotopy algebra [33].

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A Super Riemann surfaces

In this part we follow closely the exposition of [20]. A super Riemann surface Σ is a $1|1$ dimensional complex supermanifold with the additional structure of a subbundle $\mathcal{D} \subset T\Sigma$ of the tangent bundle of rank $0|1$. A Neveu-Schwartz puncture on Σ is described by a point $(z, \theta) = (z_0, \theta_0)$, whereas a Ramond puncture is described by a divisor $z = z_0$. The collection of all Ramond punctures defines the Ramond divisor. Note that the number of Ramond punctures is always even. Furthermore the subbundle $\mathcal{D}$ has to satisfy a non-degeneracy condition: for every non-zero section D of $\mathcal{D}$, the commutator $[D, D]$ has to be linearly independent of D everywhere, except along the Ramond divisor where $[D, D] = 0$. Thus a Ramond puncture is part of the structure of a super Riemann surface, in contrast to a Neveu-Schwarz puncture which merely distinguishes a point on Σ. In the following every notion in the Neveu-Schwartz sector will have its counterpart in the Ramond sector, which we will display by NS and R respectively.

A superconformal coordinate system (z, θ) is distinguished by requiring that every section D of $\mathcal{D}$ is proportional to

$$D_\theta = \partial_\theta + \theta\partial_z \,, \qquad \text{NS}$$

and

$$D^*_\theta = \partial_\theta + z\theta\partial_z \,, \qquad \text{R} \,,$$

where the coordinate system for R covers a subset of Σ containing a single R puncture at $z = 0$.[6]

[6]For several R punctures z_i, we would have $D^*_\theta = \partial_\theta + w(z)\theta\partial_z$ with $w(z) = \Pi_i(z - z_i)$.

A superconformal transformation is a change of superconformal coordinates. The general form of such a transformation is

$$\begin{aligned} z' &= u \pm \theta\alpha\sqrt{u'} \\ \theta' &= \alpha \pm \theta\sqrt{u'}\left(1+\tfrac{\alpha\alpha'}{2u'}\right) \end{aligned}, \qquad \text{NS} \tag{A.1}$$

and

$$\begin{aligned} z' &= u \pm \theta\alpha\sqrt{zuu'} \\ \theta' &= \alpha \pm \theta\sqrt{\tfrac{zu'}{u}}\left(1+\tfrac{u\alpha\alpha'}{2u'}\right) \end{aligned}, \qquad \text{R}\,, \tag{A.2}$$

where $u = u(z)$ is an even function and $\alpha = \alpha(z)$ is odd. The signs in equation (A.1) and (A.2) are determined by a choice of branch for the square root of u' and $\frac{zu'}{u}$, respectively.

Primary fields of superconformal weight h are defined to be sections of $\mathcal{D}^{-2h}$. Consider for example a function $f \in C^\infty(\Sigma)$, then $D_\theta f = (D_\theta\theta')D_{\theta'}f$ transforms as a primary of superconformal weight $1/2$. In general a primary ϕ of superconformal weight h can be expanded as $\phi = \varphi_0 + \theta\varphi_1$, where φ_0 has conformal weight h and φ_1 has conformal weight $h + 1/2$.

Finally, a superconformal vectorfield X is a vector field that preserves the subbundle $\mathcal{D}$, that is for every section D of $\mathcal{D}$

$$[X, D] \propto D\,.$$

We can choose a basis for the space of superconformal vectorfields which obeys the super Witt algebra:

$$\begin{aligned} l_n &= -z^{n+1}\partial_z - \frac{1}{2}(n+1)z^n\theta\partial_\theta\,, && n\in\mathbb{Z} \\ g_n &= z^{n+1/2}(\partial_\theta - \theta\partial_z)\,, && n\in\mathbb{Z}+1/2 \end{aligned}, \qquad \text{NS} \tag{A.3}$$

$$\begin{aligned} l_n &= -z^{n+1}\partial_z - \frac{1}{2}nz^n\theta\partial_\theta\,, && n\in\mathbb{Z} \\ g_n &= z^n(\partial_\theta - \theta z\partial_z)\,, && n\in\mathbb{Z} \end{aligned}, \qquad \text{R}\,, \tag{A.4}$$

$$\begin{aligned} [l_m, l_n] &= (m-n)l_{m+n} \\ [l_m, g_n] &= \left(\tfrac{m}{2}-n\right)g_{m+n} \\ [g_m, g_n] &= 2l_{m+n} \end{aligned}, \qquad \text{NS and R}\,.$$

In the NS sector the vectors

$$g_{-1/2}\,,\quad g_{1/2}\,,\quad l_{-1}\,,\quad l_0\,,\quad l_1$$

form a closed subalgebra and generate the $3|2$ complex dimensional NS Möbius group $\mathrm{Aut}(S^{2|1}_{NS})$. The general form of a NS Möbius transformation is given by

$$\begin{pmatrix} z' \\ \theta' \end{pmatrix} = \begin{pmatrix} \dfrac{az+b}{cz+d} \pm \theta\dfrac{\gamma z+\delta}{(cz+d)^2} \\[2ex] \dfrac{\gamma z+\delta}{cz+d} \pm \theta\dfrac{1+\frac{1}{2}\delta\gamma}{cz+d} \end{pmatrix}, \tag{A.5}$$

where $a, b, c, d \in \mathbb{C}^{1|0}$, $\gamma, \delta \in \mathbb{C}^{0|1}$ and $ad - bc = 1$.

The maximal non-trivial subalgebra in the R sector is spanned by

$$l_{-1}\,, \quad l_0\,, \quad l_1$$

and generates the $3|0$ complex dimensional R Möbius group $\mathrm{Aut}(S_R^{2|1})$. A generic element of $\mathrm{Aut}(S_R^{2|1})$ takes the form

$$\begin{pmatrix} z' \\ \theta' \end{pmatrix} = \begin{pmatrix} \dfrac{az+b}{cz+d} \\ \pm\theta \left(\dfrac{z}{(az+b)(cz+d)} \right)^{1/2} \end{pmatrix}, \tag{A.6}$$

with $a, b, c, d \in \mathbb{C}^{1|0}$ and $ad - bc = 1$.

B Superconformal field theory of type II string

The field content of the superconformal field theory of type II string theory is composed of matter fields and ghost fields. The matter sector is described by scalars

$$X^\mu(z, \tilde{z}, \theta, \tilde{\theta})\,,$$

and the ghost sector contains the holomorphic ghosts

$$B = \beta + \theta b \qquad \text{and} \qquad C = c + \theta\gamma\,,$$

of superconformal weight $(3/2, 0)$ and $(-1, 0)$, respectively, and the antiholomorphic ghosts

$$\tilde{B} = \tilde{\beta} + \tilde{\theta}\tilde{b} \qquad \text{and} \qquad \tilde{C} = \tilde{c} + \tilde{\theta}\tilde{\gamma}\,,$$

of superconformal weight $(0, 3/2)$ and $(0, -1)$, respectively. Let ϕ be a holomorphic local operator of superconformal weight h in the NS sector and $(z, \theta) = (z_{\text{radial}}, \theta_{\text{radial}})$ the standard coordinate system of radial quantization, then the mode expansion of ϕ reads

$$\phi(z, \theta) = \sum_{n\in\mathbb{Z}} \frac{\phi_n^0}{z^{n+h}} + \theta \sum_{n\in\mathbb{Z}+1/2} \frac{\phi_n^1}{z^{n+h+1/2}}\,. \tag{B.1}$$

Now consider a holomorphic local operator ϕ of superconformal weight h in the R sector. The coordinate system of radial quantization is not a good coordinate system in the R sector - it involves a branch cut [14]. We obtain a superconformal coordinate system in the sense of (A.2) by defining new coordinates $(z, \theta) = (z_{\text{radial}}, \theta_{\text{radial}} z_{\text{radial}}^{-1/2})$. In these coordinates, the mode expansion reads

$$\phi(z, \theta) = \sum_{n\in\mathbb{Z}} \frac{\phi_n^0}{z^n} + \theta \sum_{n\in\mathbb{Z}} \frac{\phi_n^1}{z^n}\,. \tag{B.2}$$

The sewing maps (2.3) and (2.4) define the bpz conjugation

$$\mathrm{bpz}_{NS}(\phi)(z, \theta) = (I^*_{(+,+)}\phi)(z, \theta) \qquad \text{and} \qquad \mathrm{bpz}_R(\phi)(z, \theta) = (I^*_{(+,-)}\phi)(z, \theta)\,.$$

From the mode expansion (B.1) and (B.2) we can infer that

$$\begin{aligned} \text{bpz}_{NS}(\phi^0_n) &= (-1)^{n+h}\phi^0_{-n}\,, & \text{bpz}_{NS}(\phi^1_n) &= (-1)^{n+h+1/2}\phi^1_{-n}\,, \\ \text{bpz}_{R}(\phi^0_n) &= (-1)^{n+h}\phi^0_{-n}\,, & \text{bpz}_{R}(\phi^1_n) &= (-1)^{n+h+1/2}\phi^1_{-n}\,, \end{aligned} \tag{B.3}$$

which is indeed the same for every sector and every type of mode.

The operator state correspondence is formulated in terms of the coordinates of radial quantization, so there is no problem in the NS sector. In the R sector, in contrast, the coordinates of radial quantization are ill defined. To resolve this problem, one introduces spin fields which map the NS ground state to the R ground state [34]. We denote the spin fields in the matter sector by

$$S_m^{s_1,\dots,s_5}(z)\,,$$

and in the ghost sector by

$$S_g^{\pm}(z)\,,$$

such that

$$S_g^- S_m^{s_1,\dots,s_5}\,|0\rangle_{NS} = |s_1,\dots,s_5\rangle_R$$

describes the R ground state. Furthermore, the ghost spin field satisfies [34, 35]

$$\begin{aligned} \beta(z_1)S_g^{\pm}(z_2) &\sim z_{12}^{\pm 1/2} : \beta S_g^{\pm} : (z_2) \\ \gamma(z_1)S_g^{\pm}(z_2) &\sim z_{12}^{\mp 1/2} : \gamma S_g^{\pm} : (z_2)\,. \end{aligned} \tag{B.4}$$

The operator state correspondence together with (B.4) determines the creation operators in the ghost sector to be

$$\begin{aligned} &\dots, \gamma_{-1/2}, \gamma_{1/2} \\ &\dots, \beta_{-5/2}, \beta_{-3/2} \\ &\dots, c_0, c_1 \\ &\dots, b_{-3}, b_{-2} \end{aligned}$$

in the NS sector and

$$\begin{aligned} &\dots, \gamma_{-1}, \gamma_0 \\ &\dots, \beta_{-2}, \beta_{-1} \\ &\dots, c_0, c_1 \\ &\dots, b_{-3}, b_{-2} \end{aligned}$$

in the R sector. The creation operators whose bpz conjugate is also a creation operator are called zero modes. This determines the ghost zero modes

$$\begin{aligned} &\gamma_{-1/2}, \gamma_{1/2} \\ &c_{-1}, c_0, c_1 \end{aligned}$$

in the NS sector and

$$\gamma_0$$
$$c_{-1}, c_0, c_1$$

in the R sector. In order to obtain a non-vanishing correlator, one has to saturate these zero modes. This requires an insertion

$$c_{-1}c_0c_1\delta(\gamma_{-1/2})\delta(\gamma_{1/2})$$

and

$$c_{-1}c_0c_1\delta(\gamma_0)$$

in the NS and R sector, respectively. A geometric interpretation of delta functions of ghost operators has first been given in [13, 16], which we review in appendix C together with the rules how to manipulate such expressions. Furthermore, the geometric interpretation suggests a grading which differs from the conventional ghost number and picture grading: we define ghost number by assigning ghost number one to c, γ and ghost number minus one to b, β. Picture number is associated with delta functions of Grassmann even ghosts, that is $\delta(\gamma_n)$ carries picture number one and $\delta(\beta_n)$ carries picture number minus one. Finally, we set the ghost number and picture for both the NS and the R groundstate to be zero. We will denote ghost number and picture collectively by $g|p$.

Thus

$$\deg(|0\rangle_{NS}) = 0|0\,, \qquad \deg(\,|s_1,\ldots,s_5\rangle_R) = 0|0$$

implies

$$\deg(S_g^-) = 0|0\,.$$

The bpz inner product of states Φ_1 and Φ_2 is defined by

$$\mathrm{bpz}_\alpha(\Phi_1,\Phi_2) := \langle (I_\alpha^*\Phi_1)\Phi_2\rangle\;, \tag{B.5}$$

where I_α is the sewing map defined in (2.7). Thus we conclude that

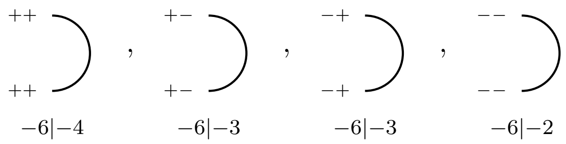
$$\begin{aligned} \deg\left(\mathrm{bpz}_{NS-NS}\right) &= -6|-4 \\ \deg\left(\mathrm{bpz}_{R-NS}\right) = \deg\left(\mathrm{bpz}_{NS-R}\right) &= -6|-3 \\ \deg\left(\mathrm{bpz}_{R-R}\right) &= -6|-2\,, \end{aligned}$$

Moreover we have

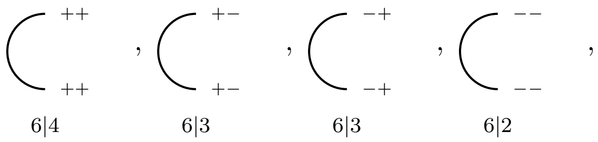
$$S_g^-(z_1)S_g^-(z_2) \sim \frac{1}{z_{12}^{1/4}}\delta(\gamma)(z_2)\,,$$

which implies that the OPE of two R vertex operators carries degree $0|1$.

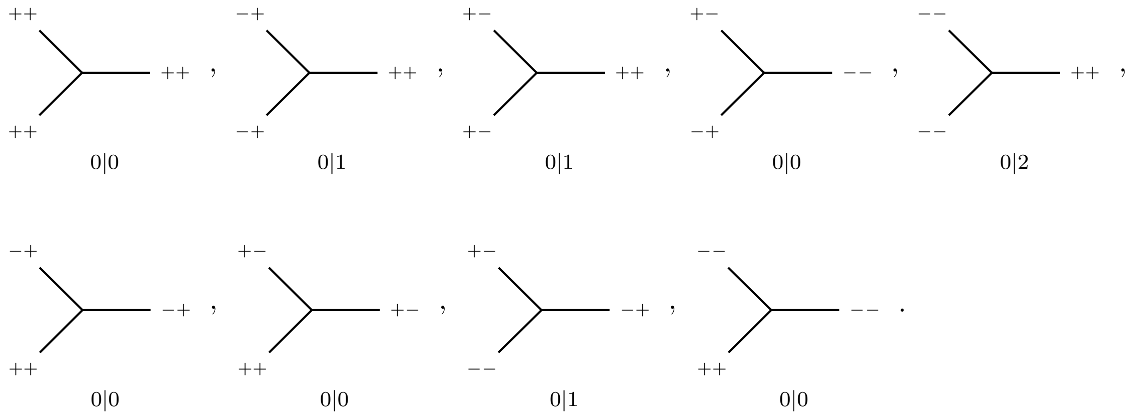
To proceed, we depict maps on the state space of the CFT by directed graphs, where the direction which distinguishes inputs and outputs points from left to right. Thus, the bpz inner product in the corresponding sectors is represented by



and its inverse by



where we abbreviate NS and R as $+$ and $-$, respectively, and also indicate the degree. Similarly, the OPE is depicted by



Now one can construct arbitrary surfaces from these elementary ones, and thus determine the degree of a correlation function $Z(\mathbf{\Sigma}_{g,\vec{n}})$ on a type II world sheet $\mathbf{\Sigma}_{g,\vec{n}}$ to be

$$\deg\left(Z(\mathbf{\Sigma}_{g,\vec{n}})\right) = 6g-6|4g-4+n_{R-R}+\frac{1}{2}(n_{R-NS}+n_{NS-R})\,. \tag{B.6}$$

Finally, the typical form of a vertex operator is given by

$$\begin{aligned}
c\delta(\gamma)\,\tilde{c}\delta(\tilde{\gamma})\,V\,, &\qquad 2|2\,, \qquad \text{NS-NS}\,,\\
c\delta(\gamma)\,\tilde{c}\tilde{S}_g^-\,\tilde{S}_m^{\tilde{s}_1,\ldots,\tilde{s}_5}\,V\,, &\qquad 2|1\,, \qquad \text{NS-R}\,,\\
cS_g^-\,S_m^{s_1,\ldots,s_5}\,\tilde{c}\delta(\tilde{\gamma})\,V\,, &\qquad 2|1\,, \qquad \text{NS-R}\,,\\
cS_g^-\,S_m^{s_1,\ldots,s_5}\,\tilde{c}\tilde{S}_g^-\,\tilde{S}_m^{\tilde{s}_1,\ldots,\tilde{s}_5}\,V\,, &\qquad 2|0\,, \qquad \text{R-R}\,,
\end{aligned} \tag{B.7}$$

where V represents some matter vertex operator.

C Forms in supergeometry and relation to string theory

The superconformal ghosts of superstring theory can be interpreted as operations acting on differential forms [13, 16]. To illustrate this analogy, we will start with a brief review of geometric integration theory on supermanifolds [13, 15, 16, 19].

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Let $M^{m|n}$ be a $m|n$ dimensional supermanifold. A differential form $\omega \in \Omega^{r|s}(M^{m|n})$ is a function of r even and s odd tangent vectors, which satisfies

$$\omega(g\mathbf{V}) = \mathrm{sdet}(g)\omega(\mathbf{V})\,, \qquad \forall g \in \mathrm{GL}(r|s) \tag{C.1}$$

and

$$\left(\partial_{V_A^M}\partial_{V_B^N} - (-1)^{AB+N(A+B)}\partial_{V_B^M}\partial_{V_A^N}\right)\omega(\mathbf{V}) = 0\,,$$

where $\mathbf{V} = (v_1,\ldots,v_r|\nu_1,\ldots,\nu_s)$ denotes a collection of tangent vectors and V_A^M is the M-th component of the A-th tangent vector, i.e. $A, B \in \{1,\ldots,r|s\}$ and $M, N \in \{1,\ldots,m|n\}$. The exterior derivative $d : \omega^{r|s}(M^{m|n}) \to \omega^{r+1|s}(M^{m|n})$ is defined by

$$(d\omega)(v_1,\ldots,v_r,v_{r+1},\nu_1,\ldots,\nu_s) = (-1)^r v_{r+1}^M(\delta_{x^M}\,\omega)(v_1,\ldots,v_r,\nu_1,\ldots,\nu_s)\,,$$

where

$$(\delta_{x^M}\,\omega)(\mathbf{V}) = \partial_{x^M}\omega(\mathbf{V}) - (-1)^{MA}V_A^N\partial_{x^N}\partial_{V_A^M}\omega(\mathbf{V})\,,$$

and x^M are coordinates on $M^{m|n}$. Let V be a vector field on $M^{m|n}$. The interior product $i_V : \omega^{r|s}(M^{m|n}) \to \omega^{r-1|s}(M^{m|n})$ is defined by

$$(i_V\omega)(v_1,\ldots,v_{r-1}|\nu_1,\ldots,\nu_s) = \omega(V,v_1,\ldots,v_{r-1}|\nu_1,\ldots,\nu_s)\,.$$

The space of differential forms is preserved under multiplication with functions. Thus, imposing the Leibniz rule w.r.t. d makes $\Omega^{r|s}(M^{m|n})$ a module over $\Omega^{r'|0}(M^{m|n})$, in particular over 1-forms. We denote the operation of multiplying a 1-form $\alpha \in \Omega^{1|0}(M^{m|n})$ by $e_\alpha : \Omega^{r|s}(M^{m|n}) \to \Omega^{r+1|s}(M^{m|n})$, which explicitly reads

$$(e_\alpha\omega)(v_1,\ldots,v_r,v_{r+1}|\nu_1,\ldots,\nu_s) = (-1)^r\left(\alpha(v_{r+1})\omega(\mathbf{V}) - (-1)^{MA}\alpha(V_A)V_{r+1}^M\partial_{V_A^M}\omega(\mathbf{V})\right)\,.$$

The operations introduced so far just affect the number of even vectors, but there are also operations witch change the number of odd vectors: let ν be an odd vector field on $M^{m|n}$. The operation $\delta(i_\nu) : \Omega^{r|s}(M^{m|n}) \to \Omega^{r|s-1}(M^{m|n})$ is defined by

$$(\delta(i_\nu)\omega)\,(v_1,\ldots,v_r|\nu_1,\ldots,\nu_{s-1}) = \omega(v_1,\ldots,v_r|\nu,\nu_1,\ldots,\nu_{s-1}).^{7} \tag{C.2}$$

Similarly, for an odd 1-form β, there is an operation $\delta(e_\beta) : \Omega^{r|s}(M^{m|n}) \to \Omega^{r|s+1}(M^{m|n})$,

$$\delta(e_\beta)(v_1,\ldots,v_r|\nu_1,\ldots,\nu_s,\nu_{s+1}) = \frac{1}{\beta(\nu_{s+1})}\omega\left(\ldots,V_A - \frac{\beta(V_A)}{\beta(\nu_{s+1})}\nu_{s+1},\ldots\right)\,. \tag{C.3}$$

The grading is defined by $r|s$ plus the Grasssmann parity $p \in \mathbb{Z}_2$, which we denote collectively by $r|s|p$. To summarize, we have five basic operations on the space of differential forms, listed in table 4 together with the corresponding degrees.

Differential $r|s$-forms on $M^{m|n}$ are the natural objects for integrating $r|s$ dimensional submanifolds of $M^{m|n}$. But as in the even case, one needs an orientation on the submanifold to carry out the integration unambiguously. The general linear group $\mathrm{GL}(m|n)$ has four normal subgroups, which determine the possible notions of orientability:

[7] In (C.2), we use a different sign convention than in the original work [16], which is more natural in the context of superstring theory.

d	i_V	e_α	$\delta(i_\nu)$	$\delta(e_\beta)$
$1\|0\|0$	$-1\|0\|V$	$1\|0\|\alpha$	$0\|-1\|1$	$0\|1\|1$

Table 4. Basic operations on differential forms. The Grassmann parity of V and α is undetermined, whereas β and ν are odd.

(i) $[++]$ orientation: $\det(g_{00}) > 0$ and $\det(g_{11}) > 0$

(ii) $[+-]$ orientation: $\det(g_{00}) > 0$

(iii) $[-+]$ orientation: $\det(g_{11}) > 0$

(iv) $[--]$ orientation: $\det(g_{00})\det(g_{11}) > 0$

where

$$\mathrm{GL}(m|n) \ni g = \begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix}.$$

Due to the symmetry properties of differential forms (C.1) and the fact that $\int d\theta_1 d\theta_2 = -\int d\theta_2 d\theta_1$ whereas $\int dx_1 dx_2 = \int dx_2 dx_1$, the appropriate orientation for integrating differential forms is the $[+-]$ orientation.

Let A_1 and A_2 be some operators on the space of differential forms of degree $r_1|s_1|p_1$ and $r_2|s_2|p_2$, respectively. We define the commutator to be

$$[A_1, A_2] = A_1 \circ A_2 - (-1)^{A_1 A_2} A_2 \circ A_1 ,$$

where

$$(-1)^{A_1 A_2} = (-1)^{r_1 r_2 + p_1 p_2} . \tag{C.4}$$

Note that s_1 and s_2 do not occur in equation (C.4), which is in accordance with the $[+-]$ orientation. Thus the part s of the grading does not produce a sign upon permutation, as it has been already observed in section 2 in the context of the oriented singular chain complex of moduli spaces.

In the following we describe some operations generated from the basic operations of table 4. The Lie derivative w.r.t. a vector field V is defined by

$$\mathcal{L}_V = [d, i_V] , \qquad \deg(\mathcal{L}_V) = 0|0|V .$$

Furthermore

$$[e_\beta, \delta(i_\nu)] = -\beta(\nu)\delta'(i_\nu) ,$$

and more generally

$$[e_\beta, \delta^{(n)}(i_\nu)] = -\beta(\nu)\delta^{(n+1)}(i_\nu) , \qquad \deg(\delta^{(n)}(i_\nu)) = n|-1|n+1 .$$

Similarly

$$[i_\nu, \delta^{(n)}(e_\beta)] = \beta(\nu)\delta^{(n+1)}(e_\beta) , \qquad \deg(\delta^{(n)}(e_\beta)) = -n|1|n+1 .$$

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The picture changing operator Γ_ν of degree $0|1|0$ associated to an odd vector field ν is defined by [16]

$$\begin{aligned}\Gamma_\nu &= \frac{1}{2}\left(\mathcal{L}_\nu\,\delta(i_\nu) - \delta(i_\nu)\,\mathcal{L}_\nu\right) \\ &= \mathcal{L}_\nu\,\delta(i_\nu) + \frac{1}{2}i_{[\nu,\nu]}\,\delta'(i_\nu) \\ &= -\delta(i_\nu)\,\mathcal{L}_\nu - \frac{1}{2}i_{[\nu,\nu]}\delta'(i_\nu)\end{aligned} \tag{C.5}$$

where the second and the third line of equation (C.5) are derived by using relations of (C.6).

The following identities hold:

$$\begin{aligned}[e_\alpha, i_V] &= \alpha(V)\,\mathrm{id} \\ [\mathcal{L}_{V_1}, i_{V_2}] &= i_{[V_1,V_2]} \\ [\mathcal{L}_{V_1}, \mathcal{L}_{V_2}] &= \mathcal{L}_{[V_1 V_2]} \\ [\delta(i_{\nu_1}), \delta(i_{\nu_2})] = [\delta(e_{\alpha_1}), \delta(e_{\alpha_2})] &= 0 \\ [\delta(i_\nu), i_V] = [\delta(e_\beta), e_\alpha] &= 0 \\ [i_{V_1}, i_{V_2}] = [e_{\alpha_1}, e_{\alpha_2}] &= 0 \\ [d, \mathcal{L}_V] &= 0 \\ [d, \Gamma_\nu] &= 0 \\ [\mathcal{L}_\nu, \delta^{(n)}(i_\nu)] &= -i_{[\nu,\nu]}\delta^{(n+1)}(i_\nu) \\ [d, \delta^{(n)}(i_\nu)] &= -\mathcal{L}_\nu\delta^{(n+1)}(i_\nu) - \frac{1}{2}i_{[\nu,\nu]}\delta^{(n+2)}(i_\nu)\,.\end{aligned} \tag{C.6}$$

Finally,

$$\delta(i_\nu)\delta(e_\beta) = -\frac{1}{\beta(\nu)}\,, \qquad \text{on forms annihilated by } i_\nu\,,$$

and similarly

$$\delta(e_\beta)\delta(i_\nu) = \frac{1}{\beta(\nu)}\,, \qquad \text{on forms annihilated by } e_\beta\,.$$

The relation to superstring theory is established by the identifications [16]

$$\begin{aligned}b_n &\leftrightarrow i_{l_n} \\ c_n &\leftrightarrow e_{l^*_{-n}} \\ \beta_n &\leftrightarrow i_{g_n} \\ \gamma_n &\leftrightarrow e_{g^*_{-n}} \\ B(V) &\leftrightarrow i_V \\ T(V) &\leftrightarrow \mathcal{L}_V \\ X_\nu &\leftrightarrow \Gamma_\nu \\ Q &\leftrightarrow d\,,\end{aligned} \tag{C.7}$$

where $\{l^*_n\}$ and $\{g^*_n\}$ represents the dual basis of $\{l_n\}$ and $\{g_n\}$, i.e. $l^*_m(l_n) = \delta_{m,n}$ and $g^*_m(g_n) = \delta_{m,n}$. The identities (C.6) hold also with the replacements of (C.7). The grading in the superconformal field theory is traditionally denoted by $g|p|\alpha$, rather than $r|s|p$, referring to ghost number, picture and Grassmann parity respectively.

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