Non-linear Massive Gravity

Lāsma Alberte

Fakultät für Physik

LMU München

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Lāsma Alberte

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LMU München

Zusammenfassung

Massive Gravitation ist ein theoretisches Modell, welches Gravitation auf kosmologischen Längenskalen modifiziert, und das so eine dynamische Erklärung für die beobachtete Beschleunigung der Expansion des Universums liefern könnte. In dieser Arbeit untersuchen wir verschiedene theoretische Probleme der massiven Gravitation, die wichtig bezüglich der Konsistenz und phänomenologischen Viabilität der Theorie sind.

Es ist bekannt, dass die Vorhersagen der massiven Gravitation auf linearer Ordnung den Vorhersagen der allgemeinen Relativitätstheorie widersprechen. Dies ist jedoch ein Artefakt, das vom Zusammenbruch der perturbativen Entwicklung im masselosen Limes verursacht wird. In unserer Arbeit untersuchen wir dieses Problem in der Diffeomorphismen-invarianten Formulierung der massiven Gravitation, in der der Graviton-Massenterm mit vier skalare Feldern ausgedrückt wird. Wir bestimmen die sogenannte Vainshtein-Skala, unterhalb derer sich die skalaren Moden des massiven Gravitons nichtperturbativ verhalten, für eine große Klasse möglicher Massenterme. Wir finden die asymptotischen Lösungen des sphärisch symmetrischen Gravitationsfeldes inner- und außerhalb des Vainshtein-Radiuses und zeigen, dass massive Gravitation sich unterhalb dieser Skala kontinuierlich der Allgemeinen Relativitätstheorie annähert. Außerdem bestimmen wir die resultierenden Korrekturen zum Newton-Potential.

Im Allgemeinen propagiert in jeder Theorie mit einer nichtlinearen Erweiterung des quadratischen Graviton-Massenterms ein Boulware–Deser Geist. Die einzige solche Theorie, in der der Geist im Hochenergie-Entkopplungslimes nicht propagiert, ist das de Rham–Gabadadze–Tolley Modell. Hier zeigen wir, dass der Geist selbst in dieser Theorie außerhalb des Entkopplungslimes in vierter Ordnung Störungstheorie erscheint. Wir argumentieren dann jedoch, dass der Geist in der voll nichtlinearen Theorie vermeiden werden kann, wenn nicht alle Skalarfelder unabhängige Freiheitsgrade darstellen. In dieser Hinsicht untersuchen wir das einfache Beispiel $(1 + 1)$-dimensionaler massiver Gravitation und finden, dass diese Theorie eine Eichsymmetrie enthält, die die Anzahl der Freiheitsgrade reduziert.

Schließlich verallgemeinern wir den Diffeomorphismen-invarianten Formalismus massiver Gravitation auf allgemeine gekrümmte Hintergründe. Wir finden, dass auf bestimmten Hintergründen die resultierende allgemein kovariante massive Gravitation eine Symmetrie im Konfigurationsraum der skalaren Felder aufweist. Die Symmetrietransformationen der skalaren Felder sind durch die Isometrien der Referenzmetrik gegeben. Insbesondere untersuchen wir massive Gravitation auf de Sitter-Raum in diesem Formalismus. Wir bestätigen das bekannte Ergebnis, dass, im Falle einer Gravitonmasse im Verhältnis zur kosmologischen Konstante von $m^2 = 2\Lambda/3$, die Theorie teilweise masselos ist. Dadurch propagieren in diesem Fall nur vier Freiheitsgrade.

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LMU München

Massive gravity is a particular theoretical model that modifies gravity on cosmological scales and therefore could provide a dynamical explanation for the observed accelerated expansion of our Universe. In this thesis we investigate various theoretical problems of massive gravity, important for its consistency and phenomenological viability.

It is known that the predictions from the linearized massive gravity contradict the predictions of General Relativity. It is, however, an artifact due to the breakdown of the perturbative expansion in the massless limit. In our work we investigate this problem in the diffeomorphism invariant formulation of massive gravity in which the graviton mass term is written in terms of four scalar fields. We determine the so-called Vainshtein scale below which the scalar modes of the massive graviton enter the non-perturbative regime for a wide class of non-linear mass terms. We find the asymptotic solutions of the spherically symmetric gravitational field below and above the Vainshtein radius, and show that massive gravity goes smoothly to the General Relativity below this scale. We also determine the corresponding corrections to the Newton potential.

In general, any non-linear extension of the quadratic graviton mass term propagates the Boulware–Deser ghost. The only theory in which the ghost is not propagating in the high energy decoupling limit, is the de Rham–Gabadadze–Tolley theory. Here we show that the ghost arises in the fourth order of perturbations in this theory away from the decoupling limit. However, we further argue that the ghost can be avoided in the full non-linear theory if not all four scalar fields propagate independent degrees of freedom. In particular, we investigate the simple example of $(1+1)$-dimensional massive gravity and find that the theory exhibits a gauge symmetry, which reduces the number of degrees of freedom.

We also generalize the diffeomorphism invariant formalism of massive gravity to arbitrary curved backgrounds. We find that, given a specific background metric, the resulting generally covariant massive gravity exhibits an internal symmetry in the configuration space of the scalar fields. The symmetry transformations of the scalar fields are given by the isometries of the reference metric. In particular, we investigate massive gravity on de Sitter space in this formalism. We confirm the known result that, in the case when the graviton mass is related to the cosmological constant as $m^2 = 2\Lambda/3$, the theory is partially massless and propagates only four degrees of freedom.

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Contents

1 Introduction **1**

1.1 Dark energy and Cosmological Constant Problem 1

1.2 Infrared modifications of gravity 2

1.3 Massive gravity 3

1.4 Classicalization 6

1.5 Summary 7

2 Construction of the dRGT massive gravity **9**

2.1 Stückelberg trick for massive gravity 9

2.2 Effective field theory for massive gravitons 10

2.2.1 Fierz–Pauli mass term 11

2.2.2 Non-linear graviton mass term and quantum corrections . . 14

2.3 Decoupling limit: raising the cutoff and avoiding the ghosts 15

2.4 Resummation 19

3 Vainshtein mechanism **21**

3.1 vDVZ discontinuity 21

3.2 Vainshtein mechanism in decoupling limit 24

3.3 Smooth limit to General Relativity in Λ_3 theories 27

4 Boulware–Deser ghost **33**

4.1 Appearance of ghost-like terms 34

4.2 Absence of the ghost in unitary gauge 39

4.3 Absence of ghost in Stückelberg formulation 41

4.4 (1+1)-dimensional dRGT gravity 43

5 Massive gravity on curved background **47**

5.1 Arbitrary reference metric 48

5.1.1 Minkowski background 48

5.1.2 Arbitrary background 49

5.2 Massive graviton in de Sitter space 51

5.3 Partially massless graviton 52

6 Summary and Outlook **55**

Publications **59**

A “Massive Gravity: Resolving the Puzzles,”
with A. Chamseddine and V. Mukhanov **61**

B “Massive Gravity: Exorcising the Ghost,”
with A. Chamseddine and V. Mukhanov **81**

C “Reduced Massive Gravity with Two Stückelberg Fields,”
with A. Khmelnitskiy **97**

D “Massive Gravity on Curved Background” **121**

E “Semiclassical Calculation of Multiparticle Scattering Cross Sections in Classicalizing Theories,”
with F. Bezrukov **141**

Curriculum Vitae **153**

Bibliography **159**

1

Introduction

1.1 Dark energy and Cosmological Constant Problem

The advances in precision cosmology since the late 1990s provide us with very precise measurements of the cosmological parameters governing the evolution and the present state of our Universe. In particular, the energy content of the Universe is firmly established by observations. It is known that the usual baryonic matter contribute only a small fraction of the total energy density today, while roughly 95% of the overall energy density is in the form of 'dark' components. The dark component is composed of dark matter, a very weakly interacting form of matter with negligible pressure ($\approx 25\%$), and dark energy, a non-clustering form of energy density with negative pressure ($\approx 70\%$). The best fit model for the observational data based on the recent measurements by the *Planck* experiment gives the values $\Omega_m = 0.314 \pm 0.020$ and $\Omega_\Lambda = 0.686 \pm 0.020$ for the respective contributions to the current energy density of cold dark matter (CDM) together with baryons, and dark energy [1].

The discovery of dark energy was made by measurements of the luminosity-redshift dependence of type IA supernovae which allowed to see that the expansion of the Universe is accelerating and thus has to be driven by an energy component with negative pressure [2, 3]. Since then the existence of dark energy has been further confirmed by the measurements of the anisotropies of the cosmic microwave background (CMB) [1, 4] and measurements of galaxy clustering [5]. The model that provides the so far best fit of all observational data is the ΛCDM model which assumes that the dark energy component is a vacuum energy density or cosmological constant Λ, which has the equation of state $p = -\varepsilon$.

One of the most serious shortcomings of the ΛCDM model is the cosmological constant problem. This problem was known in quantum field theory even before the discovery of dark energy and relies in the fact that anything that contributes to the vacuum energy density behaves as a cosmological constant. In particular, by summing up the zero-point energies of all modes of a free scalar field up to an ultraviolet wavenumber cutoff $\Lambda_{\rm UV}$ yields a vacuum energy density $\sim \Lambda_{\rm UV}^4$. If we impose the ultraviolet cutoff to be of the order of Planck scale then we obtain a vacuum energy density of order $M_{\rm Pl}^4 \sim 10^{76}$ GeV^4. Before the discovery of the accelerated expansion of the Universe the cosmological constant problem was formulated as: "Why does the observed cosmological constant equal to zero?" After the discovery of dark energy, corresponding to a cosmological constant of order $10^{-120} M_{\rm Pl}^4$, the cosmological constant problem is reformulated as: "Why is the observed cosmological constant so small?"

1.2 Infrared modifications of gravity

There are two conceptually different ways to address the cosmological constant problem: degravitation and self-acceleration. Both of them rely on the idea of modifying gravity on cosmological scales. In the degravitation approach the vacuum energy keeps its huge natural value, but the gravity is modified in infrared so that this large wavelength source gravitates very weakly [6, 7, 8]. In the meantime, the short wavelength sources such as matter and radiation gravitate normally. In [6] the graviton propagator is modified non-locally so that the effective Newton's constant becomes wavelength dependent and for long-wavelength sources is tiny. In such case the Newton's constant acts as a high-pass filter by shutting off the gravitational effects (such as curvature) of the vacuum energy.

For the self-acceleration approach, the vacuum energy is instead postulated to be equal to zero, and the gravity is modified in infrared, leading to a dynamic cosmic acceleration at late times [9]. An infrared modification of gravity, in general, invokes new dynamical degrees of freedom which become strongly coupled in the vicinity of a classical source. It is natural for the scenarios of such modifications to introduce additional degrees of freedom of the gravitational field itself. The first successful realization of such a scenario is the Dvali–Gabadadze–Porrati (DGP) brane-world model, which consists of a 4D brane embedded in a 5D Minkowski space [10]. This theory admits a self-accelerating solution with a constant Hubble parameter in the absence of an external matter source [11, 12]. From the point of view of a four-dimensional observer the effective 4D Friedmann equation receives an additional contribution in the form of a cosmological constant which dominates at late times. Unfortunately, the self-accelerating branch of the DGP model is plagued by negative energy ghost-like states [13, 14]. A comprehensive review on the DGP model and other infrared modifications of gravity can be found in [15].

1.3 Massive gravity

In this thesis we focus on massive gravity – a particular theoretical model of infrared modification of gravity which attempts to provide a dynamical explanation for the late time acceleration of our Universe. Studying massive gravity is also motivated by such a fundamental question like whether it is at all possible to consistently modify the Einstein's General Relativity (GR) so that to give a tiny mass to the graviton. Here we give a brief overview of massive gravity pointing out the main consistency problems in historical order. The solutions and current state of these issues will be presented in detail in the main body of the thesis.

The quadratic graviton mass term was proposed already in 1939 by Fierz and Pauli (FP) who found the unique mass term which ensures a unitary propagation at quadratic level [16]. For the metric perturbations around Minkowski background $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$ it takes the form

$$\mathcal{L}_{\rm FP} = \frac{m^2}{8} \int d^4x \, \left(h^2 - h^{\mu\nu} h_{\mu\nu}\right) , \tag{1.1}$$

where $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$. In 1970, it was observed by van Dam, Veltman, and Zakharov (vDVZ) that in linearized Fierz–Pauli massive gravity there is no continuous transition to the General Relativity in the limit of vanishing graviton mass. This effect is known today as the vDVZ discontinuity. It was shown that the helicity-0 component of massive graviton remains coupled to the trace of the energy-momentum tensor of external matter sources even in the limit when graviton mass is sent to zero. Hence, the naive predictions of the linearized Fierz–Pauli theory contradict with the Solar System observations related to the motion of massive objects, like the precession of the Mercury perihelion. In the meantime, the predictions for the light bending by the Sun coincide with General Relativity since the energy-momentum tensor of light is traceless.

It was, however, pointed out by Vainshtein in 1972 that the discontinuity should not persist in the full non-linear theory [17]. The reason for this is that around heavy sources the perturbative expansion in terms of the Newton's constant is singular in the limit of vanishing mass. Therefore the next-to-leading order terms become relevant in this limit, and the truncated theory cannot be trusted anymore. Vainshtein also pointed out that around a static spherically symmetric source of mass M the linear regime breaks down at the distance $R_V = (M/(M_{\rm Pl}^2 m^4))^{1/5}$. It was suggested that the scalar mode of the massive graviton decouples by entering the non-perturbative regime at distances $r < R_V$, and General Relativity is restored in the vicinity of the source. This is known as the Vainshtein mechanism, and R_V is the so-called Vainshtein radius. Although Vainshtein argued that there exist two different expansion regimes above and below Vainshtein radius, his argument does not give a real proof of the fact that General Relativity is indeed restored. It was almost immediately pointed out by Boulware and Deser in 1973 that it is necessary to show that a global solution of the non-linear theory exist which matches both asymptotic regimes [18]. That such a solution does exist for a certain non-linear massive gravity theory was shown only in 2009 by Babichev et al. [19, 20, 21].

In the same paper Boulware and Deser also pointed out that a general non-linear theory of massive gravity propagates six degrees of freedom [18]. This contradicts the well-known fact that a massive spin-2 particle propagates five degrees of freedom according to the representations of the Poincaré group. Moreover, it was shown that this sixth mode inevitably propagates negative norm states, and is therefore called the Boulware–Deser (BD) ghost. For a long time this was thought to be a no-go theorem for massive gravity since it forbids a successful implementation of the Vainshtein mechanism for which the presence of the non-linear interaction terms is crucial. An important step in understanding the ghost problem and the closely related strong coupling problem of massive gravity was made by Arkani-Hamed et al. in 2002 [22]. They implemented the effective field theory view point for the theory of a massive spin-2 field. The general covariance of General Relativity is broken in massive gravity by the graviton mass term. By analogy to the effective theory of massive spin-1 fields it was shown how the general covariance can be restored by introducing four additional fields, the so-called Stückelberg fields. [1] This method allows for a better understanding of the origin of the BD ghost, as well as makes it easy to determine the strong coupling scale of the effective field theory. Since the Stückelberg fields can be viewed as the analogue of the Goldstone bosons in gauge theories, it makes it possible to separate the relevant interactions at different energy scales. In particular, an appropriate decoupling limit capturing the leading interactions of the longitudinal modes of the massive graviton was proposed in this language. As a result, both the BD ghost and the strong coupling of the non-linear effective theory was traced to arise due to the higher derivative interaction terms of the helicity-0 mode of the graviton. The strong coupling scale of the particular massive gravity theory investigated in [22] was shown to be a disappointingly low scale $\Lambda_5 \equiv (m^4 M_{\rm Pl})^{1/5} \sim (10^{11}\,{\rm km})^{-1}$. The authors pointed out a procedure of how an order-by-order addition of higher interactions would raise the strong coupling scale of massive gravity to the highest possible scale $\Lambda_3 = (m^2 M_{\rm Pl})^{1/3} \sim (10^3\,{\rm km})^{-1}$.

The order-by-order construction of the non-linear theory of massive gravity with a higher cutoff scale was performed in [27] up to the quintic order in $h_{\mu\nu}$. Moreover, it was shown that the remaining scalar-tensor interaction terms in the decoupling limit contain at most two time derivatives and are thus free of the Boulware–Deser ghost. This theory was resummed in terms of infinite series by de Rham, Gabadadze, and Tolley (dRGT) in 2010 and up to now is the only potentially healthy non-linear massive gravity theory [28].

In [18] it was also pointed out that any massive gravity theory contains a given reference metric $f_{\mu\nu}$ as an absolute, non-dynamical object. The most natural choice of the reference metric is the Minkowski metric, which is employed also in the dRGT theory. However, around any other background the quadratic Fierz–Pauli theory propagates the Boulware–Deser ghost [18, 29]. This makes it impossible to use a theory, constructed with a Minkowski reference metric to describe a massive graviton, propagating five degrees of freedom with equal mass around an arbitrary curved background. Instead, the reference metric $f_{\mu\nu}$ has to be chosen to coincide

[1] In [22], however, the set of the four fields was mistakenly said to form a vector field. That the four Stückelberg fields are actually four space-time scalars was clarified in [23, 24, 25, 26].

with the preferred background metric. In particular, for applications in cosmology the homogeneous and isotropic Friedmann-Robertson-Walker (FRW) backgrounds are of great importance. The quadratic massive gravity on the maximally symmetric de Sitter background was first studied in 1986 by Higuchi [30]. It was found that this theory has the interesting feature that the propagated number of degrees of freedom can vary throughout the parameter space of the graviton mass m and the cosmological constant Λ. More precisely, the scalar mode of the massive graviton ceases to propagate at the special point $m^2 = 2\Lambda/3$ leaving only four propagating degrees of freedom [30, 31]. The reason for this phenomenon is that at this point in the parameter space the theory enjoys an additional gauge symmetry reducing the number of degrees of freedom [32, 33]. This theory is referred to as *partially massless*, and the existence of the gauge symmetry bounds the value of the cosmological constant to the value of the graviton mass. If the additional symmetry could be extended to the full non-linear theory, the partially massless gravity could reduce the problem of a small cosmological constant to a less severe problem of a small graviton mass [34]. It is therefore a very interesting and open subject which we will discuss in a later chapter.

1.4 Classicalization

Another interesting topic we would like to mention briefly in this thesis is the ultraviolet (UV) completion of non-renormalizable derivatively coupled effective field theories. In such theories the self-coupling of the degrees of freedom grow with the inverse wave-length. Quantum mechanically the coupling of these quanta becomes strong at the center of mass energies larger than the inverse of the cutoff length L_*. Thus the scattering of two highly energetic particles violate perturbative unitarity. The standard Wilsonian approach to UV-completion is based on the existence of weakly coupled degrees of freedom at all scales. The strong coupling is then considered to be an artifact of missing weakly coupled degrees of freedom. Recently an alternative concept of non-Wilsonian self completion was proposed by Dvali et al. in [35], and further investigated in [36, 37, 38, 39]. According to this point of view a given theory may self-complete without the need of new weakly coupled degrees of freedom. Instead their role is played by a multi-particle state composed of soft original quanta. This phenomenon is termed *classicalization*, and theories which are expected to exhibit this behavior we term *classicalizing* theories. One of the simplest examples of a classicalizing theory is the scalar Dirac-Born-Infeld (DBI) theory, as suggested in [37]. The idea of classicalization suggests that the scattering process of two highly energetic particles is dominated by production of a state with many soft quanta of wave-length r_*. This length scale r_* can be understood classically as the shortest distance down to which a spherical wave can propagate freely before being rescattered by self-interaction. The defining property of classicalizing theories is that the r_* grows with the energy.

In our work [40] we have attempted to check the conjecture that high energy scatterings should be dominated by multi-particle production in classicalizing theories. In particular, we apply a semiclassical technique in order to calculate the total transition rate from an initial few particle state to a state with large number of particles in the DBI theory. We find that for a fixed above-cutoff total energy $E > L_*^{-1}$ in the final state, the scattering process with large number of particles in the final state N is exponentially suppressed. Unexpectedly, the semiclassical method gives exponentially large cross section for small particle numbers in the final state $N < N_{\rm crit} = (EL_*)^{4/3}$. Interestingly, we see that this transition happens for the particles of wavelength $\lambda \sim N_{\rm crit}/E$ which coincides with the r_* radius predicted by Dvali et al. [35, 37]. Since the topic of classicalization is not directly related to the main subject of this thesis, we will not discuss this part of our research in greater detail here. The corresponding paper [40] can be found in appendix E.

1.5 Summary

In this thesis we investigate the various theoretical aspects of massive gravity. The subsequent chapters give a more detailed account on the consistency problems in massive gravity introduced above and their solutions. The chapter 2 is devoted to the construction of the dRGT theory of massive gravity since it is the theoretical framework of the rest of the thesis. The further chapters contain a review of the recent developments in the most important theoretical topics in massive gravity. We shall also accordingly present the main results obtained in our papers [41, 42, 43, 44]. In particular, in chapter 3 the different interpretations of the Vainshtein mechanism in massive gravity are discussed, and our results based on the publication [41] are presented. We determine the Vainshtein radius in non-linear massive gravity theories with different strong coupling scales and argue in which energy region the classical non-linear regime is reliable. We also calculate the corresponding corrections to the Newton's potential within the Vainshtein region around static spherically symmetric sources. The appearance of the Boulware–Deser ghost in the perturbative expansion of the dRGT massive gravity is investigated in chapter 4. In particular, we show how ghost-like terms appear in the fourth order of the dRGT theory away from the decoupling limit. Although the mass of the ghost is shown to be lighter than the corresponding quartic order strong coupling scale, we argue how this problem can be avoided by adding higher order interaction terms. We also present the arguments for the absence of the BD ghost in Stückelberg formalism available in the literature. To illustrate the absence of the ghost degree of freedom we perform a full Hamiltonian analysis in the simple example of $(1+1)$-dimensional dRGT massive gravity. Our results presented in this chapter are based on our findings in [42, 43]. The chapter 5 is devoted to massive gravity on curved backgrounds, and presents our results obtained in [44]. We first discuss how the general covariance can be restored in massive gravity around arbitrary background and how the dRGT theory should be adjusted in this case. We then demonstrate that the resulting diffeomorphism invariant theory exhibits different internal symmetries in the scalar field space and thus corresponds to a theory fundamentally different from the original dRGT theory. As an example, we show how to construct the generally covariant massive gravity on de Sitter space. As a consistency check we recover the previously known results concerning the partial masslessness of the de Sitter massive gravity. Chapter 6 provides a short summary of our results and the current state of the dRGT massive gravity. The full versions of our papers [40, 41, 42, 43, 44] are attached in the appendices A,B,C,D,E respectively.

2

Construction of the dRGT massive gravity

2.1 Stückelberg trick for massive gravity

The unique quadratic graviton mass term which ensures unitary propagation of the graviton is the Fierz–Pauli (FP) mass term, and for metric fluctuations around Minkowski background $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$ it takes the form [16]

$$\mathcal{L}_{FP} = \frac{m^2 M_{\rm Pl}^2}{8} \left(h^2 - h^{\mu\nu} h_{\mu\nu} \right) . \tag{2.1}$$

When added to the Einstein-Hilbert action this mass term breaks the diffeomorphism invariance of General Relativity due to the explicit dependence on the background reference metric $\eta_{\mu\nu} = \text{diag}\,(+,-,-,-)$. The general covariance can be restored by the so-called Stückelberg trick which relies on the idea of introducing four scalar fields $\phi^A,\ A = 0, 1, 2, 3$ corresponding to the four broken diffeomorphism transformations [22, 23, 24, 25, 26]. The mass term for metric perturbations is then built from various combinations of the variables [23, 26]

$$\bar{h}^{AB} = g^{\mu\nu} \partial_\mu \phi^A \partial_\nu \phi^B - \eta^{AB}, \tag{2.2}$$

where η^{AB} is the Minkowski metric in the configuration space of the scalar fields. This composite field is a scalar with respect to diffeomorphism transformations. On Minkowski background the scalar fields ϕ^A acquire vacuum expectation values proportional to Cartesian spacetime coordinates $\phi_0^A = x^\mu \delta_\mu^A$. The diffeomorphism invariance is thus spontaneously broken and the scalar field perturbations $\chi^A \equiv \phi^A - \phi_0^A$ induce four additional degrees of freedom. In combination with the two degrees of freedom of the massless graviton the theory in general propagates six degrees of freedom. Five of them constitute the five degrees of freedom of a massive spin-2 particle in agreement with the Poincaré invariance. The sixth degree of freedom is ghost-like and is the famous Boulware–Deser ghost [18]. In quadratic order the ghost is canceled by the special choice of the Fierz–Pauli mass term as given below.

In unitary gauge, when $\chi^A = 0$, the variables $\bar{h}^{AB}$ are equal to metric perturbations since $\bar{h}^{AB} = \delta_\mu^A \delta_\nu^B h^{\mu\nu}$. Thus the diffeomorphism invariance of General Relativity is restored by replacing $h^{\mu\nu} \to \bar{h}^{AB}$ in the FP mass term (2.1). This leads to the following action of the scalar fields:

$$S_\phi = \frac{m^2 M_{\rm Pl}^2}{8} \int d^4x \sqrt{-g} \left(\bar{h}^2 - \bar{h}_B^A \bar{h}_A^B \right), \tag{2.3}$$

which around the symmetry breaking background gives the quadratic FP mass term for metric perturbations. Since the field $\bar{h}_B^A \equiv \bar{h}^{AC} \eta_{BC}$ transforms as a scalar under general coordinate transformations, this Lagrangian is manifestly diffeomorphism invariant. Moreover, since the Latin indices in the action are contracted, it is invariant also under the isometries of the metric η_{AB}, namely the Lorentz transformations Λ_B^A in the scalar field space. Hence the scalar field indices A, B are raised and lowered with η_{AB}.

It is important to notice that due to the definition (5.2) the above action contains terms up to the sixth order in perturbations $h^{\mu\nu}$ and χ^A. It is therefore the simplest diffeomorphism invariant non-linear graviton mass term, which in quadratic order gives the FP mass term for metric perturbations. Henceforth, we will refer to (2.3) as the *non-linear Fierz–Pauli mass term.*

2.2 Effective field theory for massive gravitons

From the field theoretic point of view massive gravity can be regarded as an effective field theory for an interacting massive spin-2 particle. As any effective field theory, also massive gravity is not valid up to arbitrary large energy scales, but has a UV cutoff, which as we will see below depends on the exact form of the non-linear completion of the quadratic Fierz–Pauli mass term. In this section we shall follow closely the work of Arkani-Hamed et al. in which massive gravity was discussed in the effective field theory framework for the first time [22].

2.2.1 Fierz–Pauli mass term

We shall start by discussing the effective theory of graviton in flat Minkowski space with the quadratic Fierz–Pauli mass term given in (2.1). As discussed in the previous section the diffeomorphism invariance, broken by the mass term, is restored by introducing the four Stückelberg fields corresponding to the four coordinate transformations. As in [22] we define a spacetime tensor

$$H_{\mu\nu} = g_{\mu\nu} - \eta_{AB}\partial_\mu\phi^A\partial_\nu\phi^B \tag{2.4}$$

which, similarly to $\bar{h}^{AB}$, around the Minkowski background reduces to the metric perturbations $h_{\mu\nu}$ in the unitary gauge where $\phi^A = \delta^A_\mu x^\mu$. In distinction from (5.2), here the Stückelberg trick is implemented by parametrizing the absolute background metric $\eta_{\mu\nu}$ as

$$\eta_{\mu\nu} \to \partial_\mu\phi^A\partial_\nu\phi^B\eta_{AB} \tag{2.5}$$

Thus, $H_{\mu\nu}$ is a spacetime tensor and its indices are raised and lowered with the spacetime metric $g_{\mu\nu}$. In the meantime $\bar{h}^{AB}$ is a spacetime scalar and its indices are moved with the Minkowski metric η_{AB}. The traces of the fields $H_{\mu\nu}$ and $\bar{h}^{AB}$ are defined as

$$[H] = g^{\mu\nu}H_{\mu\nu}, \quad \left[H^2\right] = g^{\mu\nu}g^{\alpha\beta}H_{\mu\alpha}H_{\nu\beta}, \ldots \tag{2.6}$$

$$\left[\bar{h}\right] = \bar{h}^{AB}\eta_{AB}, \quad \left[\bar{h}\right]^2 = \bar{h}^A_B\bar{h}^B_A = \bar{h}^{AC}\bar{h}^{BD}\eta_{BC}\eta_{AD}. \tag{2.7}$$

and coincide up to a sign $\left[\bar{h}^n\right] = (-1)^n\left[H^n\right]$ [45]. The diffeomorphism invariant Fierz–Pauli theory of massive graviton is then given by the action

$$S = -\frac{1}{2}M_{\rm Pl}^2\int d^4x\,\sqrt{-g}R + \frac{m^2M_{\rm Pl}^2}{8}\int d^4x\,\sqrt{-g}\left([H]^2 - \left[H^2\right]\right). \tag{2.8}$$

Since this mass term is equivalent to (2.3) we shall also refer to this as the *non-linear Fierz–Pauli massive gravity*. The Stückelberg (or, alternatively, also called Goldstone) formulation turns out to be very useful to illuminate the interactions of the longitudinal helicity-1 and helicity-0 components of the massive graviton. To see this we expand the scalar fields as $\phi^A = x^\mu \delta^A_\mu + \chi^A$, where χ^A under spacetime diffeomorphisms transform as perturbations of scalar fields. Hence, (2.8) describes a massless graviton $h_{\mu\nu}$ together with a set of fields χ^A. If the spacetime metric is expanded around the Minkowski background the theory given by (2.8) enjoys a global spacetime Lorentz symmetry. Under this symmetry the field $\chi^\mu \equiv \delta^\mu_A \chi^A$ transforms as a vector. By having this in mind one can decompose χ^μ into the transverse helicity-1 and helicity-0 modes. In turn, the scalar perturbations χ^A are decomposed as

$$\chi^A = \delta^A_\mu \chi^\mu \equiv \delta^A_\mu \eta^{\mu\nu} \chi_\nu \equiv \delta^A_\mu \eta^{\mu\nu}(A_\nu + \partial_\nu \pi). \tag{2.9}$$

We note that the fields A_ν and π in this decomposition are not well-defined from the point of view of the spacetime diffeomorphisms. Moreover, this decomposition involves time derivative of the field π and might lead to the appearance of additional time derivatives in the action. More precisely, the spacetime tensor $H_{\mu\nu}$ defined in (2.4) now becomes

$$\begin{aligned} H_{\mu\nu} = h_{\mu\nu} &+ \partial_\mu A_\nu + \partial_\nu A_\mu + 2\partial_\mu \partial_\nu \pi - \partial_\mu A^\alpha \partial_\nu A_\alpha - \\ &- \partial_\mu A^\alpha \partial_\nu \partial_\alpha \pi - \partial_\mu \partial^\alpha \pi \partial_\nu A_\alpha - \partial_\mu \partial^\alpha \pi \partial_\nu \partial_\alpha \pi, \end{aligned} \tag{2.10}$$

and it is apparent that the field π here involves second order derivatives. However, there is an accidental $U(1)$ symmetry of $H_{\mu\nu}$, given as $A_\mu \to A_\mu + \partial_\mu \Lambda$, $\pi \to \pi - \Lambda$. Hence, the field π is pure gauge and can be set back to zero. Therefore, the total number of degrees of freedom propagated by χ^A and $\{A_\mu, \pi\}$ should be the same. In the rest of this chapter we will formally work with the decomposition (2.9), (2.10) ignoring the fact that A_μ is not a *spacetime* vector. In our work [41], we have, however, shown that all the results obtained in this chapter can also be found working in terms of the well-defined fields χ^A with no use of the splitting (2.9).

The field $H_{\mu\nu}$ in (2.10) coincides[1] with the helicity decomposition of a spin-2

[1] Up to a term needed for the diagonalization of the kinetic terms, for details see [46].

field into a helicity-2, helicity-1 and helicity-0 modes. The fields A_μ and π are analogous to the Goldstone fields in gauge theories carrying the degrees of freedom of the broken diffeomorphism invariance. In analogy to the equivalence theorem in gauge theories [47], the dynamics of the longitudinal vector and scalar polarizations of the massive graviton are described by the dynamics of the fields A_μ and π, at energies much higher than the graviton mass.

In what follows we shall focus on the strongest interactions that cause the breakdown of perturbative unitarity. At energies much higher than the graviton mass these are the interactions of the longitudinal graviton modes or, equivalently, the interactions of the Goldstone fields A_μ and π defined in (2.9). The field A_μ has a kinetic term $(mM_{\rm Pl})^2(\partial_\nu A_\mu - \partial_\mu A_\nu)^2$ whereas the field π acquires a kinetic term only through the mixing with the graviton $m^2 M_{\rm Pl}^2\,(h\Box\pi - h^{\mu\nu}\partial_\mu\partial_\nu\pi)$. The scalar and tensor sector can be un-mixed by a field redefinition of the graviton $h_{\mu\nu} \to h_{\mu\nu} - \eta_{\mu\nu} m^2\pi$. After doing this, in order to focus on the strongest interactions we set $h_{\mu\nu} = 0$ in the expansion of the action (2.8). The leading interactions are schematically

$$\mathcal{L}_\pi \supset -\frac{3}{4}\pi\Box\pi + \frac{(\partial^2\pi)^3}{m^4 M_{\rm Pl}} + \frac{(\partial^2\pi)^4}{(m^3 M_{\rm Pl})^2} + \frac{\partial^2\pi\partial A\partial A}{m^2 M_{\rm Pl}}, \tag{2.11}$$

where we have canonically normalized the fields by replacing

$$A_\mu \to \frac{A_\mu}{mM_{\rm Pl}}, \quad \pi \to \frac{\pi}{m^2 M_{\rm Pl}}. \tag{2.12}$$

The lowest strong coupling scale is that of the cubic self-interactions of the π field and is $\Lambda_5 \equiv (m^4 M_{\rm Pl})^{1/5}$. This is a very low scale which for the graviton mass of the order of the Hubble scale $m \sim (10^{28}\ \text{cm})^{-1}$ corresponds to the scale $\Lambda_5 \sim 10^{-20}\ \text{eV} \sim (10^{11}\ \text{km})^{-1}$. Hence this effective theory of massive gravity breaks down below the distances slightly larger than the size of the Solar System ($\sim 4.5 \cdot 10^9$ km). Around heavy sources the effective theory of massive gravity breaks down at even larger distances. This means that the effective theory (2.8) cannot describe gravity within our Solar System. Moreover, the presence of the higher order derivative terms in (2.11) implies ghosts and violations of unitarity in the non-linear Fierz–Pauli theory (2.8). Hence, this theory besides of having a very low UV cutoff is plagued by the Boulware–Deser ghost and is not a satisfactory non-linear massive gravity theory.

2.2.2 Non-linear graviton mass term and quantum corrections

It was pointed out in [22], that by adding to the Fierz–Pauli mass term higher order interactions in $h_{\mu\nu}$ the self-interaction terms for π could be canceled. Indeed, after Stückelberg-alizing the theory via the substitution $h_{\mu\nu} \rightarrow H_{\mu\nu}$ the cubic self-interactions of π can be canceled by tuning the coefficients in front of the H^3 terms in the graviton mass potential. After order-by-order elimination of the self-interactions of the longitudinal helicity-0 graviton mode, the strongest interactions are of the form $(\partial A)^p(\partial^2\pi)^q$. After canonical normalization one sees that the interactions become strongly coupled at the scale [22]

$$(m^{p+2q-2}M_{\rm Pl}^{p+q-2})^{\frac{1}{3q+2p-4}}. \tag{2.13}$$

Due to the $U(1)$ symmetry of $H_{\mu\nu}$ the field ∂A can only appear in the graviton mass term in the anti-symmetric combination $F_{\mu\nu}$. Hence, there are no interaction terms with a single ∂A and $(\partial^2\pi)^q$, i.e. $p \geq 2$. The highest possible cutoff scale can therefore be achieved in the full non-linear theory when all the infinitely many expansion terms are known, i.e. when $q \rightarrow \infty$. This corresponds to the scale $\Lambda_3 = (m^2 M_{\rm Pl})^{1/3}$. At this scale infinitely many operators are generated in the effective field theory at quantum level. One therefore has to include all the operators consistent with the symmetries. In unitary gauge it means that one should include operators of the form

$$c_{p,q}\partial^q h^p \tag{2.14}$$

where the coefficients $c_{p,q}$ give the strength of interactions. In order to establish the size of the coefficients $c_{p,q}$ we write down the allowed structure of the operators for the longitudinal helicity-0 mode of the graviton π. Due to the shift symmetry of the Stückelberg fields ϕ^A, the helicity-0 field always appears with two derivatives and hence the allowed operators are of the form [22, 48]

$$\frac{\partial^q(\partial^2\pi^c)^p}{\Lambda_3^{3p+q-4}}, \tag{2.15}$$

where the superscript c indicates that we mean the canoncially normalized field $\pi^c = \Lambda_3^3 \pi$. In Lagrangian the field π arises from different powers of $H_{\mu\nu} \supset \partial_\mu \partial_\nu \pi$. In unitary gauge $H_{\mu\nu} = h_{\mu\nu}$, and thus by substituting $\partial^2 \pi^c \to \Lambda_3^3 h$ in the operators (2.15) one can determine the size of the coefficients $c_{p,q}$ in (2.14) to be

$$c_{p,q} \sim \Lambda_3^{4-q} = (m^2 M_{\text{Pl}})^{(4-q)/3}. \tag{2.16}$$

The quantum operators (2.14) involve also a general quadratic mass terms for h with $q = 0$, $p = 2$, which would disturb the special form of the Fierz–Pauli mass term. However, the coefficient in front of these quadratic quantum operators, $c_{2,0} = (m^2 M_{\text{Pl}})^{4/3}$, is much smaller than the Fierz–Pauli coefficient $m^2 M_{\text{Pl}}^2$, and the unitary violating effect hits in only above the strong coupling cutoff Λ_3 [22]. The same holds also for all the further specific choices of the coefficients in front of the higher order interaction terms. Hence, the effect from the quantum operators not of the special form is small, and the special choices made for the coefficients in the non-linear potential are therefore said to be *technically natural*. We thus have a reasonable effective theory for a massive graviton below the energy cutoff Λ_3. Although it is a higher energy scale than the initial Λ_5, it is still quite low, i.e. $\Lambda_3 \sim 10^{-10}$ eV $\sim (10^3 \text{ km})^{-1}$. Hence a UV completion describing the short distance physics as successfully as General Relativity is needed for the effective field theory of massive gravity. The transition to General Relativity in the presence of spherically symmetric static sources is maintained via the Vainshtein mechanism [17] which will be the subject of the next chapter.

2.3 Decoupling limit: raising the cutoff and avoiding the ghosts

The explicit construction of the above mentioned non-linear theory of massive gravity with the strong coupling scale given by Λ_3 was done by de Rham, Gabadadze, and Tolley (dRGT). The non-linear potential of the massive graviton $\mathcal{U}(g, H)$ up to quintic order in $H_{\mu\nu}$ was found by an explicit order-by-order construction in the decoupling limit [27]. The full non-linear resummation of the theory was later found in [28]. Here we shall briefly present the main steps and resulting formulae.

The most general diffeomorphism invariant Lagrangian of a massive spin-2 field can be written as infinite series in the tensor field $H_{\mu\nu}$ defined in (2.4) and takes the form

$$\mathcal{L} = -\frac{1}{2}M_{\rm Pl}^2\sqrt{-g}R - \frac{M_{\rm Pl}^2 m^2}{8}\sqrt{-g}\left[\mathcal{U}_2(g,H) + \mathcal{U}_3(g,H) + \mathcal{U}_4(g,H) + \dots\right] \quad (2.17)$$

where the most general potential terms $\mathcal{U}_n$ at the n-th order in the field $H_{\mu\nu}$ read[1]

$$\begin{aligned} \mathcal{U}_2(g,H) &= \left[H^2\right] - [H]^2, \\ \mathcal{U}_3(g,H) &= c_1\left[H^3\right] + c_3[H]\left[H^2\right] + c_3[H], \\ \mathcal{U}_4(g,H) &= d_1\left[H^4\right] + d_2[H]\left[H^3\right] + d_3\left[H^2\right]\left[H^2\right] + d_4[H]^2\left[H^2\right] + d_5[H]^4, \\ &\vdots \end{aligned}$$

The square brackets here represent the traces as in (2.6), and c_i, d_i are arbitrary coefficients, which need to be determined. In what follows we shall focus only on the interactions of the helicity-0 and helicity-2 modes by setting the vector modes of the field to zero. The expansion (2.10) of the tensor field $H_{\mu\nu}$ in terms of the canonically normalized modes $h_{\mu\nu} \to h_{\mu\nu}/M_{\rm Pl}$ and $\pi \to \pi/\Lambda_3^3$ then becomes

$$H_{\mu\nu} = \frac{h_{\mu\nu}}{M_{\rm Pl}} + \frac{2}{\Lambda_3^3}\partial_\mu\partial_\nu\pi - \frac{\partial_\mu\partial^\alpha\pi\partial_\nu\partial_\alpha\pi}{\Lambda_3^6}. \quad (2.18)$$

The self-interactions of the helicity-0 mode π at the n-th order in non-linearities are schematically of the form

$$\mathcal{L}_\pi^{(n)} \sim \frac{(\partial^2\pi)^n}{M_{\rm Pl}^{n-2}m^{2(n-1)}}. \quad (2.19)$$

The corresponding energy scale below which the different interaction terms are suppressed grows with the order of interactions as $\Lambda_5 = (M_{\rm Pl}m^4)^{1/5}$, $\Lambda_4 = (M_{\rm Pl}m^3)^{1/4}$, $\Lambda_{11/3} = (M_{\rm Pl}^3 m^8)^{1/11}$, etc. As discussed in the previous section the highest possible strong coupling scale in non-linear massive gravity is achieved in the absence of all the self-interactions of π and is the $\Lambda_3 = (M_{\rm Pl}m^2)^{1/3}$. In [27] it was shown that it is possible to fix, order-by-order, the coefficients c_i, d_i in the

[1] When written in terms of the field $\bar{h}^{AB}$ the odd coefficients c_i need to be taken of the opposite sign since $\left[\bar{h}^n\right] = (-1)^n\left[H^n\right]$.

potentials $\mathcal{U}_3$, $\mathcal{U}_4$ so that the interactions (2.19) form a total derivative at the corresponding order. By doing so the energy cutoff scale at which the leading helicity-0 self-interactions arise is raised. At each order n there is a unique combination, $\mathcal{L}^{(n)}_{\rm tot}$, giving a total derivative. Written in terms of $\Pi_{\mu\nu} \equiv \partial_\mu\partial_\nu\pi$, the total derivative combinations are

$$\mathcal{L}^{(0)}_{\rm tot} = 1, \qquad \mathcal{L}^{(1)}_{\rm tot} = \langle\Pi\rangle\,, \tag{2.20}$$

$$\mathcal{L}^{(2)}_{\rm tot} = \langle\Pi\rangle^2 - \langle\Pi^2\rangle\,, \tag{2.21}$$

$$\mathcal{L}^{(3)}_{\rm tot} = \langle\Pi\rangle^3 - 3\langle\Pi\rangle\langle\Pi^2\rangle + 2\langle\Pi^3\rangle\,, \tag{2.22}$$

$$\mathcal{L}^{(4)}_{\rm tot} = \langle\Pi\rangle^4 - 6\langle\Pi^2\rangle\langle\Pi\rangle^2 + 8\langle\Pi^3\rangle\langle\Pi\rangle + 3\langle\Pi^2\rangle^2 - 6\langle\Pi^4\rangle\,, \tag{2.23}$$

where $\langle\Pi\rangle = \eta^{\mu\nu}\Pi_{\mu\nu}$, $\langle\Pi^2\rangle = \eta^{\mu\nu}\eta^{\alpha\beta}\Pi_{\mu\alpha}\Pi_{\nu\beta}$, etc. Equivalently these terms can be written as the contractions with the totally antisymmetric tensors as [49, 50]

$$\mathcal{L}^{(2)}_{\rm tot} = \frac{1}{2}\varepsilon^{\mu_1\mu_2\alpha\beta}\varepsilon^{\nu_1\nu_2}{}_{\alpha\beta}\Pi_{\mu_1\nu_1}\Pi_{\mu_2\nu_2}\,, \tag{2.24}$$

$$\mathcal{L}^{(3)}_{\rm tot} = \varepsilon^{\mu_1\mu_2\mu_3\alpha}\varepsilon^{\nu_1\nu_2\nu_3}{}_{\alpha}\Pi_{\mu_1\nu_1}\Pi_{\mu_2\nu_2}\Pi_{\mu_3\nu_3}\,, \tag{2.25}$$

$$\mathcal{L}^{(4)}_{\rm tot} = \varepsilon^{\mu_1\mu_2\mu_3\mu_4}\varepsilon^{\nu_1\nu_2\nu_3\nu_4}\Pi_{\mu_1\nu_1}\Pi_{\mu_2\nu_2}\Pi_{\mu_3\nu_3}\Pi_{\mu_4\nu_4}\,, \tag{2.26}$$

where $\varepsilon_{\mu_1\mu_2\mu_3\mu_4}$ is the Levi-Cvita tensor in Minkowski space. In four spacetime dimensions, all the higher order terms vanish identically due to the antisymmetry properties of the Levi-Civita tensor, giving $\mathcal{L}^{(n>4)}_{\rm tot} \equiv 0$. By tuning the potential so that at each order the interactions of π form a total derivative, all the dangerous self-interactions of π at energy scale below Λ_3 disappear. Hence, in the final theory the high energy behavior of the helicity-0 mode of the graviton is captured in the following decoupling limit

$$m \to 0, \quad M_{\rm Pl} \to \infty, \quad \Lambda_3 = \text{fixed}. \tag{2.27}$$

It was found in [27] that up to the quartic order the strong interactions arising at the scales $\Lambda < \Lambda_3$ can be removed by the following choice of coefficients

$$c_1 = 2c_3 + \frac{1}{2}, \quad c_2 = -3c_3 - \frac{1}{2}, \tag{2.28}$$

$$d_1 = -6d_5 + \frac{1}{16}(24c_3 + 5), \quad d_2 = 8d_5 - \frac{1}{4}(6c_3 + 1), \tag{2.29}$$

$$d_3 = 3d_5 - \frac{1}{16}(12c_3 + 1), \quad d_4 = -6d_5 + \frac{3}{4}c_3. \tag{2.30}$$

One then arrives at the following *exact* decoupling limit Lagrangian

$$\mathcal{L}_{\rm DL} = \mathcal{L}_h^{(2)} + \mathcal{L}_{h\pi}, \tag{2.31}$$

where $\mathcal{L}_h^{(2)}$ is the quadratic Einstein-Hilbert term

$$\mathcal{L}_h^{(2)} = \frac{1}{4}\left[\frac{1}{2}\partial_\lambda h^{\mu\nu}\partial^\lambda h_{\mu\nu} - \frac{1}{2}\partial_\lambda h^\mu_\mu \partial^\lambda h^\nu_\nu - \partial_\lambda h^{\lambda\nu}\partial^\mu h_{\mu\nu} + \partial^\nu h^\lambda_\lambda \partial^\mu h_{\mu\nu}\right], \tag{2.32}$$

and $\mathcal{L}_{h\pi}$ describes the interactions between the helicity-2 and helicity-0 modes of the graviton. As a result of the above choice of coefficients $\mathcal{L}_{h\pi}$ is given by

$$\mathcal{L}_{h\pi} = \frac{1}{2}h^{\mu\nu}\left(X^{(1)}_{\mu\nu} + \frac{1}{\Lambda_3^3}X^{(2)}_{\mu\nu} + \frac{1}{\Lambda_3^6}X^{(3)}_{\mu\nu}\right), \tag{2.33}$$

where

$$X^{(1)}_{\mu\nu} = \frac{1}{2}\frac{\partial\mathcal{L}^{(2)}_{\rm tot}}{\partial\Pi^{\mu\nu}}, \quad X^{(2)}_{\mu\nu} = \frac{1}{6}(6c_3 - 1)\frac{\partial\mathcal{L}^{(3)}_{\rm tot}}{\partial\Pi^{\mu\nu}}, \quad X^{(3)}_{\mu\nu} = -\frac{1}{4}(c_3 + 8d_5)\frac{\partial\mathcal{L}^{(4)}_{\rm tot}}{\partial\Pi^{\mu\nu}}.$$

We notice that

$$X^{(n)}_{\mu\nu} \propto \frac{\partial\mathcal{L}^{(n+1)}_{\rm tot}}{\partial\Pi^{\mu\nu}}, \tag{2.34}$$

and in combination with the earlier remark that in four dimensions $\mathcal{L}^{(n>4)}_{\rm tot} \equiv 0$, we conclude that the fifth and higher order interactions vanish in the decoupling limit, i.e. $X^{(n>3)}_{\mu\nu} \equiv 0$. Hence the decoupling limit interaction term (2.33) is *exact*.

One can check that the interaction terms $X^{(n)}_{\mu\nu}$ are conserved, i.e. that $\partial^\mu X^{(n)}_{\mu\nu} = 0$. Moreover, each of the components of $X^{(n)}_{\mu\nu}$ bears no more than two time derivatives. This ensures that there are no ghost instabilities arising in the decoupling limit of the non-linear massive gravity. Hence, by the above construction one has achieved the following. First, by tuning the non-linear interactions of $H_{\mu\nu}$ up to quartic order so that all the scalar self-interactions enter the Lagrangian only in the combinations of total derivatives the UV cutoff is raised to $\Lambda_{11/3} = (m^8 M_{\rm Pl}^3)^{1/11}$. Moreover, the resulting decoupling limit action describing the interactions of the helicity-2 and helicity-0 modes of the massive graviton is ghost-free due to the cancelation of the dangerous higher order derivative interaction terms.

2.4 Resummation

In [28] the non-linear massive gravity potential $\mathcal{U}(g,H)$ was resummed in terms of the field

$$\mathcal{K}^\mu_\nu = \delta^\mu_\nu - \sqrt{\delta^\mu_\nu - H^\mu_\nu} = -\sum_{n=1}^{\infty} d_n \left[H^n\right]^\mu_\nu \equiv \delta^\mu_\nu - \sqrt{g^{\mu\lambda} f_{\lambda\nu}}, \tag{2.35}$$

where $f_{\mu\nu} \equiv \partial_\mu \phi^A \partial_\nu \phi^B \eta_{AB}$, and the polynomial coefficients are given by the coefficients of the Taylor expansion of the square root $\sqrt{1-x}$ in the powers of x by

$$d_n = \frac{(2n)!}{(1-2n)(n!)^2 2^{2n}}. \tag{2.36}$$

The square root matrix is defined so that $\left(\sqrt{g^{-1}f}\right)^\mu_\lambda \left(\sqrt{g^{-1}f}\right)^\lambda_\nu = g^{\mu\lambda} f_{\lambda\nu}$. The resulting full non-linear action for massive gravity takes the form

$$\mathcal{L}_{\rm dRGT} = -\frac{1}{2} M_{\rm Pl}^2 \sqrt{-g} \left(R - m^2\, \mathcal{U}(g,\mathcal{K})\right) \tag{2.37}$$

where the potential $\mathcal{U}(g,\mathcal{K}) = \mathcal{U}_2 + \alpha_3 \mathcal{U}_3 + \alpha_4 \mathcal{U}_4$ is expressed in terms of the field $\mathcal{K}^\mu_\nu$ as

$$\mathcal{U}_2(g,\mathcal{K}) = [\mathcal{K}]^2 - \left[\mathcal{K}^2\right], \tag{2.38}$$

$$\mathcal{U}_3(g,\mathcal{K}) = [\mathcal{K}]^3 - 3\,[\mathcal{K}]\left[\mathcal{K}^2\right] + 2\left[\mathcal{K}^3\right], \tag{2.39}$$

$$\mathcal{U}_4(g,\mathcal{K}) = [\mathcal{K}]^4 - 6\left[K^2\right][\mathcal{K}]^2 + 8\left[\mathcal{K}^3\right][\mathcal{K}] + 3\left[\mathcal{K}^2\right]^2 - 6\left[\mathcal{K}^4\right], \tag{2.40}$$

with the coefficients $\alpha_3 = -2c_3$, $\alpha_4 = -4d_5$.

The potentials $\mathcal{U}_{2,3,4}(g,\mathcal{K})$ are given by the characteristic polynomials of the matrix $\mathcal{K}^\mu_\nu$ and can therefore be rewritten in terms of the eigenvalues of $\mathcal{K}^\mu_\nu$. Alternatively the dRGT potential of massive graviton can be expressed through the characteristic polynomials of the square root matrix $\sqrt{g^{-1}f}$ which is sometimes simpler for calculations [50]. In any case, the Lagrangian (2.37) with the potential given in (2.38)-(2.40) defines the so-called *dRGT massive gravity*. It is the only known non-linear completion of massive gravity which is ghost-free in the decoupling limit and which has the strong coupling scale $\Lambda_3 = (m^2 M_{\rm Pl})^{1/3}$. In the following chapters we shall discuss how the agreement with General Relativity is restored in the vicinity of massive sources, whether the Boulware–Deser ghost is absent also away from the decoupling limit, and how this theory can be generalized for arbitrary curved backgrounds. Since most of our discussion will be perturbative then instead of the full non-linear theory we shall often use the expansion in terms of powers of $H_{\mu\nu}$ given in (2.17) with the choice of coefficients (2.28)-(2.30).

3

Vainshtein mechanism

In 1970 van Dam, Veltman, and Zakharov (vDVZ) made an observation which initially appeared to be a no-go theorem for massive gravity due to its incompatibility with well-established Solar System tests of gravity [51, 52]. They noticed that in the linearized massive gravity the helicity-0 mode of the massive graviton does not decouple from the matter in the zero mass limit, but instead remains coupled to the trace of the stress-energy tensor. Due to this effect the predictions of massive gravity for such well-tested gravitational effects as the bending of light by the Sun or the precession of the Mercury perihelion differ from the predictions of General Relativity. This occurence is known as the vDVZ discontinuity. It was, however, shown by Vainshtein that this discontinuity is an artifact of the perturbative expansion, since around heavy sources the expansion becomes singular in the limit of vanishing graviton mass [17]. Vainshtein showed that below a certain distance from the massive body, the so-called Vainshtein radius R_V, the classical non-linear terms become important. In turn, the scalar mode of graviton, propagating the apparent "fifth force", enters the non-perturbative regime and decouples. It was therefore conjectured that due to this behavior in the vicinity of heavy sources, i.e. below the Vainshtein radius, the General Relativity is restored. This is known as the Vainshtein mechanism and is the subject of this chapter.

3.1 vDVZ discontinuity

It easy to see the manifestation of the vDVZ discontinuity at the linear level by comparing the tree-level propagators of the massive and massless graviton. The propagator of the massless graviton in momentum space takes the form

$$D_{\mu\nu,\lambda\sigma} = \frac{1}{2}\frac{\eta_{\mu\lambda}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\lambda} - \eta_{\mu\nu}\eta_{\lambda\sigma}}{k^2 + i\varepsilon} \tag{3.1}$$

while the propagator for Fierz–Pauli massive graviton of mass m is given by

$$D^{(m)}_{\mu\nu,\lambda\sigma} = \frac{1}{2}\frac{\tilde{\eta}_{\mu\lambda}\tilde{\eta}_{\nu\sigma} + \tilde{\eta}_{\mu\sigma}\tilde{\eta}_{\nu\lambda} - \frac{2}{3}\tilde{\eta}_{\mu\nu}\tilde{\eta}_{\lambda\sigma}}{k^2 - m^2 + i\varepsilon}. \tag{3.2}$$

Here k^μ is the four-momentum, and $\tilde{\eta}_{\mu\nu} \equiv \eta_{\mu\nu} - k_\mu k_\nu/m^2$. Since the graviton is coupled to conserved matter sources with $k_\mu T^{\mu\nu} = 0$ then we can replace $\tilde{\eta}_{\mu\nu} \to \eta_{\mu\nu}$. We are interested in the interaction potential between two static massive sources, described by their energy-momentum tensors $T^{00}_{(1)} = M_1\delta^{(3)}(x - x_1)$ and $T^{00}_{(2)} = M_2\delta^{(3)}(x - x_2)$ in the cases of massive and massless graviton. In quantum scattering theory the interaction potential between two sources is given by the Fourier transform of the scattering amplitude for the graviton exchange between these two probes:

$$V(r) \sim \int d^3x\, d^3x'\, d^3k\, T^{\mu\nu}_{(1)}(x) D_{\mu\nu,\lambda\sigma}(k) T^{\lambda\sigma}_{(2)}(x') \mathrm{e}^{ik^i(x-x')^i}, \tag{3.3}$$

where $r = |\vec{x}_1 - \vec{x}_2|$, and we have set $M^2_{\mathrm{Pl}} = (8\pi G_N)^{-1} \equiv 1$. From the two different numerical coefficients in (3.1), (3.2) it is easy to see that there is an additional coupling of the massive graviton to the trace of the energy-momentum tensor. This results in different expressions for the interaction potentials in the two cases

$$V(r)_{m=0} \sim -\frac{M_1 M_2}{r}, \tag{3.4}$$

$$V(r)_{m\neq 0} \sim -\frac{4}{3}\frac{M_1 M_2}{r}\mathrm{e}^{-mr}. \tag{3.5}$$

This leads to different predictions for the motion of massive bodies in gravitational potential in massless and massive gravity.

In our work [41] we have shown how the same results for the gravitational potential can be derived in a purely classical way. For this we use the method usually applied in the theory of cosmological perturbations and classify the metric perturbations according to the irreducible representations of the three-dimensional rotation group [53]. The gravitational interaction between two massive bodies is then entirely due to the static gravitational potentials ϕ and ψ defined as the scalar metric perturbations in the Newtonian (longitudinal) gauge, where the line element is

$$ds^2 = (1 + 2\phi)dt^2 - (1 - 2\psi)\delta_{ik}dx^i dx^k. \tag{3.6}$$

The ocurrence of the vDVZ discontinuity can then be seen in the Fierz–Pauli massive gravity written through the diffeomorphism invariant variables $\bar{h}^A_B = g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi_B - \delta^A_B$ introduced in the previous chapter. We consider the Fierz–Pauli action

$$S = -\frac{1}{2}\int d^4x\,\sqrt{-g}R + \frac{m^2}{8}\int d^4x\,\sqrt{-g}\left[\bar{h}^2 - \bar{h}^A_B\bar{h}^B_A\right] \tag{3.7}$$

and expand it up to the second order in perturbations in the presence of a static spherically symmetric matter source

$$^{(2)}\delta S_{\mathrm{matter}} = -\frac{1}{2}h_{\mu\nu}T^{\mu\nu} = -\phi T^{00}. \tag{3.8}$$

It is clear that the discontinuity is entirely due to the scalar interactions. At quadratic level in the action the scalar, vector and tensor perturbations decouple and can be analyzed separately. We therefore extract the part of the perturbations of the scalar fields $\chi^A \equiv \phi^A - x^\mu\delta^A_\mu$, which transforms as a scalar under the three-dimensional spatial rotations, by

$$\chi^0 = \chi^0, \quad \chi^i = \pi_{,i} \tag{3.9}$$

and focus only on the scalar part of the action. For the detailed calculations please see [41], attached in the appendix A. After eliminating the redundant fields χ^0 and π, the relevant equations of motion for the scalar components of the graviton are

$$\Delta(\phi + \psi) = 3m^2\psi + T^{00}, \qquad 2\psi - \phi = 0. \tag{3.10}$$

The latter equation gives a relation between the gravitational potentials ϕ and ψ which is by order one different from what is known in the Einstein theory where for an adiabatic matter distribution both potentials are equal in the longitudinal gauge. The combination of both equations (3.10) for a spherically symmetric matter source of mass M gives the potential

$$\phi = -\frac{4}{3}\frac{G_N M}{r}\mathrm{e}^{-mr}, \tag{3.11}$$

where we have restored the Newton's constant G_N. The exponential Yukawa-type suppression factor in the potential accounts for the finite range of the gravity due to the graviton mass. The prefactor 4/3 coincides with the result (3.5) obtained in quantum theory. It shows that at distances much shorter than the inverse of the graviton mass the gravitational potential ϕ has increased by a factor of 4/3 in comparison to the Newton potential $\phi_N = -G_N M/r$. This additional contribution survives even in the limit of vanishing graviton mass and would modify, for example, the motion of planets in comparison to what we know from General Relativity. In the meantime the bending of light is determined by the combination $\psi + \phi$ of the gravitational potentials which in General Relativity equals to $2\phi_N$. It is straightforward to check that this combination of static potentials is not changed in massive gravity. Hence the bending of light is described equally in massive and massless gravity. If the effect of the graviton mass would appear for both motion of light and motion of massive objects then one could solve it by simply redefining the Newton's constant. This is not possible in the case of massive gravity since it would then disturb the predictions for the bending of light. This is a purely classical manifest indication of the van Dam, Veltman, Zakharov discontinuity [51, 52].

3.2 Vainshtein mechanism in decoupling limit

One way to understand how the Vainshtein mechanism works is to focus on the helicity-0 mode of the graviton since this is the mode which remains coupled to matter in the limit of vanishing graviton mass. We start by considering the non-linear Fierz–Pauli action (3.7). The dRGT non-linear completion of massive gravity will be considered in the next section. In analogy to the Goldstone equivalence theorem for massive gauge bosons, at high energies the physics of the longitudinal graviton modes is governed by the physics of the Goldstone modes [47]. As we clarified in chapter 2 the scale at which the strongest scalar self-interactions arise in (3.7) is $\Lambda_5 = (m^4 M_{\rm Pl})^{1/5}$. We therefore focus on the interactions of the longitudinal helicity-0 mode by taking the decoupling limit

$$m \to 0, \quad M_{\rm Pl} \to \infty, \quad T^{\mu\nu} \to \infty, \quad \frac{T^{\mu\nu}}{M_{\rm Pl}}, \Lambda_5 = \text{fixed}. \tag{3.12}$$

The decoupling limit Lagrangian in terms of the canonically normalized field $\pi_c = m^2 M_{\rm Pl}\pi$ takes the form

$$\mathcal{L}_\pi = -\left\{\frac{3}{4}\pi^c \Box \pi^c - \frac{1}{2\Lambda_5^5}\left[(\Box\pi^c)^3 - (\Box\pi^c)(\partial_\mu\partial_\nu\pi^c)^2\right] - \frac{1}{2}\frac{1}{M_{\rm Pl}}\pi^c T\right\} \tag{3.13}$$

where $T = \eta_{\mu\nu}T^{\mu\nu}$ is the trace of the energy momentum tensor. Henceforth we drop the superscript of the field π^c and keep in mind that we are working with the canonically normalized field. In the presence of a massive spherically symmetric static source with energy momentum tensor $T^{00} = M\delta^{(3)}(x)$, the field π develops a background profile $\pi_0 \sim -M/r$.

By comparing the quadratic and cubic kinetic terms of π in the Lagrangian (3.13) we see that on the background configuration π_0 the non-linear terms become comparable with the quadratic terms at the scale

$$R_V = \left(\frac{M}{m^4 M_{\rm Pl}^2}\right)^{1/5}. \tag{3.14}$$

This coincides with the Vainshtein radius found in [17]. In order to understand whether in the non-linear Vainshtein regime below $r < R_V$ the theory (3.13) can give reliable predictions, we shall study the stability of the background profile π_0. To do the stability analysis of the perturbations around this solution we expand the action (3.13) in terms of $\delta\pi = \pi - \pi_0$. Schematically it takes the form

$$\mathcal{L}_{\delta\pi} = (\partial\delta\pi)^2 - \frac{(\partial^2\pi_0)}{\Lambda_5^5}(\partial^2\delta\pi)^2, \tag{3.15}$$

and one sees that the fluctuation $\delta\pi$ acquires a four derivative kinetic term. Such a term indicates that there are two scalar degrees of freedom propagating, and one of them is necessarily a ghost. The inverse mass of the ghost is estimated as the factor in front of the four derivative term, i.e.

$$m_{\rm ghost}^2 \sim \frac{\Lambda_5^5}{\partial^2\pi_0(x)}, \tag{3.16}$$

and it is coordinate dependent. In the absence of source when $\pi_0 = 0$, the ghost becomes infinitely heavy and decouples. On a non-trivial background the ghost is however propagating. Since we are considering an effective field theory with a strong coupling cutoff Λ_5 then the ghost is not harmful until the moment when its mass drops below Λ_5. This happens at the radius

$$r_{\text{ghost}} \sim \frac{1}{\Lambda_5}\left(\frac{M}{M_{\text{Pl}}}\right)^{1/3}, \tag{3.17}$$

which is much larger than the Vainshtein radius (3.14), i.e $r_{\text{ghost}} \gg R_V$. We will comment on the implications of this later.

The presence of the additional ghost degree of freedom provides an interesting interpretation of the Vainshtein mechanism proposed in [54]. It was notices that the Lagrangian for the strongly coupled scalar mode π can be rewritten as a system of two fields that classically can be treated perturbatively, but one of them is a ghost. This is done by appropriately modifying the Lagrangian (3.13) and by substituting $\pi = \varphi - \psi$. The result is an action schematically of the form

$$\mathcal{L}_\pi = -\frac{1}{2}\varphi\Box\varphi + \frac{1}{2}\psi\Box\psi - \mathcal{O}(1)\psi^{3/2}\Lambda_5^{5/2} - \frac{1}{M_{\text{Pl}}}\varphi T + \frac{1}{M_{\text{Pl}}}\psi T, \tag{3.18}$$

where ϕ is the healthy scalar mode and ψ is the ghost mode. As before, one can trust this Lagrangian perturbatively within the Vainshtein region as long as the mass of the ghost does not drop below the strong coupling scale Λ_5. By studying the equations of motion of the two fields below and above the Vainshtein radius R_V one finds the different asymptotic solutions for the helicity-0 mode of the graviton π/M_{Pl}:

$$\pi/M_{\text{Pl}} \sim \frac{G_N M}{r} + \mathcal{O}(1)\frac{M^2}{M_{\text{Pl}}^4 m^4}\frac{1}{r^6} = \frac{G_N M}{r}\left(1 + \mathcal{O}\left(\frac{R_V}{r}\right)^5\right), \quad r \gg R_V; \tag{3.19}$$

$$\pi/M_{\text{Pl}} \sim \mathcal{O}(1)m^2\sqrt{\frac{M}{M_{\text{Pl}}^2}}r^{3/2} = \mathcal{O}(1)\frac{G_N M}{r}\left(\frac{r}{R_V}\right)^{5/2}, \quad r \ll R_V. \tag{3.20}$$

We see that within the Vainshtein radius the helicity-0 mode is suppressed by a factor of $(r/R_V)^{5/2}$ in comparison to the Newton potential. This is due to the fact that at the leading order the ghost field ψ cancels the contribution of the scalar φ

and thus screens the "fifth force" within the Vainshtein region.

Since we are working in the effective field theory one should take care how the scale at which the quantum corrections hit in changes in the presence of heavy sources. In flat space the effective field theory is valid up to the energy scale Λ_5 while afterwards the quantum corrections have to be taken into account. In decoupling limit the corrections take the form

$$\sim \frac{\partial^q(\partial^2\pi)^p}{\Lambda_5^{3p+q-4}} \tag{3.21}$$

and around heavy sources they become comparable to the kinetic term $(\partial\pi)^2$ at the scale [48]:

$$r_{p,q} \sim \left(\frac{M}{M_{\rm Pl}}\right)^{\frac{p-2}{3p+q-4}} \frac{1}{\Lambda_5} \tag{3.22}$$

The highest value of this distance equals $r_{\rm quantum} \sim (M/M_{\rm Pl})^{1/3} \cdot 1/\Lambda_5$ which coincides with the scale (3.17) below which the ghost mass is lighter than the cutoff scale. Hence the ghost screening mechanism for interpreting the Vainshtein mechanism is not reliable and is spoiled by the quantum corrections. Therefore, in principle, the calculations of [54] for the corrections to the longitudinal graviton mode due to the classical non-linearities within the Vainshtein regime (3.19) cannot be trusted.

3.3 Smooth limit to General Relativity in Λ_3 theories

The energy cutoff scale at which the effective field theory becomes strongly coupled can be raised by adding higher order interaction terms in $h_{\mu\nu}$ such that after performing the Stückelberg trick the highest order self-interactions of the helicity-0 mode π vanish. The strong coupling scale is then set by the remaining highest interaction terms and is the $\Lambda_3 = (m^2 M_{\rm Pl})^{1/3}$ scale. The construction of the non-linear dRGT theory [28], in which this cutoff scale is achieved, was discussed in detail in the previous chapter. The purpose of this section is to see how General Relativity is restored in Λ_3 theories, as well as to find the corrections to the gravitational potential within the Vainshtein radius and to see what are the relevant scales for which these calculations are reliable.

In order to see how the General Relativity is restored within the Vainshtein radius we decompose the metric and scalar field perturbations in the irreducible representations of the three-dimensional rotation group. The four scalar fields describing the scalar metric perturbations of the metric and scalar fields are defined in equations (3.6) and (3.9) respectivley. We start by expending the Fierz–Pauli action (3.7) up to cubic order in scalar perturbations. We then look for the static solutions of the equations of motion. By doing so we use $\partial^2\pi \ll 1$ as our expansion parameter and neglect the cubic terms such as $\phi(\Delta\pi)^2, \psi(\Delta\pi)^2$ in comparison to $\phi\Delta\pi, \psi\Delta\pi$. That this is a good expansion parameter both within and outside Vainshtein radius is an assumption and was checked to hold in both cases in our work [41]. We also neglect the cubic in gravitational potentials $\phi^3, \phi\psi^2, \ldots$ over the quadratic terms like ϕ^2 etc., as well as the subdominant terms like $\phi^2\Delta\pi$ over $\phi\Delta\pi$ etc. We do so because the gravitational potentials we are considering are always much smaller than unity. In other words, the only higher order corrections we consider come purely from the scalar field perturbations χ^0 and π. Moreover, we also set $\chi^0 = 0$ due to its linear equation of motion. For all the details see our paper [41] in appendix A. The resulting leading cubic order action is

$$
{}^{(3)}\delta S = \int d^4x \left[-\psi\Delta\psi + \phi\left(2\Delta\psi - \Delta\pi\right) + 2\psi\Delta\pi - \frac{1}{M_{\rm Pl}}\phi T^{00} \right. \\ \left. + 3m^2\psi(\psi-\phi) + \frac{1}{2}\frac{1}{m^4 M_{\rm Pl}}\left(\Delta\pi\pi_{,ik}\pi_{,ik} - \pi_{,ki}\pi_{,ij}\pi_{,jk}\right)\right] \qquad (3.23)
$$

where we have rewritten everything in terms of the canonically normalized fields $\phi \to \phi/M_{\rm Pl}, \psi \to \psi/M_{\rm Pl}$ and $\pi \to \pi/(m^2 M_{\rm Pl})$. We see that the strong coupling scale in the Lagrangian is the $\Lambda_5 = (m^4 M_{\rm Pl})^{1/5}$ scale and arises from the cubic self-interaction of the π field. It is important to stress that the approximation we are working in is not equivalent to the decoupling limit in which the first term in the second line of the above equation would be absent due to the limit $m \to 0$. This term is however crucial in finding the exact form of the Yukawa-type potential ϕ in equation (3.11) and to demonstrate how the General Relativity is restored in the limit $m \to 0$ via the Vainshtein mechanism.

By analyzing the equation of motion for π we find that around a spherically symmetric static source the solution for π can be found in two different regimes.

The crossover scale is found to be $R_V = (M/m^4 M_{\rm Pl}^2)^{1/5}$ and coincides with Vainshtein radius [17]. The resulting first-order solutions for the gravitational potential are

$$\psi - \phi = -\psi \left[1 - \mathcal{O}\left(\left(\frac{R_V}{r}\right)^5\right)\right], \qquad r \gg R_V, \tag{3.24}$$

$$\psi - \phi = \mathcal{O}(1)\psi \left(\frac{r}{R_V}\right)^{5/2}, \qquad r \ll R_V. \tag{3.25}$$

We see that outside the Vainshtein radius we obtain $\phi = \psi/2$ in agreement with the linearized result (3.10). Inside the Vainshtein radius we recover the General Relativity relation $\psi = \phi$ up to corrections of order $\psi(r/R_V)^{5/2}$. Moreover, the equation of motion for the gravitational potential ϕ can be solved for $r \ll R_V$ and one can show that the Newton potential is recovered up to the corrections

$$\frac{\delta\phi}{\phi} \sim \left(\frac{r}{R_V}\right)^{5/2}. \tag{3.26}$$

The strong coupling scale of massive gravity can be raised by adding higher order terms in $h_{\mu\nu}$ in the potential. So for example the cubic self-interaction term in the Fierz–Pauli action can be removed by adding the cubic Lagrangian $\mathcal{L}_\phi^{(3)} \propto \bar{h}^A_B \bar{h}^B_C \bar{h}^C_A - \bar{h}^A_B \bar{h}^B_A \bar{h}$. This in turn modifies not only the cutoff scale of the effective field theory but it also diminishes the Vainshtein scale. For the Lagrangian in which the highest order self-interaction terms are of the order $(\partial^2\pi)^n$ the Vainshtein scale is determined as

$$R_V = (M^{n-2} m^{2(1-n)} M_{\rm Pl}^{2(2-n)})^{\frac{1}{3n-4}}. \tag{3.27}$$

The corresponding corrections to the Newton potential are given by

$$\frac{\delta\phi}{\phi} \sim \left(\frac{r}{R_V}\right)^{\frac{3n-4}{n-1}}. \tag{3.28}$$

The limit when $n \to \infty$ corresponds to the Λ_3 energy cutoff when all the higher order self-interactions are cancelled. The correposnding Vainshtein scale is

$$R_V^{(\Lambda_3)} = (M/m^2 M_{\rm Pl}^2)^{1/3} = \frac{1}{\Lambda_3}\left(\frac{M}{M_{\rm Pl}}\right)^{1/3} \tag{3.29}$$

and the corrections to the gravitational potential are given by

$$\frac{\delta\phi_{\Lambda_3}}{\phi} \sim \left(\frac{r}{R_V}\right)^3 . \tag{3.30}$$

It is interesting to note that these corrections are different from the corrections in the DGP theory [55, 56] and in the decoupling limit of the dRGT theory [48].

In the light of the discussion on the quantum corrections and ghosts in the previous section, one should check whether our classical results for the non-linear corrections around heavy sources are reliable. Again, we perform these estimates in the decoupling limit. By construction, in dRGT theory there are no higher order derivative terms in this limit so the only potentially dangerous scale of the theory is the scale at at which the quantum corrections around heavy sources become important. In order to find this scale we will use the decoupling limit Lagrangian given in chapter 2 in equations (2.31)-(2.33). This Lagrangian does not contain a normal kinetic term for the scalar mode π. The kinetic term is acquired through the kinetic mixing of the helicity-0 and helicity-2 modes of the graviton $h^{\mu\nu}X^{(1)}_{\mu\nu}$ and can be disentagled by performing the transformation $h_{\mu\nu} \to h_{\mu\nu} - \eta_{\mu\nu}m^2\pi$. Due to the shift symmetry of the Stückelberg scalar fields ϕ^A, we expect that only quantum operators with at least two derivatives of the field π, suppressed by the cutoff scale Λ_3, are present. The form of these operators is given in (2.15). By comparing these operators with the kinetic term $(\partial\pi)^2$ we find that in the presence of a heavy source, both terms become comparable at the radius $r \sim \left(\frac{M}{M_{\rm Pl}}\right)^{1/3}\frac{1}{\Lambda_3}$. This length scale coincides with the Vainshtein radius (3.29) at which the classical non-linearities hit in. If this indeed would be the right scale, at which the quantum effects become important, then our classical demonstration of how GR is restored in the limit of vanishing graviton mass would again be unreliable due to quantum effects. However, it was found in [27], that a further transformation of the canonically normalized field $h_{\mu\nu}$ can be made in order to eliminate the cubic $h\pi\pi$ couplings as $h_{\mu\nu} \to h_{\mu\nu} + \frac{2(6c_3-1)}{\Lambda_3^3}\partial_\mu\pi\partial_\nu\pi$. The scalar self-interactions of π after these transformations are given solely by the Galileon terms of the form [57]

$$\sim \frac{(\partial\pi)^2(\partial^2\pi)^p}{\Lambda_3^{3p}} . \tag{3.31}$$

These are the terms responsible for the classical non-linear effects in the Λ_3 theory since all other self-interaction terms are removed by construction. Comparison to the quantum operators show that the quantum operators (2.15) are suppressed in comparison to the Galileon terms by the powers of ∂/Λ_3. Hence the quantum effects become important at the scales $r_{\text{quantum}} \sim \frac{1}{\Lambda_3}$ [48]. This length scale is way below the Vainshtein radius which means that one can trust the classical Vainshtein mechanism in the region $1/\Lambda_3 \ll r \ll R_V$. Hence we have shown that in this region a reliable smooth limit from massive gravity to General Relativity exists, and General Relativity is restored up to corrections given in (3.30).

Another important question to address is whether a continuous global spherically symmetric solution of the non-linear theory matching the two asymptotic regions below and above Vainshtein radius exists. The first modified gravity model where such a transition was demonstrated is the DGP model [55]. In the non-linear Fierz–Pauli Λ_5 theory, an everywhere non-singular asymptotically flat solution was found numerically in both decoupling limit in [19] and in the full-theory in [20, 21]. In the dRGT theory the Vainshtein mechanism was investigated numerically in [58, 59, 60]. The numerical findings for the corrections to the Newton potential within the Vainshtein regime confirms our analytic result (3.30).

4

Boulware–Deser ghost

The Boulware–Deser ghost is usually associated with the helicity-0 mode of the massive graviton. More precisely, in a general non-linear polynomial graviton mass term the helicity-0 mode receives higher derivative self-interactions. According to the Ostrogradsky's argument the appearance of higher derivatives requires additional initial data for the complete formulation of the Cauchy problem. Moreover it can be shown that the additional initial data corresponds to a ghost degree of freedom. In order to see the self-interactions of the scalar mode of the massive graviton, the Stückelberg decomposition of the metric perturbations introduced in chapter 2 proves to be particularly useful. This decomposition allows to determine the scale at which the self-interactions become important and the effective field theory becomes strongly coupled. Due to the presence of higher derivatives in the self-interactions of the helicity-0 mode, it was believed that by tuning the graviton mass term so that these interactions vanish would not only raise the cutoff scale of the effective theory, but also eliminate the BD ghost. In [27], a non-linear massive gravity action was found which evades the non-linear self-interactions of the helicity-0 mode of graviton up to the fifth order in the composite field $\bar{h}^A_B \equiv g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi_B - \delta^A_B$ (or, equivalently, in the tensor field $H_{\mu\nu}$ defined in (2.4)). In the decoupling limit (DL) which is used to focus on the longitudinal modes of the massive graviton it was shown that the remaining action contains at most two derivatives and is thus free of the BD ghost. However, the decoupling limit is only the high-energy limit of the effective field theory of massive graviton and only captures the dynamics of the helicity-0 and helicity-1 graviton modes. The decoupling limit (2.27) therefore reflects only the gravitational interactions high above the graviton mass scale. In our work [42] we have investigated the appearance of the higher derivative terms in this non-linear action for massive gravity away from the decoupling limit. Contrary to the claims of [27] we have found that the higher derivatives reappear in the fourth order of perturbations away from the DL, when the vector modes, neglected in [27], are also taken into account.

4.1 Appearance of ghost-like terms

We consider perturbations around the Minkowski background

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}, \qquad \phi^A = x^A + \chi^A, \tag{4.1}$$

so that the field $\bar{h}^A_B$ can be expanded as

$$\begin{aligned} \bar{h}^A_B = {} & h^A_B + \partial^A \chi_B + \partial_B \chi^A + \partial_C \chi^A \partial^C \chi_B \\ & + h^A_C \partial^C \chi_B + h^C_B \partial_C \chi^A + h^C_D \partial^D \chi_B \partial_C \chi^A. \end{aligned} \tag{4.2}$$

This is an exact expression, and the field $\bar{h}^A_B$ is diffeomorphism invariant. In order to see how the higher order derivatives of the scalar components of the metric and scalar fields appear it is instructive to start with the simplest action giving the quadratic Fierz-Pauli mass term for the metric perturbations:

$$S = -\frac{1}{2} \int d^4x \, \sqrt{-g} R + \frac{m^2}{8} \int d^4x \, \sqrt{-g} \left[\bar{h}^2 - \bar{h}^A_B \bar{h}^B_A \right]. \tag{4.3}$$

This action has been investigated in great detail in numerous works and is known to have higher derivative self-interactions of the helicity-0 mode of the graviton even in the decoupling limit. We choose to work in the Newtonian gauge where the metric takes the form [61]

$$ds^2 = (1 + 2\phi)\, dt^2 + 2 S_i dt dx^i - \left[(1 - 2\psi)\, \delta_{ik} + \tilde{h}_{ik} \right] dx^i dx^k, \tag{4.4}$$

where $S_{i,i} = \tilde{h}_{ij,i} = \tilde{h}_{ii} = 0$. As in previous chapter, we only consider the perturbations of the scalar fields that transform as scalars under the three-dimensional rotation group, i.e. χ^0 and $\chi^i = \pi_{,i}$. The ghost can then be easily traced as the dynamical degree of freedom of the field χ^0. In [41] we have shown that among the four linear equations of motion, obtained by variation of the quadratic action with respect to the four scalar perturbations ϕ, ψ, χ^0, π, there are two linear constraint equations:

$$\pi = \frac{2\Delta - 3m^2}{m^2\Delta}\psi, \tag{4.5}$$

$$\chi^0 = -\frac{2\Delta + 3m^2}{m^2\Delta}\dot{\psi}. \tag{4.6}$$

Moreover the field ϕ enters the action as a Lagrange multiplier. As a result, at quadratic level the action (4.3) for the scalar perturbations can be expressed entirely in terms of the metric perturbation ψ as

$$\delta_2 S = -3\int d^4x \left[\psi(\partial_t^2 - \Delta + m^2)\psi\right]. \tag{4.7}$$

It is interesting to note that on the Minkowski background the field χ^0 is not propagating. This occurs due to the accidental $U(1)$ symmetry for the set of the scalar fields χ^A which behave as components of a vector field around Minkowski background [26]. This makes the scalar field χ^0 to be non-dynamical. However, this symmetry is not preserved on a background which slightly deviates from Minkowski. Therefore, χ^0 starts to propagate in the cubic order, and the ghost reappears. To see this we consider only the terms involving $\dot{\chi}^0$ in the cubic order action. We neglect the third order terms linear in $\dot{\chi}^0$ since they only change the constraint equations (4.5),(4.6) to second order in perturbations. Nevertheless, the cubic action also contains a term proportional to $(\dot{\chi}^0)^2$:

$$\delta_3 S = \frac{m^2}{4}\int d^4x \left[\bar{h}^i_i \left(\dot{\chi}^0\right)^2 + ...\right], \tag{4.8}$$

which induces the propagation of χ^0 on the background for which $\bar{h}^i_i = 6\psi + 3\Delta\pi + \mathcal{O}(h^2) \neq 0$. To see that, due to the appearance of the term $(\dot{\chi}^0)^2$, at non-linear level there appears an extra ghost degree of freedom, we express this cubic term entirely in terms of the gravitational potential ψ:

$$\delta_3 S = \int d^4x \left[\Delta\psi \left(\frac{2\Delta + 3m^2}{m^2\Delta}\ddot{\psi}\right)^2 + ...\right]. \tag{4.9}$$

Let us further expand the field ψ around some background configuration ψ_0 as $\psi = \psi_0 + \delta\psi$ and combine the above action with the quadratic action (4.7). By retaining the terms up to second order in $\delta\psi$ we find

$$\delta S = -3 \int d^4x \left\{ \delta\psi \left(\partial_t^2 - \Delta + m^2\right) \delta\psi + \frac{1}{m_{ghost}^2} \left[\left(\partial_t^2 \delta\psi\right)^2 + 2\frac{\ddot{\psi}_0}{\Delta\psi_0} \left(\Delta\delta\psi\right) \left(\partial_t^2 \delta\psi\right) \right] + ... \right\},$$

where

$$m_{ghost}^2 = -\frac{3m^4}{4\Delta\psi_0}. \tag{4.10}$$

Let us take for the background field the scalar mode of gravitational wave with the wave-number $k \sim m$, for which $\ddot{\psi}_0 \sim \Delta\psi_0 \sim m^2\psi_0$ and $m_{ghost}^2 \sim m^2/\psi_0$. By considering, in turn, perturbations $\delta\psi$ with wave-numbers $m_{ghost}^2 \gg k^2 \gg m^2$ and skipping subdominant terms, we can rewrite the action above as

$$\delta S \approx -\frac{3}{m_{ghost}^2} \int d^4x \, \delta\psi \left(\partial_t^2 + ...\right) \left(\partial_t^2 + m_{ghost}^2 + ...\right) \delta\psi. \tag{4.11}$$

The perturbation propagator is given then by

$$\frac{m_{ghost}^2}{\partial^2 \left(\partial^2 + m_{ghost}^2\right)} \simeq \frac{1}{\partial^2} - \frac{1}{\partial^2 + m_{ghost}^2}, \tag{4.12}$$

and it describes the scalar mode of the graviton together with the Boulware–Deser ghost of mass $m_{ghost} \sim m/\sqrt{\psi_0}$. Under the assumptions made this estimate coincides with the ghost mass (3.16) established in [62]. Indeed, the expression

$$m_{ghost}^2 \sim \frac{\Lambda_5^5}{\partial^2\pi_0} \sim \frac{m^4}{\partial^2\pi_0}, \tag{4.13}$$

where we have set $M_{\text{Pl}}^2 \equiv 1$, coincides with (4.10) under the assumption that the background configurations $\pi_0 \sim \psi_0$. For strong enough background the ghost becomes light with the mass $m < m_{ghost} < \Lambda_5$, and, hence, the non-linear ghost appears even below the strong coupling scale of the theory.

Let us now consider the quartic action proposed by de Rham et al. [27]

$$S_\phi = \frac{m_g^2}{8}\int d^4x\,\sqrt{-g}\left[\bar{h}^2 - \bar{h}^2_{AB} + \frac{1}{2}\left(\bar{h}^3_{AB} - \bar{h}\bar{h}^2_{AB}\right) - \frac{5}{16}\bar{h}^4_{AB} + \frac{1}{4}\bar{h}\bar{h}^3_{AB} + \frac{1}{16}\left(\bar{h}^2_{AB}\right)^2\right.$$
$$+c_3\left(2\bar{h}^3_{AB} - 3\bar{h}\bar{h}^2_{AB} + \bar{h}^3 + \frac{3}{4}\left(2\bar{h}^3_{AB}\bar{h} - 2\bar{h}^4_{AB} + \left(\bar{h}^2_{AB}\right)^2 - \bar{h}^2_{AB}\bar{h}^2\right)\right)$$
$$\left.+d_5\left(6\bar{h}^4_{AB} - 8\bar{h}^3_{AB}\bar{h} - 3\left(\bar{h}^2_{AB}\right)^2 + 6\bar{h}^2_{AB}\bar{h}^2 - \bar{h}^4\right)\right], \tag{4.14}$$

where c_3 and d_5 are arbitrary coefficients introduced in chapter 2. This action corresponds to the strong coupling scale $\Lambda = m^{8/11}$ and is ghost free in the decoupling limit. Moreover, due to the total derivative structure of the decoupling limit, higher order terms in $\bar{h}^A_B$ do not contribute to the decoupling limit action [27]. As before we shall trace all fourth order terms in perturbations which contain time derivatives of χ^0. By expanding the action (4.14) and inserting the linear constraint equations (4.5),(4.6) one obtains the relevant action

$$\delta_3 S_\phi + \delta_4 S_\phi = \frac{m_g^2}{8}\int d^4x\left[F\left(\delta g,\chi\right)\dot{\chi}^0 + \frac{1}{2}\left(\dot{\chi}^i + S_i + g^{0i} + \chi^0_{,i}\right)^2\left(\dot{\chi}^0\right)^2 + ...\right].$$

One sees immediately that the special form of the action (4.14) leads to the cancellations of all the third and fourth order terms $\left(\dot{\chi}^0\right)^3$, $\left(\dot{\chi}^0\right)^4$. This is in agreement with the decoupling limit construction as a result of which all the cubic and quartic self-interaction terms of the helicity-0 mode of the graviton are cancelled in (4.14). The function $F(\delta g,\chi)$ depends on terms of second and third order in perturbations, but does not depend on $\dot{\chi}^0$. This term does not induce dynamics of χ^0 and can be neglected. We also note that the third order terms containing $(\dot{\chi}^0)^2$ vanish. Hence, the ghost does not appear in the third order, but only reappears in the fourth order in perturbations. It is also easy to check that after skipping the vector modes the ghost disappears in the decoupling limit ($m^2 \to 0$) [42]. After decomposing $\chi^i = \pi_{,i} + \tilde{\chi}^i$ and performing the same manipulations as above for the action (4.8), we find that the action (4.14), along with the scalar mode of the graviton, also describes a ghost of mass

$$m^2_{ghost} = -12m^2\left(\dot{\tilde{\chi}}^i_0 + S_i - \frac{6}{\Delta}\dot{\psi}_{0,i}\right)^{-2}, \tag{4.15}$$

provided that the ghost mass satisfies the condition $\partial_t^2 m_{ghost}^2 \ll 1$. Here $\tilde{\chi}^i$ is the vector mode of the perturbations of the scalar fields with $\tilde{\chi}^i_{,i} = 0$, and $\tilde{\chi}^i_0$, ψ_0 describe the background configuration around which the ghost propagates. In the background of a scalar gravitational wave ψ_0 with $k \sim m$, the ghost mass simplifies to

$$m_{ghost}^2 \sim \frac{m^2}{\psi_0^2} \sim \frac{\Lambda^{11/4}}{\psi_0^2}, \tag{4.16}$$

where we have substituted the strong coupling scale $\Lambda = m^{8/11}$. If the time dependent background fields are strong enough the mass of this ghost is smaller than the strong coupling scale of the action (4.14). Thus we have shown that in the full theory away from the decoupling limit the non-linear ghost survives in the fourth order of perturbation theory. The same observation was also made in the vierbein formulation of massive gravity in [63].

It is, however, important to make a couple of remarks on the conclusion that the action (4.14) propagates a ghost in the fourth order in perturbation theory. In particular, we wish to understand whether it implies that the Boulware–Deser ghost is also present in the full non-linear dRGT theory (2.37). It is therefore necessary to see at which scales the ghost (4.15) propagates. For weak background fields, i.e. for $\psi_0 \ll 1$, the ghost propagates above the cutoff Λ. This implies that in order to draw conclusions about the full dRGT theory with cutoff $\Lambda_3 > \Lambda$, all the higher order terms in $\bar{h}^A_B$ have to be taken into account. The contribution of these terms can become important at the same scale as where the ghost propagates and can, thus, change the conclusion. On other hand, for strong enough background fields, i.e. for $\psi_0 > m/\Lambda = (m/M_{\rm Pl})^{3/11}$ the mass of the ghost becomes lighter than the cutoff $\Lambda = m^{8/11}$. Notice that the background can still satisfy $\psi_0 \ll 1$, and, hence, the perturbative expansion is still valid. In this case, however, one can still argue that there might exist a non-linear field redefinition such that the ghost is moved from the quartic order to some higher order [46]. In this case we cannot make any conclusive statement about the violation of unitarity in the full theory until we have computed the Lagrangian to sufficiently high order. As we will see, there are indications that this is what actually happens in the dRGT theory.

4.2 Absence of the ghost in unitary gauge

The question of the propagation of the sixth degree of freedom can also be addressed in the unitary gauge, in which the perturbations of the scalar fields are set to zero, i.e. $\chi^A \equiv 0$, and all degrees of freedom are propagated by the metric perturbations. The counting of degrees of freedom in unitary gauge is done in the Hamilton formalism following Dirac [64]. For this it is convenient to introduce the ADM variables as [65]

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^i}{N^2} \\ \frac{N^i}{N^2} & \gamma^{ij} - \frac{N^i N^j}{N^2} \end{pmatrix}. \tag{4.17}$$

It is known that in General Relativity the lapse N and the shift N^i are non-dynamical. The reason for this is that N and N^i enter the Einstein-Hilbert action with no time derivatives, and thus their conjugated momenta are identically zero. This leaves γ_{ij} and its conjugated momenta π^{ij} as the only dynamical fields propagating at most 12 phase space degrees of freedom. Moreover, the lapse and shift appear in the Hamiltonian linearly as Lagrange multipliers and thus generate 4 constraints, $\mathcal{H}_0$ and $\mathcal{H}_i$, on the components of γ_{ij} and π^{ij}. These constraints are first-class constraints and are well-understood in GR: the constraints $\mathcal{H}_i$ generate the spatial diffeomorphisms, while the Hamiltonian constraint $\mathcal{H}_0$ generates the dynamics [66]. Hence, there are in total $12 - 4 \times 2$(first-class) $= 4$ phase space degrees of freedom, corresponding to the two polarizations of the massless graviton. Alternatively, a similar observation can be made by considering the quadratic Einstein-Hilbert Lagrangian for the metric perturbations $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$. This Lagrangian does not contain time derivatives of the components h_{00} and h_{0i}. However, only h_{00} enters the quadratic action as a Lagrange multiplier, whereas the equations of motion for the h_{0i} components allow to express h_{0i} in terms of h_{ik}. After integrating out h_{0i}, the only propagating degrees of freedom are the transverse traceless modes of the spatial metric perturbations h_{ik}, carrying two independent degrees of freedom.

In massive gravity, with some general non-linear graviton mass term of the form of potential $\mathcal{U}(g, \bar{h})$, the lapse and shift enter the Lagrangian in polynomial form with no time derivatives. It was, however, pointed out in [18] (and later in [62]) that, in general, N and N_i appear non-linearly in the Lagrangian. In this case their equations of motion are not constraint equations, but instead are used in order to express themselves in terms of the dynamical fields γ_{ij}, π^{ij}. Hence, the total number of phase space degrees of freedom is 12, leading to the propagation of six physical degrees of freedom in massive gravity. How this problem is avoided in the quadratic Fierz-Pauli mass term is easy to understand by writing out the mass term explicitly:

$$h^2 - h^{\mu\nu}h_{\mu\nu} = h_{ii}^2 - h_{ik}h_{ik} + 2h_{0i}h_{0i} - 2h_{00}h_{ii}. \tag{4.18}$$

One sees that the h_{00} component enters linearly meaning that the Hamiltonian constraint, present in the quadratic Einstein-Hilbert action, is preserved in this case. This removes one of the six degrees of freedom, leaving five degrees of freedom describing a massive spin-2 particle. For a non-linear mass term this is not anymore the case.

It was suggested in [28] that the terms appearing in the dRGT mass term, which are non-linear in lapse N, can be absorbed by an appropriate redefinition of the shift variable N_i. In other words, the terms non-linear in lapse, were suggested to disappear after integration over the shift, leaving the Lagrangian linear in lapse N and hence preserving the Hamiltonian constraint. This procedure of keeping the lapse as a Lagrange multiplier up to quartic order in non-linearities was done in [28]. The full-nonlinear ADM analysis in the unitary gauge was performed in [67]. The existence of the Hamiltonian constraint can also be understood as due to the fact that the four $N_\mu = \{N, \gamma_{ij}N^j\}$ equations of motion depend only on three independent functions n^i. The functions n_i can be found by demanding that (i) after the change of the shift variables $N^i \to n^i$ the action is linear in the lapse N; and (ii) the equations of motion for n^i are independent of N thus allowing to integrate out n^i [67]. The new shift-like variables were found to satisfy

$$N^i = (\delta^i_j + ND^i_j)n^j, \tag{4.19}$$

where the matrix D is to be determined from the matrix equation

$$\left(\sqrt{1 - n^T \mathbb{I} n}\right) D = \sqrt{(\gamma^{-1} - Dnn^T D^T)\,\mathbb{I}}. \tag{4.20}$$

Here n denotes the column vector n^i, n^T stands for its transpose, and $\mathbb{I}$ is the identity matrix. After this change of the shift variables the Lagrangian becomes linear in N and, hence, the Hamiltonian constraint $\mathcal{C}$ arises (for the precise form of the constraint please see [67]). Since this is a second-class constraint, then in order to eliminate one physical degree of freedom, existence of a secondary constraint arising from the preservation of the Hamiltonian constraint in time, $\mathcal{C}_{(2)} = d\mathcal{C}/dt$, is a necessary condition. The existence of the secondary constraint was initially doubted in [68], but was later confirmed in [69]. This pair of the two second-class constraints eliminate the Boulware–Deser ghost from the dRGT massive gravity. Several other proofs for the absence of ghost are available in the literature, but will not be discussed here [70, 71, 72].

4.3 Absence of ghost in Stückelberg formulation

Although the proof of the absence of Boulware–Deser ghost in unitary gauge provides a clear way to count the number of degrees of freedom in dRGT massive gravity, it has several disadvantages. First, in this approach the energy scale of the interactions of the different helicity modes is not transparent. This was one of the main reasons why the Stückelberg trick was applied to the effective field theory of massive gravity [22]. It not only restores the diffeomorphism invariance in the graviton mass term but also allows to determine the strong coupling scale of the effective theory. Another reason to search for alternative proofs for the absence of ghosts in dRGT massive gravity is that the redefinition of the shifts is non-linear, and cannot be written explicitly. In this case the coupling to matter becomes obscure, and the physical interpretation of this degeneracy in the lapse and shift variables is not clear. This suggests, in fact, that one should look for an alternative reformulation of the theory in terms of the new non-linearly redefined shift variables. This is still an open problem. Moreover, the Hamiltonian analysis in the unitary gauge does not provide an explanation for the appearance of the ghost-like terms in the fourth order in perturbations found in our work [42].

In our work [44], we therefore discuss how the absence of the sixth degree of freedom in the dRGT massive gravity would manifest itself in the Stückelberg formulation. We take the point of view that dRGT massive gravity is a theory of Stückelberg scalar fields ϕ^A coupled to the Einstein-Hilbert gravity. It is clear that the dRGT action written in Stückelberg formulation is reparametrization invariant and that the scalar fields are coupled to gravity minimally, i.e. only through the terms $g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^B$. It is therefore legitimate to count the number of degrees of freedom propagated by the scalar field action and the Einstein-Hilbert action separately. In such a diffeomorphism invariant theory of gravity and minimally coupled scalar fields, the Hamiltonian vanishes on the constraint surface, and both the lapse and the shift enter the Hamiltonian linearly [44]. In $(d+1)$-dimensional spacetime, this implies the appearance of in total $2(d+1)$ first-class constraints[1], which can be used to reduce the number of gravitational degrees of freedom to $(d-2)(d+1)/2$. The dynamics of the scalar fields is then generated by the usual Hamiltonian of the scalar field action alone, contained in the Hamiltonian constraint of the full theory. Therefore, the scalar field dynamics in such a theory can be considered separately from gravity. Then naively one would expect that the number of degrees of freedom propagated by any dRGT-type massive gravity in $(d+1)$-dimensional space-time (with $d \geq 2$) equals to

$$\#\text{ d.o.f. } = \frac{1}{2}(d-2)(d+1) + N \tag{4.21}$$

where the first term accounts for the degrees of freedom propagated by the massless graviton, and the second term is just the number of scalar fields N. This naive counting demonstrates why, in (3+1)-dimensional space-time, a general non-linear massive gravity theory with four Stückelberg fields propagates six degrees of freedom.

As we discussed above, it has been demonstrated that in the dRGT subclass of massive gravity theories, in unitary gauge at most five degrees of freedom propagate due to the special structure of the graviton mass term [28, 67]. In Stückelberg language it is clear that, in order for the assertion to be true, the scalar field Lagrangian $\mathcal{L}_\phi = 1/2 \cdot m^2 M_{\text{Pl}}^2 \mathfrak{U}(g, \bar{h})$ with $\mathfrak{U}$ given in (2.38)-(2.40), has to have a very special structure such that it propagates less degrees of freedom than the number of fields. It was therefore suggested in [73] that in the non-linear dRGT massive gravity the four Stückelberg fields do not correspond to four independent degrees of freedom. This can be seen from the vanishing of the determinant of the

[1]Including the constraints due to the absence of the time derivatives of the lapse and the shift, i.e. that the conjugated momenta $\pi^\mu \equiv 0$.

kinetic (Hessian) matrix of the scalar field Lagrangian

$$\mathcal{A}_{AB} \equiv \frac{\partial^2 \mathcal{L}_\phi}{\partial \dot{\phi}^A \partial \dot{\phi}^B} \,. \tag{4.22}$$

Hence the equations of motion of the scalar fields are not independent from each other, and there exists (at least) one combination of the equations of motion which gives a constraint equation relating the canonical momenta of the scalar fields. As a result, in dRGT massive gravity the four scalar fields propagate at most three degrees of freedom. That this conjecture is true has so far been checked explicitly only for the so-called *minimal* dRGT action introduced in [50] with a special choice of coefficients α_3, α_4. The full Hamiltonian analysis in this case was performed with the help of introducing additional auxiliary fields in [74]. Later the analysis was done also in the presence of the Stückelberg fields only [75]. In both works, an additional primary second-class constraint, relating the conjugated momenta of the four scalar fields was found, thus confirming the assertion of only three independent degrees of freedom. An interesting and still open problem is the elimination of the redundant scalar field from the dRGT action. However, the conjugated momenta are non-linear functions of the temporal and spatial derivatives of the scalar fields. This obstacle makes the implementation of the primary constraint in the action of the scalar fields non-trivial. Moreover, it has been recently claimed that this diagonalization of the action of the scalar fields would lead to second order time derivatives on the space-time metric [76]. If true, it would imply that the Boulware–Deser ghost remains present in the dRGT action. However, this hypothesis needs further investigation.

4.4 (1+1)-dimensional dRGT gravity

An illustrative example of the absence of ghost in the Stückelberg language is the $(1+1)$-dimensional dRGT gravity. This case is somewhat degenerate since in $(1+1)$ dimensions the massive graviton propagates no degrees of freedom. It therefore needs to be shown that the two Stückelberg scalar fields propagate no degrees of freedom. We have studied this case in great detail in our work [44]. For simplicity we present here the analysis on the Minkowski background. The

Lagrangian density in this case takes the form

$$\mathcal{L}_\phi = 2\sqrt{\dot{\psi}_+ + \psi'_+}\sqrt{\dot{\psi}_- - \psi'} \tag{4.23}$$

where $\psi_\pm \equiv \phi^0 \pm \phi^1$. It is easy to see that the two conjugated momenta $\pi_\pm$ are not independent and satisfy the primary constraint

$$\mathcal{C}_0 \equiv \pi_+ - \frac{1}{\pi_-} = 0. \tag{4.24}$$

For the consistency of the Hamiltonian equations of motion with the Lagrangian equations of motion one has to impose the secondary constraint, namely that the primary constraint $\mathcal{C}_0$ is preserved in time:

$$\mathcal{C}_1 \equiv \frac{1}{2}\frac{d}{dt}\mathcal{C}_0 = -\left(\frac{1}{\pi_-}\right)' = 0. \tag{4.25}$$

The extended Hamiltonian for the system of the two scalar fields then reads

$$\mathcal{H}_E = \pi_-\psi'_- - \frac{1}{\pi_-}\psi'_+ + u_0\mathcal{C}_0 + u_1\mathcal{C}_1 \tag{4.26}$$

where u_0, u_1 are arbitrary functions of space-time coordinates, i.e. they are the Lagrange multipliers. One can check that both constraints are first-class constraints and generate transformations of the canonical fields

$$\delta\psi_+ = \varepsilon_0, \quad \delta\psi_- = \frac{1}{\pi_-^2}(\varepsilon_0 - \varepsilon'_1), \tag{4.27}$$

leaving the conjugated momenta unchanged. In order to find the symmetry of the original scalar field action (4.23), one rewrites the above transformations by expressing the conjugated momentum π_- according to its definition. By demanding that the action (4.23) remains invariant under the above transformation we find a relation between the gauge parameters $\varepsilon_0, \varepsilon_1$. The resulting gauge symmetry of the Lagrangian is

$$\psi_- \mapsto \psi_- - \frac{1}{2}(\varepsilon' + \dot{\varepsilon})\frac{\dot{\psi}_- - \psi'_-}{\dot{\psi}_+ + \psi'_+}\ , \tag{4.28}$$

$$\psi_+ \mapsto \psi_+ + \frac{1}{2}(\varepsilon' - \dot{\varepsilon})\ . \tag{4.29}$$

Since the above transformation involves both, the gauge parameter ε and its time derivative, then the number of degrees of freedom in the theory are reduced by two which coincides with the total number of first-class constraints [77]. Another way to see that there are no propagating degrees of freedom is by performing the gauge fixing in the extended action $S_E = \int d^2x \left[\pi_+\dot{\psi}_+ + \pi_-\dot{\psi}_- - \mathcal{H}_E\right]$. The equations of motion which can be derived from this action are different from those derived from (4.23). This is so due to the fact that we have introduced an additional Lagrange multiplier u_1 for the secondary constraint. However, the evolution of gauge invariant variables can be equally well described by both actions. Since there are two constraints on the momenta and two gauge symmetries on the canonical fields it is evident that the action S_E is pure gauge and propagates no degrees of freedom.

The same analysis can be done in an arbitrary curved space-time. As before there exist two first-class constraints which generate a gauge symmetry. In distinction from the Minkowski background, the symmetry transformations of the Lagrangian (analogous to (4.28),(4.29)) cannot be written in a local form. It does not, however, change the counting of degrees of freedom since the gauge fixing can be done in the first order action as described above. For details please see our paper [44], attached in appendix C. The Hamiltonian analysis in $(1+1)$-dimensional case was previously also done in [73, 78]. However, in both references the dRGT action was rewritten with the help of additional Lagrange multipliers. As a result the gauge symmetry revealed in our work remained hidden.

5

Massive gravity on curved background

It is known for a long time that the quadratic Fierz–Pauli mass term for the metric perturbations around Minkowski background propagates a ghost around any other background [18, 29]. In general a graviton mass term always involves an arbitrary reference metric $\hat{g}_{\mu\nu}$ as an absolute, non-dynamical object. This reference metric is most naturally chosen to coincide with the background metric so that the FP mass term for the metric perturbations on this background $h^{\mu\nu} \equiv g^{\mu\nu} - \hat{g}^{\mu\nu}$ can be written as

$$\mathcal{L}_{\text{FP}} = h^{\mu\nu} h^{\alpha\beta} \left(\hat{g}_{\mu\nu} \hat{g}_{\alpha\beta} - \hat{g}_{\mu\alpha} \hat{g}_{\nu\beta} \right) . \tag{5.1}$$

As in the case for Minkowski background this mass term breaks the diffeomorphism invariance of general relativity due to the explicit dependence on the absolute background metric $\hat{g}_{\mu\nu}$. We show in [43] that the above mass term can be regarded as the gauge fixed version of the full diffeomorphism invariant graviton mass term. In this chapter we restore the general covariance by introducing the four scalar fields corresponding to the four broken coordinate transformations. We show that these scalar fields preserve a given symmetry in the configuration space of the scalar fields. This symmetry needs to be postulated by hand in dependence on the chosen spacetime metric. We therefore conclude that for each chosen background metric such a construction of the four scalar fields corresponds to different generally covariant massive gravity theory with different internal symmetries.

5.1 Arbitrary reference metric

5.1.1 Minkowski background

As already presented in earlier chapters, in order to give mass to graviton in a diffeomorphism invariant way on Minkowski space we employ four scalar fields ϕ^A, $A = 0, 1, 2, 3$. In addition we introduce a Lorentz transformation Λ^A_B in the scalar field space. Hence the scalar field indices A, B are raised and lowered with the Minkowski metric η^{AB}, and the transformation $\phi^A \to \Lambda^A_B \phi^B$ is the isometry of η_{AB}. We then build the mass term for metric perturbations from the combinations of the variables

$$\bar{h}^{AB} = g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^B - \eta^{AB}. \tag{5.2}$$

This field transforms as a scalar under the spacetime diffeomorphisms and as a tensor under the Lorentz transformations in the internal space of the scalar fields. In the unitary gauge where $\phi^A = x^\mu\delta^A_\mu$ the field $\bar{h}^{AB} = h^{\mu\nu}\delta^A_\mu\delta^B_\nu$ equals the metric perturbations. It is therefore almost trivial to write the generally covariant form of the Fierz–Pauli mass term by replacing the metric perturbations through their diffeomorphism invariant version as $h^{\mu\nu} \to \bar{h}^{AB}$. The most simple generally covariant Fierz–Pauli mass term is

$$S_\phi = \int d^4x\,\sqrt{-g}\left(\bar{h}^2 - \bar{h}^A_B\bar{h}^B_A\right). \tag{5.3}$$

Higher order terms in $\bar{h}^A_B$ can be added to the action in order to solve the problems of very low strong coupling scale and Boulware–Deser ghost as discussed in previous chapters. The full non-linear dRGT theory of massive gravity is obtained by resummation of all the infinite number of terms in terms of the tensor field

$$\mathcal{K}^\mu_\nu = \delta^\mu_\nu - \sqrt{g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^B\eta_{AB}}. \tag{5.4}$$

The dRGT theory (2.37) constructed in terms of the field $\mathcal{K}^\mu_\nu$ admits the background solution

$$\hat{g}_{\mu\nu} = \eta_{\mu\nu}, \quad \hat{\phi}^A = x^\mu\delta^A_\mu, \tag{5.5}$$

around which the metric perturbations have a Fierz–Pauli mass term and propagate five degrees of freedom.

5.1.2 Arbitrary background

Besides the solution (5.5) the dRGT theory admits various other exact cosmological solutions (for recent reviews see [79, 80, 81]). However, the metric perturbations around the various solutions of massive gravity, in general, do not have a mass term of the Fierz–Pauli form. This can be understood if one considers an arbitrary background solution for the metric $g_{\mu\nu} = \hat{g}_{\mu\nu}$ and scalar fields $\phi^A = \hat{\phi}^A$. The tensor field $\mathcal{K}^\mu_\nu$ can then be splitted as $\mathcal{K}^\mu_\nu = \hat{\mathcal{K}}^\mu_\nu + \delta\mathcal{K}^\mu_\nu$ where

$$\hat{\mathcal{K}}^\mu_\nu = \delta^\mu_\nu - \sqrt{\hat{g}^{\mu\lambda}\partial_\lambda\hat{\phi}^A\partial_\nu\hat{\phi}^B\eta_{AB}} \tag{5.6}$$

and $\delta\mathcal{K}^\mu_\nu$ denotes a perturbation. For the Minkowski solution (5.5) the background value of $\mathcal{K}^\mu_\nu$ vanishes, and to linear order $\delta\mathcal{K}^\mu_\nu = -\frac{1}{2}h^\mu_\nu$. After substituting this in the action (2.37), from the quadratic potential $\mathcal{U}_2(g,\mathcal{K}) = [\mathcal{K}]^2 - [\mathcal{K}^2]$ one obtains a FP mass term for the metric perturbations. However for solutions of dRGT theory with $\hat{\mathcal{K}}^\mu_\nu \neq 0$ also the cubic and quartic potentials $\mathcal{U}_3, \mathcal{U}_4$ in $\mathcal{K}^\mu_\nu$ contribute to the quadratic terms in metric perturbations. For the simplest example when $\hat{\mathcal{K}}^\mu_\nu = f(t)\delta^\mu_\nu$ this gives

$$S_\phi = \int d^4x\,\sqrt{-g}\,\left[a(t) + b(t)\mathcal{U}_2(\delta K) + \mathcal{O}\left((\delta K)^3\right)\right], \tag{5.7}$$

where $a(t)$ and $b(t)$ depend on $f(t)$, α_3 and α_4. One sees that for this specific solution the Fierz–Pauli structure for the quadratic perturbations is preserved. However, such a simple spatially flat isotropic and homogeneous solution does not exist in the dRGT theory [82]. For background solutions $\hat{\mathcal{K}}^\mu_\nu$ not proportional to δ^μ_ν the Fierz–Pauli structure of the mass term for metric perturbations is lost. This statement has been confirmed for some specific background solutions by detailed analysis of metric perturbations in [83, 84, 85, 86, 87, 88, 89, 90]. We therefore conjecture that the form of the FP mass term is most likely preserved only for the solutions with $\hat{\mathcal{K}}^\mu_\nu = 0$. It is easy to see that this is equivalent to the condition that the background value of $\bar{h}^{AB}$ vanishes. This translates into an equation for the background of the scalar fields $\hat{\phi}^A$:

$$\hat{g}^{\mu\nu}(x)\frac{\partial\hat{\phi}^A}{\partial x^\mu}\frac{\partial\hat{\phi}^B}{\partial x^\nu} = \eta^{AB}, \tag{5.8}$$

which states that given a curved background $\hat{g}^{\mu\nu}$, the scalar fields $\hat{\phi}^A$ have to be a coordinate transformation from the flat Minkowski metric η^{AB} to the curved metric $\hat{g}^{\mu\nu}$. Such a coordinate transformation, valid at each point of the spacetime, does not exist. We therefore conclude that in dRGT theory the only background, around which the metric fluctuations have a Fierz–Pauli mass term is the Minkowski background.

In order to generalize the dRGT around some arbitrary fixed background $\hat{g}^{\mu\nu}$ in a diffeomorphism invariant way we generalize the diffeomorphism invariant variables $\bar{h}^{AB}$ as

$$\bar{h}^{AB}_{\text{curved}} \equiv g^{\mu\nu}(x)\partial_\mu\phi^A\partial_\nu\phi^B - \bar{f}^{AB}(\phi), \tag{5.9}$$

where $\bar{f}^{AB}(\phi)$ is a set of scalar functions, depending on the four scalar fields ϕ^A. If the functional dependence of $\bar{f}^{AB}$ is set by the desired background metric as

$$\bar{f}^{AB}(\phi) \equiv \hat{g}^{\mu\nu}(\phi)\delta^A_\mu\delta^B_\nu \tag{5.10}$$

then the background value of $\bar{h}^{AB}_{\text{curved}}$ vanishes for $\hat{\phi}^A = x^\mu\delta^A_\nu$. The $\bar{f}^{AB}(\phi)$ can be interpreted as the metric in the configuration space of the scalar fields, so that the Latin indices are raised and lowered as

$$\phi_B \equiv \bar{f}_{AB}\phi^A. \tag{5.11}$$

We then obtain the Fierz–Pauli mass term for metric perturbations around the curved background $\hat{g}_{\mu\nu}$ by substituting $h^{\mu\nu} \to \bar{h}^{AB}_{\text{curved}}$ in the FP mass term. Moreover, the resulting action is invariant under the isometry transformations of $\bar{f}^{AB}(\phi)$. This is analogous to massive gravity with the Minkowski scalar field metric η^{AB}, where the mass term is invariant under the Lorentz transformations of the scalar fields $\phi^A \to \Lambda^A_B\phi^B$. It is now straightforward to generalize the nonlinear dRGT theory (2.37) written in terms of the flat space fields $\mathcal{K}^\mu_\nu$ defined in (5.4) by simply redefining

$$\mathcal{K}^\mu_\nu = \delta^\mu_\nu - \sqrt{g^{\mu\lambda}\partial_\lambda\phi^A\partial_\nu\phi^B\bar{f}_{AB}}. \tag{5.12}$$

Depending of the choice of $\bar{f}_{AB}(\phi)$, the dRGT potential $\mathcal{U}(g,\mathcal{K})$ describes a graviton with Fierz–Pauli mass term around flat or curved background. Instead of $\bar{f}_{AB}(\phi)$ in the literature one often uses the so-called reference metric $f_{\mu\nu}(x)$ [91]. This can be defined in terms of the metric in the internal space of the scalar fields as

$$f_{\mu\nu} = \frac{\partial \phi^A}{\partial x^\mu}\frac{\partial \phi^B}{\partial x^\nu}\bar{f}_{AB}(\phi). \tag{5.13}$$

The reference metric $f_{\mu\nu}$ is said to be *flat* if $\bar{f}_{AB} = \eta_{AB}$. In unitary gauge when $\phi^A = x^\mu \delta^A_\mu$ both metrics coincide. However, the advantage of using $\bar{f}_{AB}(\phi)$ is that the definition of a *flat* reference metric is unambiguously given by $\bar{f}_{AB} = \eta_{AB}$. The reference metric $f_{\mu\nu}$ can also be made dynamical by adding to the dRGT theory an Einstein-Hilbert kinetic term for $f_{\mu\nu}$. These bimetric theories first proposed in [92] possess various interesting cosmological solutions and form an independent active field of research.

5.2 Massive graviton in de Sitter space

As an example we consider the Einstein action with cosmological constant and the generally covariant FP mass term

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}\,(R + 2\Lambda) + \frac{m^2}{8}\int d^4x\sqrt{-g}\,\left(\bar{h}^2 - \bar{h}^A_B\bar{h}^B_A\right) \tag{5.14}$$

where the scalar field tensor $\bar{h}^{AB}$ is defined as (5.9). In absence of the graviton mass term the background solution obtained from the Einstein-Hilbert action is de Sitter universe. We write the spatially flat de Sitter universe in terms of the conformal time η as $\hat{g}^{\mu\nu} = a^{-2}(\eta)\eta^{\mu\nu}$ with $a(\eta) = -1/(H\eta)$, where the Hubble scale $H^2 = \Lambda/3$ is set by the cosmological constant. Hence the scalar field metric is given by $\bar{f}^{AB} = (H\phi^0)^2\eta^{AB}$, and the diffeomorphism invariant FP mass term can be written explicitly as

$$\begin{aligned} S_\phi =& \frac{m^2}{8}\int d^4x\sqrt{-g}\,\{g^{\mu\nu}g^{\alpha\beta}\partial_\mu\phi^A\partial_\nu\phi^B\partial_\alpha\phi^C\partial_\beta\phi^D\,[\eta_{AB}\eta_{CD} - \eta_{BC}\eta_{AD}] - \\ &- 6(H\phi^0)^2 g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^B\eta_{AB} + 12(H\phi^0)^4\}\,. \end{aligned} \tag{5.15}$$

By construction this action gives rise to the quadratic Fierz–Pauli mass term for metric perturbations around de Sitter background. To see this we consider metric and scalar field perturbations

$$g^{\mu\nu} = a^{-2}(\eta)\left(\eta^{\mu\nu} + h^{\mu\nu}\right), \quad \phi^A = x^A + \chi^A. \tag{5.16}$$

In this case $\bar{h}^{AB}$ takes the exact form

$$\begin{aligned}\bar{h}^{AB} =& a^{-2}(\eta)\left\{\eta^{AB} - \frac{a^{-2}(\phi^0)}{a^{-2}(\eta)}\eta^{AB} + h^{AB} + \partial_\mu\chi^B\eta^{\mu A} + \partial_\mu\chi^A\eta^{\mu B} + \right. \\ &\left. + h^{B\mu}\partial_\mu\chi^A + h^{A\mu}\partial_\mu\chi^B + \partial_\mu\chi^A\partial_\nu\chi^B\eta^{\mu\nu} + \partial_\mu\chi^A\partial_\nu\chi^B h^{\mu\nu}\right\}.\end{aligned} \tag{5.17}$$

As in our previous work [41] we decompose the metric and scalar field perturbations according to the irreducible representations of the spatial rotation group and expand the action (5.14) up to second order in perturbations. We find that there are five dynamical degrees of freedom - two tensor modes, two vector modes and one scalar mode. They all satisfy the same equation of motion and thus have the same dispersion relation. Written in conformal time the equation of motion reads

$$\left(\partial_\eta^2 + 2\mathcal{H}\partial_\eta - \Delta\right) q + m^2 a^2 q = 0, \tag{5.18}$$

where we collectively denote the degrees of freedom of massive graviton as $q = \{q_s, q_v, q_t\}$, and $\mathcal{H} = a'/a$, $' \equiv \partial_\eta$. When written in terms of the physical time the equation of motion turns into

$$\ddot{\tilde{q}} - \frac{\Delta}{a^2}\tilde{q} + m_{eff}^2\tilde{q} = 0, \tag{5.19}$$

where $\tilde{q} = a^{3/2}q$, and the effective mass is defined as $m_{eff}^2 = m^2 - \frac{9}{4}H^2$, in agreement with the earlier work [31]. For the precise expressions of the propagating degrees of freedom see our paper [43].

5.3 Partially massless graviton

The properties of massive graviton in de Sitter universe have been first studied in unitary gauge in [30, 31]. It has been shown that for a specific choice of the graviton mass parameter m and cosmological constant Λ the helicity-0 mode of the massive graviton ceases to be dynamical at quadratic level. For the graviton masses below this value, i.e. $m^2 < 2\Lambda/3$, the theory allows the propagation of the states with negative norm. The unitarily allowed region for massive graviton in de Sitter space is therefore restricted to $m^2 \geq 2\Lambda/3$. This is known as the Higuchi bound [30]. The theory at the special point when $m^2 = 2\Lambda/3$ is dubbed as "partially massless" and has been studied in both earlier and recent literature [32, 33, 34, 43, 93, 94, 95, 96, 97]. At this point the quadratic massive gravity on de Sitter space acquires an additional symmetry which removes the scalar degree of freedom leaving the graviton with only four propagating modes. Around the de Sitter background $\hat{g}_{\mu\nu} = a^2(\eta)\eta_{\mu\nu}$ written in conformal time, the symmetry transformation on the metric perturbations defined as $g_{\mu\nu} = a^2(\eta)\left(\eta_{\mu\nu} + h_{\mu\nu}\right)$ takes the form[1]

$$\delta h_{\mu\nu} = a^{-2}\left[\hat{\nabla}_\mu\hat{\nabla}_\nu\alpha + \frac{m^2}{2}\alpha\hat{g}_{\mu\nu}\right], \tag{5.20}$$

where $\hat{\nabla}_\mu$ is the covariant derivative with respect to the background metric $\hat{g}_{\mu\nu}$, and α is the transformations parameter. For the scalar metric perturbations defined as

$$h_{00} = 2\phi\,, \qquad h_{0i} = B_{,i}\,, \qquad h_{ij} = 2\psi\delta_{ij} + 2E_{,ij}$$

the transformation laws read

$$\begin{aligned}
\delta\phi &= \frac{1}{2a^2}\left(\alpha'' - \mathcal{H}\alpha' + \mathcal{H}^2\alpha\right), \\
\delta B &= \frac{1}{a^2}\left(\alpha' - \mathcal{H}\alpha\right), \\
\delta\psi &= -\frac{1}{2a^2}\left(\mathcal{H}\alpha' + \mathcal{H}^2\alpha\right), \\
\delta E &= \frac{\alpha}{2a^2},
\end{aligned}$$

where we have used the relation $\mathcal{H}^2 = (ma)^2/2$ valid at the special point when $m^2 = 2\Lambda/3$. The full set of linear equations for the scalar metric components can be found in our work [43]. At the partially massless point all the equations are invariant under the above transformations and can be shown to propagate no dynamical degrees of freedom in the scalar sector. A similar analysis was later performed also in [95].

[1] The form of the transformation remains the same also for the metric perturbations around the de Sitter background written in physical time. Additional care should be taken when defining the metric perturbations.

The possibility of generalizing the symmetry transformation (5.20) to a non-linear gauge symmetry of the full dRGT theory with de Sitter reference metric is appealing for various reasons [34]. First, the existence of a non-linear gauge symmetry in the partially massless gravity would allow one to fix the form of the low-energy form of the theory. Second, in partially massless gravity the value of the cosmological constant is related to the graviton mass by the gauge symmetry as $\Lambda = 3m^2/2$. This would imply that the quantum corrections to the value of the cosmological constant would arise due to the quantum corrections to the graviton mass. As was discussed in chapter 2 a small graviton mass of order $m^2 \ll M_{\rm Pl}^2$ receives relatively small quantum corrections and is therefore considered to be *technically natural*. This would provide a technically natural solution to the cosmological constant problem. Third, in the so called "candidate theory", which is the dRGT theory with the special choice of coefficients [93]:

$$\alpha_3 = -\frac{1}{2}, \qquad \alpha_4 = \frac{1}{8}, \tag{5.21}$$

the helicity-0 component of the massive graviton vanishes completely from the scalar-tensor sector. If the scalar mode would vanish from the full non-linear theory, then it would mean that the theory would have a higher cutoff scale $\Lambda_2 \sim (mM_{\rm Pl})^{1/2}$ than the dRGT theory where the cutoff is set by $\Lambda_3 \sim (m^2 M_{\rm Pl})^{1/3}$. Unfortunately, although the generalization of (5.20) leaving the cubic order action invariant was found in [34], it was also shown that the symmetry cannot be generalized to the quartic order action. Therefore it seems that no fully non-linear partially massless gravity exists. However, further investigations in this direction, for example, by allowing for a more general form of the non-linear gauge transformation, are possible.

6

Summary and Outlook

Since massive gravity modifies the Einstein's General Relativity it is extremely constrained by the requirements of observational viability. The main reason for this is that till today General Relativity remains the standard theory of gravity satisfying all the observational tests performed on Earth, in Solar System and other astrophysical systems. It is the recent discovery of dark matter and the late-time accelerated expansion of our Universe that has led to the speculations that General Relativity might not be the correct description of our Universe on cosmological scales. Massive gravity is known to weaken the gravitational force between massive bodies on the distances larger than the inverse scale of the graviton mass. In order to be compatible with observations the graviton mass therefore cannot be much less than the inverse size of the Universe, i.e. $m \sim 10^{-28}$ cm. Since this estimate also coincides with the characteristic scale of the present day value of cosmological constant, massive gravity is thought to be able to provide a dynamical explanation for the dark energy.

In this thesis, however, we focus on the theoretical consistency of massive gravity. The reason for this is that although the quadratic Fierz–Pauli mass term was found already in 1939, the theory of massive gravity has been constantly plagued by different theoretical problems. Most of these problems have been properly understood only recently. The only candidate for a consistent model of non-linear massive gravity today is the dRGT theory [28]. In our research we have investigated the most fundamental aspects of massive gravity – the van Dam-Veltman-Zakharov discontinuity, the Vainshtein mechanism, the Boulware–Deser ghost, and the generalization to arbitrary curved backgrounds.

In particular, we restore the diffeomorphism invariance of massive gravity by introducing four scalar fields corresponding to the four broken diffeomorphism transformations. In this framework we demonstrate how the vDVZ discontinuity manifests itself in a purely classical form without any reference to the scattering theory. We also show how the General Relativity is restored in the limit of vanishing graviton mass even from the Fierz–Pauli massive gravity, once the general covariance is reintroduced. This happens via the Vainshtein mechanism due to the non-linear interactions of the scalar perturbations of the metric and scalar fields. We find that below the Vainshtein scale R_V the scalar perturbations enter the non-perturbative regime and decouple. Thus in the vicinity of massive spherically symmetric sources the General Relativity is restored. In our framework we have determined the corresponding Vainshtein scale for a wide range of non-linear massive gravity theories. We have also found the asymptotic solutions for the spherically symmetric gravitational field below and above the Vainshtein radius. Moreover, we have found the corrections to the Newton potential below the Vainshtein radius for different models of massive gravity, including the dRGT massive gravity. To conclude, we believe that the Vainshtein mechanism works in massive gravity and that the General Relativity is restored in the vicinity of massive sources. The numerical studies of global static spherically symmetric solutions in dRGT massive gravity show the existence of asymptotically flat solutions that match to the solutions which exhibit the Vainhstein mechanism in the vicinity of the source [60]. However, other recent works claim that only the static spherically symmetric solutions with cosmological asymptotics are stable [98, 99]. Moreover, some of the existent analytic black hole solutions exhibit curvature singularities on the horizon [98, 100, 101]. A further analysis and search for exact static spherically symmetric solutions is therefore of great importance in order to test the Vainshtein mechanism. Recent reviews on the Vainshtein mechanism and spherically symmetric solutions in massive gravity can be found in [80, 81, 102].

Another important problem we have studied in this thesis is the Boulware–Deser ghost in massive gravity. We have investigated the propagating scalar degrees of freedom in the dRGT theory, which is known to be ghost-free in the decoupling limit in the absence of the vector modes. We have shown that an additional propagating ghost-like scalar mode arises in the fourth order perturbation theory away from the decoupling limit. Moreover, we find that for strong enough bakcground fields the mass of this mode can become smaller than the cutoff scale $\Lambda = m^{8/11}$ of the fourth order dRGT theory. However, since there have appeared several non-perturbative proofs for the absence of the ghost in the full non-linear theory [67, 75], we have reconsidered this question. In particular, we have explored the claim of [73] that not all four Stückelberg scalar fields correspond to independent degrees of freedom in the dRGT theory. For this we have pointed out the obvious facts that the theory is diffeomorphism invariant and that the scalar fields are coupled to gravity only minimally, i.e. only through the combinations $g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^B$. This allows us to analyze the dynamics of the scalar sector separately from gravity which considerably simplifies the task. We have performed the full Hamiltonian analysis in the case of the $(1+1)$-dimensional massive gravity where there are only two scalar fields. In this special case we have found that the theory exhibits a gauge symmetry which reduces the number of degrees of freedom propagated by scalar fields to zero. This coincides with the previous works on this topic and, in addition, reveals the gauge symmetry of the theory not found previously. The analysis of the (3+1)-dimensional case, however, was left for future work. An interesting and important open problem is to see what are the actual dynamical degrees of freedom of the dRGT theory. In the case of the absence of the Boulware–Deser ghost in the full non-linear theory it should, in principle, be possible to rewrite the theory in terms of these dynamical variables only.

We have also addressed the question of how massive gravity can be generalized to arbitrary curved backgrounds in a diffeomorphism invariant way. We have shown how this can be done for an arbitrary given background metric by the use of the four scalar fields. Similarly as in the case of the Minkowski background we introduce a reference metric in the internal space of the scalar fields, $f_{AB}(\phi)$. This internal metric coincides with the chosen background metric of the space-time. As a result there is an additional symmetry in the configuration space of the scalar fields given by the isometry transformations of the scalar field reference metric. Hence, the resulting diffeomorphism invariant massive gravity theory is invariant under different symmetry transformations of the scalar fields depending on the chosen background metric. As a specific example, we analyze the quadratic perturbations in the Fierz–Pauli massive gravity with de Sitter reference metric in this formalism. We show how the previously known properties of de Sitter massive gravity are recovered from the diffeomorphism invariant approach. In particular, we show that at the special point when the mass of the graviton is related to the cosmological constant as $m^2 = 2\Lambda/3$, the linearized theory propagates only four degrees of freedom. This occurs due to an additional gauge symmetry, present at the linearized level. If this symmetry would persist in the full non-linear theory, it would provide a natural bound between the value of the cosmological constant and the graviton mass. Since this relation would be imposed by the gauge symmetry, such a possibility is very interesting. Further investigations in this direction are therefore needed.

In conclusion, we would like to point out that not all of the results of our works [40, 41, 42, 43, 44] were presented in detail in this thesis. A more detailed summary of the obtained results can be find in the conclusions of the corresponding publications attached in the appendices A,B,C,D,E. We would also like to make a remark concerning the fact whether the dRGT massive gravity can, indeed, provide an explanation for the dark energy. It has been shown that the theory admits isotropic self-accelerating solutions in the vacuum, in which the size of the cosmological constant is bound to the value of the graviton mass [58, 82, 89, 103, 104, 105]. However, the cosmological perturbations around these solutions exhibit instabilities [79, 80]. Instead it has been suggested that anisotropic cosmological solutions need to be considered. Hence, the question about cosmological solutions in massive gravity is still open and demands further investigations. Another still unsolved problem of the dRGT theory is the recent claim that the theory allows for superluminal propagation with respect to the spacetime metric [106]. Whether the acausal behavior persists in all the parameter space of the theory, and whether the model has better causal properties with respect to the reference metric $f_{\mu\nu}$, are questions that still need to be clarified.

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Massive gravity: resolving the puzzles

Lasma Alberte,[a] **Ali H. Chamseddine**[b,c] **and Viatcheslav Mukhanov**[a,d]

[a] *Theoretical Physics, Ludwig Maxmillians University, Theresienstr. 37, 80333 Munich, Germany*

[b] *American University of Beirut, Physics Department, P.O. Box 113-6044, Beirut, Lebanon*

[c] *I.H.E.S. F-91440 Bures-sur-Yvette, France*

[d] *Department of Physics, New York University, NY 10003, U.S.A.*

E-mail: Lasma.Alberte@physik.uni-muenchen.de, chams@aub.edu.lb, Viatcheslav.Mukhanov@physik.uni-muenchen.de

Abstract: We consider the massless limit of Higgs gravity, where the graviton becomes massive when the scalar fields acquire expectation values. We determine the Vainshtein scale and prove that massive gravity smoothly goes to General Relativity below this scale. We find that the Vainshtein scale depends on the particular action of scalar fields used to give mass to the graviton.

Keywords: Classical Theories of Gravity, Spontaneous Symmetry Breaking, Gauge Symmetry

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Contents

1 Introduction 1

2 Higgs for graviton: basics 3

3 Physical degrees of freedom of the massive graviton 4

4 Vainshtein scale and continuous limit 8

5 How universal is the Vainshtein scale? 12

6 Conclusions 15

1 Introduction

In the recent paper [1] we (A.Ch.,V.M) have proposed a Higgs mechanism for gravity. In our model the graviton becomes massive as a result of spontaneous symmetry breaking, where four scalar fields acquire non-vanishing expectation values. As a result, three out of four degrees of freedom of scalar fields are absorbed producing a massive graviton with five degrees of freedom, while one degree of freedom remains strongly coupled. Our model is explicitly diffeomorphism invariant and, in distinction from bigravity theories, it is simply given by General Relativity supplemented with the action of four extra scalar fields. Therefore it is completely analogous to the standard Higgs mechanism used to give masses to the gauge fields, where masses are acquired as a result of the interaction with external classical scalar fields. For instance, in the standard electroweak theory one also uses four (real) scalar fields to give masses to three vector bosons, and one remaining degree of freedom becomes a Higgs boson. However, in distinction from electroweak theory, in our case the analogue of the Higgs boson remains strongly coupled and hence completely decouples from gravity and other matter.

The theory with four scalar fields was exploited before by several authors (see [15–17] and references therein). In our case we have found the Lagrangian which resolved the problems that faced finding a consistent theory for massive gravitons. On one hand the model produces a graviton mass term with explicitly invariant form even for finite diffeomorphisms, and on the other hand, keeps the dangerous mode which could produce a ghost, in the strong coupling regime where it is completely harmless. In the linear order the mass term is of the Fierz-Pauli form [2], which is uniquely fixed by the requirements of the absence of extra scalar degree of freedom. The analysis by Deser and Boulware [3] however lead to the conclusion that in the massive theory the extra scalar degree of freedom reappears at nonlinear level and does not decouple, thus making massive gravity to be an ill-behaved theory. In distinction from [3], where diffeomorphism invariance is explicitly

JHEP12(2010)023

spoiled, our theory is diffeomorphism invariant and therefore the $g_{0\alpha}$ components of the metric remain always the Lagrange multipliers, while as we will show later, the scalar fields are always in the strong coupling regime above so called Vainshtein energy scale. This corresponds to extremely small energy and therefore the possible ghost is irrelevant.

There were many interesting attempts to extend massive gravity beyond the linear approximation in a way where one can avoid the extra mode and ghost, also at the nonlinear level (see, for instance, [13, 18, 19, 21, 23, 24] and references there). In particular, in the recent interesting papers [23, 24] an extension of the Fierz-Pauli action was found for which the ghosts are absent even at nonlinear level in the decoupling limit.

The main purpose of this paper is to investigate the existence of a smooth limit of our model to Einstein gravity, when the mass of the graviton vanishes. It was noticed long ago by van Dam, Veltman and Zakharov [4, 5] that in linearized massive gravity the extra scalar mode of the graviton did not disappear and remained coupled to matter even in the limit of a vanishing graviton mass. In turn, this spoils predictions of General Relativity either for the perihelion precession or deflection of starlight. This effect is known as the van Dam-Veltman-Zakharov (vDVZ) discontinuity and was first thought to be a no-go theorem for massive theories of gravity [4, 5]. However, it was pointed out by Vainshtein that the discontinuity could be an artifact due to the breakdown of the perturbation theory of massive gravity in the massless limit [6]. He has shown that in the case of gravitational field produced by a source of mass M_0 the nonlinear corrections become important at scales $r < R_V \equiv M_0^{1/5} m_g^{-4/5}$ (in Planck units) and conjectured that in the strong coupling regime General Relativity is restored. When the mass of the graviton m_g vanishes the Vainshtein radius R_V grows and becomes infinite, thus providing a continuous limit to General Relativity in case the Vainshtein conjecture is correct. At distances $r \ll R_V$, around a static spherically symmetric massive source of mass M_0 the full non-linear strongly coupled massive gravity has to be considered in order to recover the Einstein theory, which makes the proof of the Vainshtein conjecture non trivial. The question of continuous matching of the solutions below and above the Vainshtein radius have been extensively addressed in recent literature. The first model where such a transition was demonstrated is Dvali-Gabadadze-Porrati (DGP) model which imitates many features of massive gravity [13, 20]. There was a claim that in the bigravity version of massive graviton the corresponding solutions do not match [7], but it was recently shown that this claim is not justified [8–10].

In this paper we will find the Vainshtein scale and will prove Vainshtein conjecture in the Higgs model of massive gravity in the case when the gravitational field is produce by a source of mass M_0. Moreover, we will find how the concrete value of the Vainshtein scale depends on the nonlinear extension of the Pauli Fierz term, or in other words on the interactions of scalar fields used to produce massive gravity. As a result we will determine possible Vainshtein scales for a wide class of Higgs gravity models. We will also derive in our model the leading corrections to the gravitational potential within Vainshtein scale, which are similar, but not identical to this type of correction obtained in the framework of the DGP model in [20–22].

Finally, we will discuss the implications of our results obtained in classical theory when extended to quantum theory. In particular we argue that in quantum theory there must be

JHEP12(2010)023

a cutoff scale at energies $m_g^{4/5}$, above which the scalar fields enter strong coupling regime and completely decouple from gravity and other matter. Because this scale is extremely small for the realistic mass of the graviton it makes the problem of ghost which could appear only below this scale completely irrelevant. For the scalar and vector modes of the massive graviton the cutoff scale is an analog of the Planckian scale for the tensor graviton modes, which also become strongly coupled above Planck scale. The obtained cutoff scale is in agreement with results of [11, 12, 20].

2 Higgs for graviton: basics

We employ four scalar fields $\phi^A, A = 0, 1, 2, 3$ to play the role of Higgs fields. These will acquire a vacuum expectation value proportional to the space-time coordinates, thus giving mass to the graviton. Let us introduce the "composite metric"

$$H^{AB} = g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^B, \tag{2.1}$$

which is scalar with respect to diffeomorphism transformations. The field indices $A, B, \cdots$, are raised and lowered with the Minkowski metric η_{AB}. The diffeomorphism invariant action which will be used as our model, is given by

$$S = -\frac{1}{2}\int d^4x\,\sqrt{-g}R + \frac{M^2}{8}\int d^4x\,\sqrt{-g}\left[3\left(\left(\frac{1}{4}H\right)^2 - v^2\right)^2 - v^2\tilde{H}^A_B\tilde{H}^B_A\right], \tag{2.2}$$

where

$$\tilde{H}^A_B = H^A_B - \frac{1}{4}\delta^A_B H, \tag{2.3}$$

is the traceless part of the "composite metric" and where we have set $8\pi G = 1$. The parameter v controls the symmetry breaking scale. As will be seen later, the induced mass of the graviton is equal to $m_g = Mv^2$ and hence when $v \to 0$ gravity becomes massless. It is clear that in this limit the only surviving term in action (2.2) is Einstein gravity and M^2H^4 for the four scalar fields, which are in the regime of strong coupling and do not possess linear propagators. In the phase with restored symmetry the total number of degrees of freedom is six: two of them describe massless graviton and four correspond to scalar fields which are decoupled from gravity at linear level.

We show next that when the symmetry is broken, three out of four scalar fields are "eaten" and produce the massive graviton with five degrees of freedom, while the "surviving" degree of freedom will remain strongly coupled. In case when $v \neq 0$, the unique Minkowski vacuum solution of the equations of motion, $g_{\mu\nu} = \eta_{\mu\nu}$, corresponds to the fields, which linearly grow with coordinates, that is, $\phi^A = \sqrt{v}\delta^A_\beta x^\beta$. Let us consider perturbations around Minkowski background,

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}, \qquad \phi^A = \sqrt{v}\left(x^A + \chi^A\right) \tag{2.4}$$

and define

$$\begin{aligned}\bar{h}^A_B \equiv \frac{1}{v}H^A_B - \delta^A_B &= h^A_B + \partial^A\chi_B + \partial_B\chi^A \\ &+ \partial_C\chi^A\partial^C\chi_B + h^A_C\partial^C\chi_B + h^C_B\partial_C\chi^A + h^C_D\partial^D\chi_B\partial_C\chi^A,\end{aligned} \tag{2.5}$$

JHEP12(2010)023

where indices are moved with the Minkowski metric, in particular, $\chi_B = \eta_{BC}\chi^C$ and $h^A_B = \eta_{BC}\delta^A_\mu\delta^C_\nu h^{\mu\nu}$. We point out that we have included a factor $\sqrt{v}$ as coefficient of χ^A to obtain simpler expressions. In reality in all our results that will subsequently follow we have to make the replacement

$$\chi^A \to \chi^A \frac{1}{\sqrt{v}} = \left(\frac{M}{m_g}\right)^{\frac{1}{4}} \chi^A.$$

This, however, will not effect most of our conclusions, and we will thus comment on it only when necessary. With the help of the expressions

$$H = v\left(\bar{h}+4\right), \quad \tilde{H}^A_B \tilde{H}^B_A = v^2\left(\bar{h}^A_B\bar{h}^B_A - \frac{1}{4}\bar{h}^2\right),$$

we can rewrite the action for the scalar fields in the following form

$$S_\phi = \frac{M^2 v^4}{8}\int d^4x\sqrt{-g}\left[\bar{h}^2 - \bar{h}^A_B\bar{h}^B_A + \frac{3}{4^2}\bar{h}^3 + \frac{3}{4^4}\bar{h}^4\right]. \tag{2.6}$$

We would like to stress that we did not use any approximations to derive (2.6), and $\bar{h}^A_B$ are diffeomorphism invariant combinations of the scalar fields and metric up to an arbitrary order.

3 Physical degrees of freedom of the massive graviton

We consider now small perturbations of the metric and scalar fields and neglect higher order terms. In this case

$$\bar{h}^A_B = h^A_B + \partial^A\chi_B + \partial_B\chi^A + \mathcal{O}(h^2,\chi^2), \tag{3.1}$$

and in the leading order, action (2.6) describes Fierz-Pauli massive gravity, where the mass of the graviton is equal to $m_g = Mv^2$. However, we have to stress that in distinction from the Fierz-Pauli theory our model does not break diffeomorphism invariance and coincides with this theory only in the unitary gauge where all $\chi^A = 0$. In turn, imposing these gauge conditions completely fixes the coordinate system making the interpretation of the results rather obscure. If one would try to treat χ^A as Stückelberg "vector" field and consider *the diffeomorphism transformations for the vectors* rather than some obscure "fictitious" symmetries, then one unavoidably would conclude that the "vector components" *must* be treated as the perturbations of four scalar fields with nonzero background values, thus arriving at our model. As we will see in the next section the difference between the noncovariant Fierz-Pauli approach and our model becomes even more dramatic at higher orders. However, we first study the linearized theory using Lorentz-violating approach to explicitly reveal the true physical degrees of freedom of the massive graviton. Namely, we use the method usually applied in cosmological perturbation theory and classify the metric perturbations according to the irreducible representations of the spatial rotation group [25]. The h_{00} component of the metric behaves as a scalar under these rotations and hence

$$h_{00} = 2\phi, \tag{3.2}$$

JHEP12(2010)023

where ϕ is a 3-scalar. The space-time components h_{0i} can be decomposed into a sum of the spatial gradient of some 3-scalar B and a vector S_i with zero divergence:

$$h_{0i} = B_{,i} + S_i, \tag{3.3}$$

where $B_{,i} = \partial B/\partial x^i = \partial_i B$ and $\partial^i S_i = 0$.

In a similar way h_{ij} can be written as

$$h_{ij} = 2\psi\delta_{ij} + 2E_{,ij} + F_{i,j} + F_{j,i} + \tilde{h}_{ij}, \tag{3.4}$$

where $\partial^i F_i = 0$ and $\partial^i \tilde{h}_{ij} = 0 = \tilde{h}^i_i$. The irreducible *tensor perturbations* $\tilde{h}_{ij}$ have two independent components and describe the graviton with two degrees of freedom in a diffeomorphism invariant way. The *scalar perturbations* are characterized by the four scalar functions ϕ, ψ, B, and E. In empty space they vanish and are induced entirely by matter, which in our case are the scalar fields. The *vector perturbations* of the metric S_i and F_i are also due to the matter inhomogeneities The matter perturbations can also be decomposed into scalar and vector parts:

$$\chi^0 = \chi^0,\ \chi^i = \tilde{\chi}^i + \pi_{,i} \tag{3.5}$$

where $\partial_i \tilde{\chi}^i = 0$. In the linear approximation, scalar, vector and tensor perturbations are decoupled and can be analyzed separately.

Scalar perturbations. Up to first order in perturbations we have $h^{\alpha\beta} = -\eta^{\alpha\nu}\eta^{\beta\mu}h_{\mu\nu}$ and using the definition of $\bar{h}^A_B$ in (2.5) we find that in the leading order approximation

$${}^{(S)}\bar{h}^0_0 = -2\phi + 2\dot{\chi}^0,\ {}^{(S)}\bar{h}^0_i = -B_{,i} - \dot{\pi}_{,i} + \chi^0_{,i},\ \ {}^{(S)}\bar{h}^i_k = 2\psi\delta_{ik} + 2E_{,ik} + 2\pi_{,ik}. \tag{3.6}$$

Substituting these expressions in (2.6), keeping only second order terms, and expanding the Einstein action up to second order in metric perturbations we obtain the following action for the scalar perturbations:

$$\begin{aligned} {}^{(S)}\delta_2 S = \int d^4x \Big\{ & -3\dot{\psi}^2 + \psi_{,i}\psi_{,i} + \phi\left[2\Delta\psi - m_g^2(3\psi + \Delta(E+\pi))\right] \\ & + 2\dot{\psi}\Delta\left(B - \dot{E}\right) + m_g^2\Big[3\psi\left(\psi + \dot{\chi}^0\right) + \left(2\psi + \dot{\chi}^0\right)\Delta(E+\pi) \\ & + \frac{1}{4}\left(\chi^0 - B - \dot{\pi}\right)_{,i}\left(\chi^0 - B - \dot{\pi}\right)_{,i}\Big]\Big\}, \end{aligned} \tag{3.7}$$

where $m_g^2 = M^2 v^4$ and the dot denotes derivative with respect to time. We see that ϕ is a Lagrangian multiplier which implies the constraint

$$\Delta\psi = \frac{m_g^2}{2}(3\psi + \Delta(E+\pi)). \tag{3.8}$$

Another constraint is obtained by variation with respect to B:

$$\dot{\psi} = -\frac{m_g^2}{4}\left(\chi^0 - B - \dot{\pi}\right). \tag{3.9}$$

JHEP12(2010)023

To simplify further the calculations we select the longitudinal gauge $B = E = 0$, which when used in conjunction with (3.8), simplifies the action (3.7) to

$$
{}^{(S)}\delta_2 S = \int d^4x \left[-3\dot{\psi}^2 + \psi_{,i}\psi_{,i} \right.
$$
$$
\left. + m_g^2 \left(3\psi\left(\psi + \dot{\chi}^0\right) + \left(2\psi + \dot{\chi}^0\right)\Delta\pi + \frac{1}{4}\left(\chi^0 - \dot{\pi}\right)_{,i}\left(\chi^0 - \dot{\pi}\right)_{,i} \right) \right].
$$

Using constraints (3.8) and (3.9) with $B = E = 0$, imply

$$
m_g^2 \Delta\pi = \left(2\Delta - 3m_g^2\right)\psi \tag{3.10}
$$
$$
m_g^2 \Delta\chi^0 = -\left(2\Delta + 3m_g^2\right)\dot{\psi} \tag{3.11}
$$

which can be inverted to express π and χ^0 in terms of ψ:

$$
\pi = \left(\frac{2}{m_g^2} - \frac{3}{\Delta}\right)\psi, \tag{3.12}
$$
$$
\chi^0 = -\left(\frac{2}{m_g^2} + \frac{3}{\Delta}\right)\dot{\psi} \tag{3.13}
$$

Substituting these relations in the action above we obtain

$$
\begin{aligned}
{}^{(S)}\delta_2 S &= \int d^4x \left[-3\dot{\psi}^2 + \psi_{,i}\psi_{,i} + m_g^2\left(\frac{6}{m_g^2}\dot{\psi}^2 - \frac{4}{m_g^2}\psi_{,i}\psi_{,i} - 3\psi^2\right)\right] \\
&= -3\int d^4x \left[\psi\left(\partial_t^2 - \Delta + m_g^2\right)\psi\right].
\end{aligned} \tag{3.14}
$$

Note that the potential ψ is gauge invariant with respect to infinitesimal diffeomorphism transformations: $x^\alpha \to \tilde{x}^\alpha = x^\alpha + \xi^\alpha$. Therefore the derived result does not depend on the particular gauge we used to simplify the calculations of the action. First of all we see that the scalar mode which was non-propagating in the absence of the scalar fields has become dynamical. The variable $u = \sqrt{6}\psi$ is the canonical quantization variable for the scalar degree of freedom of metric perturbations. It is entirely induced by perturbation of the scalar fields π and χ^0. In the linear approximation we have to be careful in taking the limit $m_g \to 0$ because of the inverse mass dependence in the relations (3.12) and (3.13). In reality we have to consider instead equations (3.10) and (3.11) which implies that $\psi = 0$ as in the vacuum case. Thus the famous vDVZ discontinuity [4, 5] is not present. In addition, as mentioned before, when taking the limit $m_g \to 0$ we have to replace the fields π and χ^0 with $\left(\frac{M}{m_g}\right)^{\frac{1}{4}}\pi$ and $\left(\frac{M}{m_g}\right)^{\frac{1}{4}}\chi^0$ but this leads to the same result that $\psi = 0$. We note, however, that in the $m_g \to 0$ the Higgs action reduces to the M^2H^4 term, and there are higher order non-linear contributions to ψ. In the next section we will show that above a certain energy scale the scalar mode ceases to propagate and becomes confined due to nonlinear corrections to the equations. As a result the vDVZ discontinuity is avoided completely and we obtain a smooth limit to General Relativity when symmetry is restored and the graviton becomes massless.

Vector perturbations. For the vector perturbations

$$^{(V)}\bar{h}_i^0 = -S_i - \dot{\tilde{\chi}}^i, \quad ^{(V)}\bar{h}_k^i = F_{i,k} + F_{k,i} + \tilde{\chi}^k_{,i} + \tilde{\chi}^i_{,k}. \tag{3.15}$$

Up to second order in perturbations the action for the vector modes is

$$\begin{aligned} ^{(V)}\delta_2 S =& \frac{1}{4}\int d^4x \left[\left(\dot{F}_i - S_i\right)_{,k}\left(\dot{F}_i - S_i\right)_{,k}\right. \\ & \left. + m_g^2\left(\left(\dot{\tilde{\chi}}^i + S_i\right)\left(\dot{\tilde{\chi}}^i + S_i\right) - \left(F_i + \tilde{\chi}^i\right)_{,k}\left(F_i + \tilde{\chi}^i\right)_{,k}\right)\right]. \end{aligned} \tag{3.16}$$

Variation of this action with respect to S_i gives the constraint equation

$$\Delta\left(\dot{F}_i - S_i\right) = -m_g^2\left(\dot{\tilde{\chi}}^i + S_i\right),$$

which allows us to express S_i as

$$S_i = \frac{1}{\Delta - m_g^2}\left(\Delta\dot{F}_i + m_g^2\,\dot{\tilde{\chi}}^i\right). \tag{3.17}$$

Substituting this expression into (3.16) we obtain

$$^{(V)}\delta_2 S = -\frac{1}{2}\int d^4x\,\frac{m_g^2\Delta}{2\left(\Delta - m_g^2\right)}\left[\left(F_i + \tilde{\chi}^i\right)\left(\partial_t^2 - \Delta + m_g^2\right)\left(F_i + \tilde{\chi}^i\right)\right]. \tag{3.18}$$

In the limit $m_g \to 0$ the action for the vector modes vanishes even after replacing $\tilde{\chi}^i \to \left(\frac{M}{m_g}\right)^{\frac{1}{4}}\tilde{\chi}^i$. The canonical gauge invariant quantization variable in this case is the 3-vector

$$V^i = \sqrt{\frac{m_g^2\Delta}{2(\Delta - m_g^2)}}\left(F_i + \tilde{\chi}^i\right), \tag{3.19}$$

which describes two physical degrees of freedom as this vector satisfies an extra condition $\partial_i V^i = 0$.

Tensor perturbations. For the tensor perturbations the result is straightforward

$$^{(T)}\delta_2 S = -\frac{1}{8}\int d^4x\left[\tilde{h}_{ij}\left(\partial_t^2 - \Delta + m_g^2\right)\tilde{h}_{ij}\right]. \tag{3.20}$$

This action describes the pure gravitational degrees of freedom which have become massive. Because $\tilde{h}_{ij}$ satisfies four extra conditions $\partial^i\tilde{h}_{ij} = 0 = \tilde{h}_i^i$ the tensor perturbations have two physical degrees of freedom.

Thus, we have decomposed the massive graviton with five degrees of freedom into physical gauge invariant components: a scalar part ψ (with one degree of freedom), a vector part V^i (2 degrees of freedom) and a tensor part $\tilde{h}_{ij}$ (2 degrees of freedom). After quantization they acquire their independent gauge invariant propagators.

The metric components are the subject of minimal vacuum quantum fluctuations. In particular, the amplitude of the vacuum fluctuations of ψ and $\tilde{h}_{ij}$ at scales $\lambda \ll 1/m_g$ are about

$$\psi \sim \tilde{h}_{ij} \sim \frac{1}{\lambda},$$

JHEP12(2010)023

in Planck units. They become of the order of one at the Planck scale $l_{\mathrm{Pl}} \simeq 10^{-33}$ cm where non perturbative quantum gravity becomes important. The amplitude of the vector vacuum metric fluctuations is much smaller. In fact, for $\lambda \ll 1/m_g$, their amplitude in the gauge $S_i = 0$ is scale independent and is equal to

$$^{(V)}h_{ij} \sim F_{i,j} \sim m.$$

These results are valid only in linearized theory. While the result for the tensor fluctuations remains the same, we will show in what follows that the scalar and vector modes reach the strong coupling regime at the energy scale which is much below the Planck scale.

4 Vainshtein scale and continuous limit

Let us first consider how the static interaction between two massive bodies is modified in the Higgs model with massive graviton. In quantum field theory this interaction is interpreted as due to the exchange by gravitons with corresponding quantum propagators. This interpretation is very obscure from the physical point of view because the Newtonian force is not directly related to the propagation of gravitons. It is, however, the price to be paid in order to preserve explicit Lorentz invariance of the theory. In our approach one does not need to go to quantum theory to answer this question. The interaction is entirely due to the static potentials ϕ and ψ which are present due to the massive body. Let us take the Newtonian gauge [25], where $B = E = 0$ so that the metric takes the form

$$ds^2 = (1 + 2\phi)\, dt^2 - (1 - 2\psi)\, \delta_{ik} dx^i dx^k \tag{4.1}$$

First we have to derive the equations that this metric should satisfy in massive gravity. We consider only static solutions so all time derivatives vanish and action (3.7) simplifies to

$$\begin{aligned} ^{(S)}\delta_2 S = \int d^4x \, \Big\{ & \psi_{,i}\psi_{,i} + \phi \left[2\Delta\psi - m_g^2(3\psi + \Delta\pi) - T^{00}\right] \\ & + m_g^2 \left[3\psi^2 + 2\psi\Delta\pi + \frac{1}{4}\chi^0_{,i}\chi^0_{,i}\right] \Big\} . \end{aligned} \tag{4.2}$$

We have added a term which describes the interaction with an external source of matter for which only the T^{00} component of the energy momentum tensor does not vanish. Varying this action with respect to ϕ, ψ, χ^0 and π we arrive to the following equations:

$$\Delta\psi = \frac{m_g^2}{2}\,(3\psi + \Delta\pi) + \frac{T^{00}}{2}, \qquad \Delta\left(\psi - \phi - m_g^2\pi\right) = 0, \tag{4.3}$$

$$\Delta\chi^0 = 0, \qquad \Delta\,(2\psi - \phi) = 0. \tag{4.4}$$

It immediately follows from (4.4) that $\chi^0 = 0$ and $\psi = \phi/2$, while equations (4.3) simplify to

$$\Delta\,(\phi + \psi) = 3m_g^2\psi + T^{00}, \tag{4.5}$$

or taking into account that $\psi = \phi/2$ we obtain

$$\left(\Delta - m_g^2\right)\phi = \frac{4}{3}\left(\frac{T^{00}}{2}\right). \tag{4.6}$$

For the central source of mass M_0 the solution of this equation is

$$\phi = -\frac{4}{3}\frac{M_0}{r}e^{-m_g r} = \frac{4}{3}\phi_N e^{-m_g r}, \tag{4.7}$$

where $\phi_N = -M_0/r$ is the Newtonian gravitational potential. At scales $r \ll 1/m_g$ the metric takes the form

$$ds^2 = \left(1 + \frac{4}{3}(2\phi_N)\right) dt^2 - \left(1 - \frac{4}{3}\phi_N\right)\delta_{ik}dx^i dx^k. \tag{4.8}$$

The bending of light is determined by the $\phi+\psi$ combination of the metric components. In General Relativity, where $\psi = \phi_N$, this combination is equal to $2\phi_N$. In the case of massive gravity

$$\phi + \psi = \frac{4}{3}\phi_N + \frac{2}{3}\phi_N = 2\phi_N, \tag{4.9}$$

i.e. we obtain the same prediction for the bending of light. However, the gravitational potential ϕ which, for instance, determines the motion of planets has increased by factor 4/3 compared to the Newtonian potential, independently of the mass of the graviton. This extra contribution survives even in the limit of zero mass. If one would redefine the gravitational constant to get the correct Newtonian potential then obviously the bending of light would be wrong. This is a manifestation of vDVZ discontinuity, which in quantum field theory is interpreted as due to the propagation of the extra scalar mode in addition to the two tensor degrees of freedom. Because this scalar mode is coupled to the trace of the matter the result remains unchanged for photons, but changes by the corresponding factor for non-relativistic matter. Note that we have re-derived this result in a purely classical theory without any reference to the tensor degrees of freedom or the "true" graviton.

The paradox with vDVZ discontinuity, which implies that the graviton must be strictly massless was resolved when Vainshtein found a new scale R_V in massive gravity and suggested that for $r < R_V$ the scalar mode decouples and General Relativity is restored.

We will now show how this happens in our theory, and prove that General Relativity is smoothly restored below the Vainshtein scale. For that we will need to consider the higher order corrections to the action (4.2). First of all we notice that because the gravitational potentials with which we are dealing are always much smaller than unity, we can safely ignore the terms of order $\phi^3, \phi\psi^2$ etc. compared to $\phi^2, \ldots$ because they cannot change the solutions of the equations drastically. We will also ignore the terms $\phi^2(\Delta\pi)$ compared to $\phi(\Delta\pi)$ etc. because they are subdominant. Therefore, the only contribution to the higher order corrections which we will take into account will come purely from the matter scalar fields. In addition we will skip all terms with χ^0 since they vanish in the leading order. Hence, the only relevant terms of the third order, which should be added to the action (4.2) are:

$$^{(S)}\delta_3 S = m_g^2 \int d^4x \left[\frac{1}{2}(\Delta\pi\pi_{,ik}\pi_{,ik} - \pi_{,ki}\pi_{,ij}\pi_{,jk}) + \frac{3}{16}(\Delta\pi)^3 - \frac{1}{2}(\phi+2\psi)\pi_{,ik}\pi_{,ik}\right.$$
$$\left. +2\psi(\Delta\pi)^2 + \frac{9}{16}(3\psi-\phi)(\Delta\pi)^2 + O\left(\psi^3, \psi^2\phi, \psi^2\Delta\pi, \phi\psi\Delta\pi\ldots\right)\right]. \tag{4.10}$$

These third order corrections modify the equations obtained by variation with respect to ψ and π in the following way:

$$\Delta\left(\psi-\phi-m_g^2\pi\right)+m_g^2\left[\frac{3}{2}\left(\phi-2\psi\right)+\frac{1}{2}\pi_{,ik}\pi_{,ik}-\frac{59}{64}\left(\Delta\pi\right)^2\right]=0, \tag{4.11}$$

and

$$\Delta\left(2\psi-\phi\right)+\left(\Delta\pi\pi_{,ik}\right)_{,ik}+\frac{1}{2}\Delta\left(\pi_{,ik}\pi_{,ik}\right)-\frac{3}{2}\left(\pi_{,ij}\pi_{,jk}\right)_{,ik} \\ +\frac{9}{16}\Delta\left(\Delta\pi\right)^2+O\left(\phi_{,ik}\pi_{,ik},\Delta\psi\Delta\pi,\ldots\right)=0. \tag{4.12}$$

Equation (4.12) is the main equation where non-linearities begin to play an important role allowing us to avoid the condition $\Delta\left(2\psi-\phi\right)=0$, and thus resolve the problem of vDVZ discontinuity. In fact, this condition means that the scalar perturbations of the curvature must vanish, $\delta R=0$, and this was the main obstacle leading to the troubles with restoring General Relativity in the limit of vanishing graviton mass in the paper [3]. Assuming that $\pi_{,ik}$, $\Delta\pi\ll 1$ (this assumption will be checked a posteriori), and keeping only the leading terms in equations (4.11) and (4.12) we obtain

$$\Delta\left(\psi-\phi-m_g^2\pi\right)=0,\qquad \Delta\left(2\psi-\phi\right)+\partial^6\pi^2=0, \tag{4.13}$$

where by $\partial^6\pi^2$ we denoted all quadratic π terms in (4.12). Using the first equation in (4.13) to solve for $\Delta\phi$, the second one simplifies to

$$\Delta\left(\psi+m_g^2\pi\right)+\partial^6\pi^2=0. \tag{4.14}$$

Taking into account that $\Delta\sim\partial^2$ and estimating $\partial^6\pi^2$ in spherically symmetric field as $O\left(1\right)\pi^2/r^6$, this equation becomes

$$\psi+m_g^2\pi+O\left(1\right)r^{-4}\pi^2\simeq 0, \tag{4.15}$$

The behavior of π as a function of r crucially depends on whether the second or third term in this equation is dominating. To estimate the scale when both terms are comparable, which is called the Vainshtein scale R_V, we set

$$m_g^2\pi\sim O\left(1\right)r^{-4}\pi^2\sim\psi,$$

and from here find that

$$-\left.\psi\right|_{r=R_V}=m_g^4R_V^4. \tag{4.16}$$

In the case of a gravitational field produced by the mass M_0 in the vacuum $\psi\simeq-M_0/r$, the Vainshtein scale is equal to

$$R_V\simeq\left(\frac{M_0}{m_g^4}\right)^{1/5}. \tag{4.17}$$

For $r \gg R_V$ the last term in (4.15) is small compared to the second one and we obtain

$$\pi = \frac{\psi}{m_g^2}\left[-1 + \mathcal{O}\left(\left(\frac{R_V}{r}\right)^5\right)\right]. \tag{4.18}$$

In this limit the quadratic terms in the second equation in (4.13) are negligible and from the first equation in (4.13) we find that

$$\psi - \phi = -\psi\left[1 - \mathcal{O}\left(\left(\frac{R_V}{r}\right)^5\right)\right]. \tag{4.19}$$

This implies that in the leading order $\psi = \phi/2$ in complete agreement with the result which we have obtained above in linearized massive gravity. It is easy to check that the condition $\partial^2\pi \ll 1$ which we have used to simplify equations (4.11) and (4.12) is also satisfied. In fact,

$$\partial^2\pi \sim -\frac{\psi}{r^2 m_g^2} \sim \frac{M_0}{r^5 m_g^4} r^2 m_g^2 \sim \left(\frac{R_V}{r}\right)^5 \left(\frac{r}{1/m_g}\right)^2, \tag{4.20}$$

and hence $\partial^2\pi \ll 1$ for all $r > R_V$ if $R_V \ll 1/m_g$.

At scales smaller than Vainshtein radius, that is for $r \ll R_V$ the third term in (4.15) is larger than the second one and hence

$$\begin{aligned}\pi &\simeq O(1)\, r^2\sqrt{-\psi}\left[1 + O(1)\,\frac{m_g^2 r^2}{\sqrt{-\psi}} + \dots\right] \\ &\simeq O(1)\,\frac{\psi}{m_g^2}\left(\frac{r}{R_V}\right)^{5/2}\left[1 + O(1)\left(\frac{r}{R_V}\right)^{5/2} + \dots\right].\end{aligned} \tag{4.21}$$

Using this expression in the first equation of (4.13) we then find that in the leading order

$$\psi - \phi = O(1)\,\psi\left(\frac{r}{R_V}\right)^{5/2} + \dots \tag{4.22}$$

For $r \ll R_V$ we find that $\psi = \phi$ up to corrections of order $\psi\,(r/R_V)^{5/2}$. Because of $\partial^2\pi \sim \sqrt{-\psi}$ the condition $\partial^2\pi \ll 1$ is always satisfied. The dominating quadratic corrections to equation (4.5) is of order $m_g^2\psi \sim m_g^2\left(\partial^2\pi\right)^2$ so they change only the mass term which is irrelevant within the Vainshtein scale. Taking into account that $\psi = \phi$ for $r \ll R_V$ and neglecting the mass term, equation (4.5) in the leading order is reduced to

$$\Delta\phi = \frac{T^{00}}{2}, \tag{4.23}$$

and thus General Relativity is restored within Vainshtein scales up to the corrections

$$\frac{\delta\phi}{\phi} \sim \left(\frac{r}{R_V}\right)^{5/2}, \tag{4.24}$$

which are much smaller than the corresponding corrections in DGP model [22]. One could ask whether any higher order corrections would be able to spoil the obtained results? The

JHEP12(2010)023

most dangerous of these corrections in every next order will come as the corrections to the previous order multiplied by $\partial^2\pi \ll 1$. Therefore they are completely negligible.

We now consider the implementation of our results derived classically in quantum field theory. In the explicitly Lorentz invariant approach the change of the interaction strength at scales exceeding the Vainshtein scale is interpreted as due to exchange by the scalar mode ψ of the massive graviton in addition to the two tensor modes of the massless graviton. As we will argue, this scalar mode becomes strongly coupled below Vainshtein scale and as a result completely decouples from the gravity and matter entering the confinement regime. This is similar to QCD, where the "soft" modes do not participate in the interactions of highly energetic quarks below the confinement scale. Although for quantum fluctuations one cannot neglect the time derivatives as in the static case, we can, however, estimate the time derivatives to be of the same order of magnitude as spatial derivatives and use the formulae derived for the static case. Keeping in mind that the amplitude of the scalar quantum fluctuations at the length scale λ is about $\psi \simeq 1/\lambda$ from (4.16) we obtain that at scales smaller than

$$\Lambda_s \simeq m_g^{-4/5},$$

these scalar modes should be in the strong coupling regime, where nonlinear corrections cannot be neglected. Note that the metric fluctuations which are of order $\psi \sim m_g^{4/5}$ still remain small at this scale. In distinction from the case when gravitational field is produced by an external source the estimate $\partial^2\pi \sim \sqrt{-\psi}$ is not justified for quantum fluctuations for $\lambda \ll \Lambda_s$. However, assuming that at the scales which are just a bit smaller than Λ_s one can still use this estimate to find that the last term in action (3.7), which is of order $\partial^4\pi^2 \sim \psi$, becomes dominant compared to the terms of order $\psi^{3/2}$ and ψ^2. As a result the scalar mode ψ loses its linear propagator and decouples, entering the strong coupling regime where nonlinear corrections will prevent its unbounded growth for every $\lambda < \Lambda_s$ as $m_g \to 0$. As a result the terms proportional to m_g^2 in the action (3.7) will vanish and General Relativity is smoothly restored in this limit. A similar thing happens with the vector modes. Therefore in the limit $m_g \to 0$ only the tensor modes $\tilde{h}_{ik}$ with two degrees of freedom survive. They enter the strong coupling regime at the Planckian scale. The energy scale Λ_s^{-1} should be taken as a cutoff scale for the scalar mode ψ of graviton in all diagrams where this scalar mode participates. Above this scale our scalar fields π and χ^0 which were producing the extra degrees of freedom for the massive graviton are also in the confined regime and the symmetry is restored. These strongly coupled fields are completely decoupled from gravity and the rest of the matter. In the case when the mass of the graviton is of the order of present Hubble scale the cutoff scale is extremely small of order 10^{-18} eV. At higher energies the ghost, even if it would exist, completely decouples. Therefore the question about ghosts at the nonlinear level becomes irrelevant.

5 How universal is the Vainshtein scale?

The expression (4.17) for the Vainshtein scale was derived first in the case of Fierz-Pauli mass term which is unique in four dimensions, because only in this case there are no ghosts propagating at the linear level. We have obtained the same result in our Higgs model with

JHEP12(2010)023

the action (2.2). It is natural to ask whether it is the unique universal scale for all models with Fierz-Pauli mass term or it depends on a particular nonlinear extension of this term. Let us show that in our theory the Vainshtein scale, in fact, depends on the nonlinear completion of the theory and determine all possible extensions of the model which lead to different Vainshtein scales. With this purpose we first consider instead of (2.2) the following action for the scalar fields

$$S_\phi = \frac{M^2}{8}\int d^4x\,\sqrt{-g}\left[12\left(\frac{H}{4}-1\right)^2 + 4^3\beta\left(\frac{H}{4}-1\right)^3 - \tilde{H}^A_B\tilde{H}^B_A + O\left((H-4)^4\right)\right] \quad (5.1)$$

where without loss of generality we have set the parameter of the symmetry breaking to unity. The terms $O\left((H-4)^4\right)$ must be taken in such a way as to avoid the appearance of other vacua, besides $H=4$. One can easily verify that there are infinitely many extensions of the required type. This action, when rewritten in terms of $\bar{h}^A_B$ variables defined in (2.5), with $v=1$, takes the form

$$S_\phi = \frac{m_g^2}{8}\int d^4x\,\sqrt{-g}\left[\bar{h}^2 - \bar{h}^A_B\bar{h}^B_A + \beta\bar{h}^3 + O\left(\bar{h}^4\right)\right], \quad (5.2)$$

where $m_g^2 = M^2$. For $\beta \gg 1$, the main contribution to the cubic action (4.10) is of order $\beta\left(\Delta\pi\right)^3$ and the second equation in (4.13) is modified to

$$\Delta\left(2\psi-\phi\right) + 3\beta\Delta\left(\Delta\pi\right)^2 = 0. \quad (5.3)$$

Then using the first equation in (4.13) and considering the spherically symmetric case we find

$$\psi + m_g^2\pi + O\left(1\right)\beta r^{-4}\pi^2 \simeq 0, \quad (5.4)$$

and correspondingly the Vainshtein scale in this case is

$$R_V \simeq \left(\frac{\beta M_0}{m_g^4}\right)^{1/5}. \quad (5.5)$$

Thus, we see that taking large enough β in action (5.1) we can obtain an arbitrarily large Vainshtein scale for given masses of the source M_0 and the graviton m_g.

Next we would like to address the question whether one can obtain a smaller Vainshtein scale compared to (4.17). For that let us first consider the action

$$S_\phi = \frac{M^2}{8}\int d^4x\,\sqrt{-g}\left[-6\left(\frac{H}{4}-1\right)^2\left(\frac{H}{4}-3\right) - \frac{1}{2}\tilde{H}^A_B\tilde{H}^B_A + \right.$$
$$\left. + \frac{1}{2}\tilde{H}^A_B\tilde{H}^B_C\tilde{H}^C_A - \frac{1}{8}H\tilde{H}^A_B\tilde{H}^B_A + O\left((H-4)^4\right),\right] \quad (5.6)$$

where the terms $O\left((H-4)^4\right)$ are taken in such a way as to avoid the vacuum at $H=12$. Rewritten in terms of $\bar{h}^A_B$, action (5.6) becomes

$$S_\phi = \frac{m_g^2}{8}\int d^4x\,\sqrt{-g}\left[\bar{h}^2 - \bar{h}^A_B\bar{h}^B_A + \frac{1}{2}\left(\bar{h}^A_B\bar{h}^B_C\bar{h}^C_A - \bar{h}^A_B\bar{h}^B_A\bar{h}\right) + O\left(\bar{h}^4\right)\right], \quad (5.7)$$

where $m_g^2 = M^2$. It is clear that in the lowest order it reproduces the Fierz-Pauli term, but in higher orders it is quite different from (2.6). The action (5.7) concides with the action first derived in [23, 24] from the requirement of the absense of ghost in decoupling regime up to the third order. If we consider the case of the static gravitational field we find that in the third order the action does not contain terms of the form $\partial^6\pi^3$. Hence, by keeping only the leading terms we find that the second equation in (4.13) will be modified to

$$\Delta\left(2\psi - \phi\right) + \partial^8\pi^3 = 0. \tag{5.8}$$

Considering the spherically symmetric case and using the first equation in (4.13), which is still valid up to the leading order, we find that equation (4.15) has to be replaced by

$$\psi + m_g^2\pi + O\left(1\right)r^{-6}\pi^3 \simeq 0. \tag{5.9}$$

The Vainshtein scale will be determined by the condition that all three terms in this equation become comparable, that is,

$$\psi \sim m_g^2\pi \sim r^{-6}\pi^3 \tag{5.10}$$

and hence the expression determining this scale is

$$-\left.\psi\right|_{r=R_V} = m_g^3 R_V^3. \tag{5.11}$$

In particular, in the case of static field produced by mass M_0, we have

$$R_V \simeq \left(\frac{M_0}{m_g^3}\right)^{1/4}. \tag{5.12}$$

To obtain the correction to the Newtonian potential at $r \ll R_V$ we note that at these scales $\pi \sim r^2\psi^{1/3}$ and use of the first equation in (4.13) leads to

$$\frac{\delta\phi}{\phi} \sim \left(\frac{r}{R_V}\right)^{8/3}. \tag{5.13}$$

If we set the mass of the source in (5.12) to be equal to the Planck mass, the corresponding cutoff scale in quantum theory for the decoupling of the scalar mode is obtained: $\Lambda_s = m_g^{-3/4}$.

In principle, there are enough different combinations of $\bar{h}^A_B$ which can be added to the action (5.7) to remove all the terms of the form $\left(\partial^2\pi\right)^k$ for all $k < n$, so that the first survived terms of this structure are $\left(\partial^2\pi\right)^{n.}$. Notice that such action is unique up to the order $\bar{h}^n$. In this case, the Vainshtein scale is determined by the condition

$$-\left.\psi\right|_{r=R_V} = \left(m_g R_V\right)^{\frac{2(n-1)}{n-2}}. \tag{5.14}$$

In the case of static gravitational field due to a massive source M_0 this yields

$$R_V = \left(M_0^{n-2} m_g^{2(1-n)}\right)^{\frac{1}{3n-4}} \tag{5.15}$$

and the correction to the gravitational potential for $r \ll R_V$ is of order

$$\frac{\delta\phi}{\phi} \sim \left(\frac{r}{R_V}\right)^{\frac{3n-4}{n-1}} \tag{5.16}$$

in agreement with [26]. In the limit when $n \to \infty$ the Vainshtein scale is $R_V = M_0^{1/3} m_g^{-2/3}$. It coincides with the corresponding scale in the DGP model. However, the corrections to the gravitational potential which decay as $(r/R_V)^3$ seem different. In this limit the theory is unambiguous, but one could write it only as an infinite series. In turn this indicates that such theory is most probably nonlocal. Moreover, because $\partial^2\pi \to 1$ we completely lose control of higher order corrections and hence the results become completely unreliable.

6 Conclusions

We have addressed the most fundamental question of all theories of massive gravity - can massive gravity be a consistent theory not contradicting to current experimental and theoretical knowledge? In this paper we have treated gravity mostly as a classical field theory and have explicitly investigated the issue of a smooth limit of massive gravity to General Relativity. With this purpose we first determined the physical degrees of freedom of the massive graviton generated via Higgs mechanism. This was done in the framework of irreducible representations of the three dimensional rotation group, where the five degrees of freedom of the graviton are described in terms of a tensor mode with two degrees of freedom and vector and scalar perturbations due to the scalar fields. The propagator for each of these five constituents of massive gravity was derived separately. In the linear approximation the origin of the well-known vDVZ discontinuity at the zero mass limit was traced to the constraint equations and it was shown how the scalar and vector modes of metric perturbations become non-dynamical in this limit.

It has been suggested long ago that the linear perturbation theory of massive gravity fails at length scales below the Vainshtein scale and one has to consider the full nonlinear theory to recover General Relativity below this scale. We have determined the Vainshtein scale in Higgs gravity, with Fierz-Pauli mass term, and found the explicit solution for the spherically symmetric gravitational field. We have shown that the massive gravity solution outside the Vainshtein scale smoothly goes to the General Relativity solution in the region deep inside the Vainshtein scale. Thus the classical results and predictions of General Relativity are recovered inside the Vainshtein scale and at distances exceeding the Vainshtein radius, massive gravity strongly differs from Einstein theory. This means that the scalar mode of massive graviton decouples at Vainshtein scale and enters the strong coupling regime. In the limit of vanishing mass, when Vainshtein radius becomes infinite, the symmetry is restored and our theory is reduced to General Relativity with four scalar fields which are confined and thus decoupled from gravity and other matter. Based on these results we have argued that in quantum theory there is a cutoff energy scale above which the scalar fields responsible for the scalar and vector modes of the massive graviton are strongly coupled and confined and hence harmless. For the realistic graviton mass

JHEP12(2010)023

this scale is extremely low. Therefore, the question about extra scalar mode and ghost instability seems to be irrelevant in our model.

We have found how the Vainshtein scale depends on the particular Higgs model or, in other words, on the nonlinear extension of the Fierz-Pauli mass term. In particular, we have shown that for given masses of the graviton and source, the Vainshtein length scale depends on the Lagrangian of the scalar fields and can be made arbitrary large. On the other hand, we have also constructed Lagrangians, which produce smaller scales compared to the standard one. However, the smallest possible scale seems to be larger than $M_0^{1/3} m_g^{-2/3}$.

Finally, we have calculated the corrections to General Relativity within the Vainshtein scale which could, in principle, be interesting from experimental point of view.

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Massive gravity: exorcising the ghost

Lasma Alberte,[a] **Ali H. Chamseddine**[b,c] **and Viatcheslav Mukhanov**[a,d]

[a] *Theoretical Physics, Ludwig Maxmillians University, Theresienstr. 37, 80333 Munich, Germany*
[b] *American University of Beirut, Physics Department, Beirut, Lebanon*
[c] *I.H.E.S., F-91440 Bures-sur-Yvette, France*
[d] *Department of Physics, New York University, New York, NY 10003, U.S.A.*

E-mail: lasma.alberte@physik.uni-muenchen.de, chams@aub.edu.lb, Viatcheslav.Mukhanov@physik.uni-muenchen.de

Abstract: We consider Higgs massive gravity [1, 2] and investigate whether a nonlinear ghost in this theory can be avoided. We show that although the theory considered in [10, 11] is ghost free in the decoupling limit, the ghost nevertheless reappears in the fourth order away from the decoupling limit. We also demonstate that there is no direct relation between the value of the Vainshtein scale and the existence of nonlinear ghost. We discuss how massive gravity should be modified to avoid the appearence of the ghost.

Keywords: Classical Theories of Gravity, Spontaneous Symmetry Breaking, Gauge Symmetry

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JHEP04(2011)004

Contents

1 Introduction 1

2 Higgs massive gravity 2

3 Boulware-Deser nonlinear ghost 3

4 Ghost in nonlinear extensions of massive gravity 5

5 Can we avoid the nonlinear ghost? 7

6 How dangerous are ghosts? 10

7 Conclusions 11

JHEP04(2011)004

1 Introduction

In [1, 2] we have devised a Higgs mechanism for massive gravity and demonstrated how this theory goes smoothly to General Relativity below the Vainshtein radius [3], thus resolving the problem of van Dam, Veltman and Zakharov discontinuity [4, 5]. This result, obtained in Higgs massive gravity, is in agreement with the results derived in bigravity theories in [6–8]. Moreover, we have found that the corresponding Vainshtein scale depends on the nonlinear extension of the Fierz-Pauli term [9]. In particular, it was shown that the Vainshtein scale can be changed within the range $M_0^{1/3} m_g^{-2/3} < R_V < M_0^{1/5} m_g^{-4/5}$, where M_0 and m_g are, respectively, the mass of the external source and the mass of the graviton in Planck units. The class of actions which lead to different Vainshtein scales R_V coincide with the actions derived in [10, 11]. These were obtained from the requirement of absence of the nonlinear ghost [12] in the corresponding order of perturbation theory, in the decoupling limit when both the graviton mass and the gravitational constant simultaneously vanish in such a way that the appropriate Vainshtein scale is kept fixed. Moreover, there is a unique action (up to total derivatives), corresponding to $R_V^{\infty} = M_0^{1/3} m_g^{-2/3}$, in the decoupling limit, for which the Boulware-Deser ghost does not appear at all below Vainshtein energy scale up to an arbitrary order in perturbation theory [10, 11]. Therefore, a natural interesting question arises as to whether this result could be sustained if we consider instead of the decoupling limit (which is not physical), the full nonlinear theory of massive gravity. The answer to this question will also help us understand whether there is any deep connection between the absence of nonlinear ghost at a certain order in perturbation theory and the corresponding value of the Vainshtein scale.

The main purpose of this note is to show that in the theories considered in [10, 11] away from the decoupling limit the nonlinear ghost inevitably arises in the fourth order of the perturbative expansion. The Vainshtein scale value becomes therefore unrelated to the absence of ghost if one does not consider the unrealistic decoupling limit of massive gravity.

The inevitable appearance of nonlinear ghost in "Lorentz invariant" massive gravity theories agrees with an independent argument of [13, 14] based on helicity decomposition.

Furthermore we argue that the nonlinear ghost can easily be avoided in General Relativity with only three scalar fields, which imitate "Lorentz violating" massive gravity around Minkowski background. This agrees with the results of the papers [15, 16] where Lorentz-violating graviton mass terms have been introduced by hand from the very beginning. Therefore in the theories considered in [15, 16] it was not so simple to keep under control such quite exotic phenomena like different maximal velocities for different particle species, superluminal propagation of particles, and violation of the no-hair theorems for the black hole solutions (see [17, 18] and references therein). On the other hand our approach allows us to preselect theories as General Relativity with three scalar fields which imitate Lorentz violating gravity but do not lead to dangerous consequences.

2 Higgs massive gravity

We employ four scalar fields ϕ^A, $A = 0, 1, 2, 3$, to play the role of Higgs fields. They will acquire a vacuum expectation value proportional to the space-time coordinates $\phi^A = \delta^A_\beta x^\beta$ giving mass to the graviton. Let us consider perturbations around Minkowski background,

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}, \qquad \phi^A = x^A + \chi^A \tag{2.1}$$

and define

$$\begin{aligned}\bar{h}^A_B \equiv \eta_{BC} g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^C - \delta^A_B = h^A_B + \partial^A\chi_B + \partial_B\chi^A \\ + \partial_C\chi^A\partial^C\chi_B + h^A_C\partial^C\chi_B + h^C_B\partial_C\chi^A + h^C_D\partial^D\chi_B\partial_C\chi^A,\end{aligned} \tag{2.2}$$

where indices are moved with the Minkowski metric $\eta_{AB} = (1, -1, -1, -1)$, in particular, $\chi_B = \eta_{BC}\chi^C$ and $h^A_B = \eta_{BC}\delta^A_\mu\delta^C_\nu h^{\mu\nu}$. After introducing the diffeomorphism invariant variable $\bar{h}^A_B$ it becomes almost trivial to write the terms that produce massive gravity. In the unitary gauge where $\chi^A = 0$, we have $\bar{h}^A_B = h^A_B = \eta_{BC}\delta^A_\mu\delta^C_\nu h^{\mu\nu}$, and hence the Fierz-Pauli term for the graviton mass around broken symmetry background can immediately be obtained from the quadratic term of the following action for the scalar fields

$$S_\phi = \frac{m_g^2}{8}\int d^4x\,\sqrt{-g}\left[\bar{h}^2 - \bar{h}^A_B\bar{h}^B_A + O\left(\bar{h}^3, \dots\right)\right]. \tag{2.3}$$

where by $O\left(\bar{h}^3, \dots\right)$ we denote the terms which are of the third and higher orders in $\bar{h}^A_B$. In distinction from the Fierz-Pauli action which was introduced by explicit spoiling of the diffeomorphism invariance, our action is manifestly diffeomorphism invariant and only coincides, to leading order, with the Fierz-Pauli action, in the unitary gauge where all perturbations of the scalar fields are set to zero.

JHEP04(2011)004

3 Boulware-Deser nonlinear ghost

One could, in principle, skip all higher order terms and consider the action

$$S=-\frac{1}{2}\int d^4x\,\sqrt{-g}R+\frac{m_g^2}{8}\int d^4x\,\sqrt{-g}\left[\bar{h}^2-\bar{h}^A_B\bar{h}^B_A\right], \tag{3.1}$$

where we set $8\pi G=1$, as an exact action for massive gravity. The problem then is either the presence of a ghost around the trivial background $\phi^A=0$ or the appearance of nonlinear ghost in the broken symmetry phase. To trace the latter one it is convenient to work in some gauge where the scalar field perturbations are not equal to zero. A good choice is the Newtonian gauge in which the metric $g_{\mu\nu}$ takes the form [19]

$$ds^2=(1+2\phi)\,dt^2+2S_idtdx^i-\left[(1-2\psi)\,\delta_{ik}+\tilde{h}_{ik}\right]dx^idx^k, \tag{3.2}$$

where $S_{i,i}=0$ and $\tilde{h}_{ij,i}=\tilde{h}_{ii}=0$. Then the ghost can easily be traced as a dynamical degree of freedom of the scalar field χ^0. The field χ^0 enters only the $\bar{h}^0_0$ and $\bar{h}^i_0$ components, which can be written explicitly as

$$\bar{h}^0_0=g^{00}-1+2g^{00}\dot{\chi}^0+g^{00}\left(\dot{\chi}^0\right)^2+2g^{0i}\chi^0_{,i}+2g^{0i}\dot{\chi}^0\chi^0_{,i}+g^{ik}\chi^0_{,i}\chi^0_{,k}, \tag{3.3}$$

and

$$\bar{h}^i_0=g^{0i}+g^{00}\dot{\chi}^i+g^{ik}\chi^0_{,k}+\left(g^{0i}+g^{00}\dot{\chi}^i+g^{k0}\chi^i_{,k}\right)\dot{\chi}^0+g^{k0}\chi^i_{,k}+g^{lk}\chi^i_{,l}\chi^0_{,k}+g^{0k}\chi^0_{,k}\dot{\chi}^i. \tag{3.4}$$

Let us consider only the scalar mode of the massive graviton for which $\chi^i=\pi_{,i}$. It was shown in [2] that by using constraints one can express the linear perturbations of the scalar fields in terms of the metric potential ψ as

$$\pi=\frac{2\Delta-3m_g^2}{m_g^2\Delta}\psi \tag{3.5}$$

$$\chi^0=-\frac{2\Delta+3m_g^2}{m_g^2\Delta}\dot{\psi}. \tag{3.6}$$

Then the action (3.1) up to second order in perturbations simplifies to

$$^{(S)}\delta_2S=-3\int d^4x\,\left[\psi\left(\partial_t^2-\Delta+m_g^2\right)\psi\right]. \tag{3.7}$$

The nonlinear ghost appears in the third order in metric and scalar field perturbations. This is due to the fact that the accidental $U(1)$ symmetry, which makes the scalar field χ^0 to be the Lagrange multiplier around Minkowski background, is not preserved on a background slightly deviating from Minkowski space [1]. To prove this it is enough to consider only the third order terms in the action (3.1) which involve the powers of $\dot{\chi}^0$. By substituting (3.3) and (3.4) into (3.1) we obtain

$$\delta_3S=\frac{m_g^2}{2}\int d^4x\left\{\left[\left(g^{00}-1+\sqrt{-g}\right)\bar{h}^i_i+\left(g^{0i}+\dot{\chi}^i-\chi^0_{,i}\right)\left(g^{0i}+\dot{\chi}^i\right)\right]\dot{\chi}^0+\frac{1}{2}\bar{h}^i_i\left(\dot{\chi}^0\right)^2+\dots\right\}, \tag{3.8}$$

where by dots we have denoted all other terms not containing time derivatives of χ^0. The term, linear in $\dot{\chi}^0$, does not induce dynamics for the mode χ^0 and simply modifies the constraint equations to second order in perturbations. However, the term proportional to $\left(\dot{\chi}^0\right)^2$ induces the propagation of χ^0 on the background deviating from Minkowski space for which $\bar{h}_i^i \neq 0$. Thus at nonlinear level there appears an extra scalar degree of freedom which is a ghost. To see this let us express the relevant term in (3.8) entirely in terms of the gravitational potential ψ. Taking into account that, to linear order, $\bar{h}_i^i = 6\psi + 2\Delta\pi$ and using constraint equations (3.5) and (3.6) we find

$$\delta_3 S = \frac{m_g^2}{4}\int d^4x \left[\bar{h}_i^i \left(\dot{\chi}^0\right)^2 + \ldots\right] = \int d^4x \left[\Delta\psi\left(\frac{2\Delta + 3m_g^2}{m_g^2\Delta}\ddot{\psi}\right)^2 + \ldots\right]. \tag{3.9}$$

By considering inhomogeneities with $\Delta\psi \gg m_g^2\psi$ and combining this contribution to the action (3.7) we obtain

$$\delta S = -3\int d^4x \left[\psi\left(\partial_t^2 - \Delta + m_g^2\right)\psi - \frac{4}{3m_g^4}\Delta\psi\left(\ddot{\psi}\right)^2 + \ldots\right]. \tag{3.10}$$

Let us assume that there is a background field ψ_b and consider small perturbations around this background, that is, $\psi = \psi_b + \delta\psi$. Expanding (3.10) to second order in $\delta\psi$ we find that the behavior of linear perturbations is determined by the action

$$\delta S = -3\int d^4x \left\{\delta\psi\left(\partial_t^2 - \Delta + m_g^2\right)\delta\psi + \frac{1}{m_{Gh}^2}\left[\left(\partial_t^2\delta\psi\right)^2 + 2\frac{\ddot{\psi}_b}{\Delta\psi_b}\left(\Delta\delta\psi\right)\left(\partial_t^2\delta\psi\right)\right] + \ldots\right\}, \tag{3.11}$$

where

$$m_{Gh}^2 = -\frac{3m_g^4}{4\Delta\psi_b}, \tag{3.12}$$

Let us take for the background field the scalar mode of gravitational wave with the wave-number $k \sim m_g$, for which $\ddot{\psi}_b \sim \Delta\psi_b \sim m_g^2\psi_b$ and $m_{Gh}^2 \sim m_g^2/\psi_b$. By considering perturbations $\delta\psi$ with wave-numbers $m_{Gh}^2 \gg k^2 \gg m_g^2$ and skipping subdominant terms we can rewrite the action above as

$$\delta S \approx -\frac{3}{m_{Gh}^2}\int d^4x\delta\psi\left(\partial_t^2 + \ldots\right)\left(\partial_t^2 + m_{Gh}^2 + \ldots\right)\delta\psi. \tag{3.13}$$

The perturbation propagator is given then by

$$\frac{1}{\partial^2\left(\partial^2 + m_{Gh}^2\right)} \simeq \frac{1}{m_{Gh}^2}\left(\frac{1}{\partial^2} - \frac{1}{\partial^2 + m_{Gh}^2}\right), \tag{3.14}$$

and it obviously describes the scalar mode of the graviton together with non-perturbative Boulware-Deser ghost of mass $m_{Gh} \sim m_g/\sqrt{\psi_b}$. It is clear that when ψ_b vanishes the mass m_{Gh} becomes infinite and ghost disappears. We have argued in [2] that at energies above Vainshtein scale $\Lambda_5 = m_g^{4/5}$ the linearized consideration above breaks down and the scalar fields enter the strong coupling regime. Therefore, if m_{Gh} would be larger than Λ_5 then this

JHEP04(2011)004

ghost would not be essential. However, in strong enough background $m_g < m_{Gh} < \Lambda_5$ and therefore the nonlinear ghost appears below the Vainshtein scale, where the perturbative expansion is trustable.

Thus, the action (3.1) considered as describing massive gravity has two problems with ghosts: first, there is a linear ghost around the trivial background $\phi^A = 0$, and second, there is nonlinear ghost around broken symmetry background.

The first ghost is dangerous, because it leads to a strong instability. However, as we have shown in [1], it can be easily avoided by adding to the action (3.1) third and higher order terms in $\bar{h}$. This modification is ambiguous and there is a whole class of theories which reproduce the Fierz-Pauli theory in the lowest order, avoiding linear ghosts around trivial background.

The nonlinear ghost exists only at scales below the Vainshtein energy scale which for the realistic graviton mass is extremely low, about $10^{-20}eV$. The existence of ghost would allow, for example, a process where ghost-photon pairs are spontaneously produced from the vacuum due to the gravitational interactions [20]. The energies of such photons would be of the order of cutoff scale Λ. The measurements of the differential photon flux in the diffuse gamma ray emission lead to the bound $\Lambda \leq 3\,\mathrm{MeV}$ [21]. Therefore, taking into account that the Vainshtein scale serves as the cutoff scale in Lorentz violating background, where the nonlinear ghost propagates, we conclude that this ghost is completely harmless. Nevertheless, some interesting questions remain. One could inquire whether there is any nonlinear extension of the action (3.1) which is free of the Boulware-Deser ghost and how the absence of the ghost in the corresponding order of a perturbative expansion is related with the concrete value of the Vainshtein scale?

4 Ghost in nonlinear extensions of massive gravity

Contrary to [13, 22, 23], it was claimed recently in [10, 11], that there is a unique ghost free nonlinear extension of massive gravity and that this extension is related with $\Lambda_3 = m_g^{2/3}$ Vainshtein scale. This claim was proved in [10, 11] in the decoupling limit neglecting the vector modes of the graviton. The decoupling limit, while simplifying the calculations, is not physically justified. Therefore, we will determine whether the nonlinear ghost really disappears away from the decoupling limit. The Lagrangian in [10, 11] is expressed in terms of the invariants built out of

$$H_{\mu\nu} = g_{\mu\nu} - \eta_{AB}\partial_\mu\phi^A\partial_\nu\phi^B. \tag{4.1}$$

It is easy to see (as was also noted in [24]) that the invariants built out of $H_{\mu\nu}$, up to sign, coincide with the invariants made of $\bar{h}^A_B$, in particular,

$$g^{\mu\nu}H_{\mu\nu} = -\bar{h}, \quad H_{\mu\nu}H^{\mu\nu} = \bar{h}^A_B\bar{h}^B_A, \quad \ldots \tag{4.2}$$

Let us consider the action [10, 11]:

$$
\begin{aligned}
S_\phi =& \frac{m_g^2}{8}\int d^4x\,\sqrt{-g}\left[\bar{h}^2-\bar{h}^2_{AB}+\frac{1}{2}\left(\bar{h}^3_{AB}-\bar{h}\bar{h}^2_{AB}\right)-\frac{5}{16}\bar{h}^4_{AB}+\frac{1}{4}\bar{h}\bar{h}^3_{AB}+\frac{1}{16}\left(\bar{h}^2_{AB}\right)^2\right.\\
&+c_3\left(2\bar{h}^3_{AB}-3\bar{h}\bar{h}^2_{AB}+\bar{h}^3+\frac{3}{4}\left(2\bar{h}^3_{AB}\bar{h}-2\bar{h}^4_{AB}+\left(\bar{h}^2_{AB}\right)^2-\bar{h}^2_{AB}\bar{h}^2\right)\right)\\
&\left.+d_5\left(6\bar{h}^4_{AB}-8\bar{h}^3_{AB}\bar{h}-3\left(\bar{h}^2_{AB}\right)^2+6\bar{h}^2_{AB}\bar{h}^2-\bar{h}^4\right)\right], \qquad (4.3)
\end{aligned}
$$

where c_3 and d_5 are arbitrary coefficients and we have introduced the shortcut notations

$$
\bar{h}^2_{AB}=\bar{h}^A_B\bar{h}^B_A,\quad \bar{h}^3_{AB}=\bar{h}^A_B\bar{h}^B_C\bar{h}^C_A,\quad \bar{h}^4_{AB}=\bar{h}^A_B\bar{h}^B_C\bar{h}^C_D\bar{h}^D_A.
$$

It was proved [10, 11] that this theory is ghost free to fourth order in perturbations in the decoupling limit. The action above corresponds to the Vainshtein scale $\Lambda = m_g^{8/11}$ [2]. Let us investigate whether the ghost really disappears in non-decoupling limit. For this purpose we have to trace all fourth order terms in perturbations which contain time derivatives of χ^0. As we have noticed above, the time derivatives of χ^0 come only from $\bar{h}^0_0$ and $\bar{h}^i_0$ components. Therefore the only terms in (4.3), which survive and could be relevant for a possible ghost are the following

$$
\begin{aligned}
S_\phi =& \frac{m_g^2}{8}\int d^4x\,\sqrt{-g}\left[\left(2\bar{h}^0_0-\frac{1}{2}\left(\bar{h}^0_0\right)^2+\frac{1}{4}\left(\bar{h}^0_0\right)^3\right)\bar{h}^i_i-\frac{1}{4}\left(2\bar{h}^0_0-\frac{1}{2}\left(\bar{h}^0_0\right)^2\right)\bar{h}^2_{ik}\right.\\
&\left.+2\bar{h}^i_0\bar{h}^i_0-\frac{1}{2}\bar{h}^0_0\bar{h}^i_0\bar{h}^i_0+\frac{1}{4}\left(\bar{h}^0_0\right)^2\bar{h}^i_0\bar{h}^i_0+\frac{3}{2}c_3\left(2\bar{h}^0_0-\frac{1}{2}\left(\bar{h}^0_0\right)^2\right)\left(\left(\bar{h}^i_i\right)^2-\bar{h}^2_{ik}\right)+\dots\right]. \qquad (4.4)
\end{aligned}
$$

We have skipped here the terms which are linear in $\dot{\chi}^0$ because they only modify the constraints without inducing the dynamics for χ^0. We would like to stress that the particular choice of action (4.3) has lead to nontrivial cancellations of many terms which could have caused the appearance of a ghost. In particular, all contributions which induce the terms proportional to $\left(\dot{\chi}^0\right)^2$, $\left(\dot{\chi}^0\right)^3$, $\left(\dot{\chi}^0\right)^4$ are canceled in the d_5, term in (4.3). Further nontrivial cancellations happen when we substitute (3.3) and (3.4) in (4.4), and the final result is

$$
\delta_3 S_\phi+\delta_4 S_\phi=\frac{m_g^2}{8}\int d^4x\left[F\left(\delta g,\chi\right)\dot{\chi}^0+\frac{1}{2}\left(\dot{\chi}^i+S_i+g^{0i}+\chi^0_{,i}\right)^2\left(\dot{\chi}^0\right)^2+\dots\right], \qquad (4.5)
$$

where we denote by dots the terms which do not depend on $\dot{\chi}^0$. Note that the third and fourth powers of $\dot{\chi}^0$ are cancelled. The function $F\left(\delta g,\chi\right)$ is some rather long and complicated expression which depends on terms of second and third order in perturbations but does not depend on $\dot{\chi}^0$. Because this term does not induce the dynamics of χ^0, but simply modifies the constraints, we do not need the explicit form of F. Note that the third order terms with second and third powers of $\dot{\chi}^0$ are canceled and hence the ghost does not appear in the third order even if we do not consider the decoupling limit. However, in the fourth order in perturbations the nonlinear ghost survives. It is easy to see that this ghost disappears in the decoupling limit in agreement with [10, 11]. In fact, after skipping the

JHEP04(2011)004

vector modes, we have $\chi^i = \pi_{,i}$, $S_i = 0$ and considering the decoupling limit $(m_g^2 \to 0)$ we obtain from (3.5) and (3.6) that $\chi^0 \to -\dot{\pi}$ and hence the second term in (4.5) vanishes. However, without taking this limit, action (4.5) becomes

$$\delta_3 S_\phi + \delta_4 S_\phi = \frac{m_g^2}{16} \int d^4x \left[\left(\dot{\tilde{\chi}}^i + S_i + \left(\dot{\pi} + \chi^0 \right)_{,i} \right)^2 \left(\dot{\chi}^0 \right)^2 + \ldots \right]$$
$$= \frac{m_g^2}{16} \int d^4x \left[\left(\dot{\tilde{\chi}}^i + S_i - \frac{6}{\Delta} \dot{\psi}_{,i} \right)^2 \left(\frac{2\Delta + 3m_g^2}{m_g^2 \Delta} \ddot{\psi} \right)^2 + \ldots \right] \tag{4.6}$$

where we have taken into account that $\chi^i = \pi_{,i} + \tilde{\chi}^i$ and $\tilde{\chi}^i$ is a vector mode of graviton. Considering small perturbations $\delta\psi$ with wave-numbers $k^2 \gg m_g^2$ around some background ψ_b and $\tilde{\chi}_b^i$ we find as in the previous considerations (see (3.10)–(3.12)) that this action describes along with the scalar mode of graviton also a ghost of mass

$$m_{Gh}^2 = -12 m_g^2 \left(\dot{\tilde{\chi}}_b^i + S_i - \frac{6}{\Delta} \dot{\psi}_{b,i} \right)^{-2} \tag{4.7}$$

provided that m_{Gh}^2 satisfies the condition $\partial_t^2 m_{Gh}^{-2} \ll 1$. In the background of the scalar gravitational wave ψ_b with $k^2 \simeq m_g^2$ we have $m_{Gh} \sim m_g/\psi_b$. If the time dependent background fields are strong enough the mass of this ghost is smaller than the Vainshtein scale and can be even as small as the graviton mass. Thus, if one does not consider the decoupling limit of the theory the action (4.3) has a nonlinear ghost in the fourth order of perturbation theory. This ghost cannot be removed by adding fifth and higher order terms and it is inevitable in the theories considered in [10, 11].

5 Can we avoid the nonlinear ghost?

The theory described by action (4.3) could be a unique candidate for a ghost free massive gravity (to fourth order in perturbations) because it is the only theory which does not have a ghost in the decoupling limit [10, 11]. It's higher order extension which removes ghost to an arbitrary order is also uniquely determined by the requirement of the absence of ghost in decoupling limit. Thus the theory satisfies the necessary condition to be a ghost free theory. However, this condition is not sufficient to avoid ghost in the full fourth order nonlinear theory. Unfortunately, as we have shown, the theory considered above inevitably has an unremovable nonlinear ghost beginning with the fourth order in perturbations. One can wonder whether there is any way of avoiding this no-go theorem? It is clear that using the variables $H_{\mu\nu}$ defined in (4.1) one is forced to use only the invariants present in (4.3) because otherwise the fundamental diffeomorphism invariance of the theory will be spoiled. On the other hand in our approach the variables H^{AB} are scalars under diffeomorphism transformations. The "internal Lorentz invariance" under scalar fields transformations $\phi^A \to \tilde{\phi}^A = \phi^B \Lambda_B^A$ was simply used to imitate massive gravity with Fierz-Pauli term. However, we are not obliged to preserve this fake Lorentz invariance. In fact, there is nothing wrong from the point of view of symmetries to consider for instance the Lagrangian

$$S_\phi = \frac{m_g^2}{8} \int d^4x \sqrt{-g} \left(g^{\mu\nu} \partial_\mu \phi^0 \partial_\nu \phi^0 - 1 \right)^2 = \frac{m_g^2}{8} \int d^4x \sqrt{-g} \left(\bar{h}_0^0 \right)^2 \tag{5.1}$$

which is diffeomorphism and Lorentz invariant and simply describes the scalar field ϕ^0 with an unusual kinetic term similar to k-inflation [25, 26]. Therefore, without spoiling any fundamental invariances we could modify the action above by adding to it terms of the form $\left(\bar{h}_i^0\right)^2 \bar{h}_0^0$, $\left(\bar{h}_i^0\right)^2$, etc. It is easy to verify that the only terms in (4.4) responsible for the ghost are

$$\delta S_{Ghost} \equiv \frac{m_g^2}{8} \int d^4x \left[2\bar{h}_0^i \bar{h}_0^i - \frac{1}{2} \bar{h}_0^i \bar{h}_0^i \bar{h}_0^0 + \frac{1}{4} \bar{h}_0^i \bar{h}_0^i \left(\bar{h}_0^0\right)^2 \right]. \tag{5.2}$$

Therefore subtracting these terms from action (4.3) removes the ghost in the fourth order. In turn this also inevitably modifies the quadratic part of the action and instead of Fierz-Pauli term we obtain

$$\begin{aligned} S_\phi &= \frac{m_g^2}{8} \int d^4x \sqrt{-g} \left[\bar{h}^2 - \bar{h}_B^A \bar{h}_A^B - 2\bar{h}_0^i \bar{h}_0^i + O\left(\bar{h}^3, \ldots\right) \right] \\ &= \frac{m_g^2}{8} \int d^4x \sqrt{-g} \left[\left(\bar{h}_i^i\right)^2 - \bar{h}_k^i \bar{h}_i^k + 2\bar{h}_0^0 \bar{h}_i^i + O\left(\bar{h}^3, \ldots\right) \right]. \end{aligned} \tag{5.3}$$

As a result both scalar and vector modes of the graviton disappear and the action above describes the massive transverse graviton with two degrees of freedom. Note that this result does not contradict Wigner's theorem about the number of degrees of freedom of massive particle with spin-two because in this case the scalar fields background in the broken symmetry phase is not Lorentz invariant. Nevertheless, we would like to stress that in Higgs gravity which produces the massive graviton with two degrees of freedom there is no violation of the fundamental space time Lorentz invariance in distinction from [15, 16] where the spacetime Lorentz invariance is violated explicitly by adding distinct mass terms for time and spatial components of the metric perturbations $h_{\mu\nu}$ like $m_0^2 h_{00}^2$, $m_2^2 h_{ij}^2, \ldots$. However, effectively the graviton mass term (5.3) could be identified with the theory considered in [15] with $m_0^2 = m_1^2 = 0$ and $m_2^2 = m_3^2 = m_4^2 = m_g^2$. Although this configuration of mass parameters was not conisdered there, the authors would arrive at the same conclusion regarding the number of degrees of freedom of the massive graviton. The effective violation of the spacetime Lorentz invariance in our approach is simply due to the existence of a background scalar field in Minkowski space in a way similar to the violation of this invariance by the cosmic microwave background radiation in our universe. In the case when we had imposed the extra "Lorentz invariance" in the configuration space of the scalar fields we were able to imitate the space-time Lorentz invariance for the graviton mass term simply via redefinition of the scalar fields in the unitary gauge. However, in general when this invariance is absent any scalar fields background violates space-time Lorentz invariance explicitly.

The "Lorentz violating" procedure of removing the nonlinear ghost in Higgs gravity can be extended to any higher orders in the theory considered in [10, 11]. However, if we allow the "Lorentz violating" terms then there is no need anymore for such extension. We can simply consider

$$S_\phi = \frac{m_g^2}{8} \int d^4x \sqrt{-g} \left[\left(\bar{h}_i^i\right)^2 - \bar{h}_k^i \bar{h}_i^k \right], \tag{5.4}$$

as an exact action of massive gravity on a Lorentz violating background. It is obvious that this action depends only on three scalar fields and does not have any linear and nonlinear ghosts around any background. The transverse gravitational degrees of freedom $\tilde{h}_{ik}$ become massive and one could wonder how it will modify the usual Newtonian interaction between massive objects. To answer this question let us consider a static gravitational field produced by a matter for which only T^{00} component of the energy-momentum tensor does not vanish. The metric in this case can be written as

$$ds^2 = (1 + 2\phi)\, dt^2 - (1 - 2\psi)\, \delta_{ik} dx^i dx^k, \tag{5.5}$$

and the action for static perturbations derived in [2] (see formulae (28) and (36) there) in the case of (5.4) simplifies to

$$\begin{aligned} {}^{(S)}\delta S = \int d^4x \Big\{ &\psi_{,i}\psi_{,i} + \phi\left[2\Delta\psi - T^{00}\right] + \frac{m_g^2}{2}\left[6\psi^2 + 4\psi\Delta\pi + \right. \\ &\left. + \left(\Delta\pi\pi_{,ik}\pi_{,ik} - \pi_{,ki}\pi_{,ij}\pi_{,jk}\right) - 2\psi\left(\pi_{,ik}\pi_{,ik} - 2\left(\Delta\pi\right)^2\right)\right] \\ &+ O\left(\psi^3, \psi^2\phi, \psi^2\Delta\pi, \phi\psi\Delta\pi\ldots\right)\Big\} \end{aligned} \tag{5.6}$$

Varying this action with respect to ϕ, ψ and π, and assuming that $\Delta\pi \ll 1$ we obtain the following equations

$$\Delta\psi = \frac{T^{00}}{2}, \qquad \Delta\left(\psi - \phi - m_g^2\pi\right) - 3m_g^2\psi = 0, \tag{5.7}$$

$$\Delta\psi + \frac{1}{2}\left(\Delta\pi\pi_{,ik}\right)_{,ik} + \frac{1}{4}\Delta\left(\pi_{,ik}\pi_{,ik}\right) - \frac{3}{4}\left(\pi_{,ij}\pi_{,jk}\right)_{,ik} = 0. \tag{5.8}$$

For consistency, we have to include the higher order terms in $\Delta\pi$ because otherwise the first equation in (5.7) would contradict to the equation (5.8). The reason is that the scalar fields in this case are always in strong coupling regime. In particular, given ψ which is induced by the matter source according to Poisson equation and remains unmodified at all, we obtain from (5.8) the following estimate for induced scalar fields

$$\partial\partial\pi \sim \Delta\pi \sim \sqrt{\psi}. \tag{5.9}$$

Then considering the spherically symmetric source of mass M_0 from the second equation in (5.7) one derives

$$\psi - \phi \simeq O(1)\, \psi \left(\frac{r}{R_V}\right)^{5/2}. \tag{5.10}$$

At distances much smaller than Vainshtein radius $R_V = \left(M_0/m_g^4\right)^{1/5}$ we have $\psi = \phi$ with high accuracy and thus we recover General Relativity with corrections which are the same as in the case of Fierz-Pauli mass term (see [2]). However, for $r \gg R_V$ the gravitational potential ϕ grows as $r^{3/2}$, while ψ decays exactly as in Newtonian theory. This is due to the fact that the contribution of the energy of the field π, induced by the external source of the matter, becomes comparable with the energy of this source at the scales larger than the Vainshtein radius. To find a solution in this range we have to solve exactly the complete nonlinear system of equations. However it is obvious that at distances larger than Vainshtein radius we do not reproduce the results of massive gravity with the Fierz-Pauli mass term (see [2]). For the realistic graviton mass, Vainshtein radius for the Sun is huge and before we cross it the contribution of the other mass sources in the universe become important. Smearing the matter distribution and considering the homogeneous universe we find that for $m_g \simeq H_0$, where H_0 is the present value of the Hubble constant, the Vainshtein radius in this case is of order of horizon scale H_0^{-1}. Therefore massive gravity with action (5.4) is in agreement with experiment. An interesting question that needs investigation is to determine how General Relativity will be modified on the horizon scale (a question which could be relevant for the dark energy problem).

6 How dangerous are ghosts?

It is clear that the linear ghost around trivial background with $\phi^A = 0$ is extremely dangerous because it leads to a catastrophic instability of the vacuum and drastically reduces the lifetimes of the particles. We have shown in [1] how this ghost can be easily avoided. In distinction from it the nonlinear ghost seems to be unavoidable in all Lorentz invariant versions of massive gravity. This nonlinear ghost inevitably arises at latest in the fourth order of perturbation theory on a background which slightly deviates from the Minkowski space. How dangerous is this ghost? There exist different opinions on this subject. The main reason why those who think that it is catastrophic is the integration over the Lorentz boosts in order to insure Lorentz invariant cutoff. Leaving the question of the need to integrate over boosts aside we note however that anyway the nonlinear ghost appears only on the background which deviates from the Minkowski space. In turn this background selects the preferable coordinate system where we have a Lorentz violating cutoff on the energy scale below which the ghost exists. This cutoff is the corresponding Vainshtein energy scale, which is extremely low, of order of 10^{-20} eV for the realistic graviton mass. It is clear that the ghost with such energies is completely harmless from the point of view of agreement with experiments [20]. Therefore we believe that the nonlinear ghost in any theory of massive gravity is irrelevant. In such case one could wonder if we can avoid the requirement that the only possible Lorentz invariant graviton mass term is the Fierz-Pauli one? To answer this question let us consider the theory with the action

$$S_\phi = \frac{m_g^2}{8} \int d^4x \sqrt{-g} \left[\bar{h}^2 - \bar{h}^A_B \bar{h}^B_A + \alpha \bar{h}^2 + O\left(\bar{h}^3, \dots\right)\right]. \tag{6.1}$$

It is easy to see that if α is different from zero then already at quadratic order in the action there appears the term $\alpha \left(\dot{\chi}^0\right)^2$ which inevitably leads to a dangerous linear ghost. Moreover, for $\alpha \sim O(1)$ the Vainshtein scale disappears in this theory. This can be easily seen if we rewrite equations (31), (39) and (41) from our previous paper [2] taking into account the relevant contributions from $\alpha \bar{h}^2$ term in action (6.1)

$$\Delta(\phi + \psi) + \frac{\alpha}{3\alpha + 2} \Delta(\phi - \psi) = T^{00} + m_g^2 \times (\dots), \tag{6.2}$$

$$(2\psi - \phi) + \frac{\alpha}{\alpha + 1}(\psi + \Delta\pi) + \partial^4 \pi^2 = 0, \tag{6.3}$$

$$(1 + 2\alpha)\psi + \frac{(3\alpha + 2)(\alpha + 1)}{2} m_g^2 \pi + \alpha \Delta\pi + \partial^4 \pi^2 = 0. \tag{6.4}$$

The nonlinear Vainshtein scale was determined before by the requirement that in equation (6.4) the linear term in π is equal to the last non-linear term. However, we now have also an extra linear term in this equation which is always larger than the non-linear term if $\Delta\pi \ll 1$. Hence the non-linear term in this equation is negligible and we always remain in the week coupling regime. By considering the scales for which $k^2 \gg m_g^2$ it follows from (6.4) that

$$\Delta\pi = -\frac{(1+2\alpha)}{\alpha}\psi. \tag{6.5}$$

Substituting this expression in (6.3) we find that up to the leading order $\psi = \phi$ and hence as it follows from (6.2) curiously enough General Relativity is restored (at least in the leading approximation) without having problem with vDVZ discontinuity [4, 5]. Nevertheless the above theory is unacceptable because of the linear ghost which exists at all scales up to the Planckian one.

7 Conclusions

We have investigated the problem of the non-linear Boulware-Deser ghost in massive gravity. In particular, we have used the gravity Higgs mechanism to study whether the unique theory proposed in [10, 11] remains ghost free away from the decoupling regime. Although we have confirmed the result of [10, 11] in decoupling limit, we unfortunately find by explicit calculations that a non-linear unremovable ghost reappears in this theory below Vainshtein energy scale in fourth order of perturbation theory away from the decoupling limit. At the same time, as was shown in [1, 2], the theories considered in [10, 11], can discretely change the Vainshtein scale within the range $M_0^{1/3} m_g^{-2/3} < R_V < M_0^{1/5} m_g^{-4/5}$. Thus, the claim that massive gravity with Vainshtein scale $M_0^{1/3} m_g^{-2/3}$ is ghost free is not confirmed in the full theory and moreover the nonlinear ghost problem does not seem to be directly related to the concrete value of the Vainshtein scale.

Higgs massive gravity [1, 2] is equivalent to the formulation in [10, 11] provided one preserves the "internal Lorentz invariance" in the space of the scalar field configurations. We have argued that in Higgs gravity, in distinction from [10, 11], the ghost can be cancelled because the diffeomorphism invariance of the variables $\bar{h}^A_B$ allows to add appropriate counterterms to cancel the undesired negative energy mode in the action (5.3). This however can only be done if we abandon the Lorentz invariance in the scalar field configuration space without violating the fundamental space-time Lorentz invariance and diffeomorphism invariance of the action in distinction from the Lorentz-violating actions of massive gravity considered in [15, 16]. As a result the propagators for the scalar and vector modes of the massive graviton vanish and the action (5.3) describes a massive graviton with two physical degrees of freedom.

To summarize, we have shown that even for the simplest action, which at leading order reproduces the Fierz-Pauli mass term and ignoring the higher order terms in $\bar{h}^A_B$, the Boulware-Deser ghost will arise in third order of perturbation theory. Moving away from the decoupling limit, while keeping the contributions of the vector modes in the action, we have established the existence of the ghost state. We calculated the mass of the ghost mode

JHEP04(2011)004

m_{Gh} in the short wavelength approximation for perturbations around some locally Lorentz violating background. Moreover, with strong enough background fields it is possible to make the negative energy mode as light as needed within the interval $m_g < m_{Gh} < \Lambda_5$. However, as was argued in [2], above the Vainshtein energy scale Λ_5 the scalar metric perturbations ψ as well as the scalar field perturbations χ^A are in the strong coupling regime and possess no propagator. Therefore, the ghost is propagating on the locally nontrivial background only below the Vainshtein energy scale which for a graviton mass of the order of the present Hubble scale is extremely low and hence the ghost is harmless.

Further, we have shown that by adding terms of higher order in $\bar{h}^A_B$ to the action with the choice of coefficients corresponding to the Vainshtein scale $\Lambda = m_g^{8/11}$ the nonlinear ghost disappears from the third order of perturbations. However, away from the decoupling limit the Boulware-Deser ghost, although harmless, appears at the fourth order of perturbation theory and cannot be removed by adding higher order terms to the Lagrangian. This allows us to conclude that the value of the Vainshtein scale which tells us up to which energy scale a perturbation theory of a given order is trustable and the presence of the nonlinear ghost in the theory are two separate issues which do not have to be correlated.

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Reduced Massive Gravity with Two Stückelberg Fields

Lasma Alberte[a,1], **Andrei Khmelnitsky**[a,2]

[a] *Arnold Sommerfeld Center for Theoretical Physics, Ludwig Maximilians University, Theresienstr. 37, 80333 Munich, Germany*

Abstract: We consider the non-linear massive gravity as a theory of a number of Stückelberg scalar fields minimally coupled to the Einstein-Hilbert gravity and argue that the counting of degrees of freedom can be done for scalar theory and gravity separately. In this paper we investigate the system with only two Stückelberg scalar fields. In this case we find the analytic expression for the determinant of the kinetic matrix of the scalar field Lagrangian and perform the full constraint analysis. In $1+1$ space-time dimensions the theory corresponds to the full non-linear massive gravity, and this determinant vanishes identically. In this case we find two first-class constraints, and present the corresponding gauge symmetry of the theory which eliminates both scalar degrees of freedom. In $3+1$ dimensions the determinant of the kinetic matrix does not vanish identically and, for generic initial conditions, both scalar fields are propagating.

[1] Email: lasma.alberte@physik.lmu.de
[2] Email: khmelnitskiy@physik.lmu.de

Contents

1 Introduction **1**

2 Non-linear massive gravity in Stückelberg formulation **3**

3 Reduced massive gravity **5**
- 3.1 Number of degrees of freedom 6
- 3.2 Determinant of the kinetic matrix 7
- 3.3 Hamiltonian analysis away from $\det S = 0$ 9
- 3.4 Time evolution in the vicinity of the singular surface 11

4 The singular surface with $\det S = 0$ **12**
- 4.1 1+1 dimensions: massive gravity 12
- 4.2 Minkowski background 13
  - 4.2.1 Constraint algebra 13
  - 4.2.2 Gauge symmetry 14
- 4.3 Arbitrary background 16
- 4.4 3+1 dimensions 17

5 Conclusions **18**

1 Introduction

The observation of the accelerated expansion of our universe is the driving motivation for various infrared modifications of general relativity. One of the theoretically most natural infrared modification would be to give a small mass to the graviton. Since the early discovery of the quadratic Fierz-Pauli mass term for metric perturbations in [1], there has been an ongoing search for a healthy non-linear completion of massive gravity. The construction of the non-linear graviton mass term is based on the use of an auxiliary non-dynamical reference metric, which as an absolute object would break the diffeomorphism invariance of general relativity. The diffeomorphism invariance can be restored by introducing four Stückelberg scalars, corresponding to the four coordinate transformations [2–4]. However, a generic theory of four Stückelberg scalars together with the two degrees of freedom of massless graviton propagates six degrees of freedom in total. It is one degree of freedom too much in comparison to the five degrees of freedom expected from the massive spin-2 representations of the Poincaré group. Moreover, the additional degree of freedom is sick and represents the (in)famous Boulware-Deser (BD) ghost [5].

After an order-by-order construction of a non-linear theory which is ghost-free in the decoupling limit in [6], a full resummed theory of non-linear massive gravity was proposed by de Rham, Gabadadze, and Tolley (dRGT) [7]. In unitary gauge this theory has been shown to propagate five degrees of freedom [8, 9]. The Hamiltonian analysis of the full diffeomorphism invariant theory including the four Stückelberg fields also seems to confirm the expectation that the dRGT theory propagates at most five degrees of freedom [10–12] (for recent counterarguments see [13]). However, the canonical analysis of dRGT theory in the presence of the four scalar fields is intricate, and in the existent literature it is often obscured either by mixing the gravitational and scalar degrees of freedom or by introduction of new auxiliary fields.

In the present paper we take a different point of view and treat dRGT massive gravity as a theory of Stückelberg scalar fields ϕ^A coupled to the Einstein-Hilbert gravity. Since the theory is reparametrization invariant, and the scalars are coupled to gravity minimally, we shall count the degrees of freedom propagated by the metric and by the scalar fields separately. Hence the absence of the sixth mode in dRGT theory should manifest itself as the feature of the scalar fields Lagrangian alone.

Motivated by these considerations we study the dynamics of the Stückelberg scalar fields given by the dRGT mass term [7]. We observe that, if seen as a particular scalar field theory, the dRGT scalar field Lagrangian allows for an arbitrary number of scalar fields in it. In particular, the number of scalar fields N can be chosen to be less than the space-time dimension $d+1$ without affecting the diffeomorphism nor the space-time Lorentz invariance of the theory. We dub the dRGT theories of gravity with reduced number $N < d + 1$ of Stückelberg scalar fields as "reduced massive gravity".

The simplest particular cases of such dRGT inspired scalar theories include, for $d = 0$, the action of a massive relativistic particle in N dimensions and, for $N = 1$, the single "k-essence" field with DBI-like action [14]. Another "simple" choice is arbitrary N fields in $1 + 1$ dimensions, and gives the action of a relativistic string in N-dimensional target space-time. In the case $N = 3$, with three scalar fields living in a configuration space diffeomorphic to $\mathbb{R}^3$, the reduced dRGT action can be regarded as a particular effective field theory of homogeneous solid [15]. The degree of symmetry of the solid depends on the isometries of the metric $f_{AB}(\phi)$ in the internal space of scalar fields. If the metric is symmetric under the $SO(3)$ group, and the action contains only the term, invariant under the volume preserving diffeomorphisms, then it describes a perfect fluid. The case with the number of scalar fields $N \geq d + 1$ has been recently discussed in [16, 17] as a theory of multiple Galileon fields covariantly coupled to the dRGT massive gravity.

In general, the solutions of the reduced massive gravity theories are expected to break Lorentz and rotational symmetries and lead to anisotropic cosmologies. The pattern of such breaking is determined by the number of scalar fields and the signature and isometries of the reference metric. The connection of reduced massive gravity theories to the Lorentz violating massive gravity theories will be discussed in more detail in the main body of the paper. Another possible application of reduced massive gravity theories could be found in modeling the translational symmetry breaking and momentum dissipation in holography. In particular, in [18] the conductivity in the boundary theory was calculated in the presence of a Lorentz violating graviton mass term in the bulk, that originated from the dRGT-like action with two Stückelberg fields and Euclidean reference metric. The models discussed in our paper could be further used in holographic constructions.

In this paper we consider the case of reduced massive gravity with two Stückelberg fields. It is the simplest case with several scalar fields involved, in which we can write the Hamiltonian and constraint structure explicitly. We perform the full Hamiltonian analysis of the scalar field sector and find that, in distinction from the dRGT massive gravity the determinant of the kinetic matrix does not vanish. Hence the scalar field Lagrangian in general propagates two degrees of freedom. We formulate the condition for the scalar field configurations on which the determinant vanishes and investigate the different regions in the phase space of scalar fields. We show that on the singular surface, where the determinant of the kinetic matrix vanishes, the theory is equivalent to $1+1$-dimensional massive gravity and thus has no dynamical degrees of freedom. We also show that the regular solutions away but in close vicinity of the singular surface approach the singular surface but can never reach it in finite time. At the same time any perturbation of the singular solution drives the system away from this singular surface. In quantum theory the vanishing of the determinant signals the strong coupling regime for the scalar fields, and the dynamics in the vicinity of the singular surface are highly affected by quantum corrections. Whether or not the two dynamical degrees of freedom away from the singular surface contain ghost modes might depend on the particular choice of the reference metric in the configuration space of the scalar fields. We do not address this question in the present paper, but leave it for future studies.

The paper is organized as follows. In section 2 we recall the formulation of dRGT massive gravity. In section 3 we formulate the theory of reduced massive gravity and perform the Hamiltonian analysis away from the singularity surface. In section 4 we consider the behaviour of the system on the singular surface, and show that it is equivalent to $1+1$ dimensional massive gravity. We perform the canonical analysis in this case and find the gauge symmetry of the scalar fields, eliminating both scalar degrees of freedom. Section 5 is devoted to conclusions.

2 Non-linear massive gravity in Stückelberg formulation

The non-linear massive gravity action can be written in terms of the variables

$$\mathcal{K}^\mu_\nu = \delta^\mu_\nu - \left(\sqrt{g^{-1}f}\right)^\mu_\nu \ , \tag{2.1}$$

where $g^{\mu\nu}$ is the inverse space-time metric, and $f_{\mu\nu}$ is an auxiliary reference metric. The full dRGT action is given by

$$\mathcal{L}_{EH} + m^2\mathcal{L}_\phi = \frac{M_P^2}{2}\sqrt{-g}R + m^2\sqrt{-g}\sum_{n=0}^{4}\tilde{\alpha}_n \mathsf{e}_n(\mathcal{K}) \ , \tag{2.2}$$

where the characteristic polynomials $\mathsf{e}_n(\mathbb{X})$ of a 4×4 matrix $\mathbb{X}$ are

$$\mathsf{e}_0(\mathbb{X}) = 1 \ , \qquad \mathsf{e}_1(\mathbb{X}) = [\mathbb{X}] \ , \qquad \mathsf{e}_2(\mathbb{X}) = \frac{1}{2}\left([\mathbb{X}]^2 - [\mathbb{X}^2]\right) \ ,$$
$$\mathsf{e}_3(\mathbb{X}) = \frac{1}{6}\left([\mathbb{X}]^3 - 3[\mathbb{X}][\mathbb{X}^2] + 2[\mathbb{X}^3]\right) \ , \qquad \mathsf{e}_4(\mathbb{X}) = \det\mathbb{X} \ .$$

The squared brackets denote the traces, and the coefficients $\tilde{\alpha}_n$ are arbitrary. It is also possible to rewrite the mass term in terms of the characteristic polynomials of the square root matrix $\left(\sqrt{\Omega}\right)^\mu_\nu \equiv \left(\sqrt{g^{-1}f}\right)^\mu_\nu$ as

$$\mathcal{L}_\phi = \sqrt{-g}\sum_{k=0}^{4}\tilde{\beta}_k \mathsf{e}_k(\sqrt{\Omega}) \;, \tag{2.3}$$

with the coefficients $\tilde{\beta}_k$ given by

$$\tilde{\beta}_k = \sum_{n=k}^{4}(-1)^k \begin{pmatrix} 4-k \\ n-k \end{pmatrix} \tilde{\alpha}_n \;. \tag{2.4}$$

The characteristic polynomials of an $n \times n$ matrix $\mathbb{X}$ can be rewritten as the characteristic polynomials of its eigenvalues λ_i as $\mathsf{e}_n(\mathbb{X}) = \mathsf{e}_n(\lambda_i)$ [19, 20], where

$$\begin{aligned}
&\mathsf{e}_0(\lambda_i) = 1 \;, \qquad \mathsf{e}_1(\lambda_i) = \sum_i \lambda_i = [\mathbb{X}] \;, \\
&\mathsf{e}_2(\lambda_i) = \sum_{i<j}\lambda_i\lambda_j \;, \\
&\qquad \vdots \\
&\mathsf{e}_n(\lambda_i) = \lambda_1\lambda_2\ldots\lambda_n = \det\mathbb{X} \;.
\end{aligned}$$

Since here the matrix $\mathbb{X} = \sqrt{\Omega}$ is a square root matrix, we note that its eigenvalues are, by definition, equal to the square root $\sqrt{\lambda_i}$ of the eigenvalues of the matrix Ω. Hence the mass term (2.3) can be rewritten in terms of the eigenvalues of the matrix Ω^μ_ν, without the need of finding the explicit expression of the square root matrix itself, as

$$\mathcal{L}_\phi = \sqrt{-g}\sum_{k=0}^{4}\tilde{\beta}_k \mathsf{e}_k(\sqrt{\lambda_i}) \;. \tag{2.5}$$

Since the mass term (2.3) explicitly depends on the auxiliary metric $f_{\mu\nu}$ it breaks the diffeomorphism invariance of general relativity. It can be fully restored by introducing four Stückelberg scalar fields ϕ^A, $A = 0, 1, 2, 3$, corresponding to the four coordinate transformations as $f_{\mu\nu} = \partial_\mu\phi^A\partial_\nu\phi^B\eta_{AB}$ [3]. In addition, in this parametrization the auxiliary metric is invariant under the Lorentz transformations Λ^A_B in the scalar field space [4]. Hence the scalar field indices A, B are raised and lowered with the Minkowski metric $\eta_{AB} = \mathrm{diag}\,(-\;+\;+\;+)$. In this case the reference metric $f_{\mu\nu}$ is said to be 'flat' since it is simply a coordinate transformation from the flat Minkowski metric η_{AB}. An arbitrary 'curved' reference metric $f_{\mu\nu}$ can be obtained by replacing the flat metric η_{AB} with some arbitrary scalar field metric $f_{AB}(\phi)$ [21]. Then the Lorentz transformations in the scalar field configuration space are replaced by the isometries of the metric $f_{AB}(\phi)$.

The Stückelberg formulation of the massive gravity allows for an equivalent form of the mass term (2.3) by introducing a diffeomorphism invariant matrix

$$\mathcal{I}^A_B \equiv g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^C f_{BC} \;. \tag{2.6}$$

Since the traces and eigenvalues of the matrices $\mathcal{I}^A_B$ and $\Omega^\mu_\nu = g^{\mu\rho}\partial_\rho\phi^A\partial_\nu\phi^B f_{AB}$ are equal then the mass term (2.5) can be equivalently written in terms of the eigenvalues of $\mathcal{I}^A_B$. This rewriting makes manifest that any non-linear massive gravity theory can be viewed as a theory of a number of scalar fields minimally coupled to gravity.

3 Reduced massive gravity

In the present paper we adopt the point of view that the mass term Lagrangian $\mathcal{L}_\phi$ used in the non-linear dRGT massive gravity is a Lagrangian describing four Stückelberg scalar fields coupled to gravity. The motivation of restricting the number of scalar fields to the number of space-time dimension in the context of non-linear massive gravity is that around the background solution $g_{\mu\nu} = \eta_{\mu\nu}$, $\phi^A = x^\mu\delta^A_\mu$ the metric perturbations have a Lorentz invariant mass term of the Fierz-Pauli form at the quadratic level. However, if seen as describing a theory of scalar fields, the action

$$\mathcal{L}_\phi = \sqrt{-g}\sum_{n=0}^{4}\alpha_n \mathsf{e}_n(\mathbb{I}-\sqrt{\mathcal{I}}), \qquad \mathcal{I}^A_B \equiv g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^C f_{BC}(\phi) \tag{3.1}$$

describes just some particular theory of derivatively coupled scalar fields, and depends only on their first derivatives. This theory is diffeomorphism invariant even when the number of scalar fields is not equal to the space-time dimension. Therefore from the scalar field theory point of view the number of scalar fields N can be chosen arbitrary, both less or greater than $d+1$. In the case $N \neq d+1$ the matrices $\mathcal{I}^A_B$ and Ω^μ_ν have different dimensions, $N\times N$ and $(d+1)\times(d+1)$ respectively. Nevertheless, the non-vanishing eigenvalues of these matrices are equal, and both formulations (2.2) and (3.1) of the action are still equivalent, even though the formulation in terms of the smaller matrix is evidently simpler.[1]

In this work we focus on the case of two scalar fields $\phi^A = \{\phi^0, \phi^1\}$ in $3+1$ dimensions as the simplest non-trivial case inspired by the dRGT massive gravity. Using the diffeomorphism invariant variables $\mathcal{I}^A_B$ proves to be particularly useful for this setup since the matrix $\mathcal{I}^A_B$ is a 2×2 matrix in this case whereas Ω^μ_ν in a $(3+1)$-dimensional space-time is a 4×4 matrix. The action of the scalar fields in any $d \geq 1$ then takes the simple form

$$\mathcal{L}_\phi = \sqrt{-g}\left(\alpha_0 + \alpha_1 \mathrm{Tr}(\mathbb{I}-\sqrt{\mathcal{I}}) + \alpha_2 \det(\mathbb{I}-\sqrt{\mathcal{I}})\right), \tag{3.2}$$

where we have used the fact that for any 2×2 matrix $\mathbb{X}$ the polynomials $\mathsf{e}_{3,4}(\mathbb{X})$ vanish. In the case $\alpha_0 = \alpha_1 = 0$ and for the scalar field metric taken to be the Minkowski metric η_{AB}, the full theory $\mathcal{L} = \mathcal{L}_{EH} + m^2\mathcal{L}_\phi$ has the solution

$$g^{\mu\nu} = \eta^{\mu\nu}, \quad \phi^0 = x^0, \quad \phi^1 = x^1. \tag{3.3}$$

The quadratic action for the perturbations

$$h^{\mu\nu} \equiv g^{\mu\nu} - \eta^{\mu\nu}, \quad \chi^A \equiv \left(\phi^0 - x^0\right)\delta^A_0 + \left(\phi^1 - x^1\right)\delta^A_1 \tag{3.4}$$

[1] The coefficients $\tilde{\alpha}_n$, α_n in (2.2) and (3.1) respectively coincide only when the number of fields equals the space-time dimension, i.e. when $N = d+1$.

then reads

$$\mathcal{L}_{\phi}^{(2)} = 2\left[(h^{01})^2 - h^{00}h^{11}\right] + 4h^{AB}\left[\eta_{AB}\partial_C\chi^C - \partial_A\chi_B\right] + 2\left[(\partial_A\chi^A)^2 - \partial^A\chi^B\partial_B\chi_A\right], \quad (3.5)$$

and the indices $A, B = 0, 1$. In $1 + 1$ space-time dimensions this action coincides with the action for metric and scalar field perturbations around the Minkowski background in massive gravity. However, in $3 + 1$ space-time dimensions, this corresponds to a Lorentz-violating Fierz-Pauli-type mass term for metric perturbations. A thorough analysis of Lorentz-violating graviton mass terms, preserving the Euclidean symmetry of the three-dimensional space was carried out in [22] (see also an earlier work [23]). In our case, the symmetry of the three-dimensional rotations is in general not preserved by the ground state of the theory. Therefore, the possible mass terms of our theory go beyond the considerations of [22]. Out of their investigated mass terms, only the mass term with $m_2 = m_3 \neq 0$, $m_{0,1,4} = 0$ can be obtained in reduced massive gravity (if the number of scalars $N = 3$). However, the stability analysis of [22] does not directly apply to our case since the number of Goldstone fields is different.

A particular instance when the (3+1)-dimensional dRGT theory of massive gravity reduces to the special case of two Stückelberg fields is the case of a degenerate reference metric. To see this one can consider the spherically symmetric ansatz $\phi^0 = f(t, r)$, $\phi^i = g(t, r)$, $i = 1, 2, 3$.[2] For the flat auxiliary metric $f_{\mu\nu} = \partial_\mu\phi^A\partial_\nu\phi^B\eta_{AB}$ in spherical coordinates, this gives a matrix with the only non-zero entries in the upper-left 2×2 matrix, and it can be easily reparametrized by using only two Stückelberg fields. This illustrates our point that the reduced massive gravity with the number of scalar fields N less than the space-time dimension is equivalent to dRGT theory with a degenerate reference metric $f_{\mu\nu}$ (or f_{AB} equivalently). However, the spherically symmetric ansatz given above reduces to the degenerate reference metric only in the absence of perturbations.

3.1 Number of degrees of freedom

We would now like to estimate the total number of degrees of freedom propagated by the full non-linear theory of gravity and two scalar fields. For this we will use the Dirac's approach to the Hamiltonian analysis of constrained systems [24, 25].

As was already mentioned, due to the fact that the action (3.2) is reparametrization invariant and that the scalar fields are coupled to gravity minimally, i.e. only through the terms $g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^B$, it is legitimate to count the number of degrees of freedom propagated by the scalar field action and the Einstein-Hilbert action separately. In such a diffeomorphism invariant theory of gravity and minimally coupled scalar fields, the Hamiltonian vanishes on the constraint surface, and both the lapse and the shift enter the Hamiltonian linearly. This implies the appearance of in total $2(d + 1)$ first-class constraints, which can be used to reduce the number of gravitational degrees of freedom to $d(d + 1)/2 - 2(d + 1)$. The dynamics of the scalar fields then shall be generated by the usual Hamiltonian of the scalar field action alone, contained in the Hamiltonian constraint of the full theory. Therefore, the scalar field dynamics in such a theory can be considered separately from gravity.

[2]We note that this is not the ansatz usually studied in the context of the spherically symmetric solutions of dRGT theory. Instead the common ansatz is $\phi^0 = g(t, r)$, $\phi^i = f(r, t)x^i/r$.

Then naively one would expect that the number of degrees of freedom propagated by any dRGT-type massive gravity in $(d+1)$-dimensional space-time (with $d \geq 2$) equals to

$$\#\ \text{d.o.f.}\ = \frac{1}{2}(d-2)(d+1) + N \tag{3.6}$$

where the first term accounts for the degrees of freedom propagated by the massless graviton, and the second term is just the number of scalar fields. This naive counting demonstrates why, in (3+1)-dimensional space-time, a general non-linear massive gravity theory with four Stückelberg fields propagates six degrees of freedom. It has been demonstrated that in the dRGT subclass of massive gravity theories at most five degrees of freedom propagate due to the special structure of the graviton mass term (2.2) [7, 8]. In Stückelberg language it is clear that, in order for the assertion to be true, the scalar field Lagrangian (2.3) has to have a very special structure such that it propagates less degrees of freedom than the number of fields. In other words, in the non-linear dRGT massive gravity the four Stückelberg fields do not correspond to four independent degrees of freedom [10] (see also [11, 12]). This can be seen from the vanishing of the determinant of the kinetic (Hessian) matrix of the scalar field Lagrangian

$$\mathcal{A}_{AB} \equiv \frac{\partial^2 \mathcal{L}_\phi}{\partial \dot{\phi}^A \partial \dot{\phi}^B} \,. \tag{3.7}$$

Hence the equations of motion of the scalar fields are not independent from each other, and there exists (at least) one combination of the equations of motion which gives a constraint equation relating the canonical momenta of the scalar fields. As a result, in dRGT massive gravity the scalar fields propagate at most $N-1=3$ degrees of freedom.

Our ultimate goal is to find the constraint structure of the scalar field part of the full dRGT massive gravity while keeping the space-time metric arbitrary. In this paper we start with the case of the reduced massive gravity (3.2) with two scalar fields. For this we explicitly calculate the determinant of the kinetic matrix of the theory. Curiously, we show that the naive expectation, that also in the case of two scalar fields the determinant vanishes and the theory propagates $N-1=1$ degree of freedom, is not met. Instead we find that, in general, the determinant is not equal to zero, and thus there are two dynamical degrees of freedom in the scalar field sector.

3.2 Determinant of the kinetic matrix

In the case of two scalar fields, the only non-vanishing characteristic polynomials of the square root matrix can be explicitly expressed in terms of the $\operatorname{Tr}\mathcal{I}$ and $\det\mathcal{I}$ as

$$\mathsf{e}_0(\sqrt{\mathcal{I}}) = 1\,, \qquad \mathsf{e}_1(\sqrt{\mathcal{I}}) = \operatorname{Tr}\sqrt{\mathcal{I}} = \left(\operatorname{Tr}\mathcal{I} + 2\sqrt{\det\mathcal{I}}\right)^{1/2}\,, \tag{3.8}$$

$$\mathsf{e}_2(\sqrt{\mathcal{I}}) = \det\sqrt{\mathcal{I}} = \sqrt{\det\mathcal{I}}\,. \tag{3.9}$$

Then the scalar field action (3.2) reads

$$\mathcal{L}_\phi = \sqrt{-g}\left[\beta_0 + \beta_1\left(\operatorname{Tr}\mathcal{I} + 2\sqrt{\det\mathcal{I}}\right)^{1/2} + \beta_2\sqrt{\det\mathcal{I}}\right]. \tag{3.10}$$

Since the β_0 term does not affect the dynamics of the scalar fields, in what follows we set $\beta_0 = 0$. We also note that in dRGT massive gravity case, where the number of scalar fields N coincides with the number of space-time dimensions, the highest order term with β_N is usually dropped since it is a total derivative. In the reduced massive gravity, however, the term with $\beta_{N=2}$ does contribute to the dynamics of the scalars and, in general, cannot be neglected.

In order to separate the time derivatives of the scalar fields while keeping the space-time metric arbitrary, we employ the ADM formalism [26]. In ADM variables for the metric components

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^i}{N^2} \\ \frac{N^i}{N^2} & \gamma^{ij} - \frac{N^i N^j}{N^2} \end{pmatrix} \tag{3.11}$$

the matrix $\mathcal{I}^A_B$ can be expressed as

$$\mathcal{I}^A_B = \left(-D\phi^A D\phi^C + S^{AC}\right) f_{BC} \ , \tag{3.12}$$

where $D \equiv \frac{1}{N}\left(\partial_0 - N^i \partial_i\right)$, and the matrix $S^{AC} \equiv \gamma^{ij}\partial_i\phi^A\partial_j\phi^C$ depends only on the spatial derivatives of the scalar fields. The canonical momenta conjugated to the scalar fields are given by

$$\pi_A \equiv \frac{1}{N}\frac{\partial \mathcal{L}_\phi}{\partial D\phi^A} = -\sqrt{\gamma}\left(\frac{\beta_1}{\left(\mathrm{Tr}\,\mathcal{I} + 2\sqrt{\det \mathcal{I}}\right)^{1/2}}\left[D\phi_A + \frac{1}{\sqrt{\det \mathcal{I}}}\left(S\, f_{AB} - S_{AB}\right) D\phi^B\right] + \right.$$
$$\left. + \frac{\beta_2}{\sqrt{\det \mathcal{I}}}\left(S\, f_{AB} - S_{AB}\right) D\phi^B \right) \ . \tag{3.13}$$

Here $S \equiv \mathrm{Tr}\, S^A_B$, and the $\mathrm{Tr}\,\mathcal{I}$ and $\det \mathcal{I}$ also depend on the time derivatives $D\phi^A$ as

$$\mathrm{Tr}\,\mathcal{I} \equiv \mathrm{Tr}\,\mathcal{I}^A_B = S - D\phi^A D\phi_A \ ,$$
$$\det \mathcal{I} \equiv \det \mathcal{I}^A_B = \det S - D\phi^A D\phi^B \left(S\, f_{AB} - S_{AB}\right) \ ,$$

where $\det S \equiv \det S^A_B = \det f \det S^{AB}$. The determinant of the kinetic matrix is given by

$$\det \mathcal{A}^A_B = -\det g \frac{\det S}{(\det \mathcal{I})^2}\left(\frac{\beta_1^2}{\left(\mathrm{Tr}\,\mathcal{I} + 2\sqrt{\det \mathcal{I}}\right)^2}\left[\sqrt{\det \mathcal{I}}\,\mathcal{I}^A_B(3S\delta^B_A - 2S^B_A) - \right.\right.$$
$$\left. - \mathcal{I}^A_B\left(\mathrm{Tr}\,\mathcal{I} + 2S\right)\left(S^B_A - S\delta^B_A\right) - 2S\det S\right] +$$
$$+ \frac{\beta_1\beta_2}{\left(\mathrm{Tr}\,\mathcal{I} + 2\sqrt{\det \mathcal{I}}\right)^{3/2}}\left[\sqrt{\det \mathcal{I}}\left(S\,\mathrm{Tr}\mathcal{I} + 4\det S\right) - \right.$$
$$\left.\left. - \left(2S\mathcal{I}^A_B + S^A_B \mathrm{Tr}\,\mathcal{I}\right)\left(S^B_A - S\delta^B_A\right) - 2S\det S\right] + \beta_2^2 \det S\right) \ . \tag{3.14}$$

This expression is valid for any choice of the scalar field metric $f_{AB}(\phi)$ as long as it does not involve the time derivatives of the scalar fields. The determinant depends on temporal (contained within the matrix $\mathcal{I}^A_B$) and spatial derivatives of the scalar fields. In general it has a non-zero value which depends on the choice of initial conditions. The only special case when the determinant vanishes identically is if we are considering a two-dimensional space-time where the matrix $S^{AB} = \gamma^{11}\partial_1\phi^A\partial_1\phi^B$ is a matrix of rank one, and $\det S \equiv 0$. This case corresponds to the two-dimensional massive gravity and our findings are in agreement with the previous work by de Rham et al. [10]. If the $\det S$ factor appears also in the theory with four scalar fields, then the full dRGT massive gravity in $3+1$ dimensions also has the identically vanishing kinetic matrix, and thus at most five degrees of freedom in total.

We thus conclude that in general the action (3.10) describes two independent dynamic fields. However, on the surface $\det S = 0$ in the configuration space, the Lagrangian equations of motion are degenerate and determine the second time derivative only for one independent combination of fields. In the theory of partial differential equations the solutions that entirely belong to the $\det S = 0$ subspace are called *singular solutions* (cf. [27]). In other words, singular solutions of a system of differential equations are the solutions which belong to the surface where the number of independent highest time derivatives is less than the number of the fields. Such solutions in general are the envelopes of families of regular solutions of the system, and at each fixed moment of time coincide with some regular solution (or the whole family of regular solutions). It means that the initial conditions on this surface do not specify a unique solution since there are other solutions of the theory which are touching the $\det S = 0$ surface at the initial moment of time. We note that this discussion holds only classically. In the full quantum theory the vanishing of the determinant of the kinetic matrix signals that the scalar fields are infinitely strongly coupled, and the quantum effects are crucial for the dynamics of the system near the singular surface.

It is interesting to note that the trivial solution (3.3) with $\phi^A = x^A$ is on the surface $\det S = 0$. However, any perturbations around this solution defined as $\chi^A = \phi^A - x^A$ will no longer be on the singular surface and will propagate two degrees of freedom. In order to understand the dynamics of such field perturbations it is instructive to study the behaviour of the system in a close vicinity of the singular surface. Note that for $\beta_2 \neq 0$ the condition $\det S = 0$ is also a necessary condition for $\det \mathcal{A} = 0$. Therefore in our discussion of singular solution we will focus on the singular surface $\det S = 0$. Although in general the determinant of the kinetic matrix could vanish also in some other regions of the phase space.

3.3 Hamiltonian analysis away from $\det S = 0$

For any initial conditions away from the surface $\det S = 0$ the expression (3.13) for momenta is invertible, and the system contains two propagating degrees of freedom. In order to qualitatively understand the dynamics of the system in the vicinity of the singular surface we fix the scalar fields metric to be flat $f_{AB} = \eta_{AB}$ and construct the Hamiltonian for the limiting cases, when only one of the terms in Lagrangian (3.10) is present.

First, we consider the case when $\beta_2 = 0$ and $\beta_1 = 1$. The action with only β_1 term present, in the case of four scalar fields, was already studied as a special case of the dRGT theory and is named as "minimal non-linear massive gravity". Our Hamiltonian is in agreement with the previous results (cf. [12]). The expression (3.13) for the momenta can be inverted to give

$$D\phi^A = -\frac{\left[\operatorname{Tr}\mathcal{I} + 2\sqrt{\det \mathcal{I}}\right]^{1/2}}{\sqrt{\gamma}\left(1 + \frac{S}{\sqrt{\det \mathcal{I}}} + \frac{\det S}{\det \mathcal{I}}\right)} \left(\pi^A + \frac{1}{\sqrt{\det \mathcal{I}}} S^{AB}\pi_B\right) . \tag{3.15}$$

It still does not allow to express the velocities in the terms of momenta completely, but it turns out to be enough in order to obtain the Hamiltonian in terms of S^{AB} and π_A. After some algebra and with the help of the relation $\frac{\det S}{\det \mathcal{I}} = 1 + \gamma^{-1}\pi^A\pi_A$, the Hamiltonian takes the following form:

$$\mathcal{H} = -N\sqrt{\gamma}\left(S + \gamma^{-1}\,\pi_A\pi_B S^{AB} + 2\sqrt{\det S\,\left(1 + \gamma^{-1}\,\pi_A\pi^A\right)}\right)^{1/2} + N^i\,\partial_i\phi^A\pi_A\,. \tag{3.16}$$

This Hamiltonian has the form $\mathcal{H} = N\,\mathcal{H}_0 + N^i\,\mathcal{H}_i$, linear in the ADM lapse and shift, as it should be in any minimally coupled theory where the scalar fields enter the action only through different combinations of $g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^B$. When considered together with gravity, $\mathcal{H}_0$ and $\mathcal{H}_i$ simply contribute to the Hamiltonian constraint and to the generators of spatial diffeomorphisms respectively. Doing so does not change the dynamics in the scalar field sector, and merely reflects the reparametrization invariance of the action. We therefore feel free to consider the scalar fields separately from gravity. At last, we note also that the scalar fields Hamiltonian $\mathcal{H}_0$ can be written as a trace of the square root matrix $\mathcal{H}_0 = -\operatorname{Tr}\sqrt{S^A_C\left(\delta^C_B + \gamma^{-1}\,\pi^C\,\pi_B\right)}$, very similar to the structure of the Lagrangian $\mathcal{L}_\phi = \sqrt{-g}\,\operatorname{Tr}\sqrt{\mathcal{I}}$.

In the case when $\beta_1 = 0$ and $\beta_2 = 1$ the velocities can be expressed as

$$D\phi^A = -\frac{\sqrt{\det \mathcal{I}}}{\sqrt{\gamma}\,\det S} S^{AB}\pi_B\,. \tag{3.17}$$

Using the relation $\frac{\det S}{\det \mathcal{I}} = 1 + \frac{S^{AB}\pi_A\pi_B}{\gamma\,\det S}$ one can obtain the scalar fields Hamiltonian as

$$\mathcal{H}_0 = -\sqrt{\gamma}\,\sqrt{\det S + \gamma^{-1}\,\pi_A\pi_B S^{AB}} \equiv -\sqrt{\gamma}\,\det\sqrt{S^A_C\left(\delta^C_B + \frac{S^C_D\pi^D\,\pi_B}{\gamma\,\det S}\right)}\,. \tag{3.18}$$

The form of this Hamiltonian is also similar to the form of the original Lagrangian, $\mathcal{L}_\phi = \sqrt{-g}\,\det\sqrt{\mathcal{I}}$, and can be written as a determinant of some square root matrix.

In order to look at the dynamics we focus on the former case with $\beta_2 = 0$. The equations of motion for the scalar fields in Hamiltonian form read

$$D\phi^A = \frac{1}{\mathcal{H}_0}\left(\frac{\det S}{\sqrt{\det S\,\left(1 + \gamma^{-1}\,\pi_A\pi^A\right)}}\,\pi^A + S^{AB}\pi_B\right)\,, \tag{3.19}$$

$$\dot{\pi}_A - \partial_i\left(N^i\,\pi_A\right) = \partial_i\left(\frac{\gamma\,N}{\mathcal{H}_0}\left[\eta_{AB} + \gamma^{-1}\,\pi_A\,\pi_B + \sqrt{\det S\,\left(1 + \gamma^{-1}\,\pi_A\pi^A\right)}\,S^{-1}_{AB}\right]\partial^i\phi^B\right), \tag{3.20}$$

where the inverse of the spatial derivative matrix is $S^{-1}_{AB} = (S\eta_{AB} - S_{AB})/\det S$. These equations of motion describe the evolution of the scalar fields ϕ^A and their conjugated momenta π_A for any initial conditions with $\det S \neq 0$. In general one expects that all the regular solutions, i.e. the solutions specified with the initial conditions with $\det S \neq 0$, are tangential to the surface $\det S = 0$ at some point of time. In other words, the singular solutions, for which $\det S = 0$ at any time, are the envelopes of the families of regular solutions. Choosing the conditions in vicinity of the singular surface and following the infinitesimal evolution in time, one can study the phase portrait of the system near the singular surface and the way regular solutions are connected to the singular ones. We also note that the Hamiltonian (3.16) cannot be used to study the singular solutions themselves since it relies on the assumption $\det S \neq 0$. The solutions with $\det S = 0$ shall therefore be studied separately.

3.4 Time evolution in the vicinity of the singular surface

In order to illustrate the behaviour of regular solutions in the vicinity of the $\det S = 0$ surface let us choose some initial conditions that are infinitesimally close to the known trivial solution $\phi^0 = t$, $\phi^1 = x^1$, but have non-vanishing $\det S$ and its time derivative. Simplest way to write such initial conditions is to provide a small x^2 (or x^3) dependence to the ϕ^0:

$$\phi^0(t_0) = t_0 + \epsilon_0\, x^2\,, \quad \dot\phi^0(t_0) = 1 + \dot\epsilon_0\, x^2\,, \quad \phi^1(t_0) = x^1\,, \quad \dot\phi^1(t_0) = 0\,, \tag{3.21}$$

where ϵ_0, and $\dot\epsilon_0$ are arbitrary constants, vanishing in the case of the critical solution $\phi^A = x^A$. Since $\det S = -\epsilon_0^2$ and $\frac{d}{dt}\det S = -2\dot\epsilon_0\epsilon_0$, these constants characterize the displacement from the singular surface and its time derivative at the initial moment. Using the Hamiltonian equation (3.20) for the momenta one can follow the infinitesimal evolution of fields ϕ^A in time. Moreover it happens to be possible to find an exact solution in the case of initial conditions (3.21). It can be obtained by promoting the x^2 dependence of the initial conditions to be valid at all times. By plugging the ansatz $\phi^0(t) = \xi(t) + \epsilon(t)\, x^2$ into the equations of motion (3.20) one obtains two equations for the functions $\xi(t)$ and $\epsilon(t)$

$$\ddot\xi(t) = 2\,\dot\xi(t)\,\frac{\dot\epsilon(t)}{\epsilon(t)}\,, \qquad \ddot\epsilon(t) = 2\,\frac{\dot\epsilon(t)^2}{\epsilon(t)}\,. \tag{3.22}$$

The general solution for the fields $\phi^A(t)$ is given by

$$\phi^0(t) = \frac{\dot\xi_0\,(t-t_0) + \epsilon_0\, x^2}{1 - \frac{\dot\epsilon_0}{\epsilon_0}(t-t_0)} + \xi_0\,, \qquad \phi^1(t) = x^1\,. \tag{3.23}$$

It happens that this family of solutions never approaches the singular trivial solution $\phi^0 = t$ independently of how close are the initial conditions to it, i.e. how small is ϵ_0. Instead, the solutions (3.23) are asymptotically approaching a different set of singular solutions $\phi^0 = \dot\xi_0\,\frac{\epsilon_0}{\dot\epsilon_0} + \xi_0 = const$ in the limit $t \to \pm\infty$. Therefore, for any given constant there is a three parameter subfamily of regular solutions that approach it in the $t \to \pm\infty$ limit. Figure 1 illustrates this behaviour for the singular solution $\phi^0 = 0$. For simplicity we

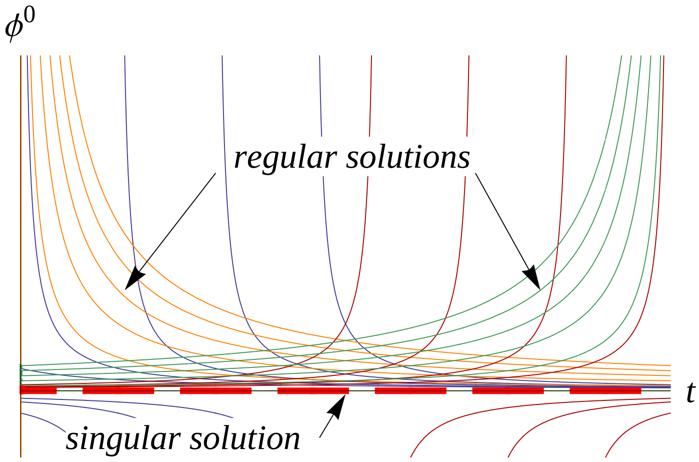

Figure 1. Some members of the family of regular solutions (3.23) (solid), which approach given singular solution $\phi^0 = 0$ (dashed) in the limit $t \to \pm\infty$.

have suppressed the x^2 dependence of the regular solutions, and each line on the figure 1 corresponds to the one parameter family of solutions, which are different from each other by the constant rescaling of $\epsilon(t)$. From the solution (3.23) one can see that $\det S \propto \frac{1}{t^2}$, and therefore the $\det S = 0$ surface cannot be reached along the discussed trajectory at any finite time. Note that apart from the solutions that start at the finite distance from the $\det S = 0$ surface and approach it in the future there exist solutions that start as a small perturbations of the singular solution and leave the $\det S = 0$ surface. We would also like to remark that all the solutions with a given x^2 dependence have a singularity at the finite time $t = t_0 + \frac{\epsilon_0}{\dot{\epsilon}_0}$. Hopefully there are other solutions in this theory that are free of singularities.

4 The singular surface with $\det S = 0$

In this section we discuss the most general scalar field configurations which satisfy the condition $\det S = 0$ and show that the dynamics of the scalar fields in this subspace are equivalent to the dynamics of scalar fields in the case of $1 + 1$ space-time dimensions. To see this we first discuss the $1 + 1$ dimensional case separately.

4.1 1+1 dimensions: massive gravity

In $1 + 1$ dimensions the scalar field Lagrangian (3.10) reduces to

$$\mathcal{L}_\phi = \beta_1 \sqrt{-g} \left(\mathrm{Tr}\, \mathcal{I} + 2\sqrt{\det \mathcal{I}} \right)^{1/2}, \tag{4.1}$$

since the β_2 term is a total derivative term. The above Lagrangian coincides with the dRGT mass term and was previously analyzed in [10, 28]. Here we shall follow a different approach of Hamiltonian analysis which enables us to find the gauge symmetry of the scalar field action. We show that the scalar field action of non-linear massive gravity propagates no degrees of freedom in $1 + 1$ dimensions, in agreement with [10, 28].

We first note that in this case the determinant of the matrix $\mathcal{I}^A_B$ is a full square

$$\det \mathcal{I}^A_B = \frac{\det f}{\det g}\left(\frac{1}{2}\bar{\varepsilon}_{AB}\bar{\varepsilon}^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^B\right)^2 , \tag{4.2}$$

where the bared Levi-Civita tensors $\bar{\varepsilon}^{\mu\nu}$, $\bar{\varepsilon}_{AB}$ denote the flat space antisymmetric tensors defined as $\bar{\varepsilon}^{01} = -\bar{\varepsilon}^{10} = 1$, etc. in every coordinate frame. In this section for simplicity we will consider the flat Minkowski scalar field metric $f_{AB} = \eta_{AB}$ for which the factor $\det f = -1$. The scalar field action then becomes (up to a constant factor)

$$\mathcal{L}_\phi = 2\frac{N}{\sqrt{\gamma^{11}}}\left[D\psi_+ + \sqrt{\gamma^{11}}\psi'_+\right]^{1/2}\left[D\psi_- - \sqrt{\gamma^{11}}\psi'_-\right]^{1/2} , \tag{4.3}$$

where $\psi_\pm \equiv \phi^0 \pm \phi^1$, $D \equiv (\partial_0 - N^1\partial_1)/N$, and $\psi'_\pm \equiv \partial_1\psi_\pm$. In the following we perform the full Hamiltonian analysis of this system according to the constraint analysis proposed by Dirac and extended by Henneaux et al. [24, 25].

4.2 Minkowski background

We start with the case of a flat Minkowski background metric $g_{\mu\nu} = \eta_{\mu\nu}$ since the generalization to an arbitrary background is straightforward, as we shall see below.

4.2.1 Constraint algebra

In flat space the Lagrangian takes the simple form

$$\mathcal{L}_\phi = 2\sqrt{\dot{\psi}_+ + \psi'_+}\sqrt{\dot{\psi}_- - \psi'_-} , \tag{4.4}$$

and the conjugated momenta to the fields $\psi_\pm$ are

$$\pi_+ = \frac{\sqrt{\dot{\psi}_- - \psi'_-}}{\sqrt{\dot{\psi}_+ + \psi'_+}} , \qquad \pi_- = \frac{\sqrt{\dot{\psi}_+ + \psi'_+}}{\sqrt{\dot{\psi}_- - \psi'_-}} . \tag{4.5}$$

It is obvious that the momenta are not independent. Instead, they satisfy the primary constraint

$$\mathcal{C}_0 \equiv \pi_+ - \frac{1}{\pi_-} = 0 . \tag{4.6}$$

The total Hamiltonian density of the theory is obtained by adding the primary constraint to the Hamiltonian as

$$\begin{aligned}\mathcal{H}_T &= \pi_+\dot{\psi}_+ + \pi_-\dot{\psi}_- - \mathcal{L}_\phi + u_0\mathcal{C}_0 \\ &= \pi_-\psi'_- - \frac{1}{\pi_-}\psi'_+ + u_0\mathcal{C}_0 ,\end{aligned} \tag{4.7}$$

where the Lagrange multiplier $u_0 = u_0(t,x)$ is an arbitrary function of the space-time coordinates. For the analysis of the dynamics of the system we define the equal-time Poisson bracket as

$$\{f(x), g(x')\} = \int dz\left(\frac{\delta f(x)}{\delta\psi^i(z)}\frac{\delta g(x')}{\delta\pi_i(z)} - \frac{\delta g(x')}{\delta\psi^i(z)}\frac{\delta f(x)}{\delta\pi_i(z)}\right) . \tag{4.8}$$

The time evolution of a functional $f(t,x) = f(t,x,\,\psi^i(t,x),\,\pi_i(t,x))$ is then given by

$$\frac{d}{dt}f(t,x) = \frac{\partial f(t,x)}{\partial t} + \int dx'\{f(t,x), \mathcal{H}_T(t,x')\}\ . \tag{4.9}$$

For the consistency of the Hamiltonian equations of motion with the Lagrangian equations of motion one has to impose an additional constraint to the system, namely that the primary constraint is preserved in time. This in general leads either to secondary (and tertiary, ...) constraints or determines the arbitrary function $u_0(t,x)$ [24]. In our case we obtain a secondary constraint

$$\frac{d}{dt}\mathcal{C}_0(t,x) = -2\left(\frac{1}{\pi_-}\right)' \equiv 2\mathcal{C}_1(t,x)\ . \tag{4.10}$$

It is straightforward to check that the time evolution of $\mathcal{C}_1$ does not imply any new constraints since

$$\frac{d}{dt}\mathcal{C}_1(t,x) = \int dx'\{\mathcal{C}_1(t,x), \mathcal{H}_T(t,x')\} = -\left(\frac{1}{\pi_-}\right)'' = \mathcal{C}_1' \tag{4.11}$$

is a spatial derivative of the secondary constraint itself. Since both constraints mutually commute, i.e. $\{\mathcal{C}_0(x), \mathcal{C}_0(x')\} = \{\mathcal{C}_0(x), \mathcal{C}_1(x')\} = \{\mathcal{C}_1(x), \mathcal{C}_1(x')\} = 0$, and since there are no further constraints, we conclude that the constraint algebra is closed and our system has two first class constraints. $\mathcal{C}_0$ is a primary first-class constraint and $\mathcal{C}_1$ is a secondary first-class constraint.

4.2.2 Gauge symmetry

The existence of first-class constraints indicates that there is a gauge symmetry in our theory. The purpose of this section is therefore to identify the gauge symmetries of the original Lagrangian (4.4) and find the number of degrees of freedom described by it.

Since the total Hamiltonian (4.7) contains an arbitrary function of space-time coordinates u_0, a given set of initial conditions for the canonical variables ψ^i, π_i after some time interval will evolve to different values of the canonical variables for different choices of u_0. Any two such set of values describe the same physical state related by a gauge transformation. In order to find the generators of the transformation one considers the evolution of a given set of initial data over a *finite* time interval. This is reached by *multiple* Poisson brackets of the canonical variables and total Hamiltonian, each of them transforming the system infinitesimally. Hence after a finite time interval two different sets of canonical variables obtained from the same initial data will differ by a gauge transformation generated by *all* first-class constraints. It is therefore why all the first-class constraints should be put on the same footing and the Hamiltonian should be extended by adding to it also the secondary (and tertiary, ...) first-class constraints [24]. This makes the full symmetry of the theory manifest. In our case the extended Hamiltonian looks like

$$\mathcal{H}_E = \pi_-\psi_-' - \frac{1}{\pi_-}\psi_+' + u_0\mathcal{C}_0 + u_1\mathcal{C}_1\ , \tag{4.12}$$

where we have introduced another arbitrary function $u_1(t,x)$. Under the transformations generated by the two constraints the canonical variables $q = \{\psi^i, \pi_i\}$ transform according to the law

$$q \mapsto q + \delta q \,, \qquad \delta q(x) = \left\{ q(x), \int dx' \left[\varepsilon_0(x') C_0(x') + \varepsilon_1(x') C_1(x') \right] \right\} . \tag{4.13}$$

This for the transformations of the canonical fields gives

$$\delta\psi_+ = \varepsilon_0 \,, \qquad \delta\psi_- = \frac{1}{\pi_-^2}(\varepsilon_0 - \varepsilon_1') \,, \tag{4.14}$$

while the conjugated momenta stay unchanged. The corresponding *extended* first order action

$$S_E = \int d^2x \left[\pi_+ \dot\psi_+ + \pi_- \dot\psi_- - \mathcal{H}_E \right] \tag{4.15}$$

is invariant under the above gauge transformations if also the Lagrange multipliers u_0, u_1 transform. Their transformation laws are not of any need in the present work, therefore we shall not give their explicit form and instead refer the reader to [25]. Due to the fact that in (4.15) we have introduced an additional arbitrary function u_1, the equations of motion which can be derived from (4.15) do not coincide with the equations of motion following from the action $S_T = \int d^2x \left(\pi_i \dot\psi^i - \mathcal{H}_T \right)$ or equivalently from the original action (4.4). Moreover, the original Lagrangian is not invariant under the gauge transformations (4.14). The reason for this is that the extended Hamiltonian formalism introduces an additional redundancy in the description. However, the time evolution of the gauge invariant fields can be equally well described by both the total Hamiltonian $\mathcal{H}_T$ and the extended Hamiltonian $\mathcal{H}_E$.

In order to obtain the symmetry of the original scalar field action, one can rewrite the transformations (4.14) by expressing the conjugated momenta according to their definitions (4.5) and demand that the action remains unchanged. This leads to the following relation between the gauge parameters

$$\varepsilon_0 = \frac{1}{2} \left(\varepsilon_1' - \dot\varepsilon_1 \right) \,. \tag{4.16}$$

Hence the gauge symmetry of the Lagrangian is

$$\psi_- \mapsto \psi_- - \frac{1}{2}(\varepsilon' + \dot\varepsilon) \frac{\dot\psi_- - \psi_-'}{\dot\psi_+ + \psi_+'} \,, \tag{4.17}$$

$$\psi_+ \mapsto \psi_+ + \frac{1}{2}(\varepsilon' - \dot\varepsilon) \,. \tag{4.18}$$

Since the above symmetry transformation involves both, the gauge parameter ε and its time derivative, then the number of degrees of freedom in the theory are reduced by two which coincides with the total number of first class constraints [25]. It is so, because the gauge parameter and its time derivatives are independent functions in the sense of independent initial data which can be chosen arbitrarily at the initial moment of time. [3] Another way

[3] A familiar example where exactly the same approach of counting the degrees of freedom can be applied is electrodynamics. There the gauge transformation of the vector field $A_\mu \to A_\mu + \partial_\mu\lambda$ also involves both the gauge parameter λ and its time derivative. The constraint analysis of the theory also shows that there is one primary and one secondary first-class constraint removing two out of four degrees of freedom.

to see that there are no propagating degrees of freedom is by performing the gauge fixing in the extended action (4.15). Since there are two constraints on the momenta and two gauge symmetries (4.14) on the canonical fields it is evident that the action is pure gauge and propagates no degrees of freedom. The same conclusion could have been drawn also from the analysis of the Lagrangian equations of motion.

4.3 Arbitrary background

The scalar field action in an arbitrary curved $1+1$ dimensional space-time can be written as

$$S_\phi = 2\int d^2x\,\sqrt{\gamma_{11}}\left[\dot{\psi}_+ + a_+\psi'_+\right]^{1/2}\left[\dot{\psi}_- - a_-\psi'_-\right]^{1/2}\,, \tag{4.19}$$

where we have introduced the notations $a_\pm = N\sqrt{\gamma^{11}} \mp N^1$. The conjugated momenta are defined as

$$\pi_+ = \sqrt{\gamma_{11}}\frac{\sqrt{\dot{\psi}_- - a_-\psi'_-}}{\sqrt{\dot{\psi}_+ + a_+\psi'_+}}\,, \qquad \pi_- = \sqrt{\gamma_{11}}\frac{\sqrt{\dot{\psi}_+ + a_+\psi'_+}}{\sqrt{\dot{\psi}_- - a_-\psi'_-}}\,, \tag{4.20}$$

and the Hamiltonian analysis of the system can be carried out in complete analogy to the case of Minkowski background. The extended Hamiltonian and the closed set of constraints can be found to be

$$\mathcal{H}_E = a_-\pi_-\psi'_- - a_+\frac{\gamma_{11}}{\pi_-}\psi'_+ + u_0\mathcal{C}_0 + u_1\mathcal{C}_1\,, \tag{4.21}$$

$$\text{with}\quad \mathcal{C}_0 = \pi_+ - \frac{\gamma_{11}}{\pi_-}\,,\quad \mathcal{C}_1 = -\left(\frac{1}{\pi_-}\right)'\,. \tag{4.22}$$

As before the constraints $\mathcal{C}_0$ and $\mathcal{C}_1$ are first class constraints and generate the gauge transformations of the canonical variables $\psi_\pm \mapsto \psi_\pm + \delta\psi_\pm$ with

$$\delta\psi_+ = \varepsilon_0\,, \qquad \delta\psi_- = \frac{1}{\pi_-^2}(\gamma_{11}\varepsilon_0 - \varepsilon'_1)\,. \tag{4.23}$$

By inserting them in the Lagrangian (4.19) one obtains the following condition on the gauge variables

$$\varepsilon'_0\gamma_{11}(a_+ + a_-) - \varepsilon_0\left(\partial_0 - a_-\partial_1 - 2a'_-\right)\gamma_{11} + \left(\partial_0 - a_-\partial_1 - 2a'_-\right)\varepsilon'_1 = 0\,, \tag{4.24}$$

under which the Lagrangian remains invariant under the transformations (4.23). This condition can be rewritten in metric components by using the relations $\gamma_{11} = g^{00}\det g$, $\gamma_{11}(a_+ + a_-) = 2\sqrt{-g}$, and

$$a_\pm = \frac{1}{g^{00}}\left(\varepsilon^{01} \pm g^{01}\right)\,, \qquad \partial_0 \pm a_\pm\partial_1 = \frac{1}{g^{00}}\left(g^{0\mu} \pm \varepsilon^{0\mu}\right)\partial_\mu\,, \tag{4.25}$$

where the only non-zero components of the Levi-Civita tensor are $\varepsilon^{01} = -\varepsilon^{10} = -(\sqrt{-g})^{-1}$. Unfortunately, for generic background metric it is impossible to solve (4.24) for ε_0 in local form. The gauge transformation is therefore in general non-local.

4.4 3+1 dimensions

For the two scalar fields in 3+1 dimensions the determinant of the matrix of spatial derivatives $S^{AB} \equiv \gamma^{ij}\partial_i\phi^A\partial_j\phi^B$ reads

$$\det S^{AB} = \left[\varepsilon^{ijk}\partial_j\phi^0\partial_k\phi^1\right]\gamma_{il}\left[\varepsilon^{lmn}\partial_m\phi^0\partial_n\phi^1\right] . \tag{4.26}$$

Hence the condition $\det S = 0$ translates into requirement that the norm of the cross product of the spatial gradients of the scalar fields ϕ^0 and ϕ^1 vanishes. In other words it means that both gradients of the scalar fields have to lie along the same spatial direction and thus can be used to parametrize only one spatial direction. Therefore the most general scalar field configuration satisfying $\det S = 0$ can be parametrized as

$$\phi^0 = \phi^0(t,\ f(t,x^i)) , \tag{4.27}$$
$$\phi^1 = \phi^1(t,\ f(t,x^i)) , \tag{4.28}$$

where $f(t,x^i)$ is an arbitrary function of space-time coordinates.

In order to see that this ansatz for the scalar fields makes the dynamics of the $3+1$ dimensional theory equivalent to the dynamics of the $1+1$ dimensional theory it is useful to introduce the short hand notations $N = \tilde{N}$, $\quad N^i\partial_i f - \partial_0 f = \tilde{N}^f$, $\quad \partial_i f\partial_j f\gamma^{ij} = \tilde{\gamma}^{ff}$. In terms of these variables the $3+1$ dimensional field $\mathcal{I}^{AB}$ takes the form

$$\begin{aligned}\mathcal{I}^{AB} \equiv g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^B &= -\frac{1}{\tilde{N}^2}\left(\partial_t - \tilde{N}^f\partial_f\right)\phi^A\left(\partial_t - \tilde{N}^f\partial_f\right)\phi^B + \tilde{\gamma}^{ff}\partial_f\phi^A\partial_f\phi^B \\ &= \mathcal{I}^{AB}_{(2)} \equiv \tilde{g}^{\tilde{\mu}\tilde{\nu}}\partial_{\tilde{\mu}}\phi^A\partial_{\tilde{\nu}}\phi^B ,\end{aligned} \tag{4.29}$$

where the tilded indices take the values $\tilde{\mu} = 0, f$. We recognize the tilded variables $\tilde{N}$, $\tilde{N}^f$, $\tilde{\gamma}^{ff}$ as the ADM variables of an effective $1+1$ dimensional metric $\tilde{g}^{\tilde{\mu}\tilde{\nu}}$. Indeed, for the components of the effective metric

$$\begin{aligned}\tilde{g}^{00} &= g^{tt} , \quad \tilde{g}^{0f} = g^{ti}\partial_i f + g^{tt}\partial_t f , \\ \tilde{g}^{ff} &= g^{tt}\partial_t f\partial_t f + 2g^{ti}\partial_t f\partial_i f + g^{ij}\partial_i f\partial_j f ,\end{aligned}$$

they satisfy

$$\tilde{g}^{00} = -\frac{1}{\tilde{N}^2} , \qquad \tilde{g}^{0f} = \frac{\tilde{N}^f}{\tilde{N}^2} , \qquad \tilde{g}^{ff} = \tilde{\gamma}^{ff} - \left(\frac{\tilde{N}^f}{\tilde{N}}\right)^2 . \tag{4.30}$$

As in the $1+1$ dimensional case, the determinant $\det\mathcal{I}$ can be rewritten as a full square

$$\det\mathcal{I} \equiv \det\mathcal{I}^A_B = \frac{\det f}{\det\tilde{g}}\left[\frac{1}{2}\,\bar{\varepsilon}_{AB}\bar{\varepsilon}^{\tilde{\mu}\tilde{\nu}}\partial_{\tilde{\mu}}\phi^A\partial_{\tilde{\nu}}\phi^B\right]^2 \tag{4.31}$$

with $(\det\tilde{g})^{-1} = -\tilde{\gamma}^{ff}/\tilde{N}^2$, and $\bar{\varepsilon}^{\tilde{\mu}\tilde{\nu}}$, $\bar{\varepsilon}_{AB}$ denoting the flat space antisymmetric tensors. Hence all the terms in the Lagrangian containing the scalar fields can be rewritten in terms of an effective two-dimensional metric $\tilde{g}^{\tilde{\mu}\tilde{\nu}}$. We would like to emphasize that this rewriting is merely cosmetic and has the meaning only as the simplification of notations in the scalar field action.

In order to simplify the analysis of the equations of motion of the scalars, we rewrite the integration measure of the Lagrangian density in another coordinate system $\{\tilde{x}^\mu\}$, where $\tilde{x}^0 = t$, $\tilde{x}^1 = f(t, x^i)$, and $\tilde{x}^2 = \tilde{x}^2(x^\mu)$, $\tilde{x}^3 = \tilde{x}^3(x^\mu)$ some arbitrary non-degenerate coordinate transformations. In this case the metric components transform according to the usual transformation laws, and the components $\tilde{g}^{00}$, $\tilde{g}^{01}$, $\tilde{g}^{11}$ coincide with the components of the effective $1+1$ dimensional metric $\tilde{g}^{\tilde{\mu}\tilde{\nu}}$, $\tilde{\mu} = \{0, f\}$ given above. Hence the Lagrangian of the scalar fields can be rewritten in terms of the metric $\tilde{g}^{\mu\nu}$ as

$$S_\phi = 2 \int d\tilde{x}^2\, d\tilde{x}^3 \int dt\, df\, \sqrt{-\tilde{g}} \frac{1}{\tilde{N}} \left[\dot{\psi}_+ + \tilde{a}_+ \partial_f \psi_+\right]^{1/2} \left[\dot{\psi}_- - \tilde{a}_- \partial_f \psi_-\right]^{1/2} \tag{4.32}$$

where $\tilde{a}_\pm = \tilde{N}\sqrt{\tilde{\gamma}^{ff}} \mp \tilde{N}^f$. The variables $\tilde{N}$, $\tilde{N}^f$, $\tilde{\gamma}^{ff}$, used for notational simplicity only, can be expressed in terms of the metric $\tilde{g}^{\mu\nu}$ as in (4.30). By comparing this action with (4.19) one sees that the only difference is the volume factor and the prefactor $\sqrt{-\tilde{g}}/\tilde{N} \neq \sqrt{\gamma_{11}}$, which depends on all four space-time coordinates. Under the assumption that the volume spanned by $\tilde{x}^2$, $\tilde{x}^3$ is finite, the Hamiltonian analysis of the scalar field dynamics coincides with that in section 4.3.

We thus conclude that the ansatz for the scalar fields (4.27), (4.28) such that the condition $\det S = 0$ is satisfied leads to a theory which is equivalent to the $1+1$ dimensional case and thus propagates no degrees of freedom. In Hamiltonian language, on this subspace of the scalar field configurations the theory has two first class constraints.

5 Conclusions

Any diffeomorphism invariant formulation of massive gravity inevitably contains a number of scalar fields minimally coupled to the dynamical metric field and can be viewed as just some particular scalar field theory coupled to general relativity. Therefore we argue that the Hamiltonian structure and the counting of degrees of freedom can be done for gravity and scalar fields separately. In other words, the absence of the sixth degree of freedom in the dRGT non-linear massive gravity [7] can be seen as a feature of the scalar field action, and can be studied in the scalar field theory given by the dRGT mass term.

While the full dRGT scalar action contains the number of fields equal to the space-time dimension, in this paper we have focused on the reduced case with two scalar fields, which coincides with the full theory only in $1+1$ dimensions. We have calculated the determinant of the kinetic matrix $\partial^2\mathcal{L}_\phi/\partial\dot{\phi}^A\partial\dot{\phi}^B$ of the non-linear theory and have found that in $d > 1$ dimensions it does not vanish for generic initial conditions. Thus in more than $1 + 1$ dimensions both of the fields are, in general, propagating. However there exists a subspace of the configuration space where the Hessian is vanishing. It corresponds to the case where the coordinate transformation represented by the scalar fields $\phi^A(x^i)$ is singular on any two-dimensional space-like surface, or, equivalently, when both of the fields depend only on one independent space-like direction. In this case the scalars effectively live on the $1+1$ dimensional space-time, and the theory is equivalent to the $1+1$ dimensional dRGT massive gravity, where there is only single spatial direction available. For the latter constrained theory the full Hamiltonian analysis reveals two first-class constraints which generate one gauge transformation that leaves the action invariant. Since the transformation involves two independent parameters, then after fixing the gauge the theory does not contain any degrees of freedom. This is in agreement with the previous findings in the $1+1$ dimensional dRGT massive gravity [10, 28]. For the theory in more than $1+1$ dimensions the effectively $1+1$ dimensional solutions with vanishing Hessian correspond to the so-called *singular solutions*. On such a singular solution at each moment in time there exist infinitely many other regular solutions of the theory which are tangential to the singular solution, i.e. with coinciding $\phi^A(x^i)$ and $\dot{\phi}^A(x^i)$. Therefore, there is no choice of initial conditions that uniquely specifies such a solution, and any perturbation in the initial conditions leads to the regular solution with two degrees of freedom and non-vanishing Hessian. We note that our findings do not allow us to draw conclusions about the behaviour of the dRGT-like theories with more than two scalar fields, but the proposed method can be extended to include arbitrary number of scalar fields.

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MASSIVE GRAVITY ON CURVED BACKGROUND

LASMA ALBERTE
*Arnold Sommerfeld Center for Theoretical Physics,
Ludwig Maximilians University, Theresienstr. 37,
80333 Munich, Germany
lasma.alberte@physik.lmu.de*

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We investigate generally covariant theories which admit a Fierz–Pauli mass term for metric perturbations around an arbitrary curved background. For this we restore the general covariance of the Fierz–Pauli mass term by introducing four scalar fields which preserve a certain internal symmetry in their configuration space. It is then apparent that for each given spacetime metric this construction corresponds to a completely different generally covariant massive gravity theory with different symmetries. The proposed approach is verified by explicit analysis of the physical degrees of freedom of massive graviton on de Sitter space.

Keyword: Massive gravity.

1. Introduction

The first successful attempt to modify the quadratic Einstein–Hilbert action in order to describe a massive spin-2 particle in Minkowski space was made by Fierz and Pauli.[1] They found that there exists a unique quadratic graviton mass term which gives unitary evolution of massive spin-2 field with five degrees of freedom, consistent with Poincaré invariance. Much later this quadratic model of massive gravity was found to be inconsistent with observations and the need of its nonlinear extension was established.[2–4] It was only recently that a nonlinear completion of massive gravity which is ghost-free at least in the decoupling limit was proposed by de Rham, Gabadadze and Tolley (dRGT).[5,6] It is also known that the Fierz–Pauli (FP) mass term explicitly breaks diffeomorphism invariance of general relativity which however can be restored by introducing four scalar fields.[7–9] Then the graviton acquires mass around a symmetry breaking background of the scalar fields *via* the gravitational Higgs mechanism.[7]

The objective of this paper is a detailed discussion of the possibility of having a consistent diffeomorphism invariant theory of a massive graviton on arbitrary curved background. We first note that there is no unambiguous definition of a mass of a particle in a curved spacetime which is not Poincaré invariant. Since any spacetime can locally be approximated by Minkowski spacetime, one would however expect that a massive graviton in curved space has the same number of degrees of freedom as a massive graviton in flat space. We will therefore assume that a massive spin-2 particle on arbitrary background propagates five degrees of freedom with equal dispersion relations.

One way of addressing the question about a massive gravity theory on arbitrary backgrounds is to investigate the nonflat metric solutions in dRGT gravity. Since the metric perturbations around Minkowski space in this theory have a FP mass term, then one could expect that also a spin-2 particle on a non-Minkowski solution of dRGT gravity has five degrees of freedom, all of which have the same mass. There have been numerous attempts to this problem and several spherically symmetric cosmological solutions have been found in the nonlinear theory.[10–17] However, metric perturbations around these nontrivial background solutions do not, in general, have a mass term of the FP form. In Refs. 14 and 18, metric perturbations around the self-accelerating solutions of dRGT gravity were investigated. It was shown that only the transverse traceless tensor metric perturbations satisfy the equation of a minimally coupled massive scalar field. The scalar and vector part of the quadratic action was shown to coincide with the corresponding action in general relativity giving no additional dynamical degrees of freedom. This behavior is quite different from the massive graviton on Minkowski space which has in total five and not two massive degrees of freedom.

Another approach to generalizing massive gravity on curved backgrounds is the bimetric theories where an additional spin-2 field is introduced.[19–21] The spherically symmetric solutions and Friedmann–Robertson–Walker (FRW) solutions in bigravity formulation were studied in Refs. 22–25. However, bimetric theories have a different scope from the single spin-2 field massive gravity theory discussed in the present work.

In this paper, we shall adopt the convention that a massive gravity on some curved background is a theory such that the metric perturbations around this background have a mass term of FP form. Since the FP mass term explicitly depends on the background metric, it breaks the diffeomorphism invariance of general relativity and can only be regarded as the gauge fixed version of the underlying generally covariant theory. It is nevertheless important to know how the general covariance is maintained even if it is often enough to work in one particular gauge with no gauge redundancy in description. We will first reason that in dRGT theory the only spacetime in which the graviton has a FP mass term is the Minkowski space. Therefore one has to look for another generally covariant theory describing FP massive gravitons on curved backgrounds. For this we will generalize the Higgs mechanism for gravity, as introduced in Ref. 7, to arbitrary curved spacetime. In the usual Higgs gravity on flat space the graviton mass term is built out of the diffeomorphism invariant combinations of the scalar fields $\bar{h}^{AB} = g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^B - \eta^{AB}$.[7] Here we modify the variables $\bar{h}^{AB}$ to be suitable for cosmological backgrounds by replacing the Minkowski metric η^{AB} in the definition of $\bar{h}^{AB}$ by some scalar functions $\bar{f}^{AB}(\phi)$. In the internal space of the scalar fields the set of the functions $\bar{f}^{AB}(\phi)$ acts as a metric.

We then demonstrate how our approach works for the special case of de Sitter spacetime. The properties of massive graviton in de Sitter universe have been studied previously in a theory where the diffeomorphism invariance is explicitly broken by the FP mass term.[26,27] It has been shown that this quadratic theory possesses a couple of properties distinctive from the massive gravity on Minkowski background. In particular, the helicity-0 component of the graviton seems to become nondynamical for a specific choice of the mass parameter m and cosmological constant Λ.[26,27] For graviton masses below this value, i.e. $m^2 < 2\Lambda/3$, the theory admits negative norm states. The unitarily allowed region for massive graviton in de Sitter space is therefore restricted to $m^2 \geq 2\Lambda/3$, and is known as the Higuchi bound. Generalization of this bound to arbitrary FRW universe has been found in Refs. 28–31 (for extension to Lorentz violating graviton mass terms see Ref. 32). This motivates us to verify that the same results can be obtained from the diffeomorphism invariant Higgs massive gravity on de Sitter space proposed in this paper.

A consistent description of massive graviton on FRW spacetime is of particular interest also from the phenomenological point of view. Conventionally a spatially flat FRW spacetime is used to approximate various stages of the history of the universe. A nonvanishing graviton mass inevitably modifies the evolution of cosmological perturbations and could thus leave observable imprints in the cosmic microwave background (CMB) spectrum. The analysis of the effects of massive tensor perturbations under the assumption that the scalar and vector perturbations of the metric coincide with general relativity was done in Ref. 33. It was shown that in the graviton mass range between 10^{-30} and 10^{-27} eV the characteristic feature of massive tensor perturbations for the CMB is a plateau in the B-mode spectrum for multipoles $l \leq 100$. For even larger graviton masses $m \gg 10^{-27}$ eV the tensor perturbations are strongly suppressed. Thus nondetection of the B-mode signal in the near future could serve as a hint towards a nonvanishing graviton mass. In this paper, we introduce a diffeomorphism invariant model of massive gravity on arbitrary curved background with five massive gravitational degrees of freedom which could also affect the evolution of scalar density perturbations. This theory thus provides a theoretical framework for studying the effects of a nonvanishing graviton mass to the CMB spectra, and therefore deserves a further investigation which is, however, beyond the scope of the present work.

The paper is organized as follows: in Sec. 2, we discuss how the diffeomorphism invariance of massive gravity can be maintained on arbitrary background. We review the gravitational Higgs mechanism in Minkowski space and discuss the nonlinear dRGT completion of the quadratic FP mass term. We briefly comment on the nonlinear cosmological solutions in this theory and argue that the dRGT gravity cannot simultaneously admit a curved background solution for the metric and a FP mass term for metric perturbations. We point out the crucial points of failure and with this knowledge we generalize the gravitational Higgs mechanism to arbitrary curved spacetimes. In Sec. 3, we work out in detail the proposed model for de Sitter universe and recover the results obtained in previous literature.[26,27] We conclude in Sec. 4.

2. Diffeomorphism Invariant Massive Gravity

Let us consider the Einstein–Hilbert action with some matter Lagrangian $\mathcal{L}_m$ and FP mass term

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}R + \int d^4x\sqrt{-g}\,\mathcal{L}_m(g_{\mu\nu},\psi) + S_{\rm FP}, \qquad (1)$$

where ψ denotes a set of matter fields and we have set $8\pi G \equiv 1$. The FP mass term for metric perturbations $h^{\mu\nu} \equiv g^{\mu\nu} - {}^{(0)}g^{\mu\nu}$ can be written as

$$S_{\rm FP} = \frac{m^2}{8}\int d^4x\sqrt{-g}\,h^{\alpha\beta}h^{\mu\nu}({}^{(0)}g_{\mu\nu}{}^{(0)}g_{\alpha\beta} - {}^{(0)}g^{(0)}_{\mu\alpha}g_{\nu\beta}), \qquad (2)$$

where the background metric ${}^{(0)}g^{\mu\nu}(x)$ satisfies the Einstein equations and is determined by the matter Lagrangian $\mathcal{L}_m$. In this section, we will generalize the FP mass term in a diffeomorphism invariant way for arbitrary background. We will show that the resulting generally covariant theory is different for each background metric ${}^{(0)}g^{\mu\nu}$.

2.1. *On Minkowski background*

In order to give mass to graviton in a diffeomorphism invariant way we employ four scalar fields $\phi^A, A = 0,1,2,3$ and introduce a Lorentz transformation Λ^A_B in the scalar field space. Hence the scalar field indices A, B are raised and lowered with the Minkowski metric $\eta^{AB} = \mathrm{diag}(+1,-1,-1,-1)$. We then build the mass term for metric perturbations from the combinations of the variables

$$\bar{h}^{AB} = H^{AB} - \eta^{AB} \quad \text{where } H^{AB} = g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^B \qquad (3)$$

is a composite field space tensor.[7] On Minkowski background the scalar fields ϕ^A acquire vacuum expectation values proportional to Cartesian spacetime coordinates ${}^{(0)}\phi^A = x^\mu\delta^A_\mu$. The diffeomorphism invariance is thus spontaneously broken and the scalar field perturbations $\chi^A \equiv \phi^A - {}^{(0)}\phi^A$ induce four additional degrees of freedom. In combination with the two degrees of freedom of the massless graviton the scalar field perturbations constitute the five degrees of freedom of a massive spin-2 particle and a ghost. The ghost in quadratic order is canceled by the choice of the FP mass term.

In unitary gauge, when $\chi^A = 0$, the variables $\bar{h}^{AB}$ are equal to metric perturbations since $\bar{h}^{AB} = \delta^A_\mu\delta^B_\nu h^{\mu\nu}$. Thus the diffeomorphism invariance of general relativity is restored by replacing $h^{\mu\nu} \to \bar{h}^{AB}$ in the FP mass term. This leads to the following action of the scalar fields which around the symmetry breaking background gives the FP mass term for metric perturbations:

$$S_\phi = \frac{m^2}{8} \int d^4x\sqrt{-g}(\bar{h}^2 - \bar{h}^A_B \bar{h}^B_A). \tag{4}$$

Since the field $\bar{h}^{AB}$ transforms as a scalar under general coordinate transformations, this Lagrangian is manifestly diffeomorphism invariant. Moreover, as the Latin indices in the action are contracted, it is invariant also under the isometries of the metric η_{AB}, namely the Lorentz transformations Λ^A_B introduced above.

It is known that the action (4) propagates the Boulware–Deser ghost in cubic order in perturbations and have to be supplemented with higher-order terms in $\bar{h}^{AB}$. It was shown by dRGT in Refs. 5 and 6 that the massive gravity potential, which in Minkowski space is ghost-free in decoupling limit, can be resummed in terms of a new field

$$\mathcal{K}^\mu_\nu = \delta^\mu_\nu - \sqrt{g^{\mu\lambda}\partial_\lambda\phi^A\partial_\nu\phi^B\eta_{AB}}. \tag{5}$$

The nonlinear dRGT massive gravity can thus be written in a closed nonperturbative form as[a]

$$S_{\mathrm{dRGT}} = S_{\mathrm{GR}} + S_\phi = -\frac{1}{2}\int d^4x\sqrt{-g}R + \frac{m^2}{2}\int d^4x\sqrt{-g}([\mathcal{K}]^2 - [\mathcal{K}^2]). \tag{6}$$

By construction this theory admits the solution

$$g_{\mu\nu} = \eta_{\mu\nu} \quad \text{and} \quad \phi^A = x^\mu\delta^A_\mu \tag{7}$$

around which the metric perturbations have a quadratic FP mass term. Other so called empty space solutions of the model (6) have been studied in numerous papers.[10–15] More solutions have been found in the presence of external matter sources described by some Lagrangian density $\mathcal{L}_m$ in Refs. 15–17.

The metric perturbations around the various solutions of dRGT theory, in general, do not have a mass term of the FP form. This can be understood by considering some arbitrary background solution for the metric ${}^{(0)}g_{\mu\nu}$ and scalar fields ${}^{(0)}\phi^A$. The tensor field $\mathcal{K}^\mu_\nu$ can then be splitted as $\mathcal{K}^\mu_\nu = {}^{(0)}\mathcal{K}^\mu_\nu + \delta\mathcal{K}^\mu_\nu$ with

$${}^{(0)}\mathcal{K}^\mu_\nu = \delta^\mu_\nu - \sqrt{{}^{(0)}g^{\mu\lambda}\partial_\lambda{}^{(0)}\phi^A\partial_\nu{}^{(0)}\phi^B\eta_{AB}} \tag{8}$$

and $\delta\mathcal{K}^\mu_\nu$ denoting a perturbation. For the solution (7) the background value of $\mathcal{K}^\mu_\nu$ vanishes and $\delta\mathcal{K}^\mu_\nu = -\frac{1}{2}h^\mu_\nu + \mathcal{O}(\delta\phi, h^2, \ldots)$. After substituting this in the action (6) one obtains a FP mass term for the metric perturbations. However for solutions of dRGT theory with ${}^{(0)}\mathcal{K}^\mu_\nu \neq 0$ the quadratic potential of (6) gives not only terms

[a] Also special combinations of cubic and quartic terms in $\mathcal{K}^\mu_\nu$ can be added to this action. We shall keep this in mind, but here we skip them in order not to clutter the notations. For the additional terms see Ref. 5.

quadratic in $\delta\mathcal{K}^\mu_\nu$ but also zeroth- and first-order terms like $({}^{(0)}\mathcal{K}^\mu_\mu)^2$ and ${}^{(0)}\mathcal{K}^\nu_\mu\delta\mathcal{K}^\mu_\nu$. This implies that also the additional cubic and quartic terms in $\mathcal{K}^\mu_\nu$ contribute to the quadratic terms in metric perturbations. Therefore, the FP structure of the mass term for metric perturbations is most probably lost. A fully general proof of this statement is still lacking, but for some specific background solutions it has been confirmed by detailed analysis of metric perturbations in Refs. 14 and 18. In other words the form of the FP mass term is most likely preserved only for the solutions with ${}^{(0)}\mathcal{K}^\mu_\nu = 0$.

Another general feature of the dRGT theory is the appearance of an effective energy–momentum tensor of the scalar fields, $T^{(\phi)}_{\mu\nu}$, arising from the mass term:

$$T^{(\phi)}_{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = -\frac{m^2}{2}g_{\mu\nu}([\mathcal{K}]^2 - [\mathcal{K}^2]) + \frac{m^2}{2}\mathcal{K}^\alpha_\beta\frac{\delta\mathcal{K}^\lambda_\rho}{\delta g^{\mu\nu}}[\delta^\beta_\alpha\delta^\rho_\lambda - \delta^\rho_\alpha\delta^\beta_\lambda]. \tag{9}$$

The contributions from the mass term thus inevitably modify the background solutions of general relativity (GR) which in the absence of graviton mass term is determined by the matter stress energy tensor. Even such important GR solutions as Schwarzschild metric and spatially flat FRW metric are not solutions of dRGT theory if ${}^{(0)}\mathcal{K}^\mu_\nu \neq 0$. Therefore, in order to recover GR from the action (6), the effect of the energy–momentum tensor due to ${}^{(0)}\mathcal{K}^\mu_\nu \neq 0$ should be negligible at least in Vainshtein regime. Basing on these observations we claim that the dRGT theory can be interpreted as a phenomenologically viable modification of gravity, such that the metric perturbations around a given background have a Fierz–Pauli mass term, only around the solutions with ${}^{(0)}\mathcal{K}^\mu_\nu = 0$.

It is easy to see that this is equivalent to the condition ${}^{(0)}\bar{h}^{AB} = 0$. In this case the quadratic mass term for metric perturbations is determined by the action quadratic in $\bar{h}^{AB}$ with no need to specify the nonlinear completion of the theory. We will therefore consider only the generally covariant quadratic action (4) and require that ${}^{(0)}\bar{h}^{AB} = 0$ for some non-Minkowski background metric ${}^{(0)}g^{\mu\nu} \neq \eta^{\mu\nu}$. This translates into an equation for the background values of the scalar fields ${}^{(0)}\phi^A$:

$${}^{(0)}g^{\mu\nu}(x)\frac{\partial^{(0)}\phi^A}{\partial x^\mu}\frac{\partial^{(0)}\phi^B}{\partial x^\nu} = \eta^{AB}. \tag{10}$$

By identifying $\tilde{x}^\mu \equiv {}^{(0)}\phi^A\delta^\mu_A$ this can be interpreted as a metric transformation law under general coordinate transformations $x^\mu \to \tilde{x}^\mu$ such that the transformed metric is $\tilde{g}^{\mu\nu} = \delta^\mu_A\delta^\nu_B\eta^{AB}$. Such a coordinate transformation which transforms a curved spacetime into a flat spacetime does not exist. Therefore, for arbitrary curved metric ${}^{(0)}g^{\mu\nu}$ there is no solution for the scalar fields ${}^{(0)}\phi^A$ such that (10) is satisfied at every point of the spacetime. Hence, in order to describe a FP massive graviton on a curved background one has to modify the diffeomorphism invariant variables $\bar{h}^{AB}$ so that the requirement ${}^{(0)}\bar{h}^{AB} = 0$ is fulfilled.

2.2. *On curved spacetimes*

In this section, we will generalize the diffeomorphism invariant field space variables $\bar{h}^{AB}$ so that in the unitary gauge when $\phi^A = x^\mu \delta^A_\mu$ the field $\bar{h}^{AB}$ would coincide with the metric perturbations $h^{\mu\nu} \equiv g^{\mu\nu} - {}^{(0)}g^{\mu\nu}$ around an arbitrary background metric ${}^{(0)}g^{\mu\nu}$. In analogy to the definition (3) we generalize $\bar{h}^{AB}$ as

$$\bar{h}^{AB} \equiv H^{AB} - \bar{f}^{AB}(\phi) \tag{11}$$

with some arbitrary scalar function $\bar{f}^{AB}(\phi)$. Independently of the function $\bar{f}^{AB}$ this variable is invariant under spacetime diffeomorphisms for $\bar{f}^{AB}$ depends only on the four scalar fields ϕ^A. We then notice that if the functional dependence of $\bar{f}^{AB}(\cdot)$ is set by the solution of Einstein equations as $\bar{f}^{AB}(\cdot) \equiv {}^{(0)}g^{\mu\nu}(\cdot)\delta^A_\mu \delta^B_\nu$ then the background value of $\bar{h}^{AB}$ vanishes. For example, if ${}^{(0)}g^{\mu\nu}(x) = a^{-2}(\eta)\eta^{\mu\nu}$ is the Friedmann metric, written by using the conformal time $x^0 \equiv \eta$, then $\bar{f}^{AB}(\phi) = a^{-2}(\phi^0)\eta^{AB}$. We have simply replaced the spacetime coordinate x^0 with the scalar field ϕ^0.

Hence, given the background solution of the Einstein equations ${}^{(0)}g^{\mu\nu}(x)$ it is straightforward to write down the quadratic FP mass term for metric perturbations around this background in a diffeomorphism invariant way. For this one simply has to perform the substitution $h^{\mu\nu} \to \bar{h}^{AB}$ in the FP mass term (2), where the latter is defined as

$$\bar{h}^{AB}_{\text{curved}} \equiv g^{\mu\nu}(x)\partial_\mu \phi^A \partial_\nu \phi^B - \bar{f}^{AB}(\phi), \quad \bar{f}^{AB}(\phi) \equiv {}^{(0)}g^{\mu\nu}(\phi)\delta^A_\mu \delta^B_\nu. \tag{12}$$

The scalar fields then admit the solution ${}^{(0)}\phi^A = x^\mu \delta^A_\mu$ and on the scalar field background the diffeomorphism invariance is spontaneously broken giving mass to the graviton. However the condition $\bar{f}^{AB} \equiv {}^{(0)}g^{\mu\nu}\delta^A_\mu \delta^B_\nu$ has to be imposed by hand depending on the matter content of the initial theory without the graviton mass term.

We note that the only distinction between the definition of $\bar{h}^{AB}$ in flat spacetime (3) and the generalized definition (11) in curved spacetime is that we have replaced the Minkowski metric $\eta^{AB} \to \bar{f}^{AB}(\phi)$. Hence the "distances" in the scalar field space are now measured by the metric $\bar{f}^{AB}$, and the scalar field space indices have to be raised and lowered as

$$\phi_B \equiv \bar{f}_{AB}\phi^A. \tag{13}$$

In particular, $\bar{h}^A_B \equiv \bar{f}_{BC}\bar{h}^{AC}$. There is however a crucial difference between the Higgs mechanism for gravity on curved background presented in this paper and massive gravity with a general reference metric investigated in Refs. 21 and 34. In these works the dRGT graviton mass term has been rewritten in terms of the square root of a matrix $g^{\mu\lambda}f_{\lambda\nu}$, where $g^{\mu\nu}$ is the physical metric of the spacetime and $f_{\mu\nu}$ is an auxiliary reference metric. The metric $f_{\mu\nu}$ explicitly depends on the spacetime coordinates, and setting $f_{\mu\nu} = \eta_{\mu\nu}$ is equivalent to going to the unitary gauge in dRGT picture. We can relate the auxiliary reference metric $f_{\mu\nu}$ to the metric $\bar{f}_{AB}(\phi)$ in the scalar field space by the parametrization

$$f_{\mu\nu} = \bar{f}_{AB}(\phi)\frac{\partial\phi^A}{\partial x^\mu}\frac{\partial\phi^B}{\partial x^\nu}. \tag{14}$$

In Ref. 21 dynamics of the reference metric $f_{\mu\nu}$ is invoked by adding to the Lagrangian a standard Einstein–Hilbert kinetic term for the metric $f_{\mu\nu}$. This gives rise to a bimetric theory of two spin-2 fields, one massive and one massless. In our work the spacetime tensor field $f_{\mu\nu}$ becomes a dynamical object since it is a function of the scalar fields ϕ^A. The scalar field metric $\bar{f}_{AB} = \bar{f}_{AB}(\phi)$ is however simply a set of functions of the scalar fields ϕ^A and should not be interpreted as an independent spin-2 field.

In the case when the background spacetime is flat the definition (12) reduces to (3). The diffeomorphism invariant FP mass term on a curved background can be written as before in Eq. (4) with $\bar{h}^A_B$ defined in (12). The resulting FP mass term (2) is invariant under the isometries of the metric $\bar{f}_{AB}$ on the configuration space of the scalar fields.

To summarize, given a certain matter Lagrangian $\mathcal{L}_m$ and a corresponding solution of Einstein equations ${}^{(0)}g^{\mu\nu}(x)$ in a specific coordinate frame $\{x^\mu\}$, it is always possible to construct a diffeomorphism invariant FP mass term (4) with (12). When setting the scalar field perturbations $\chi^A \equiv \phi^A - {}^{(0)}\phi^A$ to zero one recovers the FP mass term around the solution ${}^{(0)}\phi^A = x^\mu\delta^A_\mu$. Moreover, it is straightforward to make use of the nonlinear dRGT completion written in terms of the flat space fields $\mathcal{K}^\mu_\nu$ by simply substituting $\mathcal{K}^\mu_\nu = \delta^\mu_\nu - \sqrt{g^{\mu\lambda}\partial_\lambda\phi^A\partial_\nu\phi^B\bar{f}_{AB}}$. The resulting nonlinear theory for metric perturbations $h^{\mu\nu} = g^{\mu\nu} - {}^{(0)}g^{\mu\nu}$ should possess the same properties. However, for every given background the diffeomorphism invariant FP Lagrangian corresponds to a different theory for the four scalar fields. It is therefore not possible to have a unique massive gravity theory such that metric perturbations around any arbitrary background would have a FP mass term. Instead one can choose and fix one particular theory such that around one particular background the metric perturbations have mass term of the FP form.

3. Massive Gravity in de Sitter Universe

In the second part of this paper, we work out in detail the Higgs massive gravity model for curved backgrounds presented in previous section in the special case of de Sitter universe. We write the diffeomorphism invariant Lagrangian explicitly in terms of the scalar fields. In unitary gauge we reproduce the results obtained in previous studies of theories where the general covariance is broken explicitly by the FP mass term.[26,27]

We consider the Einstein action with cosmological constant and generally covariant FP mass term

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}\,(R + 2\Lambda) + \frac{m^2}{8}\int d^4x\sqrt{-g}(\bar{h}^2 - \bar{h}^A_B\bar{h}^B_A), \tag{15}$$

where the scalar field tensor $\bar{h}^{AB}$ is defined as

$$\bar{h}^{AB} = g^{\mu\nu}(x)\frac{\partial\phi^A}{\partial x^\mu}\frac{\partial\phi^B}{\partial x^\nu} - \bar{f}^{AB}(\phi^A). \tag{16}$$

In spatially flat de Sitter universe the background metric can be written as ${}^{(0)}g^{\mu\nu} = a^{-2}(\eta)\eta^{\mu\nu}$ with $a(\eta) = -1/(H\eta)$, where the Hubble scale $H^2 = \Lambda/3$ is set by the cosmological constant. Hence the scalar field metric entering in (16) is given by $\bar{f}^{AB} = (H\phi^0)^2\eta^{AB}$ and the diffeomorphism invariant FP mass term can be written as

$$S_{\rm FP} = \frac{m^2}{8}\int d^4x\sqrt{-g}\{g^{\mu\nu}g^{\alpha\beta}\partial_\mu\phi^A\partial_\nu\phi^B\partial_\alpha\phi^C\partial_\beta\phi^D[\eta_{AB}\eta_{CD} - \eta_{BC}\eta_{AD}]$$
$$- 6(H\phi^0)^2 g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^B\eta_{AB} + 12(H\phi^0)^4\}. \tag{17}$$

We see that this mass term has a very specific dependence on the scalar field ϕ^0 which we introduced by hand after setting $\bar{f}^{AB} = a^{-2}(\phi^0)\eta^{AB}$. This breaks the translational invariance of ϕ^0, whereas the flat space massive gravity, discussed in Sec. 2.1, is invariant under the shifts of the scalar fields. It is therefore clear that massive gravity on de Sitter spacetime and massive gravity on Minkowski spacetime are two fundamentally different theories.

In order to show that the Lagrangian (15) describes a spin-2 particle with five degrees of freedom on de Sitter background let us consider perturbations around the backgrounds

$$g^{\mu\nu} = a^{-2}(\eta)(\eta^{\mu\nu} + h^{\mu\nu}), \quad \phi^A = x^A + \chi^A. \tag{18}$$

Then $\bar{h}^{AB}$ takes the exact form

$$\bar{h}^{AB} = a^{-2}(\eta)\left\{\eta^{AB} - \frac{a^{-2}(\phi^0)}{a^{-2}(\eta)}\eta^{AB} + h^{AB} + \partial_\mu\chi^B\eta^{\mu A} + \partial_\mu\chi^A\eta^{\mu B}\right.$$
$$\left. + h^{B\mu}\partial_\mu\chi^A + h^{A\mu}\partial_\mu\chi^B + \partial_\mu\chi^A\partial_\nu\chi^B\eta^{\mu\nu} + \partial_\mu\chi^A\partial_\nu\chi^B h^{\mu\nu}\right\}. \tag{19}$$

Here additional care must be taken since the Latin and Greek indices are raised with $\bar{f}^{AB}$ and $g^{\mu\nu}$, respectively, in particular $\bar{h}^A_B \equiv \bar{f}_{BC}\bar{h}^{AC} = a^2(\phi^0)\eta_{BC}\bar{h}^{AC}$. Meanwhile the Greek indices of the metric perturbations $h^{\mu\nu}$ are raised and lowered with the Minkowski metric $\eta^{\mu\nu}$. In order to find the explicit perturbative expansion of $\bar{h}^A_B$ we have to evaluate the ratio $a^2(\phi^0)a^{-2}(\eta)$. On the scalar field background $\phi^0 = \eta$ and $a^2(\phi^0)a^{-2}(\eta) = 1$, but due to perturbations of the scalar fields this ratio deviates from one. For small scalar field perturbations $\chi^0 = \phi^0 - \eta$ the scale factor $a^2(\phi^0)$ can be expanded up to second-order in χ^0 as

$$a^2(\phi^0) = a^2(\eta) + 2aa'\chi^0 + 3(a')^2(\chi^0)^2, \tag{20}$$

where the scale factor and its derivatives are evaluated at $\phi^0 = \eta$. Hence for $\bar{h}^A_B$ one obtains

$$\bar{h}^A_B = h^A_B + \partial_B\chi^A + \partial_\mu\chi^C\eta^{A\mu}\eta_{BC} + 2\frac{a'}{a}\chi^0\delta^A_B + \mathcal{O}(h^2, \chi^2). \tag{21}$$

The linearized transformation laws under infinitesimal diffeomorphisms $x^\mu \to x^\mu + \xi^\mu$ are

$$h^A_B \equiv h^{\mu\nu}\delta^A_\mu \delta^C_\nu \eta_{BC} \to \left[h^{\mu\nu} + \eta^{\mu\alpha}\partial_\alpha \xi^\nu + \eta^{\nu\alpha}\partial_\nu \xi^\alpha + 2\frac{a'}{a}\xi^0\right]\delta^A_\mu \delta^C_\nu \eta_{BC},$$

$$\chi^A \to \chi^A - \xi^A \tag{22}$$

and hence $\bar{h}^A_B$ in (21) is indeed gauge invariant. It is therefore always possible to go to unitary gauge where $\chi^A = 0$, $\bar{h}^A_B = h^{\mu\nu}\delta^A_\mu \delta^C_\nu \eta_{BC}$, and the action (15) reduces to the FP action (2). In what follows we will consider only small metric and scalar field fluctuations and neglect higher-order terms in (21).

As in our previous work, we will classify the metric perturbations according to the irreducible representations of the spatial rotation group[35,36]:

$$h_{00} = 2\phi, \tag{23}$$

$$h_{0i} = B_{,i} + S_i, \tag{24}$$

$$h_{ik} = 2\psi\delta_{ik} + 2E_{,ik} + F_{i,k} + F_{k,i} + \tilde{h}_{ik} \tag{25}$$

with $B_{,i} \equiv \partial B/\partial x^i$ and $S_i{}^{,i} = F_i{}^{,i} = \tilde{h}^{;i}_{ik} = \tilde{h}^i_i = 0$. The fields ϕ, ψ, E, B and the fields S_i, F_i describe scalar and vector metric perturbations, respectively. In empty space scalar and vector perturbations are nondynamical, and the dynamics of $h_{\mu\nu}$ is fully characterized by the transverse traceless tensor field $\tilde{h}_{ik}$. It has two independent degrees of freedom corresponding to the massless graviton. However, in the presence of matter inhomogeneities the propagation of scalar and vector metric perturbations can be induced. We also decompose the scalar field perturbations into scalar and vector parts as

$$\chi^0 = \chi^0, \quad \chi^i = \chi^i_\perp + \pi_{,i} \tag{26}$$

with $\chi^i_{\perp,i} = 0$.

The equations of motion for metric perturbations in de Sitter universe in the presence of any matter perturbations δT^μ_ν follow from the linearized Einstein equations.[35] In the absence of any additional external matter sources the effective energy–momentum tensor arises only due to the mass term and can be obtained by varying the scalar field part of the action (15):

$$T^{(\phi)}_{\mu\nu} = \frac{m^2}{2}\bar{h}^{AB}\partial_\mu \phi^C \partial_\nu \phi^D \left[\bar{f}_{AB}\bar{f}_{CD} - \bar{f}_{AD}\bar{f}_{BC}\right] - \frac{m^2}{8}g_{\mu\nu}[\bar{h}^2 - \bar{h}^A_B \bar{h}^B_A]. \tag{27}$$

In general $T_{\mu\nu}$ can be split into a background and perturbations as $T_{\mu\nu} = {}^{(0)}T_{\mu\nu} + \delta T_{\mu\nu}$. For arbitrary FRW spacetime the expression for $\delta T_{\mu\nu}$ would depend on the coordinate frame. However the linearized stress tensor due to the FP mass term on de Sitter universe is gauge invariant. The reason for this is that by construction there are no zeroth-order contributions to this energy–momentum tensor and it is nonvanishing only at perturbative level, hence $T^{(\phi)}_{\mu\nu} \equiv \delta T^{(\phi)}_{\mu\nu}$. The only contribution to the background energy tensor comes from the cosmological constant

${}^{(0)}T_{\mu\nu} = \Lambda\eta_{\mu\nu}$, implying the equation of state $p = -\rho$. At quadratic level in the action the scalar, vector and tensor perturbations decouple from each other and can be analyzed separately.

3.1. *Scalar perturbations*

Up to linear order in perturbations the scalar part of the variables $\bar{h}^A_B$ can be determined from the expression (21) as

$$
\begin{aligned}
{}^{(S)}\bar{h}^0_0 &= -2\phi + 2(\chi^0)' + 2\frac{a'}{a}\chi^0, \\
{}^{(S)}\bar{h}^0_i &= -B_{,i} + \chi^0_{,i} - (\pi')_{,i}, \\
{}^{(S)}\bar{h}^i_k &= 2\psi\delta_{ik} + 2E_{,ik} + 2\pi_{,ik} + 2\frac{a'}{a}\chi^0\delta_{ik},
\end{aligned}
$$

where $' \equiv \partial/\partial\eta$. The explicit expressions for the scalar components of the energy–momentum tensor are

$$
{}^{(S)}T_{00} = m^2a^2\left[3\frac{a'}{a}\chi^0 + 3\psi + \Delta E + \Delta\pi\right], \tag{28}
$$

$$
{}^{(S)}T_{0i} = -\frac{m^2}{2}a^2[-B_{,i} + \chi^0_{,i} - (\pi')_{,i}], \tag{29}
$$

$$
\begin{aligned}
{}^{(S)}T_{ik} = -\frac{m^2}{2}a^2 \Bigg[&\left(-2\phi + 2(\chi^0)' + 6\frac{a'}{a}\chi^0 + 4\psi + 2\Delta E + 2\Delta\pi\right)\delta_{ik} \\
&- 2(E+\pi)_{,ik}\Bigg].
\end{aligned} \tag{30}
$$

Although ${}^{(S)}T_{\mu\nu}$ is itself gauge invariant, each of the perturbations $\phi, \psi, E, B, \chi^0, \pi$ on the right-hand side of the above equations separately is not gauge invariant. Under infinitesimal coordinate transformations $x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu$, with the scalar components of the diffeomorphism ${}^{(S)}\xi^\alpha \equiv (\xi^0, \partial_i\zeta)$, the perturbations transform as:

$$
\begin{aligned}
\phi \to \tilde{\phi} &= \phi - \frac{1}{a}(a\xi^0)', & \psi \to \tilde{\psi} &= \psi + \frac{a'}{a}\xi^0, \\
E \to \tilde{E} &= E + \zeta, & B \to \tilde{B} &= B + \zeta' - \xi^0, \\
\chi^0 \to \tilde{\chi}^0 &= \chi^0 - \xi^0, & \pi \to \tilde{\pi} &= \pi - \zeta.
\end{aligned} \tag{31}
$$

Since we are free to choose the two functions ξ^0 and ζ, we can impose two gauge conditions on scalar perturbations. This corresponds to choosing a specific coordinate system. We can always switch from one coordinate system to another by performing a further coordinate transformation. Here we will study the linearized equations of motion in unitary gauge where $\tilde{\chi}^0 = \tilde{\pi} = 0$. This gauge can be obtained from (31) by a diffeomorphism ${}^{(S)}\xi^\alpha = (\chi^0, \partial_i\pi)$. We denote the perturbations in this gauge by tilded variables. The linearized Einstein equations for scalar perturbations then become

$$\Delta\Psi - 3\mathcal{H}(\Psi' + \mathcal{H}\Phi) = \frac{1}{2}m^2a^2(3\Psi - 3\mathcal{H}(\tilde{B} - \tilde{E}') + \Delta\tilde{E}), \tag{32}$$

$$\Psi' + \mathcal{H}\Phi = \frac{1}{4}m^2a^2\tilde{B}, \tag{33}$$

$$\Psi - \Phi = m^2a^2\tilde{E}, \tag{34}$$

$$\Psi'' + \mathcal{H}(2\Psi + \Phi)' + 3\mathcal{H}^2\Phi + \frac{1}{2}\Delta(\Phi - \Psi)$$

$$= -\frac{1}{2}m^2a^2(2\Psi - \Phi - 3\mathcal{H}(\tilde{B} - \tilde{E}') - (\tilde{B} - \tilde{E}')' + \Delta\tilde{E}), \tag{35}$$

where $\mathcal{H} = a'/a$ and on both sides of the equations we have expressed the metric perturbations $\tilde{\phi}, \tilde{\psi}$ with the gauge invariant scalar perturbations Φ and Ψ defined as

$$\Phi = \phi - \frac{1}{a}[a(B - E')]', \quad \Psi = \psi + \frac{a'}{a}(B - E'). \tag{36}$$

Equations (33) and (34) are nondynamical and can be used as constraints. After eliminating the gauge-dependent metric perturbations $\tilde{B}$ and $\tilde{E}$ Eqs. (32) and (35) can be brought in the form

$$\Box_g(a^{-2}[\Psi + \Phi]) + m^2a^2(a^{-2}[\Psi + \Phi]) = 0, \tag{37}$$

$$\frac{\Delta}{2}(\Psi + \Phi) + \frac{3}{2}\mathcal{H}(\Psi' + \Phi') = 3\Psi\left(\frac{m^2a^2}{2} - \mathcal{H}^2\right), \tag{38}$$

where $\Box_g \equiv \partial_\eta^2 + 2\mathcal{H}\partial_\eta - \Delta$ denotes the covariant d'Alambertian in de Sitter space.

In order to determine the number of degrees of freedom propagated by this system of equations together with their dispersion relations, we calculate the determinant of this system in Fourier representation. As a result we obtain

$$\text{Det} = 3\left(\frac{m^2a^2}{2} - \mathcal{H}^2\right)(-\omega^2 + 2\mathcal{H}^2 - 2\mathcal{H}i\omega + \mathbf{k}^2 + m^2a^2) \tag{39}$$

with conformal time frequency ω and 3-momentum $\mathbf{k}$. The second bracket corresponds to the equation of motion (37). It is therefore clear that the four Eqs. (32)–(35) describe only one massive scalar degree of freedom corresponding to the helicity-0 component of a massive spin-2 particle. In the special case when $\mathcal{H}^2 = \frac{m^2a^2}{2}$, or equivalently $2\Lambda/3 = m^2$, the determinant vanishes identically. In other words, in this case Eq. (38) establishes a relation between the scalar mode $a^{-2}(\Phi + \Psi)$ and its time derivative. This reduces the order of the equation of motion (37). Hence the scalar mode ceases to be dynamical and the massive graviton has only vector and tensor degrees of freedom in agreement with Refs. 26 and 27. This is due to the fact that, when $\mathcal{H}^2 = \frac{m^2a^2}{2}$, the fields Ψ and Φ enter Eqs. (37) and (38) in the combination $\Phi + \Psi$ only while $\Psi - \Phi$ remains arbitrary. However this result is most likely valid only at the linear level as we have suppressed higher-order terms which would otherwise contribute to the Eq. (38). The special value of the graviton mass $m^2 = 2\Lambda/3$ corresponds to the so called Higuchi bound.[26,27] If the graviton mass is smaller, i.e. $m^2 < 2\Lambda/3$, then the sign in the helicity-0 mode propagator flips with respect to the helicity-1 and helicity-2 modes. Hence below the Higuchi bound the graviton on de Sitter background is unstable and propagates a ghost.

In order to find the effective mass of the canonical variables we rewrite the Eq. (37) with respect to the physical time t. By defining the helicity zero component of the metric perturbation as $\tilde{q}_s \equiv a^{-1/2}[\Psi + \Phi]$ the equation of motion becomes

$$\ddot{\tilde{q}}_s - \frac{\Delta}{a^2}\tilde{q}_s + m_{\text{eff}}^2 \tilde{q}_s = 0. \tag{40}$$

This allows to describe the dynamics of the scalar perturbations as if they would propagate in Minkowski space with a Laplacian taken with respect to the physical space coordinates ax^i. The effective mass is $m_{\text{eff}}^2 = m^2 - \frac{9}{4}H^2$, in agreement with Ref. 27.

3.2. *Vector perturbations*

The vector components of $\bar{h}^A_B$ in linear order are equal to

$$^{(V)}\bar{h}^0_i = -S_i - (\chi^i_\perp)', \quad ^{(V)}\bar{h}^i_k = F_{i,k} + F_{k,i} + \chi^i_{\perp,k} + \chi^k_{\perp,i} \tag{41}$$

with $S^{;i}_i = F^{;i}_i = \chi^{;i}_{\perp i} = 0$. Under infinitesimal coordinate transformation $x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu$ with the vector components of the diffeomorphism $^{(V)}\xi^\mu = (0, \xi^i_\perp)$, $\xi^i_{\perp,i} = 0$, the perturbations transform as

$$S_i \to \tilde{S}_i = S_i + (\xi^i_\perp)', \quad F_i \to \tilde{F}_i = F_i + \xi^i_\perp, \quad \chi^i_\perp \to \tilde{\chi}^i_\perp = \chi^i_\perp - (\xi^i_\perp)'. \tag{42}$$

As for scalar perturbations we will work in the unitary gauge where $\tilde{\chi}^i_\perp = 0$. This gauge can be obtained from (42) by a diffeomorphism $^{(V)}\xi^\mu = (0, \chi^i_\perp)$. In order to find the variables of vector perturbations which satisfy equation of motion of the form (40) it is convenient to consider the action and find the canonically normalized variables. In unitary gauge the quadratic action becomes

$$\delta^{(2)}S = -\frac{1}{2}\int d^4x a^2 \left(\frac{1}{2}F_i'\Delta F_i' - S_i \Delta F_i' + \frac{1}{2}S_i \Delta S_i\right) + \frac{m^2}{4}\int d^4x a^4 (S_i S_i + F_i \Delta F_i). \tag{43}$$

Variation with respect to the field S_i gives a constraint equation which allows to express

$$S_i = \frac{\Delta F_i'}{\Delta - m^2 a^2}. \tag{44}$$

After substitution of this constraint and transformation to physical time $dt_{\rm phys} = ad\eta$, and a field redefinition $F_i \to q \equiv \sqrt{-\Delta}a^{3/2}F_i$ the action becomes[b]

$$\delta^{(2)}S = \frac{1}{4}\int d^4x \frac{m^2}{\left(m^2 - \frac{\Delta}{a^2}\right)}\left[\dot{q}^2 - 3Hq\dot{q} + q\left(\frac{\Delta}{a^2} - m^2 + \frac{9}{4}H^2\right)q\right]. \tag{45}$$

We further add a total time derivative $+2\int d^4xHq\dot{q}$ to the action and define the conjugated momenta as $p \equiv \frac{\partial\mathcal{L}}{\partial\dot{q}}$. By using the definition of p the action can be put in the form

$$S = \frac{1}{2}\int d^4x \left\{ 2p\dot{q} - \left[2p\left(\frac{-\frac{\Delta}{a^2} + m^2}{m^2}\right)p - Hp\left(\frac{-8\frac{\Delta}{a^2} + 5m^2}{m^2}\right)q \right.\right.$$
$$\left.\left. + \frac{1}{2}q\left(m^2 + 4H^2\frac{-4\frac{\Delta}{a^2} + m^2}{m^2}\right)q\right]\right\} \tag{46}$$

in agreement with Ref. 27. By another field redefinition

$$\frac{q}{\sqrt{2}} \equiv \frac{3H\tilde{q}_v + 2\tilde{p}_v}{2m}, \quad \frac{p}{\sqrt{2}} \equiv \frac{4H\tilde{p}_v - (m^2 - 6H^2)\tilde{q}_v}{2m} \tag{47}$$

we arrive at the diagonal form of the action

$$S = \int d^4x\left\{\tilde{p}_v\dot{\tilde{q}}_v - \left[\frac{1}{2}\tilde{p}_v^2 + \frac{1}{2}\tilde{q}_v\left(-\frac{\Delta}{a^2} + m^2 - \frac{9}{4}H^2\right)\tilde{q}_v\right]\right\}. \tag{48}$$

This action describes two dynamical degrees of freedom of vector perturbations. The equation of motion for the canonically normalized vector modes $\tilde{q}_v$ then coincides with the equation for the scalar modes and is

$$\ddot{\tilde{q}}_v - \frac{\Delta}{a^2}\tilde{q}_v + m_{\rm eff}^2\tilde{q}_v = 0 \tag{49}$$

with the effective mass $m_{\rm eff}^2 = m^2 - \frac{9}{4}H^2$.

3.3. *Tensor perturbations*

The linearized Einstein equation for tensor perturbations is

$$\tilde{h}_{ij}'' + 2\mathcal{H}\tilde{h}_{ij}' - \Delta\tilde{h}_{ij} = -16\pi G\delta T^i_j \tag{50}$$

which with ${}^{(T)}\bar{h}^i_k = \tilde{h}_{ik}$ and ${}^{(T)}T_{ik} = \frac{m^2a^2}{2}\tilde{h}_{ik}$ immediately yields

$$\tilde{h}_{ij}'' + 2\mathcal{H}\tilde{h}_{ij}' - \Delta\tilde{h}_{ij} + m^2a^2\tilde{h}_{ij} = 0. \tag{51}$$

[b]The spatial index i is suppressed in the definition of the new variable q as the indices of vector perturbations F_i can only be contracted in an obvious way, i.e. F_iF_i. We keep in mind, however, that the variable q has two independent components.

After field redefinition $\tilde{h}_{ij} \to \tilde{q}_t \equiv a^{3/2}\tilde{h}_{ij}$ and transformation to physical time the above equation takes the form

$$\ddot{\tilde{q}}_t - \frac{\Delta}{a^2}\tilde{q}_t + m_{\text{eff}}^2 \tilde{q}_t = 0 \tag{52}$$

with effective mass $m_{\text{eff}}^2 = m^2 - \frac{9}{4}H^2$ which coincides with the effective mass of scalar and vector modes of the graviton. Hence we conclude that all canonically normalized helicity-0, ± 1, ± 2 modes of massive graviton on de Sitter universe satisfy wave equation for a massive scalar field of the form (52) with the same effective mass. In other words, all five degrees of freedom have the same dispersion relations.

4. Conclusions

In this paper, we have investigated the diffeomorphism invariant theories of massive gravity on curved backgrounds. With this we understand theories for which the metric perturbations in some curved spacetime have a quadratic FP-like mass term and thus propagate in total five degrees of freedom with equal dispersion relations.

We have argued that Minkowski metric is the only solution of the nonlinear dRGT massive gravity around which the metric perturbations have a mass term of FP form. Therefore we have generalized the gravitational Higgs mechanism[7] and restored the diffeomorphism invariance of the quadratic FP mass term for metric perturbations around arbitrary curved background. Our approach involves a set of scalar functions $\bar{f}_{AB}(\phi)$ which act as a metric on the internal space of the scalar fields ϕ^A. The functional dependence of $\bar{f}^{AB}$ is determined by the background solution of the Einstein equations as $\bar{f}^{AB} = {}^{(0)}g^{\mu\nu}\delta^A_\mu \delta^B_\nu$. This condition has to be imposed by hand and therefore the generally covariant FP action takes a different form depending on the external matter content of the theory. Moreover each massive gravity action has distinct symmetries in the scalar field space, namely, the isometries of the scalar field metric $\bar{f}_{AB}$. In other words for each background metric this mechanism corresponds to a different diffeomorphism invariant theory. In our model the scalar fields ϕ^A enter the action not only through their derivatives, but also through $\bar{f}_{AB}(\phi)$ which involves explicit dependence on ϕ^A. Hence the shift symmetry of scalar fields present in the dRGT theory is broken. This stresses clearly that the theories are fundamentally different. We therefore conclude that there does not exist one single theory of massive gravity such that the metric perturbations around any arbitrary background have a FP mass term. Instead we have shown that one can construct by hand an infinite number of massive gravity theories, each of them corresponding to one particular background metric.

In the second part of this work we have demonstrated how our approach works for de Sitter universe explicitly by investigating the equations of motion for metric perturbations in the unitary gauge. As expected we find that one scalar, two vector and two tensor modes are propagating constituting the five degrees of freedom of massive graviton with the same effective mass $m_{\text{eff}}^2 = m^2 - \frac{9}{4}H^2$.

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"Semiclassical Calculation of Multiparticle Scattering Cross Sections in Classicalizing Theories,"
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Semiclassical calculation of multiparticle scattering cross sections in classicalizing theories

Lasma Alberte[1,*] and Fedor Bezrukov[2,3,†]

[1]*Arnold Sommerfeld Center for Theoretical Physics, Fakultät für Physik Ludwig-Maximilians-Universität München, Theresienstr. 37, 80333 München, Germany*

[2]*Physics Department, University of Connecticut, Storrs, Connecticut 06269-3046, USA*

[3]*RIKEN-BNL Research Center, Brookhaven National Laboratory, Upton, New York 11973, USA*

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It has been suggested that certain derivatively coupled nonrenormalizable scalar field theories might restore the perturbative unitarity of high-energy hard scatterings by classicalization, i.e., formation of multiparticle states of soft quanta [2]. Here we apply the semiclassical method of calculating the multiparticle production rates to the scalar Dirac-Born-Infeld theory, which is suggested to classicalize. We find that the semiclassical method is applicable for the energies in the final state above the cutoff scale of the theory, L_*^{-1}. We encounter that the cross section of the process $2 \to N$ ceases to be exponentially suppressed for the particle number in the final state N smaller than a critical particle number $N_{\rm crit} \sim (EL_*)^{4/3}$. It coincides with the typical particle number produced in two-particle collisions at high energies predicted by classicalization arguments.

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I. INTRODUCTION

A traditional approach to field theory proposes that the fundamental field theories at high energies (allowing for predictive calculations) are the renormalizable ones.[1] A nonrenormalizable effective theory at a lower energy may have two different kinds of behavior at high energy. It can either complete itself at UV by additional weakly coupled perturbative degrees of freedom (Wilsonian completion) and become a renormalizable theory, or it can match to a strongly coupled phase of an asymptotically free theory. A well-known example for the Wilsonian UV completion is the four-fermion theory of weak interactions at low energy becoming a gauge theory with the Higgs mechanism above the Fermi scale. Alternatively, the effective theory describing baryons and mesons at low energy is completed at high energies by the asymptotically free QCD with gluons and quarks. Recently, an alternative mechanism, termed "classicalization," was suggested in Ref. [2] for theories with nonrenormalizable derivative self-couplings. This mechanism may work in such a fundamental theory as gravity [3,4].

The simplest example of a classicalizing theory is a scalar theory with a leading nonlinear derivative interaction of the form

$$L_*^4(\partial_\mu\phi\partial^\mu\phi)^2. \tag{1}$$

A particularly convenient example of a scalar field theory with such a leading interaction term is given by the Dirac-Born-Infeld (DBI)-type action

$$S = \epsilon_2 \int d^4x \frac{1}{2L_*^4}\sqrt{1 + 2\epsilon_2 L_*^4(\partial_\mu\phi)^2}, \tag{2}$$

with $\epsilon_2 = \pm 1$. According to the standard picture, the perturbative unitarity in such theories is violated at energies above the cutoff L_*^{-1} due to the derivative self-interactions of the scalar field. Instead, it was suggested in Refs. [2,5] that in such theories a transcutoff scattering process of two particles is dominated by low momentum transfer $\sim r_*^{-1}$, where the length scale $r_*(E)$ depends on the energy and $r_*(E) \gg L_*$. As a result, the leading contribution to the scattering process of two hard particles with high center-of-mass energy $E \gg L_*^{-1}$ comes from the production of a multiparticle quantum state of $N \sim Er_*$ soft particles. This state is called "classicalon," and in the semiclassical limit

$$L_* \to 0, \qquad N \to \infty, \qquad r_* = \text{fixed},$$

it should correspond to a classical configuration of size r_*, which is a solution of the theory [6]. The length scale $r_*(E)$ is called the "classicalization radius." In this way, the theory self-unitarizes by prohibiting the probing of small distances $r \ll L_*$ in high-energy scattering processes.

The focus of the present work is the semiclassical calculability of multiparticle production in such theories. For the convenience of calculations, we will focus on the scalar DBI action [Eq. (2)]. In conventional weakly coupled scalar field theories with a dimensionless coupling constant g, it is known that the perturbative methods fail to describe the scattering amplitudes for processes with a large particle number N in the final state. This happens when the multiplicity of the final state N becomes of the order of the inverse coupling constant $1/g^2$. Therefore, in the limit when $g \to 0$ and $N \sim 1/g^2$, nonperturbative methods are used to calculate the cross sections of multiparticle productions from a few (hard) initial particles. However, in

*lasma.alberte@physik.lmu.de
†Fedor.Bezrukov@uconn.edu

[1]Another option may be asymptotic safety, corresponding to theories with a nontrivial renormalization group fixed point in the UV, proposed in Ref. [1]. However, there are no reliable calculations for such theories in most cases at present.

the scalar DBI action [Eq. (2)], which is the main focus of interest in the present paper, the coupling constant L_* has the dimension of length. The above estimate of the critical multiplicity of the final state does not immediately generalize to theories with dimensionful couplings. One of the side results of this paper is that we modify the semiclassical technique for the calculation of the multiparticle cross sections developed in Refs. [7–11] so that it can be applied also to theories with dimensionful coupling constants. We will then use this semiclassical technique to calculate the multiparticle production rates for the theory of Eq. (2), which might exhibit the classicalization phenomena. This method is very similar to the method used to calculate high-energy instanton-like transitions in the electroweak theory (for details, see Refs. [7,8,12,13]). Using the coherent state formalism [14] allows one to reduce the problem of calculating the cross section to solving a classical boundary value problem for the scalar field. A distinctive feature of the multiparticle processes from instanton transitions is that the field configuration saturating the scattering cross section is singular at the origin [9]. This approach of singular solutions has been previously applied to the $\lambda\phi^4$ theory in Refs. [9–11]. It has successfully reproduced all the results known from perturbative tree-level calculations, as well as the exponentiated part of the leading loop contributions [9]. For a review of multiparticle processes and semiclassical analysis in generic scalar field theories, see Ref. [15].

The purpose of this paper is to apply this semiclassical technique to calculate the transition rate of the process few $\to N$ in the scalar DBI theory, which was suggested to classicalize in Ref. [2]. The paper is organized as follows: In Sec. II, we review the semiclassical method used for the calculation of the multiparticle cross sections, and we briefly present the previous results for the $\lambda\phi^4$ theory in Sec. III. We apply the technique to the DBI theory in Sec. IV. We first discuss the semiclassical limit for this theory and find that to any given energy E, one can associate a length scale $r_*(E)$ such that it remains constant in the semiclassical limit. We show that this length scale is $r_* = L_*(L_*E)^{1/3}$ and that it coincides with the classicalization radius of Refs. [2,5]. We then report the results for the scattering cross section. For a fixed above-cutoff total energy $E > L_*^{-1}$ in the final state, we find that the scattering processes with a large number of particles in the final state $N > N_{\rm crit} \sim (EL_*)^{4/3}$ are exponentially suppressed. For particle numbers $N < N_{\rm crit}$, the exponent of the scattering cross section becomes positive. We thus see an emergent critical length scale $r_{\rm crit} \equiv N_{\rm crit}/E$ which also coincides with the classicalization radius r_*. We conclude in Sec. V.

II. SEMICLASSICAL FORMALISM

Here we briefly reproduce the derivation of the semiclassical approach to calculating the multiparticle production rates of Refs. [9,10]. For further details on the formalism of Sec. II A, see Ref. [9]; for Sec. II B, see Ref. [10].

A. Generic boundary value problem

The total scattering cross section from an initial few-particle state to all possible final states with given total energy E and particle number N can be calculated as

$$\sigma(E, N) = \sum_f |\langle f|P_E P_N \hat{S}\hat{A}|0\rangle|^2, \tag{3}$$

where the operator $\hat{A}$ creates an initial state from the vacuum (see the discussion on the next page), $\hat{S}$ is the S matrix, and P_E and P_N are the projection operators to states with energy E and number of particles N, respectively. The sum runs over all final states $|f\rangle$. By using the coherent state formalism [14], Eq. (3) can be written as [7]

$$\begin{aligned}\sigma(E,N) = \int & db^*_{\mathbf{k}} db_{\mathbf{k}} d\xi d\eta D\phi D\phi' \\ & \times \exp\Big(-\int d\mathbf{k}\, b^*_{\mathbf{k}} b_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\xi + i\eta} + iE\xi + iN\eta \\ & + B_i(0,\phi_i) + B_f(b^*_{\mathbf{k}},\phi_f) + B^*_i(0,\phi'_i) + B^*_f(b_{\mathbf{k}},\phi'_f) \\ & + iS[\phi] - iS[\phi'] + J\phi(0) + J\phi'(0)\Big),\end{aligned} \tag{4}$$

where J is some arbitrary number defining the initial few-particle state as $|i\rangle \equiv \hat{A}|0\rangle = e^{J\phi(0)}|0\rangle$, and the boundary terms are

$$\begin{aligned} B_i(0,\phi) &= -\frac{1}{2}\int d\mathbf{k}\,\omega_{\mathbf{k}}\phi_i(\mathbf{k})\phi_i(-\mathbf{k}), \\ B_f(b^*_{\mathbf{k}},\phi_f) &= -\frac{1}{2}\int d\mathbf{k}\, b^*_{\mathbf{k}} b^*_{-\mathbf{k}} e^{2i\omega_{\mathbf{k}}T_f} \\ &\quad + \int d\mathbf{k}\sqrt{2\omega_{\mathbf{k}}}\, b^*_{\mathbf{k}}\phi_f(\mathbf{k}) e^{i\omega_{\mathbf{k}}T_f} \\ &\quad - \frac{1}{2}\int d\mathbf{k}\,\omega_{\mathbf{k}}\phi_f(\mathbf{k})\phi_f(-\mathbf{k}). \end{aligned}$$

Here $\omega_{\mathbf{k}} = \sqrt{m^2 + \mathbf{k}^2}$, T_f denotes some final moment of time, and $\phi_i(\mathbf{k})$ and $\phi_f(\mathbf{k})$ are the spatial Fourier transformations of the field in the initial and final asymptotic regions. The complex variables $b_{\mathbf{k}}$ characterize a set of coherent states $|\{b\}\rangle$, which are eigenstates of the annihilation operators $\hat{b}_{\mathbf{k}}$: i.e., $\hat{b}_{\mathbf{k}}|\{b\}\rangle = b_{\mathbf{k}}|\{b\}\rangle$ for all $\mathbf{k}$.

According to Ref. [8], the integral in Eq. (4) is of the saddle-point type for any scalar field theory with some dimensionless coupling constant g, provided that the constant $J \sim 1/g$, and that under the change of variables $\phi = \Phi/g$ the action has the following property:

$$S(\phi, g) = S(\Phi/g, g) = \frac{1}{g^2}s(\Phi). \tag{5}$$

In this case, after the change of variables $\phi = \Phi/g$ and $(b, b^*) = 1/g(\beta, \beta^*)$, the transition rate of Eq. (4) takes the form

$$\sigma(E, N) \sim \int db^*_{\mathbf{k}} db_{\mathbf{k}} d\xi d\eta \mathcal{D}\phi\mathcal{D}\phi' \exp W, \tag{6}$$

with $W = (1/g^2)F$, where F depends on Φ, Φ', β, β^*, gJ, g^2E, g^2N, but does not explicitly depend on g. For the sake of clarity, it is useful to define a new set of variables $j \equiv gJ$, $\varepsilon \equiv g^2E$, $n \equiv g^2N$ such that in the semiclassical limit $g \to 0$, they stay fixed. We will refer to these quantities as "semiclassical variables." In the limit $g \to 0$, j, ε, $n =$ fixed, the integral of Eq. (4) can be taken in the saddle-point approximation. We note here that the semiclassical parameter g emerges naturally in the conventional scalar field theories with a *dimensionless* coupling constant g. We will see in Sec. IV that this is not the case in theories with *dimensionful* couplings. In such theories, the semiclassical parameter has to be introduced by hand by demanding that the requirement of Eq. (5) be satisfied.

Here we have to remark that the few-particle initial state is chosen to be of the form $|i\rangle = e^{J\phi(0)}|0\rangle$ in order to formally avoid the fact that an initial hard particle state is not semiclassical.[2] In Ref. [8], it was suggested that the few-particle initial state can be recovered by first evaluating the integral in saddle-point approximation in the limit $j \equiv gJ =$ fixed, and then taking the limit $j \to 0$. The assertion is that in this limit one recovers an initial state with a small number of particles. It is, however, not obvious that this limit indeed reproduces the amplitude for the process with a few-particle initial state. For the $\lambda\phi^4$ theory a direct semiclassical calculation of the tree-level amplitudes and the exponentiated leading loop corrections was done in Ref. [9]. Comparison with the perturbative calculations confirmed the hypothesis about the correct form of the initial state (for perturbative calculations, see e.g., Refs. [16,18]). We will assume that this is also true for the theory at hand, keeping in mind that this check should, in principle, be repeated.

Thus, the dominant contribution to the scattering cross section [Eq. (4)] is given by the saddle-point field configuration. The classical field equations and boundary conditions for the field ϕ are obtained by varying the exponent of Eq. (4) with respect to ϕ, $\phi_i(\mathbf{k})$, $\phi_f(\mathbf{k})$ and $b_\mathbf{k}$. The explicit form of the boundary conditions can be found in Ref. [9]. Here we write the boundary value problem for the scalar field ϕ in a simplified form:

$$\frac{\delta S}{\delta\phi} = iJ\delta^{(4)}(x), \tag{7}$$

$$\phi_i(\mathbf{k}) = \frac{a^*_{-\mathbf{k}}}{\sqrt{2\omega_\mathbf{k}}} e^{i\omega_\mathbf{k} t}, \qquad t \to -\infty, \tag{8}$$

$$\phi_f(\mathbf{k}) = \frac{1}{\sqrt{2\omega_\mathbf{k}}}(b_\mathbf{k} e^{\omega_\mathbf{k} T - \theta - i\omega_\mathbf{k} t} + b^*_{-\mathbf{k}} e^{i\omega_\mathbf{k} t}), \qquad t \to +\infty, \tag{9}$$

[2]In principle, a different initial state can be chosen. However, perturbative calculations for the $\lambda\phi^4$ theory suggest that different choices of the initial state do not change the exponent of the scattering cross section [16]. The same is true also for generic scalar field theories with canonical kinetic terms [17].

where we have assumed that the integration variables ξ and η in Eq. (4) are purely imaginary and have substituted $T \equiv i\xi$ and $\theta \equiv -i\eta$ [9]. The complex variables $a_\mathbf{k}$ and $b_\mathbf{k}$ characterize the spatial Fourier components of the initial and final field asymptotics, respectively. This result is independent on the exact form of the nonlinear scalar field interaction terms in the Lagrangian, as long as the action satisfies the condition of Eq. (5) and one can assume that nonlinearities can be neglected for asymptotic solutions in 3 + 1 dimensions.

There are two more saddle-point equations obtained by the variation of the exponent in Eq. (4) with respect to the parameters T and θ:

$$E = \int d\mathbf{k}\,\omega_\mathbf{k} b^*_\mathbf{k} b_\mathbf{k} e^{\omega_\mathbf{k} T - \theta}, \tag{10}$$

$$N = \int d\mathbf{k}\, b^*_\mathbf{k} b_\mathbf{k} e^{\omega_\mathbf{k} T - \theta}. \tag{11}$$

This gives the physical interpretation of E and N as the energy and the number of particles in the final asymptotics. Due to the presence of a δ-functional source located at the coordinate origin $x^\mu = 0$, the energy of the system has a discontinuity at the point $t = 0$. This can be seen easily from the boundary conditions [Eqs. (8) and (9)], since at times $t < 0$, the field ϕ has only positive frequency modes and the energy vanishes; while at times $t > 0$, the energy is determined by Eq. (10). Another expression for the energy in the final state can be obtained from the Lagrangian

$$\begin{aligned} E &= \int_{t=0_+}^{t=0_-} dt \frac{d}{dt} \int d\mathbf{x}\left(\frac{\partial\mathcal{L}}{\partial\dot\phi}\dot\phi - \mathcal{L}\right) \\ &= -iJ\int_{t=0_+}^{t=0_-} dt\,\dot\phi(t,0)\delta(t) = -iJ\dot\phi(0). \end{aligned} \tag{12}$$

Let us discuss the limitations of the allowed field configurations ϕ after taking the limit $J \to 0$. One sees that for the energy jump to stay finite in this limit, the derivative $\dot\phi(0)$ has to go to infinity. Hence, the field has a singularity at the point $t = \mathbf{x} = 0$. Therefore, in order to evaluate the scattering cross section for the process few $\to N$, one has to find the solution for the boundary value problem [Eqs. (7)–(9)] which is singular at $x^\mu = 0$ but regular elsewhere in Minkowski space-time. A more detailed discussion about the limit $J \to 0$ and the correct choice of the singular solution can be found in Ref. [9]. Henceforth, we will not mention the source J anymore. We will, nevertheless, keep in mind that the condition that we are only looking for singular field configurations arises from the limit to the few-particle initial state, which is equivalent to the limit of a vanishing source.

As a result, the scattering cross section is saturated by the saddle point of the integral in Eq. (4) and has the following form:

$$\sigma(E,N) \sim e^{W(E,N)}, \tag{13}$$

where

$$W(E,N) = \frac{1}{g^2}F(n,\varepsilon) = ET - N\theta - 2\,\mathrm{Im}S[\phi]. \qquad (14)$$

The saddle-point relations between θ and T, and between E and N can be obtained by variation of the exponent [Eq. (14)] with respect to T and θ:

$$2\frac{\partial \mathrm{Im}S}{\partial T} = E, \qquad -2\frac{\partial \mathrm{Im}S}{\partial \theta} = N. \qquad (15)$$

Hence, the problem of calculating the scattering cross section for multiparticle production [Eq. (3)] is reduced to solving the classical boundary value problem for the field ϕ stated in Eqs. (7)–(9). Due to the requirement of a few-particle initial state, only the solution singular at the origin $x^\mu = 0$ needs to be considered. After substituting this solution in Eq. (14) and using Eq. (15) in order to eliminate the unphysical parameters T and θ, one arrives at an expression for the scattering cross section $\sigma(E,N)$.

B. Euclidean version of the boundary value problem for tree-level contributions

In general, solving Eqs. (7)–(9) for the singular field configuration is a complicated problem which can have more than one possible solution. However, the process of solving the boundary value problem for the field ϕ is greatly simplified if the condition $n = g^2N \ll 1$ is imposed [9]. Then one needs to find only the initial Euclidean part of the solution. In this case, the resulting "saddle-point value" of the Euclidean action is

$$\mathrm{Im}S[\phi] = S_E[\phi] = \frac{1}{2}e^{-\theta}\int d\mathbf{k}\, a^*_{\mathbf{k}} a_{\mathbf{k}} a^{\omega_{\mathbf{k}} T} \equiv \frac{1}{2}e^{-\theta} I(T), \qquad (16)$$

where the last equality defines the function $I(T)$, and $a_{\mathbf{k}}$ are the Fourier components of the initial field asymptotics [Eq. (8)] rewritten as

$$\phi(\mathbf{k}) = \frac{a^*_{\mathbf{k}}}{\sqrt{2\omega_k}} e^{-\omega_{\mathbf{k}}\tau}, \qquad (17)$$

for $\tau = -it \to +\infty$. The saddle-point equations [Eq. (15)] then allow us to express the parameters T and θ in terms of the average energy $\epsilon = E/N$ (considering massless particles) and particle number N as

$$\epsilon = \frac{I'(T)}{I(T)}, \qquad \theta = -\ln N + \ln I(T). \qquad (18)$$

Finally, for the scattering cross section we obtain [9,11]

$$\sigma(E,N) = \exp(N\ln g^2 N - N + Nf(\epsilon)), \qquad (19)$$

$$f(\epsilon) = \epsilon T(\epsilon) - \ln g^2 I(T), \qquad (20)$$

where by writing $T = T(\epsilon)$ we stress that T should be expressed through ϵ by solving Eq. (18). The energy dependence of the scattering cross section is contained in the function $f(\epsilon)$.

To summarize, this semiclassical approach allows one to determine the exponent of the scattering cross section for the multiparticle process few $\to N$; see Ref. [11]. To do this, one first has to find a set of solutions of the Euclidean equations of motion $\delta S_E/\delta\phi = 0$, singular on the surface $\tau_s(\mathbf{x}) \le 0$, $\tau_s(0) = 0$, with initial asymptotics [Eq. (17)]. Then one has to extremize the integral $I(T)$ for some fixed value of T over all values of $a_{\mathbf{k}}$ (or, equivalently, extremize over the singularity surfaces). Finally, from Eq. (18), one obtains the value of ϵ corresponding to the given T (equivalent to extremization over T for a given ϵ) and uses Eqs. (19) and (20) to calculate the cross section. This method applies to any scalar field theory with a dimensionless parameter g such that under the change of variables $\phi = \Phi/g$, the action transforms as in Eq. (5). Then, in the semiclassical limit

$$g^2 \to 0, \qquad \varepsilon \equiv g^2 E = \text{fixed}, \qquad n \equiv g^2 N = \text{fixed} \ll 1, \qquad (21)$$

the scattering cross section for the multiparticle process with the total energy E and the particle number N in the final state can be obtained as described above. We note that the condition $n \ll 1$ is not essential for the applicability of the saddle-point approximation [Eqs. (3)–(5)]. This condition allowed us to simplify the original boundary value problem to a solution of only the Euclidean part of Eqs. (3)–(5), leading to the simple prescription described in Eqs. (16)–(20); see Ref. [9]. The terms of order $\mathcal{O}(n^2 = g^4N^2)$ in the exponent of the scattering cross section arise only from loop corrections. It means that this approximation is equivalent to considering only the tree-level contribution to the scattering cross section.

Note also that, if we perform extremization over only a subclass of the singularity surfaces (e.g., only $O(4)$-symmetric ones), then the resulting cross section provides a lower bound on the cross section, analogously to a Rayleigh-Ritz extremization procedure; see Ref. [11] for the detailed proof.

III. $\lambda\phi^4$ THEORY

The semiclassical approach to the calculation of the cross section for the process few $\to N$ for large N was previously applied in Refs. [9,11,19] to the $\lambda\phi^4$ theory with the action

$$S = \int d^4x\left(\frac{1}{2}(\partial_\mu\phi)^2 - \frac{\lambda}{4}\phi^4\right). \qquad (22)$$

In this Lagrangian, the coupling constant λ is dimensionless, and thus the dimensionless saddle-point parameter is simply $g^2 = \lambda$. Indeed, it is straightforward to check that the action satisfies the condition of Eq. (5). Thus, the multiparticle scattering cross section in the limit of

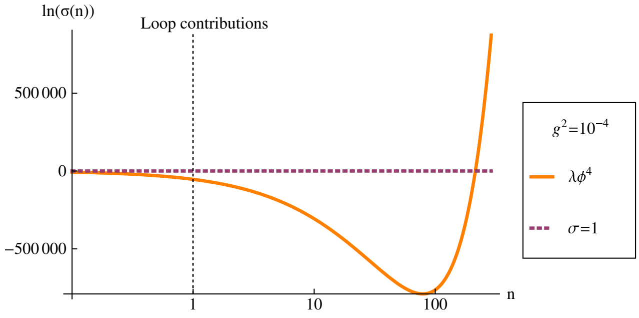

FIG. 1 (color online). The scattering cross section for multiparticle production in $\lambda\phi^4$ theory, depending on the semiclassical particle number $n \equiv \lambda N$, evaluated for $\lambda = g^2 = 10^{-4}$. For the values $n > 1$, the loop contributions have to be taken into account.

Eq. (21) can be evaluated semiclassically by using Eqs. (19) and (20), with $g^2 = \lambda$. The saddle-point value of the Euclidean action [Eq. (16)] for the $O(4)$-symmetric case can be found analytically [9,19]. In the more complicated case of the massive $\lambda\phi^4$ theory, the saddle point has been found numerically in Ref. [11]. As expected, in the high-energy region it was shown to reproduce the results of the massless case.

In the massless $\lambda\phi^4$ theory, the function $f(\epsilon)$, and consequently also the scattering cross section, is independent of energy; i.e., it is simply a constant,[3] $f(\epsilon) = \ln(1/8\pi^2)$. The scattering cross section as a function of particle number N in the final state for any value of energy E is then

$$\sigma(E, N) = \sigma(N) = \exp\left[N \ln\left(\frac{\lambda N}{8e\pi^2}\right)\right]. \tag{23}$$

In terms of the semiclassical variable $n \equiv \lambda N$, this becomes

$$\sigma(n) = \exp\frac{1}{\lambda}\left[n \ln\left(\frac{n}{8e\pi^2}\right)\right]. \tag{24}$$

Figure 1 shows the exponent of the scattering cross section.

We see that the multiparticle production is exponentially suppressed till the particle number reaches the critical value $n = 8e\pi^2 \approx 215$, above which the exponent in Eq. (24) becomes positive. This means that the result obtained in the saddle-point approximation cannot be trusted beyond this point. However, we know that the result for the scattering cross section was obtained in the limit $n \ll 1$, and hence the loop contributions become important in the region where the values of the semiclassical variable $n > 1$. The positivity of the exponent for the semiclassical particle number values $n > 8e\pi^2$ is thus well outside the validity region of our tree-level approximation.

[3]This numerical value of the function $f(\epsilon)$ coincides with that given by Son [9], but might differ from other authors, e.g., Ref. [11], due to the alternative definition of $f(\epsilon)$ in Eq. (20).

IV. SCALAR DBI THEORY

Let us now consider the following Euclidean DBI-type action:

$$S_E = \epsilon_2 \int d^4x \frac{1}{2L_*^4}\sqrt{1 - 2\epsilon_2 L_*^4(\partial_\mu\phi)^2}, \tag{25}$$

where all the quantities are dimensionful; i.e., $[\phi] = L^{-1}$, and the coupling constant has the dimension of length $[L_*] = L$. The parameter ϵ_2 can take values of ± 1. In order to make use of the semiclassical approach described in previous sections, one has to introduce a dimensionless parameter, which would play the role of the saddle-point expansion parameter g. For this we perform the following rescaling of the scalar fields: $\phi \to \phi/g$, where the parameter g is arbitrary. The action transforms as

$$S_E(\phi/g) = \epsilon_2 \frac{1}{g^2}\int d^4x \frac{1}{2l^4}\sqrt{1 - 2\epsilon_2 l^4(\partial_\mu\phi)^2} = \frac{1}{g^2} s(\phi, l^4), \tag{26}$$

where

$$l^4 \equiv \frac{L_*^4}{g^2}. \tag{27}$$

We see that the parameter $1/g^2$ factors out in front of the action $s(\phi, l^4)$, and the action becomes dependent on the new parameter l^4. It is useful to separate the parameters of the theory into two groups: the physical and semiclassical. The physical parameters of the theory are the dimensionful coupling constant L_* and the energy and particle number in the final state—E and N, respectively. The semiclassical variables were introduced in Sec. II as quantities which remain fixed in the semiclassical limit, when $g \to 0$. Besides the semiclassical energy $\varepsilon \equiv g^2 E$ and semiclassical particle number $n \equiv g^2 N$, in DBI theory, there is an additional quantity which has to stay constant in the limit $g \to 0$. We see this from the action of Eq. (26), since it explicitly depends on the new parameter l^4. It is clear that, in order to evaluate the action in saddle-point approximation, the l^4 also has to remain fixed. The corresponding limit [Eq. (21)], in which the tree-level multiparticle scattering cross section in DBI theory can be evaluated in saddle-point approximation, is then

$$g \to 0, \quad \varepsilon \equiv g^2 E = \text{fixed}, \quad l^4 \equiv \frac{L_*^4}{g^2} = \text{fixed}, \quad n \equiv g^2 N = \text{fixed} \ll 1. \tag{28}$$

The conditions ε, n, l^4 = fixed define the region of applicability of the saddle-point approximation to the scattering problem, whereas the condition $n \ll 1$ is needed in order to simplify calculations by neglecting the possible loop contributions.

There are two interesting features of the semiclassical limit of the scalar DBI theory. First, we observe that the product $EL_* = \varepsilon l/g^{3/2}$ becomes large in the limit $g \to 0$, corresponding to the interesting case of energies exceeding the cutoff scale, i.e., $E > L_*^{-1}$. The second observation is that it is possible to introduce a length scale associated with a given energy such that this length remains constant in the semiclassical limit. Indeed, by setting $r_*(E) = E^\alpha L_*^{1+\alpha}$ and replacing physical variables with the semiclassical ones, we obtain the condition

$$r_*(E) = \varepsilon^\alpha l^{1+\alpha} g^{-2(1-3\alpha)} = \text{fixed} \Rightarrow \alpha = \frac{1}{3}. \quad (29)$$

This determines the parameter α uniquely, and we obtain that $r_* = L_*(EL_*)^{1/3}$. Hence, the semiclassical length scale coincides with the classicalization radius introduced in Ref. [2].

Let us present the results for the scattering cross section of the process few $\to N$. For simplicity we limit the extremization procedure to the $O(4)$-symmetric singularity surfaces of the classical solution. As we will see, in DBI theory the derivative of the field is singular, in distinction from the $\lambda\phi^4$ theory where the field itself was singular. Nevertheless, the previous conditions for the finiteness of the energy [Eq. (12)] are still satisfied for the singularity in the first derivative.[4] The equation of motion obtained by varying the action [Eq. (26)], in terms of the four-dimensional radial coordinate ρ, is

$$\partial_\rho\left[\rho^3 \frac{\partial_\rho \phi}{\sqrt{1 - 2\epsilon_2 l^4 (\partial_\rho \phi)^2}}\right] = 0, \quad (30)$$

and hence

$$\frac{d\phi}{d\rho} = \frac{R_s^3}{\sqrt{2}l^2} \frac{1}{\sqrt{\rho^6 + \epsilon_2 R_s^6}}. \quad (31)$$

For $\epsilon_2 = -1$, the derivative becomes singular at the singularity radius $\rho = R_s$. In order to obtain the solution, which is singular at the coordinate origin $\tau = |\mathbf{x}| = 0$, one has to choose another coordinate system where the Euclidean time coordinate is shifted as $\tau \to \tau + R_s$, so that

$$\rho^2 = (\tau + R_s)^2 + \mathbf{x}^2. \quad (32)$$

For $\epsilon_2 = +1$, the derivative is regular everywhere. Hence, due to the lack of a singular $O(4)$-symmetric Euclidean solution, the semiclassical method for calculation of multiparticle scattering cross sections cannot be restricted to this subclass of solutions in this case. Instead, for the $\epsilon_2 = +1$ branch of the DBI theories, some more generic subclass of singularity surfaces should be considered, which is, however, beyond the scope of the present work. Henceforth, we will therefore investigate the $\epsilon_2 = -1$ case.

It is interesting to note that according to a recent paper by Dvali *et al.* [6], the classicalization at all UV energy scales should occur in the $\epsilon_2 = +1$ case. For the $\epsilon_2 = -1$ case, the classicalization, if present at all, is expected to happen in some finite energy range $E_* < E < \bar{\omega}$ [6,20]. The "declassicalization" scale $\bar{\omega}$ is model dependent and, in general, depends on the scale at which some new weakly coupled degrees of freedom should be integrated in, and the theory is UV completed in the usual Wilsonian sense.

After setting $\epsilon_2 = -1$, the solution of the equation of motion [Eq. (31)] for $\phi(\rho)$ is

$$\phi(\rho) = \frac{1}{\sqrt{2}l^2} \int^\rho \frac{d\rho'}{\sqrt{\left(\frac{\rho'}{R_s}\right)^6 - 1}}. \quad (33)$$

In the asymptotic region $\rho \to \infty$, the integral can be approximately taken as

$$\phi(\rho) = \frac{-R_s^3}{\sqrt{2}l^2} \frac{1}{2\rho^2}, \quad (34)$$

and the Fourier components have the following asymptotics at $\tau \to \infty$:

$$\phi(\tau, \mathbf{k}) = \frac{a_\mathbf{k}^*}{\sqrt{2\omega_\mathbf{k}}} e^{-\omega_\mathbf{k}\tau}, \quad \text{where } \omega_\mathbf{k} = |\mathbf{k}| \quad \text{and}$$

$$a_\mathbf{k}^* = \frac{R_s^3}{2l^2} \sqrt{\frac{\pi}{2\omega_\mathbf{k}}} e^{-\omega_\mathbf{k} R_s}. \quad (35)$$

The saddle-point value of the Euclidean action $I(T)$ in Eq. (16) is then

$$I(T) = \frac{R_s^6}{L_*^4} \frac{\pi^2}{2} \frac{1}{(2R_s - T)^2}. \quad (36)$$

After extremizing the function $I(T)$ over all R_s, we obtain for the function $f(\epsilon)$ the following expression:

$$f(\epsilon) = 4 + \ln\frac{2^3}{\pi^2 3^6} + 4\ln(\epsilon), \quad (37)$$

where we have used $g^2 = L_*^4/l^4$. In distinction from the $\lambda\phi^4$ case, this function grows with the energy density $\epsilon \equiv E/N$ as shown in Fig. 2.

After substituting this expression of $f(\epsilon)$ in Eq. (19), we find the tree-level scattering cross section

$$\sigma(E, N) = \exp\left[3N \ln\frac{N_{\text{crit}}}{N}\right] = \left(\frac{N_{\text{crit}}}{N}\right)^{3N}, \quad (38)$$

where we have defined the "critical particle number" in the final state N_{crit} as

$$N_{\text{crit}}^3 \equiv c^3 (L_* E)^4, \qquad c^3 = \frac{(2e)^3}{\pi^2 3^6}. \quad (39)$$

We see that for fixed total energy E, the scattering process few $\to N$ is only suppressed for $N > N_{\text{crit}}$. The notion of

[4] In the truncated DBI theory with only the $L_*^4(\partial_\mu \phi \partial^\mu \phi)^2$ self-interaction term, the singularity appears in the second derivative of ϕ. In order to apply the semiclassical technique to this case, the initial state should be chosen as $|i\rangle = \exp(J\dot{\phi}(0))|0\rangle$.

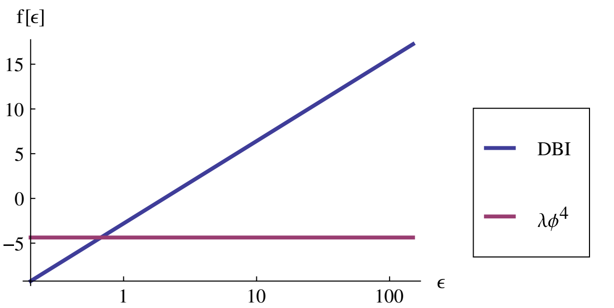

FIG. 2 (color online). Function $f(\epsilon)$ for $\lambda\phi^4$, and DBI theory evaluated at $g^2 = \lambda = 10^{-4}$.

the critical particle number allows one to define a "critical length scale" such that for given energy E, it satisfies

$$r_{\rm crit}^{-1} \equiv \frac{E}{N_{\rm crit}}, \Rightarrow r_{\rm crit} = cL_*(L_*E)^{1/3}. \qquad (40)$$

In other words, $r_{\rm crit}^{-1}$ corresponds to the maximal allowed energy per particle and coincides with the classicalization radius r_* defined in Refs. [2,5]. Hence, we have shown that the classicalization radius r_* emerges as the critical length scale at which the behavior of the scattering cross section drastically changes.

We will discuss the behavior of the transition rate in dependence of the particle number N in the final state for fixed energy E in two separate energy regions.

A. Strong coupling region: $E > L_*^{-1}$

It is useful to rewrite the expression for the scattering cross section [Eq. (38)] in terms of the semiclassical variables defined above:

$$\sigma(\varepsilon, n) = \exp\frac{1}{g^2}\left[3n\ln\left(\frac{c(\varepsilon l)^{4/3}}{n}\right)\right]. \qquad (41)$$

We see that the functional dependence of the cross section is very different from the $\lambda\phi^4$ theory in Eq. (24). In $\lambda\phi^4$ theory, the scattering is exponentially suppressed for small values of the semiclassical particle number n. Meanwhile, in DBI theory, the exponent of the scattering cross section becomes *positive* for small values of $n < c(\varepsilon l)^{4/3}$, and thus the expression in Eq. (41) cannot be trusted for these values of n. A comparison of the dependence of the scattering cross section on the semiclassical particle number n in DBI theory and in $\lambda\phi^4$ theory is shown in Fig. 3.

We recall here that a similar breakdown of the saddle-point approximation is observed in $\lambda\phi^4$ theory for large values of n. However, that is an artifact of the tree-level approximation $n \ll 1$, since we have neglected all terms of order $O(n^2)$. The behavior of the scattering cross section at larger values of n is changed by the loop contributions [9]. The same logic also applies to the DBI theory, but as is shown in Fig. 3, the higher-order corrections become relevant only at values of $n > 1$. Hence the breakdown of the

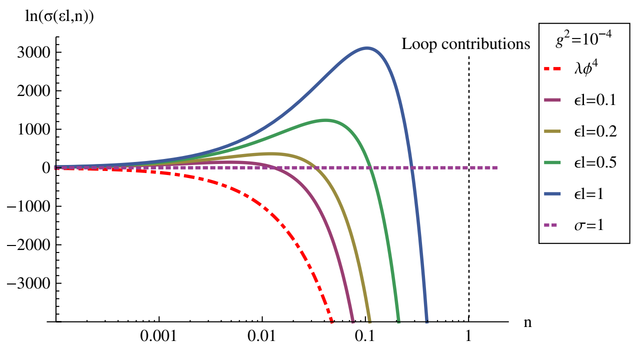

FIG. 3 (color online). The exponent of the scattering cross section as a function of the semiclassical particle number in the final state n for DBI theory with $\epsilon_2 = -1$ evaluated at different values of the parameters $\varepsilon l = EL_* g^{3/2}$. The numerical value of the semiclassical parameter $g^2 = 10^{-4}$, and hence the parameter region $\varepsilon l > g^{3/2} = 10^{-3}$ corresponds to the high-energy region $EL_* > 1$. For values of $n > 1$, the loop contributions have to be taken into account.

semiclassical approach cannot be cured by adding higher-order corrections to the exponent of the scattering cross section [Eq. (41)].

In terms of the physical particle number, this means that the semiclassical method does not allow us to make conclusive statements about the scattering cross sections for the processes where few initial particles scatter into $N < N_{\rm crit} = c(EL_*)^{4/3}$ particles with the total energy $E > L_*^{-1}$. Remarkably, the saddle-point method gives a reliable result for the transition rates to final states with a particle number larger than the critical. In this region, the scattering processes are exponentially suppressed. The scattering cross section as a function of the physical particle number in the final state is shown in Fig. 4. We note that with perturbative methods, this energy region is completely unaccessible.

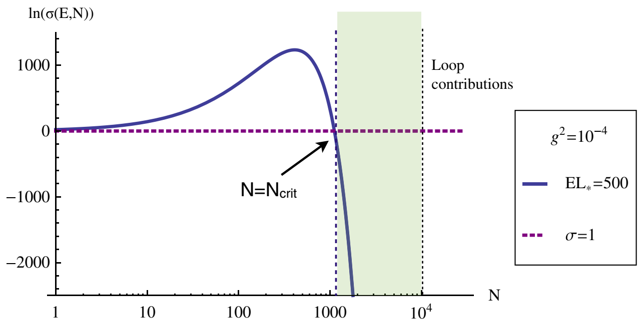

FIG. 4 (color online). The exponent of the scattering cross section as a function of the physical particle number in the final state N for DBI theory with $\epsilon_2 = -1$ evaluated at energy $EL_* = 500 \gg 1$. The numerical value of the semiclassical parameter $g^2 = 10^{-4}$. The saddle-point method breaks down for $N < N_{\rm crit} = c(EL_*)^{4/3}$, while the loop contributions become important at $g^2 N \sim 1$. The region in which the semiclassical method gives a reliable result is shaded green.

It is therefore interesting to find that there exists a region for large particle numbers in the final state $N > N_{crit}$ where the nonrenormalizable theory behaves semiclassically. The energy per particle in the final state of N_{crit} particles equals $r_{crit}^{-1} = [cL_*(L_*E)^{1/3}]^{-1} \ll L_*^{-1}$ and thus the final state is composed of very soft particles, as suggested by the classicalization arguments in Refs. [2,5]. For even larger particle numbers, the energy per particle further decreases.

B. Perturbative region: $E \lesssim L_*^{-1}$

The region of the physical particle numbers where the exponent of the scattering cross section is positive has no physical meaning as soon as the critical particle number $N_{crit} = c(EL_*)^{4/3}$ becomes less than 1. In this case, the scattering process is exponentially suppressed for all physically reasonable values of the particle number in the final state $N > 1$. This happens for low energies $(EL_*) < c^{-3/4} = 2.57$. Strictly speaking, this requirement translates into a condition on the semiclassical parameters $EL_* = \varepsilon l/g^{3/2} < c^{-3/4}$ which is not satisfied in the semiclassical limit when $g \to 0$ (however, for some numerically small values of the parameters g and εl, the condition can still be fulfilled). Nevertheless, the obtained result is physically reasonable, since the exponent of the scattering process is negative. Hence, formally the semiclassical method can also be applied for the energy values which are below the nonrenormalizability cutoff. However, the obtained results should be compared with results from perturbative calculations. The perturbative check for the exponentiation of the scattering amplitude for $2 \to N$ transitions in $\lambda\phi^4$ theory was done in Refs. [16,18]. The same procedure could be applied also to DBI theory.

V. CONCLUSIONS

We have applied the semiclassical approach to the calculation of the scattering cross sections for multiparticle production from a few-particle initial state in a classicalizing theory. A reliable result is obtained in two parameter regions for the energy E and particle number N in the final state. First, exponential suppression is observed below the energy cutoff $E \lesssim L_*^{-1}$ for any number of particles $N > 1$. This corresponds to the parameter region also accessible with perturbative methods. The second range of parameters leading to trustable results lies above the energy cutoff $E > L_*^{-1}$ but is restricted to large particle numbers $N > N_{crit}$ only. This result is obtained in the region where the theory is strongly coupled and perturbation theory cannot be used. No information about hard high-energy scattering processes, few $\to$ $N < N_{crit}$, is obtained from the semiclassical approach.

Let us discuss how robust is the failure of the applied semiclassical procedure at $E > L_*^{-1}$, $N < N_{crit}$. First, we limited the analysis here to the $O(4)$-symmetric singularity surfaces, while at least in the case of the $\lambda\phi^4$ theory it is known that the true extremum of the boundary value problem is reached on a generic surface [11]. However, this in general should lead to even larger cross sections, and thus should not resolve the breakdown of the saddle-point approximation. Another option can be that the $O(4)$-symmetric family of the semiclassical solutions passed through a bifurcation point at the typical energy $E \sim L_*^{-1}$, making it an irrelevant subclass of the classical solutions at high energies. Another promising reason may be related to the fact that the limit of the vanishing source $j \to 0$, leading to the singular solutions, no longer commutes properly with the semiclassical limit and is not imitating a few-particle initial state (the conjecture is checked by explicit comparison with the perturbation theory only in the normal renormalizable theories; see Refs. [16,17]). In this regard, an alternative approach to using the initial expression [Eq. (3)] may prove valuable. However, previous attempts to get a real-time classical solution corresponding to the high-energy spherical collisions led to the development of singularities at high energies [21–24]. Hence it is not clear if nonsingular relevant semiclassical solutions exist. Thus, further study of the real-time solutions is needed to get a useful insight into the classicalization phenomena.

We therefore do not have conclusive statements about the presence or absence of the classicalization phenomena, since this demands a better understanding of hard scattering processes with a small particle number in both the initial and final states. The critical behavior of transcutoff multiparticle production was observed at the number of particles which corresponds to N_{crit} very soft particles with the energy per particle given as $r_*^{-1} = [L_*(EL_*)^{1/3}]^{-1}$. This coincides with the inverse of the classicalization radius introduced in Refs. [2,5]. With this, we have shown the emergence of this critical length scale in the semiclassical approach, which is conceptually completely different from the classical perturbative estimates of Ref. [5].

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Curriculum Vitae

Lāsma Alberte

Contact Information

Ludwig-Maximilians University Munich
Arnold Sommerfeld Center of Theoretical Physics
Chair on Cosmology
Theresienstrasse 37
80333 Munich, Germany

office: +49 89 2180-4599
mobile: +49 1522 2678111
e-mail: lasma.alberte@physik.lmu.de

Personal Information

Nationality: Latvian
Gender: Female
Date of Birth: 27 Sep 1986 in Riga, Latvia

Research Interests

Fundamental aspects of gravity, theoretical cosmology, and quantum field theory

Education

Ph.D. in Physics **Jul 2010 – present**

- Ludwig-Maximilians-University Munich (LMU),
  Arnold Sommerfeld Center for Theoretical Physics (ASC), Germany
- Expected graduation date: July 2013
- Thesis title: "Non-linear Massive Gravity"
- Advisor: Prof. Dr. Viatcheslav Mukhanov

M.Sc. with distinction **Oct 2008 - Jun 2010**

- Masters Programm for Theoretical and Mathematical Physics,
  Ludwig-Maximilians-University Munich (LMU), Germany
- Thesis title: "On Diffeomorphism Invariance and Black Hole Quantization"
- Advisor: Prof. Dr. Viatcheslav Mukhanov

B.Sc. with distinction **Sep 2004 - Jun 2008**

- University of Latvia
- Thesis title: "Stochastic Processes in Soft Matter Physics"
- Advisor: Prof. Andrejs Cēbers

Teaching Experience

Apr 2013 - present	Teaching assistant for Cosmology at LMU
Oct 2011 - Feb 2012	Teaching assistant for General Relativity at LMU
Apr 2011 - Aug 2011	Teaching assistant for Theoretical Mechanics at LMU
Oct 2010 - Feb 2011	Teaching assistant for General Relativity at LMU
Apr 2010 - Aug 2010	Teaching assistant for Theoretical Mechanics at LMU
Oct 2009 - Feb 2010	Teaching assistant for Quantum Mechanics I at LMU
Apr 2009 - Aug 2009	Teaching assistant for Theoretical Mechanics at LMU

Talks

10 Dec 2012	Seminar: *"Semiclassical Cross Sections in Classicalizing Theories"*, EPFL, Lausanne, Switzerland
7 Dec 2012	Seminar: *"Massive Gravity on Cosmological Backgrounds"*, University of Geneva, Switzerland
29 Nov 2012	Seminar: *"Semiclassical Cross Sections in Classicalizing Theories"*, ICTP, Trieste, Italy
5 Jun 2012	Invited talk: *"Semiclassical Calculation of Multiparticle Cross Sections in Classicalizing Theories"*, Quarks 2012, Yaroslavl, Russia
29 May 2012	Parallel session talk: *"Massive Gravity on Arbitrary Backgrounds"*, Planck 2012, Warsaw, Poland
26 Sep 2011	Invited talk: *"Massive Gravity on Curved Background"*, Workshop on Infrared Modifications of Gravity, Trieste
6 Jul 2011	Parallel session talk: *"Ghosts in Massive Gravity"*, PASCOS 2011, Cambridge, UK
29 Jun 2011	'New Talent' talk: *"Massive Gravity: Exorcising the Ghost"* , International School of Subnuclear Physics, Erice, Italy
23 Nov 2010	Seminar: *"Massive Gravity"*, LMU Particle Physics / Cosmology Retreat, Herrsching, Germany
4 Sep 2010	'New Talent' talk: *"Massive Gravity: Restoring GR"*, International School of Subnuclear Physics, Erice, Italy
10 Sep 2008	Parallel Session talk: *"Thermal fluctuations of magnetotactic bacteria in AC magnetic fields"*, 7th International Pamir Conference, Giens, France

Conferences and Schools Attended

Dec 2012	Visitor at CERN TH Division, Geneva
Nov 2012	Critical Tests of Inflation Using Non-Gaussianity, Munich
Sep 2012	New Methods for Field Theory Amplitudes, Munich, Germany
Jun 2012	The 17th Itzykson Meeting - Heart of Darkness: Dark Energy and Modified Gravity, Paris, France
Jun 2012	International School of Subnuclear Physics, Erice, Italy
Jun 2011	International School of Subnuclear Physics, Erice, Italy
Sep 2010	International School of Subnuclear Physics, Erice, Italy
Jun 2010	Yukawa International Seminar (YKIS), Kyoto, Japan
Apr 2010	Workshop on Fundaments of Gravity, Munich, Germany
Jun 2008	58th Meeting of Nobel Laureates in Lindau, Germany

Academic Experience

Nov 2011 - now — Referee for Journal of High Energy Physics (JHEP)

Awards

Yuval Ne'eman Diploma, International School of Subnuclear Physics, Erice, Italy (2011)

Sidney Coleman Diploma, International School of Subnuclear Physics, Erice, Italy (2010)

DAAD Research assistantship, Munich, Germany (2010)

DAAD Scholarship for further academic studies in Munich (2008-2010)

Bachelor Thesis Award, Riga, Latvia (2008)

DAAD Stibet Scholarship for exchange studies in Hannover, Germany (2006)

Erasmus Scholarship for exchange studies in Hannover, Germany (2006)

Skills

- Languages: Latvian (native), English (fluent), German (fluent), Russian (fluent), French (conversant).
- Computer Skills: Mathematica, Matlab, C++, LaTeX.

Referees

Prof. Viatcheslav Mukhanov
Ludwig-Maximilians University (LMU)
Munich, Germany
e-mail: *viatcheslav.mukhanov@physik.lmu.de*

Prof. Georgi Dvali
Center for Cosmology and Particle Physics (CCPP),
New York University, USA;
Ludwig-Maximilians University (LMU)
Munich, Germany;
Max-Planck-Institute for Physics, Munich, Germany
e-mail: *georgi.dvali@cern.ch*

Prof. Fedor Bezrukov
University of Connecticut
Storrs, CT, USA
e-mail: *fedor.bezrukov@uconn.edu*

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