# Universal Moduli Spaces in Gromov-Witten Theory

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**Abstract**

The construction of manifold structures and fundamental classes on the (compactified) moduli spaces appearing in Gromov-Witten theory is a long-standing problem. Up until recently, most successful approaches involved the imposition of topological constraints like semi-positivity on the underlying symplectic manifold to deal with this situation. One conceptually very appealing approach that removed most of these restrictions is the approach by K. Cieliebak and K. Mohnke via complex hypersurfaces, [CM07]. In contrast to other approaches using abstract perturbation theory, it has the advantage that the objects to be studied still are spaces of holomorphic maps defined on Riemann surfaces.

In this thesis this approach is generalised from the case of surfaces of genus 0 dealt with in [CM07] to the general case.
In the first section the spaces of Riemann surfaces are introduced, that take the place of the Deligne-Mumford spaces in order to deal with the fact that the latter are orbifolds. Also, for use in the later parts, the interrelations of these for different numbers of marked points are clarified.
After a preparatory section on Sobolev spaces of sections in a fibration, the results presented there are then used, after a short exposition on Hamiltonian perturbations and the associated moduli spaces of perturbed curves, to construct a decomposition of the universal moduli space into smooth Banach manifolds. The focus there lies mainly on the global aspects of the construction, since the local picture, i. e. the actual transversality of the universal Cauchy-Riemann operator to the zero section, is well understood.
Then the compactification of this moduli space in the presence of bubbling is presented and the later construction is motivated and a rough sketch of the basic idea behind it is given.
In the last part of the first chapter, the necessary definitions and results are given that are needed to transfer the results on moduli spaces of curves with tangency conditions from [CM07]. There also the necessary restrictions on the almost complex structures and Hamiltonian perturbations from [IP03] are incorporated, that later allow the use of the compactness theorem proved in that reference.

In the last part of this thesis, these results are then used to give a definition of a Gromov-Witten pseudocycle, using an adapted version of the moduli spaces of curves with additional marked points that are mapped to a complex hypersurface from [CM07]. Then a proof that this is well-defined is given, using the compactness theorem from [IP03] to get a description of the boundary and the constructions from the previous parts to cover the boundary by manifolds of the correct dimensions.

**Zusammenfassung**

Die Konstruktion von Mannigfaltigkeitsstrukturen und Fundamentalklassen auf den in der Gromov-Witten Theorie auftretenden (kompaktifizierten) Modulräumen ist ein lange währendes Problem. Bis vor kurzem beinhalteten die meisten erfolgreichen Lösungsansätze die Auferlegung topologischer Einschränkungen, wie zum Beispiel Semipositivität, an die dem Problem zu Grunde liegende symplektische Mannigfaltigkeit. Ein konzeptuell sehr interessanter Zugang der die meisten dieser Einschränkungen unnötig machte ist der Zugang von K. Cieliebak und K. Mohnke mit Hilfe komplexer Hyperflächen, [CM07]. Im Unterschied zu anderen Zugängen unter Verwendung von abstrakter Störungstheorie hat dieser den zusätzlichen Vorteil dass die betrachteten Objekte immer noch Räume holomorpher Abbildungen auf Riemannschen Flächen sind.

In dieser Arbeit wird dieser Zugang von der Betrachtung von Flächen von Geschlecht 0 auf den allgemeinen Fall verallgemeinert.
Im ersten Abschnitt werden die Räume von Riemannschen Flächen eingeführt die die Stelle der Deligne-Mumford Räume einnehmen, um mit der Tatsache umgehen zu können dass die letzteren Orbifaltigkeiten darstellen. Des weiteren werden zur späteren Verwendung die Beziehungen zwischen diesen Räumen für unterschiedliche Anzahlen von markierten Punkten beleuchtet.
Im Anschluss an einen vorbereitenden Abschnitt über Sobolev Räume von Schnitten in einer Faserung werden diese Resultate dann, nach einer kurzen Darstellung über Hamiltonsche Störungen und die zugehörigen Modulräume gestörter Kurven, verwendet um eine Zerlegung des universellen Modulraums in glatte Banachmannigfaltigkeiten zu konstruieren. Der Blick wird hierbei vor allem auf die globalen Aspekte der Konstruktion gerichtet, da das lokale Bild, d. h. die eigentliche Transversalität des universellen Cauchy-Riemann Operators gut verstanden ist.
Danach wird die Kompaktifizierung dieses Modulraumes unter Berücksichtigung der Blasenbildung vorgestellt und die spätere Konstruktion wird motiviert sowie ein grober Umriss der zugrundeliegenden Idee gegeben.
Im letzten Teil des ersten Kapitels werden die benötigten Definitionen und Ergebnisse für die Übertragung der Resultate aus [CM07] präsentiert. Ebenfalls werden dort die notwendigen Einschränkungen an die fast komplexen Strukturen und Hamiltonschen Störungen aus [IP03] berücksichtigt, die später die Verwendung des dort bewiesenen Kompaktheitssatzes ermöglichen.

Im letzten Teil dieser Arbeit werden diese Resultate verwendet um eine Definition eines Gromov-Witten Pseudozykels zu geben, unter Verwendung einer angepassten Version des Modulraumes von Kurven mit zusätzlichen markierten Punkten die in eine komplexe Hyperfläche abgebildet werden, wie in [CM07]. Hierauf wird ein Beweis geführt dass dies wohldefiniert ist, unter Verwendung des Kompaktheitssatzes aus [IP03] für die Beschreibung des Randes wie

auch unter Verwendung der Ergebnisse der vorhergehenden Abschnitte um eine Überdeckung des Randes mit Mannigfaltigkeiten der korrekten Dimension zu erhalten.

# Contents

# CHAPTER I

## Introduction

A much studied question in contemporary symplectic geometry concerns the existence of holomorphic curves. In its simplest form, this means that given a closed symplectic manifold $(X, \omega)$ and an $\omega$-compatible (or tame) almost complex structure $J$ on $X$, as well as a Riemann surface $(S, j)$ and a homology class $A \in H_2(X)$, does there exist a holomorphic map $u : S \to X$, i. e. $J \circ \mathrm{d}u = \mathrm{d}u \circ j$, that represents the homology class $A$ (if $A = 0$, then a trivial answer to this question is provided by the constant maps)? The usual strategy to answer this question is the following: Find a way to "count" holomorphic curves (in homology class $A$) for a set of almost complex structures on $X$ that are dense (at least in a connected neighbourhood of the given $J$) in $\mathcal{J}_\omega(X)$ (the set of $\omega$-compatible almost complex structures on $X$) and in a way that is invariant under deformations of the almost complex structures. Invariance here means that for a homotopy/deformation $(J_t)_{t\in[0,1]}$, the counts of $J_0$- and of $J_1$-holomorphic curves coincide. Then by Gromov's compactness theorem, cf. [Hum97] and the references therein, one can conclude the existence of an, although broken, $J$-holomorphic curve. The way this question is studied is usually the following:
Fix numbers $g, n \in \mathbb{N}_0$ with $2g - 2 + n > 0$. Then (see Definitions II.1 and II.2 for the notation used in the following)

$$\begin{aligned}\mathcal{M}_{g,n}(X, A, J) := \{(S, j, r_*, u) \mid\ & (S, j, r_*) \text{ smooth marked Riemann surface} \\ & \text{of type } (g, n),\ u : S \to X\ j\text{-}J\text{-holomorphic}, \\ & [u] = A\}/_\sim,\end{aligned}$$

where $(S, j, r_*, u) \sim (S', j', r'_*, u')$ iff there exists a diffeomorphism $\phi \in \mathrm{Diff}((S, j, r_*), (S', j', r'_*))$ with $\phi^* u' = u$. This comes with two maps

$$\begin{aligned}\mathrm{ev} : \mathcal{M}_{g,n}(X, A, J) &\to X^n \\ [(S, r_*, j, u)] &\mapsto [u(r_1), \dots, u(r_n)],\end{aligned}$$

and

$$\begin{aligned}\pi_M^{\mathcal{M}} : \mathcal{M}_{g,n}(X, A, J) &\to M_{g,n}\\ [(S, j, r_*, u)] &\mapsto [(S, j, r_*)],\end{aligned}$$

where $M_{g,n}$ is the moduli space of smooth marked Riemann surfaces of type $(g, n)$, defined by

$$\begin{aligned}M_{g,n} := \{(S, j, r_*) \mid{} & (S, j, r_*) \text{ smooth marked Riemann}\\ & \text{surface of type } (g, n)\}/\sim,\end{aligned}$$

where $(S, j, r_*) \sim (S', j', r'_*)$ iff $\mathrm{Diff}((S, j, r_*), (S', j', r'_*)) \neq \emptyset$.
"Counting invariant under deformations" then usually refers to the question of whether, for a dense subset of $J$ in $\mathcal{J}_\omega(X)$, $\mathcal{M}_{g,n}(X, A, J)$ is an oriented manifold of a certain expected dimension that carries a fundamental class s. t.

$$\pi_M^{\mathcal{M}} \times \mathrm{ev} : \mathcal{M}_{g,n}(X, A, J) \to M_{g,n} \times X^n \tag{I.1}$$

defines a (singular or otherwise) chain and hence homology class in the image. One asks that for any two such almost complex structures $J_0, J_1$ there exists a deformation $(J_t)_{t\in[0,1]}$ s. t. $\bigcup_{t\in[0,1]} \mathcal{M}_{g,n}(X, A, J_t)$ defines a cobordism between $\mathcal{M}_{g,n}(X, A, J_0)$ and $\mathcal{M}_{g,n}(X, A, J_1)$ that via $\pi_M^{\mathcal{M}} \times \mathrm{ev}$ induces a chain equivalence between the chains defined by these two spaces so that the corresponding homology classes coincide. Assuming that one can construct a well-defined homology class in this way, one would then like to use Poincaré-duality in the image, in the form of intersection theory in homology, to define numerical invariants.
Unfortunately none of the above is true if taken literally. The two most blatantly obvious reasons the above can't work (for any $X$) are that neither is $M_{g,n}$ a manifold nor is it compact, so one can't expect there to be Poincaré-duality in singular homology. This also applies, e. g. by taking $X$ to be a point, to $\mathcal{M}_{g,n}(X, A, J)$, since in general only closed oriented manifolds can be expected to carry a fundamental class in singular homology.
To fix the second problem, one has to compactify $M_{g,n}$ and $\mathcal{M}_{g,n}(X, A, J)$. For $M_{g,n}$ this is done via the Deligne-Mumford compactification

$$\begin{aligned}\overline{M}_{g,n} := \{(S, j, r_*, \nu) \mid{} & (S, j, r_*, \nu) \text{ stable marked nodal}\\ & \text{Riemann surface of type } (g, n)\}/\sim,\end{aligned}$$

where $(S, j, r_*, \nu) \sim (S', j', r'_*, \nu')$ iff $\mathrm{Diff}((S, j, r_*, \nu'), (S', j', r'_*, \nu')) \neq \emptyset$. The compactification of $\mathcal{M}_{g,n}(X, A, J)$ by Gromov is a more difficult concept that requires some more preparation. But a first step is to define the moduli space of nodal holomorphic curves in $X$,

$$\begin{aligned}\overline{\mathcal{M}}_{g,n}(X, A, J) := \{(S, j, r_*, \nu, u) \mid{} & (S, j, r_*, \nu) \text{ stable marked nodal Riemann surface}\\ & \text{of type } (g, n),\ u : S \to X\ j\text{-}J\text{-holomorphic},\\ & u(n_1) = u(n_2)\ \forall\, \{n_1, n_2\} \in \nu,\ [u] = A\}/\sim,\end{aligned}$$

where $(S, r_*, j, \nu, u) \sim (S', r'_*, j', \nu', u')$ iff there exists a diffeomorphism $\phi \in \mathrm{Diff}((S, j, r_*, \nu), (S', j', r'_*, \nu'))$ with $\phi^* u' = u$.
Analogously to before there are then also canonical extensions

$$\pi_M^{\mathcal{M}} : \overline{\mathcal{M}}_{g,n}(X, A, J) \to \overline{M}_{g,n}$$

and

$$\mathrm{ev} : \overline{\mathcal{M}}_{g,n}(X, A, J) \to X^n.$$

This still leaves the first problem, namely that (for $g > 1$) $\overline{M}_{g,n}$ (as well as $M_{g,n}$) is not a manifold but only a complex orbifold, as is shown in [RS06]. So $\overline{M}_{g,n}$ (as a topological space) can be decomposed in two ways: By signature, i. e. by homeomorphism type of the underlying surface, and via the stratification coming from the orbifold structure. Since the morphisms in the groupoids (from [RS06]) defining the orbifold structure are given by ismorphisms of nodal surfaces, which in particular preserve the signature, this stratification is compatible with the decomposition by orbit type. More explicitely, if as in [RS06], esp. Definitions 6.2 and 6.4, $(\pi : \Sigma \to M, R_*)$ is a universal marked nodal family of type $(g, n)$ and $(M, \Gamma, s, t, e, i, m)$ is the associated groupoid, then $M$ has a stratification by locally closed submanifolds. Here, two points $b, b' \in M$ lie on the same stratum iff $\Sigma_b$ and $\Sigma_{b'}$ have the same signature (as marked nodal Riemann surfaces). If an orbit of the groupoid structure on $M$ intersects a stratum of this stratification, then it is completely contained in that stratum. Although this gives $\overline{M}_{g,n}$ a stratification with a connected top-dimensional stratum and all other strata of codimension at least two, this does not suffice to have Poincaré-duality in singular homology (examples for this can be found e. g. in [Mac90]). The standard way, started in [Mum83], to remedy this is to regard, instead of $\overline{M}_{g,n}$, certain closed complex manifolds $\overline{M}^\lambda$ with maps $\pi^\lambda : \overline{M}^\lambda \to \overline{M}_{g,n}$ that are, in a certain sense, branched coverings (for existence results, see e. g. [Loo94] or [BP00] and the references therein). Since one of the goals in this text is to keep to manifolds and smooth maps, esp. to the description of $\overline{M}_{g,n}$ provided in [RS06], it is hard to make this precise. But at least the part $M^\lambda$ of such a manifold $\overline{M}^\lambda$ that maps to $M_{g,n} \subseteq \overline{M}_{g,n}$ has an easy description: Remembering that if $\mathcal{T}_{g,n}$ denotes Teichmüller space and $\Gamma_{g,n}$ denotes the mapping class group (both of smooth surfaces of type $(g, n)$), then $M_{g,n} \cong \mathcal{T}_{g,n}/_{\Gamma_{g,n}}$. If $\Gamma^\lambda \subseteq \Gamma_{g,n}$ is a finite index normal subgroup that operates freely on $\mathcal{T}_{g,n}$, then $M^\lambda := \mathcal{T}_{g,n}/_{\Gamma^\lambda_{g,n}}$ is a smooth manifold on which the finite group $G^\lambda := \Gamma_{g,n}/_{\Gamma^\lambda_{g,n}}$ operates and the canonical projection $M^\lambda = \mathcal{T}_{g,n}/_{\Gamma^\lambda_{g,n}} \to (\mathcal{T}_{g,n}/_{\Gamma^\lambda_{g,n}})/_{G^\lambda} = \mathcal{T}_{g,n}/_{\Gamma_{g,n}} = M_{g,n}$ is an orbifold covering. Now assume that such a manifold $M = \overline{M}^\lambda$ has been picked and let $\upsilon : M \to \overline{M}_{g,n}$ be the projection.
This requires to also modify the definition of $\overline{\mathcal{M}}_{g,n}(X, A, J)$, for there is no a priori reason for the map $\pi_M^{\mathcal{M}} : \overline{\mathcal{M}}_{g,n}(X, A, J) \to \overline{M}_{g,n}$ to factor through $M$. Also, since the goal is to define a manifold of maps, it stands to reason to first of all fix the domains on which the maps that are the elements of this manifold are defined. Since $\overline{M}_{g,n}$ contains equivalence classes of surfaces of different

homeomorphism types, one first of all has to define a notion of smooth family of such nodal surfaces. The notion used in this text is that of a (regular) marked nodal family of Riemann surfaces as in [RS06]. So the goal is not only to have a manifold $M$ as above together with a map $\upsilon : M \to \overline{M}_{g,n}$, but for this map to be defined via a regular marked nodal family of Riemann surfaces $(\pi : \Sigma \to M, R_*)$ i. e. the map $\upsilon : M \to \overline{M}_{g,n}$ is supposed to map $b \in M$ to the equivalence class of the fibre $\Sigma_b$ of $\Sigma$ over $b$. Or, in the reverse direction, $(\pi : \Sigma \to M, R_*)$ is a smooth choice of a marked nodal Riemann surface in the equivalence class $\upsilon(b)$ for each $b \in M$. Collecting the basic definitions for and properties of such families is done at the beginning of this thesis in Section II.1. Aside from this, that section also contains two results, Propositions II.2 and II.1, that are not found in [RS06], but will be important in the later parts of this text, esp. in the definition of the Gromov compactification in Section II.4. Namely first there is a natural operation on a stable marked nodal Riemann surface of type $(g, n + 1)$, that forgets the last marked point and stabilises, i. e. contracts every component that becomes unstable after removing the last marked point. This provides a well-defined map

$$f^{n+1}_{\text{stab}} : \overline{M}_{g,n+1} \to \overline{M}_{g,n}.$$

And second, there is an action

$$\mathcal{S}_n \times \overline{M}_{g,n} \to \overline{M}_{g,n}$$

of the permutation group $\mathcal{S}_n$ of $\{1, \ldots, n\}$ on $\overline{M}_{g,n}$ by permuting the labels of the marked points of a marked nodal Riemann surface. The question addressed in Propositions II.2 and II.1 then is, assuming that for every $n$ a marked nodal family $(\pi^n : \Sigma^n \to M^n, R^n_*)$ with induced map $\upsilon^n : M^n \to \overline{M}_{g,n}$ as above has been chosen, of whether or not one can lift these maps and actions to smooth ones on the manifolds $M^n$ which are covered by bundle morphisms on the $\Sigma^n$, i. e.

$$\begin{tikzcd}
\Sigma^{n+1} \arrow[r] \arrow[d, "\pi^{n+1}"'] & \Sigma^n \arrow[d, "\pi^n"] \\
M^{n+1} \arrow[r] \arrow[d, "\upsilon^{n+1}"'] & M^n \arrow[d, "\upsilon^n"] \\
\overline{M}_{g,n+1} \arrow[r, "f^{n+1}_{\text{stab}}"] & \overline{M}_{g,n}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{S}_n \times \Sigma^n \arrow[r] \arrow[d, "\text{id}\times\pi^n"'] & \Sigma^n \arrow[d, "\pi^n"] \\
\mathcal{S}_n \times M^n \arrow[r] \arrow[d, "\text{id}\times\upsilon^n"'] & M^n \arrow[d, "\upsilon^n"] \\
\mathcal{S}_n \times \overline{M}_{g,n} \arrow[r] & \overline{M}_{g,n}
\end{tikzcd}$$

This has the additional advantage that along the way the question of existence of the regular marked nodal family of Riemann surfaces $(\pi : \Sigma \to M, R_*)$ defining $\upsilon$ is reduced to the case $n = 0$ and given such a choice, for all other values of $n$ there is then a natural one. Also, it gives concrete differential-geometric meaning to the adages that "the universal curve over $\overline{M}_{g,n}$ is isomorphic to $\overline{M}_{g,n+1}$" and that adding marked points to a marked nodal Riemann surface kills automorphisms and doesn't add new ones. Section II.1 concludes with a remark about the construction of invariants, given the data that has been established so far.

Now given a nodal family of marked Riemann surfaces $(\pi : \Sigma \to M, R_*)$, one can for $b \in M$ and a desingularisation $\hat{\imath} : S \to \Sigma_b \subseteq \Sigma$ make the definition

$$\begin{aligned}\mathcal{M}_b(\Sigma, X, A, J) &:= \{u : \Sigma_b \to X \mid \hat{\imath}^* u : S \to \hat{X} \text{ is } j\text{-}J\text{-holomorphic, } [u] = A\},\\ \mathcal{M}(\Sigma, X, A, J) &:= \coprod_{b \in M} \mathcal{M}_b(\Sigma, X, A, J).\end{aligned}$$

The important difference to the definitions from before is that the elements of $\mathcal{M}(\Sigma, X, A, J)$ now are actual maps defined on the fibres of $\Sigma$ and not equivalence classes of maps any more ("all automorphisms have been fixed"). By definition there are canonical maps

$$\begin{tikzcd} \mathcal{M}(\Sigma, X, A, J) \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{g,n}(X, A, J) \arrow[d] \\ M \arrow[r] & \overline{M}_{g,n}. \end{tikzcd}$$

Now that one has an actual set of maps to work with, there is a better chance to equip this set with a manifold structure using the usual methods from the Fredholm-theory of the Cauchy-Riemann operator.

To do so, in Section II.2 the technical results needed for this are presented. That section is largely independent of the rest of the text. It mainly deals with the necessary analytical results that need to be proved in order to be able to give a rigorous definition of Banach manifolds of sections of a Riemannian submersion. It is actually easily possible to skip that section and just take notice of the main results in Subsection II.2.3. In case of a trivial (topologically and geometrically) bundle, i. e. when dealing with maps from one Riemannian manifold to another, this has been done e. g. in [Eic07]. It is most likely actually possible to use this to define spaces of sections via the implicit function theorem as subspaces of the space of maps from the base to the total space that when composed with the projection to the base give the identity. This is not done here in this way for a couple of reasons. For one, it is usally nicer to have intrinsic definitions that make use of a naturally given structure instead of making noncanonical choices and these results may be of independent interest. Also, when done as suggested above, one does not get an explicit description for the charts on this manifolds. First, this makes it harder to calculate the coordinate expressions and their linearisations, of the Cauchy-Riemann operator. And second, charts on the moduli space of holomorphic sections as zero set of the Cauchy-Riemann operator are now given via the implicit function theorem applied to an operator defined on a manifold that is defined via the implicit function theorem itself. When dealing with questions of elliptic regularity which constitute a large part of the construction of the manifold structure on the moduli spaces of curves studied later, this causes some unwanted complications, since one has to compare Sobolev spaces of different types. The intrinsic definition from Section II.2 on the other hand allows for rather straightforward proofs, which usually boil down to calculating some coordinate expression and then applying some result from the theory of linear Cauchy-Riemann operators (on vector bundles).

The fact that all the manifolds of sections constructed are subsets of the same topological space and the manifold structures are all defined using transition functions that all come from the set-theoretically same maps, then makes the transition from these local coordinate calculations to global statements work.

Using this setup, in Section II.3 the construction of a smooth structure on $\mathcal{M}(\Sigma, X, A, J)$, or rather a generalisation of that space, is examined. First of all, remember that on $M$ there is the stratification by signature, where a stratum is defined by the condition that the homeomorphism type of the fibres does not change. Since general gluing results are quite difficult to prove and outside of the scope of the methods employed in this text, smooth structures will only be defined on the restrictions of the (universal) moduli spaces to these strata. Over one of these strata the situation then basically can be reduced to the consideration of a smooth fibre bundle $\rho : S \to B$ with typical fibre a fixed smooth surface. Also, a smooth bundle endomorphism $j : VS \to VS$ ($VS$ is the vertical tangent bundle) with $j^2 = -\mathrm{id}$ is given, that turns every fibre $S_b$ into a Riemann surface $(S_b, j_b)$, together with sections $R_i : B \to S$, $i = 1, \dots, n$. If this bundle is (topologically) trivial, then the construction follows the lines of the discussion in [MS04] or [CM07] rather closely: For a fixed Riemann surface (i. e. the case where $B$ is a point), one constructs the universal Cauchy-Riemann operator w. r. t. an appropriately chosen Banach manifold of perturbations and hence the universal moduli space just as in these references. At this point some familiarity with (universal) Cauchy-Riemann operators, and this line of argument via the Sard-Smale theorem is assumed. Since we allow surfaces of arbitrary genus, this necessitates the use of Hamiltonian perturbations as in Chapter 8 in [MS04]. For the constant maps are always holomorphic, w. r. t. to any holomorphic structure on the target and it is easy to see that this also holds for domain dependent complex structures as used in [CM07]. But the Fredholm index of the Cauchy-Riemann operator at a constant map in the case of genus greater than 1 is negative, which contradicts transversality. So instead of the space $\mathcal{M}(\Sigma, X, A, J)$ one considers spaces $\mathcal{M}(\tilde{X}, A, J, H)$, where $\tilde{X} := \Sigma \times X$ is the trivial bundle and $H$ is a Hamiltonian perturbation on $\tilde{X}$ as defined in Subsection II.3.1 and the references therein.
If $B$ is not a point but the bundle $S$ over $B$ is topologically trivial, then the construction of the universal moduli space is essentially a parametrised version of the previous one.
In the case of varying complex structures that is not dealt with in [MS04] (which only deals with a fixed complex structure and varying marked points and [CM07] restricts to the genus 0 case, where there is essentially only one complex structure) one has to consider the case of a topologically nontrivial family of surfaces. The problem here is that there no longer is a globally defined Banach manifold on which to define a universal Cauchy-Riemann operator (see the explanation on page 72 and the references there) due to the failure of the diffeomorphism group of the base to act smoothly on the Sobolev spaces of sections of a fibre bundle over that base. This requires one to patch together universal moduli spaces obtained via a trivialisation after restricting to an open subset of $B$ “by hand”. This is done in the discussion leading up to Corollary

II.5. Similar but slightly less difficult problems also arise for the smoothness of the evaluation maps at the varying marked points, which are dealt with in Subsection II.3.4.

At that point, what one has achieved is the following: A universal moduli space $\mathcal{M}(\tilde{X}, A, J, \mathcal{H}(\tilde{X}))$ has been defined that comes together with three maps

$$\begin{aligned} \pi_M^{\mathcal{M}} &: \mathcal{M}(\tilde{X}, A, J, \mathcal{H}(\tilde{X})) \to M, \\ \mathrm{ev} &: \mathcal{M}(\tilde{X}, A, J, \mathcal{H}(\tilde{X})) \to X^n \end{aligned}$$

and

$$\pi_{\mathcal{H}}^{\mathcal{M}} : \mathcal{M}(\tilde{X}, A, J, \mathcal{H}(\tilde{X})) \to \mathcal{H}(\tilde{X})$$

s. t. if $B \subseteq M$ is a stratum of the stratification on $M$ by signature, then $(\pi_M^{\mathcal{M}})^{-1}(B)$ is a smooth Banach manifold and the restriction of $\pi_{\mathcal{H}}^{\mathcal{M}}$ to $(\pi_M^{\mathcal{M}})^{-1}(B)$ is a Fredholm map of the correct expected index $\dim_{\mathbb{C}}(X)\chi + 2c_1(A) + \dim_{\mathbb{R}}(B)$, where $\chi$ is the Euler characteristic of the surfaces in the family $\Sigma$ (which is $2(1-g)$).
Section II.4 then first of all equips this space with a topology that makes all of the above maps continuous, which is basically a variation of the classical Gromov topology.
Unfortunately, with this topology $\mathcal{M}(\tilde{X}, A, J, H)$ is not compact, due to the well-known bubbling phenomena. Usually, these are dealt with by imposing topological conditions like semipositivity on $X$, see e. g. [MS04], Section 6.4. In [CM07] a different approach was first introduced for the genus 0 case, which in this text will be extended to the case of positive genus. To do so first of all a description of the problem is given: Remember that there were the operations of forgetting the last marked point and stabilising and permuting the marked points on the Deligne-Mumford moduli spaces $\overline{M}_{g,n}$. These lift to maps and actions, for $\tilde{\ell} \geq \ell$,

$$\begin{tikzcd} \Sigma^{\tilde{\ell}} \arrow[r, "\hat{\pi}_{\ell}^{\tilde{\ell}}"] \arrow[d, "\pi^{\tilde{\ell}}"'] & \Sigma^{\ell} \arrow[d, "\pi^{\ell}"] \\ M^{\tilde{\ell}} \arrow[r, "\pi_{\ell}^{\tilde{\ell}}"] & M^{\ell} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{S}_{\ell} \times \Sigma^{\ell} \arrow[r, "\hat{\sigma}^{\ell}"] \arrow[d, "\mathrm{id}\times\pi^{\ell}"'] & \Sigma^{\ell} \arrow[d, "\pi^{\ell}"] \\ \mathcal{S}_{\ell} \times M^{\ell} \arrow[r, "\sigma^{\ell}"] & M^{\ell}, \end{tikzcd}$$

where $\pi^{\ell} : \Sigma^{\ell} \to M^{\ell}$ is obtained from $\pi : \Sigma \to M$ by adding $\ell \geq 0$ additional marked points. There are then induced maps

$$(\hat{\pi}_{\ell}^{\tilde{\ell}})^* : (\pi_{\ell}^{\tilde{\ell}})^* \mathcal{M}((\hat{\pi}_0^{\ell})^* \tilde{X}, A, J, (\hat{\pi}_0^{\ell})^* \mathcal{H}(\tilde{X})) \to \mathcal{M}((\hat{\pi}_0^{\tilde{\ell}})^* \tilde{X}, A, J, (\hat{\pi}_0^{\tilde{\ell}})^* \mathcal{H}(\tilde{X}))$$

and actions

$$\tilde{\sigma}^{\ell} : \mathcal{S}_{\ell} \times \mathcal{M}((\hat{\pi}_0^{\ell})^* \tilde{X}, A, J, (\hat{\pi}_0^{\ell})^* \mathcal{H}(\tilde{X})) \to \mathcal{M}((\hat{\pi}_0^{\ell})^* \tilde{X}, A, J, (\hat{\pi}_0^{\ell})^* \mathcal{H}(\tilde{X})).$$

Using these structures one can define the Gromov compactification $\overline{\mathcal{M}}(\tilde{X}, A, J, \mathcal{H}(\tilde{X}))$ of $\mathcal{M}(\tilde{X}, A, J, \mathcal{H}(\tilde{X}))$ as the colimit of the spaces $\mathcal{M}((\hat{\pi}_0^{\ell})^* \tilde{X}, A, J, (\hat{\pi}_0^{\ell})^* \mathcal{H}(\tilde{X}))$

over the above maps and actions (cf. Definition II.26 and Remark II.11). More explicitely, this compactification consists of the union over all the spaces $\mathcal{M}((\hat{\pi}_0^\ell)^*\tilde{X}, A, J, (\hat{\pi}_0^\ell)^*\mathcal{H}(\tilde{X}))$ for $\ell \geq 0$, where two holomorphic sections $u'$ and $u''$ with domains $\Sigma_{b'}^{\ell'}$ and $\Sigma_{b''}^{\ell''}$ are identified if there exists the following: An $\tilde{\ell} \geq \ell', \ell''$ and a $b \in M^{\tilde{\ell}}$ as well as a holomorphic section $u$ with domain $\Sigma_b^{\tilde{\ell}}$ s. t. $\Sigma_{b'}^{\ell'}$ is obtained from $\Sigma_b^{\tilde{\ell}}$ by forgetting the last $\tilde{\ell} - \ell'$ marked points and the corresponding map $\Sigma_b^{\tilde{\ell}} \to \Sigma_{b'}^{\ell'}$ pulls $u'$ back to $u$. Also, after possibly reordering the last $\tilde{\ell}$ marked points, $\Sigma_{b''}^{\ell''}$ is obtained from $\Sigma_b^{\tilde{\ell}}$ by forgetting the last $\tilde{\ell} - \ell''$ marked points and the corresponding map $\Sigma_b^{\tilde{\ell}} \to \Sigma_{b''}^{\ell''}$ pulls $u''$ back to $u$.
As before, $\overline{\mathcal{M}}(\tilde{X}, A, J, \mathcal{H}(\tilde{X}))$ comes with natural maps $\pi_M^{\overline{\mathcal{M}}} : \overline{\mathcal{M}}(\tilde{X}, A, J, \mathcal{H}(\tilde{X})) \to M$ and $\pi_{\mathcal{H}}^{\overline{\mathcal{M}}} : \overline{\mathcal{M}}(\tilde{X}, A, J, \mathcal{H}(\tilde{X})) \to \mathcal{H}(\tilde{X})$. Roughly, the transversality problem then is that the Hamiltonian perturbations $(\hat{\pi}_0^\ell)^*H \in (\hat{\pi}_0^\ell)^*\mathcal{H}(\tilde{X})$ vanish on ghost components, i. e. those components of $\Sigma^\ell$ that are mapped to a point under $\hat{\pi}_0^\ell$ or equivalently those that become unstable after forgetting the last $\ell$ marked points. The solution to this problem, first applied in the genus 0 case in [CM07] and which will be extended to the present situation in the rest of this text, can now roughly be described as follows:
Construct subsets $\mathcal{K}^\ell \subseteq \mathcal{H}((\hat{\pi}_0^\ell)^*\tilde{X})$ of Hamiltonian perturbations, compatible under $\hat{\pi}_\ell^\ell$ in the sense that $(\hat{\pi}_\ell^\ell)^*\mathcal{K}^\ell \subseteq \mathcal{K}^\ell$, and for every $\ell$ sufficiently large a subset $\mathcal{N}^\ell(\mathcal{K}^\ell) \subseteq \mathcal{M}((\hat{\pi}_0^\ell)^*\tilde{X}, A, J, \mathcal{K}^\ell)$ with $\pi_M^{\mathcal{M}}(\mathcal{N}^\ell(\mathcal{K}^\ell)) \subseteq \mathring{M}^\ell$ (the part corresponding to smooth curves, as in Section II.1) s. t. the closure of $\mathcal{N}^\ell(\mathcal{K}^\ell)$ in $\overline{\mathcal{M}}((\hat{\pi}_0^\ell)^*\tilde{X}, A, J, \mathcal{K}^\ell)$, which then in particular is compact, lies in $\mathcal{M}((\hat{\pi}_0^\ell)^*\tilde{X}, A, J, \mathcal{K}^\ell)$.
Since over $\mathring{M}^\ell$, $\hat{\pi}_0^\ell$ is an isomorphism on every fibre, for every $H \in \mathcal{K}^0$ there is a well-defined map $(\hat{\pi}_0^\ell)_* : \mathcal{N}^\ell((\hat{\pi}_0^\ell)^*H) \to \mathcal{M}(\tilde{X}, A, J, H)$ (the left-hand side is defined in the obvious way) given by $u \mapsto ((\hat{\pi}_{0,b}^\ell)^{-1})^*u$, where $\pi_M^{\mathcal{M}}(u) = b$.
Then for generic $H \in \mathcal{K}^0$ the above will be s. t. $\mathcal{N}^\ell((\hat{\pi}_0^\ell)^*H)$ is a manifold of the correct dimension, invariant under the $\mathcal{S}_\ell$-action and the map $(\hat{\pi}_0^\ell)_*$ is an $\ell!$-sheeted covering on the complement of a subset of codimension at least 2 (see Lemma III.1). For Hamiltonian perturbations of this form, apart from compactness, unfortunately not much can be said about the closure of $\mathcal{N}^\ell((\hat{\pi}_0^\ell)^*H)$. But for generic $H \in \mathcal{K}^\ell$ it will be shown that the boundary of $\mathcal{N}^\ell(H)$ can be covered by manifolds of real dimension at least 2 less than that of $\mathcal{N}^\ell(H)$, which suffices for the definition of a pseudocycle.

Roughly speaking, the $\mathcal{N}^\ell(\mathcal{K}^\ell)$ will be defined as follows:
Under the assumption that $[\omega] \in H^2(X; \mathbb{Z})$, $\mathcal{N}^\ell(\mathcal{K}^\ell)$ and $\mathcal{K}^\ell$ depend on a choice of $J \in \mathcal{J}_\omega(X)$ and a closed $J$-complex submanifold $Y \subseteq X$ of real codimension 2 with $\mathrm{PD}(Y) = D[\omega]$ for some integer $D \in \mathbb{N}$. Then for $\ell := D\omega(A)$, let $\tilde{X}^\ell := \Sigma^\ell \times X$, $\tilde{Y}^\ell := \Sigma^\ell \times Y$. The $\mathcal{K}^\ell$ then are spaces of Hamiltonian perturbations on $\tilde{X}^\ell$ that are compatible with $\tilde{Y}^\ell$ in a certain way, see Definition II.28. If $\mathring{\Sigma}^\ell$ and $\mathring{M}^\ell$ denote the parts of $\Sigma^\ell$ and $M^\ell$, respectively, that correspond to the smooth curves, then the $\mathcal{N}^\ell(\mathcal{K}^\ell)$ are defined to be those holomorphic sections with domains in $\mathring{\Sigma}^\ell$ that map the last $\ell$ markings to $\tilde{Y}^\ell$.

One then has to show that the thus defined spaces $\mathcal{N}^\ell(\mathcal{K}^\ell)$ satisfy the properties above. A major point in showing this is the positivity of intersection numbers of a holomorphic curve with a complex hypersurface. Namely one can show that a (connected) holomorphic curve either has only a finite number of intersection points with a complex hypersurface or is completely contained in the hypersurface. Furthermore, at each intersection point, the holomorphic curve is tangent to the hypersurface of some finite order $k$ and each such intersection point contributes by $k+1$ to the (homological) intersection number. That all this still holds in a suitable sense in the presence of a Hamiltonian perturbation that satisfies suitable compatibility conditions is shown in Subsection II.4.2.
Since for a holomorphic curve $u$ in the homology class $A$, $[Y]\cdot[u] = [Y]\cdot A = \mathrm{PD}(Y)(A) = D\omega(A) = \ell$, it follows that if there are $\ell$ disjoint intersection points, then these are unique up to reordering. So for $H \in \mathcal{K}^0$, $\mathcal{N}^\ell((\hat{\pi}_0^\ell)^*H)$ defines an $\ell!$-sheeted covering of its image in $\mathcal{M}(X, A, J, H)$. To show that, after a suitable perturbation, the complement of this image has codimension at least 2, one has to consider spaces of holomorphic curves that intersect $Y$ in fewer than $\ell$ points. But, as was stated above, these then need to have a tangency of higher order at one of the intersection points. It was shown in [CM07] that these tangency conditions cut out, again after a suitable perturbation, submanifolds of the moduli space of holomorphic curves that have the correct (i. e. high enough) codimensions.

Another major point is that, extending a result from the same reference, one can show that for suitably chosen $Y$, $J$ and $H$, $\mathcal{K}^\ell(H)$ has compact closure in $\mathcal{M}(\tilde{X}^\ell, A, J, H)$. The boundary of $\mathcal{K}^\ell(H)$ in $\mathcal{M}(\tilde{X}^\ell, A, J, H)$ can then be described in terms of nodal holomorphic curves that have some components mapped into $Y$ and some components intersecting the complement of $Y$ in $X$. Via a transversality argument, one then has to show that the spaces of such curves can be covered by manifolds of codimension at least 2. To do so, one first of all shows that, again for suitably chosen $H$, any component that lies in $Y$ needs to represent homology class 0.
In the genus 0 case this suffices, for a result in [CM07] shows that one can choose $J$ s. t. any holomorphic sphere with image in $Y$ is constant (which is used in the proof of the compactness statement above). This means that one can actually replace each such component with a point, i. e. such a curve factors through a nodal curve with fewer components. It is then shown in [CM07] that this implies a tangency condition to $Y$ for this curve which suffices to give the necessary estimates on the dimension.
In the case of higher genus curves, this argument does not suffice for the following reason:
Assume the domain $S$ of a curve in the boundary of $\mathcal{N}^\ell(H)$ has several components, some of which are mapped to $Y$, denoted by $S_i^Y$, say, and the others, denoted $S_j^X$, intersect $Y$ only in a finte number of points. Then this curve lies in a moduli space that is the subset, cut out by the matching conditions at the nodes, of the product of the moduli spaces of curves defined on the $S_i^Y$ with target $Y$ and of the moduli spaces of curves defined on the $S_j^X$ with target $X$. The reason one has to regard moduli spaces of curves in $Y$ (naively, a curve in

$Y$ is in particular a curve in $X$) is that because of the compatibility condition of the Hamiltonian perturbations with $Y$, one otherwise can't achieve transversality.
If the genus of $S_i^Y$ is $g_i^Y$ then the contribution to the dimension formula of the moduli space of curve on $S_i^Y$ in $Y$ by the Riemann-Roch theorem is (for vanishing homology class) given by $\dim_{\mathbb{C}}(Y)(2-2g_i^Y) = \dim_{\mathbb{C}}(X)(2-2g_i^Y)+2g_i^Y-2$, which is larger than that for curves in $X$. Hence although these moduli spaces then cover the boundary of $\mathcal{N}^\ell(H)$, their dimensions are too large.
A further problem is that some of the additional $\ell$ marked point may lie on a component that is mapped to $Y$. This means that the condition that these marked points lie on $Y$ does not provide for a nontrivial condition on these curves and does not serve to cut down the dimension of the moduli space any more.
The solution to this problem is to use an SFT-type compactness theorem, in this text from [IP03], for related results see also [BEH$^+$03], esp. the "stretching of the neck" construction. This provides a more detailed description of the boundary of $\mathcal{N}^\ell(H)$. The important consequence of this result here is that every component that is mapped to $Y$ comes together with a nonvanishing meromorphic section of the normal bundle of $Y$ in $X$ along the image of the curve. First of all this provides an additional condition on the moduli spaces associated to the parts of a curve that are mapped to $Y$, which serves to cut down the dimension by exactly the factors $2(1-g_i^Y)$ above by which these were too large. Additionally, these meromorphic sections are known to have zeroes only at the nodes and at the additional marked points and to have poles only at the nodes. Also these satisfy the following matching conditions: If at a node, both components of the curve that border on the node are mapped into $Y$ and the meromorphic section over one has a zero of order $k$, then the other has a pole of order $k$ and vice versa. If one component is mapped to $Y$ and the other intersects $Y$ only in a finite number of points, then the meromorphic section over the first has a pole of some order $k$ and the other has a tangency to $X$ at the node of order $k$. Since every component in $Y$ represents homology class 0, the first Chern number of the pullback of the normal bundle to $Y$ in $X$ under the holomorphic map vanishes. Hence the total order of the poles equals the total order of the zeroes of a meromorphic section on every component. The matching conditions above then imply that the total order of tangency to $Y$ of the part of the curve that is not mapped into $Y$ is still given by $\ell$.

# CHAPTER II

## Construction of smooth structures and the main transversality results

### II.1 Families of complex curves

When regarding moduli spaces of holomorphic curves in a symplectic manifold, where the complex structure on the domain is not fixed, as e. g. in [MS04], Chapter 8, but is allowed to vary, before one can hope to define a smooth structure on such a moduli space, first of all one has to decide on a smooth space over which the complex structure on the domain is allowed to vary. To a certain extent this is a matter of choice, the following constructions certainly work for an arbitrary family $\rho : S \to B$, where $B$ is any manifold, $S \to B$ is a smooth fibre bundle and $j \in \Gamma(\mathrm{End}(VS))$ is a smooth family of (almost) complex structures on the vertical tangent bundle $VS = \ker \rho_*$ of $S$. On the other hand, usually one would like to use the "universal family" of Riemann surfaces of a given genus $g$ and a given number of marked points $n$, the moduli space $M_{g,n}$ of Riemann surfaces of genus $g$ with $n$ marked points, or to get a compact moduli space, the Deligne-Mumford moduli space $\overline{M}_{g,n}$ of nodal Riemann surfaces. But unless one is in the genus $g = 0$ case, neither $M_{g,n}$, nor $\overline{M}_{g,n}$ is a smooth manifold (not even a set in certain interpretations), but depending on point of view an orbifold, Deligne-Mumford-stack, etc. To make a definite choice in notation, without further qualification $\overline{M}_{g,n}$ will always denote the (compact Hausdorff) topological space underlying the Deligne-Mumford orbifold. Then, at least locally, a function $B \to \overline{M}_{g,n}$ for a manifold $B$ should be given by a family of (nodal) Riemann surfaces of genus $g$ over $B$ together with $n$ sections defining the marking. Regarding $\overline{M}_{g,n}$ simply as the quotient space of the groupoid with objects all nodal Riemann surfaces of genus $g$ with $n$ marked points and morphisms biholomorphic maps that respect the markings, the map corresponding to a family simply maps a point in $B$ to the equivalence class of the fibre over $b$. The for the present purpose best way to make the above precise can be found in [RS06] and hence all the notions of (proper étale) Lie groupoid, (universal, marked) nodal family and related concepts used in this text are exactly the ones from [RS06], Sections 2–6. More explicitely, the following are the basic notions to be dealt with here, all taken from [RS06]:

**Definition II.1.**

1. A *surface* is a closed oriented 2-dimensional manifold $S$.
2. A *nodal surface* is a pair $(S, \nu)$, consisting of a surface $S$ together with a set of unordered pairs
$$\nu = \{\{n_1^1, n_1^2\}, \ldots, \{n_d^1, n_d^2\}\}$$
of pairwise distinct points, called the *nodal points*, $n_1^1, \ldots, n_d^2 \in S$. The points $n_i^1$ and $n_i^2$ defining one of the unordered pairs in $\nu$ will be said to correspond to the same node. Note that $S$ in this definition is still a smooth surface.
A surface $S$ is considered as the nodal surface $(S, \emptyset)$.
3. A *marked nodal surface* is a triple $(S, r_*, \nu)$, where $(S, \nu)$ is a nodal surface and
$$r_* = (r_1, \ldots, r_n)$$
is an ordered tuple of pairwise distinct points on $S$, called the *marked points*, that are disjoint from all the nodal points.
The marked and nodal points are also called *special points*.
A nodal surface $(S, \nu)$ is considered as the nodal surface $(S, \emptyset, \nu)$.
4. The *signature* of a marked nodal surface $(S, r_*, \nu)$ is the labelled graph with vertices $\{S_i\}_{i \in I}$ the connected components of $S$ and for every pair of nodal points $n_j^1, n_j^2$ corresponding to the same node an edge from $S_{i_1}$ to $S_{i_2}$, where $n_j^1 \in S_{i_1}$ and $n_j^2 \in S_{i_2}$. Each vertex $S_i$ is labelled by the genus $g_i$ of $S_i$ and the subset $\{r_j \in \{r_1, \ldots, r_n\} \mid r_j \in S_i\}$.
5. The *Euler characteristic* $\chi(S, \nu)$ of a nodal surface $(S, \nu)$ is defined as the Euler characteristic of the smooth surface obtained by removing disk neighbourhoods of each pair of nodal points corresponding to the same node and gluing the resulting boundary components by an orientation reversing diffeomorphism. If that same smooth surface is connected, then $(S, \nu)$ is called connected.
6. A marked nodal surface $(S, r_*, \nu)$ is said to be of type $(g, n)$, where $g, n \in \mathbb{N}_0$, if $(S, \nu)$ is connected, $\chi(S, \nu) = 2(1 - g)$ and $r_* = (r_1, \ldots, r_n)$.
Its signature is then also said to be of type $(g, n)$.
7. An *isomorphism of marked nodal surfaces* $(S, r_*, \nu)$ and $(S', r'_*, \nu')$ is an orientation preserving diffeomorphism $\phi : S \to S'$ s.t. $\phi(r_*) = r'_*$ and

$\phi_*\nu = \nu'$ in the sense that if $r_* = (r_1, \ldots, r_n)$, then $r'_* = (\phi(r_1), \ldots, \phi(r_n))$ and $\phi$ maps each pair of nodal points on $S$ correponding to the same node to a pair of nodal points on $S'$ corresponding to the same node.
An automorphism of $(S, r_*, \nu)$ is an isomorphism from this marked nodal surface to itself.
The sets consisting of these will be denoted by $\mathrm{Diff}((S, r_*, \nu), (S', r'_*, \nu'))$ and $\mathrm{Aut}(S, r_*, \nu)$ (which is a group), respectively.

*Remark* II.1. 1. Two marked nodal surfaces are isomorphic iff their signatures are isomorphic as labelled graphs.

2. If the number of pairs of nodal points of a marked nodal surface $(S, r_*, \nu)$ is $d \in \mathbb{N}_0$ and $\{S_i\}_{i\in I}$ are the connected components of $S$, then $\chi(S, \nu) = \sum_{i\in I} \chi(S_i) - 2d = \sum_{i\in I} 2(1 - g_i) - 2d$, where $g_i$ is the genus of $S_i$.

**Definition II.2.** 1. A *marked nodal Riemann surface* is a tuple $(S, j, r_*, \nu)$ consisting of a marked nodal surface $(S, r_*, \nu)$ together with a complex structure $j \in \Gamma(\mathrm{End}(TS))$, $j^2 = -\mathrm{id}$, that induces the given orientation on $S$.

2. An *isomorphism of marked nodal Riemann surfaces* $(S, j, r_*, \nu)$ and $(S', j', r'_*, \nu')$ is an isomorphism $\phi$ of the marked nodal surfaces $(S, r_*, \nu)$ and $(S', r'_*, \nu')$ s. t. $\phi_*j = j'$. The set of these will be denoted

$$\mathrm{Diff}((S, j, r_*, \nu), (S', j', r'_*, \nu)).$$

An automorphism of $(S, j, r_*, \nu)$ is an isomorphism of this marked nodal Riemann surface to itself. The group of automorphisms of $(S, j, r_*, \nu)$ will be denoted by $\mathrm{Aut}(S, j, r_*, \nu)$.

3. A marked nodal Riemann surface is called *stable*, if $\mathrm{Aut}(S, j, r_*, \nu)$ is finite. This is the case iff every component of $S$ of genus zero contains at least three special points and every component of $S$ of genus one contains at least one special point.
The signature of a stable marked nodal Riemann surface is called a *stable signature*.

4. For $g, n \in \mathbb{N}_0$ with $n > 2(1 - g)$, as a set, the *Deligne-Mumford moduli space (of type $(g, n)$) $\overline{M}_{g,n}$* is the set of isomorphism classes of stable marked nodal Riemann surfaces of type $(g, n)$.

*Remark* II.2. That $\overline{M}_{g,n}$ indeed is a set is shown by picking, for every isomorphism class of stable signature of type $(g, n)$, a marked nodal surface of this signature. There are only finitely many choices of ismorphism classes of stable signatures of fixed type. For each such choice one then considers ismorphism classes of complex structures on a fixed surface, which, as sections of a bundle, form a set.

The above only defines $\overline{M}_{g,n}$ as a set, so next a description of the smooth (or holomorphic) structure is required. One way to define such a structure is by describing holomorphic functions from complex manifolds into $\overline{M}_{g,n}$. Because $\overline{M}_{g,n}$ is supposed to serve as a kind of moduli space for marked nodal Riemann surfaces, a holomorphic map into $\overline{M}_{g,n}$ should correspond to holomorphic families of marked nodal Riemann surfaces, where by family of marked nodal Riemann surfaces, the following is meant:

**Definition II.3.** 1. A *marked nodal family of Riemann surfaces* is a pair $(\pi : \Sigma \to B, R_*)$, where $\Sigma$ and $B$ are complex manifolds with $\dim_{\mathbb{C}}(\Sigma) = \dim_{\mathbb{C}}(B) + 1$, $\pi : \Sigma \to B$ is a proper holomorphic map and $R_* = (R_1, \dots, R_n)$ is a sequence of pairwise disjoint complex submanifolds of $\Sigma$ s. t. the following hold:
For every $z \in \Sigma$, there exist holomorphic coordinates $(t_0, \dots, t_s)$, $s := \dim_{\mathbb{C}}(B) = \dim_{\mathbb{C}}(\Sigma - 1)$, around $z$ in $\Sigma$ and $(v_1, \dots, v_s)$ around $\pi(z)$ in $B$, mapping $z$ to $0 \in \mathbb{C}^{s+1}$ and $\pi(z)$ to $0 \in \mathbb{C}^s$, respectively, s. t. in these coordinates, $\pi$ is given by either

$$(t_0, \dots, t_s) \mapsto (t_1, \dots, t_s) \tag{II.1}$$

or

$$(t_0, \dots, t_s) \mapsto (t_0 t_1, t_2, \dots, t_s). \tag{II.2}$$

In the first case, $p$ is called a *regular point*, in the second case, $p$ is called a *node* of *nodal point*.
Furthermore, for each $i = 1, \dots, n$, $\pi|_{R_i} : R_i \to B$ is assumed to be a diffeomorphism. Each $R_i$ hence defines a section of $\pi : \Sigma \to B$, with which it will usually be identified.

2. A *desingularisation* of a fibre $(\Sigma_b, R_{*,b})$, for $b \in B$ and $R_{*,b} := R_* \cap \Sigma_b$, of a marked nodal family of Riemann surfaces $(\pi : \Sigma \to B, R_*)$ is a marked nodal Riemann surface $(S, j, r_*, \nu)$ together with a surjective holomorphic immersion $\hat{\iota} : S \to \Sigma_b \subseteq \Sigma$, that is an embedding from the complement of the nodal points on $S$ onto the complement of the nodes on $\Sigma_b$ and maps every pair of nodal points on $S$ corresponding to the same node to a node on $\Sigma_b$. Furthermore, if $R_* = (R_1, \dots, R_n)$, then $r_* = (r_1, \dots, r_n)$ and for each $i = 1, \dots, n$, $\hat{\iota}(r_i) = \Sigma_b \cap R_i$.

3. A *morphism* between marked nodal families of Riemann surfaces $(\pi : \Sigma \to B, R_*)$ and $(\pi' : \Sigma' \to B', R'_*)$ is a pair of holomorphic maps $\phi : B \to B'$ and $\Phi : \Sigma \to \Sigma'$ s. t. $\pi' \circ \Phi = \phi \circ \pi : \Sigma \to B'$. Furthermore, for every $b \in B$, if $(S, j, r_*, \nu)$ is a marked nodal Riemann surface and $\hat{\iota} : S \to \Sigma_b$ is a desingularisation of the fibre of $\Sigma$ over $b$, then $\Phi \circ \hat{\iota} : S \to \Sigma'_{\phi(b)}$ is a desingularisation of the fibre of $\Sigma'$ over $\phi(b)$.

4. The *signature* of a fibre $(\Sigma_b, R_{*,b})$, for $b \in B$, of a marked nodal family of Riemann surfaces $(\pi : \Sigma \to B, R_*)$ is the (isomorphism class of the) signature of a desingularisation of $(\Sigma_b, R_{*,b})$.
$(\Sigma_b, R_{*,b})$ is said to be *stable* (*of type* $(g, n)$), if a desingularisation of $(\Sigma_b, R_{*,b})$ is stable (of type $(g, n)$).

$(\pi : \Sigma \to b, R_*)$ is called stable (of type $(g, n)$), if every fibre is stable (of type $(g, n)$).

The above is well-defined by Lemma 4.3 in [RS06], i. e. every fibre of a marked nodal family of Riemann surfaces has a desingularisation and for any two desingularisations of the same fibre, there is a unique isomorphism of the marked nodal Riemann surfaces that commutes with the maps to the fibre.
Hence every stable marked nodal family of Riemann surfaces of type $(g, n)$ comes with a well-defined map to $\overline{M}_{g,n}$, mapping a point in the base to the isomorphism class of a marked nodal Riemann surface of a desingularisation of the fibre over the point. The requirement that the maps obtained in this way are smooth then gives a criterion by which one can define a topology on $\overline{M}_{g,n}$, namely the finest one s. t. all the maps of this form are continuous. This abstract way of defining the topology does not provide a way to deal with the usual questions of topology like the verification of the Hausdorff property, $2^{\mathrm{nd}}$-countability and compactness. To deal with these, one singles out a special type of stable marked nodal family that serve as charts for an orbifold structure on $\overline{M}_{g,n}$ and define the topology as well:

**Definition II.4.** Let $(S, j, r_*, \nu)$ be a stable marked nodal Riemann surface of type $(g, n)$. A *(nodal) unfolding* of $(S, j, r_*, \nu)$ is a stable marked nodal family of Riemann surfaces of type $(g, n)$ $(\pi : \Sigma \to B, R_*)$ together with a point $b \in B$ and a desingularisation $\hat{\iota} : S \to \Sigma_b \subseteq \Sigma$ of the fibre over $b$.
The unfolding is called *universal*, iff for every other nodal unfolding $(\pi' : \Sigma' \to B', R'_*), b' \in B', \hat{\iota}' : S \to \Sigma'_{b'}$, there exists a unique germ of a morphism $(\Phi, \phi) : (\pi : \Sigma \to B, R_*) \to (\pi' : \Sigma' \to B', R'_*)$ s. t. $\phi(b) = b'$ and $\Phi \circ \hat{\iota} = \hat{\iota}'$.

Some of the main theorems from [RS06] can now be summed up as follows:

**Theorem II.1.**
1. *A marked nodal Riemann surface admits a universal unfolding iff it is stable.*
2. *If $(\pi : \Sigma \to B, R_*), b \in B, \hat{\iota} : S \to \Sigma_b$ is a universal nodal unfolding of the marked nodal Riemann surface $(S, j, r_*, \nu)$, then there exists a neighbourhood $U \subseteq B$ of $b$ s. t. it is a universal unfolding of every desingularisation of every fibre $\Sigma_{b'}$ for $b' \in U$.*

**Definition II.5.** A *local universal marked nodal family of Riemann surfaces of type* $(g, n)$ is a stable marked nodal family of Riemann surfaces $(\pi : \Sigma \to B, R_*)$ of type $(g, n)$ with the property that for every $b \in B$ and every desingularisation $\hat{\iota} : S \to \Sigma_b$ of $\Sigma_b$ by a stable marked nodal Riemann surface $(S, j, r_*, \nu)$ of type $(g, n)$, $(\pi : \Sigma \to B, R_*), b, \hat{\iota} : S \to \Sigma_b$ is a universal unfolding of $(S, j, r_*, \nu)$.
If the canonical map $B \to \overline{M}_{g,n}$ is surjective, then it is called a *universal marked nodal family of Riemann surfaces of type* $(g, n)$.

A further important result about universal unfoldings, apart from the existence result above and uniqueness result built into the definition is that one can actually give a fairly explicit construction for them. The relevant results can be found in the proof of Theorem 5.6 in [RS06], which comes in two parts, in Section 8 in the proof of Theorem 8.9 for the case of a marked Riemann surface without nodes and in Section 12 in the presence of nodes:

**Construction II.1.** 1. For a marked (nodal) Riemann surface $(S, j, r_*, \emptyset)$ of type $(g, n)$ with $S$ connected and $g \geq 2$, one can choose $(\pi : \Sigma \to B, R_*), b \in B, \hat{\iota} : S \to \Sigma_b$ in the following way:

- $B = \mathbb{D}^{3(g-1)} \times \mathbb{D}^n \cong \mathbb{D}^{3(g-1)+n}$;
- $b = \{0, 0\}$;
- $\Sigma = B \times S$;
- The complex structure on $\Sigma$ is of the form $T_{(b,z)}\Sigma = T_b B \times T_z S \ni (X, \xi) \mapsto (\mathbf{i}X, \hat{j}(b_0)\xi)$, for $b = (b_0, (b_1, \ldots, b_n)) \in B = \mathbb{D}^{3(g-1)} \times \mathbb{D}^n$, where $\mathbf{i}$ is the standard complex structure on $D^{3(g-1)} \times \mathbb{D}^n$ and $\hat{j} : \mathbb{D}^{3(g-1)} \to \mathcal{J}(S)$ is a holomorphic map to the set of complex structures on $S$ with $\hat{j}(0) = j$.
- The markings are of the form $R_i(b) = (b, \iota_i(b_0, b_i))$, for $b = (b_0, (b_1, \ldots, b_n)) \in B = \mathbb{D}^{3(g-1)} \times \mathbb{D}^n$, where $\iota_i(b_0, 0) = r_i$ and for every $b_0 \in \mathbb{D}^{3(g-1)}$, the $\iota_i(b_0, \cdot) : \mathbb{D} \to S$ are $\hat{j}(b_0)$-holomorphic embeddings with pairwise disjoint images.

2. For a marked (nodal) Riemann surface $(S, j, r_*, \emptyset)$ of type $(1, n)$ with $S$ connected and $n \geq 1$, one can choose $(\pi : \Sigma \to B, R_*), b \in B, \hat{\iota} : S \to \Sigma_b$ in the following way:

   - $B = \mathbb{D} \times \mathbb{D}^{n-1} \cong \mathbb{D}^{3(g-1)+n}$;
   - $b = \{0, 0\}$;
   - $\Sigma = B \times S$;
   - The complex structure on $\Sigma$ is of the form $T_{(b,z)}\Sigma = T_b B \times T_z S \ni (X, \xi) \mapsto (\mathbf{i}X, \hat{j}(b_0)\xi)$, for $b = (b_0, (b_1, \ldots, b_{n-1})) \in B = \mathbb{D} \times \mathbb{D}^{n-1}$, where $\mathbf{i}$ is the standard complex structure on $D \times \mathbb{D}^{n-1}$ and $\hat{j} : \mathbb{D} \to \mathcal{J}(S)$ is a holomorphic map to the set of complex structures on $S$ with $\hat{j}(0) = j$.
   - The markings are of the form $R_1(b) = (b, r_1)$ and for $i = 2, \ldots, n$, $R_i(b) = (b, \iota_i(b_0, b_i))$, for $b = (b_0, (b_1, \ldots, b_n)) \in B = \mathbb{D} \times \mathbb{D}^{n-1}$, where $\iota_i(b_0, 0) = r_i$ and for every $b_0 \in \mathbb{D}$, the $\iota_i(b_0, \cdot) : \mathbb{D} \to S$ are $\hat{j}(b_0)$-holomorphic embeddings with pairwise disjoint images that do not contain $r_1$ in their closures.

3. For a marked (nodal) Riemann surface $(S, j, r_*, \emptyset)$ of type $(0, n)$ with $S$ connected and $n \geq 3$, one can choose $(\pi : \Sigma \to B, R_*), b \in B, \hat{\iota} : S \to \Sigma_b$ in the following way:

   - $B = \mathbb{D}^{n-3} \cong \mathbb{D}^{3(g-1)+n}$;
   - $b = \{0\}$;
   - $\Sigma = B \times S$;

- The complex structure on $\Sigma$ is the product of the standard complex structure on $\mathbb{D}^{n-3}$ and $j$.
- The markings are of the form $R_i(b) = (b, r_i)$ for $i = 1, 2, 3$ and for $i = 4, \ldots, n$, $R_i(b) = (b, \iota_i(b_i))$, for $b = (b_1, \ldots, b_n) \in B = \mathbb{D}^{n-3}$, where $\iota_i(0) = r_i$ and the $\iota_i : \mathbb{D} \to S$ are $j$-holomorphic embeddings with pairwise disjoint images that do not contain $r_1, r_2, r_3$ in their closures.

4. In the general case $(S, j, r_*, \nu)$, choose a numbering $\nu = \{\{n_1^1, n_1^2\}, \ldots, \{n_d^1, n_d^2\}\}$ and consider the marked Riemann surface (without nodes) $(S, j, (r_*, n_*^1, n_*^2), \emptyset)$ where all the nodes have been replaced by marked points. Denote by $\{S_i\}_{i\in I}$ the connected components of $S$ and by $g_i$ their genera. Then for every $i \in I$, $(S_i, j|_{S_i}, (r_*^i, n_*^{1,i}, n_*^{2,i}))$ is a marked Riemann surface of one of the types above, where $r_*^i$ consists of those $r_j$ with $r_j \in S_i$ and analogously for $n_*^{1,i}$ and $n_*^{2,i}$. Let $(\pi_i : \Sigma_i \to B_i, (R_*^i, N_*^{1,i}, N_*^{2,i})), 0 \in B_i, \hat{\iota}_i : S_i \to \Sigma_{i,0}$ be the corresponding universal unfolding from above. If $n_i := |r_*^i|$, $d^{1,i} := |n^{1,i}|$, $d^{2,i} := |n^{2,i}|$, then $\dim_{\mathbb{C}}(B_i) = 3(g_i - 1) + n_i + d^{1,i} + d^{2,i}$. Define $B := \left(\bigtimes_{i\in I} B_i\right) \times \mathbb{D}^d$ and $\hat{\Sigma} := \coprod_{i\in I} \mathrm{pr}_i^* \Sigma_i$, where $\mathrm{pr}_i : B \to B_i$ is the projection. $B$ has dimension $\dim_{\mathbb{C}}(B) = \sum_{i\in I} \dim_{\mathbb{C}}(B_i) + d = \sum_{i\in I}(3(1 - g_i) + n_i + d^{1,i} + d^{2,i}) + d = 3(\sum_{i\in I}(g_i - 1) + d) + n = 3(g-1) + n$. Denote by $\hat{\pi} : \hat{\Sigma} \to B$ the obvious projection. This comes with markings $\hat{R}_*, \hat{N}_*^1, \hat{N}_*^2$, which are the pullbacks of the markings of the $R_*^i, N_*^{1,i}, N_*^{2,i}$ above. Also, one can choose disjoint open sets $U_i, V_i \subseteq \hat{\Sigma}$, $i = 1, \ldots, d$ that are tubular neighbourhoods of the $\hat{N}_*^1, \hat{N}_*^2$ that do not meet the $\hat{R}_*$ and come with holomorphic functions $x_i : U_i \to \mathbb{D}$ and $y_i : V_i \to \mathbb{D}$ s. t. $x_i(\hat{N}_i^1) = 0$, $y_i(\hat{N}_i^2) = 0$ and $(\hat{\pi}, x_i)$ and $(\hat{\pi}, y_i)$ are coordinates on $\hat{\Sigma}$. For each $i = 1, \ldots, d$, let $K_i := \{\xi \in U_i \mid x_i(\xi) \leq |z_i|, \hat{\pi}(\xi) = (b, z_1, \ldots, z_d), z_i \neq 0\}$ and $L_i := \{\xi' \in V_i \mid y_i(\xi') \leq |z_i|, \hat{\pi}(\xi') = (b, z_1, \ldots, z_d), z_i \neq 0\}$. Also let $\hat{\Sigma}' := \hat{\Sigma} \setminus \bigcup_{i=1}^d K_i \cup L_i$. Now define $\Sigma := \hat{\Sigma}'/_\sim$, where the equivalence relation on $\hat{\Sigma}'$ is generated by the following identification, for $\xi \in U_i, \xi' \in V_i$:

$$\begin{aligned} \xi \sim \xi' \quad :\Leftrightarrow \quad & \hat{\pi}(\xi) = \hat{\pi}(\xi') = (b, z_1, \ldots, z_d) \\ & \text{and either } x_i(\xi) y_i(\xi') = z_i \neq 0 \\ & \text{or } x_i(\xi) = y_i(\xi') = z_i = 0. \end{aligned}$$

The projection $\pi : \Sigma \to B$ is given by $\pi([\xi]) := \hat{\pi}(\xi)$ and the markings are given by the images of the $\hat{R}_*$ under the projection onto the quotient.
The above differs from the construction in the proof of Theorem 5.6 in [RS06] by the removal of the subsets $K_i$ and $L_i$ from $\hat{\Sigma}$. But otherwise it seems to me the map $\pi : \Sigma \to B$ thus constructed does not have as fibres nodal surfaces.

The existence and explicit construction of the universal unfoldings above is useful for a number of reasons:

1. Let $\pi : \Sigma \to B$ be the unfolding of a marked nodal Riemann surface $(S, j, r_*, \nu)$ with $d$ nodes from case 4. above. Then $B$ is of the form

$B = B_0 \times \mathbb{D}^d$, so has coordinates $(b_0, z_1, \dots, z_d)$ and is stratified by the following locally closed submanifolds: Let $N \subseteq \{1, \dots, d\}$ be a subset. Then one can look at the subset $B^N := \{(b_0, z_1, \dots, z_d) \in B \mid z_i = 0 \text{ for } i \in N\}$. These are precisely the subsets for which all $\Sigma_b$, $b \in B^N$, have the same signature. Since the signatures of the fibres are preserved under morphisms of nodal families, these stratifications of the universal unfoldings of all stable marked nodal Riemann surfaces of type $(g, n)$ induces a stratification of $\overline{M}_{g,n}$, called the *stratification by signature*.
Also, if $(\pi : \Sigma \to B, R_*)$ is any local universal marked nodal family of Riemann surfaces of type $(g, n)$, it also carries an induced stratification by signature.

2. If $(\pi : \Sigma \to B, R_*)$ is a local universal marked nodal family of Riemann surfaces of type $(g, n)$, then over every stratum of the stratification by signature one has the following parametrised version of a desingularisation. Namely let $b \in B$ and let $(S, j, r_*, \nu), \hat{\iota} : S \to \Sigma_b$ be desingularisation of $\Sigma_b$. Associated to this desingularisation is the universal unfolding from 4. above, which defines a smooth (trivial) fibre bundle $\hat{\pi} : \hat{\Sigma} \to C$, where $C = C_0 \times \mathbb{D}^d$, $d$ being the number of nodes on $\Sigma_b$. Making $B$ small enough, this comes with a unique pair of maps $\phi : C \to B$ and $\Phi : \hat{\Sigma} \to \Sigma$. If $\hat{\Sigma}/_\sim$ is the quotient that defines the universal unfolding as in 4. above, then there is a unique morphism $(\Phi', \phi)$ from $\hat{\Sigma}/_\sim$ to $\Sigma$ s. t. $\phi$ maps $(0, 0) \in B_0 \times \mathbb{D}^d$ to $b \in B$ and one can define $\Phi$ as the composition of $\Phi'$ with the projection from $\hat{\Sigma}$ to $\hat{\Sigma}/_\sim$. Then $C' := C_0 \times \{0\} \subseteq C$ is precisely the part of $C$ that gets mapped to the stratum $B'$ of the stratification by signature on $B$ that corresponds to the signature of $(S, j, r_*, \nu)$. Also, the restriction $\hat{S} := \hat{\Sigma}|_{C'}$ is a holomorphic family $\rho : \hat{S} \to C'$ of smooth Riemann surfaces, with a complex structure $\hat{j}$ on $\hat{S}$, that comes with $n$ sections $\hat{R}_*$ corresponding to the markings on $S$ and $d$ pairs of section $\hat{N}^1_*, \hat{N}^2_*$ corresponding to the nodes. Furthermore, it comes with canonical maps $\iota : C' \to B$ and $\hat{\iota} : \hat{S} \to \Sigma$ that have the property that for every $c \in C'$, $(\hat{S}_c, \hat{j}_c, \hat{R}_{*,c}, \{\{\hat{N}^1_{i,c}, \hat{N}^2_{i,c}\}\}_{i=1}^d)$ together with $\hat{\iota}_c : \hat{S} \to \Sigma_{\iota(c)}$ is a desingularisation of $\Sigma_{\iota(c)}$. By the universal properties of a universal unfolding and local universal marked nodal family, one can do this for every $b \in B'$, and the resulting (trivial) fibre bundles as above patch together to a fibre bundle over $\rho : \hat{S} \to B'$ with fibres smooth Riemann surfaces and that comes with $n$ sections $\hat{R}_*$. Furthermore, the $N^1_*, N^2_*$ define a discrete subbundle $\hat{N} \subseteq \hat{S}$ with structure group $\mathcal{S}(d, 2)$ defined to be the subgroup of permutations of a set $(n^1_1, n^2_1, \dots, n^1_d, n^2_d)$, generated by the permutations in the lower indices, $(n^1_1, n^2_1, \dots, n^1_d, n^2_d) \mapsto (n^1_{\sigma(1)}, n^2_{\sigma(1)}, \dots, n^1_{\sigma(d)}, n^2_{\sigma(d)})$ for $\sigma \in \mathcal{S}(d)$ and switching a pair of upper indices, $(n^1_1, n^2_1, \dots, n^1_d, n^2_d) \mapsto (n^1_1, n^2_1, \dots, n^{\tau(1)}_j, n^{\tau(d)}_j, \dots, n^1_d, n^2_d)$ for $\tau \in \mathcal{S}(2)$. So
$$(\rho : \hat{S} \to C', \hat{R}_*, \hat{N})$$
is a triple consisting of a smooth fibre bundle with fibre $S$ and structure group $\mathrm{Aut}(S, r_*, \nu)$, an $n$-tuple of sections of $\hat{S}$ and a discrete subbundle with fibre a $d$-tuple of pairs of points and structure group $\mathcal{S}(d, 2)$.

**Definition II.6.** A *(parametrised) desingularisation* of a marked nodal family of Riemann surfaces is a tuple $(\rho : \hat{S} \to C', \hat{R}^*, \hat{N}, \iota, \hat{\iota})$, where $\rho : \hat{S} \to C'$ is a smooth fibre bundle equipped with a smooth family of complex structures $j$. $\hat{R}_* = (\hat{R}_1, \ldots, \hat{R}_n)$ is an $n$-tuple of sections of $\rho : \hat{S} \to C'$, $N \subseteq \hat{S}$ is a $\mathcal{S}(d,2)$-subbundle and $\iota : C' \to B$ is an embedding of $C'$ as a locally closed submanifold of $B$. Furthermore, for every $b \in C'$, $(\hat{S}_b, j_b, \hat{R}_*(b), N_b), \iota(b), \hat{\iota}_b : \hat{S}_b \to \Sigma_{\iota(b)}$ is a desingularisation in the original sense.

3. It allows to single out "especially nice" maps to $\overline{M}_{g,n}$ that come from nodal families. The most desirable case here would be the (local) universal marked nodal families. Unfortunately, for the definition of invariants, one would like for the base of the (universal) family of marked nodal Riemann surfaces to be compact, which in general is not possible. The next best kind of maps are the following: Let $\pi : \Sigma \to B$ be a nodal family, $b \in B$ and let $(S, j, r_*, \nu), \kappa : S \to \Sigma_b$ be a desingularisation of $\Sigma_b$. Associated to $(S, j, r_*, \nu)$ is a universal unfolding $(\tilde{\pi} : \tilde{\Sigma} \to \tilde{B}, \tilde{R}_*), \tilde{b} = (0,0) \in \tilde{B}, \tilde{\iota} : S \to \tilde{\Sigma}_b$, where $\tilde{B} = \mathbb{D}^{3(g-1)+n-d} \times \mathbb{D}^d$ and $d$ is the number of nodes on $\Sigma_b$. By the universal property there then exists a neighbourhood $U \subseteq B$ of $b$ and a morphism

$$\begin{array}{ccc} \Sigma|_U & \xrightarrow{\Phi} & \tilde{\Sigma} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\tilde{\pi}} \\ U & \xrightarrow{\phi} & \tilde{B}. \end{array}$$

Choosing $U$ to be a coordinate neighbourhood of $b$, holomorphically equivalent to $\mathbb{D}^r$, $r := \dim_{\mathbb{C}}(B)$, with complex coordinates $(z_1, \ldots, z_r)$, $\phi$ is equivalent to a map $\mathbb{D}^r \to \mathbb{D}^{3(g-1)+n-d} \times \mathbb{D}^d$. A requirement one can then pose on the nodal family $\pi : \Sigma \to B$ is that $\dim_{\mathbb{C}}(B) = 3(g-1) + n$ and that around every point $b \in B$ one can choose the coordinate system as above s. t. in these coordinates $\phi$ is given by the map

$$\mathbb{D}^{3(g-1)+n} \to \mathbb{D}^{3(g-1)+n-d} \times \mathbb{D}^d$$
$$(z_1, \ldots, z_{3(g-1)+n}) \mapsto ((z_1, \ldots, z_{3(g-1)+n-d}), (z^{l_1}_{3(g-1)+n-d+1}, \ldots, z^{l_d}_{3(g-1)+n}))$$

for some constants $l_1, \ldots, l_d \in \mathbb{N}^d$ (depending on $b \in B$), or in other words a branched covering that branches exactly over the strata of the stratification by signature.

**Definition II.7.** A marked nodal family of Riemann surfaces of type $(g, n)$ with the properties above is called an *orbifold branched covering of $\overline{M}_{g,n}$ that branches over the Deligne-Mumford boundary.*

This implies that on $B$ there also is a well-defined stratification by signature, where each stratum is a locally closed submanifold of complex codimension given by the number of nodes of a surface of that signature (i. e. the number of edges of the graph). If $\phi$, $U$ and $\tilde{B}$ are as above, then the restriction of $\phi$ to every stratum of the stratification by signature on $U$ is a (non-branched) covering of the corresponding stratum on $\tilde{B}$. Also, one can pull back the parametrised desingularisations from 2. above over the strata on each $\tilde{B}$ to the strata on $B$ to get over each such stratum $B_i$ a parametrised desingularisation $(\rho : \hat{S} \to B_i, \hat{R}_*, \hat{N})$.

4. Last, one can examine the interactions between universal families of type $(g, n)$, where $g$ is fixed, but for different values of $n$, in these local models. In the genus $g = 0$ case, it is well known that $\overline{M}_{0,n}$ is a closed complex manifold itself (follows from the results in [RS06] because a stable sphere carries no nontrivial automorphisms) and there is a well-defined smooth map $\overline{M}_{0,n+1} \to \overline{M}_{0,n}$ that is defined by forgetting the $(n+1)^{\text{st}}$ marked point and stabilising. Furthermore, this map $\overline{M}_{0,n+1} \to \overline{M}_{0,n}$ defines a universal marked nodal family, see [RS06], Example 6.7.
In the higher genus case, the situation is built around the following model: Let $(S, j, r_*, \nu)$ and $(\tilde{S}, \tilde{j}, \tilde{r}_*, \tilde{\nu})$ be stable marked nodal Riemann surfaces of types $(g, n)$ and $(g, n+1)$, respectively. $(S, j, r_*, \nu)$ is said to be *obtained from* $(\tilde{S}, \tilde{j}, \tilde{r}_*, \tilde{\nu})$ *by forgetting the last marked point and stabilising*, if the following holds: Let $\tilde{S}_i$ be the connected component of $\tilde{S}$ with $\tilde{r}_{n+1} \in \tilde{S}_i$. One has to distinguish three cases:

   (a) If $\tilde{S}_i$ together with the special points on it other than $\tilde{r}_{n+1}$ is still stable, then define $\tilde{S}' := \tilde{S}$, $\tilde{j}' := \tilde{j}$, $\tilde{r}'_* := (\tilde{r}_1, \ldots, \tilde{r}_n)$ and $\tilde{\nu}' := \tilde{\nu}$.

   Otherwise, define $\tilde{S}' := \tilde{S} \setminus \tilde{S}_i$ and $\tilde{j}' := \tilde{j}|_{\tilde{S}'}$. $\tilde{S}_i$ then is a sphere with three special points, for if the genus of $\tilde{S}_i$ is $\geq 2$, then it is stable without any special points and if the genus is 1, then because $(g, n)$ is also a stable type, i. e. $n \geq 1$, and $\tilde{S}$ is connected, $\tilde{S}_i$ either contains a marked point other than $\tilde{r}_{n+1}$ (if $\tilde{S} = \tilde{S}_i$ is connected) or a nodal point. The other two special points apart from $\tilde{r}_{n+1}$ then are either a nodal point and another marked point or two nodal points.

   (b) In the first case, let $\tilde{r}_l$ be the second marked point on $\tilde{S}_i$ and let $\tilde{n}_d^2$ be the nodal point on $\tilde{S}_i$. Define $\tilde{r}'_* = (\tilde{r}_1, \ldots, \tilde{n}_d^1, \ldots, \tilde{r}_n)$, where $\tilde{n}_d^1$ replaces $\tilde{r}_l$, and $\tilde{\nu}' := \{\{\tilde{n}_1^1, \tilde{n}_1^2\}, \ldots, \{\tilde{n}_{d-1}^1, \tilde{n}_{d-1}^2\}\}$.

   (c) In the second case, the two nodal points cannot correspond to the same node, for that would imply by connectedness of $\tilde{S}$ that $\tilde{S} = \tilde{S}_i$, so $g = 1$ and there would be at least two marked points. So assume $\tilde{\nu} = \{\{\tilde{n}_1^1, \tilde{n}_1^2\}, \ldots, \{\tilde{n}_d^1, \tilde{n}_d^2\}\}$ and that the two nodal points on $\tilde{S}_i$ are $\tilde{n}_{d-1}^2$ and $\tilde{n}_d^1$. Define $\tilde{r}'_* := (\tilde{r}_1, \ldots, \tilde{r}_n)$ and

$$\tilde{\nu}' := \{\{\tilde{n}_1^1, \tilde{n}_1^2\}, \ldots, \{\tilde{n}_{d-2}^1, \tilde{n}_{d-2}^2\}, \{\tilde{n}_{d-1}^1, \tilde{n}_d^2\}\}.$$

   In all of these cases, $(\tilde{S}', \tilde{j}', \tilde{r}'_*, \tilde{\nu}')$ is a stable marked nodal surface of type $(g, n)$. If $(\tilde{S}', \tilde{j}', \tilde{r}'_*, \tilde{\nu}')$ and $(S, j, r_*, \nu)$ are isomorphic, then the latter is said to be obtained from the former by forgetting the last marked point and stabilising.
   Furthermore, the choice of such an isomorphism defines a (open and

closed) holomorphic embedding of $S$ into $\tilde{S}$ that maps special points to special points (but may map a marked point to a nodal point). Also, this inclusion defines an injection of $\mathrm{Aut}(\tilde{S}, \tilde{j}, \tilde{r}_*, \tilde{\nu})$ into $\mathrm{Aut}(S, j, r_*, \nu)$ (because the automorphism group of a sphere with three special points is trivial). More precisely, there is a one-to-one correspondence between points on $S$ that are not nodal points or pairs of nodal points corresponding to the same node and stable marked nodal surfaces $(\tilde{S}, \tilde{j}, \tilde{r}_*, \tilde{\nu})$ of type $(g, n+1)$ up to unique equivalence as above:
If $z \in S$ is neither a marked point nor a nodal point, define $\tilde{S} := S$, $\tilde{j} := j$, $\tilde{r}_i := r_i$ for $i = 1, \ldots, n$, $r_{n+1} := z$ and $\tilde{\nu} := \nu$. This corresponds to case (a) above, which conversely defines $z := \tilde{r}_{n+1}$.
If $z = r_l \in S$ for some $l \in \{1, \ldots, n\}$, define $\tilde{S} := S \amalg S^2$, where $S^2 = \mathbb{C} \cup \{\infty\}$, $\tilde{j}|_S = j$ and $\tilde{j}|_{S^2}$ is the standard complex structure, $\tilde{r}_i = r_i$ for $i = 1, \ldots, n$ with $i \neq l$, $\tilde{r}_l = \infty \in S^2$, $\tilde{r}_{n+1} := 1 \in S^2$ and $\tilde{\nu} := \nu \cup \{\{r_l, 0\}\}$ ($0 \in S^2$). This corresponds to case (b) above, which conversely defines $z := \tilde{r}_l$.
If w. l. o. g. $\nu = \{\{n_1^1, n_1^2\}, \ldots, \{n_{d-1}^1, n_{d-1}^2\}\}$ and $z = \{n_{d-1}^1, n_{d-1}^2\}$, define $\tilde{S} := S \amalg S^2$, $\tilde{j}|_S = j$ and $\tilde{j}|_{S^2}$ the standard complex structure, $\tilde{r}_i := r_i$ for $i = 1, \ldots, n$, $r_{n+1} := 1 \in S^2$ and

$$\tilde{\nu} := \{\{n_1^1, n_1^2\}, \ldots, \{n_{d-2}^1, n_{d-2}^2\}, \{n_{d-1}^1, 0\}, \{\infty, n_{d-1}^2\}\}.$$

This corresponds to case (c) above, which conversely defines $z := \{\tilde{n}_{d-1}^1, \tilde{n}_d^2\}$.

Marked nodal families of Riemann surfaces of type $(g, n)$ that define an orbifold branched covering of $\overline{M}_{g,n}$ that branches over the Deligne-Mumford boundary (hence in particular local universal marked nodal families) are a special case of a type of marked nodal family that is called regular in [RS06] (Definition 12.1) and for which the above construction of forgetting the last marked point and stabilising has a global generalisation.

**Definition II.8.** Let $(\pi : \Sigma \to B, R_*)$ be a marked nodal family of Riemann surfaces. Let $C \subseteq \Sigma$ be the submanifold of nodal points, which comes with the immersion $\pi|_C : C \to B$. Given $b \in B$, $(\pi : \Sigma \to B, R_*)$ is called *regular at* $b$ if all self-intersections of $\pi(C)$ in $b$ are transverse in the following sense: Either $b \notin \pi(C)$ or if $b \in \pi(C)$, let $C_b := C \cap \Sigma_b = \{n_1, \ldots, n_d\}$, a finite set of points. Then for all $1 \leq m \leq d$, $1 \leq i_1 < \cdots < i_m \leq d$

$$\dim_{\mathbb{C}}(\mathrm{im}(\pi_{*,n_{i_1}}) \cap \cdots \cap \mathrm{im}(\pi_{*,n_{i_m}})) = \dim_{\mathbb{C}}(B) - m.$$

$(\pi : \Sigma \to B, R_*)$ is called *regular* if it is regular at all points $b \in B$.

By definition of a marked nodal family of Riemann surfaces, in the notation of the previous definition and if $b \in \pi(C)$, the following hold: For $i = 1, \ldots, d$ there exist neighbourhoods $U_i \subseteq \Sigma$ of the $n_i$ not containing any of the marked points, neighbourhoods $V_i \subseteq B$ of $b$ and holomorphic maps $x_i, y_i : U_i \to \mathbb{D}$, $z_i : V_i \to \mathbb{D}$ obtained from a nodal coordinate system as in Equation II.2 s. t. $(x_i, y_i) : U_i \to \mathbb{D}^2$ and $z_i : V_i \to \mathbb{D}$ are submersions and $z_i \circ \pi|_{U_i} = x_i y_i$ :

$U_i \to \mathbb{D}$. Also, $C \cap U_i = (x_i, y_i)^{-1}(0,0)$, $\pi_* : \ker((x_i, y_i)_*) \to \ker(z_{i,*})$ is an isomorphism and $\operatorname{im}(\pi_{*,n_i}) = \ker(z_{i,*})$. Making the $U_i$ and $V_i$ smaller, one can assume that $V_1 = \cdots = V_d =: V$. The transversality condition above then states that the $z_{i,*,b} : T_bB \to T_0\mathbb{D}$ are linearly independent. By the implicit function theorem, after possibly making $V$ and the $U_i$ smaller, one hence can find holomorphic functions $t_1, \dots, t_k : V \to \mathbb{D}$, $k := \dim_{\mathbb{C}}(B) - d$, s. t. $(z_1, \dots, z_d, t_1, \dots, t_k) : V \to \mathbb{D}^{\dim_{\mathbb{C}}(B)}$ is a holomorphic coordinate system on $B$ and s. t. $(z_1 \circ \pi|_{U_i}, \dots, z_{i-1} \circ \pi|_{U_i}, x_i, y_i, z_{i+1} \circ \pi|_{U_i}, \dots, z_d \circ \pi|_{U_i}, t_1 \circ \pi|_{U_i}, \dots, t_k \circ \pi|_{U_i}) : U_i \to \mathbb{D}^{\dim_{\mathbb{C}}(\Sigma)}$ is a holomorphic coordinate system on $\Sigma$.

**Lemma II.1.** *Let $(\pi : \Sigma \to B, R_*)$ be a regular marked nodal family of Riemann surfaces of type $(g, n)$. Then there exists a regular marked nodal family of Riemann surfaces $(\tilde{\pi} : \tilde{\Sigma} \to \Sigma, \tilde{R}_*)$ of type $(g, n+1)$ together with a holomorphic map $\hat{\pi} : \tilde{\Sigma} \to \Sigma$ with the following properties:*

$$\begin{tikzcd} \tilde{\Sigma} \arrow[r, "\hat{\pi}"] \arrow[d, "\tilde{\pi}"'] \arrow[dr, phantom, "\circlearrowleft"] & \Sigma \arrow[d, "\pi"] \\ \Sigma \arrow[r, "\pi"] & B \end{tikzcd}$$

*commutes. Also, let $(S, j, r_*, \nu), b \in B, \hat{\iota} : S \to \Sigma$ be a desingularisation of $\Sigma_b$ and let $(\tilde{S}, \tilde{j}, \tilde{r}_*, \tilde{\nu})$ be a stable marked nodal Riemann surface of type $(g, n+1)$ s. t. $(S, j, r_*, \nu)$ is obtained from $(\tilde{S}, \tilde{j}, \tilde{r}_*, \tilde{\nu})$ by forgetting the last marked point and stabilising. Let $\kappa : S \to \tilde{S}$ be the resulting embedding. Then there exists a unique $z \in \Sigma_b$ and a unique $\tilde{\iota} : \tilde{S} \to \tilde{\Sigma}_z \subseteq \tilde{\Sigma}$ s. t. $(\tilde{S}, \tilde{j}, \tilde{r}_*, \tilde{\nu}), z \in \Sigma, \tilde{\iota} : \tilde{S} \to \tilde{\Sigma}$ is a desingularisation and $\hat{\pi} \circ \tilde{\iota} \circ \kappa = \hat{\iota}$:*

$$\begin{tikzcd} \tilde{S} \arrow[r, "\tilde{\iota}"] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{\Sigma} \arrow[d, "\hat{\pi}"] \\ S \arrow[u, "\kappa"] \arrow[r, "\hat{\iota}"] & \Sigma. \end{tikzcd}$$

*The stratification by signature on $\Sigma$ as base space of the marked nodal family $(\tilde{\pi} : \tilde{\Sigma} \to \Sigma, \tilde{R}_*)$ is given in the following way: For every stratum $C \subseteq B$ of the stratification by signature on $B$ consider the following subsets of $\pi^{-1}(C)$: The complement of the markings and nodes in $\pi^{-1}(C)$, for every marking $R_i$ the subset $R_i(C)$ and the connected components of the set of nodes in $\pi^{-1}(C)$. In particular, the restriction of $\pi$ to each of these is a submersion onto $C$.*
*If $(\pi : \Sigma \to B, R_*)$ is a local universal family or defines an orbifold branched covering of $\overline{M}_{g,n}$, then so does $(\tilde{\pi} : \tilde{\Sigma} \to \Sigma, \tilde{R}_*)$ (of $\overline{M}_{g,n+1}$).*

*Proof.* Let $(\pi : \Sigma \to B, R_*)$ be as in the statement of the lemma. The goal is to show that for every $z \in \Sigma$ there exists a neighbourhood $U' \subseteq \Sigma$ of $z$ that is the domain of a (nodal) coordinate system as in Definition II.3, 1. and is also the base of a marked nodal family of the type indicated in the statement of the lemma. I will only indicate the definitions of $\tilde{\Sigma}$, $\tilde{\pi}$, $\hat{\pi}$ and the $\tilde{R}_i$, which are a variation of the constructions in the proof of Theorem 5.6 in [RS06]. The smooth structure on $\tilde{\Sigma}$ is then also defined analogously to the smooth structures

defined in that reference and the other properties of $\hat{\pi}$ follow from the remarks in 4. preceeding Definition II.8.
The statements about local universal families and orbifold branched coverings then follow because applying the construction below to the explicit local models from Construction II.1 produces again one of those local models.

If $z \in \Sigma_b$ is not one of the marked or nodal points, let $U' \subseteq \Sigma$ be a neighbourhood of $z$ disjoint from all the marked or nodal points and s. t. $\pi|_{U'} : U' \to B$ is a holomorphic submersion onto $B$. Define $\tilde{\Sigma}|_{U'} := (\pi|_{U'})^*\Sigma$, $\tilde{\pi}|_{\tilde{\Sigma}|_{U'}} : \tilde{\Sigma}|_{U'} \to U'$ is the canonical projection, $\tilde{R}_i := (\pi|_{U'})^* R_i$ for $i = 1, \dots, n$ and $\tilde{R}_{n+1}(z') := z' \in \Sigma_{\pi(z')} = \tilde{\Sigma}_{z'}$. The restriction $\hat{\pi}|_{\tilde{\Sigma}|_{U'}} : \tilde{\Sigma}|_{U'} \to \Sigma$ is given by the canonical map $(\pi|_{U'})^*\Sigma \to \Sigma$.

If $z = R_l(b)$ for some $l \in \{1, \dots, n\}$, then there exists a neighbourhood $U' \subseteq \Sigma$ of $z$ that does not contain any nodal points or marked points aside from those of the form $R_l(b')$ for $b' \in B$. Also, as in Construction II.1, 4., one can assume that there exists a holomorphic function $x' : U' \to \mathbb{D}$ s. t. $(\pi|_{U'}, x') : U' \to B \times \mathbb{D}$ is a holomorphic coordinate system on $U'$ and that $x'(R_l) = \{0\}$. Define $U := (\pi|_{U'})^* U' \subseteq (\pi|_{U'})^*\Sigma$ and $x := x' \circ \Phi : U \to \mathbb{D}$, where $\Phi : U \to U'$ is the canonical bundle map covering $\pi|_{U'} : U' \to B$. Consider $V := U' \times \mathbb{D} \subseteq U' \times S^2$ and the function $y : V \to \mathbb{D}$ given by projection onto the second factor. Let $q_1 : (\pi|_{U'})^*\Sigma \to U'$, $q_2 : U' \times S^2 \to U'$ be the projections and let $K := \{\xi \in U \mid |x(\xi)| \leq |x'(q_1(\xi))| \neq 0\}$, $L := \{\xi' \in V \mid |y(\xi')| \leq |x'(q_2(\xi))| \neq 0\}$. Denoting $\hat{\Sigma}_1 := (\pi|_{U'})^*\Sigma \setminus K$, $\hat{\Sigma}_2 := (U' \times S^2) \setminus L$ one can define $\tilde{\Sigma}|_{U'} := \hat{\Sigma}_1 \amalg \hat{\Sigma}_2/_\sim$, where the equivalence relation is defined as in Construction II.1, 4. Namely $\xi \sim \xi'$ for $\xi \in U$, $\xi' \in V$ with $q_1(\xi) = q_2(\xi')$ and either $x(\xi)y(\xi') = x'(q_1(\xi)) \neq 0$ or $x(\xi) = y(\xi') = x'(q_1(\xi)) = 0$.
The projection $\tilde{\pi}|_{\tilde{\Sigma}|_{U'}} : \tilde{\Sigma}|_{U'} \to U'$ is then induced by the map $q_1 \amalg q_2 : \hat{\Sigma}_1 \amalg \hat{\Sigma}_2 \to U'$.
The markings $\tilde{R}_i$ for $i \in \{1, \dots, n\} \setminus \{l\}$ are defined by $\tilde{R}_i := (\pi|_{U'})^* R_i \subseteq (\pi|_{U'})^*\Sigma \setminus U \subseteq \tilde{\Sigma}|_{U'}$. $\tilde{R}_l := U' \times \{\infty\} \subseteq U' \times S^2 \setminus V \subseteq \tilde{\Sigma}|_{U'}$ and $\tilde{R}_{n+1} := U' \times \{1\} \subseteq U' \times S^2 \setminus V \subseteq \tilde{\Sigma}|_{U'}$.
The restriction $\hat{\pi}|_{\tilde{\Sigma}|_{U'}} : \tilde{\Sigma}|_{U'} \to \Sigma$ is given as follows: On $(\pi|_{U'})^*\Sigma \setminus U$, $\hat{\pi}|_{\tilde{\Sigma}|_{U'}}$ is given by the canonical morphism $(\pi|_{U'})^*\Sigma \to \Sigma$. To define $\hat{\pi}|_{\tilde{\Sigma}|_{U'}}$ on the remaining part of $\tilde{\Sigma}|_{U'}$, let $\zeta \in U'$. If $x'(\zeta) = 0$, $\tilde{\Sigma}_\zeta$ is the union of $\Sigma_{\pi(\zeta)}$ with $S^2$, with $R_l(\pi(\zeta)) \in \Sigma_{\pi(\zeta)}$ and $0 \in S^2$ identified. Let $\hat{\pi}|_{\tilde{\Sigma}_\zeta}$ be the identity on $\Sigma_{\pi(\zeta)}$ and on $S^2$ the constant map to $R_l(\pi(\zeta))$. If $x'(\zeta) \neq 0$, $\tilde{\Sigma}_\zeta$ is given by the union of $\Sigma_{\pi(\zeta)} \setminus \{z' \in \Sigma_{\pi(\zeta)} \mid |x'(z')| \leq |x'(\zeta)|\}$ with $S^2 \setminus \{z' \in S^2 \mid |z'| \leq |x'(\zeta)|\}$, where $w \in \mathbb{D} \setminus \{z' \in S^2 \mid |z'| \leq |x'(\zeta)|\} \subseteq S^2 \setminus \{z' \in S^2 \mid |z'| \leq |x'(\zeta)|\}$ is identified with $(x'|_{U'_{\pi(\zeta)}})^{-1}\left(\frac{x'(\zeta)}{w}\right)$, where $U'_{\pi(\zeta)} := U' \cap \Sigma|_{\pi(\zeta)}$. Let $\hat{\pi}|_{\tilde{\Sigma}|_{U'}}$ be the identity on $\Sigma_{\pi(\zeta)} \setminus \{z' \in \Sigma_{\pi(\zeta)} \mid |x'(z')| \leq |x'(\zeta)|\}$ and on $S^2 \setminus \{z' \in S^2 \mid |z'| \leq |x'(\zeta)|\}$ be given by the map $w \mapsto (x'|_{U'_{\pi(\zeta)}})^{-1}\left(\frac{x'(\zeta)}{w}\right)$. This is then a well-defined holomorphic diffeomorphism that maps $\infty \in S^2$ to $(x'|_{U'_{\pi(\zeta)}})^{-1}(0) = R_l(\pi(\zeta))$ and $1 \in S^2$ to $(x'|_{U'_{\pi(\zeta)}})^{-1}(x'(\zeta)) = \zeta$.

Finally, for $z$ one of the nodes, let the notation be as in the remark just before the statement of the lemma and assume w. l. o. g. that $z = n_1$. Denote $U' := U_1$, $(x, y) := (x_1, y_1) : U' \to \mathbb{D}^2$, $z' := z_1 : V \to \mathbb{D}$. Let $C_i := C \cap U_i$, $C' := C_1$. Note that $(\pi|_{U'})^*(\Sigma \setminus C')$ is a well-defined complex manifold and the projection onto $U'$ at every point is either a holomorphic submersion or has a neighbourhood that is the domain of nodal coordinates as in II.2. I. e. $(\pi|_{U'})^*(\Sigma \setminus C') \to U'$ satisfies the definition of a marked nodal family of Riemann surfaces, apart from the properness condition and the fibres are punctured marked nodal surfaces instead of marked nodal surfaces. This is clear away from the subsets $(\pi|_{U'})^*C_i$, for the projection $\pi$ is a submersion away from the nodes. In a neighbourhood of one of the $(\pi|_{U'})^*C_i$ for $i \geq 2$, $i = 2$, say, w. r. t. the coordinates described before the statement of the lemma, an explicit description of $(\pi|_{U'})^*(\Sigma \setminus C') \to U'$ is the following: $\pi|_{U'} : U' \to V$ in coordinates is the map $f_1 : \mathbb{D}^{k+1} \to \mathbb{D}^k$, $(x, y, z_2, \ldots, z_k) \mapsto (xy, z_2, \ldots, z_k)$, $k := \dim_{\mathbb{C}}(B)$, whereas $\pi|_{U_2} : U_2 \to V$ in coordinates is the map $f_2 : \mathbb{D}^{k+1} \to \mathbb{D}^k$, $(z_1, x_2, y_2, z_3, \ldots, z_k) \mapsto (z_1, x_2 y_2, z_3, \ldots, z_k)$. The pullback of the latter by the former hence explicitely is given by the map with domain

$$\begin{aligned}\{(w_1, w_2) &\in \mathbb{D}^{k+1} \times \mathbb{D}^{k+1} \mid f_1(w_1) = f_2(w_2)\} \\ &= \{((x, y, x_2 y_2, z_3, \ldots, z_k), (xy, x_2, y_2, z_3, \ldots, z_k)) \in \mathbb{D}^{k+1} \times \mathbb{D}^{k+1} \mid \\ &\qquad\qquad (x, y, x_2, y_2, z_3, \ldots, z_k) \in \mathbb{D}^{k+2}\} \cong \mathbb{D}^{k+2}\end{aligned}$$

and projection given by $(x, y, x_2, y_2, z_3, \ldots, z_k) \mapsto (x, y, x_2 y_2, z_3, \ldots, z_k)$. Now to turn $(\pi|_{U'})^*(\Sigma \setminus C')$ into a marked nodal family, work in the local coordinates as before and consider the subset

$$\begin{aligned}K := \{(\zeta, z') \in U' \times S^2 \mid x(\zeta) \neq 0, |z'| \leq |x(\zeta)|\} \\ \cup \{(\zeta, z') \in U' \times S^2 \mid y(\zeta) \neq 0, |z'| \geq \frac{1}{|y(\zeta)|}\}\end{aligned}$$

of $U' \times S^2$. Then $\tilde{\Sigma}|_{U'} := (\pi|_{U'})^*(\Sigma \setminus C') \amalg (U' \times S^2 \setminus K)/_\sim$, where the equivalence relation is defined as follows:
If $\zeta \in U'$ has coordinates $(x(\zeta), y(\zeta), z_1(\zeta), \ldots, z_k(\zeta))$ with $x(\zeta) \neq 0$, $y(\zeta) \neq 0$, then $z' \in \{\zeta\} \times S^2 \setminus K$ is identified with the point on $((\pi|_{U'})^*(\Sigma \setminus C'))_\zeta = \Sigma_{\pi(\zeta)} \setminus C'$ with coordinates $\left(\frac{x(\zeta)}{z'}, z' y(\zeta), z_2(\zeta), \ldots, z_k(\zeta)\right)$.
If $\zeta \in U'$ has coordinates $(x(\zeta), y(\zeta), z_1(\zeta), \ldots, z_k(\zeta))$ with $x(\zeta) = 0$, $y(\zeta) \neq 0$, then $z' \in \{\zeta\} \times S^2 \setminus K$ with $z' \neq 0$ is identified with the point on $((\pi|_{U'})^*(\Sigma \setminus C'))_\zeta = \Sigma_{\pi(\zeta)} \setminus C'$ with coordinates $(0, z' y(\zeta), z_2(\zeta), \ldots, z_k(\zeta))$.
Analogously, if $\zeta \in U'$ has coordinates $(x(\zeta), y(\zeta), z_1(\zeta), \ldots, z_k(\zeta))$ with $x(\zeta) \neq 0$, $y(\zeta) = 0$, then $z' \in \{\zeta\} \times S^2 \setminus K$ with $z' \neq \infty$ is identified with the point on $((\pi|_{U'})^*(\Sigma \setminus C'))_\zeta = \Sigma_{\pi(\zeta)} \setminus C'$ with coordinates $\left(\frac{x(\zeta)}{z'}, 0, z_2(\zeta), \ldots, z_k(\zeta)\right)$.
Finally, if $\zeta \in U'$ has coordinates $(0, 0, z_1(\zeta), \ldots, z_k(\zeta))$, then no identification takes place.
The projection $\tilde{\pi}|_{\tilde{\Sigma}|_{U'}} : \tilde{\Sigma}|_{U'} \to U'$ is induced by the projections $(\pi|_{U'})^*(\Sigma \setminus C') \to U'$ and $U' \times S^2 \to U'$.

The markings $\tilde{R}_i$ for $i = 1, \dots, n$ are given by the images of the pullbacks $(\pi|_{U'})^* R_i$ under the projection to the quotient $\tilde{\Sigma}|_{U'}$ and $\tilde{R}_{n+1}$ is the image of $U' \times \{1\}$ under the projection to $\tilde{\Sigma}|_{U'}$.
The restriction $\hat{\pi}|_{\tilde{\Sigma}|_{U'}} : \tilde{\Sigma}|_{U'} \to \Sigma$ to the image of $(\pi|_{U'})^*(\Sigma \setminus C')$ is given by the canonical map to $\Sigma$. This covers all of $\tilde{\Sigma}|_{U'}$ apart from the points $\{(\zeta, 0) \in U' \times S^2 \mid x(\zeta) = 0\}$, $\{(\zeta, \infty) \in U' \times S^2 \mid y(\zeta) = 0\}$ and $\{(\zeta, z') \in U' \times S^2 \mid x(\zeta) = y(\zeta) = 0\}$. Each such point $(\zeta, z')$ is mapped to the point $\Sigma_{\pi(\zeta)} \cap C'$ in $\Sigma_{\pi(\zeta)}$.
Note that by construction, under $\hat{\pi}|_{\tilde{\Sigma}|_{U'}}$, the point corresponding to $(\zeta, 1) \in U' \times S^2$, i. e. $\tilde{R}_{n+1}(\zeta)$ is mapped to $\zeta$.
The important thing here is the following: In the notation from before, locally the projection in a neighbourhood of the first node looks like the map $f_1 : \mathbb{D}^{k+1} \to \mathbb{D}^k$, $(x, y, z_2, \dots, z_k) \mapsto (xy, z_2, \dots, z_k)$, and analogously for the other nodes. In these local coordinates, the pullback of $f_i$ for $i \geq 2$ by $f_1$ gave a well-defined nodal coordinate system. But for $i = 1$ this is not the case, because both the subset $\{x = 0\}$ and the subset $\{y = 0\}$ get mapped to $\{0\} \times \mathbb{D}^{k-1}$, the stratum along which the first node perseveres. So the set of nodes in the naive pullback of $f_1$ by itself would have a set of nodes that looks like two hyperplanes intersecting transversely at the origin, which is not a submanifold, hence there can't exist a nodal coordinate system at this intersection.
The construction above then "resolves" this intersection by inserting a sphere, producing two different nodes at $(0, 0, z_2, \dots, z_k)$, one corresponding to the one which perseveres along $(0, y, z_2, \dots, z_k)$, the other to the one that perseveres along $(x, 0, z_2, \dots, z_k)$. □

As long as one does not impose any compactness condition, the existence of a local universal family s. t. the induced map to $\overline{M}_{g,n}$ is surjective is shown in [RS06], Proposition 6.3. In the genus $g = 0$ case, one can also find such a family even with compact base space, for $\overline{M}_{0,n}$ itself is a complex manifold. In the case of genus $g > 0$, such a result will not hold true. But one can ask instead for the existence of a marked nodal family $(\pi : \Sigma \to B, R_*)$ that defines an orbifold branched covering of $\overline{M}_{g,n}$ that branches over the Deligne-Mumford boundary, maps $B$ surjectively onto $\overline{M}_{g,n}$ and has a compact base space $B$. By the previous lemma, if one can show such a result for Riemann surfaces of type $(g, n)$, then the result also holds for all $(g, n')$ with $n' \geq n$. First results in this direction were proved by Looijenga in [Loo94], where it is shown that $\overline{M}_g$ has a finite branched covering by a smooth projective variety. The difference to the result that I would like to use here is that this covering morphism does not come from a marked nodal family (which requires in particular the total space $\Sigma$ to be smooth), which is not the case for the branched covering constructed in [Loo94]. But, although the construction in [Loo94] doesn't produce the desired result, see Proposition 1.4 in [BP00], there seems (to the author's limited understanding of algebraic geometry) to be a generalisation of that construction, see Theorem 3.9 in op. cit. This shows, in conjunction with the previous lemma, i. e. apply Theorem 3.9 in [BP00] to get the marked nodal family $(\pi : \Sigma \to M, R_*)$ below and then apply the previous Lemma to get the families $(\pi^\ell : \Sigma^\ell \to M^\ell, R^\ell_*, T^\ell_*)$

for $\ell \geq 1$, the following conjecturally stated result.

**Proposition II.1.** *There exists a sequence of marked nodal families* $(\pi^\ell : \Sigma^\ell \to M^\ell, R_*^\ell, T_*^\ell)$ *for* $\ell \geq 0$ *of Riemann surfaces of type* $(g, n+\ell)$, *with markings* $R_1^\ell, \ldots, R_n^\ell$, $T_1^\ell, \ldots, T_\ell^\ell$ *s. t.* $\Sigma^\ell = M^{\ell+1}$ *for all* $\ell \geq 0$, *together with maps* $\hat{\pi}^\ell : \Sigma^{\ell+1} \to \Sigma^\ell$ *s. t.*

$$\begin{tikzcd}
\cdots \arrow[r, "\hat\pi^{\ell+1}"] & \Sigma^{\ell+1} \arrow[r, "\hat\pi^{\ell}"] \arrow[d, "\pi^{\ell+1}"] & \Sigma^{\ell} \arrow[r, "\hat\pi^{\ell-1}"] \arrow[d, "\pi^{\ell}"] & \Sigma^{\ell-1} \arrow[r, "\hat\pi^{\ell-2}"] \arrow[d, "\pi^{\ell-1}"] & \cdots \arrow[r, "\hat\pi^{1}"] & \Sigma^{1} \arrow[r, "\hat\pi^{0}"] \arrow[d, "\pi^{1}"] & \Sigma^{0} \arrow[r, equal] \arrow[d, "\pi^{0}"] & \Sigma \arrow[d, "\pi"] \\
\cdots \arrow[r, "\pi^{\ell+1}"] & M^{\ell+1} \arrow[r, "\pi^{\ell}"] & M^{\ell} \arrow[r, "\pi^{\ell-1}"] & M^{\ell-1} \arrow[r, "\pi^{\ell-2}"] & \cdots \arrow[r, "\pi^{1}"] & M^{1} \arrow[r, "\pi^{0}"] & M^{0} \arrow[r, equal] & M
\end{tikzcd}$$

$$\begin{tikzcd}
\Sigma^\ell \arrow[r, "\hat\pi^{\ell-1}"] \arrow[d, "\pi^\ell"'] & \Sigma^{\ell-1} \arrow[d, "\pi^{\ell-1}"] \\
M^\ell \arrow[r, "\pi^{\ell-1}"] \arrow[u, bend left, "R_j^\ell"] & M^{\ell-1} \arrow[u, bend right, "R_j^{\ell-1}"']
\end{tikzcd} \quad \forall\, j = 1, \ldots, n$$

$$\begin{tikzcd}
\Sigma^\ell \arrow[r, "\hat\pi^{\ell-1}"] \arrow[d, "\pi^\ell"'] & \Sigma^{\ell-1} \arrow[d, "\pi^{\ell-1}"] \\
M^\ell \arrow[r, "\pi^{\ell-1}"] \arrow[u, bend left, "T_j^\ell"] & M^{\ell-1} \arrow[u, bend right, "T_j^{\ell-1}"']
\end{tikzcd} \quad \forall\, j = 1, \ldots, \ell-1$$

*all commute,*

$$\hat{\pi}^{\ell-1} \circ T_\ell^\ell = \mathrm{id} : \Sigma^{\ell-1} \to M^\ell$$

*and where* $M$ *is assumed to be closed, and hence so are the* $M^\ell$ *for all* $\ell \geq 0$. *Furthermore, for all* $\ell \geq 0$, $(\pi^\ell : \Sigma^\ell \to M^\ell, R_*^\ell, T_*^\ell)$ *defines an orbifold branched covering of* $\overline{M}_{g,n+\ell}$ *that branches over the Deligne-Mumford boundary and for every* $z \in \Sigma^\ell = M^{\ell+1}$, *putting* $b := \pi^\ell(z) \in M^\ell$, *the map*

$$\begin{aligned}\hat{\pi}_z^{\ell-1} : (\Sigma_z^\ell, R_1^\ell(z), \ldots, R_n^\ell(z), T_1^\ell(z), \ldots, T_{\ell-1}^\ell(z)) \to \\ \to (\Sigma_b^{\ell-1}, R_1^{\ell-1}(b), \ldots, R_n^{\ell-1}(b), T_1^{\ell-1}(b), \ldots, T_{\ell-1}^{\ell-1}(b))\end{aligned}$$

*is stabilising, i. e. biholomorphic on every stable component of*

$$(\Sigma_z^\ell, R_1^\ell(z), \ldots, R_n^\ell(z), T_1^\ell(z), \ldots, T_{\ell-1}^\ell(z))$$

*and constant on every unstable component (of which there is at most one). For* $\ell > k$ *denote the compositions*

$$\hat{\pi}_k^\ell := \hat{\pi}^k \circ \hat{\pi}^{k+1} \circ \cdots \circ \hat{\pi}^{\ell-1} : \Sigma^\ell \to \Sigma^k$$

*and*

$$\pi_k^\ell := \pi^k \circ \pi^{k+1} \circ \cdots \circ \pi^{\ell-1} : M^\ell \to M^k.$$

*If* $M_i^\ell$ *is a stratum of the stratification of* $M^\ell$ *by signature, then for* $k \leq \ell$ *there exists a signature* $j(i)$ *s. t.* $\pi_k^\ell|_{M_i^\ell} : M_i^\ell \to M_{j(i)}^k$ *is a submersion.*

**Definition II.9.** In the notation of the proposition above, a component of $\Sigma^\ell_b$, for any $b \in M^\ell$ and $\ell \in \mathbb{N}$, that is mapped to a point under $\hat{\pi}^\ell_0$ is called a *ghost component.*

On the spaces in Proposition II.1 there are also canonical actions of permutation groups of the last $\ell$ markings, as follows directly from the construction of the spaces $\Sigma^\ell$, $M^\ell$ and maps $\pi^\ell$, $\hat{\pi}^\ell$.

**Proposition II.2.** *In the notation of the previous proposition:*
*For $\ell \geq 1$, let $\mathcal{S}_\ell$ be the group of permutations $\{1,\dots,\ell\}$. Then there exist actions $\sigma^\ell$ and $\hat{\sigma}^\ell$ of $\mathcal{S}_\ell$ on $M^\ell$ and $\Sigma^\ell$ s. t.*

$$\begin{tikzcd} \mathcal{S}_\ell \times \Sigma^\ell \arrow[r, "\hat{\sigma}^\ell"] \arrow[d, "\mathrm{id}\times\pi^\ell"'] & \Sigma^\ell \arrow[d, "\pi^\ell"] \\ \mathcal{S}_\ell \times M^\ell \arrow[r, "\sigma^\ell"] & M^\ell \end{tikzcd} \tag{II.3}$$

*commutes.*
*Furthermore, for any $g \in \mathcal{S}_\ell$ and $k \in \{1,\dots,\ell\}$,*

$$\hat{\sigma}^\ell_{g^{-1}} \circ T^\ell_k \circ \sigma^\ell_g = T^\ell_{g(k)} \tag{II.4}$$

*and under the inclusion $\mathcal{S}_\ell \subseteq \mathcal{S}_{\ell+1}$ as permutations of $\{1,\dots,\ell+1\}$ leaving $\ell+1$ fixed and the identification $\Sigma^\ell = M^{\ell+1}$,*

$$\begin{tikzcd} \Sigma^{\ell+1} \arrow[r, "\hat{\sigma}^{\ell+1}_g"] \arrow[d, "\hat{\pi}^\ell"'] & \Sigma^{\ell+1} \arrow[d, "\hat{\pi}^\ell"] \\ \Sigma^\ell \arrow[r, "\hat{\sigma}^\ell_g"] & \Sigma^\ell \end{tikzcd} \qquad \begin{tikzcd} M^{\ell+1} \arrow[r, "\sigma^{\ell+1}_g"] \arrow[d, "\pi^\ell"'] & M^{\ell+1} \arrow[d, "\pi^\ell"] \\ M^\ell \arrow[r, "\sigma^\ell_g"] & M^\ell \end{tikzcd} \tag{II.5}$$

*commute and*

$$\hat{\sigma}^\ell = \sigma^{\ell+1}|_{\mathcal{S}_\ell \times M^{\ell+1}} : \mathcal{S}_\ell \times \Sigma^\ell \to \Sigma^\ell. \tag{II.6}$$

*Denoting by $\tau_{\ell,\ell+1} \in \mathcal{S}_{\ell+1}$ the transposition exchanging $\ell$ and $\ell+1$,*

$$\hat{\pi}^{\ell-1} = \pi^\ell \circ \sigma^{\ell+1}_{\tau_{\ell,\ell+1}} : M^{\ell+1} = \Sigma^\ell \to M^\ell = \Sigma^{\ell-1}. \tag{II.7}$$

*Proof.* Again, not a complete proof, just a description of the construction of the actions.
Actually, the above characterisation serves at the same time as definition of these actions by induction: Because $\mathcal{S}_1 = \{\mathrm{id}\}$, the actions on $M^1$ and $\Sigma^1$ are automatically the identity. Now assume that the actions of $\mathcal{S}_k$ for $k = 1,\dots,\ell$ have been defined. Equation II.6 defines the restriction of $\sigma^{\ell+1}$ to $\mathcal{S}_\ell \times M^{\ell+1}$. Equation II.4 requires that, for $g \in \mathcal{S}_\ell$, i.e. $g(\ell+1) = \ell+1$, and $b \in M^{\ell+1}$, $\hat{\sigma}^{\ell+1}_g(T^{\ell+1}_{\ell+1}(b)) = T^{\ell+1}_{\ell+1}(\sigma^{\ell+1}_g(b))$. Equation II.5 then gives $\hat{\sigma}^\ell_g \circ \hat{\pi}^\ell(T^{\ell+1}_{\ell+1}(b)) = \hat{\pi}^\ell(T^{\ell+1}_{\ell+1}(\sigma^{\ell+1}_g(b))) \in \Sigma^\ell_{\pi^\ell(b)}$. Defining $z := \hat{\pi}^\ell(T^{\ell+1}_{\ell+1}(b)) \in \Sigma^\ell_{\pi^\ell(b)}$, this shows that $\Sigma^\ell_{\pi^\ell(b)}$ is obtained from $\Sigma^{\ell+1}_b$ by forgetting the last marked point and stabilising

associated to the point $z$ and $\Sigma^\ell_{\pi^\ell(\sigma^{\ell+1}_g(b))}$ is obtained from $\Sigma^{\ell+1}_{\sigma^{\ell+1}_g(b)}$ by forgetting the last marked point and stabilising associated to the point $\hat{\sigma}^\ell_g(z)$. Lemma II.1 then gives a unique lift $\hat{\sigma}^{\ell+1}_g : \Sigma^{\ell+1}_b \to \Sigma^{\ell+1}_{\sigma^{\ell+1}_g(b)}$ of $\hat{\sigma}^\ell_g : \Sigma^\ell_{\pi^\ell(b)} \to \Sigma^\ell_{\sigma^\ell_g(\pi^\ell(b))}$ that maps $T^{\ell+1}_{\ell+1}(b)$ to $T^{\ell+1}_{\ell+1}(\sigma^{\ell+1}_g(b))$.
Because $\mathcal{S}_{\ell+1}$ is generated by $\mathcal{S}_\ell$ and $\tau_{\ell,\ell+1}$, it suffices to define $\sigma^{\ell+1}_{\tau_{\ell,\ell+1}}$ and $\hat{\sigma}^{\ell+1}_{\tau_{\ell,\ell+1}}$. Now $M^{\ell+1}$ by definition in Lemma II.1 is the union of $(\Sigma^\ell \times_{\pi^\ell, M^\ell, \pi^\ell} \Sigma^\ell) \setminus C$, where $C$ is the diagonal in this fibre product over the nodes and markings in $\Sigma^\ell$, with a collection of spheres. The action of $\sigma^{\ell+1}_{\tau_{\ell,\ell+1}}$ is then the one induced by the action on $\Sigma^\ell \times_{\pi^\ell, M^\ell, \pi^\ell} \Sigma^\ell$ exchanging the factors, and the identity on the spheres filling in $C$.
$\hat{\sigma}^{\ell+1}_{\tau_{\ell,\ell+1}}$ is then defined analogously to $\hat{\sigma}^{\ell+1}_g$ for $g \in \mathcal{S}_\ell$ before. □

The compactness statement in Proposition II.1 is important for the following reason: In the genus $g = 0$ case, $\overline{M}_{0,n}$ is a compact complex manifold, hence in particular it is oriented and carries a fundamental class in its top homology group (with any coefficient group). Hence any smooth (or continuous) map from $\overline{M}_{0,n}$ to another manifold defines a homology class in that manifold. Now in the case of positive genus this holds no longer true for $\overline{M}_{g,n}$ itself. But for any universal marked nodal family $(\pi : \Sigma \to M, R)$ of type $(g, n)$, $\overline{M}_{g,n}$ is the quotient space of the associated groupoid as in Definition 6.4 in [RS06]. As such both $\overline{M}_{g,n}$ as its quotient space and $M$ as the space of objects of this groupoid carry a stratification by orbit type, see [PPT10], esp. Section 5. A stratum of $M$ in this stratification is a connected component of an equivalence class of the relation on $M$ given by abstract isomorphism of automorphism groups. The stratification on $\overline{M}_{g,n}$ is then the one induced by the quotient map. Since the morphisms of the associated groupoid are given by isomorphisms of nodal surfaces, this stratification respects the stratification by signature. Now let $(\pi : \Sigma \to M, R_*)$ be a marked nodal family of Riemann surfaces of type $(g, n)$ with $M$ closed and s. t. the induced map $\upsilon : M \to \overline{M}_{g,n}$ defines an orbifold branched covering that branches over the Deligne-Mumford boundary. Let $\mathring{M}$ be the top-dimensional part of the stratification by signature, i. e. the set of those $b \in M$ s. t. $\Sigma_b$ is a smooth Riemann surface. Let correspondingly $M_{g,n}$ be the part of $\overline{M}_{g,n}$ consisting of the equivalence classes of smooth Riemann surfaces. Then $M_{g,n}$ is an orbifold, an orbifold structure (in the sense of Definition 2.4 in [RS06]) being defined by the restriction of the orbifold structure for $\overline{M}_{g,n}$ constructed in [RS06]. By definition, $\mathring{\upsilon} := \upsilon|_{\mathring{M}} : \mathring{M} \to M_{g,n}$ defines a (finite non-branched) orbifold covering. Defining $\mathring{\Sigma} := \Sigma|_{\mathring{M}}$, $\mathring{R}_* := R_*|_{\mathring{M}}$, one can hence form the associated groupoid to the marked family of Riemann surfaces $(\mathring{\Sigma} \to \mathring{M}, \mathring{R}_*)$ as in Definition 6.4 in [RS06], which defines a groupoid structure on $M_{g,n}$. $\upsilon$ as a branched orbifold covering is an open map and since $M$ is assumed compact, it is also a closed map. Since $\overline{M}_{g,n}$ is connected, the restriction of $\upsilon$ to every connected component of $M$ is surjective and one can assume w. l. o. g. that $M$ is connected as well. Since the complement of $\mathring{M}$ in $M$ consists of submanifolds of real codimension at least two, $\overset{\circ}{M}$ is then connected as well. Since the groupoid associated to $(\overset{\circ}{\Sigma} \to \overset{\circ}{M}, \overset{\circ}{R}_*)$, being complex is oriented, the stratification by orbit type on $\overset{\circ}{M}$ has a unique connected top-dimensional stratum and all other strata have codimension at least two in $\overset{\circ}{M}$. Denote this top-dimensional stratum by $\overset{\circ\circ}{M}$.

Because $M$ is compact, one can assign two well-defined numbers, $|\mathcal{O}(\overset{\circ\circ}{M})|$, the length of the orbit $\mathcal{O}(b)$ of any point $b \in \overset{\circ\circ}{M}$ (by compactness of $M$ this is a finite number) and $|\operatorname{Aut}(\overset{\circ\circ}{M})|$, the order of the automorphism group $\operatorname{Aut}(b)$ of any point $b \in \overset{\circ\circ}{M}$ (which is a finite number by properness of the groupoid, irrespective of whether $M$ is compact or not). With the help of these, to any map $f : \overline{M}_{g,n} \to X$, where $X$ is any manifold, s. t. $f \circ \pi^M_{M_{g,n}} : M \to X$ is smooth, $\pi^M_{M_{g,n}}$ being the quotient projection, one can assign a well-defined rational pseudocycle (as defined in Section 1 of [CM07])

$$\frac{1}{|\operatorname{Aut}(\overset{\circ\circ}{M})||\mathcal{O}(\overset{\circ\circ}{M})|} f \circ \pi^M_{M_{g,n}}|_{\overset{\circ\circ}{M}}.$$

# II.2 Spaces of sections

This section is preparatory in nature. Since it is also largely independent of the rest of the text, it is suggested to only read the necessary definitions in Subsection II.2.1 and then skip to the definitions of the Sobolev spaces of sections defined in Subsections II.2.3 and II.2.4.

## II.2.1 Riemannian submersions and the vertical exponential map

**Definition II.10.** A *Riemannian submersion* is a surjective submersion $\pi : W \to \Sigma$ between Riemannian manifolds $(W, g)$ and $(\Sigma, h)$ s. t. the distribution $HW := VW^\perp \subseteq TW$ (i. e. $H_wW := (V_wW)^\perp \subseteq T_wW$ for all $w \in W$) given by the orthogonal complement to the vertical distribution $VW = \ker D\pi$ has the property that $D\pi|_{HW} : HW \to T\Sigma$ is a fibrewise isometry and hence in particular defines the horizontal distribution of a connection on $\pi : W \to \Sigma$.
Denote by $\mathrm{pr}^{TW}_{VW} : TW \to VW$ the orthogonal projection along $HW$.
For $0 \leq k \leq \infty$, denote by $\Gamma^k(W)$ the space of sections of $\pi : W \to \Sigma$ of class $C^k$, $\Gamma(W) := \Gamma^\infty(W)$.

*Remark* II.3. Denote by $\pi^{TW}_W : TW \to W$ the tangent bundle projection and by $\pi^{VW}_W := \pi^{TW}_W|_{VW} : VW \to W$ that of $VW$. The latter then defines a Riemannian vector bundle equipped with a Riemannian connection: The metric is simply the restriction of the metric on $TW$ and the covariant derivative is given by $\nabla^\perp_X \xi := \mathrm{pr}^{TW}_{VW}(\nabla^W_X \xi)$, for $\xi \in \Gamma(VW)$, $X \in TW$, where on the right hand side the Levi-Civita covariant derivative on $W$ features. This gives a well-defined covariant derivative that is compatible with the metric: Linearity is immediate, the Leibniz rule follows from $\mathrm{pr}^{TW}_{VW}|_{VW} = \mathrm{id}$ and for $X \in TW$, $\xi, \zeta \in \Gamma(VW)$,

$$\begin{aligned} X\langle \xi, \zeta \rangle &= \langle \nabla^W_X \xi, \zeta \rangle + \langle \xi, \nabla^W_X \zeta \rangle \\ &= \langle \mathrm{pr}^{TW}_{VW}(\nabla^W_X \xi), \zeta \rangle + \langle \xi, \mathrm{pr}^{TW}_{VW}(\nabla^W_X \zeta) \rangle \\ &= \langle \nabla^\perp_X \xi, \zeta \rangle + \langle \xi, \nabla^\perp_X \zeta \rangle. \end{aligned}$$

It follows that for $u \in \Gamma(W)$, $u^*VW$ is a Riemannian vector bundle equipped with a Riemannian connection. The covariant derivative on sections $\Gamma(u^*VW) = \{\xi : \Sigma \to VW \mid \pi^{VW}_W \circ \xi = u\}$ is given by $\nabla_X \xi = \mathrm{pr}^{TW}_{VW}(\nabla^W_{Du(X)} \xi)$. This is well-defined (remember that $u$, being a section, is an embedding), for if $\gamma : (-\varepsilon, \varepsilon) \to \Sigma$ is a path with $\dot\gamma(0) = X$, then $u \circ \gamma$ is a path in $W$ with $\dot\gamma(0) = Du(X)$ and $\xi \circ \gamma$ defines a vector field along this path.

The main goal in the following is to define Sobolev spaces of sections of a Riemannian submersion $\pi : W \to \Sigma$. The rough guide to this is to define charts around smooth sections $u : \Sigma \to W$ of $\pi$ by considering the Sobolev space of sections of the pullback vector bundle $u^*VW$. The Riemannian structure is as defined above and the Sobolev class high enough for this Sobolev space to consist of continuous sections. These are mapped to $\Gamma^0(W)$ via the vertical exponential map $\exp^\perp : VW \to W$, which for $\xi \in V_wW$, $w \in W$, $\pi(w) = z \in \Sigma$, i. e. $\xi \in T_wW_z$, is given by the exponential map in the fibre $W_z$ with the induced metric. The main result to be proved for this to be well-defined is smoothness (or differentiability of class $C^k$ for some $k \geq 0$) of the transition functions.

**Definition II.11.** Let $\pi : W \to \Sigma$ be a Riemannian submersion. The *vertical exponential map* is the map

$$\begin{aligned} \exp^\perp : VW &\to W \\ \xi_w &\mapsto \exp_w^{W_{\pi(w)}}(\xi_w), \end{aligned}$$

where $\exp_w^{W_{\pi(w)}} : T_wW_{\pi(w)} = V_wW \to W_{\pi(w)} \subseteq W$ is the exponential map on the Riemannian manifold (with the induced metric as a submanifold of $W$) $W_{\pi(w)} = \pi^{-1}(\pi(w))$.

To compute the differentials of the transition functions mentioned above, one needs to consider the differential of this vertical exponential map, $D\exp^\perp : TVW \to TW$. The range $TW$ of this map, as a bundle over $W$, decomposes into the subbundles $HW$ and $VW$ given by the structure of a Riemannian submersion. The domain $TVW$ of this map, as a vector bundle over $VW$, decomposes into a number of subbundles, the decomposition being given by the connection on $VW$ and structure of Riemannian submersion in the following way: First, because the vector bundle $\pi_W^{VW} : VW \to W$ carries a connection, $TVW \cong HVW \oplus VVW \cong ((\pi_W^{VW})^*TW) \oplus ((\pi_W^{VW})^*VW)$. Second, because one has the decomposition $TW = HW \oplus VW$, one can further decompose $(\pi_W^{VW})^*TW \cong ((\pi_W^{VW})^*HW) \oplus ((\pi_W^{VW})^*VW)$ and hence

$$TVW \cong ((\pi_W^{VW})^*HW) \oplus ((\pi_W^{VW})^*VW) \oplus ((\pi_W^{VW})^*VW). \tag{II.8}$$

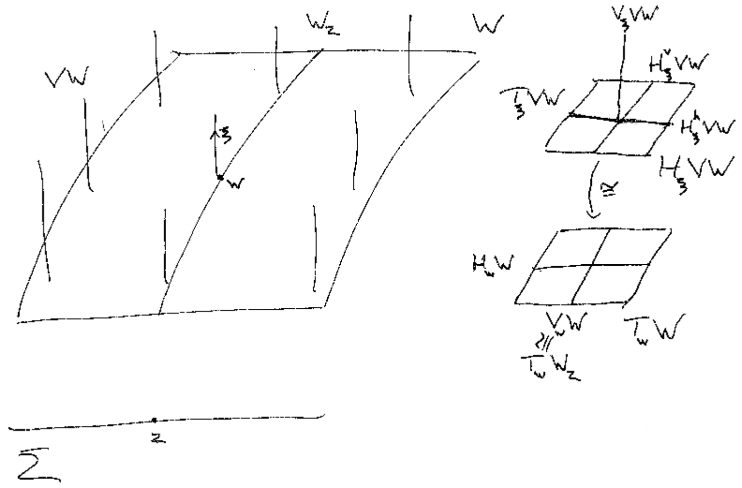
Denote the first of these summands by $H^{\mathrm{h}}HW$, the second one by $H^{\mathrm{v}}HW$, i. e.

$$\begin{aligned} H^{\mathrm{h}}HW &:= \{X \in HVW \mid (\pi_W^{VW})_*(X) \in HW\} \\ H^{\mathrm{v}}HW &:= \{X \in HVW \mid (\pi_W^{VW})_*(X) \in VW\}. \end{aligned}$$

For $z \in \Sigma$, one has the inclusion $W_z \hookrightarrow W$ which results in an inclusion $TW_z \hookrightarrow TW$, with image (by definition of $VW$) $VW|_{W_z} \subseteq TW$ and hence another inclusion $TTW_z \hookrightarrow TVW$. Now $TTW_z$, $W_z$ being a Riemannian manifold with the induced metric, and hence equipped with the Levi-Civita connection, decomposes into horizontal and vertical subspaces $HTW_z$ and $VTW_z$ as well. A basic result about Riemannian submersions, see e. g. [Sak96], Section II.6, Proposition 6.1, p. 75, shows that this decomposition coincides with the second and third summand of the above decomposition:

$$TVW|_{W_z} \cong H^{\mathrm{h}}VW|_{TW_z} \oplus HTW_z \oplus VTW_z, \tag{II.9}$$

or in other words, $TTW_z \cong H^{\mathrm{v}}VW|_{TW_z} \oplus VVW|_{TW_z}$ Also, the second and third summand above, under $D\exp^\perp : TVW \to TW \cong HW \oplus VW$, get mapped to




$VW$.
The reason this is relevant is the following: Let $u : \Sigma \to W$ be a smooth section and let $\xi \in \Gamma(u^*VW)$, i. e. $\xi : \Sigma \to VW$ with $\pi_W^{VW} \circ \xi = u$. Then one in particular wants to compute $D(\exp^\perp \circ \xi) = D\exp^\perp \circ D\xi : T\Sigma \to TW$. Now $D\xi : T\Sigma \to TVW$ with respect to decomposition II.8 is given by $(D^{\mathrm{h}}u, D^{\mathrm{v}}u, \nabla_{\cdot}\xi)$, where $D^{\mathrm{h}}u := \mathrm{pr}^{TW}_{HW} \circ Du$ is given by horizontal lift from $T\Sigma$ to $TW$, because $u$ is a section, and $D^{\mathrm{v}}u := \mathrm{pr}^{TW}_{VW} \circ Du$. That this decomposition coincides with decomposition II.9 means that for $X \in T_z\Sigma$, to evaluate $D\exp^\perp$ on $D\xi(X) = (D^{\mathrm{h}}u(X), D^{\mathrm{v}}u(X), \nabla_X\xi)$, one can regard $(D^{\mathrm{v}}u(X), \nabla_X\xi) \in HTW_z \oplus VTW_z = TTW_z$, where $W_z$ is regarded as a Riemannian manifold by itself and evaluate the differential of its exponential map on this vector.
Also, because $\exp^\perp : VW \to W$ is a fibrewise map, i. e. $\pi \circ \exp^\perp = \pi \circ \pi_W^{VW} : VW \to \Sigma$, for $z \in \Sigma$, $w \in W$ with $\pi(w) = z$ and $\xi \in VW$ with $\pi_W^{VW}(\xi) = w$, if $X \in T_z\Sigma$ with horizontal lift $\tilde{X}_w \in H_wW$ and further horizontal lift $\tilde{\tilde{X}}_\xi \in H^{\mathrm{h}}VW$ as well as horizontal lift $\tilde{X}_{\exp^\perp(\xi)} \in H_{\exp^\perp(\xi)}W$, then $\mathrm{pr}^{TW}_{VW}\left(D\exp^\perp(\tilde{\tilde{X}}_\xi)\right) = \tilde{X}_{\exp^\perp(\xi)}$.
To summarise:

**Lemma II.2.** *Let $\pi : W \to \Sigma$ be a Riemannian submersion and let $z \in \Sigma$. Then under the decompositions*

$$\begin{aligned} TVW|_{TW_z} &\cong H^{\mathrm{h}}VW|_{TW_z} \oplus H^{\mathrm{v}}VW|_{TW_z} \oplus VVW|_{TW_z} \\ &\cong H^{\mathrm{h}}VW|_{TW_z} \oplus HTW_z \oplus VTW_z \\ &\cong H^{\mathrm{h}}VW|_{TW_z} \oplus TTW_z \end{aligned}$$

*and*

$$TW \cong HW \oplus VW,$$

*the differential*

$$D\exp^\perp : TVW \to TW$$

*of the vertical exponential map* $\exp^\perp : VW \to W$ *satisfies* $D\exp^\perp|_{TTW_z} \subseteq VW|_{W_z} \cong TW_z$ *and*

$$D\exp^\perp|_{TTW_z} : TTW_z \to TW_z$$

*is given by the differential of the exponential map*

$$\exp^{W_z} : TW_z \to W_z$$

*on the Riemannian manifold* $W_z$ *(with the induced metric).*
*Furthermore, for* $\xi \in VW$ *with* $\pi(\pi_W^{VW}\xi) = z = \pi(\exp^\perp(\xi)) \in \Sigma$*, under the identifications* $H^{\mathrm{h}}_\xi VW \cong T_z\Sigma$ *and* $H_{\exp^\perp(\xi)}W \cong T_z\Sigma$ *given by the differentials of the projections and horizontal lifts,*

$$\mathrm{pr}^{TW}_{HW} \circ D\exp^\perp|_{H^{\mathrm{h}}VW} : H^{\mathrm{h}}VW \to HW$$

*corresponds to the identity.*

**Definition II.12.** Denote by

$$\tau := \mathrm{pr}^{TW}_{VW} \circ D\exp^\perp|_{H^{\mathrm{h}}VW} : H^{\mathrm{h}}VW \to VW$$

the annoying part of the differential of the vertical exponential map.
For $z \in \Sigma$, $u \in W_z$, $\xi \in V_uW$ and $X \in T_z\Sigma$ with horizontal lift $\tilde{X} \in HW|_{W_z}$ and second horizontal lift $\tilde{\tilde{X}}_\xi \in H^{\mathrm{h}}_\xi VW$, denote

$$\tau_\xi(X) := \tau(\tilde{\tilde{X}}_\xi).$$

**Corollary II.1.** *Let* $u \in \Gamma^1(W)$*,* $\xi \in \Gamma^1(u^*VW)$*. Then the section* $\exp^\perp_u(\xi) \in \Gamma^1(W)$ *satisfies, for* $z \in \Sigma$*,* $X \in T_z\Sigma$*,*

$$(D^{\mathrm{v}}\exp^\perp_u(\xi))(X) = D^{\mathrm{v}}\exp^{W_z}_{u(z)}(\nabla_X\xi) + D^{\mathrm{h}}\exp^{W_z}_{u(z)}(D^{\mathrm{v}}u(X)) + \tau_\xi(X).$$

Now let the following be given: $u \in \Gamma(W)$ a section and a neighbourhood $U \subseteq W$ of $u(\Sigma)$ s. t. for every $z \in \Sigma$, $U_z := U \cap W_z$ is a neighbourhood of $u(z)$. Assume that $U_z$ is the diffeomorphic image under the exponential map on $W_z$ of a ball around $0 \in T_{u(z)}W_z = VW|_{W_z}$, i. e. $U$ is diffeomorphic to a neighbourhood of the zero section in $u^*VW$ via $\exp^\perp$. Then one can consider the map $(\exp^\perp_u)^{-1} : W \supseteq U \to VW$, $w \mapsto (\exp^{W_{\pi(w)}}_{u(\pi(w))})^{-1}(w)$ and the next goal is to compute its differential

$$D(\exp^\perp_u)^{-1} : TW|_U \cong HW|_U \oplus VW|_U \to TVW \cong H^{\mathrm{h}}VW \oplus H^{\mathrm{v}}VW \oplus VVW.$$

Actually, one is only interested in $\mathrm{pr}^{TVW}_{VVW} \circ D(\exp^\perp_u)^{-1} : TW|_U \to VVW$. If one denotes by $D^{\mathrm{h}}\exp^{W_z} : HTW_z \to TW_z \cong VW|_{W_z}$ and $D^{\mathrm{v}}\exp^{W_z} : VTW_z \to TW_z$ the horizontal and vertical parts, respectively, of $D\exp^{W_z} : TTW_z \to TW_z$ (again using $H^{\mathrm{v}}VW|_{TW_z} \cong HTW_z$, $VVW|_{TW_z} \cong VTW_z$ and $TW_z \cong VW|_{W_z}$), then one can use that $D^{\mathrm{v}}\exp^{W_z}$ at a point $\xi \in TW_z$ with $\pi^{TW_z}_W(\xi) = u(z)$ is given by the differential $(D\exp^{W_z}_{u(z)})_\xi : T_\xi T_{u(z)}W_z \cong V_\xi TW_z \to TW_z$ of the map

$\exp^{W_z}_{u(z)} : T_{u(z)}W_z \to W_z$. This map is a diffeomorphism on a neighbourhood of 0, hence if $\xi \in V_{u(z)}W$, then $(D^{\mathrm{v}} \exp^{W_z})_\xi$ is invertible if $\xi \in V_{u(z)}W = T_{u(z)}W_z$ is s. t. $\exp^\perp(\xi) \in U \subseteq W$.
Now first of all, one can observe that $D(\exp^\perp_u)^{-1}|_{VW|_U}$ takes values in $VVW$ and is given by the differential $D(\exp^{W_z}_{u(z)})^{-1} : TW_z|_{U_z} \to VTW_z$ on every fibre $W_z$. Now let $\xi \in V_{u(z)}W$ be s. t. $w_0 := \exp^\perp(\xi) \in U$ and let $\tilde{X}_{w_0} \in H_wW$ be the horizontal lift of $X \in T_z\Sigma$. Let $w : [0,1] \to W$ be a horizontal path with $w(0) = w_0$ and $\dot{w}(0) = \tilde{X}_w$. Then $\exp^\perp \circ (\exp^\perp_u)^{-1}(w) = w$ is horizontal, so $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \exp^\perp \circ (\exp^\perp_u)^{-1}(w) = D\exp^\perp \circ D(\exp^\perp_u)^{-1}(\tilde{X}_{w_0})$ has vanishing vertical part. From the calculations in the previous lemma, hence,

$$\begin{aligned} 0 = {} & D^{\mathrm{v}} \exp^{W_z}(\mathrm{pr}^{TVW}_{VVW} \circ D(\exp^\perp_u)^{-1}(\tilde{X}_{w_0})) + \\ & + D^{\mathrm{h}} \exp^{W_z}(D^{\mathrm{v}}u(X)_\xi) + \\ & + \tau_\xi(X). \end{aligned}$$

It follows that

$$\mathrm{pr}^{TVW}_{VVW} \circ D(\exp^\perp_u)^{-1}(\tilde{X}_{w_0}) = -(D^{\mathrm{v}} \exp^{W_z})^{-1} \left( D^{\mathrm{h}} \exp^{W_z}(D^{\mathrm{v}}u(X)_\xi) + \tau_\xi(X) \right).$$

Again summarising,

**Lemma II.3.** *Let $\pi : W \to \Sigma$ be a Riemannian submersion and let $u \in \Gamma(W)$. Let furthermore $U \subseteq W$ be a neighbourhood of $u(\Sigma)$ which is diffeomorphic under $\exp^\perp$ to a neighbourhood of the zero section in $u^*VW$. Let $\xi \in V_{u(z)}W$, for some $z \in \Sigma$, be from this neighbourhood of 0 s. t. $\exp^\perp(\xi) = w \in U$. Then for $(\tilde{X}_w, \zeta) \in T_wW \cong H_wW \oplus V_wW$,*

$$\begin{aligned} & \mathrm{pr}^{TVW}_{VVW} \circ (D(\exp^\perp_u)^{-1})_w(\tilde{X}_w, \zeta) = \\ & \qquad ((D^{\mathrm{v}} \exp^{W_z})_\xi)^{-1} \left( \zeta - D^{\mathrm{h}} \exp^{W_z}(D^{\mathrm{v}}u(X)_\xi) - \tau_\xi(X) \right) \end{aligned}$$

Combining these results yields the following:

**Proposition II.3.** *Let $\pi : W \to \Sigma$ be a Riemannian submersion, let $u, v \in \Gamma(W)$ and let $\zeta \in \Gamma(v^*VW)$ be s. t. $\exp^\perp(\zeta)$ lies in a neighbourhood of $u(\Sigma)$ which is the diffeomorphic image under $\exp^\perp$ of a neighbourhood of the zero section in $u^*VW$. Thus*

$$\xi := (\exp^\perp_u)^{-1} \circ \exp^\perp(\zeta) \in \Gamma(u^*VW)$$

*is well-defined and for $X \in T\Sigma$,*

$$\begin{aligned} \nabla_X \xi = {} & ((D^{\mathrm{v}} \exp^{W_z})_\xi)^{-1} \Big( (D^{\mathrm{v}} \exp^{W_z})_\zeta(\nabla_X \zeta) + \\ & + (D^{\mathrm{h}} \exp^{W_z})_\zeta(D^{\mathrm{v}}v(X)) - (D^{\mathrm{h}} \exp^{W_z})_\xi(D^{\mathrm{v}}u(X)) + \\ & + \tau_\zeta(X) - \tau_\xi(X) \Big) \end{aligned}$$

### II.2.2 Jacobi equations and the higher derivatives of the vertical exponential map

The sole purpose of this subsection is to calculate or rather estimate the higher derivatives of the map $(\exp_u^\perp)^{-1} \circ \exp_v^\perp$ for $u, v \in \Gamma(W)$, mapping an appropriately defined subset of $\Gamma(v^*VW)$ to $\Gamma(u^*VW)$. To do so, a number of constants have to be introduced, that are the supremums-norms of certain curvature quantites associated to the Riemannian submersion and will be defined on the following pages. Because this subsection is very technical, the main result and the definition needed for its formulation are presented here, so the reader can skip the rest of this subsection more easily.

**Definition II.13.** Let $\pi : W \to \Sigma$ be a Riemannian submersion. Define for $r \in \mathbb{N}_0$, $z \in \Sigma$ and any $u \in \Gamma^r(W)$,

$$\begin{aligned}
C_z^{\sim,r} &:= \sup\left\{ \frac{(\nabla^\top)^r \tilde{X}(\xi_1, \dots, \xi_r)}{\|X\|\|\xi_1\| \cdots \|\xi_r\|} \;\middle|\; X \in T_z\Sigma, \xi_i \in V_wW, w \in W_z \right\} \\
C_z^{R^\Sigma,r} &:= \|(\nabla^\Sigma)^r R_z^\Sigma\| \\
C_z^{R^W,r} &:= \sup\{\|(\nabla^r R^W)_w\| \mid w \in W_z\} \\
C_z^{\Omega^\perp,r} &:= \sup\{\|(\nabla^r \Omega^\perp)_w\| \mid w \in W_z\} \\
C_z^{T,r} &:= \sup\{\|(\nabla^r T)_w\| \mid w \in W_z\} \\
C_z^{A,r} &:= \sup\{\|(\nabla^r A)_w\| \mid w \in W_z\} \\
C_z^{u,r} &:= \|((\nabla^\perp)^{r-1} D^{\mathrm{v}} u)_z\|,
\end{aligned}$$

where $\tilde{X}$ denotes the horizontal lift of $X \in T\Sigma$, $\nabla^\Sigma$ and $R^\Sigma$ denote the Levi-Civita covariant derivative and Riemannian curvature tensor on $\Sigma$, respectively. $\nabla^\perp$ and $\nabla^\top$ are as in Equations II.10, $\nabla$ denotes the induced covariant derivative on tensors, $R^W$ is the Riemannian curvature on $W$, $\Omega^\perp$ is the curvature of $VW \to W$ and $T$ and $A$ are as in II.11. If $U \subseteq \Sigma$ is any subset and $C_z^*$ denotes any of the constants above, define $C_U^* := \sup\{C_z^* \mid z \in U\}$ and $C^* := C_\Sigma^*$.
Additionally, define the following quantities:
$R^\perp := \Omega^\perp|_{VW}$, i. e. for $w \in W_z$ is $R_w^\perp$ the Riemannian curvature tensor of $W_z$ at the point $w$ under $V_wW \cong T_wW_z$.
$R^\top$ is as in Proposition II.5, explicitely given in terms of $\mathrm{pr}_{VW}^{TW} R^W$, $T$ and $\nabla^\perp T$ in Lemma II.6.
With these define

$$\begin{aligned}
\kappa_z^\perp &:= \sup\{\|R_w^\perp\| \mid w \in W_z\} \\
\kappa_z^\top &:= \sup\{\|R_w^\top\| \mid w \in W_z\} \\
\delta_z &:= \min\left( \frac{1}{\sqrt{\varkappa_z^\perp}}, \frac{1}{\sqrt{\varkappa_z^\top}} \right)
\end{aligned}$$

and

$$\delta := \inf\{\delta_z \mid z \in \Sigma\}.$$

*Remark* II.4. Note that the above constants are not mutually independent, see e. g. Lemma II.4.

*Remark* II.5. If $W_z$ is compact, then the $C_z^{*,r} \in [0,\infty]$ are all finite. If $\Sigma$ and $W$ are both compact, then the $C^{*,r} \in [0,\infty]$ and $C_z^{*,r} \in [0,\infty]$ for all $z \in \Sigma$ are finite.

**Proposition II.4.** *Let $\pi : W \to \Sigma$ be a submersion.*
*Let $g, \tilde{g}$ be Riemannian metrics on $W$ that each turn $\pi : W \to \Sigma$ into a Riemannian submersion. Denote all quantities associated to $\tilde{g}$ by adding $\tilde{}$ to the symbol.*
*Consider the identity as a map $\mathrm{id}_W : (W,g) \to (W,\tilde{g})$ and define for $z \in \Sigma$ and $r \in \mathbb{N}_0$*

$$C_z^{g,\tilde{g},r} := \sup\left\{\|(\nabla^r D\mathrm{id}_W)_w\| \mid w \in W_z\right\}.$$

*$D\mathrm{id}_W : TW \to TW$ here is considered as a bundle morphism, where the left-hand side is equipped with the metric $g$ and Levi-Civita connection associated to $g$ and the right hand side is equipped with the metric $\tilde{g}$ and Levi-Civita connection associated to $\tilde{g}$. Assume that both $(W_z, g_z)$ and $(W_z, \tilde{g}_z)$ are complete. Let $u, v \in \Gamma^k(W)$, let $z \in \Sigma$ and let $\zeta \in \Gamma^k(v^*VW)$. Assume that $\delta_z, \tilde{\delta}_z > 0$ and that $\|\zeta(z)\| < \delta_z$, $\tilde{d}(\exp^\perp(\zeta(z)), u(z)) < \tilde{\delta}_z$.*
*Then $\xi := (\tilde{\exp}_u^\perp)^{-1} \circ \exp^\perp(\zeta)$ is a well-defined section of $u^*VW$ in a neighbourhood of $z$. Assume w. l. o. g. that $\xi$ is well-defined everywhere (otherwise restrict to an open subset of $\Sigma$).*
*Then there are constants $E_z^{k,\ell} \in [0,\infty]$ that are universal expressions in the constants $(\nabla^\Sigma)^r R^\Sigma$, $(\tilde{\nabla}^\Sigma)^r \tilde{R}^\Sigma$ for $0 \le r \le k+\ell-3$, $C_z^{u,r}$, $C_z^{v,r}$ for $1 \le r \le k$, $C_z^{g,\tilde{g},r}$, $C_z^{R^W,r}$, $C_z^{R^\perp,r}$, $C_z^{A,r}$ and $\tilde{C}_z^{R^W,r}$, $\tilde{C}_z^{R^\perp,r}$, $\tilde{C}_z^{A,r}$ for $0 \le r \le k+\ell$ and $C_z^{\sim,r}$, $C_z^{T,r}$ and $\tilde{C}_z^{\sim,r}$, $\tilde{C}_z^{T,r}$ for $0 \le r \le k+\ell+1$ s. t. at the point $z \in \Sigma$*

$$\left\|\frac{\partial^\ell}{\partial\eta_\ell\cdots\partial\eta_1}\tilde{\nabla}^k\xi\right\| \le E_z^{k,\ell}\sum_{k'=0}^{k}\sum_{\substack{k_1'+\cdots+k_r'+\\ +\, i_1+\cdots+i_\ell=k'\\ k_j'>0,\, i_j\ge 0,\, r\in\mathbb{N}_0}}\prod_{s=1}^{r}\left\|\nabla^{k_s'}\xi\right\|\prod_{s=1}^{\ell}\left\|\nabla^{i_s}\eta_s\right\|.$$

Here, $\frac{\partial^\ell}{\partial\eta_\ell\cdots\partial\eta_1}\tilde{\nabla}^k\xi$ is defined recursively by

$$\frac{\partial^{\ell+1}}{\partial\eta_{\ell+1}\cdots\partial\eta_1}\tilde{\nabla}^k\xi := \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0}\frac{\partial^\ell}{\partial\eta_\ell\cdots\partial\eta_1}\tilde{\nabla}^k(\xi+\lambda\eta_{\ell+1}).$$

The rest of this subsection is dedicated to a sketch of the proof of this proposition, which will then be used in the next subsection to define Sobolev spaces of sections.

To do so, the first goal is to compute the terms appearing in Proposition II.3, namely $D^{\mathrm{v}}\exp^{W_z}$ and its inverse, $D^{\mathrm{h}}\exp^{W_z}$ and $\tau$. It is well known that $D^{\mathrm{v}}\exp^{W_z}$ and $D^{\mathrm{h}}\exp^{W_z}$ satisfy the usual Jacobi equation. It will be shown that $\tau$ satisfies a Jacobi equation, too, but this time an inhomogeneous one.

This is a lengthy calculation, laid out over the course of the next three lemmas, which the reader is strongly advised to skip over and to proceed to Proposition II.5 straight away, in which the main consequence, namely that the maps above all satisfy Jacobi equations with coefficients expressions in the tensors that appeared in Definition II.13, is summarised.

But for the actual calculation of $\tau$, first, a bit of notation has to be introduced, taken from [Sak96], Section II.6, p. 74 f. Given a Riemannian submersion $\pi : W \to \Sigma$ and vertical vector fields $\xi, \eta \in \mathfrak{X}(W)$ (i. e. $\xi_w, \eta_w \in V_w W$ for all $w \in W$), for every $z \in \Sigma$, one can regard $\xi|_{W_z}, \eta|_{W_z}$ as tangent vector fields to $W_z$. Proposition 6.1 in [Sak96] then shows that $\nabla^\perp_\xi \eta$ coincides with the Levi-Civita derivative on the fibres. As before, $\nabla^W$ denotes the Levi-Civita derivative of the total space $W$ of the fibration, so for a vertical vector field $\xi$ and any vector $Z \in TW$, by definition

$$\nabla^\perp_Z \xi := \mathrm{pr}^{TW}_{VW} \nabla^W_Z \xi \tag{II.10a}$$

and one can analogously define for a horizontal vector field $X$ and any vector $Z \in TW$,

$$\nabla^\top_Z X := \mathrm{pr}^{TW}_{HW} \nabla^W_Z X. \tag{II.10b}$$

Then two tensor fields $A$ and $T$ are defined via

$$T_Y Z := \mathrm{pr}^{TW}_{HW}(\nabla^W_{\mathrm{pr}^{TW}_{VW} Y} \mathrm{pr}^{TW}_{VW} Z) + \mathrm{pr}^{TW}_{VW}(\nabla^W_{\mathrm{pr}^{TW}_{VW} Y} \mathrm{pr}^{TW}_{HW} Z) \tag{II.11a}$$

$$A_Y Z := \mathrm{pr}^{TW}_{HW}(\nabla^W_{\mathrm{pr}^{TW}_{HW} Y} \mathrm{pr}^{TW}_{VW} Z) + \mathrm{pr}^{TW}_{VW}(\nabla^W_{\mathrm{pr}^{TW}_{HW} Y} \mathrm{pr}^{TW}_{HW} Z) \tag{II.11b}$$

and if $\xi, \eta$ are vertical, $X, Y$ horizontal, vector fields, then

$$\nabla^W_\xi \eta = \nabla^\perp_\xi \eta + T_\xi \eta \qquad \nabla^W_\xi X = T_\xi X + \nabla^\top_\xi X \tag{II.12a}$$

$$\nabla^W_X \xi = \nabla^\perp_X \xi + A_X \xi \qquad \nabla^W_X Y = A_X Y + \nabla^\top_X Y \tag{II.12b}$$

hold, see Proposition 6.1 in [Sak96]. Let also $R^\perp$ denote the Riemannian curvature tensor in the fibres, i. e. for $w \in W$, $R^\perp_w$ is the curvature tensor of $W_{\pi(w)}$ at $w \in W_{\pi(w)}$, let $R^W$ denote the Riemannian curvature tensor on $W$ and let finally $\Omega^\perp$ denote the curvature tensor of $\nabla^\perp$ on $VW \to W$.

**Lemma II.4.** *Let $Y, Z \in T_w W$ for some $w \in W$. Then*

$$\begin{aligned}(\nabla^W_Y \mathrm{pr}^{TW}_{VW})(Z) = &\underbrace{-A_{\mathrm{pr}^{TW}_{HW} Y}(\mathrm{pr}^{TW}_{HW} Z) - T_{\mathrm{pr}^{TW}_{VW} Y}(\mathrm{pr}^{TW}_{HW} Z)}_{\in VW} + \\ &+ \underbrace{A_{\mathrm{pr}^{TW}_{HW} Y}(\mathrm{pr}^{TW}_{VW} Z) + T_{\mathrm{pr}^{TW}_{VW} Y}(\mathrm{pr}^{TW}_{VW} Z)}_{\in HW},\end{aligned}$$

*in particular $\nabla^W_Y \mathrm{pr}^{TW}_{VW}(HW) \subseteq VW$ and $\nabla^W_Y \mathrm{pr}^{TW}_{VW}(VW) \subseteq HW$. Also, $\nabla^W \mathrm{pr}^{TW}_{HW} = -\nabla^W \mathrm{pr}^{TW}_{VW}$. If in addition $\xi \in \Gamma(VW)$, then the curvature $\Omega^\perp$ of $\nabla^\perp$ is given* *by*

$$\begin{aligned}\Omega^{\perp}(Y,Z)\xi &= \mathrm{pr}^{TW}_{VW} R^W(Y,Z)\xi - \\ &- \Big( A_{\mathrm{pr}^{TW}_{HW}Y}(T_{\mathrm{pr}^{TW}_{VW}Z}\xi + A_{\mathrm{pr}^{TW}_{HW}Z}\xi) - A_{\mathrm{pr}^{TW}_{HW}Z}(T_{\mathrm{pr}^{TW}_{VW}Y}\xi + A_{\mathrm{pr}^{TW}_{HW}Y}\xi) \\ &\quad + T_{\mathrm{pr}^{TW}_{VW}Y}(T_{\mathrm{pr}^{TW}_{VW}Z}\xi + A_{\mathrm{pr}^{TW}_{HW}Z}\xi) - T_{\mathrm{pr}^{TW}_{VW}Z}(T_{\mathrm{pr}^{TW}_{VW}Y}\xi + A_{\mathrm{pr}^{TW}_{HW}Y}\xi)\Big)\end{aligned}$$

*Proof.* For the claim on $\nabla^W \mathrm{pr}^{TW}_{VW}$, by the formulas II.12, for $Y, Z \in HW$,

$$\begin{aligned}(\nabla^W_Y \mathrm{pr}^{TW}_{VW})(Z) &= \nabla^W_Y (\underbrace{\mathrm{pr}^{TW}_{VW} Z}_{=0}) - \mathrm{pr}^{TW}_{VW}(A_Y Z + \mathrm{pr}^{TW}_{HW}(\nabla^W_Y Z)) \\ &= -A_Y Z,\end{aligned}$$

for $Y \in HW$, $Z \in VW$,

$$\begin{aligned}(\nabla^W_Y \mathrm{pr}^{TW}_{VW})(Z) &= \nabla^W_Y (\underbrace{\mathrm{pr}^{TW}_{VW} Z}_{=Z}) - \mathrm{pr}^{TW}_{VW}(\nabla^W_Y Z) \\ &= \mathrm{pr}^{TW}_{HW} \nabla^W_Y Z \\ &= A_Y Z,\end{aligned}$$

for $Y \in VW$, $Z \in HW$,

$$\begin{aligned}(\nabla^W_Y \mathrm{pr}^{TW}_{VW})(Z) &= \nabla^W_Y (\underbrace{\mathrm{pr}^{TW}_{VW} Z}_{=0}) - \mathrm{pr}^{TW}_{VW}(T_Y Z + \mathrm{pr}^{TW}_{HW}(\nabla^W_Y Z)) \\ &= -T_Y Z\end{aligned}$$

and for $Y \in VW$, $Z \in VW$,

$$\begin{aligned}(\nabla^W_Y \mathrm{pr}^{TW}_{VW})(Z) &= \nabla^W_Y (\underbrace{\mathrm{pr}^{TW}_{VW} Z}_{=Z}) - \mathrm{pr}^{TW}_{VW}(\nabla^W_Y Z) \\ &= \mathrm{pr}^{TW}_{HW} \nabla^W_Y Z \\ &= T_Y Z.\end{aligned}$$

The second claim follows immediately from $\mathrm{pr}^{TW}_{HW} = \mathrm{id} - \mathrm{pr}^{TW}_{VW}$.
Now let $Y, Z \in \mathfrak{X}(W)$ with $[Y, Z] = 0$ and let $\xi \in \Gamma(VW)$. Then

$$\begin{aligned}\Omega^{\perp}(Y,Z)\xi &= (\nabla^{\perp}_Y \nabla^{\perp}_Z - \nabla^{\perp}_Z \nabla^{\perp}_Y)\xi \\ &= \mathrm{pr}^{TW}_{VW}(\nabla^W_Y \mathrm{pr}^{TW}_{VW} \nabla^W_Z - \nabla^W_Z \mathrm{pr}^{TW}_{VW} \nabla^W_Y)\xi \\ &= \mathrm{pr}^{TW}_{VW}\big((\nabla^W_Y \mathrm{pr}^{TW}_{VW})\nabla^W_Z + \mathrm{pr}^{TW}_{VW}\nabla^W_Y \nabla^W_Z - \\ &\quad - (\nabla^W_Z \mathrm{pr}^{TW}_{VW})\nabla^W_Y - \mathrm{pr}^{TW}_{VW}\nabla^W_Z \nabla^W_Y\big)\xi \\ &= \mathrm{pr}^{TW}_{VW} R^W(Y,Z)\xi + \mathrm{pr}^{TW}_{VW}(\nabla^W_Y \mathrm{pr}^{TW}_{VW})\mathrm{pr}^{TW}_{HW}\nabla^W_Z \xi - \\ &\quad - \mathrm{pr}^{TW}_{VW}(\nabla^W_Z \mathrm{pr}^{TW}_{VW})\mathrm{pr}^{TW}_{HW}\nabla^W_Y \xi\end{aligned}$$

by the first part of the statement

$$\begin{aligned}&= \mathrm{pr}^{TW}_{VW} R^W(Y,Z)\xi + \mathrm{pr}^{TW}_{VW}(\nabla^W_Y \mathrm{pr}^{TW}_{VW})(T_{\mathrm{pr}^{TW}_{VW}Z}\xi + A_{\mathrm{pr}^{TW}_{HW}}Z\xi) - \\ &- \ \mathrm{pr}^{TW}_{VW}(\nabla^W_Z \mathrm{pr}^{TW}_{VW})(T_{\mathrm{pr}^{TW}_{VW}Y}\xi + A_{\mathrm{pr}^{TW}_{HW}Y}\xi),\end{aligned}$$

by formulas II.12. Applying the first part of the statement to this shows the formula for $\Omega^\perp$. □

**Lemma II.5.** *Let $\pi : W \to \Sigma$ be a Riemannian submersion, $w \in W$, $\pi(w) = z \in \Sigma$, $X \in T_z\Sigma$ and $\xi \in V_wW$. Let $\tilde{X}$ be the horizontal lift of $X$ to $HW$ along $W_z$, $\tilde{\tilde{X}}_\xi$ that to $H^{\mathrm{h}}_\xi VW$ and denote by $\gamma : [0,\infty) \to W$, $\gamma(t) := \exp^\perp(t\xi)$ the vertical geodesic in the direction of $\xi$. Then $\tau_{t\xi}(X) = \mathrm{pr}^{TW}_{VW} J(t)$, where $J : [0,\infty) \to TW$ is a vector field along $\gamma$ that satisfies the following homogeneous $2^{nd}$-order linear ordinary differential equation:*

$$\begin{aligned}&\nabla^W_{\dot\gamma}\nabla^W_{\dot\gamma} J = T_{\nabla^W_{\dot\gamma}J}\dot\gamma + T_{\dot\gamma}(\nabla^W_{\dot\gamma}J) + R^W(\dot\gamma, J)\dot\gamma + (\nabla^W_J T)_{\dot\gamma}\dot\gamma \\ &\qquad J(0) = \tilde{X}_w, \quad \nabla^W_{\dot\gamma} J(0) = 0.\end{aligned}$$

*Proof.* One proceeds as in the standard derivation of the Jacobi equation. Notation is as in the statement of the lemma. Let $w' : (-\varepsilon,\varepsilon) \to W$ be an integral curve of $\tilde{X}'$, where $X'$ is an extension of $X$ to a vector field on $\Sigma$, with $w'(0) = w$ and $\dot{w}'(0) = \tilde{X}_w$. Let $J_0 : (-\varepsilon,\varepsilon) \to VW$ be parallel transport of $\xi$ along $w'$. Consider the 2-parameter family $\alpha : (-\varepsilon,\varepsilon) \times [0,\infty) \to W$, $(s,t) \mapsto \exp^\perp(tJ_0(s))$. Then with $J(t) := \frac{\partial\alpha}{\partial s}(0,t)$, by definition, $\mathrm{pr}^{TW}_{VW} J(t) = \tau_{t\xi}(X)$. To show the initial conditions, first note that $\alpha(s,0) = w'(s)$, hence $J(0) = \frac{\partial w'}{\partial s}\Big|_{s=0} = \dot{w}'(0) = \tilde{X}_w$. To show the second initial condition, note that

$$\begin{aligned}\nabla_{\dot\gamma} J(0) &= \nabla_{\dot\gamma} \frac{\partial}{\partial s}\Big|_{s=0} \exp^\perp(tJ_0(s))\Big|_{t=0} \\ &= \nabla_{\frac{\partial\alpha}{\partial s}}\Big|_{s=0} \frac{\partial}{\partial t}\Big|_{t=0} \exp^\perp(tJ_0(s))\end{aligned}$$

by a standard result from Riemannian geometry, see [Sak96], Lemma 2.2 from Chapter II, p. 35.

$$\begin{aligned}&= \nabla_{\frac{\partial\alpha}{\partial s}}\Big|_{s=0} J_0(s) \\ &= 0.\end{aligned}$$

Now again using [Sak96], Lemma 2.2 from Chapter II, p. 35, and denoting for shortness $\nabla_t := \nabla^W_{\frac{\partial\alpha}{\partial t}}$ and $\nabla_s := \nabla^W_{\frac{\partial\alpha}{\partial s}}$,

$$\begin{aligned}
\nabla_t\nabla_t J &= \nabla_t\nabla_t \left.\frac{\partial\alpha}{\partial s}\right|_{s=0} \\
&= \nabla_t\nabla_s \left.\frac{\partial\alpha}{\partial t}\right|_{s=0} \\
&= \nabla_s\nabla_t \left.\frac{\partial\alpha}{\partial t}\right|_{s=0} + R^W\left(\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial s}\right)\left.\frac{\partial\alpha}{\partial t}\right|_{s=0} \\
&= \nabla_s \left.\left(T_{\frac{\partial\alpha}{\partial t}}\frac{\partial\alpha}{\partial t}\right)\right|_{s=0} + R^W(\dot\gamma, J)\dot\gamma
\end{aligned}$$

by [Sak96], Lemma 6.1 from Chapter II, p. 75, since $\frac{\partial\alpha}{\partial t}$ is vertical and $\nabla^\perp_{\frac{\partial\alpha}{\partial t}}\frac{\partial\alpha}{\partial t} = 0$, the curves $t \mapsto \alpha(t, s)$ being vertical geodesics

$$\begin{aligned}
&= (\nabla_s T)_{\dot\gamma}\dot\gamma|_{s=0} + T_{\nabla_s\frac{\partial\alpha}{\partial t}}\left.\frac{\partial\alpha}{\partial t}\right|_{s=0} + T_{\frac{\partial\alpha}{\partial t}}\nabla_s\left.\frac{\partial\alpha}{\partial t}\right|_{s=0} + R^W(\dot\gamma, J)\dot\gamma \\
&= (\nabla^W_J T)_{\dot\gamma}\dot\gamma + T_{\nabla^W_{\dot\gamma}J}\dot\gamma + T_{\dot\gamma}\nabla^W_{\dot\gamma}J + R^W(\dot\gamma, J)\dot\gamma,
\end{aligned}$$

using $\nabla_s\frac{\partial\alpha}{\partial t} = \nabla_t\frac{\partial\alpha}{\partial s}$ as before. □

Alternatively, one can derive $\tau$ from an inhomogeneous Jacobi equation:

**Lemma II.6.** *Let $\pi : W \to \Sigma$ be a Riemannian submersion, $w \in W$, $\pi(w) = z \in \Sigma$, $X \in T_z\Sigma$ and $\xi \in V_wW$. Let $\tilde X$ be the horizontal lift of $X$ to $HW$ along $W_z$, $\tilde{\tilde X}_\xi$ that to $H^{\mathrm{h}}_\xi VW$ and denote by $\gamma : [0, \infty) \to W$, $\gamma(t) := \exp^\perp(t\xi)$ the vertical geodesic in the direction of $\xi$. Then $\tau_{t\xi}(X) = J^\perp(t)$, where $J^\perp : [0, \infty) \to VW$ is the vector field along $\gamma$ that satisfies the following inhomogeneous Jacobi equation:*

$$\begin{aligned}
\nabla^\perp_{\dot\gamma}\nabla^\perp_{\dot\gamma}J^\perp = \mathrm{pr}^{TW}_{VW}\Big(&R^W(\dot\gamma, J^\perp)\dot\gamma + (\nabla^W_{J^\perp}T)_{\dot\gamma}\dot\gamma + (\nabla^W_{\dot\gamma}T)_{\dot\gamma}J^\perp + \\
&+ T_{T_{\dot\gamma}\dot\gamma}J^\perp + 2T_{\dot\gamma}(T_{\dot\gamma}J^\perp)\Big) + \\
+ \mathrm{pr}^{TW}_{VW}\Big(&R^W(\dot\gamma, \tilde X)\dot\gamma + (\nabla^W_{\tilde X}T)_{\dot\gamma}\dot\gamma - \\
&- (\nabla^W_{\dot\gamma}T)_{\dot\gamma}\tilde X - T_{T_{\dot\gamma}\dot\gamma}\tilde X + T_{\dot\gamma}\nabla^\top_{\dot\gamma}\tilde X\Big) \\
J^\perp(0) = 0, \quad \nabla^\perp_{\dot\gamma}J^\perp(0) = -T_\xi\tilde X,&
\end{aligned}$$

*Proof.* Using the notation of the previous lemma, let $J^\perp := \mathrm{pr}^{TW}_{VW}J$. Because by construction, for every $t \in [0, \infty)$, $s \mapsto \alpha(s, t)$ is a section of $W$ along the path $s \mapsto \pi(\alpha(t, s))$ in $\Sigma$, $J(t) = J^\perp(t) + \tilde X$. With Lemma II.4, one calculates

$$\begin{aligned}
\nabla^W_{\dot\gamma}J^\perp &= \nabla^W_{\dot\gamma}(\mathrm{pr}^{TW}_{VW}J) \\
&= (\nabla^W_{\dot\gamma}\mathrm{pr}^{TW}_{VW})J + \mathrm{pr}^{TW}_{VW}\nabla^W_{\dot\gamma}J \\
&= T_{\dot\gamma}(J^\perp - \tilde X) + \mathrm{pr}^{TW}_{VW}\nabla^W_{\dot\gamma}J
\end{aligned}$$

using $J = J^\perp + \tilde{X}$ and Lemma II.4;

$$\begin{aligned}\nabla^W_{\dot\gamma}\nabla^W_{\dot\gamma}J^\perp &= (\nabla^W_{\dot\gamma}T_{\dot\gamma})(J^\perp - \tilde{X}) + T_{\dot\gamma}(\nabla^W_{\dot\gamma}(J^\perp - \tilde{X})) + \\ &+ (\nabla^W_{\dot\gamma}\mathrm{pr}^{TW}_{VW})\nabla^W_{\dot\gamma}J + \mathrm{pr}^{TW}_{VW}\nabla^W_{\dot\gamma}\nabla^W_{\dot\gamma}J\end{aligned}$$

hence

$$\begin{aligned}\mathrm{pr}^{TW}_{VW}\nabla^W_{\dot\gamma}\nabla^W_{\dot\gamma}J^\perp &= \mathrm{pr}^{TW}_{VW}(\nabla^W_{\dot\gamma}T_{\dot\gamma})(J^\perp - \tilde{X}) + \mathrm{pr}^{TW}_{VW}T_{\dot\gamma}(\nabla^W_{\dot\gamma}(J^\perp - \tilde{X})) + \\ &+ \mathrm{pr}^{TW}_{VW}T_{\dot\gamma}(\underbrace{\nabla^W_{\dot\gamma}(J^\perp + \tilde{X})}_{=\nabla^W_{\dot\gamma}J}) + \mathrm{pr}^{TW}_{VW}\nabla^W_{\dot\gamma}\nabla^W_{\dot\gamma}J \\ &= \mathrm{pr}^{TW}_{VW}(\nabla^W_{\dot\gamma}T_{\dot\gamma})(J^\perp - \tilde{X}) + 2\mathrm{pr}^{TW}_{VW}T_{\dot\gamma}(\underbrace{\nabla^\perp_{\dot\gamma}J^\perp + T_{\dot\gamma}J^\perp}_{=\nabla^W_{\dot\gamma}J^\perp}) + \\ &+ \mathrm{pr}^{TW}_{VW}\nabla^W_{\dot\gamma}\nabla^W_{\dot\gamma}J \\ &= \mathrm{pr}^{TW}_{VW}(\nabla^W_{\dot\gamma}T_{\dot\gamma})(J^\perp - \tilde{X}) + 2\mathrm{pr}^{TW}_{VW}T_{\dot\gamma}T_{\dot\gamma}J^\perp + \mathrm{pr}^{TW}_{VW}\underbrace{T_{\nabla^W_{\dot\gamma}J}\dot\gamma}_{\in HW} + \\ &+ \mathrm{pr}^{TW}_{VW}T_{\dot\gamma}(\nabla^W_{\dot\gamma}J) + \mathrm{pr}^{TW}_{VW}R^W(\dot\gamma, J)\dot\gamma + \mathrm{pr}^{TW}_{VW}(\nabla^W_J T)_{\dot\gamma}\dot\gamma\end{aligned}$$

for $T_{\dot\gamma}\nabla^\perp_{\dot\gamma}J^\perp \in HW$ and by Lemma II.5

$$\begin{aligned}&= \mathrm{pr}^{TW}_{VW}\Big(\nabla^W_{\dot\gamma}T_{\dot\gamma})(J^\perp - \tilde{X}) + 2T_{\dot\gamma}T_{\dot\gamma}J^\perp + R^W(\dot\gamma, J)\dot\gamma + \\ &+ (\nabla^W_J T)_{\dot\gamma}\dot\gamma\Big) + \mathrm{pr}^{TW}_{VW}\underbrace{T_{\dot\gamma}(\underbrace{\nabla^\perp_{\dot\gamma}J^\perp}_{\in VW})}_{\in HW} + \mathrm{pr}^{TW}_{VW}T_{\dot\gamma}T_{\dot\gamma}J^\perp + \\ &+ \mathrm{pr}^{TW}_{VW}T_{\dot\gamma}(\nabla^W_{\dot\gamma}\tilde{X})\end{aligned}$$

because $\nabla^W_{\dot\gamma}J = \nabla^W_{\dot\gamma}(J^\perp + \tilde{X}) = \nabla^\perp_{\dot\gamma}J^\perp + T_{\dot\gamma}J^\perp + \nabla^W_{\dot\gamma}\tilde{X}$, hence

$$\begin{aligned}\nabla^\perp_{\dot\gamma}\nabla^\perp_{\dot\gamma}J^\perp &= \mathrm{pr}^{TW}_{VW}\nabla^W_{\dot\gamma}\mathrm{pr}^{TW}_{VW}\nabla^W_{\dot\gamma}J^\perp \\ &= \mathrm{pr}^{TW}_{VW}(\nabla^W_{\dot\gamma}\mathrm{pr}^{TW}_{VW})\nabla^W_{\dot\gamma}J^\perp + \mathrm{pr}^{TW}_{VW}\nabla^W_{\dot\gamma}\nabla^W_{\dot\gamma}J^\perp \\ &= -T_{\dot\gamma}T_{\dot\gamma} + \mathrm{pr}^{TW}_{VW}\nabla^W_{\dot\gamma}\nabla^W_{\dot\gamma}J^\perp\end{aligned}$$

because $\nabla^W_{\dot\gamma}J^\perp = \underbrace{\nabla^\perp_{\dot\gamma}J^\perp}_{\in VW} + \underbrace{T_{\dot\gamma}J^\perp}_{\in HW}$

$$\begin{aligned}&= \mathrm{pr}^{TW}_{VW}\Big((\nabla^W_{\dot\gamma}T_{\dot\gamma})J^\perp + (\nabla^W_{J^\perp}T)_{\dot\gamma}\dot\gamma + R^W(\dot\gamma, J^\perp)\dot\gamma + \\ &+ 2T_{\dot\gamma}T_{\dot\gamma}J^\perp\Big) + \mathrm{pr}^{TW}_{VW}\Big((\nabla^W_{\tilde{X}}T)_{\dot\gamma}\dot\gamma - (\nabla^W_{\dot\gamma}T_{\dot\gamma})\tilde{X} + \\ &+ R^W(\dot\gamma, \tilde{X})\dot\gamma + T_{\dot\gamma}\mathrm{pr}^{TW}_{HW}(\nabla^W_{\dot\gamma}\tilde{X})\Big).\end{aligned}$$

Finally, note that $\nabla^W_{\dot\gamma}\dot\gamma = \nabla^\perp_{\dot\gamma}\dot\gamma + T_{\dot\gamma}\dot\gamma = T_{\dot\gamma}\dot\gamma$ and hence $\nabla^W_{\dot\gamma}T_{\dot\gamma} = (\nabla^W_{\dot\gamma}T)_{\dot\gamma} + T_{T_{\dot\gamma}\dot\gamma}$. The lemma now follows from the Formulas II.12. □

**Proposition II.5.** *Let $\pi : W \to \Sigma$ be a Riemannian submersion. There are tensors*

$$\begin{aligned} R^\top &\in \Gamma(\mathrm{Hom}(VW \otimes VW \otimes VW, VW)) \\ V_1 &\in \Gamma(\mathrm{Hom}(VW \otimes VW \otimes HW, VW)) \\ V_2 &\in \Gamma(\mathrm{Hom}(VW \otimes HW, VW)) \end{aligned}$$

*s. t. for every $w \in W$, $\pi(w) = z \in \Sigma$, $X \in T_z\Sigma$ and $\xi \in V_wW$, if $\gamma : [0,\infty) \to W$, $\gamma(t) := \exp^\perp(t\xi)$, denotes the vertical geodesic in the direction of $\xi$, then for every $\eta \in V_wW$, there are vertical vector fields $J^{\mathrm{h}}_\eta, J^{\mathrm{v}}_\eta, J^\tau$ along $\gamma$ with*

$$\begin{aligned} (D^{\mathrm{h}} \exp^{W_z})_{t\xi}(\eta) &= J^{\mathrm{h}}_\eta(t) \\ \nabla^\perp_{\dot\gamma}\nabla^\perp_{\dot\gamma} J^{\mathrm{h}}_\eta &= R^\perp(\dot\gamma, J^{\mathrm{h}}_\eta)\dot\gamma \\ J^{\mathrm{h}}_\eta(0) &= \eta \qquad \nabla^\perp_{\dot\gamma} J^{\mathrm{h}}_\eta(0) = 0 \\ (D^{\mathrm{v}} \exp^{W_z})_{t\xi}(\eta) &= \frac{1}{t} J^{\mathrm{v}}_\eta(t) \\ \nabla^\perp_{\dot\gamma}\nabla^\perp_{\dot\gamma} J^{\mathrm{v}}_\eta &= R^\perp(\dot\gamma, J^{\mathrm{v}}_\eta)\dot\gamma \\ J^{\mathrm{v}}_\eta(0) &= 0 \qquad \nabla^\perp_{\dot\gamma} J^{\mathrm{v}}_\eta(0) = \eta \\ \tau_{t\xi}(X) &= J^\tau(t) \\ \nabla^\perp_{\dot\gamma}\nabla^\perp_{\dot\gamma} J^\tau &= R^\top(\dot\gamma, J^\tau)\dot\gamma + V_1(\dot\gamma, \dot\gamma, \tilde X) + V_2(\dot\gamma, \nabla^\top_{\dot\gamma}\tilde X) \\ J^\tau(0) &= 0 \qquad \nabla^\perp_{\dot\gamma} J^\tau(0) = -T_\xi \tilde X \end{aligned}$$

*Proof.* This just sums up the discussion of the previous lemmas together with the well known Jacobi equation for the derivative of the exponential map (applied here in one fibre $W_z$ as a Riemannian manifold), which can be found in any book on Riemannian geometry. Observe here, that the curvature $R^\perp$ of the fibre $W_z$ is given by $\Omega^\perp|_{TW_z}$. □

The following corollary, the proof of which can be found in the Appendix, Section A.2, now allows to give estimates on the Jacobi fields in the above theorem and hence on the derivatives of the (vertical) exponential map.

**Corollary II.2.** *Let $M$ be a complete Riemannian manifold, $p \in M$ and $\xi \in T_pM$. Let $\gamma : [0,1] \to M$, $t \mapsto \exp_p(t\xi)$, be the geodesic through $p$ in the direction of $\xi$. Let $\kappa \geq 0$ be s. t. $\|R(X,Y)Z\| \leq \kappa\|X\|\|Y\|\|Z\|$ for all $t \in [0,1]$, $X, Y, Z \in T_{\gamma(t)}M$. Denote $\dot X := \nabla_{\dot\gamma} X$ for a vector field $X : [0,1] \to TM$ along $\gamma$ and let $V : [0,1] \to TM$ be a vector field along $\gamma$. Assume that $J : [0,1] \to TM$ is another vector field along $\gamma$ satisfying the inhomogeneous Jacobi equation*

$$\ddot J + R(J, \dot\gamma)\dot\gamma = V.$$

*Then for $\|\xi\| \leq \frac{1}{\sqrt{\kappa}}$,*

$$\|J(t) - (\|_0^t \gamma)(J(0) + t\dot J(0))\| \leq t^2 \left( \|J(0)\| + t\|\dot J(0)\| + \frac{1+t^2}{t} \int_0^t \|V(s)\| \,\mathrm{d}s \right)$$

*and in particular,*

$$\|J(t)\| \leq (1+t^2)\left(\|J(0)\| + t\|\dot{J}(0)\| + t\int_0^t \|V(s)\|\,\mathrm{d}s\right).$$

*Furthermore,*

$$\|\dot{J}(t) - (\|_0^t\,\gamma)\dot{J}(0)\| \leq t(1+t^2)\left(\|J(0)\| + t\|\dot{J}(0)\| + \frac{1}{t}\int_0^t \|V(s)\|\,\mathrm{d}s\right)$$

*and in particular*

$$\|\dot{J}(t)\| \leq \|\dot{J}(0)\| + t(1+t^2)\left(\|J(0)\| + t\|\dot{J}(0)\| + \frac{1}{t}\int_0^t \|V(s)\|\,\mathrm{d}s\right).$$

*Finally, if $J(0) = 0$, then*

$$\left\|\left(\tfrac{1}{t}J(t)\right)^{\cdot}\right\| \leq 2(1+t^2)\left(t\|\dot{J}(0)\| + \frac{1}{t}\int_0^t \|V(s)\|\,\mathrm{d}s\right).$$

Before the estimates, one can simplify matters in the following way: In the notation and under the assumptions of Proposition II.4, for $z \in \Sigma$ and $X_1, \ldots, X_k \in T_z\Sigma$, one needs to calculate $\frac{\partial^\ell}{\partial\eta_\ell\cdots\partial\eta_1}\tilde{\nabla}^k_{X_1,\ldots,X_k}\xi$.. Because $\frac{\partial^\ell}{\partial\eta_\ell\cdots\partial\eta_1}$ is symmetric in $\eta_1, \ldots, \eta_\ell$, by polarisation one can write this as a sum over terms of the form $\frac{\partial^\ell}{\partial\overline{\eta}_i^\ell}$ for certain $\overline{\eta}_i \in \Gamma(v^*VW)$ that are linear combinations of the $\eta_i$ with coefficients that are universal constants (depending on $\ell$ but nothing else).
$\tilde{\nabla}^k_{X_1,\ldots,X_k}$ on the other hand is not symmetric in the $X_1, \ldots, X_k$. But, it is "symmetric up to lower order terms involving the curvature". This actually holds for a general vector bundle $E \to M$, where $(M, g)$ is a Riemannian manifold with Levi-Civita covariant derivative $\nabla^M$ and Riemannian curvature tensor $R$, and $\nabla$ is the covariant derivative associated to a connection on $E$, with curvature form $\Omega$. In the case $k = 2$ this then is simply the definition of the curvature

$$\nabla^2_{X,Y} - \nabla^2_{Y,X} = \Omega(X, Y),$$

for $X, Y \in T_xM$ for some $x \in M$, or more explicitely for a section $\xi \in \Gamma(E)$,

$$\begin{aligned}
\nabla^2_{X,Y}\xi - \nabla^2_{Y,X}\xi &= \nabla_X\nabla_Y\xi - \nabla_{\nabla^M_X Y}\xi - (\nabla_Y\nabla_X\xi - \nabla_{\nabla^\Sigma_Y X}\xi) \\
&= \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{\nabla^\Sigma_X Y - \nabla^\Sigma_Y X}\xi \\
&= \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi \\
&= \Omega(X, Y)\xi.
\end{aligned}$$

This generalises to the general curvature identities for all $k \in \mathbb{N}$ in the sense that for $\sigma \in \mathcal{S}_k$, the group of permutations of $\{1, \ldots, k\}$,

$$\nabla^k_{X_1,\ldots,X_k} - \nabla^k_{X_{\sigma(1)},\ldots,X_{\sigma(k)}} = A(X_1, \ldots, X_k),$$

where $A$ is given by a universal expression in $\nabla^r$, $\nabla^r\Omega$ and $(\nabla^M)^r R$ for $0 \leq r \leq k-2$. Although it is possible to give explicit formulas for the tensors $A$, they are exceptionally long and here only the consequences stated above are needed. From this, by repeated polarisation, one can show the following lemma:

**Lemma II.7.** *Let $E \to M$ be a vector bundle over a Riemannian manifold $(M,g)$ with Levi-Civita covariant derivative $\nabla^M$ and Riemannian curvature tensor $R$. Let also $\nabla$ be the covariant derivative associated to a connection on $E$, with curvature form $\Omega$. Then for any $k \geq 2$ and $X_1,\dots,X_k \in T_xM$ for some $x \in M$ and $\xi \in \Gamma(E)$,*

$$\nabla^k_{X_1,\dots,X_k}\xi = \sum_{i=1}^{2^k} C_i^k \nabla^k_{Y_i,\dots,Y_i}\xi + \\ + \sum_{r=0}^{k-2} \sum_{\substack{\{1,\dots,k\} = \\ \{i_1,\dots,i_r\} \amalg \\ \{j_1,\dots,j_{k-r}\}}} \sum_{s=1}^{N_r^{i_*,j_*}} A_{r,s}^{i_*,j_*}(X_{j_1},\dots,X_{j_{k-r}})\nabla^r_{Y_{r,s}^{i_*,j_*},\dots Y_{r,s}^{i_*,j_*}}\xi,$$

*where the $C_i^k$ are constants that only depend on $k$ and the $Y_i$ are linear combinations of the $X_1,\dots,X_k$ with coefficients only depending on $k$. Also, the $N_r^{i_*,j_*}$ only depend on $k$, $r$ and the partition $i_*, j_*$ of $\{1,\dots,k\}$ and the $A_{r,s}^{i_*,j_*}$ are universal expressions in the tensors $\nabla^t\Omega$ for $0 \leq t \leq k-2-r$. Finally, the $Y_{r,s}^{i_*,j_*} = Y_{r,s}^{i_*,j_*}(X_{i_1},\dots,X_{i_r})$ are images of the $X_{i_1},\dots,X_{i_r}$ under multilinear maps that are universal expressions in the $(\nabla^M)^tR$ for $0 \leq t \leq k-3-r$.*

*Proof.* Omitted. □

A consequence of the above is that to give the estimates in Proposition II.4, instead of $\frac{\partial^\ell}{\partial\eta_\ell\cdots\partial\eta_1}\tilde{\nabla}^k_{X_1,\dots,X_k}\xi$ for all $k,\ell \in \mathbb{N}_0$ and general tuples $\eta_1,\dots,\eta_\ell \in \Gamma^k(v^*VW)$ and $X_1,\dots,X_k \in T_z\Sigma$, it suffices to estimate $\frac{\partial^\ell}{\partial\eta^\ell}\tilde{\nabla}^k_{X,\dots,X}\xi$ for a single $\eta \in \Gamma^k(v^*VW)$ and a single $X \in T_z\Sigma$. Furthermore, one can assume that $X \in T_z\Sigma$ is the evaluation at $z$ of a vector field on $\Sigma$ with the property that $((\nabla^\Sigma_X)^rX)_z = 0$ for all $r \geq 1$, i. e. by assuming that the flow line of $X$ through $z$ is a geodesic for some short time.
This has the added advantage that one can easily calculate $\tilde{\nabla}^k_{X,\dots,X}\xi = (\tilde{\nabla}_X)^k\xi$.

**Lemma II.8.** *Given a Riemannian submersion $\pi : W \to \Sigma$, $k,\ell \in \mathbb{N}_0$, $u \in \Gamma^k(W)$, $\xi,\eta,\rho \in \Gamma^k(u^*VW)$ and $X \in \mathfrak{X}(\Sigma)$ s. t. $((\nabla^\Sigma_X)^rX)_z = 0$ for all $r \geq 1$. Assume that $W_z$ is complete for all $z \in \Sigma$ and let*

$$u_\xi := \exp^\perp(\xi) \in \Gamma^k(W).$$

*Also define families of sections for $\xi' \in \Gamma^k(u^*VW)$ by*

$$\begin{aligned}
\Phi^{\mathrm{v}}_{\xi'}(\rho)_{\exp^\perp(\xi'(z'))} &:= (D^{\mathrm{v}}\exp^{W_z})_{\xi'(z')}(\rho(z')),\\
\Phi^{\mathrm{h}}_{\xi'}(\rho)_{\exp^\perp(\xi'(z'))} &:= (D^{\mathrm{h}}\exp^{W_z})_{\xi'(z')}(\rho(z')),\\
\Phi^{\tau}_{\xi'}(X)_{\exp^\perp(\xi'(z'))} &:= \tau_{\xi'}(X)_{\exp^\perp(\xi'(z'))}
\end{aligned}$$

*and*

$$\Phi_{\xi'}(X)_{\exp^\perp(\xi'(z'))} := D^{\mathrm{v}}u_\xi(X)_{\exp^\perp(\xi'(z'))}$$

*along* $\exp^\perp(\xi')$. *Then for such a family of sections* $\rho_{\xi'}$ *for* $\xi' \in \Gamma^k(W)$ *define*

$$\partial^\perp_\eta \rho_\xi := \left.\frac{\nabla^\perp}{\mathrm{d}\lambda}\right|_{\lambda=0} \rho_{\xi+\lambda\eta},$$

*where* $\left.\frac{\nabla^\perp}{\mathrm{d}\lambda}\right|_{\lambda=0}$ *denotes the covariant derivative along the path* $\exp^\perp(\xi + \lambda\eta)$.
*Let* $z \in \Sigma$ *and assume that* $\delta_z > 0$ *and that* $\|\xi(z)\| < \delta_z$.
*Then there are constants* $C_z^{k,\ell} \in [0,\infty]$ *that are universal expressions in the constants* $C_z^{R^W,r}$, $C_z^{\Omega^\perp,r}$, $C_z^{A,r}$ *for* $0 \le r \le k+\ell$ *and* $C_z^{\sim,r}$, $C_z^{T,r}$ *for* $0 \le r \le k+\ell+1$ *and* $C_z^{u,r}$ *for* $1 \le r \le k$.
*With these, at the point* $\exp^\perp(\xi(z))$,

$$\|(\partial^\perp_\eta)^\ell (\nabla^\perp_{Du_\xi(X)})^k \Phi^{\mathrm{v}}_\xi(\rho)\| \le C_z^{k,\ell} \sum_{r=0}^{k} \sum_{\substack{k_1+\cdots+k_r \\ +i_1+\cdots+i_\ell \\ +i=k,\ k_s\ge 1, \\ i,i_s\ge 0}} \prod_{s=1}^{r} \left\|(\nabla_X)^{k_s}\xi\right\| \prod_{s=1}^{\ell} \|(\nabla_X)^{i_s}\eta\| \, \|(\nabla_X)^i \rho\|$$

$$\|(\partial^\perp_\eta)^\ell (\nabla^\perp_{Du_\xi(X)})^k \Phi^{\mathrm{h}}_\xi(\rho)\| \le C_z^{k,\ell} \sum_{r=0}^{k} \sum_{\substack{k_1+\cdots+k_r \\ +i_1+\cdots+i_\ell \\ +i=k,\ k_s\ge 1, \\ i,i_s\ge 0}} \prod_{s=1}^{r} \left\|(\nabla_X)^{k_s}\xi\right\| \prod_{s=1}^{\ell} \|(\nabla_X)^{i_s}\eta\| \, \|(\nabla_X)^i \rho\|$$

$$\|(\partial^\perp_\eta)^\ell (\nabla^\perp_{Du_\xi(X)})^k \Phi^{\tau}_\xi(X)\| \le C_z^{k,\ell} \sum_{r=0}^{k} \sum_{\substack{k_1+\cdots+k_r \\ +i_1+\cdots+i_\ell \\ +i=k,\ k_s\ge 1, \\ i,i_s\ge 0}} \prod_{s=1}^{r} \left\|(\nabla_X)^{k_s}\xi\right\| \prod_{s=1}^{\ell} \|(\nabla_X)^{i_s}\eta\|$$

$$\|(\partial^\perp_\eta)^\ell (\nabla^\perp_{Du_\xi(X)})^k \Phi_\xi(X)\| \le C_z^{k,\ell} \sum_{r=0}^{k} \sum_{\substack{k_1+\cdots+k_r \\ +i_1+\cdots+i_\ell \\ +i=k,\ k_s\ge 1, \\ i,i_s\ge 0}} \prod_{s=1}^{r} \left\|(\nabla_X)^{k_s}\xi\right\| \prod_{s=1}^{\ell} \|(\nabla_X)^{i_s}\eta\| \left(\left\|(\nabla_X)^i \nabla_X \xi\right\| + \left\|(\nabla_X)^i D^{\mathrm{v}} u(X)\right\| + 1\right).$$

*Proof.* I will give only a very rough sketch of the proof, for doing all the details would get out of hand very quickly.
The last inequality follows immediately from the previous three and Corollary II.1.
The proof of the first three inequalities is by double induction over $k$ and $\ell$.

The case $k = \ell = 0$ for $\Phi^{\mathrm{v}}(\rho)$, $\Phi^{\mathrm{h}}(\rho)$ and $\Phi^{\tau}(X)$ follows directly from Proposition II.5 and Corollary II.2.
Now for the induction step: Given $\eta$, $X$ as above, let, for $\epsilon > 0$ some constant, $z(\cdot) : (-\epsilon, \epsilon) \to \Sigma$ be a flowline of $X$. Depending on whether one is dealing with $\eta$ or $X$, let $\alpha : [0,1] \times (-\epsilon, \epsilon) \to W$ be either the map $\alpha(t,\lambda) := \exp^{\perp}(t(\xi + \lambda\eta))$ or $\alpha(t,\lambda) := \exp^{\perp}(t\xi(z(\lambda)))$. In either case, there are two commuting vector fields along $\alpha$, $\dot\gamma := \frac{\partial\alpha}{\partial t}$ and $\rho := \frac{\partial\alpha}{\partial\lambda}$. By definition, $\rho(t,0)$ is either $t\Phi^{\mathrm{v}}_{t\xi}(\eta)$ or $Du_{t\xi}(X)$.

*Claim.* Let $J$ be a vertical vector field on $W$ along $\alpha$ that satisfies an inhomogeneous Jacobi equation of the form

$$\begin{aligned} \nabla^{\perp}_{\dot\gamma}\nabla^{\perp}_{\dot\gamma} J &= A(\dot\gamma,\dot\gamma)J + V(\dot\gamma,\dot\gamma) \\ J(0,\lambda) &= J_0(\lambda), \qquad (\nabla^{\perp}_{\dot\gamma} J)(0,\lambda) = \dot J_0(\lambda). \end{aligned}$$

Then $\nabla^{\perp}_{\rho} J$ satisfies an inhomogeneous Jacobi equation of the form

$$\begin{aligned} \nabla^{\perp}_{\dot\gamma}\nabla^{\perp}_{\dot\gamma}(\nabla^{\perp}_{\rho} J) &= A(\dot\gamma,\dot\gamma)(\nabla^{\perp}_{\rho} J) + V_0(\dot\gamma,\dot\gamma,\rho,J) + \\ &\quad + V_1(\dot\gamma,\nabla^{\perp}_{\dot\gamma}\rho,J) + V_2(\dot\gamma,\rho,\nabla^{\perp}_{\dot\gamma}J) + (\nabla^{\perp}_{\rho}V)(\dot\gamma,\dot\gamma) \\ \nabla^{\perp}_{\rho}J(0,\lambda) &= \nabla^{\perp}_{\rho}J_0(\lambda), \qquad (\nabla^{\perp}_{\dot\gamma}(\nabla^{\perp}_{\rho}J))(0,\lambda) = \nabla^{\perp}_{\rho}\dot J_0(\lambda) + \Omega^{\perp}(\dot\gamma,\rho)J_0(\lambda), \end{aligned}$$

where the $V_i$ are tensorial expressions in $V$, $A$, $\nabla^{\perp}A$, $\Omega^{\perp}$ and $\nabla^{\perp}\Omega^{\perp}$.

*Proof.* One calculates using that $[\rho,\dot\gamma] = 0$, in particular $\nabla^{\perp}_{\dot\gamma}\rho = \nabla^{\perp}_{\rho}\dot\gamma$, and the Jacobi equation for $J$

$$\begin{aligned} \nabla^{\perp}_{\dot\gamma}\nabla^{\perp}_{\dot\gamma}\nabla^{\perp}_{\rho} J &= \nabla^{\perp}_{\dot\gamma}(\nabla^{\perp}_{\rho}\nabla^{\perp}_{\dot\gamma} J + \Omega^{\perp}(\dot\gamma,\rho)J) \\ &= \nabla^{\perp}_{\rho}\nabla^{\perp}_{\dot\gamma}\nabla^{\perp}_{\dot\gamma} J + \Omega^{\perp}(\dot\gamma,\rho)\nabla^{\perp}_{\dot\gamma}J + (\nabla^{\perp}_{\dot\gamma}\Omega^{\perp})(\dot\gamma,\rho)J + \\ &\quad + \Omega^{\perp}(\dot\gamma,\nabla^{\perp}_{\dot\gamma}\rho)J + \Omega^{\perp}(\dot\gamma,\rho)\nabla^{\perp}_{\dot\gamma}J \\ &= (\nabla^{\perp}_{\rho}A)(\dot\gamma,\dot\gamma)J + A(\nabla^{\perp}_{\dot\gamma}\rho,\dot\gamma)J + A(\dot\gamma,\nabla^{\perp}_{\dot\gamma}\rho)J + \\ &\quad + A(\dot\gamma,\dot\gamma)(\nabla^{\perp}_{\rho}J) + (\nabla^{\perp}_{\rho}V)(\dot\gamma,\dot\gamma) + V(\nabla^{\perp}_{\dot\gamma}\rho,\dot\gamma) + \\ &\quad + V(\dot\gamma,\nabla^{\perp}_{\dot\gamma}\rho) + \Omega^{\perp}(\dot\gamma,\rho)\nabla^{\perp}_{\dot\gamma}J + (\nabla^{\perp}_{\dot\gamma}\Omega^{\perp})(\dot\gamma,\rho)J + \\ &\quad + \Omega^{\perp}(\dot\gamma,\nabla^{\perp}_{\dot\gamma}\rho)J + \Omega^{\perp}(\dot\gamma,\rho)\nabla^{\perp}_{\dot\gamma}J. \end{aligned}$$

Now resort all the terms above. □

For $k,\ell \geq 0$, the estimates then follow by induction using the claim above and Corollary II.2 for the estimate, where in each step the estimates from the previous step are used to estimate all the terms appearing in the inhomogeneous Jacobi equations. When doing this for $\Phi^{\tau}_{\xi}(X)$, because of the initial condition $\nabla^{\perp}_{\dot\gamma}J^{\tau}(0) = -T_{\xi}\tilde X$ in Proposition II.5, one has to calculate $\nabla^{\top}_{Du_{t\xi}(X)}\tilde X = \nabla^{\top}_{D^{\mathrm{v}}u_{t\xi}(X)+\tilde X}\tilde X = \nabla^{\top}_{D^{\mathrm{v}}u_{t\xi}(X)}\tilde X + \nabla^{\top}_{\tilde X}\tilde X$. Now $\nabla^{\top}_{\tilde X}\tilde X = \widetilde{\nabla^{\Sigma}_X X}$ (see Proposition 6.1 in [Sak96]), which was assumed to vanish at $z$. The other term $\nabla^{\top}_{D^{\mathrm{v}}u_{t\xi}(X)}\tilde X$, and similar ones obtained by repeating this procedure, then are

the reason for the appearance of the constants $C_z^{\sim,r}$ in Definition II.13 and Proposition II.4.
Making all of the above precise is then just an extremely long, tedious and annoying matter of bookkeeping.
In the induction step for $\ell \geq 0$ there is one caveat, because $\Phi^{\mathrm{v}}(\rho)_{t\xi}$ is given by $\frac{1}{t}J_\rho^{\mathrm{v}}(t)$, where $J_\rho^{\mathrm{v}}(t)$ is as in Proposition II.5. Then $\nabla^\perp_{Du_{t\xi}(X)}\left(\frac{1}{t}J_\rho^{\mathrm{v}}(t)\right) = \frac{1}{t}\nabla^\perp_{Du_{t\xi}(X)}J_\eta^{\mathrm{v}}(t)$, because $Du_{t\xi}(X)$ is by definition parallel to $\{t = \text{const}\}$. The same holds for $\Phi^{\mathrm{v}}_{t\xi}(\eta)$ as long as $\eta$ and $\rho$ are perpendicular, by Gauss' lemma. The problem is hence the case where $\xi$ and $\eta$ are parallel. Then as before, $(\nabla^\perp_{Du_{t\xi}(X)})^k\Phi^{\mathrm{v}}_\xi(\rho)$ is given by $\frac{1}{t}J(t)$, where $J(t)$ solves an inhomogeneous Jacobi equation. Furthermore, one can show that $J(0) = 0$ and that $J(t)$ has bounded derivatives in $t$ of all orders. The term to be estimated is then $(\nabla^\perp_{\dot\gamma})^\ell(\frac{1}{t}J(t))$, which can be done via Taylor's expansion. □

Using the lemma above, one can then prove Proposition II.4.
Again, I will give a sketch of the proof here for $g = \tilde{g}$, making the general case precise is then a straightforward, although quite substantial matter of bookkeeping.
In the notation and under the assumptions of the proposition, let $\xi = \xi(\zeta) := (\exp_u^\perp)^{-1} \circ \exp_v^\perp(\zeta)$ and let $u_\xi := \exp_u^\perp(\xi)$, $v_\zeta := \exp_v^\perp(\zeta)$. Then $u_\xi = v_\zeta$ and hence $D^{\mathrm{v}}u_\xi(X) = D^{\mathrm{v}}v_\zeta(X)$. The estimation process then consists of two steps which in turn each consist of an induction over three steps:
First, by induction over $k$, one gives the following estimates, where the induction start $k = 0$ is just the assumption $\|\xi(z)\| < \delta_z$ and the induction step is as follows:
One estimates $(\nabla^\perp_{Du_\xi(X)})^k D^{\mathrm{v}}u_\xi(X) = (\nabla^\perp_{Dv_\zeta(X)})^k D^{\mathrm{v}}v_\zeta(X)$ in terms of the $\|(\nabla_X)^i\zeta\|$ for $i = 0, \ldots, k+1$ using Lemma II.8, applied to $v$ and $\zeta$ instead of $u$ and $\xi$.
Then one estimates $(\nabla^\perp_{Du_\xi(X)})^k D^{\mathrm{v}}u_\xi(X) - (\nabla^\perp_{Du_\xi(X)})^k\Phi^{\mathrm{v}}_\xi(\nabla_X\xi)$, which is given by $(\nabla^\perp_{Du_\xi(X)})^k\Phi^{\mathrm{h}}_\xi + (\nabla^\perp_{Du_\xi(X)})^k\Phi^\tau_\xi(X)$, again using Lemma II.8, but this time the estimate is in terms of the $\|(\nabla_X)^i\xi\|$, only for $i = 0, \ldots, k$. By the induction hypothesis, the $\|(\nabla_X)^i\xi\|$ then are estimated in terms of the $\|(\nabla_X)^i\zeta\|$.
And third, one estimates $(\nabla^\perp_{Du_\xi(X)})^k\Phi^{\mathrm{v}}_\xi(\nabla_X\xi) - \Phi^{\mathrm{v}}_\xi((\nabla_X)^k\nabla_X\xi)$, again using Lemma II.8, in terms of the $\|(\nabla_X)^i\xi\|$, only for $i = 0, \ldots, k$.
And again, by the induction hypothesis, the $\|(\nabla_X)^i\xi\|$ then are estimated in terms of the $\|(\nabla_X)^i\zeta\|$. Because of the assumption $\|\xi(z)\| < \delta_z$, $\|((D^{\mathrm{v}}\exp^{W_z})_\xi)^{-1}\| \leq 2$ (see Corollary 4.6.1 in [Jos02]), so this gives an estimate for $(\nabla_X)^{k+1}\xi$ in terms of the $\|(\nabla_X)^i\zeta\|$ for $i = 0, \ldots, k+1$.

These results one then uses as induction start for an induction over $\ell$, basically going over the same three steps above, but this time involving $\frac{\partial}{\partial\eta}$ and $\partial^\perp_\eta$.

## II.2.3 Definition of manifolds of sections of a Riemannian submersion

For similar notions of geometric boundedness in the linear case and for trivial bundles, see e. g. [Aub82] or [Eic07].

**Definition II.14.** A Riemannian submersion $\pi : W \to \Sigma$ is said to be of *bounded geometry up to order* $r$, if the following hold for the constants from Definition II.13:

$$\operatorname{inj}(\Sigma) > 0, \iota := \inf\{\operatorname{inj}(W_z) \mid z \in \Sigma\} > 0,$$

$C^{\Sigma,s} < \infty$ for $s = 0, \ldots, r-3$, $C^{R^W,s}, C^{\Omega^\perp,r}, C^{A,s} < \infty$ for $s = 0, \ldots, r$, $C^{\sim,s}, C^{T,s} < \infty$ for $s = 0, \ldots, r+1$ and $\delta > 0$.
If $\pi : W \to \Sigma$ is of bounded geometry up to order $r$ for all $r \geq 0$, then $\pi : W \to \Sigma$ is said to be of *bounded geometry*.

*Remark* II.6. The lower bound on the injectivity radii of the fibres in the definition above implies in particular that $\operatorname{inj}(W_z) > 0$ for all $z \in \Sigma$ and hence that every fibre $W_z$ is a complete Riemannian manifold.

**Definition II.15.** Let $(\Sigma, h)$ be an oriented Riemannian manifold of dimension $n$ and let $k \in \mathbb{N}$, $1 < p < \infty$ with $kp > n$ (so $L^{k,p}(\Sigma) \subseteq C^0(\Sigma, \mathbb{R})$). A $(k,p)$-*Sobolev function on* $\Sigma$ is a continuous function $s^{\Sigma,k,p} : \Sigma \to (0, \infty)$, $z \mapsto s_z^{\Sigma,k,p}$, s. t.

$$|f(z)| \leq s_z^{\Sigma,k,p} \|f\|_{L^{k,p}} \quad \forall f \in L^{k,p}(\Sigma), z \in \Sigma.$$

If this function is constant, then its image, again denoted by $s^{\Sigma,k,p} \in (0,\infty)$, is called a $(k,p)$-*Sobolev constant*.

**Lemma II.9.** *Let $(\Sigma, h)$ be an oriented Riemannian manifold of dimension $n$ and let $k \in \mathbb{N}$, $1 \leq p \leq \infty$ with $kp > n$ (so $L^{k,p}(\Sigma) \subseteq C^0(\Sigma, \mathbb{R})$). Then a $(k,p)$-Sobolev function $s^{\Sigma,k,p}$ exists and has the property that for every Riemannian vector bundle $\rho : E \to \Sigma$ equipped with a Riemannian connection,*

$$\|\xi(z)\| \leq s_z^{\Sigma,k,p} \|\xi\|_{L^{k,p}} \quad \forall \xi \in L^{k,p}(E), z \in \Sigma.$$

*Proof.* See [MS04], Remark 3.5.1, p. 67 f. □

**Construction II.2.** Let $\pi : W \to \Sigma$ be a Riemannian submersion of bounded geometry, let $k \in \mathbb{N}_0$, $p > 1$ with $kp > \dim \Sigma$ and let $s^{\Sigma,k,p} \in (0,\infty)$ be a $(k,p)$-Sobolev constant.
The basic idea behind the construction of Sobolev spaces of sections of $\pi : W \to \Sigma$ is to define $L^{k,p}(W, \pi, g)$ as a subset of $\Gamma^0(W)$ (with the compact-open topology) s. t. the inclusion is continuous. Having this explicit embedding of the Sobolev spaces in the spaces of continuous sections already present in the construction, as opposed to more abstract definitions of $L^{k,p}(W, \pi, g)$, has the added advantage that one can always more easily compare the Sobolev spaces for different $k, p$ and their topologies and smooth structures. So take a section

$u \in \Gamma^k(W)$ and consider the Riemannian vector bundle $u^*VW \to \Sigma$. In addition, assume that $u$ has bounded derivatives up to order $k$, i. e. $C^{u,r} < \infty$ for all $r = 1, \dots, k$. There is then a Sobolev space $L^{k,p}(u^*VW, h, g, \nabla^\perp)$ that is defined as a Banach space in one of the usual ways (completion of the space of smooth section or via weak derivatives). The important properties these have to satisfy are the following, irrespective of the precise definition:
First of all, the smooth sections of finite $L^{k,p}$-norm form a dense linear subspace, cf. Proposition 3.2, p. 15, in [Eic07].
Second, the Sobolev embedding theorems hold, cf. Theorem 3.4, p. 16, in [Eic07].
And third, the module structure theorem holds, cf. Theorem 3.12, p. 20, in [Eic07].
Apart from the fact that the boundedness assumption on the covariant derivatives of $D^{\mathrm{v}}u$ are necessary for the embedding and module structure theorems quoted above to hold, they are also necessary to be able to use Proposition II.4 in the construction of the transition functions later on.
But on the other hand, these are the only results about the (linear) Sobolev space needed for the construction below, so one can easily replace them by weighted Sobolev spaces, for example.

To shorten notation, in the following, $L^{k,p}(u^*VW)$ will be written instead of $L^{k,p}(u^*VW, h, g, \nabla^\perp)$.
Since $kp > n := \dim \Sigma$, the Sobolev embedding theorem shows the existence of a canonical continuous injection $\Psi_u : L^{k,p}(u^*VW) \hookrightarrow \Gamma^0(u^*VW)$. As a consequence, the map $\Phi_u : L^{k,p}(u^*VW) \to \Gamma^0(W)$, $\xi \mapsto (z \mapsto \exp^\perp_{u(z)}(\xi(z)))$, is well defined and continuous as it factors through $\Psi_u$. The next step is to find a neighbourhood $V_u$ of the zero section in $L^{k,p}(u^*VW)$ that gets mapped injectively by $\Phi_u$ into $\Gamma^0(W)$, to serve as a chart of a Banach manifold structure. Here a problem arises if $\Sigma$ is noncompact. Ideally, injectivity should result from pointwise injectivity, which requires that for every $z \in \Sigma$, $\|\Psi_u(\xi)(z)\| \le \mathrm{inj}_{W_z}$, where $\mathrm{inj}_{W_z}$ denotes the injectivity radius of the fibre $W_z$ over the point $z \in \Sigma$. Because the existence of a Sobolev constant $s^{\Sigma,k,p}$ was assumed and remembering that the constant $\delta$ from Definition II.13 bounds the injectivity radii of the $W_z$ from below, since the $W_z$ are complete, one can take the open subset $V_u := \{\xi \in L^{k,p}(u^*VW) \mid \|\xi\|_{L^{k,p}} < \delta/(3s^{\Sigma,k,p})\}$. Anyway, one can tentatively define

$$L^{k,p}(W, \pi, g) := \bigcup_{\substack{u \in \Gamma^k(W) \\ C^{u,r} < \infty, \\ r=1,\dots,k}} \Phi_u(V_u) \subseteq \Gamma^0(W).$$

The topology on this set is defined to be the topology generated by the union of the induced topology on $L^{k,p}(W, \pi, g)$ as a subset of $\Gamma^0(W)$ and the topologies on the subsets $\Phi_u(V_u)$ induced by the Banach space topologies on the $V_u \subseteq L^{k,p}(u^*VW)$. This is clearly a Hausdorff space, the topology being finer than the (Hausdorff) topology on $\Gamma^0(W)$. But it is not yet clear that it is $2^{\mathrm{nd}}$-countable. At this point a caveat has to be issued, for it is not clear that the natural candidates for coordinate maps, the $\Phi_u$, are homeomorphisms onto their image, with continuity possibly failing (they are continuous w. r. t. the $C^0$-topology, but this does not imply continuity w. r. t. the finer topology on $L^{k,p}(W,\pi,g)$), although they are open injections by definition of the topology on $L^{k,p}(W,\pi,g)$. Another way to phrase this is by noting that the natural candidate for a transition function, $\Phi_u^{-1}\circ\Phi_v$ for $\Phi_u(V_u)\cap\Phi_v(V_v)\neq\emptyset$, restricted to $\Phi_v^{-1}(\Phi_u(V_u))\subseteq V_v$, need not be a homeomorphism between open subsets of $V_v$ and $V_u$.

So assume that there exist $\zeta\in V_v$, $\xi\in V_u$ with $\Phi_v(\zeta)=\Phi_u(\xi)$. As before, denote $\Phi_u(\xi)=: u_\xi = v_\zeta := \Phi_v(\zeta)\in\Gamma^0(W)$. Then for all $\xi'\in V_u$ and $z\in\Sigma$, $d(v(z),\Phi_u(\xi')(z))\leq d(v(z),v_\zeta(z))+d(u_\xi(z),u(z))+d(u(z),\Phi_u(\xi')(z))<\delta$ by definition of $V_u$. And analogously for $\zeta'\in V_v$ with the roles of $u$ and $v$ exchanged. This implies that there are well-defined maps, by abuse of notation,

$$\Phi_u^{-1}\circ\Phi_v : V_v\to\Gamma^0(u^*VW)$$

and

$$\Phi_v^{-1}\circ\Phi_u : V_u\to\Gamma^0(v^*VW).$$

Denote $V_v^k := V_v\cap\Gamma^k(v^*VW)$ and analogously $V_u^k := V_u\cap\Gamma^k(u^*VW)$. Then for $\eta\in L^{k,p}(v^*VW)\cap\Gamma^k(v^*VW)$ small enough, denote $\zeta'=\zeta'(\eta):=\zeta+\eta\in V_v^k$. Then from Proposition II.4 follow pointwise estimates

$$\left\|\frac{\partial}{\partial\eta}\nabla^m\Phi_u^{-1}\circ\Phi_v(\zeta')\right\|^p\leq C\sum_{k'=0}^{m}\sum_{r,i,k'_j}\prod_{s=1}^{r}\|\nabla^{k'_s}\zeta'\|^p\|\nabla^i\eta\|^p,$$

where $C$ is a constant that depends on all the bounds above, esp. $C^{u,r}$ and $C^{v,r}$ for $r=1,\dots,k$, but not on anything else. Integration over $\Sigma$ and applying the module structure theorem, for $m=0,\dots,k$, then gives a global estimate

$$\left\|\frac{\partial}{\partial\eta}\Phi_u^{-1}\circ\Phi_v(\zeta')\right\|_{L^{k,p}}\leq C'\|\zeta'\|_{L^{k,p}}\|\eta\|_{L^{k,p}}.$$

The mean value theorem then implies that for $\zeta'\in V_u^k$, $\Phi_u^{-1}\circ\Phi_v(\zeta')\in L^{k,p}(u^*VW)$. In particular, for $\eta$ small enough, i. e. $\zeta'$ close enough to $\zeta$, in $L^{k,p}$-norm, $\Phi_u^{-1}\circ\Phi_v(\zeta')\in V_u^k$. The same clearly also holds with the roles of $u$ and $v$ interchanged. This implies that

$$\Phi_u^{-1}\circ\Phi_v : \Phi_v^{-1}(\Phi_u(V_u^k))\to V_u^k$$

is a well-defined Lipschitz continuous map that hence has a well-defined Lipschitz continuous completion to a map

$$\Phi_u^{-1}\circ\Phi_v : \Phi_v^{-1}(\Phi_u(V_u))\to V_u.$$

In particular, $\Phi_v^{-1}(\Phi_u(V_u))\subseteq V_v$ is an open subset. Again, the same holds with the roles of $u$ and $v$ interchanged and the resulting maps are inverses to each

other.
Now the same line of arguments above using Proposition II.4 and the module structure theorem implies that the map

$$\Phi_u^{-1} \circ \Phi_v : \Phi_v^{-1}(\Phi_u(V_u^k)) \to V_u^k$$

has Lipschitz continuous derivatives of all orders. By Corollary A.1, this in turn implies that the completion

$$\Phi_u^{-1} \circ \Phi_v : \Phi_v^{-1}(\Phi_u(V_u)) \to V_u.$$

is smooth.

So far, this turns $L^{k,p}(W, \pi, g)$ into an a priori non-2$^{\text{nd}}$-countable Banach manifold. If $\Sigma$ is compact, then the standard argument via choosing a $C^k$-dense countable subset $\{u_i\}_{i\in\mathbb{N}}$ of $\Gamma^k(W)$ and in the charts above around each $u_i$ a $L^{k,p}$-dense countable subset applies, for the boundedness condition on the covariant derivatives of the $D^{\mathrm{v}} u_i$ are automatically satisfied and any smooth section of $u^*VW$ lies automatically in $L^{k,p}(u^*VW)$.
In general, for the above to carry over, one immediate condition is that the Banach spaces $L^{k,p}(u^*VW)$ need to be separable. But if $u, u' \in \Gamma^k(W)$ satisfy $C^{u,r}, C^{u',r} < \infty$ for $r = 1, \dots, r$, then $u$ and $u'$ $C^k$-close does not imply that one lies in the chart around the other. So it does not suffice to take dense subsets of the $V_u$ for $u$ in a $C^k$-dense subset of $\Gamma^k(W)$ as before.
In practice, it is then easier to just restrict the set of $C^k$-sections around which $L^{k,p}(W, \pi, g)$ is constructed to a countable subset, tailored to the concrete problem.
An example for this would be the maps with cylindrical ends used in SFT-Fredholm theory.

In the compact case, i. e. $W$ and hence $\Sigma$ compact, all of these problems vanish, because all the assumptions on finiteness of the constants $C^{*,r}$ are automatically satisfied. Also in this case the constants $C^{g,\tilde{g},r}$ appearing in Proposition II.4, in case $\pi : W \to \Sigma$ is equipped with two different structures of Riemannian submersion, are automatically finite, which implies that the Banach manifold structure on $L^{k,p}(W, \pi, g)$ is independent of $g$.

**Lemma II.10.** *If $\pi : W \to \Sigma$ is a Riemannian submersion and $W$ is compact, then $L^{k,p}(W, \pi, g)$ is a $2^{nd}$-countable Hausdorff Banach manifold and the underlying set as well as the Banach manifold structure on this set are independent of $g$.*

*Proof.* Only independence of the Riemannian structure needs to be shown.
For this, what one wants to show is that the set-theoretic identity defines a smooth map between the Banach manifolds built with respect to two different choices of Riemannian structures, given by metrics $g$ and $\tilde{g}$ on $W$. Expressing the identity in local charts around a point $u \in \Gamma^k(W)$ means that one has to look at maps of the form $\tilde{\Phi}_u^{-1} \circ \Phi_u$, where $\tilde{\Phi}_u$ is as in the construction above,

but w. r. t. the metric $\tilde{g}$. The proof that this defines a smooth map between open subsets then proceeds literally as the corresponding proof outlined in the construction above using Proposition II.4. □

One last easy consequence of the construction of $L^{k,p}(W,\pi,g)$ above is the nonlinear version of the Sobolev embedding theorem. For this, note that one can construct Banach manifolds of sections of class $C^k$ very analogously to the $L^{k,p}$-spaces above. Given a vector bundle $\rho : E \to B$ together with a Riemannian metric $h$ on $B$, a fibre metric $g$ on $E$ and a metric connection $\nabla$, the space of sections of class $C^k$, for $k \in \mathbb{N}_0$, is the Banach space ($\|\cdot\|_\infty$ denotes the usual supremums-norm on functions)

$$\Gamma^k(E,h,g,\nabla) := \{\xi \in \Gamma^k(E) \mid \sum_{i=0}^{k} \||\nabla^i\xi|\|_\infty < \infty\},$$

where $|\nabla^i\xi| \in C^k(B)$ denotes the norm on $(\Lambda^i T^*B) \otimes E$ induced by $h$ and $g$.

**Construction II.3.** If $\pi : W \to \Sigma$ is a Riemannian submersion of bounded geometry, then in the notation of Definition II.13, for $u \in \Gamma^k(W)$ with $C^{u,r} < \infty\ \forall\, r = 1,\ldots,k$, let

$$U_u := \{\xi \in \Gamma^k(u^*VW,h,g,\nabla^\perp) \mid \||\xi|\|_\infty < \delta\}.$$

Then analogously to the previous construction, there are well-defined injective maps

$$\Psi_u : U_u \to \Gamma^k(W).$$

Using these, one defines the space of sections of class $C^k$ of $\pi : W \to \Sigma$ as

$$\Gamma^k(W,\pi,g) := \bigcup_{\substack{u\in\Gamma^k(W)\\ C^{u,r}<\infty,\\ r=1,\ldots,k}} \Phi_u(V_u) \subseteq \Gamma^0(W).$$

It follows directly from Proposition II.4 that the maps $\Psi_u$ define an atlas for a Banach manifold structure on $\Gamma^k(W,\pi,g)$.

Then the following nonlinear version of the Sobolev embedding theorem is an immediate consequence of the linear Sobolev embedding theorem and the constructions of the Banach manifolds involved.

**Lemma II.11.** *Let $\pi : W \to \Sigma$ be a Riemannian submersion of bounded geometry. Let furthermore $k,\ell \in \mathbb{N}_0$ an $p \in (1,\infty)$ with $k - \frac{\dim\Sigma}{p} > \ell$. Then there is a smooth embedding*

$$L^{k,p}(W,\pi,g) \hookrightarrow \Gamma^\ell(W,\pi,g),$$

*defined by the restriction of the set-theoretic identity on $\Gamma^0(W)$.*

### II.2.4 Bundles of sections of a vector bundle

Another important concept used in the next part of this thesis are the Banach space bundles of sections of a vector bundle one constructs over the spaces of sections of a Riemannian submersion from the previous subsection. So let $\pi : W \to \Sigma$ be a Riemannian submersion of bounded geometry and let $\rho : E \to W$ be a vector bundle equipped with a fibre metric $g^E$ and a metric connection $\nabla^E$.

**Definition II.16.** Let $\rho : E \to W$ be a vector bundle over a Riemannian manifold equipped with a fibre metric $g^E$ and a metric connection $\nabla^E$. Let $\Omega^E$ be the curvature of $\nabla^E$ and denote for $s \in \mathbb{N}_0$

$$C^{\Omega^E,s} := \sup\{\|(\nabla^E)^s\Omega^E_w\| \mid w \in W\}.$$

$\rho : E \to W$ is said to be of *bounded geometry up to order* $r \in \mathbb{N}_0$, if $C^{\Omega^E,s} < \infty$ for all $s = 0, \dots, r$.
If this holds for all $r \in \mathbb{N}_0$, then $\rho : E \to W$ is said to be of *bounded geometry.*

**Construction II.4.** Assume now that $\rho : E \to W$ as well as $\pi : W \to \Sigma$ are of bounded geometry. For $k, \ell \in \mathbb{N}_0$, $p \in (1, \infty)$ with $k - \frac{\dim \Sigma}{p} > \ell$, by Lemma II.11, $L^{k,p}(W, \pi, g) \subseteq \Gamma^\ell(W, \pi, g)$. In particular, for $u \in L^{k,p}(W, \pi, g)$, $u^*E \to \Sigma$ is a vector bundle of class $C^\ell$, equipped with a fibre metric and a connection of class $C^\ell$, as well. Furthermore, because by assumption ($\pi : W \to \Sigma$ geometrically bounded) there is a lower bound on the injectivity radius of $\Sigma$ and the $C^{\Sigma,r}$ are finite. So by Proposition 3.2, p. 15 in [Eic07], the sections of class $C^\ell$ in $L^{\ell,p}(u^*E) := L^{\ell,p}(u^*E, u^*g^E, u^*\nabla^E)$ are dense. So one can define, as a set,

$$L^{\ell,p}(E, g^E, \nabla^E, W, \pi, g) := \coprod_{u \in L^{k,p}(W,\pi,g)} L^{\ell,p}(u^*E),$$

which comes with a canonical projection

$$\Pi : L^{\ell,p}(E, g^E, \nabla^E, W, \pi, g) \to L^{k,p}(W, \pi, g).$$

The goal is now to turn this into a smooth Banach space bundle. The standard way to do so is to define local trivialisations over the charts on $L^{k,p}(W, \pi, g)$ defined in Construction II.2 via parallel transport:
In the notation used there, let for $u \in \Gamma^k(W)$ with $C^{u,r} < \infty$ for $r = 0, \dots, k$,

$$\Phi_u : L^{k,p}(u^*VW) \supseteq V_u \to L^{k,p}(W, \pi, g) \subseteq \Gamma^0(W)$$

a chart. Then the trivialisation of $L^{\ell,p}(E, g^E, \nabla^E, W, \pi, g)$ over $\Phi_u(V_u)$,

$$\begin{tikzcd} V_u \times L^{\ell,p}(u^*E) \arrow[r, "\hat{\Phi}_u"] \arrow[d] & L^{\ell,p}(E, g^E, \nabla^E, W, \pi, g)|_{\Phi_u(V_u)} \arrow[d] \\ V_u \arrow[r, "\Phi_u"] & \Phi_u(V_u), \end{tikzcd}$$

is given by

$$\hat{\Phi}_u(\xi,\sigma) := (\|_{s=0}^1 \Phi_u(s\xi))\sigma,$$

where for $z \in \Sigma$,

$$(\|_{s=0}^1 \Phi_u(s\xi))_z : (u^*E)_z = E_{u(z)} \to E_{(\Phi_u(\xi))(z)} = ((\Phi_u(\xi))^*E)_z$$

denotes parallel transport along the path $((\Phi_u(s\xi)(z)))_{s\in[0,1]}$ from $u(z)$ to $(\Phi_u(\xi))(z) = \exp^{\perp}(\xi(z))$.

For the above to be well-defined, it has to be proved that the associated transition maps are smooth. In detail, for $u, v \in \Gamma^k(W)$ with $C^{u,r}, C^{v,r} < \infty \; \forall\, r = 0,\dots,k$, let $V_{uv} := \Phi_u^{-1}(\Phi_v(V_v)) \subseteq V_u$ and analogously $V_{vu} := \Phi_v^{-1}(\Phi_u(V_u)) \subseteq V_v$. Then smoothness of the map

$$\hat{\Phi}_u^{-1} \circ \hat{\Phi}_v : V_{vu} \times L^{\ell,p}(v^*E) \to V_{uv} \times L^{\ell,p}(u^*E)$$

has to be proven, or equivalently, that of the map

$$\hat{\Phi}_{vu} := \mathrm{pr}_2 \circ \hat{\Phi}_u^{-1} \circ \hat{\Phi}_v : V_{vu} \times L^{\ell,p}(v^*E) = L^{\ell,p}(u^*E).$$

This means that for $\eta_1,\dots,\eta_m \in L^{k,p}(v^*VW)$, $\xi \in V_{vu}$ and $\sigma \in L^{\ell,p}(v^*E)$, one needs to estimate at some fixed $z \in \Sigma$, in analogy to Proposition II.4,

$$\frac{\partial^m}{\partial\eta_m \cdots, \partial\eta_1} \nabla^i \hat{\Phi}_{vu}(\xi,\sigma).$$

The basic procedure here is the same as previously for the Banach manifold structure on $L^{k,p}(W,\pi,g)$. One first of all makes pointwise estimates for $\sigma \in L^{\ell,p}(v^*E)$ of class $C^\ell$ and $\xi \in V_{vu}$ of class $C^k$ by setting up an induction scheme. To do so, one first of all follows the line of reasoning on page 43 f. to reduce to the calculation of

$$\frac{\partial^m}{\partial\eta^m} (\nabla_X)^i \hat{\Phi}_{vu}(\xi,\sigma) \tag{II.13}$$

for $\eta \in L^{k,p}(v^*VW)$ and $X \in T_z\Sigma$ for a fixed $z \in \Sigma$. Then, by integrating and using the module structure theorem one deduces the corresponding estimates in $L^{k,p}$- and $L^{\ell,p}$-norms. And finally one uses the density argument provided by Corollary A.1 for the general case.
The main step in setting up the induction scheme, that takes the place of Proposition II.4 is the following calculation, which replaces Proposition II.5:
Let $\xi' : (-\varepsilon,\varepsilon) \to VW$ for some $\varepsilon > 0$ be a path. The two cases of interest here are the following:
First, for some $\xi \in V_v$ and some $\eta \in L^{k,p}(v^*VW)$ as well as some $z \in \Sigma$, $\xi'(\lambda) := \xi(z) + \lambda\eta(z)$.
And second, for some $\xi \in V_v$ and a path $z : (-\varepsilon,\varepsilon) \to \Sigma$, $\xi'(\lambda) := \xi(z(\lambda))$.
Denote also for $\rho \in VW$,

$$\alpha^\rho : [0,1] \to W, \quad t \mapsto \exp^{\perp}(t\rho).$$

Then one has the following property for the covariant derivative of a section of $E$ along the path $\exp^{\perp} \circ \xi' : (-\varepsilon,\varepsilon) \to W$, where $\frac{\nabla}{\mathrm{d}\lambda}$ denotes the total covariant derivative along a path:

**Lemma II.12.** *Let $\xi' : (-\varepsilon, \varepsilon) \to VW$ for some $\varepsilon > 0$ be a path. Then for any section $\sigma$ of $E$ along $\exp^\perp \circ \xi' : (-\varepsilon, \varepsilon) \to W$,*

$$\frac{\nabla}{\mathrm{d}\lambda}\bigg|_{\lambda=0} (\|_0^1\, \alpha^{\xi'(\lambda)})^{-1}\sigma - (\|_0^1\, \alpha^{\xi'(0)})^{-1} \frac{\nabla}{\mathrm{d}\lambda}\bigg|_{\lambda=0} \sigma = \mathcal{H}(\xi'(0), \tfrac{\partial \xi'}{\partial\lambda}(0))(\|_0^1\, \alpha^{\xi'(0)})^{-1}\sigma,$$

*where*

$$\mathcal{H}(\xi'(0), \tfrac{\partial \xi'}{\partial \lambda}(0)) := \int_0^1 sH(s)\,\mathrm{d}s$$

*and*

$$H(s) := \left(\|_0^1\, \alpha^{s\xi'(0)}\right)^{-1} \Omega^E\Big((D\exp^\perp)_{s\xi'(0)}(\tfrac{\partial\xi'}{\partial\lambda}(0)), (D^{\mathrm{v}}\exp^\perp)_{s\xi'(0)}(\xi'(0))\Big)\left(\|_0^1\, \alpha^{s\xi'(0)}\right)$$

*Proof.* Let $u : (-\varepsilon, \varepsilon) \to W$ be defined by $u(\lambda) := \mathrm{pr}_W^{VW} \circ \xi'$. Then by definition,

$$\frac{\nabla}{\mathrm{d}\lambda}\bigg|_{\lambda=0} (\|_0^1\, \alpha^{\xi'(\lambda)})^{-1}\sigma = \frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=0} (\|_{r=0}^1\, v(r\lambda))^{-1}(\|_{r=0}^1 \exp^\perp(r\xi'(\lambda)))^{-1}\sigma$$

and one can write

$$\begin{aligned}
(\|_{r=0}^1\, u(r\lambda))^{-1}(\|_{r=0}^1 \exp^\perp(r\xi'(\lambda)))^{-1} \; &= \\
&= (\|_{r=0}^1\, u(r\lambda))^{-1}(\|_{r=0}^1 \exp^\perp(r\xi'(\lambda)))^{-1} \\
&\quad (\|_{r=0}^1 \exp^\perp(\xi'(r\lambda)))(\|_{r=0}^1 \exp^\perp(r\xi'(0))) \\
&\quad (\|_{r=0}^1 \exp^\perp(r\xi'(0)))^{-1}(\|_{r=0}^1 \exp^\perp(\xi'(r\lambda)))^{-1} \\
&= (\|_{r=0}^4\, \gamma^\lambda(r))(\|_{r=0}^1 \exp^\perp(r\xi'(0)))^{-1}(\|_{r=0}^1 \exp^\perp(\xi'(r\lambda)))^{-1},
\end{aligned}$$

where $\gamma^\lambda : [0,4] \to W$ is the piecewise smooth closed loop defined by

$$\gamma^\lambda(r) := \begin{cases} \exp^\perp(r\xi'(0)) & 0 \le r \le 1 \\ \exp^\perp(\xi'((r-1)\lambda))) & 1 \le r \le 2 \\ \exp^\perp((3-r)\xi'(\lambda)) & 2 \le r \le 3 \\ u((4-r)\lambda) & 3 \le r \le 4 \end{cases}.$$

This loop is contractible via a piecewise smooth contraction $H^\lambda : [0,1] \times [0,4] \to W$, $(s,r) \mapsto H_s^\lambda(r)$, where

$$H_s^\lambda(r) := \begin{cases} \exp^\perp(sr\xi'(0)) & 0 \le r \le 1 \\ \exp^\perp(s\xi'((r-1)s\lambda)) & 1 \le r \le 2 \\ \exp^\perp(s(3-r)\xi'(s\lambda)) & 2 \le r \le 3 \\ u((4-r)s\lambda) & 3 \le r \le 4 \end{cases}$$

and $H_1^\lambda = \gamma^\lambda$. By a result which can be found e.g. in [RW06], Corollary 3, $\|_{r=0}^4\, \gamma^\lambda(r)$ can be expressed as

$$\|_{r=0}^4\, \gamma^\lambda(r) = \mathrm{id}_{E_{u(0)}} + \int_0^1 \int_0^4 (\|_{t=r}^4\, H_s^\lambda(t)) \circ \Omega^E(H_*^\lambda \frac{\partial}{\partial r}, H_*^\lambda \frac{\partial}{\partial s}) \circ (\|_{t=0}^r\, H_s^\lambda(t))\,\mathrm{d}r\mathrm{d}s,$$

where $\Omega^E \in \Omega^2(W, \mathrm{End}(E))$ denotes the curvature 2-form of the connection on $E$. Now for $0 \leq r \leq 1$ and $3 \leq r \leq 4$, $H_* \frac{\partial}{\partial r}$ and $H_* \frac{\partial}{\partial s}$ are linearly dependent, so for these values of $r$, the curvature term in the integrand of the above formula vanishes. For $1 \leq r \leq 3$ one calculates

$$H_*^\lambda \frac{\partial}{\partial s} = \begin{cases} D^{\mathrm{v}} \exp^\perp(\xi'((r-1)s\lambda)) + s(r-1)\lambda D \exp^\perp(\frac{\partial \xi'}{\partial \lambda}((r-1)s\lambda) & 1 < r < 2 \\ (3-r)D^{\mathrm{v}} \exp^\perp(\xi'(s\lambda)) + s(3-r)\lambda D \exp^\perp(\frac{\partial \xi'}{\partial \lambda}(s\lambda)) & 2 < r < 3 \end{cases}$$

$$H_*^\lambda \frac{\partial}{\partial r} = \begin{cases} s^2 \lambda D \exp^\perp(\frac{\partial \xi'}{\partial \lambda}((r-1)s\lambda)) & 1 < r < 2 \\ -sD^{\mathrm{v}} \exp^\perp(\xi'(s\lambda)) & 2 < r < 3 \end{cases}.$$

Hence after reparametrising in $r$ and again using that $\Omega^E(X, Y) = 0$ for $X$ and $Y$ parallel as well as the antisymmetry of $\Omega^E$,

$$\|_{r=0}^4 \gamma^\lambda = \mathrm{id}_{E_{u(0)}} + \lambda \int_0^1 s^2 \Big( \int_0^1 (\|_{t=r+1}^4 H_s^\lambda(t)) \circ \Omega^E(D \exp^\perp(\tfrac{\partial \xi'}{\partial \lambda}(rs\lambda)), D^{\mathrm{v}} \exp^\perp(\xi'(rs\lambda))) \circ (\|_{t=0}^{r+1} H_s^\lambda(t)) \,\mathrm{d}r + $$
$$+ \int_0^1 (1-r)(\|_{t=r+2}^4 H_s^\lambda(t)) \circ \Omega^E(D \exp^\perp(\tfrac{\partial \xi'}{\partial \lambda}(s\lambda)), D^{\mathrm{v}} \exp^\perp(\xi'(s\lambda))) \circ (\|_{t=0}^{r+2} H_s^\lambda(t)) \,\mathrm{d}r \Big) \,\mathrm{d}s.$$

One also calculates for $0 \leq r \leq 1$ and $\lambda = 0$,

$$\begin{aligned} \|_{t=0}^{r+1} H_s^0(t) &= \|_{t=0}^1 \exp^\perp(ts\xi'(0)) \\ \|_{t=0}^{r+2} H_s^0(t) &= \|_{t=0}^{1-r} \exp^\perp(ts\xi'(0)) \\ \|_{t=r+1}^4 H_s^0(t) &= \|_{t=0}^1 \exp^\perp((1-t)s\xi'(0)) \\ \|_{t=r+2}^4 H_s^0(t) &= \|_{t=r}^1 \exp^\perp((1-t)s\xi'(0)). \end{aligned}$$

So defining the path $\alpha^{s\xi'(0)}(t) := \exp^\perp(ts\xi'(0))$,

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} \|_{r=0}^4 \gamma^\lambda = & \int_0^1 s^2 \left(\|_0^1 \alpha^{s\xi'(0)}\right)^{-1} \Omega^E\Big((D \exp^\perp)_{s\xi'(0)}(\tfrac{\partial \xi'}{\partial \lambda}(0)), \\ & (D^{\mathrm{v}} \exp^\perp)_{s\xi'(0)}(\xi'(0))\Big) \left(\|_0^1 \alpha^{s\xi'(0)}\right) \mathrm{d}s + \\ & \int_0^1 \int_0^1 s^2 r \left(\|_0^r \alpha^{s\xi'(0)}\right)^{-1} \Omega^E\Big((D \exp^\perp)_{rs\xi'(0)}(\tfrac{\partial \xi'}{\partial \lambda}(0)), \\ & (D^{\mathrm{v}} \exp^\perp)_{rs\xi'(0)}(\xi'(0))\Big) \left(\|_0^r \alpha^{s\xi'(0)}\right) \mathrm{d}r \,\mathrm{d}s \\ = & \int_0^1 s^2 \left(\|_0^1 \alpha^{s\xi'(0)}\right)^{-1} \Omega^E\Big((D \exp^\perp)_{s\xi'(0)}(\tfrac{\partial \xi'}{\partial \lambda}(0)), \\ & (D^{\mathrm{v}} \exp^\perp)_{s\xi'(0)}(\xi'(0))\Big) \left(\|_0^1 \alpha^{s\xi'(0)}\right) \mathrm{d}s + \\ & \int_0^1 \int_0^1 rs \left(\|_0^1 \alpha^{rs\xi'(0)}\right)^{-1} \Omega^E\Big((D \exp^\perp)_{rs\xi'(0)}(\tfrac{\partial \xi'}{\partial \lambda}(0)), \\ & (D^{\mathrm{v}} \exp^\perp)_{rs\xi'(0)}(\xi'(0))\Big) \left(\|_0^1 \alpha^{rs\xi'(0)}\right) s\mathrm{d}r \,\mathrm{d}s \end{aligned}$$

and denoting for simplicity

$$
\begin{aligned}
H(t) :=& \left(\|_0^1 \alpha^{t\xi'(0)}\right)^{-1} \Omega^E\Big((D\exp^\perp)_{t\xi'(0)}(\tfrac{\partial \xi'}{\partial\lambda}(0)),\\
&(D^{\mathrm{v}}\exp^\perp)_{t\xi'(0)}(\xi'(0))\Big)\left(\|_0^1 \alpha^{t\xi'(0)}\right),\\
\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=0} \|_{r=0}^4 \gamma^\lambda =& \int_0^1 s^2 H(s)\,\mathrm{d}s + \int_0^1\int_0^1 rsH(rs)\,s\mathrm{d}r\,\mathrm{d}s\\
=& \int_0^1 \underbrace{\left(s(sH(s)) + \int_0^s tH(t)\,\mathrm{d}t\right)}_{\frac{\mathrm{d}}{\mathrm{d}s} s\int_0^s tH(t)\,\mathrm{d}t} \mathrm{d}s\\
=& \int_0^1 sH(s)\,\mathrm{d}s.
\end{aligned}
$$

□

Using this lemma together with Proposition II.4, one can now estimate the norm of II.13, analogously to Proposition II.4. Since the details are quite analogous to the discussion before, simpler, actually, they will be left out.

**Example II.1.** A canonical example of this is gained by applying the above construction to the vector bundle $\mathrm{Hom}(T\Sigma, VW)$ with $\mathrm{Hom}(T\Sigma, VW)_w = \mathrm{Hom}(T_{\pi(w)}\Sigma, V_w W) \cong T^*_{\pi(w)}\Sigma \otimes V_w W$ with the induced metric $h^* \otimes g|_{VW}$ and connection $\nabla^\perp$.

Finally, note the following two results, which follow fairly easily from the definitions.

**Lemma II.13.** *The vertical derivative $D^{\mathrm{v}}$ defines a section*

$$D^{\mathrm{v}} : L^{k,p}(W,\pi,g) \to L^{k-1,p}(\mathrm{Hom}(T\Sigma, VW), h^*\otimes g|_{VW}, \nabla^\perp, W, \pi, g).$$

*If $u \in \Gamma^r(W,\pi,g)$ then in the chart around $u$ in $L^{k,p}(W,\pi,g)$, this section is given by the formula from Lemma II.2, i. e. for $\xi \in L^{k,p}(u^*VW)$, choosing any measurable section representing $\xi$ and any measurable section representing $\nabla\xi$, for $z \in \Sigma$, $X \in T_z\Sigma$, $D^{\mathrm{v}}\exp_u^\perp(\xi)$ is represented by*

$$X \mapsto (D^{\mathrm{v}}\exp_u^\perp(\xi))(X) = D\exp^{W_z}(D^{\mathrm{v}}u(X), \nabla_X\xi) + \tau_\xi(X).$$

**Lemma II.14.** *Let $(\rho : E \to W, g^E, \nabla^E)$ and $(\sigma : F \to W, g^F, \nabla^F)$ be Riemannian vector bundles over $W$ of bounded geometry. Let $\Phi : E \to F$ be a (linear) bundle morphism (covering the identity on $W$) s. t. $\|\nabla^s\Phi\|_\infty < \infty$ for all $s \in \mathbb{N}_0$. Then the map*

$$
\begin{aligned}
L^{\ell,p}(E, g^E, \nabla^E, W, \pi, g) &\to L^{\ell,p}(F, g^F, \nabla^F, W, \pi, g)\\
e &\mapsto \Phi\circ e
\end{aligned}
$$

*is a Banach bundle morphism.*

## II.3 Construction of smooth structures on moduli spaces

Throughout this section, fix a marked nodal family $(\pi : \Sigma \to M, R_*)$ of type $(g, n)$ and choose a metric $h$ on $\Sigma$ that is hermitian on every fibre of $\Sigma$. Furthermore, let $(\kappa : X \to M, \omega)$ be a family of symplectic manifolds with fibres symplectomorphic to a closed symplectic manifold $(X_0, \omega_0)$ (in other words a fibre bundle with fibre $X_0$ and structure group $\mathrm{Symp}(X_0, \omega_0)$, the symplectomorphism group of $(X_0, \omega_0)$). Define $\tilde{\kappa} : \tilde{X} \to \Sigma$ as the pullback of $\kappa : X \to M$ to $\Sigma$ via $\pi$. As before, $\mathcal{J}_\omega(X)$ is the set of $\omega$-compatible vertical almost complex structures on $X$, i. e. bundle morphisms $J \in \mathrm{End}(VX)$ with $J^2 = -\mathrm{id}$ and s. t. $\omega(\cdot, J\cdot)$ defines a metric on $VX$. In other words, for any $b \in M$, $J_b$ is a compatible almost complex structure on the symplectic manifold $(X_b, \omega_b)$. Such a $J \in \mathcal{J}_\omega(X)$ is chosen and $\tilde{X}$ is equipped with the almost complex structure given by the pullback of $J$ to $\Sigma$ via the projection onto $M$ (and again denoted by $J$), and the metric $g^J$ on $V\tilde{X}$ defined by $\omega$ and $J$. Finally, a locally trivial family $A$ of $2^{\mathrm{nd}}$ homology classes $(A_b)_{b\in M}$, $A_b \in H_2(X_b; \mathbb{Z})$, in the fibres of $X$ is fixed in the sense that there exists a covering $(U_i)_{i\in I}$ of $M$ and trivialisations $\phi_i : X|_{U_i} \cong U_i \times X_0$ s. t. $(\mathrm{pr}_2)_* \circ (\phi_i|_{X_b})_* A_b \in H_2(X_0; \mathbb{Z})$ is independent of $b \in U_i$.

### II.3.1 Hamiltonian perturbations

For almost all of the notions and results on Hamiltonian perturbations, see [MS04], Section 8.1.

The basic Banach space from which all perturbations will be chosen is defined in analogy with [CM07], Section 3.

**Definition II.17.** Let $\varepsilon = (\varepsilon_i)_{i\in\mathbb{N}_0}$ be a fixed sequence of positive numbers. Denote by $\tilde{\kappa} : \tilde{X} \to \Sigma$ the projection. The space of Floer's $C^\varepsilon$-sections of $\tilde{\kappa}^*T^*\Sigma$ is

$$\Gamma^\varepsilon(\tilde{\kappa}^*T^*\Sigma) := \{H \in \Gamma(\tilde{\kappa}^*T^*\Sigma) \mid \sum_{i=0}^{\infty} \varepsilon_i \|H\|_{C^i} < \infty\}.$$

Let $C \subseteq \Sigma$ be the set of special points, i. e. the union of all the markings and nodal points, and define $\tilde{C} := \tilde{\kappa}^{-1}(C) \subseteq \tilde{X}$. $C \subseteq \Sigma$ is a submanifold that intersects every fibre of $\Sigma$ in a finite number of points. Define (cl denotes the closure)

$$\Gamma^\varepsilon_0(\tilde{\kappa}^*T^*\Sigma) := \mathrm{cl}\{H \in \Gamma^\varepsilon(\tilde{\kappa}^*T^*\Sigma) \mid \mathrm{supp}(H) \subseteq \tilde{X} \setminus \tilde{C}\}.$$

Let $0 < \delta < \frac{1}{4}$. The space of *Hamiltonian perturbations* is defined to be the open ball of radius $\delta$ in $\Gamma^\varepsilon_0(\tilde{\kappa}^*T^*\Sigma)$, i. e.

$$\mathcal{H}_{\varepsilon,\delta}(\tilde{X}) := \{H \in \Gamma^\varepsilon_0(\tilde{\kappa}^*T^*\Sigma) \mid \sum_{i=0}^{\infty} \varepsilon_i \|H\|_{C^i} < \delta\},$$

where $\varepsilon$ is chosen as in [Flo88], Lemma 5.1. The subscripts $\varepsilon$ and $\delta$ will usually be dropped, i. e. $\mathcal{H}(\tilde{X}) := \mathcal{H}_{\varepsilon,\delta}(\tilde{X})$.

The reason for the appearance of the constant $\delta$ in the above definition is so one can apply Exercise 8.1.3 from [MS04], and for any desingularisation $\hat{\imath}_b : S_b \to \Sigma_b \subseteq \Sigma$, for $b \in M$, of a fibre of $\Sigma$, equip the total space $\hat{X}_b$ of the pullback of the fibration $\tilde{X}$ to $S_b$, with a symplectic form.

**Construction II.5.** Let $(S, j, r_*, \nu), b \in M, \hat{\imath} : S \to \Sigma_b$ be a desingularisation of $\Sigma_b$ and let $\hat{X} := \hat{\imath}^*\tilde{X}$ with projection $\hat{\kappa} : \hat{X} \to S$. Using $\hat{X} \cong S \times X_b$, an element $H \in \mathcal{H}(\tilde{X})$ defines a linear map $H_b : TS \to \coprod_{z\in S} C^\infty(\hat{X}_z, \mathbb{R}) \cong S \times C^\infty(X_b, \mathbb{R})$ in the following way: If $\zeta_z \in T_zS_b = V_zS \subseteq T_zS$, then $H_b(\zeta_z) : \hat{X}_z = X_b \to \mathbb{R}$, $x \mapsto H_{\tilde{\imath}(x)}(\hat{\imath}_*\zeta_z)$, where $\tilde{\imath} : \hat{X} \to \tilde{X}$ is the canonical map covering $\hat{\imath} : S \to \Sigma$. In this way, $H_b$ is considered as a 1-form on $S$ with values in the smooth functions on the fibres of $\hat{X}$.
Furthermore, for $\zeta_z \in T_zS$, to the function $H_b(\zeta_z) \in C^\infty(\hat{X}_z, \mathbb{R})$ corresponds a Hamiltonian vector field $X_{H_b(\zeta_z)} \in \mathfrak{X}(\hat{X}_z)$. In this way one gets a fibrewise linear function $X_H : TS \to \coprod_{z\in S} \mathfrak{X}(\hat{X}_z)$, i. e. a 1-form with values in the space of Hamiltonian vector fields on the fibres of $\hat{X}$.
$H_b$ then defines a connection on $\hat{X}$ with projection onto the vertical tangent bundle given by

$$\begin{aligned}\pi^{T\hat{X}}_{V\hat{X}} : T\hat{X} \cong TS \times TX_b &\to V\hat{X} \cong S \times TX_b\\ (\zeta_z, v_x) &\mapsto (z, v_x) + (z, X_{H_b(\zeta_z)}(x)).\end{aligned}$$

**Definition II.18.** For $H \in \mathcal{H}(\tilde{X})$ and $b \in M$ as above, $X_{H_b} : TS \to \mathfrak{X}(X_b)$ from the previous construction is called the *Hamiltonian vector field* on $S$ associated to $H$ and

$$X^{0,1}_{H_b} := \frac{1}{2}(X_{H_b} + J_b \circ X_{H_b} \circ j_b)$$

is its complex antilinear part.

**Definition II.19.** Let $J \in \mathcal{J}_\omega(X)$ and let $H \in \mathcal{H}(\tilde{X})$, $b \in M$. Using the notation from the previous construction, the *almost complex structure* $\hat{J}^{H_b}$ *on* $\hat{X}$ *defined by* $J$ *and* $H$ is given by $\hat{J}^{H_b}|_{V\hat{X}} = J_b$, using the canonical identification $V\hat{X} \cong S \times V_bX$ and $\hat{J}^{H_b}|_{H\hat{X}} = \hat{\imath}^*j$ w. r. t. the decomposition $T\hat{X} \cong V\hat{X} \oplus H\hat{X}$ defined by the connection associated to $H$.

*Remark* II.7. For $(w, v) \in T\hat{X} \cong TS \times TX_b$,

$$\hat{J}^{H_b}(w, v) = (jw, J_bv + 2J_bX^{0,1}_{H_b(w)}).$$

The main existence result for Hamiltonian perturbations:

**Lemma II.15.** *Let* $(S, j, r_*, \nu), b \in M, \hat{\imath} : S \to \Sigma_b$ *be a desingularisation of* $\Sigma_b \subseteq \Sigma$. *Given any* $z \in S \backslash \left(\bigcup_{i=1}^n r_i \cup \bigcup_{i=1}^d \{n_i^1, n_i^2\}\right)$, *where* $\nu = \{\{n_1^1, n_1^2\}, \ldots, \{n_d^1, n_d^2\}\}$, *any* $x \in \hat{\imath}^*\tilde{X}$ *with* $\hat{\imath}^*\tilde{\kappa}(x) = z$, *any* $\eta \in \mathrm{Hom}(T_zS, V_x\hat{\imath}^*\tilde{X})$ *and any neighbourhood* $U$ *of* $x$ *in* $\hat{\imath}^*\tilde{X}$, *there exists an* $H \in \mathcal{H}(\tilde{X})$ *s. t.* $\hat{\imath}^*H \in \mathcal{H}(\hat{\imath}^*\tilde{X})$ *has support in* $U$ *and satisfies* $(X_{\hat{\imath}^*H})_x = \eta$.

*Proof.* Choose a coordinate neighbourhood $V \subseteq M$ of $B$ and a symplectic trivialisation $X|_V \cong V \times X_0$ of $X$. Because $z$ does not coincide with any of the special points, there exists a coordinate neighbourhood $\tilde{V} \subseteq \Sigma$ that is mapped by $\pi$ onto $V$ and s.t. there exist coordinates $(t_0, \dots, t_k) \in \mathbb{C} \times \mathbb{C}^k$, $k := \dim_{\mathbb{C}}(M)$ on $\tilde{V}$ and coordinates on $V$ s.t. $\pi|_{\tilde{V}} : \tilde{V} \to V$ in these coordinates is the map $(t^0, t^1, \dots, t^k) \mapsto (t^1, \dots, t^k)$ and $z$ corresponds to the point $(0, \dots, 0)$. Furthermore, let $x' := \tilde{\imath}(x) \in \tilde{X}$, where $\tilde{\imath} : \hat{\imath}^*\tilde{X} \to \tilde{X}$ is the canonical map covering $\hat{\imath}$. Then there exists a neighbourhood of $x'$ in $\tilde{X}|_{\tilde{V}}$ of the form $\tilde{V} \times W$, where $W \subseteq X_0$ is a coordinate neighbourhood with coordinates $(x^1, \dots, x^l) \in \mathbb{R}^l$, $l := \dim_{\mathbb{R}}(X_0)$, mapping $x'$ to zero. One can assume that $\tilde{V} \times W \cap \tilde{\kappa}^{-1}(\pi^{-1}(b)) \subseteq \hat{\imath}(U)$, so $\hat{\imath}^{-1}(\tilde{V} \times W) \subseteq U$. Choose two smooth cutoff functions $\delta_X : \mathbb{R}^{\dim X} \to [0,1]$ and $\delta_\Sigma : \mathbb{C} \times \mathbb{C}^k \to [0,1]$ which are identically 1 in a neighbourhood of 0 and have compact support inside the neighbourhoods of 0 corresponding to $W$ and $\tilde{V}$, respectively. Let $\frac{\partial}{\partial t^i_j}$, $i = 0, \dots, k$, $j = 1, 2$, be the coordinate vector fields for the real coordinates associated to the complex coordinates $t^i$. Then $t^0$ defines a complex coordinate in a neighbourhood of $z$ in $S$ and one can evaluate $\omega(\eta(\frac{\partial}{\partial t^0_j}), \cdot) = \sum_m \lambda_{j,m} \mathrm{d}x^m$ for some $\lambda_{j,m}$ and define $H \in \mathcal{H}(\tilde{X})$ as the 1-form that vanishes identically outside $\tilde{V} \times W$ and in the above coordinates maps the $\frac{\partial}{\partial t^i_j}$ for $i > 0$ to zero and maps the $\frac{\partial}{\partial t^0_j}{}_{(t^0, t^1, \dots, t^k)}$ to the function that vanishes identically outside $W$ and maps

$$(x^1, \dots, x^l) \mapsto \delta_\Sigma(t^0, t^1, \dots, t^k)\delta_X(x^1, \dots, x^l) \sum_m \lambda_{j,m} x^m.$$

□

For such Hamiltonian perturbations $H \in \mathcal{H}(\tilde{X})$ and points $b \in M$, one can define the moduli spaces of holomorphic curves in the family $\Sigma$ with values in $X$. These are the main objects to be studied in this thesis:

**Definition II.20.** Let $H \in \mathcal{H}(\tilde{X})$, let $b \in M$ and let $(S, j, r_*, \nu), b \in M, \hat{\imath} : S \to \Sigma_b$ be a desingularisation, $\hat{\imath}^*\tilde{X} := \hat{X}$. Then

$$\begin{aligned} \mathcal{M}_b(\tilde{X}, A, J, H) := \{u : \Sigma_b \to \tilde{X} \mid \tilde{\kappa} \circ u = \mathrm{id}_{\Sigma_b}, [\mathrm{pr}_2 \circ u] = A_b \in H_2(X_b; \mathbb{Z}), \\ \hat{\imath}^* u : S \to \hat{X} \text{ is } j\text{-}\hat{J}^{H_b}\text{-holomorphic}\}, \end{aligned}$$

which is independent of the choice of desingularisation and where $\mathrm{pr}_2 : \tilde{X}|_{\Sigma_b} \cong \Sigma_b \times X_b \to X_b$ is the projection. Hence

$$\mathcal{M}(\tilde{X}, A, J, \mathcal{H}(\tilde{X})) := \coprod_{\substack{b \in M \\ H \in \mathcal{H}(\tilde{X})}} \mathcal{M}_b(\tilde{X}, A, J, H)$$

is well-defined and comes with two projections

$$\pi^{\mathcal{M}}_M : \mathcal{M}(\tilde{X}, A, J, \mathcal{H}(\tilde{X})) \to M$$

and

$$\pi^{\mathcal{M}}_{\mathcal{H}} : \mathcal{M}(\tilde{X}, A, J, \mathcal{H}(\tilde{X})) \to \mathcal{H}(\tilde{X}).$$

The remainder of this chapter consists of defining (Banach) manifold structures over certain subsets of this space (although not on all of $\mathcal{M}(\tilde{X}, A, J, \mathcal{H}(\tilde{X}))$) in such a way that it reflects the stratified structure of $M$ by the stratification by signature (where well-defined) and to define a topology on $\mathcal{M}(\tilde{X}, A, J, \mathcal{H}(\tilde{X}))$ compatible with the manifold topologies on these parts.

## II.3.2 The case of a fixed Riemann surface

In this first subsection, the case of a fixed Riemann surface and a fixed trivial symplectic fibre bundle over this surface, equipped with (fixed) almost complex and Hamiltonian structures, is treated. First, the respective Fredholm problem is set up, i. e. a Banach space bundle $\mathcal{E} \to \mathcal{B}$ and a Cauchy-Riemann operator $\overline{\partial}$ as a section of this bundle are defined. Then the linearisation of this Cauchy-Riemann operator is calculated and using this, first, it is shown that this operator is Fredholm of the expected index (Corollary II.3). Then a condition is derived for when the linearisation of $\overline{\partial}$ is complex linear (Corollary II.4), which will mainly be needed in the last third of this text. Finally, the well-known elliptic regularity result will be derived, namely that the elements in the solution set $\overline{\partial}^{-1}(0)$ of the Fredholm problem actually consist of smooth sections. Most of these are rather well-known results, but first of all, they are all crucial for the later discussion, and second, using the results from the previous chapter and assuming the standard results for linear Cauchy-Riemann operators, the proofs are actually rather short. Most of the proofs here actually follow the same scheme: By expressing everything in a chart for $\mathcal{B}$ and a trivialisation for $\mathcal{E}$, the problem is reduced to a known result about linear Cauchy-Riemann operators.

**Construction II.6.** Let, for now, $(X, \omega)$ be a fixed closed symplectic manifold and let $A \in H_2(X)$. Let $(S, j)$ be a smooth Riemann surface of Euler characteristic $\hat{\chi}$ equipped with a hermitian metric $h$, let $J \in \mathcal{J}_\omega(X)$ and let $H \in \mathcal{H}(\hat{X})$, where $\hat{X} := S \times X$. Then there are the connection defined by $H$ as in Construction II.5, together with $h$ and the metric on $\hat{X}$ defined via the connection by the metric $g^J := \omega(\cdot, J\cdot)$ on the fibres of $\hat{X}$ and the pullback of $h$ via the projection on the horizontal tangent bundle. These turn $\mathrm{pr}_1 : \hat{X} \to S$ into a Riemannian submersion. The covariant derivative on vertical vector fields will be denoted by $\nabla^H$. Now over $\hat{X}$ there are the two vector bundles $\mathrm{Hom}(TS, V\hat{X})$ and its subbundle of complex antilinear morphisms

$$\overline{\mathrm{Hom}}_{(j,J)}(TS, V\hat{X}) := \{\eta \in \mathrm{Hom}(TS, V\hat{X}) \mid \eta \circ j = -J \circ \eta\}.$$

Both of these inherit a metric from the metrics $h$ and $g^J$ on $TS$ and $V\hat{X}$, respectively. $\mathrm{Hom}(TS, V\hat{X})$ also inherits a connection from the connections on $TS$ and $V\hat{X}$. But in general, this connection does not restrict to a well defined connection on $\overline{\mathrm{Hom}}_{(j,J)}$, since this subbundle is not invariant under parallel transport. The problem here being that the Levi-Civita connection on $X$ (coming from $g^J$) is not hermitian (the metric $h$ on $S$ is automatically

Kähler, $S$ being twodimensional). This can be solved by replacing the Levi-Civita connection on $X$ by the hermitian connection $\tilde{\nabla}$ defined by $g^J$ and $J$. It is shown in [MS04], Appendix C.7, that

$$\tilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2} J(\nabla_X J) Y.$$

$\tilde{\nabla}$ preserves $J$ and the metric $g^J$, but it is not torsion free, its torsion being given by

$$T^{\tilde{\nabla}}(X,Y) = -\frac{1}{4} N_J(X,Y),$$

where $N_J$ denotes the Nijenhuis tensor of $J$. Also, the map

$$\begin{aligned} \pi_{\overline{\mathrm{Hom}}_{(j,J)}}^{\mathrm{Hom}} : \mathrm{Hom}(TS, V\hat{X}) &\to \overline{\mathrm{Hom}}_{(j,J)}(TS, V\hat{X}) \\ \eta &\mapsto \frac{1}{2}(\eta + J \circ \eta \circ j) \end{aligned}$$

defines a smooth bundle morphism.
Using these structures, one can make the following definitions:
Fix, once and for all, a real number $p > 2$. Furthermore, let $k \in \mathbb{N}$.

$$\mathcal{B}^{k,p}(\hat{X}, A, J, H) := \{u \in L^{k,p}(\hat{X}, \mathrm{pr}_1, g^J) \mid [\mathrm{pr}_2 \circ u] = A\},$$

where $L^{k,p}(\hat{X}, \mathrm{pr}_2, g^J)$ is the Sobolev space of sections of $\hat{X}$, defined in Construction II.2. This is a Banach manifold, since it is a union of connected components, hence an open subset, of $L^{k,p}(\hat{X}, \mathrm{pr}_1, g^J)$. For a continuous path in this space via the Sobolev embedding theorem defines a continuous path of continuous functions, hence two sections in the same connected component define the same homology class. Proceeding,

$$\begin{aligned} \mathcal{E}^{k-1,p}(\hat{X}, A, J, H) := \{(\eta, u) \in L^{k-1,p}(&\overline{\mathrm{Hom}}_{(j,J)}(TS, V\hat{X}), h^* \otimes g^J, \nabla^S \otimes \nabla^H, \\ &\hat{X}, \mathrm{pr}_1, g^J) \mid u \in \mathcal{B}^{k,p}(\hat{X}, A, J, H)\}, \end{aligned}$$

which, as a restriction of the Banach space bundle $L^{k-1,p}(\overline{\mathrm{Hom}}_{(j,J)}(TS, V\hat{X}), h^* \otimes g^J, \nabla^S \otimes \nabla^H, \hat{X}, \mathrm{pr}_1, g^J)$ from Construction II.4 and Example II.1 to an open subset, is a Banach space bundle over $\mathcal{B}^{k,p}(\hat{X}, A, J, H)$. The projection of this bundle will be denoted by $\varkappa^{k,p} : \mathcal{E}^{k-1,p}(\hat{X}, A, J, H) \to \mathcal{B}^{k,p}(\hat{X}, A, J, H)$.
To define the Cauchy-Riemann operator, note that just the same way, there also is the Banach space bundle

$$\begin{aligned} \mathcal{F}^{k-1,p}(\hat{X}, A, J, H) := \{(\eta, u) \in L^{k-1,p}(&\mathrm{Hom}(TS, V\hat{X}), h^* \otimes g^J, \nabla^S \otimes \nabla^H, \\ &\hat{X}, \mathrm{pr}_1, g^J) \mid u \in \mathcal{B}^{k,p}(\hat{X}, A, J, H)\} \end{aligned}$$

over $\mathcal{B}^{k,p}(\hat{X}, A, J, H)$ which by Lemma II.13 comes with the section $D^{\mathrm{v}}$ : $\mathcal{B}^{k,p}(\hat{X}, A, J, H) \to \mathcal{F}^{k-1,p}(\hat{X}, A, J, H)$. Additionally, by Lemma II.14, the bundle morphism $\pi_{\overline{\mathrm{Hom}}_{(j,J)}}^{\mathrm{Hom}}$ from above induces a morphism of Banach space bundles from $\mathcal{F}^{k-1,p}(\hat{X}, A, J, H)$ to $\mathcal{E}^{k-1,p}(\hat{X}, A, J, H)$, hence the composition of the section $D^{\mathrm{v}}$ with this morphism defines a section of $\mathcal{E}^{k-1,p}(\hat{X}, A, J, H)$.
Finally, note that $J$ induces almost complex structures on both $V\hat{X}$ and $\overline{\mathrm{Hom}}_{(j,J)}(TS, V\hat{X})$, which via Lemma II.14 turn both $T\mathcal{B}^{k,p}(\hat{X}, A, J, H)$ and $\mathcal{E}^{k-1,p}(\hat{X}, A, J, H)$ into complex Banach space bundles over $\mathcal{B}^{k,p}(\hat{X}, A, J, H)$.

**Definition II.21.** The *(nonlinear) Cauchy-Riemann operator* on $\hat{X}$ is the section

$$\begin{aligned}\overline{\partial}_S^{J,H} : \mathcal{B}^{k,p}(\hat{X}, A, J, H) &\to \mathcal{E}^{k-1,p}(\hat{X}, A, J, H)\\ u &\mapsto \left(\frac{1}{2}(D^{\mathrm{v}}u + J \circ D^{\mathrm{v}}u \circ j), u\right)\end{aligned}$$

of the Banach space bundle

$$\varkappa^{k,p} : \mathcal{E}^{k-1,p}(\hat{X}, A, J, H) \to \mathcal{B}^{k,p}(\hat{X}, A, J, H).$$

The next result is very technical, but necessary, for later the exact form of the linearisation of the Cauchy-Riemann operator is needed. The relevant result here is Corollary II.4. Still somewhat relevant to note may be the fact that this lemma shows that the linearisation of the Cauchy-Riemann operator is a (compact) perturbation of a linear Cauchy-Riemann operator (of some Sobolev class) at all points, not just the differentiable ones.

The statement, as well as all the other statements in the corollaries and lemmas following it, is to be read as follows: Pick measurable sections representing $\xi$, $\nabla\xi$, $\eta$ and $\nabla\eta$. Then there is a measurable section $\rho$ of $\mathrm{Hom}(TS, u^*V\hat{X})$ s. t. for $Z \in T_zS$, $\rho_z(Z) \in V_{u(z)}\hat{X}$ is given by the formula in the lemma. The equivalence class of this section in the relevant Sobolev space gives a well-defined element in $\mathcal{E}^{k-1,p}(\mathrm{Hom}(TS, V\hat{X}), A, J, H)$.
Alternatively, one can take the formula literally on differentiable sections and use a standard density argument for differentiable sections in $L^{k,p}$-sections.

**Lemma II.16.** *Let $u \in \Gamma^k(\hat{X}, \mathrm{pr}_1, g) \cap \mathcal{B}^{k,p}(\hat{X}, A, J, H)$, $\xi, \eta \in L^{k,p}(u^*V\hat{X})$ and assume that $\xi$ is small enough that it lies in the chart around $u$. Then w. r. t. the chart for $\mathcal{B}^{k,p}(\hat{X}, A, J, H)$ and the trivialisation of $\mathcal{E}^{k-1,p}(\mathrm{Hom}(TS, V\hat{X}), A, J, H)$ around $u$ from Constructions II.2 and II.4, respectively, the linearisation of $\overline{\partial}_S^{J,H}$ at $\exp_u^\perp(\xi)$ in the direction $\eta$ and evaluated on $Z \in T_zS$ is given by*

$$\begin{aligned}\left(\|_{s=0}^{s=1} \exp_u^\perp(s\xi(z))\right)^{-1} \Big\{ &(D^{\mathrm{v}} \exp^{\hat{X}_z})_{\xi(z)} \frac{1}{2}(\nabla_Z\eta + J_\xi \nabla_{jZ}\eta) + \\ &+ \frac{1}{2}(\nabla_{\eta_\xi} A^{u,\xi}(\cdot, Z) + J\nabla_{\eta_\xi} A^{u,\xi}(\cdot, jZ)) - \\ &- \frac{1}{2} J(\nabla_{\eta_\xi} J)(A^{u,\xi}(\xi, Z) - JA^{u,\xi}(\xi, jZ)) + \\ &+ \overline{\mathcal{H}}(\eta, \xi) \overline{\partial}_S^{J,H} \exp_u^\perp(\xi)(Z) \Big\},\end{aligned}$$

*where*

$$\begin{aligned}
J_\xi(z) &:= (D^{\mathrm{v}} \exp^{\hat{X}_z})^{-1}_{\xi(z)} \circ J \circ (D^{\mathrm{v}} \exp^{\hat{X}_z})_{\xi(z)} \\
\eta_\xi(z) &:= (D^{\mathrm{v}} \exp^{\hat{X}_z})_{\xi(z)}(\eta(z)) \\
A^{u,\xi}(\zeta, Z) &:= (D \exp^{\hat{X}_z})_\zeta (D^{\mathrm{v}} u(Z), \nabla_Z \xi) + \tau_\zeta(Z) \\
\overline{\mathcal{H}}(\eta,\xi)(z) &:= \int_0^1 t \left( \|_{s=t}^{s=1} \exp_u^\perp(s\xi) \right) \circ R^{\tilde{\nabla}} \Big( (D^{\mathrm{v}} \exp^{\hat{X}_z})_{t\xi(z)} \eta(z), \\
&\qquad (D^{\mathrm{v}} \exp^{\hat{X}_z})_{t\xi(z)} \xi(z) \Big) \circ \left( \|_{s=t}^{s=1} \exp_u^\perp(s\xi) \right)^{-1} \mathrm{d}t.
\end{aligned}$$

*Proof.* Let $u$, $\xi$ and $\eta$ be as in the statement of the Lemma and let $\lambda \in \mathbb{R}$ be so small that $\xi + \lambda\eta$ lies in the open subset of $L^{k,p}(u^*V\hat{X}, h, g^J, \nabla)$ on which the chart around $u$ is defined. For a vertical path $\alpha : [0,1] \to \hat{X}$, i. e. $\mathrm{pr}_1 \circ \alpha \equiv \mathrm{const} : [0,1] \to S$, denote by $\|_{s=0}^{s=1} \alpha(s) : V_{\alpha(0)}\hat{X} \to V_{\alpha(1)}\hat{X}$ parallel transport w. r. t. the connection $\tilde{\nabla}$ on $X$ that respects $J$. Here, the fibre $\hat{X}_z$ of $\hat{X}$ over a point $z \in S$ is identified with $X$. Note also that over each such fibre, the connection on $\overline{\mathrm{Hom}}_{(j,J)}(TS, V\hat{X})|_{\hat{X}_z}$ is given by composition with $\tilde{\nabla}$, i. e. $(\tilde{\nabla}\eta)(Z) = \tilde{\nabla}(\eta(Z))$ for $\eta$ a section of $\overline{\mathrm{Hom}}_{(j,J)}(TS, V\hat{X})|_{\hat{X}_z}$ and $Z \in T_zS$. For this fibre is canonically identified with $T_z^*S \times TX$, i. e. the covariant derivative in the first factor is the trivial one.
Then by definition, in the chart around $u$ and the trivialisation of $\mathcal{E}^{k-1,p}(\hat{X}, A, J, H)$ over this chart, the derivative of $(D\overline{\partial}_S^{J,H})$ at the point $\exp_u^\perp(\xi) \in \mathcal{B}^{k,p}(\hat{X}, A, J, H)$ in the direction $\eta$ and evaluated on $Z \in T_zS$ is given by

$$\begin{aligned}
&\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} \left( \|_{s=0}^{s=1} \exp_u^\perp(s(\xi+\lambda\eta)) \right)^{-1} \overline{\partial}_S^{J,H} \exp_u^\perp(\xi+\lambda\eta)(Z) = \\
&= \left( \|_{s=0}^{s=1} \exp_u^\perp(s\xi) \right)^{-1} \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} \left( \|_{s=0}^{s=1} \exp_u^\perp(\xi+s\lambda\eta) \right)^{-1} \overline{\partial}_S^{J,H} \exp_u^\perp(\xi+\lambda\eta)(Z) + \\
&\qquad + \mathcal{H}(\eta,\xi) \left( \|_{s=0}^{s=1} \exp_u^\perp(s\xi) \right)^{-1} \overline{\partial}_S^{J,H} \exp_u^\perp(\xi)(Z).
\end{aligned}$$

The right hand side is calculated in Lemma II.12, with $\Omega^E$ in this case given by the curvature of $\tilde{\nabla}$. Here and in the following, a number of evaluations of $\xi$ and $\eta$ on $z \in S$ will be omitted, for otherwise the formulas become completely unmanageable. E. g. above, instead of

$$\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} \left( \|_{s=0}^{s=1} \exp_u^\perp(s(\xi+\lambda\eta)) \right)^{-1} \overline{\partial}_S^{J,H} \exp_u^\perp(\xi+\lambda\eta)(Z)$$

it should actually read

$$\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} \left( \|_{s=0}^{s=1} \exp_u^\perp(s(\xi(z)+\lambda\eta(z))) \right)^{-1} \overline{\partial}_S^{J,H} \exp_u^\perp(\xi+\lambda\eta)(Z).$$

The rule here is that $\xi$ and $\eta$ should be replaced by $\xi(z)$ and $\eta(z)$ unless they appear behind a differential operator such as e. g. $\nabla_Z\xi$ or $\overline{\partial}_S^{J,H} \exp_u^\perp(\xi)(Z)$.

Proceeding,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=0} \Big(\|_{s=0}^{s=1} \exp_u^\perp(\xi + s\lambda\eta)\Big)^{-1} \overline{\partial}_S^{J,H} \exp_u^\perp(\xi+\lambda\eta)(Z)\\
&\qquad = \frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=0} \Big(\|_{s=0}^{s=1} \exp_u^\perp(\xi + s\lambda\eta)\Big)^{-1} \frac{1}{2}\big(D^{\mathrm{v}} \exp_u^\perp(\xi+\lambda\eta)(Z) +\\
&\qquad\qquad\qquad\qquad + \; J \circ D^{\mathrm{v}} \exp_u^\perp(\xi+\lambda\eta)(jZ)\big)\\
&\qquad = \frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=0} \Big(\|_{s=0}^{s=1} \exp_u^\perp(\xi + s\lambda\eta)\Big)^{-1} \frac{1}{2} D^{\mathrm{v}} \exp_u^\perp(\xi+\lambda\eta)(Z) +\\
&\qquad\qquad \frac{1}{2} J \circ \frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=0} \Big(\|_{s=0}^{s=1} \exp_u^\perp(\xi + s\lambda\eta)\Big)^{-1} D^{\mathrm{v}} \exp_u^\perp(\xi+\lambda\eta)(jZ),
\end{aligned}
$$

because parallel transport w. r. t. the connection $\tilde{\nabla}$ preserves $J$ by definition. Hence it suffices to calculate (and then use the same result with $Z$ replaced by $jZ$)

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=0} \Big(\|_{s=0}^{s=1} \exp_u^\perp(\xi + s\lambda\eta)\Big)^{-1} D^{\mathrm{v}} \exp_u^\perp(\xi+\lambda\eta)(Z) =\\
&= \frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=0} \Big(\|_{s=0}^{s=1} \exp_u^\perp(\xi + s\lambda\eta)\Big)^{-1} \Big((D\exp^{\hat{X}_z})_{\xi+\lambda\eta}(D^{\mathrm{v}}u(Z), \nabla_Z(\xi+\lambda\eta)) +\\
&\qquad + \; \tau_{\xi+\lambda\eta}(Z)\Big)\\
&= \frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=0} \Big(\|_{s=0}^{s=1} \exp_u^\perp(\xi + s\lambda\eta)\Big)^{-1} \Big((D\exp^{\hat{X}_z})_{\xi+\lambda\eta}(D^{\mathrm{v}}u(Z), \nabla_Z\xi) +\\
&\qquad + \; \tau_{\xi+\lambda\eta}(Z)\Big) + \frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=0} \lambda\Big(\|_{s=0}^{s=1} \exp_u^\perp(\xi + s\lambda\eta)\Big)^{-1} (D^{\mathrm{v}}\exp^{\hat{X}_z})_{\xi+\lambda\eta}(\nabla_Z\eta)\\
&= \tilde{\nabla}_{(D^{\mathrm{v}}\exp^{\hat{X}_z})_\xi(\eta)} \Big((D\exp^{\hat{X}_z})_\cdot(D^{\mathrm{v}}u(Z), \nabla_Z\xi) + \tau_\cdot(Z)\Big) + (D^{\mathrm{v}}\exp^{\hat{X}_z})_\xi(\nabla_Z\eta)\\
&= \tilde{\nabla}_{\eta_\xi} A(\cdot, Z) + (D^{\mathrm{v}}\exp^{\hat{X}_z})_\xi(\nabla_Z\eta)\\
&= \nabla_{\eta_\xi} A(\cdot, Z) - \frac{1}{2} J(\nabla_{\eta_\xi} J) A(\xi, Z) + (D^{\mathrm{v}}\exp^{\hat{X}_z})_\xi(\nabla_Z\eta).
\end{aligned}
$$

Now it only remains to sort all the different terms, noting that $J(\nabla_{\eta_\xi} J) = -(\nabla_{\eta_\xi} J)J$ (differentiate $J^2 = -\mathrm{id}$), and to relate $\mathcal{H}$ to $\overline{\mathcal{H}}$ via an easy calculation using the composition property of parallel transport to finish the proof. □

**Lemma II.17.** *Let $u \in \Gamma^k(\hat{X}, \mathrm{pr}_1, g^J) \cap \mathcal{B}^{k,p}(\hat{X}, A, J, H)$. Then w. r. t. the chart for $\mathcal{B}^{k,p}(\hat{X}, A, J, H)$ and the trivialisation of $\mathcal{E}^{k-1,p}(\hat{X}, A, J, H)$ around $u$ from Constructions II.2 and II.4, respectively, the linearisation of $\overline{\partial}_S^{J,H}$ at $u$ is* *given by (for $Z \in TS$)*

$$
\begin{aligned}
(D\overline{\partial}_S^{J,H})_u : T\mathcal{B}^{k,p}(\hat{X}, A, J, H) &\to \mathcal{E}^{k-1,p}(\hat{X}, A, J, H)_u \\
((D\overline{\partial}_S^{J,H})_u)\eta(Z) &= \nabla_Z^{0,1}\eta - \frac{1}{2}J\left((\nabla_\eta J)\partial u(Z) + \pi_{V\hat{X}}^{T\hat{X}}(\hat{\nabla}_\eta^H \hat{J}^H)\tilde{Z}\right) \\
&= \nabla_Z^{0,1}\eta - \pi_{V\hat{X}}^{T\hat{X}}\left(\frac{1}{2}\hat{J}^H\left(\hat{\nabla}_\eta^H \hat{J}^H\right)(\partial u(Z) + \tilde{Z})\right) \\
&= \nabla_Z^{0,1}\eta - K_{\hat{J}^H}(\eta, \partial u(Z) + \tilde{Z}) - \\
&\quad - \frac{1}{8}\pi_{V\hat{X}}^{T\hat{X}} N_{\hat{J}^H}(\eta, \partial u(Z) + \tilde{Z}),
\end{aligned}
$$

*where*

$$
\begin{aligned}
\nabla_Z^{0,1}\eta &:= \frac{1}{2}(\nabla_Z\eta + J\nabla_{jZ}\eta), \\
\partial u(Z) &:= \frac{1}{2}(D^{\mathrm{v}}u(Z) - JD^{\mathrm{v}}u(jZ)),
\end{aligned}
$$

*$K_{\hat{J}^H}$ is the symmetric part of the bundle morphism*

$$
T\hat{X} \otimes T\hat{X} \to V\hat{X}, \quad (\eta, \xi) \mapsto \pi_{V\hat{X}}^{T\hat{X}}\left(\frac{1}{2}\hat{J}^H\left(\hat{\nabla}_\eta^H \hat{J}^H\right)\xi\right)
$$

*and where $\tilde{Z}$ denotes the horizontal lift of $Z \in TS$ to $\hat{X}$, $\hat{\nabla}^H$ denotes the Levi-Civita connection on $\hat{X}$ and $\hat{J}^H$ denotes the almost complex structure on $\hat{X}$ defined by $J$, $j$ and the connection given by $H$ as in Definition II.19.*

*Proof.* This is the special case of Lemma II.16 for $\xi = 0$. One checks easily that in this case $J_\xi = J$, $\eta_\xi = \eta$ and $\overline{\mathcal{H}}(\eta, \xi) = 0$ as well as $(\|_{s=0}^{s=1} \exp_u^\perp(s\xi(z)))^{-1} = (D\exp_{u(z)}^{\hat{X}_z})_{\xi(z)} = \mathrm{id}$. Also, by elementary properties of the differential of the (full) exponential map together with Lemma II.5, $A^{u,0}(0, Z) = D^{\mathrm{v}}u(Z)$. This accounts for the first and second summand in the formula. Again by elementary properties of the differential of the exponential map together with Lemma II.6, $\nabla_\eta A^{u,0}(0, Z) = -\pi_{V\hat{X}}^{T\hat{X}}\hat{\nabla}_\eta^H\tilde{Z}$, where $\tilde{Z}$ denotes the horizontal lift. So

$$
\begin{aligned}
\frac{1}{2}(\nabla_\eta A^{u,0}(\cdot, Z) + J\nabla_\eta A^{u,0}(\cdot, jZ) &= \frac{1}{2}(-\pi_{V\hat{X}}^{T\hat{X}}\hat{\nabla}_\eta^H\tilde{Z} - J\pi_{V\hat{X}}^{T\hat{X}}\hat{\nabla}_\eta^H\widetilde{jZ}) \\
&= -\pi_{V\hat{X}}^{T\hat{X}}\frac{1}{2}(\hat{\nabla}_\eta^H\tilde{Z} + \hat{J}^H\hat{\nabla}_\eta^H\widetilde{jZ}).
\end{aligned}
$$

Now

$$
\begin{aligned}
\hat{J}^H\hat{\nabla}_\eta^H\widetilde{jZ} &= \hat{\nabla}_\eta^H(\underbrace{\hat{J}^H\widetilde{jZ}}_{=\widetilde{jjZ}=-\tilde{Z}}) - (\hat{\nabla}_\eta^H\hat{J}^H)\widetilde{jZ} \\
&= -\hat{\nabla}_\eta^H\tilde{Z} - (\hat{\nabla}_\eta^H\hat{J}^H)\hat{J}^H\tilde{Z}.
\end{aligned}
$$

So

$$\begin{aligned}\frac{1}{2}(\nabla_\eta A^{u,0}(\cdot,Z)+J\nabla_\eta A^{u,0}(\cdot,jZ)) &= \frac{1}{2}\pi^{T\hat{X}}_{V\hat{X}}(\hat{\nabla}^H_\eta \hat{J}^H)\hat{J}^H\tilde{Z}\\ &= -\frac{1}{2}\pi^{T\hat{X}}_{V\hat{X}}\hat{J}^H\tilde{Z}(\hat{\nabla}^H_\eta \hat{J}^H)\\ &= -\frac{1}{2}J\pi^{T\hat{X}}_{V\hat{X}}(\hat{\nabla}^H_\eta \hat{J}^H)\tilde{Z},\end{aligned}$$

which accounts for the remaining term in the formula.
The last equality follows from the decomposition of a linear morphism $T\hat{X}\otimes T\hat{X}\to V\hat{X}$ into its symmetric and antisymmetric part together with the right formula in line (C.7.5) of Lemma C.7.1, p. 566, in [MS04]. □

This result seems to differ by the term involving $\tilde{Z}$ from the corresponding formula in [MS04], Section 8.3, p. 257 f., esp. Remark 8.3.8. Although that is not a real argument for why the formula above is correct, Corollary II.4 and Lemma II.18 at least show that it produces the consequences one (or at least the author) would hope for.

**Corollary II.3.**

$$\overline{\partial}^{J,H}_S : \mathcal{B}^{k,p}(\hat{X},A,J,H)\to \mathcal{E}^{k-1,p}(\hat{X},A,J,H)$$

*is a Fredholm operator of index*

$$\dim_{\mathbb{C}}(X)\hat{\chi}+2c_1(A).$$

*Proof.* By Lemma II.16, the differential of $D\overline{\partial}^{J,H}_S$ at a point $\exp^\perp_u(\xi)$ in $\mathcal{B}^{k,p}(\hat{X},A,J,H)$ with $u$ differentiable and $\xi$ a $L^{k,p}$-section of $u^*V\hat{X}$, in the trivialisations around $u$, is given by the operator defined, on $\eta\in L^{k,p}(u^*V\hat{X})$, by

$$\begin{aligned}T_zS\ni Z\mapsto \left(\|_{s=0}^{s=1}\exp^\perp_u(s\xi(z))\right)^{-1}\Big\{&(D^{\mathrm{v}}\exp^{\hat{X}_z})_{\xi(z)}\frac{1}{2}(\nabla_Z\eta+J_\xi\nabla_{jZ}\eta)+\\ &+\frac{1}{2}(\nabla_{\eta_\xi}A^{u,\xi}(\cdot,Z)+J\nabla_{\eta_\xi}A^{u,\xi}(\cdot,jZ))-\\ &-\frac{1}{2}J(\nabla_{\eta_\xi}J)(A^{u,\xi}(\xi,Z)-JA^{u,\xi}(\xi,jZ))+\\ &+\overline{\mathcal{H}}(\eta,\xi)\overline{\partial}^{J,H}_S\exp^\perp_u(\xi)(Z)\Big\}.\end{aligned}$$

*Claim.* The expression $z\mapsto\left(\|_{s=0}^{s=1}\exp^\perp_u(s\xi(z))\right)^{-1}\circ(D^{\mathrm{v}}\exp^{\hat{X}_z})_{\xi(z)}$ defines an element $\Psi$ of $L^{k,p}(\mathrm{Hom}(u^*V\hat{X},u^*V\hat{X}))$, with image in the bundle isomorphisms.

*Proof.* That for fixed $z\in S$ this defines an isomorphism of $V_{u(z)}\hat{X}$ is clear from the standing assumption on $\|\xi\|$ in the chart for $\mathcal{B}^{k,p}(\hat{X},A,J,H)$ around $u$.
For $\xi=0$, $\Psi$ is clearly the identity. Using Lemma II.8 and Lemma II.12 and the line of argument used in Subsection II.2.3, one then shows that, in the notation used there, $\frac{\partial}{\partial\rho}\nabla^i\Psi$, for $\xi,\rho\in\Gamma^k(u^*\hat{X})$ and $i=0,\dots,k$, can be bounded in

$L^p$-norm by a multiple of the $L^{k,p}$-norms of $\xi$ and $\rho$. Hence the $L^{k,p}$-norm of $\frac{\partial}{\partial\rho}\Psi$ can be bounded by a multiple of the $L^{k,p}$-norms of $\xi$ and $\rho$ as well. The claim then follows by the density argument via the mean value theorem used before. □

By the Sobolev multiplication theorem (remember that $kp > 2$), such a section defines an isomorphism $L^{k-1,p}(u^*V\hat{X}) \to L^{k-1,p}(u^*V\hat{X})$. One can hence disregard this part of the first summand. The second part of the first summand, $\eta \mapsto \frac{1}{2}(\nabla.\eta + J_\xi\nabla_j.\eta)$, defines a linear Cauchy-Riemann operator of class $L^{k,p}$, by the following claim:

*Claim.* If $\xi$ is of class $L^{k,p}$, then so is $J_\xi$.

*Proof.* The proof follows along the same lines of argument as the previous one. □

All the remaining summands factor through the compact inclusion of $L^{k,p}$ in $C^0$ (by the Sobolev embedding theorem, see [MS04], Theorem B.1.11, p. 517), hence the above operator is a compact perturbation of a linear Cauchy-Riemann operator of class $L^{k,p}$. By the Riemann-Roch theorem, see [MS04], Theorem C.1.10, p. 545, this is a Fredholm operator of the given index. □

**Corollary II.4.** *Let $u \in \mathcal{B}^{k,p}(\hat{X}, A, J, H)$, $\eta \in T_u\mathcal{B}^{k,p}(\hat{X}, A, J, H)$. If $\overline{\partial}_S^{J,H} u = 0$, then*

$$(((D\overline{\partial}_S^{J,H})_u)(J\eta) - J((D\overline{\partial}_S^{J,H})_u)\eta)(Z) = \pi_{V\hat{X}}^{T\hat{X}} N_{\hat{J}^H}(\eta, Du(Z)),$$

*where $Du : TS \to T\hat{X}$ is the usual differential. In particular, if $N_{\hat{J}^H}(\eta, v) = 0$ for all $\eta \in V\hat{X}|_{\mathrm{im}\, u}$ and $v \in \mathrm{im}\, Du$, then*

$$(D\overline{\partial}_S^{J,H})_u : T_u\mathcal{B}^{k,p}(\hat{X}, A, J, H) \to \mathcal{E}^{k-1,p}(\hat{X}, A, J, H)_u$$

*is a complex linear operator.*

*Proof.* First, assume that $\eta$ is of class $C^k$. Then by definition, $\nabla_Z\eta = \pi_{V\hat{X}}^{T\hat{X}}\hat{\nabla}^H_{Du(Z)}\eta$ (in case $k = 1$, the right hand side of this formula does not make any literal sense for sections of class $L^{k,p}$, whereas the left hand side does by definition of the $L^{k,p}$-spaces), where one considers $\eta$ as a vertical vector field on $\hat{X}$ on the image of $u$. Furthermore, because $\eta$ is a vertical vector field, $J\eta = \hat{J}^H\eta$ and $\pi_{V\hat{X}}^{T\hat{X}}\hat{J}^H = J$. Also, by definition of $\overline{\partial}_S^{J,H} u$ and $\partial u$, $Du(Z) = \overline{\partial}_S^{J,H} u(Z) + \partial u(Z) + \tilde{Z}$, in particular $Du(Z) = \partial u(Z) + \tilde{Z}$ if $\overline{\partial}_S^{J,H} u = 0$. With this, by the second formula for

$(D\overline{\partial}_S^{J,H})_u$ from Lemma II.17,

$$\begin{aligned}
(((D\overline{\partial}_S^{J,H})_u)(J\eta) - J((D\overline{\partial}_S^{J,H})_u)\eta)(Z) &= \pi^{T\hat{X}}_{V\hat{X}}(((D\overline{\partial}_S^{J,H})_u)(\hat{J}^H\eta) - \hat{J}^H((D\overline{\partial}_S^{J,H})_u)\eta)(Z) \\
&= \pi^{T\hat{X}}_{V\hat{X}}\left(\hat{\nabla}^H_{Du(Z)}(\hat{J}^H\eta) - \frac{1}{2}\hat{J}^H\left(\hat{\nabla}^H_{\hat{J}^H\eta}\hat{J}^H\right)Du(Z) - \right. \\
&\quad \left. - \hat{J}^H\hat{\nabla}^H_{Du(Z)}\eta - \frac{1}{2}\left(\hat{\nabla}^H_\eta\hat{J}^H\right)Du(Z)\right) \\
&= \pi^{T\hat{X}}_{V\hat{X}}\left(\left(\hat{\nabla}^H_{Du(Z)}\hat{J}^H\right)\eta - \left(\hat{\nabla}^H_\eta\hat{J}^H\right)Du(Z)\right)
\end{aligned}$$

by the Leibniz rule and the left formula in line (C.7.5) of Lemma C.7.1, p. 566, in [MS04]. The claim for $\eta$ of class $C^k$ now follows from the right formula in line (C.7.5) of Lemma C.7.1, p. 566, in [MS04].
The general case ($\eta$ of class $L^{k,p}$) then follows by the standard density argument. □

The following lemma should motivate the appearance of the almost complex structure $\hat{J}^H$ in the lemma and corollary above.

**Lemma II.18.** *In the notation of the above construction, for a section $u \in \Gamma^k(\hat{X}, h, g^J, \nabla^H) \cap \mathcal{B}^{k,p}(\hat{X}, A, J, H)$, $\pi^{T\hat{X}}_{H\hat{X}} \circ \overline{\partial}_S^{\hat{J}^H} u = 0$ and $\pi^{T\hat{X}}_{V\hat{X}} \circ \overline{\partial}_S^{\hat{J}^H} u = \overline{\partial}_S^{J,H} u$, where $\hat{J}^H$ is the almost complex structure on $\hat{X}$ as in Construction II.5 and $\overline{\partial}_S^{\hat{J}^H}$ is the standard Cauchy-Riemann operator on functions between the almost complex manifolds $S$ and $\hat{X}$. In particular, $u$ satisfies $\overline{\partial}_S^{J,H} u = 0$ iff $u : S \to \hat{X}$ is a $(j, \hat{J}^H)$-holomorphic map.*

*Proof.* By definition of $\hat{J}^H$,

$$\begin{aligned}
\overline{\partial}_S^{\hat{J}^H} u &= \frac{1}{2}(Du + \hat{J}^H \circ Du \circ j) \\
&= \frac{1}{2}(\pi^{T\hat{X}}_{V\hat{X}} \circ Du + J \circ \pi^{T\hat{X}}_{V\hat{X}} \circ Du \circ j + \\
&\quad + \pi^{T\hat{X}}_{H\hat{X}} \circ Du + (\pi_*|_{H\hat{X}})^{-1} \circ j \circ \pi_* \circ (\pi^{T\hat{X}}_{H\hat{X}})_* \circ Du \circ j) \\
&= \overline{\partial}_S^{J,H} u + \frac{1}{2}((\pi_*|_{H\hat{X}})^{-1} + (\pi_*|_{H\hat{X}})^{-1} \circ j \circ j)
\end{aligned}$$

because by definition of a connection $\pi_* \circ (\pi^{T\hat{X}}_{H\hat{X}})_* = \pi_*$ and $\pi_* \circ Du = \mathrm{id}$ as well as $\pi^{T\hat{X}}_{H\hat{X}} \circ Du = (\pi_*|_{H\hat{X}})^{-1}$

$$= \overline{\partial}_S^{J,H} u + 0.$$

□

**Lemma II.19.** *Let $v \in \mathcal{B}^{k,p}(\hat{X}, A, J, H)$ with $\overline{\partial}_S^{J,H} v = 0$. Then $v$ is smooth, i. e. $v \in \Gamma(\hat{X}, \mathrm{pr}_1, g^J)$.*

*Proof.* In a chart around an element $u \in \Gamma^k(\hat{X}, h, g, \nabla)$, $v$ is given by $v = \exp_u(\xi)$ for some $\xi \in L^{k,p}(u^*V\hat{X})$. By the formula given in Lemma II.13, for $Z \in T_zS$,

$$\overline{\partial}_S^{J,H} v(Z) = \frac{1}{2}\big(D\exp_\xi^{\hat{X}_z}(D^{\mathrm{v}}u(Z), \nabla_Z\xi) + \tau_\xi(Z) + \\ + \ JD\exp_\xi^{\hat{X}_z}(D^{\mathrm{v}}u(jZ), \nabla_{jZ}\xi) + J\tau_\xi(jZ)\big),$$

hence $\overline{\partial}_S^{J,H} v(Z) = 0 \Leftrightarrow$

$$\frac{1}{2}(D^{\mathrm{v}}\exp^{\hat{X}_z}(\nabla_Z\xi) + JD^{\mathrm{v}}\exp^{\hat{X}_z}(\nabla_{jZ}\xi)) = \\ -\frac{1}{2}(\tau_\xi(Z) + J\tau_\xi(jZ) + \big(D^{\mathrm{h}}\exp^{\hat{X}_z}\big)_\xi(D^{\mathrm{v}}u(Z)) + J\big(D^{\mathrm{h}}\exp^{\hat{X}_z}\big)_\xi(D^{\mathrm{v}}u(jZ))).$$

Composing from the left with $(D^{\mathrm{v}}\exp^{\hat{X}_z})^{-1}_{\xi(z)}$ and defining $J_\xi := (D^{\mathrm{v}}\exp^{\hat{X}_z})^{-1}_{\xi(z)} \circ J \circ (D^{\mathrm{v}}\exp^{\hat{X}_z})_{\xi(z)}$ as before, yields

$$\frac{1}{2}(\nabla_Z\xi + J_\xi \circ \nabla_{jZ}\xi) = -\frac{1}{2}(D^{\mathrm{v}}\exp^{\hat{X}_z})^{-1}_{\xi(z)}\big(\tau_\xi(Z) + J\tau_\xi(jZ)c; + \\ + \ \big(D^{\mathrm{h}}\exp^{\hat{X}_z}\big)_\xi(D^{\mathrm{v}}u(Z)) + J\big(D^{\mathrm{h}}\exp^{\hat{X}_z}\big)_\xi(D^{\mathrm{v}}u(jZ))\big).$$

By the second claim in the proof of Corollary II.3, $J_\xi$ is an almost complex structure on the vector bundle $u^*V\hat{X}$ of the same class as $\xi$ (here, a priori $L^{k,p}$) and the right hand side of the above equation defines a section of $\mathrm{Hom}(TS, u^*V\hat{X})$ also of the same class (again a priori $L^{k,p}$) as $\xi$. This is shown using the same proof as in that of smoothness of the transition functions in Subsection II.2.3. After going to local charts of this bundle, one can apply the bootstrapping procedure from Appendix B.4 in [MS04], esp. Lemma B.4.6 and Proposition B.4.9, to show the Lemma. □

### II.3.3 The case of a smooth family of Riemann surfaces

**Construction II.7.** In the course of this construction, it will very soon be necessary to work with universal moduli spaces, in particular to fix some space of perturbations. Hence it is easier to start out with two families, namely a nodal family over which the perturbations are defined and a smooth desingularisation of this family over some locally closed submanifold. So let $(\pi : \Sigma \to M, R)$ be a nodal family of Riemann surfaces of Euler characteristic $\chi$ with $n$ markings and let $(\rho : S \to B, \hat{R}, N, \iota, \hat{\iota})$ be a desingularisation of $\Sigma$ over $B$ as in Definition II.6. Also, fix a metric $h$ on $S$ that induces a hermitian metric $h_b$ on every fibre $S_b := \rho^{-1}(b)$ over a point $b \in B$. As stated in the beginning of this section, let $(\kappa : X \to M, \omega)$ be a family of symplectic manifolds with fibres symplectomorphic to a closed symplectic manifold $(X_0, \omega_0)$. Define $\tilde{\kappa} : \tilde{X} \to \Sigma$ as the pullback of $\kappa : X \to M$ to $\Sigma$ via $\pi$ and as before, let $A$ be a locally trivial family of 2$^{\text{nd}}$ homology classes in the fibres of $X$. Assume that $M$ is connected, hence there is a well-defined first Chern class $c_1(A) := c_1^{TX_b}(A_b)$ for any $b \in M$. Let finally $H \in \mathcal{H}(\tilde{X})$ be a Hamiltonian connection on $\tilde{X}$. Using $\iota : B \to M$ and $\hat{\iota} : S \to \Sigma$, one can pull all these structures back to $B$ and $S$, i. e. $\hat{\rho} : \hat{X} := \hat{\iota}^*\tilde{X} = S \times X \to S$ is again a symplectic fibre bundle on which $\iota^*J$ and $\hat{\iota}^*H$ define almost complex and Hamiltonian structures, respectively. For simplicity and by abuse of notation, $\iota^*J$ and $\hat{\iota}^*H$ will be denoted by $J$ and $H$, respectively. For $b \in B$, denote by $J_b$, $g_b^J$ and $H_b$ the pullbacks of $J$, $g^J$ and $H$ to $\hat{X}_b := \hat{X}|_{S_b}$, considered as a symplectic fibration over $S_b$ via the restriction $\hat{\rho}_b$ of $\hat{\rho}$. Also denote by $j_b$ the complex structure on the Riemann surface $S_b$. Denote

$$\mathcal{B}_b^{k,p}(\hat{X}, A, J, H) := \mathcal{B}^{k,p}(\hat{X}_b, A, J_b, H_b).$$

Also, denote

$$\mathcal{E}_b^{k-1,p}(\hat{X}, A, J, H) := \mathcal{E}^{k-1,p}(\hat{X}_b, A, J_b, H_b)$$

and denote the bundle projection by

$$\varkappa_b^H : \mathcal{E}_b^{k-1,p}(\hat{X}, A, J, H) \to \mathcal{B}_b^{k,p}(\hat{X}, A, J, H).$$

The reason for the additional superscript $H$ in comparison to the previous notation will become clearer a little bit later.
With this define

$$\mathcal{B}^{k,p}(\hat{X}, A, J, H) := \coprod_{b \in B} \mathcal{B}_b^{k,p}(\hat{X}, A, J, H),$$
$$\mathcal{E}^{k-1,p}(\hat{X}, A, J, H) := \coprod_{b \in B} \mathcal{E}_b^{k-1,p}(\hat{X}, A, J, H)$$

and

$$\varkappa^H := \coprod_{b \in B} \varkappa_b^H : \mathcal{E}^{k-1,p}(\hat{X}, A, J, H) \to \mathcal{B}^{k,p}(\hat{X}, A, J, H).$$

Furthermore, one can define a section (set-theoretically, at this point) of $\varkappa^H$ by

$$\overline{\partial}^{J,H} := \coprod_{b \in B} \overline{\partial}_{S_b}^{J_b,H_b} : \mathcal{B}^{k,p}(\hat{X}, A, J, H) \to \mathcal{E}^{k-1,p}(\hat{X}, A, J, H).$$

**Definition II.22.** The *moduli space of* $(J, H)$*-holomorphic curves* in the family $S$ and representing the homology class $A$ is defined as the subset

$$\mathcal{M}(\hat{X}, A, J, H) := \left(\overline{\partial}^{J,H}\right)^{-1}(0)$$

of $\mathcal{B}^{k,p}(\hat{X}, A, J, H)$ for any $k \in \mathbb{N}$, $p > 1$ with $kp > 2$, where 0 denotes the image of the zero section in $\mathcal{E}^{k-1,p}(\hat{X}, A, J, H)$. This is well-defined by Lemma II.19.

The goal now is to equip this set with a manifold structure. Following the usual course of action, to achieve this one wants to turn $\varkappa^H : \mathcal{E}^{k-1,p}(\hat{X}, A, J, H) \to \mathcal{B}^{k,p}(\hat{X}, A, J, H)$ into a Banach space bundle and $\overline{\partial}^{J,H}$ into a Fredholm section.

The straightforward way to attempt to define charts on $\mathcal{B}^{k,p}(\hat{X}, A, J, H)$ would be, for a point $a \in B$, to pick an open neighbourhood $U \subseteq B$ of $a$ and a smooth trivialisation

$$\phi_a : U \times S_a \xrightarrow{\cong} S|_U \subseteq S,$$

inducing maps

$$\phi_{ab} : S_a \to S_b, \quad z \mapsto \phi_a(b, z)$$

for $b \in U$. Defining

$$\mathcal{B}^{k,p}_U(\hat{X}, A, J, H) := \coprod_{b\in U} \mathcal{B}^{k,p}_b(\hat{X}, A, J, H),$$

there is a bijection

$$\begin{aligned}\overline{\phi}_a : \mathcal{B}^{k,p}_U(\hat{X}, A, J, H) &\to U \times \mathcal{B}^{k,p}_a(\hat{X}, A, J, H)\\ \mathcal{B}^{k,p}_b(\hat{X}, A, J, H) \ni u &\mapsto (b, \phi^*_{ab}u),\end{aligned}$$

where for $z \in S_b$, if $u(z) = (z, \overline{u}(z)) \in S_b \times X$, then for $w \in S_a$, $\phi^*_{ab}u(w) = (w, \overline{u}(\phi_{ab}(w)))$. This map is well-defined, because first of all, it is clear that for $u \in \mathcal{B}^{k,p}(\hat{X}, A, J, H)$, $\phi^*_{ab}u \in \mathcal{B}^{k,p}(\hat{X}|_{(S_a,(\phi^*_a j)_b)}, A, (\phi^*_a J)_b, (\phi^*_a H)_b)$, where $\hat{X}|_{(S_a,(\phi^*_a j)_b)}$ denotes $\hat{X}_a$, but with the base space $S_a$ now equipped with the complex structure $(\phi^*_a j)_b$ instead of $j_a$. But by Lemma II.10, as Banach manifolds,

$$\mathcal{B}^{k,p}(\hat{X}|_{(S_a,(\phi^*_a j)_b)}, A, (\phi^*_a J)_b, (\phi^*_a H)_b) = \mathcal{B}^{k,p}(\hat{X}_a, A, J_a, H_a),$$

in the sense of a literal equality of sets as well as of equivalence classes of Banach manifold atlases. This raises the question why then to use the notation $\mathcal{B}^{k,p}(\hat{X}_a, A, J_a, H_a)$, and analogously $\mathcal{E}^{k-1,p}(\hat{X}_a, A, J_a, H_a)$, instead of the much shorter $\mathcal{B}^{k,p}(\hat{X}_a, A)$ and $\mathcal{E}^{k-1,p}(\hat{X}_a, A, J_a)$ (in the latter case $J_a$ is actually part of the definition). The reason is mainly due to the next construction where a copy of $\mathcal{B}^{k,p}(\hat{X}_a, A, J, H)$ appears for every $H \in \mathcal{H}(\tilde{X})$, which would then necessitate notation such as $\{H\} \times \mathcal{B}^{k,p}(\hat{X}_a, A)$. Also this notation serves as a reminder that every $\mathcal{B}^{k,p}(\hat{X}_a, A, J_a, H_a)$ comes with a distinguished atlas.

Now for another such chart given by an open subset $V \subseteq B$, trivialisation $\psi_c : V \times S_c \cong S|_V$ and corresponding trivialisation

$$\begin{aligned}\overline{\psi}_c : \mathcal{B}^{k,p}_V(\hat{X}, A, J, H) &\to V \times \mathcal{B}^{k,p}_c(\hat{X}, A, J, H)\\ \mathcal{B}^{k,p}_b(\hat{X}, A, J, H) \ni u &\mapsto (b, \psi^*_{cb}u),\end{aligned}$$

the transition functions would be given by

$$\begin{aligned}(U \cap V) \times \mathcal{B}^{k,p}_c(\hat{X}, A, J, H) &\to (U \cap V) \times \mathcal{B}^{k,p}_a(\hat{X}, A, J, H)\\ (b, u) &\mapsto (b, \phi^*_{ab}(\psi_{cb*}u)),\end{aligned}$$

where if $u(w) = (w, \overline{u}(w)) \in \hat{X}_c$ for $w \in S_c$, then for $z \in S_a$, $\phi^*_{ab}(\psi_{cb*}u)(z) = (z, \overline{u}(\psi^{-1}_{bc}\phi_{ab}(z)))$. In other words, there is a map $U \cap V \to \mathrm{Diff}(S_c, S_a)$, $b \mapsto \psi^{-1}_{bc} \circ \phi_{ab}$ and the transition functions are given in terms of the action of this map. But as is explained e. g. in [Weh09] or [Weh12], the induced action of the

diffeomorphism group on Sobolev spaces simply is not smooth. So the charts that have just been defined do *not* patch together to give an atlas and it does not even make sense to ask whether or not $\overline{\partial}^{J,H}$ defines a smooth (Fredholm) section. At this point one has to make a decision on how to proceed. The more definitive way would be to use the sc-manifold/polyfold framework of Hofer, Wysocki and Zehnder, for an introduction see e. g. the introduction by the inventors themselves [HWZ10], the slides cited above, or [FFGW12].
Here, I will take a slightly different route. Namely remember that it is actually the spaces of holomorphic curves one is interested in, i. e. the zero set of $\overline{\partial}^{J,H}$ and one should actually look at the restriction of the transition functions above to this set. By this what is meant is the following: In analogy to $\mathcal{B}^{k,p}_U(\hat{X}, A, J, H)$, define

$$\mathcal{E}^{k-1,p}_U(\hat{X}, A, J, H) := \coprod_{b\in U} \mathcal{E}^{k-1,p}_b(\hat{X}, A, J, H),$$

$$\varkappa^H_U := \varkappa^H|_{\mathcal{E}^{k-1,p}_U(\hat{X},A,J,H)} : \mathcal{E}^{k-1,p}_U(\hat{X}, A, J, H) \to \mathcal{B}^{k,p}_U(\hat{X}, A, J, H)$$

and

$$\overline{\partial}^{J,H}_U := \overline{\partial}^{J,H}|_{\mathcal{B}^{k,p}_U(\hat{X},A,J,H)} : \mathcal{B}^{k,p}_U(\hat{X}, A, J, H) \to \mathcal{E}^{k-1,p}_U(\hat{X}, A, J, H).$$

Consequently,

$$\mathcal{M}_U(\hat{X}, A, J, H) := \left(\overline{\partial}^{J,H}_U\right)^{-1}(0) = \mathcal{M}(\hat{X}, A, J, H) \cap \mathcal{B}^{k,p}_U(\hat{X}, A, J, H).$$

$\mathcal{B}^{k,p}_U(\hat{X}, A, J, H)$ can be turned into a Banach manifold, giving it the product manifold structure of $U \times \mathcal{B}^{k,p}_a(\hat{X}, A, J, H)$ via the bijection above. This structure then obviously depends on a choice of trivialisation $\phi_a$ of $S|_U$ and will be denoted by $\mathcal{B}^{k,p}_{U,\phi_a}(\hat{X}, A, J, H)$. Analogously, $\mathcal{E}^{k-1,p}_U(\hat{X}, A, J, H)$ can be given a smooth Banach space bundle structure over $\mathcal{B}^{k,p}_{U,\phi_a}(\hat{X}, A, J, H)$ by identifying it with $U \times \mathcal{E}^{k-1,p}_a(\hat{X}, A, J, H)$ via the map

$$\begin{aligned}\hat{\phi}_a : \mathcal{E}^{k-1,p}_U(\hat{X}, A, J, H) &\to U \times \mathcal{E}^{k-1,p}_a(\hat{X}, A, J, H)\\ \mathcal{E}^{k-1,p}_b(\hat{X}, A, J, H) \ni (\eta, u) &\mapsto (b, (\pi^{\overline{\mathrm{Hom}}((\phi_a^* j)_b,(\phi_a^* J)_b)}_{\overline{\mathrm{Hom}}(j_a,J_a)} \phi^*_{ab}\eta, \phi^*_{ab}u)).\end{aligned}$$

For this to be well-defined it is assumed that $U$ is small enough s. t.

$$\pi^{\overline{\mathrm{Hom}}((\phi_a^* j)_b,(\phi_a^* J)_b)}_{\overline{\mathrm{Hom}}(j_a,J_a)} : \overline{\mathrm{Hom}}_{((\phi_a^* j)_b,(\phi_a^* J)_b)}(TS_a, V\hat{X}_a) \to \overline{\mathrm{Hom}}_{(j_a,J_a)}(TS_a, V\hat{X}_a)$$

is an isomorphism for all $b \in U$. Again, $\mathcal{E}^{k-1,p}_U(\hat{X}, A, J, H)$ equipped with this smooth structure will be denoted $\mathcal{E}^{k-1,p}_{U,\phi_a}(\hat{X}, A, J, H)$.
Finally, defining

$$\begin{aligned}\overline{\partial}^{J,H}_{U,\phi_a} : U \times \mathcal{B}^{k,p}_a(\hat{X}, A, J, H) &\to U \times \mathcal{E}^{k-1,p}_a(\hat{X}, A, J, H)\\ (b, u) &\mapsto (b, \pi^{\overline{\mathrm{Hom}}((\phi_a^* j)_b,(\phi_a^* J)_b)}_{\overline{\mathrm{Hom}}(j_a,J_a)} \overline{\partial}^{(\phi_a^* J)_b,(\phi_a^* H)_b}_{(S_a,(\phi_a^* j)_b)} u),\end{aligned}$$

all of the above fit into a commutative diagram

$$\begin{tikzcd} \mathcal{E}^{k-1,p}_{U,\phi_a}(\hat{X},A,J,H) \arrow[r, "\hat{\phi}_a"] \arrow[d, "\varkappa_U^H"] & U \times \mathcal{E}^{k-1,p}_a(\hat{X},A,J,H) \arrow[d, "\mathrm{id}_U\times\varkappa_a^H"'] \\ \mathcal{B}^{k,p}_{U,\phi_a}(\hat{X},A,J,H) \arrow[r, "\overline{\phi}_a"] \arrow[u, bend left, "\overline{\partial}^{J,H}_U"] & U \times \mathcal{B}^{k,p}_a(\hat{X},A,J,H) \arrow[u, bend right, "\overline{\partial}^{J,H}_{U,\phi_a}"'] \end{tikzcd} \tag{II.14}$$

With this setup, $\overline{\partial}^{J,H}_U$ is a parametrised version of a Cauchy-Riemann operator, which hence is a Fredholm operator itself and the Fredholm index can be computed fairly easily. $c_1(A)$ here is the first Chern number as in the beginning of this subsection.

**Lemma II.20.** *In the notation of the previous construction,*

$$\overline{\partial}^{J,H}_U : \mathcal{B}^{k,p}_{U,\phi_a}(\hat{X},A,J,H) \to \mathcal{E}^{k-1,p}_{U,\phi_a}(\hat{X},A,J,H)$$

*is a Fredholm section of index*

$$\operatorname{ind}(\overline{\partial}^{J,H}_U) = \dim_{\mathbb{C}}(X)\hat{\chi} + 2c_1(A) + \dim_{\mathbb{R}}(U).$$

*Proof.* (Sketch only) The result will follow from the following functional analytic claim:

*Claim.* Let $X, Y$ be Banach spaces, let $V \subseteq X$ and $U \subseteq \mathbb{R}^n$ be open subsets and let $F : V \times U \to Y$ be a continuously differentiable map with the property that for every $b \in U$, the map $F(\cdot, b) : V \to Y$ is a (nonlinear) Fredholm map of index $d$. Then $F$ is Fredholm of index $d + n$.

*Proof.* Let $(u, b) \in V \times U$. Denote by $D_1F_{(u,b)}$ and $D_2F_{(u,b)}$ the (partial) derivatives of $F$ at $(u, b)$ in the direction of $V$ and $U$, respectively. By assumption, $D_1F_{(u,b)} : X \to Y$ is a Fredholm operator of index $d$. It follows that $D_1F_{(u,b)} \circ \mathrm{pr}_1 : X \times \mathbb{R}^n \to Y$ is a Fredholm operator of index $d + n$ (it clearly has the same image as $D_1F_{(u,b)}$ and its kernel is $\ker(D_1F_{(u,b)}) \times \mathbb{R}^n$. The operator $D_2F_{(u,b)} \circ \mathrm{pr}_2 : X \times \mathbb{R}^n \to Y$ is compact, for the image of the unit ball in $X \times \mathbb{R}^n$ is just the image of the (compact) unit ball in $\mathbb{R}^n$, hence compact. Hence $DF_{(u,b)} = D_1F_{(u,b)} \circ \mathrm{pr}_1 + D_2F_{(u,b)} \circ \mathrm{pr}_2$ is the sum of a Fredholm operator of index $d + n$ and a compact operator, hence a Fredholm operator of index $d + n$ by a standard result about Fredholm operators. □

To apply this claim, around a point $a \in B$, consider diagram II.14 and the definition of $\overline{\partial}^{J,H}_{U,\phi_a}$ from the previous construction above. In that definition, every

$$\overline{\partial}^{(\phi_a^*J)_b,(\phi_a^*H)_b}_{(S_a,(\phi_a^*j)_b)} : \mathcal{B}^{k,p}_a(\hat{X},A,J,H) = \mathcal{B}^{k,p}(\hat{X}_a,A,J_a,H_a) \to \mathcal{E}^{k-1,p}(\hat{X}|_{(S_a,(\phi_a^*j)_b)}, A, (\phi_a^*J)_b, (\phi_a^*H)_b)$$

is a Fredholm operator of index $d = \dim_{\mathbb{C}}(X)\chi + 2c_1(A)$ by Corollary II.3. Composing with the bundle isomorphism $\mathcal{E}^{k-1,p}(\hat{X}|_{(S_a,(\phi_a^* j)_b)}, A, (\phi_a^* J)_b, (\phi_a^* H)_b) \to \mathcal{E}_a^{k-1,p}(\hat{X}, A, J, H)$ defined by $\pi^{\overline{\mathrm{Hom}}_{((\phi_a^* j)_b,(\phi_a^* J)_b)}}_{\overline{\mathrm{Hom}}_{(j_a,J_a)}}$ does not change this. Choosing a chart for $\mathcal{B}_a(\hat{X}, A, J, H)$ and a local trivialisation for $\mathcal{E}_a(\hat{X}, A, J, H)$ around a given point then brings one to the situation of the claim above. $\square$

**Construction II.8.** Using the same notation as in the previous construction, define

$$\begin{aligned}
\mathcal{B}^{k,p}(\hat{X}, A, J, \mathcal{H}(\tilde{X})) &:= \coprod_{H \in \mathcal{H}(\tilde{X})} \mathcal{B}^{k,p}(\hat{X}, A, J, H) \\
\mathcal{E}^{k-1,p}(\hat{X}, A, J, \mathcal{H}(\tilde{X})) &:= \coprod_{H \in \mathcal{H}(\tilde{X})} \mathcal{E}^{k-1,p}(\hat{X}, A, J, H)
\end{aligned}$$

and

$$\begin{aligned}
\varkappa^{\mathcal{H}} &:= \coprod_{H \in \mathcal{H}(\tilde{X})} \varkappa^H : \mathcal{E}^{k-1,p}(\hat{X}, A, J, \mathcal{H}(\tilde{X})) \to \mathcal{B}^{k,p}(\hat{X}, A, J, \mathcal{H}(\tilde{X})) \\
\overline{\partial}^{J,\mathcal{H}} &:= \coprod_{H \in \mathcal{H}(\tilde{X})} \overline{\partial}^{J,H} : \mathcal{B}^{k,p}(\hat{X}, A, J, \mathcal{H}(\tilde{X})) \to \mathcal{E}^{k-1,p}(\hat{X}, A, J, \mathcal{H}(\tilde{X})).
\end{aligned}$$

There are natural projections

$$\pi^{\mathcal{B}}_{\mathcal{H}} : \mathcal{B}^{k,p}(\hat{X}, A, J, \mathcal{H}(\tilde{X})) \to \mathcal{H}(\tilde{X})$$

and

$$\pi^{\mathcal{E}}_{\mathcal{H}} : \mathcal{E}^{k-1,p}(\hat{X}, A, J, \mathcal{H}(\tilde{X})) \to \mathcal{H}(\tilde{X}).$$

**Definition II.23.**

$$\mathcal{M}(\hat{X}, A, J, \mathcal{H}(\tilde{X})) := \left(\overline{\partial}^{J,\mathcal{H}}\right)^{-1}(0).$$

Again, given an open neighbourhood $U \subseteq B$ of $a \in B$ and a smooth trivialisation $\phi_a : U \times S_a \cong S|_U$, define

$$\begin{aligned}
\mathcal{B}_U^{k,p}(\hat{X}, A, J, \mathcal{H}(\tilde{X})) &:= \coprod_{H \in \mathcal{H}(\tilde{X})} \mathcal{B}_U^{k,p}(\hat{X}, A, J, H) \\
\mathcal{E}_U^{k-1,p}(\hat{X}, A, J, \mathcal{H}(\tilde{X})) &:= \coprod_{H \in \mathcal{H}(\tilde{X})} \mathcal{E}_U^{k-1,p}(\hat{X}, A, J, H)
\end{aligned}$$

and

$$\begin{aligned}
\varkappa_U^{\mathcal{H}} &:= \coprod_{H \in \mathcal{H}(\tilde{X})} \varkappa^H : \mathcal{E}_U^{k-1,p}(\hat{X}, A, J, \mathcal{H}(\tilde{X})) \to \mathcal{B}_U^{k,p}(\hat{X}, A, J, \mathcal{H}(\tilde{X})) \\
\overline{\partial}_U^{J,\mathcal{H}} &:= \coprod_{H \in \mathcal{H}(\tilde{X})} \overline{\partial}_U^{J,H} : \mathcal{B}_U^{k,p}(\hat{X}, A, J, \mathcal{H}(\tilde{X})) \to \mathcal{E}_U^{k-1,p}(\hat{X}, A, J, \mathcal{H}(\tilde{X})) \\
\mathcal{M}_U(\hat{X}, A, J, \mathcal{H}(\tilde{X})) &:= \mathcal{M}(\hat{X}, A, J, \mathcal{H}(\tilde{X})) \cap \mathcal{B}_U^{k,p}(\hat{X}, A, J, \mathcal{H}(\tilde{X}))
\end{aligned}$$

as sets. Denote by $\mathcal{B}^{k,p}_{U,\phi_a}(\hat{X},A,J,\mathcal{H}(\tilde{X}))$ the set $\mathcal{B}^{k,p}_U(\hat{X},A,J,\mathcal{H}(\tilde{X}))$ equipped with the product Banach manifold structure of $\mathcal{H}(\tilde{X})\times\mathcal{B}^{k,p}_{U,\phi_a}(\hat{X},A,J,H)$ for any fixed chosen $H\in\mathcal{H}(\tilde{X})$, again identifying all the $\mathcal{B}^{k,p}_U(\hat{X},A,J,H)$ for different $H$ by the set theoretic identity. $\mathcal{E}^{k-1,p}_{U,\phi_a}(\hat{X},A,J,\mathcal{H}(\tilde{X}))$ is defined as a Banach manifold in the same way. In the trivialisations of these spaces defining their smooth structures, $\overline{\partial}^{J,\mathcal{H}}_U$ is given by

$$\overline{\partial}^{J,\mathcal{H}}_{U,\phi_a} := \coprod_{H\in\mathcal{H}(\tilde{X})} \overline{\partial}^{J,H}_{U,\phi_a} : \\ U\times\mathcal{B}^{k,p}_a(\hat{X},A,J,H)\times\mathcal{H}(\tilde{X}) \to U\times\mathcal{E}^{k-1,p}_a(\hat{X},A,J,H)\times\mathcal{H}(\tilde{X}).$$

**Lemma II.21.** *In the notation of the above construction,*

$$\overline{\partial}^{J,\mathcal{H}}_U : \mathcal{B}^{k,p}_{U,\phi_a}(\hat{X},A,J,\mathcal{H}(\tilde{X})) \to \mathcal{E}^{k-1,p}_{U,\phi_a}(\hat{X},A,J,\mathcal{H}(\tilde{X}))$$

*is smooth.*
*Given $(b,u,H)\in U\times\mathcal{B}^{k,p}_a(\hat{X},A,J,H)\times\mathcal{H}(\tilde{X})$ with $u\in\Gamma^k(\hat{X}|_{S_a})$, w. r. t. the charts on $\mathcal{B}^{k,p}_{U,\phi_a}(\hat{X},A,J,\mathcal{H}(\tilde{X}))$ from the previous construction and the standard chart for $\mathcal{B}^{k,p}_a(\hat{X},A,J,H)$ around $u$, the linearisation of $\overline{\partial}^{J,\mathcal{H}}_U$ at $\phi^*_{ba}u\in\mathcal{B}^{k,p}_b(\hat{X},A,J,H)$ in the direction $(e,\xi,h)$, where $e\in T_bU$, $\xi\in T_u\mathcal{B}^{k,p}_a(\hat{X},A,J,H)$ and $h$ is a $C^\varepsilon$-section of $\mathrm{pr}_1^*T^*\Sigma$, is given by*

$$\left(D\overline{\partial}^{J,\mathcal{H}}_{U,\phi_a}\right)_{(b,u,H)}(e,\xi,h) = \\ \pi^{\overline{\mathrm{Hom}}_{((\phi_a^*j)_b,(\phi_a^*J)_b)}}_{\overline{\mathrm{Hom}}_{(j_a,J_a)}}\left(\left(D\overline{\partial}^{(\phi_a^*J)_b,(\phi_a^*H)_b}_{(S_a,(\phi_a^*j)_b)}\right)_u\xi + K_{(b,u,H)}(e) + (\phi_a^*X^{0,1}_h)_b\right),$$

*with*

$$K_{(b,u,H)}(e) := \frac{1}{2}D_b(\phi_a^*J)(e)\circ D^{\mathrm{v}}u\circ(\phi_a^*j)_b + \\ +\ \frac{1}{2}\left(X_{D_b(\phi_a^*H)(e)} + (\phi_a^*J)_b\circ X_{D_b(\phi_a^*H)(e)}\circ(\phi_a^*j)_b\right) + \\ +\ \frac{1}{2}(\phi_a^*J)_b\circ D^{\mathrm{v}}u\circ\left(D_b(\phi_a^*j)(e)\right),$$

*where $D^{\mathrm{v}}u$ denotes the vertical derivative of $u$ w. r. t. the connection on $S_a\times X$ defined by $(\phi_a^*H)_b$ and $b\mapsto(\phi_a^*J)_b$, $b\mapsto(\phi_a^*H)_b$ and $b\mapsto(\phi_a^*j)_b$ are regarded as maps from $U$ to the space of $\omega$-compatible almost complex structures on $X$, the space of Hamiltonian structures on $S_a\times X$ and the space of complex structures on $S_a$, respectively.*

*Remark* II.8. For the moment the only two important things about the map $K_{(b,u,H)}$ above are that it defines a compact operator, for it factors through the finite dimensional space $T_bU$ and that its image consists of $C^{r-1}$-sections if $u$ is of class $C^r$.

**Lemma II.22.** *In the same situation as in the previous lemma, let $V\subseteq\phi_a^*\hat{X}$ be an open subset and let $W\subseteq u^{-1}(V)$ be an open subset that intersects every* *connected component of* $\{b\} \times S_a$ *nontrivially. Let* $\mathcal{K} \subseteq T_H\mathcal{H}(\tilde{X})$ *be the closure of the span of those Hamiltonian perturbations that have support in* $\mathrm{pr}_1^{-1}(W) \cap V$ *(as sections of* $\mathrm{pr}_1^* T^*\Sigma$*). Let furthermore* $z_i \in S_a$*,* $i = 1, \dots, r$ *be a collection of points on* $S_a$*. Then the following maps are surjective:*

*a) The restriction of* $\left(D\overline{\partial}^{J,\mathcal{H}}_{U,\phi_a}\right)_{(b,u,H)}$ *to* $\{0\} \times \{\xi \in T_U\mathcal{B}^{k,p}_a(\hat{X}, A, J, H) \mid \xi(z_i) = 0\ \forall\, i = 1, \dots, r\} \times \mathcal{K}$.

*b) The map*

$$\left(D\overline{\partial}^{J,\mathcal{H}}_{U,\phi_a}\right)_{(b,u,H)} \times \mathrm{ev}_{1*} \times \cdots \times \mathrm{ev}_{r*} \times \left(\pi^{\mathcal{B}}_U\right)_* :$$
$$T_bU \times T_u\mathcal{B}^{k,p}_a(\hat{X}, A, J, H) \times \mathcal{K} \to$$
$$\mathcal{E}^{k-1,p}_a(\hat{X}, A, J, H) \times (u^*V\hat{X})_{z_1} \times \cdots \times (u^*V\hat{X})_{z_r} \times T_bU$$
$$(e, \xi, h) \mapsto \left(\left(D\overline{\partial}^{J,\mathcal{H}}_{U,\phi_a}\right)_{(b,u,H)}(e, \xi, h), \xi(z_1), \dots, \xi(z_r), e\right)$$

*Proof.* b) follows immediately from a) and the proof of a) follows exactly the same line of argument that appears several times in [MS04], e. g. Proposition 3.2.1, Proposition 3.4.2, Proposition 6.2.7, or the most closely related Theorem 8.3.1, or in [CM07] Lemma 4.1. □

**Definition II.24.** For a closed affine subspace $\mathcal{K} \subseteq \mathcal{H}(\tilde{X})$, meaning the intersection of a closed affine subspace of the space of $C^\varepsilon$-sections of $\mathrm{pr}_1^* T^*\Sigma$ with $\mathcal{H}(\tilde{X})$, see Definition II.17, define

$$\begin{aligned}\mathcal{M}(\hat{X}, A, J, \mathcal{K}) &:= \left(\pi^{\mathcal{M}}_{\mathcal{H}}\right)^{-1}(\mathcal{K}) \\ &\subseteq \mathcal{M}(\hat{X}, A, J, \mathcal{H}(\tilde{X})),\end{aligned}$$

where $\pi^{\mathcal{M}}_{\mathcal{H}} := \pi^{\mathcal{B}}_{\mathcal{H}}|_{\mathcal{M}(\hat{X},A,J,\mathcal{H}(\tilde{X}))}$ and analogously $\mathcal{M}_b(\hat{X}, A, J, \mathcal{K})$ for $b \in B$ and $\mathcal{M}_U(\hat{X}, A, J, \mathcal{K})$ for $U \subseteq B$ open.
Furthermore, given any open subset $V \subseteq \tilde{X}$, define

$$\mathcal{H}^V(\tilde{X})$$

to be the closure of the set of those $H \in \mathcal{H}(\tilde{X})$ that have support in $V$ and

$$\begin{aligned}\mathcal{M}^V(\hat{X}, A, J, \mathcal{K}) := \{u \in \mathcal{M}_b(\hat{X}, A, J, \mathcal{K}) \mid u(S_{b,i}) \cap \tilde{\iota}^{-1}(V) \neq \emptyset \text{ for every} \\ \text{connected component } S_{b,i} \text{ of } S_b\},\end{aligned}$$

where $\tilde{\iota} : \hat{X} \to \tilde{X}$ is the canonical map and analogously $\mathcal{M}^V_b(\hat{X}, A, J, \mathcal{K})$ and $\mathcal{M}^V_U(\hat{X}, A, J, \mathcal{K})$.

**Lemma II.23.** *In the notation of the above construction,*

$$\overline{\partial}^{J,\mathcal{H}}_U : \mathcal{B}^{k,p}_{U,\phi_a}(\hat{X}, A, J, \mathcal{H}(\tilde{X})) \to \mathcal{E}^{k-1,p}_{U,\phi_a}(\hat{X}, A, J, \mathcal{H}(\tilde{X}))$$

*is split transverse to the zero section and* $\mathcal{M}_U(\hat{X}, A, J, \mathcal{H}(\tilde{X}))$ *is a split Banach submanifold of* $\mathcal{B}^{k,p}_{U,\phi_a}(\hat{X}, A, J, \mathcal{H}(\tilde{X}))$.
*Furthermore, with respect to this Banach manifold structure,* $\pi^{\mathcal{M}}_{\mathcal{H}} : \mathcal{M}_U(\hat{X}, A, J, \mathcal{H}(\tilde{X})) \to \mathcal{H}(\tilde{X})$ *is a Fredholm map of index*

$$\mathrm{ind}(\pi^{\mathcal{M}}_{\mathcal{H}}) = \dim_{\mathbb{C}}(X)\hat{\chi} + 2c_1(A) + \dim_{\mathbb{R}}(U).$$

*Given an open subset* $V \subset \tilde{X}$*, for any* $H \in \mathcal{H}(\tilde{X})$*,*

$$\mathcal{M}^V_U(\hat{X}, A, J, H + \mathcal{H}^V(\tilde{X}))$$

*inherits a Banach manifold structure from* $\mathcal{M}_U(\hat{X}, A, J, \mathcal{H}(\tilde{X}))$ *s. t. the projection onto* $H + \mathcal{H}^V(\tilde{X})$ *is a Fredholm map of the same index as before.*

*Proof.* Lemma A.3.6 in [MS04], and the previous Lemma together with Lemma A.6. □

The set $\mathcal{M}_U(\hat{X}, A, J, \mathcal{H}(\tilde{X}))$ equipped with the Banach manifold structure from the previous lemma, which a priori does depend on $k$, $p$ and $\phi_a$, will be denoted by

$$\mathcal{M}^{k,p}_{U,\phi_a}(\hat{X}, A, J, \mathcal{H}(\tilde{X})).$$

The goal now is to show that the Banach manifold structure on $\mathcal{M}^{k,p}_{U,\phi_a}(\hat{X}, A, J, \mathcal{H}(\tilde{X}))$ does not depend on the choice of $k \geq 1$ and $p > 1$ with $kp > 2$ nor on $\phi_a : U \times S_a \cong S|_U$. Hence writing $\mathcal{M}_U(\hat{X}, A, J, \mathcal{H}(\tilde{X}))$ makes sense, and consequently given any trivialisation $(U_i, \phi_{a_i})_{i\in I}$ of $S$, the Banach manifolds $\mathcal{M}_U(\hat{X}, A, J, \mathcal{H}(\tilde{X}))$ patch together to a Banach manifold structure on $\mathcal{M}(\hat{X}, A, J, \mathcal{H}(\tilde{X}))$. To sum the argument up in two words: Elliptic regularity.

**Lemma II.24.** *Let* $k, \ell \in \mathbb{N}$*,* $1 < p, q < \infty$ *with* $kp, \ell q > 2$ *and assume that* $k > \ell$ *and* $k - \frac{2}{p} > \ell - \frac{2}{q}$*. Then the inclusion* $\mathcal{B}^{k,p}_{U,\phi_a}(\hat{X}, A, J, \mathcal{H}(\tilde{X})) \to \mathcal{B}^{\ell,q}_{U,\phi_a}(\hat{X}, A, J, \mathcal{H}(\tilde{X}))$ *defined via the Sobolev embedding theorem induces a diffeomorphism* $\mathcal{M}^{k,p}_{U,\phi_a}(\hat{X}, A, J, \mathcal{H}(\tilde{X})) \cong \mathcal{M}^{\ell,q}_{U,\phi_a}(\hat{X}, A, J, \mathcal{H}(\tilde{X}))$.

*Proof.* By Lemma II.19, one has the set-theoretic identity $\mathcal{M}^{k,p}_{U,\phi_a}(\hat{X}, A, J, \mathcal{H}(\tilde{X})) = \mathcal{M}^{\ell,q}_{U,\phi_a}(\hat{X}, A, J, \mathcal{H}(\tilde{X}))$.
Fair warning: In the following I will prove that the identity is a diffeomorphism, so I likely am missing something obvious.
To show that this map also induces a diffeomorphism, one has to express it in charts defining the differentiable structures on $\mathcal{M}^{k,p}_{U,\phi_a}(\hat{X}, A, J, \mathcal{H}(\tilde{X}))$ and $\mathcal{M}^{\ell,q}_{U,\phi_a}(\hat{X}, A, J, \mathcal{H}(\tilde{X}))$, respectively. Said charts are given via the implicit function theorem, Theorem A.3, which unfortunately means that one has to go through the proof of said theorem, since the standard formulation does not provide much control over the implicitely defined function. Given any $(b, u, H) \in \mathcal{M}^{\ell,q}_U(\hat{X}, A, J, \mathcal{H}(\tilde{X})) = \mathcal{M}^{k,p}_U(\hat{X}, A, J, \mathcal{H}(\tilde{X}))$, first of all observe that the standard charts for the surrounding spaces $\mathcal{B}^{\ell,q}_{U,\phi_a}(\hat{X}, A, J, \mathcal{H}(\tilde{X}))$ and $\mathcal{B}^{k,p}_{U,\phi_a}(\hat{X}, A, J, \mathcal{H}(\tilde{X}))$

around $(b, u, H)$ have the property that the coordinate map for the latter space is just the restriction of the coordinate map of the former, with target $U \times T_u\mathcal{B}^{\ell,q}_a(\hat{X}, A, J, H) \times \mathcal{H}(\tilde{X}) = U \times L^{\ell,q}(u^*V\hat{X}, \dots) \times \mathcal{H}(\tilde{X})$ restricted to the (non-closed) subspace $U \times L^{k,p}(u^*V\hat{X}, \dots) \times \mathcal{H}(\tilde{X})$. Also, the Cauchy-Riemann operator on the latter space is just the restriction of the Cauchy-Riemann operator on the former space. To shorten notation, the situation can also be described as follows: Given Banach spaces $\mathbb{R}^n, X, Y, Z$ taking the roles of $U$, $L^{k-1,p}(u^*V\hat{X}, \dots)$, $\mathcal{H}(\tilde{X})$ and $L^{\ell-1,q}(\overline{\mathrm{Hom}}_{(j_a,J_a)}(TS_a, u^*V\hat{X}), \dots)$, respectively, and linear subspaces $X' \subseteq X$, $Z' \subseteq Z$ (corresponding to $L^{k,p}(u^*V\hat{X}, \dots)$ and $L^{k-1,p}(\overline{\mathrm{Hom}}_{(j_a,J_a)}(TS_a, u^*V\hat{X}), \dots)$), but equipped with a finer topology than the induced one, and a smooth map $f : \mathbb{R}^n \times X \times Y \to Z$ that restricts to a well-defined smooth map $f' : \mathbb{R}^n \times X' \times Y \to Z'$ (corresponding to the Cauchy-Riemann operator). Furthermore, $f^{-1}(0) = (f')^{-1}(0)$. Both $f$ and $f'$ are split surjective, so around every point $(b, x, y) \in f^{-1}(0)$ there exist open neighbourhoods $V \subseteq X$ and $V' \subseteq X'$ around $(b, x, y)$ together with smooth maps $\psi : V \to X$ and $\psi' : V' \to X'$ fixing $(b, x, y)$ and that are diffeomorphisms onto their images. Furthermore, $\psi$ maps $f^{-1}(0) \cap V$ to $\ker D_{(b,x,y)}f$ and $\psi'$ maps $(f')^{-1}(0) \cap V'$ to $\ker D_{(b,x,y)}f'$. $D_{(b,x,y)}f$ is of the form $(e, \xi, h) \mapsto D_bf(e) + D_xf(\xi) + D_yf(h)$, where in the notation of Lemma II.23 in the formula for the linearisation of the universal Cauchy-Riemann operator, $D_xf(\xi)$ corresponds to the term involving $D\overline{\partial}^{(\phi_a^*J)_b,(\phi_a^*H)_b}_{(S_a,(\phi_a^*j)_b)}$, $D_bf(e)$ to that involving $K_{(b,u,H)}(e)$ and $D_yf(h)$ to that involving $(\phi_a^*X_h^{0,1})_b$ and correspondingly for $f'$, where actually $D_bf'(e) = D_bf(e)$ and $D_yf'(h) = D_y(h)$. For any $(e, \xi, h) \in \ker D_{(b,x,y)}f$, $\xi$ satisfies the equation $D_xf(\xi) = D_bf(e) + D_yf(h)$ and by Lemma II.23, Remark II.8 and Lemma II.19, the right hand side is smooth. Because $D_xf$ is a smooth Cauchy-Riemann operator by Lemma II.16 and Lemma II.19, it follows from the linear elliptic regularity theorem, that $\xi$ is smooth. Hence $\ker D_{(b,x,y)}f = \ker D_{(b,x,y)}f'$ with the norms on both sides being equivalent as well. The final piece of data needed for the construction of $\psi$ and $\psi'$ are right inverses $Q : Z \to X$ and $Q' : Z' \to X'$ of $D_{(b,x,y)}f$ and $D_{(b,x,y)}f'$, respectively. If $Q'$ can be chosen as the restriction of $Q$, then the construction presented on p. 138 f. shows that indeed the resulting $\psi'$ is the restriction of $\psi$. Now these splitting maps are produced via Lemma A.3.6 of [MS04] in the following way, see the proof of said lemma: Consider the map $D_bf + D_xf : \mathbb{R}^n \times X \to Z$. This is a Fredholm operator, for the second term is the Cauchy-Riemann operator and the first term is compact, since it is defined on a finite dimensional domain. So one can choose complements $\tilde{X} \subseteq \mathbb{R}^n \times X$ and $\tilde{Z} \subseteq Z$ of $\ker(D_bf + D_xf) \subseteq \mathbb{R}^n \times X$ and $\mathrm{im}(D_bf + D_xf)$, respectively. Since $D_bf + D_xf + D_yf$ is surjective, one can choose a subspace $\tilde{Y} \subseteq Y$ s. t. $D_yf$ defines an isomorphism from $\tilde{Y}$ to $\tilde{Z}$. Then define $Q := \big((D_bf + D_xf)|_{\tilde{X}}\big)^{-1} \circ \mathrm{pr}^Z_{\mathrm{im}(D_bf+D_xf)} + \big(D_yf|_{\tilde{Y}}\big)^{-1} \circ \mathrm{pr}^Z_{\tilde{Z}}$. Similarly, $Q'$ is defined by choosing $\tilde{X}' \subseteq X'$ and $\tilde{Z}' \subseteq Z'$. The proof now finishes by observing that, because by elliptic regularity as above and because $D_bf = D_bf'$, $\ker(D_bf' + D_xf') = \ker(D_bf + D_xf)$. Hence given a choice of $\tilde{X}$, $\tilde{X}' := \tilde{X} \cap X'$ is first of all an algebraic complement and because the topology on $X'$ is finer than the topology on $X$, it is a closed subspace as well. Also, given a choice of $\tilde{Z}'$, since the Fredholm indices of $D_b f + D_x f$ and $D_b f' + D_x f'$ coincide and their kernels are the same, by dimension reasons $\tilde{Z}'$ is an algebraic complement of $\mathrm{im}(D_b f + D_x f)$, as well. Being finite dimensional it is also a closed subspace of $Z$. With these choices, $Q'$ becomes the restriction of $Q$. Note that the above argument via Fredholm indices in a sense turns the line of argument upside down, for the fact that $\tilde{Z}$ and $\tilde{Z}'$ can be chosen as the same space is actually used to show that their Fredholm indices coincide, see the proof of the Riemann-Roch theorem, Theorem C.1.10 in [MS04]. □

**Lemma II.25.** *Using the notation of Construction II.8, let $\phi_a, \psi_a : U \times S_a \xrightarrow{\cong} S|_U$ be two smooth trivialisations and let $r \in \mathbb{N}$. Then the set-theoretic inclusion*

$$\mathcal{B}^{k+r,p}_{U,\phi_a}(\hat{X}, A, J, \mathcal{H}(\tilde{X})) \hookrightarrow \mathcal{B}^{k,p}_{U,\psi_a}(\hat{X}, A, J, \mathcal{H}(\tilde{X}))$$

*is a map of class $C^{r-1}$.*

*Proof.* Let $\rho := \mathrm{pr}_2 \circ \psi_a^{-1} \circ \phi_a : U \times S_a \to S_a$. In other words, $\rho$ is a family $\rho_b : S_a \to S_a$, $b \in U$, of diffeomorphisms of $S_a$. Fix any $H \in \mathcal{H}(\tilde{X})$. Then in the trivialisations $\mathcal{B}^{k+r,p}_{U,\phi_a}(\hat{X}, A, J, \mathcal{H}(\tilde{X})) \cong U \times \mathcal{B}^{k+r,p}_a(\hat{X}, A, J, H) \times \mathcal{H}(\tilde{X})$ and $\mathcal{B}^{k,p}_{U,\psi_a}(\hat{X}, A, J, \mathcal{H}(\tilde{X})) \cong U \times \mathcal{B}^{k,p}_a(\hat{X}, A, J, H) \times \mathcal{H}(\tilde{X})$ defining their smooth structures, the coordinate expression for the inclusion is the map

$$\begin{aligned} U \times \mathcal{B}^{k+r,p}_a(\hat{X}, A, J, H) \times \mathcal{H}(\tilde{X}) &\to U \times \mathcal{B}^{k,p}_a(\hat{X}, A, J, H) \times \mathcal{H}(\tilde{X}) \\ (b, u, H) &\mapsto (b, u \circ \rho_b, H). \end{aligned}$$

The only question about differentiability of this map arises from the middle component, the map

$$\begin{aligned} \Psi : U \times \mathcal{B}^{k+r,p}_a(\hat{X}, A, J, H) &\to \mathcal{B}^{k,p}_a(\hat{X}, A, J, H) \\ (b, u) &\mapsto u \circ \rho_b. \end{aligned}$$

Fix a point $(b, u) \in U \times \mathcal{B}^{k+r,p}_a(\hat{X}, A, J, H)$ with $u$ of class $C^{k+r}$. We want to express $\Psi$ in coordinates around $(b, u)$ and $\Psi(b, u) = u \circ \rho_b$. First, assume that $U$ is an open subset of some $\mathbb{R}^d$. Then the coordinate expression $\tilde{\Psi} : U \times L^{k+r,p}(u^* V\hat{X}_a) \to L^{k,p}((u \circ \rho_b)^* V\hat{X}_a)$ of $\Psi$ is given by the string of maps

$$(b', \xi) \mapsto (b', \exp^\perp_u(\xi)) \mapsto \exp^\perp_u(\xi) \circ \rho_{b'} \mapsto (\exp^\perp_{u\circ\rho_b})^{-1}(\exp^\perp_u(\xi) \circ \rho_{b'}).$$

For simplicity from now on I will drop the subscript $a$ on $S_a$ and consequently $\hat{X}_a$ and denote by $S$ the Riemann surface $S_a$ and by $\hat{X}$ the trivial fibre bundle $S \times X$ over $S$ with fibres $\hat{X}_z \cong X$ at the points $z \in S$. Then the above formula can be evaluated at some point $z \in S$ and the definition of $\exp^\perp$ for the fibre bundle $\hat{X}$ can be inserted to give

$$\tilde{\Psi}(b', \xi)(z) = \left(\exp^{\hat{X}_z}_{u\circ\rho_b(z)}\right)^{-1} \left(\exp^{\hat{X}_{\rho_{b'}(z)}}_{u\circ\rho_{b'}(z)}(\xi \circ \rho_{b'}(z))\right).$$

First, note that the right hand side is well-defined for $\|\xi\|_{L^{1,p}(u^* V\hat{X})}$ small enough, independent of $b'$, because by compactness, $\sup\{\mathrm{inj}(\hat{X}_z) \mid z \in S\}$ is finite and an $L^{1,p}$-bound on $\xi$ implies a pointwise bound by the Sobolev embedding theorem. Second, this can be written as

$$\left(\exp^{\hat{X}_z}_{u\circ\rho_b(z)}\right)^{-1}\left(\exp^{\hat{X}_{\rho_{b'}(z)}}_{u\circ\rho_{b'}(z)}(\xi\circ\rho_{b'}(z))\right) =$$
$$\underbrace{\left(\left(\exp^{\hat{X}_z}_{u\circ\rho_b(z)}\right)^{-1}\circ\exp^{\hat{X}_{\rho_{b'}(z)}}_{u\circ\rho_b(z)}\right)}_{(*)}\circ\underbrace{\left(\left(\exp^{\hat{X}_{\rho_{b'}(z)}}_{u\circ\rho_b(z)}\right)^{-1}\circ\exp^{\hat{X}_{\rho_{b'}(z)}}_{u\circ\rho_{b'}(z)}\right)}_{(**)}(\xi\circ\rho_{b'}(z)),$$

which can be interpreted as follows: Over $U\times S$, consider the two fibre bundles $\rho^*\hat{X}$ and $\mathrm{pr}_2^*\hat{X}$, where $\mathrm{pr}_2 : U\times S\to S$ is the projection. Both of these bundles are canonically identified with the trivial one, but carry two different structures of Riemannian submersion. Furthermore $u$ is a section of $\hat{X}$, and so is $u\circ\rho_b$. Hence $\rho^*u$ and $\mathrm{pr}_2^*(u\circ\rho_b)$ are sections of $\rho^*\hat{X}$ and $\mathrm{pr}_2^*\hat{X}$, respectively, and $\rho^*\xi$ is a section of $V\rho^*\hat{X} = \rho^*V\hat{X}$ (along $\rho^*u$). Then the first term $(*)$ above is the coordinate expression for the identification $L^{k+r,p}(\mathrm{pr}_2^*\hat{X})\cong L^{k+r,p}(\rho^*\hat{X})$ induced by the canonical identification of $\mathrm{pr}_2^*\hat{X}\cong\rho^*\hat{X}$ in charts around the section $\mathrm{pr}_2^*(u\circ\rho_b)$, whereas the second one $(**)$ is the usual coordinate transformation on $L^{k,p}(\rho^*\hat{X})$ from the chart around $\rho^*u$ to the chart around $\mathrm{pr}_2^*(u\circ\rho_b)$. So the above map $\tilde{\Psi}$ can be interpreted as mapping $\xi$ to $\rho^*\xi\in L^{k+r,p}((\rho^*u)^*V\rho^*\hat{X})$, then applying the two coordinate transformations above and finally restricting to the slice $\{b\}\times\hat{X}\subseteq\mathrm{pr}_2^*\hat{X}$. A derivative of $\tilde{\Psi}(b',\xi)$ in the first variable $b'$ then corresponds to a covariant derivative of $\rho^*\xi$ in a direction tangent to the first factor of $U\times S$. The maps $(*)$ and $(**)$ have bounded derivatives of all orders after restricting to $V\times S$, where $V\subseteq U$ is a precompact open subset of $U$, by Lemma II.10.
Now $\nabla^s\rho^*\xi$ can be expressed (by the Leibniz rule, basically) as a linear combination of $\xi,\ldots,\nabla^s\xi$ with coefficients depending on the $s$-jet of $\rho$. Again after restricting to a precompact subset $V\subseteq U$, these coefficients can be bounded. Combining the above, at least for $\xi\in\Gamma^s(u^*V\hat{X})$ and $b'\in V$, via these pointwise estimates one can estimate $\|(D^s\tilde{\Psi})(b',\xi)\|_{L^{k,p}}\leq\sum_{j=0}^s c_j\|\nabla^j\xi\|_{L^{k,p}}\leq c\|\xi\|_{L^{k+s,p}}$ for some constants $c_j, c$. Applying the usual density argument, Lemma A.1, which causes the loss of one derivative (hence it says $C^{r-1}$, not $C^r$, in the statement), shows the lemma. □

**Corollary II.5.** *The set-theoretic identity defines a diffeomorphism*

$$\mathcal{M}_{U,\phi_a}(\hat{X},A,J,\mathcal{H}(\tilde{X}))\overset{\cong}{\to}\mathcal{M}_{U,\psi_a}(\hat{X},A,J,\mathcal{H}(\tilde{X})).$$

*In particular, any choice of covering $(U_i)_{i\in I}$ of the base $B$ of $S$ and trivialisations $(\phi_i : U_i\times S_{a_i}\xrightarrow{\cong} S|_{U_i})_{i\in I}$ defines a cocycle for a Banach manifold structure on $\mathcal{M}(\hat{X},A,J,\mathcal{H}(\tilde{X}))$ independent of these choices.*
*If $\mathcal{C}$ is any other Banach manifold and $f:\mathcal{B}^{k_0,p}(\hat{X},A,J,\mathcal{H}(\tilde{X}))\to\mathcal{C}$, for some $k_0\in\mathbb{N}$ and $p>1$ with $k_0p>2$, a map with the property that there exists an $r\in\mathbb{Z}$, $r\leq k_0$ s. t. $f|_{\mathcal{B}^{k,p}(\hat{X},A,J,\mathcal{H}(\tilde{X}))}:\mathcal{B}^{k,p}(\hat{X},A,J,\mathcal{H}(\tilde{X}))\to\mathcal{C}$ is of class $C^{k-r}$ for every $k\geq k_0$, then $f|_{\mathcal{M}(\hat{X},A,J,\mathcal{H}(\tilde{X}))}:\mathcal{M}(\hat{X},A,J,\mathcal{H}(\tilde{X}))\to\mathcal{C}$ is smooth.*
*With respect to this Banach manifold structure,*

$$\pi^{\mathcal{M}}_{\mathcal{H}}:\mathcal{M}(\hat{X},A,J,\mathcal{H}(\tilde{X}))\to\mathcal{H}(\tilde{X})$$

*is a Fredholm map of index*

$$\operatorname{ind}(\pi^{\mathcal{M}}_{\mathcal{H}}) = \dim_{\mathbb{C}}(X)\hat{\chi} + 2c_1(A) + \dim_{\mathbb{R}}(B).$$

*Given an open subset* $V \subseteq \tilde{X}$ *and any* $H \in \mathcal{H}(\tilde{X})$, *the same holds for* $\mathcal{M}^V(\hat{X}, A, J, H + \mathcal{H}^V(\tilde{X}))$ *and the projection onto* $H + \mathcal{H}^V(\tilde{X})$.

*Proof.* Immediate from the preceding three lemmas. □

## II.3.4 Evaluation maps and nodal families

Of interest are two kinds of evaluation maps: Evaluation at the marked points $\hat{R}^i$ and at the points corresponding to the nodes of $\Sigma|_B$. While the former can be defined as maps on $\mathcal{M}(\hat{X}, A, J, \mathcal{H}(\tilde{X}))$, the latter can not. For the nodes only form a discrete subbundle of $\Sigma|_B$ or their desingularisations one of $S$. The evaluations at these points are of importance since in the desingularisation $S$ of $\Sigma|_B$ all the nodes are resolved to pairs of points and hence the space $\mathcal{M}(\hat{X}, A, J, \mathcal{H}(\tilde{X}))$ contains "too many" curves in the sense that one is only interested in those which map each pair of points corresponding to a node to a single point. For only on this set does there exist an inclusion into $\mathcal{M}(\tilde{X}, A, J, \mathcal{H}(\tilde{X}))$. But one can still choose a covering $(U_i)_{i\in I}$ and trivialisations $(\phi_i : U_i \times S_{a_i} \to S|_{U_i})_{i\in I}$ with the property that $\phi_i$ trivialises $N|_{U_i}$ as well, i.e. after choosing some numbering $N^{i,1}_j(a_i), N^{i,2}_j(a_i)$, $j = 1, \dots, d$, of $N_{a_i}$ s.t. $N^{i,1}_j(a_i)$ and $N^{i,2}_j(a_i)$ correspond to the same node and defining $N^{i,1}_j(b) := \phi_i(b, N^{i,1}_1(a_i))$, $N^{i,2}_j(b) := \phi_i(b, N^{i,2}_j(a_i))$, for $b \in U_i$, $j = 1, \dots, d$, one has $N_b = \{N^{i,1}_j(b), N^{i,2}_j(b) \mid j = 1, \dots, d\}$. $N^{i,1}_j(b)$ and $N^{i,2}_j(b)$ here naturally are always supposed to correspond to the same node. This allows the definition of evaluation maps (as always, $kp > 2$)

$$\begin{aligned}
\operatorname{ev}^{N^{i,1},N^{i,2}} : \mathcal{B}^{k,p}_{U_i}(\hat{X}, A, J, \mathcal{H}(\tilde{X})) &\to (X \oplus X)^{\oplus d} \\
\mathcal{B}^{k,p}_b(\hat{X}, A, J, H) \ni u &\mapsto ((\mathrm{pr}_2(u(N^{i,1}_1(b))), \mathrm{pr}_2(u(N^{i,1}_1(b)))), \dots, \\
&\qquad (\mathrm{pr}_2(u(N^{i,1}_d(b))), \mathrm{pr}_2(u(N^{i,1}_d(b))))).
\end{aligned}$$

In contrast, the marked points $\hat{R}_1, \dots, \hat{R}_n : B \to S$ allow the definition of a globally defined evaluation map

$$\begin{aligned}
\operatorname{ev}^{\hat{R}} : \mathcal{B}^{k,p}(\hat{X}, A, J, \mathcal{H}(\tilde{X})) &\to \hat{R}^*_1\hat{X} \oplus \dots \oplus \hat{R}^*_n\hat{X} \\
\mathcal{B}^{k,p}_b(\hat{X}, A, J, H) \ni u &\mapsto (u(\hat{R}_1(b)), \dots, u(\hat{R}_n(b)))
\end{aligned}$$

with a well-defined restriction to $\mathcal{M}(\hat{X}, A, J, \mathcal{H}(\tilde{X}))$. The target space of the above map is the fibre bundle over $B$ which is the Whitney sum of the fibre bundles $\hat{R}^*_i\hat{X}$. Writing $u \in \mathcal{B}^{k,p}_b(\hat{X}, A, J, H)$ in the form $z \mapsto (z, \overline{u}(z)) \in S_b \times X = \hat{X}_b$, then $\operatorname{ev}^{\hat{R}}(u) = (b, \overline{u}(\hat{R}_1(b)), \dots, \overline{u}(\hat{R}_n(b)))$. As before for the $N^{i,1}_j, N^{i,2}_j$, assume that the $\phi_i$ preserve the markings in the sense that $\phi_i(b, \hat{R}_j(a_i)) = \hat{R}_j(b)$

for all $b \in U_i$, $i \in I$. The reason for this is the following: If $f : M \to N$ is a map between manifolds, and $\gamma : \mathbb{R} \to M$ is a path in $M$, then $\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)) = df(\dot\gamma(t))$ depends on the first derivative of $f$, and correspondingly for the higher derivatives. If $f$ is of some Sobolev class, then this is only well defined by the Sobolev embedding theorem as long as $f$ has enough weak derivatives. This problem is circumvented here, because with the choices of $\phi_i$ above, the markings and nodal points under the $\phi_i$ correspond to constant points on the $S_{a_i}$. Hence the restriction $\mathrm{ev}^{\hat R} : \mathcal{B}^{k,p}_{U_i,\phi_i}(\hat X, A, J, \mathcal{H}(\tilde X)) \to \hat R_1^* \hat X \oplus \cdots \oplus \hat R_n^* \hat X|_{U_i}$ is actually smooth w. r. t. to the smooth structure defined via $\phi_i$ and hence by the previous corollary

$$\mathrm{ev}^{\hat R} : \mathcal{M}(\hat X, A, J, \mathcal{H}(\tilde X)) \to \hat R_1^* \hat X \oplus \cdots \oplus \hat R_n^* \hat X$$

is a smooth map. Analogously, all the restrictions

$$\mathrm{ev}^{N^{i,1},N^{i,2}} : \mathcal{M}_{U_i}(\hat X, A, J, \mathcal{H}(\tilde X)) \to (X \oplus X)^{\oplus d}$$

are smooth maps.

Letting $\Delta := \{(x,x) \in X \oplus X \mid x \in X\}$, the space of holomorphic curves, at least over one of the $U_i$, is the preimage $\left(\mathrm{ev}^{N^{i,1},N^{i,2}}\right)^{-1}(\Delta^d)$, which is the space of those curves mapping each pair of points in a desingularisation corresponding to a node to a single point. Furthermore, $\left(\mathrm{ev}^{N^{i,1},N^{i,2}}\right)^{-1}(\Delta^d)$ is independent of the choice of the $N^{i,1}_j$ and $N^{i,2}_j$, since any compatible reordering (in $j$ or switching $N^{i,1}_j$ and $N^{i,2}_j$ for a fixed $j$) leaves the set $\Delta^d$ invariant. Hence there are well-defined sets

$$\mathcal{M}_{U_i}(\tilde X|_B, A, J, \mathcal{H}(\tilde X)) := \left(\mathrm{ev}^{N^{i,1},N^{i,2}}\right)^{-1}(\Delta^d)$$

which patch together to a well-defined set

$$\mathcal{M}(\tilde X|_B, A, J, \mathcal{H}(\tilde X)) := \bigcup_{i\in I} \mathcal{M}_{U_i}(\tilde X|_B, A, J, \mathcal{H}(\tilde X))$$

in the sense that for any $i, j \in I$ and for any $b \in U_i \cap U_j$, the sets of those points in $\mathcal{M}_{U_i}(\tilde X|_B, A, J, \mathcal{H}(\tilde X))$ and $\mathcal{M}_{U_j}(\tilde X|_B, A, J, \mathcal{H}(\tilde X))$ lying over $b$ coincide and furthermore, $\mathcal{M}(\tilde X|_B, A, J, \mathcal{H}(\tilde X))$ is independent of choices. Given $V \subseteq \tilde X$ and $H \in \mathcal{H}(\tilde X)$, there are analogously defined sets

$$\mathcal{M}^V_{U_i}(\tilde X|_B, A, J, H + \mathcal{H}^V(\tilde X)) \text{ and } \mathcal{M}^V(\tilde X|_B, A, J, H + \mathcal{H}^V(\tilde X)).$$

Also, one can restrict $\mathrm{ev}^{\hat R}$ to the above subsets. At this point it also makes sense to introduce what is mainly a change in notation. Namely remember that $\hat X$ was the pullback of $\tilde X$ under the desingularisation $\hat\imath : S \to \Sigma$ of the restriction of the nodal family $\Sigma$ to the subset $\iota : B \to M$, where $M$ was the base of the family $\Sigma$. Also, the markings $\hat R$ of $S$ were the pullbacks of the markings $R$ of $\Sigma$. So one can canonically identify $\hat R_1^* \hat X \oplus \cdots \oplus \hat R_n^* \hat X \cong R_1^* \tilde X \oplus \cdots \oplus R_n^* \tilde X|_B$. Note that because $\tilde X$ was defined to be the pullback $\pi^* X$ and because the $R_i$

are sections, every $R_i^*\tilde{X}$ is canonically identified with $X$. But to distinguish the factors, the above notation is kept. Using this, write

$$\mathrm{ev}^R := \mathrm{ev}^{\hat{R}}|_{\mathcal{M}(\tilde{X}|_B, A, J, \mathcal{H}(\tilde{X}))} : \mathcal{M}(\tilde{X}|_B, A, J, \mathcal{H}(\tilde{X})) \to R_1^*\tilde{X} \oplus \cdots \oplus R_n^*\tilde{X}|_B$$

**Lemma II.26.** *For any choice of $U_i$, $\phi_i$ and $N^{i,1}, N^{i,2}$ as above, the maps*

$$\mathrm{ev}^{N^{i,1},N^{i,2}} \times \mathrm{ev}^{\hat{R}} : \mathcal{M}_{U_i}(\hat{X}, A, J, \mathcal{H}(\tilde{X})) \to (X^2)^d \times \hat{R}_1^*\hat{X} \oplus \cdots \oplus \hat{R}_n^*\hat{X}$$

*are submersions.*
*The sets $\mathcal{M}_{U_i}(\tilde{X}|_B, A, J, \mathcal{H}(\tilde{X}))$ are split submanifolds of $\mathcal{M}_{U_i}(\hat{X}, A, J, \mathcal{H}(\tilde{X}))$ that define a cocycle that equips $\mathcal{M}(\tilde{X}|_B, A, J, \mathcal{H}(\tilde{X}))$ with the structure of a split Banach submanifold of $\mathcal{M}(\hat{X}, A, J, \mathcal{H}(\tilde{X}))$ of codimension $\dim_{\mathbb{R}}(X)\, d = \dim_{\mathbb{C}}(X)\, 2d$.*
*Furthermore,*

$$\mathrm{ev}^R : \mathcal{M}(\tilde{X}|_B, A, J, \mathcal{H}(\tilde{X})) \to R_1^*\tilde{X} \oplus \cdots \oplus R_n^*\tilde{X}|_B$$

*is a submersion and in particular so are*

$$\pi_B^{\mathcal{M}} : \mathcal{M}(\tilde{X}|_B, A, J, \mathcal{H}(\tilde{X})) \to B,$$

*the composition of $\mathrm{ev}^R$ with the projection $R_1^*\tilde{X} \oplus \cdots \oplus R_n^*\tilde{X}|_B \to B$, and every*

$$\mathrm{ev}_i^R : \mathcal{M}(\tilde{X}|_B, A, J, \mathcal{H}(\tilde{X})) \to R_i^*\tilde{X}|_B$$

*for $1 \leq i \leq n$, the composition of $\mathrm{ev}^R$ with the projection $R_1^*\tilde{X} \oplus \cdots \oplus R_n^*\tilde{X}|_B \to R_i^*\tilde{X}|_B$.*
*Finally,*

$$\pi_{\mathcal{H}}^{\mathcal{M}} : \mathcal{M}(\tilde{X}|_B, A, J, \mathcal{H}(\tilde{X})) \to \mathcal{H}(\tilde{X})$$

*is a Fredholm map of index*

$$\mathrm{ind}(\pi_{\mathcal{H}}^{\mathcal{M}}) = \dim_{\mathbb{C}}(X)\chi + 2c_1(A) + \dim_{\mathbb{R}}(B).$$

*Given an open subset $V \subseteq \tilde{X}$ and any $H \in \mathcal{H}(\tilde{X})$, the same statements hold with $\mathcal{M}_{U_i}(\hat{X}, A, J, \mathcal{H}(\tilde{X}))$ replaced by $\mathcal{M}_{U_i}^V(\hat{X}, A, J, H + \mathcal{H}^V(\tilde{X}))$, $\mathcal{M}(\tilde{X}|_B, A, J, \mathcal{H}(\tilde{X}))$ replaced by $\mathcal{M}^V(\tilde{X}|_B, A, J, H + \mathcal{H}^V(\tilde{X}))$ and $\mathcal{H}(\tilde{X})$ replaced by $H + \mathcal{H}^V(\tilde{X})$.*

*Proof.* Lemma II.22, Lemma A.6, Lemma A.7. □

**Corollary II.6.** *For generic $H \in \mathcal{H}(\tilde{X})$, $\mathcal{M}(\tilde{X}|_B, A, J, H)$ is a manifold of dimension*

$$\dim \mathcal{M}(\tilde{X}|_B, A, J, H) = \dim_{\mathbb{C}}(X)\chi + 2c_1(A) + \dim_{\mathbb{R}}(B).$$

*Proof.* Sard-Smale and Lemma II.26. □

# II.4 Compactification and transversality via hypersurfaces

## II.4.1 The Gromov compactification of $\mathcal{M}(\tilde{X}, A, J, \mathcal{H}(\tilde{X}))$

So far, a topology has only been defined on $\mathcal{M}(\tilde{X}|_B, A, J, \mathcal{H}(\tilde{X}))$, where $B \subseteq M$ is a locally closed submanifold over which there exists a desingularisation of $\Sigma$. But even if $M$ has a well-defined stratification by signature, this does not define a well-behaved topology on all of $\mathcal{M}(\tilde{X}, A, J, \mathcal{H}(\tilde{X}))$. Well-behaved here is to mean at least that the maps $\pi^{\mathcal{M}}_M : \mathcal{M}(\tilde{X}, A, J, \mathcal{H}(\tilde{X})) \to M$ and $\pi^{\mathcal{M}}_{\mathcal{H}} : \mathcal{M}(\tilde{X}, A, J, \mathcal{H}(\tilde{X})) \to \mathcal{H}(\tilde{X})$ are to be continuous.
Furthermore, to be able to apply the compactness results from [Hum97] and [BEH$^+$03], this topology has to be chosen to be compatible in a sense to that of Deligne-Mumford convergence. The relevant construction here can be found in the proof of Theorem 13.6 in [RS06] (the direction (ii) $\Rightarrow$ (i)). The implication of this theorem can be stated as saying that the map $M \to \overline{M}_{g,n}$ from the base space of a marked nodal family of Riemann surfaces of type $(g, n)$ to the Deligne-Mumford space equipped with the topology of Deligne-Mumford convergence (as defined e. g. in [Hum97] or [BEH$^+$03]) is continuous. The topology on $\mathcal{M}(\tilde{X}, A, J, \mathcal{H}(\tilde{X}))$ will be described in terms of convergence of sequences as in Section 5.6 of [MS04]. To do so the following result from [RS06] will be used, where still $(\pi : \Sigma \to M, R_*)$ is an arbitrary family of marked nodal Riemann surfaces of type $(g, n)$:

**Construction II.9.** Let $b \in M$. Then there exists a neighbourhood $U \subseteq M$ of $b$ with the following properties: Let $n_1, \ldots, n_d \in \Sigma_b$ be the nodal points on $\Sigma_b$. For $i = 1, \ldots, d$ there are pairwise disjoint neighbourhoods $N_i \subseteq \Sigma$ of the $n_i$ with $\pi(N_i) = U$ and s. t. $R_j \cap N_i = \emptyset \; \forall\, j = 1, \ldots n, i = 1, \ldots, d$ and holomorphic maps

$$(x_i, y_i) : N_i \to \mathbb{D}^2, \quad z_i : U \to \mathbb{D}, \quad t_i : U \to \mathbb{D}^{\dim_{\mathbb{C}}(M)-1}$$

s. t.

$$(z_i, t_i) : U \to \mathbb{D}^{\dim_{\mathbb{C}}(M)}$$
$$(x_i, y_i, t_i \circ \pi|_{N_i}) : N_i \to \mathbb{D}^{\dim_{\mathbb{C}}(\Sigma)}$$

are holomorphic coordinate systems with

$$(x_i, y_i)(n_i) = (0, 0) \quad \text{and} \quad x_i y_i = z_i \circ \pi|_{N_i}.$$

Denote for $b' \in U$ and $i = 1, \ldots, d$

$$\Gamma_i(b') := \{z \in \Sigma_{b'} \cap N_i \mid |x_i(z)| = |y_i(z)| = \sqrt{|z_i(b')|}\}$$

and

$$\Gamma(b') := \bigcup_{i=1}^{d} \Gamma_i(b'), \quad \Gamma := \bigcup_{b' \in U} \Gamma(b').$$

Then each $\Gamma(b')$ is a disjoint union of nodal points (one for each $i$ with $z_i(b') = 0$) and pairwise disjoint embedded circles (one for each $i$ with $z_i(b') \neq 0$) disjoint from all the nodal and marked points. Especially $\Gamma_i(b) = n_i$ and hence $\Gamma(b) = \{n_1, \ldots, n_d\}$. Also, for every $b' \in U$ there exists a continuous map

$$\psi_{b'} : \Sigma_{b'} \to \Sigma_b$$

with the following properties:

- $\psi_{b'}(\Gamma_i(b')) = n_i$ for all $i = 1, \ldots, d$.
- $\psi_{b'}|_{\Sigma_{b'} \setminus \Gamma(b')} : \Sigma_{b'} \setminus \Gamma(b') \to \Sigma_b \setminus \{n_1, \ldots, n_d\}$ is a diffeomorphism.
- The map

$$\begin{aligned} \psi : \Sigma|_U \setminus \Gamma &\to U \times (\Sigma_b \setminus \{n_1 \ldots, n_d\}) \\ z &\mapsto (\pi(z), \psi_{\pi(z)}(z)) \end{aligned}$$

  is a diffeomorphism.

These maps have the property that if $(b_i)_{i \in \mathbb{N}} \subseteq U$ is a sequence converging to $b$, then the sequence $(j_i)_{i \in \mathbb{N}}$ of complex structures on $\Sigma_b \setminus \{n_1, \ldots, n_d\}$ defined by $j_i := \psi_{b_i,*} j_{b_i}$, where $j_{b_i}$ denotes the complex structure on $\Sigma_{b_i} \setminus \Gamma(b')$, converges in the $C^\infty$-topology to the restriction of $j_b$ to $\Sigma_b \setminus \{n_1, \ldots, n_d\}$.

With this one can define sequential convergence in $\mathcal{M}(\tilde{X}, A, J, \mathcal{H}(\tilde{X}))$.

**Definition II.25.** Let $(u_i)_{i \in \mathbb{N}} \subseteq \mathcal{M}(\tilde{X}, A, J, \mathcal{H}(\tilde{X}))$ be a sequence. Then $u_i$ converges to $u \in \mathcal{M}(\tilde{X}, A, J, \mathcal{H}(\tilde{X}))$ iff the following hold:

- Let $b_i := \pi_M^{\mathcal{M}}(u_i)$ and $b := \pi_M^{\mathcal{M}}(u)$. Then $b_i \overset{i \to \infty}{\longrightarrow} b$ in $M$.
- Let $H_i := \pi_{\mathcal{H}}^{\mathcal{M}}(u_i)$ and $H := \pi_{\mathcal{H}}^{\mathcal{M}}(u)$. Then $H_i \overset{i \to \infty}{\longrightarrow} H$ in $\mathcal{H}(\tilde{X})$.
- In the notation of Construction II.9, for $b' \in U$, let $\phi_{b'} := (\psi_{b'}|_{\Sigma_{b'} \setminus \Gamma(b')})^{-1} : \Sigma_b \setminus \{n_1, \ldots, n_d\} \to \Sigma_{b'}$. Let $N \in \mathbb{N}$ be s. t. $b_i \in U$ for all $i \geq N$. Then $u_i \circ \phi_{b_i} : \Sigma_b \setminus \{n_1, \ldots, n_d\} \to \tilde{X}$, for $i \geq N$, converges uniformly to $u|_{\Sigma_b \setminus \{n_1, \ldots, n_d\}}$.

Due to bubbling, see [Hum97] Section V.3, even for $M$ compact, the moduli space $\mathcal{M}(\tilde{X}, A, J, \mathcal{H}(\tilde{X}))$ will not be compact. To remedy this situation and still get a compact moduli space, the Gromov compactification of $\mathcal{M}(\tilde{X}, A, J, \mathcal{H}(\tilde{X}))$ has to be introduced. To describe this space, first of all assume that the marked nodal family of Riemann surfaces $(\pi : \Sigma \to M, R_*)$ is regular. For $\ell \geq 0$, let $(\pi^\ell : \Sigma^\ell \to M^\ell, R_*^\ell, T_*^\ell)$, $\Sigma^\ell = M^{\ell+1}$, and $\hat{\pi}^{\ell-1} : \Sigma^\ell \to \Sigma^{\ell-1}$, $\hat{\pi}_k^\ell : \Sigma^\ell \to \Sigma^k$, $\pi_k^\ell : M^\ell \to M^k$ be the marked nodal families and maps from Lemma II.1 and Proposition II.1. Also, for all $\ell \geq 1$, let $\sigma^\ell$ and $\hat{\sigma}^\ell$ be the actions of $\mathcal{S}_\ell$, by reordering the last $\ell$ marked points, on $M^\ell$ and $\Sigma^\ell$, from Proposition II.2.

Then the following is proved in the monograph [Hum97], Chapter V (esp. Theorem 1.2, Theorem 3.3, Proposition 1.1 and the proofs of these results) as well as, in a generalised version, in [BEH+03].

**Proposition II.6.** *Let $(u_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{M}(\tilde{X}, A, J, \mathcal{H}(\tilde{X}))$, $b_i := \pi_M^{\mathcal{M}}(u_i)$, $H_i := \pi_{\mathcal{H}}^{\mathcal{M}}(u_i)$, s. t. $b_i \underset{i\to\infty}{\longrightarrow} b$ for some $b \in M$ and $H_i \underset{i\to\infty}{\longrightarrow} H$ for some $H \in \mathcal{H}(\tilde{X})$. Then there exist the following:*

- *an integer $\ell \in \mathbb{N}_0$,*
- *a subsequence $(u_{i_j})_{j\in\mathbb{N}}$ of $(u_i)_{i\in\mathbb{N}}$,*
- *$\hat{b}_{i_j} \in \mathring{M}^\ell$ with $\pi_0^\ell(\hat{b}_{i_j}) = b_{i_j}$*
- *and an element $\hat{u} \in \mathcal{M}_{\hat{b}}((\hat{\pi}_0^\ell)^*\tilde{X}, A, J, (\hat{\pi}_0^\ell)^*H)$,*

*for some $\hat{b} \in M^\ell$ with $\pi_0^\ell(\hat{b}) = b$ and*

$$(\hat{\pi}_0^\ell)^*_{\hat{b}_{i_j}} u_{i_j} \underset{j\to\infty}{\longrightarrow} \hat{u}.$$

*Furthermore, the $\hat{b}_{i_j}$, $\hat{b}$ and $\hat{u}$ can be chosen s. t. the following holds: Let $\Sigma_{i,\hat{b}}^\ell$ be a component of $\Sigma_{\hat{b}}^\ell$ on which $\hat{\pi}_0^\ell$ is not a homeomorphism, i. e. either $\hat{\pi}_0^\ell(\Sigma_{i,\hat{b}}^\ell) = \{n_i\}$, for some node $n_i \in \Sigma_b$ or $\hat{\pi}_0^\ell(\Sigma_{i,\hat{b}}^\ell) = \{R_j(b)\}$ for some $j = 1, \ldots, n$. Then $\hat{u}|_{\Sigma_{i,\hat{b}}^\ell}$ has nonvanishing vertical homology class and hence defines a nonconstant $J$-holomorphic sphere in $\tilde{X}_{n_i}$ or $\tilde{X}_{R_j(b)}$.*

**Definition II.26.** Let

$$\begin{aligned}(\hat{\pi}_0^\ell)^*\mathcal{H}(\tilde{X}) &:= \{(\hat{\pi}_0^\ell)^*H \mid H \in \mathcal{H}(\tilde{X})\}\\ &\subseteq \mathcal{H}((\hat{\pi}_0^\ell)^*\tilde{X})\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}((\hat{\pi}_0^\ell)^*\tilde{X}, A, J, (\hat{\pi}_0^\ell)^*\mathcal{H}(\tilde{X})) &:= (\pi_{\mathcal{H}}^{\mathcal{M}})^{-1}((\hat{\pi}_0^\ell)^*\mathcal{H}(\tilde{X}))\\ &\subseteq \mathcal{M}((\hat{\pi}_0^\ell)^*\tilde{X}, A, J, \mathcal{H}((\hat{\pi}_0^\ell)^*\tilde{X})).\end{aligned}$$

Then for any $\ell, \tilde{\ell} \in \mathbb{N}_0$ with $\ell \leq \tilde{\ell}$ there is a canonical map

$$(\hat{\pi}_\ell^{\tilde{\ell}})^* : (\pi_\ell^{\tilde{\ell}})^*\mathcal{M}((\hat{\pi}_0^\ell)^*\tilde{X}, A, J, (\hat{\pi}_0^\ell)^*\mathcal{H}(\tilde{X})) \to \mathcal{M}((\hat{\pi}_0^\ell)^*\tilde{X}, A, J, (\hat{\pi}_0^{\tilde{\ell}})^*\mathcal{H}(\tilde{X}))$$

of topological spaces, where

$$(\pi^{\tilde{\ell}}_{\ell})^*\mathcal{M}((\hat{\pi}^{\ell}_0)^*\tilde{X}, A, J, (\hat{\pi}^{\ell}_0)^*\mathcal{H}(\tilde{X})) = \mathcal{M}((\hat{\pi}^{\ell}_0)^*\tilde{X}, A, J, (\hat{\pi}^{\ell}_0)^*\mathcal{H}(\tilde{X})) \times_{\pi^{\mathcal{M}}_M, M^\ell, \pi^{\tilde{\ell}}_{\ell}} M^{\tilde{\ell}}$$

is the fibred product of topological spaces.
Furthermore, for all $\ell \geq 1$, the actions $\sigma^\ell$ and $\hat{\sigma}^\ell$ of $\mathcal{S}_\ell$ on $M^\ell$ and $\Sigma^\ell$, respectively, induce actions

$$\tilde{\sigma}^\ell : \mathcal{S}_\ell \times \mathcal{M}((\hat{\pi}^{\ell}_0)^*\tilde{X}, A, J, (\hat{\pi}^{\ell}_0)^*\mathcal{H}(\tilde{X})) \to \mathcal{M}((\hat{\pi}^{\ell}_0)^*\tilde{X}, A, J, (\hat{\pi}^{\ell}_0)^*\mathcal{H}(\tilde{X})),$$

compatible via $\pi^{\mathcal{M}}_M$ with the actions $\sigma^\ell$ in the obvious way.
Together, the spaces $\mathcal{M}((\hat{\pi}^{\ell}_0)^*\tilde{X}, A, J, (\hat{\pi}^{\ell}_0)^*\mathcal{H}(\tilde{X}))$, maps $(\hat{\pi}^{\tilde{\ell}}_{\ell})^*$ and actions $\tilde{\sigma}^\ell$ form a system of topological spaces, whose colimit is called the *Gromov compactification of* $\mathcal{M}(\tilde{X}, A, J, \mathcal{H}(\tilde{X}))$ and denoted by

$$\overline{\mathcal{M}}(\tilde{X}, A, J, \mathcal{H}(\tilde{X})).$$

It is equipped with canonical maps

$$\pi^{\overline{\mathcal{M}}}_M : \overline{\mathcal{M}}(\tilde{X}, A, J, \mathcal{H}(\tilde{X})) \to M$$

and

$$\pi^{\overline{\mathcal{M}}}_{\mathcal{H}} : \overline{\mathcal{M}}(\tilde{X}, A, J, \mathcal{H}(\tilde{X})) \to \mathcal{H}(\tilde{X}).$$

*Remark* II.9. For $u \in \mathcal{M}((\hat{\pi}^{\ell}_0)^*\tilde{X}, A, J, (\hat{\pi}^{\ell}_0)^*\mathcal{H}(\tilde{X}))$ with $\pi^{\mathcal{M}}_M(u) = b$, $\pi^{\mathcal{M}}_{\mathcal{H}}(u) = H$ and $\hat{b} \in M^{\tilde{\ell}}$ s. t. $b = \pi^{\tilde{\ell}}_{\ell}(\hat{b})$,

$$(\hat{\pi}^{\tilde{\ell}}_{\ell})^* u \in \mathcal{M}_{\hat{b}}((\hat{\pi}^{\tilde{\ell}}_0)^*\tilde{X}, A, J, (\hat{\pi}^{\tilde{\ell}}_{\ell})^* H) \subseteq \mathcal{M}((\hat{\pi}^{\tilde{\ell}}_0)^*\tilde{X}, A, J, (\hat{\pi}^{\tilde{\ell}}_0)^*\mathcal{H}(\tilde{X})).$$

For $\tilde{\ell} = \ell + 1$ this is clear, for on every component of $\Sigma^{\tilde{\ell}}_{\hat{b}}$, $\hat{\pi}^{\tilde{\ell}}_{\ell}$ is either a diffeomorphism or a constant map onto a point. On each component on which $\hat{\pi}^{\tilde{\ell}}_{\ell}$ is constant (which then is diffeomorphic to a sphere), $(\hat{\pi}^{\tilde{\ell}}_{\ell})^* H$ vanishes, so the restriction of a section $u \in \mathcal{M}((\hat{\pi}^{\tilde{\ell}}_0)^*\tilde{X}, A, J, (\hat{\pi}^{\tilde{\ell}}_{\ell})^* H)$ to such a component is just given by a $J$-holomorphic map to $X_b$. In particular, the constant map corresponding to the restriction of $(\hat{\pi}^{\tilde{\ell}}_{\ell})^* u$ to such a component is holomorphic. For $\ell \geq 1$, the claim follows by induction.

*Remark* II.10. Note that in the above definition, $(\hat{\pi}^{\ell}_0)^* : \mathcal{H}(\tilde{X}) \to \mathcal{H}((\hat{\pi}^{\ell}_0)^*\tilde{X})$ is an injection.

*Remark* II.11. The colimit over the above system of topological spaces is the quotient space

$$\coprod_{\ell \geq 0} \mathcal{M}((\hat{\pi}^{\ell}_0)^*\tilde{X}, A, J, (\hat{\pi}^{\ell}_0)^*\mathcal{H}(\tilde{X}))/\sim,$$

where

$$\mathcal{M}((\hat{\pi}^{\ell'}_0)^*\tilde{X}, A, J, (\hat{\pi}^{\ell'}_0)^*\mathcal{H}(\tilde{X})) \ni u' \sim u'' \in \mathcal{M}((\hat{\pi}^{\ell''}_0)^*\tilde{X}, A, J, (\hat{\pi}^{\ell''}_0)^*\mathcal{H}(\tilde{X}))$$

iff there exists an $\tilde{\ell} \geq \ell', \ell''$, an $g \in \mathcal{S}_{\tilde{\ell}}$ and $b \in M^{\tilde{\ell}}$ s. t. $\pi^{\tilde{\ell}}_{\ell'}(b) = \pi^{\mathcal{M}}_{M}(u')$, $\pi^{\tilde{\ell}}_{\ell''}(\sigma^{\tilde{\ell}}_{g^{-1}}(b)) = \pi^{\mathcal{M}}_{M}(u'')$ and

$$(\hat{\pi}^{\tilde{\ell}}_{\ell'})^*_b u' = \tilde{\sigma}^{\tilde{\ell}}_g \left( (\hat{\pi}^{\tilde{\ell}}_{\ell''})^*_{\sigma^{\tilde{\ell}}_{g^{-1}}(b)} u'' \right) \in \mathcal{M}((\hat{\pi}^{\tilde{\ell}}_0)^* \tilde{X}, A, J, (\hat{\pi}^{\tilde{\ell}}_0)^* \mathcal{H}(\tilde{X})).$$

In particular, this equivalence relation has the following property: If $u \in \mathcal{M}((\hat{\pi}^{\ell}_0)^* \tilde{X}, A, J, (\hat{\pi}^{\ell}_0)^* \mathcal{H}(\tilde{X}))$ is constant on a ghost component (as in Definition II.9) of its underlying nodal Riemann surface, then there exists a $k < \ell$ and a $u' \in \mathcal{M}((\hat{\pi}^{k}_0)^* \tilde{X}, A, J, (\hat{\pi}^{k}_0)^* \mathcal{H}(\tilde{X}))$, s. t. $u$ and $u'$ define the same point in $\overline{\mathcal{M}}(\tilde{X}, A, J, \mathcal{H}(\tilde{X}))$.

*Remark* II.12. Also, directly from the definition, there is a canonical injection

$$\mathcal{M}(\tilde{X}, A, J, \mathcal{H}(\tilde{X})) \hookrightarrow \overline{\mathcal{M}}(\tilde{X}, A, J, \mathcal{H}(\tilde{X})).$$

**Corollary II.7.**

$$\pi^{\overline{\mathcal{M}}}_M \times \pi^{\overline{\mathcal{M}}}_{\mathcal{H}} : \overline{\mathcal{M}}(\tilde{X}, A, J, \mathcal{H}(\tilde{X})) \to M \times \mathcal{H}(\tilde{X})$$

*is a proper map and* $\overline{\mathcal{M}}(\tilde{X}, A, J, \mathcal{H}(\tilde{X}))$ *is a Hausdorff topological space. In particular, if* $M$ *is compact, then for any* $H \in \mathcal{H}(\tilde{X})$,

$$\overline{\mathcal{M}}(\tilde{X}, A, J, H) := (\pi^{\overline{\mathcal{M}}}_M \times \pi^{\overline{\mathcal{M}}}_{\mathcal{H}})^{-1}(M \times \{H\})$$

*is a compact Hausdorff topological space.*

**Lemma II.27.** *If* $M$ *is compact, then there exists an* $\ell \in \mathbb{N}_0$ *s. t. the canonical map* $\mathcal{M}((\hat{\pi}^{\ell}_0)^* \tilde{X}, A, J, (\hat{\pi}^{\ell}_0)^* \mathcal{H}(\tilde{X})) \to \overline{\mathcal{M}}(\tilde{X}, A, J, \mathcal{H}(\tilde{X}))$ *is surjective.*

*Proof.* By the definition of $\mathcal{M}((\hat{\pi}^{\ell}_0)^* \tilde{X}, A, J, (\hat{\pi}^{\ell}_0)^* \mathcal{H}(\tilde{X}))$ and the definition of $\mathcal{H}(\tilde{X})$, there exists a universal bound on the vertical energy of every element of $\mathcal{M}((\hat{\pi}^{\ell}_0)^* \tilde{X}, A, J, (\hat{\pi}^{\ell}_0)^* \mathcal{H}(\tilde{X}))$ independent of $\ell$, by Lemma 8.2.9 in [MS04], where the vertical energy is defined as in Section 8.2 in [MS04], p. 249. By the usual Gromov-Schwarz and Monotonicity lemmas, this implies a universal bound on the number of components on which an element of $\mathcal{M}((\hat{\pi}^{\ell}_0)^* \tilde{X}, A, J, (\hat{\pi}^{\ell}_0)^* \mathcal{H}(\tilde{X}))$ can be nonconstant, which by definition of the equivalence relation in the definition of $\overline{\mathcal{M}}(\tilde{X}, A, J, \mathcal{H}(\tilde{X}))$ implies the lemma. □

Furthermore, for $\ell \leq \tilde{\ell}$ denote by $M^{\tilde{\ell},\ell}$ the set

$$M^{\tilde{\ell},\ell} := \{b \in M^{\tilde{\ell}} \mid \hat{\pi}^{\tilde{\ell}}_{\ell,b} : \Sigma^{\tilde{\ell}}_b \to \Sigma^{\ell}_{\pi^{\tilde{\ell}}_0(b)} \text{ is a homeomorphism}\}$$

and by $\Sigma^{\tilde{\ell},\ell} := \Sigma^{\tilde{\ell}}|_{M^{\tilde{\ell},\ell}}$.

In addition, let $\pi^{\tilde{\ell},\ell} := \pi^{\tilde{\ell}}|_{M^{\tilde{\ell},\ell}} : M^{\tilde{\ell},\ell} \to M^{\ell}$ and $\hat{\pi}^{\tilde{\ell},\ell} := \hat{\pi}^{\tilde{\ell}}_{\ell}|_{\Sigma^{\tilde{\ell},\ell}} : \Sigma^{\tilde{\ell},\ell} \to \Sigma^{\ell}$.

**Lemma II.28.** *$\pi^{\tilde{\ell},\ell} : M^{\tilde{\ell},\ell} \to M^\ell$ is a surjective submersion of complex fibre dimension $\tilde{\ell} - \ell$ and $\Sigma^{\tilde{\ell},\ell} \cong (\pi^{\tilde{\ell},\ell})^*\Sigma^\ell$ via $\hat{\pi}^{\tilde{\ell},\ell}$.*
*Furthermore,*

$$\mathcal{M}((\hat{\pi}^{\tilde{\ell},\ell})^*(\hat{\pi}_0^\ell)^*\tilde{X}, A, J, (\hat{\pi}_\ell^{\tilde{\ell},\ell})^*(\hat{\pi}_0^\ell)^*\mathcal{H}(\tilde{X})) \cong (\pi^{\tilde{\ell},\ell})^*\mathcal{M}((\hat{\pi}_0^\ell)^*\tilde{X}, A, J, (\hat{\pi}_0^\ell)^*\mathcal{H}(\tilde{X}))$$

*via $(\hat{\pi}^{\tilde{\ell},\ell})^*$.*

*Proof.* This again follows by induction from the case $\tilde{\ell} = \ell + 1$. But in this case $M^{\tilde{\ell}} = M^{\ell+1} = \Sigma^\ell$ and $M^{\tilde{\ell},\ell}$ by Lemma II.1 is the complement of the nodes and markings in $\Sigma^\ell$. The restriction $\Sigma^{\tilde{\ell},\ell}$ of $\Sigma^{\tilde{\ell}}$ to this subset, from the proof of Lemma II.1, is by definition the pullback of $\Sigma^\ell$ via $\pi^\ell$ and the restriction of $\hat{\pi}_\ell^{\tilde{\ell}}$ is by definition the canonical map covering $\pi^\ell$.
The second claim follows directly from the definitions. □

To make sense of the following remark, remember that by definition $\mathcal{M}((\hat{\pi}_0^\ell)\tilde{X}, A, J, \mathcal{K})$, for any subset $\mathcal{K} \subseteq \mathcal{H}((\hat{\pi}_0^\ell)\tilde{X})$, is a disjoint union of subsets that are mapped to the strata of $M^\ell$ in the stratification by signature under $\pi_M^{\mathcal{M}}$. By abuse of language I will call these subsets strata, even though in general it is not claimed that they are (Banach) manifolds or form any kind of reasonable stratification. Also, by the codimension of such a subset I will mean the codimension of the corresponding stratum in $M^\ell$.

The *transversality problem* now can be formulated as follows:
Does there exist a (generic subset of) $H \in \mathcal{H}(\tilde{X})$ s. t. for every $\ell \geq 0$ (and for all generic $H$), $\mathcal{M}((\hat{\pi}_0^\ell)^*\tilde{X}, A, J, (\hat{\pi}_0^\ell)^*H)$ is stratified by smooth manifolds as in the previous section, induced from the stratification by signature on $M^\ell$. And in such a way that $\overline{\mathcal{M}}(\tilde{X}, A, J, H)$ has a stratification by smooth manifolds, induced by the canonical maps $\mathcal{M}((\hat{\pi}_0^\ell)^*\tilde{X}, A, J, (\hat{\pi}_0^\ell)^*H) \to \overline{\mathcal{M}}(\tilde{X}, A, J, H)$. So that the stratification in particular coincides with the one from before on $\mathcal{M}(\tilde{X}, A, J, H)$ under the inclusion from Remark II.12? Furthermore, there should be a top-dimensional stratum which coincides with the top-dimensional stratum in $\mathcal{M}(\tilde{X}, A, J, H)$, corresponding to the smooth curves, and the codimension of every other stratum should coincide with the codimension of the stratum in $\mathcal{M}((\hat{\pi}_0^\ell)^*\tilde{X}, A, J, (\hat{\pi}_0^\ell)^*H)$ from which it arises.

In general, it is known that the answer to this question is no, for all the Hamiltonian perturbations of the form $(\hat{\pi}_0^\ell)^*H$ vanish on ghost components, so the Banach space $(\hat{\pi}_0^\ell)^*\mathcal{H}(\tilde{X})$ is "too small" to achieve the transversality results in Lemma II.22.
One hence is faced with two conflicting aims: On the one hand one would like to enlarge the spaces of perturbations in the construction of the universal moduli spaces from $(\hat{\pi}_0^\ell)^*\mathcal{H}(\tilde{X}) \cong \mathcal{H}(\tilde{X})$ to $\mathcal{H}((\hat{\pi}_0^\ell)^*\tilde{X})$ to achieve transversality, on the other hand one needs to restrict to perturbations coming from $\mathcal{H}(\tilde{X})$ so that the equivalence relation is preserved and the conditions on the dimensions of the strata of the stratification one wants to construct have any chance of holding true.

The solution to this problem, first applied in the genus 0 case in [CM07] and which will be extended to the present situation in the rest of this text, can now roughly be described as follows (all these notions will be made precise later on):
For every $\ell \geq 0$ there exists a subset $\mathcal{K}^\ell \subseteq \mathcal{H}((\hat{\pi}_0^\ell)^* \tilde{X})$ s. t. $(\hat{\pi}_\ell^{\tilde{\ell}})^* \mathcal{K}^\ell \subseteq \mathcal{K}^{\tilde{\ell}}$.
There also exists an $\ell \in \mathbb{N}_0$ and for every $\tilde{\ell} \geq \ell$ a subset $\mathcal{N}^{\tilde{\ell}}(\mathcal{K}^{\tilde{\ell}}) \subseteq \mathcal{M}((\hat{\pi}_0^{\tilde{\ell}})^* \tilde{X}, A, J, \mathcal{K}^{\tilde{\ell}})$ with $\pi_M^{\mathcal{M}}(\mathcal{N}^{\tilde{\ell}}(\mathcal{K}^{\tilde{\ell}})) \subseteq \mathring{M}^{\tilde{\ell}}$ (the part corresponding to smooth curves, as in Section II.1) s. t. the closure of $\mathcal{N}^{\tilde{\ell}}(\mathcal{K}^{\tilde{\ell}})$ in $\overline{\mathcal{M}}((\hat{\pi}_0^{\tilde{\ell}})^* \tilde{X}, A, J, \mathcal{K}^{\tilde{\ell}})$ lies in $\mathcal{M}((\hat{\pi}_0^{\tilde{\ell}})^* \tilde{X}, A, J, \mathcal{K}^{\tilde{\ell}})$.
Since $\mathring{M}^{\tilde{\ell}} \subseteq M^{\tilde{\ell},\ell}$ for all $\ell \leq \tilde{\ell}$, for every $H \in \mathcal{K}^0$ there is a well-defined map $(\hat{\pi}_0^{\tilde{\ell}})_* : \mathcal{N}^{\tilde{\ell}}((\hat{\pi}_0^{\tilde{\ell}})^* H) \to \mathcal{M}(\tilde{X}, A, J, H)$ (the left-hand side is defined in the obvious way) given by $u \mapsto ((\hat{\pi}_{0,b}^{\tilde{\ell}})^{-1})^* u$, where $\pi_M^{\mathcal{M}}(u) = b$.
Then for generic $H \in \mathcal{K}^0$ the above will be s. t. $\mathcal{N}^\ell((\hat{\pi}_0^\ell)^* H)$ is invariant under the $\mathcal{S}_\ell$-action and the map $(\hat{\pi}_0^\ell)_*$ is an $\ell!$-sheeted covering on the complement of a subset of codimension at least 2 (see Lemma III.1).

Roughly speaking, the $\mathcal{N}^{\tilde{\ell}}(\mathcal{K}^{\tilde{\ell}})$ will be defined as spaces of holomorphic sections that map the first $\ell$ additional marked points to a subbundle $\tilde{Y} \subseteq \tilde{X}$ with real codimension 2 fibres and the sets $\mathcal{K}^{\tilde{\ell}}$ will be spaces of Hamiltonian perturbations satisfying a set of compatibility conditions with this subbundle. Making these notions precise and showing the properties above will be pretty much the rest of this work.

## II.4.2 Hypersurfaces and tangency

Throughout this section, let $(\pi : \Sigma \to M, R)$ be a stable marked nodal family Riemann surfaces of type $(g, n)$ and denote their Euler characteristic by $\chi$. Furthermore, let $(\kappa : X \to M, \omega)$ be a family of symplectic manifolds together with a family $(\kappa|_Y : Y \to M, \omega|_Y)$ of symplectic hypersurfaces in $X$. Define $\tilde{\kappa} : \tilde{X} \to \Sigma$ as the pullback of $\kappa : X \to M$ to $\Sigma$ via $\pi$ and likewise for $\tilde{Y}$. As before, $\mathcal{J}_\omega(X)$ is the set of $\omega$-compatible vertical almost complex structures on $X$, i. e. bundle morphisms $J \in \mathrm{End}(VX)$ with $J^2 = -\mathrm{id}$ and s. t. $\omega(\cdot, J\cdot)$ defines a metric on $VX$. In other words, for any $b \in M$, $J_b$ is a compatible almost complex structure on the symplectic manifold $(X_b, \omega_b)$.

To define the sets $\mathcal{K}^\ell$ from the previous subsection, almost complex structures and Hamiltonian perturbations compatible with the family of symplectic hypersurface $Y$ in the sense of [IP03], Definition 3.2 are needed. The almost complex structures are treated exclusively as parameters, i. e. they are never chosen by applying the Sard-Smale theorem.

**Definition II.27.** The set of *$Y$-compatible vertical almost complex structures* on $X$ is defined as

$$\mathcal{J}_\omega(X, Y) := \{J \in \mathcal{J}_\omega(X) \mid J(VY) = VY\}.$$

The set of *normally integrable* $Y$-compatible almost complex structures on $X$

is defined as

$$\mathcal{J}_{\omega,\mathrm{ni}}(X,Y) := \{J \in \mathcal{J}_\omega(X,Y) \mid \pi^{VX}_{VY^{\perp_\omega}} N_J(v,\xi) = 0 \ \forall v \in V_yY, \xi \in V_yY^{\perp_\omega}, y \in Y\},$$

where $N_J$ denotes the Nijenhuis tensor of $J$, $V_yY^{\perp_\omega} \subseteq V_yX$ denotes the symplectic orthogonal complement and $\pi^{VX}_{VY^{\perp_\omega}} : VX \to VY^{\perp_\omega}$ denotes the projection along $VY$.

One considers $\mathcal{J}_\omega(X,Y)$ and $\mathcal{J}_{\omega,\mathrm{ni}}(X,Y)$ as subsets of $\mathcal{J}_\omega(\tilde{X},\tilde{Y})$ and $\mathcal{J}_{\omega,\mathrm{ni}}(\tilde{X},\tilde{Y})$, respectively, via pullback.

The proof of Theorem A.2 in [IP03] shows:

**Lemma II.29.** $\mathcal{J}_{\omega,\mathrm{ni}}(X,Y)$ *is nonempty and path-connected.*

Now remember that if for $b \in M$, $S_b$ is a smooth Riemann surface and $\iota_b : S_b \to \Sigma_b \subseteq \Sigma$ a desingularisation of the fibre of $\Sigma$ over $b$, then $\iota_b^*\tilde{X} = (\pi \circ \iota_b)^*X$ is a trivial bundle, for $\pi \circ \iota_b$ is the constant map to $b$. Likewise for the subbundle $Y \subseteq X$. Making the identification with the trivial bundle, $\hat{X}_b := S_b \times X_b$ and $\hat{Y}_b := S_b \times Y_b$, one can pull back any $H \in \mathcal{H}(\tilde{X})$ to $H_b \in \mathcal{H}(\hat{X}_b)$. Given such $H$ and any $J \in \mathcal{J}_\omega(X)$, which induces a vertical almost complex structure on every $\hat{X}_b$, one hence gets an almost complex structure $\hat{J}^H_b$ on $\hat{X}_b$ as in Definition II.19.

**Definition II.28.** Let $H \in \mathcal{H}(\tilde{X})$. $H$ is called a *$Y$-compatible Hamiltonian perturbation*, $H \in \mathcal{H}(\tilde{X},\tilde{Y})$, if for every $b \in M$ and every desingularisation $\iota_b : S_b \to \Sigma_b \subseteq \Sigma$, $\hat{Y}_b \subseteq \hat{X}_b$ is $H_b$-parallel, i. e. $\operatorname{im} X_{H_b(\zeta)}|_{\hat{Y}_b} \subseteq V\hat{Y}_b \ \forall \zeta \in TS_b$.
Given $J \in \mathcal{J}_{\omega,\mathrm{ni}}(X,Y)$, if furthermore for every $b \in M$ and every desingularisation $\iota_b : S_b \to \Sigma_b \subseteq \Sigma$,

$$\pi^{V\hat{X}_b}_{V\hat{Y}_b^{\perp_\omega}} N_{\hat{J}^H_b}(\hat{v},\hat{\xi}) = 0 \ \forall \, \hat{v} \in V_{\hat{y}}\hat{Y}_b, \hat{\xi} \in V_{\hat{y}}\hat{Y}^{\perp_\omega}, \hat{y} \in \hat{Y}_b,$$

where $V_{\hat{y}}\hat{Y}^{\perp_\omega} := \{0\} \times TY_b^{\perp_\omega}$, then $H$ is called a *$J$-compatible normally integrable Hamiltonian perturbation*, $H \in \mathcal{H}_{\mathrm{ni}}(\tilde{X},\tilde{Y},J)$.
This space has the two subspaces

$$\begin{aligned} \mathcal{H}^0_{\mathrm{ni}}(\tilde{X},\tilde{Y},J) &:= \{H \in \mathcal{H}_{\mathrm{ni}}(\tilde{X},\tilde{Y},J) \mid H|_{\tilde{Y}} = 0\} \\ \mathcal{H}^{00}(\tilde{X},\tilde{Y}) &:= \mathrm{cl}\left(\{H \in \mathcal{H}(\tilde{X}) \mid \mathrm{supp}(H) \subseteq \tilde{X} \setminus \tilde{Y} \text{ compact}\}\right), \end{aligned}$$

where cl denotes the closure in $\mathcal{H}(\tilde{X})$.

In the course of the ensuing construction, Hamiltonian perturbations will be chosen with increasing specialisation in the form $H + H^0 + H^{00}$, starting with some $H \in \mathcal{H}(\tilde{X},\tilde{Y},J)$ (which actually lies in some other yet to be defined subspace of $\mathcal{H}(\tilde{X},\tilde{Y},J)$) and then modifying it to $H+H^0$ for $H^0 \in \mathcal{H}^0_{\mathrm{ni}}(\tilde{X},\tilde{Y},J)$ and subsequently to $H + H^0 + H^{00}$ for some $H^{00} \in \mathcal{H}^{00}(\tilde{X},\tilde{Y})$.

*Remark* II.13. If one endows, for $b \in M$, $\hat{X}_b$ with a symplectic form $\hat{\omega}_b$ that is of the form $\mathrm{pr}_1^*\sigma + \mathrm{pr}_2^*\omega_b$ for a symplectic form $\sigma_b$ on $S_b$, then $\{0\} \times T_y Y_b^{\perp_\omega} = T_{\hat{y}} \hat{Y}_b^{\perp_{\hat{\omega}_b}}$ for $\hat{y} = (z, y) \in \hat{Y}_b$, so the definition of $\mathcal{H}_{\mathrm{ni}}(\tilde{X}, \tilde{Y}, J)$ is in complete analogy to that of $\mathcal{J}_{\omega,\mathrm{ni}}(X, Y)$.

**Lemma II.30.**

$$\mathcal{H}(\tilde{X}, \tilde{Y}) = \{H \in \mathcal{H}(\tilde{X}) \mid \mathrm{d}(H(\zeta))(v) = 0 \,\forall\, \zeta \in T\Sigma, v \in TY^{\perp_\omega}\}.$$

*defines a closed linear subspace of* $\mathcal{H}(\tilde{X})$.
*Given any* $J \in \mathcal{J}_\omega(\tilde{X})$, *for any* $b \in M$ *and every desingularisation* $\iota_b : S_b \to \Sigma_b \subseteq \Sigma$, $\hat{Y}_b \subseteq \hat{X}_b$ *is a* $\hat{J}_b^H$*-complex hypersurface.*

The following is the reason for the above definition, which recovers Lemma 3.3 from [IP03] in the present notation:

**Corollary II.8.** *Let* $J \in \mathcal{J}_{\omega,\mathrm{ni}}(X, Y)$, *let* $H \in \mathcal{H}_{\mathrm{ni}}(\tilde{X}, \tilde{Y}, J)$, *let* $b \in M$ *and* $\iota_b : S_b \to \Sigma_b \subseteq \Sigma$ *a desingularisation. Then for any* $u \in \mathcal{M}(\hat{X}_b, A, J_b, H_b)$ *with* $\mathrm{im}(u) \subseteq \hat{Y}_b$,

$$\pi^{V\hat{X}_b}_{V\hat{Y}_b^{\perp_\omega}} \circ \left(D\overline{\partial}^{J_b,H_b}_{S_b}\right)_u : L^{k,p}(u^* V\hat{Y}_b^{\perp_\omega}) \to L^{k-1,p}(\overline{\mathrm{Hom}}_{(j_b,J_b)}(TS_b, u^* V\hat{Y}_b^{\perp_\omega}))$$

*is complex linear (for any* $k \in \mathbb{N}, p > 1$ *with* $kp > 2$*).*

*Proof.* This is a special case of Lemma II.4. □

The following lemma and remark recover formulas (3.3) (b) and (c) from [IP03], which will be used in Lemma II.32 below showing the existence of "enough" normally integrable Hamiltonian perturbations:

**Lemma II.31.** *Let* $J \in \mathcal{J}_\omega(X)$, $H \in \mathcal{H}(\tilde{X})$ *and assume that* $\tilde{X} = \Sigma \times X$ *is a trivial bundle. Then w. r. t. the decomposition* $T\tilde{X} = T\Sigma \times TX$, *for* $(w, v), (0, \xi) \in T\tilde{X}$,

$$N_{\tilde{J}^H}((w, v), (0, \xi)) = (0, N_J(v, \xi)) - (0, 2(L_{JX^{0,1}_{H(w)}} J)\xi).$$

*In particular, for* $J \in \mathcal{J}_{\omega,\mathrm{ni}}(X, Y)$,

$$\begin{aligned}\mathcal{H}_{\mathrm{ni}}(\tilde{X}, \tilde{Y}, J) = \{H \in \mathcal{H}(\tilde{X}, \tilde{Y}) \mid \pi^{TX}_{TY^{\perp_\omega}}(L_{JX^{0,1}_{H(w)}} J)\xi = 0 \\ \forall\, w \in T\Sigma, \xi \in TY^{\perp_\omega}\}.\end{aligned}$$

*Also, for* $J \in \mathcal{J}_{\omega,\mathrm{ni}}(X, Y)$,

$$\begin{aligned}\pi^{TX}_{TY^{\perp_\omega}}(L_{JX^{0,1}_{H(w)}} J)\xi = \frac{1}{2}\pi^{TX}_{TY^{\perp_\omega}}\Big([X_{H(w)}, \xi] + J[X_{H(w)}, J\xi] + \\ + J\big([X_{H(jw)}, \xi] + J[X_{H(jw)}, J\xi]\big)\Big).\end{aligned}$$

*Proof.* By definition of the Nijenhuis tensor and Remark II.7

$$
\begin{aligned}
N_{\tilde{J}^H}((w,v),(0,\xi)) &= [(w,v),(0,\xi)] + \tilde{J}^H[\tilde{J}^H(w,v),(0,\xi)] + \\
&\quad + \tilde{J}^H[(w,v),\tilde{J}^H(0,\xi)] - [\tilde{J}^H(w,v),\tilde{J}^H(0,\xi)] \\
&= (0,[v,\xi]) + \tilde{J}^H[(jw, Jv + 2JX^{0,1}_{H(w)}),(0,\xi)] + \\
&\quad + \tilde{J}^H[(w,v),(0,J\xi)] - [(jw, Jv + 2JX^{0,1}_{H(w)}),(0,J\xi)] \\
&= (0,[v,\xi]) + \tilde{J}^H(0,[Jv,\xi] + 2[JX^{0,1}_{H(w)},\xi]) + \\
&\quad + \tilde{J}^H(0,[v,J\xi]) - (0,[Jv,J\xi] + 2[JX^{0,1}_{H(w)},J\xi]) \\
&= (0,[v,\xi] + J[Jv,\xi] + J[v,J\xi] - [Jv,J\xi) + \\
&\quad + 2(0,J[JX^{0,1}_{H(w)},\xi] - [JX^{0,1}_{H(w)},J\xi]) \\
&= (0,N_J(v,\xi)) - 2(0,[JX^{0,1}_{H(w)},J\xi] - J[JX^{0,1}_{H(w)},\xi]) \\
&= (0,N_J(v,\xi)) - (0,2(L_{JX^{0,1}_{H(w)}}J)\xi),
\end{aligned}
$$

for $[JX^{0,1}_{H(w)},J\xi] - J[JX^{0,1}_{H(w)},\xi] = L_{JX^{0,1}_{H(w)}}(J\xi) - JL_{JX^{0,1}_{H(w)}}\xi = (L_{JX^{0,1}_{H(w)}}J)\xi + JL_{JX^{0,1}_{H(w)}}\xi - JL_{JX^{0,1}_{H(w)}}\xi = (L_{JX^{0,1}_{H(w)}}J)\xi$.
To show the last equation, one can explicitely write out $X^{0,1}_{H(w)}$ to get

$$
\begin{aligned}
2([JX^{0,1}_{H(w)},J\xi] - J[JX^{0,1}_{H(w)},\xi]) &= [JX_{H(w)},J\xi] - J[JX_{H(w)},\xi] - \\
&\quad - ([X_{H(jw)},J\xi] - J[X_{H(jw)},\xi]).
\end{aligned}
$$

Now using that $J \in \mathcal{J}_{\omega,\mathrm{ni}}(X,Y)$, hence $\pi^{TX}_{TY^{\perp\omega}}N_J(v,\xi) = \pi^{TX}_{TY^{\perp\omega}}([v,\xi]+J[v,J\xi]-([Jv,J\xi]-J[Jv,\xi])) = 0$ for $v \in TY$, $\xi \in TY^{\perp\omega}$, with $v = X_{H(w)}$, shows the last equation in the statement of the lemma. □

*Remark* II.14. If $\nabla$ denotes *any* torsion-free connection on $X$, then the second part in the above formula for the Nijenhuis tensor can also be written as

$$
2(L_{JX^{0,1}_{H(w)}}J)\xi = 2J(\nabla_\xi(JX^{0,1}_{H(w)}) + J\nabla_{J\xi}(JX^{0,1}_{H(w)}) - J(\nabla_{JX^{0,1}_{H(w)}}J)\xi),
$$

which recovers formula (3.3) (c) in Definition 3.2 from [IP03], although it will not be used in this form in this text. For starting with the second to last line in the string of equalities in the above proof, because $\nabla$ is torsion-free,

$$
\begin{aligned}
[JX^{0,1}_{H(w)},J\xi] - J[JX^{0,1}_{H(w)},\xi] &= \nabla_{JX^{0,1}_{H(w)}}(J\xi) - \nabla_{J\xi}(JX^{0,1}_{H(w)}) - \\
&\quad - J\nabla_{JX^{0,1}_{H(w)}}\xi - \nabla_\xi(JX^{0,1}_{H(w)}) \\
&= (\nabla_{JX^{0,1}_{H(w)}}J)\xi + J\nabla_{JX^{0,1}_{H(w)}}\xi - \nabla_{J\xi}(JX^{0,1}_{H(w)}) - \\
&\quad - J\nabla_{JX^{0,1}_{H(w)}}\xi + J\nabla_\xi(JX^{0,1}_{H(w)}) \\
&= J(\nabla_\xi(JX^{0,1}_{H(w)}) + J\nabla_{J\xi}(JX^{0,1}_{H(w)}) - J(\nabla_{JX^{0,1}_{H(w)}}J)\xi).
\end{aligned}
$$

**Lemma II.32.** *There exists a continuous linear right inverse $\iota : \mathcal{H}(\tilde{Y}) \to \mathcal{H}(\tilde{X}, \tilde{Y})$ to the restriction map $\mathcal{H}(\tilde{X}, \tilde{Y}) \to \mathcal{H}(\tilde{Y})$, $H \mapsto (\zeta \mapsto H(\zeta)|_Y)$, i. e. the restriction map $\mathcal{H}(\tilde{X}, \tilde{Y}) \to \mathcal{H}(\tilde{Y})$ is a split surjection.*
*Furthermore, $\iota$ can be chosen s. t. $\operatorname{im}\iota \subseteq \mathcal{H}_{\mathrm{ni}}(\tilde{X}, \tilde{Y}, J)$ for any $J \in \mathfrak{J}_{\omega,\mathrm{ni}}(X, Y)$.*

*Proof.* First, choosing a locally finite covering of $M$ over which $X$ and $Y$ are trivial and a subordinate partition of unity, one can reduce to the case that $X$ and $Y$ are trivial bundles, so assume that to be the case.
By the Weinstein symplectic neighbourhood theorem, Theorem 3.30, p. 101, in [MS98], there exists a neighbourhood $N(Y)$ of $Y$ in $X$, symplectomorphic to an open neighbourhood $V$ of the zero section in $TY^{\perp_\omega}$ and mapping the zero section to $Y$ via the inclusion. $\omega$ turns $TY^{\perp_\omega}$ into a symplectic vector bundle. Choose any $\omega$-compatible Riemannian metric $g$ on $TY^{\perp_\omega}$ and let $\varepsilon > 0$ be so small that for all $y \in Y$, the ball of radius (w. r. t. $g$) in $T_yY^{\perp_\omega}$ lies in $V$. Now choose a smooth cutoff-function $\rho : [0, \infty) \to [0, 1]$ s. t. $\rho(r) = 1$ for all $0 \le r \le \varepsilon/3$ and $\rho(r) = 0$ for all $r \ge 2\varepsilon/3$. Let $\tau : TY^{\perp_\omega} \to Y$ the projection. Given $H_0 \in C^\infty(Y, \mathbb{R})$, define $\hat{H}_0 : TY^{\perp_\omega} \to \mathbb{R}$, $\hat{H}_0(v) := \rho(\|v\|)\tau^*H(v)$. Then $\hat{H}_0$ has compact support in $V$ and by identifying $V$ with $N(Y)$, $\hat{H}_0$ hence defines a function $H \in C^\infty(X, \mathbb{R})$. Furthermore, for $v \in TY^{\perp_\omega}$, $\mathrm{d}H(v) = 0$, for again identifying $N(Y)$ with $V$, by construction and since $\rho$ is constant in a neighbourhood of zero, $\mathrm{d}H(v) = \mathrm{d}H_0(\tau_* v) = \mathrm{d}H_0(0) = 0$, for $v \in TY^{\perp_\omega} = \ker\tau_*$. Also, by definition, $H|_Y = H_0$. Denote the resulting map $\eta : C^\infty(Y, \mathbb{R}) \to C^\infty(X, \mathbb{R})$. One can now define $\iota : \mathcal{H}(\tilde{Y}) \to \mathcal{H}(\tilde{X}, \tilde{Y})$ by $H_0 \mapsto (\zeta \mapsto \eta(H_0(\zeta)))$. By Lemma II.30, this defines a right inverse to the restriction map. To show the second statement, let $J \in \mathfrak{J}_{\omega,\mathrm{ni}}(X, Y)$ be arbitrary. By Lemma II.31 it has to be shown that for the $H$ just constructed

$$\pi^{TX}_{TY^{\perp_\omega}}\Big([X_{H(w)}, \xi] + J[X_{H(w)}, J\xi] + J\big([X_{H(jw)}, \xi] + J[X_{H(jw)}, J\xi]\big)\Big) = 0$$

for all $w \in T\Sigma$, $\xi \in TY^{\perp_\omega}$. I will show that each of the four summands $[X_{H(w)}, \xi], [X_{H(w)}, J\xi], [X_{H(jw)}, \xi], [X_{H(jw)}, J\xi]$ vanishes separately for a suitably chosen extension of $\xi$ to a locally defined vector field. Here and in the following it is used that $J$ leaves $TY$ and $TY^{\perp_\omega}$ invariant and so in particular $\pi^{TX}_{TY^{\perp_\omega}} \circ J = J \circ \pi^{TX}_{TY^{\perp_\omega}}$. Let $\xi \in T_yY^{\perp_\omega}$, $y \in Y$, and $w \in T\Sigma$. Choose local coordinates around $y$ in $X$ of the form $(y^1, \dots, y^{2n-2}, x^1, x^2)$ by use of the Weinstein symplectic neighbourhood theorem. By a smooth change of trivialisation in the corresponding trivialisation of $TY^{\perp_\omega}$ over this neighbourhood one can assume that $J$ is the standard complex structure along $Y$ in the coordinates $x^1$ and $x^2$, i. e. $J\frac{\partial}{\partial x^1}|_{x^1=x^2=0} = \frac{\partial}{\partial x^2}$ and $J\frac{\partial}{\partial x^2}|_{x^1=x^2=0} = -\frac{\partial}{\partial x^1}$. Extend $\xi = a^1\frac{\partial}{\partial x^1} + a^2\frac{\partial}{\partial x^2}$, with $a^1, a^2 \in \mathbb{R}$, locally by the same formula. Then $X_{H(w)}$ can be written in these coordinates as $X_{H(w)} = \sum_j b^j \frac{\partial}{\partial y^j}$ with $\frac{\partial}{\partial x^i} b^j = 0$ by construction of $H$. Then

$$[X_{H(w)}, \xi]|_{x^1=x^2=0} = \sum_i \sum_j b^j \frac{\partial}{\partial y^j}|_{x^1=x^2=0} a^i \frac{\partial}{\partial x^i} - \sum_j \sum_i a^i \frac{\partial}{\partial x^i}|_{x^1=x^2=0} b^j \frac{\partial}{\partial y^j} = 0.$$

Similarly,

$$[X_{H(w)}, J\xi]|_{x^1=x^2=0} = \sum_j b^j \frac{\partial}{\partial y^j}|_{x^1=x^2=0} a^1 \frac{\partial}{\partial x^2} - \sum_j b^j \frac{\partial}{\partial y^j}|_{x^1=x^2=0} a^2 \frac{\partial}{\partial x^1} - \\ - \sum_j (a^1 \frac{\partial}{\partial x^2} - a^2 \frac{\partial}{\partial x^1})|_{x^1=x^2=0} b^j \frac{\partial}{\partial y^j} = 0.$$

The other two cases are completely analogous. □

For $Y$-compatible almost complex structures and Hamiltonian perturbations one can now define the sets $\mathcal{N}^\ell$ from the previous subsection. The main observation used in the definition is the following, which for convenience subsequently is summarised from Section 7, in [CM07].

**Definition II.29.** Let $(S, j)$ be a Riemann surface, $f : S \to X$ a differentiable map. An *isolated intersection* of $f$ with $Y$ is a point $z \in f^{-1}(Y)$ s. t. there exists a closed disk $D \subseteq S$ around $z$ and a closed disk $B \subseteq Y$ around $f(z)$ with $f^{-1}(B) \cap D = \{z\}$.
Given such an isolated intersection $z \in f^{-1}(Y)$, the *local intersection number* $\iota(f, Y; z)$ of $f$ with $Y$ at $z$ is defined as follows: Assume that $f$ intersects $Y$ in $z$ transversely. Then $\iota(f, Y; z) = 1$, if the orientation on $T_{f(z)}X$ agrees with the orientation induced (via $T_{f(z)}X \cong (f_* T_z S) \oplus T_{f(z)}Y$) by the orientations on $T_z S$ and $T_{f(z)}Y$, and $\iota(f, Y; z) = -1$, otherwise. In general, choose a differentiable perturbation $f_t : S \to X$, $t \in [0, 1]$, of $f$ with compact support in the interior of $D$ and s. t. $f_1|_D$ is transverse to $B$. Then

$$\iota(f, Y; z) := \sum_{z' \in f_1^{-1}(B) \cap D} \iota(f_1, Y; z').$$

If $S$ is compact and all intersections of $f$ with $Y$ are isolated (in particular by compactness there are only finitely many), then the intersection number of $f$ with $Y$ is defined as

$$\iota(f, Y) := \sum_{z \in f^{-1}(Y)} \iota(f, Y; z).$$

The adaptation of Proposition 7.1, in [CM07] to the present situation.

**Lemma II.33.** *Let $\tilde{u} \in \mathcal{M}(\tilde{X}, A, J, \mathcal{H}(\tilde{X}, \tilde{Y}))$. Define $u := \mathrm{pr}_2 \circ \tilde{u} : \Sigma_b \to X$. Then for every component (i. e. connected component of a desingularisation) $\Sigma_b^i$ of $\Sigma_b$, either $u(\Sigma_b^i) \subseteq Y$ or $(u|_{\Sigma_b^i})^{-1}(Y)$ is finite. In the latter case,*

$$\iota(u|_{\Sigma_b^i}, Y) = [u|_{\Sigma_b^i}] \cdot [Y],$$

*i. e. the intersection number of $u|_{\Sigma_b^i}$ with $Y$ coincides with the topological intersection number of the homology classes in $X$ defined by $u|_{\Sigma_b^i}$ and $Y$. Furthermore, at each intersection point $z \in (u|_{\Sigma_b^i})^{-1}(Y)$, $u$ is tangent to $Y$ of some finite order $s \geq 0$ with*

$$\iota(u|_{\Sigma_b^i}, Y; z) = s + 1.$$

*In particular, each local intersection number $\iota(u|_{\Sigma_b^i}, Y; z)$ is positive.*

*Proof.* $(\tilde{X}|_{\Sigma_b^i}, \tilde{J}^H)$ is a complex manifold with $\tilde{Y}|_{\Sigma_b^i}$ as a complex submanifold by definition of $\mathcal{H}(\tilde{X}, \tilde{Y})$. Furthermore, $\tilde{u}|_{\Sigma_b^i} : \Sigma_b^i \to \tilde{X}|_{\Sigma_b^i}$ is a holomorphic map. Now observe that $\tilde{u}(z) \in \tilde{Y}$ iff $u(z) \in Y$, and the orders of tangency coincide. Now apply Proposition 7.1, in [CM07] to $\tilde{u}|_{\Sigma_b^i}$. □

This allows for the following definition:

**Definition II.30.** Let $(\ell_1, \ldots, \ell_n) \in (\mathbb{Z}_{\geq -1})^n$ and denote $\ell'_j := \min\{0, \ell_j\}$. Given any open subset $V \subseteq \tilde{X} \setminus \tilde{Y}$ and $H \in \mathcal{H}(\tilde{X}, \tilde{Y})$, define

$$\tilde{Y}^{\ell'_j} := \begin{cases} \tilde{X} & \ell'_j = -1 \\ \tilde{Y} & \ell'_j = 0 \end{cases}$$

and note that $\mathcal{H}^V(\tilde{X}) \subseteq \mathcal{H}(\tilde{X}, \tilde{Y})$. Then

$$\mathcal{M}(\tilde{X}, \tilde{Y}^{(\ell'_1, \ldots, \ell'_n)}, A, J, H + \mathcal{H}^V(\tilde{X})) := \left(\mathrm{ev}^R\right)^{-1} \left(R_1^* \tilde{Y}^{\ell'_1} \oplus \cdots \oplus R_n^* \tilde{Y}^{\ell'_n}\right),$$

for

$$\mathrm{ev}^R : \mathcal{M}^V(\tilde{X}, A, J, H + \mathcal{H}^V(\tilde{X})) \to R_1^* \tilde{X} \oplus \cdots \oplus R_n^* \tilde{X}.$$

Furthermore for any subset $B \subseteq M$,

$$\begin{aligned} &\mathcal{M}^V(\tilde{X}|_B, \tilde{Y}^{(\ell_1, \ldots, \ell_n)}, A, J, H + \mathcal{H}^V(\tilde{X})) := \\ &\{u \in \mathcal{M}_b^V(\tilde{X}, \tilde{Y}^{(\ell'_1, \ldots, \ell'_n)}, A, J, H + \mathcal{H}^V(\tilde{X})) \mid b \in B, \iota(u, \tilde{Y}|_{\Sigma_b}; R_j(b)\} = \ell_j\}. \end{aligned}$$

By the previous lemma, if $u$ is a holomorphic curve in $\tilde{X}$ s. t. $u$ intersects $\tilde{Y}$ at each of $\ell$ different marked points, the last $\ell$, say, $u$ is not contained completely in $\tilde{Y}$ and $[u] \cdot [Y] = \ell$, then $u$ intersects $\tilde{Y}$ transversely. Unfortunately one cannot expect this behaviour to persevere under limits of sequences of such maps. For example even for a fixed complex structure on the underlying curve, two of the last $\ell$ marked points could converge on the domain forming a nodal curve, built up of the original curve together with a sphere component that gets mapped to $\tilde{Y}$. Since the restriction of $\tilde{Y}$ to every fibre $\Sigma_b$ of $\Sigma$ is trivial by definition, it makes sense to say that the sphere component is constant. In this case this map actually factors through a map from the original surface, but with the two converging marked points replaced by the point at which the sphere component is attached and which gets mapped to $\tilde{Y}$. At this new point, the curve no longer needs to be transverse to $\tilde{Y}$, but the previous lemma states that, if the curve does not lie completely in $\tilde{Y}$, the limit map can only have tangencies of second order. So apart from moduli spaces of curves with marked points lying on a given submanifold, a case already dealt with in Lemma II.26, one should also construct moduli spaces of curves with tangencies to a given complex hypersurface of (at least) a given order. The tangency of order 1-condition is easy enough to define, if $u \in \mathcal{M}_b(\tilde{X}, A, J, H)$ with $u(R_i(b)) \in \tilde{Y}$, then $u$ is tangent to $\tilde{Y}$ at $R_i(b)$ to first order simply if $\operatorname{im}(D^{\mathrm{v}}u)_{R_i(b)} \subseteq V\tilde{Y}$. For $J \in \mathcal{J}_\omega(X,Y)$, $V\tilde{Y}^{\perp_\omega}$ is a $J_b$-complex subspace of complex dimension 1. If $H \in \mathcal{H}(\tilde{X},\tilde{Y})$, then since $\overline{\partial}_b^{J,H}u = 0$, $\pi^{V\tilde{X}}_{V\tilde{Y}^{\perp_\omega}}(D^{\mathrm{v}}u)_{R_i(b)}$ is a $j_b$-$J_b$-complex linear map from $V_{R_i(b)}\Sigma$ to $V_{u(R_i(b))}\tilde{Y}^{\perp_\omega}$. Hence over the subset of elements of $\mathcal{M}(\tilde{X},A,J,\mathcal{H}(\tilde{X},\tilde{Y}))$ that map the $i^{\text{th}}$ marked point to $\tilde{Y}$ (a submanifold by Lemma II.26), one can consider the complex line bundle with line over $u$ given by $\operatorname{Hom}_{(j,J)}(V_{R_i}\Sigma, V_{u(R_i)}V\tilde{Y}^{\perp_\omega})$ and the section $u \mapsto \pi^{V\tilde{X}}_{V\tilde{Y}^{\perp_\omega}}(D^{\mathrm{v}}u)_{R_i}$. In case of transversality of this section to the zero section, the moduli space of curves tangent to $\tilde{Y}$ at the $i^{\text{th}}$ marked point then has complex codimension one in the submanifold of those curves that map the $i^{\text{th}}$ marked point to $\tilde{Y}$. Unfortunately the higher order tangency conditions do not seem to admit such an easy description as global sections of a globally defined complex vector bundle (of the "correct" rank) over the universal moduli space. [CM07], which allows to use the transversality result (or rather a slight variation of its proof) from [CM07].

**Construction II.10.** Let $(\rho : S \to B, \hat{R}, \iota, \hat{\iota})$ be a desingularisation of $\Sigma$ over $B \subseteq M$ and as before denote $\hat{X} := \rho^*\iota^*X = \hat{\iota}^*\tilde{X}$ and $\hat{Y} := \rho^*\iota^*Y = \hat{\iota}^*\tilde{Y}$. For $a \in B$, let $U \subseteq B$ be an open neighbourhood of $a$ s. t. both $X|_U$ and $Y|_U$ are trivial, and hence so are $\hat{X}|_U$ and $\hat{Y}|_U$. Also let $\phi_a : U \times S_a \to S|_U$ be a trivialisation that preserves the marked points and nodes. Assume that there are pairwise disjoint open neighbourhoods $D_j \subseteq S_a$ of the marked points $\hat{R}_j(a) \in S_a$, biholomorphically equivalent to the unit disk $\mathbb{D} \subseteq \mathbb{C}$ and disjoint from all the nodal points. These are assumed to have the property that for all $b \in U$, $\phi_{ab}|_{D_j} : S_a \supseteq D_j \to S_b$ is a biholomorphic map from $D_j$ onto a neighbourhood of $\hat{R}_j(b) \in S_b$. Let $u_0 \in \mathcal{M}_a(\tilde{X}|_B, A, J, H)$ for some $H \in \mathcal{H}(\tilde{X},\tilde{Y})$. Fix some $i \in \{1,\dots,n\}$ and assume that $\mathrm{ev}_i^{\hat{R}}(u_0) \in \hat{Y}$, but that the component of $\Sigma_a$ containing $\hat{R}_i(a)$ does not get mapped completely to $\hat{Y}$ by $u_0$. Using triviality of $X$ and $Y$ over $U$, pick a neighbourhood $W \subseteq \hat{X}$ of $\mathrm{ev}_i^{\hat{R}}(u_0)$ diffeomorphic to $U \times S_a \times \mathbb{C}^r$, where $r := \dim_{\mathbb{C}}(X)$, via a diffeomorphism $\Psi$ that maps $\hat{Y} \cap W$ to $U \times S_a \times \mathbb{C}^{r-1} \times \{0\}$. Also assume that this diffeomorphism covers $\phi_a$. On the right hand side then for any $H \in \mathcal{H}(\tilde{X},\tilde{Y})$ and $b \in U$, $\{b\} \times S_a \times \mathbb{C}^r$ is equipped with the pullback complex structure $\overline{J}_b^H$ of $\hat{J}^H$ which turns $\{b\} \times S_a \times \mathbb{C}^r$ into an almost complex manifold and $\{b\} \times S_a \times \mathbb{C}^{r-1} \times \{0\}$ into a complex submanifold. Remember that the topology on $\mathcal{M}_U(\tilde{X}|_B, A, J, \mathcal{H}(\tilde{X},\tilde{Y}))$ is finer than the topology induced by that on $U \times \mathcal{B}_a^{k,p}(\hat{X}|_B, A, J, H) \times \mathcal{H}(\tilde{X},\tilde{Y})$ (for some $k,p$ with $kp > 2$) by the chart defined via $\phi_a$ from Construction II.8. And that the topology on $\mathcal{B}_a^{k,p}(\hat{X}|_B, A, J, H)$ in turn is finer than the $C^0$-topology (which was part of the definition of the topology on $\mathcal{B}_a^{k,p}(\hat{X}|_B, A, J, H)$ in Construction II.2). Also, the intersection of $u_0$ with $\hat{Y}$ at $R_i(a)$ is isolated by Lemma II.33. Hence there is a neighbourhood $\mathcal{V}$ of $u_0$ in $\mathcal{M}(\tilde{X}|_B, A, J, \mathcal{H}(\tilde{X},\tilde{Y}))$ s. t. $u(\phi_{ab}(D_j)) \subseteq W$ for all $u \in \mathcal{V}$, $\pi_B^{\mathcal{M}}(u) = b$. With the help of the above one can now assign, for every $j = 1,\dots,n$ and to every $u \in \mathcal{V}$ with $\pi_B^{\mathcal{M}}(u) = b$ and $\pi_{\mathcal{H}}^{\mathcal{M}}(u) = H$ an ($i$ here is the standard complex structure on $D_j \cong \mathbb{D}$) $i$-$\overline{J}_b^H$-holomorphic map $D_j \to \{b\} \times \mathbb{D}_j \times \mathbb{C}^r$. Now one is pretty much exactly in (a parametrised version of) the situation of Section 6 of [CM07] and can follow the discussion leading up to Proposition 6.9 almost to the letter, dropping the simplicity requirement and replacing the space of perturbations of the almost complex structures by the space of Hamiltonian perturbations used in this text, esp. in Lemma 6.6, to show the following result:

**Lemma II.34.** *Let $V \subseteq \tilde{X}$ be an open subset s. t. $V \cap \tilde{Y} = \emptyset$, let $H \in \mathcal{H}(\tilde{X}, \tilde{Y})$ and let $B \subseteq M$ be a stratum over which $\Sigma$ has a desingularisation. Then for any $n$-tuple $(\ell_1, \dots, \ell_n) \in (\mathbb{Z}_{\geq -1})^n$, $\mathcal{M}^V(\tilde{X}|_B, \tilde{Y}^{(\ell_1, \dots, \ell_n)}, A, J, H + \mathcal{H}^V(\tilde{X}))$ is a Banach submanifold of $\mathcal{M}^V(\tilde{X}|_B, A, J, H + \mathcal{H}^V(\tilde{X}))$ of real codimension $2\sum_{i=1}^n (\ell_i + 1)$.*

# CHAPTER III

## Construction of the rational pseudocycle

In this final part, it is made precise in which sense the map I.1 from the introduction to this thesis defines a homology class, after suitable modifications.

To do so, the notion of a pseudocycle from [MS04], Section 6.5, will be used. Also remember that in order to have a smooth moduli space of Riemann surfaces, the Deligne-Mumford space was replaced by a finite (branched) covering. To get a well-defined count, the order of this covering has to be divided out, so instead of integral pseudocycles, rational pseudocycles will be used, as in [CM07].

Since quite a few different notions are involved in this definition, for convenience they are presented in the first subsection.

After that, the definition is given and a few basic properties are shown.

The compactness result presented then is the first step in showing that this indeed does define a pseudocycle.
Most of the rest of this text is concerned with showing that (after imposing some restrictions on $J$ and $H$) the $\Omega$-limit set described by this compactness result can be covered by manifolds of real codimension 2, hence showing that the pseudocycle is indeed well-defined.

The thesis then concludes with a few words about independence of this definition of the choices made.

# III.1 Definition of the pseudocycle and questions of compactness

## III.1.1 The data involved in the definition

Given the following data (leaving questions of existence for the moment aside):

1. A closed symplectic manifold $(X, \omega)$ with integer symplectic form, $[\omega] \in H^2(X, \mathbb{Z})$.
2. $0 \neq A \in H_2(X; \mathbb{Z})$, $E := \omega(A) + 1$.
3. An orbifold branched covering $(\pi : \Sigma \to M)$ of $\overline{M}_{g,n}$ and consequently a sequence of marked nodal families $(\pi^\ell : \Sigma^\ell \to M^\ell, R^\ell, T^\ell)$ for $\ell \geq 0$ of Euler characteristic $\chi$ with $n + \ell$ marked points $R_1^\ell, \dots, R_n^\ell$, $T_1^\ell, \dots, T_\ell^\ell$, as in Lemma II.1 and Proposition II.1:

$$\begin{tikzcd}
\cdots \arrow[r, "\hat\pi^{\ell+1}"] & \Sigma^{\ell+1} \arrow[r, "\hat\pi^{\ell}"] \arrow[d, "\pi^{\ell+1}"] & \Sigma^{\ell} \arrow[r, "\hat\pi^{\ell-1}"] \arrow[d, "\pi^{\ell}"] & \Sigma^{\ell-1} \arrow[r, "\hat\pi^{\ell-2}"] \arrow[d, "\pi^{\ell-1}"] & \cdots \arrow[r, "\hat\pi^{1}"] & \Sigma^{1} \arrow[r, "\hat\pi^{0}"] \arrow[d, "\pi^{1}"] & \Sigma^{0} \arrow[r, equal] \arrow[d, "\pi^{0}"] & \Sigma \arrow[d, "\pi"] \\
\cdots \arrow[r, "\pi^{\ell+1}"] & M^{\ell+1} \arrow[r, "\pi^{\ell}"] \arrow[ur, equal] & M^{\ell} \arrow[r, "\pi^{\ell-1}"] \arrow[ur, equal] & M^{\ell-1} \arrow[r, "\pi^{\ell-2}"] & \cdots \arrow[r, "\pi^{1}"] & M^{1} \arrow[r, "\pi^{0}"] \arrow[ur, equal] & M^{0} \arrow[r, equal] & M
\end{tikzcd}$$

$$\begin{tikzcd}
\Sigma^\ell \arrow[r, "\hat\pi^{\ell-1}"] \arrow[d, "\pi^\ell"] & \Sigma^{\ell-1} \arrow[d, "\pi^{\ell-1}"'] \\
M^\ell \arrow[r, "\pi^{\ell-1}"] \arrow[u, bend left, "R_j^\ell"] & M^{\ell-1} \arrow[u, bend right, "R_j^{\ell-1}"']
\end{tikzcd} \quad \forall\, j = 1, \dots, n$$

$$\begin{tikzcd}
\Sigma^\ell \arrow[r, "\hat\pi^{\ell}"] \arrow[d, "\pi^\ell"] & \Sigma^{\ell-1} \arrow[d, "\pi^{\ell-1}"'] \\
M^\ell \arrow[r, "\pi^{\ell-1}"] \arrow[u, bend left, "T_j^\ell"] & M^{\ell-1} \arrow[u, bend right, "T_j^{\ell-1}"']
\end{tikzcd} \quad \forall\, j = 1, \dots, \ell - 1$$

where $M$ is assumed to be closed, and hence so are the $M^\ell$ for all $\ell \geq 0$. Furthermore, for every $b \in M^\ell$, putting $b' := \pi^{\ell-1}(b) \in M^{\ell-1}$, the map

$$\begin{aligned}
\hat\pi_b^{\ell-1} : (\Sigma_b^\ell, R_1^\ell(b), \dots, R_n^\ell(b), T_1^\ell(b), \dots, T_{\ell-1}^\ell(b)) \to \\
\to (\Sigma_{b'}^{\ell-1}, R_1^{\ell-1}(b'), \dots, R_n^{\ell-1}(b'), T_1^{\ell-1}(b'), \dots, T_{\ell-1}^{\ell-1}(b'))
\end{aligned}$$

is assumed to be stabilising, i. e. to be biholomorphic on every stable component of $(\Sigma_b^\ell, R_1^\ell(b), \dots, R_n^\ell(b), T_1^\ell(b), \dots, T_{\ell-1}^\ell(b))$ and constant on every unstable component. For $\ell > k$ denote the compositions

$$\hat\pi_k^\ell := \hat\pi^k \circ \hat\pi^{k+1} \circ \dots \circ \hat\pi^{\ell-1} : \Sigma^\ell \to \Sigma^k$$

and

$$\pi_k^\ell := \pi^k \circ \pi^{k+1} \circ \cdots \circ \pi^{\ell-1} : M^\ell \to M^k.$$

By the same argument as in Section II.1, assume that $M$ and hence all the $M^\ell$ are connected.

4. Metrics $h^\ell$ on the $\Sigma^\ell$, restricting to a hermitian metric on every $\Sigma_b^\ell$, $b \in M^\ell$.

5. $D \in \mathbb{N}$.

6. A Donaldson pair $(Y, J_0)$ of degree $D$, i. e. $J_0 \in \mathcal{J}_\omega(X)$ and $Y \subseteq X$ an, in the sense of [CM07], Section 8, approximately $J_0$-holomorphic, in particular symplectic, hypersurface with $\mathrm{PD}(Y) = D[\omega]$.

7. $\ell := D\omega(A)$.

The existence of the marked nodal families $(\pi^\ell : \Sigma^\ell \to M^\ell)$ from 3. above had already been dealt with in Section II.1.
The necessary existence and uniqueness results for hypersurfaces as in 6. can be found in Theorem 8.1 from [CM07] and the references quoted there.

Also, for later reference, introduce the following notation:
Denote by $M^{\ell,i}$ the strata (which are not assumed to be connected) of $M^\ell$ by signature. The top stratum corresponding to the smooth surfaces will be denoted by $\mathring{M}^\ell = M^{\ell,0}$. Over every $M^{\ell,i}$, there exists a desingularisation $(\rho^{\ell,i} : S^{\ell,i} \to M^{\ell,i}, \hat{R}^{\ell,i}, \hat{T}^{\ell,i}, \iota^{\ell,i}, \hat{\iota}^{\ell,i})$ of $\Sigma^{\ell,i} := \Sigma^\ell|_{M^{\ell,i}}$. Cover each $M^{\ell,i}$ by finitely many open subsets $U_j^{\ell,i} \subseteq M^{\ell,i}$ s. t. there exist trivialisations $\phi_j^{\ell,i} : U_j^{\ell,i} \times S_j^{\ell,i} \to S^{\ell,i}|_{U_j^{\ell,i}}$ and assume that the $\phi_j^{\ell,i}$ have all the properties from the previous chapter: There exist points $R_{j,s}^{\ell,i} \in S_j^{\ell,i}$, $s = 1, \dots, n$, and $T_{j,s}^{\ell,i} \in S_j^{\ell,i}$, $s = 1, \dots, \ell$, s. t. $\phi_j^{\ell,i}(b, R_{j,s}^{\ell,i}) = \hat{R}_s^{\ell,i}(b)$ as well as $\phi_j^{\ell,i}(b, T_{j,s}^{\ell,i}) = \hat{T}_s^{\ell,i}(b)$ and pairs of points $N_{j,r}^{\ell,i,1}, N_{j,r}^{\ell,i,2}$, $r = 1, \dots, d^{\ell,i}$ (for some $d^{\ell,i} \in \mathbb{N}_0$), s. t. for all $b \in U_j^i$, $\phi_j^{\ell,i}(b, N_{j,r}^{\ell,i,1})$, $\phi_j^{\ell,i}(b, N_{j,r}^{\ell,i,1})$ is a pair of points corresponding to a single node on $\Sigma_b^\ell$. Furthermore, all the $R_{j,s}^{\ell,i}$, $N_{j,r}^{\ell,i,1}$, $N_{j,r}^{\ell,i,2}$ are assumed to have mutually disjoint neighbourhoods all biholomorphically equivalent to the open unit disk in $\mathbb{C}$ and s. t. for all $b \in U_j^{\ell,i}$, the restriction of $(\phi_j^{\ell,i})_b : S_j^{\ell,i} \to S_b^{\ell,i}$ to every such neighbourhood is holomorphic. Finally, assume that the $U_j^{\ell,i}$ are compatible for varying $\ell$ in the sense that for every $\ell \geq k$, given $i$ and $j$ there exist $i'$ and $j'$ s. t. $\pi_{k-1}^{\ell-1}|_{U_j^{\ell,i}} : U_j^{\ell,i} \to U_{j'}^{k,i'}$ is a submersion.

### III.1.2 The definition of the pseudocycle

Let $\tilde{X}^\ell := \Sigma^\ell \times X$, $\tilde{Y}^\ell := \Sigma^\ell \times Y$, where $\ell = D\omega(A)$. The pseudocycle in question will be of the following form, for appropriately chosen $H$:

**Definition III.1.** For $H \in \mathcal{H}(\tilde{X}^\ell)$, define

$$\begin{aligned}\mathring{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H) := \{u \in \mathcal{M}(\tilde{X}^\ell|_{\mathring{M}^\ell}, A, J, H) \mid\ & \mathrm{im}(u \circ T_j^\ell) \subseteq \tilde{Y}^\ell,\ j = 1, \dots, \ell, \\ & \mathrm{im}(u) \cap \tilde{X}^\ell \setminus \tilde{Y}^\ell \neq \emptyset\}\end{aligned}$$

and as before, for any (affine) subspace $\mathcal{K} \subseteq \mathcal{H}(\tilde{X}^\ell)$,

$$\mathring{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, \mathcal{K}) := \bigcup_{H \in \mathcal{K}} \mathring{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H).$$

Also denote by $\overline{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H)$ the closure in $\mathcal{M}(\tilde{X}^\ell, A, J, H)$ of $\mathcal{M}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H)$.

**Lemma III.1.** *For $A \neq 0$ and $D$ large enough, for generic $H^Y \in \mathcal{H}(\tilde{Y}) \subseteq \mathcal{H}(\tilde{X}, \tilde{Y})$ there exists a generic subset (depending on $H^Y$) of $\mathcal{H}^{00}(\tilde{X}, \tilde{Y})$ s. t. for each $H^{00}$ in this subset, outside a subset of $\mathcal{M}(\tilde{X}, A, J, H^Y + H^{00})$ with complement of codimension at least 2,*

$$\begin{aligned}\mathring{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, (\hat{\pi}_0^\ell)^*(H^Y + H^{00})) &\to \mathcal{M}(\tilde{X}, A, J, H^Y + H^{00}) \\ u &\mapsto u \circ ((\hat{\pi}_0^\ell)^*_{\pi_M^{\mathcal{M}}(u)})^{-1}\end{aligned}$$

*is an $\ell!$-sheeted covering.*

*Proof.* As will be shown in Subsection III.2.2, for $A \neq 0$ and $D$ large enough, for generic $H^Y \in \mathcal{H}(\tilde{Y})$ and for any $H^{00} \in \mathcal{H}(\tilde{X}, \tilde{Y})$ one can assume that no section in $\mathcal{M}(\tilde{X}, A, J, H^Y + H^{00})$ lies completely in $\tilde{Y}$. Note that over $\mathring{M}^\ell$, $\hat{\pi}_0^\ell$ is a fibrewise isomorphism, hence the space $(\hat{\pi}_0^\ell)^*(H^Y + \mathcal{H}^{00}(\tilde{X}, \tilde{Y}))$ is "large enough" for all the transversality results in the following to hold for generic $H := H^Y + H^{00}$, in particular all the strata of $\mathcal{M}(\tilde{X}, A, J, H)$ can be assumed to be manifolds of the correct dimensions. Now every element of $\mathcal{M}(\tilde{X}, A, J, H)$ has intersection number $\ell$ with $\tilde{Y}$, so $\mathcal{M}(\tilde{X}, A, J, H)$ is covered by a union of spaces

$$\mathcal{M}(\tilde{X}^{\ell'}|_{\mathring{M}^{\ell'}}, (\tilde{Y}^{\ell'})^{(0,\dots,0,t_1,\dots,t_{\ell'})}, A, J, (\hat{\pi}_0^{\ell'})^* H)$$

for $1 \leq \ell' \leq \ell$ and $\sum_{i=1}^{\ell'}(t_i + 1) = \ell$. For $\ell' = \ell$ this is just the space $\mathring{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, (\hat{\pi}_0^\ell)^* H)$, where the fibre over a point is given by the $\ell!$ choices to label the $\ell$ intersection points with $\tilde{Y}$. And for $\ell' < \ell$, by Lemma II.34, this space has dimension at least two less than $\mathring{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, (\hat{\pi}_0^\ell)^* H)$. □

Remember from Section II.1 that on every $M^{\ell,i}$ there is the induced stratification from the stratification of $M^\ell$ given by the groupoid structure. In particular, because the $M^\ell$ were assumed connected, $\mathring{M}^\ell$ has a unique open and dense connected stratum $\mathring{\mathring{M}}^\ell$ as well as a number of strata $\overset{\circ c}{M}{}^{\ell,j}$ of codimension at least 2. Then for every $H_0 \in \mathcal{H}(\tilde{X}^\ell)$,

$$\pi_M^{\mathcal{M}} : \mathring{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H_0 + \mathcal{H}^{00}(\tilde{X}^\ell, \tilde{Y}^\ell)) \to \mathring{M}^\ell$$

is a submersion, so for every $j$, $\left(\pi^{\mathcal{M}}_{M}\right)^{-1}(\overset{\circ c}{M}{}^{\ell,j})$ is a codimension at least 2 split submanifold of $\overset{\circ}{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H_0 + \mathcal{H}^{00}(\tilde{X}^\ell, \tilde{Y}^\ell))$ by Lemma A.6. Hence by Lemma A.7 and the Sard-Smale theorem, for generic $H'$ in $\mathcal{H}^{00}(\tilde{X}^\ell, \tilde{Y}^\ell)$, one can assume that for every $j$,

$$\overset{\circ\circ}{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H_0 + H') := \{u \in \overset{\circ}{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H_0 + H') \mid \pi^{\mathcal{M}}_{M}(u) \in \overset{\circ\circ}{M}{}^\ell\}$$

and

$$\overset{\circ c}{\mathcal{M}}{}^j(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H_0 + H') := \{u \in \overset{\circ}{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H_0 + H') \mid \pi^{\mathcal{M}}_{M}(u) \in \overset{\circ c}{M}{}^{\ell,j}\}$$

are manifolds of the expected dimensions.
Furthermore, note that for such $H := H_0 + H'$, $\overset{\circ\circ}{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H)$ carries a natural orientation: First, note that $\overset{\circ}{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, \mathcal{H}(\tilde{X}^\ell, \tilde{Y}^\ell))$ carries a natural coorientation as split submanifold of $\overset{\circ}{\mathcal{M}}(\tilde{X}^\ell, A, J, \mathcal{H}(\tilde{X}^\ell, \tilde{Y}^\ell))$, since it is the preimage under the evaluation map at the last $\ell$ marked points of $\tilde{Y}^\ell$, which is cooriented in $\tilde{X}^\ell$.
Second, for the Fredholm map

$$\pi^{\mathcal{M}}_{\mathcal{H}} : \overset{\circ}{\mathcal{M}}(\tilde{X}^\ell, A, J, \mathcal{H}(\tilde{X}^\ell, \tilde{Y}^\ell)) \to \mathcal{H}(\tilde{X}^\ell, \tilde{Y}^\ell),$$

at a regular point the kernel of its differential is canonically oriented, since it is identified with the kernel of the corresponding Cauchy-Riemann operator, by Lemma A.3.6 in [MS04]. This in turn is oriented by the usual argument as in the proof of Theorem 3.1.5, p. 50, in [MS04]. Hence the kernel of the restriction

$$\pi^{\mathcal{M}}_{\mathcal{H}} : \overset{\circ}{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell A, J, \mathcal{H}(\tilde{X}^\ell, \tilde{Y}^\ell)) \to \mathcal{H}(\tilde{X}^\ell, \tilde{Y}^\ell),$$

at a regular point carries an induced orientation as well.

With this, the definition is as in Section II.1 ($H = H_0 + H'$):

**Definition III.2.** The pseudocycle associated to the above data is the rational pseudocycle

$$\frac{1}{\ell!}\frac{1}{|\operatorname{Aut}(\overset{\circ\circ}{M})||\mathcal{O}(\overset{\circ\circ}{M})|}\operatorname{ev}^{R^\ell} : \overset{\circ\circ}{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H) \to M \times X^n,$$

where on the right-hand side $\tilde{X}^\ell = \Sigma^\ell \times X$ and the canonical map $\pi^\ell_0 : M^\ell \to M$ were used.

The remainder of this chapter now consists of the proof that, for appropriately chosen $H$, the above is a well-defined pseudocycle. Following the course of action as set out in [CM07], first it will be shown that for an appropriate choice of $J$ and any choice of $H$, $\overline{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H)$ is compact. The next order of business then is to choose $H$ appropriately s. t. $\overset{\circ}{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H)$ is a smooth manifold of dimension $\dim_{\mathbb{C}}(X)\chi + 2c_1(A) + \dim_{\mathbb{R}} M$ and s. t. $\overline{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H) \setminus \overset{\circ}{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H)$ can be covered by finitely many manifolds of dimension at most $\dim_{\mathbb{C}}(X)\chi + 2c_1(A) + \dim_{\mathbb{R}} M - 2$.

### III.1.3 The main compactness result

The appropriate conditions for compactness of $\overline{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H)$ to hold have been formulated in [CM07], Section 8:

**Definition III.3.** Let, for $E > 0$, $\mathfrak{J}_\omega(X, Y; E) \subseteq \mathfrak{J}_\omega(X, Y)$ be the subset of almost complex structures $J \in \mathfrak{J}_\omega(X, Y)$ s. t.

1. All $J$-holomorphic spheres of energy $\leq E$ contained in $Y$ are constant.
2. Every nonconstant $J$-holomorphic sphere of energy $\leq E$ in $X$ intersects $Y$ in at least 3 distinct points in the domain.

Also, define

$$\mathfrak{J}_{\omega,\mathrm{ni}}(X, Y; E) := \mathfrak{J}_\omega(X, Y; E) \cap \mathfrak{J}_{\omega,\mathrm{ni}}(X, Y).$$

In the same reference, in Corollary 8.14, it has been shown that this condition is non-void, which needs to be adapted to include the condition of normal integrability:

**Lemma III.2.** *There exists a constant $D^* = D^*(X, \omega, J_0)$ and a nonempty $C^0$-neighbourhood $\mathfrak{J}_\omega(X; J_0) \subseteq \mathfrak{J}_\omega(X)$ of $J_0$ s. t. if $D \geq D^*$, then*

$$\mathfrak{J}_{\omega,\mathrm{ni}}(X, Y; J_0, E) := \mathfrak{J}_{\omega,\mathrm{ni}}(X, Y; E) \cap \mathfrak{J}_\omega(X; J_0)$$

*is nonempty for every $E > 0$.*
*Moreover, any two elements in $\mathfrak{J}_{\omega,\mathrm{ni}}(X, Y; J_0, E)$ can be connected by a path in $\mathfrak{J}_{\omega,\mathrm{ni}}(X, Y; E)$.*

*Proof.* Let $\mathfrak{J}_\omega(X; J_0)$ be the $C^0$-ball around $J_0$ in $\mathfrak{J}_\omega(X)$ of radius $\theta_2$, where $\theta_2$ is as in Corollary 8.14 of [CM07]. Then by that reference, there exists a $J' \in \mathfrak{J}_\omega(X, Y; E) \cap \mathfrak{J}_\omega(X; J_0)$. Applying the procedure in the proof of Theorem A.2 in [IP03] yields an arbitrarily $C^0$-close (to $J'$, the endomorphism $K$ in equation (A.2) in said proof can be chosen arbitrarily small in $C^0$, but not in $C^1$) $J'' \in \mathfrak{J}_{\omega,\mathrm{ni}}(X, Y)$, in particular $J'' \in \mathfrak{J}_\omega(X; J_0)$, with $J''|_Y = J'$. Hence $J''$ still satisfies condition 1. in Definition III.3. Now observe that condition (ii) of Proposition 8.11 in [CM07] can be achieved by a perturbation $J$ of $J''$ s. t. $J - J''$ lies in the closure of those endomorphisms of $TX$ that have compact support in the complement of $Y$. But such perturbations still lie in $\mathfrak{J}_{\omega,\mathrm{ni}}(X, Y)$. Now if $J_0, J_1 \in \mathfrak{J}_{\omega,\mathrm{ni}}(X, Y; J_0, E)$, then by Corollary 8.14 in op. cit. they can be connected by a path $(J'_\tau)_{\tau \in [0,1]}$ in $\mathfrak{J}_\omega(X, Y; E)$. Again applying the procedure from Theorem A.2 in [IP03] produces a path $(J''_\tau)_{\tau \in [0,1]}$, arbitrarily close to $(J'_\tau)_{\tau \in [0,1]}$ in $C^0$-topology, that coincides with $(J'_\tau)_{\tau \in [0,1]}$ along $Y$ and satisfies $J''_0 = J'_0 = J_0$ as well as $J''_1 = J'_1 = J_1$. So in particular $(J''_\tau)_{\tau \in [0,1]}$ still satisfies condition 1. in Definition III.3. Now proceed as before: Condition (ii) of Proposition 8.12 in [CM07] can be achieved by a perturbation $(J_\tau)_{\tau \in [0,1]}$ of

$(J''_\tau)_{\tau\in[0,1]}$ s. t. $J''_\tau - J_\tau$ vanishes for $\tau = 0, 1$ and for $\tau \in (0, 1)$ lies in the closure of those endomorphisms of $TX$ that have compact support in the complement of $Y$. □

Finally, again in [CM07], Proposition 9.5, the necessary compactness result is given, which can easily be adapted to the present situation to show:

**Lemma III.3.** *Let $J \in \mathcal{J}_\omega(X, Y; E)$ and let $\ell = [Y] \cdot A$. Then*

$$\pi_M^{\mathcal{M}} \times \pi_{\mathcal{H}}^{\mathcal{M}} : \overline{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, \mathcal{H}(\tilde{X}^\ell, \tilde{Y}^\ell)) \to M^\ell \times \mathcal{H}(\tilde{X}^\ell, \tilde{Y}^\ell)$$

*is a proper map.*

*Proof.* Let $(u_i)_{i\in\mathbb{N}} \subseteq \mathcal{M}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, \mathcal{H}(\tilde{X}^\ell, \tilde{Y}^\ell))$ be a sequence s. t. $b_i := \pi_M^{\mathcal{M}}(u_i) \to b \in M^\ell$ and $H_i := \pi_{\mathcal{H}}^{\mathcal{M}}(u_i) \to H \in \mathcal{H}(\tilde{X}^\ell, \tilde{Y}^\ell)$. By Proposition II.6, there then exists an $\ell' \in \mathbb{N}_0$ and a subsequence $(u_{i_j})_{j\in\mathbb{N}}$ together with a sequence $\hat{b}_{i_j} \in (M^\ell)^{\circ\ell'}$ and an element $\hat{u} \in \mathcal{M}((\hat{\pi}_\ell^{\ell'})^*\tilde{X}^\ell, A, J, \mathcal{H}((\hat{\pi}_\ell^{\ell'})^*\tilde{X}^\ell))$, $\hat{b} := \pi_M^{\mathcal{M}}(\hat{u})$ s. t. $\pi_{\mathcal{H}}^{\mathcal{M}}(\hat{u}) = H$, $\pi_\ell^{\ell'}(\hat{b}) = b$ and

$$(\hat{\pi}_\ell^{\ell'})^* u_{i_j} \underset{j\to\infty}{\longrightarrow} \hat{u}.$$

Furthermore, either $\hat{b} \in (M^\ell)^{\circ\ell'}$ and hence $\hat{u}$ defines an element $(u, b, H) \in \overline{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, \mathcal{H}(\tilde{X}^\ell, \tilde{Y}^\ell))$, or there exists a component $(\Sigma^\ell)^{\circ\ell'}_{i,\hat{b}}$ with $\hat{\pi}_\ell^{\ell'}((\Sigma^\ell)^{\circ\ell'}_{i,\hat{b}}) = z$ for some $z \in \Sigma_b^\ell$ either a node or a marked point. Furthermore, $\hat{u}$ defines a nonconstant $J$-holomorphic sphere in $\tilde{X}_z^\ell \cong X$. Because $J \in \mathcal{J}_\omega(X, Y; E)$, the image of this sphere is not contained in $Y$ and intersects $Y$ in at least 3 distinct points, at least one of which is not one of the last $\ell$ marked points $T_j^\ell(b)$. Now proceed literally as in the proof of Proposition 9.5 in [CM07]. □

## III.2 The pseudocycle is well-defined

### III.2.1 A description of the boundary

The next order of business is a description of the boundary $\partial\overline{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H) := \overline{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H) \setminus \mathring{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H)$, where from now on it will be assumed that $D \geq D^*$ with $D^*$ from Lemma III.2 and that $J \in \mathcal{J}_{\omega,\mathrm{ni}}(X, Y; J_0, E)$. As in [IP03], $\partial\overline{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H)$ decomposes into a number of subsets. First of all, the subsets defined analogously to $\mathring{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H)$, but for the remaining strata of $M^\ell$ as in Definition III.1:

$$\begin{aligned}\mathcal{M}^i(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H) := \{u \in \mathcal{M}_b(\tilde{X}^\ell, A, J, H) \mid\ & b \in M^{\ell,i}\\ & u(T_j^\ell(b)) \in \tilde{Y}^\ell,\ j = 1, \ldots, \ell,\\ & \mathrm{im}(u|_{\Sigma_{b,s}}) \cap \tilde{X}^\ell \setminus \tilde{Y}^\ell \neq \emptyset \text{ for}\\ & \text{every component } \Sigma_{b,s} \text{ of } \Sigma_b\}.\end{aligned}$$

Note that $\mathcal{M}^0(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H) = \mathring{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H)$ by the convention $M^{\ell,0} = \mathring{M}^\ell$. This case is the easiest to deal with, because it already has been: From Lemma II.22 and the ensuing discussion leading up to and including Lemma II.26 shows the following lemma ($H + \mathcal{H}^{00}(\tilde{X}^\ell, \tilde{Y}^\ell)$ denotes the affine subspace, for the definition of $\mathcal{H}^{00}(\tilde{X}^\ell, \tilde{Y}^\ell)$ see Definition II.28):

**Lemma III.4.** *Let $H \in \mathcal{H}(\tilde{X})$. Then $\mathcal{M}^i(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H + \mathcal{H}^{00}(\tilde{X}^\ell, \tilde{Y}^\ell))$ is a smooth Banach manifold and the projection*

$$\pi_{\mathcal{H}}^{\mathcal{M}} : \mathcal{M}^i(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H + \mathcal{H}^{00}(\tilde{X}^\ell, \tilde{Y}^\ell)) \to H + \mathcal{H}^{00}(\tilde{X}^\ell, \tilde{Y}^\ell)$$

*is a Fredholm map of index*

$$\mathrm{ind}(\pi_{\mathcal{H}}^{\mathcal{M}}) = \dim_{\mathbb{C}}(X)\chi + 2c_1(A) + \dim_{\mathbb{R}}(M^{\ell,i}) - 2\ell.$$

*Proof.* Let

$$\begin{aligned}\mathcal{V} := \{u \in \mathcal{M}_b(\tilde{X}^\ell, A, J, H') \mid\ & b \in M^{\ell,i}, H' \in H + \mathcal{H}^{00}(\tilde{X}^\ell, \tilde{Y}^\ell)\\ & \mathrm{im}(u|_{\Sigma_{b,s}}) \cap \tilde{X}^\ell \setminus \tilde{Y}^\ell \neq \emptyset \text{ for}\\ & \text{every component } \Sigma_{b,s} \text{ of } \Sigma_b\}.\end{aligned}$$

By Lemma II.22 and the lemmas up to and including Lemma II.26, $\mathcal{V}$ is a Banach manifold and the map

$$\mathrm{ev}^{R^\ell,T^\ell}|_{\mathcal{V}} : \mathcal{V} \to (R_1^\ell)^*\tilde{X}^\ell \oplus \cdots \oplus (R_n^\ell)^*\tilde{X}^\ell \oplus (T_1^\ell)^*\tilde{X}^\ell \oplus \cdots \oplus (T_\ell^\ell)^*\tilde{X}^\ell$$

is a submersion. So by Lemma A.6 and Lemma A.7,

$$\mathcal{M}^i(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H + \mathcal{H}^{00}(\tilde{X}^\ell, \tilde{Y}^\ell)) = \\ \left(\mathrm{ev}^{R^\ell, T^\ell}|_{\mathcal{V}}\right)^{-1}\left((R_1^\ell)^*\tilde{X}^\ell \oplus \cdots \oplus (R_n^\ell)^*\tilde{X}^\ell \oplus (T_1^\ell)^*\tilde{Y}^\ell \oplus \cdots \oplus (T_\ell^\ell)^*\tilde{Y}^\ell\right)$$

and $\pi_{\mathcal{H}}^{\mathcal{M}}$ have the properties stated. □

The next type of holomorphic sections in $\partial\overline{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H)$ is given by those sections $u$ in $\overline{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H)$ that do not satisfy the conditions defining the $\mathcal{M}^i(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H)$. The condition $\mathrm{im}(u \circ T_j^\ell) \subseteq \tilde{Y}^\ell \ \forall j = 1, \ldots, \ell$ is clearly closed. So that leaves those sections with one or more components ending up in $\tilde{Y}^\ell$.

### III.2.2 Reduction to the case of vanishing homology classes

Although probably not strictly necessary, the first goal is to show that one can choose $H$ s. t. every component of a section in $\overline{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H)$ with image contained in $\tilde{Y}^\ell$ needs to represent a vanishing homology class. Assuming $A \neq 0$, this in particular implies that no section over a smooth curve in $\overline{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H)$ has image contained in $\tilde{Y}^\ell$. The way this will be proved is by following the line of argument in [CM07] leading up to Proposition 8.11. First, the analogue of Lemma 8.9 in [CM07]:

**Lemma III.5.** *Let $\Sigma$ be a fixed smooth Riemann surface equipped with a compatible volume form $\mathrm{dvol}_\Sigma$ s. t. $\mathrm{vol}_\Sigma(\Sigma) = 1$, let $(X, \omega)$ be a closed symplectic manifold, $\hat{X} := \Sigma \times X$, let $J$ be an $\omega$-compatible almost complex structure and let $H \in \mathcal{H}(\hat{X})$ be a hamiltonian perturbation with corresponding almost complex structure $\hat{J}^H$ on $\hat{X}$. Let $u = (\mathrm{id}, u_2) : \Sigma \to \hat{X}$ be a $\hat{J}^H$-holomorphic section of $\hat{X}$ with $[u_2] = A \in H_2(W; \mathbb{Z})$. Then for constants $\rho, \kappa \geq 0$ there exists a constant $D_* = D_*(X, \omega, J, \rho, \kappa)$ s. t. if $\|H\|_{C^1} < \rho$ and $R_H < \kappa$, where $R_H : \hat{X} \to \mathbb{R}$ is s. t. $R_H \mathrm{dvol}_\Sigma$ is the curvature of the connection defined by $H$, then $\langle c_1(TW), A\rangle \leq D_*(\omega(A) + \kappa)$.*

*Proof.* Let $H$ be as in the statement of the lemma. Then by [MS04], $\hat{\omega}_\kappa := \mathrm{pr}_2^*\omega + \mathrm{pr}_1^*(\kappa \mathrm{dvol}_\Sigma)$ is a symplectic form on $\hat{X}$ s. t. $\hat{J}^H$ is $\hat{\omega}_\kappa$-compatible. Now proceed as in the proof of Lemma 8.9 in [CM07]: Let $\alpha \in \Omega^2(X)$ be a closed 2-form that represents $c_1(TX)$. Then

$$\begin{aligned} \langle c_1(TX), A\rangle &= \int_\Sigma u_2^*\alpha \\ &= \int_\Sigma u^*\mathrm{pr}_2^*\alpha \\ &\leq \|\mathrm{pr}_2^*\alpha\|_{\hat{\omega}_\kappa, \hat{J}^H} \int_\Sigma u^*\hat{\omega}_\kappa \end{aligned}$$

as in op. cit., because $u$ is $\hat{J}^H$-holomorphic and $\hat{J}^H$ is $\hat{\omega}_\kappa$-compatible

$$\begin{aligned}
&= \|\mathrm{pr}_2^*\alpha\|_{\hat{\omega}_\kappa,\hat{J}^H}\left(\int_\Sigma u_2^*\omega + \kappa\right)\\
&= \|\mathrm{pr}_2^*\alpha\|_{\tilde{\omega}_\kappa,J_H}(\omega(A)+\kappa).
\end{aligned}$$

$\|\mathrm{pr}_2^*\alpha\|_{\hat{\omega}_\kappa,\hat{J}^H}$ here denotes the norm w.r.t. the metric defined by $\hat{\omega}_\kappa$ and $\hat{J}^H$. The claim now follows, because $\|\mathrm{pr}_2^*\alpha\|_{\hat{\omega}_\kappa,\hat{J}^H}$ depends continuously on $\kappa$ and on $\hat{J}^H$, which in turns depends continuously on the $C^1$-norm of $H$, and coincides with $\|\alpha\|_{\omega,J}$ for $H=0$. □

Now consider one of the open subsets $U_j^{\ell,i} \subseteq M^{\ell,i}$. Assume that $U_j^{\ell,i}$ is connected and to simplify notation, drop the indices $i$ and $j$, i.e. assume that $(\rho^\ell = \mathrm{pr}_1 : \hat{S}^\ell := U^\ell \times S^\ell \to U^\ell, \hat{R}^\ell, \hat{T}^\ell, \iota^\ell : U^\ell \to M^\ell, \hat{\iota}^\ell : U^\ell \times S^\ell \to \Sigma^\ell)$ is a desingularisation, where $S^\ell$ is a smooth 2-dimensional manifold. Also, let $N_r^{\ell,1}, N_r^{\ell,2} : U^\ell \to \hat{S}^\ell$ be sections parametrising the nodal points. Denote by $S_i^\ell$, $i=1,\dots,s$, the connected components of $S^\ell$ and correspondingly $\hat{S}_i^\ell := U^\ell \times S_i^\ell$. Then any

$$\begin{aligned}
u' \in \overline{\mathcal{M}}_b(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, \mathcal{H}(\tilde{X}^\ell,\tilde{Y}^\ell)) &:= \\
&\overline{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, \mathcal{H}(\tilde{X}^\ell,\tilde{Y}^\ell)) \cap \mathcal{M}_b(\tilde{X}^\ell, A, J, \mathcal{H}(\tilde{X}^\ell,\tilde{Y}^\ell))
\end{aligned}$$

for $b \in \iota(U^\ell)$ pulls back to a $u \in \mathcal{M}(\hat{X}^\ell, A, J, \mathcal{H}(\tilde{X}^\ell,\tilde{Y}^\ell))$, where $\hat{X}^\ell := (\hat{\iota}^\ell)^*\tilde{X}^\ell$. Denoting $\hat{X}_i^\ell := \hat{X}^\ell|_{\hat{S}_i^\ell}$ and correspondingly $\hat{Y}_i^\ell := \hat{Y}^\ell|_{\hat{S}_i^\ell}$, one can identify

$$\begin{aligned}
\mathcal{M}(\hat{X}^\ell, A, J, \mathcal{H}(\tilde{X}^\ell,\tilde{Y}^\ell)) \cong \coprod_{\sum_{i=1}^s A_i = A} &\mathcal{M}(\hat{X}_1^\ell, A_1, J, \mathcal{H}(\tilde{X}^\ell,\tilde{Y}^\ell)) \times \cdots \\
&\cdots \times \mathcal{M}(\hat{X}_s^\ell, A_s, J, \mathcal{H}(\tilde{X}^\ell,\tilde{Y}^\ell))
\end{aligned}$$

by mapping $u \in \mathcal{M}(\hat{X}^\ell, A, J, \mathcal{H}(\tilde{X}^\ell,\tilde{Y}^\ell))$ to

$$\begin{aligned}
(u|_{S_1^\ell}, \dots, u|_{S_s^\ell}) \in \mathcal{M}(\hat{X}_1^\ell, [\mathrm{pr}_2(u|_{S_1^\ell})], J, \mathcal{H}(\tilde{X}^\ell,\tilde{Y}^\ell)) \times \cdots \\
\cdots \times \mathcal{M}(\hat{X}_s^\ell, [\mathrm{pr}_2(u|_{S_s^\ell})], J, \mathcal{H}(\tilde{X}^\ell,\tilde{Y}^\ell)).
\end{aligned}$$

Denoting $u_i := u|_{S_i^\ell}$, $u'$ as above having a component lying in $\tilde{Y}^\ell$ then means that its pullback $u$ has one of its components $u_i$ lying in

$$\mathcal{M}(\hat{Y}_i^\ell, [\mathrm{pr}_2(u_i)], J, \mathcal{H}(\tilde{X}^\ell,\tilde{Y}^\ell)) \subseteq \mathcal{M}(\hat{X}_i^\ell, [\mathrm{pr}_2(u_i)], J, \mathcal{H}(\tilde{X}^\ell,\tilde{Y}^\ell)).$$

One would now like to reproduce the argument in [CM07], Proposition 8.11 (a), to show that for generic $H$, $\mathcal{M}(\hat{Y}_i^\ell, A_i, J, H) = \emptyset$ for $D$ large enough. Doing just that, by Lemma II.26, $\pi_{\mathcal{H}}^{\mathcal{M}} : \mathcal{M}(\hat{Y}_i^\ell, A_i, J, \mathcal{H}(\tilde{X}^\ell,\tilde{Y}^\ell)) \to \mathcal{H}(\tilde{X}^\ell,\tilde{Y}^\ell)$ has Fredholm index given by

$$\begin{aligned}
\mathrm{ind}(\pi_{\mathcal{H}}^{\mathcal{M}}) &= \dim_{\mathbb{C}}(Y)\chi(S_i^\ell) + 2c_1^{TY}(A_i) + \dim_{\mathbb{R}}(U^\ell)\\
&\le 2\dim_{\mathbb{C}}(Y) + 2(c_1^{TX}(A_i) - D\omega(A_i)) + \dim_{\mathbb{R}}(M^\ell)\\
&\le 2\dim_{\mathbb{C}}(Y) + 2D_*\kappa + 2(D_* - D)\omega(A_i) + \dim_{\mathbb{R}}(M^\ell),
\end{aligned}$$

where $D_*$ and $\kappa$ are as in Lemma III.5. But $\dim_{\mathbb{R}}(M^\ell) = \dim(M) + 2\ell = \dim(M) + 2[Y] \cdot A = \dim(M) + 2D\omega(A)$, choosing $\ell = [Y] \cdot A$ to satisfy Lemma III.3. So while the middle term in the above index formula decreases with increasing $D$, the last term increases just as quickly, at least for $A_i = A$. This is a case that one definitely would like to deal with in this way. But observe that if $S_i^\ell$ is a component of genus zero (a sphere) and $H \in \mathcal{H}(\tilde{X}^\ell, \tilde{Y}^\ell)$ satisfies $(\hat{\iota}^\ell)^* H|_{\hat{S}_i^\ell} \equiv 0$, then any $u_i \in \mathcal{M}(\hat{Y}_i^\ell, A_i, J, H)$ defines a $J$-holomorphic sphere in $Y$. But for $J \in \mathcal{J}(X, Y; E)$, the only such spheres are the constant ones, implying $A_i = 0$. This allows for the following construction, which first of all requires the introduction of quite a bit of notation to signify the two parts of a curve in the family $\Sigma^\ell|_{U^\ell}$ that lie in $\tilde{Y}^\ell$ and those that intersect $\tilde{X}^\ell \setminus \tilde{Y}^\ell$:

1. Let $I$ be the index set for the connected components of $S^\ell$, i.e. $S^\ell = \coprod_{i \in I} S_i^\ell$ and let $I = I_X \coprod I_Y$ be an arbitrary decomposition of $I$ into two subsets.
2. Then $S^\ell = S^{\ell,X} \amalg S^{\ell,Y}$, where $S^{\ell,X} := \coprod_{i \in I_X} S_i^\ell$ and $S^{\ell,Y} := \coprod_{i \in I_Y} S_i^\ell$. Correspondingly $\hat{S}^\ell = \hat{S}^{\ell,X} \amalg \hat{S}^{\ell,Y} = \left(\coprod_{i \in I_X} \hat{S}_i^\ell\right) \amalg \left(\coprod_{i \in I_Y} \hat{S}_i^\ell\right)$.
3. Denote by $\Sigma_{U^\ell}^\ell$ the restriction of $\Sigma^\ell$ to $U^\ell$ and by $\Sigma_{U^\ell}^{\ell,X}$ and $\Sigma_{U^\ell}^{\ell,Y}$ the image of $\hat{S}^{\ell,X}$ and $\hat{S}^{\ell,Y}$ under $\hat{\iota}^\ell$, respectively, so that $\Sigma_{U^\ell}^\ell = \Sigma_{U^\ell}^{\ell,X} \amalg \Sigma_{U^\ell}^{\ell,Y}$.
4. Denote by $\chi^X$ and $\chi^Y$ the Euler characteristics of the fibres of $\Sigma_{U^\ell}^{\ell,X}$ and $\Sigma_{U^\ell}^{\ell,Y}$, respectively.
5. Let $\{1, \ldots, \ell\} = K_X \amalg K_Y$ be the decomposition s. t. $T_j^\ell(b) \in \Sigma_{U^\ell}^{\ell,X}$ for all $j \in K_X$ and $b \in U^\ell$ and $T_j^\ell(b) \in \Sigma_{U^\ell}^{\ell,Y}$ for all $j \in K_Y$ and $b \in U^\ell$.
6. Among the nodal points on $\hat{S}^\ell$, there is a subset of those pairs, where one of the two points corresponding to a node lies on $\hat{S}^{\ell,X}$ and the other lies on $\hat{S}^{\ell,Y}$. Denote these by $N_r^{\ell,XY,X}, N_r^{\ell,XY,Y}$, $r = 1, \ldots, d$, the first one lying on $\hat{S}^{\ell,X}$, the second one on $\hat{S}^{\ell,Y}$.
7. Denote by $N_r^{\ell,Y,1}, N_r^{\ell,Y,2}$, $r = 1, \ldots, d'$, the nodal points where both lie on $\hat{S}^{\ell,Y}$.
8. Regard both $\Sigma_{U^\ell}^{\ell,X}$ and $\Sigma_{U^\ell}^{\ell,Y}$ as families of nodal Riemann surfaces with marked points $((T_j^\ell)_{j \in K_X}, (N_r^{\ell,XY,X})_{r=1,\ldots,d})$ and $((T_j^\ell)_{j \in K_Y}, (N_r^{\ell,XY,Y})_{r=1,\ldots,d})$, respectively.

Now fix some $b \in U^\ell$. Under $\hat{\pi}_0^\ell \circ \hat{\iota}^\ell|_{\hat{S}_b^{\ell,Y}} : \hat{S}_b^{\ell,Y} \to \Sigma_b$, a certain number of genus zero components of $\hat{S}_b^{\ell,Y}$ are mapped to points. This happens if and only if a component contains fewer than three special points apart from the $\hat{T}_j^\ell$, i.e. fewer than three nodal points or marked points among the $\hat{R}_j^\ell(b)$. These can be grouped together into "collapsed subtrees" as in Section 2 in [CM07] in the following way: Call two components of $\hat{S}_b^{\ell,Y}$ connected if there exists an $r$ s. t. $N_r^{\ell,Y,1}(b)$ lies on one of them, $N_r^{\ell,Y,2}(b)$ on the other. Now take the equivalence relation this generates on the set of components of $\hat{S}_b^{\ell,Y}$ on which $\hat{\pi}_0^\ell$ is constant. Since $U^\ell$ was assumed to be connected, this is independent of $b \in U^\ell$. An equivalence class of this equivalence relation then corresponds to a collapsed subtree.

9. Denote the set of equivalence classes from above by $\mathcal{C}$. This gives a decomposition $I_Y = I_{Y,0} \amalg \coprod_{C\in\mathcal{C}} I_{Y,C}$, s. t. $\hat{\pi}_0^\ell\left(\coprod_{i\in I_{Y,C}} \hat{S}_i^{\ell,Y}\right) = \text{const}$, for every $C \in \mathcal{C}$, and $\hat{\pi}_0^\ell|_{\hat{S}_{i,b}^\ell}$ is a biholomorphic map onto its image for every $i \in I_{Y,0}$, $b \in U^\ell$.
10. $\mathcal{C}$ can be further decomposed into subsets $\mathcal{C}_0$ and $\mathcal{C}_1$, where every $C \in \mathcal{C}_0$ has the property that there exists at least one (and at most two) $i \in I_{Y,C}$ s. t. for every $b \in U$, $\hat{S}_{i,b}^{\ell,Y}$ is connected to $\hat{S}_{j,b}^{\ell,Y}$ for some $j \in I_{Y,0}$ and $\mathcal{C}_1 := \mathcal{C} \setminus \mathcal{C}_0$. Let $\hat{S}^{\ell,Y,0}$ be the subfamily of $\hat{S}^{\ell,Y}$ consisting of the components in $I_{Y,0} \cup \coprod_{C\in\mathcal{C}_0} I_{Y,C}$.
11. Denote by $\Sigma_{U^\ell}^{\ell,Y,0}$ the image of $\hat{S}^{\ell,Y,0}$ in $\Sigma_{U^\ell}^\ell$ under $\hat{\imath}^\ell$, by $\chi_0^Y$ the Euler characteristic of the fibres of $\Sigma_{U^\ell}^{\ell,Y,0}$ and denote by $U$ the open subset of the stratum of $M$ to which $U^\ell$ gets mapped under $\pi_{-1}^{\ell-1}$.
12. Then $\hat{\pi}_0^\ell$ is a well-defined map from $\Sigma_{U^\ell}^{\ell,Y,0}$ to a subfamily of $\Sigma_U$ (the restriction of $\Sigma$ to $U$), which will be denoted by $\Sigma_U^Y$ and has fibres of Euler characteristic $\chi_0^Y$ as well.

One can now for any $B \in H_2(Y)$ look at the moduli spaces $\mathcal{M}(\tilde{Y}|_{\Sigma_U^Y}, B, J, \mathcal{H}(\tilde{Y}))$, which are equipped with the smooth structure from Lemma II.26. The calculation from before then shows that the Fredholm index of the projection $\pi_{\mathcal{H}}^{\mathcal{M}} : \mathcal{M}(\tilde{Y}|_{\Sigma_U^Y}, B, J, \mathcal{H}(\tilde{Y})) \to \mathcal{H}(\tilde{Y})$ can be bounded from above by

$$\dim_{\mathbb{C}}(Y)\chi_0^Y + 2D_*\kappa + 2(D_* - D)\omega(B) + \dim_{\mathbb{R}}(M).$$

In particular, taking a bound for $\chi_0^Y$ depending only on $g$ and $n$, there is a constant $D_0$ only depending on $g$, $n$ and $D_*$ but not depending on $\ell$ s. t. for $D \geq D_0$ this is negative, provided that $B \neq 0$, due to integrality of $\omega$. So from now on assume that $D \geq D_0$. Also, due to the choices made, one has an isomorphism

$$\mathcal{M}(\tilde{Y}^\ell|_{\Sigma_{U^\ell}^{\ell,Y,0}}, B, J, (\hat{\pi}_0^\ell)^*\mathcal{H}(\tilde{Y})) \cong (\pi_0^\ell)^*\mathcal{M}(\tilde{Y}|_{\Sigma_U^Y}, B, J, \mathcal{H}(\tilde{Y})).$$

This means that by the Sard-Smale theorem there is a generic subset of $\mathcal{H}(\tilde{Y})$ s. t. for every $H$ in this subset, if $B \neq 0$, then

$$\mathcal{M}(\tilde{Y}|_{\Sigma_U^Y}, B, J, H) = \mathcal{M}(\tilde{Y}^\ell|_{\Sigma_{U^\ell}^{\ell,Y,0}}, B, J, (\hat{\pi}_0^\ell)^*H) = \emptyset$$

and if $B = 0$, then $\mathcal{M}(\tilde{Y}^\ell|_{\Sigma_{U^\ell}^{\ell,Y,0}}, 0, J, (\hat{\pi}_0^\ell)^*H)$ is a smooth manifold of dimension $\dim_{\mathbb{C}}(Y)\chi_0^Y + \dim_{\mathbb{R}}(U^\ell)$ that comes with a canonical map to the manifold

$\mathcal{M}(\tilde{Y}|_{\Sigma_U^Y}, 0, J, H)$ of dimension $\dim_{\mathbb{C}}(Y)\chi_0^Y + \dim_{\mathbb{R}}(U)$. Analogously, for $C \in \mathcal{C}_1$, let $\hat{S}^{\ell,Y,C} := \coprod_{i \in I_{Y,C}} \hat{S}_i^{\ell,Y}$ be the subfamily of $\hat{S}^{\ell,Y}$ consisting of the components in $I_{Y,C}$ and $\Sigma_{U^\ell}^{\ell,Y,C}$ its image in $\Sigma^\ell$ under $\hat{\iota}$. Then for any $H \in \mathcal{H}(\tilde{Y})$, for $B \neq 0$ again $\mathcal{M}(\tilde{Y}^\ell|_{\Sigma_{U^\ell}^{\ell,Y,C}}, B, J, (\hat{\pi}_0^\ell)^* H) = \emptyset$ and for $B = 0$,

$$\mathcal{M}(\tilde{Y}^\ell|_{\Sigma_{U^\ell}^{\ell,Y,C}}, 0, J, (\hat{\pi}_0^\ell)^* H) \cong \tilde{Y}^\ell|_{U^\ell} \cong (\hat{\pi}_0^\ell)^*(\tilde{Y}|_U),$$

the isomorphism given by evaluation at any special point on $\Sigma_{U^\ell}^{\ell,Y,C}$. Note that the Euler characteristic $\chi_C^Y$ of any fibre of $\hat{S}^{\ell,Y,C}$ is 2. So

$$\dim_{\mathbb{R}}(\mathcal{M}(\tilde{Y}^\ell|_{\Sigma_{U^\ell}^{\ell,Y,C}}, 0, J, (\hat{\pi}_0^\ell)^* H)) = \dim_{\mathbb{C}}(Y)\chi_C^Y + \dim_{\mathbb{R}}(U^\ell).$$

Finally, one can take the intersection of all the generic subsets one gets via the construction above, for all the countably many choices of data as above (i. e. $U^\ell$, $I_X$ and $I_Y$, $B \in H_2(Y)$, and so on), to get a generic subset $\mathcal{H}_{\mathrm{reg}}(\tilde{Y}) \subseteq \mathcal{H}(\tilde{Y})$. So finally, one can summarise the results from this subsection:

**Lemma III.6.** *There exists an integer $D_0$ depending only on $g$, $n$ and $D_*$ s. t. for $D \geq D_0$ there exists a generic subset $\mathcal{H}_{\mathrm{reg}}(\tilde{Y}) \subseteq \mathcal{H}(\tilde{Y})$ with the property that for every $H \in \mathcal{H}_{\mathrm{reg}}(\tilde{Y})$ and for any choice of data $U^\ell, I_X, I_Y$ as above, for $0 \neq B \in H_2(Y)$,*

$$\mathcal{M}(\tilde{Y}^\ell|_{\Sigma_{U^\ell}^{\ell,Y}}, B, J, (\hat{\pi}_0^\ell)^* H) = \emptyset$$

*and $\mathcal{M}(\tilde{Y}^\ell|_{\Sigma_{U^\ell}^{\ell,Y}}, 0, J, (\hat{\pi}_0^\ell)^* H)$ is a smooth manifold diffeomorphic to*

$$(\pi_0^\ell)^* \left( \mathcal{M}(\tilde{Y}|_{\Sigma_U^Y}, 0, J, H) \amalg \coprod_{C \in \mathcal{C}_1} Y|_U \right)$$

*and hence of dimension*

$$\dim_{\mathbb{R}} \left( \mathcal{M}(\tilde{Y}^\ell|_{\Sigma_{U^\ell}^{\ell,Y}}, 0, J, (\hat{\pi}_0^\ell)^* H) \right) = \dim_{\mathbb{C}}(Y)\chi^Y + \dim_{\mathbb{R}}(U^\ell).$$

*Furthermore this manifold comes with the smooth evaluation map*

$$\mathrm{ev}^{N^{\ell,XY,Y}} : \mathcal{M}(\tilde{Y}^\ell|_{\Sigma_{U^\ell}^{\ell,Y}}, 0, J, (\hat{\pi}_0^\ell)^* H) \to \bigoplus_{r=1}^{d} \left( N_r^{\ell,XY,Y} \right)^* \tilde{X}^\ell.$$

### III.2.3 Construction of the manifolds covering the boundary

From now on, pick some $H \in \mathcal{H}_{\mathrm{reg}}(\tilde{Y})$ and use the inclusion from Lemma II.32 to find an $H^Y \in \mathcal{H}_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J)$ restricting to $(\hat{\pi}_0^\ell)^* H$ along $\tilde{Y}^\ell$.
One now has to turn ones attention to the second part $\Sigma_{U^\ell}^{\ell,X}$ of $\Sigma_{U^\ell}^\ell$. Noting

that for $V := \tilde{X}^\ell \setminus \tilde{Y}^\ell$, $\mathcal{H}^V(\tilde{X}^\ell) = \mathcal{H}^{00}(\tilde{X}^\ell, \tilde{Y}^\ell)$, by Lemma II.26, the moduli space $\mathcal{M}(\tilde{X}^\ell|_{\Sigma^{\ell,X}_{U^\ell}}, A, J, H^Y + \mathcal{H}^{00}(\tilde{X}^\ell, \tilde{Y}^\ell))$ comes with a submersion

$$\mathrm{ev}^{T^\ell}_{K_X} \times \mathrm{ev}^{N^{\ell,XY,X}} : \mathcal{M}(\tilde{X}^\ell|_{\Sigma^{\ell,X}_{U^\ell}}, A, J, H^Y + \mathcal{H}^{00}(\tilde{X}^\ell, \tilde{Y}^\ell)) \to \bigoplus_{j\in K_X} \left(T^\ell_j\right)^* \tilde{X}^\ell \oplus \bigoplus_{r=1}^{d} \left(N^{\ell,XY,X}_r\right)^* \tilde{X}^\ell.$$

Then

$$\begin{aligned}&\mathrm{ev}^{T^\ell}_{K_X} \times \mathrm{ev}^{N^{\ell,XY,X}} \times \mathrm{ev}^{N^{\ell,XY,Y}} :\\ &\quad\mathcal{M}(\tilde{X}^\ell|_{\Sigma^{\ell,X}_{U^\ell}}, A, J, H^Y + \mathcal{H}^{00}(\tilde{X}^\ell, \tilde{Y}^\ell)) \times \mathcal{M}(\tilde{Y}^\ell|_{\Sigma^{\ell,Y}_{U^\ell}}, 0, J, (\hat{\pi}^\ell_0)^* H) \to\\ &\qquad \bigoplus_{j\in K_X} \left(T^\ell_j\right)^* \tilde{X}^\ell \oplus \bigoplus_{r=1}^{d} \left(\left(N^{\ell,XY,X}_r\right)^* \tilde{X}^\ell \oplus \left(N^{\ell,XY,Y}_r\right)^* \tilde{X}^\ell\right)\end{aligned} \tag{III.1}$$

is transverse to $\bigoplus_{j\in K_X} \left(T^\ell_j\right)^* \tilde{Y}^\ell \oplus \bigoplus_{r=1}^d \Delta$, where $\Delta$ denotes the diagonal. So

$$\begin{aligned}&\mathcal{M}_{I_X,I_Y}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H^Y + \mathcal{H}^{00}(\tilde{X}^\ell, \tilde{Y}^\ell)) :=\\ &\quad\left(\mathrm{ev}^{T^\ell}_{K_X} \times \mathrm{ev}^{N^{\ell,XY,X}} \times \mathrm{ev}^{N^{\ell,XY,Y}}\right)^{-1} \left(\bigoplus_{j\in K_X} \left(T^\ell_j\right)^* \tilde{Y}^\ell \oplus \bigoplus_{r=1}^{d} \Delta\right)\end{aligned} \tag{III.2}$$

is a split submanifold of codimension $\dim_{\mathbb{R}}(U^\ell){+}2|K_X|{+}2d\dim_{\mathbb{C}}(X)$ by Lemma A.6 and by Lemma A.7,

$$\pi^{\mathcal{M}}_{\mathcal{H}} : \mathcal{M}_{I_X,I_Y}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H^Y + \mathcal{H}^{00}(\tilde{X}^\ell, \tilde{Y}^\ell)) \to H^Y + \mathcal{H}^{00}(\tilde{X}^\ell, \tilde{Y}^\ell)$$

is a Fredholm map of index

$$\begin{aligned}\mathrm{ind}(\pi^{\mathcal{M}}_{\mathcal{H}}) &= \dim_{\mathbb{C}}(X)\chi^X + 2c_1(A) + \dim_{\mathbb{R}}(U^\ell) +\\ &\quad+\ \dim_{\mathbb{C}}(Y)\chi^Y + \dim_{\mathbb{R}}(U^\ell) -\\ &\quad-\ (\dim_{\mathbb{R}}(U^\ell) + \dim_{\mathbb{C}}(X)2d + 2|K_X|)\\ &= \dim_{\mathbb{C}}(X)\chi^X + \dim_{\mathbb{C}}(X)\chi^Y - \dim_{\mathbb{C}}(X)2d + 2c_1(A) +\\ &\quad+\ \dim_{\mathbb{R}}(U^\ell) - 2|K_X| - \chi^Y\\ &= \dim_{\mathbb{C}}(X)\chi + 2c_1(A) + \dim_{\mathbb{R}}(U^\ell) - 2|K_X| - \chi^Y.\end{aligned} \tag{III.3}$$

So again by the Sard-Smale theorem there exists a generic subset of $\mathcal{H}^{00}(\tilde{X}^\ell, \tilde{Y}^\ell)$ s. t. for any $H^{00}$ in this subset, $\mathcal{M}_{I_X,I_Y}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H^Y + H^{00})$ is a smooth manifold of dimension $\mathrm{ind}(\pi^{\mathcal{M}}_{\mathcal{H}})$. The above can now be done for every $U^\ell = U^{\ell,i}_j$ for $i \neq 0$, and all partitions $I_X, I_Y$ of the set of components of a fibre of $\Sigma^\ell_{U^\ell}$ and one can take the intersection of the generic subsets of $\mathcal{H}^{00}(\tilde{X}^\ell, \tilde{Y}^\ell)$ from above to get a perturbation $H^{00}$ of $H^Y$ s. t. $\mathcal{M}_{I_X,I_Y}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H^Y + H^{00})$ is a smooth manifold of the dimension above for all $U^\ell$ and $I_X, I_Y$ as above. Also note that the Banach manifolds from Lemma III.4 are actually the special case of the above for $I_Y = \emptyset$. These sets, for all $U^\ell$ and $I_X, I_Y$ now cover $\partial\overline{M}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H^Y + H^{00})$. Unfortunately, this does not suffice to show that $\mathring{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H^Y + H^{00})$ is a pseudocycle, because by formula III.3, the dimensions of the above manifolds covering the boundary are not of small enough dimension, i. e. real dimension at least 2 less than that of $\mathring{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H^Y + H^{00})$. Comparing the dimension formula III.3 with the formula from Lemma III.4 (in the case $i = 0$), the failure of this is due to two effects:

1. The Euler characteristic $\chi^Y$ of the fibres of $\Sigma^{\ell,Y}_{U^\ell}$ might be strictly negative, so the term $-\chi^Y$ in the formula above contributes positively to the dimension.
2. It can happen that $|K_X| < \ell$, e. g. the case $|K_X| = 0$ (all the marked points $T^\ell_j$ lie on components that are mapped completely into $\tilde{Y}^\ell$) and $d = 1$ can't be ruled out.

To deal with the first problem, if one denotes by $g_i$ the genus of the component $S^\ell_i$ of the surface $S^\ell$, the desingularisation of a fibre of $\Sigma^\ell_{U^\ell}$, then

$$\chi^Y = \sum_{i \in I_Y} 2(1 - g_i) - 2d'.$$

Two kinds of terms contribute negatively to $\chi^Y$, the term $d'$ and the terms $2(1 - g_i)$. The term $2d'$ is not an issue, because the codimension of $U^\ell$ in $M^\ell$ is given by two times the total number of nodes in a fibre of $\Sigma^\ell_{U^\ell}$, which is at least $d' + d$. And one can assume $d \geq 1$, for $A \neq 0$, because by construction of $H$, no curve in the family $\Sigma^\ell_{U^\ell}$ is mapped completely into $\tilde{Y}^\ell$. Also, if $g_i$ is zero or one, then $2(1 - g_i) \geq 0$, so the corresponding term contributes non-negatively to $\chi^Y$. This leaves the components of genus $g_i \geq 2$. Remember that the contribution of these to the dimension formula arises in the following way: $\mathcal{M}(\tilde{Y}^\ell|_{\Sigma^{\ell,Y}_{U^\ell}}, 0, J, (\hat{\pi}^\ell_0)^* H)$ is the preimage in $\mathcal{M}(\hat{Y}^\ell|_{\hat{S}^{\ell,Y}}, 0, J, (\hat{\pi}^\ell_0)^* H)$, whereas $\hat{Y}^\ell$ always is the pullback of $\tilde{Y}^\ell$ to $\hat{S}^\ell$, of the diagonal under the evaluation map at the nodal points,

$$\mathrm{ev}^{N^{\ell,Y,1}, N^{\ell,Y,2}} : \mathcal{M}(\hat{Y}^\ell|_{\hat{S}^{\ell,Y}}, 0, J, (\hat{\pi}^\ell_0)^* H) \to \bigoplus_{r=1}^{d'} \left(N^{\ell,Y,1}_r\right)^* \hat{Y}^\ell \oplus \left(N^{\ell,Y,2}_r\right)^* \hat{Y}^\ell.$$

Because $\hat{S}^{\ell,Y}$ is the disjoint union of the $\hat{S}^{\ell,Y}_i$ for $i \in I_Y$, $\mathcal{M}(\hat{Y}^\ell|_{\hat{S}^{\ell,Y}}, 0, J, (\hat{\pi}^\ell_0)^* H)$ is the fibre product over $U^\ell$ of terms $\mathcal{M}(\hat{Y}^\ell|_{\hat{S}^{\ell,Y}_i}, 0, J, (\hat{\pi}^\ell_0)^* H)$ for $i \in I_Y$, each of which has dimension

$$\begin{aligned}\dim_{\mathbb{R}}\left(\mathcal{M}(\hat{Y}^\ell|_{\hat{S}^{\ell,Y}_i}, 0, J, (\hat{\pi}^\ell_0)^* H)\right) &= \dim_{\mathbb{C}}(Y) 2(1 - g_i) + \dim_{\mathbb{R}}(U^\ell) \\ &= \dim_{\mathbb{C}}(X) 2(1 - g_i) + \dim_{\mathbb{R}}(U^\ell) - 2(1 - g_i).\end{aligned}$$

The important point now (which will be proved in the remainder of this section) is that the image of $\overline{\mathcal{M}}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H')$, for any $H' \in \mathcal{H}(\tilde{X}^\ell, \tilde{Y}^\ell)$ that restricts to $(\hat{\pi}_0^\ell)^* H$ on $\tilde{Y}^\ell$, in $\mathcal{M}(\hat{Y}^\ell|_{\hat{S}_i^{\ell,Y}}, 0, J, (\hat{\pi}_0^\ell)^* H)$ under the restriction can be covered by countably many manifolds of codimension $-2(1 - g_i) + 2$. The proof of this uses a refined compactness result, of the type that, among others, has been studied in [BEH$^+$03] and in [IP03] (which, as is stated in the introduction of [BEH$^+$03], is a special case of the "stretching of the neck" construction in [BEH$^+$03]). But in the following I will use a different transversality result from [IP03]. The setup of the formulation of the compactnes results above is actually quite involved and will never be used in full generality in this text. So instead of reciting the whole story, I will only describe a corollary of this, which sums up the results as needed in the following. To do so, first observe that for any $u \in \mathcal{M}_b(\hat{Y}^\ell|_{\hat{S}_i^{\ell,Y}}, 0, J, (\hat{\pi}_0^\ell)^* H)$, one can form the complex line bundle $u^*(V\hat{Y}_b^\ell)^{\perp_\omega}$ over the Riemann surface $\hat{S}_{i,b}^{\ell,Y}$. If now $H' \in \mathcal{H}_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J)$ is so that it restricts to $H^Y$ along $\tilde{Y}^\ell$, then by Corollary II.8, the operator (as usual for some $kp > 2$)

$$\overline{D}_{i,u}^{H_b'} := \pi_{(V\hat{Y}_b^\ell)^{\perp_\omega}}^{V\hat{X}_b^\ell} \circ \left( D\overline{\partial}_{\hat{S}_{i,b}^{\ell,Y}}^{J_b, H_b'} \right)_u :$$
$$L^{k,p}(u^*(V\hat{Y}_b^\ell)^{\perp_\omega}) \to L^{k-1,p}(\overline{\mathrm{Hom}}_{(j_b,J_b)}(T\hat{S}_{i,b}^{\ell,Y}, u^*(V\hat{Y}_b^\ell)^{\perp_\omega}))$$

is complex linear. By the Koszul-Malgrange integrability theorem, this means that $u^*(V\hat{Y}_b^\ell)^{\perp_\omega}$ is actually (can be identified with) a holomorphic line bundle over $\hat{S}_{i,b}^{\ell,Y}$ with $\overline{D}_{i,u}^{H_b'}$ as Cauchy-Riemann operator. Since $[\mathrm{pr}_2(u)] = 0 \in H_2(Y)$, the bundle $u^*(V\hat{Y}_b^\ell)^{\perp_\omega}$ has vanishing first Chern class and it follows that every meromorphic section of this bundle has the same order of poles as of zeroes. The compactness result from [BEH$^+$03] or [IP03] then implies the following: Let $(u_j)_{j\in\mathbb{N}} \subseteq \mathcal{M}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H')$ be a sequence that converges to $u \in \overline{\mathcal{M}}_b(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H')$, where $b \in U^\ell$. Assume that there are $I_X, I_Y$ as above s. t. the components of $\Sigma_b^\ell$ in $I_X$ intersect $\tilde{X}^\ell \setminus \tilde{Y}^\ell$ nontrivially and the components of $\Sigma_b^\ell$ in $I_Y$ are mapped into $\tilde{Y}^\ell$. Assume that $I_Y \neq \emptyset$. Then there exist the following:

1. An integer $k \in \mathbb{N}$ and partitions $I_Y = I_Y^{-k} \amalg I_Y^{-k+1} \amalg \cdots \amalg I_Y^{-1}$ as well as $K_Y = K_Y^{-k} \amalg \cdots \amalg K_Y^{-1}$.
2. Up to reordering of the nodes $N_r^{\ell,Y,1}, N_r^{\ell,Y,2}$, i. e. reordering of the index set $\{1, \dots, d'\}$ and exchanging $N_r^{\ell,Y,1}$ and $N_r^{\ell,Y,2}$ for a fixed $r$, a partition $\{1, \dots, d'\} = D^{-k} \amalg \cdots \amalg D^{-2} \amalg E^{-k} \amalg \cdots \amalg E^{-1}$, $D^{-1} := \{1, \dots, d\}$.
3. For every $j = -k, \dots, -1$, $r \in D^j$, an integer $p_r^j \in \mathbb{N}$.

For these, the following hold:

a) For all $j = -k, \dots, -1$, $r \in K_Y^j$, $T_r^\ell$ lies on $\hat{S}_i^{\ell,Y}$ for some $i \in I_Y^j$.

b) For all $j = -k, \dots, -1$, $r \in E^j$, there are $i, i' \in I_Y^j$ s. t. $N_r^{\ell,Y,1}$ lies on $\hat{S}_i^{\ell,Y}$ and $N_r^{\ell,Y,2}$ lies on $\hat{S}_{i'}^{\ell,Y}$.

c) For all $j = -k, \dots, -2$, $r \in D^j$, $N_r^{\ell,Y,1}$ lies on $\hat{S}_i^{\ell,Y}$ for some $i \in I_Y^j$ and $N_r^{\ell,Y,2}$ lies on $\hat{S}_i^{\ell,Y}$ for some $i \in I_Y^{j+1}$. The $N_r^{\ell,XY,Y}$, $r \in D^{-1}$, all lie on $\hat{S}_i^{\ell,Y}$ for some $i \in I_Y^{-1}$.

d) For each $j = -k, \dots, -1$, denote $\hat{S}_{I_Y^j}^{\ell,Y} := \coprod_{i \in I_Y^j} \hat{S}_i^{\ell,Y}$ and by $u_{I_Y^j}$ the restriction of $u$ to $\hat{S}_{I_Y^j,b}^{\ell,Y}$. Analogously, denote by $u_{I_X}$ the restriction of $u$ to $\hat{S}_b^{\ell,X}$. Then there exists a meromorphic section $\xi_u^j$ of $u_{I_Y^j}^*(V\hat{Y}_b^\ell)^{\perp_\omega}$ with the following properties:

   i) For all $j = -k, \dots, -1$, $\xi_u^j$ has simple zeroes at the points $T_r^\ell$ for $r \in K_Y^j$.
   ii) For all $j = -k, \dots, -2$, $r \in D^j$, $\xi_u^{j+1}$ has a zero of order $p_r^j$ at the point $N_r^{\ell,Y,2}$.
   iii) For all $j = -k, \dots, -2$, $r \in D^j$, $\xi_u^j$ has a pole of order $p_r^j$ at the point $N_r^{\ell,Y,1}$.
   iv) For every $r \in D^{-1}$, $\xi_u^{-1}$ has a pole of order $p_r^{-1}$ at the point $N_r^{\ell,XY,Y}$.
   v) Other than the above, the $\xi_u^j$ have no zeroes or poles.
   vi) For every $r \in D^{-1}$, $u_{I_X}$ is tangent to $\tilde{Y}^\ell$ to order $p_r^{-1} - 1$ at $N_r^{\ell,XY,X}$.

Note that the above gives a countable number of choices: For the integer $k$, the partitions $I_Y^*$, $K_Y^*$ and $D^* \amalg E^*$ and the orders of the zeros and poles $p_*^*$ of the $\xi_u^j$. Also note that as remarked above, every $\xi_u^j$ has the same (total) order of zeroes as of poles, i. e. at each level $j$ the total order of zeroes of $\xi_u^j$ is given by the total order of poles of $\xi_u^{j-1}$, plus the number of marked points $T_r^\ell$ and this is the same as the total order of poles of $\xi_u^j$:

$$\sum_{r \in D^{-k}} p_r^j = |K_Y^{-k}|$$

$$\sum_{r \in D^j} p_r^j = |K_Y^j| + \sum_{r \in D^{j-1}} p_r^{j-1} \quad \forall\, j = -k+1, \dots, -1$$

In particular, the total order of poles of $\xi_u^{-1}$ is given by $|K_Y|$.

This partially solves the problem in formula III.3 above of the term $|K_X|$ being smaller than $\ell$: In the definition of $\mathcal{M}_{I_X,I_Y}(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H^Y + \mathcal{H}^{00}(\tilde{X}^\ell, \tilde{Y}^\ell))$, the part $\mathcal{M}(\tilde{X}^\ell|_{\Sigma_{U^\ell}^{\ell,X}}, A, J, H^Y + \mathcal{H}^{00}(\tilde{X}^\ell, \tilde{Y}^\ell))$ corresponding to the part of a curve that does not get mapped into $\tilde{Y}^\ell$, with the zero order matching condition at the nodal points $N^{\ell,XY,X}$, see formulas III.1 and III.2, can now be replaced by the subspace of those curves that also are tangent to $\tilde{Y}^\ell$ at the $N^{\ell,XY,X}$ of total order given by $|K_Y|$, see Lemma II.34.

**Lemma III.7.** *Let $u \in \overline{\mathcal{M}}_b(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H')$ for some $b \in U^\ell$, $H' \in \mathcal{H}_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J)$. Then there exist $I_X, I_Y, p_*^*$ as above s. t. the restriction $u|_{\Sigma_{U^\ell,b}^{\ell,X}}$ lies in*

$$\mathcal{M}_b^V(\tilde{X}^\ell|_{\Sigma_{U^\ell}^{\ell,X}}, (\tilde{Y}^\ell|_{\Sigma_{U^\ell}^{\ell,X}})^{(0,\dots,0,p_1^{-1},\dots,p_d^{-1})}, A, J, H'),$$

*for* $V := \tilde{X}^\ell \setminus \tilde{Y}^\ell$, *with* $\sum_{r=1}^d p_r^{-1} = |K_Y| - d$.
*Here* $\Sigma^{\ell,X}_{U^\ell}$ *has marked points* $T^\ell_j$, $j \in K_X$ *and* $N^{\ell,XY,X}_r$, $r = 1, \dots, d$.

Furthermore, it also allows for a solution of the problem of the terms $2(1-g_i)$ for $g_i \geq 2$ contributing negatively to the Euler characteristic $\chi^Y$:
For every $U^\ell$, $I_X$ and $I_Y$, $k \in \mathbb{N}$, partitions $I^*_Y$, $K^*_Y$, $D^*$, $E^*$ and integers $p^*_*$ as above, consider the family of Riemann surfaces $\rho : P \to B$, where the base $B$ is given by $\mathcal{M}(\tilde{Y}^\ell|_{\Sigma^{\ell,Y}_{U^\ell}}, 0, J, (\hat{\pi}^\ell_0)^* H)$ and the family $P$ of (disconnected smooth) Riemann surfaces over $B$ is given by $(\pi^{\mathcal{M}}_{U^\ell})^* \hat{S}^{\ell,Y}$. Fibrewise deleting the nodal points $N^{\ell,Y,1}_r$ and $N^{\ell,Y,2}_r$ for $r \in D^j$ as well as the $T^\ell_r$ for $r \in K^j_Y$, where applicable, gives a family of punctured Riemann surfaces $\dot{\rho} : \dot{P} \to B$. Over $P$ and by restriction over $\dot{P}$, there is a complex line bundle $Z \to P$, where for $u \in B$, $Z|_{P_u} = (u_{I_Y})^*(V\hat{Y}^\ell)^{\perp_\omega}$. The complex structure is given by the restriction of $J$ to $(V\hat{Y}^\ell)^{\perp_\omega}$ and is compatible with the restriction of $\omega$. By abuse of notation, both these structures will be denoted by $J$ and $\omega$ again. This complex line bundle can hence be regarded as a symplectic fibre bundle with real 2-dimensional fibres and deleting the zero-section also gives a symplectic fibre bundle $\dot{Z} \to \dot{P}$. An important property of this bundle is that it comes with a free action of $(\mathbb{C}^*)^{I_Y}$ ($\mathbb{C}^* := \mathbb{C} \setminus \{0\}$) on the fibres of $\dot{Z}$. For $u \in B$, $\pi^{\mathcal{M}}_{U^\ell}(u) = b$, the $i$-th component of $(\mathbb{C}^*)^{I_Y}$ acts fibrewise on $(u_i)^*(V\hat{Y}^\ell)^{\perp_\omega}$, $u_i := u|_{\hat{S}^{\ell,Y}_{i,b}}$. The restriction of the $\xi^j_u$ as above to the components of $P_u$ then defines a section of $\dot{Z}$ over $\dot{P}_u$. Finally, this bundle also comes with a connection, induced by the Levi-Civita connection on $V\hat{X}^\ell$. Next, observe that the operator $\overline{D}^{H'_b}_{i,u}$ above is a complex linear Cauchy-Riemann operator in the sense of [MS04], Appendix C.1: Let $u \in \mathcal{M}_b(\hat{Y}^\ell|_{\hat{S}^{\ell,Y}_i}, 0, J, (\pi^\ell_0)^* H)$ for any component given by $i \in I_Y$, where $H' \in \mathcal{H}_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J)$ restricts to $H^Y$ along $\tilde{Y}^\ell$. Then because $J \in \mathcal{J}_{\omega,\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell)$, $H' \in \mathcal{H}_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell)$, by Lemma II.17, for a section $\xi$ of $u^*(V\hat{Y}^\ell)^{\perp_\omega}$, $Z \in T\hat{S}^{\ell,Y}_{i,b}$,

$$(\overline{D}^{H'_b}_{i,u}\xi)(Z) = \pi^{V\hat{X}^\ell_b}_{(V\hat{Y}^\ell_b)^{\perp_\omega}} \left( \nabla^{0,1}_Z \xi - K_{\hat{J}H'}(\xi, Du(Z)) \right). \tag{III.4}$$

These observations allow one to define a Fredholm problem whose solutions are the meromorphic sections from above. Basically, one now goes through the steps of the previous chapter. What was the bundle $\pi : \Sigma \to B$ there now is the bundle $\rho : \dot{P} \to B$, and what was the bundle $\tilde{X} \to \Sigma$ there now is the bundle $\dot{Z} \to \dot{P}$. For the most part, the noncompactness of $\dot{P}$ and $\dot{Z}$ is not a problem, for the analytical results in Section II.2 never referenced any compactness but only required universal bounds for the different curvatures and their covariant derivatives, as well as for the covariant derivatives for the smooth curves around which the charts are defined. These clearly are still satisfied, for the latter see the definition below. One just has to be more careful in the definition of the (linear) Sobolev spaces, see e.g. [Loc81] and [Loc87], but as long as the definition is such that the usual embedding theorems, elliptic estimates for the Cauchy-Riemann operator, etc., hold, the details are not that important. Also remember that $\hat{S}^\ell \to U^\ell$ was a (topologically but not holomorphically) trivial bundle and that there are tubular neighbourhoods of all the marked points and nodal points on which the trivialisation preserves the complex structure in the fibres of $\hat{S}^\ell$. This allows one to use the SFT Fredholm theory from [Cie06]. To this end, denote pullbacks to $P$ of the $N_r^{\ell,Y,1}$, $N_r^{\ell,Y,2}$, for $r \in D^j$, and $T_r^\ell$ for $r \in K_Y^j$ for some $j$, by $N_r^+$, $r \in I^+$, and $N_r^-$, $r \in I^-$, the latter incorporating the $T_r^\ell$, for index sets $I^\pm$. Furthermore, denote by $p_r^+ \in \mathbb{N}$, $r \in I^+$, and $p_r^- \in \mathbb{N}$, $r \in I^-$, the orders of poles and zeroes at the $N^+$ and $N^-$, respectively. For this part of the discussion, the matching conditions from above on the poles and zeroes are irrelevant. Because $\hat{S}^\ell \to U^\ell$ was holomorphically trivial in a neighbourhood of all the nodes, and hence so is $P \to B$, one can pick holomorphic coordinates defined on $[0,\infty) \times S^1$ to punctured disk neighbourhoods $D_r^\pm$ of the $N_r^\pm$ in $\dot{P}$ that are preserved under the (smooth) identification of the fibres of $\dot{P}$. Denote the resulting maps $\sigma_r^\pm : B \times [0,\infty) \times S^1 \to \dot{P}$. Also note that as was remarked above, by the Koszul-Malgrange integrability theorem (and because everything extends from a punctured disk to a disk), for every $b \in B$, over $D_{r,b}^\pm$ one can find a holomorphic trivialisation of $\dot{Z}|_{D_{r,b}^\pm}$ with fibre $E_{r,b}^\pm \cong \mathbb{C} \setminus \{0\}$, i. e. $\dot{Z}|_{D_{r,b}^\pm} \cong D_{r,b}^\pm \times E_{r,b}^\pm$. This gives maps $\overline{\sigma}_r^\pm : B \times ([0,\infty) \times S^1) \times (\mathbb{C} \setminus \{0\}) \to \dot{Z}$ covering the maps $\sigma_r^\pm$. If $[0,\infty) \times S^1 \to \mathbb{D} \setminus \{0\}$, $(s,\theta) \mapsto e^{-(s+i\theta)}$ is the standard identification, then a zero or pole of order $p$ that is given in standard coordinates on $\mathbb{D}$ by $z \mapsto cz^p$, for some $c = e^{-(a+i\vartheta)} \in \mathbb{C}$, under this identification is the map $(s,\theta) \mapsto ce^{-p(s+i\theta)} = e^{-(ps+a+i(p\theta+\vartheta))}$. Fix some $b \in B$, $l \in \mathbb{N}$, $q > 1$ with $lq > 2$, and a weight $\delta > 0$. Furthermore, fix a smooth function $s : \dot{P} \to (0,\infty)$ that in all the coordinates $\sigma_r^\pm$ from above is given by the projection onto the factor $[0,\infty)$. Then for any metric vector bundle with connection $E \to \dot{P}_b$, one can define the weighted Sobolev space

$$L^{l,q,\delta}(E) := \{\eta \in L^{l,q}_{\mathrm{loc}}(E) \mid e^{\delta s}\eta \in L^{l,q}(E)\}.$$

With these choices, let (cf. the first definition in Section 3 of [Cie06])

$$\begin{aligned}\mathcal{B}_b := \{\xi : \dot{P}_b \to \dot{Z}_b \mid \xi \text{ a section of } \dot{Z}_b \to \dot{P}_b \text{ of class } L^{l,q}_{\mathrm{loc}} \text{ s.t. } \forall\, r \in I^\pm,\\ &(\mathrm{pr}_2 \circ (\overline{\sigma}_{r,b}^\pm)^{-1} \circ \xi \circ \sigma_{r,b}^\pm)(s,\theta) :\\ &(s,\theta) \mapsto e^{-((t(s,\theta)-(p_r^\pm s+a_r^\pm))+i(\varphi(s,\theta)-(p_r^\pm\theta+\vartheta_r^\pm)))}\\ &\in L^{l,q,\delta}([0,\infty) \times S^1, \mathbb{C}) \text{ for some}\\ &(a_r^\pm, \vartheta_r^\pm) \in [0,\infty) \times S^1 \text{ and for all } r\}.\end{aligned}$$

This is a Banach manifold, that around a smooth $\xi \in \mathcal{B}_b$ is modelled on the Banach space $L^{l,q,\delta}(\xi^* V\dot{Z}_b)$, just with the Sobolev spaces used in the previous chapter replaced by the weighted Sobolev spaces. Analogously to the situation in the previous chapter there is then also a Banach space bundle $\mathcal{E}_b$ over $\mathcal{B}_b$, with fibre

$$(\mathcal{E}_b)_\xi := L^{l-1,q,\delta}(\overline{\mathrm{Hom}}_{(j_b,J_b)}(T\dot{P}_b, \xi^* V\dot{Z}_b))$$

over $\xi \in \mathcal{B}_b$. $\mathcal{E}_b \to \mathcal{B}_b$ comes with a section $\overline{\nabla}_b$, defined by the Cauchy-Riemann operator from Equation III.4 and for an appropriate choice of $\delta > 0$, this is a Fredholm operator. By definition of $\mathcal{B}_b$, the zero set of $\overline{\nabla}_b$ is given by the meromorphic sections of $Z_b \to P_b$ that have zeros and poles at the $N^-$ and $N^+$, of orders given by the numbers $p^-$ and $p^+$, respectively. Because $\overline{\nabla}_b$ is a linear Cauchy-Riemann operator on $Z_b$, its linearisation $D\overline{\nabla}_b$ in the sense of Section 3 in [Cie06] is (modulo canonical identifications) given by $\overline{\nabla}_b$ itself. In particular the operators $S_i(t)$ in op. cit. vanish identically, and the paths of symplectic matrices $\Phi_i(t)$, as in the same reference, are the constant paths at the identity. By Corollary 3.6 in [Cie06], again for $\delta > 0$ sufficiently small, $\overline{\nabla}_b$ is a Fredholm operator of index $\chi(P_b) = \sum_{i\in I_Y} 2(1-g_i)$ (not $\chi(\dot{P}_b)$), as was expected from the classical Riemann-Roch theorem from the start. Now as in the previous chapter, remembering that $P$ and hence $\dot{P}$ were trivial, one can take the union over all $b \in B$ to get a Banach manifold $\mathcal{B}$, together with a projection to $B$, and a Banach space bundle $\mathcal{E}$ over $\mathcal{B}$, together with a Fredholm section $\overline{\nabla} : \mathcal{B} \to \mathcal{E}$ of index $\sum_{i\in I_Y} 2(1-g_i) + \dim_{\mathbb{R}}(B)$. Remembering that $\overline{\nabla}$ depends on the choice of $H' \in \mathcal{H}_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J)$, whereas $B = \mathcal{M}(\tilde{Y}^\ell|_{\Sigma^{\ell,Y}_{U^\ell}}, 0, J, (\hat{\pi}_0^\ell)^* H)$ and hence $P$ and $Z$ only depend on the restriction of $H'$ to $\tilde{Y}^\ell$. So one can look at the affine subspace $H^Y + \mathcal{H}^0_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J)$ of $\mathcal{H}_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J)$. Making the dependence on $H'$ explicit and writing $\mathcal{B}^{H'}$, $\mathcal{E}^{H'}$ and $\overline{\nabla}^{H'}$, one can then as in the previous chapter look at the spaces

$$\mathcal{B}(H^Y + \mathcal{H}^0_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J)) := \coprod_{H' \in H^Y + \mathcal{H}^0_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J)} \mathcal{B}^{H'}$$

$$\mathcal{E}(H^Y + \mathcal{H}^0_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J)) := \coprod_{H' \in H^Y + \mathcal{H}^0_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J)} \mathcal{E}^{H'}$$

and the map

$$\overline{\nabla}^{\mathcal{H}} := \coprod_{H' \in H^Y + \mathcal{H}^0_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J)} \overline{\nabla}^{H'}$$

Finally, note that because $H' \in \mathcal{H}_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J)$, $\overline{\nabla}^{\mathcal{H}}$ is equivariant w. r. t. the free $(\mathbb{C}^*)^{I_Y}$-actions on $\mathcal{B}(H^Y + \mathcal{H}^0_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J))$ and $\mathcal{E}(H^Y + \mathcal{H}^0_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J))$ induced by the one on $\dot{P}$. Furthermore, the projections to $B$ and $H^Y + \mathcal{H}^0_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J)$ are invariant under this action.

**Definition III.4.** Abbreviate $\mathcal{D} := (U^\ell, I_X, I_Y^*, K_Y^*, D^*, E^*, p_*^*)$.

$$\mathcal{M}^{\mathcal{D}}_Y(\tilde{X}^\ell, \tilde{Y}^\ell, J, H^Y + \mathcal{H}^0_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J)) := (\overline{\nabla}^{\mathcal{H}})^{-1}(0)$$

and for $H^0 \in \mathcal{H}^0_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J)$,

$$\mathcal{M}^{\mathcal{D}}_Y(\tilde{X}^\ell, \tilde{Y}^\ell, J, H^Y + H^0) := (\overline{\nabla}^{H^Y + H^0})^{-1}(0).$$

The proof (not the statement) of Proposition 6.4 in [IP03] shows that $\overline{\nabla}^{\mathcal{H}}$ is transverse to the zero section. One should note here that for any $H \in \mathcal{H}^0_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J)$, $\mathrm{d}H|_{V\tilde{X}^\ell}$ vanishes along $\tilde{Y}^\ell$, because the condition that $H$ lies in $\mathcal{H}(\tilde{X}^\ell, \tilde{Y}^\ell)$ implies that $\mathrm{d}H$ vanishes on $(V\tilde{Y}^\ell)^{\perp_\omega}$ and the condition that

$H$ vanishes along $\tilde{Y}^\ell$ implies that $\mathrm{d}H$ vanishes on $V\tilde{Y}^\ell$. From this it follows that in Formula III.4 for $\overline{\nabla}^{H'}$, the term involving $\nabla^{0,1}$ is independent of $H' \in H^Y + \mathcal{H}^0_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J)$ since it only depends on the restriction of $\mathrm{d}H'$ to $V\tilde{X}^\ell|_{\tilde{Y}^\ell}$. For transversality, the crucial term is the second one, involving $K_{\hat{J}^{H'}}$, i.e. the symmetric part of the morphism $\frac{1}{2}\hat{J}^{H'}(\hat{\nabla}^{H'}\hat{J}^{H'})$. Together with the vanishing of certain components of its antisymmetric part, which is given by the Nijenhuis tensor, to satisfy normal integrability, this gives a number of conditions on the Hessian of $H$ along $\tilde{Y}^\ell$. By the usual line of argument using Lemma A.3.6 in [MS04], the universal moduli space $(\overline{\nabla}^{\mathcal{H}})^{-1}(0)$ hence is a smooth Banach manifold and the projection onto $H^Y + \mathcal{H}^0_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J)$ is a Fredholm map of index $\sum_{i\in I_Y} 2(1-g_i) + \dim_{\mathbb{R}}(B)$. So by the Sard-Smale theorem, for generic $H' \in H^Y + \mathcal{H}^0_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J)$, $(\overline{\nabla}^{H'})^{-1}(0)$ is a smooth manifold of dimension $\sum_{i\in I_Y} 2(1-g_i) + \dim_{\mathbb{R}}(B)$. Also, it comes with a free $(\mathbb{C}^*)^{I_Y}$-action and projection/forgetful map (smooth for the generic $H'$ above) to $\mathcal{M}(\tilde{Y}^\ell|_{\Sigma^{\ell,Y}_{U^\ell}}, 0, J, (\hat{\pi}^\ell_0)^*H)$ invariant under this action.
The above discussion is summed up in the following two lemmas.

**Lemma III.8.** *Let $u \in \overline{\mathcal{M}}_b(\tilde{X}^\ell, \tilde{Y}^\ell, A, J, H^Y + H^0)$ for some $b \in U^\ell$, $H^0 \in \mathcal{H}^0_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J)$. Then there exists $\mathcal{D} = (U^\ell, I_X, I^*_Y, K^*_Y, D^*, E^*, p^*_*)$ as above s. t. the restriction $u|_{\Sigma^{\ell,Y}_{U^\ell,b}}$ lies in the image of the forgetful map of $\mathcal{M}^{\mathcal{D}}_Y(\tilde{X}^\ell, \tilde{Y}^\ell, J, H^Y + H^0)/_{(\mathbb{C}^*)^{I_Y}}$ in $\mathcal{M}(\tilde{Y}^\ell|_{\Sigma^{\ell,Y}_{U^\ell}}, 0, J, (\hat{\pi}^\ell_0)^*H)$.*

**Lemma III.9.** *Given any $H \in \mathcal{H}_{\mathrm{reg}}(\tilde{Y})$, with extension $H^Y \in \mathcal{H}_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J)$ of $(\hat{\pi}^\ell_0)^*H$, there exists a generic subset $\mathcal{H}^0_{\mathrm{reg}}(\tilde{X}^\ell, \tilde{Y}^\ell, J; H^Y)$ of $\mathcal{H}^0_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J)$ s. t. for every $H^0 \in \mathcal{H}^0_{\mathrm{reg}}(\tilde{X}^\ell, \tilde{Y}^\ell, J; H^Y)$ and every choice of $\mathcal{D} = (U^\ell, I_X, I^*_Y, K^*_Y, D^*, E^*, p^*_*)$,*

$$\mathcal{M}^{\mathcal{D}}_Y(\tilde{X}^\ell, \tilde{Y}^\ell, J, H^Y + H^0)/_{(\mathbb{C}^*)^{I_Y}}$$

*is a smooth manifold of dimension*

$$\dim_{\mathbb{R}}\left(\mathcal{M}^{\mathcal{D}}_Y(\tilde{X}^\ell, \tilde{Y}^\ell, J, H^Y + H^0)/_{(\mathbb{C}^*)^{I_Y}}\right) = \dim_{\mathbb{C}}(X)\chi^Y + \dim_{\mathbb{R}}(U^\ell) + 2d' - 2|I_Y|.$$

So finally, again for $\mathcal{D} = (U^\ell, I_X, I^*_Y, K^*_Y, D^*, E^*, p^*_*)$ and $H^Y$ and $H^0$ regular, one can first of all look at the construction in Equations III.1 and III.2, where by abuse of notation the pullback of $\mathrm{ev}^{N^{\ell,XY,Y}}$ to $\mathcal{M}^{\mathcal{D}}_Y(\tilde{X}^\ell, \tilde{Y}^\ell, J, H^Y + H^0)/_{\mathbb{C}^*}$ via the forgetful map is denoted by the same symbol again:

$$\begin{aligned} &\mathrm{ev}^{T^\ell}_{K_X} \times \mathrm{ev}^{N^{\ell,XY,X}} \times \mathrm{ev}^{N^{\ell,XY,Y}} : \\ &\mathcal{M}(\tilde{X}^\ell|_{\Sigma^{\ell,X}_{U^\ell}}, A, J, H^Y + H^0 + \mathcal{H}^{00}(\tilde{X}^\ell, \tilde{Y}^\ell)) \times \mathcal{M}^{\mathcal{D}}_Y(\tilde{X}^\ell, \tilde{Y}^\ell, J, H^Y + H^0)/_{(\mathbb{C}^*)^{I_Y}} \to \\ &\qquad \bigoplus_{j\in K_X} \left(T^\ell_j\right)^* \tilde{X}^\ell \oplus \bigoplus_{r=1}^{d} \left(\left(N^{\ell,XY,X}_r\right)^* \tilde{X}^\ell \oplus \left(N^{\ell,XY,Y}_r\right)^* \tilde{X}^\ell\right) \end{aligned} \tag{III.5}$$

is transverse to $\bigoplus_{j\in K_X}\left(T_j^\ell\right)^*\tilde{Y}^\ell\oplus\bigoplus_{r=1}^d\Delta$, where $\Delta$ denotes the diagonal. So

$$\begin{aligned}&\tilde{\mathcal{M}}^{\mathcal{D}}(\tilde{X}^\ell,\tilde{Y}^\ell,A,J,H^Y+H^0+\mathcal{H}^{00}(\tilde{X}^\ell,\tilde{Y}^\ell)):=\\&\qquad\left(\mathrm{ev}_{K_X}^{T^\ell}\times\mathrm{ev}^{N^{\ell,XY,X}}\times\mathrm{ev}^{N^{\ell,XY,Y}}\right)^{-1}\left(\bigoplus_{j\in K_X}\left(T_j^\ell\right)^*\tilde{Y}^\ell\oplus\bigoplus_{r=1}^d\Delta\right)\end{aligned}\tag{III.6}$$

is a split submanifold of codimension $\dim_{\mathbb{R}}(U^\ell)+2|K_X|+2d\dim_{\mathbb{C}}(X)$ by Lemma A.6 and by Lemma A.7,

$$\pi_{\mathcal{H}}^{\mathcal{M}}:\tilde{\mathcal{M}}^{\mathcal{D}}(\tilde{X}^\ell,\tilde{Y}^\ell,A,J,H^Y+H^0+\mathcal{H}^{00}(\tilde{X}^\ell,\tilde{Y}^\ell))\to H^Y+H^0+\mathcal{H}^{00}(\tilde{X}^\ell,\tilde{Y}^\ell)$$

is a Fredholm map of index

$$\begin{aligned}\pi_{\mathcal{H}}^{\mathcal{M}}&=\dim_{\mathbb{C}}(X)\chi^X+2c_1(A)+\dim_{\mathbb{R}}(U^\ell)+\\&\quad+\ \dim_{\mathbb{C}}(X)\chi^Y+\dim_{\mathbb{R}}(U^\ell)+2d'-2|I_Y|-\\&\quad-\ (\dim_{\mathbb{R}}(U^\ell)+2|K_X|+2d\dim_{\mathbb{C}}(X))\\&=\dim_{\mathbb{C}}(X)\chi+2c_1(A)+\dim_{\mathbb{R}}(U^\ell)+2d'-2|K_X|-2|I_Y|.\end{aligned}$$

By the same reasoning leading to Lemma II.34,

$$\begin{aligned}&\mathcal{M}^{\mathcal{D}}(\tilde{X}^\ell,\tilde{Y}^\ell,A,J,H^Y+H^0+\mathcal{H}^{00}(\tilde{X}^\ell,\tilde{Y}^\ell)):=\\&\qquad\{u\in\tilde{\mathcal{M}}_b^{\mathcal{D}}(\tilde{X}^\ell,\tilde{Y}^\ell,A,J,H^Y+H^0+\mathcal{H}^{00}(\tilde{X}^\ell,\tilde{Y}^\ell))\mid\\&\qquad\qquad b\in U^\ell,\iota(u,\tilde{Y}^\ell|_{\Sigma_b};N_r^{\ell,XY,X}(b))=p_r^{-1}\ \forall\, r=1,\dots,d\}\end{aligned}$$

is a smooth submanifold of real codimension $2\sum_{r=1}^d p_r^{-1}=2(|K_Y|-d)$ and the projection

$$\pi_{\mathcal{H}}^{\mathcal{M}}:\mathcal{M}^{\mathcal{D}}(\tilde{X}^\ell,\tilde{Y}^\ell,A,J,H^Y+H^0+\mathcal{H}^{00}(\tilde{X}^\ell,\tilde{Y}^\ell))\to H^Y+H^0+\mathcal{H}^{00}(\tilde{X}^\ell,\tilde{Y}^\ell)$$

is Fredholm of index ($|K_X|+|K_Y|=\ell$)

$$\mathrm{ind}(\pi_{\mathcal{H}}^{\mathcal{M}})=\dim_{\mathbb{C}}(X)\chi+2c_1(A)+\dim_{\mathbb{R}}(U^\ell)+2d'+2d-2\ell-2|I_Y|.$$

Because $\dim_{\mathbb{R}}(U^\ell)$ is $\dim_{\mathbb{R}}(M^\ell)=\dim_{\mathbb{R}}(M)+2\ell$ minus 2 times the total number of nodes, which is at least $2(d'+d)$,

$$\mathrm{ind}(\pi_{\mathcal{H}}^{\mathcal{M}})\le\dim_{\mathbb{C}}(X)\chi+2c_1(A)+\dim_{\mathbb{R}}(M)-2|I_Y|.$$

So by the Sard-Smale theorem, there exists a generic subset of $\mathcal{H}^{00}(\tilde{X}^\ell,\tilde{Y}^\ell)$ s. t. for every $H^{00}$ in this subset, $\mathcal{M}^{\mathcal{D}}(\tilde{X}^\ell,\tilde{Y}^\ell,A,J,H^Y+H^0+H^{00})$ is a smooth manifold of dimension at most $\dim_{\mathbb{C}}(X)\chi+2c_1(A)+\dim_{\mathbb{R}}(M)-2|I_Y|$. Taking the intersection of all these generic subsets for the countably many choices of $\mathcal{D}$ as well as the countably many generic subsets one gets via the Sard-Smale theorem from Lemma III.4, making all the $\mathcal{M}^i(\tilde{X}^\ell,\tilde{Y}^\ell,A,J,H^Y+H^0+H^{00})$ smooth manifolds of the expected dimension, one gets a generic subset

$$\mathcal{H}^{00}_{\mathrm{reg}}(\tilde{X}^\ell,\tilde{Y}^\ell;H^Y+H^0)\subseteq\mathcal{H}^{00}(\tilde{X}^\ell,\tilde{Y}^\ell).$$

**Theorem III.1.** *There exists an integer $\overline{D}$ s. t. for every $D \geq \overline{D}$ and $A \in H_2(X)$ with $\omega(A) > 0$, $E := \omega(A) + 1$, there exists a symplectic hypersurface $Y \subseteq X$, $\mathrm{PD}(Y) = D[\omega]$, and $J \in \mathcal{J}_{\omega,\mathrm{ni}}(X, Y; E)$ s. t. the following hold:*

1. *Let $\pi^j : \Sigma^j \to M^j$, $j \geq 0$ be as in Subsection III.1.1 and let $\ell := D\omega(A)$. Then there exist*
   - *a generic subset $\mathcal{H}^{\mathrm{reg}}(\tilde{Y})$, identified with a subset of $\mathcal{H}_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J)$ by taking, for $H^Y \in \mathcal{H}_{\mathrm{reg}}(\tilde{Y})$, the image of $(\hat{\pi}_0^\ell)^* H^Y$ in $\mathcal{H}_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J)$ under the inclusion from Lemma II.32.*
   - *For every $H^Y \in \mathcal{H}_{\mathrm{ni}}(\tilde{Y})$, a generic subset $\mathcal{H}^0_{\mathrm{reg}}(\tilde{X}^\ell, \tilde{Y}^\ell, J; H^Y) \subseteq \mathcal{H}^0_{\mathrm{ni}}(\tilde{X}^\ell, \tilde{Y}^\ell, J)$.*
   - *For every $H^Y \in \mathcal{H}_{\mathrm{reg}}(\tilde{Y})$ and $H^0 \in \mathcal{H}^0_{\mathrm{reg}}(\tilde{X}^\ell, \tilde{Y}^\ell, J; H^Y)$ a generic subset $\mathcal{H}^{00}_{\mathrm{reg}}(\tilde{X}^\ell, \tilde{Y}^\ell; H^Y + H^0) \subset \mathcal{H}^{00}(\tilde{X}^\ell, \tilde{Y}^\ell)$.*

   *And, for $H^Y \in \mathcal{H}_{\mathrm{reg}}(\tilde{Y})$, $H^0 \in \mathcal{H}^0_{\mathrm{reg}}(\tilde{X}^\ell, \tilde{Y}^\ell, J; H^Y)$ and $H^{00} \in \mathcal{H}^{00}_{\mathrm{reg}}(\tilde{X}^\ell, \tilde{Y}^\ell; H^Y + H^0)$, $H := H^Y + H^0 + H^{00}$, the pseudocycle from Definition III.2 is well-defined.*

2. *Furthermore, let $Y$ be as above and let $J_t \in \mathcal{J}_\omega(X)$, $t \in \mathbb{R}$, be a family of almost complex structures s. t. $J_t \in \mathcal{J}_{\omega,\mathrm{ni}}(X, Y; E)$ for all $t \in \mathbb{R}$ and $J_t = J_0$ for $t \leq 0$ as well as $J_t = J_1$ for $t \geq 1$. Then for any choice of $H_i^Y \in \mathcal{H}_{\mathrm{reg}}(\tilde{Y})$, $H_i^0 \in \mathcal{H}^0_{\mathrm{reg}}(\tilde{X}^\ell, \tilde{Y}^\ell, J_i; H_i^Y)$ and $H_i^{00} \in \mathcal{H}^{00}_{\mathrm{reg}}(\tilde{X}^\ell, \tilde{Y}^\ell; H_i^Y + H_i^0)$, for $i = 1, 2$, the pseudocycles associated to $Y, J_0, H_0^Y + H_0^0 + H_0^{00}$ and $Y, J_1, H_1^Y + H_1^0 + H_1^{00}$ are rationally cobordant.*
   *In particular, given $Y$ and $J$ as above, the pseudocycle is independent of the choice of Hamiltonian perturbation.*

*Proof.* 1. Is just a summary of the results of this chapter so far.

2. For this consider the marked nodal families $(\pi'^\ell : \Sigma'^\ell \to M'^\ell, R'^\ell)$, where $\Sigma'^\ell := \Sigma^\ell \times \mathbb{R}$, $M'^\ell := M^\ell \times \mathbb{R}$, $\pi'^\ell := \pi^\ell \times \mathrm{id}_{\mathbb{R}}$, $(R'^\ell)_j := (R_j^\ell \times \mathrm{id}_{\mathbb{R}})$. These space are stratified by taking the product of a stratum of the original space with $\mathbb{R}$. Correspondingly, define $\tilde{X}'^\ell := \Sigma'^\ell \times X$ and $\tilde{Y}'^\ell := \Sigma'^\ell \times Y$ so that $J_\cdot$ defines an $\omega$-compatible vertical almost complex structure on $\tilde{X}'^\ell$. But, instead of the spaces $\mathcal{H}(\tilde{Y}')$, $\mathcal{H}^0_{\mathrm{ni}}(\tilde{X}'^\ell, \tilde{Y}'^\ell, J_\cdot)$ and $\mathcal{H}^{00}(\tilde{X}'^\ell, \tilde{Y}'^\ell)$, now consider the spaces

$$\mathcal{H}(\tilde{Y}', H_i^Y) := \left\{ H^Y \in \mathcal{H}(\tilde{Y}') \mid H^Y|_{\tilde{Y} \times \{t\}} = \begin{cases} H_0^Y & t \leq 0 \\ H_1^Y & t \geq 1 \end{cases} \right\}$$

$$\mathcal{H}^0_{\mathrm{ni}}(\tilde{X}'^\ell, \tilde{Y}'^\ell, J, H_i^0) := \left\{ H^0 \in \mathcal{H}^0_{\mathrm{ni}}(\tilde{X}'^\ell, \tilde{Y}'^\ell, J) \mid H^0|_{\tilde{X}^\ell \times \{t\}} = \begin{cases} H_0^0 & t \leq 0 \\ H_1^0 & t \geq 1 \end{cases} \right\}$$

$$\mathcal{H}^{00}(\tilde{X}'^\ell, \tilde{Y}'^\ell, H_i^{00}) := \left\{ H^{00} \in \mathcal{H}^{00}(\tilde{X}'^\ell, \tilde{Y}'^\ell) \mid H^{00}|_{\tilde{X}^\ell \times \{t\}} = \begin{cases} H_0^{00} & t \leq 0 \\ H_1^{00} & t \geq 1 \end{cases} \right\}.$$

These spaces of Hamiltonian perturbations are large enough for all the transversality results to hold, because for $t \leq 0$ or $t \geq 1$, transversality holds by choice of $H_i^Y$, $H_i^0$ and $H_i^{00}$ and for $0 < t < 1$ one is free in the choice of perturbation. For the analogue of Lemma III.6 to hold, one possibly has to replace $\overline{D}$ by $\overline{D}+1$. So for generic choices of perturbations in these spaces, as above, one gets strata-wise cobordisms between the moduli spaces associated to $H_0^Y, H_0^0, H_0^{00}$ and $H_1^Y, H_1^0, H_1^{00}$.

□

Finally, one should say a few words about independence of the remaining choices made. An easy consequence, which follows immediately from the way the pseudocycle was constructed is the following:

**Lemma III.10.** *Let $(\pi : \Sigma \to M, R_*)$, $(\pi' : \Sigma' \to M', R'_*)$ and $(\pi'' : \Sigma'' \to M'', R''_*)$ be orbifold branched coverings of $\overline{M}_{g,n}$ that branch over the Deligne-Mumford boundary and assume that there are morphisms*

$$\begin{tikzcd} & \Sigma \arrow[dl, "\Phi'"'] \arrow[d, "\pi"] \arrow[dr, "\Phi''"] & \\ \Sigma' \arrow[d, "\pi'"'] & M \arrow[dl, "\phi'"] \arrow[dr, "\phi''"'] & \Sigma'' \arrow[d, "\pi''"] \\ M' & & M'' \end{tikzcd}$$

*of marked nodal families. Then the rational pseudocycles from Definition III.2 associated to $(\pi : \Sigma \to M, R_*)$ and $(\pi' : \Sigma' \to M', R'_*)$ , but with all other data as in Subsection III.1.1 the same, are equivalent in the sense that the maps ($\tilde{X}^\ell = \Sigma^\ell \times X$, etc.)*

$$\begin{tikzcd} \overset{\circ\circ}{\mathcal{M}}(\tilde{X}^\ell, A, J, \Phi'^* H) \arrow[r, "\mathrm{ev}^{R^\ell}"] & M \times X^n \arrow[d, "\phi' \times \mathrm{id}"] \\ \overset{\circ\circ}{\mathcal{M}}(\tilde{X}'^\ell A, J, H) \arrow[u, "\Phi'^*"] \arrow[r, "\mathrm{ev}^{R'^\ell}"] & M' \times X^n \end{tikzcd}$$

*induce equivalences between the corresponding pseudocycles.*
*By the analogous statement for $(\pi : \Sigma \to M, R_*)$ and $(\pi'' : \Sigma'' \to M'', R''_*)$, the rational pseudocycles associated to $(\pi' : \Sigma' \to M', R'_*)$ and $(\pi'' : \Sigma'' \to M'', R''_*)$ then are equivalent as well.*

The final choice that has been made and that one would like to show independence of is that of the integer $D \in \mathbb{N}$ and hypersurface $Y$. Since most of this consists of adapting the methods from [CM07], Section 10, by methods that have been presented in this chapter before, I will only present a sequence of steps one has to take to show this.

The first step is to describe in which sense the choice of Donaldson hypersurface and adapted $\omega$-compatible almost complex structure is unique.

**Lemma III.11.** *Let $(Y_i, J_i)$ be Donaldson pairs of degrees $D_i$, $i = 0, 1$. Then there exist*

- *an isotopy $\phi_\cdot : [0,1] \times X \to X$, $\phi_0 = \mathrm{id}$, through symplectomorphisms,*
- *an integer $\overline{D} \in \mathbb{N}$,*
- *a hypersurface $\overline{Y} \subseteq X$ of degree $\overline{D}$,*
- *a path $(\overline{J}_t)_{t\in[0,1]} \subseteq \mathcal{J}_\omega(X)$ s. t. $\overline{Y}$ is approximately $\overline{J}_t$-holomorphic for all $t \in [0,1]$,*
- *a constant $\varepsilon > 0$,*

*s. t. the following hold:*

1. *$\mathcal{J}_{\omega,\mathrm{ni}}(X, \overline{Y}; \overline{J}_t, E) \neq \emptyset$ for all $t \in [0,1]$,*
2. *$\mathcal{J}_{\omega,\mathrm{ni}}(X, Y_0; J_0, E) \cap \mathcal{J}_{\omega,\mathrm{ni}}(X, \overline{Y}; \overline{J}_0, E) \neq \emptyset$ and $Y_0$ and $\overline{Y}$ intersect $\varepsilon$-transversely,*
3. *$\mathcal{J}_{\omega,\mathrm{ni}}(X, \phi_1(Y_1); (\phi_1)_*J_1, E) \cap \mathcal{J}_{\omega,\mathrm{ni}}(X, \overline{Y}; \overline{J}_1, E) \neq \emptyset$ and $\phi_1(Y_1)$ and $\overline{Y}$ intersect $\varepsilon$-transversely.*

Here, $\mathcal{J}_{\omega,\mathrm{ni}}(X, Y_0; J_0, E)$, etc., are as in Lemma III.2. For the proof, use the methods from the proof of Lemma III.2 to adapt Corollary 8.18 in [CM07] and the relevant steps laid out in the proof of Theorem 1.3 in Chapter 10 of that reference.

The remaining steps are then to show that from 1.–3. above it follows that the pseudocycles associated to $(Y_0, J_0)$ and $(Y_1, J_1)$ are equivalent.

First of all one can note that the pseudocycles associated to $(Y_1, J_1)$ and $(\phi_1(Y_1), (\phi_1)_*J_1)$ are equivalent. For $(\phi_1)_*A = A$, since $\phi_1$ is isotopic to the identity, and $\phi_1$ induces a well defined map between the corresponding moduli spaces. Hence one can assume that $\phi_\cdot \equiv \mathrm{id}$.

Next, one shows that it follows from 1. that the pseudocycles associated to $(\overline{Y}, \overline{J}_0)$ and $(\overline{Y}, \overline{J}_1)$ are equivalent. To do so, one chooses a subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$ of $[0,1]$ s. t. there exist $J'_i \in \mathcal{J}_{\omega,\mathrm{ni}}(X, \overline{Y}; \overline{J}_{t_i}, E) \cap \mathcal{J}_{\omega,\mathrm{ni}}(X, \overline{Y}; \overline{J}_{t_{i+1}}, E)$, $i = 0, \ldots, k-1$, and then connects, by Lemma III.2, $J'_i$ and $J'_{i+1}$ by a path in $\mathcal{J}_{\omega,\mathrm{ni}}(X, \overline{Y}; E)$. Theorem III.1 then shows equivalence of the pseudocycles associated to $(\overline{Y}, \overline{J}_0)$ and $(\overline{Y}, \overline{J}_1)$.

Finally, and most difficult, one shows that from 2. in the lemma above follows equivalence of the pseudocycles associated to $(Y_0, J_0)$ and $(\overline{Y}, \overline{J}_0)$, and analogously for $(Y_1, J_1)$ and $(\overline{Y}, \overline{J}_1)$ from 3.
To simplify notation, write from now on $(Y, J)$ for $(Y_0, J_0)$ and $(\overline{Y}, \overline{J})$ for $(\overline{Y}, \overline{J}_0)$. Also, let $\ell := D\omega(A)$, $\overline{\ell} := \overline{D}\omega(A)$ and $\hat{\ell} := \ell + \overline{\ell}$.

Then define $\mathcal{H}(\tilde{X}^{\hat{\ell}}, \tilde{Y}^{\hat{\ell}}, \tilde{\overline{Y}}^{\hat{\ell}}) := \mathcal{H}(\tilde{X}^{\hat{\ell}}, \tilde{Y}^{\hat{\ell}}) \cap \mathcal{H}(\tilde{X}^{\hat{\ell}}, \tilde{\overline{Y}}^{\hat{\ell}})$ and analogously for $\mathcal{H}^{00}(\tilde{X}^{\hat{\ell}}, \tilde{Y}^{\hat{\ell}}, \tilde{\overline{Y}}^{\hat{\ell}})$ as well as with $\hat{\ell}$ replaced by $\ell$. With these definitions, $(\hat{\pi}^{\hat{\ell}}_{\ell})^*\mathcal{H}(\tilde{X}^{\ell}, \tilde{Y}^{\ell}, \tilde{\overline{Y}}^{\ell}) \subseteq \mathcal{H}(\tilde{X}^{\hat{\ell}}, \tilde{Y}^{\hat{\ell}}, \tilde{\overline{Y}}^{\hat{\ell}})$ and analogously for $\mathcal{H}^{00}$.
Then, one considers the moduli spaces

$$\begin{aligned}\mathring{\mathcal{M}}(\tilde{X}^{\hat{\ell}}, \tilde{Y}^{\hat{\ell}}, \tilde{\overline{Y}}^{\hat{\ell}}, A, J, H') := \{u \in \mathcal{M}(\tilde{X}^{\hat{\ell}}|_{\mathring{M}^{\hat{\ell}}}, A, J, H') \mid\ & \mathrm{im}(u \circ T^{\hat{\ell}}_j) \subseteq \tilde{Y}^{\hat{\ell}},\ j = 1, \dots, \ell, \\ & \mathrm{im}(u \circ T^{\hat{\ell}}_j) \subseteq \tilde{\overline{Y}}^{\hat{\ell}},\ j = \ell+1, \dots, \hat{\ell}, \\ & \mathrm{im}(u) \cap \tilde{X}^{\hat{\ell}} \setminus (\tilde{Y}^{\hat{\ell}} \cup \tilde{\overline{Y}}^{\hat{\ell}}) \neq \emptyset\}\end{aligned}$$

and defines pseudocycles (yet to be shown to be well-defined for generic choices of $H$ $H'$ and $H''$ as below)

$$\frac{1}{\overline{\ell}!\ell!}\frac{1}{|\operatorname{Aut}(\mathring{\mathring{M}})||\mathcal{O}(\mathring{\mathring{M}})|}\operatorname{ev}^{R^{\hat{\ell}}} : \mathring{\mathring{\mathcal{M}}}(\tilde{X}^{\hat{\ell}}, \tilde{Y}^{\hat{\ell}}, \tilde{\overline{Y}}^{\hat{\ell}}, A, J, H) \to M \times X^n, \tag{III.7}$$

as well as the ones from before,

$$\frac{1}{\ell!}\frac{1}{|\operatorname{Aut}(\mathring{\mathring{M}})||\mathcal{O}(\mathring{\mathring{M}})|}\operatorname{ev}^{R^{\ell}} : \mathring{\mathcal{M}}(\tilde{X}^{\ell}, \tilde{Y}^{\ell}, A, J, H') \to M \times X^n \tag{III.8}$$

and

$$\frac{1}{\overline{\ell}!}\frac{1}{|\operatorname{Aut}(\mathring{\mathring{M}})||\mathcal{O}(\mathring{\mathring{M}})|}\operatorname{ev}^{R^{\overline{\ell}}} : \mathring{\mathcal{M}}(\tilde{X}^{\overline{\ell}}, \tilde{Y}^{\overline{\ell}}, A, J, H'') \to M \times X^n \tag{III.9}$$

for $J \in \mathcal{J}_{\omega,\mathrm{ni}}(X, Y; J, E) \cap \mathcal{J}_{\omega,\mathrm{ni}}(X, \overline{Y}; \overline{J}, E)$, $H \in \mathcal{H}(\tilde{X}^{\hat{\ell}}, \tilde{Y}^{\hat{\ell}}, \tilde{\overline{Y}}^{\hat{\ell}})$, $H' \in \mathcal{H}(\tilde{X}^{\ell}, \tilde{Y}^{\ell}, \tilde{\overline{Y}}^{\ell})$ and $H'' \in \mathcal{H}(\tilde{X}^{\overline{\ell}}, \tilde{Y}^{\overline{\ell}}, \tilde{\overline{Y}}^{\overline{\ell}})$.
It is important to note here that the factors $\frac{1}{|\operatorname{Aut}(\mathring{\mathring{M}})||\mathcal{O}(\mathring{\mathring{M}})|}$ in all three cases coincide, i. e. do not depend $\ell$, $\overline{\ell}$ and $\hat{\ell}$.

Compactness of the closures of $\mathcal{M}(\tilde{X}^{\hat{\ell}}, A, J, H)$, $\mathcal{M}(\tilde{X}^{\ell}, A, J, H)$ and $\mathcal{M}(\tilde{X}^{\overline{\ell}}, A, J, H)$, respectively, has already been shown in the latter two cases and in the former, $\mathring{\mathcal{M}}(\tilde{X}^{\hat{\ell}}, \tilde{Y}^{\hat{\ell}}, \tilde{\overline{Y}}^{\hat{\ell}}, A, J, H) \subseteq \mathcal{M}(\tilde{X}^{\hat{\ell}}, \tilde{\overline{Y}}^{\hat{\ell}}, A, J, H)$ by definition, which in turn has compact closure in $\mathcal{M}(\tilde{X}^{\hat{\ell}}, A, J, H)$. To be more exact w. r. t. the last statement, one may note that $\Sigma^{\hat{\ell}}$ and $M^{\hat{\ell}}$ are canonically identified with $(\Sigma^{\ell})^{\overline{\ell}}$ and $(M^{\ell})^{\overline{\ell}}$, respectively, so one can literally apply Lemma III.3, where $(\pi : \Sigma \to M)$ is replaced by $(\pi^{\ell} : \Sigma^{\ell} \to M^{\ell})$.

Then as before, the proof being the same as that of Lemma III.1, one can show that given any $H'_0 \in \mathcal{H}(\tilde{X}^{\ell}, \tilde{Y}^{\ell}, \tilde{\overline{Y}}^{\ell})$ for generic $H' \in \mathcal{H}^{00}(\tilde{X}^{\ell}, \tilde{Y}^{\ell}, \tilde{\overline{Y}}^{\ell})$,

$$\mathring{\mathcal{M}}(\tilde{X}^{\hat{\ell}}, \tilde{Y}^{\hat{\ell}}, \tilde{\overline{Y}}^{\hat{\ell}}, A, J, (\hat{\pi}^{\hat{\ell}}_{\ell})^*(H'_0 + H')) \to \mathring{\mathcal{M}}(\tilde{X}^{\ell}, \tilde{Y}^{\ell}, A, J, H'_0 + H')$$

is an $\overline{\ell}!$-sheeted covering outside a subset of codimension at least 2.
Because this is symmetric in $Y$ and $\overline{Y}$, in the same way given any $H''_0 \in$

$\mathcal{H}(\tilde{X}^{\bar\ell}, \tilde{Y}^{\bar\ell}, \tilde{\overline{Y}}^{\bar\ell})$ for generic $H'' \in \mathcal{H}^{00}(\tilde{X}^{\bar\ell}, \tilde{Y}^{\bar\ell}, \tilde{\overline{Y}}^{\bar\ell})$,

$$\mathring{\mathcal{M}}(\tilde{X}^{\hat\ell}, \tilde{Y}^{\hat\ell}, \tilde{\overline{Y}}^{\hat\ell}, A, J, (\hat\pi^{\hat\ell}_{\bar\ell})^*(H_0'' + H'')) \to \mathring{\mathcal{M}}(\tilde{X}^{\bar\ell}, \tilde{Y}^{\bar\ell}, A, J, H_0'' + H'')$$

is an $\ell!$-sheeted covering outside a subset of codimension at least 2.

The difficulty then is to show the following two things:
First, that for generic $H \in \mathcal{H}(\tilde{X}^{\hat\ell}, \tilde{Y}^{\hat\ell}, \tilde{\overline{Y}}^{\hat\ell})$, III.7 is a well-defined pseudocycle and that for every two choices of generic $H$, one can find a generic path connecting them that gives a cobordism between the corresponding pseudocycles. And second, that the statements in Theorem III.1 also hold when restricting to Hamiltonian perturbations from $\mathcal{H}(\tilde{X}^{\ell}, \tilde{Y}^{\ell}, \tilde{\overline{Y}}^{\ell})$.

Once these two statements have been shown, one can finish the proof of equivalence of the pseudocycles III.8 and III.9:
First, choose generic $H' \in \mathcal{H}(\tilde{X}^{\ell}, \tilde{Y}^{\ell}, \tilde{\overline{Y}}^{\ell})$ and $H'' \in \mathcal{H}(\tilde{X}^{\bar\ell}, \tilde{Y}^{\bar\ell}, \tilde{\overline{Y}}^{\bar\ell})$ s. t. III.8 and III.9 define pseudocycles and s. t. (possibly after perturbing further by elements of $\mathcal{H}^{00}(\tilde{X}^{\ell}, \tilde{Y}^{\ell}, \tilde{\overline{Y}}^{\ell})$ and $\mathcal{H}^{00}(\tilde{X}^{\bar\ell}, \tilde{Y}^{\bar\ell}, \tilde{\overline{Y}}^{\bar\ell})$, respectively)

$$\mathring{\mathcal{M}}(\tilde{X}^{\hat\ell}, \tilde{Y}^{\hat\ell}, \tilde{\overline{Y}}^{\hat\ell}, A, J, (\hat\pi^{\hat\ell}_{\ell})^* H') \to \mathring{\mathcal{M}}(\tilde{X}^{\ell}, \tilde{Y}^{\ell}, A, J, H')$$

and

$$\mathring{\mathcal{M}}(\tilde{X}^{\hat\ell}, \tilde{Y}^{\hat\ell}, \tilde{\overline{Y}}^{\hat\ell}, A, J, (\hat\pi^{\hat\ell}_{\bar\ell})^* H'') \to \mathring{\mathcal{M}}(\tilde{X}^{\bar\ell}, \tilde{Y}^{\bar\ell}, A, J, H'')$$

define $\bar\ell!$- and $\ell!$-sheeted coverings, respectively. In particular for the two choices $H = (\hat\pi^{\hat\ell}_{\ell})^* H', (\hat\pi^{\hat\ell}_{\bar\ell})^* H''$, III.7 defines pseudocycles, since the corresponding evaluation maps factor through those of III.8 and III.9. But $(\hat\pi^{\hat\ell}_{\ell})^* H'$ and $(\hat\pi^{\hat\ell}_{\bar\ell})^* H''$ may not be generic choices in general, but only for the stratum over $\mathring{M}^{\hat\ell}$. Now connect $(\hat\pi^{\hat\ell}_{\ell})^* H'$ and $(\hat\pi^{\hat\ell}_{\bar\ell})^* H''$ by a path $(H_t)_{t\in[0,1]}$ in $\mathcal{H}(\tilde{X}^{\hat\ell}, \tilde{Y}^{\hat\ell}, \tilde{\overline{Y}}^{\hat\ell})$ with $H_0 = (\hat\pi^{\hat\ell}_{\ell})^* H'$ and $H_1 = (\hat\pi^{\hat\ell}_{\bar\ell})^* H''$ that induces a cobordism $\coprod_{t\in[0,1]} \mathring{\mathcal{M}}(\tilde{X}^{\hat\ell}, \tilde{Y}^{\hat\ell}, \tilde{\overline{Y}}^{\hat\ell}, A, J, H_t)$ between $\mathring{\mathcal{M}}(\tilde{X}^{\hat\ell}, \tilde{Y}^{\hat\ell}, \tilde{\overline{Y}}^{\hat\ell}, A, J, (\hat\pi^{\hat\ell}_{\ell})^* H')$ and $\mathring{\mathcal{M}}(\tilde{X}^{\hat\ell}, \tilde{Y}^{\hat\ell}, \tilde{\overline{Y}}^{\hat\ell}, A, J, (\hat\pi^{\hat\ell}_{\bar\ell})^* H'')$ and is generic for $t \in (0,1)$.
I. e. the boundary of $\coprod_{t\in[0,1]} \mathring{\mathcal{M}}(\tilde{X}^{\hat\ell}, \tilde{Y}^{\hat\ell}, \tilde{\overline{Y}}^{\hat\ell}, A, J, H_t)$ is the union of three parts. Namely those parts for $t = 0, 1$, where the evaluation maps factor through the boundary of $\mathring{\mathcal{M}}(\tilde{X}^{\hat\ell}, \tilde{Y}^{\hat\ell}, \tilde{\overline{Y}}^{\hat\ell}, A, J, (\hat\pi^{\hat\ell}_{\ell})^* H')$ and $\mathring{\mathcal{M}}(\tilde{X}^{\hat\ell}, \tilde{Y}^{\hat\ell}, \tilde{\overline{Y}}^{\hat\ell}, A, J, (\hat\pi^{\hat\ell}_{\bar\ell})^* H'')$, which can be covered by manifolds of codimension at least 2. And the part of the boundary for $t \in (0,1)$, which can be covered by manifolds of codimension at least 2 by a generic choice of $(H_t)_{t\in[0,1]}$.
This finishes the proof of the equivalence of the pseudocycles III.8 and III.9, modulo showing the two claims above.

The reason this is more difficult is that in contrast to the situation before, one can only use Hamiltonian perturbations in the smaller sets $\mathcal{H}(\tilde{X}^{\ell}, \tilde{Y}^{\ell}, \tilde{\overline{Y}}^{\ell})$ and

$\mathcal{H}(\tilde{X}^{\hat{\ell}}, \tilde{Y}^{\hat{\ell}}, \tilde{\overline{Y}}^{\hat{\ell}})$, for which the transversality statements from before no longer hold true. This is due to the fact that along the intersection $\tilde{Y}^{\ell} \cap \tilde{\overline{Y}}^{\ell}$ all Hamiltonian perturbations need to be compatible with both $\tilde{Y}^{\ell}$ and $\tilde{\overline{Y}}^{\ell}$. So in the boundary of the pseudocycles one wishes to construct, where before one had to deal with curves that have components lying in $\tilde{Y}^{\ell}$, one now has to deal with curves that have components lying in $\tilde{Y}^{\ell}$, components lying in $\tilde{\overline{Y}}^{\ell}$ and components lying in $\tilde{Y}^{\ell} \cap \tilde{\overline{Y}}^{\ell}$. Before, the dimensions of the corresponding moduli spaces were cut down by the existence of (meromorphic) sections of the normal bundle, which also provided the necessary matching conditions for the tangencies of the part of the curve that does not lie in $\tilde{Y}^{\ell}$. And this still suffices to deal with components in $\tilde{Y}^{\ell}$ and $\tilde{\overline{Y}}^{\ell}$, but do not lie completely in $\tilde{Y}^{\ell} \cap \tilde{\overline{Y}}^{\ell}$. But to deal with components that lie in $\tilde{Y}^{\ell} \cap \tilde{\overline{Y}}^{\ell}$ and to achieve the necessary matching conditions that provide the correct order of tangency of the part of the curve that lies in the complement of $\tilde{Y}^{\ell} \cup \tilde{\overline{Y}}^{\ell}$, a refined compactness theorem as in [Ion11] is needed. Once that is established, the methods used here before should extend in a rather straightforward way and the main challenge should be in keeping the notation in check.

APPENDIX A

# Notation and technical results

## A.1 Notation and basic results on Banach manifolds and -bundles

### A.1.1 Banach manifolds, Banach bundles and tangent spaces

For the following basic results about differentiable maps between normed spaces, see [Wer00], Section III.5.

For Banach spaces $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$, $B(X,Y)$ is the Banach space of bounded linear operators $T : X \to Y$ equipped with the operator norm $\|T\| := \sup\{\frac{\|T(x)\|_Y}{\|x\|_X} \mid 0 \neq x \in X\}$. Their product $(X \times Y, \|\cdot\|_{X\times Y})$ is the Banach space given by the vector space $X \times Y$ equipped with the norm $\|(x,y)\|_{X\times Y} := \max(\|x\|_X, \|y\|_Y)$. With this choice, for another Banach space $(Z, \|\cdot\|_Z)$, the canonical map $B(Z, X \times Y) \to B(Z,X) \times B(Z,Y)$, $f \mapsto (\mathrm{pr}_1 \circ f, \mathrm{pr}_2 \circ f)$, is an isometry.

**Definition A.1.** Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be Banach spaces and let $U \subseteq X$ be an open subset. A map $f : U \to Y$ is called (Fréchet-)differentiable at a point $x_0 \in U$, if there exists a continuous linear operator $Df_x \in B(X,Y)$ s. t. the continuous map $r : U - x \to Y$, where $U - x := \{u - x \mid u \in U\}$, defined by

$$f(x+v) = f(x) + Df_x(v) + r(v), \quad \text{for } v \in U - x$$

satisfies $\lim\limits_{v\to 0} \frac{r(v)}{\|v\|} = 0$.
$f$ is called differentiable if it is differentiable at every point $x \in U$. If the resulting map $Df : U \to B(X,Y)$, $x \mapsto Df_x$, is continuous, then $f$ is called continuously differentiable.

Recursively one defines: $f$ is $n$-times (continuously) differentiable if it is differentiable and $Df$ is $(n-1)$-times (continuously) differentiable.
$f$ is called smooth, if it is $n$-times (continuously) differentiable for all $n \in \mathbb{N}$.
For $n \in \mathbb{N}$, let $B^{(n)}(X,Y) := \{T \in B(\underbrace{X \times \cdots \times X}_{n\text{-times}}, Y) \mid T \text{ multilinear}\}$ be the Banach space of multilinear maps from $X^n$ to $Y$. Then $B(X, B^{(n-1)}(X,Y)) \cong B^{(n)}(X,Y)$ via $T \mapsto ((x, y_1, \ldots, y_{n-1}) \mapsto (T(x))(y_1, \ldots, y_{n-1}))$ and one can inductively define the higher derivatives of a $n$-times differentiable map $f : U \to Y$ as

$$D^n f : U \to B^{(n)}(X,Y)$$

via $D^n f = D(D^{n-1}f) : U \to B(X, B^{(n-1)}(X,Y)) \cong B^{(n)}(X,Y)$.

*Remark* A.1. Differentiable maps are continuous, so the above definition makes sense.

For the following, see [Wer00], Satz III.5.4, p. 120.

**Theorem A.1.** *Let $X, Y, Z$ be normed spaces $U \subseteq X$, $V \subseteq Y$, be open subsets.*

1. *If $f, g : U \to Y$ are $n$-times (continuously) differentiable, then so are $f+g$ and $\lambda f$ ($\lambda \in \mathbb{R}$) with*

$$D(f+g) = Df + Dg, \quad D(\lambda f) = \lambda Df.$$

2. *If $f : U \to Y$, $g : V \to Z$ are $n$-times (continuously) differentiable with $f(U) \subseteq V$, then so is $g \circ f$ with*

$$D(g \circ f)_x = Dg_{f(x)} \circ Df_x \qquad \forall\, x \in U.$$

3. *A map $f : U \to Y \times Z$ is $n$-times (continuously) differentiable iff the maps $\mathrm{pr}_1 \circ f : U \to Y$ and $\mathrm{pr}_2 \circ f : U \to Z$ are.*

4. *The evaluation map*

$$\begin{aligned} \mathrm{ev} : B(X,Y) \times X &\to Y \\ (T, x) &\mapsto T(x) \end{aligned}$$

*is smooth.*

5. *(Mean value theorem) Let $f : U \to Y$ be differentiable, $x \in U$ and let $u \in X$ be s. t. $x + \lambda u \in U\ \forall\, \lambda \in [0,1]$. Then*

$$\|f(x+u) - f(x)\|_Y \le \sup\{\|Df_{x+\lambda u}\| \mid \lambda \in [0,1]\} \|u\|_X.$$

6. *(Taylor's theorem) Let $f : U \to Y$ be $(n+1)$-times differentiable, $x \in U$ and let $u \in X$ be s. t. $x + \lambda u \in U\ \forall\, \lambda \in [0,1]$. Then*

$$\|f(x+u) - \sum_{k=0}^{n} \frac{1}{k!} D^k f_x(u, \ldots, u)\|_Y \le \frac{1}{(n+1)!} \sup\{\|D^{n+1} f_{x+\lambda u}\| \mid \lambda \in [0,1]\} \|u\|_X^{n+1}.$$

The following lemma will be used repeatedly to construct differentiable maps between open subsets of Banach spaces.

**Lemma A.1.** *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be Banach spaces, let $U \subseteq X$ be an open subset and let $X_0 \subseteq X$ be a dense normed subspace. Define $U_0 := U \cap X_0$ (which is an open subset of the normed space $(X_0, \|\cdot\|_{X_0} := \|\cdot\|_X|_{X_0})$) and let a map $f_0 : U_0 \to Y$ be given. If $f_0$ is 2-times differentiable with bounded first and second derivatives, then there exists a unique (Lipschitz) continuously differentiable map $f : U \to Y$ with $f|_{U_0} = f_0$. Under the canonical isometry $\iota : B(X_0, Y) \cong B(X, Y)$, with $\overline{Df}_0 := \iota \circ Df_0 : U_0 \to B(X, Y)$, $Df|_{U_0} = \overline{Df}_0$.*

*Proof.* Let $x \in U$. Then for $r > 0$ small enough, the ball (in $X$) of radius $r$ around $x$ is contained in $U$, and since the statement of the lemma is local (on $X$!), one can replace $U$ by this ball. In particular one can assume that $U$, and hence $U_0$, is convex. Then by the mean value theorem above, since the derivative of $f_0$ is bounded, $f_0$ is Lipschitz continuous and hence has a unique Lipschitz continuous completion to $f : U \to Y$. It remains to show that $f$ is continuously differentiable. Now via the canonical isometry $B(X_0, Y) \cong B(X, Y)$, given by the completion of a bounded linear operator in one direction and restriction to a subspace in the other, one can regard $Df_0$ as a map $\overline{Df}_0 : U_0 \to B(X, Y)$. Again by the mean value theorem and because $f_0$ is assumed to have a bounded second derivative, this map is Lipschitz continuous and has a unique Lipschitz continuous completion $\widetilde{Df}_0 : U \to B(X, Y)$. It remains to show that $\widetilde{Df}_0$ is the derivative $Df$ of $f$. So let $x \in U$ and let $u \in X$ be so small that $x + u \in U$. Pick sequences $(x_n)_{n\in\mathbb{N}} \subseteq U_0$, $(u_m)_{m\in\mathbb{N}} \subseteq X_0$, s. t. $x_n + u_m \in U_0$ and $x_n \to x$ as well as $u_m \to u$. Then

$$\begin{aligned} f(x+u) - f(x) - (\widetilde{Df}_0)_x(u) = {} & f(x+u) - f(x_n + u_m) + f(x_n + u_m) - \\ & - f(x_n) + f(x_n) - f(x) - \\ & - (Df_0)_{x_n}(u_m) + (Df_0)_{x_n}(u_m) - (\overline{Df}_0)_{x_n}(u) + \\ & + (\overline{Df}_0)_{x_n}(u) - (\widetilde{Df}_0)_x(u) \end{aligned}$$

and so

$$\begin{aligned} \|f(x+u) - f(x) - (\widetilde{Df}_0)_x(u)\|_Y \le {} & \|f_0(x_n + u_m) - f_0(x_n) - (Df_0)_{x_n}(u_m)\|_Y + \\ & + \|f(x+u) - f(x_n + u_m)\|_Y + \\ & + \|f(x_n) - f(x)\|_Y + \\ & + \|(Df_0)_{x_n}(u_m) - (\overline{Df}_0)_{x_n}(u)\|_Y + \\ & + \|(\overline{Df}_0)_{x_n}(u) - (\widetilde{Df}_0)_x(u)\|_Y. \end{aligned}$$

In the above expression on the right hand side, because the second derivative of $f_0$ is assumed to be bounded by a constant $c > 0$, say, by Taylor's theorem the first summand on the right hand side can be estimated from above by $\frac{c}{2}\|u_m\|_X^2$, independent of $x_n$. Now first taking the limit $m \to \infty$ and then the limit $n \to \infty$, the first summand on the right hand side is estimated from above by $\frac{c}{2}\|u\|_X^2$, whereas the $2^{\text{nd}}$ to $5^{\text{th}}$ summand vanish, by continuity of $f$, the definition of $\overline{Df}_0$ and the definition of $\widetilde{Df}_0$. In conclusion, $\|f(x+u)-f(x)-(\widetilde{Df}_0)_x(u)\|_Y \leq \frac{c}{2}\|u\|_X^2$, showing that $f$ is differentiable with differential given by $Df = \widetilde{Df}_0$. $\square$

**Corollary A.1.** *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be Banach spaces, let $U \subseteq X$ be an open subset and let $X_0 \subseteq X$ be a dense normed subspace. Define $U_0 := U \cap X_0$ (which is an open subset of the normed space $(X_0, \|\cdot\|_{X_0} := \|\cdot\|_X|_{X_0})$) and let a map $f_0 : U_0 \to Y$ be given. If for some $k \in \mathbb{N}$, $f_0$ is $r+1$-times differentiable with bounded first and second derivatives, then there exists a unique $r$-times (Lipschitz) continuously differentiable map $f : U \to Y$ with $f|_{U_0} = f_0$.*

*Proof.* Follows from the lemma by induction, noting that $B^{(n)}(X_0, Y)$ is canonically isomorphic to $B^{(n)}(X, Y)$ just as in the case $n = 1$. $\square$

**Definition A.2.** Let $\mathcal{B}$ be a topological space. A *(smooth) Banach manifold atlas* on $\mathcal{B}$ is given by the following data:

1. A covering $(U_i)_{i\in I}$ of $\mathcal{B}$ by open sets,
2. a collection $(B_i, \|\cdot\|_i)_{i\in I}$ of separable Banach spaces and
3. a collection $(\phi_i)_{i\in I}$ of homeomorphisms $\phi_i : U_i \to V_i \subseteq B_i$ onto open subsets $V_i \subseteq B_i$,

s. t. for all $i, j \in I$, $\phi_{ij} := \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ is a smooth map (i. e. infinitely many times Fréchet-differentiable) between open subsets of Banach spaces.
The maps $\phi_i : \mathcal{B} \supseteq U_i \to V_i \subseteq B_i$ are called *charts*.
A continuous map $f : \mathcal{B} \to \mathcal{B}'$ between topological spaces equipped with Banach manifold atlases $(U_i, (B_i, \|\cdot\|_i), \phi_i)_{i\in I})$ and $(U'_j, (B'_j, \|\cdot\|'_j), \phi'_j)_{j\in J}$ is called smooth, if for all $i \in I, j \in J$, $\phi'_j \circ f \circ \phi_i^{-1} : \phi_i(f^{-1}(U'j)) \to B'_j$ is a smooth map between open subsets of Banach spaces.
A diffeomorphism is a smooth map between topological spaces equipped with Banach manifold atlases that has a smooth inverse.
Two atlases on the same topological space are called equivalent if the identity is a diffeomorphism, where the space is equipped with one atlas on the domain and the other atlas on the image.

The above defines an equivalence relation on the class of Banach manifold atlases on a given topological space. A smooth map between topological spaces equipped with Banach manifold atlases still defines a smooth map if any of the two atlases (on the domain or target) are replaced by an equivalent one. So the following makes sense:

**Definition A.3.** A *(smooth) Banach manifold* is a $2^{\text{nd}}$-countable Hausdorff topological space together with an equivalence class of Banach manifold atlases.

*Remark* A.2. Clearly, open subsets of Banach manifolds are Banach manifolds in a canonical way.

**Construction A.1.** Given a Banach manifold $\mathcal{B}$ with atlas $(U_i, (B_i, \|\cdot\|_i), \phi_i)_{i\in I}$. Let $V_i := \phi_i(U_i) \subseteq B_i$ and $V_{ij} := \phi_j(U_i \cap U_j) \subseteq V_j$. Define $\tilde{\mathcal{B}} := \coprod_{i\in I} V_i/_\sim$, where for $v_i \in V_i$, $v_j \in V_j$, $v_i \sim v_j \;:\Leftrightarrow\; v_i \in V_{ji}, v_j \in V_{ij}$ and $\phi_{ij}(v_j) = v_i$. Then there is a canonical homeomorphism $\rho : \tilde{\mathcal{B}} \to \mathcal{B}$ induced by the map $\coprod_{i\in I} V_i \to \mathcal{B}$, $V_i \ni v_i \mapsto \phi_i^{-1}(v_i)$.
Now define (as a topological space) $T\mathcal{B} := \coprod_{i\in I} V_i \times B_i/_\sim$, where $V_i \times B_i \ni (v_i, e_i) \sim (v_j, e_j) \in V_j \times B_j \;:\Leftrightarrow\; v_i \in V_{ji}, v_j \in V_{ij}$ and $(\phi_{ij}(v_j), D(\phi_{ij})_{v_j}(e_j)) = (v_i, e_i)$. This topological space is second countable Hausdorff by general point set topology.
Define $TU_i := [V_i \times B_i] \in T\mathcal{B}$, $TB_i := B_i \times B_i$, and $d\phi_i : TU_i \to V_i \times B_i \subseteq TB_i$ as the inverse of the canonical map $V_i \times B_i \to T\mathcal{B}$ on its image $TU_i$. This defines a Banach manifold atlas on $T\mathcal{B}$, making it a Banach manifold.
Furthermore, the canonical map $\coprod_{i\in I} V_i \times B_i \to \coprod_{i\in I} V_i$ induces a map $\tilde{\pi} : T\mathcal{B} \to \tilde{\mathcal{B}}$ and hence a smooth map $\pi := \rho \circ \tilde{\pi} : T\mathcal{B} \to \mathcal{B}$ of Banach manifolds.
The fibres $T_b\mathcal{B} := \pi^{-1}(b)$, called the *tangent space at the point* $b \in \mathcal{B}$, for $b \in \mathcal{B}$ are topological vector spaces in a canonical way, but in general, over points in different connected components, are nonisomorphic. Furthermore, there is no a priori distinguished norm on the fibre $T_b\mathcal{B}$ making it a Banach space, but only an equivalence class of norms making it a topological vector space.
The above definition depends on the choice of atlas, but if $\tilde{\pi} : \widetilde{T\mathcal{B}} \to \mathcal{B}$ is defined by a different choice of atlas, then one can see that there exists a canonical diffeomorphism $\rho : \widetilde{T\mathcal{B}} \to T\mathcal{B}$ making

$$\begin{tikzcd}
\widetilde{T\mathcal{B}} \arrow[r, "\rho"] \arrow[d, "\tilde{\pi}"'] & T\mathcal{B} \arrow[d, "\pi"] \\
\mathcal{B} \arrow[r, equal] & \mathcal{B}
\end{tikzcd}$$

commute and that is linear on each fibre. One can hence think of these choices for different atlases on $\mathcal{B}$ as giving different but equivalent atlases on one fixed space $T\mathcal{B}$.

**Definition A.4.** A *Banach space bundle* over a Banach manifold $\mathcal{B}$ is a Banach manifold $\mathcal{E}$ together with the following:

1. A smooth map $\pi : \mathcal{E} \to \mathcal{B}$,
2. for every $b \in \mathcal{B}$ a vector space structure on $\mathcal{E}_b := \pi^{-1}(b)$ and
3. a continuous map (the norm) $\|\cdot\| : \mathcal{E} \to \mathbb{R}$,

s. t. the following hold:

1. For every $b \in \mathcal{B}$, $\|\cdot\|_b := \|\cdot\||_{\mathcal{E}_b} : \mathcal{E}_b \to \mathbb{R}$ makes $(\mathcal{E}_b, \|\cdot\|_b)$ a Banach space;

2. for every open subset $U \subseteq \mathcal{B}$ and every section $\sigma : U \to \pi^{-1}(U)$ (i. e. smooth map $\sigma : U \to \mathcal{E}$ with $\pi \circ \sigma = \mathrm{id}_U$), the map $U \to \mathbb{R}$, $b \mapsto \|\sigma(b)\|_b$ is smooth;

3. there exists a covering $(U_i)_{i\in I}$ of $\mathcal{B}$ together with a collection $((E_i, \|\cdot\|_i))_{i\in I}$ of Banach spaces and a collection $(\psi_i : \pi^{-1}(U_i) \to U_i \times E_i)_{i\in I}$ of diffeomorphisms making

$$\begin{tikzcd}
\pi^{-1}(U_i) \arrow[r, "\psi_i"] \arrow[d, "\pi"'] & U_i \times E_i \arrow[d, "\mathrm{pr}_1"] \\
U_i \arrow[r, equal] & U_i
\end{tikzcd}$$

commute s. t. $\psi_{i,b} := \mathrm{pr}_2 \circ \psi_i|_{\mathcal{E}_b} : \mathcal{E}_b \to E_i$ defines a linear map for each $b \in U_i$.

The covering $(U_i)_{i\in I}$ together with the Banach spaces $(E_i)_{i\in I}$ and the diffeomorphisms $(\psi_i)_{i\in I}$ is called a *trivialisation* of the Banach space bundle.
A diffeomorphism $\psi : \pi^{-1}(U) \to U \times E$ where $U$ is an open subset of $\mathcal{B}$ and $E$ a Banach space, that appears as a member of a trivialisation is called a *local trivialisation.*

**Definition A.5.** Let $\pi : \mathcal{E} \to \mathcal{B}$ and $\rho : \mathcal{F} \to \mathcal{C}$ be Banach space bundles. A *(smooth) morphism* between them is a pair $(f, \hat{f})$ of smooth maps $f : \mathcal{B} \to \mathcal{C}$ and $\hat{f} : \mathcal{E} \to \mathcal{F}$ making the diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\hat{f}"] \arrow[d, "\pi"'] & \mathcal{F} \arrow[d, "\rho"] \\
\mathcal{B} \arrow[r, "f"] & \mathcal{C}
\end{tikzcd}$$

commute and s. t. for every $b \in \mathcal{B}$ the induced map $\hat{f}_b := \hat{f}|_{\mathcal{E}_b} : \mathcal{E}_b \to \mathcal{F}_{f(b)}$ is linear.
Composition of morphisms and isomorphisms are defined the usual way.

*Remark* A.3. Any Banach space bundle is in particular a topological vector bundle.

**Lemma A.2.** *1. Let $\pi_i : \mathcal{E}_i \to \mathcal{B}$, for $i = 1, 2$, be Banach space bundles. Their Whitney sum as topological vector bundles, $\mathcal{E}_1 \oplus \mathcal{E}_2 \to \mathcal{B}$ is a Banach space bundle and the canonical maps $\mathrm{pr}_i : \mathcal{E}_1 \oplus \mathcal{E}_2 \to \mathcal{E}_i$ define morphisms $(\mathrm{pr}_i, \mathrm{id}_B)$ between $\mathcal{E}_1 \oplus \mathcal{E}_2 \to \mathcal{B}$ and $\mathcal{E}_i \to \mathcal{B}$ which for every $b \in \mathcal{B}$ induce an isometry $(\mathcal{E}_1 \oplus \mathcal{E}_2)_b \cong (\mathcal{E}_1)_b \times (\mathcal{E}_2)_b$.*

*2. Let $\pi : \mathcal{E} \to \mathcal{B}$ be a Banach space bundle and let $f : \mathcal{C} \to \mathcal{B}$ be a smooth map of Banach manifolds. Then the pullback bundle as a topological vector bundle, $f^*\pi : f^*\mathcal{E} \to \mathcal{C}$ is a Banach space bundle and the induced* *morphism $\hat{f}$, where*

$$\begin{tikzcd} f^*\mathcal{E} \arrow[r, "\hat{f}"] \arrow[d, "f^*\pi"'] & \mathcal{E} \arrow[d, "\pi"] \\ \mathcal{C} \arrow[r, "f"'] & \mathcal{B} \end{tikzcd}$$

*commutes, is a fibrewise isometry.*

**Construction A.2.** Let $\mathcal{B}, \mathcal{B}'$ be Banach manifolds and let $f : \mathcal{B} \to \mathcal{B}'$ be a smooth map. Then there exists, as in the finite dimensional case, a map $Df : T\mathcal{B} \to T\mathcal{B}'$ defined in the usual way: Let $(U_i, (B_i, \|\cdot\|_i), \phi_i)_{i\in I}$ be an atlas on $\mathcal{B}$, $(U'_j, (B'_j, \|\cdot\|'_j, \phi'_j)_{j\in J}$ an atlas on $\mathcal{B}'$ and assume that for every $i \in I$ there exists a $j_i \in J$ s. t. $f(U_i) \subseteq U'_{j_i}$ and for $i \neq i'$, $j_i \neq j_{i'}$ (otherwise replace the atlases on $\mathcal{B}$ and $\mathcal{B}'$ by compatible ones). Define $V_i := \phi_i(U_i) \subseteq B_i$, $V'_j := \phi'_j(U'_j) \subseteq B'_j$ and $f_i := \phi'_{j_i} \circ f|_{U_i} \circ \phi_i^{-1} : V_i \to V'_{j_i}$. Then there is an induced map $\coprod_{i\in I} V_i \times B_i \to \coprod_{i\in I} V'_{j_i} \times B'_{j_i} \to \coprod_{j\in J} V'_j \times B'_j$ given on each summand by $U_i \times B_i \ni (x, e) \mapsto (f_i(x), (Df_i)_x(e)) \in V_{j_i} \times B'_{j_i}$. One can check that this map is compatible with the equivalence relation on these disjoint unions as in Construcion A.1, hence inducing a smooth map $Df : T\mathcal{B} \to T\mathcal{B}'$.
Furthermore, $Df$ induces, for every $b \in \mathcal{B}$, a linear map $Df_b : T_b\mathcal{B} \to T_{f(b)}\mathcal{B}'$.

**Lemma A.3.** *If $\mathcal{B}, \mathcal{B}'$ are Banach manifolds equipped with compatible Banach norms and $f : \mathcal{B} \to \mathcal{B}'$ is a smooth map then the pair $(f, Df)$ defines a morphism between the Banach space bundles $T\mathcal{B} \to \mathcal{B}$ and $T\mathcal{B}' \to \mathcal{B}'$.*

**Lemma A.4.** *Let $\mathcal{B}$ be a Banach manifold. For every $b \in \mathcal{B}$ and $\xi_b \in T_b\mathcal{B}$ there exists an $\varepsilon > 0$ and a smooth map $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{B}$ s. t. $\gamma(0) = b$ and $\dot{\gamma}(0) := (D\gamma)_0 \frac{\partial}{\partial t} = \xi_b$.*

**Construction A.3.** Let $\pi : \mathcal{E} \to \mathcal{B}$ be a Banach space bundle. For $e \in \mathcal{E}$ let $V_e\mathcal{E} := \ker D\pi_e \subseteq T_e\mathcal{E}$. There is the usual canonical identification $V_e\mathcal{E} \cong \mathcal{E}_{\pi(e)}$, where $\mathcal{E}_{\pi(e)} \ni v \mapsto \dot{\gamma}^v(0)$, with $\gamma^v(t) := e + tv$. Hence $V_e\mathcal{E}$ carries an induced Banach norm and $V\mathcal{E} := \coprod_{e\in E} V_e\mathcal{E} \subseteq T\mathcal{E}$ becomes a Banach space bundle.

**Lemma A.5.** *Let $\pi : \mathcal{E} \to \mathcal{B}$ be a Banach space bundle. Then there is a canonical isomorphism $\pi^*\mathcal{E} \to V\mathcal{E}$ which is a fibrewise isometry.*

### A.1.2 Submanifolds, transversality and Fredholm maps

**Definition A.6.** Let $\mathcal{B}$ be a Banach manifold. A subset $\mathcal{C} \subseteq \mathcal{B}$ is called a *(Banach) submanifold* if for every $b \in \mathcal{C}$ there exists a chart $\phi : U \to V \subseteq B$ defined on an open neighbourhood $U \subseteq \mathcal{B}$ of $b$, mapping onto an open neighbourhood $V$ of 0 in a Banach space $B$, and a closed subspace $C \subseteq B$ s. t. $\phi(U \cap \mathcal{C}) = V \cap C$.
If in addition $C$ splits $B$, i. e. there exists a closed subspace $C' \subseteq B$ s. t. $B \cong C \oplus C'$, then $\mathcal{C}$ is called a *split submanifold* of $\mathcal{B}$. If furthermore the dimension of $C'$ is a finite number $k$, independent of the point $b \in \mathcal{C}$, then $\mathcal{C}$ is called a *submanifold of codimension* $\operatorname{codim}_{\mathcal{B}} \mathcal{C} = k$.

*Remark* A.4. Dropping the adjective "split" in the definition of a submanifold of codimension $k < \infty$ is consistent, for if, in the notation of the definition above, the Banach space $B/_C$ is finite dimensional, then one can choose a finite number of $e_i \in B$ s. t. the $[e_i] \in B/_C$ are a basis. Then $C' := \operatorname{span}\{e_i\}$ is a complement of $C$ in $B$ which is closed by virtue of being finite dimensional.

*Remark* A.5. Note that the usual holds: A Banach submanifold $\mathcal{C}$ of a Banach manifold $\mathcal{B}$ is a Banach manifold itself, the inclusion $\iota : \mathcal{C} \to \mathcal{B}$ is a smooth map that is an embedding of topological spaces and there is a corresponding inclusion $T\mathcal{C} \subseteq T\mathcal{B}$ s. t. for every $b \in \mathcal{C}$, $T_b\mathcal{C}$ is (more precisely can be identified with) a closed subspace of the topological vectorspace $T_b\mathcal{B}$. In case of a split submanifold, the subspace $T_b\mathcal{C}$ splits $T_b\mathcal{B}$ along a closed subspace $N_b\mathcal{C}$ and one can fit these together to a subbundle $N\mathcal{C} \subseteq \iota^*T\mathcal{B}$.

**Example A.1.** If $\pi : \mathcal{E} \to \mathcal{B}$ is a Banach space bundle and $\sigma : \mathcal{B} \to \mathcal{E}$ is a section, then $\sigma(\mathcal{B}) \subseteq \mathcal{E}$ is a split submanifold with $N\sigma(\mathcal{B}) \cong \iota^*V\mathcal{E}$.

**Definition A.7.** Let $\mathcal{B}, \mathcal{B}'$ be Banach manifolds of class $C^r$ and let $f : \mathcal{B} \to \mathcal{B}'$ be a $C^r$-smooth map. Let $\mathcal{C}' \subseteq \mathcal{B}'$ be a submanifold. Then $f$ is called a *split transverse to* $\mathcal{C}'$, if for all $b \in f^{-1}(\mathcal{C}')$

a) $T_{f(b)}\mathcal{B}' = T_{f(b)}\mathcal{C}' + \operatorname{im} Df_x$.

b) $Df_x^{-1}(T_{f(x)}\mathcal{C}')$ splits $T_b\mathcal{B}$, i. e. there exists a closed subspace $N_b \subseteq T_b\mathcal{B}$ s. t. $T_b\mathcal{B} \cong Df_x^{-1}(T_{f(b)}\mathcal{C}') \oplus N_b$.

*Remark* A.6. Note that condition b) is redundant in case that $\mathcal{C}' \subseteq \mathcal{B}'$ is a split submanifold of finite codimension $\operatorname{codim}_{\mathcal{B}'} \mathcal{C}' < \infty$, for one can then choose, for all $b \in f^{-1}(\mathcal{C}')$, a basis $(e_i)$ of $N_{f(b)}\mathcal{C}'$ and lift the $e_i$ by condition a) to $\tilde{e}_i \in T_b\mathcal{B}$. Then define $N_b := \operatorname{span}\{\tilde{e}_i\}$, which is a closed subspace, because it is finite dimensional.

**Lemma A.6.** *Let $\mathcal{B}, \mathcal{B}'$ be Banach manifolds of class $C^r$ and let $f : \mathcal{B} \to \mathcal{B}'$ be a $C^r$-smooth map that is split transverse to a submanifold $\mathcal{C}' \subseteq \mathcal{B}'$. Then $\mathcal{C} := f^{-1}(\mathcal{C}')$ is a split submanifold of $\mathcal{B}$ s. t. for $b \in \mathcal{C}$, $T_b\mathcal{C} = Df_b^{-1}(T_{f(b)}\mathcal{C}')$. In particular, if $\mathcal{C}' \subseteq \mathcal{B}'$ is a split submanifold of codimension $\operatorname{codim}_{\mathcal{B}'} \mathcal{C}' < \infty$, then $\mathcal{C} \subseteq \mathcal{B}$ is a split submanifold of codimension $\operatorname{codim}_{\mathcal{B}} \mathcal{C} = \operatorname{codim}_{\mathcal{B}'} \mathcal{C}'$.*

*Proof.* The same as in finite dimensions, but using the implicit function theorem from the next subsection. □

**Definition A.8.** Let $\mathcal{B}, \mathcal{B}'$ be Banach manifolds and let $f : \mathcal{B} \to \mathcal{B}'$ be a $C^r$-map, $r \geq 1$. $f$ is called *Fredholm of index* $\operatorname{ind}(f) \in \mathbb{Z}$ if for every $b \in \mathcal{B}$, $Df_b : T_b\mathcal{B} \to T_{f(b)}\mathcal{B}'$ is a Fredholm operator of index $\operatorname{ind}(f)$.
If $\pi : \mathcal{E} \to \mathcal{B}$ is a Banach space bundle equipped with a connection $H\mathcal{E} \subseteq T\mathcal{E}$, then a section $\sigma : \mathcal{B} \to \mathcal{E}$ is called a *Fredholm section of index* $\operatorname{ind}(\sigma) \in \mathbb{Z}$, if for every $b \in \mathcal{B}$, $(\mathrm{d}\sigma)_b : T_b\mathcal{B} \to V_{\sigma(b)}\mathcal{E}$ is a Fredholm operator of index $\operatorname{ind}(\sigma)$.

**Lemma A.7.** *Let $f : \mathcal{B} \to \mathcal{B}'$ be a Fredholm map. Let $\mathcal{C} \subseteq \mathcal{B}$ be a split submanifold of codimension $\operatorname{codim}_{\mathcal{B}} \mathcal{C} < \infty$. Then $f|_{\mathcal{C}} : \mathcal{C} \to \mathcal{B}'$ is a Fredholm map of index $\operatorname{ind} f|_{\mathcal{C}} = \operatorname{ind} f - \operatorname{codim}_{\mathcal{B}} \mathcal{C}$.*

*Proof.* Let $X, Y$ be Banach spaces and let $\phi : X \to Y$ be a Fredholm operator. Assume that $X \cong X_0 \oplus X_1$ splits into two closed subspaces $X_0, X_1 \subseteq X$ with $\dim X_1 < \infty$. Then $\phi|_{X_0} : X_0 \to Y$ is Fredholm of index $\operatorname{ind} \phi|_{X_0} = \operatorname{ind} f - \dim X_1$. For if $\iota : X_0 \to X$ is the inclusion, then $\iota$ is Fredholm iff $\dim X_1 < \infty$, in which case $\operatorname{ind} \iota = -\dim X_1$ (for $\ker \iota = \{0\}$ and $\operatorname{coker} \iota = X/_{X_0} \cong X_1$). Hence in this case $\operatorname{ind} f|_{X_0} = \operatorname{ind}(f \circ \iota) = \operatorname{ind} f + \operatorname{ind} \iota = \operatorname{ind} f - \dim X_1$ by a standard result about the composition of Fredholm operators. □

Note the following trivial consequence/extension of Lemma A.3.6 in [MS04]:

**Lemma A.8.** *Let $X, Y, Z$ be Banach spaces, where $Y = Y_0 \oplus Y_1$ splits into two closed subspaces with projections $\mathrm{pr}_i : Y \to Y_i$. Let $D : X \to Y$ be a bounded operator s. t. $D_1 := \mathrm{pr}_1 \circ D : X \to Y_1$ is Fredholm. Let furthermore $L : Z \to Y$ be a bounded linear operator s. t. $(D \oplus L)_1 = D_1 \oplus L_1 : X \oplus Z \to Y_1$ is onto, where $D \oplus L : X \oplus Z \to Y$, $(x, z) \mapsto Dx + Lz$. Then $(D \oplus L)_1$ has a right inverse. Moreover, the projection $\Pi : (D \oplus L)^{-1}(Y_0) = \ker(D \oplus L)_1 \to Z$ is Fredholm with $\ker \Pi \cong \ker D_1 = D^{-1}(Y_0)$ and $\operatorname{coker} \Pi \cong \operatorname{coker} D_1$. In particular,*

$$\operatorname{ind} \Pi = \operatorname{ind} D_1.$$

*Proof.* Apply Lemma A.3.6 from [MS04] to $D_1 : X \to Y_1$ and $L_1 : Z \to Y_1$. □

*Remark* A.7. In the following, two applications of this lemma will be relevant. First of all, of course, the special case $Y_0 = \{0\}$, reproducing Lemma A.3.6 from [MS04] and second the case $\dim X, \dim Y_1 < \infty$ s. t. $\operatorname{ind} D_1 = \dim X - \dim Y_1$.

### A.1.3 The implicit function theorem in Banach spaces

Since heavily used in the construction of moduli spaces, and since the proof is referenced in Lemma II.24, here is a short layout of the implicit function theorem in Banach spaces.

The one recurring theme in this text is the construction of charts for (i. e. diffeomorphisms onto open subsets of) the zero set of a section of a vector bundle. In its simplest form, the relevant result is the following corollary of the constant rank theorem (which in turn follows from the inverse function theorem):

**Theorem A.2.** *Let $\pi : E \to B$ be a vector bundle over a manifold $B$, equipped with a linear connection and hence a covariant derivative $\nabla$. Let $\sigma : B \to E$ be a section and let $b \in B$. If the map $D_b : T_bB \to E_b$, $X_b \mapsto \nabla_{X_b}\sigma$, is surjective, then there exists an open neighbourhood $U \subseteq B$ of $b$, an open neighbourhood $V \subseteq T_bB$ of $0 \in T_bB$ and a bundle trivialisation $(\phi, \Phi)$ of $E$ over $U$ mapping $b$ to $0$ and making the following diagram commute:*

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\Phi"] \arrow[d, "\pi"] & V \times E_b \arrow[d, "\mathrm{pr}_1"'] \\
U \arrow[r, "\phi"] \arrow[u, bend left, "\sigma"] & V \arrow[u, bend right, "(\mathrm{id}, D_b + \sigma_b)"']
\end{tikzcd}$$

*Here,* $(\mathrm{id}, D_b+\sigma_b) : T_bB \to T_bB \times E_b$*,* $X_b \mapsto (X_b, D_b(X_b)+\sigma_b)$*.*

An easy consequence of this is that if $\sigma \pitchfork 0$, then $\sigma^{-1}(0)$ is a closed submanifold of $B$ of dimension $n-k$, where $n := \dim B$, $k := \operatorname{rank} E$, a chart around a point $b \in \sigma^{-1}(0)$ being given (in the notation of the theorem) by $\phi : U \cap \sigma^{-1}(0) \to V \cap \ker D_b \subseteq \ker D_b \cong \mathbb{R}^{n-k}$.
The main goal of this chapter is a quantitative version of this result in a Banach manifold setting. To explain what is meant by "quantitative", take a look at the result in finite dimensions, first:
Let again $\pi : E \to B$ be a vector bundle with linear connection and covariant derivative $\nabla$. Let $\sigma : B \to E$ be a section and for $b \in B$ let $D_b : T_bB \to E_b$, $X_b \mapsto \nabla_{X_b}\sigma$.

In some applications (specifically when it comes to gluing of holomorphic curves), one is not only interested in charts for $\sigma^{-1}(0)$ as above, given by the theorem and centered around a point $b \in \sigma^{-1}(0)$, but rather around a point $b \in B$ with "$\sigma(b)$ small". Here is how this works: Let $b \in B$ be such that $D_b$ is surjective and let $(\phi, \Phi), U, V$ be as in the theorem. Define $s := D_b + \sigma_b : V \to E_b$, i. e. $\mathrm{pr}_2 \circ \Phi \circ \sigma|_U = s \circ \phi|_U$, in particular $\phi(U \cap \sigma^{-1}(0)) = s^{-1}(0)$. Pick a right inverse $Q_b : E_b \to T_bB$ of $D_b$. Then $\operatorname{im} Q_b \subseteq T_bB$ is a complementary subspace to $\ker D_b$ in $T_bB$ and $Q_b \circ D_b : T_bB \to T_bB$ defines the projection of $T_bB$ onto $\operatorname{im} Q_b$ along $\ker D_b$. Likewise, $\mathrm{id} - Q_b \circ D_b$ defines the projection of $T_bB$ onto $\ker D_b$ along $\operatorname{im} Q_b$. If $\xi_b := -Q_b(\sigma_b)$, then $P_b : T_bB : T_bB$, $v \mapsto (1-QD)v + \xi_b$ defines the projection of $T_bB$ onto $s^{-1}(0)$ along $Q_b$. Now if $\xi_b \in V$, then $V_0 := s^{-1}(0) \cap V$ is a neighbourhood of $\xi_b$ in $s^{-1}(0) \cap V$, $V' := P_b^{-1}(V_0) \cap V$ is a neighbourhood of $0$ in $V$, and $P_b|_{V'}$ defines the projection $V' \to V_0$ along $Q_b$. In particular, if $f : M \to V'$ is a map from a manifold to $V'$ that is transverse to $f(x) + \operatorname{im} Q_b$ for some $x \in M$, then $\phi^{-1} \circ P_b \circ f$ defines a diffeomorphism from a neighbourhood of $x$ in $M$ onto an open neighbourhood of $\phi^{-1}(P_b(f(x)))$ in $\sigma^{-1}(0)$. "Quantitative" now refers to giving conditions for this (i. e. for $\xi_b \in V$) to hold.

As a first step and for future reference, the model situation for a smooth map between Banach spaces:
Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be Banach spaces, let $U \subseteq X$ be open and let $f : U \to Y$ be a smooth map. W. l. o. g. assume that $U = B_d(0)$ for some $d > 0$ and assume that the differential of $f$ at $0$, $D := Df_0 : X \to Y$, has a bounded right inverse $Q : Y \to X$. The goal is to find open neighbourhoods $U' \subseteq U, V \subseteq X$ of $0$ and a diffeomorphism $\phi : V \to U'$ with $f \circ \phi(x) = D(x) + f(0)$. Furthermore, one wants to find a good (i. e. depending on as little as possible) estimate for $\sup\{r > 0 \mid B_r(0) \subseteq V\}$. The construction of $\phi$ proceeds as follows: Let $\hat{f} : U \to Y$, $\hat{f}(x) := f(x) - f(0)$. Then $\hat{f}(0) = 0$ and $D\hat{f}_x = Df_x$. Consider the function $\tilde{\phi} : U \to X$, $x \mapsto \underbrace{(Q \circ \hat{f})(x)}_{\in \operatorname{im} Q} + \underbrace{(\mathrm{id} - Q \circ D)(x)}_{\in \ker D}$. This function satisfies $\tilde{\phi}(0) = 0$ and $D\tilde{\phi}_0 = \mathrm{id}_X$, so by the inverse function theorem there exist open neighbourhoods $U' \subseteq U, V \subseteq X$ of $0$ s. t. $\tilde{\phi}|_{U'} : U' \to V$ is a diffeomorphism. Define $\phi := (\tilde{\phi}|_{U'})^{-1} : V \to U'$. Furthermore, $\tilde{\phi}(x) \in \operatorname{im} Q \;\Leftrightarrow\; x \in \operatorname{im} Q$

and $\tilde{\phi}(x) \in \ker D \Leftrightarrow \hat{f}(x) = 0$. This implies that $\tilde{\phi}$ restricts to a map $\tilde{\phi}|_{\operatorname{im} Q} : \operatorname{im} Q \to \operatorname{im} Q$ and that (note that $Q : Y \to \operatorname{im} Q$ is bijective, although $Q : Y \to X$ is not) $\hat{f} \circ \phi(x) = \begin{cases} Q^{-1}(x) = D(x) & , x \in \operatorname{im} Q \\ 0 & , x \in \ker D \end{cases}$, so together with the definition of $\hat{f}$, $f \circ \phi(x) = D(x) + f(0)$, as desired.
To arrive at the desired estimate for $\sup\{r > 0 \mid B_r(0) \subseteq V\}$ one needs to examine the proof of the inverse function theorem, as presented e. g. in [Con01], Appendix B:
Let $\psi := \mathrm{id} - \tilde{\phi}$. Then $V$ can be taken to be $B_{\eta/2}(0)$, where $\eta > 0$ is so small that (a) $B_\eta(0) \subseteq U$, (b) $D\tilde{\phi}_x$ is nonsingular for $x \in B_\eta(0)$ and that (c) $\|\psi(x_1) - \psi(x_2)\|_X \leq \frac{1}{2}\|x_1 - x_2\|_X$ for $x_1, x_2 \in B_\eta(0)$. Condition (a) just says that $\eta \leq d$ ($d$ as above). Condition (b) is satisfied (by a general result about operators on Banach spaces) if $\|\mathrm{id}_X - D\tilde{\phi}_x\| = \|D\psi_x\| < 1$ ($\|\cdot\|$ now denotes the norm on $B(X,Y)$) and condition (c) is satisfied, by the mean value theorem, if $\|D\psi_x\| \leq \frac{1}{2}$ for $x \in B_\eta(0)$, which also implies (b). To examine this further, calculate $D\tilde{\phi}_x = Q \circ Df_x + (\mathrm{id}_X - Q \circ D) = \mathrm{id}_X - Q \circ (D - Df_x)$ and hence $D\psi_x = \mathrm{id}_X - D\tilde{\phi}_x = Q \circ (D - Df_x)$. The condition hence becomes $\|Q(D - Df_x)\| \leq \frac{1}{2}$. For this to hold it is sufficient that $\|D - Df_x\| \leq \frac{1}{2\|Q\|}$.
Again applying the mean value theorem, this time to the function $Df(\eta) : U \to B(X,Y)$, $x \mapsto Df_x(\eta)$, and remembering that $D = Df_0$, gives $\|(D - Df_x)(\eta)\| \leq \sup\{\|D(Df(\eta))_{\lambda x}(x)\| \mid \lambda \in [0,1]\}$. Hence if there is a bound on the second derivative of $f$, i.e. a constant $c > 0$ s.t. $\sup\{\|D(Df(\eta))_{\lambda x}(x)\| \mid \lambda \in [0,1]\} \leq c\|x\|\|\eta\|$ for all $x \in U$, $\lambda \in [0,1]$, or more generally if there is a constant $c > 0$ s.t. $\|D - Df_x\| < c\|x\|$ for all $x \in U$, then one can choose any $0 < \eta \leq \frac{1}{2c\|Q\|}$.
Putting everything together, one arrives at the following theorem:

**Theorem A.3.** *Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be Banach spaces, let $U \subseteq X$ be an open neighbourhood of $0$ and let $f : U \to Y$ be a smooth map. Assume that $Df_0 : X \to Y$, has a bounded right inverse $Q : Y \to X$ and that there exists a constant $c > 0$ s. t. $\|Df_0 - Df_x\| < c\|x\|$ for all $x \in U$. Let $d := \sup\{r > 0 \mid B_r(0) \subseteq U\}$ and define*

$$\rho := \min\left(\frac{1}{4c\|Q\|}, d\right).$$

*Then there exists a map $\phi : B_\rho(0) \to U$ that is a diffeomorphism onto an open neighbourhood $U' \subseteq U$ of $0$ with $\phi(0) = 0$ and that satisfies $\|D\phi_x\| \leq 2$ and $f \circ \phi(x) = Df_0(x) + f(0)$ for all $x \in B_\rho(0)$.*

The aforementioned condition $\xi_b := -Q_b\sigma_b \in V$ in the terminology of the theorem then becomes (here, $\sigma, Q_b, V, \xi_b$ get replaced by $f, Q, B_\rho(0), \xi := -Qf(0)$) $\|Qf(0)\| < \rho$, which is satisfied if $\|f(0)\| < \frac{1}{4c\|Q\|^2}$ and $\|f(0)\| < \frac{d}{\|Q\|}$.

## A.2 Jacobi estimates

In the following, a variant of Theorems 4.5.2 and 4.5.3 of [Jos02], for inhomogeneous Jacobi equations instead of homogeneous ones, is proved. Actually, the proofs (and statements) of the following lemma and theorem are pretty much literally taken from the proof of Theorem 4.5.2 of [Jos02].

**Lemma A.9.** *Let $\tau > 0$ and let $g_1, g_2 \in C^0([0,\tau),\mathbb{R})$ be continuous functions. Let furthermore $\rho \geq 0$ and let $f_1, f_2 \in C^2([0,\tau),\mathbb{R})$ be solutions to the ODEs*

$$\ddot{f}_i - \rho f_i = g_i, \quad i = 1, 2,$$

*with $f_1(0) = f_2(0)$, $\dot{f}_1(0) = \dot{f}_2(0)$. Then if $g_1(t) \leq g_2(t)$ for all $t \in [0,\tau)$, then $f_1(t) \leq f_2(t)$ for all $t \in [0,\tau)$.*

*Proof.* Let $s_\rho : \mathbb{R} \to \mathbb{R}$, $t \mapsto \begin{cases} t & \rho = 0 \\ \frac{1}{\sqrt{\rho}}\sinh(\sqrt{\rho}t) & \rho > 0 \end{cases}$, be the solution to the equation $\ddot{f} - \rho f = 0$ with $f(0) = 0$, $\dot{f}(0) = 1$. Define $d := (f_1 - f_2)^{\cdot} s_\rho - (f_1 - f_2)\dot{s}_\rho$. Then on $[0,\tau)$,

$$\begin{aligned} \dot{d} &= (f_1 - f_2)^{\cdot\cdot} s_\rho - (f_1 - f_2)\ddot{s}_\rho \\ &= (\rho(f_1 - f_2) + g_1 - g_2)s_\rho - (f_1 - f_2)\rho s_\rho \\ &= (g_1 - g_2)s_\rho \\ &\leq 0, \end{aligned}$$

since for $t \in [0,\tau)$, $s_\rho(t) \geq 0$ and $g_1(t) \leq g_2(t)$ by assumption. It follows that for $t \in (0,\tau)$ (hence $s_\rho(t) > 0$), $\frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{s_\rho}(f_1 - f_2)) = \frac{d}{s_\rho^2} \leq 0$ and since $(f_1 - f_2)(0) = 0$ as well as $\frac{\mathrm{d}}{\mathrm{d}t}(f_1 - f_2)(0) = 0$, $\lim_{t \searrow 0} \frac{1}{s_\rho(t)}(f_1 - f_2)(t) = 0$. This shows that $\frac{1}{s_\rho}(f_1 - f_2) \leq 0$ and since $s_\rho > 0$ on $(0,\tau)$, $f_1 - f_2 \leq 0$ on $[0,\tau)$. □

**Theorem A.4.** *Let $M$ be a complete Riemannian manifold, $p \in M$ and $\xi \in T_pM$. Let $\gamma : [0,1] \to M$, $t \mapsto \exp_p(t\xi)$, be the geodesic through $p$ in the direction of $\xi$. Let $\kappa \geq 0$ be s. t. $\|R(X,Y)Z\| \leq \kappa\|X\|\|Y\|\|Z\|$ for all $t \in [0,1]$, $X, Y, Z \in T_{\gamma(t)}M$. Denote $\dot{X} := \nabla_{\dot{\gamma}}X$ for a vector field $X : [0,1] \to TM$ along $\gamma$ and let $V : [0,1] \to TM$ be a vector field along $\gamma$. Assume that $J : [0,1] \to TM$ is another vector field along $\gamma$ satisfying the inhomogeneous Jacobi equation*

$$\ddot{J} + R(J,\dot{\gamma})\dot{\gamma} = V.$$

*Then for $\kappa = 0$*

$$\|J(t) - (\|_0^t \gamma)(J(0) + t\dot{J}(0))\| \leq \int_0^t \|V(s)\|(t-s)\,\mathrm{d}s$$

*and for* $\kappa > 0$

$$\begin{aligned}\|J(t) - (\|_0^t \gamma)(J(0) + t\dot J(0))\| \le{}& (\cosh(\sqrt{\kappa}\|\xi\|t) - 1)\|J(0)\| + \\ &+ \frac{1}{\sqrt{\kappa}\|\xi\|}\left(\sinh(\sqrt{\kappa}\|\xi\|t) - \sqrt{\kappa}\|\xi\|t\right)\|\dot J(0)\| + \\ &+ \int_0^t \|V(s)\|\frac{1}{\sqrt{\kappa}\|\xi\|}\sinh(\sqrt{\kappa}\|\xi\|(t-s))\,\mathrm{d}s.\end{aligned}$$

*Proof.* Let $A : [0,1] \to TM$ be the vector field along $\gamma$ defined by $A(t) := (\|_0^t \gamma)(t\dot J(0) + J(0))$, i.e. $\ddot A = 0$, $A(0) = J(0)$, $\dot A(0) = \dot J(0)$ and the goal is to estimate $\|J(t) - A(t)\|$. Let furthermore $a : [0,1] \to \mathbb{R}$ be the solution of

$$\ddot a - \kappa\|\xi\|^2 a = \kappa\|\xi\|^2\|A\| + \|V\|, \quad a(0) = \dot a(0) = 0$$

and let $b : [0,1] \to \mathbb{R}$ be the solution of

$$\ddot b = \kappa\|\xi\|^2\|J\| + \|V\|, \quad b(0) = \dot b(0) = 0.$$

If $P : [0,1] \to TM$ is a parallel vector field along $\gamma$ with $\|P\| = 1$, then

$$\langle J - A, P\rangle^{\cdot\cdot} = \langle \ddot J, P\rangle = -\langle R(J,\dot\gamma)\dot\gamma + V, P\rangle \le \kappa\|\xi\|^2\|J\| + \|V\|.$$

So by the previous lemma, $\langle J - A, P\rangle \le b$, and hence $\|J - A\| \le b$ on $[0,1]$. It follows that

$$\begin{aligned}\ddot b &= \kappa\|\xi\|^2\|J\| + \|V\| \\ &\le \kappa\|\xi\|^2\|J - A\| + \kappa\|\xi\|^2\|A\| + \|V\| \\ &\le \kappa\|\xi\|^2 b + \kappa\|\xi\|^2\|A\| + \|V\|,\end{aligned}$$

so

$$\ddot b - \kappa\|\xi\|^2 b \le \kappa\|\xi\|^2\|A\| + \|V\|.$$

Again by the previous lemma, $b \le a$ and hence $\|J - A\| \le a$.

Let again $s_\rho : \mathbb{R} \to \mathbb{R}$, $t \mapsto \begin{cases} t & \rho = 0 \\ \frac{1}{\sqrt{\rho}}\sinh(\sqrt{\rho}t) & \rho > 0\end{cases}$, be the solution to the equation $\ddot f - \rho f = 0$ with $f(0) = 0$, $\dot f(0) = 1$. Then $a$ is given by $a(t) = \int_0^t (\kappa\|\xi\|^2\|A(s)\| + \|V(s)\|) s_{\kappa\|\xi\|^2}(t-s)\,\mathrm{d}s$. If $\kappa = 0$, this is just $a(t) = \int_0^t \|V(s)\|(t-s)\,\mathrm{d}s$. For $\kappa > 0$, one can estimate $\|A(s)\| \le s\|\dot J(0)\| + \|J(0)\|$ and thus

$$\begin{aligned}a(t) \le{}& \|\dot J(0)\|\int_0^t s\sqrt{\kappa}\|\xi\|\sinh(\sqrt{\kappa}\|\xi\|(t-s))\,\mathrm{d}s + \\ &+ \|J(0)\|\int_0^t \sqrt{\kappa}\|\xi\|\sinh(\sqrt{\kappa}\|\xi\|(t-s))\,\mathrm{d}s + \\ &+ \int_0^t \|V(s)\|\frac{1}{\sqrt{\kappa}\|\xi\|}\sinh(\sqrt{\kappa}\|\xi\|(t-s))\,\mathrm{d}s \\ ={}& \frac{\|\dot J(0)\|}{\sqrt{\kappa}\|\xi\|}(\sinh(\sqrt{\kappa}\|\xi\|t) - \sqrt{\kappa}\|\xi\|t) + \\ &+ \|J(0)\|(\cosh(\sqrt{\kappa}\|\xi\|t) - 1) + \\ &+ \int_0^t \|V(s)\|\frac{1}{\sqrt{\kappa}\|\xi\|}\sinh(\sqrt{\kappa}\|\xi\|(t-s))\,\mathrm{d}s.\end{aligned}$$

□

**Corollary A.2.** *Let $M$ be a complete Riemannian manifold, $p \in M$ and $\xi \in T_pM$. Let $\gamma : [0,1] \to M$, $t \mapsto \exp_p(t\xi)$, be the geodesic through $p$ in the direction of $\xi$. Let $\kappa \geq 0$ be s. t. $\|R(X,Y)Z\| \leq \kappa\|X\|\|Y\|\|Z\|$ for all $t \in [0,1]$, $X, Y, Z \in T_{\gamma(t)}M$. Denote $\dot{X} := \nabla_{\dot\gamma} X$ for a vector field $X : [0,1] \to TM$ along $\gamma$ and let $V : [0,1] \to TM$ be a vector field along $\gamma$. Assume that $J : [0,1] \to TM$ is another vector field along $\gamma$ satisfying the inhomogeneous Jacobi equation*

$$\ddot{J} + R(J, \dot\gamma)\dot\gamma = V.$$

*Then for $\|\xi\| \leq \frac{1}{\sqrt{\kappa}}$,*

$$\|J(t) - (\|_0^t \gamma)(J(0) + t\dot{J}(0))\| \leq t^2 \left( \|J(0)\| + t\|\dot{J}(0)\| + \frac{1+t^2}{t} \int_0^t \|V(s)\| \,\mathrm{d}s \right)$$

*and in particular,*

$$\|J(t)\| \leq (1+t^2) \left( \|J(0)\| + t\|\dot{J}(0)\| + t \int_0^t \|V(s)\| \,\mathrm{d}s \right).$$

*Furthermore,*

$$\|\dot{J}(t) - (\|_0^t \gamma)\dot{J}(0)\| \leq t(1+t^2) \left( \|J(0)\| + t\|\dot{J}(0)\| + \frac{1}{t} \int_0^t \|V(s)\| \,\mathrm{d}s \right)$$

*and in particular*

$$\|\dot{J}(t)\| \leq \|\dot{J}(0)\| + t(1+t^2) \left( \|J(0)\| + t\|\dot{J}(0)\| + \frac{1}{t} \int_0^t \|V(s)\| \,\mathrm{d}s \right).$$

*Finally, if $J(0) = 0$, then*

$$\left\| \left(\tfrac{1}{t} J(t)\right)^{\boldsymbol{\cdot}} \right\| \leq 2(1+t^2) \left( t\|\dot{J}(0)\| + \frac{1}{t} \int_0^t \|V(s)\| \,\mathrm{d}s \right).$$

*Proof.* For $\kappa > 0$ and $\|\xi\| \leq \frac{1}{\sqrt{\kappa}}$ one can apply the estimates $\sinh(x) - x \leq x^3$, $\cosh(x) - 1 \leq x^2$ and $\sinh(x) \leq x(1+x^2) \leq 2x$ to the formula from the previous theorem to get

$$\begin{aligned} \|J(t) - (\|_0^t \gamma)(J(0) + t\dot{J}(0))\| &\leq t^2\|J(0)\| + t^3\|\dot{J}(0)\| + \int_0^t \|V(s)\|(t-s)(1+(t-s))^2 \,\mathrm{d}s \\ &\leq t^2\|J(0)\| + t^3\|\dot{J}(0)\| + t(1+t^2) \int_0^t \|V(s)\| \,\mathrm{d}s. \end{aligned}$$

This estimate clearly also holds for $\kappa = 0$ by the previous theorem and the second inequality in the statement then simply follows by the triangle inequality.

For the following cf. the proof of Corollary 4.5.1 of [Jos02].
Let $P(t) := (\|_0^t \gamma)P(0)$ for $P(0) \in T_pM$ with $\|P(0)\| = 1$ be a parallel unit length vector field along $\gamma$. Then for $\|\xi\| \leq \frac{1}{\sqrt{\kappa}}$

$$\begin{aligned}
|\langle \dot{J}(t) - (\|_0^t \gamma)\dot{J}(0), P(t)\rangle^{\cdot}| &= |\langle -R(J,\dot{\gamma})\dot{\gamma} + V, P\rangle(t)| \\
&\leq \kappa\|\xi\|^2\|J(t)\| + \|V(t)\| \\
&\leq (1+t^2)\|J(0)\| + (t+t^3)\|\dot{J}(0)\| + 2t\int_0^t \|V(s)\|\,\mathrm{d}s + \\
&\quad + \|V(t)\|,
\end{aligned}$$

so by integration

$$\begin{aligned}
\|\dot{J}(t) - (\|_0^t \gamma)\dot{J}(0)\| &\leq (t + \frac{1}{3}t^3)\|J(0)\| + (\frac{1}{2}t^2 + \frac{1}{4}t^4)\|\dot{J}(0)\| + \\
&\quad + \int_0^t 2r \underbrace{\int_0^r \|V(s)\|\,\mathrm{d}s}_{\leq \int_0^t \|V(s)\|\,\mathrm{d}s} \mathrm{d}r + \int_0^t \|V(s)\|\,\mathrm{d}s \\
&\leq t(1+\frac{t^2}{3})\|J(0)\| + \frac{1}{2}t^2(1+\frac{t^2}{2})\|\dot{J}(0)\| + \\
&\quad + (1+t^2)\int_0^t \|V(s)\|\,\mathrm{d}s \\
&\leq (1+t^2)\Bigg( t\|J(0)\| + t^2\|\dot{J}(0)\| + \\
&\quad + \int_0^t \|V(s)\|\,\mathrm{d}s \Bigg).
\end{aligned}$$

The estimate on $\|\dot{J}(t)\|$ then also follows simply by the triangle inequality.
For the last inequality, write

$$\begin{aligned}
\left(\tfrac{1}{t}J(t)\right)^{\cdot} &= \frac{1}{t}\dot{J}(t) - \frac{1}{t^2}J(t) \\
&= \frac{1}{t^2}(t\dot{J}(t) - J(t)) \\
&= \frac{1}{t^2}(t\dot{J}(t) - t(\|_0^t \gamma)\dot{J}(0) + (\|_0^t \gamma)(\underbrace{J(0)}_{=0} + t\dot{J}(0)) - J(t)) \\
&= \frac{1}{t}(\dot{J}(t) - (\|_0^t \gamma)\dot{J}(0)) - \frac{1}{t^2}(J(t) - (\|_0^t \gamma)(J(0) + t\dot{J}(0))).
\end{aligned}$$

The last inequality then follows easily from this and the previous estimates. $\square$

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