Arnold Sommerfeld Center

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CENTER FOR THEORETICAL PHYSICS

Modified and Condensed Gravity

Doktorarbeit
von

Felix Berkhahn

Ludwig Maximilians Universität München

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Zusammenfassung

Diese Doktorarbeit befasst sich sowohl mit Infrarot-Modifikationen der Einstein'schen Relativitätstheorie als auch mit dem vor Kurzem vorgestellten mikroskopischen Bild schwarzer Löcher.

Infrarot-Modifikationen von Gravitation bezeichnen eine Klasse von Theorien die typischerweise die Gravitationskraft von Einsteins allgemeiner Relativitätstheorie bei großen (meist kosmologischen) Distanzen abschwächen und gleichzeitig die Erfolge der Einstein'schen Theorie bei kleinen Distanzen sicherstellen (und damit insbesondere für Beobachtungen innerhalb des Sonnensystems). Infrarotmodifizierte Gravitationstheorien erlauben es, Fortschritte mit dem Kosmologischen-Konstanten-Problem zu machen, da eine kosmologische Konstante als eine Quelle mit unendlicher Ausdehnung anzusehen ist. Die Resultate, die in dieser Arbeit vorgestellt werden, betreffen zwei Repräsentanten infrarotmodifizierter Gravitationstheorien: *Massive Gravitation* und *braneninduzierte Gravitation*.

Massive Gravitation wurde bisher ausgiebig auf flachen Minkowski-Hintergründen untersucht. Diese Arbeit befasst sich hingegen mit der Propagation massiver Gravitonen auf gekrümmten Hintergründen, wie beispielsweise kosmologisch relevanten FRW-Hintergründen. Tatsächlich stellt sich heraus, dass die Physik massiver Gravitonen auf gekrümmten Hintergründen außergewöhnlich reichhaltig ist. So war es uns unter Anderem möglich zu zeigen, dass die lineare massive Gravitationstheorie gegen mögliche Unitaritätsverletzungen geschützt ist, indem sie generisch in eine Phase starker Kopplung übergeht bevor die Unitaritätsverletzung der linearen Theorie auftreten könnte. Wir haben dies den *Selbstschutz-Mechanismus* getauft. Tatsächlich kann der Selbstschutz-Mechanismus als ein eindrucksvolles Beispiel des kürzlich vorgestellten Klassikalon-Mechanismus angesehen werden. In diesem Fall stellt das Klassikalon den neuen Hintergrund dar, in den die Metrik in der Phase starker Kopplung evolviert. Obwohl der Selbstschutz-Mechanismus sehr attraktiv von einer theoretischen Warte erscheint, ist er phänomenologisch als uninteressant einzustufen, da die Phase starker Kopplung notwendigerweise mit einer Zerstörung des FRW-Hintergrundes einhergeht. Dies wird umso deutlicher, da die starke Kopplung gerade zu frühen Zeiten im Universum auftritt. Daher haben wir eine komplett neue Theorie massiver Deformationen entwickelt, die den bekannten „harten Massenterm" um einen „weichen Massenterm" ergänzt. Diese Theorie ist *sowohl störungstheoretisch stabil als auch konsistent auf der kompletten FRW-*

Mannigfaltigkeit. Wenn wir die bekannte harte Masse identisch null setzen, erhalten wir eine Theorie, bei welcher ausschließlich in Raumzeitregionen nicht verschwindender Krümmung eine Modifikation hervorgerufen wird. Insbesondere folgt die bekannte Propagation masseloser Gravitonen in Regionen verschwindender Hintergrundskrümmung, weshalb diese Theorie als *komplett neuartig* hinsichtlich bekannter massiver Gravitationstheorien einzustufen ist.

Der zweite Repräsentant infrarotmodifizierter Graviationstheorien, die braneninduzierte Gravitation, wurde aufgrund früherer Arbeiten im Falle mehr als einer zusätzlichen räumlichen Extradimension zu Unrecht als eine geistbehaftete Theorie angesehen (für genau eine räumliche Extradimension folgt hingegen das konsistente DGP-Modell). Dieser Geist-Freiheitsgrad ist allerdings physikalisch komplett unverstanden, da wir braneninduzierte Gravitation als höher-dimensionale Einstein-Gravitation mit einer vierdimensionalen, konsistenten Quelle ansehen können. Daher haben wir eine vollständige Dirac-Analyse durchgeführt, die tatsächlich gezeigt hat, dass der Hamilton ausgewertet auf der Zwangsbedingungshyperfläche positiv definit ist. Wir können daher schlussfolgern, dass *braneninduzierte Gravitation eine konsistente Theorie darstellt*, trotz gegensätzlicher Behauptungen in früheren Veröffentlichungen. Wir haben das System zusätzlich auch im kovarianten Formalismus untersucht und konnten feststellen, dass diese früheren Arbeiten die 00-Einstein-Gleichung nicht berücksichtigt haben. Die 00-Einstein-Gleichung stellt allerdings eine Zwangsbedingung dar, welche den angeblichen Geistfreiheitsgrad eliminiert.

Das weitere Thema dieser Promotion befasst sich mit dem vor Kurzem von Gia Dvali und Cesar Gomez vorgestellten mikroskopischen Bild schwarzer Löcher. Dazu haben wir ein neuartiges nicht-relativistisches Skalarfeld-Modell entwickelt, welches die allgemeine Relativititästheorie in den für die Schwarze-Loch-Physik wesentlichen Eigenschaften nachahmt, aber dennoch so einfach ist, dass es umfangreiche quantitative Berechnungen erlaubt. In einem ersten Schritt haben wir dieses System perturbativ analysiert und konnten Indikationen auffinden, die darauf hindeuten, dass sich das System tatsächlich dynamisch am Punkt des Quantenphasenüberganges hält. Letztlich ist aber festzuhalten, dass nur eine nicht-lineare numerische Berechnung (die wir derzeit durchführen) Gewissheit über die genauen Eigenschaften des Modells liefern kann.

Abstract

This doctoral thesis deals with both infrared modifications of gravity and with the recently proposed microscopic picture of black holes.

The former subject, i.e. infrared modifications of gravity, denotes a class of theories that typically weaken Einsteins theory of gravity at very large (usually cosmological) distance scales while preserving its successes at smaller distances (in particular within the solar system). Infrared modified theories of gravity allow to make progress with the cosmological constant problem since the cosmological constant literally corresponds to a space-time source of infinite extent. The results presented in this thesis concern two representatives of infrared modified theories of gravity: *Massive Gravity* and *Brane Induced Gravity.*

Massive Gravity has been extensively studied for graviton propagation on a flat Minkowski background. What we will do in this thesis, however, is to study Massive Gravity on curved backgrounds such as cosmologically relevant FRW backgrounds. It actually turns out that the physics associated with the propagation of gravitons on curved spaces is enormously rich. In particular, we were able to show that the linear theory is protected from potential unitarity violations by generically entering a strong coupling regime before the unitary violation of the linear theory could have occurred. We coined this mechanism the *self-protection mechanism.* In fact, the self-protection mechanism can be understood as a striking example of the recently proposed classicalization mechanism, where the classicalon plays the role of the new background geometry that forms when entering the non-linear regime. Even though that the self-protection mechanism is very appealing from a theoretical perspective, it goes hand in hand with the destruction of the FRW background as soon as we enter the non-linear regime. This is phenomenological unacceptable as this always happens for early times in the universe. This led us to the construction of a completely new theory of massive deformations, where we supplemented the known 'hard mass' term with a new 'soft mass' term. This new theory is *both stable and consistent on the whole Friedman manifold.* A particular interesting special case can be obtained when we set the hard mass identically equal to zero, since in this case we obtain a modification that is solely operative on curved backgrounds, whereas we still have standard massless graviton propagation for regions where the background curvature is small. This modification is thus *completely orthogonal to known massive gravity theories.*

The other infrared modified theory of gravity this thesis deals with, i.e. Brane Induced Gravity, has been thought to contain a ghost within its spectrum of physical particles if we consider two or more additional spatial dimensions (whereas for one spatial dimension we would obtain the consistent DGP model). However, this ghost degree of freedom is completely unexpected physically, as we can think of Brane Induced Gravity simply as a higher dimensional Einstein gravity theory with a specific, healthy four dimensional source. Therefore, we performed a complete Dirac constraint analysis that actually showed that the Hamiltonian on the constraint surface is positive definite, and thus that *Brane Induced Gravity is consistent*, contrary to prior claims in the literature. By studying the system as well in the covariant language, we were able to understand that these previous derivations of the ghost degree of freedom did not take the 00-Einstein equation into account properly. This equation actually is a constraint that renders the would-be ghost mode non-dynamical.

The other subject of this thesis deals with the microscopic picture of black holes recently proposed by Gia Dvali and Cesar Gomez. To be concrete, we invented a novel non-relativistic scalar theory that is supposed to mimic properties of general relativity relevant for black hole physics but is simple enough to make extensive quantitative calculations. In a first step, we analyzed the system perturbatively. This allowed us to show that there is indeed indication that the system dynamically secures to stay at the point of quantum phase transition. However, only a thorough nonlinear numerical analysis that is currently under investigation will yield a definite answer.

Contents

1 Overview

Current advance in high precision cosmology allows to test Einstein's theory of general relativity at the largest distances. And even though that Einstein's theory passed every test so far brilliantly, we should be open minded to the possibility that it could be modified at distance regimes that have not yet been thoroughly investigated. In fact, we already know about one observation that cannot be addressed satisfactorily: the cosmological constant problem. Given that zero point fluctuations are a natural contributor to the cosmological constant, we would expect it to be many orders of magnitude larger than what we observe it to be. One approach to a solution of the cosmological constant problem could lie within the matter sector of our theories, such that there is some mysterious screening mechanism operating that (almost) annihilates the value of the zero point fluctuations. The other interesting approach deals with the left hand side of the Einstein equation, and it thus modifies the gravity sector itself. In these theories the cosmological constant might be as large as zero point fluctuations predict it to be, but gravitons simple do not (or at least do only weakly) couple to it. This is an appealing approach, since we can only infer the presence of the cosmological constant by gravitational measurements anyways. However, it turns out to be notoriously difficult to modify Einsteins theory consistently, that is, without introducing any ghost degrees of freedom. But being difficult does not mean impossible, and in fact the thesis at hand deals with two interesting representatives of such modifications: *Massive Gravity* and *Brane Induced Gravity*. Both are gravity theories that recover the successes of general relativity below some distance scale r_C which is commonly set to be of the order of todays Hubble parameter H, but which weaken gravity in situations where the distances invoked are larger than r_C. In particular, given that a cosmological constant corresponds to a source of infinite extent, its gravitational effect is supposed to be suppressed for both of these theories.

Let us shortly recap the history of Massive Gravity. In the year 1939, Fierz and Pauli were the first who have written down a linear theory of a massive spin-2 particle [29] propagating on a Minkowski background $\eta_{\mu\nu}$. It can be obtained by adding the combination

$$-\frac{1}{2}m^2\left(h_{\mu\nu}h^{\mu\nu}-h^2\right) \tag{1.1}$$

to the linearized Einstein Hilbert term known from general relativity. Here, $h_{\mu\nu}$ denotes the metric fluctuation of the full metric of general relativity, that is $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Only the relative minus sign between the two Lorentz invariant combinations $h_{\mu\nu}h^{\mu\nu}$ and h^2 is allowed to appear in (1.1), since else wise the theory would contain a ghost. The mass term (1.1) is now known as 'Fierz-Pauli mass term', and by adding the linearized Einstein Hilbert term we arrive at the 'Fierz Pauli theory'. The metric fluctuation field of the Fierz Pauli model propagates five degrees of freedom, as expected for a massive spin-2 field. However, only one extra degree of freedom is directly coupling to a conserved energy momentum tensor, and for distance regimes below the 'Vainshtein radius' $r_V = \left(\frac{2M}{M_{Pl}^2 m^4}\right)^{\frac{1}{5}}$ (where M_{Pl} denotes the Planck constant and M the mass of the gravitating object under consideration) this extra degree of freedom gets strongly self-coupled with the effect of being shielded and suppressed compared to the two degrees of freedom known from general relativity [33]. Therefore, massive gravity does not spoil the successes of general relativity for 'small distances', for example for solar system observations. On the other hand, for large distances, the mass term (1.1) has the effect of an extra Yukawa suppression of the gravitational potential, as expected. In particular, it has been shown for the case of a cosmological constant that the linear theory does not couple to it at all, such that gravitons become completely blind to its presence [43]. Of course, any non-linearly completed version of the linear Fierz-Pauli theory should be able to offer an explanation for a small remnant of the cosmological constant, which is what we currently observe. One possibility would be that the theory contains a 'self-accelerated' branch, so that the observed acceleration can again be completely understood in terms of modified gravitational laws. In this respect modified gravity theories could potentially offer an explanation for the nature of 'dark energy'.

The linear model (1.1) has waited for decades to become non-linearly completed, because usually every attempt to do so was spoiled by the introduction of ghost degrees of freedom. However, the work of [14, 15, 16, 17, 18, 19, 20, 21] recently accomplished this task and identified a 3 parameter family of models which non-linearly complete (1.1) (actually, already in the 70's [24] discovered a 1-parameter subfamily, but this work remained fairly unknown until the recent rediscovery for being only published in lecture notes). The non-linearly completed model of (1.1), however, does not allow for homogenous and isotropic cosmological solutions. To circumvent this shortcome, we have to introduce a second dynamical metric and arrive at a wide class of bi-metric theories. Even though that it is possible to obtain ΛCDM-like cosmologies within these models, these are not very predictive due to the large amount of extra parameters these theories have to offer.

Thus, the route that has been followed historically to arrive at non-linear cosmological models of (1.1) was to non-linearly complete them first on a Minkwoski background, and subsequently putting them in a cosmological environment. The approach advocated in this thesis is the opposite: *We will first push the linear models in the cosmological regime, and try to non-linearly complete them in the next step.* This approach allowed us to derive the most general linear generalization of (1.1) to cosmological (instead of Minkowskian) backgrounds. By doing this we actually discovered a new linear modification that is only operative at curved backgrounds [4] and which is not covered at all by the theories of [14, 15, 16, 17, 18, 19, 20, 21]. The non-linear generalization of our model, however, is still under construction.

The construction of a consistent, linear deformation term on curved backgrounds furthermore resulted in many interesting theoretical byproducts. The most important one is the discovery of the 'self-protection' mechanism [1, 3]. It shows that a potentially threatening unitarity violation of the linear theory is always accompanied by a beforehand breakdown of the perturbative calculation anyways. Therefore, it is not possible to diagnose a unitarity violation within the linear theory, and a non-linear completed version of the theory that has to take over from thereon could very well be free from any inconsistencies. We actually showed in [2] that this self-protection mechanism is nothing else but a striking example of a theory 'classicalizing' in the recently proposed fashion [71, 72, 73, 74].

The other modified gravity theory that is being investigated in this thesis is Brane Induced Gravity (often abbreviated as 'BIG'). BIG is an extra dimensional model, where our four dimensional world (the 'brane') is embedded into a $4+n$-dimensional 'bulk'. Only gravitons are allowed to propagate into the bulk space-time, and an additional four dimensional Ricci scalar $R^{(4)}$ confines the propagation of gravitons below a distance scale $r_C = M_4/M_{4+n}^2$ (with the exception of the special case $n = 1$, where the crossover distance is given by $r_C = M_4^2/M_5^3$) on the brane

$$\begin{aligned} \mathcal{S} = \mathcal{S}_{\mathrm{EH}}^{(4+n)}[g] + \mathcal{S}_{\mathrm{EH}}^{(4)}[\omega] + S_{\mathrm{matter}}^{(4)}[\omega] \;&=\; \int \mathrm{d}^{4+n}x M_{4+n}^{2+n}\sqrt{-g}\,R^{(4+n)}[g] \\ &\quad + \int \mathrm{d}^4x\,\sqrt{-\omega}\,\left(M_4^2 R^{(4)}[\omega] + \mathcal{L}_{\mathrm{matter}}^{(4)}[\omega]\right) \end{aligned} \tag{1.2}$$

where M_{4+n} is the higher dimensional Planck constant, ω the four-dimensional sub metric and $\mathcal{L}_{\mathrm{matter}}^{(4)}[\omega]$ the matter confined on the brane. For the case $n = 1$ this model is known as the DGP model, named after its first discoverers [76]. The gravitational force is again weakened when distance scales larger than r_C are being involved, since in this case the higher dimensional Ricci scalar $R^{(4+n)}[g]$ takes over the dynamics. Intuitively, this can be thought of as 'graviton leakage into the bulk'. In fact, if we again just put a cosmological constant on a $n > 1$ brane, it will not curve the brane directions at all, but only the extra dimensional part of $R^{(4+n)}[g]$ (which is known as the extrinsic curvature). Therefore, brane induced gravity is again a promising candidate to address the cosmological constant problem.

The problem historically was that (1.2) was believed to contain a ghost in its spectrum of particles for $n > 1$ [80, 81, 82] . This ghost is unexpected physically, since we can think of (1.2) as simply a higher dimensional Einstein gravity theory, where the presence of $R^{(4)}$ is induced by integrating out heavy particles confined on the brane (hence the name 'Brane *Induced* Gravity'). Therefore, we performed a full-fledged Dirac constraint analysis for the system (1.2) and actually showed that its Hamiltonian on the constraint surface is positive definite [5]. *We can thus conclude that (1.2) is a consistent theory.* We also showed that previous calculations did not take proper care of the 00-Einstein equation, which is a constraint equation that renders the would-be problematic mode non dynamical. Therefore, previous calculations have incorrectly inferred the presence of the ghost degree of freedom for the models (1.2).

Having proven the consistency of (1.2), the window for calculating cosmological solutions of this theory is wide open. We currently derive the modified Friedman equations of the model (1.2) and will likely publish first results very soon [7]. The main problem for higher co-dimensional models (that is, for $n > 1$) is that the brane necessarily radiates gravitational radiation. This makes the analysis for $n > 1$ more complicated compared to the well established cosmological theory for $n = 1$ [88, 89, 90].

Another question this thesis is elaborating on is the microscopic picture of classicalons. Very recently, Gia Dvali and Cesar Gomez put forward the idea that black holes (which are a particular representative of a classicalon) can be understood as Bose-Einstein condensates of N weakly interacting gravitons [91, 92, 93, 94] (which is called the 'quantum portrait of black holes'). This allowed them to obtain a simple microscopic understanding for the following black hole phenomena:

- Hawking radiation is explained as *quantum depletion* of the condensate, a well known phenomenon from condensed matter physics. Given that Hawking radiation is a quantum effect, the condensate is supposed to be at the point of quantum phase transition to explain its efficiency.
- Bekenstein entropy can be understood from the quantum entanglement entropy of the condensate.
- The information paradox is resolved since the no hair theorem applies only in the strict semi-classical limit $N \to \infty$, whereas away from it we obtain important $1/N$ corrections that will allow to retrieve information thrown into the black hole after

a finite amount of time.

The work of [91, 92, 93, 94] discovers the basic analogies between black hole physics and Bose-Einstein condensation. A quantitive derivation that would, for example, allow to calculate the leading $1/N$ corrections is at the moment missing due to the complexity of the setup. However, we were able to construct an explicit, calculable non-relativistic scalar toy model that is supposed to mimic general relativity as much as possible [8]. Performing perturbation theory allows us indeed to find indications that our theory dynamically secures to stay at the point of quantum phase transition, albeit we need to go beyond the perturbative approach to claim that we really understand the behavior of our system. This non-linear numerical analysis is currently in preparation [9].

Given the cumulative nature of this thesis, it will be organized as follows: Chapter 2 introduces the reader to the framework of effective field theories and provides the understanding of technical naturalness problems. In particular, it will become clear why the cosmological constant problem is the most severe problem of technical naturalness that we know. An understanding of this chapter is absolutely essential both to appreciate the motivation for considering modified theories of gravity and to follow many arguments related to the scientific results of this thesis that are based upon the framework of effective field theories. Chapter 3 provides a summary of the recently proposed 'classicalization' mechanism, which plays an important role for some of our research results. Chapter 4 summarizes the author's paper published in the field of massive deformations [1, 2, 3, 4], whereas chapter 5 contains a summary of the author's work on the ghost freeness of brane induced gravity in higher co-dimension [5]. Chapter 6 contains the published paper in their verbatim form. Note that [1] has been published during the author's diploma studies, whereas [2, 3, 4, 5] have all been published during the author's doctoral studies. Chapter 7.1 presents the novel non-relativistic classicalon theory and its microscopic interpretation. We will submit this work in the absolutely near future. Chapter 7.1 also contains a concise introduction to the quantum portrait of classicalons.

2 Technical Naturalness Problems

Technical naturalness problems are important guidelines to new physics beyond the known theories. To put it short, technical naturalness problems occur whenever we cannot explain the smallness of a measured number at a given energy scale without invoking a conspirative fine tuning between high and low energetic physical quantities. The following sections are supposed to give a solid understanding of technical naturalness, in particular we want to explain in section 2.3 why the cosmological constant problem can be considered the most severe problem of technical naturalness that we know.

This chapter follows the arguments given in [10, 11, 12, 13].

2.1 Effective Field Theory Approach

Understanding problems of technical naturalness is not possible without a knowledge of the framework of effective field theories. The current section is devoted to give such an explanation.

We can investigate nature around us using different energy scales. For example, we may look at a crystal using everyday lives energies, describing its properties by macroscopic quantities such as its stiffness, mass, refraction index, etc. Or we may take X-ray radiation, which makes it able to resolve the constituents of the crystal, its atoms. In this language we would describe the crystal by its crystal structure, mass of the atoms, etc. We would in principle expect that we can deduce the macroscopic quantities of the crystal by knowing its microscopic properties. Effective field theories are supposed to provide a framework for such situations where a physical system can be considered using two or more widely separated energy scales.

Suppose that we are given a microscopic theory operative at high energies. As a simple but instructive example we may consider the theory of quantum electrodynamics

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \bar{\Psi} \left(\not{D} + m_e \right) \Psi \tag{2.1}$$

where, as always, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor, Ψ the electron field, m_e its mass, and $\not{D} = \gamma^\mu \left(\partial_\mu + ieA_\mu \right)$ the covariant derivative operator in Dirac notation (with e the charge of the electron).

From this Lagrangian, we can in principle calculate any observable or quantity we want. For example, we may be interested in the two-point function of the Fourier transform of the field A_μ

$$\langle A_\mu(k_1)A_\nu(k_2)\rangle = \int \mathcal{D}A\mathcal{D}\Psi\mathcal{D}\bar{\Psi}\, A_\mu(k_1)A_\nu(k_2)e^{i\int d^4x(\mathcal{L}+J_\mu A^\mu)} \tag{2.2}$$

where we choose to give a path integral representation, and we have decided to couple the field A_μ to a conserved external current J_μ. We may calculate (2.2) by splitting the path integral over A into a high energetic and low energetic part

$$\begin{aligned}
&\int \mathcal{D}A\mathcal{D}\Psi\mathcal{D}\bar{\Psi}\, A_\mu(k_1)A_\nu(k_2)e^{i\int d^4x(\mathcal{L}+J_\mu A^\mu)} \\
=\ &\int (\mathcal{D}A)_{k^2<m_e^2}\, A_\mu(k_1)A_\nu(k_2)\ (\mathcal{D}A)_{k^2>m_e^2}\,\mathcal{D}\Psi\mathcal{D}\bar{\Psi}e^{i\int d^4x(\mathcal{L}+J_\mu A^\mu)} \\
\equiv\ &\int (\mathcal{D}A)_{k^2<m_e^2}\, A_\mu(k_1)A_\nu(k_2)e^{i\int d^4x\left(\mathcal{L}_{eff}+J_\mu A^\mu\right)}
\end{aligned} \tag{2.3}$$

where we distinguished between high energetic Fourier modes of A satisfying $k^2 > m_e^2$ (where in this particular case, k^2 is calculated using Euclidean signature), and low energetic Fourier modes obeying $k^2 < m_e^2$. The dividing energy scale thus taken to be the electron mass m_e. Therefore, the contributions of the electrons automatically fall within the high energetic regime. Note that we assumed that both the external current J_μ and the external legs $A_\mu(k_1)$, $A_\nu(k_2)$ only have support for $k^2 < m_e^2$. In the last line of (2.3) we have defined the effective Lagrangian $\mathcal{L}_{eff}$ by

$$e^{i\int d^4x\mathcal{L}_{eff}} = \int (\mathcal{D}A)_{k^2>m_e^2}\,\mathcal{D}\Psi\mathcal{D}\bar{\Psi}e^{i\int d^4x\mathcal{L}} \tag{2.4}$$

which means that we just have 'integrated out the heavy fields'. We can infer the principle form of $\mathcal{L}_{eff}$ already by symmetry requirements. $\mathcal{L}_{eff}$ inherits the symmetries from $\mathcal{L}$ (as long as these symmetries are not anomalous), that is, Lorentz and gauge invariance in the first place. Therefore, organizing $\mathcal{L}_{eff}$ as a series in the dimensionality of the field operators yields

$$\mathcal{L}_{eff} = \mathcal{L}_4 + \mathcal{L}_6 + \mathcal{L}_8 + \ldots \tag{2.5}$$
$$\mathcal{L}_4 = -\frac{Z}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.6}$$
$$\mathcal{L}_6 = \frac{a}{m_e^2}F_{\mu\nu}\Box F^{\mu\nu} + \frac{a'}{m_e^2}\partial_\mu F^{\mu\nu}\partial^\lambda F_{\lambda\nu} \tag{2.7}$$
$$\mathcal{L}_8 = \frac{b}{m_e^4}\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + \frac{c}{m_e^4}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2 + \left(\partial^4F^2\,\text{terms}\right) \tag{2.8}$$
$$\ldots$$

$\tilde{F}$ is the dual field strength tensor $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$. $\mathcal{L}_n$ represents the part of $\mathcal{L}_{eff}$ containing field operators of mass dimensionality n. Since the Lagrangian must be of mass dimensionality 4, these operators have to be suppressed by some mass scale. It is clear that this mass scale must be m_e, since this is the only mass scale appearing in our problem. The remaining coefficients $Z, a, a', b, c, \ldots$ are thus dimensionless quantities. Their precise number can be obtained by explicitly performing the process of integrating out the heavy particles (2.4). For example, the leading contribution in the coupling constant e to the F^4 terms in (2.8) can be obtained from the Feynman Diagram containing 4 external photons coupled to an internal electron loop, since this graph contains the right number of external photons, and the factor $1/m_e^4$ clearly signals the appearance of four internal electron propagators.

The usage of an effective Lagrangian like (2.5) has many advantages. First of all, it is often more simple performing first a perturbative series in $1/m_e$ and then calculating a low energy observable than the other way around. In addition, keeping track of the $1/m_e$ dependence of the effective field operators allows to identify those observables which are most sensitive to the high energetic world. In the case of low energy QED, these are the observables constructed out of the operators (2.7) - in this case, for example, a modified dispersion relation for photon propagation. Last but not least, we can learn in general a lot about the interplay between different energy scales for a given physical system. This last point is most important for this thesis.

Notice that the effective Lagrangian (2.5) is not renormalizable. This is a general result for effective theories, which, however, is not a problem at all. By construction, the effective field theory (2.5) is not supposed to apply for energies above m_e. On the other hand, for energies below m_e we can take the following viewpoint: We might not be able to calculate an observable at hand at absolute precision, since (2.5) is an infinity series in $1/m_e$. However, the contribution of terms with higher mass dimensionality is suppressed compared to preceding terms, since the appearance of the extra factors $1/m_e$ has to be saturated by another dimensional quantity when calculating the observable. The only logical possibility is the characteristic momentum $k \ll m_e$ of the electrons involved in the process. Therefore, the effective field theory can be considered as a perturbative series in k/m_e and as long as we only want to calculate any given observable up to an arbitrary, but fixed precision, we can truncate the effective Lagrangian and just use the operators relevant for the wanted precision.

Since the theory for the fluctuations of the metric field $h_{\mu\nu}$ in general relativity is as well not renormalizable, we should consider it precisely in this effective field theory sense. Here, the suppressing energy scale of the higher order operators is the planck mass M_{Pl}. So we know that when we want to make any sense out of a graviton scattering process for momenta $k \geq M_{Pl}$ some new physics has to enter at the latest at planckian energies. The process of integrating in new particles to replace an effective field theory with a more fundamental theory above its regime of validity is called the 'Wilsonian completion' of the effective field theory. In this sense, (2.1) is a Wilsonian completed version of (2.5). A successful example of a Wilsonian completion of recent interest is the Higgs particle. In the standard model without the Higgs, the WW scattering process violates perturbative unitarity for energies above the electroweak scale, which is due to the breakdown of the effective field theory of the Higgsless standard model.

An alternative to Wilsonian completion will be presented in chapter 3.

In fact, even the renormalizable standard model of particle physics should be considered as an effective field theory. This belief is of course mostly due to the mysteries that occur within the standard model alone (for example the Hierarchy problem, see section 2.2). But it is also no surprise at all that the standard model comes to us as a renormalizable theory at low energies, given that the non-renormalizable terms are all suppressed by the (yet unknown) high energy scale M. The theory (2.5) would as well look renormalizable (though very boring) in the extremely weak energy limit $k \lll m_e$: it would simply become the free theory $\mathcal{L}_{eff} = \mathcal{L}_4$.

In addition, the effective field theory approach allows to obtain a sophisticated viewpoint to the physics of renormalization in general. Suppose that we would have taken not the electron mass m_e but some arbitrary 'cutoff' scale $M \leq m_e$ as the dividing energy scale in (2.3). If we calculated any loop in the effective theory (2.4), the momentum integrals would obviously be cut off at $k^2 = M^2$ by definition of the effective field theory. If we changed the value of M, the coefficients of the effective field theory would also need to change, precisely in such a way that the change of the contributions of the loop integrals are neutralized. Moreover, since the dividing energy scale M is not directly associated with a physical particle in the high energy theory, it is clear that the M dependence originating from the loop integrals must be cancelled by the M dependence of the coefficients. The preceding statements must hold true since we could also calculate any given amplitude within the high energy theory right from the start - but the high energy theory does not know anything about the artificial energy scale M. The above mechanism of cancellation between the cutoff dependence of the momentum integrals and the coefficients resembles the same situation occurring for the known renormalization process of divergent integrals in field theory. Thus, it is irresistible to interpret even a renormalizable theory like (2.1) as possibly being Wilsonian completed at some high energy scale $\Lambda \gg m_e$.

2.2 Technical Naturalness

The world of nature comes to us in energy scales. Usually, we should be able to address a question (which is not a priori confined to the quantities only present at high energies) at whatever energy scale we like, see section 2.1. This should be in particular true if we ask why a considered parameter ϵ is small. A technical natural setup must provide the following explanation to this question:

(i) There must be an underlying reason for the smallness of ϵ in the fundamental theory.

(ii) Radiative corrections from integrating out heavy particles must not give contributions larger than ϵ for any effective field theory.

If both (i) and (ii) hold, we will be able to understand the smallness of ϵ at any energy scale.

We can contrast this technical natural setup with the following technical unnatural setup that is referred to as 'fine-tuning': Suppose that we only measure the smallness of ϵ in the low energy regime while the high energy regime is not yet accessible experimentally. In this case, we could allow for large radiative corrections, since the smallness of ϵ could in principle occur due to a conspiratorial cancellation of these corrections with a large value of ϵ at high energies, such that we observe the small value of ϵ at low energies. However, this approach is unphysical, since it either suggests an interplay between the high energetic and low energetic physical world, or we accidentally observe ϵ at exactly the only energy scale where it appears to be small. Both of these possibilities are unsatisfactory. Also, note that we don't know of any example where nature has invoked this fine tuning mechanism so far.

As a simple example of a technical natural quantity we can consider the hierarchy between the size of the hydrogen atom, that is its Bohr radius a_0, and the size l_n of the nucleus, that is the size of the proton. So we have $\epsilon = l_n/a_0$ in this setup. In QCD, which we consider to be the fundamental theory in this case, the size of the proton can be derived to be of order of the inverse QCD scale $l_n \sim \Lambda_{QCD}^{-1}$, and the size of the Bohr radius is given by $a_0 \sim \alpha m_e$ (where α is the electromagnetic fine structure constant). QCD thus offers a fundamental explanation for the smallness of ϵ, since both $\alpha \ll 1$ and $m_e/\Lambda_{QCD} \ll 1$. In a low energy effective field theory treatment, where the constituents of nature are reorderd to be protons and neutrons instead of quarks and gluons, the size of the nucleus would be estimated by its de Broglie wavelength $l_n \sim m_p^{-1}$. Again, the smallness of ϵ follows, since $\alpha \ll 1$ and $m_e/m_p \ll 1$.

An important problem of technical naturalness is the electroweak hierarchy problem. We will shortly discuss it, mostly in order to contrast it with the cosmological constant problem.

To be concrete, let us for example consider the Yukawa coupling of the Higgs to any fermion Ψ with mass $m = v\,y$ (where v is the vacuum expectation value of the standard model $SU(2)$ Lorentz scalar field, and y is the Yukawa coupling of the Fermion under consideration)

$$\mathcal{L}_{Yuk} = -yh\bar{\Psi}\Psi \tag{2.9}$$

We thus see that the Higgs couples most strongly at the heavy particles and is thus most sensitive to radiative corrections caused by them. The coupling (2.9) corrects the free propagation of the Higgs field due to an internal Fermion loop, with the effect of yielding a mass shift of the order

$$\delta\mu^2 = a_0 M^2 + b_0 m^2 + \mathcal{O}(\frac{m}{M}) \tag{2.10}$$

within the low energy effective field theory with cutoff $M \leq m$. As it was explained in section 2.1, the cutoff dependence necessarily has to cancel with a corresponding bare term $-a_0 M^2 h^2$ in the low energy effective theory. This cancellation is nothing mysterious and just reflects our artificial decision of dividing the fundamental theory in a high ($k^2 > M^2$) and low ($k^2 < M^2$) energy part. Choosing a cutoff independent regularization such as dimensional regularization, this cutoff dependence would even not show up in the result for the radiative corrections. However, the contribution $b_0 m^2$ is a physical contribution caused by the presence of the heavy particles with mass m and there is no fundamental reason why it should cancel with any bare term. Therefore we generically expect that the Higgs mass is of the same order as the heaviest particle coupled to it. Assuming that there is new physics above the electroweak scale then necessarily raises the question 'Why is the higgs mass (and thus the electroweak scale) so small compared to any physics scale beyond the standard model?'. Even if there is a mechanism at the yet unknown new physics beyond the standard model that explains the smallness of the Higgs mass in the fundamental theory (which would mean that we satisfy requirement (i) above), the radiative corrections to the Higgs mass at a lower energy scale are huge, invalidating requirement (ii) above. This is the famous Hierarchy problem of electroweak interactions.

To avoid this problem of technical naturalness, there has to be a mechanism to set $b_0 = 0$. One possibility could have been that there is an extra (non-anomalous) symmetry occurring in the theory for vanishing Higgs mass. In this case, the radiative corrections have to vanish in the limit of vanishing Higgs mass $m_H \to 0$, which would forbid a term like $b_0 m^2$ in (2.10). The leading term would then have been a logarithm like $c_0 m_H^2 \log(m/m_H)$. This would have furnished a technical natural setup due to the very mild dependence of the logarithm on its argument. This described setup of having an extra symmetry for vanishing parameter to be radiatively corrected is called a technical natural setup 'in the t'Hooft sense'. The electron mass is an example for such a protection as there is an extra chiral symmetry $U(1)_L \times U(1)_R$ for vanishing mass of the electron. Unfortunately, there is no such extra symmetry in the standard model for vanishing Higgs mass, which means that we have to find something else.

A popular solution for the electroweak hierarchy problem is supersymmetry broken slightly above the electroweak scale. In this case, every fermion (boson) contributing in a way like (2.10) is accompanied by a superpartner, a boson (fermion). Given that fermions and bosons contribute with different signs to their respective coefficients b_0 every pair of super partners contributes as

$$\delta\mu^2 \sim b_0 \left(m_F^2 - m_B^2\right) \sim M_{SUSY}^2 \tag{2.11}$$

where m_F and m_B is the mass of the fermion and boson, respectively. M_{SUSY} is the supersymmetry breaking scale. If we take M_{SUSY} to be relatively close to the electroweak scale the theory will be technical natural since we would expect a Higgs mass at this order of magnitude. The only thing that we needed to do in order to obtain a technically natural theory is to assume that there is some new physics operative above the energy scales which had been probed so far (this is also true for other approaches to solve the hierarchy problem, such as technicolor). This, however, is exactly the way how high energy physics has always progressed in the past - pushing the energy frontier to discover new particles and smaller constituents.

Note that another possible solution to the hierarchy problem is to assume that there simply is no new physics beyond the electroweak scale (or, at least, no new physics coupled to the Higgs). In this case there is no physical $b_0 m^2$ contribution to (2.10). However, by adding gravity, we know that something has to happen at the Planck scale, which makes this viewpoint maybe less attractive.

2.3 The Cosmological Constant Problem

Experimentally, we measure a value of the vacuum energy of $\rho_{vac} \sim 10^{-3} eV^4$. From a theoretical viewpoint, however, we know that quantum fluctuations of all fields around us should contribute to the vacuum energy, that is, to the cosmological constant, since they furnish a space-time independent energy source. We also know that gravitons should couple to these fields, given that hydrogen atoms obey the equivalence principle even though that their energy levels are subject to the Lamb shift.

The contribution of these quantum fluctuations can be calculated by summing over their corresponding zero-point energies

$$\int_0^M d^3p \, \frac{1}{2}\sqrt{m^2+p^2} = \frac{\pi}{2}M^4 + \frac{\pi}{2}M^2m^2 + \frac{\pi}{16}m^4 + \frac{\pi}{8}m^4 \log\left(\frac{m^2}{4M^2}\right) + \mathcal{O}\left(\frac{m}{M}\right) \quad (2.12)$$

Again, the M-dependent terms have to drop out of (2.12) because physical quantities cannot depend on the cutoff. This just reflects our artificial choice of dividing the high from the low energy regime. After renormalization, there will thus be a physical contribution of the order $\sim m^4$ to the cosmological constant. Suppose now we consider a very low energy effective field theory, with the only constituents being photons, gluons, gravitons and neutrinos. Already the electron, which is the lightest particle above these, contributes an amount of

$$m_e^4 \sim 10^{23} eV^4 \quad (2.13)$$

which is off by 25 orders of magnitude compared to the measured value! So even if we understood the smallness of the cosmological constant in an effective theory incorporating the electron, we would never be able to satisfy requirement (ii) of section 2.2. The problem already occurred for energy scales of the electron mass (not to speak about if we include particles of even higher mass), which is a drastic result, since we thought that we understood nature up to energy scales of the electron fairly well. Instead, the cosmological constant problem seems to push us in the opposite direction: There seems to be some yet undiscovered physics operative at very low energy scales, likely about the energy scale given by the measured cosmological constant (and thus, by the Hubble scale). This is exactly where infrared modified theories of gravity come in. They change the gravity sector at cosmological distances, more precisely at the Hubble scale such that a cosmological constant is not gravitating as strong as you would suggest.

This thesis deals with two representatives of such modifications: Massive deformations in chapter 4 and Brane Induced Gravity in chapter 5.

Note, in particular, that we cannot solve the cosmological constant problem by assuming that new physics kicks in for UV energies. In this sense, the cosmological constant problem is much more severe than the electroweak hierarchy problem.

3 Classicalization

This chapter follows closely the lines of [71, 72, 73, 74] and will thus present the recently suggested idea of classicalization. Classicalisation is a possibility of a how a non-renormalizable theory 'self-defends' from unitary violations in the UV. As opposed to the Wilsonian mechanism, Classicalization does not introduce any new particles showing up at high energies. Instead, it is argued that in a classicalizing theory with cutoff $L_\star$ (here, the choice of name for $L_\star$ is alluding to a 'length', which makes sense in the physical context of the present chapter) the problematic high energy scattering processes such as $2 \to 2$ are suppressed and instead the final state becomes a classical object of size $r_\star \gg L_\star$, coined 'classicalon'.

To be concrete, let us consider a prototype example of a classicalizing scalar theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{L_\star^4}{4}\left(\partial_\mu\phi\partial^\mu\phi\right)^2 \tag{3.1}$$

Naively, we might be afraid that for scattering of particles with momentum transfer $k \gg L_\star^{-1}$ the above theory violates perturbative unitarity, since the amplitudes grow like $kL_\star$ in some potency. However, to have a momentum transfer k, the particles (or, let's say their corresponding wave packets) must come at least as close as $L = k^{-1}$. But we can prove that this is not possible for the theory (3.1). The non-linear term $\frac{L_\star^4}{4}\left(\partial_\mu\phi\partial^\mu\phi\right)^2$ self-sources the field ϕ with source

$$J = L_\star\partial^\mu\left(\partial_\mu\phi\left(\partial\phi\right)^2\right) \tag{3.2}$$

This gives the following representation for the equation of motion

$$\Box\phi + J = 0 \tag{3.3}$$

We will solve (3.3) perturbatively $\phi = \phi_0 + \phi_1$: ϕ_0 contains two initially free wave packets of center-of-mass energy $\sqrt{s} = 1/L \gg 1/L_\star$ which come as close as $r_\star^3$. We can thus estimate

$$\sqrt{s} = 1/L \sim \int d^3x\phi_0\Box\phi_0 \sim r_\star\phi_0^2 \tag{3.4}$$

This gives the following estimate for the integral of the source J (since $\partial_\mu \sim 1/r_\star$ and $\phi_0 \sim 1/(r_\star L)^{1/2}$)

$$\int d^3xJ \sim \frac{L_\star^4}{r_\star}\frac{1}{(r_\star L)^{\frac{3}{2}}} \tag{3.5}$$

Considering this source from far away thus amounts to replace it by its leading short distance approximation

$$J \sim \delta(\vec{r}) \frac{L_\star^4}{r_\star} \frac{1}{(r_\star L)^{\frac{3}{2}}} \tag{3.6}$$

Therefore, we can derive the solution for the perturbed field ϕ_1 using (3.3) with $J = J(\phi_0)$

$$\phi_1 \sim \frac{1}{r} \frac{L_\star^4}{r_\star} \frac{1}{(r_\star L)^{\frac{3}{2}}} \tag{3.7}$$

Evaluating this 'outside' solution at the boundary of the collision region $r = r_\star$ gives the maximum value ϕ acquires within our short distance approximation. This becomes of order ϕ_0 for

$$r_\star = L_\star (L_\star / L)^{\frac{1}{3}} \tag{3.8}$$

Therefore, we see that due to self-sourcing ϕ becomes *classically* strongly coupled at a distance scale $r_\star$ which is much larger than the supposed separation $L \ll L_\star$ of the wave packets. This means that we cannot trust the perturbative calculations to infer the breakdown of unitarity for momentum transfers $k \gg L_\star$. Instead, it rather seems that a classical stable solution has built up. This reasoning was only possible because of the $1/L^3$ term in (3.8) which in fact is a direct tracer of the derivative coupling of the scalar field. We can thus conclude that derivative couplings are absolutely essential to obtain a classicalizing theory. They lead to the characteristic behavior of a growing $r_\star$ radius when decreasing the separation L of the two wave packets - actually, $r_\star$ becomes of the order $L_\star$ precisely if the separation itself reaches the would-be problematic distance $L_\star$ and from thereon it overshoots it.

Note that we could have inferred the importance of the classicalization radius $r_\star$ already just from dimensional analysis by restoring factors of $\hbar$. In this case, $L_\star^4$ would have been replaced by a coupling constant G_ϕ of dimensionality $[G_\phi] = M^{-1} l^3$ (with M denoting a mass scale and l denoting a length scale)

$$L_\star = (G_\phi \hbar)^{\frac{1}{4}} \tag{3.9}$$

We see that $L_\star$ is an intrinsic quantum length, since it vanishes for $\hbar \to 0$. As we know, it corresponds to the scale where quantum effects would become large (and, in fact, uncontrollable in a non-renormalizable theory).

In the same way, L corresponds to a quantum length

$$L = \frac{\hbar}{\sqrt{s}} \tag{3.10}$$

which is nothing but the de-Broglie wavelength of the source.

Combining $L_\star$ and L allows to form allows to form a classical length scale that comes without a factor $\hbar$, which is precisely $r_\star$

$$r_\star = \left(G_\phi \sqrt{s}\right)^{\frac{1}{3}} \tag{3.11}$$

Given that $r_\star \gg L_\star, L$, which precisely happens for $L \ll L_\star$, the classical length scale exceeds all quantum lengths of the system. We thus conclude that the system should behave completely classical.

There is another theory which is supposed to classicalize according to [66, 96]: General relativity. Here, the theory of metric fluctuations $h_{\mu\nu}$ is as well derivatively coupled, where the Planck length L_{pl} corresponds to $L_\star$. Moreover, we already know the precise classicalon solution: It is a black hole, where $r_\star$ corresponds to the Schwarzschild radius $r_S = 2M/M_{Pl}^2 = 2L_{Pl}^2/L$. This becomes clear if we assume that the hoop conjecture holds, which states that whenever we confine an amount of energy M (in our case the energy of two colliding particles with center-of-mass energy $\sqrt{s} = M = 1/L \gg 1/L_{Pl}$) within a sphere of its corresponding Schwarzschild radius (in our case, $r_S = 2L_{Pl}^2/L \gg L_{Pl} \gg L$) we will necessarily form a black hole of mass M.

We could also consider a situation where the classicalization mechanism is combined with the standard Wilsonian situation. Consider another example of a classicalizing theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{L_\star^2}{2}\left(\partial_\mu\phi\partial^\mu\phi\right)\phi^2 \tag{3.12}$$

Suppose that the theory (3.12) is the low energy theory obtained by integrating out a particle of mass m. In this case, the leading approximation can be captured by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{L_\star^2}{2}\left(\partial_\mu\phi\partial^\mu\phi\right)\frac{m^2}{\Box + m^2}\phi^2 \tag{3.13}$$

where the factor $1/(\Box + m^2)$ is the remnant of the internal propagator of the particle which has been integrated out. Notice that (3.13) correctly reduces to (3.12) for $\Box \ll m^2$. However, for energies $\Box \gg m^2$ the fundamental high energy theory takes over the dynamics. From naively counting the derivatives of the interaction term of (3.13) we already see that we will obtain a renormalizable theory which will not classicalize. In the same sense, the coupling constant becomes the dimensionless quantity $\lambda = L_\star m$. We say that (3.13) de-classicalizes for energies above m. The crucial question is now whether $m \gg L_\star$ or not. In the latter case, the theory is purely Wilsonian completed and shows no sign of classicalization. In the former case, for intermediate energies $L_\star \ll \Box \ll m$ the theory forms a classicalon, whereas for energies above m it will be Wilsonian completed. In this latter case we thus have a finite classicalization window.

4 Massive Deformations

4.1 Massive Gravity on Minkowski space

In general relativity, the excitations $h_{\mu\nu}$ of the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ (where $\eta_{\mu\nu}$ is the Minkowski background) can be shown to describe a massless spin 2 particle. Massive Gravity consists in extending the gravitational sector such that the excitations become a spin 2 field with a mass m. Thanks to the pioneering work of Fierz and Pauli in the year 1939 [29] it is well known how to formulate a massive, free spin 2 theory. Let us shortly rederive their results in a modern language. The only two Lorentz invariant quantities available quadratic in $h_{\mu\nu}$ are h^2 and $h_{\mu\nu}h^{\mu\nu}$. So the massive gravity theory must look like

$$\mathcal{S}[H] = \frac{1}{2}\int \mathrm{d}^4x\ \sqrt{|\eta|}\, h_{\mu\nu}\mathcal{E}^{\mu\nu\alpha\beta}h_{\alpha\beta} - m^2\left(h_{\mu\nu}h^{\mu\nu} - ah^2\right) \tag{4.1}$$

Here, m is supposed to represent the mass of the graviton, whereas the relative coefficient a has to be determined by demanding that the theory is free of ghost instabilities. $\mathcal{E}^{\mu\nu\alpha\beta}$ is denoting the linearized Einstein-Hilbert operator around the Minkwoski background $\eta_{\mu\nu}$:

$$h_{\alpha\beta}\mathcal{E}^{\alpha\beta\mu\nu}h_{\mu\nu} = -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h\partial^\lambda h \tag{4.2}$$

The introduction of the mass term in (4.1) breaks the gauge invariance

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu}\zeta_{\nu)} \tag{4.3}$$

of the massless theory (the round brackets denote symmetrization). However, this gauge invariance can be restored by introducing four Stückelberg degrees of freedom

$$h_{\mu\nu} \to H_{\mu\nu} = h_{\mu\nu} + \partial_{(\mu}A^\perp_{\nu)} + \partial_\mu\partial_\nu\phi \tag{4.4}$$

where $A^\perp_\mu$ is subject to the transversality condition $\partial^\mu A_\mu = 0$. With the replacement (4.4) the theory (4.1) becomes automatically invariant under (4.3) if we assume that $A^\perp_\mu$ and ϕ transform as

$$\begin{aligned} A^\perp_\mu &\to A^\perp_\mu - \zeta^\perp_\mu & (4.5)\\ \phi &\to \phi - \zeta & (4.6) \end{aligned}$$

where $\zeta_\mu^\perp$ denotes the transversal ($\partial^\mu \zeta_\mu^\perp = 0$) and ζ the longitudinal part of ζ_μ:

$$\zeta_\mu = \zeta_\mu^\perp + \partial_\mu \zeta \tag{4.7}$$

It is obvious that we can always find a gauge where $H_{\mu\nu} = h_{\mu\nu}$ and we thus recover the original theory (4.2). We just added gauge redundancies.

Alternatively, we could have introduced five Stückelberg fields A_μ (not transversal) and ϕ. This would give us a $U(1) \times U(1)^4$ symmetry. We will use this approach later, but will stick here with the transversal decomposition.

Already just from looking at (4.4) it becomes clear that the dominant degree of freedom at high energies is given by ϕ, since it contains two extra derivatives and is thus k^2 enhanced. Actually, this statement becomes solidified by noting that ϕ is nothing else but a goldstone boson of the theory (4.2), and thus the goldstone boson equivalence theorem exactly states that it represents the dominant contribution at high energies [30, 59].

The ϕ degree of freedom is potentially introducing a higher derivative structure in (4.1). The relevant terms look like

$$-\frac{1}{2}m^2 \int d^4x \left(\partial_\mu \partial_\nu \phi \partial^\mu \partial^\mu \phi - a\,(\Box\phi)^2\right) = -\frac{1}{2}m^2 \int d^4x\,(1-a)\,(\Box\phi)^2 \tag{4.8}$$

We thus see that we can only get rid of the problematic four derivative term by setting $a = 1$. This yields precisely the Fierz-Pauli form of the mass term

$$\mathcal{L}_{FP} = -\frac{1}{2}m^2\big(h_{\mu\nu}h^{\mu\nu} - h^2\big) \tag{4.9}$$

4.1.1 Non-linear completed Massive Gravity on Minkowski space

Recently, Gababadze et al. were able to construct a particular, consistent, non-linear completion of the Fierz-Pauli theory [14, 15]. They were using a bottom-up approach to construct a non-linear version of (4.9) such that for each order in the perturbation theory the higher derivative terms of the non-linear equivalent of the Stückelberg field ϕ drop out. While this approach guaranteed the absence of ghosts in the Stückelberg sector, the work of [16, 17, 18, 19, 20, 21] proofed that this theory is in fact free of unitary violations in general by using a complete constrained analysis. Moreover, they generalized this proof for massive bi-metric theories.

To be concrete, let us consider the non-linear completed model of massive gravity on Minkowski space

$$\mathcal{L} = M_{Pl}^2 \int d^4x \sqrt{-g}\left(R[g] + 2m^2 \sum_{n=0}^{3} \beta_n e_n(\sqrt{g^{-1}\eta})\right) \tag{4.10}$$

where the $e_n(\mathbb{M})$ are the elementary symmetric polynomials of the eigenvalues of $\mathbb{M}$

$$
\begin{aligned}
e_0(\mathbb{M}) &= 1 &(4.11)\\
e_1(\mathbb{M}) &= [\mathbb{M}] &(4.12)\\
e_2(\mathbb{M}) &= \frac{1}{2}\left([\mathbb{M}]^2 - [\mathbb{M}^2]\right) &(4.13)\\
e_3(\mathbb{M}) &= \frac{1}{6}\left([\mathbb{M}]^3 - 3[\mathbb{M}][\mathbb{M}^2] + 2[\mathbb{M}^3]\right) &(4.14)\\
e_4(\mathbb{M}) &= \frac{1}{24}\left([\mathbb{M}]^4 - 6[\mathbb{M}]^2[\mathbb{M}^2] + 3[\mathbb{M}^2]^2 + 8[\mathbb{M}][\mathbb{M}^3] - 6[\mathbb{M}^4]\right) &(4.15)\\
e_k(\mathbb{M}) &= 0 \quad \text{for} \quad k > 4 &(4.16)
\end{aligned}
$$

here, the square brackets denote the trace of the corresponding matrix. Note that the Fierz-Pauli structure reappears in e_2. One combination of the 5 coefficients β_n corresponds to the cosmological constant, another to the mass and yet another has to be set to zero to avoid a linear tadpole term. Therefore, the theory (4.10) contains 3 new parameters: the mass m and two combinations of the β_n.

The main problem with this theory is that it does not allow for a homogeneous and isotropic cosmological solution [22]. In addition, the introduction of the static background reference η might feel uncomfortable.

However, as we already mentioned, (4.10) can be generalized to a bi-metric theory that contains a massive spin-2 mode within its spectrum. In this case, the above objections do not apply. The action for this theory looks like (including the coupling of both metrics g and f to an arbitrary bunch of matter fields Ψ_a)

$$
\begin{aligned}
\mathcal{L} &= \int d^4x\sqrt{-g}\left(M_{Pl}^2 R[g] + \mathcal{L}_{matter}[g,\Psi_a] + 2M_{Pl}^2 m^2 \sum_{n=0}^{3}\beta_n e_n(\sqrt{g^{-1}f})\right)\\
&+ \int d^4x\sqrt{-f}\left(M_\star^2 R[f] + \mathcal{L}_{matter}[f,\Psi_a]\right) \qquad (4.17)
\end{aligned}
$$

This is the only ghost-free possibility to construct such a bi-metric theory. In particular, any other coupling to matter than the one used in (4.17) would reintroduce ghosts (with the exception of switching off the coupling of f to matter completely).

For the case of switching off the coupling of f to matter an exhaustive parameter fit to cosmological background evolution data has been performed [25]. It has been shown that the parameter space allows for self-accelerated solutions of (4.17) (that is, switching the explicit occurrence of the cosmological constant off in (4.17)) that are in good agreement with the data - even though that ΛCDM is slightly preferred. However, the main problem with these solutions is that they are degenerate in parameter space due to the many parameters available. Therefore, the theory (4.17) is less predictive. The large parameter space of (4.17) is a negative aspect in general.

Another possibility to allow for standard cosmological evolution which simultaneously reduces the parameter space significantly is to demand to have an additional discrete symmetry $f \leftrightarrow g$. This can be achieved by coupling both metrics to matter and setting $M_{Pl} = M_\star$ and $\beta_n = \beta_{4-n}$, since

$$e_k(\mathbb{M}) = \frac{e_{4-k}(\mathbb{M}^{-\mathbb{1}})}{\det(\mathbb{M}^{-1})} \tag{4.18}$$

In this setup, we always have the solution $g = f$, which incorporates (with possibly shifted cosmological constant) the standard Friedman evolution. However, one of the implications of our findings in section 4.2 is that this approach suffers from perturbative instabilities whenever $m^2 > H^2 + \frac{1}{3}\dot{H}$ is violated.

4.1.2 Degravitation

The idea of massive gravity is that the introduction of the mass term is supposed to affect only predictions that take place at a length scale $r \gtrsim \frac{1}{m}$, and as long as m is very small (reasonably of the order of the inverse size of the universe), it would not threat the successes of general relativity on those small length scales. For example, in Fierz Pauli theory (that is, at the quadratic order) the Newtonian potential will get modified to a Yukawa potential $\sim \frac{e^{-mr}}{r}$ [27]. The extra exponential factor will thus be negligible if $mr \ll 1$, but will yield a strong suppression for $mr \gg 1$.

As it is well known, live is not that simple in massive gravity theories due to the introduction of new degrees of freedom. In fact, the Stückelberg degree of freedom ϕ which was introduced in section 4.1 couples in general to the trace of the energy-momentum tensor. However, Vainshtein discovered [33] that ϕ becomes classically strongly coupled for $m \to 0$. To be concrete, for the case of the gravitational field of a point mass M, the 'Vainshtein radius' where this strong coupling occurs is given by $r_V = \left(\frac{2M}{M_{Pl}^2 m^4}\right)^{\frac{1}{5}}$. This strong coupling shields the gravitational effect of ϕ, so that within r_V we recover standard general relativity including all of its solar system experiment successes.

Given that a cosmological constant corresponds to a space-time source with infinite extent, we intuitively expect its gravitational effect to be suppressed according to the extra exponential Yukawa factor.

In this picture, the cosmological constant might be as large as we expect it to be from zero-point fluctuations, but gravitons are coupling to it with a reduced strength. So the right question to ask suggested by this approach would not be

Why is the cosmological constant so small?

Instead, it would answer

Why is the cosmological constant gravitating so little?

Given that the mass of the graviton is protected from radiative corrections (because of the extra gauge symmetry occurring for $m = 0$), this would encompass an important step forward in finding a technical natural explanation for the value of the cosmological constant. This above idea was first put forward in [43].

We will solidify this idea with a reliable calculation in the following.

Let us start with doing something which looks stupid in the first place: Instead of incorporating the presence of a cosmological constant Λ into the background space-time, we will use it to source the fluctuations $h_{\mu\nu}$ about a Minkowski space-time

$$\mathcal{E}^{\mu\nu}_{\alpha\beta} h_{\mu\nu} = \frac{\Lambda}{M_{Pl}^2} \eta_{\alpha\beta} \tag{4.19}$$

The solutions for $h_{\mu\nu}$ are quadratically growing with spacetime coordinates and correspond to the linearized version of the deSitter metric (in different coordinate systems). One such solution, for example, is

$$h_{00} = 0,\ h_{0i} = 0,\ h_{ij} = \frac{\Lambda}{3M_{Pl}^2}\Big(t^2 \delta_{ij} + x_i x_j\Big) \tag{4.20}$$

which corresponds to the linearized version of the de Sitter metric in closed FRW slicings. We just got out what we could have expected to get: Our system evolves back into deSitter space-time and the linear approach of (4.19) will eventually loose its applicability.

However, when including a mass term in equation (4.19) the situation is entirely different

$$\mathcal{E}^{\mu\nu}_{\alpha\beta} h_{\mu\nu} - m^2\Big(h_{\alpha\beta} - \eta_{\alpha\beta} h\Big) = \frac{\Lambda}{M_{Pl}^2} \eta_{\alpha\beta} \tag{4.21}$$

Now, one obtains a solution proportional to the Minkowski metric itself

$$h_{\alpha\beta} = \frac{\Lambda}{3m^2 M_{Pl}^2} \eta_{\alpha\beta} \tag{4.22}$$

Therefore, the full metric is again given as a Minkowski solution

$$g_{\alpha\beta} = \Big(1 + \frac{\Lambda}{3m^2 M_{Pl}^2}\Big) \eta_{\alpha\beta} \tag{4.23}$$

up to an irrelevant constant rescaling. The fluctuation field $h_{\mu\nu}$ became 'blind' to the presence of the cosmological constant! This is the essence of 'degravitation'.

One might object that $\Lambda \lesssim m^2 M_{Pl}^2$ must be fulfilled so that the solution $h_{\mu\nu}$ in (4.22) is consistent with the linear approximation used. This is true, but the nice thing is that including any finite number of non-linear terms in $h_{\mu\nu}$ does not change the picture. Any solution of the form $h_{\mu\nu} = C\eta_{\mu\nu}$ (with C constant) will also be annihilated by non-linear terms originating from the Ricci scalar R, since they always come with derivatives. In addition, the non-linear terms that complete the mass term will just yield some polynomial for C, which vanishes for $C = 0$ and will become infinite for $C \to \infty$ (since the Hamiltonian must be bounded from below). Therefore, this polynomial should always admit for real solutions.

The question is what is happening for the realistic case of an infinite number of non-linear terms. Do they sum to something reasonable for some degravitating solution? Actually, this can easily be answered since we know the complete non-linear theory of massive gravity: By choosing $g_{\mu\nu} = \kappa\eta_{\mu\nu}$ (with $\kappa = const.$) we can in general solve (4.10) even if we switch on an explicit cosmological constant. However, perturbations around this solution are modified by the Stückelberg scalar ϕ if $\Lambda \gtrsim (10^{-3}eV)^4$ [23]. Thus, the degravitation mechanism directly only works for cosmological constants smaller than $\Lambda \sim (10^{-3}eV)^4$, since else wise we would spoil the successful description of solar system experiments by general relativity.

However, it remains an interesting question whether the theory (4.17) and especially a non-linear completed version of (4.71) could potentially degravitate larger cosmological constants.

Note that the introduction of the mass term broke in some sense the democratic principle of general relativity, as it does discriminate sources with respect to their extension.

4.2 Massive Deformations on Curved Space

Most of the knowledge about properties of massive gravity is confined to applications where the graviton propagates on a Minkowski or deSitter background $\eta_{\mu\nu}$. From a phenomenological point of view, given that we ultimately want to address cosmological questions, it looks very interesting to modify graviton propagation on a realistic FRW universe. This is exactly what this section at hand is supposed to do. Since we want to stick at the IR leading deformation term, the deformation should be such that we wont't introduce any new derivatives on $h_{\mu\nu}$. Thus, let us start considering a general action of the following form for the metric fluctuation field $h_{\mu\nu}$

$$\mathcal{S}[H] = \frac{1}{2}\int_M \mathrm{d}^4x\ \sqrt{|g_0|}\, h_{\mu\nu}\left[\mathcal{E}(g_0,\nabla)^{\mu\nu\alpha\beta} + \mathcal{M}(g_0)^{\mu\nu\alpha\beta}\right] h_{\alpha\beta}\,. \tag{4.24}$$

where now $\mathcal{E}^{\mu\nu\alpha\beta}$ is the perturbed Einstein-Hilbert operator constructed out of the FRW background space-time $g_{0\,\mu\nu}$ and its corresponding covariant derivative operator ∇. $\mathcal{M}$ is the most general deformation matrix constructed out of the background space-time metric g_0. The naive generalization of the Fierz-Pauli structure to curved space times would read as

$$\mathcal{M}(g_0)_{\mu\nu\alpha\beta} = m^2\left(g_0^{\mu\nu}g_0^{\alpha\beta} - g_0^{\mu\alpha}g_0^{\nu\beta}\right) \tag{4.25}$$

Let us first start by discussing this naive generalization. Even though that it will turn out that this theory is phenomenologically not viable, we can learn a lot from it conceptually.

It is well known that there exists a parametrical relation between the mass m of the graviton and the cosmological constant Λ when specializing (4.25) to a deSitter background:

$$m^2 > H^2 = \frac{1}{3}\Lambda \tag{4.26}$$

where H denotes the Hubble parameter. This relation is called the 'Higuchi bound', after its first discoverer [34]. If the Higuchi bound is violated, the corresponding quantum theory contains a state with negative norm in its spectrum, that is, a ghost. Actually, we were able to generalize this bound to an arbitrary FRW background. Note that there might be interesting physical questions associated with a generalized Higuchi bound to FRW spacetimes, since in this case the right hand side involving H will become a function of the time t and accordingly the bound will no longer just be a parametrical relation, so that possibly there will always be times where the bound is violated. Does this mean that we have to discard the massive gravity theory on FRW backgrounds?

The answer is no. In fact we found that the Higuchi bound becomes resolved into two different bounds on an FRW spacetime. These bounds are given in such a way that a possible unitarity violating negative norm state always comes along with an instability of the theory, which invalidates the derivation and demands for a non-linear completion. In this sense the theory is *self-protected*. To be concrete, the bound associated with the occurrence of negative norm states looks like

$$m^2 > H^2 + \dot{H} \tag{4.27}$$

whereas the bound which signals the breakdown of perturbation theory reads at high energies as

$$m^2 > H^2 + \frac{1}{3}\dot{H} \tag{4.28}$$

We see that for non-phantom matter, $\dot{H} < 0$, the stability bound is always stronger than the unitarity bound. Therefore, suppose we start out in a region in space-time where both bounds are satisfied. If we now evolve the system in such a way that potentially a violation of the unitarity bound could occur, we always violate the stability bound before, and thus we are leaving the applicability of the perturbation theory in which the unitarity bound has been derived. From now on, we have to use a non-linear completed version of our theory, and there is no a priori reason why there should occur any violation of unitarity within this non-linear completed theory.

The paper [1] (see section 6.1) describes a derivation of the generalized Higuchi bounds based upon the Stückelberg approach and the Goldstone Boson equivalence theorem, which allows to extract the relevant degree of freedom at high energies. We will summarize this derivation in the following.

Self-Protection in the Stückelberg sector

Let us introduce four Stückelberg fields to restore gauge invariance

$$h_{\mu\nu} \to H_{\mu\nu} = h_{\mu\nu} + \nabla_{(\mu} A^{\perp}_{\nu)} + \nabla_\mu \nabla_\nu \phi \, . \tag{4.29}$$

where $A^{\perp}_{\mu}$ denotes the transversal ($\nabla \cdot A^{\perp} = 0$) and ϕ the longitudinal part of the Stückelberg vector $A_\mu = A^{\perp}_{\mu} + \nabla_\mu \phi$. This obviously restores the invariance

$$h_{\mu\nu} \to h_{\mu\nu} + \nabla_{(\mu} \zeta_{\nu)} \tag{4.30}$$

where $A^{\perp}_{\mu}$ and ϕ absorb the transversal and longitudinal parts of ζ_μ, respectively

$$A^{\perp}_{\mu} \quad \to \quad A^{\perp}_{\mu} - \zeta^{\perp}_{\mu} \tag{4.31}$$

$$\phi \quad \to \quad \phi - \zeta \tag{4.32}$$

where $\zeta_\mu = \zeta^{\perp}_{\mu} + \nabla_\mu \zeta$ with $\nabla \cdot \zeta^{\perp}$. As it was the case for the flat background metric, at high energies, (4.29) will obviously be dominated by the Stückelberg scalar Φ due to the appearance of two extra derivatives which results in a k^2 enhancement. This can again be solidified by noting that ϕ is nothing else but a goldstone boson of the theory (4.24), and thus the goldstone boson equivalence theorem exactly states that it represents the dominant contribution at high energies [30, 59]. We will thus focus on this degree of freedom, which allows to analyze the leading behavior of the system at high energies.

The Fierz-Pauli structure of (4.25) ensures that the four derivative terms drop out of the ϕ sector of the theory. However, ϕ mixes with the metric perturbation h as

$$\frac{1}{2} H_{\mu\nu} \mathcal{M}(g_0)^{\mu\nu\alpha\beta} H_{\alpha\beta} \supset m^2 \left(\Box h - \nabla^\mu \nabla^\nu h_{\mu\nu} \right) \phi = m^2 \left(R^{(0)}_{\mu\nu}(g) h^{\mu\nu} - R^{(1)}(g,h) \right) \phi \, , \tag{4.33}$$

where we have integrated (hiddenly) by parts and introduced the Ricci tensor $R^{(0)} \equiv R$ evaluated on the background configuration g_0, as well as the Ricci scalar $R^{(1)}$ expanded to first order in h.

In order to eliminate the kinetic mixing term $R^{(1)}(g_0, h)\phi$, we carry out a conformal transformation

$$\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \equiv (1+\omega)^2 g_{\mu\nu}, \tag{4.34}$$

which, at the linear level, is equivalent to $\hat{h}_{\mu\nu} = h_{\mu\nu} + 2\omega\,\eta_{\mu\nu}$. The Einstein–Hilbert term transforms as

$$\sqrt{-g}\Omega^2 R = \sqrt{-\hat{g}}\left(\hat{R} - 6\Omega^{-2}\,\hat{g}^{ab}\partial_a\Omega\partial_b\Omega\right) . \tag{4.35}$$

In order to eliminate the mixing between $h_{\mu\nu}$ and ϕ we must choose $\Omega^2 = 1 - m^2\phi$ or, equivalently (since ϕ is a first order quantity in the expansion of $H_{\mu\nu}$), $\omega = -\frac{1}{2}m^2\phi$.

Therefore, the conformal transformation (4.35) contributes a standard kinetic term for the Goldstone–Stückelberg scalar

$$-\frac{3}{2}m^4\partial_\mu\phi\partial^\mu\phi \tag{4.36}$$

while the massive deformation (4.25) gives rise to a non-standard kinetic contribution with the metric field replaced by the background Ricci tensor,

$$m^2\left((\Box\phi)^2 - \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi\right) = m^2 R_{\mu\nu}\partial^\mu\phi\partial^\nu\phi \tag{4.37}$$

which occurred due to exchange of covariant derivatives when integrating by parts. The action for ϕ is thus given by

$$S = \int \mathrm{d}^4x\sqrt{-g_0}\,\left(A\dot{\phi}^2 + B^{ij}(\partial_i\phi)(\partial_j\phi) + \dot{\phi}D^i\partial_i\phi\right) , \tag{4.38}$$

with $A = m^2(-3m^2g^{00}+R^{00})$, $B^{ij} = m^2(-3m^2g^{ij}+R^{ij})$, and $D^i = 2m^2(-3m^2g^{i0}+R^{i0})$. In general, these coefficients are spacetime dependent functions. Note that in (4.38) we have not displayed any potential terms such as $R(g)h\phi$, since we investigate the high energy sector of the theory (4.25) anyways.

In the case of a generic Friedman background geometry the action for the Goldstone–Stückelberg scalar reduces to

$$S = \int \mathrm{d}^4x\sqrt{-g_0}\,\left(A(t)\dot{\phi}^2 + B(t)\left(\vec{\nabla}\phi/a\right)^2\right) , \tag{4.39}$$

where $A(t) = 3m^2(m^2 - \dot{H} - H^2)$, $B(t) = -m^2(3m^2 - \dot{H} - 3H^2)$, $H = H(t)$ denotes the Hubble parameter, and $a = a(t)$ the scale factor. These coefficients depend on the energy-momentum source curving the Friedman background and, in particular, can change signs during the background evolution.

We necessarily have to demand that $A > 0$ in order to avoid that ϕ propagates a ghost degree of freedom. This can be directly understood by the fact that A controls the relative sign between $\dot{\phi}$ and the conjugated momentum field $\pi = \frac{\delta \mathcal{L}}{\delta \dot{\phi}}$. Expanding the fields as usual

$$\hat{\phi}(t,\mathbf{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} C \, \Big(u(t,\mathbf{k}) \exp{(\mathrm{i}\mathbf{k}\cdot\mathbf{x})} \hat{a}(\mathbf{k}) + u^*(t,\mathbf{k}) \exp{(-\mathrm{i}\mathbf{k}\cdot\mathbf{x})} \hat{a}^\dagger(\mathbf{k}) \Big) \tag{4.40}$$

etc., (with C a normalization constant and $u(t,\mathbf{k})\exp{(i\mathbf{k}\mathbf{x})}$ and $u^\star(t,k)\exp{(-i\mathbf{k}\mathbf{x})}$ the elementary solutions of the equation of motion of ϕ) and demanding standard commutation rules

$$\left[\phi(t,\mathbf{x}), \pi(t',\mathbf{x}')\right]_{t=t'} = i\delta^{(3)}\left(\mathbf{x}-\mathbf{x}'\right) \ , \ldots \ , \tag{4.41}$$

yields directly

$$\left[a(\mathbf{k}), a^\dagger(\mathbf{k}')\right] = \mathrm{sign}(A) \ \delta^{(3)}(\mathbf{k}-\mathbf{k}'). \tag{4.42}$$

since the relative sign change between $\dot{\phi}$ and π can only be absorbed by a corresponding change of sign in the commutation relations of the creation and annihilation operators. For a detailed calculation see section 6.1. The wrong sign in the commutation relation (4.42) results directly in negative norm states. Consider for example the one-particle states $|\mathbf{k}\rangle = a^\dagger(\mathbf{k})|0\rangle$:

$$\langle \mathbf{k}'|\mathbf{k}\rangle = \langle 0|a(\mathbf{k}')a^\dagger(\mathbf{k})|0\rangle = \left[a(\mathbf{k}'), a^\dagger(\mathbf{k})\right] \langle 0|0\rangle = \mathrm{sign}(A) \ \delta^{(3)}(\mathbf{k}-\mathbf{k}') \tag{4.43}$$

Negative norm states are unacceptable, since they spoil the probabilistic interpretation of the theory. We therefore have to demand that $A > 0$. As promised, this gives directly the unitarity bound

$$m^2 > H^2 + \dot{H} \tag{4.44}$$

Once we have $A > 0$, the classical equation of motion for ϕ requires that $B < 0$ to obtain a theory that is classically stable, since else wise the sign of $\ddot{\phi}$ and $\nabla^2\phi$ would coincide, as can easily be seen from (4.39). For a detailed derivation, we once again refer the reader to section 6.1. The requirement $B < 0$ gives directly the stability bound

$$m^2 > H^2 + \frac{1}{3}\dot{H} \tag{4.45}$$

Note that we have shown in [1] that a minimal coupling to matter $h_{\mu\nu}\delta T^{\mu\nu}$ does not spoil the derivation of the bounds at high energies.

Self-Protection and cosmological perturbation theory for all energies

If we want to study the behavior of the theory (4.25) for super-Hubble scales $k \ll H$ and intermediate energy regimes $k \sim H$ we need to go beyond the above presented Stückelberg approach. We were doing this by performing a full-fledged cosmological perturbation analysis, encompassing both all degrees of freedom and all energy regimes. To this end, let us consider the following decomposition, which is compatible with the $SO(3)$ invariance of the FRW background space-time

$$
\begin{aligned}
h_{00} &= -E \,, && (4.46)\\
h_{i0} &= a\left[\partial_i F + G_i\right] \,, && (4.47)\\
h_{ij} &= a^2\left[A\delta_{ij} + \partial_i\partial_j B + \partial_{(j}C_{i)} + D_{ij}\right] \,. && (4.48)
\end{aligned}
$$

Here, E, F, A, and B denote SO(3) scalars, G_i and C_i are the components of transverse SO(3) vectors ($\partial^a G_a = 0, \partial^b C_b = 0$), and the D_{ij} denote the components of a transverse-traceless rank-2 SO(3) tensor ($\partial^a D_{ab} = 0$ and $\delta^{ab} D_{ab} = 0$).

In the same sense, we assume the energy-momentum tensor of a perfect fluid and decompose its perturbation as follows

$$
\begin{aligned}
\delta T_{00} &= \delta\rho - \bar{\rho}\, h_{00} \,, && (4.49)\\
\delta T_{0i} &= -\left(\bar{\rho} + \bar{p}\right)\left(\delta u_i^V + \partial_i u\right) + \bar{p}\, h_{0i} \,, && (4.50)\\
\delta T_{ij} &= \bar{p}\, h_{ij} + a^2 \delta_{ij}\delta p \,, && (4.51)
\end{aligned}
$$

where u denotes the longitudinal part and u_i^V its transversal part ($\partial^i u_i^V = 0$).

Due to the background isometries, the equations of motion separate into tensor, vector and scalar parts. We were able to analytically investigate these resulting equations with respect to their stability behavior, both at sub-Hubble scales $k \gg H$ (confirming the results of the Stückelberg approach) and for zero momentum $k = 0$. For the detailed calculations, see [3], where these findings have been published. Here, we will only summarize the main results. To start with, let us consider the tensor fluctuations, which are governed by the following equation

$$
-\ddot{D}_{ij} - 3H\dot{D}_{ij} + \left(\Delta/a^2\right) D_{ij} - m^2 D_{ij} = 0 \,. \qquad (4.52)
$$

This equation is obviously stable for all times and all momenta. Therefore, the tensor part of our theory does not yield any new results concerning the stability behavior. It is worth mentioning that (4.52) reduces to its counterpart in the undeformed theory in the $m \to 0$ limit. Provided the deformation parameter is small, $m^2 \lesssim H^2$, the deformation term in (4.52) will not change the dynamics very much. In particular, the frozen mode on super-Hubble scales, $-\Delta/a^2 \ll H^2$, is still present like in the undeformed theory.

To obtain a classically stable vector sector, the following relation has to be satisfied

$$\left[-\left(\Delta/a^2\right)+3\dot{H}+2m^2\right]\mathbf{G}\geq 0 \tag{4.53}$$

For high momenta $k_{\text{phys}}^2 = k^2/a^2 \gg H^2, m^2$ this relation is obviously always satisfied. This is why we were not able to obtain it with the pure Stückelberg analysis. Demanding that (4.53) is fulfilled for all momenta k_{pays} yields the new bound

$$\dot{H}+\frac{2}{3}m^2>0 \tag{4.54}$$

This new bound further supports the self-protection mechanism, as it does not come along with a potential violation of unitarity that could have set in before.

Finally, let us consider the scalar sector of the theory, which turns out to be the richest. The equations in the scalar sector are very lengthy and complicated to investigate. However, the unitarity bound directly reappears as the coefficient in front of the $\ddot{A}$ term

$$\frac{m^2-H^2-\dot{H}}{(1-w^2)^2(1+3w^2)}\ddot{A} \tag{4.55}$$

where w is the equation of state parameter $\delta p = w\delta\rho$. If this coefficient switched sign, the expression for the momentum field corresponding to A would switch sign as well, with the same consequence of occurent negative norm states (see the discussion succeeding (4.40)). Therefore, we have shown that the same unitarity bound occurs irrespective of the momenta under consideration.

The stability bound is harder to derive. We were only able to perform an analytical derivation for $k_{\text{phys}}^2 \gg m^2, H^2$ and $k_{\text{phys}} = 0$. In the former case, we directly reproduced the stability bound found using the Stückelberg analysis, as expected. The stability bound in the latter case is a more complicated expression, and it is easiest illustrated in figure 4.1. This figure should be read as follows: Starting in a regime where the theory is both stable and unitary, that is for large t, and evolving the system backwards in time, we see that we always first hit a region of classical instability before we would have hit the unitarity violating regime. In fact, the stability bound is even stronger for $k_{\text{phys}} = 0$ than it is for large k_{phys}. From looking at the picture, we intuitively infer that the stability violating region corresponding to intermediate momenta should lie in between those for $k_{\text{phys}} = 0$ and $k_{\text{phys}} \gg H^2, m^2$. We were indeed able to confirm this conjecture by numerical simulations.

Cosmological classicalons

The described self-protection mechanism is in fact a striking example of a classicalizing theory (see chapter 3 for a short introduction into classicalization). The essence

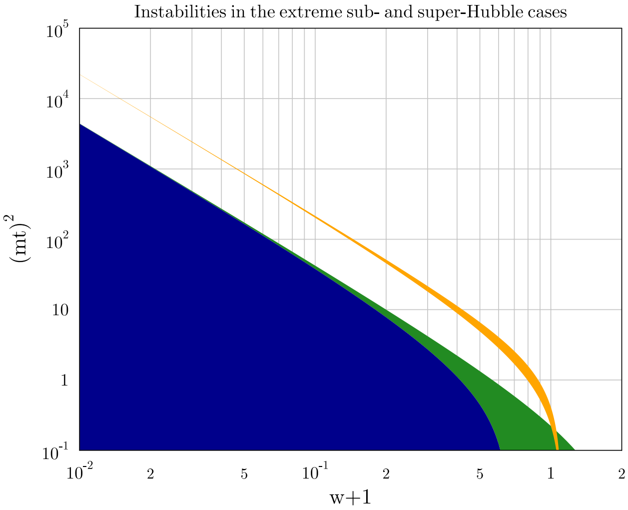

Figure 4.1: Instabilities in the extreme sub- and super-Hubble cases. In the orange region (top, detached), the system is classically unstable for $k_{\text{phys}} = 0$. The dark-blue region (bottom, left) depicts the region, where unitarity would be violated. In the green region (adjacent to the former), the system is classically unstable for large k_{phys}, that is, for $k^2_{\text{phys}} \gg m^2, H^2$.

of classicalization is that a theory protects itself from a potential violation of unitarity at high energies by entering a strong-coupling regime and consequently forming a classicalon. In the case of massive gravitons, the identification of the energy of the physical process with the energy stored in the background space-time (which is proportional to H^2) makes the link to classicalization obvious, since we also have the effect of forming a classical object (in our case, a new background space-time when the stability bound is violated) before the theory is able to enter the unitarity violating regime (which would be given by a violation of the unitarity bound). Moreover, since we have shown that the self-protection mechanism applies to the complete dynamical domain (4.2), and in particular the small momenta modes become unstable first, we should expect that the new classical background itself is predominantly populated by soft modes, as it should be the case for a generic classicalon. This new classical background literally embraces the largest classicalon possible, as it encompasses the whole size of the universe.

We can also identify the equivalents of the classicalization radius $r_\star$ and the would-be unitarity violating length scale $L_\star$. In our case, given that we are considering a homogenous but non-static background geometry, these physical quantities appear as time scales, where $t_\star$ is the time scale where the theory becomes strongly coupled (which corresponds to $r_\star$), and t_U the time scale where the would-be unitarity violation occurs (which corresponds to $L_\star$). A basic property of $r_\star$ was given by its increase with increasing energy of the physical process under consideration. In our case, this would correspond to an increase of the energy density ρ stored in the Friedman background. In the previous derivations, we always assumed a flat background. However, by doing this, we cannot alter ρ without changing a (and thus the physical time) due to the constraint of being at the critical energy density. Therefore, we will now consider the more general metric ansatz

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{1}{1-kr^2} dr^2 + r^2 d\Omega^2 \right) \tag{4.56}$$

For simplicity, we will only consider the high energy sector of the theory. The unitarity bound is still looking the same

$$m^2 > H^2 + \dot{H} \,. \tag{4.57}$$

The saturation of this bound defines t_U.
The stability bound gets modified to

$$m^2 > H^2 + \frac{1}{3}\dot{H} + \frac{2}{3}\frac{k}{a^2} \tag{4.58}$$

where its saturation defines $t_\star$. Increasing the energy stored in the Friedman background corresponds to making the universe 'more and more closed', which means that k becomes larger and larger and thus the stability bound becomes stronger and stronger, or equivalently, $t_\star$ becomes larger and larger, which proves the desired property of increasing classicalization 'radius'.

We can even prove that the massive gravity theory always possesses a finite classicalization window. For the low momenta modes $k^2_{phys} \ll m^2, H^2$ of the graviton this is obvious, since their evolution cannot be altered by new UV physics kicking in above some scale Λ (note that Λ must be above m, because else wise the effective theory of a graviton of mass m would have been incomplete). However, even the high momenta modes have a finite classicalization window. This can be seen by assuming that the mode dominating at high energies, that is, the Stückelberg mode ϕ, is UV completed in the Wilsonian sense

$$R^{\mu\nu} \, \nabla_\mu \phi \nabla_\nu \phi \to R^{\mu\nu} \, \frac{\Lambda^2}{\Box + \Lambda^2} \nabla_\mu \phi \nabla_\nu \phi \,. \tag{4.59}$$

At high energies, $\Box \gg \Lambda^2$, this term will loose its kinetic nature. Accordingly, we would be left with a standard kinetic term for ϕ, and there would neither occur a stability nor a unitarity issue. However, in the opposite regime, $\Box \ll \Lambda^2$, this modification is negligible, and there is no window for a Wilsonian cure of the theory. Instead, classicalization must occur.

The Stückelberg sector describes the theory only for $\Box \gg m^2, H^2$. For phenomenological reasons, we typically take $m^2 \approx H^2$, so that we effectively have $\Box \gg m^2$. Moreover, once again noting that any new Wilsonian heavy degrees of freedom must have a mass Λ much above m, $\Lambda^2 \gg m^2$ (otherwise, the effective theory (4.39) would have been incomplete) we see that intermediate energies $m^2 \ll \Box \ll \Lambda^2$, at which the theory classicalizes, exist always, whereas for $\Box \gg \Lambda^2$ a Wilsonian mechanism might be at work. Accordingly, Fierz–Pauli theory on FRW has always a finite classicalitzation window [74].

The similarities to the classicalization scenario are not surprising. We know that we need to consider fields that are derivatively coupled to obtain a classicalizing theory. Typically, these fields are Goldstone fields like our field ϕ, since these fields automatically come with extra derivatives. Still, we could object that we are only considering the linear theory in ϕ and no (self-)interactions, and so why should we expect to see classicalization? But the point is: We are considering 'interactions' of ϕ with the background geometry g_0, so we effectively consider higher derivative terms of the form $\partial^2 g_0^2\, \partial^2 \phi^2$. Thus, the field ϕ is sourced by the background geometry via a derivatively coupled vertex and thus classicalization is supposed to occur. In this respect we can say that our analysis showed explicitly that the classicalization paradigm extends to free field theories on curved spacetimes (while the original classicalization proposal embraced interacting field theories over Minkowski spacetime).

We further elaborated on the connection of the massive gravity theories with the classicalization mechanism in [2].

Beyond self-protection - a novel theory for massive deformations

Even though that the self-protection mechanism is appealing from a theoretical perspective, it is insufficient phenomenologically, since the breakdown of perturbation theory is accompanied by a classical instability and thus the transition to a new background space-time. To put it differently, an FRW universe cannot be a solution of (4.25) in the whole space-time region and especially at early times (4.28) would be violated due to high curvatures in the early universe. Therefore, we have to go beyond the theory (4.25). From looking at the stability bound (4.28) we already can gain an intuition of how to extent the theory: The bound cannot be fulfilled for all times since the right hand side is time dependent, whereas the left hand side is not. So what we should do is to make the left hand side time dependent, too, by somehow introducing a concept of a 'running mass'.

Therefore, we now want to find an answer to the question:
What is the most general deformation matrix $\mathcal{M}$ that leads to a stable and unitary theory

on an FRW space-time?
Note that this question manifests the direct generalization of the quality management Fierz and Pauli were applying to find their respective mass term for a Minkowski background. So in some sense we want to carry out the same program for a curved background as they did for a Minkowski background.

We will again introduce Stückelberg fields to reintroduce gauge invariance

$$h_{\mu\nu} \to H_{\mu\nu} = h_{\mu\nu} + \nabla_{(\mu}A_{\nu)} + \nabla_\mu \nabla_\nu \Phi \ . \tag{4.60}$$

In fact, this introduces a $U(1) \times U(1)^4$ invariance of the form

$$h_{\mu\nu} \to h_{\mu\nu} + \nabla_{(\mu}\zeta_{\nu)} \quad , \qquad A_\mu \to A_\mu - \zeta_\mu \tag{4.61}$$

$$A_\mu \to A_\mu + \nabla_\mu \psi \quad , \qquad \phi \to \phi - \psi \tag{4.62}$$

Again, at high energies, the $H_{\mu\nu}$ amplitude will be dominated by the scalar degree of freedom Φ. To assure that the higher time derivatives of this degree of freedom drop out, the deformation matrix needs to have a high index symmetry that will be worked out in the following:
First we notice that we can assume without loss of generality an exchange symmetry of the form

$$\mathcal{M}^{\mu\nu\alpha\beta} = \mathcal{M}^{\alpha\beta\mu\nu} \ . \tag{4.63}$$

This follows from the fact that both index tuples α, β and μ, ν are contracted with the same object in the action (4.24). Moreover, the potentially problematic term quadratic in the field Φ can be written as

$$\begin{aligned}
&\tfrac{1}{2} \int_M \mathrm{d}^4x \ \sqrt{|g_0|} \left(\nabla_\mu \nabla_\nu \Phi\right) \mathcal{M}^{\mu\nu\alpha\beta} \left(\nabla_\alpha \nabla_\beta \Phi\right) \\
&= \tfrac{1}{4} \left(\int_M \mathrm{d}^4x \ \sqrt{|g_0|} \left(\nabla_\mu \nabla_\nu \Phi\right) \mathcal{M}^{\mu\nu\alpha\beta} \left(\nabla_\alpha \nabla_\beta \Phi\right) + \left(\nabla_\alpha \nabla_\nu \Phi\right) \mathcal{M}^{\alpha\nu\mu\beta} \left(\nabla_\mu \nabla_\beta \Phi\right) \right) \\
&= -\tfrac{1}{2} \int_M \mathrm{d}^4x \ \sqrt{|g_0|} \Big(\left(\left[\nabla_\alpha, \nabla_\mu\right] \nabla_\nu \Phi \right) \mathcal{M}^{\mu\nu\alpha\beta} \nabla_\beta \Phi \\
&\quad + \tfrac{1}{2} \left(\nabla_\alpha \nabla_\mu \nabla_\nu \Phi\right) \left(\mathcal{M}^{\mu\nu\alpha\beta} + \mathcal{M}^{\alpha\nu\mu\beta}\right) \nabla_\beta \Phi \\
&\quad + \left(\nabla_\mu \nabla_\nu \Phi\right) \left(\nabla_\alpha \mathcal{M}^{\mu\nu\alpha\beta}\right) \nabla_\beta \Phi \Big)
\end{aligned} \tag{4.64}$$

Demanding that the four derivative term $\left(\nabla_\alpha \nabla_\mu \nabla_\nu \Phi\right) \left(\mathcal{M}^{\mu\nu\alpha\beta} + \mathcal{M}^{\alpha\nu\mu\beta}\right) \nabla_\beta \Phi$ vanishes yields the symmetry

$$\mathcal{M}^{\mu\nu\alpha\beta} = -\mathcal{M}^{\alpha\nu\mu\beta} \ . \tag{4.65}$$

Together with the symmetry (4.63) the same anti-symmetry follows for the exchange of ν and β. With a similar calculation as in (4.64) we can show that the remaining three derivative term in (4.64) drops out as well. The same holds to be true for the potentially dangerous mixing terms of the form

$$\frac{1}{2}\int_M \mathrm{d}^4x\,\sqrt{|g_0|}\nabla_\mu\nabla_\nu\mathcal{M}^{\mu\nu\alpha\beta}\nabla_{(\alpha}A_{\beta)} \tag{4.66}$$

Thus, we will consider the general ansatz for the deformation matrix consistent with the symmetries (4.63) and (4.65)

$$\begin{aligned}\mathcal{M}^{\mu\nu\alpha\beta} &= \big(m_0^2+\alpha R_0\big)g_0^{\mu[\nu}\,g_0^{\beta]\alpha} \\ &+ \beta\Big(R_0^{\mu[\nu}\,g_0^{\beta]\alpha}+R_0^{\alpha[\beta}\,g_0^{\nu]\mu}\Big)+\gamma\,R_0^{\mu\alpha\nu\beta}\,,\end{aligned} \tag{4.67}$$

Here, α, β and γ are dimensionless quantities. Note that we were not introducing higher curvature operators as R^2, etc., since this would necessitate the introduction of new mass scales into the theory. Moreover, from an effective field theory viewpoint, these masses would naturally suppress the higher curvature operators, since they would appear in the denominator.

We now want to confine the values of the three new unknowns α, β and γ by demanding to obtain a stable and unitary theory. Right from the start we notice that the three parameters are not independent on an FRW background due to its vanishing Weyl tensor. Therefore, the Riemann tensor can be expressed as

$$R_0^{\mu\alpha\nu\beta}=-\frac{1}{3}R_0\,g_0^{\mu[\nu}\,g_0^{\beta]\alpha}+\frac{1}{2}\Big(R_0^{\mu[\nu}\,g_0^{\beta]\alpha}+R_0^{\alpha[\beta}\,g_0^{\nu]\mu}\Big)\,. \tag{4.68}$$

Thus, the Riemann tensor term effectively just shifts the α and β parameter which means that we can set $\gamma=0$ without loss of generality.

We will use the gauge freedom (4.61) to set $h_{0\mu}=0$ and $A_0=0$. Unlike in the case of a naive Fierz-Pauli deformation, Φ now kinetically mixes with all other degrees of freedom. Therefore, we have to diagonalize the complete kinetic sector in order to obtain the leading high energy behavior. At high energies, we can treat the background evolution as adiabatic compared to the graviton propagation. It is thus justified to switch to Fourier space in the standard way, in which the pure kinetic sector looks like

$$\int d^4k F_a k_\mu \mathcal{K}^{\mu\nu}_{ab} k_\nu F_b \tag{4.69}$$

Here, F_a is a vector containing the 10 Fourier space degrees of freedom of our system: $(F_a)=(h_{ij},A_i,\Phi)$, k_μ is the four momentum associated with the gravitons propagation, and $\mathcal{K}^{\mu\nu}_{ab}$ is the kinetic matrix, which is non-diagonal in a,b due to kinetic mixing. Note that $\mathcal{K}^{\mu\nu}_{ab}$ is symmetric along the three spatial directions due to the symmetry of the Friedman background. The zeros of the determinant of the 10×10 matrix $k_\mu\mathcal{K}^{\mu\nu}_{ab}k_\nu$ will

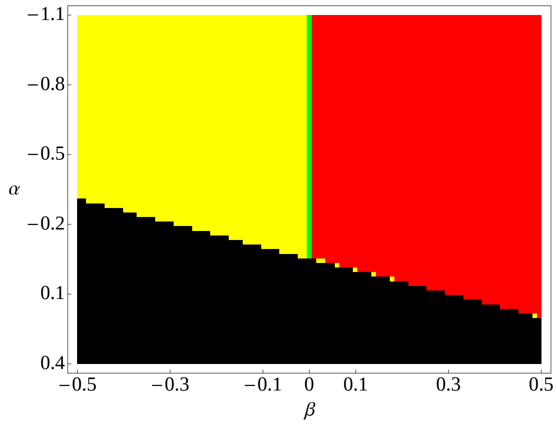

Figure 4.2: Parameter plot in the α-β plane, for $m_0 = 0$. (For $m_0 > 0$ the plot looks qualitatively the same. The black region is just traded for an extension of the yellow and red regions.)

reveal the generalized stability and unitarity bounds. To be precise, let us display the determinant as follows

$$\det(k_\mu \mathcal{K}^{\mu\nu} k_\nu) = \mathrm{ub}(H, \dot{H}, \ddot{H}) k_0^{20} + \ldots + \mathrm{sb}(H, \dot{H}, \ddot{H}, H^{(3)}) \, (k_0 k_i)^{10} + \ldots \tag{4.70}$$

where we have written only the leading behavior both in k_0 and k_i. Their pre factors now give the unitarity and stability bound, respectively, since this corresponds to setting the coefficient of ∂_i or ∂_0 equal to zero. In this case, the unitarity and stability bound are extremely lengthy expressions. But it is possible to systematically analyze them in the limits $a \to 0$ and $a \to \infty$. Demanding that both bounds are fulfilled for all values of a (and thus for all times t if we have an open or flat universe) yields that the only allowed choice (with one exception, to be discussed below) is $\beta = 0$ and α sufficiently negative. The term 'sufficient' is again a very lengthy, implicit expression. Instead of quoting it here, we will present a complete parameter plot of our theory in figure 4.2. Here, we can distinguish four different regions: The black region, where the theory violates either unitarity or stability today. The red region, where the theory is healthy today, but is at some other time entering a regime of unitarity violation within the applicability of perturbation theory. The yellow region, where the theory is healthy today, but is becoming classically unstable at some other value of the time t. This corresponds to the region of self-protection. And finally, the green region corresponding to $\beta = 0$ and α sufficiently negative, where the theory is both stable and unitary which is what we were looking for.

Note that there is a single exception to the statement that α has to be sufficiently negative. In fact we can show that there is a single point with positive α, to be precise with $\alpha = m_0^2/48\Omega_\Lambda$ and $\beta = 0$. It is precisely this point that corresponds to general relativity in the limit $m_0 \to 0$.

Thus we conclude that we succeeded in constructing the unique massive deformation term on an FRW space-time that yields a stable and unitary theory. It looks like

$$\mathcal{M}^{\mu\nu\alpha\beta} = \left(m_0^2 + \alpha R_0\right) g_0^{\mu[\nu} g_0^{\beta]\alpha} \tag{4.71}$$

where α has to be sufficiently negative (or with the single exception of $\alpha = m_0^2/48\Omega_\Lambda$). Given that the index structure is the same as in the Fierz-Pauli case, we consider this as precisely the covariant formulation of a 'running mass'. It consists of the known hard mass term m_0^2 and a new soft mass term αR_0.

Our result opens up the window for a completely new kind of modification, which is linearly achieved by setting the hard mass equal to zero $m_0 = 0$. This modification is completely orthogonal to known standard massive gravity, in the sense that it is only operative on curved space-time backgrounds and is identically vanishing for flat regions where we have $R_0 = 0$ - in contrary to standard massive gravity. We think that this modification deserves more attention, since it results in a completely sound theory on the whole Friedman manifold. In particular, the construction of a non-linear completion is one of the next logical and challenging tasks to do.

5 Brane Induced Gravity

5.1 The Model

Brane induced gravity is another example of an IR modified gravity theory, which has been invented by [76]. It is an extra-dimensional model, that is, we are introducing n additional spatial dimensions (for $n = 1$ it is known as the 'DGP model', named after the initials of its inventors). We assume that only gravitons can propagate in the 'bulk' (that is, in the extra-dimensions), whereas all other fields are confined to live on the 4-dimensional sub-manifold that is supposed to make up our world (which we will call the 'brane'). This assures that we do not spoil the success of the standard model of particle physics. However, since brane induced gravity is supposed to modify gravity only in the large distance regime, we also have to add a term that ensures the recovery of standard 4d-gravity for distances, smaller than some crossover scale r_C, which is usually taken to be of the order of today's Hubble parameter. All in all, our Lagrangian will look like

$$\begin{aligned} \mathcal{S} = \mathcal{S}_{\mathrm{EH}}^{(4+n)}[g] + \mathcal{S}_{\mathrm{EH}}^{(4)}[\omega] + S_{\mathrm{matter}}^{(4)}[\omega] \quad &= \quad \int \mathrm{d}^{4+n} x M_{4+n}^{2+n} \sqrt{-g}\, R^{(4+n)}[g] \\ &+ \int \mathrm{d}^4 x \sqrt{-\omega} \left(M_4^2 R^{(4)}[\omega] + \mathcal{L}_{\mathrm{matter}}^{(4)}[\omega] \right) \end{aligned} \tag{5.1}$$

where ω is the four-dimensional sub-metric of g, and M_6 and M_4 are the six- and four-dimensional Planck constants, respectively. The four-dimensional Ricci scalar is the piece that will restore four-dimensional gravity below the crossover distance $r_C = M_4/M_{4+n}^2$ (with the exception of the special case $n = 1$, where the crossover distance is given by $r_C = M_4^2/M_5^3$). The model (5.1) for $n > 2$ could be an important puzzle piece for making progress with respect to the cosmological constant problem, since if we consider a pure cosmological constant Λ as matter source it will not curve the brane but only the extrinsic curvature part of $R_{AB}^{(d)}$ [26] (here, capital latin indices run over all $4 + n$ space-time dimensions). Therefore, a four-dimensional observer living on the brane would not infer the presence of the cosmological constant at all. To be concrete, let us consider the case $n = 2$, in which case the solution reads

$$ds^2 = -dt^2 + dx_i dx_j \delta^{ij} + dr^2 + (2\pi - \theta) r^2 d\phi^2 \tag{5.2}$$

Here, we have introduced cylindrical coordinates in the extra dimensions, and the deficit angle $\theta = \Lambda/M_6^4$. We clearly diagnose the flat metric along the four brane dimensions.

However, the investigation of the models (5.1) with $n > 2$ have been hampered by numerous claims in the literature [80, 81, 82] stating that those models contain - for phenomenologically interesting parameter choices - a perturbative ghost when expanding the metric around a Minkowski background $\eta_{\mu\nu}$. However, we have shown in [5] that the previous calculations overlooked a constraint that renders the would-be ghost not dynamical. We have further solidified the healthiness of the models (5.1) by performing a full-fledged Hamiltonian analysis, showing that the Hamiltonian on the constraint surface is manifestly positive definite, leaving no room for a ghost like instability. We will summarize these findings in section 5.2. In the remaining of the current section, we will investigate certain aspects of the model (5.1) in more detail, in particular deriving the expressions for the crossover length scale r_C and discussing the former belief of (5.1) containing a ghost.

We will perturb the metric as usual $g_{AB} = \eta_{AB} + h_{AB}$ and expand the action (5.1) up to second order in the field fluctuations, that is, we will consider the free kinetic sector of h_{AB} and its coupling to a matter source $h_{\alpha\beta}T^{\alpha\beta}$. The coupling to matter involves only the four brane space-time dimensions (which are denoted by greek indices α, β etc.) as $\mathcal{L}^{(4)}_{\text{matter}}[\omega]$ depends on the four dimensional sub-metric ω. We will split the metric fluctuation field h_{AB} according to the $SO(3,1) \times SO(n)$ symmetry of the setup at hand

$$\begin{aligned} h_{\alpha\beta} &= D^{(\text{tt})}{}_{\alpha\beta} + \partial_{(\alpha} C^{(t)}_{\beta)} + P^{(\|)}{}_{\alpha\beta}\, B + \eta_{\alpha\beta}\, S \,, \\ h_{ab} &= d^{(\text{tt})}{}_{ab} + \partial_{(a} \hat{C}^{(t)}_{b)} + P^{(\|)}{}_{ab} \hat{B} + \delta_{ab}\, s \,, \\ h_{\alpha b} &= G^{(\text{v,v})}{}_{\alpha b} + \partial_b G^{(\text{v,s})}{}_{\alpha} + \partial_\alpha F^{(\text{s,v})}{}_{b} + \partial_\alpha \partial_b F^{(s,s)} \,, \end{aligned} \tag{5.3}$$

where we introduced the longitudinal projectors $P^{(\|)}{}_{\alpha\beta} = \frac{\partial_\alpha \partial_\beta}{\Box}$ and $P^{(\|)}{}_{ab} = \frac{\partial_a \partial_b}{\Delta}$. Small latin indices run only over the n extra-dimensional space-time dimensions. All tensor qualifications are with respect to the $SO(3,1) \times SO(n)$ symmetry underlying the space-time. Thus, $D^{(\text{tt})}$, $C^{(t)}$ and B, S transform under Lorentz transformations $SO(1,3)$ as a transverse and traceless second rank tensor, transverse vector and two scalars, respectively, while $d^{(\text{tt})}$, $\hat{C}^{(t)}_b$, $\hat{B}$ and s transform under the roation group $SO(n)$ as a transverse and traceless second rank tensor, a transverse vector and two scalars, respectively. The mixed sector involves $G^{(\text{v,v})}$, $G^{(\text{v,s})}$, $F^{(\text{s,v})}$ and $F^{(s,s)}$, which transform under the direct product $SO(1,3) \times SO(n)$ as (transverse vector, transverse vector), (transverse vector, scalar), (scalar, transverse vector) and (scalar, scalar) quantities, respectively.

The gauge symmetry $h_{AB} \to h_{AB} + \partial_{(A}\zeta_{B)}$ allows to remove $n + 4$ degrees of freedom. To be concrete, it introduces the following transformation behavior among the decomposed variables

$$C^{(t)}_\alpha \;\to\; C^{(t)}_\alpha + \zeta^{(t)}_\alpha \tag{5.4}$$
$$B \;\to\; B + \phi \tag{5.5}$$
$$G^{(v,s)}_\alpha \;\to\; G^{(v,s)}_\alpha + \zeta^{(t)}_\alpha \tag{5.6}$$
$$\hat{C}^{(t)}_a \;\to\; \hat{C}^{(t)}_a + \zeta^{(t)}_a \tag{5.7}$$
$$\hat{B} \;\to\; \hat{B} + \psi \tag{5.8}$$
$$F^{(s,v)}_a \;\to\; F^{(s,v)}_a + \zeta^{(t)}_a \tag{5.9}$$
$$F^{(s,s)} \to F^{(s,s)} + \phi + \psi \tag{5.10}$$

where we have decomposed ζ_α and ζ_a into transverse $\zeta^{(t)}_\alpha$, $\zeta^{(t)}_a$ and longitudinal parts ϕ, ψ corresponding to their transformation behavior under $SO(3,1)$ and $SO(n)$, respectively. We will use this gauge freedom to set $\hat{C} = 0$, $\hat{B} = 0$, $C = 0$ and $F^{(s,s)} = 0$ (which are $n + 4$ conditions).

The equations of motion can also be decomposed into tensor, vector and scalar parts with respect to their transformation behavior under either $SO(3,1)$ or $SO(n)$ rotations. The transverse-traceless part of the $\alpha\beta$-Einstein equation gives directly the equation of motion for the field $D^{(tt)}$

$$\frac{1}{2}M^{2+n}_{4+n}\Box D^{(\mathrm{tt})}{}_{\alpha\beta} + \frac{1}{2}M^2_4\delta^{(n)}(y)\Box_4 D^{(\mathrm{tt})}{}_{\alpha\beta} = T^{(\mathrm{tt})}{}_{\alpha\beta}\delta^{(n)}(y) \tag{5.11}$$

Here, $\Box = \Box_4 + \Delta_n$ denotes the $4+n$-dimensional d'Alembert operator and $\Box_4$ its 4-dimensional counterpart. $T^{(\mathrm{tt})}{}_{\alpha\beta}$ is the transverse-traceless part of the energy-momentum tensor, y denotes the n extra dimensional spatial coordinates. Using the scalar equations of the $\alpha\beta$- and ab-Einstein equations allows to derive a single equation for the mode S

$$\frac{1}{2}M^{2+n}_{4+n}\frac{n+2}{n-1}\Box S - \frac{1}{2}M^2_4\delta^{(n)}(y)\Box_4 S = \frac{T^{(4)}}{3}\delta^{(n)}(y) \tag{5.12}$$

We see that the sign in front of the $\Box_4$ operator differs between (5.11) and (5.12). It is precisely this negative sign which led people belief that S is a ghost degree of freedom, since it contributes with a negative sign to the brane-to-brane amplitude $\int d^4x h_{\mu\nu}T^{\mu\nu} = \int d^4x \left(D^{(\mathrm{tt})}{}_{\mu\nu} + \eta_{\mu\nu}S\right)T^{\mu\nu}$. To derive the brane-to-brane amplitude, we perform a Fourier transform of the four space-time dimensions x and leave the extra dimensional coordinates y untouched. For example, the equation of motion of the field $D^{(\mathrm{tt})}{}_{\alpha\beta}(x,y)$ will look like

$$\frac{1}{2}M^{2+n}_{4+n}\left(-k^2 + \Delta\right)D^{(\mathrm{tt})}{}_{\alpha\beta}(k,y) - \frac{1}{2}M^2_4 k^2\delta^{(n)}(y)D^{(\mathrm{tt})}{}_{\alpha\beta}(k,y) = T^{(\mathrm{tt})}{}_{\alpha\beta}\delta^{(n)}(y) \tag{5.13}$$

We will solve (5.13) with the ansatz

$$D^{(\mathrm{tt})}{}_{\alpha\beta}(k,y) = g_n(y,k)D^{(tt)}_{\alpha\beta}(k) \tag{5.14}$$

where $g_n(y,k)$ fulfills

$$\left(k^2 - \Delta_n\right)g_n(y,k) = \delta^{(n)}(y) \tag{5.15}$$

From this we can derive the solution of (5.13) since the delta functions cancel and we obtain an algebraic equation for $D_{\alpha\beta}(k)$

$$D^{(\mathrm{tt})}{}_{\alpha\beta}(k,y) = \frac{2g_n(y,k)}{M_{4+n}^{2+n} + M_4^2 k^2 g_n(0,k)} T^{(tt)}_{\alpha\beta} \tag{5.16}$$

Accordingly, the four dimensional Fourier transform of $D^{(\mathrm{tt})}{}_{\alpha\beta}(x,0)$ (which is what is needed for the brane-to-brane amplitude) reads as

$$D^{(\mathrm{tt})}{}_{\alpha\beta}(k,0) = \frac{2T^{(tt)}_{\alpha\beta}}{M_{4+n}^{2+n} g_n(0,k)^{-1} + M_4^2 k^2} \tag{5.17}$$

$g_n(y,k)$ diverges at $y = 0$ for $n > 2$, as can easily be seen by solving (5.15) using a Fourier transform in the extra dimensional space. This divergence is a direct artifact of the approximation of an infinitely thin brane. Any realistic model should give the brane a finite thickness r_0 in the extra-dimensional directions, which can be modeled by folding every term in the Lagrangian (5.1) with a localizer function $f(y)$ falling off to zero for $|y| > r_0$ (in the limit $r_0 \to 0$ we have $f(y) \to \delta^{(n)}(y)$, recovering the infinitely thin brane theory). Due to the folding theorem, this will effectively cut off all momenta above a scale $\lambda \sim 1/r_0$ in the extra dimensional Fourier integral determining the solution $g_n(0,k)$. For example, for the case $n = 2$ it will look like

$$g_2(0,k) = \int_0^\lambda \frac{d^2 k_y}{k_y^2 + k^2} = \pi \ln\left(\frac{\lambda^2}{k^2} + 1\right) \tag{5.18}$$

An upper bound on λ is the extra dimensional Planck scale M_{4+n}, since a higher curvature operator term of the schematical form R^d/M_{4+n}^{2d-4-n} would introduce higher potencies k^{2d}/M_{4+n}^{2d-4-n} which would again effectively cut the momentum integral in (5.18) above M_{4+n} [79]. That's why we often take $\lambda = M_{4+n}$.

We already see the transition from the high energy to the low energy regime by inspecting (5.17). For $k \to \infty$ the standard $M_4^2 k^2$ term will dominate, thus recovering standard 4d-gravity, whereas in the deep IR (which means $k \to 0$) the modification $M_{4+n}^{2+n} g_n(0,k)^{-1}$ will describe the dynamics of the system (for $n = 2$ this is apparent from (5.18); for $n > 2$ this can be similarly derived, where g_n turns out to contribute a hard mass term). Considering for example the case $n = 2$ and setting $\lambda = M_6$, we find that the transition occurs roughly for $k \sim M_6^2/M_4 = r_C^{-1}$, which defines the mentioned crossover scale.

Using the same technique, we can derive the brane-to-brane propagator in Fourier representation for the mode S

$$S(k,0) = \frac{\frac{2}{3}T^{(4)}}{\frac{n+2}{n-1} M_{4+n}^{2+n} g_n(0,k)^{-1} - M_4^2 k^2} \tag{5.19}$$

As mentioned, the crucial difference between (5.19) and (5.17) is the sign in front of the $M_4^2 k^2$ term. According to (5.19), the propagator of the mode S develops a pole with negative residue when the four dimensional term takes over. This was the reason why people believed that the model (5.1) contains a ghost for $n > 1$ (note that for $n = 1$ the mode S is set to zero, so that no problem can occur). We will show in the following that this interpretation and belief is wrong.

5.2 From a No-Go to a No-Ghost Theorem

5.2.1 Physical picture

The claimed appearance of a ghost for $n > 2$ seems incomprehensible from an effective field theory viewpoint, as we can think of the model (5.1) as follows: Suppose that we start with a n-dimensional Einstein-gravity model, that is, with the Ricci scalar $M_{4+n}^{2+n}\sqrt{-g}\,R^{(4+n)}[g]$. On top of that we simply put a special matter source $\delta^{(n)}(y)\mathcal{L}^{(4)}_{\text{matter}}[\omega]$, which is confined to live on a 4-dimensional subspace. If we assume that the matter fulfills the standard energy conditions, there is no reason why this setup so far should be inconsistent, simply because we know that gravity is not (even the singular nature of the source could be handled by giving the brane a certain finite width). The only possibility which could introduce inconsistencies is the additional four dimensional Ricci scalar $\delta^{(n)}(y)M_4^2R^{(4)}[\omega]$, which alters the kinetic content of the theory. However, confining matter with mass $M > \Lambda$ (where Λ is the cutoff of the effective field theory) on a four dimensional subspace induces a four dimensional Ricci scalar term in the low energy effective field theory anyways. This can be verified explicitly by integrating out loops of heavy matter particles with legs confined on the brane [77, 78], but it is also clear by symmetries, since the result must be a general covariant quantity constructed out of the four dimensional sub metric ω - which will result in a series of the effective form $R^{(4)}[\omega]^m$, where the zeroth order term represents the induced cosmological constant on the brane and the first order term the induced four dimensional Ricci scalar. This reasoning is depicted in figure 5.1. But if we can understand the model (5.1) as pure higher dimensional gravity with a special matter source, there is absolutely no reason to expect a ghost in this theory.

5.2.2 Hamiltonian analysis

Given the tension between our physical expectation outlined in 5.2.1 and the results reported in the literature (see section 5.1) we performed a full-fledged Dirac constraint analysis of the model (5.1). This machinery automatically takes all constraints into account and gives us a clean instrument to decide whether or not the theory contains a ghost: the positive definiteness of the Hamiltonian on the constrained surface, since any

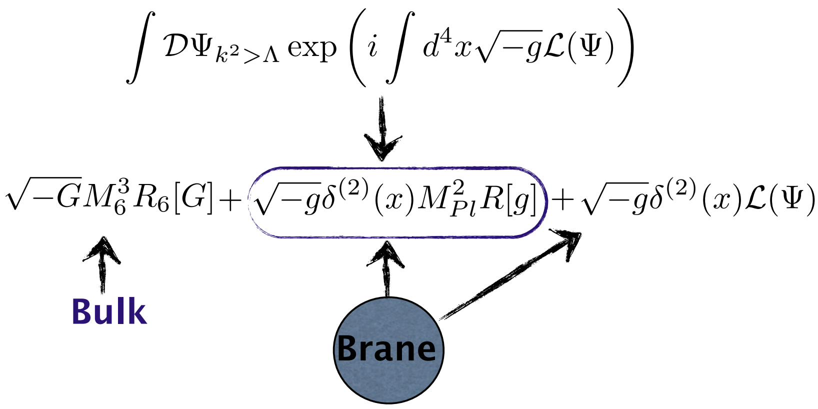

Figure 5.1: Schematical representation of the model (5.1) for $n = 2$. The induced four dimensional Ricci scalar results from integrating out particles above the cutoff Λ.

ghost degree of freedom will inevitably introduce directions along which the Hamiltonian on the constrained surface is unbounded from below. For reasons of simplicity we were only considering the case $n = 2$, but we are currently working on a generalization of the subsequent derivation to arbitrary n [6].

In the following we will summarize the derivation for the case $n = 2$. The reader interested in the details of the derivation can find these in [5].

To start with we will perform a multiple ADM-split as starting point. For concreteness, both spatial extra–dimensions and the time direction are split in the usual ADM sense:

$$g_{AB} = \begin{pmatrix} \lambda_{\mu\nu} & \Lambda_\mu \\ \Lambda_\nu & \Lambda^2 + \Lambda_\lambda \Lambda^\lambda \end{pmatrix}, \tag{5.20}$$

with

$$\lambda_{\mu\nu} = \begin{pmatrix} \omega_{\alpha\beta} & \Omega_\alpha \\ \Omega_\beta & \Omega^2 + \Omega_\gamma \Omega^\gamma \end{pmatrix}, \tag{5.21}$$

and

$$\omega_{\alpha\beta} = \begin{pmatrix} -\Gamma^2 + \Gamma_i \Gamma^i & \Gamma_i \\ \Gamma_j & \gamma_{ij} \end{pmatrix}. \tag{5.22}$$

Here, γ_{ij} denotes the submetric of the spatial hypersurface orthogonal to the normal vectors

$$\left(\hat{n}_6^A\right) = \left(-\Lambda^\mu/\Lambda, \quad 1/\Lambda\right) , \tag{5.23}$$

$$\left(\hat{n}_5^\mu\right) = \left(-\Omega^\alpha/\Omega, \quad 1/\Omega\right) , \tag{5.24}$$

$$\left(\hat{n}_4^\alpha\right) = \left(1/\Gamma, \quad -\Gamma^i/\Gamma\right) , \tag{5.25}$$

where Λ, Ω and Γ denote the three 'Lapse'–functions corresponding to the three ADM–splits, and Λ^μ, Ω^α and Γ^i are the respective 'Shift'–functions. Indices are raised and lowered with the background Minkowski metric η_{AB}. The index ranges are as follows:

A, B, C, D	$0, 1, 2, 3, 5, 6$	
α, β, γ, δ	$0, 1, 2, 3$	ADM 4+1
λ, μ, ν, ρ	$0, 1, 2, 3, 5$	ADM 5+1
a, b, c, d	$5, 6$	bulk directions
i, j, k, l	$1, 2, 3$	spatial surface directions

The usage of ADM variables allows to decompose the Ricci scalar of a d–dimensional space–time in terms of the $d-1$-dimensional Ricci scalar

$$\begin{aligned}\sqrt{-g}R^{(d)} = \sqrt{-g}\Big\{ R^{(d-1)} + (\hat{n}_d \cdot \hat{n}_d)\Big[(\mathrm{Tr}K_d)^2 - \mathrm{Tr}K_d^2 \\ + 2(\nabla \cdot ((\hat{n}_d \cdot \nabla)\hat{n}_d) - \nabla \cdot (\hat{n}_d(\nabla \cdot \hat{n}_d)))\Big]\Big\} .\end{aligned} \tag{5.26}$$

Here, the covariant derivative ∇ is compatible with the parent metric g characterizing the geometrical structure on the d–dimensional space–time. K_d denotes the extrinsic curvature tensor in d–dimensions. In terms of ADM variables,

$$(K_6)_{\mu\nu} = \frac{1}{2\Lambda}(\partial_6 \lambda_{\mu\nu} - \nabla_\mu \Lambda_\nu - \nabla_\nu \Lambda_\mu) , \tag{5.27}$$

$$(K_5)_{\alpha\beta} = \frac{1}{2\Omega}(\partial_5 \omega_{\alpha\beta} - \nabla_\alpha \Omega_\beta - \nabla_\beta \Omega_\alpha) , \tag{5.28}$$

$$(K_4)_{ij} = \frac{1}{2\Gamma}(\partial_t \gamma_{ij} - \nabla_i \Gamma_j - \nabla_j \Gamma_i) . \tag{5.29}$$

Once we outlined the variables used, we can calculate the canonical momentum fields corresponding to them. As expected for 6-dimensional gravity, it is not possible to invert the expressions for the canonical momentum fields for their respective velocities. Instead, the system of canonical momentum fields incorporates 6 primary constraints $\phi_a^{(1)}$. Demanding that these primary constraints are conserved under time evolution yields further 6 secondary constraints

$$\phi_a^{(2)} = \dot{\phi}_a^{(1)} = \{H, \phi_a^{(1)}\} . \tag{5.30}$$

where H denotes the full Hamiltonian. These secondary constraints are all conserved under time evolution, i.e. they (weakly) commute with the Hamiltonian under the Poisson bracket

$$\dot{\phi}_a^{(2)} = \{H, \phi_a^{(2)}\} \simeq 0\,. \tag{5.31}$$

Note that the last relation ($\simeq$) is a weak equality, which means that the right hand side of (6.124) is a linear combination of the constraints $\phi_a^{(p)}$ of the system ($p \in \{1,2\}$ & $a \in \{1,2,3,4,5,6\}$). According to (6.124), the system does not possess any tertiary constraints. Thus, the constraint content is given by the set of 12 primary and secondary constraints $\phi_a^{(p)}$. Moreover, it can be shown that the constraint system is completely first class:

$$\forall\, p, p', a, a' : \;\; \{\phi_a^{(p)}, \phi_{a'}^{(p')}\} \simeq 0\,. \tag{5.32}$$

As a consequence, every constraint generates a gauge transformation on any quantity Θ, that is built up out of the dynamical field variables, as follows:

$$\delta\Theta = \xi\{\Theta, \phi_a^{(p)}\}\,, \tag{5.33}$$

with a space–time dependent gauge function ξ. In this way, the 12 first class constraints allow to reduce the number of independent dynamical degrees of freedom by 24.

To make further progress we can use the extra dimensional $SO(2)$-symmetry inherent in our setup. Explicitly implementing this symmetry in our derivation still enables us to address the absence of the ghost, since the derivation of the would-be ghost in former works is solid under the assumption of such a symmetry. The $SO(2)$ symmetry is most easily implemented using polar coordinates (r, φ), where $x^5 = r\cos\varphi$ and $x^6 = r\sin\varphi$. Then the symmetry demands the extra space components of the graviton field not to depend on φ. Additionally, the $h_{\varphi r}$ and $h_{\varphi j}$ components have to vanish.

The gauge freedom, the constraints and the $SO(2)$-symmetry allow us to derive the Hamiltonian on the constraint surface

$$\begin{aligned}\mathcal{H} = &\frac{1}{M_6^4}\Pi^{(T)}_{(R)ij}\Pi^{(T)ij}_{(R)} + \frac{1}{M_4^2}\,\delta_y^{(2)}\,\Pi^{(T)}_{(I)ij}\Pi^{(T)ij}_{(I)} + \frac{1}{4M_6^4}\Pi_N^2 + \frac{1}{2M_6^4}\tilde{\Pi}_i\tilde{\Pi}^i + \frac{1}{4}M_6^4\tilde{F}_{ij}\tilde{F}^{ij}\\ &+ \frac{1}{4}M_6^4\partial_a h^{(tt)}_{ij}\partial^a h^{(tt)ij} + \frac{1}{4}\left(M_6^4 + M_4^2\,\delta_y^{(2)}\right)\partial_k h^{(tt)}_{ij}\partial^k h^{(tt)ij} + 2M_6^4\partial_a N\partial^a N\,.\end{aligned} \tag{5.34}$$

Here, the fields denoted by the symbol Π denote the canonical conjugated momentum fields. Evidently, $\mathcal{H}$ consists only of positive squares, which implies that the Hamiltonian H is positive definite, which in turn is a sufficient condition for a ghost–free theory. Note that a real ghost degree of freedom, which originates from a negative sign kinetic operator, would inevitably destroy the positive definiteness of the classical Hamiltonian.

Let us finally comment on the number of propagating degrees of freedom. Given that h_{AB} is a 6×6 matrix, we start with $2\times\frac{6\times 7}{2}=42$ phase space degrees of freedom. These are reduced by 12 first class constraints inducing 12 gauge redundancies, so that we obtain 18 phase space degrees of freedom, or 9 physical degrees of freedom. This is the same number as the number of degrees of freedom in 6-dimensional general relativity, which is no surprise, since we know from our physical picture of section 5.2.1 that we should be able to consider the model (5.1) simply as higher dimensional general relativity with a specific matter source. The special $SO(2)$-symmetry of this matter source allows a further reduction of 4 physical degrees of freedom, so that we are left with 5 degrees of freedom, which is the same number of degrees of freedom as in the DGP model. In fact, this is a general result that will persist in higher co-dimensions. Finally, from these 5 degrees of freedom only h^{tt}_{ij} is coupled to the brane energy momentum tensor, so that we are left with 2 excited degrees of freedom, which in turn is the same number as in general relativity.

5.2.3 Covariant arguments

The results of the Dirac constraint analysis in section 5.2.2 are in clear tension to the previous work [80, 81, 82] which was performed in a manifest covariant language like in section 5.1. So something must have gone wrong in this previous work. Actually, we were able to show that these treatments have overseen a constraint in the system that renders the would-be ghost mode S not dynamical, and thus it cannot threaten the probabilistic interpretation of the theory. In fact, the 00-component of the Einstein equation can be put into the form

$$\begin{aligned}\left[\partial^i\partial^j-\delta^{ij}(\Delta_3+\Delta_n)\right]D^{(\mathrm{tt})}_{ij}+\frac{n+2}{n-1}\left[\Delta_n P^{(\parallel)}{}^i_i+\Delta_3\right]S\\ =M_{4+n}^{-2-n}\delta^{(n)}(y)\left\{2T_{00}+M_4^2\left(2\Delta_3 S-\left(\partial^i\partial^j-\delta^{ij}\Delta_3\right)D^{(tt)}_{ij}\right)\right\}\ .\end{aligned}\tag{5.35}$$

Obviously, (5.35) does not contain any time derivatives, and can thus be read as constraining the scalar S with respect to the physical field $D^{(tt)}$. We might object that the appearance of the longitudinal projector $P^{(\parallel)}$ reintroduces a dynamical content due to the inverse $\Box$ operators. However, this is just a direct artifact of the variables used, see equation (5.3). Once we choose a prescription for the inverse $\Box$ operators in the decomposition (5.3), this same prescription will be used at every occurrence of $P^{(\parallel)}$. Thus, there is no hidden dependence on initial conditions in (5.35). This becomes even more clear when considering the 00-Einstein equation in the original fluctuation variable h, since there is neither any time derivative nor Green's function appearing, clearly indicating the constraint nature of (5.35). We were even able to show that (5.35) is nothing else but a specific constraint of the Hamiltonian analysis (actually, the so called

'Hamiltonian constraint') by transforming the covariant variables (5.3) into the variables of the multiple ADM split of section 5.2.2.

The constraint (5.35) clearly shows that S is not dynamical and can thus not cause any violation of unitarity, contrary to the claims of [80, 81, 82]. In particular, the usage of the brane-to-brane amplitude as a diagnostic tool for determining the occurrence of a ghost degree of freedom implicitly assumes that the quantum fields $\hat{S}$, etc., obey standard commutation relations with respect to the Poisson bracket. However, we know that in the presence of constraints we should instead replace them with their corresponding Dirac brackets. This is yet an open task, but the results of the Hamiltonian analysis show that the ghost-like pole of the mode S has to disappear when incorporating the Dirac bracket.

Additionally, note that even the counting of the degrees of freedom would not come out correctly if we assumed that S is a dynamical degree of freedom. For $n = 2$ co-dimensions, the Hamiltonian analysis shows that the system is made up in general out of 9 degrees of freedom, from which 4 are removed because of the $SO(2)$-symmetry requirement and only 2 of the remaining 5 degrees of freedom are sourced by the brane matter source. We find this number again in the covariant analysis, since $h_{\mu\nu}$ contains 21 independent fields per se, which is reduced by 6 gauge redundancies and 6 constraints originating from the 5 temporal-spatial components and the temporal-temporal component of the Einstein equations resulting in 9 degrees of freedom in principal. From these 9 degrees of freedom only $D^{(tt)}$ (5 degrees of freedom) and S (1 degree of freedom) would couple to matter, where the $0i$-Einstein equations remove 3 degrees of freedom from $D^{(tt)}$. Only if we further recognize the fact that the 00-Einstein equation removes the mode S completely we will arrive at the number of 2 sourced degrees of freedom (again, see [5] for details regarding the determination of the dynamical degrees of freedom in the covariant formulation).

Finally, note that the same problem and the same solution to the problem already occurs in standard 4-dimensional general relativity. Here, the trace and the transverse-traceless component of the Einstein equations result in

$$+\Box_4 S = \frac{1}{3M_4^2}\, T_\alpha^{\ \alpha}\,, \tag{5.36}$$

$$-\Box_4 D^{(\mathrm{tt})}{}_{\alpha\beta} = \frac{2}{M_4^2}\, T^{(\mathrm{tt})}{}_{\alpha\beta}\,, \tag{5.37}$$

which again could make us belief that general relativity contains a ghost mode S. But again, the 00-Einstein equation saves the day, resulting in the (in this case particularly simple) constraint

$$\Delta S = 0 \; . \tag{5.38}$$

It should have become clear that the pattern that we observe over and over again is: Brane induced gravity behaves simply as standard Einstein gravity (in higher dimensions, though) with a some specific, but unproblematic matter source.

6 Paper

6.1 Self-protection of massive gravitons

6.1.1 Abstract

Relevant deformations of gravity present an exciting window of opportunity to probe the rigidity of gravity on cosmological scales. For a single-graviton theory, the leading relevant deformation constitutes a graviton mass term. In this paper, we investigate the classical and quantum stability of massive cosmological gravitons on generic Friedman backgrounds. For a Universe expanding towards a de Sitter epoch, we find that massive cosmological gravitons are self-protected against unitarity violations by a strong coupling phenomenon.

6.1.2 Introduction

Technical naturalness is arguably one of the most promising pathfinders to physics beyond the standard model of particle interactions and gravity, as well. It offers many exciting windows of opportunity related to renormalisable standard model operators that share an enhanced sensitivity to the scale of new physics.

Among these, the vacuum energy density is standing out in various ways. Being the unique operator with quartic sensitivity to the ultraviolet scale, it also represents the most relevant term in the Einstein–Hilbert action. The basic observation is that the vacuum energy density is technically unnatural within the standard model already at energy scales set by the lightest measured particle masses within its spectrum. In other words, the technical naturalness facet of this challenge is not solely tied to the quantum gravity scale, unless there is an ultraviolet-infrared conspiracy operative in the vacuum sector that also respects the many high-precision successes of the standard model at lower energies.

Although collider experiments cannot measure the vacuum energy density, the challenge it poses becomes serious once the standard model of particle physics is coupled to gravity. Since gravity, being the most democratic field theory, couples to energy-momentum in a universal manner irrespective of its sources' nature, the vacuum curves spacetime and affects the Universe's expansion history. As a consequence, and opening up yet another window of opportunity, it seems attractive to reconsider gravity in the deep infrared and its consistent deformations.

The hunt for a fundamental completion of gravity is characterised by incorporating new degrees of freedom in the ultraviolet with the infrared region kept untouched (in a relevant sense), its role reduced to providing the classical benchmark tests. However, this precludes the opening up of additional gravitational degrees of freedom that could be relevant in the deep infrared on scales that are only poorly constrained by state-of-the-art cosmological observations.

Historically, this question was discouraged by a mighty no-go theorem stating the impossibility to embed gravity in a QCD-like theory with a self-interacting graviton multiplet under the spell of Yang–Mills. More precisely, assuming locality, Poincaré invariance, and a free-field limit consisting of massless gravitons, the only consistent deformations involving a multiplet of gravitons are such that the deformed gauge algebra is just a direct sum of independent diffeomorphism algebras [68].

Interestingly, by relaxing one of its conditions — allowing for new relevant degrees of freedom — this no-go theorem gave rise to a potent tool for studying consistent deformations of Einstein's gravity within the effective field theory framework. For a nonzero deformation parameter, the candidate deformations are characterised by a tremendous reduction of symmetries:

$$\mathrm{diff}(M_1) \otimes \mathrm{diff}(M_2) \otimes \cdots \longrightarrow \mathrm{diag}\left(\mathrm{diff}(M_1) \otimes \mathrm{diff}(M_2) \otimes \cdots\right) ,$$

with M_j denoting (not necessarily) different spacetime (sub-)manifolds. The resulting gauge symmetry restricts the deformation, i.e. the coupling of different geometries, to be solely constructed from invariants of the reduced symmetry [28].

Under the umbrella of this framework, many proposals for relevant deformations of gravity that have been suggested in the last decade become cousins. Even more promising, the unique ghost-free theory for a massive spin-two field propagating all degrees of freedom [29] fits under this umbrella. In the latter case, two copies of Minkowski spacetime are considered, one perfect the other perturbed, and the most relevant deformation becomes a spin-two mass term

$$S_{\mathrm{deform}} = -\frac{m^2}{2} \int \mathrm{d}^4x \; h_{\alpha\beta} \, \mathcal{M}^{\alpha\beta\mu\nu} \, h_{\mu\nu} \, . \tag{6.1}$$

The mass matrix $\mathcal{M}$ depends only on the background geometry and is constant in this case. As mentioned earlier, this matrix is uniquely determined by unitarity arguments and requires tuning.

There is a rather straightforward nonlinear completion of the leading infrared deformation [30],

$$
\begin{aligned}
S_{\text{deform}} &= -\frac{m^2}{2}\int_{M_1} \mathrm{d}^4x\sqrt{-g_1}\;\mathcal{M}^{\alpha\beta\mu\nu}(g_1)\;H_{\alpha\beta}H_{\mu\nu}(g_1,g_2)\;, \\
H_{\alpha\beta} &= g_{1\;\alpha\beta} - E_\alpha^{\;\mu}E_\beta^{\;\nu}\;g_{2\;\mu\nu}\;, \qquad (6.2)
\end{aligned}
$$

where E denotes the pullback from M_2 to M_1. Since the spacetimes (M_1, g_1) and (M_2, g_2) need not be diffeomorphic to each other (even not in the perturbative sense), $H(x)$ is in general not a fluctuation on M_1. In fact, the deformation (6.2) represents a mass term for a graviton[1] only if (M_2, g_2) is a copy of (M_1, g_1) at the background level.

Consider the case when the most relevant deformation is a mass term. Let us assume for simplicity, that some sort of Higgs mechanism is responsible for the graviton mass generation. For instance, in the Fierz–Pauli setup, a massless graviton, which has two transverse polarisation states, combines with a Goldstone vector to a massive spin-2 field, which has six polarisation states in general (but only five on Minkowski spacetime). The Goldstone vector carries three transverse polarisation states and one longitudinal polarisation state. When the massive graviton is at rest, its six polarisation states are completely equivalent. However, if it is moving, the longitudinal polarisation becomes increasingly parallel to the graviton's momentum. As a consequence, at high energies, a massive graviton might look like the longitudinal polarisation state carried by the Goldstone vector [30]. This statement is known as the Goldstone equivalence theorem and rests on the underlying gauge invariance.

The longitudinal polarisation state does not receive kinetic support from the Einstein–Hilbert term, which is precisely why it is in the focus of unitarity requirements. As a matter of fact, many distinguished features of massive gravity, like, for example the van Dam–Veltman–Zakharov discontinuity [31] [32] , the Vainshtein radius [33], or the structure of the Fierz–Pauli mass term on Minkowski spacetime [29], are captured by the longitudinal polarisation state of the Goldstone vector. Consistency invesitgations of massive gravity mostly focused on ghostly excitations at some finite perturbation level based on a Minkowski ground state, with the earliest exception being Higuchi's unitarity analysis on de Sitter spacetime [34].

Higuchi found a consistency relation between the deformation parameter, the graviton mass, and the curvature scale of de Sitter, set by the cosmological constant: In order to avoid negative norm states $m^2 > H^2$, where H denotes the Hubble constant. This bound is of great interest, since de Sitter geometry is unique in the sense that is does not require any source specification. However, from a field theoretical point of view this makes the setup special, because the background reference scale is constant here.

[1] In abuse of notation and logic, we refer to metric perturbations as gravitons irrespective of the background geometry.

In this paper, we generalise Higuchi's bound from de Sitter to general Friedman cosmologies by employing the Goldstone equivalence theorem outlined above. We find a competition between classical stability, the requirement that perturbations respect the background, and quantum stability, the requirement that the spectrum does not contain negative norm states. For the special case of a de Sitter background, both criteria coincide and give rise to the unitarity bound quoted above. The situation is richer for generic Friedman cosmologies. There, the very nature of either bound is more intriguing, since it involves a time dependent curvature scale that is monotonously increasing or decreasing in the past, depending on the specific sources that drive the background expansion. Generically, none of the bounds can be satisfied on the entire spacetime manifold. This, however, does not imply that the theory is invalidated. Indeed, it turns out that massive gravitons in generic Friedman universes are protected against unitarity violations. More precisely, it is not clear at all whether a unitarity bound really exists, because before entering the would-be unitarity violating spacetime region, the theory becomes strongly coupled [2].

We reach the following verdict: Phenomenological constraints require to choose the initial hypersurface close to the present hypersurface. In the case of a radiation or matter dominated Friedman universe, the evolution towards future hypersurfaces is guaranteed to be healthy (for consistent initial conditions) by the strictly monotonic background expansion. In the most interesting case of a matter-cosmological constant mixture, the future evolution will be sound provided the mass is large enough and consistent initial condition have been imposed. In all of the above cases, evolving backwards in time, massive cosmological gravitons will soon enter a strong coupling regime that demands a nonlinear completion of the theory. In other words, the fact that we have a sound theory on all past hypersurfaces is nontrivial and ensured by a strong coupling phenomenon that is confined to these spacetime regions. This is the advertised self-protection mechanism.

Last but not least, we show that all conclusions hold for a generic Friedman source.

6.1.3 Goldstone–Stückelberg analysis

In this section we derive the classical and quantum stability requirements for massive cosmological gravitons.

Consider two copies of a generic (background) spacetime, (M_{B}, γ) and (M_{B}, g) with $\gamma \equiv g + h$, where h denotes the metric perturbation obeying $|g(t)| \gg |h(t, \mathbf{x})|$. In this case, $H = h$ is a perturbation (where, for simplicity, we have chosen the same coordinate system on both manifolds) under the spell of diff(M_{B}) for vanishing deformation parameter.

Turning on the most relevant deformation (6.2), the gauge symmetry is deformed to the

[2]The connection between stability and strong coupling is clarified at the end of section 6.1.3.

diagonal subgroup of diff(M_{B}) $\otimes$ diff(M_{B}). In other words, the deformation removes the freedom to gauge h relative to the background geometry. The massive graviton, however, still carries six degrees of freedom, due to the second Bianchi identity

$$\nabla^{\mu} h_{\mu\nu} = \nabla_{\nu} h \ . \tag{6.3}$$

In general, this is not a gauge choice. For instance, in the undeformed theory for massless gravitons on Minkowski spacetime the constraint (6.3) is not a legitimate gauge, because the corresponding gauge shifts become singular for this choice (as a testimony of the van Dam–Veltman–Zakharov discontinuity on this background).

As is well known, in view of this explicit symmetry deformation, there are two equivalent state descriptions. In the first case, the metric perturbation is split according to

$$h_{\mu\nu} = h^{\perp}_{\mu\nu} + \nabla_{(\mu} V_{\nu)} \ , \tag{6.4}$$

where $h^{\perp}$ is covariantly conserved and carries two transverse degrees of freedom, while V is unconstrained and carries four degrees of freedom. The latter can be decomposed further, $V = V^{\perp} + \partial \Psi$. Here $\nabla \cdot V^{\perp} = 0$ and Ψ carries one degree of freedom. Using this state description, the theory can be considered as a gauge fixed theory.

The second and equivalent state description, called the Goldstone–Stückelberg completion, is based on adding four degrees of freedom, carried by a vector field π in order to restore the original gauge symmetry, diff(M_{B}) $\otimes$ diff(M_{B}). In this case, the completion is given by

$$H_{\mu\nu} \equiv h_{\mu\nu} + \nabla_{(\mu} \pi_{\nu)}, \tag{6.5}$$

where H has ten degrees of freedom, of which six are carried by h and four by the Goldstone–Stückelberg vector π, which can be further decomposed as $\pi = \pi^{\perp} + \partial \phi$. Here $\nabla \cdot \pi^{\perp} = 0$ and ϕ carries one scalar degree of freedom.

As mentioned above, the crucial point of this construction is the restored gauge symmetry that allows to shift h and π relative to the background such that the Goldstone–Stückelberg completed H itself is rendered gauge invariant. It is clear that four degrees of freedom represent gauge redundancies, leaving us with six physical degrees of freedom.

We choose to work with the Goldstone–Stückelberg completed state, for which the leading relevant deformation is exactly the celebrated Fierz–Pauli mass term,

$$\begin{aligned} S_{\text{mass}} \quad &= \quad -\frac{m^2}{2} \int_{M_B} \mathrm{d}^4 x \sqrt{-g}\, H_{\alpha\beta}\, \mathcal{M}^{\alpha\beta\mu\nu}(g)\, H_{\mu\nu} \ , \\ \mathcal{M}_{\alpha\beta\mu\nu}(g) &= \ g_{\alpha\mu} g_{\beta\nu} - \ g_{\alpha\beta} g_{\mu\nu} \ . \end{aligned} \tag{6.6}$$

The mode of the metric fluctuation corresponding to the Goldstone–Stückelberg scalar ϕ dominates scattering processes at high momenta. This is tantamount to the Goldstone boson equivalence theorem and most easily understood from the observation that this mode enters processes with at least two derivatives, $\nabla_\mu \partial_\nu \phi$, which, therefore, grows fastest in the high momentum limit. This is precisely the regime for which we are interested in studying the stability of the deformed theory.

The field ϕ mixes with the metric perturbation h through the mass term (6.6),

$$2m^2 \left(\Box h - \nabla^\mu \nabla^\nu h_{\mu\nu}\right) \phi = 2m^2 \left(R^{(0)}_{\mu\nu}(g) h^{\mu\nu} - R^{(1)}(g,h)\right) \phi \,, \tag{6.7}$$

where we have integrated by parts and introduced the Ricci tensor $R^{(0)} \equiv R$ evaluated on the background configuration g, as well as the Ricci scalar $R^{(1)}$ expanded to first order in h.

In order to eliminate the kinetic mixing term $R^{(1)}(g,h)\phi$, we carry out a conformal transformation

$$\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \equiv (1+\omega)^2 g_{\mu\nu}, \tag{6.8}$$

which, at the linear level, is equivalent to $\hat{h}_{\mu\nu} = h_{\mu\nu} + 2\omega\, \eta_{\mu\nu}$. The Einstein–Hilbert term transforms as

$$\sqrt{-g}\Omega^2 R = \sqrt{-\hat{g}} \left(\hat{R} - 6\Omega^{-2}\, \hat{g}^{ab} \partial_a \Omega \partial_b \Omega\right) \,. \tag{6.9}$$

In order to eliminate the mixing between $h_{\mu\nu}$ and ϕ we must choose $\Omega^2 = 1 - 2m^2\phi$ or, equivalently (since ϕ is a first order quantity in the expansion of $H_{\mu\nu}$), $\omega = -m^2\phi$.

The conformal transformation (6.9) contributes a standard kinetic term for the Goldstone–Stückelberg scalar, while the massive deformation (6.6) gives rise to a non-standard kinetic contribution with the metric field replaced by the background Ricci tensor,

$$2m^2 \left((\Box\phi)^2 - \nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla^\nu \phi\right) = 2m^2 R_{\mu\nu} \partial^\mu \phi \partial^\nu \phi \,. \tag{6.10}$$

The action for ϕ is given by

$$S = \int \mathrm{d}^4x \sqrt{-\hat{g}}\, \left(A\dot{\phi}^2 + B^{ij}(\partial_i \phi)(\partial_j \phi) + \dot{\phi} D^i \partial_i \phi\right) \,, \tag{6.11}$$

with $A = 2m^2(-3m^2 g^{00} + R^{00})$, $B^{ij} = 2m^2(-3m^2 g^{ij} + R^{ij})$, and $D^i = 4m^2(-3m^2 g^{i0} + R^{i0})$. In general, these coefficients are spacetime dependent functions. Note that in (6.11) we have not displayed any potential terms (self-couplings) such as $R(g)h\phi$, since quantum stability refers to the free evolution, and classical stability relies on the kinetic terms at high momenta. The corresponding Hamilton density reads

$$\mathcal{H} = \frac{\pi^2}{4\sqrt{-\hat{g}}A} - \sqrt{-\hat{g}} B^{ij} (\partial_i \phi)(\partial_j \phi) \,, \tag{6.12}$$

where $\pi \equiv \delta\mathcal{L}/\dot{\phi}$. The Hamiltonian is unbounded from below for $A < 0$ or B^{ij} positive definite.

In the case of a generic Friedman background geometry the action for the Goldstone–Stückelberg scalar reduces to

$$S = \int \mathrm{d}^4x\sqrt{-\hat{g}}\ \left(A(t)\dot{\phi}^2 + B(t)\left(\vec{\nabla}\phi/a\right)^2\right)\ , \tag{6.13}$$

where $A(t) = 6m^2(m^2 - \dot{H} - H^2)$, $B(t) = -2m^2(3m^2 - \dot{H} - 3H^2)$, $H = H(t)$ denotes the Hubble parameter, and $a = a(t)$ the scale factor. These coefficients depend on the energy-momentum source curving the Friedman background and, in particular, can change signs during the background evolution. Classical stability requires $A > 0$ and $B < 0$.

In order to study quantum stability of the Goldstone–Stückelberg scalar on a generic Friedman background, we need to transform (6.13) into normal form. This requires the following field redefinition: $\phi \to f\phi$ with $\dot{f} = -Cf/(2A)$, where $C \equiv \dot{A} - 3HB$. The Lagrangian transforms as

$$f^{-2}\,\mathcal{L} = A\,\dot{\phi}^2 - C\,\phi\dot{\phi} + B\,\left(\vec{\nabla}\phi/a\right)^2 + C^2/(4A)\,\phi^2\ . \tag{6.14}$$

The coefficient A controls the sign between $\dot{\phi}$ and $f^{-2}\pi = 2A\dot{\phi} - C\phi$ and has therefore an important impact on the quantum stability of (6.13). This can be worked out along the canonical quantisation prescription. As usual, we postulate the equal time commutation relations

$$\left[\phi(t,\mathbf{x}), \pi(t',\mathbf{x}')\right]_{t=t'} = i\delta^{(3)}\left(\mathbf{x} - \mathbf{x}'\right)\ , \ldots\ , \tag{6.15}$$

and decompose the field $\phi(x)$ into modes $U(t,\mathbf{k}) \equiv u(t,\mathbf{k})\exp\left(\mathrm{i}\mathbf{k}\cdot\mathbf{x}\right)$, where $u(t,\mathbf{k})$ satisfies

$$A\ddot{u} - \left[B\left(\frac{\mathbf{k}}{a}\right)^2 + A\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{C}{2A}\right)\frac{C}{2A}\right]u = 0\ . \tag{6.16}$$

The fact that the collection of $U(t,\mathbf{k})$ represents a complete orthonormal set of solutions (with respect to a spatial hypersurface-independent scalar product) results in a simple condition on the Wronskian of the solutions,

$$\left(u^*\dot{u} - u\dot{u}^*\right)(t,\mathbf{k}) = 1\ . \tag{6.17}$$

As a consequence, the Goldstone–Stückelberg scalar may be expanded as

$$\phi(t,\mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\frac{1}{\sqrt{2|A|}f}\ \left(u(t,\mathbf{k})\exp\left(\mathrm{i}\mathbf{k}\cdot\mathbf{x}\right)a(\mathbf{k}) + u^*(t,\mathbf{k})\exp\left(-\mathrm{i}\mathbf{k}\cdot\mathbf{x}\right)a^\dagger(\mathbf{k})\right)\ . \tag{6.18}$$

Inserting this expansion and the corresponding one for π into the canonical commutation relations (6.15) yields

$$\left[a(\mathbf{k}), a^\dagger(\mathbf{k}')\right] = \mathrm{sign}(A)\ \delta^{(3)}(\mathbf{k} - \mathbf{k}'). \tag{6.19}$$

The construction of a vacuum state and Fock space can now proceed as usual. However, whenever $A < 0$, the construction results in negative norm states, which violate unitarity.

For an arbitrary spacetime, this quantisation procedure bears conceptual challenges, because there may be no Killing vectors at all to define positive frequency modes. The situation is simpler for a Friedman background since it accommodates a restricted set of isometries, i.e. invariance under spatial rotations. Then, together with the corresponding Killing vectors there exist associated (natural) coordinates. Of course, coordinate systems are physically irrelevant — a fact that renders the particle concept somewhat arbitrary on curved spacetimes. However, this concerns the interpretation of the theory. The unitarity requirement $A < 0$ is a coordinate independent statement, since Bogolubov transformations are unitary transformations.

The most important result of this section is that stability considerations led us to require $A > 0$ and $B < 0$ in order to have a bounded Hamiltonian and, in addition, $A > 0$ to have a sound probabilistic interpretation. Hence, provided the theory respects unitarity, $B < 0$ represents the classical stability bound.

This is also clear from the equation of motion (6.16). Indeed, for $A > 0$, to have a stable solution, requires $B < 0$. Further, in order to have a damped solution at late (early) times demands $C > 0 (C < 0)$. Whenever this condition is violated, the background will be destabilised.

In order to investigate the signs of the coefficients A, B, C in a spatially flat Friedman Universe, consider the Friedman equations

$$
\begin{aligned}
3H^2 &= 8\pi G\rho + \Lambda, &(6.20)\\
3\left(\dot{H} + H^2\right) &= -4\pi G\left(\rho + 3p\right) + \Lambda, &(6.21)
\end{aligned}
$$

where ρ is the density, p the pressure, and Λ the cosmological constant. An expanding universe is characterised by $H > 0$ and $\dot{\rho} < 0$, so $\dot{H} < 0$. Furthermore, $\ddot{H} = -3H\dot{H}(1 + {c_{\mathrm{s}}}^2)$, where c_{s} denotes the isentropic sound speed. As a consequence, $\ddot{H} > 0$, for an equation of state governing an arbitrary mixture of matter and radiation.

In the absence of a cosmological constant, H and all its time derivatives vanish at late times. The late time asymptotics of the coefficients are therefore given solely by the gravitons mass, $A = 6m^4 = -B > 0$ and $C = 0$. Consequently, the modes (6.16) are well-behaved. In a de Sitter universe, $A = 6m^2(m^2 - H^2) = -B$ and $C \propto A$. The absence of negative norm states requires $m^2 > H^2$, which is nothing else but the famous Higuchi bound. We see that in this case, classical and quantum stability collapses to a single criterion between the graviton mass and the constant de Sitter curvature scale.

For a generic Friedman universe, at any moment in time, $A > -B$ and $C = 6m^2H(3m^2 - \ddot{H}/H - 3\dot{H} - 3H^2) < 3HA$. As a consequence, evolving the modes (6.16) backward in time, B and C change signs before A does (i.e. at later cosmological times). In the case of C this sign change stabilises the modes since it leads to mode damping. This stabilisation, however, is only marginal and nullified by the change of sign of B which triggers an exponential instability that dominates the large proper momentum regime.

As a consequence, evolving the modes backwards in cosmological times, the system enters first a strong coupling regime (at later cosmological time) before it would violate unitarity (at earlier cosmological times). In this sense, the strong coupling regime protects massive cosmological gravitons from unitarity violations. In other words, whenever massive gravitons in an expanding Friedman universe experience unitarity violations, then, for sure, they are already in a strong coupling regime that demands a nonlinear completion, but not vice versa. In this respect, de Sitter is a borderline geometry since both inconsistencies coincide.

It is important to appreciate that once a generic mode enters the classical instability region $B > 0$, it does not destabilise the background instantaneously. Instead, the characteristic time scale for this to happen is $T(k) \propto a/k$. As a consequence, modes with arbitrary large proper momentum become strongly coupled without further delay once $B > 0$. More precisely, for arbitrary initial conditions, there will always be a critical proper momentum k_* such that all modes with momenta $k > k_*$ enter the nonlinear regime before they would violate the unitarity bound.

Strictly speaking, the self-protection mechanism is confined to and efficient for $B \leq 0$, that is, before the fluctuations enter the unstable regime, which is characterised by the background coefficient B becoming positive, they hit a strong coupling regime. Indeed, for $B < 0$ and $B \to 0$, the proper kinetic energy density $\propto B(\nabla\phi/a)^2$ becomes subdominant on any scale as compared to the potential energy. Hence, the exponential instability discussed above, characterized by $B > 0$, is a testimony of this strong coupling regime and can legitimately be used to identify it.

Self-protection under matter impact

In the following we argue that both the stability and unitarity behaviour of the theory are generically not altered by the specific choice of a matter action $S_{\mathrm{matter}}\left[\Psi_i, g_{\mu\nu}\right]$, where Ψ_i denotes the collection of matter degrees of freedom. This is remarkable because the matter action explicitly depends on the metric field and, in particular, on the Goldstone–Stückelberg scalar ϕ. This coupling could, in principle, change the dynamics of ϕ in a relevant way such that the self-protection mechanism will be overridden. As we will show below, this is not the case.

Since the matter action is invariant under general coordinate transformations, the Goldstone–Stückelberg scalar enters the matter sector only via the conformal transformation (6.8).

Let us first expand the matter action to second order in the fluctuations,

$$S_{\text{matter}}\left[\Psi_i+\delta\Psi_i, g+h\right] \supset \int \mathrm{d}^4x\, \mathrm{d}^4y\, h_{\mu\nu}(x) \left(h_{\alpha\beta}(y)\frac{\delta}{\delta g_{\alpha\beta}(y)} + \delta\Psi_i(y)\frac{\delta}{\delta\Psi_i(y)}\right) T^{\mu\nu}(x)\ . \tag{6.22}$$

After the conformal transformation (6.8), $h = \hat{h} + 2m^2\phi g$, the first term on the right-hand side of (6.22) contributes only self-interaction terms of the form ϕ^2 and couplings $h\phi$. The coefficient of this potential term can be estimated to be of order H^2. Hence, the potential term is subdominant as compared to the kinetic term at high proper momenta.

The conformal transformation of the second term on the right-hand side of (6.22) gives rise to a coupling between the Goldstone–Stückelberg scalar and the matter degrees of freedom,

$$-\,2m^2 \int \mathrm{d}^4x\, \phi \frac{\partial T}{\partial(\partial_\mu\Psi_i)}\partial_\mu\delta\Psi_i\ , \tag{6.23}$$

where we have again neglected potential terms of the form $\phi\delta\Psi_i$. A coupling like (6.23) will not modify the momentum field conjugated to ϕ, and, hence, the unitarity bound is robust against its inclusion. Suppose now we integrate (6.23) by parts, thereby producing a time derivative acting on ϕ. This will modify the conjugated momentum field, but it will only contribute a term proportional to $\delta\Psi_i$. Again, such a coupling respects the unitarity bound.

With respect to stability, the coupling (6.22) might alter the dynamics of the Goldstone–Stückelberg field substantially, as it represents a derivative coupling to the matter degrees of freedom. However, the coefficient of this derivative coupling is a vector field constructed solely from background quantities. Due to the isometries of the Friedman geometry, this background vector has to be of the form

$$\left(\frac{\partial T}{\partial(\partial_\mu\Psi_i)}\right) \propto e_\mu \equiv (1,0,0,0)\ . \tag{6.24}$$

Therefore, no spatial derivative enters the coupling (6.23). Thus the modes experience an additional source proportional to $\partial_t\delta\Psi_i$, and as long as the matter sector takes good care of itself this source cannot alter the stability bound, in particular at large proper momenta.

6.1.4 Discussion

Let us consider the concrete expressions for the coefficients A and B and discuss the possible implications of the corresponding bounds.

De Sitter spacetime

For a de Sitter spacetime, $A = 6m^2(m^2 - H^2) = -B$. The absence of negative norm states requires the unitarity bound $m^2 > H^2$ to hold, which is just the well-known

Higuchi bound (in our conventions). Provided this bound has been satisfied, classical stability is established automatically, as $A = -B$. In fact, de Sitter geometry is unique within the class of Friedman spacetimes where classical and quantum stability requirements coincide, and where stability is solely expressed in terms of two model parameters, i.e. the graviton mass and the constant de Sitter curvature scale. Since linearisation is permissible, the stability bound, $m^2 > H^2$, should be an important consistency condition for a nonlinear completion, as well.

Generic Friedman spacetimes

For a generic Friedman spacetime, the absence of negative norm states demands $m^2 > H^2 + \dot{H}$, while classical stability requires $m^2 > H^2 + \dot{H}/3$. We were able to confirm these stability bounds by a full-fledged perturbation analysis of massive cosmological gravitons involving all degrees of freedom, as well as a complete set of couplings. (See [3].) The same bounds were also derived for the special case of scalar field matter in [36]. The findings of [49], however, do not coincide with ours, due to the unconventional matter Lagrangian that has been used in their paper.

In the case of an expanding Universe the unitarity bound will always be satisfied during radiation ($m^2 > -H^2$) or matter domination ($m^2 > -H^2/2$). Thus, the absence of negative norm states is guaranteed by the isotropic expansion. This does, however, not imply that classical stability is unchallenged during the Universe's expansion, since the classical stability bounds $m^2 > H^2/3$ (during radiation domination) and $m^2 > H^2/2$ (during matter domination) will eventually be violated when the modes are being evolved backwards in cosmological time.

The situation becomes more interesting for a universe filled with a mixture of matter and a cosmological constant. In this case, there is no unitarity issue at early times before the transition from matter to cosmological constant domination. However, after the matter-cosmological constant transition $H^2 + \dot{H}$ will become positive and eventually constant as the Universe evolves towards its de Sitter fate. If the cosmological graviton's mass is larger than the asymptotic value for the Hubble parameter set by the de Sitter curvature scale, the theory always respects unitarity. If, however, the mass parameter is chosen to be smaller than the cosmological constant, it is guaranteed that the ϕ modes first enter the epoch of classical instability, before (evolving forward in time) they hit the then would-be unitarity bound (since $\dot{H} < 0$). As a consequence, massive cosmological gravitons are protected against unitarity violation by the background expansion. We conjecture that this sort of self-protection could be robust against any nonlinear completion.

We are well aware that a non-linear completion may suffer from another inconsistency problem, the so-called Boulware–Deser ghost [27]. This definitely requires further investigations, but there are already first indications that certain non-linear deformations could be consistent [14, 50].

Similar remarks apply to more general source terms. For a general universe with $\dot{H} < 0$ the massive cosmological gravitons always first enter the strong coupling regime, rendering the would-be unitarity bound fictitious and opening an exciting window of opportunity towards a self-protection mechanism. Finally, let us state the situation is reversed in a universe with $\dot{H} > 0$. Here, negative norm ϕ states will show up in the weak coupling regime. There is no obvious self-protection mechanism in this case.

6.2 Cosmological Classicalons

6.2.1 Abstract

Generic relevant deformations of Einstein's gravity theory contain additional degrees of freedom that have a multi–facetted stabilization dynamics on curved spacetimes. We show that these relevant degrees of freedom are self–protected against unitarity violations by the formation of classical field lumps that eventually merge to a new background geometry. The transition is heralded by the massive decay of the original vacuum and evolves through a strong coupling regime. This process fits in the recently proposed classicalization mechanism and extends it further to free field dynamics on curved backgrounds.

6.2.2 Introduction

At the core of Einstein's gravity theory is a democratic principle guaranteeing any source the same coupling to spacetime, independent of its nature. Exploiting this principle has been the successful strategy sui generis to infer the vacuum's energy density by observing the Universe's expansion history.

Ever since Cosmology has advanced to a high precision science, the window of opportunity to probe gravity's rigidity on cosmological scales is wide open. Deforming Einstein's theory of gravity introduces additional relevant degrees of freedom, which have a profound impact: they violate the gravitational coupling's universality. Such proposals might be crucial to cure the spacetime impact of technically unnatural sources, such as the vacuum energy density, which is in staggering conflict with the observed expansion history.

At the linear level, a seminal mechanism for this is at work in the Fierz–Pauli theory for massive spin-2 excitations on a Minkowski background, where the relevant deformation corresponds to a mass term that is unique by consistency requirements. On generic backgrounds, the principle of equivalence demands additional geometric deformations that allow for a richer phenomenology at every level of the effective field theory description.

In this letter we consider generic relevant deformations of the Einstein–Hilbert action on arbitrary backgrounds at the linear level. Explicit results are shown for Friedmann–Robertson–Walker (FRW) spacetimes. We analyze the classical and quantum stability of the 'free theory,' which is a necessary prerequisite before completing the corresponding deformations at the nonlinear level.

We show that the stabilization dynamics for the additional relevant degrees of freedom on a curved background, even when they are 'free,' is as multi–facetted and rich as self–protection mechanisms in certain non–renormalisable interacting systems on a Minkowsk geometry.

In particular, we show that the recent classicalization proposal [72, 73, 74] is at work: the dynamics of the additional degrees of freedom protects them against unitarity violations via the formation of classical objects that eventually become the new background spacetime. The transition to the new geometric ground state is heralded by the massive decay of the original vacuum and evolves through a strong coupling regime.

6.2.3 Framework

The effective Lagrangian describing the dynamics of a single dimensionless scalar field Φ coupled to the metric field g, organized as a derivative expansion, is given by

$$-\mathcal{L}_{\text{eff}} = \sqrt{-\mathrm{g}} \sum_{n=0}^{\infty} \sum_{j=0}^{n} M^{4-2n} C_{2j}^{(2n)}(\Phi) R^{n-j} \left(\nabla\Phi\right)^{2j} . \tag{6.25}$$

For simplicity, all terms have been written down schematically to sketch their scaling with the fiducial mass M. $2n$ is the total number of derivatives at this level. The coefficient $C_0^{(0)}$ is the non–derivative part of the Φ self–interactions. The characteristic scale v of this term might be well below M, and $C_0^{(0)} \propto (v/M)^4$. At $n = 1 = j$ the kinetic terms enter. R^{n-j} stands for all possible combinations of a total of $n - j$ Ricci scalars, tensors, and Riemann tensors. Note that the fiducial mass scale M could be much smaller than the reduced Planck mass M_{P}, in which case $C_{0,2}^{(2)} \propto (M_{\mathrm{P}}/M)^2$. Further and Φ independent gravitational sources could be added.

For the purpose of studying the stability of the effective theory (6.25), we expand about classical background configurations and geometries, $\Phi = \Phi_0 + \phi/M$ and $\mathrm{g} = g + h/M$. Of particular interest is the case where the background configuration is decoupled from the background geometry, i.e., when $\Phi_0 \equiv 0$. Expanding the effective theory (6.25) up to second order in the fluctuations ϕ, the kinetic sector for ϕ becomes

$$-2\mathcal{L}_{\text{kin}} = \sqrt{-g}\,\left[g^{\mu\nu} + \mathcal{F}^{\mu\nu}(R/M^2)\right] \nabla_\mu\phi\nabla_\nu\phi \,. \tag{6.26}$$

Generically, the matrix $\mathcal{F}$ has no definite signature and thus, the perturbative consistency of (6.26) is rather sensitive to the geometrical background. It will prove to be useful to recast it in terms of a canonical kinetic term and a coupling to a ϕ dependent source $\mathcal{J}$,

$$2\mathcal{L}_{\rm kin}/\sqrt{-g} \;\;=\;\; \phi\,\Box\,\phi + \mathcal{J}(R/M^2,\phi)\phi\;, \tag{6.27}$$

with $\mathcal{J} \equiv \nabla_\mu \left(\mathcal{F}^{\mu\nu}\nabla_\nu\phi\right)$, where potential boundary contributions have been suppressed for the moment.

Starting from (6.26) we can analyze the stability of the system by looking after imaginary contributions to the one-loop effective Lagrangian (the vacuum persistence amplitude),

$$2\mathcal{L}^{(1)} = \ln \mathrm{Det}\{\nabla_\mu[(g^{\mu\nu} + \mathcal{F}^{\mu\nu})\nabla_\nu\Box^{-1}]\}, \tag{6.28}$$

which is normalised to the free part $\Box$. In the short-distance limit we get contributions of the form

$$2\mathcal{L}^{(1)} \supset \ln \mathrm{Det}[(g^{\mu\nu} + \mathcal{F}^{\mu\nu})\partial_\mu\partial_\nu\Box^{-1}]. \tag{6.29}$$

The fact that the matrix $g^{\mu\nu} + \mathcal{F}^{\mu\nu}$ can, in general, have a signature different from the metric $g^{\mu\nu}$ can lead to negative arguments of the ln and thereby, to imaginary contributions, which signal the decay of the vacuum.

However, here a signature change is heralded by a strong coupling regime, and, as a consequence, the degrees of freedom that trigger the vacuum decay are self–protected against unitarity violation.

There are two known self–protection mechanism between which a theory can choose to establish consistency. Either it allows for weakly coupled heavy degrees of freedom with masses above the original strong coupling scale, or it generates classical field configurations via energy–momentum self–sourcing, corresponding to a feedback through nonlinear terms. The latter case is the recently proposed classicalization mechanism, see below.

6.2.4 Cosmology

For concreteness, let us consider Fierz–Pauli theory on FRW spacetimes. There, five instead of two degrees of freedom propagate. One of these supplementary degrees of freedom, which, in the spirit of the Goldstone boson equivalence theorem dominates the dynamics at high energies corresponds to the field ϕ. As a consequence, Fierz-Pauli theory fits in the framework (6.25). To be more precise, the metric fluctuation $h_{\mu\nu}$ is written as

$$h_{\mu\nu} = \tilde{h}_{\mu\nu} + \nabla_{(\mu}A^T_{\nu)} + \nabla_\mu\nabla_\nu\phi \tag{6.30}$$

where $\tilde{h}_{\mu\nu}$ carries two degrees of freedom, like the massless graviton would propagate. This clarifies how ϕ feeds into the spacetime fluctuations.

In Fierz–Pauli theory over an FRW spacetime we find for the source [1]

$$m^2 \mathcal{J} = (\dot{H} + H^2)\ddot{\phi} + (\ddot{H} + 5H\dot{H} + H^3)\dot{\phi} + (\dot{H} + 3H^2)\frac{\vec{\nabla}^2}{a^2}\phi \,, \tag{6.31}$$

where $H = H(t)$ denotes the Hubble parameter, $a = a(t)$ the scale factor and m the deformation parameter of Fierz-Pauli theory, which would be interpreted as the graviton mass on a Minkowski background. Moreover, $\mathcal{F}^{\mu\nu}(R/M^2) = -R^{\mu\nu}/3m^2$. Then stability of the vacuum state requires

$$m^2 > H^2 + \dot{H}/3 \,. \tag{6.32}$$

We had already discovered this bound in [1], applying a classical stability analysis to the system (6.26) for the case of Fierz–Pauli gravity. This classical stability bound (6.41) arises when the spatial components of $g^{\mu\nu} + \mathcal{F}^{\mu\nu}$ change sign. In fact, the coefficient in front of $\vec{\nabla}^2\phi/a^2$ in (6.31), which coincides with $\mathcal{F}^{ii}$, changes its sign relative to g^{ii}.

A violation of the stability bound manifests itself in an explosion of the otherwise oscillating fluctuation solution. This is shown in Fig. 6.2, where B corresponds to ϕ. The

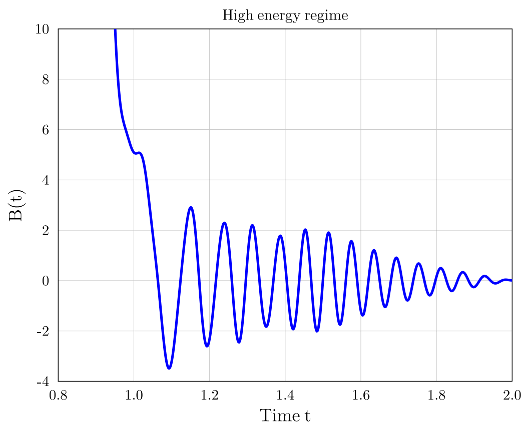

Figure 6.1: Numerical analysis in the high energy regime for a radiation dominated universe [3]. m is chosen such that the bound (6.41) is violated for $t < 1$. B corresponds to ϕ.

loss of classical stability typically signals that the system evolves into a new classical background solution. Equipped with this knowledge, we are able to reinterpret the above decay of the vacuum: Many free modes pop out of the vacuum and their superposition yields a classical object with large occupation number. In this case, the largest possible classical object is formed — a new spacetime background.

At this level in the effective field theory description, the newly formed classical field configuration evolves according to the differential operator in the numerator of (6.28).

It is then decomposed into a superposition of 'free' solutions (those of $\Box$), which present potential decay modes. Especially this last aspect will become important below.

Even though the vacuum persistence amplitude cares about quantum mechanical consistency, its non–normalizablitlity, in our case, can be traced back solely to the background decay, and bears no impact on the intrinsic consistency of the quantum theory.

For the temporal components in the one-loop effective Lagrangian the sign change is postponed to

$$m^2 > \dot{H} + H^2 \,. \tag{6.33}$$

This is linked to the coefficient of $\ddot{\phi}$ in the source (6.31), which coincides with $\mathcal{F}^{00}$, and which has to be compared to g^{00}.

In [1, 3] we found that the relation (6.40) must be satisfied to ensure the absence of negative norm states. For non–phantom matter ($\dot{H} < 0$) the classical stability bound (6.41) is stronger than the unitarity bound (6.40). Therefore, we cannot trust this derivation of the unitarity bound at all, since, whenever it looks as if the theory would contain negative norm states, it is no longer in the perturbative regime. There is thus the hope that a full non–linear theory might not contain any unitarity violating negative norm states. We call this self–defense mechanism of the linear theory 'self–protection' [1].

6.2.5 Classicalisation

The above is a striking example for the concept of classicalization [74, 73, 72]: Classicalization is a unitarization mechanism based on energy–momentum self–sourcing at variance with unitarization by weakly interacting short-distance physics (the Wilsonian mechanism). In a nutshell, at high energies, the formation of a classical object (the 'classicalon') inhibits interactions at short distances, which leads to the unitarization of the process. Remarkably, as was already noted in [74], the concept was already brought forward by Heisenberg in 1952 [75], in a closely related context. The classical object formed in the course of the unitarization process will finally decay into a large number of final states, which is the prime signal for this mechanism. As explained just above Eq. (6.40), the analysis (6.28) has also exactly this interpretation.

For the example of Fierz-Pauli theory it can be ascertained that the number of free modes is indeed large, as the bound (6.41) is independent of how short distances we regard. As a consequence, there will be contributions to the imaginary part of the determinant from arbitrarily small distances.

This brings us back to the aforementioned concept of energy/momentum self–sourcing [72, 73, 74]. Self–sourcing occurs in interacting field theories in which the interaction terms contain sufficiently many derivatives. In such a setting solutions with small amplitudes but sufficiently high four–momentum lead to a strong enhancement of exactly these interaction terms and to unitarization by classicalization. In Eq. (6.27) this is realized in the source $\mathcal{J}$: First, there occurs a function of the curvature R, which power by power contains two derivatives of the background geometry, and additionally, there are the derivatives of ϕ. Accordingly, for Fierz–Pauli gravity over an FRW spacetime, the source (6.31) contains the curvature H and its temporal derivatives as well as derivatives of ϕ. For rather generic choices of spacetime sources, the curvature part behaves as $\mathcal{J} \supset R_{\mu\nu} \propto 1/t^{\alpha}$. The increase of the self–source's strength with increasing localization is a feature characteristic for classicalization [72]. In our case, due to the background isometries, the localization scale is a time–scale.

Self–sourcing does not stop there. In a next step, the fluctuations ϕ and particularly their derivatives would become sufficiently sizable to trigger a change of the background spacetime geometry.

6.2.6 Dictionary

Restoring explicit insertions of $\hbar$ we are able to distinguish between mass and length scales. After canonically normalizing the field ϕ, the parameter in front of $R^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ has units of length squared. In the same way, the parameter in front of the Fierz–Pauli combination has units of inverse length squared. The corresponding term in (6.25) should thus be written as

$$M^{-2}R^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \to L^2 R^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \tag{6.34}$$

In Fierz–Pauli theory, L would denote the screening length of the gravitational field. The bound (6.40) defines the time t_U where the theory would violate unitarity, whereas (6.41) defines the time $t_\star$ where the classical theory becomes strongly coupled. For a cosmology that is dominated by matter with the equation of state $\partial p/\partial\rho = w$, their values are given by,

$$t_U = \frac{\theta(\hbar)}{3L}\frac{1}{1+w}\sqrt{2(-1-3w)}, \tag{6.35}$$

$$t_\star = \frac{1}{3L}\frac{1}{1+w}\sqrt{2(1-w)}\,. \tag{6.36}$$

We have explicitly restored the $\hbar$ dependence in the expression (6.35) with the convention $\theta(0) = 0$. The characteristic time scale for unitarity violation, t_U, is a direct consequence of a quantum commutation relation, which explains the appearance of the Heaviside function θ. The characteristic time scale (6.36) for violating classical stability does not contain $\hbar$, as it is solely set by classical physics.

The existence of two time scales, one characteristic for quantum instability, the other for classical instability, and their hierarchy $t_\star > t_U$ (provided $\mathrm{Im}(t_U) = 0$), is again in analogy to the classicalization concept [72, 73, 74].

Note that for many reasonable values of w the time scale t_U is imaginary. In this case, the would–be unitarity bound is absent. However, for mixtures of a cosmological constant with other FRW sources, t_U will give some finite positive number.

One might ask whether it is possible to cure the theory by a Wilsonian treatment, that is, integrating in new heavy degrees of freedom, instead of creating classical objects. To do so, we replace

$$\mathcal{F}^{\mu\nu}(R/M^2)\ \nabla_\mu\phi\nabla_\nu\phi \to \mathcal{F}^{\mu\nu}(R/M^2)\ \frac{\Lambda^2}{\Box+\Lambda^2}\nabla_\mu\phi\nabla_\nu\phi\ . \tag{6.37}$$

At high energies, $\Box \gg \Lambda^2$, this term will loose its kinetic nature. Accordingly, we would be left with a standard kinetic term for ϕ, and there would neither occur a stability nor a unitarity issue. However, in the opposite regime, $\Box \ll \Lambda^2$, this modification is negligible, and there is no window for a Wilsonian cure of the theory. Instead, classicalization must occur.

In particular, in the case of Fierz–Pauli theory, the Lagrangian (6.26) only describes the theory in the high energy regime, $\Box \gg m^2, H^2$. For phenomenological reasons, we typically take $m^2 \approx H^2$, so that we effectively have $\Box \gg m^2$. Moreover, any new Wilsonian heavy degrees of freedom must have a mass Λ much above m, $\Lambda^2 \gg m^2$. Otherwise, the effective theory (6.26) would have been incomplete. Hence, intermediate energies $m^2 \ll \Box \ll \Lambda^2$, at which the theory classicalizes, exist always, whereas for $\Box \gg \Lambda^2$ a Wilsonian mechanism might be at work. Accordingly, Fierz–Pauli theory on FRW has always a finite classicalitzation window [74].

The original classicalization proposal embraced interacting field theories over Minkowski spacetime. Here, we showed explicitly that the classicalization paradigm extends to free field theories on curved spacetimes.

6.2.7 Summary

In this letter, we have shown that generic relevant deformations of Einstein's gravity theory feature strong coupling phenomena among the additional degrees of freedom that originate from energy–momentum self–sourcing, corresponding to a feedback through nonlinear terms. Moreover, this kind of self–protection follows precisely the recently proposed classicalization paradigm, however, extending its domain to include free field dynamics on curved backgrounds.

We have demonstrated explicitly that the classicalization window is open for Fierz–Pauli like deformations of gravity on FRW spacetimes. A Wilsonian mechanism could only close it partially (at high energies), then leading to a finite classicalization window. There, classicalization proceeds through a strong coupling regime that triggers the massive decay of the original vacuum and signals the formation of the largest possible classicalon — the new background spacetime.

The consistency of relevant deformations on arbitrary background is thus implied — a necessary prerequisite for a nonlinear completion. In this respect, classicalization might present a self–protection mechanism that stabilizes the theory at the nonlinear level, provided the classicalization scale always beats the characteristic scale for unitarity violation.

6.3 Consistency of Relevant Cosmological Deformations on all Scales

6.3.1 Abstract

Generic relevant deformations of Einstein's gravity theory contain additional degrees of freedom that have a multi–facetted stabilization dynamics on curved spacetimes. We show that these relevant degrees of freedom are self–protected against unitarity violations by the formation of classical field lumps that eventually merge to a new background geometry. The transition is heralded by the massive decay of the original vacuum and evolves through a strong coupling regime. This process fits in the recently proposed classicalization mechanism and extends it further to free field dynamics on curved backgrounds.

6.3.2 Introduction and Outline

Given the tremendous progress in high-precision cosmology, in particular, the decisive character of distance indicators and structure formation probes on large scales, the time is ripe to test the rigidity of Einstein's theory of gravitation on cosmological scales. This observational challenge is preceded by theoretical efforts aiming at consistent modifications of gravity at the largest observable distances. Obviously, only consistent theories are worthy to be confronted with data.

In a classical theory, at the exact level, consistency refers to the existence of a well posed initial value formulation and continuous solutions for the underlying degrees of freedom on the entire spacetime manifold. More precisely, at the technical level, the evolution of a scalar degree of freedom Φ on a spacetime manifold $\mathcal{M}$, should be given by a quasilinear, diagonal, second order hyperbolic equation

$$q^{\mu\nu}\left(x;\Phi;\nabla\Phi\right)\nabla_\mu\nabla_\nu\Phi(x)=\mathcal{J}\left(x;R;\Phi;\nabla\Phi\right)\ , \tag{6.38}$$

where q is a smooth Lorentz metric, which, in general is not identical to the spacetime metric g, since it is permitted to depend on the scalar degree of freedom and its first derivative, and $\mathcal{J}$ is a smooth function that may have a nonlinear dependence on these variables. Moreover, the current density $\mathcal{J}$ may depend on the Ricci tensor $R(g)$.

At the perturbative level, consistency of a classical theory demands hyperbolic evolution only on a bounded spacetime region, the perturbative domain, beyond which the fluctuation dynamics requires a non-perturbative completion that is consistent in the aforementioned sense. Perturbations around a classical solution can be quantized in the usual way, given technically natural interactions. The standard requirements for a probabilistic interpretation offer yet another and distinct notion of consistency related to the quantum stability of the theory.

Classical stability at the perturbative level and quantum stability stand on quite different footings. In fact, a finite domain of validity for classical perturbations does not cause any principal obstacle provided the underlying theory is consistent. Of course, once fluctuations leave the classical stability region their background develops an instability towards a new ground state. In contrast, a quantum mechanical instability is not related to specific initial conditions but instead to the massive production of particles at no cost, which are represented by negative norm states. Therefore, the underlying theory is flawed at the fundamental level. Additionally, what here is called quantum instability already has incisive effects within the framework of a purely classical analysis, which we discuss in Sec. 6.3.4.

There are different frameworks for constructing consistent modifications of Einstein's theory of gravitation, once additional degrees of freedom are allowed in the description[3]. As an instructive example[4], consider an additional second rank tensor Ψ, not necessarily a metric, inducing the following relevant deformation of the Einstein–Hilbert action

$$\mathcal{S} = \int d^4x\sqrt{-g}\, M_{\mathrm{P}}^2 \left[R\,(g) - 2\Lambda - m^2 HMH/2\right] + \dots\ , \tag{6.39}$$

where $H \equiv g - \Psi$, m has mass dimension one and sets the characteristic scale for the deformation, and $M(g)$ denotes the de Witt bimetric. Note that the de Witt bimetric is the most relevant albeit not unique choice for M, and we have neither written down explicitly the Ψ kinetic and potential self-interaction terms nor the matter sector.

Assuming that Ψ is locked into the Minkowski metric, for one reason or another[5], the interpretation of the deformation parameter follows from perturbing the metric around the Minkowski geometry, $g = \eta + h$. Expanding the action (6.39) to second order in the fluctuations h, the Fierz–Pauli theory [29] is rediscovered, for which the de Witt bimetric with respect to the background spacetime is the unique unitary choice. This justifies to think of the deformation as a mass term with the deformation parameter being the graviton mass. Of course, this interpretation hinges on the background geometry.

[3] Hence, strictly speaking, these modifications are not faithful deformations in the BRST terminology. This is known as the statement that multi-diffeomorphic theories have no Yang–Mills analogue.

[4] For a bi-diffeomorphic construction see [28, 30].

[5] For the moment it is not important to specify a dynamical mechanism that would give rise to the locking process.

The deformation presented in (6.39) was primarily investigated on Minkowski and de Sitter background geometries for the following reasons: Given the interpretation of the deformation parameter on a Minkowski background, (6.39) has been used to study consequences of a graviton mass for the principle of equivalence, in particular, how the impact of seemingly technically unnatural sources on the background geometry could be weakened. Higuchi [34] showed that an intriguing relation between the deformation parameter and the cosmological constant needs to be fulfilled, $m^2 > 1/3\Lambda \equiv H^2$ (H stands for the Hubble constant), in order to render the free dynamics of h on a de Sitter geometry unitary. If this bound is violated, unitarity violating negative norm states are introduced in the respective Hilbert space.

Both backgrounds are special in that no source specifications based on radiation or matter fields are required. This is of course different for generic Friedmann cosmologies for which the Hubble parameter varies in time and, thus, the right-hand side of Higuchi's bound generalized to sourced Friedmann geometries can be expected to become time dependent. In particular, it seems that for any deformation parameter at early enough times unitarity violation is inevitable. The observation that the Hubble parameter's flow backwards in time seems to induce quantum instabilities is a serious challenge for the viability of the considered deformation. In fact, it is not clear whether the theory (6.39) makes sense at all.

The question of generalizing the Higuchi bound was first addressed in [36]. Here, the generalized bounds were derived for the special case of scalar field matter. In [1] we were able to proof that these bounds are valid for all matter Lagrangians which do not explicitly depend on derivatives of $g_{\mu\nu}$. Moreover, we discussed the self-protection aspect of these bounds, as well its interpretation in terms of classicalon formation [2]. The investigation of [1] relied on the usual Stückelberg completion of h in conjunction with the Goldstone boson equivalence theorem[6] [30]. We found that the theory (6.39) is, naively, subjected to two distinct bounds on Friedmann cosmologies characterized by time dependent Hubble parameters. One of them,

$$m^2 > H^2 + \dot{H} , \tag{6.40}$$

enforces the absence of negative norm states (unitarity bound), whereas the second,

$$m^2 > H^2 + \dot{H}/3 , \tag{6.41}$$

describes the region where hyperbolic evolution of the fluctuations is guaranteed (stability bound). Beyond this region, hyperbolicity breaks down. But this is no principal

[6]This theorem has not yet been rigorously proofed for the spin-2 case, but in [1] we gave very reasonable arguments for its applicability. Moreover, the fact that the results of this paper coincide with [1] in the high energy limit underlines its applicability.

problem, since the breakdown is triggered by a strong coupling regime that simply invalidates the perturbative approach, demanding for a nonlinear completion. Now, for all reasonable Friedmann sources, $\dot{H} < 0$. As an important consequence, the (classical) stability bound imposes a stronger requirement on the deformation parameter than the unitarity bound. For concreteness, we assign a value to the deformation parameter such that the stability bound is satisfied for times $t > t_*$. Evolving backwards in time, the (classical) stability bound will eventually be invalidated since the deformation parameter is constant for the most relevant deformation (6.39). This signals the onset of the nonlinear regime. Thus, the would be unitarity bound lies beyond the perturbative domain and its derivation using perturbation theory cannot be trusted. In this precise sense, the theory is self-protected against unitarity violations, and moreover, there is an open window of opportunity for a consistent nonlinear completion.

Even though the Goldstone boson equivalence theorem represents a powerful diagnostic tool that allows to extract the leading short-distance behavior (and, futhermore, many interesting phenomena related to the most relevant deformation of the Einstein–Hilbert term can be understood by employing it, as for example the structure of the Fierz–Pauli mass term, the vDVZ discontinuity [31, 32] or the Vainshtein radius [33], see also [30]), it applies only in normal neighborhoods characterized by sub-Hubble distances $\ll 1/m$.

The main purpose of the present paper is to extend our consistency analysis to the intermediate and low energy regime. The prime framework to achieve this is a full-fledged cosmological perturbation theory for all degrees of freedom. As usual, the metric fluctuations are decomposed into irreducible SO(3) tensors in accordance with the isometries of Friedmann geometries. Compared to the $m = 0$ case, the equation of motion for the second rank SO(3) tensor modes is deformed only by an additional hard mass term. This is due to the fact that the degrees of freedom carried by the second rank SO(3) tensor are gauge invariant in the undeformed theory. The equations of motion for the first (vector) and zeroth (scalar) rank SO(3) tensors change considerably in the deformed theory. This is a testimony of the fact that the deformed theory (6.39) apparently has no gauge redundancy. It should be noted, however, that the deformed theory has an equal amount of constraints compared to the gauge freedom possessed by the undeformed theory (and in fact could be understood as the gauge fixed version of the Stückelberg extended theory).

The importance of these efforts is easily illustrated by the following results: From the SO(3) vector sector arises a stability criterion that cannot be recognized by employing the Goldstone boson equivalence theorem. This additional criterion signals the presence of a tachyonic instability whenever

$$\mathbf{k}_{\text{phys}}^2 + 3\dot{H} + 2m^2 \geq 0 \tag{6.42}$$

is not satisfied. Here, $\mathbf{k}_{\text{phys}} \equiv \mathbf{k}/a(t)$ denotes the physical wavenumber. On sub-Hubble scales, this criterion is always fulfilled and, thus, the dynamics extracted by employing the Goldstone boson equivalence is not affected by the tachyonic instability in the vector sector. In fact, the equivalence theorem does not cover this sector at all, as it is subdominant compared to the scalar sector. In order to preserve stability on super-Hubble scales, however, we find the new bound

$$m^2 > -3/2\,\dot{H}\ . \tag{6.43}$$

For any choice of the deformation parameter, this bound will be violated in the sufficiently early Universe, and, as a consequence, the vector modes will develop a tachyonic instability, thereby triggering the transition to a new ground state. This result supports the self-protection mechanism found and analyzed in [1]. The vector sector, thus, plays an important part in the stability analysis, although it does not participate in the Goldstone boson equivalence.

The cosmological perturbation theory of (6.39) reveals more insight into the stability dynamics, even in the scalar sector. Most importantly, the unitarity bound (6.40) seems at work on all scales and not just on extreme sub-Hubble scales. Isolating the scalar sector, this poses a potential threat for the self-protection mechanism, since it is a priori not clear whether a strong coupling regime self-protects the theory also on super-Hubble scales. We have, however, shown analytically that the scalar sector is protected against unitarity violations for $\mathbf{k} = \mathbf{0}$ in the same sense as it was for sub-Hubble domains. To be more precise, we again find a stability violating region that occurs before the system enters the would-be unitarity violating region when evolved backwards in time. Compared to the sub-Hubble case, this region is simply shifted to larger values of the time t, so it seems reasonable to assume that there exists such a stability violating region for all values of $\mathbf{k}$. This conjecture is also confirmed by a numerical analysis. Moreover, as we have discussed, we know that the vector sector will become unstable whenever (6.43) is violated, and thus contributes importantly to the self-protection of the system.

6.3.3 The evolution of small fluctuations in the deformed theory

The deformed equations of motion for the metric field g following from (6.39) are given by

$$G_{\mu\nu}(g) - m^2 M_{\mu\nu}^{\ \ \alpha\beta}(g)\ H_{\alpha\beta} = -8\pi M_{\mathrm{P}}^{-2}\ T_{\mu\nu}(g,\chi)\ , \tag{6.44}$$

where again $H = g - \Psi$. Ψ is assumed to be locked into some reference metric, by one mechanism or another. T denotes the energy-momentum source, which depends on matter and radiation fields χ, the metric field, and, in principle, an effective cosmological constant, as well. Any solution of the undeformed Einstein equations will be respected by the deformation, provided Ψ is locked into the appropriate tensor.

The Bianchi identity of the undeformed theory together with energy-momentum conservation of the source implies the following four exact constraints on the combination

$H = g - \Psi$ in the deformed theory,

$$\nabla^\mu H_{\mu\nu} - \nabla_\nu H = 0 \ . \tag{6.45}$$

Consider now metric perturbations $h = g - \gamma$ around a Friedmann background γ compatible with T. Assume Ψ to be locked into the respective Friedmann metric and to be inert to the extend that it can be considered a fixed reference metric. Then $H = h$ and the equations of motion for small h-fluctuations following from (6.44) are

$$\delta R_{\mu\nu}(\gamma, h) - m^2 \left(h_{\mu\nu} + \frac{1}{2} h \ \gamma_{\mu\nu} \right) = -8\pi M_{\mathrm{P}}^{-2} \ M_{\mu\nu}{}^{\alpha\beta}(\gamma) \ \delta T_{\alpha\beta} \ , \tag{6.46}$$

to linear order in h. Here, δR and δT are the linearized Ricci and energy-momentum tensors, respectively. To this order, the four constraints are given by

$$\nabla^\mu(\gamma) h_{\mu\nu} - \nabla_\nu(\gamma)\gamma^{\alpha\beta} h_{\alpha\beta} = 0 \ , \tag{6.47}$$

which looks like a gauge constraint, but in fact is not.

The spatial isotropy and homogeneity of Friedmann backgrounds allow us to decompose the metric fluctuation h into irreducible tensors with respect to these isometries,

$$\begin{aligned} h_{00} &= -E \ , & (6.48)\\ h_{i0} &= a\left[\partial_i F + G_i\right] \ , & (6.49)\\ h_{ij} &= a^2\left[A\delta_{ij} + \partial_i\partial_j B + \partial_{(j} C_{i)} + D_{ij}\right] \ . & (6.50) \end{aligned}$$

Here, E, F, A, and B denote SO(3) scalars, G_i and C_i are the components of a transverse SO(3) vectors ($\partial^a G_a = 0, \partial^b C_b = 0$), and the D_{ij} denote the components of a transverse-traceless rank-2 SO(3) tensor ($\partial^a D_{ab} = 0$ and $\delta^{ab} D_{ab} = 0$).

The appropriate source for a Friedmann spacetime is the energy-momentum tensor of a perfect fluid. Its perturbations can be decomposed in the same spirit

$$\begin{aligned} \delta T_{00} &= \delta\rho - \bar\rho \, h_{00} \ , & (6.51)\\ \delta T_{0i} &= -\left(\bar\rho + \bar p\right)\delta u_i + \bar p \, h_{0i} \ , & (6.52)\\ \delta T_{ij} &= \bar p \, h_{ij} + a^2 \delta_{ij}\delta p \ , & (6.53) \end{aligned}$$

where the normalization condition $g(u,u) = -1$ and the background equation $\bar u^\mu = \delta^\mu_0$ have been used. The three-velocity field $\delta\mathbf{u}$ will be decomposed in a gradient and a curl, $\delta u_a = \partial_a \delta u + \delta u_a^V$.

Using the irreducible SO(3) tensors from (6.48-6.50), the constraint (6.47) can be decomposed accordingly,

$$\begin{aligned} -3\dot A - \dot{\tilde B} + (\Delta/a^2) aF + 3HE - 3HA - H\tilde B &= 0 \ , & (6.54)\\ \partial_j\left[-\dot{(aF)} - 3H(aF)\right] - \partial_j\left[E + 2A\right] &= 0 \ , & (6.55)\\ -\dot{(aG_j)} + \Delta C_j - 3H(aG_j) &= 0 \ , & (6.56) \end{aligned}$$

where $\tilde{B} \equiv \Delta B$. The constraint (6.54) is obtained from the $\nu = 0$ part of (6.47), (6.55) from its $\nu = i$ part proportional to a gradient of a scalar, and (6.56) from its $\nu = i$ part given by a transverse vector.

Now, we have all ingredients to linearize Eq. (6.46) and to equate the rank-2,1,0 SO(3) tensor contributions separately.

Rank-2 contribution

The rank-2 SO(3) tensor contribution results from the transverse-traceless part of the spatial-spatial components of (6.46), and is given by

$$-\ddot{D}_{ij} - 3H\dot{D}_{ij} + \left(\Delta/a^2\right) D_{ij} - m^2 D_{ij} = 0 \,. \tag{6.57}$$

It is worth mentioning that (6.57) reduces to its counterpart in the undeformed theory in the $m \to 0$ limit. This is a manifestation of the fact that the constraint (6.47) cannot support transverse-traceless modes and, as a result, general relativity can be continuously recovered in this sector. Provided the deformation parameter is small, $m^2 \lesssim H^2$, the deformation term in (6.57) will not change the dynamics very much. In particular, the frozen mode on super-Hubble scales, $-\Delta/a^2 \ll H^2$, is still present like in the undeformed theory.

Concerning stability, the equation of motion (6.57) always yields stable solutions, since the coefficients of both, the D_{ij} and $\dot{D}_{ij}$ terms coincide with the sign of the coefficient in front of $\ddot{D}_{ij}$. As a consequence, displacements will always be pulled back to the equilibrium position.

In the following, we will always use the same symbol for both the real space and Fourier space amplitudes of any dynamical variable like D_{ij}.

Rank-1 contribution

The deformed equations of motion (6.46) contribute two equations in the SO(3) vector sector of the theory, one from equating the spatial-temporal components, the other from equating the spatial-spatial components. As we will see, it suffices to consider the spatial-temporal equation together with the constraint (6.56) and momentum conservation to solve the vector sector. The vector part of the spatial-temporal equation is given by

$$16\pi M_{\mathrm{P}}^{-2} \left(\bar{\rho} + \bar{p}\right) \delta\mathbf{u}^{\mathbf{V}}/a = \left(\Delta/a^2 - 2m^2\right) \mathbf{G} - (\Delta/a^2) a\dot{\mathbf{C}} \,. \tag{6.58}$$

For convenience, let us define $\widetilde{G}_j \equiv aG_j$. From the constraints (6.56), it then follows that

$$\Delta\dot{\mathbf{C}} = \ddot{\widetilde{\mathbf{G}}} + \left(3H\widetilde{\mathbf{G}}\right)^{\cdot} \,. \tag{6.59}$$

Inserting this equation into (6.58) yields

$$16\pi M_{\mathrm{P}}^{-2}\,(\bar{\rho}+\bar{p})\,\delta\mathbf{u}^{\mathbf{V}} = -\ddot{\widetilde{\mathbf{G}}} - \left(3H\widetilde{\mathbf{G}}\right)^{\cdot} - 2m^2\widetilde{\mathbf{G}} + (\Delta/a^2)\widetilde{\mathbf{G}}\ . \tag{6.60}$$

A solution for the divergence-free part or the three-velocity field $\delta\mathbf{u}^{\mathbf{V}}$ can be obtained from the momentum conservation statement in the corresponding sector, which is given by

$$\left((\bar{\rho}+\bar{p})\,\delta\mathbf{u}^{\mathbf{V}}\right)^{\cdot} + 3H\,(\bar{\rho}+\ \bar{p})\,\delta\mathbf{u}^{\mathbf{V}} = 0\ . \tag{6.61}$$

This shows that the quantity $(\bar{\rho}+\bar{p})\delta\mathbf{u}^{\mathbf{V}} \propto 1/a^3$ decays and can therefore be neglected at late times. As a consequence, the equation of motion for $\mathbf{G}$ (6.60) is source-free at late times.

Investigating the stability of (6.60), we see that the Hubble-friction enters with the correct sign, whereas the terms with no time derivatives on $\mathbf{G}$ need to satisfy

$$\left[-\left(\Delta/a^2\right) + 3\dot{H} + 2m^2\right]\mathbf{G} \geq 0 \tag{6.62}$$

to give a stable solution for $\mathbf{G}$. Surely, in certain kinematical regions and for particular values of the deformation parameter, the bound (6.62) will be violated, and, as a consequence, a tachyonic instability will be generated. Indeed, for sufficiently early times, there will be such an instability for all three-momenta, provided that $\dot{H}$ increases faster than $-\Delta/a^2$ for decreasing t. This is the case, for instance, during radiation and matter domination, but not for the epoch when the cosmological constant dominates. In the latter case, the vector modes are always stable for arbitrary three-momenta.

On extreme super-Hubble scales, $-\Delta \ll (aH)^2$, the system develops instabilities whenever the bound

$$m^2 \geq -3/2\,\dot{H}\ . \tag{6.63}$$

is violated. This bound is a new result that has not been obtained in the previous work [1] based on the Goldstone boson equivalence. The bound (6.63) is instrumental for the self-defense of the theory against unitarity violations: Consider an equation of state of the form $p(\rho) = w\rho\ , w=$const. For $w < 10/3$, the bound (6.63) is even stronger than (6.41) and, furthermore, supports the self-protection mechanism described in [1].

Once the equation of motion (6.60) for $\widetilde{\mathbf{G}}$ is solved, the constraint (6.56) allows to solve for $\mathbf{C}$ up to a spatially homogeneous contribution which, anyhow, does not contribute to the spatial-spatial components of the metric perturbation, since $\mathbf{C}$ enters only with spatial derivatives. This clearly shows that the vector sector contains exactly one independent divergence-free three-vector field, and, thus, is inhabited by two independent degrees of freedom.

Rank-0 contribution

Like in the undeformed theory, the scalar sector is the most intricate. It contains as geometric ingredients the scalars A, B, E as well as F, and from the source $\delta\rho$, δp, and δu. Not all of these variables are, however, independent. Indeed, assuming a source with equation of state $p = p(\rho)$ allows to reduce the dynamics to a set of two coupled second-order differential equations for A and $\widetilde{B} = \Delta B$:

$$\begin{aligned}
\ddot{A} &= -3(1-w)H\dot{A} + w\left(\Delta/a^2\right)A - \left[2m^2 - 6w\left(H^2 - m^2/2\right)\right]A + \\
&\quad + wH\dot{\widetilde{B}} + 2w\left(H^2 - m^2/2\right)\widetilde{B} + \\
&\quad + H\dot{E} - m^2 E(A,B)\ ,
\end{aligned} \tag{6.64}$$

$$\begin{aligned}
\ddot{\widetilde{B}} &= -7H\dot{\widetilde{B}} - 4\left(H^2 + m^2/2\right)\widetilde{B} + \\
&\quad -12H\dot{A} - 3\left(\Delta/a^2\right)A - 12H^2 A + \\
&\quad + \left(12H^2 - \Delta/a^2\right)E(A,B)\ ,
\end{aligned} \tag{6.65}$$

where E is expressed in terms of A and B,

$$\begin{aligned}
&\left[\dot{H} + (2-3w)\,H^2 - m^2\right]E(A,B) = \\
&-(w-1/3)\,H\dot{A} - (w-1/3)\left(\Delta/a^2\right)A - \left[\dot{H} + (1+6w)\,H^2 - (2+3w)\,m^2\right]A + \\
&-(w-1/3)\,H\dot{\widetilde{B}} - (1/3)\left[\dot{H} + (1+6w)\,H^2 - (2+3w)\,m^2\right]\widetilde{B}\ .
\end{aligned} \tag{6.66}$$

The remaining geometrical SO(3) scalar F can be obtained using the deformation constraint (6.54). Then $\delta\rho$ can be derived from the temporal-temporal component of the linearized deformed equations of motion (6.46), and δu can be derived from the spatial-temporal components of (6.46) by extracting the spatial gradient contributions. Finally, δp follows from the equation of state $\delta p = c_s^2\delta\rho$ where c_s denotes the isentropic sound speed in the source. The details of this calculation can be found in the appendix.

6.3.4 Stability analysis in the scalar sector

In [1] we have already discussed some qualitative differences between the two bounds (6.40) and (6.41): The former leads to negative norm states, which spoils the probabilistic interpretation of the theory, while the latter signals the breakdown of perturbation theory. In Sec. 6.3.4 we reiterate on the issue by presenting further arguments for the physical difference of both bounds, based purely on the classical evolution. After Sec. 6.3.4 we continue with the stability analysis in the scalar sector.

Classical effects of the different types of instabilities

As already mentioned above, what here is called quantum instability (that is the appearance of negative norm states in the quantized theory) already has an incisive effect within the framework of a purely classical analysis: Let us have a look at a setup, which is actually capable of capturing all the relevant physics at the linear level for sub-Hubble scales [1], based on the classical equation of motion for a scalar ϕ, $\alpha\ddot{\phi} + \epsilon\dot{\phi} + \beta\phi = 0$. Here, the coefficients α, β, and ϵ are functions of time. For $\alpha, \beta, \epsilon > 0$ the system is stable. The classical stability bound manifests itself in a change of the sign of β while α is still positive, which triggers an exponential instability, and the perturbative analysis breaks down. For a gradual zero-crossing the spring constant is already small before the hard bound is hit and the oscillations might enter the nonlinear regime already before the exponential instability is triggered.

Nevertheless, we can still choose initial conditions that allow us to evolve the system for a small amount of time inside the region $\beta < 0$ until the fluctuation grows large. We can, however, not use this approach to try to cross the point where α turns negative as well, as close to this point, α is already small, and the effective spring constant has an extremely negative value, which goes to $-\infty$ just at the zero-crossing. Hence, in its vicinity, the time for which we can evolve the system in the just described fashion goes to zero. As a consequence, there is no reason why a change of sign of α *after* a change of sign of β should have any physical relevance for the full system. This is a manifestation of the self-protection mechanism.

Let us now consider the opposite case when α changes its sign before β does. In this case, the effective spring constant β/α grows big before the zero-crossing of α, confining the oscillations of ϕ to small values even more. The equation of motion, however, runs into a singularity because the term with two time derivatives (thus terminating the time evolution of the system) vanishes. Hence, this case would be much more severe, as the system cannot even be evolved across the point where α vanishes. A possible counterargument to this reasoning is that the system enters the strong coupling regime whenever $\alpha \to 0$. We will argue, however, that the described singular behavior of the equation of motion persists in the same way in the non-linear theory: Consider the non-linear term $\gamma\phi\ddot{\phi}$ that will become important once $\alpha \sim \phi\gamma$. In fact, this is the only relevant non-linear contribution, since any other term containing two time derivatives but more fields, such as $\phi^2\ddot{\phi}$, will be subdominant due to the fact that ϕ itself is small, as explained. Thus the combination $(\alpha + \gamma\phi)\ddot{\phi}$ will determine the time evolution of the system, with the equation of motion

$$(\alpha + \gamma\phi)\ddot{\phi} + \epsilon\dot{\phi} + \beta\phi = 0, \tag{6.67}$$

Again, as long as $(\alpha + \gamma\phi) > 0$, the effective spring constant of the system grows large and confines ϕ to small values. At best, $\gamma\phi$ might have some positive value, so that α can become negative, but now α eventually drops to large negative values and will certainly overshoot the contribution $\gamma\phi$ which is still small due to the small ϕ fluctuations. Hence, even the sum $\alpha + \gamma\phi$ will pass through zero and result in a singularity of the system.

Let us elaborate a little bit more on the question why a vanishing coefficient $\alpha + \gamma\phi$ in front of the $\ddot{\phi}$ term entails an unacceptable singularity. We will name the time of zero crossing t_0, that is

$$\alpha(t_0) + \gamma(t_0)\phi(t_0) = 0. \tag{6.68}$$

Assuming that $\ddot{\phi}$ is regular at t_0 yields the constraint $\epsilon(t_0)\dot{\phi}(t_0) + \beta(t_0)\phi(t_0) = 0$ by virtue of the equation of motion (6.67). Moreover, (6.68) yields the additional constraint $\phi(t_0) = -\alpha(t_0)/\gamma(t_0)$. These constraints completely spoil the Cauchy problem as they allow only one particular choice of initial conditions. This clearly illustrates the singular behavior of (6.67) under the assumption of regular $\ddot{\phi}$.

Thus, we try to abandon the assumption of regularity of $\ddot{\phi}$, and instead assume that $\ddot{\phi} \sim (\alpha+\gamma\phi)^{-1}$ around t_0. Taylor expansion of the vanishing coefficient gives the leading behavior $\ddot{\phi} \sim (t-t_0)^{-\delta}$. The case $\delta = 2$ results in $\phi \sim \ln(|t-t_0|)$ which is singular at $t = t_0$ and thus unacceptable. The same is true for $\delta > 2$, for which we obtain $\phi \sim (t-t_0)^{-\delta+2}$. If instead we have $\delta = 1$, ϕ would behave as $\phi \sim (t - t_0)\ln(|t - t_0|) - (t - t_0)$, which would be well-defined at $t = t_0$. The term $\epsilon\dot{\phi}$ in (6.67), however, would still be singular for this behavior of ϕ, such that this behavior cannot give a solution to the equation (6.67).

Unitarity Bound

At the level of the action for the SO(3) scalar A, the sign of the prefactor in front of the $\dot{A}^2$ term is crucial for the absence of negative norm states. (See [1] for details.) At the level of the equation of motion, this sign is determined by the prefactor of the $\ddot{A}$ term which can be derived from combining equation (6.64) with the corresponding prefactor in the $\dot{E}$ term from (6.66). Combining both prefactors gives

$$\frac{m^2 - H^2 - \dot{H}}{(1 - c_s^2)^2(1 + 3c_s^2)}\ddot{A}. \tag{6.69}$$

Evidently, in the scalar sector unitarity seems to require that $m^2 > H^2 + \dot{H}$, which is precisely the bound (6.40) found in [1] by employing Goldstone boson equivalence. As an important result, we re-derived this unitarity bound in a full-fledged cosmological perturbation analysis, with a very important qualification: we find that the unitarity bound applies at all energies, and not just in the high-energy regime considered in [1].

In the following we solve the coupled equations of motions (6.65, 6.64) for the scalars A, B numerically, and analyze the stability of these solutions. For clarity, we subdivide the kinematical domain in three subdomains: extreme sub-Hubble scales ($\mathbf{k}^2/a^2 \gg m^2, H^2$), intermediate scales, and extreme super-Hubble scales ($\mathbf{k}^2/a^2 \ll m^2, H^2$).

Extreme sub-Hubble scales

This regime has been investigated previously [1] employing the Goldstone boson equivalence as a diagnostic tool to extract the leading short-distance dynamics.

From the full, coupled set of linear differential equations (6.64-6.66) these dynamics can be recovered by means of the adiabatic ansatz $A, B, E \propto e^{\mu t}$, which is best for large $\mathbf{k}_{\rm phys}$: Introducing the ansatz into the system of equations and solving the (biquadratic) secular equation $c_4\lambda^4 + c_2\lambda^2 + c_0 = 0$, which results to leading order in large $\mathbf{k}_{\rm phys}$, yields $\sqrt{2}\lambda = \pm\sqrt{(-c_2 \pm \sqrt{c_2^2 - 4c_0c_4})/c_4}$, where $\lambda^2 = \mu^2|\mathbf{k}_{\rm phys}|^2$. (Upper and lower signs can be chosen independently, which leads to four combinations.) In order to have a stable system, none of the eigenvalues may have a positive real part. Therefore, the presence of the outer $\pm$ implies that all eigenvalues must be purely imaginary. That necessitates that $c_2^2 \leq 4c_0c_4$ and that c_0, c_2, and c_4 must have the same sign. Unitarity requires further that $c_4 = 2(H^2 + \dot{H} - m^2)$ is negative, reproducing Eq. (6.40). Hence, the system is stable when all coefficients are negative and $c_2^2 \leq 4c_0c_4$. Then, from $c_0 = (H^2 + \dot{H}/3 - m^2)w$ we reproduce Eq. (6.41) for $w > 0$.

For $w < 0$ this relation would be exactly the other way round, implying that the system would never be stable. This phenomenon is known already from *unmodified* general relativity [60], where a system filled by a perfect fluid with $w < 0$ is always unstable as long as $\mathbf{k}_{\rm phys}$ is not very small. As it is already present in general relativity, this instability cannot have anything to do with the degree of freedom used in the Goldstone boson equivalence analysis, which is absent in general relativity. This explains why said instability goes unnoticed in this case. It is important to notice that in this respect a scalar field does not correspond to a perfect fluid [61], which explains why this bound is also not obtained in [36].

Coming back to $w > 0$, the bound derived from c_2 is always weaker than the stability bound (6.41), which follows from c_0, or the requirement $c_2^2 \leq 4c_0c_4$. For $w > 1/3$ the requirement that $c_2^2 \leq 4c_0c_4$ would be stronger than the bound (6.41). A numerical analysis in the regime where $c_2^2 \leq 4c_0c_4$ shows, however, that there is no instability as in the case where (6.41) is violated. While the latter leads to a clear exponential explosion forwards and backwards in time, the latter manifests itself in a beat with an amplitude of the envelope that grows relatively mildly backwards in time. Here the requirement $c_2^2 \leq 4c_0c_4$ obtained in the framework of the adiabatic analysis does not seem to give a relevant bound. Also in the case where the condition $c_2 < 0$ is violated, numerically no instability can be detected.

Figure 6.2 shows the numerical solution for the scalars A and B in a radiation-dominated universe ($c_s^2 = 1/3$). The parameters were chosen such that (units unspecified) $k_{\rm phys} = 250/\sqrt{t}$, $m = 1/\sqrt{12}$, and $H = 1/(2t)$. Hence, $\mathbf{k}^2/a^2 \gg m^2, H^2$ is guaranteed for times $t \in [0.8, 2]$. The initial conditions have been chosen at $t = 2$, such that the system is evolved backwards in time.

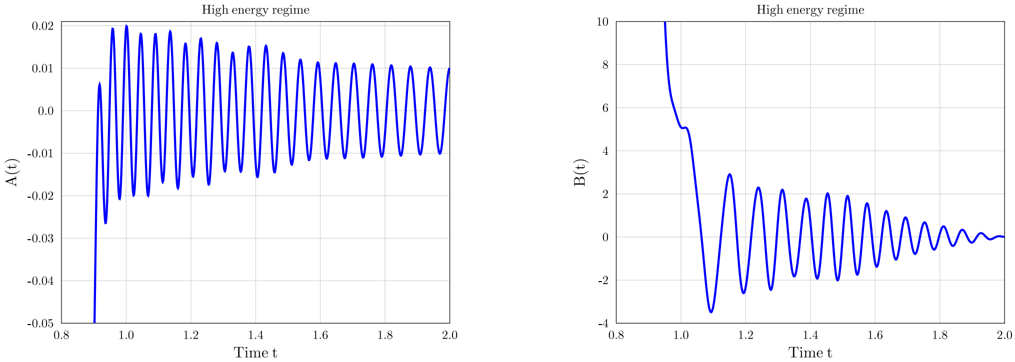

Figure 6.2: Scalars A and B during radiation domination deep inside the Hubble radius.

Let us first investigate the behavior of B. For times $t > 1$, B is oscillating with a Hubble-damped amplitude, clearly showing a healthy hyperbolic evolution forward in time. Evolving backward in time, however, B develops an instability for $t < 1$. Indeed, the parameters have been chosen such that the stability bound (6.41) is violated for $t < 1$ This confirms the results of [1]. The behavior of A is similar, except that it develops the instability at an earlier cosmological time scale (which is later from the point of view of the system evolving backwards in time), and oscillates with a higher frequency as compared to the scalar B.

The basic properties of the solution are independent of the source's equation of state in the interval $0 \leq c_s^2 \leq 1$. The case of a de Sitter source ($c_s^2 = -1$) is borderline, since the parameter range for which the classical instability is triggered coincides precisely with the range of parameters for which unitarity gets violated. Hence, the strong coupling regime goes hand in hand with negative norm states. (See [1] for details.)

Intermediate scales

Figures 6.3 and 6.4 show the solutions for the scalars A and B during radiation domination from intermediate to extreme super-Hubble scales, that is, for different values of the comoving wavenumber k or, equivalently, for the physical wavenumber $k/a(t)$ at time $t = 1$. For convenience and clarity, the other parameters have been chosen precisely as in the previous section. Like in the previous case, the initial conditions have been chosen at $t = 2$ and the scalar modes have been evolved backwards in time. For concreteness, the initial conditions are given by $A = 0.01$, $B = 0.01$, and $dA/dt = 0$, $dB/dt = 0$ at $t = 2$. Note that the qualitative behavior of this dynamical system is quite insensitive

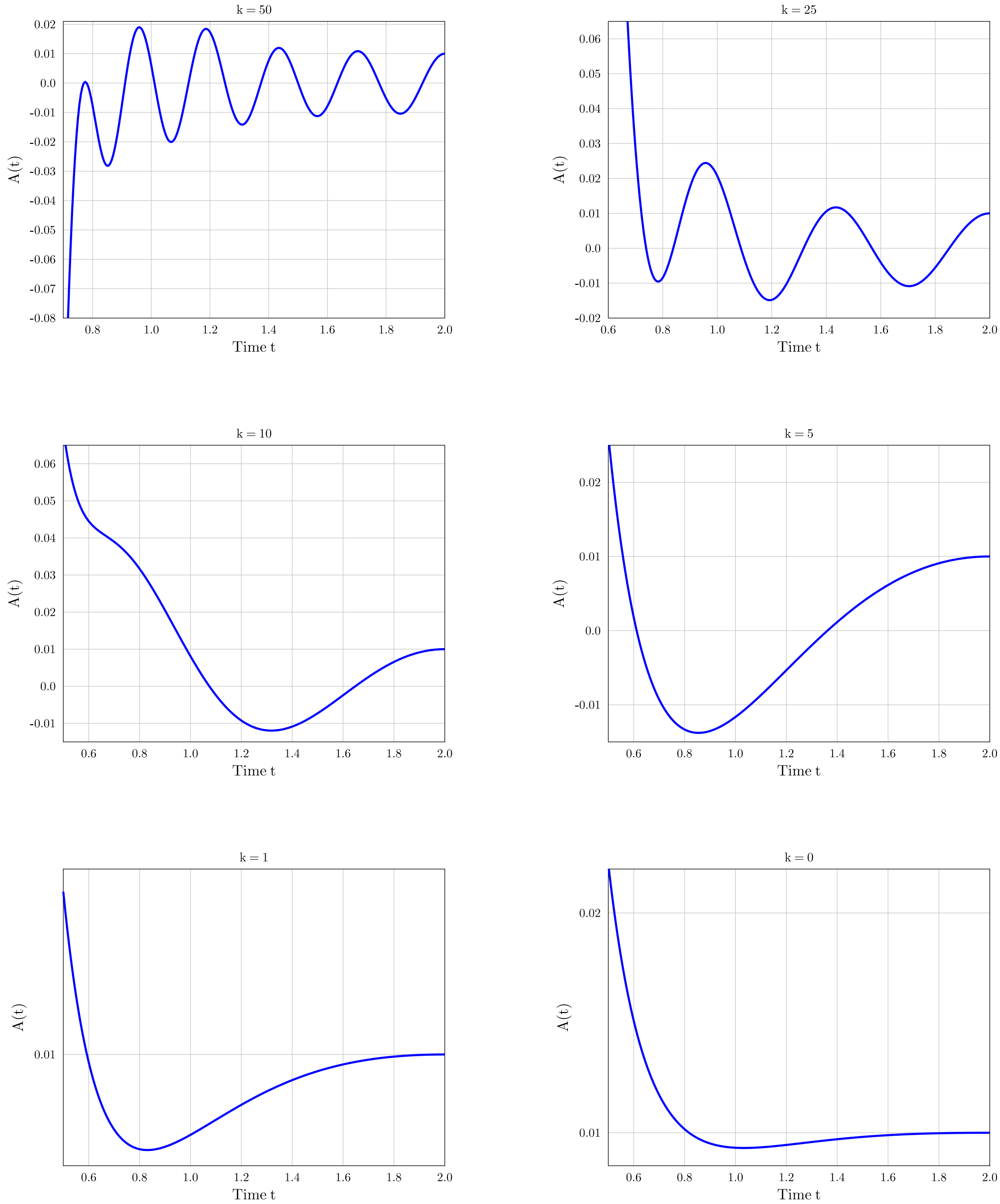

Figure 6.3: Numerical solution of $A(t)$ during radiation domination for different values of $k_{\text{phys}}(t = 1)$. The other parameters have been chosen to be the same as for Figure 1.

to the choice of initial conditions, in particular, with respect to the stability analysis. It can be seen that the scalar modes' behavior on intermediate scales (and also on

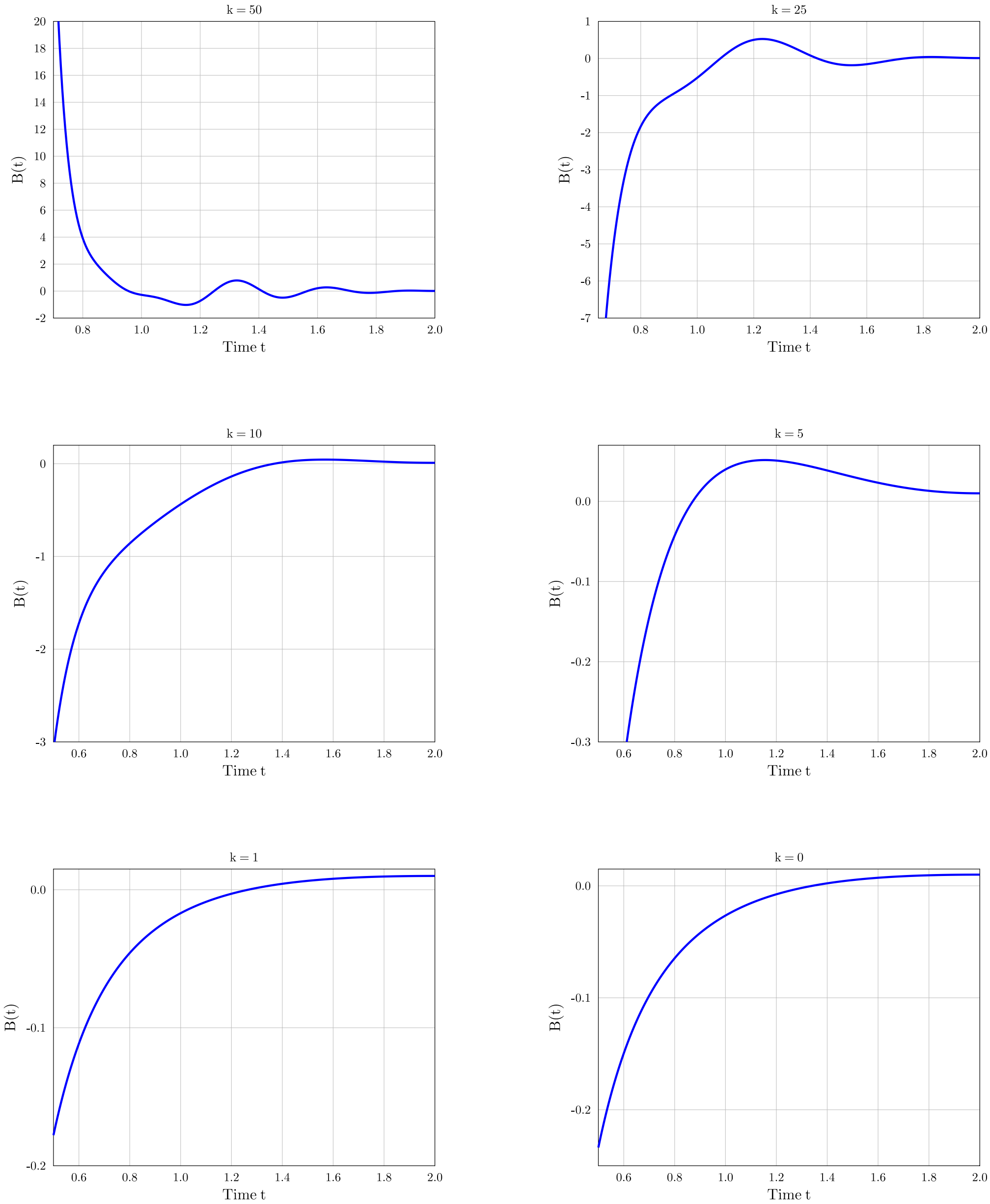

Figure 6.4: Numerical solution for $B(t)$ during radiation domination for different values of $k_{\text{phys}}(t = 1)$. The other parameters have been chosen to be the same as in Figure 1.

extreme super-Hubble scales, see next section) is very different from the dynamics in a normal neighborhood (see previous section). Compared to the latter case, the instability triggered at $t = 1$ becomes less and less pronounced with decreasing wavenumber. In order to appreciate this fact, notice the different ranges of mode amplitudes covered on the y-axes in figures 6.3 and 6.4 as compared to figure 6.2. In fact, scalar fluctuations on super-Hubble scales show a power law behavior which is triggered by the cosmological singularity (i.e. by the singular coefficients $H \propto 1/t$ etc), and which is clearly distinct from an instability triggered by a non-hyperbolic evolution.

Extreme super-Hubble scales

In order to elucidate further this result, let us analyze the stability of the scalar zero modes, which can be performed analytically. The zero modes of A and B satisfy (6.64-6.66),

$$\begin{aligned}\ddot{A} &= -3(1-w)H\dot{A} - \left[2m^2 - 6w\left(H^2 - m^2/2\right)\right]A + \\ &\quad +wH\dot{\widetilde{B}} + 2w\left(H^2 - m^2/2\right)\widetilde{B} + \\ &\quad +H\dot{E} - m^2 E(A,B)\ ,\end{aligned} \tag{6.70}$$

$$\begin{aligned}\ddot{\widetilde{B}} &= -7H\dot{\widetilde{B}} - 4\left(H^2 + m^2/2\right)\widetilde{B} + \\ &\quad -12H\dot{A} - 12H^2 A + \\ &\quad +12H^2 E(A,B)\ ,\end{aligned} \tag{6.71}$$

where E is expressed in terms of A and B as follows,

$$\begin{aligned}&\left[\dot{H} + (2-3w)\,H^2 - m^2\right]E(A,B) = \\ &-(w-1/3)\,H\dot{A} - \left[\dot{H} + (1+6w)\,H^2 - (2+3w)\,m^2\right]A + \\ &-(w-1/3)\,H\dot{\widetilde{B}} - (1/3)\left[\dot{H} + (1+6w)\,H^2 - (2+3w)\,m^2\right]\widetilde{B}\ .\end{aligned} \tag{6.72}$$

As a consequence, in this limit, the system of two coupled differential equations for A and B reduces to a single equation of motion for the linear combination $S \equiv A + \tilde{B}/3$,

$$\left[C_2(w;t)\partial_t{}^2 + C_1(w;t)\partial_t + C_0(w;t)\right]S = 0\ , \tag{6.73}$$

where the coefficients $C_{2,1,0}$ depend on the equation of state parameter w of the source and on time via the Friedmann background evolution. Explicit expressions for these coefficients can be found in the appendix.

A sufficient condition for hyperbolic evolution on the entire Friedmann manifold and thus, for classical stability, is given by $C_1/C_2 > 0$ and $C_0/C_2 > 0$ for all times, for a given source equation of state parameter w. We can analyze how these stability conditions depend on the parameter w and time t. The result is shown in Fig. 6.5, where the

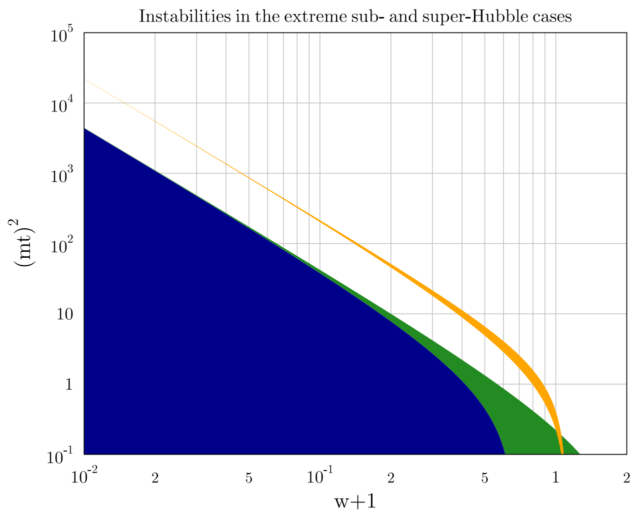

Figure 6.5: Instabilities in the extreme sub- and super-Hubble cases. In the orange region (top, detached), the system is classically unstable for $\mathbf{k}_{\text{phys}} = \mathbf{0}$. The dark-blue region (bottom, left) depicts the region, where unitarity would be violated. In the green region (adjacent to the former), the system is classically unstable for large $\mathbf{k}_{\text{phys}}$.

orange region corresponds to the classical instability region for the zero mode S, and inside the dark-blue region unitarity would be violated. Figure 6.5 shows that for a source with equation of state parameter $w \gtrapprox 0.11$, the zero mode's dynamics is always stable, confirming our explicit numerical result for a radiation dominated Friedmann universe discussed in the previous section. For smaller values of w, when evolved backwards in time, the zero mode will always first enter the region of classical instability (orange), which signals the breakdown of perturbation theory. Evidently, it cannot enter the unitarity violating region (dark-blue), without passing through the strong coupling regime (orange). For large momenta, the area of classical instability moves downwards and comes to rest exactly on top of the area where unitarity would be violated, which, thus, still cannot be reached without first crossing the former (green). Hence, in this sense the strong coupling regime self-protects the scalar zero mode from unitarity violation, as well. As a consequence, it is not clear at all whether the thus diagnosed unitarity violating region is of physical relevance, as it lies well outside the perturbative regime. We can turn this argument around and conclude that no inconsistency is present within the perturbative regime.

6.3.5 Conclusion

In summary, using cosmological perturbation theory, we have proven the consistency of the most relevant Einstein–Hilbert deformation in the perturbative regime. The deformation itself achieves consistency via a self-protection mechanism that pushes potential unitarity violations beyond the weak coupling regime. This confirms previous studies concerning the deformation's nontrivial stability dynamics, based on a Stückelberg completion of the deformation in conjunction with the Goldstone boson equivalence [1]. Most importantly, this work extends the self-protection mechanism to encompass the entire kinematical domain, ranging from sub- to super-Hubble scales.

It would be interesting to study the proposed non-linear theories [14, 15, 16] with a rigid FRW background to see whether they non-linearly exhibit the self-protection mechanism.

As discussed in great detail in [2], the self-protection phenomenon is a prime example for the recently conceived classicalization mechanism [62, 74, 73, 72, 66] and extends it further to free field dynamics on curved backgrounds.

6.4 Island of Stability for Consistent Deformations of Einstein's Gravity

6.4.1 Abstract

We construct deformations of general relativity that are consistent and phenomenologically viable, since they respect, in particular, cosmological backgrounds. These deformations have unique symmetries in accordance with their Minkowski cousins (Fierz-Pauli theory for massive gravitons) and incorporate a background curvature induced self-stabilizing mechanism. Self-stabilization is essential in order to guarantee hyperbolic evolution in and unitarity of the covariantized theory, as well as the deformation's uniqueness. We show that the deformation's parameter space contains islands of absolute stability that are persistent through the entire cosmic evolution.

6.4.2 Introduction and Overview

Cosmology encompasses the relativistic domain of gravity and allows to investigate the rigidity of Einstein's theory. Consistent deformations of general relativity have been investigated at the level of linear perturbations on a frozen Minkowski background. Fierz and Pauli showed that this system allows for a unique deformation satisfying all stability requirements for the prize of introducing new degrees of freedom corresponding to additional helicities of a massive graviton.

The new degrees of freedom consistently violate the principle of equivalence by constituting a source filter that decreases the vacuum's weight on space-time in a technical natural way, albeit due to a delicate mass term. This offers a dynamical mechanism to address the staggering conflict between naive but educated expectations for our vacuum's energy density and a plethora of data probing the background expansion history and the evolution of density perturbations in our Universe at various epochs.

In this Letter we covariantize the Fierz-Pauli mass term to a deformation that is capable of coexisting with generic cosmological backgrounds. Requiring hyperbolic evolution and unitarity allows the covariantized theory to inherit the uniqueness property of its Minkowski cousin (Fierz-Pauli theory). We show that absolute stability is guaranteed via a background induced self-sourcing (feedback) mechanism that is already operational at the linear level.

For realistic cosmological backgrounds the deformation of Einstein's theory is characterized by three parameters and its symmetries do agree with those of the Fierz-Pauli mass term. The parameter space features a multi-facetted stability dynamics: It includes strictly forbidden regions, regions that are consistent but challenged through strong coupling, and parameter islands that support absolutely stable deformations.

6.4.3 Framework

At the linear level the leading relevant deformation of general relativity can be written as a field theory for the combination

$$H_{\mu\nu} = h_{\mu\nu} + \nabla_{(\mu} A_{\nu)} + \nabla_\mu \nabla_\nu \Phi \ . \tag{6.74}$$

Here, h, A, Φ are rank-2,1,0 tensors, respectively, under full background diffeomorphisms; round brackets around indices stand for symmetrization. This parametrization corresponds to two successive Stückelberg completions and introduces a $U(1)^4 \times U(1)$ gauge symmetry among the fields h, A, Φ.

The action for H on a spacetime (M, g_0) reads

$$\mathcal{S}[H] = \frac{1}{2} \int_M \mathrm{d}^4x \ \sqrt{|g_0|} \, H^\mathsf{T} \left[\mathcal{E}(g_0, \nabla) + \mathcal{M}(g_0)\right] H \ . \tag{6.75}$$

Here, g_0 denotes the background metric, $\mathcal{E}(g_0, \nabla)$ is the kinetic operator for h, obtained from linearizing the Einstein tensor around the background g_0, and ∇ denotes the g_0-compatible covariant derivative. Note that the Stückelberg combination in (6.74) is effectively an element of $\ker\left[\mathcal{E}(g_0, \nabla)\right]$, once sources have been supplied. The deformation operator is denoted by $\mathcal{M}(g_0)$ and is, at this stage, of second adiabatic order (given by the number of derivatives acting on the background metric) barring parameters with inverse mass dimension.

6.4.4 Uniqueness

The Goldstone-Stückelberg field Φ enters the gauge invariant combination H with two derivatives and, therefore, the action (6.75) with four derivatives. Without further restricting $\mathcal{M}(g_0)$, the short distance behavior of the deformation would be governed by a higher-derivative theory that violates unitarity. A similar conclusion holds for the field A. Now, the necessary and sufficient condition on $\mathcal{M}(g_0)$ to yield only second order equations of motion for Φ and A is $\mathcal{M}^{\mu\nu\alpha\beta} = -\mathcal{M}^{\alpha\nu\mu\beta}$, in addition to the previously-discussed symmetries.

As a result, to second adiabatic order, the deformation operator can be expanded uniquely as

$$\begin{aligned} \mathcal{M}^{\mu\nu\alpha\beta} &= \big(m_0^2 + \alpha R_0\big) g_0^{\mu[\nu} g_0^{\beta]\alpha} \\ &+ \beta\Big(R_0^{\mu[\nu} g_0^{\beta]\alpha} + R_0^{\alpha[\beta} g_0^{\nu]\mu}\Big) + \gamma\, R_0^{\mu\alpha\nu\beta}\,, \end{aligned} \tag{6.76}$$

where the subscript 0 indicates a background quantity, α, β, γ are real dimensionless parameters, and square brackets around indices stand for anti-symmetrization. Including terms of higher adiabatic order requires introducing further parameters with appropriate inverse mass dimension to compensate for the additional derivatives acting on g_0.

To lowest adiabatic order ($\alpha, \beta, \gamma = 0$), $\mathcal{M}$ coincides with a naïvely covariantized Fierz-Pauli term and reduces precisely to the well-known Fierz-Pauli mass deformation on the Minkowski background [29].

6.4.5 Stability Analysis

The stability analysis only requires to determine the roots of the determinant of the kinetic operator, c.f. (6.75), which signal the saturation of the stability or unitarity bounds [1], respectively. In order to calculate the determinant of the kinetic operator it is useful to completely fix the gauge to $h_{0\mu} = 0$ and $A_0 = 0$. The saturation of the unitary bound is marked by the zero crossing of the coefficient in front of the highest power in the temporal component of the momentum, which here is ${k_0}^{20}$. The stability bound is determined by the zero crossing of the coefficient in front of the highest power in the spatial components of the momentum, which here is $(k_0\, k_j)^{10}$.

In general, we distinguish the following four cases. *Case 1*: Both bounds are satisfied on the entire spacetime. Hence, the deformation is well-defined at the perturbative level. *Case 2*: Regions that support both bounds are separated from areas where the unitarity bound is violated by regions in which the stability bound is violated. This situation is called 'self-protected' [1]. *Case 3*: There are spacetime regions on which both bounds are satisfied, and these regions have a common border with regions where unitarity is violated. In this case, the theory must be dismissed, as the unitarity violation diagnosed

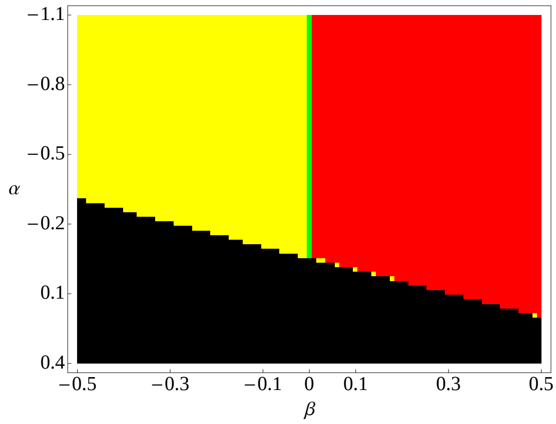

Figure 6.6: Parameter plot in the α-β plane, for $m_0 = 0$. The green (top center) line corresponds to *Case 1*, the yellow (top left) region to *Case 2*, the red (right) to *Case 3*, and the black (bottom) region to *Case 4*. (For $m_0 = H_0$ the plot looks essentially the same.)

at the linear level cannot be cured by a nonlinear completion. *Case 4*: The theory is unstable or unitarity violating in the observer's spacetime region.

Let us specialize to a Friedmann spacetime, $g_0 = \text{diag}(-1, a^2, a^2, a^2)$, where $a = a(t)$ is the scale factor. *Case 2* corresponds to a situation where a healthy region at late times t ("today") is preceded by a stability violating one, which always separates the former from a potentially present but even earlier unitarity violating region. Correspondingly, in *Case 3* the healthy is preceded by a unitarity violating regime without an intermediate unstable phase. For $m_0 \neq 0$ but $\alpha, \beta, \gamma = 0$ this case never occurs [1]. For the Friedmann metric, the Riemann tensor can be expressed through the Ricci tensor, the Ricci scalar, and the background metric. Thus, without loss of generality, we can set γ to zero in (6.76). Remarkably, then, and *this is our main result*, an appropriate choice of α and β makes the deformation absolutely stable, corresponding to *Case 1*. (See Fig. 6.6.)

To proceed, we parametrize time with the scale factor a, which is determined using the observed mixture of matter and radiation densities, and the cosmological constant [69] as sources for the cosmological concordance model. Parametrizing time with the scale factor gives the unitarity and stability bound as a polynomial in a.

We have shown analytically that the interplay between both bounds results in $\beta = 0$ as a necessary condition for obtaining absolute stability over the entire cosmological expansion history. It turns out that the radiation dominated epoch restricts the stability dynamics considerably. Moreover, we find as a condition for absolute stability $\alpha < \alpha_{\max} < 0$, where $\alpha_{\max}$ depends on the precise mixture of cosmological sources. The expression for $\alpha_{\max}$ is rather involved and will be presented elsewhere. In addition, there is an isolated point of absolute stability in parameter space, given by $\alpha = m_0^2/(48\,\Omega_\Lambda)$,

$\beta = 0$, with Ω_Λ denoting the current relative density parameter of the cosmological constant.

6.4.6 Covariantized Deformation Parameter

Our results show that the models with

$$\mathcal{M}^{\mu\nu\alpha\beta}(g_0) = \left(m_0^2 + \alpha\, R_0\right)\left({g_0}^{\mu\nu}\, {g_0}^{\alpha\beta} - {g_0}^{\mu\beta}\, {g_0}^{\alpha\nu}\right) \tag{6.77}$$

yield a completely stable theory if $\alpha < \alpha_{\max} < 0$. It is tempting, albeit not quite correct, to think of (6.77) as a 'running mass' deformation.

The leading short-distance behavior of the minimal deformation (6.77) is captured by the action

$$\mathcal{S}[\Phi] \simeq \int \mathrm{d}^4x\, \sqrt{|g_0|}\, \Phi\, \mathcal{O}^{\mu\nu} \nabla_\mu \nabla_\nu\, \Phi\,, \tag{6.78}$$

with the pseudo metric

$$\begin{aligned}\mathcal{O}^{\mu\nu} :=& \left[4\, \frac{\left(\nabla^\alpha \mathrm{m}^2\right)\left(\nabla_\alpha \mathrm{m}^2\right)}{\mathrm{m}^2} - 3\,\mathrm{m}^4 - 2\left(\Box \mathrm{m}^2\right)\right] {g_0}^{\mu\nu} \\ &+ 2\left[\mathrm{m}^2\, {R_0}^{\mu\nu} + \left(\nabla^\mu \nabla^\nu \mathrm{m}^2\right) - 2\, \frac{\left(\nabla^\mu \mathrm{m}^2\right)\left(\nabla^\nu \mathrm{m}^2\right)}{\mathrm{m}^2}\right],\end{aligned} \tag{6.79}$$

where $\mathrm{m}^2 = m_0^2 + \alpha R_0$. For a Friedmann spacetime, the action (6.78) can be brought into the form

$$\mathcal{S}[\Phi] \simeq \int \mathrm{d}^4x\, \left[A(t)\big(\dot{\Phi}\big)^2 + B(t)\left(\nabla\Phi\right)^2\right], \tag{6.80}$$

with known functions $A(t)$ and $B(t)$. A violation of unitarity/stability is heralded by a sign flip of A/B.

Fig. 6.7 shows the dominant short distance degree of freedom $\Phi(t)$ for the specified deformation parameters, corresponding to a naïvely covariantized Fierz-Pauli term and the covariantized deformation parameter model, respectively, as an example for *Case 1.* As can be seen, a static deformation parameter ($\alpha = 0$) yields a theory that is self-protected against unitarity violations by a strong coupling regime at the linear level, but is unfit from a phenomenological point of view. The running deformation parameter results in a model that is absolutely stable and, hence, potentially phenomenologically viable.

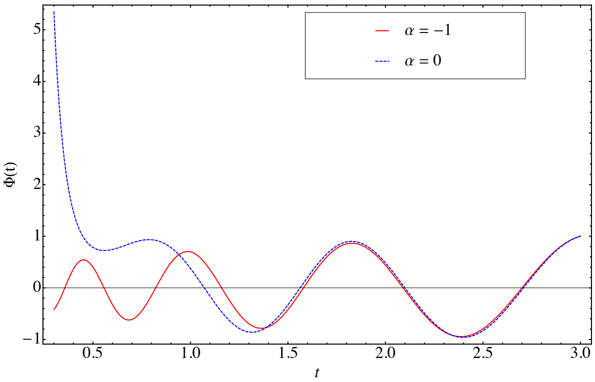

Figure 6.7: $\Phi(t)$ on a purely matter dominated background, $m_0 = H_0$, and $\beta, \gamma = 0$. The dashed blue curve corresponds to $\alpha = 0$, while the solid red line corresponds to the absolutely stable situation with $\alpha = -1$.

6.4.7 Phenomenology and Applications

The phenomenology of our theory in the solar system will be the same as the one of standard massive gravity with mass m_0 [70], since in this environment we have $R = 0$. For example, mercurys perihelion advance per orbit $\delta\phi$ due to the gravity modification will be given by $\delta\phi = \pi r \frac{d}{dr}\left(r^2 \frac{d}{dr}(r^{-1}\epsilon)\right)$, with $\epsilon = e^{-m_0 r}$ (r being the mean distance of mercury to the sun). The theory with $m_0 = 0$ will yield the same phenomenological predictions on the linear level as general relativity, and is thus unconstrained from solar system experiments.

However, we will get a modification on cosmological scales where the Friedmann expansion applies. Taking as a reasonable scale $-\alpha \sim 1$, the effective graviton mass will automatically be in the interesting cosmological domain $m^2 \sim H^2$. In our theory, this scale is naturally set by a dynamical mechanism and does not have to be put in by hand.

As an application, we consider gravitational waves on a de Sitter background with cosmological constant Λ, the equation of motion for the rank-2 tensor is given by

$$\begin{aligned} \hat{\mathcal{E}}_{\mu\nu}{}^{\alpha\beta}(g^0, \nabla) h_{\alpha\beta} - \frac{1}{3}\Lambda\left(h_{\mu\nu} + \frac{1}{2} g^0_{\mu\nu} h\right) + \\ - \left(m_0^2 + 4\alpha\Lambda\right)\left(h_{\mu\nu} - g^0_{\mu\nu} h\right) = \delta T_{\mu\nu} \,, \end{aligned} \tag{6.81}$$

where $\hat{\mathcal{E}}$ is the part of the linearized Einstein tensor containing covariant derivatives acting on h, and δT denotes the perturbation of a covariantly conserved background source. In our case, $\delta T = \delta\Lambda\ g^0$, where $\delta\Lambda$ is an additional de Sitter source. Clearly, $h = C\ g^0$ is a solution of (6.81), provided $C = -\delta\Lambda/[(1-12\alpha)\Lambda - 3m_0^2]$, resulting in the total metric field $g = (1+C)g^0$. The total curvature is related to the background curvature as $R = R_0/(1+C) \approx (1-C)R_0$. In general relativity, this is $R = 4(\Lambda + \delta\Lambda)$,

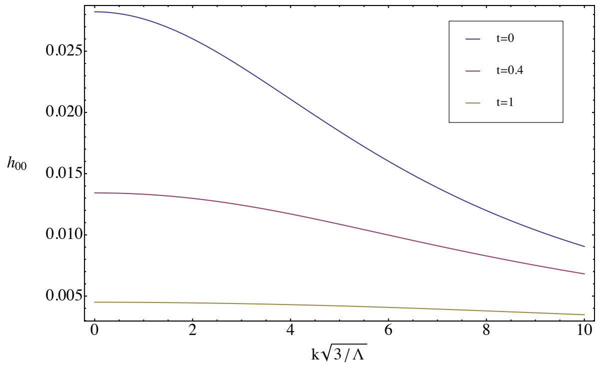

Figure 6.8: The evolution of $h_{00}(k)$ at different times for $\mu = \sqrt{\Lambda/3} = 1$ and $\mathrm{m} = 5$. The blue line (upper one) is at $t = 0$, the red line (middle one) at $t = 0.4$ and yellow line (lower one) at $t = 1$.

as expected. In contrast, if the deformation is operative, then $|C|$ can be smaller (note that $\alpha < 0$ on the stable island), and the resulting curvature can be smaller as compared to the previous case. The effect of $\delta\Lambda$ on the curvature is partially degravitated [43].

As a second application, let us calculate the gravitational potential of a point particle with mass μ on a de Sitter background. Parametrizing the scale factor as $a(t) \equiv \exp(\sqrt{\Lambda/3}\, t)$ and defining $\hat{h}_{00} = a^{7/2}(t)\, h_{00}$, we have

$$\left(\partial_t^2 + \omega(k_\mathrm{p})\right) \hat{h}_{00} = 4\sqrt{a}\mu/3 \, , \tag{6.82}$$

where

$$\omega^2(k_\mathrm{p}) = k_\mathrm{p}^2(t) + \mathrm{m}^2 - 3\Lambda/4, \tag{6.83}$$

and k_p denotes the physical wavenumber, $k_\mathrm{p}(t) \equiv k/a(t)$ in terms of the comoving wavenumber k.

Using the WKB approximation, we recover the Yukawa potential at early times, $t \ll 1/\sqrt{\Lambda/3}$,

$$V(r_\mathrm{p}) \propto \frac{\mu \, \exp\left(-\mathrm{m}\, r_\mathrm{p}(t)\right)}{4\pi r_\mathrm{p}(t)} \, , \tag{6.84}$$

with r_p denoting the physical distance, $r_\mathrm{p}(t) \equiv a(t) r$ in terms of the comoving distance r.

At late times, $t \sim 1/\sqrt{\Lambda/3}$, ω becomes k_p independent and the gravitational potential becomes $V(r_\mathrm{p}) \sim \delta(r_\mathrm{p})$. This shows that in the deformed theory the effective interaction range of two point particles generically is much smaller than their physical distance at late times, see Fig. 3.

6.4.8 Conclusion

In this letter we have constructed a unique deformation of gravity corresponding to a covariantized Fierz-Pauli theory for massive gravitons that posses islands of absolute stability in its parameter space over the entire cosmic expansion history. The uniqueness property is a legacy of the deformation's Minkowski cousin when requiring classical stability and unitarity. We are hopeful that this deformation represents an exciting window of opportunity for studying consistent modifications of gravity on the largest observable distances.

Certainly, the very important question about a possible nonlinear completion of this unique deformation remains. However, we achieved a consistent covariantization of Fierz-Pauli theory on realistic cosmological backgrounds that is kept healthy via a background induced self-stabilization mechanism. We are currently working on a nonlinear completion. On a nonlinear completion of the hard mass deformation ($\alpha = 0$) has been recently constructed in a different framework .

6.5 Brane Induced Gravity: From a No-Go to a No-Ghost Theorem

6.5.1 Abstract

Numerous claims in the literature suggest that gravity induced on a higher co–dimensional surface violates unitarity in the weak coupling regime. However, it remained unclear, why a conserved source localized on this surface and giving rise to an induced gravity term at low energies would absorb and emit the associated ghost, given a consistent source–free theory. In this article it is shown that the appearance of the induced Einstein Hilbert term does not threaten the unitarity of the theory. It is shown that the would–be ghost highlighted in previous works is non–dynamical and therefore not associated with a state in the Hilbert space. The physics arguments behind this statement are presented in a semi–covariant language, but the detailed proof is given using Dirac's constraint analysis. The Hamiltonian on the constraint surface of the linearized theory is derived and turns out to be manifestly positive definite.

As a result of these investigations, brane induced gravity (BIG) goes without a ghost, opening an exciting window of opportunity for consistent deformations of gravity at the largest observable distances.

6.5.2 Introduction

Gravity induced on a surface $\mathcal{M}_4$ of co–dimension n that is equipped with a weakly coupled source, embedded in a $d = 4 + n$–dimensional Minkowski space–time $\mathcal{M}_d$, is defined as the following effective theory

$$\mathcal{S} = \mathcal{S}^{(d)}_{\rm EH}\,[h] + \mathcal{S}^{(4)}_{\rm M}\,[\psi] + \mathcal{V}^{(4)}_{\rm dyn}[h,\psi] + \mathcal{V}^{(4)}_{\rm ext}[h] + \lambda\,\mathcal{S}^{(4)}_{\rm EH}\,[h]\ , \tag{6.85}$$

where the first line represents the free graviton action on $\mathcal{M}_d$, and the action describing the free dynamics of all other weakly coupled fields ψ localized on $\mathcal{M}_4$, respectively. The second line collects all effective vertex contributions, i.e. the minimal coupling of gravitons to the dynamically resolved degrees of freedom ψ, and the minimal coupling to external graviton absorbers and emitters. The last term is due to weakly coupled fields Ψ that qualify as heavy with respect to some finite cut–off scale and that have been integrated out in the low energy effective theory (6.85). This term is customarily referred to as the induced Einstein–Hilbert term, giving the effective field theory (6.85) its name. In the case of $n = 1$ this is the well studied DGP model [76, 79].

It is very important to distinguish between external sources and the induced Einstein–Hilbert term. The former represent sources that are absolutely inert against backreaction, while the latter is, in fact, the dynamical fingerprint for the principal presence of Ψ in nature.

In the literature, the $\mathcal{S}^{(4)}_{\rm EH}\,[h]$ contribution in (6.85) has often been interpreted as an ad hoc kinetic modification rather than an induced operator. This offered the possibility to probe the theory's consistency solely employing external graviton absorbers and emitters. For $n > 1$ it seemed that for phenomenologically interesting choices of the model parameters gravitational fluxes originating from the surface and ending on it violate unitarity. This unitarity violation was associated with the gauge invariant scalar carried by propagating gravitons. From the absorber or emitters viewpoint, this scalar weakened its own source, while from a dynamical point of view the unitarity violation manifested itself in a wrong sign for the induced kinetic term, as compared to the propagation of the transverse and traceless graviton excitations. In both cases the unescapable conclusion was that (6.85) exhibits a ghost [80, 82].

However, within the effective field theory framework it is clear that the induced Einstein–Hilbert term arises from some weakly coupled fields Ψ at high energies, which have been integrated out to obtain the low energy effective field theory (6.85), compare to [76, 79]. In this way $\mathcal{S}^{(4)}_{\rm EH}\,[h]$ can be understood as a pure source modification. In particular, decoupling all Ψ fields implies setting λ to zero. In this case the gravitons free field theory would be represented by $\mathcal{S}^{(d)}_{\rm EH}$. So the question arises how a legitimate source could absorb or emit a ghost–like excitation?

Legitimate sources are only required to allow for a Lorentz invariant and gauge invariant vertex as well as to respect the usual energy conditions. If unitarity violation would occur, then the source requirements listed above would be incomplete, although no further source qualifications are at the core of standard field theory.

Therefore, the physical origin of a ghost–like excitation being emitted or absorbed by a reasonable source is unclear. To resolve the tension between the physical expectation of (6.85) being healthy and the technical analysis of [80, 82] indicating a ghost in the spectrum is the main motivation for the paper.

The tool that has been used in [80, 82] to diagnose a ghost in (6.85) is the classical brane–to–brane propagator. Its calculation is performed in a manifestly covariant framework. Here, the problematic scalar mode contributes a term with a negative sign residue. The corresponding quantum propagator is derived by simply applying a Feynman prescription to this pole. It causes the vacuum persistence amplitude to grow unbounded, thereby threatening the unitarity of (6.85).

However, it is found that the (00)–Einstein equation, which is a pure constraint, renders the would-be ghost mode non-dynamical. Thus, there is no particle interpretation associated with it. This result strongly questions the reasoning in terms of the classical propagator.

To evaluate the status of (6.85) a full-fledged Hamiltonian analysis was performed incorporating all constraints of the system. This allowed to derive the Hamiltonian on the constraint surface which is *manifestly positive definite*. This is the main result of this work. It shows that the quantum theory is totally consistent, and thus, the problematic scalar mode is not threatening the unitarity of the theory. The difficulties in the covariant framework arise because of an inconsistent expression for the quantum propagator that is not taking into account the constraints of the theory properly. This problem is closely related to the conformal factor problem in GR [84, 85].

This paper is organized as follows: Section 6.5.3 presents the Hamiltonian analysis, with its main result being a positive definite Hamiltonian on the constraint surface. In the first part of Section 6.5.6 the conformal factor problem in GR is reviewed and the covariant language is set up. The second part depicts the Lagrangian approach to the model and shows that the would-be ghost mode is in fact constrained by the (00)–Einstein equation. The proper number of sourced propagating degrees of freedom is derived. The conclusion is presented in Section 6.5.8, followed by technical details in the appendices.

6.5.3 Hamiltonian Analysis

In this section the classical Hamiltonian on the constraint surface is calculated and serves as a solid diagnostic tool to evaluate the status of (6.85). To this end, a complete Dirac constraint analysis is performed for the case of a two co–dimensional surface ($n = 2$). Since the ghost was formerly derived on a linear level, it suffices to study the linear theory of fluctuations h on a Minkowski background.

However, it is convenient to start with the non-linear version of the theory and to decompose the six–dimensional Ricci scalar in its four–dimensional analogue and extrinsic curvature terms. BIG in two co–dimensions is described by the action

$$\mathcal{S} \quad = \quad \mathcal{S}^{(6)}_{\rm EH}[g] \;+\; \mathcal{S}^{(4)}_{\rm EH}[\omega] \quad = \quad \int \! \mathrm{d}^6 x M_6^4 \sqrt{-g}\, R^{(6)}_{[g]} \;+\; \int \! \mathrm{d}^4 \tilde{x}\, M_4^2 \sqrt{-\omega}\, R^{(4)}_{[\omega]} \tag{6.86}$$

where g is the six dimensional bulk metric and ω the induced metric on the brane. The brane coordinates $\tilde{x}^\alpha$ are chosen such that they coincide with the bulk coordinates x^A: $\tilde{x}^\alpha = \delta^\alpha_A x^A$. This static gauge implies $\omega_{\alpha\beta} = \delta^A_\alpha \delta^B_\beta g_{AB}$. The index ranges are specified in the table below. M_6 is the gravitational scale in the bulk and M_4 the induced scale on the brane.

Preparations

The following canonical analysis uses a multiple ADM-split as starting point. For concreteness, both spatial extra–dimensions and the time direction are split in the usual ADM sense:

$$g_{AB} = \begin{pmatrix} \lambda_{\mu\nu} & \Lambda_\mu \\ \Lambda_\nu & \Lambda^2 + \Lambda_\lambda \Lambda^\lambda \end{pmatrix}, \tag{6.87}$$

with

$$\lambda_{\mu\nu} = \begin{pmatrix} \omega_{\alpha\beta} & \Omega_\alpha \\ \Omega_\beta & \Omega^2 + \Omega_\gamma \Omega^\gamma \end{pmatrix}, \tag{6.88}$$

and

$$\omega_{\alpha\beta} = \begin{pmatrix} -\Gamma^2 + \Gamma_i \Gamma^i & \Gamma_i \\ \Gamma_j & \gamma_{ij} \end{pmatrix}. \tag{6.89}$$

Here, γ_{ij} denotes the submetric of the spatial hypersurface orthogonal to the normal vectors

$$\left(\hat{n}^A_6\right) = \left(-\Lambda^\mu/\Lambda, \quad 1/\Lambda\right), \tag{6.90}$$

$$\left(\hat{n}^\mu_5\right) = \left(-\Omega^\alpha/\Omega, \quad 1/\Omega\right), \tag{6.91}$$

$$\left(\hat{n}^\alpha_4\right) = \left(1/\Gamma, \quad -\Gamma^i/\Gamma\right), \tag{6.92}$$

where Λ, Ω and Γ denote the three 'Lapse'–functions corresponding to the three ADM–splits, and Λ^μ, Ω^α and Γ^i are the respective 'Shift'–functions. Indices are raised and lowered with the background Minkowski metric η_{AB}. The index ranges are as follows:

$A,\, B,\, C,\, D$	$0,1,2,3,5,6$	
$\alpha,\, \beta,\, \gamma,\, \delta$	$0,1,2,3$	ADM 4+1
$\lambda,\, \mu,\, \nu,\, \rho$	$0,1,2,3,5$	ADM 5+1
$a,\, b,\, c,\, d$	$5,6$	bulk directions
$i,\, j,\, k,\, l$	$1,2,3$	spatial surface directions

The relation between the Ricci scalar of a d–dimensional space–time and an imbedded space–time with either one temporal or spatial dimension reduced is given by, including all total derivative contributions,

$$\sqrt{-g}R^{(d)} = \sqrt{-g}\Big\{R^{(d-1)} + (\hat{n}_d\cdot\hat{n}_d)\Big[(\mathrm{Tr}K_d)^2 - \mathrm{Tr}K_d^2 \\ + 2(\nabla\cdot((\hat{n}_d\cdot\nabla)\hat{n}_d) - \nabla\cdot(\hat{n}_d(\nabla\cdot\hat{n}_d)))\Big]\Big\}\,. \quad (6.93)$$

Here, the covariant derivative ∇ is compatible with the parent metric g characterizing the geometrical structure on the d–dimensional space–time. In the case of a single ADM–split, the second line in (6.93) can be dropped since it gives rise to a total derivative term in the action. However, using (6.93) successively introduces covariant derivatives compatible with the induced metric on the imbedded space–times, whereas the volume measure in the action is always given by the parent metric g. In this case, the second line in (6.93) needs to be kept. Depending on whether the ADM–split amounts to separate either a temporal or a spatial dimension, the product $\hat{n}_d\cdot\hat{n}_d$ yields -1 or $+1$, respectively. K_d denotes the extrinsic curvature tensor in d–dimensions. In terms of ADM variables,

$$(K_6)_{\mu\nu} = \frac{1}{2\Lambda}(\partial_6\lambda_{\mu\nu} - \nabla_\mu\Lambda_\nu - \nabla_\nu\Lambda_\mu)\,, \quad (6.94)$$

$$(K_5)_{\alpha\beta} = \frac{1}{2\Omega}(\partial_5\omega_{\alpha\beta} - \nabla_\alpha\Omega_\beta - \nabla_\beta\Omega_\alpha)\,, \quad (6.95)$$

$$(K_4)_{ij} = \frac{1}{2\Gamma}(\partial_t\gamma_{ij} - \nabla_i\Gamma_j - \nabla_j\Gamma_i)\,. \quad (6.96)$$

6.5.4 Calculating the Hamiltonian

Performing three ADM–splits in succession to separate both spatial extra directions and the temporal dimension and using (6.93) in each succession yields the following non-linear Lagrangian:

$$\begin{aligned}\mathcal{L} &= M_6^4\sqrt{-g}R^{(6)}_{[g]} + M_4^2\sqrt{-\omega}\,\delta_y^{(2)}R^{(4)}_{[\omega]} \quad (6.97)\\ &= \big(M_6^4\sqrt{-\omega}\,\Omega\,\Lambda + M_4^2\sqrt{-\omega}\,\delta_y^{(2)}\big)\big\{R^{(3)}_{[\gamma]} + (K_4)_{ij}(K_4)^{ij} - (K^i_{4\,i})^2\big\}\\ &\quad + M_6^4\sqrt{-\omega}\,\Omega\,\Lambda\big\{(K^\alpha_{5\,\alpha})^2 - (K_5)_{\alpha\beta}(K_5)^{\alpha\beta}\big\} + M_6^4\sqrt{-\omega}\,\Omega\,\Lambda\big\{(K^\mu_{6\,\mu})^2 - (K_6)_{\mu\nu}(K_6)^{\mu\nu}\big\}\\ &\quad - 2M_6^4\sqrt{-\lambda}\,\Lambda\big\{\nabla_\mu(\hat{n}_5^\mu\nabla_\nu\hat{n}_5^\nu) - \nabla_\nu(\hat{n}_5^\mu\nabla_\mu\hat{n}_5^\nu)\big\} + 2M_6^4\sqrt{-\omega}\,\Omega\,\Lambda\big\{\nabla_\alpha(\hat{n}_4^\alpha\nabla_\beta\hat{n}_4^\beta) - \nabla_\alpha(\hat{n}_4^\beta\nabla_\beta\hat{n}_4^\alpha)\big\}\,.\end{aligned}$$

Here, ∇_λ, ∇_μ, ... denotes the covariant derivative compatible with the induced metric $\lambda_{\mu\nu}$, and ∇_α, ∇_β, ... is the covariant derivative compatible with $\omega_{\alpha\beta}$. For ease of notation $\delta_y^{(2)} = \delta^{(2)}(y)$ has been introduced, where y denotes the extra space coordinates x^5 and x^6, collectively.

Linearizing the space–time geometry around a Minkowski background, $g_{AB} = \eta_{AB} + h_{AB}$, introduces the following fluctuations for the ADM variables:

$$
\begin{aligned}
\Gamma &= 1 + \delta\Gamma & \Omega &= 1 + \delta\Omega & \Lambda &= 1 + \delta\Lambda \\
&\equiv 1 + n & &\equiv 1 + N & &\equiv 1 + L \\[1em]
\Gamma_i &= \delta\Gamma_i & \Omega_\alpha &= \delta\Omega_\alpha & \Lambda_\mu &= \delta\Lambda_\mu \\
&\equiv n_i & &\equiv N_\alpha & &\equiv L_\mu \\[1em]
\gamma_{ij} &= \delta_{ij} + h_{ij}
\end{aligned}
\tag{6.98}
$$

Expanding (6.97) up to second order in metric fluctuations, allows to derive the canonical momentum fields:

$$
\Pi_n = \frac{\partial \mathcal{L}}{\partial \dot{n}} = 0\,, \tag{6.99}
$$

$$
\Pi_{n^i} = \frac{\partial \mathcal{L}}{\partial \dot{n}^i} = 0\,, \tag{6.100}
$$

$$
\Pi_N = \frac{\partial \mathcal{L}}{\partial \dot{N}} = M_6^4\big(- \dot{h}_i^i + 2\partial_i n^i - 2\dot{L}\big)\,, \tag{6.101}
$$

$$
\hat{\Pi}_L = \frac{\partial \mathcal{L}}{\partial \dot{L}} = M_6^4\big(- \dot{h}_i^i + 2\partial_i n^i - 2\dot{N}\big)\,, \tag{6.102}
$$

$$
\Pi_0 = \frac{\partial \mathcal{L}}{\partial \dot{N}^0} = M_6^4\big(- \partial_5 h_i^i - 2\partial_5 L\big)\,, \tag{6.103}
$$

$$
\hat{\Pi}_0 = \frac{\partial \mathcal{L}}{\partial \dot{L}^0} = M_6^4\big(- \partial_6 h_i^i - 2\partial_6 N\big)\,, \tag{6.104}
$$

$$
\Pi_i = \frac{\partial \mathcal{L}}{\partial \dot{N}^i} = M_6^4\big(- \partial_5 n_i + \dot{N}_i + \partial_i N^0\big)\,, \tag{6.105}
$$

$$
\hat{\Pi}_i = \frac{\partial \mathcal{L}}{\partial \dot{L}^i} = M_6^4\big(- \partial_6 n_i + \dot{L}_i + \partial_i L^0\big)\,, \tag{6.106}
$$

$$
\hat{\Pi}_5 = \frac{\partial \mathcal{L}}{\partial \dot{L}^5} = M_6^4\big(- \partial_6 N_0 + \dot{L}_5 - \partial_5 L_0\big)\,, \tag{6.107}
$$

$$
\begin{aligned}
\Pi_{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}^{ij}} &= \Big(M_6^4 + M_4^2\,\delta_y^{(2)}\Big)\Big(\frac{1}{2}\dot{h}_{ij} - \partial_{(i}n_{j)} \\
&\quad - \frac{1}{2}\delta_{ij}\dot{h}_i^i + \delta_{ij}\partial_k n^k\Big) - M_6^4\delta_{ij}\big(\dot{N} + \dot{L}\big)\,.
\end{aligned}
\tag{6.108}
$$

Since the momentum field (6.108) conjugated to h_{ij} involves a delta function, it can be further decomposed into a regular and an irregular contribution $\Pi^{ij} = \Pi^{ij}_{(R)} + \delta_y^{(2)}\,\Pi^{ij}_{(I)}$.

Equations (6.103), (6.104), (6.99) and (6.100), constitute six primary constraints (in what follows, they will be collectively denoted by $\phi_a^{(1)}$, where a runs from 1 to 6) as they do not involve time derivatives acting on the dynamical degrees of freedom. Accordingly, it will not be possible to solve for the velocity of the fields δN^0, δn, δn^i and δL^0. The velocities of the other fields are given by

$$\dot{N}_i = \frac{1}{M_6^4}\Pi_i + \partial_5 n_i + \partial_i N_0\,, \tag{6.109}$$

$$\dot{L}_i = \frac{1}{M_6^4}\hat{\Pi}_i + \partial_6 n_i + \partial_i L_0\,, \tag{6.110}$$

$$\dot{L}_5 = \frac{1}{M_6^4}\hat{\Pi}_5 + \partial_6 N_0 + \partial_5 L_0\,, \tag{6.111}$$

$$\dot{N} = \begin{cases} \frac{1}{4M_6^4}\left\{\frac{3}{2}(\Pi_N + \hat{\Pi}_L) - \Pi_{(R)}{}^k{}_k\right\} - \frac{1}{2M_6^4}\hat{\Pi}_L\,, & \vec{y} \neq 0\,, \\ \\ \frac{1}{2M_4^2}\Pi_{(I)}{}^k{}_k - \frac{1}{2M_6^4}\hat{\Pi}_L\,, & \vec{y} = 0\,, \end{cases} \tag{6.112}$$

$$\dot{L} = \begin{cases} \frac{1}{4M_6^4}\left\{\frac{3}{2}(\Pi_N + \hat{\Pi}_L) - \Pi_{(R)}{}^k{}_k\right\} - \frac{1}{2M_6^4}\Pi_N\,, & \vec{y} \neq 0\,, \\ \\ \frac{1}{2M_4^2}\Pi_{(I)}{}^k{}_k - \frac{1}{2M_6^4}\Pi_N\,, & \vec{y} = 0\,, \end{cases} \tag{6.113}$$

$$\dot{h}_{ij} = \begin{cases} \frac{2}{M_6^4}\Pi_{(R)\,ij} - \frac{1}{4M_6^4}(\Pi_N + \hat{\Pi}_L)\delta_{ij} \\ \qquad -\frac{1}{2M_6^4}\delta_{ij}\Pi_{(R)}{}^k{}_k + 2\partial_{(i}n_{j)}\,, \quad \vec{y} \neq 0\,, \\ \\ \frac{2}{M_4^2}\Pi_{(I)\,ij} - \frac{1}{M_4^2}\delta_{ij}\Pi_{(I)}{}^k{}_k + 2\partial_{(i}n_{j)}\,, \vec{y} = 0\,. \end{cases} \tag{6.114}$$

Note that the equation determining $\dot{L}$, $\dot{N}$ and $\dot{h}_{ij}$ have been decomposed into the two equations, where one determines the field value off the surface, and the other on the surface localized at $\vec{y} = 0$. This decomposition is a reminiscence of the delta function appearing in (6.108). Equations (6.109)–(6.114) allow to derive the Hamiltonian H, which is a lengthy expression presented explicitly in Appendix 8.3. In order to prove the stability of (6.85), only the Hamiltonian on the constraint surface is required.

Canonical consistency demands that the primary constraints are conserved under time evolution. This yields the secondary constraints $\phi_a^{(2)}$:

$$\phi_a^{(2)} = \dot{\phi}_a^{(1)} = \{H, \phi_a^{(1)}\}\,. \tag{6.115}$$

In greater detail,

$$
\begin{aligned}
\phi_1^{(1)} &= \Pi_0 + M_6^4\big(\partial_5 h_i^i + 2\partial_5 L\big)\,, && (6.116)\\
\phi_2^{(1)} &= \tilde{\Pi}_0 + M_6^4\big(\partial_6 h_i^i + 2\partial_6 N\big)\,, && (6.117)\\
\phi_3^{(1)} &= \Pi_n\,, && (6.118)\\
\phi_{i+3}^{(1)} &= \Pi_{n^i}\,, && (6.119)
\end{aligned}
$$

and

$$
\begin{aligned}
\phi_1^{(2)} &= \{H, \phi_1^{(1)}\} = \partial_5 \Pi_N + \partial_i \Pi^i + 2M_6^4 \partial_6 \partial_5 L_0 + \partial_6 \hat{\Pi}_5\,, && (6.120)\\
\phi_2^{(2)} &= \{H, \phi_2^{(1)}\} = \partial_6 \hat{\Pi}_L + \partial_i \hat{\Pi}^i + 2M_6^4 \partial_6 \partial_5 N_0 + \partial_5 \hat{\Pi}_5\,, && (6.121)\\
\phi_3^{(2)} &= \{H, \phi_3^{(1)}\} = M_6^4\{(\partial_5^2 + \partial_6^2) h_i^i - 2\partial_6 \partial_i L^i - 2\partial_5 \partial_i N^i - 2\partial_5 \partial_6 L_5 && (6.122)\\
&\qquad + 2\partial_i \partial^i (N + L) + 2\partial_6^2 N + 2\partial_5^2 L\} - (M_6^4 + M_4^2 \delta_y^{(2)})\delta^1 R^{(3)}\,, &&\\
\phi_{i+3}^{(2)} &= \{H, \phi_{i+3}^{(1)}\} = -2\partial_j \Pi^{ij} - 2M_6^4(\partial_6 \partial^i L_0 + \partial_5 \partial^i N_0) - \partial_5 \Pi^i - \partial_6 \hat{\Pi}^i\,. && (6.123)
\end{aligned}
$$

Here, $\delta^1 R^{(3)} = \partial_i \partial_j h^{ij} - \partial_i \partial^i h_k^k$ is the first order variation of the Ricci-scalar for the surface's spatial dimensions. $\phi_3^{(2)}$ denotes the generalization of the so–called Hamiltonian constraint in general relativity, which can be seen by sending M_6^4 to zero. It is equivalent to the constraint given by the (00)–Einstein equation, compare to Section 6.5.6. These secondary constraints are all conserved under time evolution, i.e. they commute with the Hamiltonian under the Poisson bracket

$$
\dot{\phi}_a^{(2)} = \{H, \phi_a^{(2)}\} \simeq 0\,. \tag{6.124}
$$

These relations have been checked explicitly. Note that the last relation ($\simeq$) is a weak equality, which means that the right hand side of (6.124) is a linear combination of the constraints $\phi_a^{(p)}$ of the system ($p \in \{1, 2\}$ & $a \in \{1, 2, 3, 4, 5, 6\}$). According to (6.124), the system does not possess any tertiary constraints. Thus, the constraint content is given by the set of 12 primary and secondary constraints $\phi_a^{(p)}$. Moreover, it can be shown that the constraint system is completely first class:

$$
\forall\, p, p', a, a' : \;\; \{\phi_a^{(p)}, \phi_{a'}^{(p')}\} \simeq 0\,. \tag{6.125}
$$

As a consequence, every constraint generates a gauge transformation on any quantity Θ, that is built up out of the dynamical field variables, as follows:

$$
\delta\Theta = \xi\{\Theta, \phi_a^{(p)}\}\,, \tag{6.126}
$$

with a space–time dependent gauge function ξ. In this way, the 12 first class constraints allow to reduce the number of independent dynamical degrees of freedom by 24. This freedom allows to implement the gauges:

$$\begin{aligned}
\phi_1^{(1)} &: N_0 = 0\,, && (6.127)\\
\phi_2^{(1)} &: L_0 = 0\,, && (6.128)\\
\phi_3^{(1)} &: n = 0\,, && (6.129)\\
\phi_{3+i}^{(1)} &: n^i = 0\,, && (6.130)\\
\phi_2^{(2)} &: L_5 = 0\,, && (6.131)\\
\phi_3^{(2)} &: \Pi_i^i = 0\,, && (6.132)\\
\phi_{3+i}^{(2)} &: \partial_j h^{ij} = 0\,. && (6.133)
\end{aligned}$$

The gauge freedom given by $\phi_1^{(2)}$ will be used later.

Because all the sources in the BIG setup are four-dimensional and localized on the higher co–dimensional surface, graviton absorption and emission processes are spatially isotropic in the directions normal to it. This means that the graviton field h respects a $SO(2)$–symmetry with respect to the two extra dimensions. Moreover, the derivation of the would-be ghost in former works is solid under the assumption of such a symmetry. Thus, in order to show the absence of the ghost, it is justified to make use of this symmetry. It is most easily implemented using polar coordinates (r,φ), where $x^5 = r\cos\varphi$ and $x^6 = r\sin\varphi$. Then the symmetry demands the extra space components of the graviton field not to depend on φ. Additionally, the $h_{\varphi r}$ components have to vanish. Transforming back to Cartesian coordinates yields

$$N = \cos^2\!\varphi\, h_{rr} + \frac{\sin^2\!\varphi}{r^2}\, h_{\varphi\varphi}\,, \tag{6.134}$$

$$L = \sin^2\!\varphi\, h_{rr} + \frac{\cos^2\!\varphi}{r^2}\, h_{\varphi\varphi}\,, \tag{6.135}$$

$$L_5 = \cos\varphi\sin\varphi\, h_{rr} - \frac{\cos\varphi\sin\varphi}{r^2}\, h_{\varphi\varphi}\,. \tag{6.136}$$

The gauge choice (6.131) then implies $r^2\, h_{rr} = h_{\varphi\varphi}$, which in turn demands

$$N = L\,, \tag{6.137}$$

where $N(r,x)$ only depends on r and x. The symmetry implies as well $h_{\varphi j} = 0$. Using the same reasoning as before, one finds

$$N_i = \tilde{N}_i \cos\varphi \qquad \text{and} \qquad L_i = \tilde{N}_i \sin\varphi\,, \tag{6.138}$$

where $\tilde{N}_i(r,x)$ is a function of r and x. Similarly, for the Π-sector,

$$\Pi_N = \hat{\Pi}_L\,, \tag{6.139}$$

$$\Pi_i = \tilde{\Pi}_i \cos\varphi \qquad \text{and} \qquad \hat{\Pi}_i = \tilde{\Pi}_i \sin\varphi\,, \tag{6.140}$$

$$\hat{\Pi}_5 = 0\,. \tag{6.141}$$

Here, Π_N and $\tilde{\Pi}_i$ are φ–independent. It should be noticed that these relations are gauge dependent. The remaining gauge freedom corresponding to $\phi_1^{(2)}$ can be used to implement

$$\partial_r \partial_i \tilde{N}^i = (\Delta_3 + \Delta_2) N \,. \tag{6.142}$$

In this gauge, the Hamiltonian constraint $\phi_3^{(2)}$ simplifies to

$$\tilde{\Delta} h^i_i = 0 \,, \tag{6.143}$$

with the generalized Laplace operator defined as

$$\tilde{\Delta} \equiv \left[\Delta_2 + \Delta_3 + \frac{M_4^2}{M_6^4} \delta_y^{(2)} \, \Delta_3 \right] . \tag{6.144}$$

The only bounded solution to the constraint equation (6.143) is $h^i_i = 0$.
Using all gauge conditions, (6.127)–(6.133), as well as (6.137), (6.142) and (6.143), the Hamiltonian on the constraint surface is given by

$$\begin{aligned} \mathcal{H} = &\frac{1}{M_6^4} \Pi^{(T)}_{(R)ij} \Pi^{(T)ij}_{(R)} + \frac{1}{M_4^2} \, \delta_y^{(2)} \, \Pi^{(T)}_{(I)ij} \Pi^{(T)ij}_{(I)} + \frac{1}{4 M_6^4} \Pi_N^2 + \frac{1}{2 M_6^4} \tilde{\Pi}_i \tilde{\Pi}^i + \frac{1}{4} M_6^4 \tilde{F}_{ij} \tilde{F}^{ij} \\ &+ \frac{1}{4} M_6^4 \partial_a h^{(tt)}_{ij} \partial^a h^{(tt)ij} + \frac{1}{4} \left(M_6^4 + M_4^2 \, \delta_y^{(2)} \right) \partial_k h^{(tt)}_{ij} \partial^k h^{(tt)ij} + 2 M_6^4 \partial_a N \partial^a N \,. \end{aligned} \tag{6.145}$$

Here, $\tilde{F}_{ij} = \partial_i \tilde{N}_j - \partial_j \tilde{N}_i$, $\Pi^{(T)}_{ij}$ denotes the traceless part of the momentum field Π_{ij} and $h^{(tt)}_{ij}$ is the transverse and traceless part of h_{ij}.
Evidently, $\mathcal{H}$ consists only of positive squares, which implies that the Hamiltonian H is positive definite, which in turn is a sufficient condition for a ghost–free theory. Note that a real ghost degree of freedom, which originates from a negative sign kinetic operator, would inevitably destroy the positive definiteness of the classical Hamiltonian. It should be stressed that this result does not depend on the fact of having a perfectly localized brane. $\delta_y^{(2)}$ should rather be thought of some finite width profile function.

6.5.5 Counting Degrees of Freedom

It is very instructive to count the degrees of freedom in the effective theory (6.85) for a two co–dimensional source. The source–free theory is gravity in six dimensions, which suggests $[h] \leq 9$. Note that it could be less than nine due to the fact that the source is four–dimensional and brane–localized. This introduces an additional $SO(2)$–symmetry by which the number of independent degrees of freedom gets reduced. Moreover, some of the potential degrees of freedom could turn out not to be sourced.

Dirac's constraint analysis allows to do a solid counting. Because of the index symmetries, $[h] \leq 21$, which is doubled in phase space due to the conjugated momentum fields. The dynamical system possesses 12 first class constraints, each generating a gauge transformation. This allows to remove 24 gauge redundancies, leaving $[h] \leq 9$, which is again doubled in phase space. This is already confirming the naive assumption relying on the effective field theory argumentation.

The list containing the remaining conjugated pairs is of course a gauge dependent statement. For instance, in what follows, the gauge freedom represented by $\phi_3^{(2)}$ is used to render Π_{ij} traceless, whereas the longitudinal part of Π_{ij} is fixed employing the constraint $\phi_{3+i}^{(2)}$. As a consequence, the conjugated momentum field in this gauge becomes $\Pi_{ij}^{(tt)}$, which is the transverse and traceless part of Π_{ij}, and similarly for the other fields. A possible list after gauge fixing is:

Conjugate pairs	Degrees of freedom	Constraint
$(h_{ij}^{(tt)}, \Pi_{ij}^{(tt)})$	2	$\phi_{3+i}^{(2)}$, $\phi_3^{(2)}$
(N, Π_N)	1	
$(L, \hat{\Pi}_L)$	1	
$(N_i^{(t)}, \Pi_i^{(t)})$	2	$\phi_1^{(2)}$
$(L_i, \hat{\Pi}_i)$	3	

The background isometry $SO(2)$ in the space of directions transverse to the surface reduces this list further. Taking (6.138)–(6.141) into account, it becomes:

Conjugate pairs	Degrees of freedom	Constraint
$(h_{ij}^{(tt)}, \Pi_{ij}^{(tt)})$	2	$\phi_{3+i}^{(2)}$, $\phi_3^{(2)}$
(N, Π_N)	1	
$(\tilde{N}_i^{(t)}, \tilde{\Pi}_i^{(t)})$	2	$\phi_1^{(2)}$

The canonical counting gives five dynamical degrees of freedom. This number agrees with the degrees of freedom in the Dvali–Gabadadze–Porrati model, which corresponds to (6.85) for the special case of one co–dimension. There it has been shown that the graviton is a continuous superposition of massive spin–2 excitations each propagating five helicity components. However, as it will become more clear in Section 6.5.6, only $h_{ij}^{(tt)}$ can be sourced by a localized four-dimensional source. Therefore, the number of sourced degrees of freedom is given by $[h^{(tt)}] = 2$ coinciding with the results found in the covariant analysis below.

6.5.6 Semi–covariant analysis

In [80, 82] a covariant language was used to derive the ghost in the BIG model. The appearance of this unitarity violating mode clearly contradicts the result of Section 6.5.3.

In order to make contact to these works, the covariant approach is studied in detail here. An explanation is offered why the covariant treatment does not allow to reliably address the unitarity issue. The main argument boils down to the statement that the scalar which was always regarded as a threat to unitarity is not dynamical. Former works did not take into account this constraint nature of the scalar mode properly, since both, [80] and [82], indicate that a ghost-like scalar degree of freedom can be found amongst the physical particle content of the theory. Furthermore, in this chapter a more physical viewpoint is established in which the BIG term plays the role of a localized source modification. The source argument provides a physical indication that the theory should be healthy.

6.5.7 Conformal factor problem in GR

The seeming unitarity violation in BIG is known to be mediated by the conformal mode of the graviton. In GR there is a very similar problem, sometimes called the conformal factor problem. Here the conformal mode of the graviton is threatening the unitarity of the corresponding quantum theory, too. In GR the appearance of this problem is strongly tied to the covariant description of the system and absent in the canonical formulation. One important aim of this work is to show that both problems, the one in GR and the one in BIG, are closely related. As a warming up exercise and in order to set up the covariant language, the conformal factor problem in standard GR in a weakly coupling regime is studied first.

Let $(\mathcal{M}_4, \eta)$ be a four–dimensional space–time, equipped with a geometrical structure provided by the Minkowski metric η. The action reads

$$\mathcal{S} = \mathcal{S}^{(4)}_{\mathrm{EH}}[h] + \mathcal{V}^{(4)}[h] \; , \tag{6.146}$$

where $\mathcal{S}^{(4)}_{\mathrm{EH}}[h]$ is the perturbed Einstein-Hilbert action on a Minkowski background up to second orders in h. A source can absorb and emit gravitons h, respectively, according to the minimal coupling vertex

$$\mathcal{V}^{(4)} = \int_{\mathcal{M}_4} \mathrm{d}^4 x \; h_{\alpha\beta} \, t^{\alpha\beta} \; . \tag{6.147}$$

In order to further study the dynamics of this model, the graviton h is decomposed into its gauge invariant and gauge variant contributions

$$h_{\alpha\beta} = D^{(\mathrm{tt})}_{\;\;\alpha\beta} + \partial_{(\alpha} V^{(\perp)}_{\;\;\beta)} + P^{(\parallel)}_{\;\;\alpha\beta} \, B + \eta_{\alpha\beta} \, S \; , \tag{6.148}$$

where $D^{(tt)}$ is the transverse and traceless tensor part of the graviton and $V^{(\perp)}$ denotes it transverse vector part. B and S are the gauge variant and gauge invariant scalar parts, respectively. The field S is the aforementioned conformal mode. The corresponding projectors are specified in the Appendix 8.4. The Einstein equations then are

$$\Box_4 D^{(\mathrm{tt})}_{\;\;\alpha\beta} - 2\eta_{\alpha\beta}\Box_4 S + 2\partial_\alpha\partial_\beta S = -2\kappa_0 \, t_{\alpha\beta} \; , \tag{6.149}$$

where the pure gauge modes $V^{(\perp)}$ and B dropped out and $\Box_4 = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$. κ_0 denotes the gravitational strength with which gravitons couple to a conserved source. The energy momentum tensor t is decomposed into its transverse–traceless part $t^{(\mathrm{tt})}$ and its trace t^α_α,

$$t_{\alpha\beta} = t^{(\mathrm{tt})}{}_{\alpha\beta} - \frac{1}{3} P^{(\parallel)}{}_{\alpha\beta}\, t^\gamma_\gamma + \frac{1}{3}\eta_{\alpha\beta}\, t^\gamma_\gamma\,. \tag{6.150}$$

Taking the trace and applying the transverse-traceless projector, respectively, allows to decompose the Einstein equations (6.149)

$$+\Box_4 S = \frac{\kappa_0}{3}\, t_\alpha^{\ \alpha}\,, \tag{6.151}$$

$$-\Box_4 D^{(\mathrm{tt})}{}_{\alpha\beta} = 2\kappa_0\, t^{(\mathrm{tt})}{}_{\alpha\beta}\,. \tag{6.152}$$

These equations suggest that the scalar mode S is a ghost as it comes with a different sign for its kinetic term compared to $D^{(\mathrm{tt})}$.

A slightly different way to phrase the problem consists in considering the lorentzian functional integral of the free theory

$$\int \mathcal{D}\,[h]\, e^{i\mathcal{S}^{(4)}_{\mathrm{EH}}[h]}\,. \tag{6.153}$$

Expressed in terms of the decomposition (6.148) the real time lorentzian action is

$$\kappa_0\, \mathcal{S}^{(4)}_{\mathrm{EH}}\,[h] = \frac{1}{4}\int_{\mathcal{M}_4} \mathrm{d}^4x \left[-\left(\partial_\gamma D^{(tt)}_{\alpha\beta}\right)^2 + 6\left(\partial_\gamma S\right)^2 \right]. \tag{6.154}$$

Performing the euclidian continuation $t \to -i\tau$ yields an expression for the corresponding euclidian path integral. As the kinetic term of the conformal mode S in (6.154) comes with the wrong sign, this integral is divergent and hence ill defined. This so-called conformal factor problem is for example discussed in [84, 85] in the case of GR.

Of course, it is known that linearized GR on a Minkowski background is a healthy quantum theory. This can be proven by performing a Dirac constraint analysis showing that the Hamiltonian is a positive definite quantity, which is a sufficient condition for a theory to respect unitarity. This is exactly what is predicted by the positive energy theorem for asymptotically flat space-times [86, 87]. So the question naturally arises, why the above analysis is suggesting a different result. An answer was given in [84, 85]: The conformal mode S is no independent degree of freedom. It is constraint by the physical degrees of freedom which are contained in $D^{(\mathrm{tt})}$ and the matter sector. Therefore, the S mode cannot be a ghost as there is no state in the Hilbert space associated with it. In the case of the path integral the summation is only allowed to include the true physical degrees of freedom.

In order to reveal the constraint, it is necessary to abandon the manifestly covariant description of the system and depict the (00)–Einstein equation. Using the transversality of $D^{(\mathrm{tt})}$ and its traceless property, the (00)–component of (6.149) is

$$\Delta_3 D^{(\mathrm{tt})i}_{\ \ \ i} - \partial^i \partial^j D^{(\mathrm{tt})}_{ij} + 2\Delta_3 S = -2\kappa_0\, t_{00}\,, \tag{6.155}$$

where $\Delta_3 = \delta^{ij}\partial_i\partial_j$, with i, j running over the spatial directions. As there are no time derivatives occurring in this equation, it is a constraint. Note that $D^{(\mathrm{tt})}$ contains two physical degrees of freedom that are encapsulated in its (ij)–components. The transversality of $D^{(\mathrm{tt})}$ is constraining the (0β)–components. The 6 independent (ij)–components are further reduced by the traceless property and the $(0j)$–Einstein equations that are constraints on $D^{(\mathrm{tt})}$ as well. The dynamics of these remaining two degrees of freedom is fully captured by equation (6.152). Once the dynamical equation for $D^{(\mathrm{tt})}$ is solved, the conformal mode is totally fixed by equation (6.155).

One might ask whether this is indicating an inconsistency between the dynamical (6.151) and the constraint (6.152) equation already on a classical level. However, it can be explicitly shown that the solution S of the constraint solves the would-be dynamical equation: Substituting the solution of $D^{(\mathrm{tt})}$ in (6.155) and using the decomposition of the energy momentum tensor (6.150) yields

$$\Delta_3 S = \frac{\kappa_0}{3}\Delta_3 \frac{1}{\Box_4} t^{\alpha}_{\alpha}\,. \tag{6.156}$$

By demanding that S should fulfill appropriate fall-off conditions at spatial infinity, the Δ_3–operator may be simply dropped. The solution for S then obviously solves the would-be dynamical equation (6.151). It is clear that the non-local operator $1/\Box_4$ arises, because a solution to the equations of motion (6.152) was inserted in the constraint (6.155).

The purpose of this GR exercise was to show that a solid analysis of the unitarity issue necessitates to first extract the true dynamical content of the theory. This however is not possible in a completely covariant description and requires to study the (00)–component of the Einstein equation. It should be noted that the canonical hamiltonian description of the free theory is well defined and does not bear these difficulties. Once the Hamiltonian on the constraint surface is bounded from below, the theory is in accordance with unitarity.

In Section 6.5.7 the same semi-covariant arguments are presented for BIG in order to make contact to former works that were using the same language [80, 82]. In a first step ordinary GR in higher dimensions with a localized four–dimensional source is investigated. This allows to establish the higher dimensional framework and highlight the source arguments. In the next step this scenario is generalized to BIG. Equivalently to the GR case the worrisome S mode is not dynamical and therefore does not constitute a ghost.

Gravity of a localized source

Let $(\mathcal{M}_d, \eta)$ be a $(d = 4 + n)$–dimensional space–time, equipped with a geometrical structure provided by the Minkowski metric η. Embedded in this space–time is a source of co–dimension n, localized on a background geometry $(\mathcal{M}_4, \eta)$. The presence of the source makes it natural to consider the space–time isomorphism $(\mathcal{M}_d, \eta) \cong (\mathcal{M}_4, \eta) \times (\mathbb{R}^n, \delta)$, where $\mathbb{R}^n$ denotes an n–dimensional vector space with Euclidean geometry δ. A coordinate system covering $(\mathcal{M}_d, \eta)$ can be described as follows: $\eta = \eta_{AB}\, \mathrm{d}X^A \otimes \mathrm{d}X^B = \eta_{\alpha\beta}\, \mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta + \delta_{ab}\, \mathrm{d}y^a \otimes \mathrm{d}y^b$, with obvious index ranges. Then, the localized source is given by

$$T_{AB}(X) = L_{AB}{}^{\alpha\beta}(y_0, y)\; t_{\alpha\beta}(x, y)\;, \tag{6.157}$$

where $L(y, y_0)$ denotes a localizer concentrating the sources energy–momentum around a fixed position $y_0 \in \mathbb{R}^n$ in the directions transverse to $(\mathcal{M}_4, \eta)$. In gauge theories the localizer density must allow for a conserved source.

The action reads

$$\mathcal{S} = \mathcal{S}^{(d)}_{\mathrm{EH}}[h] + \mathcal{V}[h]\;, \tag{6.158}$$

where $\mathcal{S}^{(4)}_{\mathrm{EH}}[h]$ is the second order, perturbed Einstein-Hilbert action on a Minkowski background in d dimensions.

The source can absorb and emit gravitons h, respectively, from and in all space–time directions according to the minimal coupling vertex

$$\mathcal{V} = \int_{\mathcal{M}_4} \mathrm{d}^4 x \int_{\mathbb{R}^n} \mathrm{d}^n y\; h_{AB}\, T^{AB}\;. \tag{6.159}$$

If the localizer is ideal, i.e. a distribution describing a sharp source extension in the transverse directions, the vertex density becomes effectively four–dimensional, and Lorentz–invariance requires only to integrate over $\mathcal{M}_4$ at y_0. Invariance under gauge transformations requires a conserved localized source, which in turn implies

$$\mathcal{V} = \int_{\mathcal{M}_4} \mathrm{d}^4 x \left(D^{(\mathrm{tt})}{}_{\alpha\beta}\, t^{(\mathrm{tt})\,\alpha\beta} + S t_\gamma{}^\gamma \right)(x, y_0)\;, \tag{6.160}$$

where the decomposition (6.148) has been used for the $(\alpha\beta)$–components of the graviton. It follows that the source specifications allow for no more than six sourced degrees of freedom, distributed as follows: $\left[D^{(tt)}\right] = 5$ and $[S] = 1$. Although this counting applies to gauge invariant objects, the theory may contain further constraints that reduce the number of dynamical degrees of freedom. Exactly as in the case of four–dimensional GR, discussed before. Therefore, at this stage, the number of sourced and propagating degrees of freedom is $[h_{\alpha\beta}] \leq 6$.

The graviton flux is non–vanishing away from the source anchored at y_0. In order to extract the dynamical content of the theory, this necessitates to deconstruct the entire graviton h in its gauge invariant and gauge variant contributions. In a particular coordinate system, outlined in the Appendix, the gauge fixed deconstruction is given by

$$\begin{aligned} h_{\alpha\beta} &= D^{(\mathrm{tt})}{}_{\alpha\beta} + P^{(\parallel)}{}_{\alpha\beta}\, B + \eta_{\alpha\beta}\, S \ , \\ h_{ab} &= d^{(\mathrm{tt})}{}_{ab} + \delta_{ab}\, s \ , \\ h_{\alpha b} &= G^{(\mathrm{v,v})}{}_{\alpha b} + \partial_b G^{(\mathrm{v,s})}{}_{\alpha} + \partial_\alpha F^{(\mathrm{s,v})}{}_{b} \ . \end{aligned} \tag{6.161}$$

All tensor qualifications are with respect to the isometries underlying the space–time $(\mathcal{M}_4, \eta) \times (\mathbb{R}^n, \delta)$. Thus, $D^{(\mathrm{tt})}$ and B, S transform under Lorentz transformations $SO(1,3)$ as a transverse and traceless second rank tensor and two scalars, respectively, while $d^{(\mathrm{tt})}$ and s transform under the roation group $SO(n)$, operating on the transverse directions, as a transverse and traceless second rank tensor and a scalar, respectively. The mixed sector involves $G^{(\mathrm{v,v})}$, $G^{(\mathrm{v,s})}$ and $F^{(\mathrm{s,v})}$, which transform under the direct product $SO(1,3) \times SO(n)$ as (transverse vector, transverse vector), (transverse vector, scalar) and (scalar, transverse vector) quantities, respectively. More details can be found in the Appendix.

The a priori gauge invariant contributions are $D^{(\mathrm{tt})}$, S, $d^{(\mathrm{tt})}$, s, $G^{(\mathrm{v,v})}$, while B, $G^{(\mathrm{v,s})}$, $F^{(\mathrm{s,v})}$ resemble gauge fixed quantities. Before fixing the gauge, for an extended source and $n > 1$, $[h_{AB}] \leq (5+n)(4+n)/2$, distributed as follows: $[h_{\alpha\beta}] \leq 10$, $[h_{ab}] \leq (n+1)n/2$ and $[h_{\alpha b}] \leq 4n$. After eliminating the gauge redundancies, h_{AB} carries no more than $(4+n)(3+n)/2$ propagating degrees of freedom. In detail, $h_{\alpha\beta}$ carries no more than seven degrees of freedom, where $\left[D^{(\mathrm{tt})}\right] \leq 5$, $[B] = [S] \leq 1$, h_{ab} carries no more than $n(n-1)/2$ propagating degrees of freedom, where $\left[d^{(\mathrm{tt})}\right] \leq n(n-1)/2 - 1$, $[s] \leq 1$, and, finally, $h_{\alpha b}$ carries no more than $4n - 1$ dynamical degrees of freedom, where $\left[G^{(\mathrm{v,v})}\right] \leq 3(n-1)$, $\left[G^{(\mathrm{v,s})}\right] \leq 3$ and $\left[F^{(\mathrm{s,v})}\right] \leq n-1$. This number gets further reduced to $[h_{AB}] \leq (4+n)(1+n)/2$ by the existence of d constraints encapsulated in the $(A\,0)$–Einstein equations. These remaining degrees of freedom are all truely dynamical but not necessarily sourced.

Given the source specifications outlined above, the observable quanta requiring a unitary evolution correspond to a subset of the sourced fields $D^{(\mathrm{tt})}$ and S. For these, the dynamical equations are given by

$$\Box S = -\frac{2}{3}\, \kappa_n\, \frac{n-1}{n+2}\, \widehat{t}_\gamma^{\,\gamma} \ , \tag{6.162}$$

$$\Box D^{(\mathrm{tt})}{}_{\alpha\beta} = -2\, \kappa_n\, \widehat{t}^{(\mathrm{tt})}{}_{\alpha\beta} \ , \tag{6.163}$$

where $\Box = \Box_4 + \Delta_n$, $\Box_4 = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$, $\Delta_n = \delta^{ab}\partial_a\partial_b$, and κ_n denotes the gravitational strength with which gravitons couple to a conserved source in $d = 4+n$. It is related to the scales used in Section 6.5.3 by $M_{4+n} = \kappa_n^{-1/(2+n)}$. For ease of notation, $\widehat{t} \equiv t\,\delta^{(n)}(y-y_0)$ has been introduced. All other fields that are left after eliminating the gauge redundancies, $B, d^{(tt)}, s, G^{(v,v)}, G^{(v,s)}$ and $F^{(s,v)}$ are decoupled from the source. In the general case, a source extended in the transverse directions would couple to $d^{(tt)}, s, G^{(v,v)}$ with equal strength κ_n, while $B, G^{(v,s)}$ and $F^{(s,v)}$ are always decoupled by gauge invariance. For example, by applying the projector for the transverse vector ($SO(1,3)$) on the $(\alpha\beta)$–Einstein equations, it follows

$$\Delta_n G_\alpha^{(v,s)} = 0 \, . \tag{6.164}$$

In order to extract the true dynamical content of (6.158), (6.162, 6.163) have to be supplemented with the (00)–Einstein equation, which imposes a constraint on the dynamics of $D^{(\mathrm{tt})}$ and S. This equation is

$$\begin{aligned}\left[\partial^i\partial^j - \delta^{ij}(\Delta_3+\Delta_n)\right] D^{(\mathrm{tt})}_{ij} - \Delta_n P^{(\parallel)}{}_i^{\,i} B \\ - \left[2\Delta_3 + 3\Delta_n\right] S - \left[n\,\Delta_3 + (n-1)\Delta_n\right] s = 2\kappa_n\,\widehat{t}_{00} \, ,\end{aligned} \tag{6.165}$$

where the following definitions apply: $\Delta_3 = \delta^{ij}\partial_i\partial_j$, with i,j running over the spatial directions along the brane, and $P^{(\parallel)}{}_i^{\,i}$ is the trace over the spatial components of the longitudinal projector on $(\mathcal{M}_4,\eta)$. Note that there are no time derivatives occurring in this equation. A more rigorous analysis of (6.165) requires to study the equations for s and B. An appropriate linear combination of the trace and longitudinal–longitudinal part of the $(\alpha\beta)$–Einstein equations yields

$$\Delta_n S = -\frac{(n-1)}{3}\Delta_n s \, . \tag{6.166}$$

It follows that $S = -(n-1)s/3 + \Phi$, where Φ is a solution of the Laplace equation, $\Delta_n\Phi = 0$, and has to be taken to be zero as the fields have to respect appropriate fall-off conditions. By inserting this in the (ab)–Einstein equations describing the longitudinal–longitudinal dynamics one finds

$$\Delta_n B = -\frac{n+2}{n-1}\Delta_n S \, . \tag{6.167}$$

Equation (6.165) becomes

$$\left[\partial^i\partial^j - \delta^{ij}(\Delta_3+\Delta_n)\right] D^{(\mathrm{tt})}_{ij} + \frac{n+2}{n-1}\left[\Delta_n P^{(\parallel)}{}_i^{\,i} + \Delta_3\right] S = 2\,\kappa_n\,\widehat{t}_{00} \, . \tag{6.168}$$

Equation (6.168) does not contain any time derivatives and therefore is a constraint on S, that is sometimes referred to as the Hamiltonian constraint. Once the dynamical equation for $D^{(\mathrm{tt})}$ is solved, the conformal mode S is totally fixed by equation (6.168). One might be worried by the appearance of the Green's function G in the projector $P^{(\parallel)}{}_i^{\,i}$.

If the (00)-Einstein equation is depicted in the original h variables, all time derivatives drop out immediately and there is no doubt that this equation is a constraint. This is no surprise since the theory under consideration is simply higher dimensional GR for which this property of the (00)–equation is well known. (Compare also to the discussion after equation (6.173), where the equivalence to the Hamiltonian constraint in ADM variables is explicitly shown.) The non-local operator is just a relict of the graviton decomposition (6.161) and totally fixed by imposing some arbitrary boundary conditions. Therefore, it cannot spoil the constraint character of this equation. The important result is again that S is not an independent degree of freedom. Again it should be checked that there is no inconsistency between the constraint and the dynamical equation. In total agreement to the GR example the solution of the dynamical equation for $D^{(\mathrm{tt})}$ can be inserted in the constraint which yields

$$\Delta_3 S + P^{(\|)}{}_i^{\,i} \Delta_n S = -\frac{2}{3}\frac{n-1}{n+2}\,\kappa_n\, P^{(\|)}{}_i^{\,i}\, \widehat{t}_\gamma^{\,\gamma} \,. \tag{6.169}$$

It is consistent with (6.162) in a sense that every solution of (6.169) is a solution of (6.162) but not the other way around.

The spectrum of observable quanta is reduced to the propagating degrees of freedom carried by $D^{(\mathrm{tt})}$, i.e. $\left[D^{(\mathrm{tt})}\right] \leq 5$, due to the index symmetry and four transversality and one traceless condition. For $j \in \{1,2,3\}$, the $(0j)$–Einstein equations are constraints, eliminating further three components of $D^{(\mathrm{tt})}$ from the list of propagating degrees of freedom. Since all the gauge redundancies are fixed, all geometrical conditions exploited, and all constraints imposed, it follows that the localized source (6.157) absorbs and emits $\left[D^{(\mathrm{tt})}\right] = 2$ physical degrees of freedom, subject to a healthy dynamics (6.163).

The localized source (6.157) could be an external graviton absorber and emitter, or it could be resolved in dynamical degrees of freedom. In fact, both source types could be operational. External sources describe those absorbers and emitters that are absolutely inert against backreaction. They are not the result of integrating out dynamical fields qualifying as heavy relative to a preset finite cut–off scale. Integrating out heavy fields on $\mathcal{M}_4 \times \{y_0\}$ results in an additional Einstein–Hilbert term localized at y_0:

$$T_{AB} = L_{AB}{}^{\alpha\beta}(y_0, y)\left(t_{\alpha\beta} + \lambda\, G^{(4)}_{\alpha\beta}(h)\right)(x,y) \,. \tag{6.170}$$

Here, $G^{(4)}$ denotes the four–dimensional perturbed Einstein tensor linear in h. Note that action (6.158) together with (6.170) is exactly the linearized BIG model in n co-dimensions (6.85). The coefficient λ depends on the details of the heavy field theory. For phenomenological reasons, $\lambda = 1/\kappa_0$ is an attractive choice. In this language, t contains, in principal, both, external and dynamically resolved graviton absorbers and emitters.

The source modification (6.170) allows for a straightforward generalization of (6.162) and (6.163):

$$\Box S = \frac{2}{3}\frac{n-1}{n+2}\,\kappa_n \left(-\widehat{t}^{\,\mu}_{\mu} + 3\,\kappa_0^{-1}\,\Box_4 \widehat{S}\right)\,, \tag{6.171}$$

$$\Box D^{(\mathrm{tt})}{}_{\alpha\beta} = \kappa_n \left(-2\,\widehat{t}^{(\mathrm{tt})}{}_{\alpha\beta} - \kappa_0^{-1}\,\Box_4 \widehat{D}^{(\mathrm{tt})}{}_{\alpha\beta}\right)\,, \tag{6.172}$$

where $\widehat{S} \equiv S\delta^{(n)}(y-y_0)$ and $\widehat{D}^{(\mathrm{tt})} \equiv D^{(\mathrm{tt})}\delta^{(n)}(y-y_0)$ denote the localizations of the gauge invariant scalar and the transverse and traceless tensor on $\mathcal{M}_4 \times \{y_0\}$, respectively. Remarkably, while $\widehat{D}^{(\mathrm{tt})}$ gravitates like an ordinary energy–momentum source, $\widehat{S}$ does not. In fact, the localized gauge invariant scalar seems to weaken its own source, which is an inconsistent modification of the equivalence principle at the classical level and indicative for a strong violation of unitarity. Assuming the gauge invariant scalar and its localized cousin are dynamical, the propagator corresponding to (6.171) for the conformal mode S would exhibit a tachyon pole with a wrong sign residue.

This is precisely the result of [80, 82]. The analysis performed in [80] uses a slightly different deconstruction of the graviton, and, as a consequence, the unitarity violation was claimed to be communicated by $h_\alpha^{\ \alpha}$. This can be easily mapped to the deconstruction (6.161) since $h_\alpha^{\ \alpha} = B + 4S$, where S is gauge invariant, while B is gauge fixed. Hence, choosing as a gauge condition $B = 0$, it is straightforward to show that (6.171) agrees with the corresponding equations in former treatments.

However, as before, the (00)–Einstein equation gives a constraint on S

$$\begin{aligned}\left[\partial^i\partial^j - \delta^{ij}(\Delta_3+\Delta_n)\right] D^{(\mathrm{tt})}_{ij} + \frac{n+2}{n-1}\left[\Delta_n P^{(\parallel)}{}_i{}^i + \Delta_3\right] S \\ = \kappa_n \left\{2\widehat{t}_{00} + \kappa_0^{-1}\left(2\Delta_3 \widehat{S} - \left(\partial^i\partial^j - \delta^{ij}\Delta_3\right)\widehat{D}^{(tt)}_{ij}\right)\right\}\,, \end{aligned} \tag{6.173}$$

which agrees with (6.168) in the limit when the heavy fields on $\mathcal{M}_4 \times \{y_0\}$ are formally decoupled from gravity. Thus, exactly the same reasoning as before applies. As an important result, the effective source (6.170) does not absorb or emit propagating scalars, which therefore will no challenge its unitarity. Equation (6.173) is equivalent to the Hamiltonian constraint $\Phi_3^{(2)}$ in (6.123). This can be checked by explicitly translating the covariant variables to the ADM variables of Section 6.5.3. From (6.161) together with

(6.87) and (6.98) as well as (6.166) and (6.167) it follows

$$\begin{aligned} h_{ij} &= D^{(\mathrm{tt})}_{ij} + P^{(\parallel)}_{ij} B + \delta_{ij} S \\ &= D^{(\mathrm{tt})}_{ij} - 4 P^{(\parallel)}_{ij} S + \delta_{ij} S \;, \end{aligned} \tag{6.174}$$

$$N = \frac{1}{2} s = -\frac{3}{2} S \; (= L) \,, \tag{6.175}$$

$$L_5 = 0 \,, \tag{6.176}$$

$$N_i = G^{(\mathrm{v,v})}_{i5} + \partial_5 G^{(\mathrm{v,s})}_{i} + \partial_i F^{(\mathrm{s,v})}_{5} \,, \tag{6.177}$$

$$L_i = G^{(\mathrm{v,v})}_{i6} + \partial_6 G^{(\mathrm{v,s})}_{i} + \partial_i F^{(\mathrm{s,v})}_{6} \,, \tag{6.178}$$

where it has been used that the two dimensional transverse and traceless tensor vanishes. Inserting these relation in (6.123) yields (6.173) with $\widehat{t}_{00} = 0$ and $n = 2$ as expected. From (6.174) is becomes clear that the non-local term in the constraint is just a relict of the decomposition. As before it can be checked that the constraint (6.173) is in accordance with the would-be dynamical equation (6.171).

It should be stressed that this analysis is independent of the brane regularization. Substituting the delta function with some finite width profile function will not change the results and conclusions of this section. The (00)–Einstein equation still is a constraint equation even if the source is allowed to have some spread in the extra space directions. The delta function was only taken for the sake of convenience. (This issue is more difficult to handle if one is interested in the brane-to-brane propagator. In order to regularize its divergencies, one normally has to introduce a certain thickness of the brane to which the final expression for the propagator is sensitive in an essential way [82]. The crossover properties of the theory for example strongly depend on this choice and setting the brane width to zero corresponds to an infinite crossover length scale which would mean that GR would not be modified. A typical choice is the inverse bulk scale $M_{4+n} = \kappa_n^{-1/(2+n)}$ at which the effective field theory description of higher dimensional gravity breaks down.)

In accordance with the GR case in Section 6.5.7, this result questions the validity of former arguments claiming that the S mode is a ghost. These arguments shall be briefly reviewed. The diagnostic tool that is normally employed to highlight a ghostly absorption or emission process on $\mathcal{M}_4$ is the classical brane–to–brane propagator following from

(6.171) and (6.172), or equivalently the source–to–source amplitude

$$\int_{\mathcal{M}_4} d^4x\, h_{\alpha\beta} t^{\alpha\beta} = \int_{\mathcal{M}_4} d^4p\, t^{\alpha\beta}(p) G_{\alpha\beta\gamma\delta}(p^2) t^{\gamma\delta}(-p) \; , \tag{6.179}$$

where the propagator in Fourier space is given by

$$\begin{aligned} G_{\alpha\beta\gamma\delta}(p^2) = &\left(\eta_{\alpha\gamma}\eta_{\beta\delta} - \frac{1}{3}\eta_{\alpha\beta}\eta_{\gamma\delta}\right) G^{(\mathrm{D})}(p^2) \\ &+ \eta_{\alpha\beta}\eta_{\gamma\delta}\, G^{(\mathrm{S})}(p^2) \end{aligned} \tag{6.180}$$

with

$$G^{(\mathrm{D})}(p^2) = \frac{2}{\kappa_n^{-1} g_n^{-1}(p^2) + \kappa_0^{-1} p^2} \; , \tag{6.181}$$

$$G^{(\mathrm{S})}(p^2) = \frac{2}{\left(\frac{n+2}{n-1}\right) \kappa_n^{-1} g_n^{-1}(p^2) - 2\kappa_0^{-1} p^2} \; . \tag{6.182}$$

The function $g_n(p^2)$ is a solution to the equation

$$\left(p^2 - \Delta_n\right) g_n(p^2, y) = \delta^{(n)}(y) \tag{6.183}$$

evaluated at the brane position $y = y_0 = 0$: $g_n(p^2) \equiv g_n(p^2, 0)$. The Green's function $G^{(\mathrm{D})}$ follows from (6.171) and $G^{(S)}$ follows from (6.172), where the partially Fourier transformed ansatz

$$G^{(\mathrm{D,S})}(x, y) = \int d^4p\, e^{ipx} g_n(p^2, y) f^{(D,S)}(p) \tag{6.184}$$

has been used.

It is well known that, given phenomenological interesting choices for the parameters of the theory, the denominator of (6.182) exhibits a pole with negative sign residue. For example, in two co-dimensions

$$g_n(p^2) \propto \ln\left(1 + \frac{\kappa_2^{-1/2}}{p^2}\right) \; , \tag{6.185}$$

so that the denominator of (6.182) has a negative sign residue for $\kappa_0^{-1} \gg \kappa_2^{-1/2}$. To solve (6.183), one has to regulate an indefinite momentum space integral. Here, a cutoff $\kappa_2^{-1/2}$ was introduced, which equals the cutoff of the effective field theory (6.85). Applying the usual Feynman prescription to this pole would lead to the conclusion that the corresponding quantum mechanical amplitude contains a negative imaginary part and thus violates unitarity due to the optical theorem. However, this calculation does not take into account the fact that S is not dynamical. On the contrary, considering the spectral density of the amplitude of [82] (equivalently, equation (6.179)) actually suggests that a ghost-like scalar degree of freedom is propagating.

To guarantee that only physical degrees of freedom are generated from the vacuum, one would need to incorporate the constraints properly. This is yet an open task. Therefore, in this work a different and more solid way was chosen by performing an hamiltonian analysis.

6.5.8 Summary

In this article, the consistency of gravity induced on a higher co–dimensional surface (6.85) is investigated. These models are of great phenomenological interest and might serve as a faithful anchor for technically natural approaches to the challenge posed by the Universe's observed accelerated expansion. The prospects of these models have been threatened by claims [80, 82] questioning their quantum mechanical stability due to seemingly unitarity violating absorption and emission processes of a particular scalar degree of freedom.

However, the action (6.85) can be derived as an effective low energy description of a stable parent theory at higher energies. The heavy degrees of freedom belonging to the parent theory leave fingerprints in the effective theory (6.85) in terms of an induced Einstein–Hilbert term. Thus, assuming the heavy degrees of freedom constitute legitimate graviton absorber and emitter sources in accordance with Lorentz invariance and gauge invariance, in particular, there is no physical understanding for the presence of a ghost like excitation in the theory's spectrum.

Accordingly, in this article it has been shown that gravity induced on a surface of arbitrary co–dimension respects unitarity.

As a solid diagnostic tool the classical Hamiltonian on the constraint surface has been derived within a full–fledged canonical constraint analysis. Its positive definiteness clearly proves that the theory (6.85) is healthy because any ghost–like excitation would necessarily result in a classical instability.

This result on its own causes a tension with former results in [80, 82]. Thus, in Chapter 6.5.6 a covariant language was used in order to make contact to these works. It has been shown that the (00)–Einstein equation is a constraint on the dangerous scalar mode S rendering it non-dynamical. Therefore, the mode S cannot be excited as an independent degree of freedom. This fact has not been taken into account properly in [80, 82], since both works indicate that a ghost-like scalar degree of freedom can be found amongst the physical particle content of the theory.

These results open an exciting window of opportunity to consistently deform gravity at the largest observable distances. We leave the question of phenomenological viability for future work, where we intend to confront the deformation (6.85) with data from supernova observation campaigns as a first step [7], which already shows the richness the effective theory (6.85) has to offer.

7 Prepared for submission

7.1 Microscopic picture of non-relativistic classicalons

7.1.1 Abstract

A theory of a non-relativistic, complex scalar field with derivatively coupled interaction terms is investigated. This toy model is considered as a prototype of a classicalizing theory and in particular of general relativity, for which the black hole constitutes a prominent example of a classicalon. Accordingly, the theory allows for a non-trivial solution of the stationary Gross-Pitaevskii equation corresponding to a black hole in the case of GR. Quantum fluctuations on this classical background are investigated within the Bogoliubov approximation. It turns out that the perturbative approach is invalidated by a high occupation of the Bogoliubov modes. Recently, it was proposed that a black hole is a Bose-Einstein condensate of gravitons that dynamically ensures to stay at the verge of a quantum phase transition. Our result is understood as an indication for that claim. Furthermore, it motivates a non-linear numerical analysis of the model.

7.1.2 Introduction

Recently, Dvali and Gomez proposed a microscopic picture of black holes [91, 92, 93]. According to them, black holes can be understood as Bose-Einstein condensates of gravitons. In this picture, the Schwarzschild geometry would effectively emerge from the interaction of a test particle with the condensate of gravitons. In [94, 95] this picture was further elaborated and the authors concluded that the black hole is at the point of quantum phase transition.

Within the Schwarzschild radius, the graviton theory is strongly coupled. This necessitates to sum up a large number of equally important terms in the perturbation series. This fact and the relativistic nature of the graviton theory makes it hard to obtain any quantitive predictions along the lines of [91, 92, 93, 94, 95] within the theory of general relativity. Therefore, in this paper we propose a non-relativistic, derivatively coupled toy model that allows to quantitatively compute properties expected for black holes according to [91, 92, 93, 94, 95]. Our model is constructed such that it contains a ground state corresponding to the black hole of general relativity, which is nothing else but a non-relativistic classicalon state. For a description of the concept of classicalization in the case of gravity see [66, 96] and for its generalisation to other derivatively coupled theories compare to [72, 62, 74, 71].

We perform a quantum perturbation theory around a highly occupied classical state (so called 'Bogoliubov approximation') which is supposed to make up the classicalon. Our results indicate that the perturbative approach is not applicable, which is exactly what we expect to see if the system indeed manages to stay at the point of quantum phase transition. Therefore, we see indications for the claims of [94, 95], even though only a subsequent numerical and non-linear analysis will clearly decide about the status of our model.

Our paper is organized as follows: Section 7.1.3 summarizes the main ideas of [91]. Section 7.1.4 contains our model and results. Future prospects of our theory are discussed in Section 7.1.5.

7.1.3 Black Holes as Graviton Condensates

Quantum Portrait

The starting point in the approach of [91, 92, 93, 94] is the observation that the graviton interaction strength α_{gr} is momentum dependent due to the derivatively coupled nature of interaction terms of the metric fluctuation field with itself:

$$\alpha_{gr} = hG_N\lambda^{-2} \, , \tag{7.1}$$

where G_N is Newtons constant and λ is the typical graviton wavelength involved in a given scattering process. For the case of black holes, the characteristic wavelength is set by the Schwarzschild radius $r_g = 2\,G_N M \sim \lambda$, where M is the mass of the black hole. Accordingly, each graviton contributes an energy $\sim h/(2\,G_N M)$. The total number N of gravitons constituting a black hole is thus

$$N = \frac{2G_N M^2}{h} \sim \frac{\lambda^2}{L_P^2} \, , \tag{7.2}$$

where we have introduced the planck length $L_P = \sqrt{hG_N}$. Equation (7.2) is also true for the number of gravitons contained in the gravitational field of other objects such as planets since it can be obtained from summing up the Fourier modes of any Newtonian gravitational field $\phi = -r_g/r$. Inserting (7.2) in (7.1) yields the dependence of the coupling with N

$$\alpha_{gr} = \frac{1}{N} \, . \tag{7.3}$$

The occupation number N can be understood as the parameter measuring the classicality of a given object composed out of gravitons, in this case black holes. Intrinsic quantum processes such as the decay into a two particle state are exponentially suppressed

$\langle\text{Out}|\exp(-S)|\text{In}\rangle \sim \exp(-N)$. Additionally, the number of gravitons produced in the gravitational field of any elementary particle is negligibly small, for example for an electron we get $N = 2G_n m_e^2/h \approx 10^{-44}$. This shows why elementary particles cannot be considered as a classical gravitating object (even though they contribute a standard Newton law at large distances), and in particular it becomes clear why a single elementary particle does not collapse into a black hole.

Let us contrast black holes with the gravitational field of other objects such as planets. Assuming that the characteristic wavelength of the gravitons is in any case given by the characteristic size R of the object, we obtain as the gravitational part of the energy

$$E_{grav} \sim \frac{Nh}{R} \sim M\frac{r_g}{R} \,. \tag{7.4}$$

This shows that for objects not being a black hole (i.e., for $R > r_g$) a substantial part of the energy is carried by other constituents than gravitons. This is why the gravitational field of other objects than black holes cannot exist without an external source, for example a planet. However, once the extension of the gravitational object reaches $R = r_g$, the whole energy M of our object is stored in the gravitational field, so that an external source is not required to balance the energy budget. It is exactly at this point where the interaction of an individual graviton with the collective potential generated by the other gravitons becomes significant. This can most easily be seen by appreciating that the classical perturbation series in the metric fluctuation field h about a Minkowski background breaks down at the horizon r_g. However, the interaction of two individual gravitons is still small as long as we consider regions $r > L_P$. Given that the dominant interaction is gravity itself, the authors of [91] concluded that black holes are *self-sustained* bound states of gravitons. Moreover, black holes are *maximally packed* in the sense that the only characteristic of a black hole in the semi-classical limit is the number of gravitons N composing it, and any further increase of this number results inevitably in an increase of the size and mass of the black hole. This becomes clear since by default the extension of the black hole is no free parameter but given by r_g, and accordingly all physical black hole quantities (mass, size, entropy, etc.) can be quantified by N. This is nothing else but the famous no-hair theorem translated in the language of gravitons. An important consequence of this picture is that black holes always balance on the verge of self-sustainability, since the kinetic energy h/r_g of a single graviton is just as large as the collective binding potential $-\alpha_{gr}Nh/r_g$ produced by the remaining $N-1$ gravitons. Thus, if you give a graviton just a slight amount of extra energy, its kinetic energy will be above the escape energy of the bound state. In [91] it was therefore concluded that black holes are *leaky condensates*.

The above reasoning strictly applies only in the (semi-)classical limit $N \to \infty$. This is important, because we might wonder how a quantum effect like Hawking radiation can be understood in our picture of highly occupied graviton states, since usually we expect quantum effects to be exponentially suppressed. Actually, to explain this, the authors in [94] conjectured that the black hole is at a point of *quantum phase transition*. Thus, quantum excitations are always significant and cannot be ignored. In particular, given that black holes are leaky condensates, every quantum excitation will lead to the escape of the corresponding graviton. These escaped particles are interpreted as the Hawking radiation of the black hole.

Moreover, due to the quantum phase transition, the leading corrections to the above (semi-)classical ($N \to \infty$) picture are not exponentially but only $1/N$ suppressed. This makes it possible for any finite N to retrieve information from the black hole (for instance the Hawking spectrum contains $1/N$ corrections, making it for example possible to read out the amount of Baryons originally stored in the black hole). The famous information paradox is thus just a relict of working in the strict (semi-)classical $N \to \infty$ approach in which the hair of the black hole is negligible compared to the N graviton state.

In the next section we discuss the well known physics of quantum phase transition for the example of a non-relativistic condensed matter system. Assuming that black holes behave similar to this model, we will qualitatively discuss the implications for black hole physics, as it was done in [94].

On the Verge of Quantum Phase Transition

The discussion of this section closely follows [97], where the properties of a quantum phase transition are studied. We want to describe a system of N bosons of mass m with an attractive interaction in one dimension of size V at zero temperature. The second quantized field $\hat{\Psi}(x,t)$ in the Heisenberg representation is measuring the particle density at position x. The corresponding Hamiltonian reads

$$\hat{H} = \frac{\hbar^2}{2m}\int_0^V \mathrm{d}x (\partial_x \hat{\Psi})^\dagger (\partial_x \hat{\Psi}) - \frac{U}{2}\int_0^V \mathrm{d}x\, \hat{\Psi}^\dagger \hat{\Psi}^\dagger \hat{\Psi}\hat{\Psi} \,, \tag{7.5}$$

where U is a positive parameter of dimension [energy]$\times$[length] controlling the interaction strength. The dynamics of $\hat{\Psi}(x,t)$ are given by the Heisenberg equation

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi} = \left[\hat{\Psi}, \hat{H}\right] \tag{7.6}$$

$$= \left(-\frac{\hbar^2}{2m}\partial_x^2 - U(\hat{\Psi}^\dagger\hat{\Psi})\right)\hat{\Psi} \tag{7.7}$$

where the equal time commutation relations

$$\left[\hat{\Psi}(x,t), \hat{\Psi}^\dagger(x',t)\right] = \delta(x-x') \qquad\qquad \left[\hat{\Psi}(x,t), \hat{\Psi}(x',t)\right] = 0 \tag{7.8}$$

have been used. Applying the mean-field approximation amounts to replacing the operator $\hat{\Psi}(x,t)$ by a classical field $\Psi_0(x,t)$. This replacement is justified when the quantum ground state is highly occupied. In this case the non-commutativity of the field operator is a negligible effect. Since we are looking for stationary solutions, the time dependence is separated in the usual way

$$\Psi_0(x,t) = \Psi_0(x)\exp\left(-\frac{i\mu t}{\hbar}\right), \tag{7.9}$$

where μ is the chemical potential. Inserting this ansatz in (7.6), yields the stationary Gross-Pitaevskii equation. A trivial solution that fulfils the periodic boundary conditions $\Psi_0(0) = \Psi_0(V)$ is given by

$$\Psi_0^{(\mathrm{BE})}(x) = \sqrt{\frac{N}{V}} = const. \tag{7.10}$$

This solution corresponds to the homogenous Bose-Einstein condensate. However, this solution is the minimal energy configuration only for $U < U_c$. The critical value has been be derived in [97] to be: $U_c = \hbar^2\pi^2/(mVN)$. For $U > U_c$ the ground state is given by an inhomogenous solution $\Psi_0^{(\mathrm{sol})}(x)$ describing a soliton. By increasing the parameter U, i.e. the interaction strength, the ground state of the system undergoes a phase transition from the Bose-Einstein phase to the soliton phase once the critical point U_c is reached. As the authors in [97] have shown, this point of phase transition is characterized by a cusp in the chemical potential $\mu(U)$, the kinetic energy $\epsilon_{kin}(U)$ and the interaction energy $\epsilon_{int}(U)$ per particle as functions of U.

The main result of [97] was to show that at the point of phase transition quantum corrections to Ψ_0 become important and a purely classical description is no longer possible, therefrom the name 'quantum phase transition'. A suitable way to investigate this effect is provided by the Bogoliubov approximation in which the classical field Ψ_0 is furnished with small quantum corrections $\delta\hat{\Psi}$. A proper quantum mechanical treatment, of which the details are given in the next section, allows to derive the famous energy spectrum of the Bogoliubov excitations

$$\begin{aligned}\epsilon(k) &= \left(\left(\frac{\hbar^2\delta k^2}{2m}\right)^2 - \frac{\hbar^2 UN}{mV}\delta k^2\right)^{1/2} \\ &= \left(\left(\frac{\pi\hbar^2}{mV}\right)^2 \delta k^2\left[\left(\frac{V}{2\pi}\right)^2\delta k^2 - \frac{U}{U_c}\right]\right)^{1/2}.\end{aligned} \tag{7.11}$$

Due to the periodic boundary conditions, the momentum δk of the Bogoliubov modes is quantized in steps of $2\pi/V$. From (7.11) it is clear that once the interaction strength approaches the value U_c, the energy of the first Bogoliubov mode ($\delta k = 2\pi/V$) vanishes. Consequently, the excitation of the first mode becomes energetically favourable and the condensate is depleting very efficiently. This is the characteristic property of a quantum phase transition. This picture is further substantiated by calculating the occupation number of excited Bogoliubov modes

$$n(\delta k) = \frac{\hbar^2 \delta k^2/2m - UN/V}{2\epsilon(\delta k)} - \frac{1}{2}\,, \tag{7.12}$$

which shows that the vanishing of $\epsilon(\delta k)$ is accompanied by an extensive occupation of the corresponding quantum states. This means that the Bogoliubov approximation is no longer applicable and quantum corrections are significant. For values $U > U_c$ the energy becomes imaginary, which signals the formation of a new ground state that is given by the soliton solution $\Psi_0^{(\mathrm{sol})}(x)$, compare to the discussion in [97]. Moreover, the work of [98, 99] shows that the system becomes drastically quantum entangled at the critical point, which is yet another characterization of quantum phase transition.

By making the N dependence of U_c explicit and introducing the new dimensionless coupling parameter $\alpha = UmV/(\hbar^2\pi^2)$, the condition for the breakdown of the Bogoliubov approximation becomes

$$\alpha = \frac{1}{N}\,. \tag{7.13}$$

This is exactly the condition for self-sustainability in the case of a black hole (7.3). These considerations closely follow [94], where the authors wanted to illustrate the relation between black hole physics and Bose-Einstein condensation at the critical point. Of course, in this toy model the relation (7.13) is not generically realized, but has to be imposed by adjusting the model parameters by hand. (For a given value of N, the interaction strength U has to be chosen appropriately.) In the case of GR the left hand side of equation (7.13) is k-dependant which in principal could allow for a generic cancelation between the two terms in the squared bracket in the last line of (7.11). This cancelation is assumed to take place up to $1/N$–corrections.

The aim of our work is to present a non-relativistic scalar model that is in principle able to account for this cancelation and thus generically stays at the point of quantum phase transition independent of the chosen parameters. It is not possible to derive this result within the Bogoliubov approximation since a high occupation of quantum states is the defining property of a quantum phase transition. However, the breakdown of the perturbative approach is a necessary condition and therefore provides an indication for it.

7.1.4 Microscopic Picture of Non-Relativistic Classicalons

The Model

Non-relativistic classicalizing theories have the advantage of being computable without a resummation of infinitely many equally important terms as it would be the case for example in GR. In the following, we will consider a special non-relativistic, classicalizing theory that was constructed to mimic general relativity. As in [97], we choose to confine our theory in a 1-dimensional box of size V. To be concrete, we consider the following Hamiltonian for the second quantized field $\hat{\Psi}(x)$ measuring the particle density at position x:

$$\hat{H} = \frac{\hbar^2}{2m}\int_0^V \mathrm{d}x :(\partial_x\hat{\Psi})^\dagger(\partial_x\hat{\Psi}): +\lambda\int_0^V \mathrm{d}x :\Big((\partial_x\hat{\Psi})^\dagger(\partial_x\hat{\Psi})\Big)^2: + \\ \kappa\int_0^V \mathrm{d}x :\Big((\partial_x\hat{\Psi})^\dagger(\partial_x\hat{\Psi})\Big)^3: , \quad (7.14)$$

where : : denotes the normal ordering. We are looking for homogenous solutions of the Heisenberg equation

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi} = \left[\hat{\Psi}, \hat{H}\right] , \quad (7.15)$$

in which the field operator is again replaced by a classical field $\Psi_0(x)$. (The subscript 0 will be suppressed throughout the rest of this work.) We try to generalize the known homogenous BEC solution (7.10). We can separate the time dependence as in (7.9). Since $\Psi(x)$ is a complex field, (7.15) has in general the following class of solutions

$$\Psi_k(x) = \sqrt{\frac{N}{V}}\exp\left(ikx\right) , \quad (7.16)$$

where the momentum k is quantized in steps of $2\pi/V$ by implementing periodic boundary conditions. The number of particles is denoted by N. Inserting (7.16) in the Hamiltonian (7.14) results in the polynomial

$$\frac{H^{(0)}}{V} = \frac{\hbar^2}{2m}z + \lambda z^2 + \kappa z^3 \quad (7.17)$$

where $z = \frac{N}{V}k^2$.

However, not every solution (7.16) is a local minimum of the energy (7.17). For sure, one minimum is given by $k = 0$ (since the kinetic energy contributes positively), which would exactly correspond to the Minkowski vacuum in the case of general relativity given that this is the global energetic minimum of the theory (7.14). Moreover, by appropriately choosing the coefficients λ and κ, we can construct a second minimum of (7.17) at $z_0 = Nk_0^2/V$ with positive energy, denoted with Ψ_{k_0}, where $k_0 > 0$. It is easy to show that the corresponding solution not only minimizes (7.17) (that is, minimizing the energy within the sub-class of homogenous solutions (7.16)) but is also given as a minimum in complete field space (that is, it is a minimum for general fluctuations $\Psi = \Psi_{k_0} + \delta\Psi$). It is this solution that will turn into the classicalon which corresponds to the black hole solution of general relativity. Furthermore, it should be noted that the chemical potential is zero due to the relation $\mu \propto \partial H^{(0)}/\partial z|_{z_0}$.

Bogoliubov Theory

We will study the leading quantum perturbations $\delta\hat{\Psi}(x)$ about the classical condensate $\Psi_{k_0}(x)$. To this end, we write

$$\hat{\Psi}(x) \quad = \quad \frac{1}{\sqrt{V}}\sum_k \hat{a}(k)e^{ikx} = \frac{1}{\sqrt{V}}\hat{a}(k_0)e^{ik_0x} + \frac{1}{\sqrt{V}}\sum_{k\neq k_0}\hat{a}(k)e^{ikx} \ , \qquad (7.18)$$

where $\hat{a}(k)$ is the annihilation operator of the momentum mode k. The Bogoliubov approximation consists in treating the first term in (7.18) classically due to the large occupation of the state with momentum k_0. The second term presents a small quantum correction. On account of this, the replacement

$$\hat{a}(k_0) \to \sqrt{N_0} \qquad (7.19)$$

is introduced, which allows to identify $\Psi_{k_0}(x)$ with the first term in (7.18). The second term is simply the Fourier representation of the quantum perturbation $\delta\hat{\Psi}(x)$. We want to calculate the perturbation series up to second order in $\delta\hat{\Psi}(x)$ or $\hat{a}(k \neq k_0)$. Note that once we allow for an occupation of the momentum states with $k \neq k_0$, we have to distinguish between N_0, the number of particles in the ground state, and N, the total number of particles. Since we want to express everything in terms of N, the normalisation condition

$$\hat{a}^\dagger(k_0)\hat{a}(k_0) = N - \sum_{k\neq k_0}\hat{a}^\dagger(k)\hat{a}(k) \qquad (7.20)$$

has to be employed. This means that the zeroth order $H^{(0)}$ terms contribute to the second order $H^{(2)}$ when we express N_0 in terms of N. Inserting (7.18) and (7.20) into the Hamiltonian (7.14), results in the following quadratic order expression:

$$H^{(2)} = \sum_{\delta k\neq 0}\left[\epsilon_0^{(1)}\hat{a}^\dagger\hat{a} + \epsilon_0^{(2)}\hat{b}^\dagger\hat{b} + \epsilon_1(\hat{a}^\dagger\hat{b}^\dagger + \hat{b}\hat{a})\right] \ , \qquad (7.21)$$

where the decomposition $k = k_0 + \delta k$ has been used and the (re-)definitions

$$\hat{a}(\delta k) \equiv \hat{a}(k_0 + \delta k) \ , \qquad (7.22)$$

$$\hat{b}(\delta k) \equiv \hat{a}(k_0 - \delta k) \ , \qquad (7.23)$$

as well as

$$\epsilon_0^{(1)} = (k_0 + \delta k)^2 P_0 + \Lambda_0 \ , \qquad (7.24)$$

$$\epsilon_0^{(2)} = (k_0 - \delta k)^2 P_0 + \Lambda_0 \ , \qquad (7.25)$$

$$\epsilon_1 = (k_0^2 - \delta k^2) P_1 \ , \qquad (7.26)$$

apply. Here, the polynomials P_0, P_1 and Λ_0 are functions of the combination z_0 and the coefficients m, λ and κ:

$$P_0 = \frac{\hbar^2}{4m} + 2\lambda z_0 + \frac{9}{2}\kappa z_0^2 \tag{7.27}$$

$$\Lambda_0 = -k_0^2\left(\frac{\hbar}{4m} + \lambda z_0 + \frac{3}{2}\kappa z_0^2\right) \tag{7.28}$$

$$P_1 = \lambda z_0 + 3\kappa z_0^2 \tag{7.29}$$

Note that when using the minimal energy condition $\partial H^{(0)}/\partial z|_{z_0} = 0$, see equation (7.17), we obtain $P_0 = P_1$ and $\Lambda_0 = 0$ due to the relations $2V(P_0 - P_1) = \partial H^{(0)}/\partial z|_{z_0}$ and $2V\Lambda_0/k_0^2 = -\partial H^{(0)}/\partial z|_{z_0}$, respectively. Furthermore, it can be checked that $P_0 > 0$ if z_0 corresponds to the minimum of (7.17) because $2VP_1 = z_0\ \partial^2 H^{(0)}/\partial z^2|_{z_0}$. The Hamiltonian (7.21) is almost of the Bogoliubov form and can be diagonolised by means of the transformation

$$\hat{\alpha} = u\hat{a} + v\hat{b}^\dagger \qquad \text{and} \qquad \hat{\beta} = u\hat{b} + v\hat{a}^\dagger, \tag{7.30}$$

where $u, v \in \mathbb{R}$. Setting the off-diagonal terms to zero and requiring standard commutation relations for $\hat{\alpha}$ and $\hat{\beta}$ implies

$$\epsilon_1\left(u^2 + v^2\right) - 2u\,v\frac{\epsilon_0^{(1)} + \epsilon_0^{(2)}}{2} = 0\ , \tag{7.31}$$

as well as

$$u^2 - v^2 = 1\ . \tag{7.32}$$

These two equations are solved by

$$u = \pm\frac{1}{\sqrt{2}}\left(\frac{1}{2}\frac{\epsilon_0^{(1)} + \epsilon_0^{(2)}}{\epsilon} + 1\right)^{1/2},\quad v = \pm\frac{1}{\sqrt{2}}\left(\frac{1}{2}\frac{\epsilon_0^{(1)} + \epsilon_0^{(2)}}{\epsilon} - 1\right)^{1/2}, \tag{7.33}$$

where

$$\epsilon = \sqrt{\frac{1}{4}\left(\epsilon_0^{(1)} + \epsilon_0^{(2)}\right)^2 - \epsilon_1^2}\ . \tag{7.34}$$

Note that $\epsilon_0^{(1)}$ and $\epsilon_0^{(2)}$ are strictly positive, whereas the sign of ϵ_1 depends on the value of δk. Thus in order to fulfill (7.31), we have to choose u and v in (7.33) both positive when $\delta k < k_0$ and one of both has to be chosen negative when $\delta k > k_0$. In both cases the diagonalized version of (7.21) reads

$$H^{(2)} = \sum_{\delta k \neq 0}\left[\left(\epsilon + \frac{1}{2}(\epsilon_0^{(1)} - \epsilon_0^{(2)})\right)\hat{\alpha}^\dagger\hat{\alpha} + \left(\epsilon - \frac{1}{2}(\epsilon_0^{(1)} - \epsilon_0^{(2)})\right)\hat{\beta}^\dagger\hat{\beta} + \epsilon - \frac{1}{2}(\epsilon_0^{(1)} + \epsilon_0^{(2)})\right]\ . \tag{7.35}$$

Using the definitions (7.24), (7.25) and (7.26), we find $\epsilon = 2P_0 k_0 |\delta k|$ and $(\epsilon_0^{(1)} - \epsilon_0^{(2)})/2 = 2P_0 k_0 \delta k$. Note that ϵ is strictly positive. By employing the relation $\hat{\alpha}(\delta k) = \hat{\beta}(-\delta k)$ we find

$$H^{(2)} = \sum_{\delta k \neq 0} \left[2 \left(\epsilon + \frac{1}{2}(\epsilon_0^{(1)} - \epsilon_0^{(2)}) \right) \hat{\alpha}^\dagger \hat{\alpha} + \epsilon - \frac{1}{2}(\epsilon_0^{(1)} + \epsilon_0^{(2)}) \right] . \tag{7.36}$$

Accordingly, the vacuum $|0\rangle$ of the Fock space is defined as

$$\hat{\alpha}|0\rangle = 0 . \tag{7.37}$$

It follows from the Hamiltonian (7.36) that the combination

$$e(\delta k) \equiv 2 \left(\epsilon + \frac{1}{2}(\epsilon_0^{(1)} - \epsilon_0^{(2)}) \right) \tag{7.38}$$

is the energy of the quasi particles created by $\hat{\alpha}^\dagger(\delta k)$ with momentum $k_0 + \delta k$. Since the vacuum of our theory is defined with respect to $\hat{\alpha}$, it contains a non-vanishing amount of excited real particles associated with $\hat{a}$ (and $\hat{b}$ equivalently). This effect goes under the name quantum depletion and occurs physically due to the interactions amongst the particles which necessarily pushes some of them to excited states. Their precise number is given by

$$\langle 0|\hat{a}^\dagger(\delta k)\hat{a}(\delta k)|0\rangle = v^2(\delta k) . \tag{7.39}$$

This allows to rewrite the energy of the quasi particles associated with $\hat{\alpha}$ as

$$e(\delta k) = \begin{cases} 8P_0 k_0\, \delta k & \text{for } \delta k > 0 \\ 0 & \text{for } \delta k \leq 0 \end{cases} \tag{7.40}$$

and the number of depleted real particles with momentum $k_0 + \delta k$ as

$$v^2(\delta k) = \frac{1}{2} \left(\frac{k_0^2 + \delta k^2}{2k_0 |\delta k|} - 1 \right) . \tag{7.41}$$

The above results can easily be generalized to a derivatively coupled theory with an arbitrary number of higher order terms

$$H = \sum_{r=1}^{r_{\max}} c_r \int_0^V \mathrm{d}x \; :(\partial_x \Psi^\dagger \partial_x \Psi)^r: \; . \tag{7.42}$$

Note that the coefficients c_r have dimension [energy][length]$^{3\mathrm{r}-1}$. The standard kinetic term corresponds to $r = 1$ for which the coefficient is $c_1 = \hbar^2/(2m)$. The energy of the quasi particles and the number of depleted particles are given by (7.40) and (7.41) where P_0 now is given by the generalized expression

$$P_0 = \sum_{r=1}^{r_{\max}} c_r \, \frac{r^2}{2} \left(\frac{N}{V}\right)^{r-1} \left(k_0^2\right)^{r-1} \ , \tag{7.43}$$

and k_0 is determined as a minimum of the generalized version of (7.17)

$$\frac{H^{(0)}}{V} = \sum_{r=1}^{r_{\max}} c_r \, \left(k^2\right)^r \left(\frac{N}{V}\right)^r \ . \tag{7.44}$$

The coefficients c_r have again to be chosen such that there is a non trivial minimum.

Discussion

Our results incorporate the vanishing of the energy gap for $\delta k < 0$. This (at least partly) vanishing energy gap can be considered as an indication for the occurrence of a quantum phase transition, as we discussed in section 7.1.3. Moreover, we see that the Bogoliubov modes become highly occupied for $\delta k \gg k_0$. This in fact signals a breakdown of the Bogoliubov theory anyways, as two succeeding terms in the quantum perturbation theory compare as

$$N_0 \left(k_0 + \delta k\right)^2 k_0^2 \delta N \sim N_0^{1/2} \left(k_0 + \delta k\right)^3 k_0 \delta N^{3/2} \ , \tag{7.45}$$

where δN denotes the number of excited particles in the momentum state $k_0 + \delta k$. Equation (7.45) clearly shows that the number of excited particles should at least be suppressed as $\delta N \sim N_0 k_0^2/\delta k^2$. The result for the number of depleted particles (7.41) is, however, completely the opposite, as it is not suppressed but enhanced for large δk. Therefore, we can safely conclude that the perturbative approximation has broken down anyways. Again, this is in accordance with the expectation of being at the quantum critical point because at this point the system behaves purely quantum and cannot even approximately be described classically. Therefore, the breakdown of the Bogoliubov theory was expected, since it amounts to calculate the perturbative quantum corrections around a classical ground state.

Note that the breakdown is also intuitive from the viewpoint of a vanishing energy gap for the quasi particles with $\delta k < 0$. Of course, neither $\hat{a}$ or $\hat{b}$ particles can directly be related with the direction of $\hat{\alpha}$ or $\hat{\beta}$ particles in phase space. But the vanishing of the energy gap should somehow be transferred into the sector of physical $\hat{a}$ and $\hat{b}$ particles. Since a vanishing energy gap means that it is indefinitely easy to excite the quasi particles, we seem to recover this behavior in the high momentum sector of $\hat{a}$ and $\hat{b}$ particles.

We can also perform the Bogoliubov approximation around the global minimum of (7.17) at $k = 0$. Due to the derivatively coupled nature of the interaction terms, the higher order terms in (7.14) do not contribute, which in turn implies that the Hamiltonian (7.21) is already diagonal. Therefore, there is no depletion of the vacuum which allows us to further extend the GR analogy: This state would simply correspond to the Minkowski vacuum in the case of GR.

7.1.5 Future Prospects

Contrary to model (7.5), where the critical point is actually reached and crossed by sufficiently increasing the interaction strength U, in our model there is some indication that the system stays at the point of quantum phase transition and does not organize itself in a new classical ground state. However, this indication is only inferred from the observation of the breakdown of the Bogoliubov theory. To get some solid measures, we need to go beyond the Bogoliubov approximation in the next step [7]. This can be achieved by a full quantum mechanical treatment of the theory (7.14). The diagonalization of the Hamiltonian can be performed under the assumption that only the lowest l momentum eigenstates are significantly occupied (given that we are supposed to sit in a local minimum, this seems to be a good assumption). Therefore, it suffices to diagonalize the Hamiltonian within a Hilbert subspace containing only a finite number of states describing N bosons occupying l different momentum eigenstates. For l chosen appropriately small the calculation is numerically feasible and has been performed in the case of the non-derivativly coupled model in [97]. By means of this calculation we would be able to address quantitative questions, such as the size of the energy gap, the number and spectrum of depleted particles or the amount of quantum entanglement in the system.

The generalization of our results to a relativistic classicalon theory offers another promising prospect of future research. This necessitates to apply the ideas of the Bogoliubov approach to a relativistic theory and would be a significant step towards a more quantitative treatment of the black hole condensate in general relativity.

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8 Appendix

8.1 Derivation of the evolution equations in the scalar sector

To start with, consider the part of the momentum conservation equation $\delta\nabla^\mu T_{\mu i} = 0$ that is built up from a derivative ∂_j of a scalar variable:

$$\partial_j \left[\delta p + \partial_0 \left(\left(\bar{\rho} + \bar{p}\right) \delta u\right) + 3H \left(\bar{\rho} + \bar{p}\right) \delta u + \frac{1}{2} \left(\bar{\rho} + \bar{p}\right) E\right] = 0 \tag{8.1}$$

We will specialize to an equation of state of the simple form $\delta p = \frac{\partial p}{\partial \rho}\delta\rho$. By doing so, we restrict ourselves to the case of a one-component system. The more complicated case of multi-component systems can be investigated, but one needs further special information about the system (for example the separate energy-momentum conservation of each component if they do not interchange energy and momentum). Further, using the Friedmann equations, one easily shows that $8\pi G \left(\bar{\rho} + \bar{p}\right) = -2\dot{H}$. The fluctuation δu can be expressed in terms of metric variables using the $i0$-equations of (6.46), where one again extracts the contributions built from a derivative of scalar variables,

$$8\pi G \left(\bar{\rho} + \bar{p}\right) \partial_j \delta u = \partial_j \left[-HE + \dot{A} - m^2 (aF)\right]. \tag{8.2}$$

Using this in Eq. (8.1), together with equation (6.55), one derives

$$\partial_j \left[8\pi G \frac{\partial p}{\partial \rho}\delta\rho - H\dot{E} - \left(3H^2 + 2\dot{H}\right) E + m^2 E + \ddot{A} + 3H\dot{A} + 2m^2 A\right] = 0. \tag{8.3}$$

Since the spatial divergence of the bracket in (8.3) vanishes identically, we know that the expression in the bracket is equal to some function of time alone. As we know from the basic equation (6.46) that $h_{\mu\nu} = 0$, $T_{\mu\nu} = 0$ (which corresponds to $A = 0$, $B = 0$, $E = 0$, $\delta\rho = 0$, etc.) must be a solution, this function of time must be identically zero. Hence, we obtain

$$8\pi G \frac{\partial p}{\partial \rho}\delta\rho - H\dot{E} - \left(3H^2 + 2\dot{H}\right) E + m^2 E + \ddot{A} + 3H\dot{A} + 2m^2 A = 0. \tag{8.4}$$

Next, we will consider the ij-equations of (6.46) from which we extract the part of the form $\partial_i\partial_j S$ with S a scalar. This gives

$$\partial_i\partial_j \left[E + A - a^2\ddot{B} - 3a\dot{a}\dot{B} - 2m^2 a^2 B + 2a\dot{F} + 4\dot{a}F\right] = 0. \tag{8.5}$$

Using (6.55) we can reexpress

$$\partial_j \left(2a\dot{F} + 4\dot{a}F\right) = \partial_j \left(2\dot{(aF)} + 2\dot{a}F\right) = \, = \partial_j \left(-4\dot{a}F - 2E - 4A\right). \tag{8.6}$$

Inserting this in (8.5) and taking the trace of the result gives

$$-\ddot{\tilde{B}} - 3H\dot{\tilde{B}} - 2m^2 B - 4H\frac{\Delta}{a}F - \frac{\Delta}{a^2}E - 3\frac{\Delta}{a^2}A = 0. \tag{8.7}$$

Finally, using (6.54) we obtain,

$$-\ddot{\tilde{B}} - 7H\dot{\tilde{B}} - 4H^2\tilde{B} - 2m^2 B - 12H\dot{A} - 3\frac{\Delta}{a^2}A - 12H^2 A - \frac{\Delta}{a^2}E + 12H^2 E = 0. \tag{8.8}$$

This equation is the first of the two basic evolution equations in the scalar sector, see (6.65).

The 00-equation of (6.46) gives

$$\begin{aligned}
-4\pi G\left(1 + 3\frac{\partial p}{\partial \rho}\right)\delta\rho = \; & -\tfrac{3}{2}H\dot{E} - \tfrac{\Delta}{2a^2}E - 3\left(H^2 + \dot{H}\right)E + \tfrac{3}{2}m^2 E + \\
& + \tfrac{3}{2}\ddot{A} + 3H\dot{A} + \tfrac{3}{2}m^2 A + \\
& + \tfrac{1}{2}\ddot{\tilde{B}} + H\dot{\tilde{B}} + \tfrac{1}{2}m^2\tilde{B} \\
& - \tfrac{1}{a^2}\dot{(a\Delta F)}.
\end{aligned} \tag{8.9}$$

Using (6.54) one can eliminate F from (8.9),

$$\begin{aligned}
-4\pi G\left(1 + 3\frac{\partial p}{\partial \rho}\right)\delta\rho = \; & +\tfrac{3}{2}H\dot{E} - \tfrac{\Delta}{2a^2}E + 3H^2 E + \tfrac{3}{2}m^2 E \\
& - \tfrac{3}{2}\ddot{A} - 6H\dot{A} - 3\dot{H}A - 6H^2 A + \tfrac{3}{2}m^2 A \\
& - \tfrac{1}{2}\ddot{\tilde{B}} - 2H\dot{\tilde{B}} - \dot{H}\tilde{B} - 2H^2\tilde{B} + \tfrac{1}{2}m^2\tilde{B}.
\end{aligned} \tag{8.10}$$

The jk-equations proportional to δ_{jk} give

$$\begin{aligned}
-4\pi G\left(1 - \frac{\partial p}{\partial \rho}\right)\delta\rho = \; & \tfrac{1}{2}H\dot{E} + \left(3H^2 + \dot{H}\right)E - \tfrac{1}{4}m^2 E \\
& - \tfrac{1}{2}\ddot{A} + \tfrac{\Delta}{2a^2}A - 3H\dot{A} - \tfrac{5}{4}m^2 A + \\
& - \tfrac{1}{2}H\dot{\tilde{B}} - \tfrac{1}{4}m^2\tilde{B} \\
& + H\tfrac{\Delta}{a}F.
\end{aligned} \tag{8.11}$$

Let us again eliminate F using Eq. (6.46),

$$\begin{aligned}
-4\pi G\left(1 - \frac{\partial p}{\partial \rho}\right)\delta\rho = \; & \tfrac{1}{2}H\dot{E} + \dot{H}E - \tfrac{1}{2}m^2 E \\
& - \tfrac{1}{2}\ddot{A} + \tfrac{\Delta}{2a^2}A + 3H^2 A + -\tfrac{5}{2}m^2 A + \\
& + \tfrac{1}{2}H\dot{\tilde{B}} + H^2\tilde{B} - \tfrac{1}{2}m^2\tilde{B}.
\end{aligned} \tag{8.12}$$

Inserting this expression for $\delta\rho$ into (8.4) results in a second independent evolution equation in the scalar sector (6.64).

Equating (8.10) and (8.12) allows us to eliminate $\delta\rho$

$$\begin{aligned}\frac{1}{1-\frac{\partial p}{\partial\rho}}\Big[\dot{H}E+\frac{1}{2}H\dot{E}-\frac{1}{2}m^2E-\frac{1}{2}\ddot{A}+\frac{\Delta}{2a^2}A+3H^2A-\frac{5}{2}m^2A+\frac{1}{2}H\dot{\tilde{B}}+H^2\tilde{B}+\\ -\frac{1}{2}m^2\tilde{B}\Big]=\frac{1}{1+3\frac{\partial p}{\partial\rho}}\Big[\frac{3}{2}H\dot{E}-3H^2E+\frac{3}{2}m^2E-\frac{3}{2}\ddot{A}+\frac{3\Delta}{2a^2}A-3\dot{H}A+\frac{3}{2}m^2A\\ +\frac{3}{2}H\dot{\tilde{B}}-\dot{H}\tilde{B}+\frac{3}{2}m^2\tilde{B}\Big]\end{aligned} \quad (8.13)$$

Here, in addition we have used Eq. (8.8) to eliminate $\ddot{B}$.

Our ultimate aim is to express E in terms of A and B. In the first place, one might think that Eq. (8.8) does the job for every mode $\vec{k}_{phys}$, but the problem with this equation is that the resulting expression for E would contain $\ddot{\tilde{B}}$, so that whenever $\dot{E}$ appears one would get three time derivates on $\tilde{B}$. This is something we should, if possible, try to avoid for the sake of tractability, and indeed, this is possible. One way (among others) is first to derive an additional equation in A, B, and E by just using the constraints (6.54) and (6.55):

$$\begin{aligned}\left(\tfrac{\Delta}{a}F\right)^{\dot{}} &= \left(-3HE+3HA+H\tilde{B}+3\dot{A}+\dot{\tilde{B}}\right)^{\dot{}}=\\ =\left(\tfrac{1}{a^2}\Delta(aF)\right)^{\dot{}} &= \frac{1}{a^2}\Delta(a\dot{F})-2H\frac{\Delta}{a}F=-5H\frac{\Delta}{a}F-\frac{\Delta}{a^2}E-2\frac{\Delta}{a^2}A=\\ &=15H^2E-15H^2A-5H^2\tilde{B}-15H\dot{A}-5H\dot{\tilde{B}}-\frac{\Delta}{a^2}E+\\ &\quad -2\frac{\Delta}{a^2}A.\end{aligned} \quad (8.14)$$

Using in addition Eq. (8.8) to eliminate $\ddot{B}$ this can be cast into the form

$$\begin{aligned}3\ddot{A}+6H\dot{A}-\tfrac{\Delta}{a^2}A+3\dot{H}A+3H^2A-3H\dot{E}-3\dot{H}E+\\ -H\dot{\tilde{B}}+\dot{H}\tilde{B}-2m^2\tilde{B}+H^2\tilde{B}-3H^2E=0.\end{aligned} \quad (8.15)$$

As it happens, the ratio of the coefficients in front of $\dot{E}$ and $\ddot{A}$ coincides for equations (8.13) and (8.15). Therefore, by appropriately adding both equations, one eliminates $\dot{E}$ and $\ddot{A}$ at once, leaving an equation, which can be solved explicitly for E in terms of A and B and their first derivatives. This equation is given by Eq. (6.66).

8.2 The evolution equations for $\mathbf{k} = \mathbf{0}$

In the case $\vec{k}_{phys} = 0$ the equations of motion (6.64, 6.65, 6.66) reduce to

$$E = \frac{\left[\dot{H} + H^2\left(1+6w\right) - \frac{m^2}{2}\left(2+3w\right)\right]\left(A + \frac{1}{3}\tilde{B}\right) + H\left(\dot{A} + \frac{1}{3}\dot{\tilde{B}}\right)\left(-1+3w\right)}{-\dot{H} - H^2\left(2-3w\right) + \frac{m^2}{2}} \tag{8.16}$$

$$12H^2E - 12H^2\left(A + \frac{1}{3}\tilde{B}\right) - 12H\left(\dot{A} + \frac{1}{3}\dot{\tilde{B}}\right) - \ddot{\tilde{B}} - 3H\dot{\tilde{B}} - m^2\tilde{B} = 0 \tag{8.17}$$

$$\begin{aligned}\left[H^2(4-6w) + m^2\left(1 + \frac{3}{2}w\right)\right]\left(A + \frac{1}{3}\tilde{B}\right) + H(7-3w)\left(\dot{A} + \frac{1}{3}\dot{\tilde{B}}\right) + \\ + \left(H^2(-7+3w) - 2\dot{H} + \frac{m^2}{2}\right)E - H\dot{E} + \left(\ddot{A} + \frac{1}{3}\ddot{\tilde{B}}\right) = 0\end{aligned} \tag{8.18}$$

i.e. the equation of motion for $S \equiv A + \frac{1}{3}\tilde{B}$ (8.18) decouples, which we will abbreviate by

$$C_2(t)\ddot{S} + C_1(t)\dot{S} + C_0(t)S = 0 \tag{8.19}$$

with $C_0(t), C_1(t)$ and $C_2(t)$ given by

$$
\begin{aligned}
C_0 &= \left[H^2(4-6w)+m^2\left(1+\frac{3}{2}w\right)\right]+ \\
&+\left[H^2(-7+3w)-2\dot{H}+\frac{m^2}{2}\right]\cdot \\
&\cdot\left[\dot{H}+H^2\left(1+6w\right)-\frac{m^2}{2}\left(2+3w\right)\right]\left[-\dot{H}-H^2(2-3w)+\frac{m^2}{2}\right]^{-1} \\
&-H\left\{\left[-\dot{H}-H^2(2-3w)+\frac{m^2}{2}\right]^{-1}\left[\ddot{H}+2\dot{H}H(1+6w)\right]-\right. \\
&\left.\left[-\ddot{H}-2\dot{H}H(2-3w)\right]\left[\dot{H}+H^2(1+6w)-\frac{m^2}{2}(2+3w)\right]\right\} \qquad (8.20)\\
C_1 &= (7-3w)H+ \\
&+H\left(H^2(-7+3w)-2\dot{H}+\frac{m^2}{2}\right)(-1+3w)\,[-\dot{H}-H^2\left(2-3w\right)+\frac{m^2}{2}]^{-1} \\
&-H\left\{\dot{H}+H^2\left(1+6w\right)-\frac{m^2}{2}\left(2+3w\right)\right\}\left[-\dot{H}-H^2\left(2-3w\right)+\frac{m^2}{2}\right]^{-1} \\
&-H\left\{\dot{H}\left(-1+3w\right)(-\dot{H}-H^2\left(2-3w\right)+\frac{m^2}{2})^{-1}\right. \\
&\qquad\left.-H\left(-1+3w\right)(-\ddot{H}-2\dot{H}H\left(2-3w\right))\right\} \qquad (8.21)\\
C_2 &= 1-\frac{H^2\left(-1+3w\right)}{-\dot{H}-H^2\left(2-3w\right)+\frac{m^2}{2}}=\frac{-\dot{H}-H^2+\frac{m^2}{2}}{-\dot{H}-H^2\left(2-3w\right)+\frac{m^2}{2}} \qquad (8.22)
\end{aligned}
$$

8.3 BIG Full Hamiltonian

Here the full Hamiltonian is presented, which is neither gauge-fixed nor simplified by applying any constraints. It directly follows from substituting the velocities in the linearized version of (6.97) with (6.109)–(6.114) and calculating the Hamiltonian.

$$
\begin{aligned}
\frac{\mathcal{H}}{M_6^4} &= \frac{1}{M_6^8}\Pi_{(R)ij}\Pi^{ij}_{(R)} + \frac{1}{M_4^2 M_6^4}\,\delta_y^{(2)}\,\Pi_{(I)ij}\Pi^{ij}_{(I)} - \frac{1}{4M_6^8}(\Pi_{(R)}{}^{i}{}_{i})^2 - \frac{1}{2M_6^4 M_4^2}\,\delta_y^{(2)}\,(\Pi_{(I)}{}^{i}{}_{i})^2 \\
&-\frac{1}{4M_6^8}\Pi_{(R)}{}^{i}{}_{i}(\Pi_N+\hat{\Pi}_L)-\frac{1}{8M_6^8}\Pi_N\hat{\Pi}_L+\frac{3}{16M_6^8}(\Pi_N^2+\hat{\Pi}_L^2)+\frac{1}{2M_6^8}\Pi_5^2+\frac{1}{2M_6^8}(\Pi_i\Pi^i+\hat{\Pi}_i\hat{\Pi}^i)+\frac{2}{M_6^4}\Pi_{ij}n^{j,i} \\
&+\hat{\Pi}_i(n^{i,6}+L_0^{,i})+\Pi_5(L_{0,5}+N_{0,6})-n_{,a}h_i^{i,a}-L_{,5}h^i_{i,5}-N_{,6}h^i_{i,6}-2L_{,i}N^{,i}-2L_{,5}n_{,5}-2N_{,6}n_{,6}-2(N+L)_{,i}n^{,i} \\
&+2N_{0,6}L_{0,5}+2n^i_{,5}N_{0,i}+2n^i_{,6}L_{0,i}+(h^i_i+2n+2L)_{,5}N^i_{,i}+(h^i_i+2n+2N)_{,6}L^i_{,i}-L_{5,i}L^i_{,5}-L_{5,i}N^i_{,6}+\frac{1}{2}L_{5,i}L_5^{;i} \\
&+(2n+h^i_i)_{,6}L_{5,5}-N^i_{,6}L_{i,5}+\frac{1}{4}F_{(N)ij}F^{ij}_{(N)}+\frac{1}{4}F_{(L)ij}F^{ij}_{(L)}+\frac{1}{2}L_{i,5}L^i_{,5}+\frac{1}{2}N_{i,6}N^i_{,6}-h_{ij,5}N^{i,j}-h_{ij,6}L^{i,j} \\
&+\frac{1}{4}h_{ij,a}h^{ij,a}-\frac{1}{4}h^i_{i,a}h_j^{j,a}-(1+\frac{M_4^2}{M_6^4}\delta_y^{(2)})(\delta^1\sqrt{-\gamma}\,\delta^1R^{(3)}+\delta^2R^{(3)})-(N+L+n)\delta^1R^{(3)}-\frac{M_4^2}{M_6^4}\delta_y^{(2)}n\,\delta^1R^{(3)}
\end{aligned}
\tag{8.23}
$$

Where the following definitions are used:

$$
\begin{aligned}
\delta^1\sqrt{-\gamma}\,\delta^1R^{(3)} + \delta^2R^{(3)} &= -\frac{1}{2}h^{ij}_{,i}h^k_{k,j} + \frac{1}{2}h^{jk}_{,j}h^i_{k,i} && (8.24)\\
&\quad -\frac{1}{4}h_{jk,i}h^{jk,i} + \frac{1}{4}h^j_{j,i}h^{k,i}_k\,, && (8.25)\\
\delta^1R^{(3)} &= h^{ij}_{,ij} - h^{k,i}_{k,i}\,, && (8.26)
\end{aligned}
$$

and

$$
F^{ij}_{(N)} = N^{j,i} - N^{i,j} \qquad\qquad F^{ij}_{(L)} = L^{j,i} - L^{i,j}\,. \tag{8.27}
$$

This Hamiltonian is used to calculate the secondary constraints. On the constraint-surface, it reduces to (6.145).

8.4 BIG deconstruction

In this appendix the details of the graviton deconstruction (6.161) based on the background space–time isomorphism $(\mathcal{M}_d, \eta) \cong (\mathcal{M}_4, \eta) \times (\mathbb{R}^n, \delta)$ are presented. The deconstruction (6.148) in the case of four–dimensional GR is the trivial case when the parent space time equals $(\mathcal{M}_4, \eta)$. The global Minkowski coordinate system is split accordingly into the Cartesian product $X^A = (x^\alpha, y^a)$ with obvious index ranges. The required projectors are

$$
\begin{aligned}
P^{(\parallel)}_{\ \alpha\beta} &:= \partial_\alpha(x) \int_{\mathcal{M}_4} \mathrm{d}^4\tilde{x}\; G(x-\tilde{x})\; \partial_\beta(\tilde{x}) \;, \\
P^{(\perp)}_{\ \alpha\beta} &:= \eta_{\alpha\beta} \int_{\mathcal{M}_4} \mathrm{d}^4\tilde{x}\; \delta^{(4)}(x-\tilde{x}) - P^{(\parallel)}_{\ \alpha\beta} \;, \\
p^{(\parallel)}_{\ ab} &:= \partial_a(y) \int_{\mathbb{R}^n} \mathrm{d}^n\tilde{y}\; g(y-\tilde{y})\; \partial_b(\tilde{y}) \;, \\
p^{(\perp)}_{\ ab} &:= \delta_{ab} \int_{\mathbb{R}^n} \mathrm{d}^n\tilde{y}\; \delta^{(n)}(y-\tilde{y}) - p^{(\parallel)}_{\ ab} \;,
\end{aligned}
\tag{8.28}
$$

where $\partial_A(X) \equiv \partial/\partial X^A$, and G and g are Green's functions corresponding to the four–dimensional wave and n–dimensional Poisson equation, respectively, equipped with arbitrary but fixed boundary conditions. The decomposition of the graviton of course works for every choice. The transverse and traceless projectors are given by the composite operators:

$$
\begin{aligned}
\mathcal{O}^{(4,\mathrm{tt})\ \mu\nu}_{\ \alpha\beta} &:= P^{(\perp)\gamma}_{\ \alpha}\, P^{(\perp)\delta}_{\ \beta} - P^{(\perp)}_{\ \alpha\beta}\, P^{(\perp)\gamma\delta}/3 \;, \\
\mathcal{O}^{(n,\mathrm{tt})\ cd}_{\ ab} &:= p^{(\perp)c}_{\ a}\, p^{(\perp)d}_{\ b} - p^{(\perp)}_{\ ab}\, p^{(\perp)cd}/(n-1) \;.
\end{aligned}
\tag{8.29}
$$

The field content of the graviton in the $(\alpha\beta)$–sector is given by

$$
\begin{aligned}
h_{\alpha\beta} &= D^{(\mathrm{tt})}_{\ \alpha\beta} + \partial_{(\alpha} V^{(\perp)}_{\ \beta)} + P^{(\parallel)}_{\ \alpha\beta}\, B + \eta_{\alpha\beta}\, S \;, \\
\\
D^{(\mathrm{tt})}_{\ \alpha\beta} &= \mathcal{O}^{(4,\mathrm{tt})\ \gamma\delta}_{\ \alpha\beta}\, h_{\gamma\delta} \;, \\
\partial_\alpha V^{(\perp)}_{\ \beta} &= P^{(\parallel)\gamma}_{\ \alpha}\, P^{(\perp)\delta}_{\ \beta}\, h_{\gamma\delta} \;, \\
B &= \left(P^{(\parallel)\gamma\delta} - P^{(\perp)\gamma\delta}/3\right) h_{\gamma\delta} \;, \\
S &= P^{(\perp)\gamma\delta}\, h_{\gamma\delta}/3 \;.
\end{aligned}
\tag{8.30}
$$

In the (ab)–sector, the graviton's field content is consequently given by

$$
\begin{aligned}
h_{ab} &= d^{(\mathrm{tt})}_{\ ab} + \partial_{(a} v^{(\perp)}_{\ b)} + p^{(\parallel)}_{\ ab}\, b + \delta_{ab}\, s \;, \\
\\
d^{(\mathrm{tt})}_{\ ab} &= \mathcal{O}^{(n,\mathrm{tt})\ cd}_{\ ab}\, h_{cd} \;, \\
\partial_a v^{(\perp)}_{\ b} &= p^{(\parallel)c}_{\ a}\, p^{(\perp)d}_{\ b}\, h_{cd} \;, \\
b &= \left(p^{(\parallel)cd} - p^{(\perp)cd}/(n-1)\right) h_{cd} \;, \\
s &= p^{(\perp)cd}\, h_{cd}/(n-1) \;.
\end{aligned}
\tag{8.31}
$$

Finally, the field content in the (αb)–sector is given by

$$
\begin{aligned}
h_{\alpha b} &= G^{(\mathrm{v,v})}_{\ \ \alpha b} + \partial_b G^{(\mathrm{v,s})}_{\ \ \alpha} + \partial_\alpha F^{(\mathrm{s,v})}_{\ \ b} + \partial_\alpha \partial_b F^{(\mathrm{ss})} \ ,
\end{aligned}
$$

$$
\begin{aligned}
G^{(\mathrm{v,v})}_{\ \ \alpha b} &= P^{(\perp)\gamma}_{\ \ \alpha} p^{(\perp)c}_{\ \ b}\, h_{\gamma c} \ , \\
\partial_b G^{(\mathrm{v,s})}_{\ \ \alpha} &= p^{(\parallel)c}_{\ \ b}\, P^{(\perp)\gamma}_{\ \ \alpha}\, h_{\gamma c} \ , \\
\partial_\alpha F^{(\mathrm{s,v})}_{\ \ b} &= P^{(\parallel)\gamma}_{\ \ \alpha}\, p^{(\perp)c}_{\ \ b}\, h_{\gamma c} \ , \\
\partial_\alpha \partial_b F^{(\mathrm{ss})} &= P^{(\parallel)\gamma}_{\ \ \alpha}\, p^{(\parallel)c}_{\ \ b}\, h_{\gamma c} \ .
\end{aligned} \tag{8.32}
$$

Introducing a new coordinate system via the following infinitesimal transformations

$$
\begin{aligned}
\delta x^\alpha &= \partial^\alpha(x) \left(\int \mathrm{d}^n \tilde{y}\ g(y-\tilde{y}) b\,(x,\tilde{y})\,/2 - F^{(\mathrm{ss})}(x,y) \right) \\
&\quad - V^{(\perp)\alpha}(x,y) \ , \\
\delta y^a &= \int \mathrm{d}^n \tilde{y}\ g(y-\tilde{y}) \partial^a(\tilde{y}) b(x,\tilde{y}) - v^{(\perp)a}(x,y) \ ,
\end{aligned} \tag{8.33}
$$

results in the gauge fixed graviton deconstruction (6.161).

Bibliography

[1] F. Berkhahn, D. D. Dietrich and S. Hofmann, JCAP **1011**, 018 (2010) [arXiv:1008.0644 [hep-th]].

[2] F. Berkhahn, D. D. Dietrich and S. Hofmann, Phys. Rev. Lett. **106**, 191102 (2011) [arXiv:1102.0313 [hep-th]].

[3] F. Berkhahn, D. D. Dietrich and S. Hofmann, JCAP **1109**, 024 (2011) [arXiv:1104.2534 [hep-th]].

[4] F. Berkhahn, S. Hofmann, F. Kuhnel, P. Moyassari and D. Dietrich, Phys. Rev. Lett. **108**, 131102 (2012) [arXiv:1106.3566 [hep-th]].

[5] F. Berkhahn, S. Hofmann and F. Niedermann, Phys. Rev. D **86** (2012) 124022 [arXiv:1205.6801 [hep-th]].

[6] F. Berkhahn, L.. Eglseer, F. Niedermann, "Exorcising the ghost of Brane Induced Gravity in arbitrary co-dimensions", *to appear soon*

[7] F. Berkhahn, S.. Hofmann, F. Niedermann and R. Schneider. "Cosmological solutions of Brane Induced Gravity in higher co-dimensions", *to appear soon*

[8] F. Berkhahn, F. Niedermann and R. Schneider. "Microscopic picture of non-relativistic classicalons", arXiv:1302.6581 [hep-th]

[9] F. Berkhahn, S. Müller, F. Niedermann and R. Schneider. "Non linear analysis of non-relativistic classicalons", *in preparation*

[10] C. P. Burgess and D. London, Phys. Rev. D **48** (1993) 4337 [hep-ph/9203216].

[11] C. P. Burgess and L. van Nierop, arXiv:1108.0345 [hep-th].

[12] C. P. Burgess, arXiv:0912.4295 [gr-qc].

[13] C. P. Burgess, Ann. Rev. Nucl. Part. Sci. **57** (2007) 329 [hep-th/0701053].

[14] C. de Rham and G. Gabadadze, Phys. Rev. D **82**, 044020 (2010) [arXiv:1007.0443 [hep-th]].

[15] C. de Rham, G. Gabadadze and A. J. Tolley, Phys. Rev. Lett. **106**, 231101 (2011) [arXiv:1011.1232 [hep-th]].

[16] S. F. Hassan and R. A. Rosen, JHEP **1107** (2011) 009 [arXiv:1103.6055 [hep-th]].

[17] S. F. Hassan and R. A. Rosen, Phys. Rev. Lett. **108** (2012) 041101 [arXiv:1106.3344 [hep-th]].

[18] S. F. Hassan, R. A. Rosen and A. Schmidt-May, JHEP **1202** (2012) 026 [arXiv:1109.3230 [hep-th]].

[19] S. F. Hassan and R. A. Rosen, JHEP **1202** (2012) 126 [arXiv:1109.3515 [hep-th]].

[20] S. F. Hassan and R. A. Rosen, JHEP **1204** (2012) 123 [arXiv:1111.2070 [hep-th]].

[21] S. F. Hassan, A. Schmidt-May and M. von Strauss, Phys. Lett. B **715** (2012) 335 [arXiv:1203.5283 [hep-th]].

[22] G. D'Amico, C. de Rham, S. Dubovsky, G. Gabadadze, D. Pirtskhalava and A. J. Tolley, Phys. Rev. D **84** (2011) 124046 [arXiv:1108.5231 [hep-th]].

[23] C. de Rham, G. Gabadadze, L. Heisenberg and D. Pirtskhalava, Phys. Rev. D **83** (2011) 103516 [arXiv:1010.1780 [hep-th]].

[24] B. Zumino, "Effective Lagrangians and broken symmetries," in Brandeis Univ. Lectures on Elementary Particles and Quantum Field Theory (MIT Press Cambridge, Mass.), Vol. 2, 1970, 437.

[25] Y. Akrami, T. S. Koivisto and M. Sandstad, arXiv:1209.0457 [astro-ph.CO].

[26] O. Corradini, A. Iglesias, Z. Kakushadze and P. Langfelder, Phys. Lett. B **521** (2001) 96 [hep-th/0108055].

[27] D. G. Boulware and S. Deser, Phys. Rev. D **6** (1972) 3368.

[28] T. Damour and I. I. Kogan, "Effective Lagrangians and universality classes of nonlinear bigravity," Phys. Rev. D **66** (2002) 104024 [arXiv:hep-th/0206042].

[29] M. Fierz and W. Pauli, "On relativistic wave equations for particles of arbitrary spin in an electromagnetic field," Proc. Roy. Soc. Lond. A **173** (1939) 211.

[30] N. Arkani-Hamed, H. Georgi and M. D. Schwartz, "Effective field theory for massive gravitons and gravity in theory space," Annals Phys. **305** (2003) 96 [arXiv:hep-th/0210184].

[31] H. van Dam and M. J. G. Veltman, "Massive And Massless Yang-Mills And Gravitational Fields," Nucl. Phys. B **22** (1970) 397.

[32] V. I. Zakharov, "Linearized gravitation theory and the graviton mass," JETP Lett. **12** (1970) 312 [Pisma Zh. Eksp. Teor. Fiz. **12** (1970) 447].

[33] A. I. Vainshtein, "To the problem of nonvanishing gravitation mass," Phys. Lett. B **39** (1972) 393.

[34] A. Higuchi, "Forbidden mass range for spin-2 field theory in de sitter space-time," Nucl. Phys. B **282** (1987) 397.

[35] S. Deser and A. Waldron, "Stability of massive cosmological gravitons," Phys. Lett. B **508** (2001) 347 [arXiv:hep-th/0103255].

[36] L. Grisa and L. Sorbo, "Pauli-Fierz Gravitons on Friedmann-Robertson-Walker Background," Phys. Lett. B **686** (2010) 273 [arXiv:0905.3391 [hep-th]].

[37] S. Weinberg, "The Quantum theory of fields. Vol. 1: Foundations," *Cambridge, UK: Univ. Pr. (1995) 609 p*

[38] T. Damour, I. I. Kogan and A. Papazoglou, Phys. Rev. D **67** (2003) 064009 [arXiv:hep-th/0212155].

[39] E. Babichev, C. Deffayet and R. Ziour, JHEP **0905** (2009) 098 [arXiv:0901.0393 [hep-th]].

[40] E. Babichev, C. Deffayet and R. Ziour, Phys. Rev. Lett. **103** (2009) 201102 [arXiv:0907.4103 [gr-qc]].

[41] E. Babichev, C. Deffayet and R. Ziour, arXiv:1007.4506 [gr-qc].

[42] L. Alberte, A. H. Chamseddine and V. Mukhanov, arXiv:1008.5132 [hep-th].

[43] G. Dvali, S. Hofmann and J. Khoury, Phys. Rev. D **76** (2007) 084006 [arXiv:hep-th/0703027].

[44] C. Aragone and S. Deser, Nuovo Cim. A **3** (1971) 709.

[45] P. Creminelli, A. Nicolis, M. Papucci and E. Trincherini, JHEP **0509** (2005) 003 [arXiv:hep-th/0505147].

[46] M. Porrati, Phys. Lett. B **498** (2001) 92 [arXiv:hep-th/0011152].

[47] I. I. Kogan, S. Mouslopoulos and A. Papazoglou, Phys. Lett. B **503** (2001) 173 [arXiv:hep-th/0011138].

[48] R. Wald, "General Relativity," *University of Chicago Press (1984) 491 p*

[49] D. Blas, D. Comelli, F. Nesti and L. Pilo, Phys. Rev. D **80** (2009) 044025 [arXiv:0905.1699 [hep-th]].

[50] G. Gabadadze, Phys. Lett. B **681** (2009) 89 [arXiv:0908.1112 [hep-th]].

[51] G. 't Hooft, arXiv:0708.3184 [hep-th].

[52] Z. Kakushadze, Phys. Rev. D **77** (2008) 024001 [arXiv:0710.1061 [hep-th]].

[53] Z. Kakushadze, Int. J. Mod. Phys. A **23** (2008) 1581 [arXiv:0709.1673 [hep-th]].

[54] A. Iglesias and Z. Kakushadze, arXiv:1007.2385 [hep-th].

[55] I. Oda, Mod. Phys. Lett. A **25** (2010) 2411 [arXiv:1003.1437 [hep-th]].

[56] A. H. Chamseddine and V. Mukhanov, JHEP **1008** (2010) 011 [arXiv:1002.3877 [hep-th]].

[57] M. Maeno and I. Oda, Mod. Phys. Lett. B **22** (2009) 3025 [arXiv:0801.0827 [hep-th]].

[58] C. Aragone and S. Deser, Nuovo Cim. B **57** (1980) 33.

[59] S. Hofmann, S. Müller, "Goldstone boson equivalence theorem for massive gravitons", *to be published*

[60] J. C. Fabris, J. Martin, "Amplification of density perturbations in fluids with negative pressure," Phys. Rev. **D55** (1997) 5205-5207.

[61] R. R. Caldwell, R. Dave and P. J. Steinhardt, "Cosmological imprint of an energy component with general equation of state," Phys. Rev. Lett. **80** (1998) 1582 [arXiv:astro-ph/9708069].

[62] G. Dvali, C. Gomez and A. Kehagias, "Classicalization of Gravitons and Goldstones," arXiv:1103.5963 [hep-th].

[63] G. Dvali, "Classicalize or not to Classicalize?," arXiv:1101.2661 [hep-th].

[64] G. Dvali and D. Pirtskhalava, "Dynamics of Unitarization by Classicalization," arXiv:1011.0114 [hep-ph].

[65] G. Dvali, G. F. Giudice, C. Gomez and A. Kehagias, "UV-Completion by Classicalization," arXiv:1010.1415 [hep-ph].

[66] G. Dvali and C. Gomez, "Self-Completeness of Einstein Gravity," arXiv:1005.3497 [hep-th].

[67] G. Dvali, S. Folkerts and C. Germani, Phys. Rev. D **84** (2011) 024039 [arXiv:1006.0984 [hep-th]].

[68] N. Boulanger, T. Damour, L. Gualtieri and M. Henneaux, "No consistent cross-interactions for a collection of massless spin-2 fields," arXiv:hep-th/0009109.

[69] D. Larson *et al.*, Astrophys. J. Suppl. **192**, 16 (2011).

[70] G. Dvali, A. Gruzinov, M. Zaldarriaga, Phys. Rev. **D68**, 024012 (2003). [hep-ph/0212069].

[71] G. Dvali, A. Franca and C. Gomez, arXiv:1204.6388 [hep-th].

[72] G. Dvali, G. F. Giudice, C. Gomez and A. Kehagias, arXiv:1010.1415 [hep-ph].

[73] G. Dvali and D. Pirtskhalava, arXiv:1011.0114 [hep-ph].

[74] G. Dvali, arXiv:1101.2661 [hep-th].

[75] W. Heisenberg, Z. Phys. **133**, 65 (1952).

[76] G. R. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B **485**, 208 (2000) [hep-th/0005016].

[77] D. M. Capper, "On Quantum Corrections To The Graviton Propagator," Nuovo Cim. A25 (1975) 29.

[78] Diploma thesis of F. Niedermann, "Brane Induced Gravity in Infinite Volume Extra Dimensions", 2011, section 2.2

[79] G. Dvali, G. Gabadadze, X. -r. Hou and E. Sefusatti, Phys. Rev. D **67** (2003) 044019 [hep-th/0111266].

[80] S. L. Dubovsky and V. A. Rubakov, Phys. Rev. D **67** (2003) 104014 [hep-th/0212222].

[81] G. Gabadadze and M. Shifman, Phys. Rev. D **69** (2004) 124032 [hep-th/0312289].

[82] S. F. Hassan, S. Hofmann and M. von Strauss, JCAP **1101** (2011) 020 [arXiv:1007.1263 [hep-th]].

[83] G. R. Dvali and G. Gabadadze, Phys. Rev. D **63**, 065007 (2001) [hep-th/0008054].

[84] K. Schleich, Phys. Rev. D **36** (1987) 2342. [85]

[85] P. O. Mazur and E. Mottola, Nucl. Phys. B **341** (1990) 187.

[86] E. Witten, Commun. Math. Phys. **80** (1981) 381.

[87] R. Schon and S. -T. Yau, Commun. Math. Phys. **65** (1979) 45. [88]

[88] C. Deffayet, Phys. Lett. B **502** (2001) 199 [hep-th/0010186].

[89] P. Binetruy, C. Deffayet, U. Ellwanger and D. Langlois, Phys. Lett. B **477** (2000) 285 [hep-th/9910219].

[90] P. Binetruy, C. Deffayet and D. Langlois, Nucl. Phys. B **565** (2000) 269 [hep-th/9905012].

[91] G. Dvali and C. Gomez, "Black Hole's Quantum N-Portrait," arXiv:1112.3359 [hep-th].

[92] G. Dvali and C. Gomez, "Landau-Ginzburg Limit of Black Hole's Quantum Portrait: Self Similarity and Critical Exponent," Phys. Lett. B **716**, 240 (2012) [arXiv:1203.3372 [hep-th]].

[93] G. Dvali and C. Gomez, "Black Hole's 1/N Hair," arXiv:1203.6575 [hep-th].

[94] G. Dvali and C. Gomez, "Black Holes as Critical Point of Quantum Phase Transition," arXiv:1207.4059 [hep-th].

[95] G. Dvali and C. Gomez, "Black Hole Macro-Quantumness," arXiv:1212.0765 [hep-th].

[96] G. Dvali, S. Folkerts and C. Germani, "Physics of Trans-Planckian Gravity," Phys. Rev. D **84**, 024039 (2011) [arXiv:1006.0984 [hep-th]].

[97] R. Kanamoto, H. Saito and M. Ueda. Quantum Phase Transition in One-Dimensional Bose-Einstein Condensate with Attractive Interaction. Phys.Rev.,A67,013608

[98] L. Qian, M. Wall, S. Zhang and H. Pu. Bose-Einstein condensates on a ring with periodic scattering length: Spontaneous symmetry breaking and entanglement. Phys.Rev.,A77,013611 (2008)

[99] D. Flassig, A. Pritzel and N. Wintergerst, arXiv:1212.3344 [hep-th].