# Pseudoholomorphic Curves in Exact Courant Algebroids

The Mathematics and Physics of Topological Nonlinear Sigma-Models with H-Flux

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*To Marina*

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**Abstract**

In this dissertation we take the first steps towards a generalization of symplectic topology, which includes e.g. Gromov-Witten invariants and Fukaya categories, to generalized complex geometry. In order to facilitate this, we extend the notion of pseudoholomorphic curves to arbitrary almost generalized complex manifolds.

Our ansatz is motivated by instantons in the generalized B-model of topological string theory defined on generalized Calabi-Yau manifolds. It is shown that instantons are not invariant under $B$-transformations as geometric objects, but only modulo canonical transformations acting on the string super phase space $\Pi T^*\mathcal{L}M$ of $M$. To establish an invariant notion, we introduce generalized pseudoholomorphic pairs, or abbreviated $(E, \mathcal{J})$-holomorphic pairs. They consist of a map $\Phi : \Sigma \to M$ and an isotropic embedding $\lambda : TM \to E$. Here $(\Sigma, j_\Sigma)$ is a compact Riemann surface, $(M, \mathcal{J})$ is an almost generalized complex manifold and $(E, q, [\cdot, \cdot], \pi)$ is an exact Courant algebroid over $M$. The almost generalized complex structure $\mathcal{J}$ acts on $E$.

Moreover, the notions of tamed as well as compatible almost generalized complex structures are defined. Exploiting them we will introduce the generalized energy of a pair $(\Phi, \lambda)$ and we will show that $(E, \mathcal{J})$-holomorphic pairs admit an energy identity. Furthermore, the generalized energy is a topological invariant for a suitable choice of an isotropic embedding.

Additionally to that we will establish the local theory of generalized pseudoholomorphic pairs. In particular we will prove an identity theorem for them. The main theorem of the local theory states that a $(E, \mathcal{J})$-holomorphic pair can be locally interpreted as an ordinary pseudoholomorphic curve in a space of doubled dimension, where half of the coordinates of the map are constant.

First results concerning the deformation theory of $(E, \mathcal{J})$-holomorphic pairs are developed. We will give an explicit expression for the vertical differential of a section $\overline{\partial}_{\mathcal{J}}$ in a Banach bundle. In doing so, a map $\Phi$ is $(E, \mathcal{J})$-holomorphic, for a fixed isotropic embedding $\lambda$, precisely if $\overline{\partial}_{\mathcal{J}}(\Phi) = 0$. In order to obtain an explicit expression, torsion as well as a generalized Levi-Civita connection on an exact Courant algebroid are invented. It transpires that the vertical differential is the composition of a real linear Cauchy-Riemann operator, which is Fredholm, and an upper semi-Fredholm operator. Hence, in contrast to the almost complex case, the vertical differential is not a Fredholm operator but semi-Fredholm. To tackle this problem, admitted vector fields along a map $\Phi$ are invented.

Two additional important results are the following. First, for any Hyperkähler manifold we will give an interpolation between the Bogoliubov transformation of the A-model with symplectic structure $\omega_I$ and the B-model with complex structure $J$. This

enables one to view mirror symmetry of Hyperkähler manifolds as a continuous symmetry rather than a discrete one. Second, an intrinsically geometric formulation of non-linear sigma models is derived and the existence of a mode expansion in curved spacetimes is motivated. In order to achieve linearity, the solution space has to get enhanced by non-physical modes which are not associated to a world-sheet map. Physical states are recovered via a cohomological constraint.

**Zusammenfassung**

In dieser Dissertation werden die Grundlagen für eine Verallgemeinerung herkömmlicher symplektischer Topologie hin zu verallgemeinert komplexe Mannigfaltigkeiten ausgearbeitet. Um dies zu ermöglichen, erweitern wir den Begriff der pseudoholomorphen Kurve auf beliebige fast verallgemeinert komplexe Mannigfaltigkeiten.

Unser Ansatz wird durch Instantonen im verallgemeinerten B-Modell topologischer Stringthorie auf verallgemeinerten Calabi-Yau Mannigfaltigkeiten motiviert. Es wird gezeigt, dass Instantonen nicht als geometrische Objekte unter B-Transformationen invariant sind, sondern nur modulo kanonischer Transformationen auf dem String Super Phasenraum $\Pi T^*\mathcal{L}M$ von $M$. Um einen invarianten Begriff zu erlangen, führen wir verallgemeinert pseudoholomorphe Paare, bzw. abgekürzt $(E, \mathcal{J})$-holomorphe Paare ein. Diese bestehen aus einer Abbildung $\Phi : \Sigma \to M$ und einer isotropen Einbettung $\lambda : TM \to E$. Hierbei ist $(\Sigma, j_\Sigma)$ eine Riemannsche Fläche, $(M, \mathcal{J})$ eine fast verallgemeinert komplexe Mannigfaltigkeit und $E$ ein exakter Courant Algebroid über $M$, wobei die fast verallgemeinert komplexe Struktur $\mathcal{J}$ auf $E$ operiert.

Weiterhin werden die Begriffe der zahmen und der kompatiblen fast verallgemeinert komplexen Strukturen eingeführt. Diese ermöglichen es eine verallgemeinerte Energie eines Paares $(\Phi, \lambda)$ zu definieren. Es zeigt sich, dass $(E, \mathcal{J})$-holomorphe Paare eine Energie-Identität erfüllen und die verallgemeinerte Energie nach der Wahl einer geeigneten isotropen Einbettung eine topologische Invariante ist.

Daraufhin wird die lokale Theorie verallgemeinert pseudo-holomorpher Paare entwickelt. Insbesondere wird für diese ein Identitätssatz bewiesen. Der Hauptsatz der lokalen Theorie besagt, dass ein $(E, \mathcal{J})$-holomorphes Paar lokal als eine gewöhnliche pseudo-holomorphe Kurve in einem Raum der doppelten Dimension aufgefasst werden kann, wobei eine Hälfte der Koordinaten der Abbildung konstant ist.

Darüber hinaus werden erste Resultate bezüglich einer Deformationstheorie $(E, \mathcal{J})$-holomorpher Paare hergeleitet. Dazu wird ein expliziter Ausdruck für das vertikale Differenzial eines Schnitts $\overline{\partial}_{\mathcal{J}}$ in einem Banachbündel angegeben. Dabei ist eine Abbildung $\Phi$, für eine feste isotrope Einbettung $\lambda$, genau dann $(E, \mathcal{J})$-holomorph, wenn $\overline{\partial}_{\mathcal{J}}(\Phi) = 0$ gilt. Um das Angeben eines expliziten Ausdrucks für das vertikale Differenzial zu ermöglichen, wird die Torsion eines Zusammenhangs und der verallgemeinerte Levi-Civita Zusammenhang auf einem exakten Courant Algebroid eingeführt. Es stellt sich heraus, dass sich das vertikale Differenzial als Komposition eines reellen Cauchy-Riemann Operators, welcher Fredholm ist, und eines oberen semi-Fredholm Operators darstellen lässt. Im Gegensatz zum komplexen Fall ist das vertikale Differenzial daher kein Fredholm-Operator, sondern nur semi-Fredholm. Um sich einer Lösung dieses Problems zu nähern, wird der Begriff des zugelassenen Vektorfeldes entlang einer Ab-

bildung $\Phi$ definiert.

Zwei weitere wichtige Ergebnisse sind die Folgenden. Erstens wird für eine beliebige Hyperkähler Mannigfaltigkeit $(M, I, J, K)$ eine Interpolation zwischen dem Bogoliubov transformierten A-Modell bezüglich der symplektischen Struktur $\omega_I$ und dem B-Modell bezüglich der komplexer Struktur $J$ hergeleitet. Dies ermöglicht Spiegelsymmetrie von Hyperkähler Mannigfaltigkeiten nicht als diskrete, sondern als kontinuierliche Symmetrie aufzufassen. Zweitens wird eine intrinsisch geometrische Formulierung nichtlinearer Sigma-Modelle angegeben und die Existenz einer Modenentwicklung in gekrümmten Raumzeiten motiviert. Um Linearität zu erreichen, muss der Lösungsraum mittels nicht-physikalischer Moden angereichert werden, welche nicht einer Abbildung zwischen der Weltfläche $\Sigma$ und der Zielmannigfaltigkeit $M$ zugeordnet sind. Die Eigenschaft einer Mode physikalisch zu sein wird mittels einer kohomologen Bedingung ausgedrückt.

# Contents

# Part I.

# Prologue

*"Können wir uns dem Göttlichen auf keinem anderen Wege als durch Symbole nähern, so werden wir uns am passendsten der mathematischen Symbole bedienen, denn diese besitzen unzerstörbare Gewißheit."*

Nikolaus von Cues (1401-1464)

# 1. Introduction

Since their invention by Mikhail Gromov in his 1985 paper [Gro85], pseudoholomorphic curves have had profound influence on the development of symplectic topology. They were one of the main inspirations for the creation of Floer homology and are of great importance for other constructions like Gromov-Witten invariants and quantum cohomology. These concepts are ingredients of the so called A-side of mirror symmetry.

Mirror symmetry is part of a web of dualities of string theory and states that type IIa superstring theory compactified on a Calabi-Yau manifold $M$ and type IIb superstring theory compactified on a mirror manifold $M'$ are equivalent. If this is true, any statement of type IIa superstring theory can be translated into a statement of type IIb superstring theory and vice versa. Since it is a very difficult task to compute interesting observables on Calabi-Yau manifolds and because it is even unclear how to formulate superstring theory in a non perturbative way, it is presently too optimistic to hope for a complete and rigorous proof of such kind of conjectures relating different manifestations of string theory.

The situation improves if we restrict ourselves to topological string theory [Wit88, Wit91]. In this framework the mirror symmetry conjecture states that the A-model on a Calabi-Yau manifold $M$ is equivalent to the B-model on the mirror manifold $M'$ and vice versa.

The so called A-model and B-model of topological string theory are constructed by twisting the nonlinear sigma-model of maps $\Phi$ from a Riemann surface $\Sigma$ into a manifold $M$. In physical language, the twisting is done by shifting the spin of fermions by one half of their R-charge [MS103]. If we shift spins using a vector R-symmetry $U(1)_V$, we arrive at the so called A-model. By using an axial R-symmetry $U(1)_A$, we obtain the so called B-model. In mathematical terms, one twists the respective spin bundles of the holomorphic or anti-holomorphic components of the fermions by either the square root of the canonical bundle or the anti-canonical bundle. This can be done in two inequivalent ways, leading to the A-model and the B-model [Wit91], too. The twisted action splits into a sum of a BRST-invariant part and a part which is independent of any metric. The BRST-invariant part has no effect on correlators. Hence, the resulting theory is of topological nature.

In the closed sector, mirror symmetry predicts the equivalence of the so called anti-chiral ring of $M$, which decodes the quantum structure of supersymmetric ground states in the A-model, and the chiral ring on $M'$, which decodes the quantum structure of supersymmetric ground states in the B-model. In the following we will give some basics about (anti-) chiral rings.

The space of physical operators in the A-model can be expressed as de Rham cohomology, instantons are precisely given by pseudoholomorphic curves with respect to a specific integrable complex structure and correlators are essentially given by Gromov-Witten invariants. Consequently, the anti-chiral ring is a physical realization of quantum cohomology. This also shows that topological string theory provides us with a physical motivation of quantities which are of great importance in symplectic topology.

In the B-model, supersymmetric ground states are realized as Dolbeault cohomology and instantons are constant maps into $M$. Genus 0 correlators are given by

$$\langle O_1 \cdots O_s \rangle = \int_M \langle \omega_1 \wedge \cdots \wedge \omega_s , \Omega \rangle \wedge \Omega \,,$$

where $\Omega$ is the holomorphic n-form, $O_i$ are local physical operators represented by complex forms $\omega_i$ and

$$\langle \omega, \Omega \rangle \wedge \Omega := \omega^{i_1 \cdots i_n}_{\bar{j}_1 \cdots \bar{j}_n} \Omega_{i_1 \cdots i_n} \, d\bar{z}^{\bar{j}_1} \wedge \cdots \wedge d\bar{z}^{\bar{j}_n} \wedge \Omega \,.$$

Calculations can often be performed explicitly in the B-model. If mirror symmetry is true, this could for instance be used to compute quantum cohomology in the mirror manifold.

In the open sector, additional degrees of freedom arise in the form of branes.[1] Incorporating them there is a well known category theoretic formulation of mirror symmetry in the open sector. If we view strings which are stretched between a stack of branes as morphisms, branes form a ($A_\infty$-) category [MS209]. It transpires that B-branes are equivalent to the bounded derived category of coherent sheaves, whereas A-branes form an enriched version of the Fukaya category. Hence, in the open sector, the equivalence of the A-model and the B-model is restated as the Kontsevich homological mirror symmetry conjecture [Kon94]. It has only been proven in special cases. First, Alexander Polishchuk and Eric Zaslow proved it in [PZ98] for elliptic curves and Paul Seidel in [Sei03] for quintic surfaces.

[1] If they are defined in the A/B-model, we call them A/B-branes.

More recently [KO03], it turned out that, at least on the physical level, A-branes do not only wrap special Lagrangian submanifolds, but also coisotropic submanifolds of $M$. The construction of morphisms between coisotropic branes is not understood, yet. First examples have been given in [AZ05]. They considered the case of A-branes wrapping coisotropic submanifolds of elliptic curves. In doing so they used the mode expansion of an open string which stretches between two coisotropic branes to show that morphisms are given by a non-commutative deformation of the algebra of functions which are defined on the intersection of the two present branes. If $M$ is not flat, their proof cannot be applied. Then the classical equation of motion is not given by a Laplace equation and is highly nonlinear. Hence, the sum of two solutions is not a solution anymore and there is a priori no mode expansion at hand.

We resolve this issue by giving an intrinsically geometric formulation of nonlinear sigma models and showing that it is possible to recover some kind of mode expansion. In order to do that, one has to enhance the solution space by non-physical modes. This yields a linear problem. Later it is possible to decide which states are physical by looking at some cohomological constraint. For any smooth embedding $\Phi$ of the world sheet $\Sigma$ into the target manifold $M$, the resulting equation of motion is given by a Laplace-Beltrami equation with respect to a covariant exterior derivative which is associated to the pullback connection $\Phi^*\nabla$. This equation can be solved and we get for any embedding an enhanced solution space.[2] Expressed in physical language, this yields the statement that any physically sensible state is the linear combination of modes, but not any linear combination is a physically sensible state. This construction can potentially be applied in order to extend the proof of Aldi and Zaslow to manifolds with non-vanishing Christoffel symbols.

From the purely symplectic or purely complex point of view it is not clear how to imply coisotropic branes into the picture of mirror symmetry, which gives a deep connection between symplectic and complex expressions on a mirror pair.

Generalized complex geometry unifies symplectic and complex geometry in a more general framework. It has been invented by Nigel Hitchin in [Hit02] and has been further developed by his students Marco Gualtieri [Gua03] and Gil Cavalcanti [Cav04]. In generalized complex geometry one extends geometric structures on the tangent bundle $TM$ to the direct sum of the tangent- and the co-tangent bundle.

As an example let us consider the notion of a generalized complex structure. It is an almost complex structure on $\mathbb{T}M := TM \oplus T^*M$ which is integrable with respect to the Courant bracket[3] on $\mathbb{T}M$. It is possible to show that $\mathcal{J}$ is also orthogonal with respect

[2] In the present work we will not develop all details.

[3] Equivalently it is possible to use the Dorfman bracket.

to a naturally defined inner product $q$ on $\mathbb{T}M$ and, hence, is complex and symplectic at the same time. Therefore, it is not surprising that both usual complex and usual symplectic structures can be realized as generalized complex structures. More generally, it is possible to extend the notions of generalized complex geometry to arbitrary exact Courant algebroids. The latter had been defined in [LWX97] and [Roy99]. In contrast to usual geometry, symmetries of generalized complex geometry are not just given by diffeomorphism, but also by so called B-transformations.

Almost 20 years ago, S. James Gates et al. realized in [SJGR84] that Kähler structures are not the most general type of target space geometry admitting extended $\mathcal{N} = 2$ supersymmetry. Recall that such supersymmetry is an essential ingredient in the construction of topologically twisted sigma-models. Instead they found out that more general geometries are allowed and identified them as so called bi-Hermitian geometries. They are given by a pair $(I_1, I_2)$ of integrable almost complex structures which are Hermitian with respect to the same metric $g$ and covariantly constant with respect to some connections $\nabla^{\pm}$ with torsion being dictated by a 2-form $b$. In [Gua03] it has been shown that bi-Hermitian geometries can be expressed in the language of generalized complex geometry as so called twisted generalized Kähler manifolds. Moreover, Maxim Zabzine proved in [Zab06] that the most general target manifold $M$ such that the string super phase space $\Pi T^*\mathcal{L}M$ admits extended supersymmetry is given by a generalized complex manifold.

Extended supersymmetry of this kind of sigma models can be used to extend the construction of the A-model and the B-model of topological string theory to twisted generalized Kähler manifolds. This has been done by Kapustin and Li in [KL07]. One important point which has to be addressed, in order to be able to do the topological twist, is the need for non anomalous R-symmetries. They are present if $M$ possesses a twisted generalized Calabi-Yau metric geometry. The classical structure of local observables is given by Lie algebroid cohomology with respect to some generalized complex structure $\mathcal{J}_1$, while Instantons are given by solutions to an equation which involves $\mathcal{J}_2$. Here $\mathcal{J}_1$ and $\mathcal{J}_2$ define the generalized Kähler structure under consideration.

A Bogoliubov transformation of the A-model can be realized as a special case of the generalized B-model in the following way. The two generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$, which define the generalized Kähler structure, are associated to the Kähler form $\omega$ and its complex structure $I$, respectively. Then local observables are given by a Bogoliubov transformation of de Rham cohomology and instantons are given by pseudoholomorphic curves with respect to $I$. If we interchange the role of $I$ and $\omega$, we obtain the usual B-model. Hence, the generalized B-model includes the A-model and the B-model as special manifestations.

Up to now there is no answer to the question whether the generalized B-model is of topological nature. Some results into that direction can be found in [Zuc06], [Pes07] and [Chu08]. In this work we will encounter similar problems. They will be solved by allowing for local canonical transformations in order to get a generalized energy which is invariant under homotopy.[4]

The above considerations suggest that there exists a more general framework in which it is possible to give a unified description of the mathematics of mirror symmetry and related topics. It should be given by an extension of symplectic topology to generalized complex manifolds. Let us name this extension "generalized complex topology". As symplectic topology can be motivated by stringy considerations, we are allowed to expect that generalized complex topology will arise as a rigorous treatment of the generalized B-model. In order to initiate this development, we will introduce generalized pseudoholomorphic curves among other things. They are motivated by the instantons in the generalized B-model and interpolate between ordinary pseudoholomorphic curves and constant maps. Up to now there is no examination of instantons in the generalized B-model of topological string theory in the literature. A complete list of the topics which will be treated in this dissertation can be found in the next chapter.

It is remarkable that mirror symmetry is conjectured to be connected to other deep mathematical problems like the geometric Langlands conjecture [GW08]. Using concepts of string theory, Geometric Langlands correspondence for arbitrary reductive Lie Groups can essentially be rephrased as self mirror symmetry of the Hitchin moduli space of semi-stable Higgs bundles.

As a first application of our notions we will exploit the generalized B-model to give a smooth interpolation between the A-model and the B-model being defined on the same Hyperkähler manifold. The existence of such a smooth interpolation enables one to view mirror symmetry of Hyperkähler manifolds not as a discrete symmetry, as it is done in the literature, but as the equivalence of the two endpoints of a continuous deformation. In particular the Hitchin moduli space is a Hyperkähler manifold. This kind of construction can be applied to any pair of structures which can be connected through the moduli space of generalized complex structures.

[4]Strictly speaking, we will choose an isotropic splitting $\lambda : TM \to E \cong \mathbb{T}M$ such that the generalized energy is invariant. In physical language this corresponds to a canonical transformation.

# 2. Summary and Structure of the Dissertation

The present work is divided into five parts. Part I contains the introduction as well as a summary. Part II gives a physical motivation of the mathematical considerations in part III. One exception is chapter 4, which concerns nonlinear sigma models. It develops the foundation of a possible quantization of nonlinear sigma models on curved manifolds. As it helps to understand the physical motivation of part III and because it is connected to the computation of string states which are stretched between coisotropic branes, it is included in this work. Part III develops the theory of generalized pseudoholomorphic pairs. Part IV contains conclusions, an outlook and acknowledgments. Part V is given by an appendix treating generalized complex geometry. In the following we will give a more detailed summary.

Part II begins with a presentation of supersymmetric quantum mechanics (SQM). According to definition 3.1.1 we will call a quantum mechanics a SQM if the following conditions are fulfilled:

- Its Hilbert space has a $\mathbb{Z}_2$-grading generated by an operator $(-1)^F$ whose spectrum is $\{-1, 1\}$.
- There are odd operators $Q$, $Q^\dagger$ which square to zero.
- It is true that $2H = \{Q, Q^\dagger\}$, where $H$ is the Hamiltonian and $H$ is even with respect to $(-1)^F$.
- The supercharges $Q$ and $Q^\dagger$ are formally adjoint to each other.

Afterwards we will show that the energy spectrum of a SQM is non-negative and we will define the notion of supersymmetric ground states. They are by definition annihilated by $Q$ and $Q^\dagger$. It will transpire that supersymmetric ground states are exactly the zero-energy states. After exploring the consequences of a decomposition of the Hilbert space into energy eigenspaces we will define the Witten index. Two important results in chapter 3 are theorem 3.1.8 and the localization principle. The former states that supersymmetric ground states are given by $Q$-cohomology, which is also called BRST-cohomology. The latter shows in particular that path integration localizes to $Q$-

fixed configurations if we compute expectation values of supersymmetric operators. At the end of chapter 3 we will demonstrate the concepts of SQM using supersymmetric quantum mechanics on Riemannian manifolds.

Then we will turn to an exposition of nonlinear sigma-models on Riemannian manifolds. We will do that in an intrinsically geometric way. To this end let $(\Sigma, h)$ and $(M, g)$ be Riemannian manifolds, $\nabla$ be the Levi-Civita connection of $g$ and $\Phi : \Sigma \to M$ be a smooth embedding. It will turn out that the fundamental degrees of freedom (for fixed $\Sigma$ and $M$) of nonlinear sigma-models are not given by $\Phi$, but by $d\Phi$. Here $d\Phi$ is a $\Phi^*TM$-valued 1-form on $\Sigma$ being defined by $d\Phi(u) := T\Phi \circ u$ for $u \in \Gamma(\Sigma, T\Sigma)$.

After giving an $L^2$-norm $\langle \cdot, \cdot \rangle$ on the space of $\Phi^*TM$-valued $k$-forms, we will show that the action of nonlinear sigma-models can be globally expressed as $S[\Phi] = \langle d\Phi, d\Phi \rangle$. After denoting the exterior covariant derivative with respect to the pullback connection $\Phi^*\nabla$ by $d_{\Phi^*\nabla}$ and its formal adjoint with respect to $\langle \cdot, \cdot \rangle$ as $d^\dagger_{\Phi^*\nabla}$, we will prove that the equations of motion can be globally written as $d^\dagger_{\Phi^*\nabla} d\Phi = 0$. Moreover, we will show that $\nabla$ being torsion free implies $d_{\Phi^*\nabla} d\Phi = 0$. Hence, stationary points of $S[\Phi]$ will be given by $\Phi^*TM$-valued $d_{\Phi^*\nabla}$-harmonic 1-forms.

The fact that $d\Phi$ as well as $\Phi^*\nabla$ depend non trivially on $\Phi$ renders the equations of motion highly nonlinear. Instead of solving $d^\dagger_{\Phi^*\nabla} d\Phi = 0$, we will look at the easier linear problem $d^\dagger_{\Phi^*\nabla}\xi = 0$, where $\xi$ is an arbitrary $\Phi^*TM$-valued 1-form. We will denote the space of all $\xi$ with $d^\dagger_{\Phi^*\nabla}\xi = 0$ by $\mathcal{H}_\Phi$. It will be possible to recover solutions of the original problem as all $\xi \in \mathcal{H}_\Phi$ which obey $\xi = d\Phi$.

We will proof that $\mathcal{H}_{\Phi_1}$ and $\mathcal{H}_{\Phi_2}$ are isomorphic in the case of $\Phi_1$ and $\Phi_2$ being homotopic to each other and $M$ being flat. Moreover, we will heuristically show that this is also true in the general case. This will be done in several steps. First, we will act with the map which is an isomorphism in the flat case on $\mathcal{H}_{\Phi_1}$. Afterwards, we will deform the 1-form part of $\xi \in \mathcal{H}_{\Phi_1} \subset \Omega^1(\Sigma, \Phi_1^*TM)$ into a solution with respect at $\Phi_2$. This will be done using series methods and we will omit a proof that the present series converge. The problem in finding a general proof can be rephrased as a continuation problem. We would need to extend a section in $\Omega^1(\Phi_1(\Sigma), TM|_{\Phi_1(\Sigma)})$ to a section of $\Omega^1(\mathcal{F}(\Sigma, I), TM_{\mathcal{F}(\Sigma,I)})$, where $\mathcal{F} : \Sigma \times I \to M$ is a homotopy of $\Phi_1$ and $\Phi_2$, such that the continuation of $d_\nabla\xi$ is equal to $d_\nabla$ acting on the continuation of $\xi$. At the end of chapter 4 we will briefly state a program how to proceed towards a quantization of nonlinear sigma-models.

In chapter 5 the extension of topological string theory to generalized complex manifolds will be treated. First, we will review [KL07] and explain the construction of the generalized $B$-model of topological string theory. In particular, we will give an expression for instantons. Kapustin and Li call them twisted generalized complex maps. As we will show, their notion is not invariant under $B$-transformations. Instead, incorporating results of Zabzine [Zab06], it will transpire that instantons are invariant under $B$-transformation modulo canonical transformations on the string super phase space $\Pi T^*\mathcal{L}M$ of $M$.

We will include these additional degrees of freedom into a definition of generalized pseudoholomorphic curves. This will be done by introducing generalized pseudoholomorphic pairs in part III. They consist of a map $\Phi : \Sigma \to M$ and an isotropic embedding $\lambda : TM \to E$, where $E$ is an exact Courant algebroid. Recall that a map $\lambda : TM \to E$ is called isotropic if and only if $q(\lambda(X), \lambda(Y)) = 0$ for all $X$, $Y \in TM$. Below you can find more details on generalized pseudoholomorphic pairs. After the examination of the transformation behavior of instantons we will give a smooth interpolation between the A-model and the B-model on Hyperkähler manifolds. At the end of chapter 5 we will consider topological branes in the A-model and review that they are coisotropic submanifolds with additional structure. In the language of generalized complex geometry, A-branes can be described as generalized Lagrangian submanifolds of the generalized complex manifold $M$.

Part III develops the theory of $(E, \mathcal{J})$-holomorphic pairs. At the beginning we will recall very briefly the well known notion of $J$-holomorphic curves in symplectic manifolds. Then we will examine different possible extensions of $J$-holomorphic curves to generalized complex manifolds. Besides looking at some seemingly natural extensions, we will examine one definition given in the literature [OP11]. We will argue that none of these notions suit our purpose of finding a unified description of the mathematics of mirror symmetry/S-duality. Thereafter we will turn immediately to the development of the theory of generalized pseudoholomorphic pairs as they are motivated by string theory.

Let $(\Sigma, j_\Sigma)$ be a Riemann surface, $M$ be a real $2n$-dimensional manifold, $\Phi : \Sigma \to M$ be a sufficiently smooth map, $(E, q, [\cdot,\cdot], \pi)$ be an exact Courant algebroid over $M$, $\mathcal{J}$ be an almost generalized complex structure on $E$ and $\lambda : TM \to E$ be an isotropic embedding. The pair $(\Phi, \lambda)$ will be called $(E, \mathcal{J})$-holomorphic, or generalized pseudoholomorphic, if

$$\mathcal{J} \circ \lambda \circ T\Phi = \lambda \circ T\Phi \circ j_\Sigma \,. \tag{6.3.1}$$

In proposition 6.3.4 we will show that this definition is invariant under orthogonal automorphism of $E$ with respect to $q$. If $\lambda$ is an isotropic splitting of $E$, we will call $\Phi$ a generalized pseudoholomorphic curve (with respect to $E$ and $s$), or simply "$(E, \mathcal{J})$-holomorphic curve". If, moreover, $E$ is given by the standard Courant algebroid, we will name $\Phi$ a $\mathcal{J}$-holomorphic curve.

In order to be able to define a generalized energy, we will introduce the notions of tamed and compatible almost generalized complex structures. Thereby, an almost generalized complex structure $\mathcal{J}_2$ will be called tamed by $\mathcal{J}_1$ if

$$-q(\mathcal{J}_1\mathcal{J}_2A, A) > 0 \tag{6.3.6}$$

for all $A \in E$. They will be named compatible if they are tamed and $[\mathcal{J}_1, \mathcal{J}_2] = 0$. In a remark we will argue that $\mathcal{J}_2$ is tamed by $\mathcal{J}_1$ if and only if $\mathcal{J}_1$ is tamed by $\mathcal{J}_2$. Thus we will call two structures $\mathcal{J}_1$, $\mathcal{J}_2$ taming each other simply tamed. An interesting result is the possibility to rephrase a generalized Kähler structure as a pair of integrable and compatible generalized complex structures. Two tamed almost generalized complex structures will be used to define a metric $G$ on $E$ via

$$G(A, B) := -\frac{1}{2}q(\{\mathcal{J}_1, \mathcal{J}_2\}A, B)\,, \tag{6.3.8}$$

where $A$, $B \in E$.

In section 6.4 we will give an important technical result concerning $(E, \mathcal{J})$-holomorphic pairs. We will show that there exists an almost generalized complex structure $\mathcal{J}'$ on $\mathbb{T}M$ for any almost generalized complex structure $\mathcal{J}$ on $E$ , such that $(\Phi, \lambda)$ is an $(E, \mathcal{J})$-holomorphic pair, if and only if $\Phi$ is a $\mathcal{J}'$-holomorphic curve. This statement can be found in proposition 6.4.4. Its proof uses proposition 6.4.3, which reduces $(E, \mathcal{J})$-holomorphic pairs to $(E, \mathcal{J})$-holomorphic curves, and theorem 6.4.1. Theorem 6.4.1 states that any isotropic embedding $\lambda : TM \to E$ can be expressed as the composition of an orthogonal automorphism $\Lambda$ of $E$, as usual with respect to $q$, and an isotropic splitting $s$.

In section 6.5 we will exploit $G$ to define the generalized energy of a pair $(\Phi, \lambda)$, where $\Phi : \Sigma \to M$ is a map and $\lambda : TM \to E$ is an isotropic embedding. We will realize that the generalized energy is invariant under orthogonal transformations of $E$ (cf. proposition 6.5.2) and that generalized pseudoholomorphic pairs obey an energy identity (cf. proposition 6.5.3). Moreover, we will prove that, after choosing a suitable isotropic embedding, the generalized energy is invariant under homotopy if $\mathcal{J}_1$ is a regular integrable generalized complex structure and the Ševera class $[H]$ of $E$ vanishes. This is still general enough to cover manifolds which do not admit any integrable complex structures or symplectic structures.

Chapter 7 presents the local theory of generalized pseudoholomorphic curves. At the beginning we will compute a local expression of (6.3.1). It reads

$$\frac{\partial\phi^\mu}{\partial s}\lambda(e_\mu) + \frac{\partial\phi^\mu}{\partial t}\mathcal{J}(\phi)\lambda(e_\mu) = 0 \tag{7.1.10}$$

and we name it the "generalized nonlinear Cauchy-Riemann equation". We will exploit it to analyze the local behavior of $(E, \mathcal{J})$-holomorphic pairs. In particular we will prove an identity theorem for them. It is corollary 7.2.2. Moreover, we will show in theorem 7.3.1 that solutions of (6.3.1) obey elliptic regularity. Thereafter we will extend the notions of simple and somewhere injective curves to generalized complex manifolds and examine their properties. We will discover that $(E, \mathcal{J})$-holomorphic curves locally behave in the same way as ordinary $J$-holomorphic curves. The reason for this will be clarified in theorem 7.4.4. It states the following: If $(\Phi, \lambda)$ is an $(E, \mathcal{J})$-holomorphic pair and $\mathcal{J} \in \mathcal{C}^l$, then it is true that for every $\sigma \in \Sigma$ there exist neighborhoods $\Omega \subset \Sigma$ of $\sigma$ and $U \subset M$ of $\Phi(\sigma)$ and an almost complex structure $J$ on $U \times U$ of class $\mathcal{C}^l$ such that $(\Phi, p_0) : \Omega \to U \times U$ is a (local) $J$-holomorphic curve for any fixed $p_0 \in U$. Its proof uses theorem 6.4.1, propositions 6.4.3 and 6.4.4 as well as the existence of a geodesically convex neighborhood of any point in a smooth manifold. It is not possible in general to find an almost complex structure $J$ everywhere on $M$ such that $\Phi \times p$ has the same properties as in theorem 7.4.4.

Chapter 8 will treat first results concerning the global theory of $(E, \mathcal{J})$-holomorphic pairs. We will invent a torsion operator $T$, associated to a connection on $E$, which extends the very well known notion of torsion on $TM$ to exact Courant algebroids. It will not be a tensor in general, but its restriction to any Dirac structure in $E$ will be tensorial. This will enable us to define a generalized Levi-Civita connection with respect to $s$ on $E$. It will be defined as

$$\nabla_X A = \nabla_X(s(Y) + \pi^*(\xi)) := s(\nabla_X Y) + \pi^*(\nabla^*_X \xi) + \frac{1}{2}\pi^*(i_X i_Y H)\,. \tag{8.1.7}$$

In particular the restriction of $T$ to $s(TM)$ vanishes. Using these concepts we will be able to compute the vertical differential $\mathcal{D}_\Phi$ of

$$\overline{\partial}_{\mathcal{J}}(\Phi) := \frac{1}{2}(s \circ d\Phi + \mathcal{J} \circ s \circ d\Phi \circ j_\Sigma)\,. \tag{7.1.2}$$

In contrast to usual symplectic topology, the generalized vertical differential will not be a Fredholm operator (acting on the Sobolev completion of $\mathcal{C}^\infty(\Sigma, M)$), but semi-Fredholm. This will be true because $\mathcal{D}_\Phi$ will be the composition of a real Cauchy-Riemann operator $\mathbb{D}_\Phi$, which will be Fredholm, and a semi-Fredholm operator $s$. The operator $s$ will be induced by the isotropic splitting $s : TM \to E$. In order to keep notions simple, we will denote both with the same symbol. This will render the examination of the structure of the moduli space of pseudoholomorphic curves more involved than in the symplectic case.

We will take the first steps towards a solution of the deformation problem by introducing admissible vector fields along a map $\Phi$. They will be defined as vector fields

$\xi$ along $\Phi$ such that the tangent map of $\Phi_t$ lies inside the kernel of $B := s^* \circ \mathcal{J} \circ s$ for all $t$. The deformation $\Phi_t$ will be induced by the geodesic flow along $\xi$. By only incorporating admissible vector fields along $\Phi$, we will infer that the vertical differential of $\overline{\partial}_{\mathcal{J}}(\Phi)$ will decompose as $s \circ D_\Phi$. This will show that the excess degrees of freedom of $\Omega^{0,1}(\Sigma, \Phi^* E)$, which render $\mathbb{D}_\Phi \circ s$ to be only semi-Fredholm, will separate if we only deform in directions dictated by admissible vector fields. The operator $D_\Phi$ will formally look exactly like the vertical differential of symplectic topology. We will stop the present examination of the global structure of solutions to (6.3.1) at this point. At the end we will give some examples to clarify our notions.

# Part II.

# The Physics of Topological Sigma Models On Generalized Complex Manifolds

# 3. Supersymmetric Quantum Mechanics

In this introductory chapter we review the methods and ideas of supersymmetric quantum mechanics. Although we will give a slightly different exposition of the material, we will orientate ourselves at [MS103] and [Wit82]. This chapter is rather informal and has a lack of mathematical rigor, but its ideas and methods are quite important for this work. We will begin with the definition of a supersymmetric quantum mechanics and proceed with the exposition of the general structure of its Hilbert space. In particular we will show that supersymmetric ground states are given by BRST-cohomology and motivate the appearance of instantons in supersymmetric theories. Afterwards we will consider SQM on Riemannian manifolds $M$. We will show that in this case the supersymmetric ground states are given by de Rham cohomology of $M$. Thereafter, we will explain perturbative ground states. The end of this chapter will discuss the important notion of instanton corrections.

## 3.1. Generalities of Supersymmetric Quantum Mechanics

We start our treatment of supersymmetric quantum mechanics by stating some general remarks about the structure of the Hilbert space of a SQM. In the following we will assume that the objects we are talking about do all exist and are well defined.

### 3.1.1. Structure of Hilbert Space and Witten-Index

At the beginning we should fix our terminology. To this end let us state

**Definition 3.1.1.** *We call a quantum mechanics with a $\mathbb{Z}_2$ graded Hilbert space $\mathcal{H}$ (with positive definite inner product $\langle \cdot, \cdot \rangle$) a supersymmetric quantum mechanics (SQM), if there exist operators $H$, $Q$, $Q^\dagger$ and $(-1)^F$ acting on $\mathcal{H}$, such that*

1. *$Q$ and $Q^\dagger$ are formally adjoint operators with respect to $\langle \cdot, \cdot \rangle$,*
2. *The spectrum of $(-1)^F$ is given by $\{1, -1\}$,*

3. *the grading of $\mathcal{H}$ is defined with respect to $(-1)^F$, i.e. for*

$$\mathcal{H}^B := \{\psi \in \mathcal{H} | (-1)^F \psi = \psi\} \quad \textit{and} \quad \mathcal{H}^F := \{\psi \in \mathcal{H} | (-1)^F \psi = -\psi\} \tag{3.1.1}$$

*it is true that*

$$\mathcal{H} = \mathcal{H}^B \oplus \mathcal{H}^F \,, \tag{3.1.2}$$

4. *$H$ is an even operator in the sense that*

$$[H, (-1)^F] = 0 \,, \tag{3.1.3}$$

5. *$Q$ and $Q^\dagger$ are odd operators, i.e.*

$$\{Q, (-1)^F\} = \{Q^\dagger, (-1)^F\} = 0 \quad \textit{and} \tag{3.1.4}$$

6. *the supersymmetry algebra is given by*

$$Q^2 = (Q^\dagger)^2 = 0 \,, \tag{3.1.5}$$
$$\{Q, Q^\dagger\} = 2H \,. \tag{3.1.6}$$

*We call $H$ the Hamiltonian and $Q$, $Q^\dagger$ the supercharges.*

Now we will state three simple, but important, consequences of this definition of SQM. That is

**Proposition 3.1.2.** *Let a SQM in the sense of definition 3.1.1 be given. Then it is true that*

1. *the supercharges are conserved,*
2. *$Q$,$Q^\dagger$ define maps between the bosonic eigenspace and fermionic eigenspace,*

$$Q, Q^\dagger : \mathcal{H}^F \longrightarrow \mathcal{H}^B \tag{3.1.7}$$
$$Q, Q^\dagger : \mathcal{H}^B \longrightarrow \mathcal{H}^F \,, \tag{3.1.8}$$

3. *the spectrum of $H$ is non-negative and*
4. *A state $\psi$ has zero energy if and only if it is annihilated by $Q$ and $Q^\dagger$.*

*Proof.* From (3.1.5) and (3.1.6) it follows that the supercharges are conserved,

$$[H, Q] = [H, Q^\dagger] = 0 \,. \tag{3.1.9}$$

Second can be inferred from equation (3.1.4).

Equation (3.1.6) and the fact that there are no negative norm states implies the non-negativeness of the spectrum of $H$. Indeed, let $\psi$ be an eigenvector of $H$ with eigenvalue $\lambda$. Then it is true that

$$\lambda|\psi|^2 = \langle\psi, \lambda\psi\rangle = \langle\psi, H\psi\rangle = \left\langle \psi, \frac{1}{2}\{Q, Q^\dagger\}\psi \right\rangle = \frac{1}{2}\left(|Q\psi|^2 + |Q^\dagger\psi|^2\right) . \tag{3.1.10}$$

Since the norm of a state is positive, it follows that $\lambda > 0$. The last equation also implies that a state $\psi$ has zero energy if and only if it is annihilated by $Q$ and $Q^\dagger$. □

**Remark 3.1.3** The action of $Q$ and $Q^\dagger$ on a state $\psi$ defines the variation of $\psi$ under supersymmetry transformations. Hence, zero energy states are invariant and are called supersymmetric. Let us denote supersymmetric states with zero energy as supersymmetric ground states.

In the following we assume that $\mathcal{H}$ can be decomposed in terms of eigenspaces $\mathcal{H}_n$ of the Hamiltonian $H$ with eigenvalue $E_n$ and $E_0 < E_1 < E_2 < \ldots$, i.e.

$$\mathcal{H} = \bigoplus_n \mathcal{H}_n . \tag{3.1.11}$$

Then another easy consequence of definition 3.1.1 is

**Proposition 3.1.4.** *Let a SQM in the sense of definition 3.1.1 be given and let $\mathcal{H}$ be decomposable into energy eigenspaces. Then the grading respects the energy levels and the supercharges map fermionic states of energy $E_n$ to bosonic states of the same energy and vice versa.*

*Proof.* Since $(-1)^F$ commutes with the Hamiltonian, $(-1)^F$ maps $\mathcal{H}_n$ to $\mathcal{H}_n$ and thus the grading respects the energy levels

$$\mathcal{H}_n = \mathcal{H}_n^B \oplus \mathcal{H}_n^F . \tag{3.1.12}$$

Here $(-1)^F$ acts on $\mathcal{H}_n^B$ as $\mathbb{1}$ and on $\mathcal{H}_n^F$ as $-\mathbb{1}$. As the supercharges $Q$ and $Q^\dagger$ commute with the Hamiltonian, too, they map fermionic states of energy $E_n$ to bosonic states of the same energy and vice versa:

$$Q, Q^\dagger : \mathcal{H}_n^F \longrightarrow \mathcal{H}_n^B , \tag{3.1.13}$$
$$Q, Q^\dagger : \mathcal{H}_n^B \longrightarrow \mathcal{H}_n^F , \tag{3.1.14}$$

□

Some of our future considerations will crucially rely on the following fact: Let $\mathcal{O}$ be a supersymmetric operator, i.e. $[\![Q, \mathcal{O}]\!] = 0$ where $[\![\cdot, \cdot]\!]$ denotes the graded commutator. Then we will see that only zero energy states contribute to the expectation value of $\mathcal{O}$. The excited states pair together because of supersymmetry and cancel their respective contributions. Strictly speaking, this is true because of

**Proposition 3.1.5.** *Let a SQM in the sense of definition 3.1.1 be given and let $\mathcal{H}$ be decomposable into energy eigenspaces. Then the fermionic and bosonic eigenspaces of energy $E_n$ are isomorphic if $E_n > 0$.*

*Proof.* Let $E_n > 0$ and $Q_1 := Q + Q^\dagger$. Then $(Q_1)^2 = 2H$ acts as $2E_n$ on $\mathcal{H}_n$ and is, therefore, invertible. Hence,

$$\mathcal{H}_n^F \cong \mathcal{H}_n^B \,. \tag{3.1.15}$$

□

**Remark 3.1.6** Zero-energy eigenspaces are not isomorphic in general. In fact, the difference of the dimension of the bosonic zero-energy eigenspace and the dimension of the fermionic zero-energy eigenspace is invariant under supersymmetry preserving deformations of the theory. This will be examined in the following.

Let a SQM be given. We deform its Hamiltonian and thereby its spectrum. Let us assume that we do this in such way that the deformed theories are also SQMs with the same Hilbert space. Energy levels can be splitted, massive states can become massless, massless states can become excited and so on. But this has to happen in pairs of bosons and fermions since we assume the presence of supersymmetry during and after the deformation. Hence, the number of bosonic ground states minus the number of fermionic ground states is invariant during the deformation. This invariant is called supersymmetric index or Witten index. Let state this more precisely in

**Definition 3.1.7.** *Let a SQM in the sense of definition 3.1.1 be given. Then we call the difference* $\dim \mathcal{H}_0^B - \dim \mathcal{H}_0^F$ *the Witten index of the SQM under consideration.*

During the above mentioned deformation, excited states pair together and have different sign of $(-1)^F$. The Hamiltonian acts on zero-energy states as $0$. Hence, the Witten index can be expressed as

$$\dim \mathcal{H}_0^B - \dim \mathcal{H}_0^F = \mathrm{Tr}(-1)^F \mathrm{e}^{-\beta H} \,. \tag{3.1.16}$$

Our next aim is to find a recipe to compute the supersymmetric ground states. It will transpire that they can be represented as Q-cohomology of some complex. Then the Witten index is given by its Euler number. That is

**Theorem 3.1.8.** *Let a SQM in the sense of definition 3.1.1 be given and let $\mathcal{H}$ be decomposable into energy eigenspaces. Moreover, let there exist an operator $F$, which we call fermion-number operator, such that*

$$[F, Q] = Q \quad \textit{and} \quad (-1)^F = \mathrm{e}^{i\pi F} . \tag{3.1.17}$$

*Moreover, let $\mathcal{H}$ be decomposable into eigenspaces with respect to $F$. If the eigenvalues of $F$ are given as a representative set $\mathcal{R} = \{0, 1, \ldots, 2k-1\}$ of $\mathbb{Z}_{2k}$, with corresponding eigenspaces $\mathcal{H}^p$ for $p \in \mathcal{R}$, it is true that supersymmetric ground states are given as the cohomology of the $\mathbb{Z}_{2k}$ graded complex*

$$\cdots \xrightarrow{\;Q\;} \mathcal{H}^{p-1} \xrightarrow{\;Q\;} \mathcal{H}^{p} \xrightarrow{\;Q\;} \mathcal{H}^{p+1} \xrightarrow{\;Q\;} \cdots \tag{3.1.18}$$

*and the Witten index is given by its Euler characteristic. The cohomology is also called BRST cohomology.*

*Proof.* Using eq. (3.1.17), a straight forward computation shows that

$$[H, F] = [H, \mathrm{e}^{i\pi F}] = 0 , \tag{3.1.19}$$

as well as

$$[F, Q^\dagger] = -Q^\dagger . \tag{3.1.20}$$

In particular (3.1.19) implies (3.1.3). By assumption it is true that $\mathcal{H}$ can be decomposed as

$$\mathcal{H} = \bigoplus_{p \in \mathcal{R}} \mathcal{H}^p . \tag{3.1.21}$$

If we define

$$\mathcal{H}^B := \bigoplus_{p\,\text{even}} \mathcal{H}^p \quad \text{and} \quad \mathcal{H}^F := \bigoplus_{p\,\text{odd}} \mathcal{H}^p , \tag{3.1.22}$$

this is naturally isomorphic to (3.1.2). Because of equation (3.1.19) we are also able to decompose $\mathcal{H}^p$ into energy eigenspaces,

$$\mathcal{H}^p = \bigoplus_{n} \mathcal{H}^p_n . \tag{3.1.23}$$

Equation (3.1.17) is the defining property of $Q$ having fermion-number, or R-charge, 1. Therefore, it follows together with eq. (3.1.19) that for all $n$

$$Q : \mathcal{H}^p_n \longrightarrow \mathcal{H}^{p+1}_n \quad \text{and} \quad Q^\dagger : \mathcal{H}^p_n \longrightarrow \mathcal{H}^{p-1}_n . \tag{3.1.24}$$

which shows together with $Q^2 = 0$ that (3.1.18) defines a complex.

At each of the excited levels this complex is exact. Let $E_n > 0$, $\psi \in \mathcal{H}_n^p$ and $Q\psi = 0$. Then $\psi = QQ^\dagger(2E_n)^{-1}\psi$ and $\psi$ is exact. Now let $E_n = 0$ and $\psi \in \mathcal{H}_0^p$. Then it is by definition true that $Q\psi = 0$ and thus $[\psi] \in H^p(Q)$. On the other hand, if $[\psi] \in H^p(Q)$ and $d^\dagger\psi = 0$, it is obviously $\psi \in \mathcal{H}_0^p$. If $d\psi = 0$ and $d^\dagger\psi \neq 0$ we infer that $H\psi \neq 0$ which is a contradiction. Hence,

$$\mathcal{H}_0^p = H^p(Q) := \frac{\ker Q : \mathcal{H}^p \longrightarrow \mathcal{H}^{p+1}}{\operatorname{im} Q : \mathcal{H}^{p-1} \longrightarrow \mathcal{H}^p} \,. \tag{3.1.25}$$

Let us look at this result from a different perspective. We saw above that one necessary condition for $\psi$ being a supersymmetric ground state is $Q\psi = 0$. But since we are looking at a supersymmetric theory, we consider two states $\psi$ and $\psi + Q\xi$ for some $\xi$ to be physically equivalent.

Hence, we have shown that the supersymmetric ground states can be characterized by the cohomology of the complex (3.1.18). If we define

$$\mathcal{H}_0^B = \bigoplus_{p\in\mathcal{R}\wedge p\,\text{even}} H^p(Q) \quad \text{and} \quad \mathcal{H}_0^F = \bigoplus_{p\in\mathcal{R}\wedge p\,\text{odd}} H^p(Q) \tag{3.1.26}$$

we get for the Witten index

$$\operatorname{Tr}(-1)^F \mathrm{e}^{-\beta H} = \sum_{p\in\mathcal{R}} (-1)^p \dim H^p(Q) \,, \tag{3.1.27}$$

which is the Euler characteristic of (3.1.18). □

**Remark 3.1.9** We say that states in $\mathcal{H}^p$ have fermion-number $p$.

### 3.1.2. Localization Principle

Here we will give a short motivation for the localization principle as it has been done in [Wit91]. This is an important technical tool for computations in supersymmetric theories. It states that in order to perform a path integration, it suffices to sum over all contributions which are invariant under supersymmetry transformations. This can be seen as follows.

Let us look at an arbitrary quantum field theory and assume that we want to perform a path-integration over some space $\mathcal{F}$ of fields. Moreover, assume the existence of a group $G$ which acts on $\mathcal{F}$ from the left. The side from where $G$ operates on $\mathcal{F}$ is not important. Then it is possible to decompose $\mathcal{F}$ into orbits under $G$. Suppose $G$ acts freely and denote the orbit-space as $\mathcal{F}/G$. Path-integration over $\mathcal{F}$ is now performed by first integrating over $G$ and afterwards over $\mathcal{F}/G$. If $\mathcal{O}$ is a $G$-invariant observable, we get

$$O = \int\limits_{\mathcal{F}} [\mathscr{D}\phi]\, \mathrm{e}^{-S}\mathcal{O} = \mathrm{vol}(G) \int\limits_{\mathcal{F}/G} [\mathscr{D}\phi]\, \mathrm{e}^{-S}\mathcal{O}\,. \tag{3.1.28}$$

Let us apply this to the case where $G$ is the symmetry generated by $Q$, namely supersymmetry, and $\mathcal{O}$ is invariant under $Q$. Since supersymmetry is a fermionic symmetry, its volume is $0$. This means that as long as we consider $Q$-invariant operators, path integration yields $0$ if the symmetry acts freely on $\mathcal{F}$. In general $G$ does not act freely, but there is a fixed point set $\mathcal{F}_0$.

Let $\mathcal{F}_0 \neq \emptyset$ and $G$ be a fermionic symmetry. Furthermore, let $\mathcal{E}$ be a $G$ invariant neighborhood of $\mathcal{F}_0$, namely $G\,\mathcal{E} \subseteq \mathcal{E}$. Then $G$ acts freely on the complement of $\mathcal{E}$ and its contribution to path integration yields $0$. The neighborhood $\mathcal{E}$ can possibly be arbitrary small. Therefore, the above procedure can be viewed as a localization procedure. The details depend on the vanishing of $Q - \mathrm{id}$ on $\mathcal{F}_0$. Generically the path integral collapses to a sum over the $Q$-fixed points weighted by the determinant of the transverse degrees of freedom.

Later we will see that the set $\mathcal{F}_0$ is precisely the set of instantons and the above localization explains the appearance of instanton contributions to non-perturbative results in supersymmetric quantum theories.

## 3.2. Example: Supersymmetric Quantum Mechanics on Riemannian Manifolds

In this section we want to apply the results of the previous one to a simple example. We will consider supersymmetric quantum mechanics on a Riemannian manifold. It is a supersymmetric non-linear sigma model with one-dimensional world-sheet. We will define the Hilbert space and give expressions of $Q$, $Q^\dagger$, $H$ and the algebra of observables. We will also review the important notion of instanton corrections and show why they are important.

### 3.2.1. Supersymmetric Non-Linear Sigma Models with One-Dimensional World-Sheet

Let $(M, g)$ be an oriented, compact Riemannian manifold with coordinate maps $\xi^\mu_M$ and $\mathcal{T}$ a one-dimensional real manifold, which we use to parameterize time. Moreover, let $\Phi : \mathcal{T} \longrightarrow M$ be a differentiable map with local coordinates $\phi^\mu := \xi^\mu_M \circ \Phi$. It defines the bosonic variables. Fermions of this theory are $\phi^*TM$ valued spinors. Since the spin bundle of a one dimensional manifold is the trivial bundle $\mathcal{T} \times \mathbb{C}$,

$$\psi, \overline{\psi} \in \Gamma(\mathcal{T}, \mathbb{C} \otimes \phi^*TM)\,, \tag{3.2.1}$$

with $\psi, \overline{\psi}$ being complex conjugates of each other and locally

$$\psi = \psi^\mu \frac{\partial}{\partial x^\mu}\bigg|_\Phi\,, \quad \text{as well as} \quad \overline{\psi} = \overline{\psi}^\mu \frac{\partial}{\partial x^\mu}\bigg|_\Phi\,. \tag{3.2.2}$$

The action of SQM on a Riemannian manifold reads

$$S = \int_{\mathcal{T}} dt \left( \frac{1}{2} g_{\mu\nu} \dot{\phi}^\mu \dot{\phi}^\nu + \frac{i}{2} g_{\mu\nu} \left( \overline{\psi}^\mu D_t \psi^\nu - D_t \overline{\psi}^\mu \psi^\nu \right) - \frac{1}{2} R_{\mu\nu\rho\sigma} \psi^\mu \overline{\psi}^\nu \psi^\rho \overline{\psi}^\sigma \right)\,, \tag{3.2.3}$$

where $D_t\psi^\mu = \partial_t \psi^\mu + \Gamma^\mu_{\nu\lambda}(\partial_t \phi^\nu)\psi^\lambda$ is the Dirac operator of $\mathbb{C} \otimes \phi^*TM$, $R_{\mu\nu\rho\sigma}$ are the components of the Riemann curvature tensor $R$ with respect to the Levi-Civita connection $\nabla$ and $\Gamma^\mu_{\nu\lambda}$ the Christoffel symbols of $(M, g)$. At this point we do not want to go into the details of the construction and physical meaning of this action and postpone it to chapter 5.

The theory is invariant under infinitesimal supersymmetry transformations

$$\delta\phi^\mu = \epsilon \overline{\psi}^\mu - \overline{\epsilon} \psi^\mu\,, \tag{3.2.4}$$

$$\delta\psi^\mu = \epsilon(i\dot{\phi}^\mu - \Gamma^\mu_{\nu\lambda} \overline{\psi}^\nu \psi^\lambda)\,, \tag{3.2.5}$$

$$\delta\overline{\psi}^\mu = \overline{\epsilon}(-i\dot{\phi} - \Gamma^\mu_{\nu\lambda} \overline{\psi}^\nu \psi^\lambda)\,, \tag{3.2.6}$$

i.e. $\delta S = 0$ under (3.2.4) - (3.2.6). By a Noether theorem there are conserved charges (one for $\epsilon$ and one for $\overline{\epsilon}$) associated to this continuous symmetry, which we call supercharges. They are given by

$$Q = i g_{\mu\nu} \overline{\psi}^\mu \dot{\phi}^\nu\,, \tag{3.2.7}$$

$$Q^\dagger = -i g_{\mu\nu} \psi^\mu \dot{\phi}^\nu\,. \tag{3.2.8}$$

Equation (3.2.3) has another important symmetry. It leaves bosons unchanged while it acts on fermions as a constant phase rotation

$$\psi^\mu \longrightarrow \mathrm{e}^{-i\alpha} \psi \quad \text{and} \quad \overline{\psi}^\mu \longrightarrow \mathrm{e}^{i\alpha} \overline{\psi}^\mu\,, \quad \alpha \in \mathbb{R}\,. \tag{3.2.9}$$

The resulting Noether charge reads

$$F = g_{\mu\nu}\overline{\psi}^\mu\psi^\nu \,. \tag{3.2.10}$$

This is also our first example of a so called R-symmetry. Later we will use such kind of invariance to construct so called "topological non-linear sigma-models" via topological twisting.

Next we show that (3.2.3) defines a SQM in the sense of the last section. As is explained there, the existence of a fermion-number operator is one important ingredient and the operator associated to the charge $F$ will act as such a one.

Let us denote the conjugate momentum of $\phi^\mu$ as $p_\mu = g_{\mu\nu}\dot{\phi}^\nu$. Then the supercharges can be expressed in terms of the conjugate momentum as

$$Q = i\overline{\psi}^\mu p_\mu \quad \text{and} \quad Q^\dagger = -i\psi^\mu p_\mu \,. \tag{3.2.11}$$

We quantize the system by imposing canonical (anti-) commutation relations

$$[\phi^\mu, p_\nu] = i\delta^\mu_\nu \,, \tag{3.2.12}$$
$$\{\psi^\mu, \overline{\psi}^\nu\} = g^{\mu\nu} \,, \tag{3.2.13}$$

with all other (anti-)commutators vanishing. Using this and (3.2.10) results in

$$[F, \psi^\mu] = -\psi^\mu \quad \text{and} \quad [F, \overline{\psi}^\mu] = \overline{\psi}^\mu \,. \tag{3.2.14}$$

In accordance to section 3.1 we define

$$H := \frac{1}{2}\{Q, Q^\dagger\} \,. \tag{3.2.15}$$

Using (3.2.7), (3.2.8), (3.2.10) and (3.2.12), simple calculations show

$$[F, Q] = Q \,, \quad [F, Q^\dagger] = -Q^\dagger \quad \text{and} \tag{3.2.16}$$
$$[H, F] = 0 \,. \tag{3.2.17}$$

Therefore, $F$ really defines a fermion-number operator in the sense of section 3.1.

The next step in constructing a quantum pendant to the classical theory given by (3.2.3) is to specify a representation of the quantum algebra. For that purpose we have to define a Hilbert space, where the algebra of observables can operate. Let us define this Hilbert space to be the space of complexified differential Forms $\mathcal{H} := \Omega(M) \otimes \mathbb{C}$ on $M$ equipped with the Hermitian inner product

$$\langle \omega_1, \omega_2 \rangle := \int_M \overline{\omega_1} \wedge *\omega_2 \,. \tag{3.2.18}$$

We give a representation of the algebra of observables in local coordinates acting on $\mathcal{H}$ as

$$\phi^{\mu} = x^{\mu} \times \,, \tag{3.2.19}$$
$$p_{\mu} = -\,\mathrm{i}\mathcal{L}_{e_\mu} \,, \tag{3.2.20}$$
$$\overline{\psi}^{\mu} = dx^{\mu} \wedge \,, \tag{3.2.21}$$
$$\psi^{\mu} = g^{\mu\nu} i_{e_\mu} \,, \tag{3.2.22}$$

where $i_X$ is the substitution operator and $\mathcal{L}_{e_\mu}$ denotes the Lie-derivative in $e_\mu = \frac{\partial}{\partial x^\mu}$ direction. Using

$$i_X\,(\omega_1 \wedge \omega_2) = (i_X \omega_1) \wedge \omega_2 + (-1)^p \omega_1 \wedge i_X \omega_2 \,, \tag{3.2.23}$$

where $\omega_1 \in \Omega^p(M)$ and $\omega_2 \in \Omega^q(M)$, an easy calculation shows that (3.2.19) - (3.2.22) fulfill (3.2.12) and (3.2.13).

Since the intention of this chapter is just to give a motivation for later considerations, we do not want to address the question in which sense these operations really define operators on $\Omega(M)$, i.e. whether and in which sense they are continuous, smooth or bounded. But it is clear that the above operators are subject to coordinate transformations. We defined them with respect to certain local coordinate neighborhoods. To be well defined operators, changing coordinates and acting on elements of $\Omega(M)$ have to commute. This clearly holds. Thus the canonical (anti-) commutation relations are independently true in all coordinate systems. This can also be seen by a straight forward calculation transforming the right hand side and the left hand side of (3.2.12) and (3.2.13).

By denoting the state which is annihilated by all $\psi^\mu$ as $|0\rangle$ we obtain the correspondence

$$|0\rangle \longleftrightarrow 1 \tag{3.2.24}$$
$$\overline{\psi}^{\mu}\,|0\rangle \longleftrightarrow dx^{\mu} \tag{3.2.25}$$
$$\overline{\psi}^{\mu}_1 \overline{\psi}^{\mu}_2\,|0\rangle \longleftrightarrow dx^{\mu} \wedge dx^{\nu} \tag{3.2.26}$$
$$\vdots$$

Because of equations (3.2.14) it is obviously true that the fermion-number associated to a state which corresponds to a $p$-form is $p$. Thus we observe that $\mathcal{H}$ is graded by the fermion-number which is given by the form-degree, i.e.

$$\mathcal{H} = \bigoplus_{p=0}^{n} \Omega^p(M) \otimes \mathbb{C} \,. \tag{3.2.27}$$

Our next aim is to construct the supersymmetric ground states. To this end we have to give an expression of $Q$ acting on $\mathcal{H}$. Let $\omega \in \Omega^p(M)$. Then locally

$$Q\omega = \mathrm{i}\overline{\psi}^\mu p_\mu \omega = dx^\mu \wedge (\mathcal{L}_{e_\mu}\omega) = dx^\mu \wedge \frac{1}{p!}\frac{\partial \omega_{\mu_1 \dots \mu_p}}{dx^\mu} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} = d\omega \tag{3.2.28}$$

and

$$Q^\dagger \omega = d^\dagger \omega \,, \tag{3.2.29}$$

where $d^\dagger$ is the formal adjoint of $d$ with respect to (3.2.18). Therefore, the Hamiltonian is

$$H = \frac{1}{2}\{Q, Q^\dagger\} = \frac{1}{2}(dd^\dagger + d^\dagger d) = \frac{1}{2}\Delta \,, \tag{3.2.30}$$

where $\Delta$ is the ordinary Laplace operator on $M$. Hence, the zero energy states are precisely the harmonic forms and we get

$$\mathcal{H}_0 = \mathrm{Harm}(M, g) = \bigoplus_{p=0}^{n} \mathrm{Harm}^p(M, g) \,. \tag{3.2.31}$$

This is consistent with $[F, Q] = Q$, i.e. the space of supersymmetric ground states is graded by fermion-number, which is the form-degree here. Moreover, we know by the general structure of a Hilbert space of a supersymmetric quantum mechanics that

$$\mathcal{H}_0 = H(Q)^* = H^*_{dR} = \bigoplus_{p=0}^{n} H^p_{dR} \cong \bigoplus_{p=0}^{n} \mathrm{Harm}^p(M, g) \tag{3.2.32}$$

as graded complexes. This refines to

$$H^p_{dR} \cong \mathrm{Harm}^p(M, g) \,. \tag{3.2.33}$$

Last but not least we are are able to give an expression of the Witten index of this system. It reads

$$\mathrm{Tr}(-1)^F \mathrm{e}^{-\beta H} = \sum_{p=0}^{n} (-1)^p \dim H^p(Q) = \sum_{p=0}^{n} (-1)^p h^p(M) = \chi(M) \,, \tag{3.2.34}$$

where $h^p(M)$ are the Betti numbers and $\chi(M)$ is the Euler characteristic of $M$.

### 3.2.2. Deformation by Potential Term

One aim of this chapter is to review the connection between supersymmetric quantum mechanics and Morse theory. To gain this connection we will have to deform the Theory. This subsection gives a brief discussion what we mean by a deformation of a theory in this given context. Let $h : M \longrightarrow \mathbb{R}$ be a smooth map and a deformation of the action (3.2.3) be

$$\Delta S = \int\limits_{\mathcal{T}} \left( -\frac{1}{2} g^{\mu\nu} \partial_\mu h \partial_\nu h - (D_\mu \partial_\nu h) \overline{\psi}^\mu \psi^\nu \right) \tag{3.2.35}$$

with $D_\mu \partial_\nu h = \partial_\mu \partial_\nu h - \Gamma^\kappa_{\mu\nu} \partial_\kappa h$. This changes the supersymmetry transformations to

$$\delta\phi^\mu = \epsilon\overline{\psi}^\mu - \overline{\epsilon}\psi^\mu \, , \tag{3.2.36}$$

$$\delta\psi^\mu = \epsilon \left( i\dot{\phi}^\mu - \Gamma^\mu_{\nu\kappa} \overline{\psi}^\nu \psi^\kappa + g^{\mu\nu} \partial_\nu h \right) \, , \tag{3.2.37}$$

$$\delta\overline{\psi}^\mu = \overline{\epsilon} \left( -i\dot{\phi}^\mu - \Gamma^\mu_{\nu\kappa} \overline{\psi}^\kappa \psi^\nu + g^{\mu\nu} \partial_\nu h \right) \, . \tag{3.2.38}$$

The expression for R-symmetry remains unchanged, while the supercharges are now given by

$$Q_h = d + d\phi^\mu \wedge \partial_\mu h = d + dh\wedge = \mathrm{e}^{-h} d \mathrm{e}^h =: d_h = \mathrm{e}^{-h} Q_0 \mathrm{e}^h \tag{3.2.39}$$

$$Q_h^\dagger = (d + d\phi^\mu \wedge \partial_\mu)^\dagger h = \mathrm{e}^{-h} d^\dagger \mathrm{e}^h =: d_h = \mathrm{e}^{-h} Q_0^\dagger \mathrm{e}^h \, , \tag{3.2.40}$$

with $Q_0$ being the expression of $Q_h$ for $h = 0$. The Hamiltonian reads

$$H_h = \frac{1}{2} \{Q_h, Q_h^\dagger\} = \frac{1}{2} \left( d_h d_h^\dagger + d_h^\dagger d_h \right) \tag{3.2.41}$$

It is clear that $[F, Q_h] = Q_h$ and $[F, Q_h^\dagger] = -Q_h^\dagger$ still holds and thus $\mathcal{H}_{\mathrm{SUSY}}$ is graded by form-degree. Moreover, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^0(M) \arrow[r, "d"] \arrow[d, "\mathrm{e}^{-h}"] & \Omega^1(M) \arrow[r, "d"] \arrow[d, "\mathrm{e}^{-h}"] & \ldots \arrow[r, "d"] & \Omega^n(M) \arrow[r] \arrow[d, "\mathrm{e}^{-h}"] & 0 \\
0 \arrow[r] & \Omega^0(M) \arrow[r, "\mathrm{e}^{-h} d \mathrm{e}^h"] & \Omega^1(M) \arrow[r, "\mathrm{e}^{-h} d \mathrm{e}^h"] & \ldots \arrow[r, "\mathrm{e}^{-h} d \mathrm{e}^h"] & \Omega^n(M) \arrow[r] & 0
\end{tikzcd} \tag{3.2.42}$$

where the vertical arrows are isomorphisms. Therefore, we get isomorphic spaces

$$\mathcal{H}^p_{h,0} \cong H^p(Q_h) \cong H^p(Q_0) \cong H^p_{dR}(M) \cong \mathcal{H}^p_{0,0} \, . \tag{3.2.43}$$

In particular, $\dim \mathcal{H}^p_{h,0}$ is invariant with respect to changes of $h$ and, hence, the Witten index $\mathrm{Tr}(-1)^F \mathrm{e}^{-\beta H}$ is invariant, too.

### 3.2.3. Perturbative Ground States

In [Wit82] Edward Witten described a beautiful connection between supersymmetric quantum mechanics and Morse-theory. It appears by computing instanton corrections to perturbative ground states $\Psi_i$. Physically this means we construct states which are massless up to any order in perturbation theory. But there are non-perturbative quantum corrections to their mass which are called instantons.[1] This subsection is devoted to the construction of perturbative ground states. The next one will give a short motivation of a very important tool in supersymmetric field theories, the localization principle. After that we will identify the instantons which contribute to $\langle \Psi_i, Q\Psi_j \rangle$.

Let $(M, g)$ be a Riemannian manifold. Consider a deformation of supersymmetric quantum mechanics on $M$ by $\lambda h : M \to \mathbb{R}$ in the sense of the last subsection, where $h$ is a Morse function with critical points $x_1$, $x_2$, ..., $x_N$. This yields

$$H_\lambda = \frac{1}{2}\Delta + \frac{1}{2}\lambda^2 g^{\mu\nu}\partial_\mu h\, \partial_\nu h + \frac{1}{2}\lambda D_\mu \partial_\nu h\, [\overline{\psi}^\mu, \psi^\nu] \tag{3.2.44}$$

as the Hamiltonian. It is clear that at very large $\lambda$, low energy states localize at critical points of h.

Let us expand $h$ around its critical points $x_i$

$$h = h(x_i) + \sum_{I=1}^{n} c_I (x^I)^2 + \mathcal{O}\left((x^I)^3\right) , \tag{3.2.45}$$

where $c_I$ are the eigenvalues of the Hessian of $h$ at $x_i$. Therefore, we get in normal coordinates at $x_i$ up to leading order in perturbation theory for $H_\lambda$

$$H_0(x_i) = \sum_{I=1}^{n} \left( \frac{1}{2}{p_I}^2 + \frac{1}{2}\lambda^2 (c_I)^2 (x^I)^2 + \frac{1}{2}\lambda c_I [\overline{\psi}^I, \psi^I] \right) . \tag{3.2.46}$$

This can be done since a change in $h$ or $g$ does not affect $Q$-cohomology. If we compare this to the Hamiltonian of the supersymmetric harmonic oscillator,

$$H^I_{\text{osc}} = \frac{1}{2}{p_I}^2 + \frac{1}{2}{\omega_I}^2 x^2 + \frac{1}{2}\omega_I [\overline{\psi}, \psi], \tag{3.2.47}$$

we infer that

$$H_0(x_i) = \sum_{I=1}^{n} H^I_{\text{osc}} , \tag{3.2.48}$$

[1] Instantons are special solutions to the equations of motion which extremize the action and are located in time. Hence the name instanton.

with $\omega_I = \lambda c_I$. Now $H_{\text{osc}}$ has got eigenfunctions

$$\mathrm{e}^{-\frac{1}{2}\omega x^2}\,|0\rangle \quad \text{for} \quad \omega > 0$$

and

$$\mathrm{e}^{+\frac{1}{2}\omega x^2}\overline{\psi}\,|0\rangle \quad \text{for} \quad \omega < 0\,.$$

Since $[H_I, H_J] \neq 0$ we get for the ground-states up to leading order in perturbation theory

$$\Psi_i^{(0)} = \mathrm{e}^{-\frac{\lambda}{2}\sum_{I=1}^{n}|c_I|(x^I)^2} \prod_{J:c_J<0} \overline{\psi}^J\,|0\rangle\;. \tag{3.2.49}$$

Observe that the number of $\overline{\psi}^J$ which multiply $|0\rangle$ is the Morse index $\mu_i$ of $h$ at $x_i$ and, hence, $\Psi_i^{(0)}$ is a $\mu_i$-form. It is possible to add corrections to $\Psi_i^{(0)}$ yielding $\Psi_i$, such that $\Psi_i$ is a ground state to all orders of perturbation theory. Since the latter preserves fermion-number, it holds

$$\Psi_i \in \Omega^{\mu_i}(M) \otimes \mathbb{C}\,. \tag{3.2.50}$$

### 3.2.4. Instanton Corrections

In this subsection we will apply the localization principle to supersymmetric quantum mechanics on Riemannian manifolds. In the last subsection we constructed perturbative ground-states $\Psi_i$ corresponding to critical points $x_i$ of $h : M \to \mathbb{R}$. They are massless up to any order in perturbation theory. But there are possible quantum corrections to their masses such that some of them produce non-vanishing matrix elements of $Q_h$.[2] As for $\lambda \to \infty$ all states diverge except $\Psi_i$ for all $i$. Moreover, the considerations in subsection 3.2.2 show that the number of ground states is invariant under deformations of $h$. Therefore, the number of true ground states does not exceed the number of critical points. Calculating

$$Q_{ij} = \langle \Psi_i, Q\Psi_j\rangle \tag{3.2.51}$$

will enable us to define a quantum corrected action of $Q$ on approximate ground states,

$$Q\Psi_i = \sum_{j=1}^{N} \langle \Psi_j, Q\Psi_i\rangle\,\Psi_j = \sum_{j=1}^{N} Q_{ji}\Psi_j\,. \tag{3.2.52}$$

[2]In order to keep notations simple we drop the $h$ in the following.

Thus, we sought after

$$\int_M \overline{\Psi_j} \wedge *(d + dh\wedge)\Psi_i = \langle \Psi_j, Q\Psi_i \rangle \ . \tag{3.2.53}$$

Because differential-forms of different degree are orthogonal with respect to (3.2.18), the above equation is only non-vanishing if $\mu_j = \mu_i + 1$. Since we are not aware of the concrete form of $\Psi_i$, we compute (3.2.53) using path-integration.

In the large volume limit $\lambda \to \infty$ the perturbative ground states localize at the critical points of h. Therefore, $h$ viewed as an operator sends

$$h : \Psi_i \mapsto h(x_i)\Psi_i \, . \tag{3.2.54}$$

Now let $\mathrm{e}^{-TH}$ project on the perturbative ground state which corresponds to the critical point $x_i$ of $h$. Then it holds

$$\lim_{T\to\infty} \left\langle \Psi_j, \mathrm{e}^{TH}[Q,h]\mathrm{e}^{-TH}\Psi_i \right\rangle = \left\langle \Psi_j, \left( Qh(x_i) - h(x_j)Q + \mathcal{O}\left(\frac{1}{\lambda}\right)\right)\Psi_i \right\rangle \, . \tag{3.2.55}$$

Hence, we infer

$$\langle \Psi_j, Q\Psi_i \rangle = \frac{1}{h(x_i) - h(x_j) + \mathcal{O}\left(\frac{1}{\lambda}\right)} \lim_{T\to\infty} \left\langle \Psi_j, \mathrm{e}^{TH}[Q,h]\mathrm{e}^{-TH}\Psi_i \right\rangle \, . \tag{3.2.56}$$

To get an expression which we are able to compute we use

$$\begin{aligned}[Q,h]\omega = [d + dh\wedge, h]\omega &= dh\wedge\omega + hd\omega + hdh\wedge\omega - hd\omega - hdh\wedge\omega = \\ &= dh\wedge\omega = \partial_\mu h \, d\phi^\mu \wedge \omega = \partial_\mu h \overline{\psi}^\mu \omega\end{aligned} \tag{3.2.57}$$

to obtain

$$\langle \Psi_j, Q\Psi_i \rangle = \frac{1}{h(x_i) - h(x_j) + \mathcal{O}\left(\frac{1}{\lambda}\right)} \int_{\mathcal{F}} \mathscr{D}\phi \mathscr{D}\psi \mathscr{D}\overline{\psi} \; \mathrm{e}^{-S_E} \overline{\psi}^\mu \partial_\mu h \, . \tag{3.2.58}$$

Here $\mathcal{F}$ is the set of superfields with bosonic part given by

$$\mathcal{F}_{\text{bos}} = \{\phi : \mathcal{T} \longrightarrow M | \phi(-\infty) = x_i \wedge \phi(\infty) = x_j\} \tag{3.2.59}$$

and $S_E$ is the euclidean action obtained by wick rotation $t = -i\tau$. Its bosonic part reads

$$S_{\text{bos}} = \frac{1}{2}\int_{-\infty}^{\infty} d\tau \, \left( \left| \frac{d\phi^\mu}{d\tau} \pm \lambda g^{\mu\nu}\partial_\nu h \right|^2 \mp \lambda(h(x_j) - h(x_i)) \right) \, , \tag{3.2.60}$$

where we used $|V^\mu|^2 := V^\mu g_{\mu\nu} V^\nu$ and the boundary conditions $\phi(-\infty) = x_i$ as well as $\phi(\infty) = x_j$. Equation (3.2.60) is minimized by

$$\frac{d\phi^\mu}{d\tau} \pm \lambda g^{\mu\nu}\partial_\nu h = 0 \quad \text{for} \quad h(x_j) \lessgtr h(x_i)\,. \tag{3.2.61}$$

Later we will see that $Q$-fixed points in field-space fulfill (3.2.61), too. There a sign will be chosen naturally. Such configurations are precisely the instantons of the theory. Before we will treat that, we should look for possible deformations of an instanton. To this end let us look at a first order variation of (3.2.61),

$$\begin{aligned} \frac{d}{d\epsilon}\left(\frac{d(\phi^\mu + \epsilon\,\delta\phi^\mu)}{d\tau} \pm \lambda g^{\mu\nu}(\phi^\mu + \epsilon\,\delta\phi^\mu)\partial_\nu h(\phi^\mu + \epsilon\,\delta\phi^\mu)\right)\Bigg|_{\epsilon=0} &= \\ = \mathcal{D}_\pm \delta\phi^\mu := D_\tau \delta\phi^\mu \pm \lambda g^{\mu\nu}(D_\mu \partial_\kappa)\delta\phi^\kappa &= 0\,. \end{aligned} \tag{3.2.62}$$

This means that deformations of an instanton are given by the zero-modes of $\mathcal{D}_\pm$. In particular we get for the fermion bilinear part of $S_E$

$$S_{\psi\overline{\psi}} = -\int_{-\infty}^{\infty} d\tau\, g_{\mu\nu}(\mathcal{D}_- \overline{\psi}^\mu)\psi^\nu\,. \tag{3.2.63}$$

We saw in subsection 3.1.2 that path-integration of an operator which is invariant under $Q$ only picks up $Q$-fixed points in field-space. To be able to use this fact we have to assure that $[Q,h] = \overline{\psi}^\mu \partial_\mu h$ is invariant under $Q$. Since $[Q,h]$ has fermion-number 1 and

$$\{Q,[Q,h]\} = Q^2 h - QhQ + QhQ - hQ^2 = 0\,, \tag{3.2.64}$$

this obviously holds. The infinitesimal euclidean supersymmetry transformation generated by $Q$ ($\overline{\epsilon} = 0$) are

$$\delta\phi^\mu = \epsilon\overline{\psi}^\mu\,, \tag{3.2.65}$$

$$\delta\psi^\mu = \epsilon\left(-\frac{d\phi^\mu}{d\tau} + \lambda g^{\mu\nu}\partial_\nu h - \Gamma^\mu_{\nu\kappa}\overline{\psi}^\mu\psi^\nu\right)\,, \tag{3.2.66}$$

$$\delta\overline{\psi}^\mu = 0\,. \tag{3.2.67}$$

Thus we observe that the $Q$-fixed configurations (instantons) in field-space are given by

$$\overline{\psi}^\mu = 0 \quad \text{and} \quad \frac{d\phi^\mu}{d\tau} = \lambda g^{\mu\nu}\partial_\nu h\,, \tag{3.2.68}$$

which is exactly the instanton equation (3.2.61) for $h(x_j) > h(x_i)$. This shows that in this case an instanton is an ascending gradient flow for $h$ from $x_i$ to $x_j$ (steepest ascent).

Since there is one insertion of $\overline{\psi}^\mu$ in (3.2.58), we observe that the path-integral is non-vanishing if and only if there is one more $\mathcal{D}_-$ zero-mode of $\overline{\psi}^\mu$ than $\psi^\mu$ zero-modes. Hence $\operatorname{ind}\mathcal{D}_- = 1$. Next we will motivate that the index of the respective Dirac operator is given by the relative Morse index

$$\operatorname{ind}\mathcal{D}_- = \Delta\mu = \mu_j - \mu_i \,. \tag{3.2.69}$$

*Proof.* Let $H_h(x)$ be the Hessian of $h$ at $x$. As a map $H_h(x) : T_xM \longrightarrow T_xM$ it is represented as a symmetric $n \times n$ matrix and can be diagonalized using eigenvectors $e_I$ with corresponding eigenvalues $\lambda_J$. A family of eigenvectors and eigenvalues along a trajectory $\phi$ starting at $x_i$ and ending at $x_j$ fulfills

$$H_h(\phi(\tau))e_I(\phi(\tau)) = \lambda_I(\tau)e_I(\tau)\,. \tag{3.2.70}$$

The family of eigenvalues is called spectral flow. Now the relative Morse-index counts the net number of eigenvalues going from positive to negative values,

$$\Delta\mu = \#\{\lambda_I(-\infty) > 0 \wedge \lambda(\infty) < 0\} - \#\{\lambda_I(-\infty) < 0 \wedge \lambda(\infty) > 0\}\,. \tag{3.2.71}$$

Consider the differential operator

$$\tilde{\mathcal{D}}_\mp := \tilde{D}_\tau \mp \phi^* H_h\,, \tag{3.2.72}$$

with $e_I$ being parallel with respect to $\tilde{D}_-$. It can be written as

$$\tilde{\mathcal{D}}_\mp := \frac{d}{d\tau} \mp \begin{pmatrix} \lambda_1(\tau) & 0 & \cdots & 0 \\ 0 & \lambda_2(\tau) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n(\tau) \end{pmatrix}\,. \tag{3.2.73}$$

Since $\mathcal{D}_- = D_\tau - \phi^* H_h$ it holds $\operatorname{ind}\mathcal{D}_- = \operatorname{ind}\tilde{\mathcal{D}}_-$. Next we will compute $\operatorname{ind}\tilde{\mathcal{D}}_-$ and see that it equals the relative Morse index $\Delta\mu$. To this end we look at the equation $\tilde{\mathcal{D}}_\mp f_\mp = 0$. It is solved by

$$f_{I,\mp}(\tau) = e_I \exp\left(\pm\int_o^\tau \lambda_I(\tau')d\tau'\right)\,. \tag{3.2.74}$$

Using

$$\int_{-\infty}^{\infty} |f_{I,\mp}|^2 d\tau = \int_{-\infty}^{\infty} \exp\left(\pm 2\int_o^\tau \lambda_I(\tau')d\tau'\right) d\tau \tag{3.2.75}$$

we observe that $f_{I,-}$ is square normalizable if and only if $\lambda(\infty) < 0$ and $\lambda(-\infty) > 0$, whereas $f_{I,+}$ is square normalizable if and only if $\lambda(\infty) > 0$ and $\lambda(-\infty) < 0$. Therefore, $\operatorname{ind}\tilde{\mathcal{D}}_- = \dim\ker\tilde{\mathcal{D}}_- - \dim\ker\tilde{\mathcal{D}}_+ = \Delta\mu$. This shows the claim. □

Now we are ready to evaluate (3.2.58). We will do so by using the quadratic approximation of $S_E$, omitting terms of higher degree than two in fermion fields. This is justified since we saw that in order to perform path-integration of a supersymmetric expression we can sum over the $Q$-fixed points (3.2.61) weighted by small fluctuations around these configurations. Let us denote bosonic fluctuations by $\xi$, the fermionic fluctuations by $\psi$, $\overline{\psi}$ and the bosonic part of a $Q$-fixed configuration by $\gamma$. Then the use of (3.2.63), (3.2.60) and the natural choice of the signs therein yields

$$S_E = -\lambda(h(x_i) - h(x_j)) + \int \left( \frac{1}{2}|\mathcal{D}_-\xi|^2 - (\mathcal{D}_-\overline{\psi})\psi \right) d\tau \,. \tag{3.2.76}$$

It is clear by definition that the $\xi$ zero modes with respect to $\mathcal{D}_-$ are given by the deformations of $\gamma$. Therefore, path-integration over the zero modes of $\xi$ results in an integral over the moduli space $[\gamma]$ of $\gamma$. The kernel of $\mathcal{D}_-$ is one-dimensional and we denote deformations of $\gamma$ by $\delta\gamma$ and its coordinate $\tau_1$.[3] Physically this corresponds to a translation of the instanton in space-time. Zero modes of $\overline{\psi}^\mu$ are notated as $\overline{\psi}_0^\mu$. Thus,

$$\begin{aligned}\int\limits_{\substack{\phi(-\infty)=x_i\\ \phi(\infty)=x_j}} \mathscr{D}\xi\mathscr{D}\psi\mathscr{D}\overline{\psi}\, \mathrm{e}^{-S_E(\xi,\psi,\overline{\psi})}[Q,h] = \sum_\gamma \int\limits_{\substack{\text{non-zero}\\ \text{modes}}} \mathscr{D}\xi\mathscr{D}\psi\mathscr{D}\overline{\psi} \int\limits_{[\gamma]} d(\delta\gamma) \int d\overline{\psi}_0\, \overline{\psi}_0^\mu \times \\ \times\, \partial_\mu h(\gamma+\delta\gamma) \mathrm{e}^{\lambda(h(x_i)-h(x_j))} \mathrm{e}^{-\int\left(\frac{1}{2}|\mathcal{D}_-\xi|^2 - (\mathcal{D}_-\overline{\psi})\psi\right)d\tau}\end{aligned} \tag{3.2.77}$$

Setting $\mathscr{D}\Phi := \mathscr{D}\xi\mathscr{D}\psi\mathscr{D}\overline{\psi}\, \mathrm{e}^{-\int\left(\frac{1}{2}|\mathcal{D}_-\xi|^2 - (\mathcal{D}_-\overline{\psi})\psi\right)d\tau}$ and putting in the respective definitions yields up to order $1/\lambda$

$$(h(x_i) - h(x_j))\, Q_{ij} = \sum_\gamma \mathrm{e}^{\lambda(h(x_i)-h(x_j))} \int\limits_{\substack{\text{non-zero}\\ \text{modes}}} \mathscr{D}\Phi \int\limits_{-\infty}^{\infty} d\tau_1 \int d\overline{\psi}_0 \frac{d(\delta\gamma)}{d\tau_1} \overline{\psi}_0^\mu \partial_\mu h(\gamma+\delta\gamma) \tag{3.2.78}$$

As at the boundary of $[\gamma]$ the instanton stays at $x_j$ and $x_i$, we are able to evaluate

[3] According to $[\gamma](\tau,\tau_1) = \gamma(\tau+\tau_1) = \gamma(\tau) + \delta\gamma(\tau,\tau_1)$.

(3.2.78),

$$\sum_\gamma \mathrm{e}^{\lambda(h(x_i)-h(x_j))} \int\limits_{\substack{\text{non-zero}\\ \text{modes}}} \mathscr{D}\xi\mathscr{D}\psi\mathscr{D}\overline{\psi}\, \mathrm{e}^{-\int\left(\frac{1}{2}|\mathcal{D}_-\xi|^2-(\mathcal{D}_-\overline{\psi})\psi\right)d\tau}(h(x_j)-h(x_i)) =$$

$$= \sum_\gamma \mathrm{e}^{\lambda(h(x_i)-h(x_j))}\,(h(x_j)-h(x_i))\,\underbrace{\frac{\det'\mathcal{D}_-}{|\det'\mathcal{D}_-|}}_{n_\gamma}\,. \tag{3.2.79}$$

Therefore, the matrix-element $Q_{ji}$ is given by

$$Q_{ji} = \sum_\gamma n_\gamma\, \mathrm{e}^{-\lambda(h(x_j)-h(x_i))}\,. \tag{3.2.80}$$

Absorbing the exponential factor via $\Psi_i \longrightarrow \mathrm{e}^{\lambda h(x_i)}\Psi_i$ we conclude that

$$Q\Psi_i = \sum_{j:\Delta\mu=1}\left(\sum_\gamma n_\gamma\right)\Psi_j\,. \tag{3.2.81}$$

Next we wish to determine the precise value of $n_\gamma$. Looking at (3.2.53) and (3.2.80) we observe that the integral gets dominant contributions by a path $\gamma$ of steepest ascent of $h$. Thus $n_\gamma = 1$ if the orientation of $\Psi_i \wedge *(d + dh\wedge)\Psi_j$ coincides with the orientation of $M$ along $\gamma$ and $n_\gamma = -1$ otherwise. It can be shown [Wit82] that this coincides with the definition of the differential in Morse-homology. Since we do not need it at this point, I will not review the proof here. The important fact is that in order to get the true ground states we construct the graded space of approximate ground states with $\mu$ chains

$$C^\mu := \bigoplus_{\mu_i=\mu} \mathbb{C}\Psi_i\,, \tag{3.2.82}$$

being the free graded complex of critical points of $h$. Then $Q$ acts on this complex by (3.2.81)

$$0 \longrightarrow C^0 \xrightarrow{Q} C^1 \xrightarrow{Q} \ldots \xrightarrow{Q} C^n \xrightarrow{Q} 0\ . \tag{3.2.83}$$

This is exactly the Morse-complex and its cohomology defines true ground-states. This is in accordance with the fact that Morse-homology is isomorphic to ordinary de Rham cohomology.

We should learn the following from this considerations. To obtain the true supersymmetric ground states of a supersymmetric quantum mechanics we can do two things.

If there is a well defined operator $Q$ acting on the states in the prescribed manner, supersymmetric ground-states are given by its cohomology. The other way to get them is to start with approximate (perturbative) ground states $\Psi_i$ and compute quantum corrections to the action of $Q$. Using the localization principle they can be calculated via summing over infinitesimal neighborhoods of $Q$-fixed points, which are instantons in the theory. This will result in some kind of Morse-theory. In the A-model of topological string theory for example such considerations lead to Floer-homology. A good introduction to Floer theory can be found in [Sal97].

# 4. Nonlinear Sigma-Models on Riemannian Manifolds

In this chapter we treat nonlinear sigma-models on Riemannian manifolds. The focus will be on the formulation of the material in a thorough geometrical language. After that we will give an intuitive construction of solutions of the equations of motion, being harmonic maps, via topological methods. Although this construction seems to be very natural, the author has not found such constructions applied to this context elsewhere in the literature. One corollary of this construction is, in physical language, the proof that even in curved space, strings obey a mode-expansion. This is a surprising result, at least from the mathematical viewpoint. The resulting equations of motion are nonlinear and therefore the sum of two solutions is not a solution in general. It turns out that in contrast to euclidean space-time, in the curved case not every mode of the string is physical, but only those which are associated to a smooth map between the world-sheet $\Sigma$ and the target manifold $M$. At the end of chapter 4 we will give a short outlook of possible applications of these concepts. But as the focus of this work is on a generalization of pseudo-holomorphic curves to generalized complex manifolds, we will not go very much into details. In particular, we will state a program of how to construct the quantum theory of nonlinear sigma models on arbitrary target manifolds.

In textbooks on string theory, e.g. [Pol98a], [GSW87] and [BBS07], nonlinear sigma-models are usually introduced as a generalization of a particle to a tiny vibrating string of energy moving in space-time. While a moving particle defines a so called world line, a moving string defines an embedding of a world-sheet $\Phi(\Sigma)$[1]. It is postulated to be in such a way that the area of the embedding $\Phi : \Sigma \to M$ into Lorentzian space-time $M$ is minimal, i.e.

$$S_0 = \mathrm{i} \int_{\Sigma} \eta_{\mu\nu} \partial_a \phi^\mu \partial_b \phi^\nu \eta^{ab} d^2\sigma \,. \tag{4.0.1}$$

is minimal. This is the starting point of the formulation of string theory as a conformal quantum field theory, which we will not discuss here.

[1] The genus of $\Sigma$ corresponds to the loop expansion of ordinary QFT.

## 4.1. Action and Equations of Motion

To be more precise, let $(M, g, \nabla)$ be a Riemannian manifold with metric $g$ and Levi-Civita connection $\nabla$. Moreover, let $(\Sigma, h)$ be a Riemann surface and $\Phi : \Sigma \to M$ be a smooth embedding, i.e. an injective immersion, of $\Sigma$ into $M$. If $M$ is a manifold, we denote the coordinate maps as $\xi_M$. Let us define the action of nonlinear sigma-models to be

$$S[\Phi] := \int_\Sigma \sqrt{|h|}\, d^2\sigma\, g_{\mu\nu}(\Phi(\sigma))\, \partial_a\phi^\mu \partial_b\phi^\nu\, h^{ab}\,, \tag{4.1.1}$$

where $\phi^\mu$ are local coordinates of $\Phi$. Here we slightly abuse notation, since we omit the use of a partition of unity to integrate over the whole manifold $\Sigma$. Using the fact that $\Sigma$ is two-dimensional, it is easy to show that (4.1.1) is invariant under $\mathrm{Diff}(M)$, $\mathrm{Diff}(\Sigma)$ and $\mathrm{Weyl}(\Sigma)$.

**Proposition 4.1.1.** *Let $(M, g, \nabla)$, $(\Sigma, h)$ and $\Phi : \Sigma \to M$ be as above. Then in local coordinates the equations of motion for $\Phi$ defined by* (4.1.1) *are*

$$\Delta_\Sigma \phi^\lambda - \Gamma^\lambda{}_{\mu\nu} \frac{\partial \phi^\mu}{\partial \sigma^a} \frac{\partial \phi^\nu}{\partial \sigma^b} h^{ab} = 0\,. \tag{4.1.2}$$

*Proof.* A proof can be found in [Jos95]. □

The next section will treat a global formulation of $S[\Phi]$ and its equations of motion.

## 4.2. Geometric Formulation

In order to obtain a global and intrinsically geometric formulation of non-liner sigma-models, we have to exploit several constructions. We will see that the fundamental objects in such theories are differential forms on $\Sigma$ with values in in the pullback of the tangent bundle $\pi_{TM} : TM \to M$ under $\Phi$, which we denote as $\Phi^*TM$. Its total space is given by

$$\Phi^*TM := \{(\sigma, X) \in \Sigma \times TM | \Phi(\sigma) = \pi_{TM}(X)\} \tag{4.2.1}$$

with transition functions

$$t^*_{ij}(\sigma) := t^{TM}_{ij}(\Phi(\sigma))\,, \tag{4.2.2}$$

and projections

$$\mathrm{pr}_1 : \Phi^*TM \longrightarrow \Sigma, \quad (\sigma, X) \longmapsto \sigma\,, \tag{4.2.3}$$

$$\mathrm{pr}_2 : \Phi^*TM \longrightarrow TM, \quad (\sigma, X) \longmapsto X\,. \tag{4.2.4}$$

In particular, the forms which are needed to enable a global formulation of (4.1.1) are associated to a smooth map $\Phi : \Sigma \to M$ and defined as

$$d\Phi(v) := T\Phi \circ v\,. \tag{4.2.5}$$

Here $v \in \Gamma(\Sigma, T\Sigma)$, and $T\Phi$ is the differential of $\Phi$.[2] To see that this truly defines a $\Phi^*TM$ valued one-form, i.e. an element of $\Omega^1(\Sigma, \Phi^*TM)$, we first need

**Lemma 4.2.1.** *Let $\Sigma$ and $M$ be smooth manifolds and $\Phi : \Sigma \to M$ be a smooth map. Then*

$$\Gamma(\Sigma, \Phi^*TM) \cong \{f : \Sigma \to TM | \pi_{TM} \circ f = \Phi\}\,. \tag{4.2.6}$$

*We will call such maps vector fields along $\Phi$.*

*Proof.* Let $U \in \Gamma(\Sigma, \Phi^*TM)$. Hence, $(\mathrm{pr}_1 \circ U)(\sigma) = \sigma$. Thus, $U(\sigma) = (\sigma, X_\sigma)$ with $\pi_{TM}(X_\sigma) = \sigma$. Therefore, we define $f_U(\sigma) := X_\sigma$. By construction $f_U$ is a vector-field along $\Phi$. The map $g : U \mapsto f_U$ is clearly injective. Moreover, there is a map $h : f \mapsto U_f$ defined by $U_f(\sigma) := (\sigma, f(\sigma))$. By construction it is true that $U_f \in \Gamma(\Sigma, \Phi^*TM)$ and $g \circ h = \mathrm{id}$, as well as $h \circ g = \mathrm{id}$. $\square$

Usual one-forms $\omega \in \Omega^1(\Sigma)$ assign smoothly to every point $\sigma \in \Sigma$ a linear map $T_\sigma M \to \mathbb{R}$, which means $\omega(\sigma)(v) \in \mathbb{R}$ for $v \in \Gamma(\Sigma, T\Sigma)$. Because of $\pi_{TM} \circ d\Phi(v) = \Phi$, we infer that $d\Phi(v)$ is a vector-field along $\Phi$. Using lemma 4.2.1, $d\Phi$ assigns smoothly to every $\sigma \in \Sigma$ a linear map $T_\sigma\Sigma \to {\mathrm{pr}_1}^{-1}(\sigma) =: (\Phi^*TM)_\sigma$. Hence, it can be interpreted as a $\Phi^*TM$ valued one-form.

To give a summary, we draw the commutative diagram

$$\begin{tikzcd}
T\Sigma \arrow[r, "T\Phi"] \arrow[d, shift left, "\pi_{T\Sigma}"] & TM \arrow[d, "\pi_{TM}"] & \\
\Sigma \arrow[u, shift left, "v"] \arrow[ur, "d\Phi(v)"] \arrow[r, "\Phi"] \arrow[drr, shift left, "U"] & M & \\
& & \Phi^*TM \arrow[uul, "\mathrm{pr}_2"'] \arrow[ull, shift left, "\mathrm{pr}_1"]
\end{tikzcd} \tag{4.2.7}$$

[2]We use this convention to emphasize that $T$ is a functor from the category of smooth manifolds $\mathrm{Man}_\infty$ to the category of smooth vector bundles $\mathrm{VecB}$.

Since we are interested in the connection between $d\Phi$ and equation (4.1.1), whose integrand is a local expression, we have to compute the local coordinates of $d\Phi$. To this end recall that for any vector-bundle $\pi : E \to \Sigma$ the space of $E$-valued forms on $\Sigma$, i.e. $\Omega^*(\Sigma, E)$, is isomorphic to

$$\Omega^*(\Sigma, E) \cong \Omega^*(\Sigma) \otimes E \cong \bigoplus_{k=0}^{\dim \Sigma} \Omega^k(\Sigma) \otimes E\,. \tag{4.2.8}$$

Hence, every $\eta \in \Omega^k(\Sigma, E)$ can be decomposed as

$$\eta = \omega^\mu \otimes X_\mu \tag{4.2.9}$$
$$\eta(u) := \omega^\mu(u) \cdot X_\mu\,, \tag{4.2.10}$$

where $X_\mu \in \Gamma(\Sigma, E)$, $u \in \Gamma(\Sigma, T\Sigma)$.[3] In the case of $E$ being equipped with a linear connection $\nabla$, we define its action on $E$-valued p-forms via

$$d_\nabla \eta \equiv \nabla(\eta^\mu \otimes X_\mu) := d\eta^\mu \otimes X_\mu + (-1)^p \eta^\mu \wedge \nabla X_\mu\,. \tag{4.2.11}$$

If $p = 0$ the wedge product is just ordinary multiplication.

Let us return to the computation of the local expression of $d\Phi$. After decomposing $v \in \Gamma(\Sigma, T\Sigma)$ as $v = v^a \otimes \partial_a$, we obtain

$$d\Phi(v) = T\Phi \circ v = v^a \frac{\partial}{\partial \sigma^a}\left(\xi^\mu_M \circ \Phi \circ \xi^{-1}_\Sigma\right)(\sigma) \frac{\partial}{\partial x^\mu} = v^a\, \frac{\partial \phi^\mu}{\partial \sigma^a} \frac{\partial}{\partial x^\mu}\,. \tag{4.2.12}$$

Thus we conclude that locally

$$d\Phi = \frac{\partial \phi^\mu}{\partial \sigma^a}\, d\sigma^a \otimes \frac{\partial}{\partial x^\mu}\,. \tag{4.2.13}$$

Since $S[\Phi]$ has to be a number for all $\Phi$, we need an inner product on $\Omega^*(\Sigma, \Phi^*TM)$ to achieve this. Such an inner product can be constructed using the metric $g$ on $TM$ and $h$ on $T\Sigma$ by defining for $\alpha = \eta \otimes U \in \Gamma(\Sigma, \Phi^*TM)$ and $\beta = \omega \otimes V \in \Gamma(\Sigma, \Phi^*TM)$ a local inner product

$$(\alpha, \beta) := h(\eta, \omega)\, \tilde{g}(U, V)\,. \tag{4.2.14}$$

Here $\tilde{g}$ is the metric on $\Phi^*TM$ which is induced by $g$, i.e.

$$\tilde{g}_\sigma(U, V) := g_{\Phi(\sigma)}(\mathrm{pr}_2\, U, \mathrm{pr}_2\, V)\,. \tag{4.2.15}$$

[3]Hence, local sections $U$ of $E$ can be written as $U = U^\mu \otimes e_\mu$. This corresponds to the usual decomposition $U = U^\mu e_\mu$.

Let us define a global inner product on $\Omega^*(\Sigma, \Phi^*TM)$ by

$$\langle \alpha, \beta \rangle := \int_\Sigma (\alpha, \beta) * 1\,, \tag{4.2.16}$$

where the Hodge-star acts on $\Sigma$. This enables us to state

**Proposition 4.2.2.** *Let* $(\Sigma, h)$ *and* $(M, g)$ *be Riemannian manifolds and* $\Phi \in \mathcal{C}^1(\Sigma, M)$*. Then equation* (4.1.1) *can be written as*

$$\int_\Sigma \sqrt{|h|}\, d^2\sigma\, g_{\mu\nu}(\Phi(\sigma))\, \partial_a \phi^\mu \partial_b \phi^\nu\, h^{ab} = \langle d\Phi, d\Phi \rangle \tag{4.2.17}$$

*Proof.* Using $*1 = \mathrm{dvol}_\Sigma = \sqrt{|h|}\, d\sigma^1 \wedge \ldots \wedge d\sigma^{\dim \Sigma}$ and locally

$$(d\Phi, d\Phi) = g_{\Phi(\sigma)} \left( \frac{\partial \phi^\mu}{\partial \sigma^a} \frac{\partial}{\partial x^\mu}, \frac{\partial \phi^\nu}{\partial \sigma^b} \frac{\partial}{\partial x^\nu} \right) h_\sigma(d\sigma^a, d\sigma^b) = g_{\mu\nu} \frac{\partial \phi^\mu}{\partial \sigma^a} \frac{\partial \phi^\nu}{\partial \sigma^b} h^{ab}\,, \tag{4.2.18}$$

as well as equation (4.2.16) the statement is evident. □

Hence, we obtained a global formulation of the action of nonlinear sigma-models. It would be very natural, if we could bring the left $d$ in $\langle d\Phi, d\Phi \rangle$ on the right side to gain $\langle \Phi, d^\dagger\, d\Phi \rangle$. The operator $d^\dagger$ should be the formal adjoint of $d$. Then minimizing the action would be achieved by demanding $d^\dagger\, d\Phi = 0$. This should correspond to some harmonic element in some complex. But not surprisingly, such a naive ansatz is somehow problematic. First, $\langle \cdot, \cdot \rangle$ acts on $\Phi^*TM$ valued k-forms and $\Phi$ is a map between $\Sigma$ and $M$. Second, since $d\Phi$ is a $\Phi^*TM$ valued one-form, $d^\dagger d\Phi$ has to be a $\Phi^*TM$ valued zero-form, i.e. a section in $\Phi^*TM$, for an explanation via complexes to make sense. But then $\Phi$ has to be a section of $\Phi^*TM$, too. This is obviously not the case. One possible way out of this conflict could be to view $\Phi$ as a $\Phi^*TM$ valued zero-form, since by lemma 4.2.1 all sections of $\Phi^*TM$ encode the information on $\Phi$ as a map from $\Sigma$ to $M$.

Later we will see that the equations of motion in this global setting read $\tau(\Phi) := -d^\dagger_{\Phi^*\nabla}\, d\Phi = 0$. Here $d^\dagger_{\Phi^*\nabla}$ is the formal adjoint of $d_{\Phi^*\nabla}$ which is the covariant exterior differential on $\Omega^*(\Sigma, \Phi^*TM)$ with respect to the pullback connection $\Phi^*\nabla$ of the Levi-Civita connection $\nabla$ on $TM$. The $\Phi^*TM$ valued zero-form $\tau(\Phi)$ is sometimes called "tension-field".

This suggests that $d^\dagger$ from above should be in fact $d^\dagger_{\Phi^*\nabla}$ and there should be some $\Phi^*TM$ valued zero-form $\varphi$ such that $d\Phi = d_{\Phi^*\nabla}\varphi$. In section 4.3 we ask whether a $d^\dagger_{\Phi^*\nabla}$-closed element can be written as $d\Phi$. The existence of such a $\varphi$ would ease the search for an answer, but we will not follow this route now.

Our next goal is the global formulation of the equations of motion. It is

**Proposition 4.2.3.** *Let $(\Sigma, h)$ and $(M, g)$ be Riemannian manifolds, $\nabla$ the Levi-Civita connection on $TM$, $\Phi : \Sigma \to M$ a smooth embedding and $S[\Phi] = \langle d\Phi, d\Phi\rangle$. Then the associated equations of motion are*

$$d^{\dagger}_{\Phi^*\nabla}\, d\Phi = 0\,, \tag{4.2.19}$$

*where $\Phi^*\nabla$ is the pullback of $\nabla$ to $\Phi^*TM$.*

To be able to prove this proposition we have to recall a few facts on fiber bundles and connections.

Let $\pi : E \to \Sigma$ and $\tilde{\pi} : \tilde{E} \to \tilde{\Sigma}$ be vector bundles with connection $\nabla$ and $\tilde{\nabla}$, respectively. Moreover, let $\varphi : E \to \tilde{E}$ be a bundle map which restricts to isomorphisms $\varphi_\sigma$ in each fiber above $\sigma$ and induces a bijective map $\psi : \Sigma \to \tilde{\Sigma}$. Then for any connection $\tilde{\nabla}$ on $\tilde{E}$ there is a connection $\nabla$ on $E$ such that $\varphi$ is connection preserving, viz. the diagram

$$\begin{tikzcd}
\Gamma(\Sigma, E) \arrow[d, "\nabla"'] & \Gamma(\tilde{\Sigma}, \tilde{E}) \arrow[l, "\varphi^\sharp"'] \arrow[d, "\tilde{\nabla}"] \\
\Omega^1(\Sigma, E) & \Omega^1(\tilde{\Sigma}, \tilde{E}) \arrow[l, "\varphi^\sharp"]
\end{tikzcd} \tag{4.2.20}$$

commutes, see [GHV72] section 7.13. The map $\varphi^\sharp : \Omega^*(\tilde{\Sigma}, \tilde{E}) \to \Omega^*(\Sigma, E)$ is defined by

$$(\varphi^\sharp(U))(\sigma) := \varphi_\sigma^{-1}(U(\psi(\sigma))\,, \tag{4.2.21}$$

for $U \in \Gamma(\tilde{\Sigma}, \tilde{E})$ and more generally

$$(\varphi^\sharp \alpha)_\sigma(u_1, \dots, u_k) := \varphi_\sigma^{-1}\left(\alpha_{\psi(\sigma)}(T\psi\, u_1, \dots, T\psi\, u_k)\right)\,, \tag{4.2.22}$$

where $u_1$, ..., $u_k \in \Gamma(\Sigma, T\Sigma)$ and $\alpha \in \Omega^*(\tilde{\Sigma}, \tilde{E})$ for $k \geq 1$. The connection $\nabla$ is defined by

$$\nabla(\varphi^\sharp U) := \varphi^\sharp\left(\tilde{\nabla}(U)\right)\,. \tag{4.2.23}$$

It remains to show that for any $u \in \Gamma(\Sigma, \Phi^*TM)$ there exists a $U \in \Gamma(\Phi(\Sigma), TM|_{\Phi(\Sigma)})$ such that $u = \varphi^\sharp U$. This is

**Lemma 4.2.4.** *Let $\pi : E \to \Sigma$ and $\tilde{\pi} : \tilde{E} \to \tilde{\Sigma}$ be vector bundles and $\varphi : E \to \tilde{E}$ be a bundle map which restricts to isomorphisms $\varphi_\sigma$ in each fiber above $\sigma$ and induces a bijective map $\psi : \Sigma \to \tilde{\Sigma}$. Then $\varphi^\sharp$ is also bijective.*

*Proof.* Let $\tilde{u}_1, \tilde{u}_2 \in \Gamma(\tilde{\Sigma}, \tilde{E})$. Assume $\varphi^\sharp(\tilde{u}_1) = \varphi^\sharp(\tilde{u}_2)$. Then it follows by definition that $\varphi^{-1} \circ \tilde{u}_1 \circ \psi = \varphi^{-1} \circ \tilde{u}_2 \circ \psi$. Since both $\varphi$ and $\psi$ are bijective and in particular injective we conclude $\tilde{u}_1 = \tilde{u}_2$. To show surjectivity of $\varphi^\sharp$ define $\tilde{u} := \varphi \circ u \circ \psi^{-1}$ for $u \in \Gamma(\Sigma, E)$. It is defined for all $u$ and fulfills $\varphi^\sharp(\tilde{u}) = u$. ☐

The connection $\nabla$ is called the pullback of $\tilde{\nabla}$ and is the unique connection such that $\varphi$ is connection preserving.

Let us apply this construction to the present case of the nonlinear sigma-model by calculating the connection coefficients of $\Phi^*\nabla$. In the notation of above we have the given data

$$\Sigma = \Sigma\,, \tag{4.2.24}$$
$$\tilde{\Sigma} = \Phi(\Sigma)\,, \tag{4.2.25}$$
$$E = \Phi^* TM\,, \tag{4.2.26}$$
$$\tilde{E} = TM|_{\Phi(\Sigma)}\,, \tag{4.2.27}$$
$$\nabla = \Phi^* \nabla_{TM}\,, \tag{4.2.28}$$
$$\tilde{\nabla} = \nabla_{TM}\,, \tag{4.2.29}$$
$$\varphi = \mathrm{pr}_2\,, \tag{4.2.30}$$
$$\psi = \Phi\,. \tag{4.2.31}$$

With a slight abuse of notation we denote $\nabla_{TM} \equiv \nabla$. Let us first note that according to lemma 4.2.4 it follows that $(\mathrm{pr}_2)^\sharp e_\mu$ is a local frame of $\Phi^*TM$, interpreted as vector fields along $\Phi$, if $e_\mu$ is a local frame of $TM|_{\Phi(\Sigma)}$. Then it is true that

$$\begin{aligned}
(\Phi^*\nabla)\left(\delta^\nu_\mu \cdot (\mathrm{pr}_2)^\sharp e_\nu\right)\left(\frac{\partial}{\partial\sigma^a}\right) &= (\mathrm{pr}_2)^\sharp(\nabla e_\mu) = \\
= {\mathrm{pr}_2}^{-1} \circ \nabla e_\mu \circ T\Phi\left(\frac{\partial}{\partial\sigma^a}\right) &= \left(\sigma, \nabla_{\frac{\partial\phi^\lambda}{\partial\sigma^a}\frac{\partial}{\partial x^\lambda}} e_\mu\right) = \\
= \frac{\partial\phi^\lambda}{\partial\sigma^a}\Gamma^\kappa{}_{\lambda\mu} \cdot (\sigma, e_\kappa) &= \frac{\partial\phi^\lambda}{\partial\sigma^a}\Gamma^\kappa{}_{\lambda\nu}(\mathrm{pr}_2)^\sharp e_\kappa\,,
\end{aligned} \tag{4.2.32}$$

where $\frac{\partial}{\partial\sigma^a}$ is a local frame of $T\Sigma$ and $\Gamma^\lambda{}_{\mu\nu}$ are the Christoffel symbols. Therefore, the connection coefficients of $\Phi^*\nabla$ read[4],

$$A^\lambda{}_{a\nu} = \frac{\partial \phi^\mu}{\partial \sigma^a}\,\Gamma^\lambda{}_{\mu\nu}\,. \tag{4.2.33}$$

If we extend the action of $\Phi^*\nabla$ to $\Phi^*TM$ valued $k$-forms $\alpha$ via equation 4.2.11, we get our sought after exterior covariant differential $d_{\Phi^*\nabla}$. One lemma which we will need in a moment is

**Lemma 4.2.5.** *Let $\pi : E \to \Sigma$ and $\tilde{\pi} : \tilde{E} \to \tilde{\Sigma}$ be vector bundles, $f : E \to \tilde{E}$ a bundle map restricting to isomorphisms in each fiber and inducing a map $\psi : \Sigma \to \tilde{\Sigma}$, $g$ a Riemannian structure on $\tilde{E}$, $\tilde{\nabla}$ a linear connection on $\tilde{E}$, $\nabla$ the pullback connection of $\tilde{\nabla}$ and $h$ the pullback of $g$. Then it is true that if $\tilde{\nabla}$ is a metric connection on $\tilde{E}$ with respect to $g$, it follows that $\nabla$ is a metric connection on $E$ with respect to $h$.*

*Proof.* Let $e_\mu$ be a local frame of $\tilde{E}$. Since $f$ restricts to isomorphisms in each fiber, it follows that $f^\sharp e_\mu$ is a local frame of $E$. By definition the metric $h$ at $\sigma \in \Sigma$ is given by

$$(h_\sigma)_{\mu\nu} := h_\sigma(f^\sharp e_\mu, f^\sharp e_\nu) := g_{f(\sigma)}(e_\mu, e_\nu)\,. \tag{4.2.34}$$

To prove that $\nabla$ is a metric connection it suffices to show

$$\frac{\partial}{\partial \sigma^a}(h_\sigma)_{\mu\nu} = A^\lambda{}_{a\mu}(h_\sigma)_{\lambda\nu} + A^\lambda{}_{a\nu}(h_\sigma)_{\mu\lambda}\,. \tag{4.2.35}$$

The above calculation of the connection coefficients of $\Phi^*\nabla$ shows, mutatis mutandis, that the connection coefficients $A^\mu{}_{a\nu}$ of a general pullback connection $\nabla$ are

$$A^\lambda{}_{a\nu} = \frac{\partial \psi^\mu}{\partial \sigma^a}\,\Gamma^\lambda{}_{\mu\nu}\,, \tag{4.2.36}$$

where $\Gamma^\lambda{}_{\mu\nu}$ are the connection coefficients of $\tilde{\nabla}$. Because of the fact that $\tilde{\nabla}$ is a metric connection it follows that

$$\begin{aligned}\frac{\partial}{\partial \sigma^a} g_{f(\sigma)}(e_\mu, e_\nu) = \frac{\partial \psi^\kappa}{\partial \sigma^a}\frac{\partial}{\partial \tilde{\sigma}^\kappa} g(e_\mu, e_\nu) &= \frac{\partial \psi^\kappa}{\partial \sigma^a}\left(\Gamma^\lambda{}_{a\mu}(h_\sigma)_{\lambda\nu} + \Gamma^\lambda{}_{a\nu}(h_\sigma)_{\mu\lambda}\right) = \\ &= A^\lambda{}_{a\mu}(h_\sigma)_{\lambda\nu} + A^\lambda{}_{a\nu}(h_\sigma)_{\mu\lambda}\end{aligned} \tag{4.2.37}$$

Thus the assertion is true. $\square$

**Remarks 4.2.6** 1. We call $h := f^\sharp g$ the pullback of $g$ w.r.t. the bundle map $f : E \to \tilde{E}$. The usual pullback under a map $\psi : \Sigma \to \tilde{\Sigma}$ corresponds to $f = T\psi$, which clearly induces $\psi$. Then it is true that $f^\sharp = \psi^*$, in accordance to the usual definition.

[4] We use the convention that the connection coefficients of a general linear connection $\nabla$ are defined as $A^\lambda{}_{\mu\nu} e_\lambda = \nabla_{e_\mu} e_\nu = (\nabla e_\nu)(e_\mu) = (A^\lambda{}_\nu \otimes e_\lambda)(e_\mu) = (A^\lambda{}_\nu)_\mu e_\lambda$.

2. It is possible to lax the condition of $f$ restricting to isomorphisms in each fiber to $f$ just being injective. But since we will only need the assertion in the given form, we omit this generalization.

The next step in our preparation of the proof of proposition 4.2.3 is to get an expression of $d^{\dagger}_{\Phi^*TM}$, i.e. the formal adjoint of $d_{\Phi^*\nabla}$ w.r.t. equation (4.2.16). It is given in

**Lemma 4.2.7.** *Let $(\Sigma, h)$ be a Riemannian manifold, $\pi : E \to \Sigma$ be a vector bundle and $\nabla$ be a linear metric connection on $E$ with respect to a Riemannian structure $g$ on $E$. Then the formal adjoint of $d_\nabla$ with respect to the obvious generalization of equation* (4.2.16) *to $E$, is*

$$d^{\dagger}_{\nabla}(\eta^\mu \otimes e_\mu) = (-1)^{(k-1)\dim\Sigma+1} * \left(d * \eta^\mu + A^\mu{}_\lambda \wedge *\eta^\lambda\right) \otimes e_\mu\,, \quad \eta^\mu \in \Omega^k(\Sigma)\,, \quad (4.2.38)$$

*where $*$ is the Hodge-star operator with respect to $h$.*

*Proof.* Let $\alpha = \alpha^\mu \otimes e_\mu \in \Omega^k(E)$ and $\beta = \beta^\nu \otimes e_\nu \in \Omega^k(E)$. The proof will just be a straight forward calculation. First we need

$$\begin{aligned} d_\nabla(\alpha^\mu \otimes e_\mu) &= d\alpha^\mu \otimes e_\mu + (-1)^k \alpha^\mu \wedge (A^\mu{}_\nu \otimes e_\nu) = \\ &= d\alpha^\mu \otimes e_\mu + (-1)^{2k}(A^\mu{}_\nu \wedge \alpha^\nu) \otimes e_\mu = \\ &= (d\alpha^\mu + A^\mu{}_\nu \wedge \alpha^\nu) \otimes e_\mu\,. \end{aligned} \quad (4.2.39)$$

Moreover, let us denote the metric on $\Omega^*(\Sigma)$ by h and the given fiber metric on $E$ by $g$. Then it holds

$$\begin{aligned} \langle d_\nabla \alpha, \beta\rangle &= \int_\Sigma g_{\mu\nu}\, h(d_\nabla\alpha^\mu, \beta^\nu) * 1 = \int_\Sigma g_{\mu\nu}\, d_\nabla\alpha^\mu \wedge *\beta^\nu = \\ &= \int_\Sigma g_{\mu\nu}\, d\alpha^\mu \wedge *\beta^\nu + \int_\Sigma g_{\mu\nu}\, A^\mu{}_\lambda \wedge \alpha^\lambda \wedge *\beta^\nu\,. \end{aligned} \quad (4.2.40)$$

A simple calculation shows

$$d(g_{\mu\nu}\,\alpha^\mu \wedge *\beta^\nu) = dg_{\mu\nu} \wedge \alpha^\mu \wedge *\beta^\nu + g_{\mu\nu}\, d\alpha^\mu \wedge *\beta^\nu - g_{\mu\nu}\,\alpha^\mu \wedge *(-1)^{kd+1} * d * \beta^\nu \quad (4.2.41)$$

Using the fact that $g$ is a metric connection, i.e.

$$dg_{\mu\nu} = g_{\mu\nu}\, A^\lambda{}_\nu + g_{\lambda\nu}\, A^\lambda{}_\mu\,, \quad (4.2.42)$$

equation (4.2.41) yields

$$\begin{aligned} g_{\mu\nu}\, d\alpha^\mu \wedge *\beta^\nu = d(g_{\mu\nu}\,\alpha^\mu \wedge *\beta^\nu) + g_{\mu\nu}\alpha^\mu \wedge *(-1)^{kn+1} * d * \beta^\nu - \\ - g_{\mu\lambda} A^\lambda{}_\nu \wedge \alpha^\mu \wedge *\beta^\nu - g_{\lambda\nu} A^\lambda{}_\mu \wedge \alpha^\mu \wedge *\beta^\nu\,, \end{aligned} \quad (4.2.43)$$

where $n = \dim \Sigma$. If $(g_{\mu\nu}\, \alpha^\mu \wedge *\beta^\nu)|_{\partial\Sigma} = 0$ or $\partial\Sigma = \emptyset$, it follows that[5]

$$\langle d\alpha^\mu \otimes e_\mu, \beta^\nu \otimes e_\nu \rangle = \int\limits_\Sigma g_{\mu\nu} \alpha^\mu \wedge *(-1)^{kn+1} * d * \beta^\nu - \int\limits_\Sigma g_{\mu\lambda} A^\lambda{}_\nu \wedge \alpha^\mu \wedge *\beta^\nu - \\ - \int\limits_\Sigma g_{\lambda\nu} A^\lambda{}_\mu \wedge \alpha^\mu \wedge *\beta^\nu \,. \tag{4.2.44}$$

Moreover,

$$\left\langle A^\mu{}_\lambda \wedge \alpha^\lambda \otimes e_\mu, \beta^\nu \otimes e_\nu \right\rangle = \int\limits_\Sigma g_{\mu\nu} A^\mu{}_\lambda \wedge \alpha^\lambda \wedge *\beta^\nu \,, \tag{4.2.45}$$

which cancels the last term in (4.2.44). Hence, we conclude

$$\langle d_\nabla(\alpha^\mu \otimes e_\mu), \beta^\nu \otimes e_\nu \rangle = \left\langle \alpha^\mu \otimes e_\mu, (-1)^{kn+1} * d * \beta^\nu \otimes e_\nu \right\rangle - \int\limits_\Sigma g_{\mu\nu} A^\lambda{}_\nu \wedge \alpha^\mu \wedge *\beta^\nu \,. \tag{4.2.46}$$

It remains to analyze the last summand in equation (4.2.46). It reads

$$\begin{aligned} -\int\limits_\Sigma g_{\mu\lambda} A^\lambda{}_\nu \wedge \alpha^\mu \wedge *\beta^\nu &= \int\limits_\Sigma g_{\mu\lambda} \alpha^\mu \wedge * \,(-1)^{k(n-(k+1))+1} * A^\lambda{}_\nu \wedge *\beta^\nu \\ &= \int\limits_\Sigma g_{\mu\lambda}\, \alpha^\mu \wedge * \left((-1)^{kn+1} * A^\lambda{}_\nu \wedge *\beta^\nu\right) = \\ &= \left\langle \alpha^\mu \otimes e_\mu, (-1)^{kn+1} * A^\lambda{}_\nu \wedge *\beta^\nu \right\rangle \,. \end{aligned} \tag{4.2.47}$$

Altogether we get

$$\langle d_\nabla(\alpha^\mu \otimes e_\mu), \beta^\nu \otimes e_\nu \rangle = \left\langle \alpha^\mu \otimes e_\mu, (-1)^{kn+1} * \left(d * \beta^\lambda + A^\lambda{}_\nu \wedge *\beta^\nu\right) \otimes e_\lambda \right\rangle \,. \tag{4.2.48}$$

This is almost our result. The last step is to observe $\beta \in \Omega^{k+1}(\Sigma, E)$ and, hence, if we view $d^\dagger_\nabla$ as an operator $d^\dagger_\nabla : \Omega^k(\Sigma, E) \to \Omega^{k-1}(\Sigma, E)$, we have to change $k$ into $k-1$. This yields equation (4.2.38). □

Now we are ready for the

*Proof of proposition 4.2.3.* We have to show that the local expression of $-d^\dagger_{\Phi^*\nabla}\, d\Phi$ is given by equation (4.1.2). Since the Levi-Civita connection $\nabla$ on $TM$ is a metric connection

[5] In fact, this condition applied to $d\Phi$ yields the condition for a D-brane.

and the inner product (4.2.16) on $\Phi^*TM$ is defined with respect $\tilde{g}$, i.e. the pullback of $g$ w.r.t. $\mathrm{pr}_2$, it follows by lemma 4.2.5 that $\Phi^*\nabla$ is a metric connection, too. By lemma 4.2.7 the expression of the formal adjoint of the exterior covariant differential $d_{\Phi^*\nabla}$ in local coordinates acting on $d\Phi$ is

$$\begin{aligned}-d^\dagger_{\Phi^*\nabla}\, d\Phi &= -d^\dagger_{\Phi^*\nabla}\left(\frac{\partial\phi^\mu}{\partial\sigma^a}\, d\sigma^a \otimes \frac{\partial}{\partial x^\mu}\right) = \\ &= \left(-*d*\frac{\partial\phi^\mu}{\partial\sigma^a} d\sigma^a\right)\otimes\frac{\partial}{\partial x^\mu} + \left(-*A^\mu{}_\lambda \wedge *\frac{\partial\phi^\mu}{\partial\sigma^a} d\sigma^a\right)\otimes\frac{\partial}{\partial x^\mu}\,. \end{aligned}\tag{4.2.49}$$

Let us analyze the first summand. Its form part fulfills

$$-*d*\frac{\partial\phi^\lambda}{\partial\sigma^a} d\sigma^a = d^\dagger d\phi^\lambda\,. \tag{4.2.50}$$

Moreover, it holds $\phi^\lambda \in \Omega^0(\Sigma)$. Hence, $d^\dagger\phi^\lambda = 0$. Thus, it is true that the first summand in equation (4.2.49) reads

$$\left(-*d*\frac{\partial\phi^\lambda}{\partial\sigma^a} d\sigma^a\right)\otimes\frac{\partial}{\partial x^\lambda} = (d^\dagger d + dd^\dagger)\phi^\lambda \otimes \frac{\partial}{\partial x^\lambda} = \Delta_\Sigma\phi^\lambda\otimes\frac{\partial}{\partial x^\lambda}\,. \tag{4.2.51}$$

Now let us examine the second summand of eq. (4.2.49). Its form part is realized as

$$-*A^\mu{}_\lambda\wedge*\frac{\partial\phi^\lambda}{\partial\sigma^a} d\sigma^a = *-A^\mu{}_{c\lambda}\frac{\partial\phi^\lambda}{\partial\sigma^a} h^{ab}\frac{1}{(m-1)!}\epsilon_{bb_2\ldots b_m}\sqrt{|h|}d\sigma^c\wedge d\sigma^{b_2}\wedge\cdots\wedge d\sigma^{b_m}\,, \tag{4.2.52}$$

where $m := \dim\Sigma$ and $\epsilon$ is the totally antisymmetric tensor. Next we will show

$$\frac{1}{(m-1)!}\epsilon_{bb_2\ldots b_m}\sqrt{|h|}d\sigma^c\wedge d\sigma^{b_2}\cdots\wedge d\sigma^{b_m} = \delta^c_b * 1\,. \tag{4.2.53}$$

Interchanging $d\sigma^{b_i}$ and $d\sigma^{b_{i+1}}$ in the left hand side of (4.2.53) results in a sign change by $-1$. This change of sign is compensated by $\epsilon_{bb_2\ldots b_i b_{i+1}\ldots b_m} = -\epsilon_{bb_2\ldots b_{i+1} b_i\ldots b_m}$. If $b \neq c$ either $\epsilon_{bb_2\ldots b_m} = 0$ or $d\sigma^c\wedge d\sigma^{b_2}\wedge\cdots\wedge d\sigma^{b_m} = 0$. Hence,

$$\frac{1}{(m-1)!}\epsilon_{bb_2\ldots b_m}\sqrt{|h|}d\sigma^c\wedge d\sigma^{b_2}\cdots\wedge d\sigma^{b_m} = \sqrt{|h|}\delta^c_b\, d\sigma^1\wedge\cdots\wedge d\sigma^m = \delta^c_b * 1\,. \tag{4.2.54}$$

Combining equations (4.2.52), (4.2.53), $**1 = 1$ and $h^{ab} = h^{ba}$, we arrive at

$$\left(-*A^\lambda{}_\nu\wedge*\frac{\partial\phi^\nu}{\partial\sigma^a} d\sigma^a\right)\otimes\frac{\partial}{\partial x^\lambda} = \left(-\Gamma^\lambda{}_{\mu\nu}\frac{\partial\phi^\mu}{\partial\sigma^a}\frac{\partial\phi^\nu}{\partial\sigma^b}h^{ab}\right)\otimes\frac{\partial}{\partial x^\lambda}\,. \tag{4.2.55}$$

Altogether we showed that in local coordinates

$$-d^\dagger_{\Phi^*\nabla}\, d\Phi = \left(\Delta_\Sigma\phi^\lambda - \Gamma^\lambda{}_{\mu\nu}\frac{\partial\phi^\mu}{\partial\sigma^a}\frac{\partial\phi^\nu}{\partial\sigma^b}h^{ab}\right)\otimes\frac{\partial}{\partial x^\lambda}\,. \tag{4.2.56}$$

□

In the next section we will define the enhanced solution space of degree $k$, denoted as $\mathcal{H}^k(\Phi, \Sigma, M)$. It is the set of all $d^\dagger_{\Phi^*\nabla}$-closed $\Phi^*TM$-valued $k$-forms on $\Sigma$. The connection of this enhanced solution space and harmonic maps[6] is given by the fact that $d\Phi \in \mathcal{H}^1(\Phi, \Sigma, M)$. Moreover, it is also true that $d\Phi$ is $d_{\Phi^*\nabla}$-closed and, hence, harmonic. To see this we look at

**Proposition 4.2.8.** *Let $(\Sigma, h)$ and $(M, g)$ be Riemannian manifolds, $\nabla$ be the associated Levi-Civita connection on $TM$ and $\Phi : \Sigma \to M$ be a smooth embedding. Then it holds*

$$d_{\Phi^*\nabla}\, d\Phi = 0\,, \tag{4.2.57}$$

*Proof.* Let $\omega = \omega^\mu \otimes e_\mu \in \Omega^k(\Phi(\Sigma), TM|_{\Phi(\Sigma)})$. Then it follows that

$$\begin{aligned} d_{\Phi^*\nabla}({\mathrm{pr}_2}^\sharp\omega) &= d_{\Phi^*\nabla}\left(({\mathrm{pr}_2}^*\omega^\mu)\otimes({\mathrm{pr}_2}^\sharp\, e_\mu)\right) = \\ &= d({\mathrm{pr}_2}^*\omega^\mu)\otimes {\mathrm{pr}_2}^\sharp e_\mu + (-1)^k({\mathrm{pr}_2}^*\omega^\mu)\wedge\Phi^*\nabla({\mathrm{pr}_2}^\sharp e_\mu) = \\ &= {\mathrm{pr}_2}^\sharp d\omega^\mu \otimes {\mathrm{pr}_2}^\sharp e_\mu + (-1)^k {\mathrm{pr}_2}^*\omega^\mu\wedge{\mathrm{pr}_2}^\sharp\nabla e_\mu = \\ &= {\mathrm{pr}_2}^\sharp\left(d\omega^\mu\otimes e_\mu + (-1)^k\omega^\mu\wedge\nabla e_\mu\right) = \\ &= {\mathrm{pr}_2}^\sharp\left(d_\nabla\omega\right)\,, \end{aligned} \tag{4.2.58}$$

where we used ${\mathrm{pr}_2}^\sharp(\omega^\mu\otimes e_\mu) = ({\mathrm{pr}_2}^*\omega^\mu)\otimes({\mathrm{pr}_2}^\sharp\, e_\mu)$ and lemma 4.2.4. Moreover, in local coordinates it is true that

$$\left({\mathrm{pr}_2}^\sharp\,\mathrm{id}_{TM}\right)_\sigma(v) = {\mathrm{pr}_2}^{-1}\left(\mathrm{id}_{\Phi(\sigma)}(T\Phi\circ v)\right) = {\mathrm{pr}_2}^{-1}(d\Phi(v)) = (\sigma, d\Phi(v))\,. \tag{4.2.59}$$

Therefore, lemma 4.2.1 induces ${\mathrm{pr}_2}^\sharp(\mathrm{id}_{TM}) = d\Phi$. Thus,

$$d_{\Phi^*\nabla}d\Phi = {\mathrm{pr}_2}^\sharp\left(d_\nabla\,\mathrm{id}_{TM}\right) \tag{4.2.60}$$

and

$$(d_\nabla\,\mathrm{id})(X_1, X_2) = \nabla_{X_1}(\mathrm{id}\,X_2) - \nabla_{X_2}(\mathrm{id}\,X_1) - \mathrm{id}([X_1, X_2])\,. \tag{4.2.61}$$

The last expression equals the torsion of $\nabla$ and vanishes due to the fact that $\nabla$ is Levi-Civita. ☐

[6] The name harmonic map results from the fact that they generalize harmonic functions $f : \Sigma \to \mathbb{R}$, which is the special case $M = \mathbb{R}$.

## 4.3. Extended Solution Spaces and Flows

In the last section we proved three propositions. The first one, prop. 4.2.2, shows that the action of nonlinear sigma-models is given by the $L^2$-norm of $d\Phi$, which is a $\Phi^*TM$-valued one-form. The second and third one, namely props. 4.2.3 and 4.2.8, together show that $d\Phi$ is both $d^\dagger_{\Phi^*\nabla}$-closed and $d_{\Phi^*\nabla}$-closed and, hence, a $d_{\Phi^*\nabla}$-harmonic one-form. If we look at equations (4.1.2), we see that the corresponding equations of motion are highly nonlinear. Thus many arguments of string theory, which use the mode-expansion of a vibrating string, do not apply naively. As an example which is connected to this work I want to mention [AZ05]. There they study co-isotropic A-branes in the sigma-model on a four-torus. They find that morphisms between co-isotropic branes are given by a fundamental representation of the noncommutatively deformed algebra of functions on the intersection of the two branes. The noncommutativity parameter is expressed in terms of the bundles on the branes. They conjecture these findings to hold in general. Furthermore, they write "Noncommutativity arose from the nonexistence of solutions to the equations of motion that were linear in time. As a result, there were no separate conjugate momenta to the zero-mode oscillators - they formed their own momentum conjugates and did not simultaneously commute. This argument also depended on the decoupling of these modes from the oscillators of the open string." and "This assumption would follow from an orthogonal basis of solutions to the appropriate Laplace equation (equations of motion) with boundary conditions determined by the curvatures, with respect to an appropriately defined inner product.". We saw that the equations of motion are not given by a simple Laplace equation on curved manifolds, but it should be possible to show the existence of such a mentioned orthogonal basis. Although $d^\dagger_{\Phi^*\nabla} d\Phi = 0$ is nonlinear[7], there is a back-door to recover the existence of a mode expansion.

Let us look at the equation

$$d^\dagger_{\Phi^*\nabla}\alpha = 0\,, \tag{4.3.1}$$

where $\alpha \in \Phi^*TM$. It is a first-order linear partial differential equation and fulfills in particular the principle of super-position. That means the set of all solutions to this equation forms a vector-space. Given $(\Sigma, h)$ and $(M, g, \nabla)$, we call this space the *enhanced solution space of* $\Phi$ and denote it by $\mathcal{H}_\Phi$. It clearly depends on all given data, but to simplify notation we do not explicitly write down these dependencies.

The problem of constructing harmonic maps, which is highly nonlinear, can thus be expressed as two linear problems. First, construct to any $\Phi$ its enhanced solution space

[7] Both the operator which acts on $d\Phi$ and $d\Phi$ itself depend non-trivially on the embedding $\Phi$

and afterwards ask which of these forms belong to an embedding, i.e. $\alpha = d\Phi$ for $\alpha \in \mathcal{H}_\Phi$. This can be interpreted physically. Any harmonic map $\Phi$ gives rise to a $d_{\Phi^*\nabla}$-harmonic one-form $d\Phi$ which lies in $\mathcal{H}_\Phi$. The latter most probably has a basis and thus $d\Phi$ can be expressed as a linear combination. This is the existence of a mode-expansion, even in curved space. But there is a difference to euclidean space. There every linear combination gives raise to a physically sensible state of a vibrating string. Here not every mode is physical, only those which can be written as $d\Phi$ for some embedding $\Phi$ are. In Euclidean space, every mode simply belongs to some embedding $\Phi$. It could be asked whether we improved something, since the enhanced solution space still seems to depend on $\Phi$. Let us state

**Conjecture 1.** *Let $\Phi_1 : \Sigma \to M$ and $\Phi_2 : \Sigma \to M$ be two differentiable maps which are homotopic to each other. Then it holds $\mathcal{H}^k_{\Phi_1} \cong \mathcal{H}^k_{\Phi_2}$ for all $k$.*

As an example consider $\Sigma = S^2$. The above conjecture then tells us that there is one class of enhanced solution spaces for every homotopy-class of embeddings $S^2 \to M$, i.e. for every element of $\pi_2(M)$. Hence, in order to construct harmonic maps, we could first compute $\pi_2(M)$ and construct $\mathcal{H}_\Phi$ for every $\Phi \in \pi_2(M)$.

The conjecture can be motivated by the following argument. In fact, conjecture 1 can be inferred from the stronger statement of commutativity of

$$\begin{tikzcd}
0 \arrow[r] & \Omega^0(\Phi_0^*TM) \arrow[r, shift left, "d_{\Phi_0^*\nabla}"] \arrow[d, "J_0"] & \Omega^1(\Phi_0^*TM) \arrow[l, shift left, "d^\dagger_{\Phi_0^*\nabla}"] \arrow[r, shift left, "d_{\Phi_0^*\nabla}"] \arrow[d, "J_1"] & \Omega^2(\Phi_0^*TM) \arrow[l, shift left, "d^\dagger_{\Phi_0^*\nabla}"] \arrow[r, shift left, "d_{\Phi_0^*\nabla}"] \arrow[d, "J_2"] & \cdots \arrow[l, shift left, "d^\dagger_{\Phi_0^*\nabla}"] \\
0 \arrow[r] & \Omega^0(\Phi_1^*TM) \arrow[r, shift left, "d_{\Phi_1^*\nabla}"] & \Omega^1(\Phi_1^*TM) \arrow[l, shift left, "d^\dagger_{\Phi_1^*\nabla}"] \arrow[r, shift left, "d_{\Phi_1^*\nabla}"] & \Omega^2(\Phi_1^*TM) \arrow[l, shift left, "d^\dagger_{\Phi_1^*\nabla}"] \arrow[r, shift left, "d_{\Phi_1^*\nabla}"] & \cdots \arrow[l, shift left, "d^\dagger_{\Phi_1^*\nabla}"]
\end{tikzcd} \tag{4.3.2}$$

where $J_i$ are the sought after isomorphisms. Without loss of generality, let $\Phi_0$ and $\Phi_1$ be $\mathcal{C}^\infty$-homotopic via $\mathscr{F}_\sigma(t)$. Then vector fields $X(\sigma, 0)$ along $\Phi_0$ can be transported to vector fields $X(\sigma, 1)$ along $\Phi_1$ using $\mathcal{D}(\dot{\mathscr{F}}_\sigma(t))$, where $\mathcal{D}(X)$ is some linear differential operator depending on $X$, e.g. the covariant derivative along $X$. It remains to identify $\mathcal{D}(X)$. This is $J_0$ in diagram (4.3.2).

As a preliminary consideration let us focus on sections in $\Phi_t^*TM$, i.e. $J_0$. To examine the structure of the above idea suppose $\mathcal{D}(X) = \nabla_X$. If $U_0$ is a vector field along $\Phi_0$, there is an initial-value problem,

$$\nabla_{\dot{\mathscr{F}_\sigma}(t)} X^{U_0}(t) = \left( \frac{d\left(X^{U_0}\right)^\mu}{dt} + \Gamma^\mu{}_{\nu\lambda} \dot{\mathscr{F}_\sigma}(t)^\nu \left(X^{U_1}\right)^\lambda \right) e_\mu = 0 \,, \quad \text{with} \quad X^{U_0}(0) = U_0 \,. \tag{4.3.3}$$

Here we consider the homotopy $\mathscr{F}$ as a flow and construct its associated generating vector field with respect to the covariant derivative. Because of the theorem of Picard-Lindelöf, this system of ODEs has a unique solution $X^{U_0}(t)$. We define $J_0 : \Phi_0^*TM \to \Phi_1^*TM$ by

$$J_0\left((\sigma, U_0)\right) := \left(\sigma, X^{U_0}(1)\right) . \tag{4.3.4}$$

Clearly $J_0$ is a bundle isomorphism. The inverse map is given by the generating vector field $X^{U_1}$ of the flow $\mathscr{F}_\sigma(1-t)$. It is a diffeomorphism and is fiber preserving. Let us show that $J_0$ is linear. Additivity can be inferred from $X^{U_0}(t) + X^{U_0'}(t)$ fulfilling the definition of $X^{U_0+U_0'}$. The equation $J_0(fU_0) = fJ_0(U_0)$ for $f$ being a function on $\Sigma$ is true since $f$ does not depend on any variables in $M$ and can thus be considered as a constant in equation (4.3.3).

Let us extend the action of $J_0$ to elements $\alpha \in \Omega^k(\Phi_0^*TM)$ by

$$(J_k\eta)(v) := J_0(\eta(v)) \quad \text{i.e.} \quad J_k(\eta^\mu \otimes e_\mu) = \eta^\mu \otimes J_0 e_\mu \, . \tag{4.3.5}$$

Unfortunately, this $J_k$ does not lead to a commutative diagram (4.3.2), in general. If we would have used $*\mathcal{L}_X^\nabla*$, where $\mathcal{L}_X^\nabla := \iota_X\nabla + \nabla\iota_X$, instead of $\nabla_X$ in (4.3.3) and additionally $M$ would be flat, i.e. has vanishing Riemann tensor, $J_k$ would render (4.3.2) to be commutative and would send harmonic forms to harmonic forms. This is true since the commutator of $\mathcal{L}_X^\nabla$ and $d_\nabla$ gives the curvature of $\nabla$. This implies the vanishing of the commutator of $J_k$ and $d_\nabla^\dagger$ and, hence, $d_\nabla^\dagger$ acting on the continuation of $\omega$ is the same as the continuation of $d_\nabla^\dagger\omega$. Hence, we proved

**Theorem 4.3.1.** *Let $(\Sigma, h)$ and $(M, g)$ be two Riemannian manifolds such that the Riemann tensor of $(M, g)$ vanishes. Furthermore, let $\Phi_1$ and $\Phi_2$ be homotopic to each other. Then the enhanced solution spaces with respect to $\Phi_i$ are isomorphic, i.e. $\mathcal{H}_{\Phi_1} \cong \mathcal{H}_{\Phi_2}$.*

Next we show that there should be an isomorphism[8] between the two enhanced solution spaces of $\Phi_0$ and $\Phi_1$, respectively. Let $\eta \in \mathcal{H}^k(\Phi_0)$, i.e. $\eta \in \Omega^k(\Phi_0^*TM)$ and $d^\dagger_{\Phi_0*\nabla}\eta = 0$. Suppose

$$\Phi_t(\sigma) := \mathcal{F}_\sigma(t) = \Phi_0(\sigma) + \delta\Phi_t(\sigma) = \Phi_0(\sigma) + (\delta_1\Phi)(\sigma)t + (\delta_2\Phi)(\sigma)t^2 + \dots . \tag{4.3.6}$$

We would wish to view this as a real expansion, but if someone feels uncomfortable with that, view it as a formal deformation. Furthermore, expand

$$\begin{aligned}\Gamma^\mu{}_{\lambda\nu}(\Phi_0 + \delta\Phi(t)) = \Gamma^\mu{}_{\lambda\nu}(\Phi_0) + \Gamma^\mu{}_{\lambda\nu;\alpha_1}(\Phi_0)\,\delta\Phi(t)^{\alpha_1} + \\ + \frac{1}{2}\Gamma^\mu{}_{\lambda\nu;\alpha_1\alpha_2}(\Phi_0)\,\delta\Phi(t)^{\alpha_1}\delta\Phi(t)^{\alpha_2} + \dots ,\end{aligned} \tag{4.3.7}$$

[8] I do not call the following a proof, since we use methods which we do not prove to be e.g. convergent. Nevertheless they show how to identify the two spaces and are very constructive.

where $f_{;\alpha_1}$ denotes the partial derivative of $f$ with respect to the coordinate $x^{\alpha_1}$. At $\Phi_t$ it holds

$$d^\dagger_{\Phi_t^*\nabla} J_k \eta = d^\dagger_{\Phi_t^*\nabla}(\eta^\mu \otimes J_0 e_\mu) = (-1)^{\dim\Sigma(k-1)} \left(*d * \eta^\mu + (-1)^k * A^\mu{}_\nu(t) \wedge *\eta^\nu\right) \otimes J_0 e_\mu\,, \tag{4.3.8}$$

where we used $J_0$ from equation (4.3.4) to map a $\Phi_0^*TM$ valued $k$-form to a $\Phi_t^*TM$ valued $k$-form. Next we deform $\eta$ to $\eta(t) = \eta^\mu(t) \otimes J_0 e_\mu$ at $\Phi_t$, i.e.

$$\eta^\mu(t) = \eta^\mu + (\delta_1\eta^\mu)t + (\delta_2\eta^\mu)t^2 + \ldots \tag{4.3.9}$$

and determine $\delta_i\eta^\mu$ such that $d^\dagger_{\Phi_t^*\nabla}\eta^\mu(t) = 0$. Using (4.3.7) it follows that

$$\Gamma^\mu{}_{\lambda\nu}(\Phi(t)) = \Gamma^\mu{}_{\lambda\nu}(\Phi_0) + \sum_{k=1}^\infty \frac{1}{k!}\,\Gamma^\mu{}_{\lambda\nu}(k)t^k\,, \tag{4.3.10}$$

with

$$\Gamma^\mu{}_{\lambda\nu}(k) = \sum_{l=1}^k \Gamma^\mu{}_{\lambda\nu;\alpha_1\cdots\alpha_k} \left( \sum_{j_1,\cdots,j_l=1}^k \delta_{k,j_1+\ldots+j_l}\,(\delta_{j_1}\phi)^{\alpha_1}\cdots(\delta_{j_l}\phi)^{\alpha_l}\right)\,. \tag{4.3.11}$$

Therefore[9],

$$A^\mu{}_{a\nu}(t) = \sum_{k=0}^\infty \frac{1}{k!} A^\mu{}_{a\nu}(k)t^k = \sum_{k=0}^\infty \frac{1}{k!}\left(\sum_{l=0}^k \frac{\partial(\delta_l\phi)^\lambda}{\partial\sigma^a}\Gamma^\mu{}_{\lambda\nu}(k-l)\right)t^k\,. \tag{4.3.12}$$

Hence, equation (4.3.8) becomes after an easy calculation

$$\begin{aligned} d^\dagger_{\Phi_t^*\nabla} J_k\eta =& (-1)^{\dim\Sigma(k-1)} \sum_{i=0}^\infty \Bigg( d^\dagger_{\Phi_0^*\nabla}(\delta_i\eta)^\mu + \\ &+ (-1)^k * d\sigma^a \wedge * \sum_{j=1}^i \frac{1}{j!} A^\mu{}_{a\nu}(j)(\delta_{i-j}\eta)^\mu \Bigg) t^i \otimes J_0 e_\mu\,. \end{aligned} \tag{4.3.13}$$

Together with $d^\dagger_{\Phi_t^*\nabla} J_k\eta \stackrel{!}{=} 0$ this yields

$$d^\dagger_{\Phi_0^*\nabla}(\delta_i\eta)^\mu = (-1)^{k+1} * d\sigma^a \wedge * \sum_{j=1}^i \frac{1}{j!} A^\mu{}_{a\nu}(j)(\delta_{i-j}\eta)^\mu\,. \tag{4.3.14}$$

[9] We set $\Gamma^\mu{}_{\lambda\nu}(0) := \Gamma^\mu{}_{\lambda\nu}(\Phi_0)$.

Let us assume that $d^\dagger_{\Phi_0^*\nabla}\eta = 0$ can be solved and there is an associated Green-operator. Then (4.3.14) can be solved successively. Indeed, if we assume that we know all $\delta_j\eta$ up to $i-1$, the right hand side only depends on these $\delta_j\eta$ with $j < i$. Then the Green-operator can be used to construct $\delta_i\eta$. This motivates $\mathcal{H}^k(\Phi_0) \cong \mathcal{H}^k(\Phi_t)$. Thus we can view the enhanced solution space of $\Phi$ as a topological invariant.

The next question is to ask whether $\eta = d\Phi$ for some embedding $\Phi$. If we look at

$$d\Phi = \frac{\partial \phi^\mu}{\partial \sigma^a} d\sigma^a \otimes e_\mu = d\phi^\mu \otimes e_\mu \,, \tag{4.3.15}$$

we infer that this question is connected to the fact whether all components of $d\Phi$ are globally exact forms. Let us call the corresponding cohomology class $[0]$.

Let us summarize our results. We found a geometrical formulation of nonlinear sigma-models. They are closely connected to Hodge-theory and are in some manner a generalization of the latter. It should be noted that the Yang-Mills equations read $d_\nabla\Omega^\omega = d^\dagger_\nabla\Omega^\omega = 0$, where $\Omega^\omega$ is the curvature form of a connection $\omega$ on a principal fiber bundle, and are thus very similar to our equations of motion. If there is some differential operator $\mathcal{D}$ which commutes with $d^\dagger_\nabla$ and if $\Phi_0$ is homotopic to $\Phi_t$, we could infer that $J_k : \Omega^k(\Sigma, \Phi_0^*TM) \to \Omega^k(\Sigma, \Phi_t^*TM)$ is also a map $J_k : \mathcal{H}^k(\Phi_0) \to \mathcal{H}^k(\Phi_t)$. Here $J_k$ is given by (4.3.5) and $J_0$ is defined as the continuation of 0-forms in $\Omega^0(\Phi_0(\Sigma), TM|_{\Phi_0(\Sigma)})$ to 0-forms in $\Omega^0(\Phi_t(\Sigma), TM|_{\Phi_t(\Sigma)})$ via $\mathcal{D}$ (see above). Thus, every homotopy class $[\Phi_0]$ gives rise to an enhanced solution space $\mathcal{H}^k([\Phi_0])$ and harmonic maps $\Phi : \Sigma \to M$ with fixed $[\Phi_0]$ are given as the elements of $\mathcal{H}^1([\Phi_0]) \cap [0]$.

In particular, our considerations show that there is always a mode expansion. It depends on the homotopy class to which the embedding of the world sheet belongs to. Furthermore, not every mode is physical, in general. Only those with $d\Phi \in [0]$ are. This holds independently of conjecture 1 and could be used to determine supersymmetric ground states in supersymmetric nonlinear sigma-models with co-isotropic A-branes. This should give raise to the non-commutativization of $M$ and was shown in [AZ05] for the special case of $M$ being a torus, which is nearly literally the same as Euclidean space. The above ideas should enable a general proof.

Moreover, the above ideas may enable the quantization of nonlinear sigma models on curved manifolds. Let us briefly state a program to achieve this. First the exact structure of the enhanced solution space $\mathcal{H}^1_\Phi$ should be examined. This should lead to similar results as the solution space of the Laplace equation. Using the similar methods as in the flat case it should be possible to quantize $\mathcal{H}^1_\Phi$. The last step will be to find a quantum analog of $\xi = d\Phi$ for some map $\Phi : \Sigma \to M$. As the set of all $\xi = d\Phi$ may be some kind of variety in the enhanced solution space, the set of all states which correspond to $d\Phi$ should be some kind of variety inside the quantization of $\mathcal{H}^1_\Phi$. Another possible ansatz is to introduce some kind of cobordism between conformal field theories. Physical states are then some kind of section in the Fréchet bundle $\mathcal{E} \to \mathcal{B}$, where $\mathcal{B} = \mathcal{C}^\infty(\Sigma, M)$ and $\mathcal{E}_\Phi = \Omega^1(\Sigma, \Phi^*TM)$. Since the focus of this work is on a generalization of pseudo-holomorphic curves to generalized complex geometry, we will postpone the detailed examination of the quantization of nonlinear sigma models to the future.

# 5. Topological Sigma Models with H-Flux

This chapter is an exposition of topological sigma models with H-Flux. Most of section 5.1 is a review of [KL07], but with more care on technical details. An exposition of this material is important for this work as it is the physical motivation of the considerations in part three, "The Mathematics of Topological Sigma-Models with H-Flux". We will see that the most general target manifold allowing for $\mathcal{N} = (2,2)$ supersymmetry is a so called twisted generalized Kähler manifold. Looking at R-symmetry, which we will use to construct the topological twisted theory, we will conclude that there is an obstruction to $M$. For R-symmetry being present at the quantum level, the first Chern-class of some bundle $E \to M$ has to vanish. If we also pretend the absence of anomalies of $Q^2_{\mathrm{BRST}} = 0$, the generalized canonical bundle of the target manifold $M$ has to be trivial. Thus $M$ has to be a twisted generalized Calabi-Yau manifold [Gua03],[Hit02], i.e. $M$ has to admit a generalized Calabi-Yau metric geometry. The work [KL07] ends with a section called "Towards the Twisted Generalized Quantum Cohomology Ring". There they examine the instantons of the theory and use them to formulate the concept of a twisted generalized complex map.

We will use their concept to give a precise definition in part three.[1] At the end of section 5.1 we will investigate the behavior of their concept under B-field transformations, as physical objects, and interpret this in terms of results of [Zab06]. We will see that their instantons are invariant under B-field transformations modulo canonical transformations on the super-loop space of the target manifold. In section 5.2 we will give a smooth interpolation between the A- and the B-model of a Hyperkähler manifold through the generalized B-model. Even though they are not of immediate importance to the rest of this work, in section 5.3 we will review the concept of coisotropic branes. They play an important role in far reaching applications of topological string theory, like the physical motivation of Langlands duality. For sake of completeness and future reference these branes will be treated here as well.

[1] In analogy to their pendant in symplectic geometry we will call such maps $\mathcal{J}$-holomorphic curves or generalized pseudoholomorphic curves.

## 5.1. Topological Supersymmetric Nonlinear Sigma-Model with H-Flux

Here we will give an extension of the nonlinear sigma model of the last section to supersymmetric theories. This is mainly a review of [KL07], which is a generalization of [Kap05], footing on [SJGR84]. Although we will not give a rigorous exposition, we will present more technical details as in [KL07].

This review will be as follows. We will start by giving the action of $(1,1)$-supersymmetric nonlinear sigma-models with $H$-flux. After that we will demand the existence of another supersymmetry such that the whole theory is $(2,2)$-supersymmetric. This will lead to the consequence that the target manifold $M$ is a so called bi-hermitian manifold. In modern language a bi-hermitian manifold is a twisted generalized Kähler manifold. Relevant definitions are given in the appendix.

Thereafter we will give some facts about so called topologically twisted sigma-models and will recall the generalized complex twisted theory. In [KL07] this was called generalized B-twist. The only difference between generalized A- and B-twists is a sign change of one of the two given generalized complex structures of the generalized Kähler manifold. As noted in [KL07], the two models can be converted into each other by a Bogoliubov transformation, at least at the classical level. The description of the topologically twisted theory will also contain a treatment of anomalies.

Afterwards we will construct the vector-space structure of BRST-cohomology of operators, i.e. without its algebraic structure, which would comprise quantum corrections. The complete algebraic structure would give raise to the so called chiral ring of the theory. Using methods of chapter 3 we will explain why the space of supersymmetric ground states is given by a certain Lie algebroid cohomology of the so called canonical complex associated to $M$. Comprising instanton corrections this would lead to the construction of the quantum cohomology ring of the theory.

### 5.1.1. Action, Supersymmetry and Geometrical Implications

Let us start by demanding that the action of supersymmetric sigma-models with $H$-flux is given by[2]

$$S[\Phi] = \frac{1}{2}\int d^2\sigma d^2\theta (g_{ab}(\phi) + B_{ab}(\phi)) D_+\Phi^a D_-\Phi^b \,. \tag{5.1.1}$$

[2]Due to the presence of fields which are usually denoted by Greek letters, we use Latin letters for space-time indices.

Here $(M, g)$ is a Riemannian manifold, $H$ is a real closed three-form on $M$ with $H = dB$ locally,

$$\Phi^a = \phi^a + \theta^+ \psi^a_- + \theta^- \psi^a_- + \theta^- \theta^+ F^a \tag{5.1.2}$$

are components of a supermap from $(1,1)$ superworldsheet to $(M, g)$ and supercovariant derivatives

$$D_\pm = \frac{\partial}{\partial \theta^\pm} + i\theta^\pm \partial_\pm\,, \quad \partial_\pm = \partial_0 \pm \partial_1\,. \tag{5.1.3}$$

To be able to construct a topologically twisted theory, we need extended, i.e. $\mathcal{N} = (2,2)$ supersymmetry. In [SJGR84] it is shown that such supersymmetry transformations can be written as

$$\delta\Phi^a = -i(\epsilon^+ Q_+ + \epsilon^- Q_-)\Phi^a\,, \tag{5.1.4}$$

$$\tilde{\delta}\Phi^a = (\tilde{\epsilon}^+ + \tilde{\epsilon}^- \tilde{Q}_-)\Phi^a = \tilde{\epsilon}^+ I_+(\phi)^a{}_b D_+ \Phi^b + \tilde{\epsilon} I_-(\phi)^a{}_b D_- \Phi^b\,. \tag{5.1.5}$$

The first transformation corresponds to usual $(1,1)$ supersymmetry, while the second one is an additional supersymmetry transformation. The two infinitesimal generators $\delta$ and $\tilde{\delta}$ automatically commute. The requirement that $\tilde{\delta}$ fulfills the algebra of $(1,1)$ supersymmetry forces $I_\pm$ to be a pair of integrable almost complex structures. Invariance of $S$ under $\tilde{\delta}$ dictates $g$ to be hermitian with respect to both $I_+$ and $I_-$. Furthermore, $I_\pm$ are covariantly constant w.r.t. connections $\nabla_\pm$, i.e. $d_{\nabla_\pm} I_\pm = 0$, with torsion defined by $H$. The coefficients of $\nabla_\pm$ are

$$(A_\pm)^a{}_{bc} = \Gamma^a{}_{bc} \pm \frac{1}{2} g^{ad} H_{dbc}\,, \tag{5.1.6}$$

where $\Gamma^\mu{}_{\lambda\nu}$ denote the Christoffel symbols. Due to the presence of torsion, $(g, I_\pm)$ do not define Kähler structures. Instead they define a so called bi-Hermitian structure. In [Gua03] it is shown that these data are equivalent to specifying a pair of twisted generalized complex structures $(\mathcal{J}_1, \mathcal{J}_2)$ which define a twisted generalized Kähler structure.That means the most general target manifolds allowing $\mathcal{N} = (2,2)$ supersymmetric nonlinear sigma-models with $H$-flux are twisted generalized Kähler manifolds.

### 5.1.2. Topologically Twisted Theory

Next we will construct the topologically twisted version of $\mathcal{N} = (2,2)$ supersymmetric nonlinear sigma-models with $H$-flux. At the beginning we will give some notes on topologically twisted sigma models in general. The full theory, in particular the quantum theory, of nonlinear sigma models are quite complicated, as we saw in the last section. Topologically twisted theories are easier to describe, since in some sense they are projections to the symplectic or complex analytic aspects of the full theory. Thus it is possible to state that they are a nice framework to test the mathematics of string theory. Another reason for examining topologically twisted theories is that supersymmetric nonlinear sigma models do not admit global supersymmetry on the world-sheet, as there is no global covariantly constant spinor $\epsilon$. In contrast to this fact topologically twisted theories allow global supersymmetry [MS103]. Moreover, many mathematical applications like mirror symmetry, e.g. [Kon94, MSm99, MS103, MS209, KO04] and references therein, or the recent formulation of the geometric Langlands conjecture via the special case of mirror symmetry of Hitchin's moduli space [KW06, GW08, Fre09a] use these theories explicitly or implicitly.

For the possibility to do the topological twist, there has to be at least one non-anomalous R-symmetry, i.e. $U(1)$ symmetry. Having such a symmetry at hand, one shifts the spin of fermions by one half of the R-charge. Mathematically this corresponds to twisting, i.e. tensoring, the spin bundle with a $U(1)_R$-bundle. After twisting, all terms involving the metric become Q-exact. For the ordinary A- and B- model this has been shown in [Wit88, Wit91]. The proof in generalized geometry is still lacking. First attempts are given in [Pes07], [Zuc06] and [Chu08]. By standard arguments correlators involving such terms vanish. We face a similar problem in section 6.5. There we find that there is an isotropic embedding $\lambda : TM \to E$ which enable one to write the generalized energy of a map $\Phi$ as the sum of a term which vanishes for instantons and a topological term. Although $\lambda$ will not be involutive in general, in every neighborhood it corresponds to a canonical transformation of the super loop-space. The correspondence between $B$-transformations and canonical transformations was first mentioned in [Zab06].

There are two $U(1)$ R-symmetries (one vector and one axial). The vector like symmetry is always non-anomalous, whereas the anomaly of $U(1)_A$ depends on $c_1(M)$. For Kähler manifolds it is non-anomalous if $M$ is Calabi-Yau.

In order to apply this construction to the generalized complex case, we observe that $I_\pm$ induce two splittings of the complexified tangent bundle

$$TM_{\mathbb{C}} \cong TM_+^{(1,0)} \oplus TM_+^{(0,1)} \cong TM_-^{(1,0)} \oplus TM_-^{(0,1)}\,, \tag{5.1.7}$$

where the decompositions are the usual holomorphic and anti-holomorphic ones w.r.t. $I_\pm$. Accordingly, the fermions $\psi_\pm$ decompose into holomorphic components $\psi_\pm^h$ and anti-holomorphic components $\psi_\pm^a$, too,

$$\psi_+ = \frac{1}{2}(\mathbb{1} - \mathrm{i}I_+)\psi_+ + \frac{1}{2}(\mathbb{1} + \mathrm{i}I_+)\psi_+ = \psi_+^h + \psi_+^a\,, \tag{5.1.8}$$

$$\psi_- = \frac{1}{2}(\mathbb{1} - \mathrm{i}I_-)\psi_- + \frac{1}{2}(\mathbb{1} + \mathrm{i}I_-)\psi_- = \psi_-^h + \psi_-^a \,. \tag{5.1.9}$$

Classically, there are two inequivalent ways of assigning $U(1)$ charges to the components of $\psi_\pm$. For sake of completeness we give the following table

| component | $q_V$ | $q_A$ | $q(U(1)_E)$ | $\mathcal{L}$ | $q(U(1)_{E'})$ | $\mathcal{L}'$ | $q(U(1)_{E''})$ | $\mathcal{L}''$ |
|---|---|---|---|---|---|---|---|---|
| $\frac{1}{2}(1-\mathrm{i}I_+)\psi_+$ | $-1$ | $-1$ | $-\frac{1}{2}$ | $\sqrt{\overline{K}}$ | $-1$ | $\overline{K}$ | $-1$ | $\overline{K}$ |
| $\frac{1}{2}(1+\mathrm{i}I_+)\psi_+$ | $1$ | $1$ | $-\frac{1}{2}$ | $\sqrt{\overline{K}}$ | $0$ | $\underline{\mathbb{C}}$ | $0$ | $\underline{\mathbb{C}}$ |
| $\frac{1}{2}(1-\mathrm{i}I_-)\psi_-$ | $-1$ | $1$ | $\frac{1}{2}$ | $\sqrt{K}$ | $0$ | $\underline{\mathbb{C}}$ | $1$ | $K$ |
| $\frac{1}{2}(1+\mathrm{i}I_-)\psi_-$ | $1$ | $-1$ | $\frac{1}{2}$ | $\sqrt{K}$ | $1$ | $K$ | $0$ | $\underline{\mathbb{C}}$ |

Here $q_V$ and $q_A$ denote the charge under vector- and axial-R-symmetry, respectively. The line bundle $\mathcal{L}$ denotes the bundle of which the components are sections before the twist. One primed objects denote the A-twist, whereas two primed objects denote the B-twist. Topological twisting is achieved by shifting the spin of the fermions by one-half of the $R$-charge. This amounts to tensoring the bundles $\mathcal{L}$ by the respective $R$-symmetry bundles, as depicted in the table.

It is very important that a sign change of $I_-$ interchanges the A- and the B-twist. This is true since $q_V(\psi_-) = 0$ and changing the sign of $I_-$ produces

$$\frac{1}{2}(1 - \mathrm{i}I_-)\psi_- \longmapsto \frac{1}{2}(1 + \mathrm{i}I_-)\psi_- \,. \tag{5.1.10}$$

Next we examine what happens at the quantum level. We adopt the argumentation of [MS103], chapter 13.2.2. By the Atiyah-Singer index theorem, the difference of $\#\psi_-^h$-zero-modes and $\#\psi_-^a$-zero-modes is given by the index of the Dirac operator with respect to $I_-$, i.e.

$$\begin{aligned}\operatorname{ind} D_- = \dim\ker D_{\bar{z},-} - \dim\ker D_{z,-} &= \int\limits_\Sigma c_1\left(\phi^* TM_-^{(1,0)}\right) = \int\limits_\Sigma \phi^* c_1\left(TM_-^{(1,0)}\right) = \\ &= \left\langle \phi_*(\Sigma), c_1\left(TM_-^{(1,0)}\right)\right\rangle =: k_- \,,\end{aligned} \tag{5.1.11}$$

where $\phi_*(\Sigma)$ denotes the homology class induced by $\Sigma$. The difference of $\#\psi_+^h$-zero-modes and $\#\psi_+^a$-zero-modes is given by

$$\begin{aligned}\operatorname{ind} D_+ = \dim\ker D_{\bar{z},+} - \dim\ker D_{z,+} &= \int\limits_\Sigma c_1\left(\phi^* TM_+^{(1,0)}\right) = \int\limits_\Sigma \phi^* c_1\left(TM_+^{(1,0)}\right) = \\ &= \left\langle \phi_*(\Sigma), c_1\left(TM_+^{(1,0)}\right)\right\rangle =: k_+ \,.\end{aligned} \tag{5.1.12}$$

Let there be $a := m_- + k_-$ zero modes of $\psi_-$ and $b := m_+ + k_+$ zero modes of $\psi_+$. Now look at the expression

$$f(z_1, \dots z_{a+m_-}, w_1, \dots, w_{b+m_+}) := \big\langle \psi_-^h(z_1) \cdots \psi_-^h(z_a)\, \psi_-^a(z_{a+1}) \cdots \psi_-^a(z_{a+m_-}) \\ \times \overline{\psi}_+(w_1) \cdots \overline{\psi}_+(w_b)\, \psi_+(w_{b+1}) \cdots \psi_+(w_{b+m_+}) \big\rangle \,. \tag{5.1.13}$$

This transforms as

$$f \longmapsto \mathrm{e}^{\mathrm{i}\alpha(k_- - k_+)} f \tag{5.1.14}$$

under $U(1)_V$ and

$$f \longmapsto \mathrm{e}^{\mathrm{i}\beta(k_- + k_+)} f \tag{5.1.15}$$

under $U(1)_A$. Here $\alpha, \beta \in \mathbb{R}$ are the variables of the respective (global) R-symmetry. Correlators and in particular $f \neq 0$ are invariant under R-symmetry if $\exp(\alpha(k_- - k_+)) = 1$ and $\exp(\beta(k_- + k_+)) = 1$. This has to hold for all $\alpha$, $\beta$ and we infer together with $k_\pm \in \mathbb{Z}$ that

$$k_- - k_+ = 0 \quad \text{for } U(1)_V \quad \text{and} \tag{5.1.16}$$

$$k_- + k_+ = 0 \quad \text{for } U(1)_A \,. \tag{5.1.17}$$

Because this has to be true for all embeddings $\Sigma$, we conclude

$$c_1(TM_-^{(1,0)}) - c_1(TM_+^{(1,0)}) = 0 \quad \text{for } U(1)_V \quad \text{and} \tag{5.1.18}$$

$$c_1(TM_-^{(1,0)}) + c_1(TM_+^{(1,0)}) = 0 \quad \text{for } U(1)_A \,. \tag{5.1.19}$$

This is the same result as in [KL07] and can also be obtained using Fujikawa's method, which uses the transformation behavior of the measure of path integration. Observe that the two conditions are interchanged if we change the sign of $I_-$.

The above result can also be formulated in the language of generalized complex geometry. We saw that $\mathcal{N} = (2,2)$ supersymmetry requires $M$ to be a twisted generalized complex manifold with two commuting twisted generalized complex structures (TGC) $\mathcal{J}_1$ and $\mathcal{J}_2$ and positive definite metric $G = -q\mathcal{J}_1\mathcal{J}_2$ on $TM \oplus T^*M$. As [KL07] we use the results of section 6.4 in [Gua03]. Let $C_\pm$ be the $\pm 1$ eigenbundles of $G$. There are isomorphisms $\pi_\pm : C_\pm \to TM$ induced by the natural projection $TM \oplus T^*M \to TM$. Furthermore, let $L_1$, $L_2$ denote the $+\mathrm{i}$ eigenbundle of $\mathcal{J}_1$, $\mathcal{J}_2$, respectively. Because of the fact that $\mathcal{J}_1$ and $\mathcal{J}_2$ commute, it is true that

$$L_1 = L_1^+ \oplus L_1^- \quad \text{and} \quad L_2 = L_2^+ \oplus L_2^- \,, \tag{5.1.20}$$

where the superscript $\pm$ refers to the eigenvalue $\pm\,\mathrm{i}$ of the other TGC structure. Hence,

$$C_\pm \otimes \mathbb{C} \cong L_1^\pm \oplus (L_1^\pm)^* \cong L_2^\pm \oplus (L_2^\pm)^* . \tag{5.1.21}$$

The anomaly free conditions can now be formulated as

$$c_1(L_2) = 0 \quad \text{for } U(1)_V \quad \text{and} \tag{5.1.22}$$

$$c_1(L_1) = 0 \quad \text{for } U(1)_A . \tag{5.1.23}$$

The two conditions are interchanged if we change the sign of $I_-$, as the latter interchanges $\mathcal{J}_1$ and $\mathcal{J}_2$. This looks quite similar to the condition of a manifold being a Calabi-Yau manifold. But it differs from the one giving $M$ a twisted generalized Calabi-Yau metric geometry. The latter implies (5.1.22) and (5.1.23), but not the other way around. If we additionally demand the absence of $Q_{\text{BRST}}$ anomaly, $(M, \mathcal{J}_1, \mathcal{J}_2)$ defines a twisted generalized Calabi-Yau metric geometry [KL07].

Our next aim is to show the equivalence of the BRST-cohomology of operators and the Lie algebroid cohomology associated to $\mathcal{J}_1$, at least at the classical level. First we will give a construction of the BRST operator after the generalized complex twist. Following [KL07] we will focus on the B-twist. This is not essential, since we saw that the difference of the A-twist and the B-twist is the sign of $I_-$. Afterwards we will construct the supersymmetric ground states of the resulting theory using methods of chapter 3. As $M$ is assumed to be a TGC-CY manifold, the resulting space is isomorphic to BRST cohomology ([MS103] mutatis mutandis). We will refer the result of [KL07] that the space of Ramond-Ramond ground states is isomorphic to the Lie algebroid cohomology associated to $\mathcal{J}_1$. Recall that we used $\mathcal{J}_1$ to define the topological twist. Hence, the generalization of Quantum cohomology to generalized complex geometry should be given by a deformation of Lie algebroid cohomology.

### 5.1.3. BRST Cohomology of Operators

Let $(M, \mathcal{J}_1)$ be a twisted generalized Calabi-Yau manifold. Looking at the B-twisted theory let us define $\chi := (1 + \mathrm{i}I_+)\psi_+$ and $\lambda := (1 + \mathrm{i}I_-)\psi_-$. They are sections of $\phi^*TM_+^{(1,0)}$ and $\phi^*TM_-^{(1,0)}$, respectively, whereas $(1 - \mathrm{i}I_+)\psi_+$ and $(1 - \mathrm{i}I_-)\psi_-$ are spin one fields. Moreover, let us define

$$Q_L := \frac{1}{2}(Q_+ + \mathrm{i}\tilde{Q}_+) \quad \text{and} \quad Q_R := \frac{1}{2}(Q_- + \mathrm{i}\tilde{Q}_-) . \tag{5.1.24}$$

Since we assume $M$ to be generalized Calabi-Yau, the central charges of the supersymmetry algebra vanish and $Q_L$ and $Q_R$ are nilpotent operators.[3]. Define $Q_{\text{BRST}} :=$

[3]In the open sector this may be different due to the bubbling of holomorphic discs.

$Q_L + Q_R$. To get an expression of the action of $Q_{\text{BRST}}$ on operators, we need the supersymmetry algebra. Noting that $\delta W = \mathrm{i}\epsilon[\![Q, W]\!]$ for any field $W$ and graded commutator $[\![A, B]\!] := AB + (-1)^{k_A+k_B} BA$, the transformations of the scalar fields under $Q_L$ and $Q_R$ are

$$[\![Q_L, \phi^a]\!] = \chi^a\,, \tag{5.1.25}$$
$$[\![Q_L, \chi^a]\!] = 0\,, \tag{5.1.26}$$
$$[\![Q_L, \lambda^a]\!] = -(\Gamma_-)^a{}_{bc}\chi^b\lambda^c\,, \tag{5.1.27}$$
$$[\![Q_R, \phi^a]\!] = \lambda^a\,, \tag{5.1.28}$$
$$[\![Q_R, \chi^a]\!] = 0\,, \tag{5.1.29}$$
$$[\![Q_R, \lambda^a]\!] = -(\Gamma_+)^a{}_{bc}\lambda^b\chi^c\,. \tag{5.1.30}$$

Local observables of the topological theory have the form

$$\mathcal{O}_f = f_{a_1\ldots a_p;b_1\ldots b_q}\chi^{a_1}\cdots\chi^{a_p}\lambda^{b_1}\cdots\lambda^{b_q}\,, \tag{5.1.31}$$

where $f \in \Gamma(\Sigma, \phi^*(\Omega_+^{(0,p)} \otimes \Omega_-^{(0,q)})$. First we state the result of the action of the supercharges on the scalar fields,

$$[\![Q_L, \mathcal{O}_f]\!] = \mathcal{O}_{\overline{D}_{(-,+)}f} \quad \text{and} \tag{5.1.32}$$
$$[\![Q_R, \mathcal{O}_f]\!] = \mathcal{O}_{\overline{D}_{(+,-)}f}\,. \tag{5.1.33}$$

Here $\overline{D}_{(\pm,\mp)}$ denotes the covariantization of $\partial_\pm$ w.r.t. $\nabla_\mp$. In order to show that equation (5.1.32) is true, let us regard $f \in \phi^*\Omega_+^{(0,p)}\left(M, \phi^*\Omega_-^{(0,q)}\right)$. If $p = q = 0$ it holds

$$[\![Q_L, \mathcal{O}_f]\!] = [\![Q_L, f]\!] = f(\phi + \chi) - f(\phi) = (\overline{\partial}_+ f)_{a_1}\chi^{a_1} = \mathcal{O}_{\overline{\partial}_+ f}\,. \tag{5.1.34}$$

Using

$$[\![Q, AB]\!] = [\![Q, A]\!]B + (-1)^{k_A}A[\![Q, B]\!] \tag{5.1.35}$$

and equations (5.1.25) - (5.1.27), a straight forward calculation yields

$$\begin{aligned}[\![Q_L, \mathcal{O}_f]\!] = \Big((\overline{\partial}_+ f_{a_1\ldots a_p;b_1\ldots b_q})_{a_{p+1}} - (\Gamma_-)^c{}_{a_{p+1}b_1} f_{a_1\ldots a_p;c\ldots b_q} - \ldots \\ \ldots - (\Gamma_-)^c{}_{a_{p+1}b_q} f_{a_1\ldots a_p;a_1\ldots c}\Big)\,\chi^{a_1}\cdots\chi^{a_{p+1}}\lambda^{b_1}\cdots\lambda^{b_q} = \mathcal{O}_{\overline{D}_{(-,+)}f}\,.\end{aligned} \tag{5.1.36}$$

Equation (5.1.33) can be shown analogously. Now the space of local observables has a bi-grading by left- and right-moving R-charges, corresponding to $p$ and $q$. This gives a bi-complex

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r, "Q_L"] & \mathcal{O}^{p-1,q+1} \arrow[u, "Q_R"] \arrow[r, "Q_L"] & \mathcal{O}^{p,q+1} \arrow[u, "Q_R"] \arrow[r, "Q_L"] & \mathcal{O}^{p+1,q+1} \arrow[u, "Q_R"] \arrow[r, "Q_L"] & \cdots \\
\cdots \arrow[r, "Q_L"] & \mathcal{O}^{p-1,q} \arrow[u, "Q_R"] \arrow[r, "Q_L"] & \mathcal{O}^{p,q} \arrow[u, "Q_R"] \arrow[r, "Q_L"] & \mathcal{O}^{p+1,q} \arrow[u, "Q_R"] \arrow[r, "Q_L"] & \cdots \\
\cdots \arrow[r, "Q_L"] & \mathcal{O}^{p-1,q-1} \arrow[u, "Q_R"] \arrow[r, "Q_L"] & \mathcal{O}^{p,q-1} \arrow[u, "Q_R"] \arrow[r, "Q_L"] & \mathcal{O}^{p+1,q-1} \arrow[u, "Q_R"] \arrow[r, "Q_L"] & \cdots \\
 & \vdots \arrow[u, "Q_R"] & \vdots \arrow[u, "Q_R"] & \vdots \arrow[u, "Q_R"] &
\end{tikzcd} \tag{5.1.37}$$

and its total cohomology gives the space of physical states.

### 5.1.4. Supersymmetric Ground States

We will review that, at the classical level, the cohomology of states in nonlinear sigma-models with H-flux is isomorphic to the Lie algebroid cohomology of the canonical complex associated to $(M, \mathcal{J}_1)$. Definitions can be found in appendix A.

To this end we will show as a preliminary step that any function of the bosonic coordinates $\phi$ and fermionic scalars $\chi, \lambda$ can be rewritten as a function on $\Pi\overline{L}_1$, $\overline{L}_1$ being the eigenbundle of $\mathcal{J}_1$ with eigenvalue $-\mathrm{i}$.

Let us define

$$\psi^a := \frac{1}{\sqrt{2}}\left(\psi_+^a + \mathrm{i}\psi_-^a\right) \quad \text{and} \quad \rho_a := \frac{1}{\sqrt{2}} g_{ab}\left(\psi_+^b + \mathrm{i}\psi_-^b\right) . \tag{5.1.38}$$

They are sections of $\phi^* TM_{\mathbb{C}}$ and $\phi^* T^* M_{\mathbb{C}}$, respectively. Introducing a fermionic field

$$\Psi := \begin{pmatrix} \psi \\ \rho \end{pmatrix} \tag{5.1.39}$$

taking values in $\phi^*TM_{\mathbb{C}} \oplus \phi^*T^*M_{\mathbb{C}} \cong \phi^*(TM_{\mathbb{C}} \oplus T^*M_{\mathbb{C}})$, we get anti-commutation relations

$$[\![\Psi(\sigma)^\alpha, \Psi(\sigma')^\beta]\!] = (q^{-1})^{\alpha\beta}\delta(\sigma - \sigma') \,. \tag{5.1.40}$$

Using the explicit expression of $\mathcal{J}_1$,

$$\mathcal{J}_1 = \begin{pmatrix} I_1 & \beta_1 \\ B_1 & -I_1^* \end{pmatrix} \,, \tag{5.1.41}$$

where

$$I_1 = \frac{1}{2}(I_+ + I_-) \,, \tag{5.1.42}$$

$$\beta = \frac{1}{2}(\omega_-^{-1} - \omega_+^{-1}) \,, \tag{5.1.43}$$

$$B = \frac{1}{2}(\omega_+ + \omega_-) \,, \tag{5.1.44}$$

it is easy to check

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix} \begin{pmatrix} \chi \\ \lambda \end{pmatrix} = (1 + \mathrm{i}\mathcal{J}_1)\Psi \,. \tag{5.1.45}$$

Hence, any function of the bosonic coordinates and fermionic scalars can be rewritten as a function on $\Pi\overline{L}_1$. Since we are interested in the supersymmetric ground states in the RR sector, we are allowed to use point particle approximation. This guides us to the realm of supersymmetric quantum mechanics in the sense of chapter 3.

In this approximation the Noether charges associated to $Q_+$ and $Q_-$ are

$$Q_+ = \psi_+^a g_{ab}\dot{\phi}^b - \frac{\mathrm{i}}{6}H_{abc}\psi_+^a\psi_+^b\psi_+^c \quad \text{and} \tag{5.1.46}$$

$$Q_- = \psi_-^a g_{ab}\dot{\phi}^b + \frac{\mathrm{i}}{6}H_{abc}\psi_-^a\psi_-^b\psi_-^c \,. \tag{5.1.47}$$

We choose the BRST charges to be $Q := Q_+ + \mathrm{i}Q_-$ and $Q^\dagger := Q_+ - \mathrm{i}Q_-$. Then it is true that

$$Q^2 = (Q^\dagger)^2 = 0 \quad \text{and} \quad 4H = \{Q, Q^\dagger\} \,. \tag{5.1.48}$$

Here $H$ is the Hamiltonian of the supersymmetric quantum mechanics,

$$H = \frac{1}{2}g_{ab}\dot{\phi}^a\dot{\phi}^b - \frac{1}{4}R^{(+)}_{abcd}\psi_+^a\psi_+^b\psi_+^c\psi_+^d \,. \tag{5.1.49}$$

As we saw in theorem 3.1.8, supersymmetric ground states are in one to one correspondence with the elements of $Q_{\text{BRST}}$-cohomology. The same arguments as in chapter 3 imply

$$Q = d_H = d - H \wedge \,. \tag{5.1.50}$$

Our next step is to identify $Q_{\text{BRST}}$ in this context. The $R$-current reads

$$J = -\frac{\mathrm{i}}{2}\left(\omega_+(\psi_+, \psi_+) + \omega_-(\psi_-, \psi_-)\right) \,. \tag{5.1.51}$$

Because of the canonical anti-commutation relations the fermions $\chi$ and $\lambda$ have charge $+1$ under $J$. Expressing $\chi$ and $\lambda$ by $\psi$ and $\rho$ and quantizing the latter by

$$\psi^a \longleftrightarrow dx^a \wedge \,, \qquad \rho_a \longleftrightarrow i_{\frac{\partial}{\partial x^a}} \,, \tag{5.1.52}$$

the $R$-current becomes

$$J = -\,\mathrm{i}(B \wedge + i_\beta - i_{I_1}) \,, \tag{5.1.53}$$

where $i_{I_1} = I_1{}^a{}_b(dx^b \wedge) \circ i_{\frac{\partial}{\partial x^a}}$. Thus, $J$ obviously depends only on $\mathcal{J}_1$.
It is well known that $Q_{\text{BRST}}$ can be expressed via

$$Q_{\text{BRST}} = \frac{1}{2}(Q + [J, Q]) \,. \tag{5.1.54}$$

Hence, it remains to express $J$ in the language of generalized complex geometry. To this end let $A = X \oplus \xi \in \Gamma(M, TM_{\mathbb{C}} \oplus T^*M_{\mathbb{C}})$. It acts on $\rho \in \bigwedge^* T^*M_{\mathbb{C}}$ via

$$A \cdot \rho = i_X \rho + \xi \wedge \rho \,. \tag{5.1.55}$$

Since $\mathcal{J}_1$ is an endomorphism of $TM_{\mathbb{C}} \oplus T^*M_{\mathbb{C}}$ with eigenvalues $\pm\,\mathrm{i}$ it follows that a grading operator $R(\mathcal{J}_1)$ can be defined by

$$[R(\mathcal{J}_1), A] = -\,\mathrm{i}\mathcal{J}_1 A \,, \quad \forall A \in \Gamma(TM_{\mathbb{C}} \oplus T^*M_{\mathbb{C}}) \,. \tag{5.1.56}$$

This condition fixes $R(\mathcal{J}_1)$ up to a constant. Using the explicit form of $\mathcal{J}_1$ results in

$$\mathcal{J}_1 A \cdot \rho = \left(i_{I_1 X} - i_{\beta(\xi)} - i_X B \wedge - I_1^*(\xi) \wedge\right) \rho \,. \tag{5.1.57}$$

A solution of this equation is given by $R(\mathcal{J}_1) = -\,\mathrm{i}(B \wedge + i_\beta - i_{I_1}) = J$. Thus $J$ is a grading operator of $\bigwedge^* T^*M_{\mathbb{C}}$ with respect to $\mathcal{J}_1$. It follows that

$$Q_{\text{BRST}} = \frac{1}{2}(d_H + [R(\mathcal{J}_1), d_H)]) = \frac{1}{2}(d_H + R(\mathcal{J}_1) d_H - d_H R(\mathcal{J}_1)) = \overline{\partial}_H \,. \tag{5.1.58}$$

Moreover, it is true that $\left(\bigwedge^* T^*M_{\mathbb{C}}, \overline{\partial}_H\right)$ is a graded module over the complex of operators, $(\bigwedge^* L_1, d_L)$, i.e. [Gua03]

$$\overline{\partial}_H(s \cdot \rho) = (d_L s) \cdot \rho + (-1)^{|s|} s \cdot \overline{\partial}_H \rho \quad \text{for} \quad s \in \bigwedge^* L_1\,,\ \rho \in \bigwedge^* T^*M_{\mathbb{C}}\,. \tag{5.1.59}$$

Interpreting this equation as $Q_{\text{BRST}}(s \cdot \rho) = (Q_{\text{BRST}} s) \cdot \rho + (-1)^{|s|} s \cdot (Q_{\text{BRST}} \rho)$, it becomes clear that $Q_{\text{BRST}}$ acts on operators (i.e. $\bigwedge^* L_1$) as $d_L$. Therefore, the BRST-cohomology of operators is isomorphic at the classical level to the Lie algebroid cohomology $(\bigwedge^* L_1, d_L)$.

Strictly speaking, it remains to show that the resulting theory is a topological field theory, indeed, since it is not clear how to write it as the sum of a topological term and a $Q_{\text{BRST}}$ exact term. First results into this direction can be found in [Zuc06], [Chu08] and [Pes07]. In [KL07] it is also shown that the Frobenius structure of the resulting TQFT is in the complex case identical to the very well known one, whereas it is only isomorphic in the symplectic case. There one has to perform a Bogoliubov transformation to get the usual action of forms on themselves by wedge product. This is in accordance with 4.26 of [Gua03]. Section 6 of [KL07], "Towards the Twisted Generalized Quantum Cohomology Ring" tries to give a first discussion of quantum corrections. Using the same arguments as in section 3.2.4 of this work, they figure out the relevant instantons and call them twisted generalized complex holomorphic maps. The complete set of BRST transformations on the fields can be found in [Zuc06] and [Chu08]. For $H = 0$ the BRST-fixed points fulfill in local coordinates

$$(\delta_\mu{}^\nu + \mathrm{i}(j_\Sigma)_\mu{}^\nu)\,\frac{1}{2}(1 - \mathrm{i}I_+)^a{}_b \partial_\nu \phi^b = 0\,, \tag{5.1.60}$$

$$(\delta_\mu{}^\nu - \mathrm{i}(j_\Sigma)_\mu{}^\nu)\,\frac{1}{2}(1 - \mathrm{i}I_-)^a{}_b \partial_\nu \phi^b = 0\,, \tag{5.1.61}$$

where $j_\Sigma$ is the complex structure on $\Sigma$. Comparing the equations given in [KL07] with [Zuc06] and [Chu08] it turns out that in [KL07] there is a sign error. This sign also manifests in the formulation of equations (5.1.60)-(5.1.61) as

$$(1 + \mathrm{i}\mathcal{J}_1)\begin{pmatrix} \partial_\sigma \phi \\ g\partial_t \phi \end{pmatrix} = 0\,, \tag{5.1.62}$$

which is in accordance with [Zab06]. The error possibly originates in the fact that the A-twist in the latter references is the B-twist in the former and vice versa. The result of Kapustin and Li, i.e. eq. (4.3.3) in [Li05], is the right one, as there seems to be a second sign error in the step from (4.3.2) to (4.3.3). Expanding equations (5.1.60) and (5.1.61) into real and imaginary part we deduce that they are equivalent to

$$T\phi \circ j_\Sigma = I_+ \circ T\phi \quad \text{and} \tag{5.1.63}$$

$$T\phi \circ j_\Sigma = -I_- \circ T\phi\,. \tag{5.1.64}$$

Using the concrete form of $\mathcal{J}_1$ and $\mathcal{J}_2$ we can write these two equations compactly as

$$\mathcal{J}_2 \circ (\iota \circ T\phi) = (\iota \circ T\phi) \circ j_\Sigma \,, \tag{5.1.65}$$

where $\iota : TM \to TM \oplus T^*M =: \mathbb{T}M$ is the canonical embedding. Kapustin and Li call them twisted generalized complex maps.

Recall that the Lie algebroid cohomology with respect to $\mathcal{J}_1$ computes the supersymmetric ground states. If we want to know the instanton corrections to this classical structure, we need to solve an equation for the other generalized complex structure $\mathcal{J}_2$, which is in some sense compatible to $\mathcal{J}_1$, cf. section 6.3. This is similar to the physical motivation of quantum cohomology. There the Lie algebroid cohomology is given by (a Bogoliubov transformation of) the usual de Rham cohomology for $\mathcal{J}_1$ being just a symplectic structure $\omega$. The other structure $\mathcal{J}_2$ is given by an ordinary almost complex one, namely $I$, which renders $M$ to be Kähler, leading to the known results.

### 5.1.5. Behavior of Instantons under $B$-transformations

We should take a moment to examine the behavior of equation (5.1.65) under $B$-transformations.

One possible interpretation is that a $B$-transformation only changes $H$ to $H + dB = H$, as $B$ has to be a closed 2-form. Thus instantons are invariant under this transformation. At first sight this is a nice result and we should check whether we find an interpretation of it in terms of generalized complex geometry.

Naively, one could think that the $B$-field transformed equation (5.1.65) is given by

$$\mathrm{e}^B \mathcal{J}_2 \mathrm{e}^{-B} \circ \mathrm{e}^B(\iota \circ T\phi) = \mathrm{e}^B(\iota \circ T\phi) \circ j \,, \tag{5.1.66}$$

as $\exp(B)$ should also act on $\iota \circ T\Phi$, which maps into $\mathbb{T}M$. This equation is clearly equivalent to equation (5.1.65), in accordance with the above argument.

There are two possible issues with such a behavior.

The first one concerns the search for non-trivial examples of generalized pseudoholomorphic curves. Equation (5.1.66) tells us that if we do not find a solution for some generalized complex structure, there will not be a solution in the whole equivalence class under $B$-transformations. Recall that every type zero generalized complex structure is a $B$-transformation of a symplectic structure.

The second and more important one concerns an inconsistency in the prescribed object. Let us assume that we start with a generalized complex structure $\mathcal{J}_2$ and look at equation (5.1.65). Performing a $B$-transformation we end up with a generalized complex structure $\mathcal{J}_2'$. We saw above that (5.1.66) is the *same* equation as (5.1.65). If we would have started with $\mathcal{J}'$, the equation would not be the same. This can be rephrased as performing a $B$-transformation on the equation is not equivalent to starting with the transformed $\mathcal{J}_2'$. This is a serious problem and will be resolved in the following.

One possible way to resolve this issue is to look at equation (5.1.62). We saw that it is equivalent to (5.1.65). In [Zab06] it is shown that a canonical transformation acting on $\Pi\mathcal{L}M$ which is not originated by a diffeomorphism results in

$$(1 + \mathrm{i}\mathcal{J}_1) \begin{pmatrix} \partial_\sigma \phi \\ g\partial_t \phi + i_{\partial_\sigma \phi} b \end{pmatrix} = 0 \,, \tag{5.1.67}$$

as $g\partial_t\phi$ is the canonical momentum associated to $\partial_\sigma\phi$ for the nonlinear sigma model. This equation is true if and only if

$$\left(1 + \mathrm{i}\mathrm{e}^{-b} \mathcal{J}_1 \mathrm{e}^{b}\right) \begin{pmatrix} \partial_\sigma \phi \\ g\partial_t \phi \end{pmatrix} = 0 \,. \tag{5.1.68}$$

Therefore, a canonical transformation which acts on $\Pi T^*\mathcal{L}M$ by $b$ induces a $B$-transformation by $-b$ acting on $\mathcal{J}_1$. This induces a $B$-transformation by $-b$ acting on $\mathcal{J}_2$. Therefore, a $B$-transformation acting on $\mathbb{T}M$ also induces a canonical transformation on the super loop-space. In other words, there exists a canonical transformation, which clearly should not have any effect on physics, such that a $B$-transformation acts only on $\mathcal{J}$ and not on the embedding $\iota : TM \to \mathbb{T}M$. Using this we get that modulo canonical transformations the transformed equation is given by the equation with respect to the transformed generalized complex structure. This is in some sense $B$-field invariance. We still have the problem that there is not yet a notion of a $B$-field transformed instanton, or generalized pseudoholomorphic curve in the language of part three, such that the $B$-field transformed instanton is a solution of the $B$-field transformed instanton equation. This is the reason why we take $\iota$ not to be the canonical embedding, which is not present in the case of a general exact Courant algebroid anyway, but an arbitrary isotropic embedding $\lambda : TM \to \mathbb{T}M$ and call the resulting object a generalized pseudoholomorphic, or simply $\mathcal{J}$-holomorphic, pair. The above consideration shows that $B$-transformations on $\lambda$ correspond to canonical transformations on the super loop-space in the case of supersymmetric nonlinear sigma models.

If one wants to determine the instantons of a supersymmetric theory, one has to find the BRST fixed points and not just the purely bosonic fixed points, as explained in section 3.1.2. This might be a possible issue in the literature. Thus the instanton equation for $H \neq 0$ possibly differs from the above one. In [Li05] he writes that the moduli space of instantons has to be known to proceed the calculation of the quantum structure of supersymmetric nonlinear sigma models with H-flux. Since the former is not known, yet, he stops there. This is the starting point of our mathematical treatment of $\mathcal{J}_2$-holomorphic curves. We will examine the moduli-space of instantons to do the next step towards Quantum cohomology in generalized complex geometry and other related topics. The latter should be a deformation of the Lie algebroid cohomology associated to the canonical complex of $\mathcal{J}_1$.

## 5.2. Interpolation Between A- and B-Model on Hyperkähler Manifolds

In this section I want to give a nice interpolation between the A- and B- Model on Hyperkähler manifolds through the generalized B-model. This can for instance be used to view mirror symmetry of Hyperkähler manifolds as a continuous symmetry rather than a discrete one. Recall that a manifold $(M, I)$ is mirror to another manifold $(N, J)$ if the A-model on $M$ is equivalent to the B-model on $N$ and the B-model on $M$ is equivalent to the A-model on $N$. Hopefully, the generalizations of the Fukaya category and quantum-cohomology will turn out to be only dependent on the homotopy class of the generalized complex structures defining it. Then such a continuous interpolation between the A- and B- model could be used to prove the equivalence of the appropriate categories.[4] With this in mind, it should be possible to prove the Mirror symmetry of arbitrary Hyperkähler manifolds with respect to a pair of complex structures with their associated Kähler forms.

More concretely, let $(M, g, I, J, K)$ be a Hyperkähler manifold. Recall that this means $M$ admits a $2$-sphere of complex structures with respect to which $g$ is Hermitian. In particular it is true that there are three complex structures which obey

$$I^2 = J^2 = K^2 = IJK = -\mathbb{1}\,. \tag{5.2.1}$$

To give an interpolation between the A-model in complex structure $J$ and the B-model in complex structure $I$, we define the following two almost generalized complex structures,

$$\mathcal{J}_1(t) := \sin(t)\mathcal{J}_I + \cos(t)\mathcal{J}_{\omega_J} \quad \text{and} \tag{5.2.2}$$

$$\mathcal{J}_2(t) := \sin(t)\mathcal{J}_{\omega_I} + \cos(t)\mathcal{J}_J\,. \tag{5.2.3}$$

As usual $\mathcal{J}_I$, $\mathcal{J}_J$, $\mathcal{J}_{\omega_I}$ and $\mathcal{J}_{\omega_J}$ are the generalized complex structures associated to the complex structures $I$, $J$ and the Kähler forms $\omega_I$, $\omega_J$ (compare equations (A.4.4) and

[4] They are the enhanced Fukaya category and the bounded derived category of coherent sheaves.

(A.4.5)). Then it is true that

$$\mathcal{J}_1(0) = \mathcal{J}_{\omega_J} \quad \text{and} \quad \mathcal{J}_2(0) = \mathcal{J}_J \quad \text{(A-model in symplectic structure } \omega_J\text{)}\,, \tag{5.2.4}$$

$$\mathcal{J}_1\left(\frac{\pi}{2}\right) = \mathcal{J}_I \quad \text{and} \quad \mathcal{J}_2\left(\frac{\pi}{2}\right) = \mathcal{J}_{\omega_I} \quad \text{(B-model in complex structure } I\text{)}\,. \tag{5.2.5}$$

Therefore, the generalized B-model defined by $(\mathcal{J}_1(t), \mathcal{J}_2(t))$ interpolates between the (Bogoliubov transformation of the) A-model in complex structure $J$ and the B-model in complex structure $I$. It remains to show that $(\mathcal{J}_1(t), \mathcal{J}_2(t))$ is a generalized Kähler structure. In the language of definition 6.3.6 this means we have to show that $(\mathcal{J}_1(t), \mathcal{J}_2(t))$ is a pair of integrable generalized complex structures being tamed by each other and being compatible.

After expanding $\mathcal{J}_1(t)\mathcal{J}_2(t)$ we obtain

$$\mathcal{J}_1(t)\mathcal{J}_2(t) = \sin^2(t)\mathcal{J}_I\mathcal{J}_{\omega_I} + \sin(t)\cos(t)\mathcal{J}_I\mathcal{J}_J + \sin(t)\cos(t)\mathcal{J}_{\omega_J}\mathcal{J}_{\omega_I} + \cos^2(t)\mathcal{J}_{\omega_J}\mathcal{J}_J\,. \tag{5.2.6}$$

Because of $\omega_J^{-1}\omega_I = -Jg^{-1}gI = -JI$ and $\omega_J\omega_I^{-1} = -gJIg^{-1} = gIJg^{-1} = I^*J^*gg^{-1} = I^*J^*$ it follows that $\mathcal{J}_I\mathcal{J}_J = -\mathcal{J}_{\omega_J}\mathcal{J}_{\omega_I}$. Equation (5.2.6) now simplifies to

$$\mathcal{J}_1(t)\mathcal{J}_2(t) = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}\,. \tag{5.2.7}$$

Hence, $\mathcal{J}_2(t)$ is tamed by $\mathcal{J}_1(t)$. Interchanging $I$ and $J$ in the above calculations yields

$$\mathcal{J}_2(t)\mathcal{J}_1(t) = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} = \mathcal{J}_1(t)\mathcal{J}_2(t)\,. \tag{5.2.8}$$

This shows that $\mathcal{J}_1(t)$ and $\mathcal{J}_2(t)$ are compatible structures for all $t$.

Next we will show that they are both integrable. This will be done by demonstrating that for all $t$ they are $B$-transformations of a symplectic structure. Then it follows immediately that $\mathcal{J}_1(t)$ and $\mathcal{J}_2(t)$ are integrable almost generalized complex structures. Let us first consider $\mathcal{J}_1(t)$. It is true that

$$\begin{aligned}
&\mathrm{e}^{\tan(t)\omega_K}\mathcal{J}_1(t)\mathrm{e}^{-\tan(t)\omega_K} = \begin{pmatrix} 1 & 0 \\ \tan(t)\omega_K & 1 \end{pmatrix}\begin{pmatrix} \sin(t)I & -\cos(t)\omega_J^{-1} \\ \cos(t)\omega_J & -\sin(t)I^* \end{pmatrix} \\
&\times\begin{pmatrix} 1 & 0 \\ -\tan(t)\omega_K & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \tan(t)\omega_K & 1 \end{pmatrix}\begin{pmatrix} 0 & -\cos(t)\omega_J^{-1} \\ \sec(t)\omega_J & -\sin(t)I^* \end{pmatrix} = \\
&= \begin{pmatrix} 0 & -\left(\sec(t)\omega_J\right)^{-1} \\ \sec(t)\omega_J & 0 \end{pmatrix}\,.
\end{aligned} \tag{5.2.9}$$

Now $d\tan(t)\omega_K = \tan(t)d\omega_K = d\sec(t)\omega_J = 0$ implies that $\mathcal{J}_1(t)$ is a $B$-transformation of a symplectic structure and thus integrable. A completely analogous calculation shows that

$$\mathcal{J}_2(t) = \mathrm{e}^{-\cot(t)\omega_K} \begin{pmatrix} 0 & -(\csc(t)\omega_I)^{-1} \\ \csc(t)\omega_I & 0 \end{pmatrix} \mathrm{e}^{\cot(t)\omega_K} \,. \tag{5.2.10}$$

Therefore, $\mathcal{J}_2(t)$ is integrable for all $t$. Altogether it follows that $(\mathcal{J}_1(t), \mathcal{J}_2(t))$ defines a generalized Kähler structure and its associated generalized B-model interpolates between the Bogoliubov transformation of the A-model in complex structure $J$ and the B-model in complex structure $I$. It remains to construct the Lie-algebroid cohomology and to incorporate instantons to calculate the generalized chiral ring. In order to show that we get a physically sensible model we have to show that $(\mathcal{J}_1(t), \mathcal{J}_2(t))$ define generalized Calabi-Yau metric geometries for all $t$. The answer of these questions will be given somewhere else.

## 5.3. Coisotropic Branes and Geometry

In this section we will give a geometrical definition of classical A-branes. The definition of rank-one branes will follow the world sheet approach in [KO03] while we will give a more detailed exposure. Our aim is to develop the sufficient and necessary conditions for an object to be a (classical) A-brane, which is a Dirichlet brane, or D-brane, being defined in the topologically twisted A-theory of non-linear sigma-model with boundary. They play an important role in "Homological Mirror Conjecture" [Kon94],[KO03],[AZ05],[PZ98] and the gauge theory formulation of "Geometric Langlands Conjecture" [KW06],[Fre09b]. We will see that the classical conservation of the R-current, which underlies the topologically twisted $\mathcal{N} = (2, 2)$ non-linear sigma-model, on the endpoints of open strings ending on A-branes will give some algebraic conditions on the given data.

### 5.3.1. Setup and Characterization via Poisson-Structures

Let $X$ be a Kähler manifold with metric $G$ and Kähler form $\omega$. Then the complex structure $I_\omega$ which is associated to the Kähler form $\omega$ reads $I_\omega = G^{-1}\omega$. This follows from the fact that the $(1,1)_I$ form $\omega \in \Gamma\left(X, (T^{1,0}X)^* \otimes_{\mathbb{C}} (T^{0,1}X)^*\right) =: \mathcal{A}^{1,1}(X)$ is defined as $\omega(U,V) = g(IU,V)$ for all $U \in T^{1,0}X$ and $V \in T^{0,1}X$. Since we will use the topologically twisted $\mathcal{N} = (2,2)$ non-linear sigma-model, absence of anomalies of $R$-symmetry in the bulk sector dictates us to further demand $X$ being Ricci flat and, hence, Calabi-Yau. Therefore, let $c_1(X) = 0$.

Next, we want to specify the objects which are potential candidates for being A-branes. The strings which end on a brane may carry a charge. Hence, we look at submanifolds $Y \subset X$ equipped with a line bundle $\pi_L : L \to Y$ and a unitary connection $\nabla$.

Since A-branes are defined in the context of topologically twisted non-linear sigma models, we have to specify some additional data. Let $\Sigma$ be an open string world-sheet, i.e. a Riemann surface $\Sigma$ with boundary $\partial\Sigma$. Then we call $\Phi : \Sigma \to X$, where $\Phi(\partial\Sigma) \subset Y$, the sigma-model fields and the fermions $\psi$, $\bar{\psi}$ are sections of $\Phi^*(TX) \otimes \Pi S^{\pm}$, with $S^{\pm}$ being semi spinor line bundles and $\Pi$ is the parity-reversal functor. Furthermore, we define supercurrents

$$Q^{\pm} := \frac{i}{4\sqrt{2}} G(\psi, \partial\Phi) \pm \omega(\psi, \partial\Phi) \tag{5.3.1}$$

$$\bar{Q}^{\pm} := \frac{i}{4\sqrt{2}} G(\bar{\psi}, \bar{\partial}\Phi) \pm \omega(\bar{\psi}, \bar{\partial}\Phi) \tag{5.3.2}$$

and $U(1)$ R-currents

$$J := -\frac{i}{2}\omega(\psi, \psi) \tag{5.3.3}$$
$$\overline{J} := -\frac{i}{2}\omega(\bar{\psi}, \bar{\psi})\,. \tag{5.3.4}$$

Next, we wish to give the classical boundary conditions which $\partial\Phi$ and $\psi$ fulfill on $Y$. In order to be able to specify them, we use $G$ to decompose $TX|_Y \cong NY \oplus TY$, where $TY$ is the tangential bundle of $Y$ and $NY$ is the normal bundle of $Y$ in $X$ with respect to $G$. The boundary conditions are then given by[5]

$$\partial\Phi = R(\bar{\partial}\Phi) \quad \text{and} \tag{5.3.5}$$
$$\psi = R(\bar{\psi})\,. \tag{5.3.6}$$

Here $R$ is an endomorphism of $TX|_Y$. It is well known that it can be expressed with respect to the above decomposition of $TX|_Y$ as

$$R = (-\operatorname{id}_{NY}) \oplus (g - F)^{-1}(g + F)\ , \tag{5.3.7}$$

where $g = G|_{TY}$ and $F$ is the curvature of $(Y, L, \nabla)$. Equation (5.3.7) reflects the fact that $\Sigma$ is the trajectory of a string end-point, which is charged with respect to the gauge fields on the brane. Thus, the end-points are subjected to the Lorentz force. Equation (5.3.7) tells us that the velocity of the end-points is tangent to the brane and the Lorentz force is balanced with the tension of the moving string. Furthermore, $R$ is an orthogonal transformation with respect to $G$, i.e.

$$R^{\mathrm{T}}\, G\, R = G\,. \tag{5.3.8}$$

This can be seen by inserting (5.3.7) into (5.3.8).

So far, we do not know whether $Y$ defines a D-brane or not. To check this we observe that $Q^+ + Q^- = \bar{Q}^+ + \bar{Q}^-$ on the boundary, which can be easily checked by putting the boundary conditions into (5.3.1) and (5.3.2). Hence, $Y$ defines at least a D-brane [Pol98b].

In addition, the topologically twisted non-linear-sigma-model must preserve $\mathcal{N} = 2$ supersymmetry [Wit95]. This can be done in two inequivalent ways:

$$Q^{\pm} = \bar{Q}^{\mp} \quad \text{and} \quad J = -\bar{J} \qquad \text{(A-type)} \tag{5.3.9}$$

[5] The transformation behavior of the fermions is dictated by the presence of $\mathcal{N} = 1$ supersymmetry on the world-sheet.

and

$$Q^{\pm} = \bar{Q}^{\pm} \quad \text{and} \quad J = \bar{J} \qquad \text{(B-type)} \tag{5.3.10}$$

on the boundary. For now we concentrate on A-type boundary conditions. Combining equations (5.3.1), (5.3.2), (5.3.5), (5.3.6) and (5.3.9) we get

$$R^{\mathrm{T}}\,\omega\,R = -\omega\,. \tag{5.3.11}$$

In the following we will look at the geometric implications of (5.3.11) on $(Y, L, \nabla)$.

Since $I = G^{-1}\omega$, it is easy to check that $R\,I = I\,R$. Now let us choose a basis of $TX|_Y$, where the first $\dim_{\mathbb{R}} X - \dim_{\mathbb{R}} Y$ vectors span $NY$ and the remaining $\dim_{\mathbb{R}} Y$ vectors span $TY$. Let in this basis the inverse of the Kähler form $\omega$ be expressed as

$$\omega^{-1} = \begin{pmatrix} A & B \\ -B^{\mathrm{T}} & C \end{pmatrix}, \tag{5.3.12}$$

where $A^{\mathrm{T}} = -A$ and $C^{\mathrm{T}} = -C$ so that $\omega^{\mathrm{T}} = -\omega$. Because of $R^{-1} = -\,\mathrm{id}_{NY} \oplus (g + F)^{-1}(g - F)$ and $R^{\mathrm{T}}\omega R = -\omega \Leftrightarrow R^{-1}\omega^{-1}(R^{-1})^{\mathrm{T}} = -\omega^{-1}$, a straight forward calculation shows that (5.3.11) is *equivalent* to

$$A = 0 \tag{5.3.13}$$
$$B\,F = 0 \tag{5.3.14}$$
$$g\,C\,g = F\,C\,F \tag{5.3.15}$$

### 5.3.2. Geometric Implications and Definition of Classical Rank-One Coisotropic A-Branes

Next we look for a geometric formulation of the algebraic properties (5.3.13), (5.3.14) and (5.3.15). First, we will show that property (5.3.13) is equivalent to $Y$ being a coisotropic submanifold of $X$. Second, we will observe that (5.3.14) is equivalent to $F$ descending to a form on some sub-bundle of $TY$. Third we will use (5.3.15) to define an almost complex structure, which is distinct from $I$ and acts on the normal bundle of the foliation associated to the symplectic radical $\mathscr{L}Y$ of $TY$ with respect to $\omega$.

As for the first property, let $T_zY_\omega^\perp := \{U \in T_zX \,|\, \forall V \in T_zY : \omega(U,V) = 0\}$. It is clear that $Y \subset X \Rightarrow T_zY \subset T_zX$ for $z \in Y$.[6] Recall that $Y$ is coisotropic in $(X,\omega)$ iff

[6]Strictly speaking $T_zY$ can be embedded into $T_zX$ such that the embedding fulfills above property.

$\forall z \in Y : T_zY^{\perp}_{\omega} \subset T_zY$. Since $\omega$ is a Kähler form and in particular non-degenerate[7], it is possible to regard it as an isomorphism

$$\omega_z : T_zX \to T^*_zX\,, \quad \text{with} \tag{5.3.16}$$
$$U \mapsto \alpha_U := \omega_z(U, \cdot)\,. \tag{5.3.17}$$

Next we will examine the properties of $\alpha_U$ for $U \in T_zY^{\perp}_{\omega}$. The definition of the $\omega$-radical $T_zY^{\perp}_{\omega}$ implies that $\alpha_U(V) = 0$ for all $V \in T_zX$. Therefore, let us define $T^*_zY^{\perp}_{\omega} := \{\alpha \in T^*_zX \,|\, \forall V \in T_zY : \alpha(V) = 0\}$. To prove that $Y$ is coisotropic we need following

**Lemma 5.3.1.** *Let $T_zY^{\perp}_{\omega}$ and $T^*_zY^{\perp}_{\omega}$ be as above. Then it holds*

$$\omega_z^{-1}\left(T^*_zY^{\perp}_{\omega}\right) = T_zY^{\perp}_{\omega}\,. \tag{5.3.18}$$

*Proof.* Let $\alpha \in T^*_zY^{\perp}_{\omega}$. Then $U_\alpha := \omega_z^{-1}(\alpha)$ is defined by $\alpha(V) = \omega_z(U_\alpha, V)$ for all $V \in T_zX$. Since $\alpha(V) = \omega_z(U_\alpha, V) = 0$ for all $V \in T_zY$, it follows that $U_\alpha \in T_zY^{\perp}_{\omega}$ and, hence, $\omega_z^{-1}(T^*_zY^{\perp}_{\omega}) \subseteq T_zY^{\perp}_{\omega}$.

Let $U_\alpha \in T_zY^{\perp}_{\omega}$. Then $\alpha := \omega_z(U_\alpha, \cdot)$ fulfills $\omega_z^{-1}(\alpha) = U_\alpha$. Thus, $T_zY^{\perp}_{\omega} \subseteq \omega^{-1}(T^*_zY^{\perp}_{\omega})$.

Altogether, we infer $\omega^{-1}\left(T^*_zY^{\perp}_{\omega}\right) = T_zY^{\perp}_{\omega}$. ☐

Now we are able to show that (5.3.13) is equivalent to $Y$ being coisotropic in $(X, \omega)$. This is subject of the following

**Proposition 5.3.2.** *Let $(X, \omega, G)$ be a Kähler manifold, $Y \hookrightarrow X$ be a submanifold of $X$ and the inverse Kähler form $\omega^{-1}$ be expressed as in* (5.3.12). *Then $A = 0$ if and only if $Y$ is a coisotropic submanifold of $(X, \omega)$.*

*Proof.* Let $\alpha \in T^*_zY^{\perp}_{\omega}$ and $\frac{\partial}{\partial x^\mu}$ be basis vectors of $T_zX$, where the first $\dim_{\mathbb{R}} X - \dim_{\mathbb{R}} Y$ vectors span $N_zY$ and the remaining $\dim_{\mathbb{R}} Y$ vectors span $T_zY$. Moreover, let $dx^\mu$ be their dual basis vectors. Then

$$\langle \alpha, U \rangle = \alpha(U) = \alpha_\mu dx^\mu \left( V^\nu \frac{\partial}{\partial x^\nu} \right) = \alpha_\mu V^\nu \, dx^\mu \left( \frac{\partial}{\partial x^n u} \right) = \alpha_\mu V^\mu\,. \tag{5.3.19}$$

Hence, it is clear that in this basis $\alpha$ can be written as

$$\alpha = \begin{pmatrix} {\alpha_1}^{\mathrm{T}} & 0 \end{pmatrix}\,. \tag{5.3.20}$$

[7]Recall that $\omega_z(U, V) = g(U, IV)$ for $U, V \in T_zX$, an hermitian structure $g$ and an almost complex structure $I$.

Let $A = 0$. Lemma 5.3.1 tells us that for every vector $U \in T_zY_\omega^\perp$ there exists a 1-form $\alpha \in T_z^*Y_\omega^\perp$ such that $U = \omega^{-1}\alpha$. This implies together with (5.3.20)

$$U = \omega^{-1}\alpha = \begin{pmatrix} 0 & B \\ -B^{\mathrm{T}} & C \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -B^{\mathrm{T}}\alpha_1 \end{pmatrix} \in T_zY \tag{5.3.21}$$

for all $U \in T_zY_\omega^\perp$ and, hence, $T_zY_\omega^\perp \subseteq T_zY$ for all $z \in Y$.

Now let $Y$ be a coisotropic submanifold of $(X, \omega)$. Thus, it is true that $\omega^{-1}(T_z^*Y_\omega^\perp) = T_zY_\omega^\perp \subseteq T_zY$ for all $z \in Y$. Hence, it holds that

$$U = \omega^{-1}\alpha = \begin{pmatrix} A & B \\ -B^{\mathrm{T}} & C \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix} = \begin{pmatrix} A\alpha_1 \\ -B^{\mathrm{T}}\alpha_1 \end{pmatrix} \overset{!}{\in} T_zY \tag{5.3.22}$$

for arbitrary $U$ and associated $\alpha_1$. Therefore, $A\alpha_1 = 0$ for arbitrary $\alpha_1$ which enables us to infer that $A = 0$. This proves the proposition. $\square$

Since $Y$ is coisotropic, the rank of $\omega_z$ is constant along $Y$ and we can define the kernel of $\omega$, which is a sub-bundle of $TY$,

$$\mathscr{L}Y := \coprod_{z \in Y} T_zY_\omega^\perp \,, \tag{5.3.23}$$

where $\coprod$ denotes the disjoint union. Furthermore, we define an equivalence relation on $TY$ by

$$u_1 \sim u_2 \quad :\Leftrightarrow \quad \pi_{TY}(u_1) = \pi_{TY}(u_2) \ \wedge \ u_1 - u_2 \in \mathscr{L}Y \,, \tag{5.3.24}$$

with $\pi_{TY} : TY \to Y$ being the projection in $TY$. There exists a short exact sequence

$$0 \longrightarrow \mathscr{L}_zY \hookrightarrow T_zY \twoheadrightarrow T_zY/\mathscr{L}_zY \longrightarrow 0 \tag{5.3.25}$$

for all $z \in Y$, where $\mathscr{L}_zY := T_zY_\omega^\perp$ and $\mathscr{F}_zY := T_zY/\mathscr{L}_zY := T_zY/\!\sim$. Thus it is true that $T_zY = \mathscr{L}_zY \oplus \mathscr{F}_zY$ for all $z \in Y$ and

$$\mathscr{F}Y := \coprod_{z \in Y} \mathscr{F}_zY \tag{5.3.26}$$

is well defined. Therefore, we have a decomposition

$$TY = \mathscr{L}Y \oplus \mathscr{F}Y \,. \tag{5.3.27}$$

Later we will give a geometric interpretation of $\mathscr{F}Y$ and $\mathscr{L}Y$. To be able to interpret $BF = 0$ geometrically, we need

**Proposition 5.3.3.** *Let $(X, \omega, G)$ be a Kähler manifold, $Y$ a coisotropic submanifold of $(X, \omega)$, $\mathscr{L}Y$ be as above, $\pi_L : L \to Y$ a unitary line bundle with a unitary connection $\nabla$ and curvature $F$ and the inverse Kähler form $\omega^{-1}$ be parameterized as in* (5.3.12)*. Then $BF = 0$ if and only if $F(\mathscr{L}_z Y) = \{0\}$ for all $z \in Y$.*

*Proof.* Let $z \in Y$ and $V \in \mathscr{L}_z Y$. Then by lemma 5.3.1 there exists a unique $\alpha \in T_z^* Y_\omega^\perp$ such that $V = \omega^{-1}\alpha$. Let $F$ act on $TX|_Y$ by first projecting on $TY$ and then acting by $F$ on the result, i.e. for all $z \in Y$ it is true that $F$ on $T_z X$ can be written as

$$\tilde{F} = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \tag{5.3.28}$$

Then it follows that

$$F(V) = \tilde{F}\omega^{-1}\alpha = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \cdot \begin{pmatrix} 0 & B \\ -B^{\mathrm{T}} & C \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \left({\alpha_1}^{\mathrm{T}} BF\right)^{\mathrm{T}} \end{pmatrix} \tag{5.3.29}$$

Let $BF = 0$. Then equation (5.3.29) implies that $F(V) = 0$ for all $V \in \mathscr{L}_z Y$ and $z \in Y$ and, hence, $F(\mathscr{L}_z Y) = \{0\}$ for all $z \in Y$. Let $F(\mathscr{L}_z Y) = \{0\}$ for all $z \in Y$. Then $F(V) = 0$ for all $V \in \mathscr{L}_z Y$ and $z \in Y$. From equation (5.3.29) we infer that ${\alpha_1}^{\mathrm{T}} BF = 0$ for arbitrary $\alpha_1$. Therefore, $BF = 0$ and the proposition is proven. ☐

Proposition 5.3.3 implies that the curvature $F$ descends to a section of $\bigwedge^2 \mathscr{F}Y$, i.e. an element $f \in H^0(Y, \bigwedge^2 \mathscr{F}Y)$. Therefore, $F = 0 \oplus f$ on $T_z Y = \mathscr{L}_z Y \oplus \mathscr{F}_z Y$. Furthermore, it is clear that $\omega(\mathscr{L}_z Y) = \{0\}$. Thus, $\omega$ can be written as $\omega = 0 \oplus \sigma$ on $T_z Y = \mathscr{L}_z Y \oplus \mathscr{F}_z Y$. Later we will need

**Lemma 5.3.4.** *Let $(X, \omega, G)$ be a Kähler manifold, $Y \hookrightarrow X$ be a submanifold of $X$, $z \in Y$, $\mathscr{L}_z Y := T_z Y_\omega^\perp$ and $\mathscr{F}_z Y := T_z Y / \mathscr{L}_z Y$ . Then $\omega = 0 \oplus \sigma$ on $T_z Y = \mathscr{L}_z Y \oplus \mathscr{F}_z Y$ and $\sigma$ is non-degenerate on $\mathscr{F}_z Y$.*

*Proof.* Let $(X, \omega, G)$ be a Kähler manifold, $Y \hookrightarrow X$ be a submanifold of $X$, $\mathscr{L}_z Y := T_z Y_\omega^\perp$ and $\mathscr{F}_z Y := T_z Y / \mathscr{L}_z Y$ . Then it follows by definition that $\omega(\mathscr{L}_z Y) = \{0\}$. Thus, $\omega = 0 \oplus \sigma$. Suppose that there exists an $U \in \mathscr{F}_z Y$ such that for all $V \in \mathscr{F}_z Y$ it holds that $\omega(U, V)$. Since $\omega(U, V_{\mathscr{L}}) = 0$ for all $V_{\mathscr{L}} \in \mathscr{L}_z Y$, it holds that $\omega(U, V) = 0$ for all $V \in T_z Y$. Hence, $U \in \mathscr{L}_z Y$ and $[U] = [0] \in \mathscr{F}_z Y$. Therefore, $\sigma$ is non-degenerate on $\mathscr{F}_z Y$ for all $z \in Y$. ☐

Our next aim is to show that $\sigma^{-1} f$ is an almost complex structure on $\mathscr{F}_z Y$. To that purpose we have to assure $\omega^{-1} F : \mathscr{F}_z Y \to \mathscr{F}_z Y$ and $\omega^{-1} F$ is well defined on $\mathscr{F}_z Y$. This is part of

**Lemma 5.3.5.** *Let $(X, \omega, G)$ be a Kähler manifold, $Y \hookrightarrow X$ be a submanifold of $X$, $\pi_L : L \to Y$ a unitary line bundle with a unitary connection $\nabla$ and curvature $F$, $z \in Y$, $\mathscr{L}_z Y := T_z Y_\omega^\perp$, $\mathscr{F}_z Y := T_z Y / \mathscr{L}_z Y$ and $F(\mathscr{L}_z Y) = \{0\}$ for all $z \in Y$. Then $J := \omega^{-1} F$ is a well-defined linear map from $\mathscr{F}_z Y$ to $\mathscr{F}_z Y$.*

*Proof.* Let $U \in \mathscr{F}_z Y$. Since $T_z X = T_z Y \oplus N_z Y$ and $G(T_z X) = T_z^* X$ it follows that $T_z^* X = G(T_z Y) \oplus G(N_z Y)$. It is obvious that

$$G(T_z Y) \cong \mathrm{Ann}_{T_z Y}(T_z^* X) := \{\alpha \in T_z^* X : \alpha(N_z Y) = \{0\}\} \cong T_z^* Y\,, \tag{5.3.30}$$

where $\mathrm{Ann}_{T_z^* X}(T_z Y)$ is the annihilator of $T_z Y$ in $T_z^* X$ and analogously

$$G(N_z Y) \cong N_z^* Y := (N_z Y)^*\,. \tag{5.3.31}$$

Therefore, it is possible to embed $i_{T_z^* Y} : T_z^* Y \hookrightarrow T_z^* X$. Thus, $\omega^{-1} F$ can be defined by

$$T_z Y \xrightarrow{F} T_z^* Y \xhookrightarrow{i_{T_z^* Y}} T_z^* X \xrightarrow{\omega^{-1}} T_z X \quad . \tag{5.3.32}$$

Since $F(\mathscr{L}_z Y) = \{0\}$ it follows by proposition 5.3.3 that $BF = 0$ and, hence, $J(U) \in T_z Y$. Every vector $U$ tangent to $Y$ lies in exactly one equivalence class $[U] \in \mathscr{F}_z Y$. So $J$ can be regarded as a map $J : \mathscr{F}_z Y \to \mathscr{F}_z Y$.

Next we have to ensure that this map is well defined. This can be achieved by examining the definition of $J = \omega^{-1} F$. Recall that $U' = J(U) = \omega^{-1} F(U)$ is defined by $\omega(U', V) = F(U, V)$ for all $V \in T_z Y$. Let $U_1$, $U_2 \in [U]$ and $U_1 \neq U_2$. Then $U_2 = U_1 + U_{\mathscr{L}}$, where $\mathscr{L}_z Y \ni U_{\mathscr{L}} \neq 0$. Since $F(\mathscr{L}_z Y) = \{0\}$, it follows that

$$F(U_2, V) = F(U_1 + U_{\mathscr{L}}, V) = F(U_1, V) + F(U_{\mathscr{L}}, V) = F(U_1, V) \tag{5.3.33}$$

and, thus, $J(U_1) = J(U_2)$. Therefore, $J$ is well defined on $\mathscr{F}_z Y$. Analogously it follows that $[U'] \in \mathscr{F}_z Y$ is unique. $\square$

Since we have shown that $J : \mathscr{F}_z Y \to \mathscr{F}_z Y$ is a well defined map, it is possible to state

**Proposition 5.3.6.** *Let $(X, \omega, G)$ be a Kähler manifold, $Y \hookrightarrow X$ be a submanifold of $X$, $\pi_L : L \to Y$ a unitary line bundle with a unitary connection $\nabla$ and curvature $F$, $F(\mathscr{L}_z Y) = \{0\}$ for all $z \in Y$ and the inverse Kähler form $\omega^{-1}$ be represented as in* (5.3.12). *Furthermore, let $f$ and $\sigma$ be as in lemma 5.3.4 and above. Then $J := \sigma^{-1} f : \mathscr{F}_z Y \to \mathscr{F}_z Y$ fulfills $J^2 = -\mathrm{id}_{\mathscr{F}_z Y}$ for all $z \in Y$ if and only if $gCg = FCF$.*

*Proof.* Let $z \in Y$. Since $G^{-1}\omega$ is an almost complex structure, we infer that $G\omega^{-1}G = -\omega$ on $T_zX$ and in particular on $T_zY$. Let us decompose $T_zY = \mathscr{L}_zY \oplus \mathscr{F}_zY$. Then $F = 0 \oplus f$ and $\omega = 0 \oplus \sigma$ with respect to this decomposition. Lemma 5.3.4 tells us that $\sigma$ is non-degenerate on $\mathscr{F}_zY$ and in particular invertible. For $g = G|_{TY}$ it then follows that $G\omega^{-1}g = 0 \oplus (-\sigma)$ on $T_zY$. The proof of lemma 5.3.5 also shows that $\omega(\mathscr{F}_zY) = F(\mathscr{F}_zY)$ and it is true that $\omega(\mathscr{F}_zY) = \sigma(\mathscr{F}_zY)$ as well as $F(\mathscr{F}_zY) = f(\mathscr{F}_zY)$. Thus, it follows from lemma 5.3.5 that $J := \sigma^{-1}f : \mathscr{F}_zY \to \mathscr{F}_zY$ is a well defined map. Since $F(\mathscr{L}_zY) = \{0\}$ for all $z \in Y$, it follows by proposition 5.3.3 that $BF = 0$. This in turn implies

$$\omega^{-1}F(U) = \begin{pmatrix} 0 & B \\ -B^{\mathrm{T}} & C \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \cdot \begin{pmatrix} 0 \\ U \end{pmatrix} = \begin{pmatrix} 0 \\ CF\,U \end{pmatrix} \quad \forall U \in T_zY\,. \tag{5.3.34}$$

Moreover, it holds

$$G\omega^{-1}G(U) = G\omega^{-1}(\underbrace{g(U)}_{\in T_z^*Y}) = G\begin{pmatrix} 0 & B \\ -B^{\mathrm{T}} & C \end{pmatrix} \cdot \begin{pmatrix} 0 \\ gU \end{pmatrix} = G\underbrace{\begin{pmatrix} BgU \\ CgU \end{pmatrix}}_{\in T_zX}\,. \tag{5.3.35}$$

If we decompose $G = g_\perp \oplus g$ on $T_zX = N_zY \oplus T_zY$, it follows that

$$G\omega^{-1}G(U) = g_\perp Bg(U) \oplus gCg(U) \in N_z^*Y \oplus T_z^*Y \tag{5.3.36}$$

Restricting equation (5.3.36) to $T_zY$ yields $G\omega^{-1}G|_{T_zY} = 0 \oplus gCg$. Moreover, it holds $G\omega^{-1}G = -\omega$ and, hence, $gCg = 0 \oplus (-\sigma)$ acting on $\mathscr{L}_zY \oplus \mathscr{F}_zY$.

Let us assume $gCg = FCF$. Then it follows that $F\omega^{-1}F = G\omega^{-1}G = -\omega$ restricted to $\mathscr{F}_zY$. This allows us to infer

$$J^2 = \left(\omega^{-1}F\right)^2\Big|_{\mathscr{F}_zY} = \left(\sigma^{-1}f\right)^2 = -\,\mathrm{id}_{\mathscr{F}_zY}\,. \tag{5.3.37}$$

Now suppose $\sigma^{-1}f$ is an almost complex structure on $\mathscr{F}_zY$. Then it follows that $f\sigma^{-1}f = -\sigma$. Thus, $FCF = F\omega^{-1}F|_{\mathscr{F}_zY} = -\omega|_{\mathscr{F}_zY} = 0 \oplus (-\sigma)$. Since $G^{-1}\omega$ is a complex structure, it is also true that $gCg = G\omega^{-1}G|_{\mathscr{F}_zY} = -\omega|_{\mathscr{F}_zY} = 0 \oplus (-\sigma)$. Therefore, $gCg = FCF$. □

This completes the examination of the geometrical implications of equations (5.3.13), (5.3.14) and (5.3.15) on a submanifold $Y$ of $X$ and a line bundle $\pi_L : L \to Y$ with connection $\nabla$. These implications are given by propositions 5.3.2, 5.3.3 and 5.3.6. We showed that they are in fact equivalent to $(Y, L, \nabla)$ defining a classical rank one A-brane in the $\mathcal{N} = (2,2)$ supersymmetric topologically twisted non-linear sigma-model. This guides us to

**Definition 5.3.7.** *Let $(X, \omega, G)$ be a Kähler manifold, $Y \hookrightarrow X$ be a submanifold of $X$ and $\pi_L : L \to Y$ a unitary line bundle with a unitary connection $\nabla$ and curvature $F$. Then we call $(Y, \pi_L : L \to Y, \nabla)$ a classical rank one A-brane if and only if*

1. *$Y$ is a coisotropic submanifold of $X$,*
2. *$F$ descends to a section in $\mathscr{F}Y$, i.e. if we regard $F$ as a bundle morphism $TY \to T^*Y$ it holds $F = 0 \oplus f$ on $TY = \mathscr{L}Y \oplus \mathscr{F}Y$ and*
3. *for $\omega : TY \to T^*Y$ with $\omega = 0 \oplus \sigma$ on $TY = \mathscr{L}Y \oplus \mathscr{F}Y$, $f$ as above and $J := \sigma^{-1} f : \mathscr{F}Y \to \mathscr{F}Y$ it is true that $J^2 = -\operatorname{id}_{\mathscr{F}Y}$.*

In particular, the support of a rank-one A-brane is a coisotropic submanifold $Y \hookrightarrow X$, which we defined via the skew-complement of $TY$ in $TX$. Another definition will be useful to us, too. A submanifold is coisotropic if and only if $\omega|_Y$ has a constant rank and its kernel $\mathscr{L}Y \subset TY$ is an integrable distribution, which is equivalent to $[U, V] \in \Gamma(Y, \mathscr{L}Y)$ for all $U, V \in \Gamma(Y, \mathscr{L}Y)$. By the Frobenius theorem $\mathscr{L}Y$ arises from a regular foliation $\Phi$ of $Y$, i.e. the vector fields tangent to the leaves of the foliation are given by $\Gamma(Y, \mathscr{L}Y)$. The quotient bundle $\mathscr{F}Y$ is called the normal bundle of the foliation. We stop here our exposition of classical coisotropic A-branes. In generalized complex geometry the role of these coisotropic A-branes will be played by so called generalized Lagrangian submanifolds. If we restrict $M$ to be a symplectic manifold, a generalized Lagrangian submanifold is precisely given by a classical rank one coisotropic A-brane.

# Part III.

# The Mathematics of Topological Sigma Models On Twisted Generalized Complex Manifolds

# 6. First Look at Generalized Pseudoholomorphic Curves

This part of the work considers a definition and the properties of $\mathcal{J}$-holomorphic, or generalized pseudoholomorphic, curves. Our definition differs from a definition of generalized holomorphic maps in the literature [OP11]. This is due to the fact that we ultimately aim towards a possible generalization of Quantum Cohomology and Fukaya Category to generalized complex (GC) geometry. Such an extension would surely enable one to shed light on the mathematics of mirror symmetry and to find new or relations between known invariants of symplectic and complex geometry.

To motivate our definition of generalized pseudoholomorphic curves, let us recall the importance of pseudoholomorphic curves in topological string theory. They appear for example as instantons in the calculation of the anti-chiral ring of a manifold. The latter gives the quantum structure of states in the topological A-model under consideration. A mathematical formulation of the anti-chiral ring is given by Quantum Cohomology. Hence, a mathematical treatment of the generalized chiral ring, i.e. the quantum structure of states in the generalized B-model, should lead towards a generalization of Quantum Cohomology. This generalization should be a deformation of the Lie algebroid cohomology associated with an integrable generalized complex structure $\mathcal{J}_1$. Fukaya categories are the mathematical formulation of the quantum structure of branes in the topological A-model. Hence, a mathematical formulation of generalized B-branes will lead to a generalization of Fukaya categories to generalized complex manifolds. This motivates why we should take the results of string theorists serious and develop a mathematical treatment of instantons in the generalized B-model.

We will begin the examination of generalized pseudoholomorphic curves by reviewing very briefly some facts about $J$-holomorphic curves in ordinary complex geometry. After that we will look at possible natural generalizations of this notion and the problems in finding them. Then we will probe the definition given in [OP11] and explain why it does not suite our purpose. Thereafter we will turn to a definition of $\mathcal{J}$-holomorphic curves which suites our purpose. Furthermore, motivated by the almost complex case, we will find the almost generalized complex counterpart of tamed and compatible structures, simple curves and somewhere injective curves. Eventually we will examine the local structure of generalized pseudoholomorphic curves. In particular, we will show, using a theorem of Aronszajn, that they satisfy an identity theorem analogous to holomorphic curves and give a criterion for a generalized pseudoholomorphic curve to be simple. The main theorem of the local theory is theorem 7.4.4. It states that locally a $\mathcal{J}$-holomorphic curve can be interpreted as an ordinary pseudoholomorphic curve taking values in a space of doubled dimension while one half of the coordinates are constant. Then we will turn to the global behavior of $\mathcal{J}$-holomorphic curves. We will find the generalized complex counterpart of the vertical differential. In contrast to usual symplectic topology it is not a Fredholm operator, but an upper semi-Fredholm operator. This complicates the deformation theory of generalized pseudoholomorphic curves. In order to solve this problem we will invent admissible vector fields along a map $\Phi$. If we only incorporate infinitesimal deformations given by admissible vector fields, the excess degrees of freedom get removed. At the end we will look at some examples and show that our notion reproduces the known cases.

In the following $\Sigma$ is assumed to be a compact Riemann surface.

## 6.1. $J$-Holomorphic Curves in Complex Manifolds

We recall the well known notion of $J$-holomorphic curves. Let $M$ be a 2n-dimensional real manifold and $(\Sigma, j)$ a Riemann surface. Furthermore, let $J : TM \to TM$ be an almost complex structure on $M$, i.e. an automorphism of $TM$ which squares to $-\mathbb{1}$. A map $\Phi : \Sigma \to M$ is called $J$-holomorphic, or pseudoholomorphic, if $J \circ T\Phi = T\Phi \circ j$. Starting from this definition one can construct the moduli space of $J$-holomorphic curves, [MS04] and references therein, using the differential operator $\overline{\partial}_J(\Phi) = d\phi + J \circ d\Phi \circ j = 0$. Here we will not go into the details as we will generalize these considerations later to almost GC manifolds. Another equivalent criterion for a map $\Phi : \Sigma \to M$ to be $J$-holomorphic is

**Proposition 6.1.1.** *Let $T\Sigma^{(1,0)}$ and $TM^{(1,0)}$ be the $+\mathrm{i}$-eigenbundle of $j$ and $J$ in $T\Sigma_{\mathbb{C}}$ and $TM_{\mathbb{C}}$. Furthermore, let $T\Phi_{\mathbb{C}}$, $J_{\mathbb{C}}$ and $j_{\mathbb{C}}$ be the complexifications of $T\Phi$, $J$ and $j$, respectively. Then it is equivalent*

1. $J_{\mathbb{C}} \circ T\Phi_{\mathbb{C}} = T\Phi_{\mathbb{C}} \circ j_{\mathbb{C}}$ (6.1.1)
2. $T\Phi_{\mathbb{C}}(T\Sigma^{(1,0)}) \subseteq TM^{(1,0)}$ . (6.1.2)

*Proof.* The proof of this proposition is obvious and left to the reader. □

## 6.2. Holomorphic Curves in Generalized Complex Manifolds

It is a difficult task to find "the" appropriate generalization of pseudoholomorphic curves. In this subsection we discuss various possible definitions of generalized pseudoholomorphic curves, in particular one which can be found in the literature.

The common definition of $J$-holomorphic curves $\Phi : \Sigma \to M$ uses the existence of a naturally induced map $T\Phi$ between $T\Sigma$ and $TM$. Unluckily, there is no such naturally induced map between the associated Pontrjagin bundles $\mathbb{T}\Sigma := T\Sigma \oplus T^*\Sigma$ and $\mathbb{T}M := TM \oplus T^*M$. This relies on the fact that the construction of a push-forward of an element in $T^*\Sigma$ to $T^*M$ cannot be done without additional data.

Given metrics $h$ on $\Sigma$ and $g$ on $M$ one can exploit the isomorphism $h^{-1} : T^*\Sigma \to T\Sigma$ to produce a vector-field which can be naturally mapped to $TM$. Afterwards $g$ can be used to map the resulting section in $TM$ to a section in $T^*M$. Altogether, we get $g \circ T\Phi \circ h^{-1} : T^*\Sigma \to T^*M$. If we define $\mathbb{T}\Phi_{h,g} := T\Phi \oplus g \circ T\Phi \circ h^{-1}$, we get a map $\mathbb{T}\Phi_{h,g} : \mathbb{T}\Sigma \to \mathbb{T}M$. Clearly the property of a map being holomorphic should only depend on the map, the manifolds and the respcetive structures. Here we have some additional structures, namely the metrics. At least the special case of a map from a Riemann surface to an almost complex or almost symplectic manifold reproduces the holomorphic maps as well as the constant maps, the instantons of the A- and B- model of topological strings. In particular, the equations are independent of $g$ and $h$. Whether this holds in general is not part of our interest in this work, so we will not develop this further.

Another possible generalization of $J$-holomorphic maps to GC geometry uses proposition 6.1.1. In order to be able to do this, we need a notion of pushforward in GC geometry. It is given by the following consideration. Let $(M_1, \mathcal{J}_1)$ and $(M_2, \mathcal{J}_2)$ be almost GC manifolds and $\Phi : M_1 \to M_2$ be $\mathcal{C}^1$. An almost generalized complex structure on $M$ can either be given by an automorphism $\mathcal{J} : \mathbb{T}M \to \mathbb{T}M$ fulfilling

$$\mathcal{J}^2 = -1 \quad \text{and} \tag{6.2.1}$$

$$\forall A, B \in \mathbb{T}M \, : \, q(\mathcal{J}A, \mathcal{J}B) = q(A, B)\,, \tag{6.2.2}$$

where $q(X \oplus \xi, Y \oplus \eta) := \eta(X) + \xi(Y)$. Or it can be given by an almost Dirac-structure $L_M$ with real index zero. That is a subbundle $L_M \subseteq \mathbb{T}M_{\mathbb{C}}$ which is maximally isotropic w.r.t. $q$ and obeys $L_M \cap \overline{L_M} = \{0\}$. The bundle $L_M$ is given by the $+$i-eigenbundle of $\mathcal{J}_{\mathbb{C}}$ in $\mathbb{T}M_{\mathbb{C}}$ [Gua03]. Recall that in complex geometry $TM^{(1,0)}$ is also given by the $+$i-eigenbundle of the complexified almost complex structure. Proposition 6.1.1 then states that the pushforward of the almost complex structure $T\Sigma^{(1,0)}$ is a subbundle of the almost complex structure $TM^{(1,0)}$. Hence, for almost GC manifolds we have to look at the pushforward of almost Dirac structures. It is given by

$$(T\Phi)_* L_{M_1} := \{T\Phi(u) \oplus \xi \in TM_2 \oplus T^*M_2 \, : \, u \oplus (T\Phi)^*\xi \in L_{M_1}\} \; . \tag{6.2.3}$$

Then it seems natural to call a map $\Phi : M_1 \to M_2$ generalized holomorphic if $(T\Phi)_* L_{M_1} \subseteq L_{M_2}$. Both sides of this relation are almost Dirac structures and have, due to their maximal isotropy, in particular maximal dimension. Hence, $M_1$ and $M_2$ have to have the same dimension for this definition to be non trivial. Alternatively, one can use the pullback of almost Dirac structures,

$$(T\Phi)^* L_{M_2} := \{u \oplus (T\Phi)_*\xi \in TM_1 \oplus T^*M_1 \, : \, T\Phi u \oplus \xi \in L_{M_2}\} \; , \tag{6.2.4}$$

and call $\Phi$ generalized holomorphic if $(T\Phi)^* L_{M_2} = L_{M_1}$. If $T\Phi$ is injective, i.e. $\Phi$ is an immersion, this definition recovers the usual holomorphic maps for both $M_1$ and $M_2$ being almost complex and the iso-symplectic maps for both $M_1$ and $M_2$ being almost symplectic. If $T\Phi$ is not injective, it seems that although all maps in the complex/symplectic case are holomorphic/iso-symplectic, not all holomorphic/iso-symplectic maps are generalized holomorphic. This is very puzzeling and suggests that this kind of definition is very likely not the right one. If $M_1$ is almost complex and $M_2$ is almost symplectic, this definition renders $j$ to be compatible with the pullback of the almost symplectic structure[1] $\omega$. As in two dimensions all almost complex structures are compatible with a surface-form[2] $\Phi^*\omega$, all maps $\Phi : M_1 \to M_2$ are generalized holomorphic in this manner. Remember that we wish to apply our results to mirror symmetry and related topics. So we have to recover constant maps as holomorphic maps from a Riemann surface to an symplectic manifold, which is obviously not the case here.

The next possible definition which we want to examine is the one of [OP11]. It is given by

**Definition 6.2.1.** *Let $(\Sigma, L(D_\Sigma, \epsilon_\Sigma))$ and $(M, L_M(D_M, \epsilon_M))$ be regular almost generalized complex manifolds and $\Phi : \Sigma \to M$ be $\mathcal{C}^1$. Furthermore, let*

$$P_{\Sigma/M} := L\left(D_{\Sigma/M} \cap \overline{D_{\Sigma/M}}, \mathrm{Im}\left(\epsilon_{\Sigma/M}\big|_{D_{\Sigma/M} \cap \overline{D_{\Sigma/M}}}\right)\right) \; .$$

*Then we call $\Phi$ generalized holomorphic, iff*

1. *$T\Phi(D_\Sigma) \subseteq D_M$ and*

[1] This is a non-degenerate 2-form, not necessarily closed.

[2] I thank K. Cieliebak for clarifying this to me.

2. $(T\Phi)_*(P_\Sigma) = P_M$.

**Remarks 6.2.2** 1. Recall that an almost generalized complex structure can either be given by an automorphism $\mathcal{J}$ or an almost Dirac structure $L$ with real index zero. Any such Dirac structure is in particular a maximal isotropic subbundle of $TM \oplus T^*M$. If it is regular, i.e. $\pi(L)$ has constant rank, it is possible to write it as

$$L(D,\epsilon) := \{X \oplus \xi \in D \oplus T^*M \,:\, \xi|_D = i_X\epsilon\} \ , \tag{6.2.5}$$

where $D$ is a subbundle of $TM$ and $\epsilon \in \bigwedge^2 E^*$.

2. A map $f$ between vector spaces with Poisson structures $(V_1, \pi_1)$ and $(V_2, \pi_2)$ is called a Poisson morphism, if it maps Poisson bi-vectors to Poisson bi-vectors. This is equivalent to $f_*(\mathrm{graph}(\pi_1)) = \mathrm{graph}(\pi_2)$, [BR02].

3. Ornea and Pantilie call $D$ and $L(D\cap\overline{D}, \mathrm{Im}(\epsilon|_{D\cap\overline{D}}))$ the associated linear co-CR and Poisson structures, respectively. Thus the property of a map being generalized holomorphic can also be phrased as the differential being a co-CR linear Poisson morphism between the associated structures, as done in [OP11].

This definition has some really nice properties, as for example a generalized holomorphic map is the product of iso-symplectic and holomorphic maps, up to a $B$-transform. If $\Phi : \Sigma \to M$ is a diffeomorphism, definition 6.2.1 is equivalent to $(T\Phi)_* L_\Sigma = L_M$ as suggested and refused above. A natural question is to ask, whether the definition of Ornea and Pantilie is equivalent or a generalization of the instantons found by Kapustin and Li. This has to be answered in the negative, as we will explain now.

If the class of all generalized holomorphic maps contains the class of instantons, every instanton has to be a generalized holomorphic map. As an example consider the case of $(\Sigma, j)$ being a Riemann surface and $(M, \Omega)$ being a symplectic manifold. Then $D_\Sigma = T\Sigma^{(1,0)}$ and, obviously $D_\Sigma \cap \overline{D_\Sigma} = \{0\}$. Thus

$$P_\Sigma = L(0,0) = \{u \oplus \xi \in 0 \oplus T^*\Sigma_{\mathbb{C}} | \xi(0) = 0\} = 0 \oplus T^*\Sigma_{\mathbb{C}}\,. \tag{6.2.6}$$

On the other hand, $L_M = \mathrm{graph}(-\mathrm{i}\Omega) = \{X \oplus (-\mathrm{i}\Omega(X)) \,:\, X \in TM_{\mathbb{C}}\}$ and, hence, $D_M = TM_{\mathbb{C}}$. Therefore, $D_M \cap \overline{D_M} = TM_{\mathbb{C}}$, as $\overline{TM_{\mathbb{C}}} = TM_{\mathbb{C}}$ and $P_M = L(TM_{\mathbb{C}}, \mathrm{Im}(\epsilon_M))$. It is obvious that $T\Phi(D_\Sigma) \subseteq D_M$ and $T\Phi$ is co-CR linear. But there is a problem which is connected to the associated Poisson structures. Namely, it is true that

$$(T\Phi)_* P_\Sigma = \{(T\Phi\, 0) \oplus \xi \,|\, 0 \oplus (T\Phi)^*\xi \in P_\Sigma = 0 \oplus T^*\Sigma_{\mathbb{C}}\} = 0 \oplus T^*M_{\mathbb{C}} \neq P_M \tag{6.2.7}$$

for any $\Phi$. Hence, we conclude that there are no generalized holomorphic maps between a Riemann surface and a symplectic manifold. In particular, constant maps, which are the instantons in this case, are not generalized holomorphic. Therefore, definition 6.2.1 does not contain the instantons of the generalized B-model of topological sigma-models. That means generalized holomorphic maps do not suite our purpose. But as they seem to be a sensible notion of generalized holomorphic maps, we call the instantons of chapter 5 "generalized pseudoholomorphic" or $\mathcal{J}$-holomorphic.

## 6.3. Generalized Pseudoholomorphic Curves, Pairs, Tame and Compatible Almost Generalized Complex Structures

At the end of the last section and chapter 5 we saw that in order to guess a sensible generalization of the concept of symplectic topology to generalized complex geometry, we need to look at the generalized topological twisted nonlinear sigma-model. Instantons were found to fulfill (5.1.65). They will play a role similar to $J$-holomorphic curves in symplectic topology. Strictly speaking, this equation reproduces the instantons in every physically sensible case of the generalized B-model. We demand that this equation holds for curves in arbitrary generalized complex manifolds. For usual almost complex manifolds such an approach leads to the right concepts.

We saw that the instanton equation is not invariant under $B$-transformations, viewed as an object in generalized complex geometry. By that we mean that there is no operation acting only on $\Phi$ and dictated by the $B$-transformation such that the transformed class of generalized pseudoholomorphic curves with respect to $\mathcal{J}$ is the class of all generalized pseudoholomorphic curves with respect to the transformed structure $\mathrm{e}^{B}\mathcal{J}\mathrm{e}^{-B}$. As we wish to be able to generalize the objects of interest to arbitrary exact Courant algebroids, e.g. twisted generalized complex geometry, they should be manifestly invariant under $B$-transformations. In order to restore invariance, we introduce a second object, namely an isotropic embedding $\lambda : TM \to E$, where $E$ is an exact Courant algebroid. The case which is most relevant to the generalized B-model is $\lambda$ being the canonical inclusion into the standard Courant algebroid $\mathbb{T}M$.

Our next aim is to establish a precise definition of the objects of interest. To this end we will define $\mathcal{J}$-holomorphic, or generalized pseudoholomorphic, curves and the corresponding notions of tamed and compatible structures. Moreover, we will show that they reproduce the special cases of symplectic and complex manifolds. Let us start with

**Definition 6.3.1.** *Let $(\Sigma, j_\Sigma)$ be a Riemann surface, $M$ be a smooth manifold, $(E, q, [\cdot,\cdot], \pi)$*

*be an exact Courant algebroid over $M$ and $\mathcal{J}$ an almost generalized complex structure on $E$. Moreover, let $\Phi : \Sigma \to M$ be a map and $\lambda : TM \to E$ be an isotropic embedding (with respect to $q$). Then we call $(\Phi, \lambda)$ a generalized pseudoholomorphic pair with respect to $(E, \mathcal{J})$ or $(E, \mathcal{J})$-holomorphic pair, iff*

$$\mathcal{J} \circ (\lambda \circ T\Phi) = (\lambda \circ T\Phi) \circ j_\Sigma \, . \tag{6.3.1}$$

*If $\lambda$ is a smooth isotropic splitting $s$ of $E$, we simply call $\Phi$ a generalized pseudoholomorphic curve with respect to $E$ or an $(E, \mathcal{J})$-holomorphic curve. If $E = \mathbb{T}M$, we call $\Phi$ just a $\mathcal{J}$-holomorphic curve.*

**Remarks 6.3.2**

1. Recall that a map $\lambda : TM \to E$ is called isotropic (with respect to $q$) if and only $q(\lambda(X), \lambda(Y)) = 0$ for all $X, Y \in TM$.
2. It is easy to prove that equation (6.3.1) is equivalent to $\lambda(T\Phi(T\Sigma^{(1,0)})) \subseteq L$, where $T\Sigma^{(1,0)}$ are holomorphic tangent vectors with respect to $j_\Sigma$ and $L$ is the $+\,$i-eigenbundle of $\mathcal{J}$.

The next two examples show that definition 6.3.1 covers the well known notion of $J$-holomorphic curves and shows that their symplectic pendant are constant maps.

**Examples 6.3.3**

1. If $I$ is an almost complex structure on a smooth manifold $M$, $\mathcal{J}_I$ can be found in (6.3.13). After choosing $\lambda = \iota$, Equation (6.3.1) is equivalent to

$$\begin{pmatrix} I & 0 \\ 0 & -I^* \end{pmatrix} \begin{pmatrix} d\Phi \\ 0 \end{pmatrix} = \begin{pmatrix} I \circ T\Phi \\ 0 \end{pmatrix} = \begin{pmatrix} T\Phi \circ j_\Sigma \\ 0 \end{pmatrix} \, . \tag{6.3.2}$$

   This is the well known equation for $J$-holomorphic curves.

2. If $\omega$ is an almost symplectic structure on a smooth manifold $M$, $\mathcal{J}_\omega$ can also be found in (6.3.13). After choosing $\lambda = \iota$, Equation (6.3.1) is equivalent to

$$\begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \begin{pmatrix} T\Phi \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \omega \circ T\Phi \end{pmatrix} = \begin{pmatrix} T\Phi \circ j_\Sigma \\ 0 \end{pmatrix} \, . \tag{6.3.3}$$

   Since $\omega$ is non degenerate, we infer that $T\Phi = 0$ and $\Phi$ has to be a constant map.

3. As a direct consequence we realize that if $J$ is the product of an almost complex and an almost symplectic structure, $\Phi$ is pseudoholomorphic in the complex directions and constant in symplectic directions.

The next proposition shows that equation (6.3.1) is invariant under orthogonal automorphisms of $E$.

**Proposition 6.3.4.** *Let $(\Sigma, j_\Sigma)$ be a Riemann surface, $M$ be a smooth manifold, $(E, q, [\cdot,\cdot], \pi)$ an exact Courant algebroid and $\mathcal{J}$ an almost generalized complex structure on $E$. Moreover, let $\Phi : \Sigma \to M$ be a map, $\lambda : TM \to E$ be an isotropic embedding and $\Lambda$ be an orthogonal automorphism with respect to $q$ inducing a diffeomorphism $\chi : M \to M$. Then it is true that that equation 6.3.1 is invariant under $(\Lambda, \chi)$.*

*Proof.* Let $\Lambda$ be an orthogonal automorphism of $E$ inducing a diffeomorphism $\chi : M \to M$, i.e.

$$\begin{tikzcd} E \arrow[r, "\Lambda"] \arrow[d, "\pi_E"'] & E \arrow[d, "\pi_E"] \\ M \arrow[r, "\chi"] & M \end{tikzcd} \tag{6.3.4}$$

commutes. Under this automorphism, $\mathcal{J}$ gets mapped to $\Lambda\mathcal{J}\Lambda^{-1}$, $T\Phi$ to $T\chi \circ T\Phi$ and $\lambda$ to $\Lambda \circ \lambda \circ T\chi^{-1}$. Now let $(\Phi, \lambda)$ be an $(E, \mathcal{J})$-holomorphic pair. Performing $\Lambda$ on equation (6.3.1) yields

$$\Lambda \circ \mathcal{J} \circ \Lambda^{-1} \circ \Lambda \circ \lambda \circ T\chi^{-1} \circ T\chi \circ T\Phi = \Lambda \circ \lambda \circ T\chi^{-1} \circ T\chi \circ T\Phi \circ j_\Sigma \,. \tag{6.3.5}$$

All contributions of $\Lambda$ and $\chi$ cancel each other and we see that equation (6.3.1) is invariant under $(\Lambda, \chi)$. ☐

**Remark 6.3.5** In particular this shows together with theorem 6.4.1 that (6.3.1) is independent of the choice of an isotropic splitting. For $E = TM \oplus T^*M$ this is equivalent to the invariance of equation (6.3.1) under $B$-transformations.

It transpires that it is possible to reduce $\mathcal{J}$-holomorphic pairs to $\mathcal{J}$-holomorphic curves if one allows the almost generalized complex structure to vary. This will be our concern in the next section.

The main part of this work is the examination of the moduli space[3] of (simple) solutions to equation (6.3.1). In order to do this, especially to prepare for a theorem analogous to Gromov compactness, we need

**Definition 6.3.6.** *Let $M$ be a smooth manifold, $(E, q, [\cdot,\cdot], \pi)$ be an exact Courant algebroid over $M$ and $\mathcal{J}_1$, $\mathcal{J}_2$ be two almost generalized complex structures on $E$. Then we call $\mathcal{J}_2$ tamed by $\mathcal{J}_1$, or simply $\mathcal{J}_1$-tame, iff*

$$\forall\, 0 \neq A \in E \,:\, -q(\mathcal{J}_1\mathcal{J}_2 A, A) > 0 \,. \tag{6.3.6}$$

[3]For now we think of a moduli space as a naturally parameterized set, e.g. a subset of the Fréchet manifold of all maps from $\Sigma$ to $M$.

*Furthermore, we call $\mathcal{J}_2$ compatible with $\mathcal{J}_1$, iff $\mathcal{J}_2$ is $\mathcal{J}_1$-tame and $[\mathcal{J}_1, \mathcal{J}_2] = 0$, i.e. the two structures commute.*

**Remarks 6.3.7**

1. As $q$ has signature $(n, n)$, $n$ being the dimension of $M$, $\mathcal{J}$ is never tamed by itself and thus never compatible to itself. But, clearly, it commutes with itself.
2. After observing
$$\begin{aligned} &\forall A \in E \,:\, -q(\mathcal{J}_1\mathcal{J}_2 A, A) > 0 \Leftrightarrow \forall A \in E \,:\, q(\mathcal{J}_1 A, \mathcal{J}_2 A) > 0 \Leftrightarrow \\ &\forall A \in E \,:\, -q(A, \mathcal{J}_2\mathcal{J}_1 A) = -q(\mathcal{J}_2\mathcal{J}_1 A, A) > 0 \,, \end{aligned} \tag{6.3.7}$$
we deduce that $\mathcal{J}_1$ is tamed by $\mathcal{J}_2$ if and only if $\mathcal{J}_2$ is tamed by $\mathcal{J}_1$.
3. Moreover, $\mathcal{J}_1$ is compatible with $\mathcal{J}_2$ if and only if $\mathcal{J}_2$ is compatible with $\mathcal{J}_1$. Hence, we simply call $\mathcal{J}_1$ and $\mathcal{J}_2$ compatible.
4. If $\mathcal{J}_2$ is $\mathcal{J}_1$-tame, we get a metric on $E$,
$$G(A, B)_{\mathcal{J}_1, \mathcal{J}_2} := -\frac{1}{2} q\left(\{\mathcal{J}_1, \mathcal{J}_2\} A, B\right) \,, \tag{6.3.8}$$
where $\{\cdot, \cdot\}$ denotes the anti-commutator. To reveal the symmetry of equation (6.3.8) we observe
$$G(A, B)_{\mathcal{J}_1, \mathcal{J}_2} = \frac{1}{2} q\left(\mathcal{J}_1 A, \mathcal{J}_2 B\right) + \frac{1}{2} q\left(\mathcal{J}_2 A, \mathcal{J}_1 B\right) \,. \tag{6.3.9}$$
Positive definiteness originates in the tameness condition. Hence, $G_{\mathcal{J}_1, \mathcal{J}_2}$ defines a metric on $E$, indeed. It is obviously true that
$$G(\Lambda A, \Lambda B)_{\Lambda\mathcal{J}_1\Lambda^{-1}, \Lambda\mathcal{J}_2\Lambda^{-1}} = G(A, B)_{\Lambda^{-1}\Lambda\mathcal{J}_1\Lambda^{-1}\Lambda, \Lambda^{-1}\Lambda\mathcal{J}_2\Lambda^{-1}\Lambda} = G(A, B)_{\mathcal{J}_1, \mathcal{J}_2} \,, \tag{6.3.10}$$
where $\Lambda : E \to E$ is an automorphism of $E$ which is orthogonal with respect to $q$ (cf. the next section). Therefore, the metric is invariant under orthogonal automorphisms. In particular, it is invariant under $B$-transformations.
5. Moreover, the above notions of tamed and compatible almost generalized complex structures are invariant under orthogonal automorphisms with respect to $q$. This can be deduced from
$$-q(\Lambda\mathcal{J}_1\Lambda^{-1}\Lambda\mathcal{J}_2\Lambda^{-1}\Lambda A, \Lambda A) = -q(\mathcal{J}_1\mathcal{J}_2 A, A) \quad \text{and} \tag{6.3.11}$$
$$[\Lambda\mathcal{J}_1\Lambda^{-1}, \Lambda\mathcal{J}_2\Lambda^{-1}] = \Lambda[\mathcal{J}_1, \mathcal{J}_2]\Lambda^{-1} \,. \tag{6.3.12}$$
Using the fact that $\Lambda$ is an automorphism, this shows that $\mathcal{J}_1$ and $\mathcal{J}_2$ are tame (compatible) if and only if the transformed structures $\Lambda\mathcal{J}_1\Lambda^{-1}$ and $\Lambda\mathcal{J}_2\Lambda^{-1}$ are tame (compatible).

6. A generalized Kähler structure can be rephrased as a pair of integrable, tamed and compatible almost generalized complex structures.
7. We will denote the space of $\mathcal{J}_1$ compatible almost generalized complex structures on $M$ by $\mathfrak{J}(M,\mathcal{J}_1)$ and the space of $\mathcal{J}_1$-tame almost generalized complex structures by $\mathfrak{J}_\tau(M,\mathcal{J}_1)$.

**Examples 6.3.8** Let us give some simple examples of tame and compatible generalized complex structures.

1. The first example is the obvious one. We want to show that the above definitions reproduce the well known notion of tameness and compatibility in symplectic geometry. To this end let $(E,q,[\cdot,\cdot],\pi)$ be the standard Courant algebroid $(\mathbb{T}M,q_0,[\cdot,\cdot]_0,\mathrm{pr}_1)$, $\mathcal{J}_I$ and $\mathcal{J}_\omega$ be the associated generalized complex structures of an almost complex structure $I$ which is tamed by a symplectic structure $\omega$. Recall that the generalized complex structures can be represented as

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{J}_I = \begin{pmatrix} I & 0 \\ 0 & -I^* \end{pmatrix} . \tag{6.3.13}$$

Since $I$ is tamed by $\omega$, it follows that $\omega(X,IX) > 0\,\forall\, 0 \neq X \in TM$. Then it is also true that

$$-\omega^{-1}(\eta,I^*\eta) = -\eta(\omega^{-1}(I^*\eta)) = -\eta(X) = \omega(X,IX) > 0\,\forall\, 0 \neq \eta \in T^*M\,, \tag{6.3.14}$$

where $X = \omega^{-1}(I^*\eta)$ is defined by $I^*\eta(Y) = \omega(X,Y)\,\forall\, Y \in TM$ and we used

$$I^*\eta(Y) = \omega(X,Y) = \eta(IY)\,\forall\, Y \in TM \Leftrightarrow \eta(Y) = -\omega(X,IY)\,\forall\, Y \in TM\,. \tag{6.3.15}$$

Therefore, we immediately infer that

$$\begin{aligned} -q\left(\mathcal{J}_\omega\mathcal{J}_I(X\oplus\eta),X\oplus\eta\right) &= -q(\omega^{-1}I^*\eta\oplus\omega IX, X\oplus\eta) = \\ &= \frac{1}{2}\left(\omega(X,IX) - \omega^{-1}(\eta,I^*\eta)\right) > 0\,\forall\, X\oplus\eta \in \mathbb{T}M\,. \end{aligned} \tag{6.3.16}$$

Thus $\mathcal{J}_I$ is tamed by $\mathcal{J}_\omega$ in the sense of definition 6.3.6.

Let $\mathcal{J}_\omega$ and $\mathcal{J}_I$ be the generalized complex structures associated with a symplectic structure and an almost complex structure, respectively. If $\mathcal{J}_I$ is tamed by $\mathcal{J}_\omega$, the above calculation shows by restriction to $TM \cong \iota(TM) = TM\oplus 0$ that $I$ is tamed by $\omega$.

Now let $\mathcal{J}_\omega$ and $\mathcal{J}_I$ tame each other. Observing

$$\mathcal{J}_\omega\mathcal{J}_I = \begin{pmatrix} 0 & \omega^{-1}I^* \\ \omega I & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{J}_I\mathcal{J}_\omega = \begin{pmatrix} 0 & -I\omega^{-1} \\ -I^*\omega & 0 \end{pmatrix} , \tag{6.3.17}$$

it is easy to check that $[\mathcal{J}_\omega, \mathcal{J}_I] = 0$ is equivalent to $\omega$ and $I$ being compatible. Hence, we see that definition 6.3.6 is equivalent to tamed and compatible structures in the usual sense for the special case of an almost complex structure and a symplectic structure.

2. Next we want to answer the question whether only symplectic structures tame complex structures in the generalized sense. This is false in general and there are more structures in the setting of generalized complex geometry. In order to justify this, let $\mathcal{J}_{I_1}$ be the generalized complex structure associated with an almost complex structure $I_1$ on a $2n$-dimensional real smooth manifold $M$. Furthermore, let us represent another generalized complex structure $\mathcal{J}_2$ as

$$\mathcal{J}_2 = \begin{pmatrix} I_2 & \beta_2 \\ B_2 & -I_2^* \end{pmatrix} . \tag{6.3.18}$$

A straight forward calculation shows that

$$-q(\mathcal{J}_{I_1}\mathcal{J}_2(x \oplus \eta), x \oplus \eta) = \frac{1}{2}\left(\eta(\{I_1, I_2\}X) + B_2(X, I_1X) - \beta_2(I_1^*\eta, \eta)\right) , \tag{6.3.19}$$

where we set $\beta_2(\eta, \xi) := \xi(\beta_2(\eta))$. Since this has to be true for all $X \oplus \eta \in \mathbb{T}M$, it holds in particular for $X \oplus 0$ and $0 \oplus \eta$. This implies

$$\forall\, 0 \neq X \in TM \ : \ B_2(X, I_1X) > 0 \quad \text{and} \tag{6.3.20}$$
$$\forall\, 0^* \neq \eta \in T^*M \ : \ -\beta_2(\eta, I_1^*\eta) > 0 \,. \tag{6.3.21}$$

These equations look very similar to the case of $B_2$ being a symplectic form which tames $I_1$ and $-\beta_2$ being its associated Poisson bi-vector. Because of the presence of $I_2$ there is no need for $B_2$ being non-degenerate or closed, even if $\mathcal{J}_2$ is integrable.

Hence, we observe that there are more generalized complex structures taming $I_1$ than just symplectic structures. This might be very important if one looks e.g. at nilmanifolds. There are examples which do not admit any symplectic structure or integrable almost complex structure [Sal98]. In [Gua03] it has been shown that they admit generalized complex structures.

## 6.4. Reduction of $\mathcal{J}$-Holomorphic Pairs to $\mathcal{J}$-Holomorphic Curves

In this section we will show that it is possible to reduce $(E, \mathcal{J})$-holomorphic pairs to $\mathcal{J}$-holomorphic curves. First, we will prove theorem 6.4.1. It states that any isotropic embedding $\lambda$ of $TM$ into some exact Courant algebroid $E$ can be written as the composition of a splitting map $s$ and an orthogonal automorphism $\Lambda$ of $E$ with respect to $q$. Proposition 6.4.3 then shows that $(\Phi, \lambda)$ is an $(E, \mathcal{J})$-holomorphic pair if and only if $\Phi$ is a $(E, \mathcal{J}')$-holomorphic curve, where $\mathcal{J}' = \Lambda \mathcal{J} \Lambda^{-1}$ is the transformed almost generalized complex structure. Finally, proposition 6.4.4 states that $\Phi$ is a $(E, \mathcal{J}')$-holomorphic curve if and only if $\Phi$ is a $\mathcal{J}''$-holomorphic curve, where $\mathcal{J}''$ is the induced almost generalized complex structure on $\mathbb{T}M$.

More concretely, let us state

**Theorem 6.4.1.** *Let $(E, q, [\cdot,\cdot], \pi)$ be an exact Courant algebroid over $M$, $s \in \mathcal{C}^\infty(TM, E)$ be any choice of a smooth isotropic splitting of $E$ and $\lambda : TM \to E$ be an isotropic embedding. Then there exists an automorphism $\Lambda : E \to E$ such that*

1. $$\lambda = \Lambda \circ s\,, \tag{6.4.1}$$
2. $$\forall\, A, B \in E \,:\, q\left(\Lambda(A), \Lambda(B)\right) = q(A, B) \quad \textit{and} \tag{6.4.2}$$
3. $$\lambda \in \mathcal{C}^k(M, (TM)^* \otimes E) \;\Leftrightarrow\; \Lambda \in \mathcal{C}^k(M, E^* \otimes E)\,. \tag{6.4.3}$$

*Proof.* Let $(E, q, [\cdot,\cdot], \pi)$ be an exact Courant algebroid over $M$ and let us choose any smooth isotropic splitting $s$ of $E$. Moreover, let $\lambda : TM \to E$ be an isotropic embedding. We will show that there exists an orthogonal automorphism $\Lambda$ with respect to $q$ which fulfills (6.4.1), (6.4.2) and (6.4.3) by constructing $\Lambda$ explicitly.

Since $E$ is a exact, it fits by definition into the short exact sequence (compare appendix A.2)

$$0 \longrightarrow T^*M \underset{s^*}{\overset{\pi^*}{\rightleftarrows}} E \underset{s}{\overset{\pi}{\rightleftarrows}} TM \longrightarrow 0 \quad . \tag{6.4.4}$$

By the splitting lemma this implies that $E = s(TM) \oplus \pi^*(T^*M) \cong \mathbb{T}M$. The last isomorphism maps the bracket given on $E$ to the twisted Dorfman bracket dictated by $s$. In the usual case $E = \mathbb{T}M$ (with ordinary Courant- or Dorfman bracket), $s$ can be chosen to be the canonical inclusion $\iota : TM \to \mathbb{T}M$. Because the above sequence is a split short exact sequence, it is true that $TM \cong s(TM)$ and $T^*M \cong \pi^*(T^*M)$.

As we want to construct $\Lambda$ such that in particular (6.4.1) is true, we define $\Lambda$ on $s(TM)$ by

$$\Lambda(s(X)) := \lambda(X)\,, X \in TM\,. \tag{6.4.5}$$

Therefore, it remains to define $\Lambda$ on $\pi^*(T^*M)$. For notational reasons let us denote the restriction of $\Lambda$ to $\pi^*(T^*M)$ as $\mu$ acting on $T^*M$, i.e. $\Lambda(\pi^*(\eta)) := \mu(\eta)$.

In view of equation (6.4.2) it is more convenient to define $q(\mu(\eta))$ instead of giving a formula for $\mu$. Moreover, it transpired that the following calculations become more transparent. Recall that $q(\mu(\eta)) \in E^*$, where we set $q(A)(B) := q(A, B)$ and use the same symbol in both cases in order to keep notations simple. Using the fact that $q$ is a pseudo-metric and in particular non-degenerate, we infer that $q(\mu(\eta))$ defines $\mu(\eta)$ uniquely.

Let $A \in E$. If $A \notin \lambda(TM)$, define

$$q(\mu(\eta))(A) := 0\,. \tag{6.4.6}$$

Whereas we define

$$q(\mu(\eta))(A) = q(\mu(\eta))(\lambda(X)) := q(\pi^*(\eta), s(X)) = \eta(X)\,, \tag{6.4.7}$$

if $A \in \lambda(TM)$. Compare equation (A.2.9). As $\lambda$ is an isotropic embedding and in particular an injection, there exists only one $X$ such that $A = \lambda(X)$. Hence, $q(\mu(\eta))(A)$ is well defined and equations (6.4.6) and (6.4.7) uniquely define an element $q(\mu(\eta)) \in E^*$. This in turn gives $\mu(\eta) \in E$ via $q^{-1}$.

The following argument shows that $\mu$ is isotropic. Let $A \in \lambda(TM) \cap \mu(T^*M)$. This means that there exist $\eta \in T^*M$ and $X \in TM$ such that $A = \mu(\eta) = \lambda(X)$. Then for all $Y \in TM$ it holds

$$0 = q(\lambda(X), \lambda(Y)) = q(\mu(\eta), \lambda(Y)) = q(\pi^*(\eta), s(Y)) = \eta(Y)\,. \tag{6.4.8}$$

This implies $\eta = 0$. Hence, $\lambda(TM) \cap \mu(T^*M) = \{0\}$. Then it follows from equation (6.4.6) that $q(\mu(\eta), \mu(\xi)) = 0$.

For $A = s(X) + \pi^*(\eta)$ we define

$$\Lambda(A) := \lambda(X) + \mu(\eta)\,. \tag{6.4.9}$$

The last equation is well defined since the decomposition $A = s(X) + \pi^*(\eta)$ is unique for any choice of an isotropic splitting $s$.

There are several things left to prove. We have to show that $\Lambda$ defined by equation (6.4.9) is an automorphism, is orthogonal with respect to $q$ and is as smooth as $\lambda$.

The restriction of $\Lambda$ to $s(TM)$ is obviously linear, injective and as smooth as $\lambda$. Our next step is to show that $\Lambda$ is also linear restricted to $\pi^*(T^*M)$. We will do this again by using the image of $\mu(\eta)$ under $q$. Let $\eta_1$, $\eta_2 \in T^*M$ and $c \in \mathbb{R}$. Then it is true that

$$\begin{aligned} q(\mu(c\eta_1 + \eta_2))(\lambda(X)) &= q(\mu(c\eta_1 + \eta_2), \lambda(X)) = q\left(\pi^*(c\,\eta_1 + \eta_2)\,, s(X)\right) = \\ &= c\, q(\pi^*(\eta_1), s(X)) + q(\pi^*(\eta_2), s(X)) = \\ &= c\, q(\mu(\eta_1), \lambda(X)) + q(\mu(\eta_2), \lambda(X)) = \\ &= q\left(c\, \mu(\eta_1) + \mu(\eta_2)\right)\left(\lambda(X)\right)\,. \end{aligned} \tag{6.4.10}$$

Hence, $q(\mu(c\eta_1 + \eta_2)) = q(c\mu(\eta_1) + \mu(\eta_2))$ which implies $\mu(c\eta_1 + \eta_2) = c\mu(\eta_1) + \mu(\eta_2)$. Thus $\Lambda$ restricted to $\pi^*(T^*M)$ is linear and, therefore, $\Lambda$ is linear.

The following calculation shows that $\Lambda$ is orthogonal with respect to $q$.

$$\begin{aligned} q(\Lambda A, \Lambda B) &= q\left(\Lambda(s(X) + \pi^*(\xi)), \Lambda(s(Y) + \pi^*(\eta))\right) = q(\lambda(X), \lambda(Y)) + \\ &\quad + q(\lambda(X), \mu(\eta)) + q(\mu(\xi), \lambda(Y)) + q(\mu(\xi), \mu(\eta)) = \\ &= 0 + q(s(X), \pi^*(\eta)) + q(\pi^*(\xi), s(Y)) + 0 = q(s(X), s(Y)) + \\ &\quad + q(s(X), \pi^*(\eta)) + q(\pi^*(\xi), s(Y)) + q(\pi^*(\xi), \pi^*(\eta)) = q(A, B)\,. \end{aligned} \tag{6.4.11}$$

Next we will show that $\Lambda$ is an automorphism. Clearly it is an endomorphism $\Lambda : E \to E$. Since $E$ is a finite dimensional vector bundle, it is sufficient to show that $\Lambda$ is surjective locally, i.e. at every $p \in M$. That will be done in several steps. The first one is the proof that $\lambda(T_pM)$ and $\mu(T_p^*M)$ are both maximally dimensional isotropic subspaces of $E_p$. Then we will use $\lambda(TM) \cap \mu(T^*M) = \{0\}$ to construct a basis of the image of $\Lambda$ at $p$ which also spans $E_p$. Because of $\Lambda_p$ being an endomorphism of a finite dimensional vector space which is surjective, it then follows that $\Lambda$ is also injective.

According to our premises $\lambda$ is an isotropic embedding and so $\lambda(T_pM)$ is isotropic and $n$-dimensional. Because $E_p$ is $2n$-dimensional, it follows that $\lambda(T_pM)$ is a maximally dimensional isotropic subspace of $E_p$.

By construction $\mu$ is isotropic, too. To be able to state that it is maximally dimensional we will prove that $\mu$ is injective. To this end let $\eta_1, \eta_2 \in T^*M$ with $\mu(\eta_1) - \mu(\eta_2) = \mu(\eta_1 - \eta_2) = 0$. This implies $\forall\, X \in TM \;:\; q\left(\pi^*(\eta_1 - \eta_2), s(X)\right)) = (\eta_1 - \eta_2)(X) = 0$. We infer $\eta_1 = \eta_2$ and $\mu$ is injective. Hence, it follows that $\mu(T^*M)$ is $n$-dimensional and a maximally dimensional isotropic subspace of $E_p$.

Since both $\lambda(T_pM)$ and $\mu(T_p^*M)$ are $n$-dimensional real vector spaces, they have a basis $\mathcal{B}$ and $\mathcal{B}'$, respectively. Recall that equation (6.4.8) implies $\lambda(TM) \cap \mu(T^*M) = \{0\}$. Hence, we infer that $\mathcal{B}$ and $\mathcal{B}'$ are linearly independent. Thus, $\mathcal{B} \cup \mathcal{B}'$ is a basis consisting of $2n$ linearly independent vectors in $E_p$. As $E_p$ is $2n$-dimensional, we infer that $\Lambda(E) = \mathrm{span}(\mathcal{B} \cup \mathcal{B}') = E$ and $\Lambda$ is surjective. Since it is an endomorphism and $E_p$ is finite dimensional, it follows that $\Lambda$ is injective and, hence, an automorphism.

It remains to show that $\Lambda$ is as smooth as $\lambda$.

Let $\Lambda \in \mathcal{C}^k(M, E^* \otimes E)$, i.e. a $k$-times continuously differentiable section in the endomorphism bundle of $E$. As $\lambda = \Lambda \circ s$ and $s$ is smooth, we obtain $\lambda \in \mathcal{C}^k(M, (TM)^* \otimes E)$. Now let $\lambda \in \mathcal{C}^k(M, (TM)^* \otimes E)$ for some $k$. This means that $\lambda(TM)$ is a $\mathcal{C}^k$-differentiable isotropic distribution in $E$. Define sections $\Lambda_1$ and $\Lambda_2$ in the endomorphism bundle of $E$ by

$$\Lambda_1(p)(A) := \left(\lambda_p \circ \pi_p\right)(A) \quad \text{and} \tag{6.4.12}$$

$$\Lambda_2(p)(A) := \left(\mu_p \circ s_p^*\right)(A)\,. \tag{6.4.13}$$

Because $E$ is an exact Courant algebroid, $A \in E$ can be written as $A = s(X) + \pi^*(\eta)$ for some uniquely defined $X \in TM$ and $\eta \in T^*M$. Using $\pi \circ s = \mathrm{id}_{TM}$, $s^* \circ \pi^* = \mathrm{id}_{T^*M}$, $\pi \circ \pi^* = 0$ and $s^* \circ s = 0$ it becomes evident that $\Lambda = \Lambda_1 + \Lambda_2$. Since $\Lambda_1$ is a $\mathcal{C}^k$ section, it remains to show that $\Lambda_2$ is at least $\mathcal{C}^k$. By assumption $q$ is a smooth. Then it is true that $\mu$ is as smooth as $q \circ \mu$. If $q \circ \mu(\eta) \in \mathcal{C}^k(M, E^*)$ for any smooth one-form $\eta$, it follows that $q \circ \mu \in \mathcal{C}^k(M, T^*M \otimes E^*)$. Thus let $\eta \in \Gamma(M, T^*M)$ be an arbitrary smooth 1-form. To prove that $q \circ \mu(\eta) \in \mathcal{C}^k(M, E^*)$ we have to show that $q \circ \mu(\eta)(B) \in \mathcal{C}^k(M, \mathbb{R})$ for all sections $B$ in $E$ which are as smooth as possible. If $B \notin \lambda(TM)$, it follows from equation (6.4.6) that $q \circ \mu(\eta)(B) = 0$, which is smooth. If $B(p) = \lambda_p(X)$ it is true that

$$q \circ \mu(\eta)(B) = q(\pi^*(\eta), s(X)) = \eta(X)\,. \tag{6.4.14}$$

Hence, $q \circ \mu(\eta)(B) \in \mathcal{C}^\infty(M, \mathbb{R})$. Thus, $q \circ \mu(\eta) \in \mathcal{C}^\infty(M, E^*)$, which yields $q \circ \mu \in \mathcal{C}^\infty(M, T^*M \otimes E^*)$ and, finally, $\mu \in \mathcal{C}^\infty(M, (T^*M)^* \otimes E)$. The last assertion implies that $\Lambda_2 \in \mathcal{C}^\infty(M, E^* \otimes E)$. Recalling that $\Lambda = \Lambda_1 + \Lambda_2$ we have arrived at our result. ☐

**Remarks 6.4.2**
1. If $\mathcal{J}$ is an integrable almost generalized complex structure on $E$, obviously $\Lambda^{-1}\mathcal{J}\Lambda$ is integrable if $\Lambda$ is also an automorphism of the bracket. If $E$ is given by the standard Courant algebroid, $\Lambda$ then has to be a semi-direct product of diffeomorphisms and $B$-transformations.
2. We assume that the isotropic splitting is smooth. If it has only to be in $\mathcal{C}^l$ for some $l \in \mathbb{N}$, $\Lambda$ would be at most $\mathcal{C}^l$ differentiable.

Now we are able to reduce $(E, \mathcal{J})$-holomorphic pairs $(\Phi, \lambda)$ to $(E, \mathcal{J})$-holomorphic curves $\Phi$. That is

**Proposition 6.4.3.** *Let $(\Sigma, j_\Sigma)$ be a Riemann surface, $M$ be a smooth manifold, $(E, q, [\cdot,\cdot], \pi)$ be an exact Courant algebroid over $M$ and $\mathcal{J}$ be an almost generalized complex structure on $E$. Moreover, let $\Phi : \Sigma \to M$ be a map and $\lambda : TM \to E$ be an isotropic embedding. Then it is true that $(\Phi, \lambda)$ is an $(E, \mathcal{J})$-holomorphic pair if and only if $\Phi$ is a $(E, \Lambda^{-1} \circ \mathcal{J} \circ \Lambda)$-holomorphic curve, where $\Lambda$ is defined in theorem 6.4.1.*

*Proof.* Since $\lambda$ is an isotropic embedding and $E$ is an exact Courant algebroid with smooth isotropic splitting $s$, it follows by theorem 6.4.1 that there exists an orthogonal automorphism $\Lambda$ with respect to $q$ such that $\lambda = \Lambda \circ s$. Let us assume without loss of generality that $\Lambda$ induces $\mathrm{id}$ on $M$. If it induces $\chi$, we define $\Phi' := \chi \circ \Phi$ and omit the prime. Therefore, equation (6.3.1) is equivalent to

$$(s \circ T\Phi) \circ j_\Sigma = \left(\Lambda^{-1} \circ \mathcal{J} \circ \Lambda\right) \circ (s \circ T\Phi)\,. \tag{6.4.15}$$

It remains to show that $(\Lambda^{-1} \circ \mathcal{J} \circ \Lambda)$ is an almost generalized complex structure. Using $(\Lambda^{-1} \circ \mathcal{J} \circ \Lambda)^2 = -\mathbb{1}$ and

$$\begin{aligned} \forall\, A, B \in \mathbb{T}M \,:\, q\left(\left(\Lambda^{-1}\mathcal{J}\Lambda\right) A, \left(\Lambda^{-1}\mathcal{J}\Lambda\right) B\right) &= q\left(\mathcal{J}\Lambda A, \mathcal{J}\Lambda B\right) = q(\Lambda A, \Lambda B) = \\ &= q(A, B)\,, \end{aligned} \tag{6.4.16}$$

we infer that $(\Lambda^{-1} \circ \mathcal{J} \circ \Lambda)$ is an almost generalized complex structure. Therefore, $\Phi$ fulfills the definition of a $(\Lambda^{-1} \circ \mathcal{J} \circ \Lambda)$-holomorphic curve. By reversing the argument we prove the proposition. $\square$

The next proposition shows that $(E, \mathcal{J})$-holomorphic curves are equivalent to generalized pseudoholomorphic curves on $\mathbb{T}M$.

**Proposition 6.4.4.** *Let $(\Sigma, j_\Sigma)$ be a Riemann surface, $M$ be a smooth manifold, $(E, q, [\cdot,\cdot], \pi)$ be an exact Courant algebroid over $M$ and $\mathcal{J}$ be an almost generalized complex structure on $E$. Moreover, let $\Phi : \Sigma \to M$ be a map, $\lambda : TM \to E$ be an isotropic embedding and let $s : TM \to E$ be a smooth isotropic splitting of $E$. Furthermore, let $\psi_E : E \to \mathbb{T}M$ denote the isomorphism between $E$ and $\mathbb{T}M$ which is induced by $s$ and let $\mathcal{J}'$ be the almost generalized complex structure on $\mathbb{T}M$ which is induced by $\mathcal{J}$ and $\psi_E$ . Then it is true that $\Phi$ is an $(E, \mathcal{J})$-holomorphic curve if and only if $\Phi$ is a $(\mathbb{T}M, \mathcal{J}')$-holomorphic curve.*

*Proof.* Let us choose a smooth isotropic splitting $s$ of $E$. Then it is true that

$$\mathcal{J}' = \begin{pmatrix} I & \beta \\ B & -I^* \end{pmatrix}\,, \tag{6.4.17}$$

where

$$I = \pi \circ \mathcal{J} \circ s\,, \tag{6.4.18}$$

$$\beta = \pi \circ \mathcal{J} \circ \pi^*\,, \tag{6.4.19}$$

$$B = s^* \circ \mathcal{J} \circ s \quad \text{and} \tag{6.4.20}$$

$$-I^* = s^* \circ \mathcal{J} \circ \pi^*\,. \tag{6.4.21}$$

Let $\Phi$ be an $(E, \mathcal{J})$-holomorphic curve. Then it follows by definition that $\mathcal{J} \circ s \circ T\Phi = s \circ T\Phi \circ j_\Sigma$. This implies together with $s^* \circ s = 0$ and $\pi \circ s = \mathrm{id}_{TM}$ that

$$\pi \circ \mathcal{J} \circ s \circ T\Phi = \pi \circ s \circ T\Phi \circ j_\Sigma \Leftrightarrow I \circ T\Phi = T\Phi \circ j_\Sigma \quad \text{and} \tag{6.4.22}$$

$$s^* \circ \mathcal{J} \circ s \circ T\Phi = s^* s \circ T\Phi \circ j_\Sigma \Leftrightarrow B \circ T\Phi = 0\,. \tag{6.4.23}$$

Therefore, $\mathcal{J}' \circ \iota \circ T\Phi = \iota \circ T\Phi \circ j_\Sigma$.

Now let $\Phi$ be a $\mathcal{J}'$-holomorphic curve. By recalling that $A \in E$ can uniquely be decomposed as $A = s(X) + \pi^*(\eta)$ and $s \circ \pi$ being a smooth projection operator $s \circ \pi : E \to s(TM)$, it becomes evident that $\mathcal{J}A = s(IX)+\pi^*(B\eta)$ and in particular $\mathcal{J}s(X) = s(IX)$. Hence,

$$s \circ T\Phi \circ j_\Sigma + 0 = s \circ I \circ T\Phi + \pi^* \circ B \circ T\Phi = \mathcal{J} \circ s \circ T\Phi \,. \tag{6.4.24}$$

□

**Remark 6.4.5** The last two propositions show in particular that in order to prove generic properties of $(E, \mathcal{J})$-holomorphic pairs we can restrict ourselves to generalized pseudoholomorphic curves without loss of generality. This will be important for instance in the proof of the identity theorem 7.2.1.

## 6.5. Generalized Energy

If $\Phi : \Sigma \to M$ is at least $\mathcal{C}^1$, one defines its energy as the $L^2$-norm of $d\Phi$, compare chapter 4 and in particular proposition 4.2.2. Next we wish to define the generalized energy of a pair $(\Phi, \lambda)$ consisting of a map $\Phi : \Sigma \to M$ with values in a generalized complex manifold and an isotropic embedding $\lambda : TM \to E$. To achieve this we need remarks 6.3.7 and in particular equation (6.3.8). There we used two almost generalized complex structures taming each other to define a metric $G$ on $E$.

In analogy to equations (4.2.14) ff. a metric $G$ on $E$ induces a metric $\tilde{G}$ on the pullback bundle $\Phi^*E$,

$$\tilde{G}_\sigma(A, B) := G_{\Phi(\sigma)}(\mathrm{pr}_2\, A, \mathrm{pr}_2\, B) \,, \tag{6.5.1}$$

for $A$, $B \in (\Phi^*E)_\sigma$. Moreover, if $h$ is an ordinary metric on $\Sigma$, it extends to $k$-forms. This yields a metric on $\Omega^k(\Sigma, \Phi^*E)$, namely the product metric. For $\alpha\,, \beta \in \Omega^k(\Sigma, \Phi^*\mathbb{T}M)$ and $\alpha = \eta \otimes A$, $\beta = \xi \otimes B$, we define a $L^2$-norm by

$$\langle \alpha, \beta \rangle_{h,\mathcal{J}_1,\mathcal{J}_2} := \int\limits_\Sigma (\alpha, \beta)_{h,\mathcal{J}_1,\mathcal{J}_2} * 1 \,, \tag{6.5.2}$$

where

$$(\alpha, \beta)_{h,\mathcal{J}_1,\mathcal{J}_2} := h(\eta, \xi)\tilde{G}(A, B) \,. \tag{6.5.3}$$

Now we are ready for

**Definition 6.5.1.** *Let $(\Sigma, h)$ be a Riemann surface, $M$ be a real $2n$-dimensional smooth manifold, $(E, q, [\cdot,\cdot], \pi)$ be an exact Courant algebroid over $M$ and $\mathcal{J}_1$, $\mathcal{J}_2$ be two almost generalized complex structures on $E$ such that $\mathcal{J}_1$ and $\mathcal{J}_2$ are tamed by each other. Furthermore, let $\lambda : TM \to E$ be an isotropic embedding. Then we define the generalized energy of the pair $(\Phi, \lambda)$ to be*

$$E_{h,\mathcal{J}_1,\mathcal{J}_2}(\Phi, \lambda) := \frac{1}{2} \langle \lambda \circ d\Phi, \lambda \circ d\Phi \rangle_{h,\mathcal{J}_1,\mathcal{J}_2} \ . \tag{6.5.4}$$

An important question is whether the generalized energy is invariant under orthogonal automorphisms covering diffeomorphisms. Let us explore again what that means for the standard Courant algebroid $\mathbb{T}M$. If we restrict ourselves to automorphisms $\Lambda$ respecting the Dorfman bracket, $\Lambda$ has to be the free product of diffeomorphisms, embedded in $O(q, \mathbb{T}M) \cong O(n, n)$, and $B$-transformations. In the general case this is

**Proposition 6.5.2.** *Let $(\Sigma, h)$ be a Riemann surface, $M$ be a real $2n$-dimensional manifold, $(E, q, [\cdot,\cdot], \pi)$ be an exact Courant algebroid over $M$ and $\mathcal{J}_1$, $\mathcal{J}_2$ be two almost generalized complex structures on $E$ such that $\mathcal{J}_1$ and $\mathcal{J}_2$ are tamed by each other. Furthermore, let $\lambda : TM \to E$ be an isotropic embedding and $\Lambda : E \to E$ be an orthogonal automorphism covering a diffeomorphism $\chi : M \to M$. Then the generalized energy is invariant under the transformation $(\Lambda, \chi)$.*

*Proof.* Let $\Lambda : E \to E$ be an orthogonal automorphism covering $\chi : M \to M$. Then $\Phi$ is changed to $\chi \circ \Phi$, $\lambda$ to $\Lambda \circ \lambda \circ T\chi^{-1}$, $\mathcal{J}_i$ to $\Lambda \mathcal{J}_i \Lambda^{-1}$ and the transformed energy can be simplified as

$$\begin{aligned} & E_{h,\Lambda\mathcal{J}_1\Lambda^{-1},\Lambda\mathcal{J}_2\Lambda^{-1}}(\chi \circ \Phi, \Lambda \circ \lambda \circ T\chi^{-1}) = \\ & = \frac{1}{2} \left\langle \Lambda \circ \lambda \circ T\chi^{-1} \circ d(\chi \circ \Phi), \Lambda \circ \lambda \circ T\chi^{-1} \circ d(\chi \circ \Phi) \right\rangle_{h,\Lambda\mathcal{J}_1\Lambda^{-1},\Lambda\mathcal{J}_2\Lambda^{-1}} = \\ & = \frac{1}{2} \left\langle \Lambda \circ \lambda \circ T\chi^{-1} \circ T\chi \circ d\Phi, \Lambda \circ \lambda \circ T\chi^{-1} \circ T\chi \circ d\Phi \right\rangle_{h,\Lambda\mathcal{J}_1\Lambda^{-1},\Lambda\mathcal{J}_2\Lambda^{-1}} = \\ & = \frac{1}{2} \langle \Lambda \circ \lambda \circ d\Phi, \Lambda \circ \lambda \circ d\Phi \rangle_{h,\Lambda\mathcal{J}_1\Lambda^{-1},\Lambda\mathcal{J}_2\Lambda^{-1}} \ . \end{aligned} \tag{6.5.5}$$

Looking at equations (6.5.1) to (6.5.3) and (7.1.3), it becomes evident that

$$\begin{aligned} & \frac{1}{2} \langle \Lambda \circ \lambda \circ d\Phi, \Lambda \circ \lambda \circ d\Phi \rangle_{h,\Lambda\mathcal{J}_1\Lambda^{-1},\Lambda\mathcal{J}_2\Lambda^{-1}} = \\ & = \frac{1}{2} \langle \lambda \circ d\Phi, \lambda \circ d\Phi \rangle_{h,\Lambda^{-1}\Lambda\mathcal{J}_1\Lambda^{-1}\Lambda,\Lambda^{-1}\Lambda\mathcal{J}_2\Lambda^{-1}\Lambda} = \frac{1}{2} \langle \lambda \circ d\Phi, \lambda \circ d\Phi \rangle_{h,\mathcal{J}_1,\mathcal{J}_2} \ . \end{aligned} \tag{6.5.6}$$

□

In usual symplectic topology one uses the energy to control the $L^2$-norm of the derivative of a $J$-holomorphic curve and shows e.g. Gromov compactness of the moduli space of pseudoholomorphic curves. This should in principle be possible in the generalized setting. As the focus of this work is not on the examination of a possible generalization of Gromov compactness and the definition of topological invariants like Gromov-Witten invariants, we will not go into the details. This will hopefully be part of future work. But as a first step into that direction we state a proposition and a theorem. The proposition shows that the energy of a $\mathcal{J}_2$-holomorphic pair is related to the pullback of a certain 2-form on $M$ and the theorem shows that under certain conditions one can choose an isotropic embedding $\lambda$ such that the energy is invariant under homotopy.

**Proposition 6.5.3.** *Let $(\Sigma, h)$ be a Riemann surface, $(E, q, [\cdot,\cdot], \pi)$ be an exact Courant algebroid over $M$ and $\mathcal{J}_1$, $\mathcal{J}_2$ be almost generalized complex structures such that $\mathcal{J}_1$ and $\mathcal{J}_2$ are tamed by each other. Furthermore, let $\Phi : \Sigma \to M$ be a map and $\lambda : TM \to E$ be an isotropic embedding such that $(\Phi, \lambda)$ is a generalized pseudoholomorphic pair with respect to $\mathcal{J}_2$. Then the following generalized energy identity holds:*

$$E_{h,\mathcal{J}_1,\mathcal{J}_2}(\Phi,\lambda) = \int_\Sigma (\lambda \circ T\Phi)^* \, q \, \mathcal{J}_1 = \int_\Sigma \Phi^*(\lambda^* q \, \mathcal{J}_1) \, . \tag{6.5.7}$$

*Moreover, if $\mathcal{J}_1$ and $\mathcal{J}_2$ are compatible, it is true that*

$$E_{h,\mathcal{J}_1,\mathcal{J}_2}(\Phi,\lambda) = \left\langle \overline{\partial}_{\mathcal{J}_2}(\Phi,\lambda), \overline{\partial}_{\mathcal{J}_2}(\Phi,\lambda) \right\rangle + \int_\Sigma (\lambda \circ T\Phi)^* \, q \, \mathcal{J}_1 \tag{6.5.8}$$

*for any $\mathcal{C}^1$ map $\Phi : \Sigma \to M$ and isotropic embedding $\lambda : TM \to E$.*

*Proof.* We will show this proposition by direct computations using conformal coordinates on $\Sigma$. Since $\Sigma$ is two-dimensional, it is easy to show that $E_{h,\mathcal{J}_1,\mathcal{J}_2}(\Phi,\lambda)$ is invariant under conformal transformations on $\Sigma$. It is also true that any Riemann surface is conformally flat. Thus, we are allowed to set without loss of generality $h_{ab} = \delta_{ab}$, i.e. $h_{ss} = h_{tt} = 1$ and $h_{st} = h_{ts} = 0$. Let us start our calculation by recalling that in local coordinates it is true that (cp. equation (7.1.2))

$$\begin{aligned} \overline{\partial}_{\mathcal{J}_2}(\Phi,\lambda) = {} & ds \otimes \frac{1}{2}\left( \frac{\partial \phi^\mu}{\partial s} \lambda(e_\mu) + \frac{\partial \phi^\mu}{\partial t} \mathcal{J}_2 \, \lambda(e_\mu) \right) + \\ & + dt \otimes \frac{1}{2}\left( \frac{\partial \phi^\mu}{\partial t} \lambda(e_\mu) - \frac{\partial \phi^\mu}{\partial s} \mathcal{J}_2 \, \lambda(e_\mu) \right) \, . \end{aligned} \tag{6.5.9}$$

Using this and

$$G(A,B)_{h,\mathcal{J}_1,\mathcal{J}_2} = -\frac{1}{2}q(\{\mathcal{J}_1,\mathcal{J}_2\}A,B) = \frac{1}{2}q(\mathcal{J}_1 A,\mathcal{J}_2 B) + \frac{1}{2}q(\mathcal{J}_2 A,\mathcal{J}_1 B) \tag{6.5.10}$$

we get for the local norm of $\overline{\partial}_{\mathcal{J}_2}(\Phi,\lambda)$ the expression

$$\begin{aligned}
&\left(\overline{\partial}_{\mathcal{J}_2}(\Phi,\lambda),\overline{\partial}_{\mathcal{J}_2}(\Phi,\lambda)\right)_{h,\mathcal{J}_1,\mathcal{J}_2} = \\
&\frac{1}{8}q\left(\mathcal{J}_1\left(\frac{\partial\phi^\mu}{\partial s}\lambda(e_\mu)+\frac{\partial\phi^\mu}{\partial t}\mathcal{J}_2\,\lambda(e_\mu)\right),\mathcal{J}_2\left(\frac{\partial\phi^\nu}{\partial s}\lambda(e_\nu)+\frac{\partial\phi^\nu}{\partial t}\mathcal{J}_2\,\lambda(e_\nu)\right)\right)+ \\
&+\frac{1}{8}q\left(\mathcal{J}_2\left(\frac{\partial\phi^\mu}{\partial s}\lambda(e_\mu)+\frac{\partial\phi^\mu}{\partial t}\mathcal{J}_2\,\lambda(e_\mu)\right),\mathcal{J}_1\left(\frac{\partial\phi^\nu}{\partial s}\lambda(e_\nu)+\frac{\partial\phi^\nu}{\partial t}\mathcal{J}_2\,\lambda(e_\nu)\right)\right)+ \\
&+\frac{1}{8}q\left(\mathcal{J}_1\left(\frac{\partial\phi^\mu}{\partial t}\lambda(e_\mu)-\frac{\partial\phi^\mu}{\partial s}\mathcal{J}_2\,\lambda(e_\mu)\right),\mathcal{J}_2\left(\frac{\partial\phi^\nu}{\partial t}\lambda(e_\nu)-\frac{\partial\phi^\nu}{\partial s}\mathcal{J}_2\,\lambda(e_\nu)\right)\right)+ \\
&+\frac{1}{8}q\left(\mathcal{J}_2\left(\frac{\partial\phi^\mu}{\partial t}\lambda(e_\mu)-\frac{\partial\phi^\mu}{\partial s}\mathcal{J}_2\,\lambda(e_\mu)\right),\mathcal{J}_1\left(\frac{\partial\phi^\nu}{\partial t}\lambda(e_\nu)-\frac{\partial\phi^\nu}{\partial s}\mathcal{J}_2\,\lambda(e_\nu)\right)\right) .
\end{aligned}$$

Exploiting the fact that $q(\mathcal{J}_i A,\mathcal{J}_i B) = q(A,B)$ the above expression can be rewritten as

$$\begin{aligned}
&\left(\overline{\partial}_{\mathcal{J}_2}(\Phi,\lambda),\overline{\partial}_{\mathcal{J}_2}(\Phi,\lambda)\right)_{h,\mathcal{J}_1,\mathcal{J}_2} = -\frac{1}{2}q\left(\frac{\partial\phi^\mu}{\partial s}\mathcal{J}_1\lambda(e_\mu),\frac{\partial\phi^\nu}{\partial t}\lambda(e_\nu)\right)+ \\
&+\frac{1}{2}q\left(\frac{\partial\phi^\mu}{\partial s}\mathcal{J}_1\lambda(e_\mu),\frac{\partial\phi^\nu}{\partial s}\mathcal{J}_2\lambda(e_\nu)\right)+\frac{1}{2}q\left(\frac{\partial\phi^\mu}{\partial t}\mathcal{J}_2\lambda(e_\mu),\frac{\partial\phi^\nu}{\partial t}\mathcal{J}_1\lambda(e_\nu)\right)- \\
&-\frac{1}{2}q\left(\frac{\partial\phi^\mu}{\partial t}\mathcal{J}_2\lambda(e_\mu),\frac{\partial\phi^\nu}{\partial s}\mathcal{J}_1\mathcal{J}_2\lambda(e_\nu)\right) .
\end{aligned} \tag{6.5.11}$$

Another easy calculation which is similar as above shows that the local part of the generalized energy of $(\Phi,\lambda)$ reads

$$\begin{aligned}
&\frac{1}{2}(\lambda\circ d\Phi,\lambda\circ d\Phi)_{h,\mathcal{J}_1,\mathcal{J}_2} = \frac{1}{2}q\left(\frac{\partial\phi^\mu}{\partial s}\mathcal{J}_1\lambda(e_\mu),\frac{\partial\phi^\nu}{\partial s}\mathcal{J}_2\lambda(e_\nu)\right)+ \\
&+\frac{1}{2}q\left(\frac{\partial\phi^\mu}{\partial t}\mathcal{J}_2\lambda(e_\mu),\frac{\partial\phi^\nu}{\partial t}\mathcal{J}_1\lambda(e_\nu)\right) .
\end{aligned} \tag{6.5.12}$$

Let us now consider the difference

$$\begin{aligned}
\Delta :&= \frac{1}{2}(\lambda\circ d\Phi,\lambda\circ d\Phi)_{h,\mathcal{J}_1,\mathcal{J}_2} - \left(\overline{\partial}_{\mathcal{J}_2}(\Phi,\lambda),\overline{\partial}_{\mathcal{J}_2}(\Phi,\lambda)\right)_{h,\mathcal{J}_1,\mathcal{J}_2} = \\
&= \frac{1}{2}q\left(\frac{\partial\phi^\mu}{\partial s}\mathcal{J}_1\lambda(e_\mu),\frac{\partial\phi^\nu}{\partial t}\lambda(e_\nu)\right)+\frac{1}{2}q\left(\frac{\partial\phi^\mu}{\partial t}\mathcal{J}_2\lambda(e_\mu),\frac{\partial\phi^\nu}{\partial s}\mathcal{J}_1\mathcal{J}_2\lambda(e_\nu)\right) .
\end{aligned} \tag{6.5.13}$$

As we wish to prove equation (6.5.8), let $(\Phi, \lambda)$ be a generalized pseudoholomorphic pair. Then equation (6.5.9) implies that in local coordinates

$$\frac{\partial \phi^\mu}{\partial s}\lambda(e_\mu) = -\frac{\partial \phi^\mu}{\partial t}\mathcal{J}_2\,\lambda(e_\mu)\,. \tag{6.5.14}$$

Therefore,

$$\Delta * 1 = q\left(\frac{\partial \phi^\mu}{\partial s}\mathcal{J}_1\lambda(e_\mu), \frac{\partial \phi^\nu}{\partial t}\lambda(e_\nu)\right)\, ds\wedge dt = (\lambda\circ T\Phi)^* q\mathcal{J}_1\,, \tag{6.5.15}$$

where we used

$$\begin{aligned}(\lambda\circ T\Phi)^* q\mathcal{J}_1(u,v) &= q(\mathcal{J}_1(\lambda\circ T\Phi)(u), (\lambda\circ T\Phi)(v)) = \\ &= u^a v^b q\left(\mathcal{J}_1\frac{\partial \phi^\mu}{\partial \sigma^a}\lambda(e_\mu), \frac{\partial \phi^\nu}{\partial \sigma^b}\lambda(e_\nu)\right)\end{aligned} \tag{6.5.16}$$

for $u, v$ being vector fields on $\Sigma$. Hence, we conclude that for $(\Phi, \lambda)$ being a generalized pseudoholomorphic pair, i.e. $\overline{\partial}_{\mathcal{J}_2}(\Phi, \lambda) = 0$, its energy can be expressed as (6.5.7). Now let $\Phi : \Sigma \to M$ be an arbitrary $\mathcal{C}^1$ map and $\lambda : TM \to E$ be an isotropic embedding.[4] If $\mathcal{J}_1$ and $\mathcal{J}_2$ are compatible, we infer that (6.5.13) still implies (6.5.15). Therefore, equation (6.5.8) is true, which proves the proposition. ☐

**Remarks 6.5.4** 1. If $\lambda^* q\mathcal{J}_1$ is a closed 2-form on $M$, it is true that the energy of a generalized pseudoholomorphic pair is invariant under homotopic deformations of $\Phi$. If $\mathcal{J}_1$ and $\mathcal{J}_2$ are compatible, it follows that generalized pseudoholomorphic pairs are local minima of the generalized energy functional. That means that they are in some sense generalized harmonic maps. These maps should be defined by analogous considerations as in chapter 4. This will be done somewhere else.

2. The most important case from the viewpoint of the generalized B-model of topological string theory, compare chapter 5, is $E = \mathbb{T}M$ and $\lambda$ being the canonical embedding $\iota$. Then it is true that the generalized energy is a topological invariant for $\iota^* q\mathcal{J}_1$ being closed. If we parameterize $\mathcal{J}_1$ as

$$\mathcal{J}_1 = \begin{pmatrix} I_1 & \beta_1 \\ B_1 & -I_1^* \end{pmatrix}\,, \tag{6.5.17}$$

this corresponds to $dB_1 = 0$. It is not the case in general, even for integrable generalized complex structures, but can be achieved locally after a $B$-transformation and diffeomorphism (cp. local Darboux theorem). In physical languange this

[4] Actually, at this stage $\lambda$ can be an arbitrary embedding.

can be achieved by a local canonical transformation. For symplectic manifolds, i.e. $I_1 = 0$, $B = \omega$ and $\beta_1 = -\omega^{-1}$, where $d\omega = 0$ and $\omega$ antisymmetric and non-degenerate, the energy is always a topological invariant. Later we will show that there exists an isotropic embedding $\lambda_0$ such that $d\lambda_0^* q\mathcal{J}_1 = 0$. In the notation of chapter 5 this corresponds to the replacement of $H$ by $H + dB$ for some (not necessarily closed) two-form $B$.

In order to show that this proposition is consistent with the known cases, let us look at two examples.

**Examples 6.5.5** 1. First, we want to consider symplectic manifolds. To this end let $\omega$ be a symplectic structure on $M$ and $\mathcal{J}_\omega$ be the associated generalized complex structure on the standard Courant algebroid $E = \mathbb{T}M$. Moreover, let $\mathcal{J}_2$ be an almost generalized complex structure which is $\mathcal{J}_\omega$-tame. In particular such $\mathcal{J}_2$ which are associated with a $\omega$-tame almost complex structure $J$ do the job (see examples 6.3.8). If we take as $\lambda$ the canonical embedding $\iota : TM \to \mathbb{T}M$, it follows that the energy of a $\mathcal{J}_2$-holomorphic curve $\Phi$ is

$$E(\Phi, \iota) = \int_\Sigma \Phi^* \iota^* \mathcal{J}_\omega = \int_\Sigma \Phi^* \omega \,. \tag{6.5.18}$$

This is exactly the result for pseudo holomorphic curves in symplectic topology.

2. Next we take a look on complex manifolds. Let $J$ be a complex structure on $M$ and $\mathcal{J}_J$ be its associated generalized complex structure. Let $\mathcal{J}_2$ be any almost generalized complex structure which is $\mathcal{J}_J$-tame and let again $\lambda = \iota$. Then we obtain for the energy of a $\mathcal{J}_2$-holomorphic curve

$$E(\Phi, \iota) = \int_\Sigma \Phi^* \iota^* \mathcal{J}_J = \int_\Sigma \Phi^* 0 = 0 \,. \tag{6.5.19}$$

Hence, every generalized pseudoholomorphic curve has vanishing energy in the complex case. Observe that we may get a different energy if we take another isotropic embedding $\lambda$.

3. The last example which we want to consider here is the product of two generalized complex manifolds. If $(M, \mathcal{J}_1^M)$ and $(N, \mathcal{J}_1^N)$ are generalized complex manifolds, the product $M \times N$ is a generalized complex manifold in a natural way with generalized complex structure $\mathcal{J}_1 := \mathcal{J}_1^M \oplus \mathcal{J}_1^N$. Any map $\Phi : \Sigma \to M \times N$ can be expressed as the Cartesian product of two maps $\Phi_M : \Sigma \to M$ and $\Phi_N : \Sigma \to N$. If the isotropic embedding $\lambda$ is the direct sum of two isotropic embeddings $\lambda_M$ and $\lambda_N$, it follows that the energy of a $\mathcal{J}_2$-holomorphic pair $(\Phi_M \times \Phi_N, \lambda_M \oplus \lambda_N)$

is the sum of $E(\Phi_M, \lambda_M)$ and $E(\Phi_N, \lambda_N)$:

$$E(\Phi_M \times \Phi_N, \lambda_M \oplus \lambda_N) = \int_\Sigma (\Phi_M \times \Phi_N)^*(\lambda_M \oplus \lambda_N)^* q(\mathcal{J}_1^M \oplus \mathcal{J}_1^N) =$$
$$= \int_\Sigma (\Phi_M \times \Phi_N)^*(\lambda_M^* q\mathcal{J}_1^M \oplus \lambda_N^* q\mathcal{J}_1^N) = \int_\Sigma \Phi_M^* \lambda_M^* q\mathcal{J}_1^M + \int_\Sigma \Phi_N^* \lambda_N^* q\mathcal{J}_1^N . \tag{6.5.20}$$

In particular, if a manifold is the product of a symplectic and a complex manifold, the energy is dictated by the components of $\Phi$ in the "symplectic directions".

As mentioned above, in order to be able to construct topological invariants, we have at least to be able to control the energy of $\mathcal{J}$ holomorphic pairs. Provided that the cohomology class of the curvature $H$ vanishes and $\mathcal{J}_1$ is a regular generalized complex structure, the next theorem shows that there is a certain choice of isotropic embedding $\lambda : TM \to E$ such that the generalized energy of $\mathcal{J}_2$-holomorphic pairs is an invariant under homotopy. More precisely, we prove

**Theorem 6.5.6.** *Let $(\Sigma, h)$ be a Riemann surface, $(E, q, [\cdot,\cdot], \pi)$ be an exact Courant algebroid over $M$ with vanishing Ševera class $[H]$ and $\mathcal{J}_1$ be a regular generalized complex structure. Furthermore, let $\mathcal{J}_2$ be an almost generalized complex structure which is tamed by $\mathcal{J}_1$ and $(\Phi_0, \lambda)$, $(\Phi_1, \lambda)$ be two $(E, \mathcal{J}_2)$-holomorphic pairs. Then there exists an isotropic embedding $\lambda : TM \to E$ such that $E(\Phi_0, \lambda)_{h,\mathcal{J}_1,\mathcal{J}_2} = E(\Phi_1, \lambda)_{h,\mathcal{J}_1,\mathcal{J}_2}$ if $\Phi_0$ and $\Phi_1$ are homotopic to each other. Moreover, $\lambda$ is independent of $\mathcal{J}_2$, $\Phi_0$, $\Phi_1$ and compatible with the projections of $E$ and $TM$, i.e. $\pi_E \circ \lambda = \pi_{TM}$.*

*Proof.* The proof will have several steps. First, we will motivate our choice of isotropic embedding $\lambda = \Lambda \circ s$ for some smooth isotropic splitting $s$. This will use methods from [AB06]. Second we will demonstrate that $ds^* q\Lambda^{-1}\mathcal{J}_1\Lambda = 0$. Standard arguments then show that the energy is invariant under homotopy.

Because of the fact that by assumption $H = dB$, it follows that there exists a splitting $s$ of the exact Courant algebroid such that $H = 0$. Let us choose such a splitting $s : TM \to E$. As $\mathcal{J}_1$ is regular, the +i-eigenbundle $L$ of $\mathcal{J}_1$ can be written as $L(D, \epsilon)$ with $D$ being a subbundle of $TM$ and $\epsilon \in \Gamma(D, \wedge^2 D^*)$. From $L(D, \epsilon)$ being of real index zero, i.e. $L \cap \overline{L} = \{0\}$, it follows that $D \oplus \overline{D} = TM \otimes \mathbb{C}$ and $\omega_\Delta := \mathrm{Im}(\epsilon|_{D\cap\overline{D}})$ is non-degenerate on $\Delta \otimes \mathbb{C} := D \cap \overline{D}$. As $\mathcal{J}_1$ is integrable, we infer that $D$ is involutive (the anchor $\pi$ is bracket preserving) and

$$d_D\epsilon = d_D\iota^*(B + \mathrm{i}\omega) = \iota^* d(B + \mathrm{i}\omega) = 0\,. \tag{6.5.21}$$

The latter equation implies that $\omega_\Delta$ is closed along the leaves of the foliation given by the distribution $\Delta$. Next we will show that there is an orthogonal automomrphism $\Lambda$ of $E$ covering the identity map such that $\Lambda^{-1}\mathcal{J}_1\Lambda$ is the direct sum of a generalized complex structure of symplectic type and a generalized complex structure of complex type. The symplectic structure will be given by $\omega_\Delta$.

Let us construct $\Lambda$ explicitly. As $\mathcal{J}_2$ is tamed by $\mathcal{J}_1$, we get a natural metric $G$ on $E$ induced by $\mathcal{J}_1$ and $\mathcal{J}_2$. Recall that we chose an isotropic splitting $s$ to get rid of $H$. This splitting can be used to define a metric $g$ on $TM$ via $g(X,Y) := G(s(X), s(Y))$. Now let $\Delta'$ denote the orthogonal complement of $\Delta$ with respect to $g$. Since $\Delta$ is by assumption a smooth distribution of constant rank and $g$ is a smooth metric, we infer that $\Delta'$ is smooth, too.[5] Combining the facts $TM = \Delta \oplus \Delta'$ and $D \subseteq TM_\mathbb{C}$ we obtain a decomposition of $D$ into

$$D = \Delta_\mathbb{C} \oplus (D \cap \Delta'_\mathbb{C}) \,. \tag{6.5.22}$$

Indeed, $\Delta_\mathbb{C} \subseteq D$ and $D \cap \Delta'_\mathbb{C} \subseteq D$ implies $\Delta_\mathbb{C} \oplus (D \cap \Delta'_\mathbb{C}) \subseteq D$. Conversely, $X \in D \subseteq TM_\mathbb{C}$ can be written uniquely as $X_\Delta + X_{\Delta'}$. It is also true that $X_\Delta \in \Delta_\mathbb{C} \subseteq D$ which yields $X_{\Delta'} \in D$. Therefore, it follows that $X_\Delta \in \Delta_\mathbb{C}$ and $X_{\Delta'} \in D \cap \Delta'_\mathbb{C}$. Hence, we infer that $D \subseteq \Delta_\mathbb{C} \oplus (D \cap \Delta'_\mathbb{C})$, obtaining (6.5.22).

Thus,

$$L(D, \epsilon) = L(\Delta_\mathbb{C} \oplus (D \cap \Delta'_\mathbb{C}), \epsilon) \,. \tag{6.5.23}$$

Our next step is the construction of an orthogonal automorphism $\tilde{\Lambda}$ of $E$ such that

$$\tilde{\Lambda}_\mathbb{C}^{-1} L(D,\epsilon) = L(\Delta_\mathbb{C} \oplus (D \cap \Delta'_\mathbb{C}), \epsilon|_{\Delta_\mathbb{C}} + \epsilon|_{D\cap\Delta'_\mathbb{C}}) = L(\Delta, \epsilon_{\Delta_\mathbb{C}}) \oplus L(D \cap \Delta'_\mathbb{C}, \epsilon|_{D\cap\Delta'_\mathbb{C}}) \,. \tag{6.5.24}$$

Before proving the existence of $\tilde{\Lambda}$ let us analyze the type of structures on the right hand side of (6.5.24). The first question which arises is whether the above structures are integrable. Observing $\Delta_\mathbb{C} = D \cap \overline{D}$ and $D$ being closed under the Lie-bracket it becomes evident that $\Delta_\mathbb{C}$ is involutive and $d_\Delta(\epsilon|_{\Delta_\mathbb{C}}) = (d_D\epsilon)|_{\Delta_\mathbb{C}} = 0$ (the derivative is defined via the Cartan formula). In particular $\Delta$ defines a foliation of $M$ into submanifolds of dimension $2n - 2\text{type}\mathcal{J}_1$. This foliation is called canonical symplectic foliation. The fact that it is symplectic will be clarified in a moment. In general, $\Delta'$ need not to be involutive. Here, $\Delta'$ is just introduced for calculational reasons. The result will be independent of it.

Let us now clarify what kind of generalized complex structures they are. The complex 2-form $\epsilon|_{\Delta_\mathbb{C}}$ is closed if and only if the real and the imaginary part is closed. Hence,

[5] It would be sufficient to choose any smooth distribution which is complementary to $\Delta$

$\epsilon|_{\Delta_\mathbb{C}} = \tilde{B}_\Delta - \mathrm{i}\omega_\Delta$ defines a $B$-transformation of a symplectic structure

$$L(\Delta_\mathbb{C}, \epsilon|_{\Delta_\mathbb{C}}) = \mathrm{e}^{\tilde{B}_\Delta}\left(\mathrm{e}^{-\mathrm{i}\omega_\Delta}\Delta\right) . \tag{6.5.25}$$

The transformation $\tilde{B}_\Delta$ and the form $\omega_\Delta$ are real complex 2-forms, i.e. $\overline{\tilde{B}_\Delta} = \tilde{B}_\Delta$ and analogously for $\omega_\Delta$. To reveal the nature of the second structure on the right hand side of (6.5.24) we use the fact that $D \cap \overline{D} = \{0\}$ and infer

$$D \cap \Delta'_\mathbb{C} \cap \overline{D \cap \Delta'_\mathbb{C}} = D \cap \Delta'_\mathbb{C} \cap \overline{D} \cap \overline{\Delta'_\mathbb{C}} = (D \cap \overline{D}) \cap \Delta'_\mathbb{C} \cap \overline{\Delta'_\mathbb{C}} = \{0\} . \tag{6.5.26}$$

Therefore, it is true that

$$L(D \cap \Delta'_\mathbb{C}, \epsilon|_{D\cap\Delta'_\mathbb{C}}) = \mathrm{e}^{\epsilon_{D\cap\Delta'_\mathbb{C}}} L(D \cap \Delta'_\mathbb{C}, 0) , \tag{6.5.27}$$

is the $B$-transformation of a complex structure. Hence, we are allowed to write

$$\begin{aligned}\tilde{\Lambda}_\mathbb{C}^{-1} L(D,\epsilon) &= \mathrm{e}^{\tilde{B}_\Delta + \epsilon|_{D\cap\Delta'_\mathbb{C}}} \left(L(\Delta_\mathbb{C}, -\mathrm{i}\omega_\Delta) \oplus L(D \cap \Delta'_\mathbb{C}, 0)\right) = \\ &= \mathrm{e}^{\tilde{B}_\Delta + \epsilon|_{D\cap\Delta'_\mathbb{C}}} \left(\mathrm{e}^{-\mathrm{i}\omega_\Delta} L(\Delta_\mathbb{C} \oplus D \cap \Delta'_\mathbb{C}, 0)\right) = \\ &= \mathrm{e}^{\tilde{B}_\Delta + \epsilon|_{D\cap\Delta'_\mathbb{C}}} \left(\mathrm{e}^{-\mathrm{i}\omega_\Delta} \det \mathrm{Ann}(D)\right)\end{aligned} \tag{6.5.28}$$

In the second and third line we use the same symbol $\omega_\Delta$ for the extension of $\omega_\Delta$ to $TM$ by 0, i.e. $\omega_\Delta|_{TM\setminus\Delta} = 0$.

We turn again to the construction of $\Lambda$ and do the ansatz $\tilde{\Lambda} = \exp(B)$. Obviously, $\tilde{\Lambda}$ is an orthogonal automorphism covering the identity on $M$. Since $\epsilon$ is a complex differential form, we infer that $\overline{\epsilon}(\overline{X}, \overline{Y}) = \overline{\epsilon(X,Y)}$. Let $s \in \Gamma(\Delta)$ and $w \in \Gamma(\Delta')$. Because of the fact that $L$ has real index zero, it follows that $TM_\mathbb{C} = D \oplus \overline{D}$. Therefore, $w \in \Gamma(\Delta') \subseteq \Gamma(\Delta'_\mathbb{C}) \subseteq \Gamma(TM_\mathbb{C})$ combined with $w$ being real yields the unique (uniqueness is implied by $\Delta_\mathbb{C} \cap \Delta'_\mathbb{C} = \{0\}$) decomposition $w = d + \overline{d}$ with $d \in \Gamma(D \cap \Delta'_\mathbb{C})$. Now let $s \in \Gamma(\Delta)$ and define the real 2-form $B$ as

$$B|_\Delta := 0 , \tag{6.5.29}$$

$$B|_{D\cap\Delta'} := 0 , \tag{6.5.30}$$

$$-B(w,s) := B(s,w) := 2\,\mathrm{Re}\,\epsilon(s,d) . \tag{6.5.31}$$

It is easy to see that $\epsilon$ being smooth implies $B$ being smooth. It remains to show that equation (6.5.24) holds. Since

$$\begin{aligned}B_\mathbb{C}(s+d, s'+d') &= B_\mathbb{C}(s,s) + B_\mathbb{C}(s,d') + B_\mathbb{C}(d,s') + B_\mathbb{C}(d,d') = \\ &= B_\mathbb{C}(s,d') - B_\mathbb{C}(s',d) ,\end{aligned} \tag{6.5.32}$$

it is sufficient to prove $B_{\mathbb{C}}(s,d) = \epsilon(s,d)$ for $s \in \Delta_{\mathbb{C}}$ and $d \in D \cap \Delta'_{\mathbb{C}}$. Clearly,

$$d = \frac{d+\overline{d}}{2} + \mathrm{i}\,\frac{d-\overline{d}}{2\,\mathrm{i}}\,, \quad \text{where} \quad \frac{d+\overline{d}}{2},\ \frac{d-\overline{d}}{2\,\mathrm{i}} \in W\,. \tag{6.5.33}$$

Assume $s$ is real. Therefore, it follows by definition that

$$B_{\mathbb{C}}\left(s, \frac{d+\overline{d}}{2}\right) = 2\,\mathrm{Re}\,\epsilon\left(s, \frac{d}{2}\right) = \mathrm{Re}\,\epsilon(s,d) \tag{6.5.34}$$

and

$$B_{\mathbb{C}}\left(s, \frac{d-\overline{d}}{2\,\mathrm{i}}\right) = B\left(s, \frac{d}{2\,\mathrm{i}} + \overline{\frac{d}{2\,\mathrm{i}}}\right) = 2\,\mathrm{Re}\,\epsilon\left(s, \frac{d}{2\,\mathrm{i}}\right) = \mathrm{Im}\,\epsilon\,(s,d)\ . \tag{6.5.35}$$

Hence, $B(s,d) = \epsilon(s,d)$ for $s$ being real. Now let $s$ and $d$ be complex. Then

$$\begin{aligned} B_{\mathbb{C}}(s,d) &= B_{\mathbb{C}}\left(\frac{s+\overline{s}}{2}, d\right) + \mathrm{i}\,B_{\mathbb{C}}\left(\frac{s-\overline{s}}{2\,\mathrm{i}}, d\right) = \epsilon\left(\frac{s+\overline{s}}{2}, d\right) + \mathrm{i}\,\epsilon\left(\frac{s-\overline{s}}{2\,\mathrm{i}}, d\right) = \\ &= \epsilon(s,d)\,. \end{aligned} \tag{6.5.36}$$

Thus we obtain $B_{\mathbb{C}}(s,d) = \epsilon(s,d)$ and (6.5.28).

Now we are ready to choose our isotropic embedding $\lambda$. Define

$$\lambda = \Lambda \circ s := \tilde{\Lambda} \circ \mathrm{e}^{\tilde{B}_{\Delta}} \circ \mathrm{e}^{B'} \circ s = \mathrm{e}^{B+\tilde{B}_{\Delta}+B'} \circ s\,, \tag{6.5.37}$$

where $B'$ is the real form whose complexification is $\epsilon|_{D\cap\Delta'_{\mathbb{C}}}$. Observe that $i^*(B + \tilde{B}_{\Delta} + B')_{\mathbb{C}} = \epsilon - (-\,\mathrm{i}\omega_{\Delta})$ for $i : D \to TM$ being the inclusion. The right hand side of the former equation is in particular independent of $\Delta'$. Because of the fact that $\Lambda$ covers the identity on $M$, the isotropic embedding $\lambda$ is compatible with the projections of $E$ and $TM$, i.e. $\pi_E \circ \lambda = \pi_{TM}$.

Using proposition 6.4.3 we infer that $(\Phi, \lambda)$ is a $\mathcal{J}_2$-holomorphic pair if and only if $\Phi$ is a $\Lambda^{-1}\mathcal{J}_2\Lambda$-holomorphic curve. Moreover, the proof of proposition 6.5.2 shows that

$$E(\Phi,\lambda)_{h,\mathcal{J}_1,\mathcal{J}_2} = E(\Phi,s)_{h,\Lambda^{-1}\mathcal{J}_1\Lambda,\Lambda^{-1}\mathcal{J}_2\Lambda}\,. \tag{6.5.38}$$

We constructed $\Lambda$ in such a way that $\Lambda^{-1}\mathcal{J}_1\Lambda =: \mathcal{J}_1'$ is the direct sum of a symplectic structure and a complex structure.

The next step to prove the theorem is to show that $s^*q\mathcal{J}_1'$ is closed at every $p \in M$. This is a local statement and, hence, we are allowed to work in a local coordinate chart $U$ containing $p$. The fact that $\Delta$ integrates to a regular foliation shows that there are distinguished directions in $M$ associated with $\Delta$ which we call symplectic directions. Let $p \in M$. Then in some neighborhood $U'$ of $p$ there exist smooth complex coordinates $z_1, \ldots, z_k$ such that $\{dz^1, \ldots dz^k\}$ are linearly independent and span $\det \mathrm{Ann}(D)$, cf. proposition 3.12 in [Gua03]. Let us denote these directions as complex directions. Since $L(D, \epsilon)$ is involutive with respect to the Courant bracket and $\pi$ is bracket preserving, it follows that $D$ is involutive and induces a foliation of $M$. The above complex coordinats define an integrable complex structure transverse to $D$. By the Newlander-Nirenberg theorem there is a diffeomorphism only affecting the complex coordinates such that the complex structure $J$ induced by $\mathcal{J}_1'$ is given by the standard complex structure. Strictly speaking we need the Newlander-Nirenberg theorem for a family of complex structures. Its existence is mentioned at the end of [NN57]. The complex coordinates are constant along the leaves of the symplectic foliation. Compare this situation with the proof of the existence of a complex structure on the leaf space given in proposition 4.2 in [Gua11]. Weinstein's proof of the Darboux normal coordinate theorem for a family of symplectic structures shows that there exists a leaf preserving local diffeomorphism $\chi : V' \times V \subset \mathbb{R}^{2k} \times \mathbb{R}^{2n-2k} \to U$ and an open neighborhood $U$ of $p$ such that on each leaf

$$\chi^* \omega_\Delta = \omega_0 = dx^1 \wedge dx^2 + \ldots dx^{2n-2k-1} \wedge dx^{2n-2k} \,. \tag{6.5.39}$$

Consider from now on $U \cap U' \neq \emptyset$ and denote it for simplicity as $U$. Since the transverse complex coordinates are constant along the leaves, it follows that $J$ is unaffected by $\chi$. Hence, $\mathrm{e}^{T\chi} \mathcal{J}_1' \mathrm{e}^{T\chi^{-1}} =: \mathcal{J}_0$ is the direct sum of a constant symplectic structure in normal Darboux form and a constant complex structure in normal form on $\mathbb{C}^k$, where $k$ is the type of $\mathcal{J}_1$. Recall that the type is unchanged by a $B$-transformation. The generalized complex structure $\mathcal{J}_0$ on $V \times V'$ reads

$$\mathcal{J}_0 = \begin{pmatrix} 0 & 0 & -\omega_0^{-1} & 0 \\ 0 & J_0 & 0 & 0 \\ \omega_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -J_0^* \end{pmatrix} , \tag{6.5.40}$$

where all matrices are constant. Then $\omega_0$ being constant implies

$$0 = ds^* q \mathcal{J}_0 = ds^* q \mathrm{e}^{T\chi} \mathcal{J}_1' \mathrm{e}^{T\chi^{-1}} = ds^* \left(\mathrm{e}^{T\chi^{-1}}\right)^* q \mathcal{J}_1' = d \left(\mathrm{e}^{T\chi^{-1}} \circ s\right)^* q \mathcal{J}_1' \,. \tag{6.5.41}$$

Here we used

$$\begin{aligned} \left(\mathrm{e}^{T\chi^{-1}}\right)^* (q\mathcal{J}_1') (A, B) &= (q\mathcal{J}_1') \left(\mathrm{e}^{T\chi^{-1}} A, \mathrm{e}^{T\chi^{-1}} B\right) = q \left(\mathcal{J}_1' \mathrm{e}^{T\chi^{-1}} A, \mathrm{e}^{T\chi^{-1}} B\right) = \\ &= q \left(\mathrm{e}^{T\chi} \mathcal{J}_1' \mathrm{e}^{T\chi^{-1}} A, B\right) = \left(q \, \mathrm{e}^{T\chi} \mathcal{J}_1' \mathrm{e}^{T\chi^{-1}}\right) (A, B) \,. \end{aligned} \tag{6.5.42}$$

Recall that $\exp(T\chi^{-1})$ is the embedding of $T\chi^{-1}$ into Aut$(E)$. It can be expressed in the form

$$\mathrm{e}^{T\chi^{-1}} = \begin{pmatrix} T\chi^{-1} & 0 \\ 0 & \chi^* \end{pmatrix} , \tag{6.5.43}$$

acting on $E = s(TM) \oplus \pi^*(T^*M)$ for some splitting $s$. This implies in particular that

$$\exp(T\chi^{-1}) \circ s = s \circ T\chi^{-1} \, . \tag{6.5.44}$$

Hence, equation (6.5.42) can be simplified further into

$$0 = d(s \circ T\chi^{-1})^* q\mathcal{J}_1' = d\left(\chi^{-1}\right)^* s^* q\mathcal{J}_1' = (\chi^{-1})^* d\, s^* q\mathcal{J}_1' \, . \tag{6.5.45}$$

Since $\chi^{-1}$ is a diffeomorphism, it follows that $d\, s^* q\mathcal{J}_1' = 0$.

Now let $(\Phi_1, \lambda)$ and $(\Phi_2, \lambda)$ be two generalized pseudoholomorphic pairs with respect to $\mathcal{J}_2$. Proposition 6.5.3 together with proposition 6.5.2 shows that

$$E(\Phi_i, \lambda)_{h,\mathcal{J}_1,\mathcal{J}_2} = \int\limits_{\Sigma} \Phi_i^*(\lambda^* q\, \mathcal{J}_1) = \int\limits_{\Sigma} \Phi_i^*(s^* q\, \mathcal{J}_1') \tag{6.5.46}$$

Then standard arguments show that $E(\Phi_1, \lambda)_{h,\mathcal{J}_1,\mathcal{J}_2} = E(\Phi_2, \lambda)_{h,\mathcal{J}_1,\mathcal{J}_2}$.

□

**Remarks 6.5.7**

1. The above constructed $\lambda$ is even more than an isotropic embedding, it is a smooth isotropic splitting of $E$ for the described case, even though it is not involutive in general.
2. For such an isotropic embedding the energy is dictated by the components of $\Phi$ in direction of the canonical symplectic foliation, while the complex directions give vanishing contribution.
3. The proof makes still sense for non-regular $\mathcal{J}_1$. It is true that in every regular neighborhood the $+\,$i-eigenbundle is of the form $L(D, \epsilon)$. In general situations this is still the case. But then we have to work with generalized distributions in the sense of [Sus73]. They are sub-bundles whose rank may vary along $M$. Sections in $D$ are finitely generated, but the generating system may be linearly dependent at some points. The form part $\epsilon$ is then a section in the dual object $\wedge^2 D^*$. If we define a form by its action on vector fields, the above construction will yield an orthogonal automorphism. But as the case of regular structures is still general enough to contain manifolds which do not admit any complex or symplectic structure, cp. examples below, we restricted the above theorem to this simpler context.

4. Another problem arises if we do not assume $[H] = 0$. The differential form $\omega_\Delta$ will no longer give a symplectic form on each leaf. This is originated in the fact that $d_D\epsilon = i^*H$ for some splitting $s : TM \to E$ with curvature $H$ and $i : D \to TM$ being the inclusion. For general $\mathcal{J}_1$ acting on arbitrary exact Courant algebroids Gualtieri proved in [Gua11] that $\pi \circ \mathcal{J}_1 \circ \pi^* : T^*M \to TM$ is a Poisson bi-vector. Hence, a generalized complex manifold $M$ is always a Poisson manifold in a natural way. As such it admits a natural symplectic foliation [Wei83]. Moreover, the leaf space inherits a natural complex structure (proposition 4.2 in [Gua11]). It remains to construct an isotropic embedding such that $\mathcal{J}_1$ acts as the symplectic structure in directions of the symplectic foliation and as the complex structure in complex directions. Such an isotropic embedding would be implicitly given if we were able to find a complement $\tilde{\Delta}'$ to $\pi^*(T^*M) + \mathcal{J}_1\pi^*(T^*M)$ which is isotropic, stable under $\mathcal{J}_1$ and projects isomorphically to a complement of $\pi(\mathcal{J}_1\pi^*T^*M)$. We postpone this for future work.

**Examples 6.5.8**

1. If we take $\mathcal{J}_1$ to be a complex or symplectic manifold, the construction of $\lambda$ yields $\lambda = \iota$. Examples 6.5.5 show that this is not surprising, as the complex case yields zero energy and the symplectic case is the known one.
2. If $(M, \mathcal{J}_1^M)$ and $(N, \mathcal{J}_1^N)$ are generalized complex manifolds, the product $M \times N$ is a generalized complex manifold in a natural way with generalized complex structure $\mathcal{J}_1 := \mathcal{J}_1^M \oplus \mathcal{J}_1^N$. Obviously, the direct sum of $\lambda_M$ and $\lambda_N$, which are the results of the above construction, is an isotropic embedding rendering the energy to be invariant.
3. More non-trivial examples are given by nilmanifolds. Some of them do not admit any known symplectic or complex structures [Sal98]. But they admit generalized complex structures [CG04]. All examples listed there are generalized complex manifolds of constant type. If we take as $\mathcal{J}_1$ any of those structures, they fulfill the assumptions of theorem 6.5.6. The differential form $B + \tilde{B}_\Delta + B'$ is exactly the real part of the exponent given in table 1 in [CG04].

# 7. Local Behavior of Generalized Pseudoholomorphic Curves

In the last chapter we examined some possible definitions of generalized holomorphic maps and defined the objects of interest, namely generalized pseudoholomorphic curves, tamed structures as well as compatible structures and the generalized energy. In this chapter we will look at the local behavior of solutions to equation (6.3.1). First, we will calculate the coordinate expression of this equation. This will yield a generalization of the nonlinear Cauchy-Riemann equations in complex geometry. After that we will introduce the generalized energy of a map $\Phi : \Sigma \to M$ with values in a generalized complex manifold. Thereafter, we will use the generalized nonlinear Cauchy-Riemann equation to establish an identity theorem which tells us that two curves which coincide at one point $\sigma \in \Sigma$ up to infinite order are actually equal everywhere. Having proven this theorem we will proceed to the definition of somewhere injective and simple curves. They play a special role in symplectic topology and we expect them to do the same in generalized complex geometry. At the end of this chapter we will prove an elliptic regularity theorem for generalized pseudoholomorphic curves which states roughly that solutions to equation (6.3.1) are as smooth as $\mathcal{J}$.

## 7.1. Generalized Nonlinear Cauchy-Riemann Equation

As mentioned in the introduction of this chapter, we will establish a local expression for the generalized pseudoholomorphic equation (6.3.1). This can be done similar to the complex case. Moreover, we will look at the anti-holomorphic part of $\lambda \circ d\Phi$ and give some first indications why it is important. To this end let $(E, q, [\cdot,\cdot], \pi)$ be an exact Courant algebroid over $M$ and $(\Phi, \lambda)$ be an $(E, \mathcal{J})$-holomorphic pair. That means they fulfill

$$\mathcal{J} \circ \lambda \circ d\Phi = \lambda \circ d\Phi \circ j_\Sigma \,. \tag{7.1.1}$$

In order to get a local expression and for future purposes, we restate the last equation as

$$\overline{\partial}_{\mathcal{J}}(\Phi, \lambda) := \frac{1}{2}(\lambda \circ d\Phi + \mathcal{J} \circ (\lambda \circ d\Phi) \circ j_\Sigma) = 0\,. \tag{7.1.2}$$

Using the local expression of $d\Phi$, i.e. eq. (4.2.13), we get locally

$$\lambda \circ d\Phi = \frac{\partial \phi^\mu}{\partial \sigma^a} d\sigma^a \otimes \lambda \left(\frac{\partial}{\partial x^\mu}\right) \tag{7.1.3}$$

and

$$\begin{aligned}\overline{\partial}_{\mathcal{J}}(\Phi, \lambda) &= \frac{1}{2}\left(\frac{\partial \phi^\mu}{\partial \sigma^a} d\sigma^a \otimes \lambda(e_\mu) + \frac{\partial \phi^\mu}{\partial \sigma^a} \mathcal{J} \circ d\sigma^a \otimes \lambda(e_\mu) \circ j_\Sigma\right) = \\ &= \frac{1}{2}\left(\frac{\partial \phi^\mu}{\partial \sigma^a} d\sigma^a \otimes \lambda(e_\mu) + \frac{\partial \phi^\mu}{\partial \sigma^a} \mathcal{J} \circ j_\Sigma^*(d\sigma^a) \otimes \lambda(e_\mu)\right) = 0\,,\end{aligned} \tag{7.1.4}$$

where we denoted $\frac{\partial}{\partial x^\mu}$ as $e_\mu$. It is very well known that on a two-dimensional surface any almost complex structure is integrable. Therefore, we can choose coordinates $s$ and $t$ on $\Sigma$ such that

$$j_\Sigma\left(\frac{\partial}{\partial s}\right) = \frac{\partial}{\partial t} \quad \text{and} \quad j_\Sigma\left(\frac{\partial}{\partial t}\right) = -\frac{\partial}{\partial s}\,. \tag{7.1.5}$$

Hence, it follows that

$$j_\Sigma^*\, ds(u) = ds(ju) = ds(u^s \partial_t - u^t \partial_s) = -u^t \tag{7.1.6}$$

and

$$j_\Sigma^*\, dt(u) = dt(ju) = dt(u^s \partial_t - u^t \partial_s) = u^s\,. \tag{7.1.7}$$

This induces

$$j_\Sigma^*\, ds = -dt \quad \text{and} \quad j_\Sigma^*\, dt = ds\,. \tag{7.1.8}$$

Equation (7.1.4) then reduces to

$$\begin{aligned}&\frac{1}{2}\left(\frac{\partial \phi^\mu}{\partial s} ds \otimes \lambda(e_\mu) + \frac{\partial \phi^\mu}{\partial t} dt \otimes \lambda(e_\mu) + \mathcal{J}\left(-\frac{\partial \phi^\mu}{\partial s} dt \otimes \lambda e_\mu + \frac{\partial \phi^\mu}{\partial t} ds \otimes \lambda(e_\mu)\right)\right) = \\ &= ds \otimes \frac{1}{2}\left(\frac{\partial \phi^\mu}{\partial s}\lambda(e_\mu) + \mathcal{J}\frac{\partial \phi^\mu}{\partial t}\lambda(e_\mu)\right) + dt \otimes \frac{1}{2}\left(\frac{\partial \phi^\mu}{\partial t}\lambda(e_\mu) - \mathcal{J}\frac{\partial \phi^\mu}{\partial s}\lambda(e_\mu)\right) = 0\end{aligned} \tag{7.1.9}$$

which is equivalent to

$$\frac{\partial \phi^\mu}{\partial s}\lambda(e_\mu) + \frac{\partial \phi^\mu}{\partial t}\mathcal{J}(\phi)\lambda(e_\mu) = 0\,. \tag{7.1.10}$$

This is the local form of equation (6.3.1). It will play a crucial role in finding the local properties of $\mathcal{J}$-holomorphic pairs and we call it generalized nonlinear Cauchy-Riemann equation. Before we proceed with our local examination, we should take a closer look on the definition of $\overline{\partial}_{\mathcal{J}}(\Phi, \lambda)$. It is the anti-holomorphic part of $\lambda \circ d\Phi$ as a 1-form on $\Sigma$ with values in $\Phi^* E$. This can be seen from the following consideration.

Let $u$ be a complex vector field on $\Sigma$ and decompose it into its holomorphic and anti-holomorphic part with respect to $j_\Sigma$ as $u = u^{(1,0)} + u^{(0,1)}$. Moreover, let us decompose the complexification of $\lambda \circ d\Phi$ into its holomorphic and anti-holomorphic part with respect to $\mathcal{J}$ as $\lambda \circ d\Phi = (\lambda \circ d\Phi)^{(1,0)} + (\lambda \circ d\Phi)^{(0,1)}$. Then it is true that

$$\begin{aligned}
\overline{\partial}_{\mathcal{J}}(\Phi, \lambda) =& \frac{1}{2}(\lambda \circ d\Phi)^{(1,0)}(u^{(1,0)}) + \frac{1}{2}(\lambda \circ d\Phi)^{(1,0)}(u^{(0,1)}) + \\
&+ \frac{1}{2}(\lambda \circ d\Phi)^{(0,1)}(u^{(1,0)}) + \frac{1}{2}(\lambda \circ d\Phi)^{(0,1)}(u^{(0,1)}) + \\
&+ \frac{\mathrm{i}^2}{2}(\lambda \circ d\Phi)^{(1,0)}(u^{(1,0)}) - \frac{\mathrm{i}^2}{2}(\lambda \circ d\Phi)^{(1,0)}(u^{(0,1)}) - \\
&- \frac{\mathrm{i}^2}{2}(\lambda \circ d\Phi)^{(0,1)}(u^{(1,0)}) + \frac{\mathrm{i}^2}{2}(\lambda \circ d\Phi)^{(0,1)}(u^{(0,1)}) = \\
=& (\lambda \circ d\Phi)^{(1,0)}(u^{(0,1)}) + (\lambda \circ d\Phi)^{(0,1)}(u^{(1,0)})\,.
\end{aligned} \tag{7.1.11}$$

Therefore, $\overline{\partial}_{\mathcal{J}}(\Phi, \lambda)$ measures how far $(\lambda \circ d\Phi)$ is not mapping $+$i-eigenbundles to $+$i-eigenbundles, which we call anti-holomorphic. It will also play a crucial role in the theory of deformations of $\mathcal{J}$-holomorphic pairs. We will consider it, analogous to the ordinary complex case, as a section in a Banach-bundle and investigate its intersection with the 0-section in this bundle. Its vertical linearization with respect to some specific connection $\nabla$ will be the composition of a real Cauchy-Riemann operator and a semi-Fredholm operator. This makes further examinations more involved than in usual symplectic topology. We will resolve this issue by defining so called admissible vector fields along $\Phi$. These questions will be our concern in chapter 8.

## 7.2. Identity Theorem

In the last section we derived the generalized nonlinear Cauchy-Riemann equation. Now we will use the former to establish an identity theorem. It states that two pairs which are $\mathcal{J}$-holomorphic coincide if their difference vanishes to infinite order and their isotropic embeddings are the same. We call an integrable function $w : B_\epsilon \to \mathbb{C}^n$ vanishing to infinite order at $z = 0$ if

$$\int\limits_{|z|\leq r} |w(z)| = \mathcal{O}\left(r^k\right) \tag{7.2.1}$$

for every $k > 0$, where $B_\epsilon := \{z \in \mathbb{C} : |z| < \epsilon\}$. If $w$ is smooth, this is equivalent to the vanishing of the $\infty$-jet of $w$.

First, we will state a local version of the identity theorem. It is

**Theorem 7.2.1.** *Let $\phi, \psi \in \mathcal{C}^1(B_\epsilon, \mathbb{R}^{2n})$ such that they are both solutions to* (7.1.10) *for some almost generalized complex structure $\mathcal{J} \in \mathcal{C}^2(\mathbb{R}^{2n}, \mathrm{GL}(4n, \mathbb{R}))$ and a fixed embedding $\lambda : \mathbb{R}^{2n} \to \mathbb{R}^{4n}$. If $\mathcal{J}$ has bounded derivatives and $\phi - \psi$ vanishes to infinite order at some $z_0 \in \mathbb{C}$, it follows that $\phi \equiv \psi$.*

Before giving the proof we state the global version of the identity theorem. It is a corollary of theorem 7.2.1 and reads

**Corollary 7.2.2.** *Let $\Sigma$ be a connected Riemann surface, $M$ be a smooth $2n$-dimensional manifold, $(E, q, [\cdot,\cdot], \pi)$ be an exact Courant algebroid over $M$, $\mathcal{J}$ be an almost generalized complex structure on $E$ and $\mathcal{J}$ be at least $\mathcal{C}^2$ with bounded derivatives. Moreover, let $\Phi, \Psi : \Sigma \to M$ be two $\mathcal{J}$-holomorphic curves for some fixed isotropic embedding $\lambda : TM \to E$. If there exists a $\sigma \in \Sigma$ such that $\Phi - \Psi$ vanishes to infinite order at $\sigma$, it holds $\Phi \equiv \Psi$.*

*Proof.* Using theorem 6.4.1 and propositions 6.4.3 and 6.4.4 we are allowed to assume without loss of generality that $E = \mathbb{T}M$ and $\lambda = \iota$. Since two curves coincide to infinite order if they coincide locally to infinite order, the assertion is an immediate corollary of theorem 7.2.1. □

The proof of theorem 7.2.1 will rely on

**Theorem 7.2.3** (Aronszajn). *Let $\Omega \subset \mathbb{C}$ be a connected open set. Suppose that there is some constant $c$ such that the function $w \in W^{2,2}_{loc}(\Omega, \mathbb{R}^n)$ satisfies the pointwise estimate*

$$|\Delta w(z)| \leq c(|w| + |\partial_s w| + |\partial_t w|) \tag{7.2.2}$$

*almost everywhere in $\Omega$ and that $w$ vanishes to infinite order at some point $z_0 \in \Omega$. Then $w \equiv 0$.*

*Proof.* The theorem and its proof can be found in [Aro57]. □

Now we are able to give the

*Proof of theorem 7.2.1.* Using theorem 6.4.1 and propositions 6.4.3 and 6.4.4 we are allowed to assume without loss of generality that $E = \mathbb{T}M$ and $\lambda = \iota$. We would like to use theorem 7.2.3 to prove the theorem. That means we need to show that there exists a constant $c$ such that the difference of two $\mathcal{J}$-holomorphic curves satisfies the estimate (7.2.2). In order to do that, we need to obtain a relation between the Laplace operator acting on a local generalized pseudoholomorphic curve and its derivative with respect to $t$ and $s$. To this end let $\Omega$ be a connected subset of $\mathbb{C}$ and $\phi : \Omega \to \mathbb{R}^{4n}$ be a solution of (7.1.10) for $\lambda = \iota$. Then it is true that

$$\begin{aligned}
&(\partial_s - \mathcal{J}\partial_t)\left(\frac{\partial\phi^\mu}{\partial s}(e_\mu \oplus 0) + \mathcal{J}\frac{\partial\phi^\mu}{\partial t}(e_\mu \oplus 0)\right) = \\
&\quad = \frac{\partial^2\phi^\mu}{\partial s^2}(e_\mu \oplus 0) + \partial_s\mathcal{J}\frac{\partial\phi^\mu}{\partial t}(e_\mu \oplus 0) - \mathcal{J}\frac{\partial^2\phi^\mu}{\partial t\,\partial s}(e_\mu \oplus 0) - \mathcal{J}\partial_t\mathcal{J}\frac{\partial\phi^\mu}{\partial t}(e_\mu \oplus 0) = \\
&\quad = \left(\partial_s^2\phi^\mu + (\partial_s\mathcal{J})\partial_t\phi^\mu + \mathcal{J}\partial_s\partial_t\phi^\mu - \mathcal{J}\partial_t\partial_s\phi^\mu - \mathcal{J}\left(\partial_t\mathcal{J}\right)\partial_t\phi^\mu + \partial_t^2\phi^\mu\right) e_\mu \oplus 0 = \\
&\quad = \left(\Delta\phi^\mu + (\partial_s\mathcal{J})\partial_t\phi^\mu - (\partial_t\mathcal{J})\partial_s\phi^\mu\right) e_\mu \oplus 0
\end{aligned} \tag{7.2.3}$$

In the last step we used $0 = \partial_t\mathcal{J}^2 = (\partial_t\mathcal{J})\mathcal{J} + \mathcal{J}(\partial_t\mathcal{J})$ and equation (7.1.10). Therefore, we get for local $\mathcal{J}$-holomorphic curves

$$\Delta\phi = (\partial_t\mathcal{J})\partial_s\phi - (\partial_s\mathcal{J})\partial_t\phi\,, \tag{7.2.4}$$

where we set $\phi := (\phi^\mu\,\iota(e_\mu))$. Here it was crucial that we were able to reduce $\lambda$ to the canonical embedding $\iota : TM \to \mathbb{T}M$. If $\lambda$ is a nonconstant isotropic embedding, we have to take care of its variation along the map $\Phi$. This would result in additional terms in equation (7.2.4) involving partial derivatives of $\lambda$.

Now let $\phi$ and $\psi$ be solutions of (7.1.10). Then a simple calculation shows that

$$\begin{aligned}
\Delta(\phi^\mu - \psi^\mu)\iota(e_\mu) &= (\partial_t)(\phi)\partial_s(\phi^\mu - \psi^\mu)\iota(e_\mu) - (\partial_s\mathcal{J})(\phi)\partial_t(\phi^\mu - \psi^\mu)\iota(e_\mu) + \\
&+ \left[(\partial_t\mathcal{J})(\phi) - (\partial_t\mathcal{J})(\psi)\right]\partial_s\psi^\mu\iota(e_\mu) - \left[(\partial_s\mathcal{J})(\phi) - (\partial_s\mathcal{J})(\psi)\right]\partial_t\psi^\mu\iota(e_\mu)
\end{aligned} \tag{7.2.5}$$

Since $\mathcal{J}(p) : \mathbb{R}^{4n} \to \mathbb{R}^{4n}$, every component ${\mathcal{J}_\mu}^\nu$ fulfills at every $z \in \omega$

$${(\partial_t\mathcal{J})_\mu}^\nu(\phi) - {(\partial_t\mathcal{J})_\mu}^\nu(\psi) = \int_\gamma d{(\partial_t\mathcal{J})_\mu}^\nu\,. \tag{7.2.6}$$

Here $\gamma(\tau) = \psi(z) + \tau(\psi(z) - \phi(z))$ and $\tau \in [0,1]$. This implies

$$(\partial_t\mathcal{J})(\phi) - (\partial_t\mathcal{J})(\psi) = \int_\gamma d(\partial_t\mathcal{J})\,, \tag{7.2.7}$$

where we integrate every component for its own. Estimating the absolute value of the integral by the product of the supremum of the integrand and the length of $\gamma$, we arrive at

$$\left|\left((\partial_t \mathcal{J})(\phi) - (\partial_t \mathcal{J})(\psi)\right) \partial_s \psi^\mu \iota(e_\mu)\right| \leq \|d(\partial_t \mathcal{J})\|_\infty \left|\phi - \psi\right| \left|\partial_s \psi^\mu \iota(e_\mu)\right| . \tag{7.2.8}$$

A calculation which is completely analogous to the above one shows

$$\left|\left((\partial_s \mathcal{J})(\phi) - (\partial_s \mathcal{J})(\psi)\right) \partial_t \psi^\mu \iota(e_\mu)\right| \leq \|d(\partial_s \mathcal{J})\|_\infty \left|\phi - \psi\right| \left|\partial_t \psi^\mu \iota(e_\mu)\right| . \tag{7.2.9}$$

If we set

$$c := \mathrm{Max}\left(\|d(\partial_t \mathcal{J})\|_\infty \left|\partial_s \psi^\mu \iota(e_\mu)\right|, \|d(\partial_s \mathcal{J})\|_\infty \left|\partial_t \psi^\mu \iota(e_\mu)\right|, \left|(\partial_t \mathcal{J})(\phi)\right|, \left|(\partial_t \mathcal{J})(\psi)\right|\right) , \tag{7.2.10}$$

it follows that

$$\left|\Delta(\phi^\mu - \psi^\mu)\iota(e_\mu)\right| \leq c\left(\left|\partial_s(\phi^\mu - \psi^\nu)\iota(e_\mu)\right| + \left|\partial_t(\phi^\mu - \psi^\nu)\iota(e_\mu)\right| + \left|(\phi^\mu - \psi^\nu)\iota(e_\mu)\right|\right) \tag{7.2.11}$$

Therefore, the requirements of theorem 7.2.3 are fulfilled. This implies $(\phi^\mu - \psi^\mu)\iota(e_\mu) = 0$. Since $\iota(e_\mu)$ are linearly independent, we infer that $\phi^\mu = \psi^\mu$. □

This theorem shows that $\mathcal{J}$-holomorphic pairs and in particular $\mathcal{J}$-holomorphic curves behave much like ordinary pseudoholomorphic curves. At least locally. Later we will realize the reason for this behavior. A $\mathcal{J}$-holomorphic curve is locally an ordinary $J$-holomorphic curve into $U \times U$, where $U$ is some neighborhood in $M$. This will be discussed in details later. In ordinary symplectic topology one is interested in the moduli space of $J$-holomorphic curves. In order to get finite dimensional smooth manifolds one considers so called simple curves. As we are interested in the generalization of symplectic topology to generalized complex topology, it is interesting to consider their analog, too. This will be done in section 7.4. Before we are able to do that we have to establish elliptic regularity of solutions to eq. (7.1.10).

## 7.3. Elliptic Regularity

This section deals with elliptic regularity of $\mathcal{J}$-holomorphic pairs. It is an important technical result in the theory of generalized pseudoholomorphic pairs. It states roughly that solutions to equation (6.3.1) are as smooth as the almost generalized complex structure $\mathcal{J}$. In the theory of usual pseudoholomorphic curves it is proven using an elliptic bootstrapping argument. A treatment of these matters can be found in appendix B of [MS04]. There they also include the possibility of $\partial\Sigma \neq \emptyset$. As we will not be concerned with this in the present work, we restrict ourselves to the case where $\Sigma$ has no boundary. But it should be easy to generalize to this case, at least locally.

We will see that it is possible to trace back elliptic regularity of $\mathcal{J}$-holomorphic pairs to the respective assertions in the ordinary theory. This relies on the fact that locally $\mathcal{J}$-holomorphic curves look like ordinary pseudoholomorphic curves in a space with doubled dimension and the second half of the coordinates is set to some fixed point. Later we will give a precise statement of this fact in theorem 7.4.4. Furthermore, we will see that if $\mathcal{J}$ is integrable, it follows that after choosing a particular splitting $s$, a $\mathcal{J}$-holomorphic curve is constant in the symplectic directions and holomorphic in the complex directions, confer section 8.2.

Let us return to the examination of elliptic regularity of $\mathcal{J}$-holomorphic pairs. We will show that given an isotropic embedding $\lambda : TM \to E$, which we assume to be smooth, solutions $\Phi$ of (6.3.1) are as smooth as $\mathcal{J}$. More precisely, we will prove

**Theorem 7.3.1** (Elliptic Regularity)**.** *Let $l \geq 2$ and $p > 2$. Moreover, let $(\Sigma, j_\Sigma)$ be a Riemann surface, $M$ be a smooth $2n$-dimensional manifold, $(E, q, [\cdot,\cdot], \pi)$ an exact Courant algebroid, $\mathcal{J}$ be a $\mathcal{C}^l$ almost generalized complex structure on $E$ and $\lambda : TM \to E$ be a smooth isotropic embedding. Suppose that $\Phi : \Sigma \to M$ is a $W^{1,p}$-function such that $\lambda \circ T\Phi \circ j_\Sigma = \mathcal{J} \circ \lambda \circ T\Phi$. Then $\Phi$ is of class $W^{l,p}$. In particular $l = \infty$ implies $\Phi$ is smooth.*

*Proof.* Choose a splitting $s$ of $E$. Since $\lambda : TM \to E$ is a smooth embedding, it follows from theorem 6.4.1 that there exists a smooth orthogonal automorphism $\Lambda$ of $E$ such that $\lambda = \Lambda \circ s$. Because of proposition 6.4.3 it is then true that $(\Phi, \lambda)$ is an $(E, \mathcal{J})$-holomorphic pair if and only if $\Phi$ is a $(E, \Lambda^{-1} \circ \mathcal{J} \circ \Lambda)$-holomorphic curve. In particular $(\Lambda^{-1} \circ \mathcal{J} \circ \Lambda)$ is a $\mathcal{C}^l$ almost generalized complex structure. If we denote the almost generalized complex structure on $\mathbb{T}M$ induced by $(\Lambda^{-1} \circ \mathcal{J} \circ \Lambda)$ as $\mathcal{J}'$, proposition 6.4.4 shows that $\Phi$ is $\mathcal{J}'$-holomorphic.

In order to examine the regularity properties of the $(E, \mathcal{J})$-holomorphic pair $(\Phi, \lambda)$ it is thus sufficient to look at the regularity properties of $\mathcal{J}'$-holomorphic curves. This means $\Phi$ is a solution of (6.3.1) for $\mathcal{J}'$ and $\lambda = \iota : TM \to \mathbb{T}M$.

It suffices to prove the result in local holomorphic coordinates on $\Sigma$ and in local coordinates on $M$. Hence, we have to show elliptic regularity for solutions to (7.1.10) for $\mathcal{J}'$ and $\lambda = \iota$, i.e.

$$\partial_s \phi + \mathcal{J}'(u) \partial_t \phi = 0 \,, \tag{7.3.1}$$

where we set

$$\phi := \phi^\mu \iota(e_\mu) = \begin{pmatrix} \phi^1 \\ \phi^2 \\ \vdots \\ \phi^{2n} \\ 0 \\ \vdots \\ 0 \end{pmatrix} . \tag{7.3.2}$$

Here it is again important to absorb the dependency of $\lambda$ on the map $\phi$ into the almost generalized complex structure $\mathcal{J}'$. We have to pushforward $\mathcal{J}$ to coordinate charts in order to get an almost generalized complex structure on $\mathbb{R}^{4n}$.

Since the coordinate maps $\xi$ are diffeomorphisms, it is possible to define the pushforward of $\eta \in T_p^*M$ by $(\xi_*\eta)_{\xi(p)}(V) := \eta_p((T_p\xi)^{-1}V)$, where $V \in T_{\xi(p)}\mathbb{R}^{2n} \cong \mathbb{R}^{2n}$. This yields a map

$$\mathrm{e}^{T\xi} : T_pM \oplus T_p^*M \to T_{\xi(p)}\mathbb{R}^{2n} \oplus T^*_{\xi(p)}\mathbb{R}^{2n} \cong \mathbb{R}^{4n} . \tag{7.3.3}$$

It can be used to pushforward the almost generalized complex structure $\mathcal{J}'$ to get a $\mathcal{C}^{l-1}$ almost generalized complex structure $\mathcal{J}'$ on $\mathbb{R}^{4n}$. We slightly abuse notation and denote this local structure by the same symbol $\mathcal{J}'$. In particular it is true that $\mathcal{J}'^2 = -\mathbb{1}$.

Now we are able to adopt the proof of the corresponding theorem B.4.1 in [MS04]. This will be done by induction.

Assume that $\Omega \subset \mathbb{H}$ is an open set and $\phi$ is a $W^{1,p}_{\mathrm{loc}}$-weak solution of

$$\partial_s u + \mathcal{J}'(u)\partial_t\phi = 0 , \tag{7.3.4}$$

where $\phi^{2n+1} = \phi^{2n+1} = \ldots = \phi^{4n} = 0$, cp. eq. (7.3.2). Then it can be shown by partial integration that $\phi$ also fulfills equation (7.3.7) (there it corresponds to $u$). Then $\phi \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{4n})$ satisfies the requirements of proposition 7.3.2 with $n$ being even, $k = 1$, $\eta = 0$ and $\mathcal{J}$ being $\mathcal{J}'(\phi) = \mathcal{J}' \circ \phi \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{4n\times 4n})$. Therefore, $\phi \in W^{2,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{4n})$ and $\mathcal{J}' \circ \phi \in W^{2,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{4n\times 4n})$. This argument can be repeated to show that $\phi \in W^{k,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{4n})$ implies $\phi \in W^{k+1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{4n})$ for $k = 1, 2, \ldots l-1$. By induction we infer that $\phi \in W^{l,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{4n})$. □

The above proof used an elliptic bootstrapping argument based on

**Proposition 7.3.2** ([MS04]). *Let $\Omega' \subset \Omega \subset \mathbb{H}$ be open sets such that $\overline{\Omega'} \subset \Omega$. Moreover, let $l$ be a positive integer and $p > 2$. Then for every constant $c_0$ there exists a constant $c > 0$ with the following significance. Assume $J \in W^{l,p}(\Omega, \mathbb{R}^{2n\times 2n})$ satisfies $J^2 = -\mathbb{1}$ and*

$$J(s,0) = J_0 := \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{7.3.5}$$

*as well as*

$$\|J\|_{W^{l,p}(\Omega)} \leq c_0 \,. \tag{7.3.6}$$

*Then the following holds for every $k \in \{0, \dots, l\}$.*

1. *If $\phi \in L^p_{loc}(\Omega, \mathbb{R}^{2n})$ and $\eta \in W^{k,p}_{loc}(\Omega, \mathbb{R}^{2n})$ are such that*

$$\varphi(\Omega \cap \mathbb{R}) \subset \mathbb{R}^n \times \{0\} \Rightarrow \int_\Omega \left\langle \partial_s\varphi + J^T\partial_t\varphi, \phi \right\rangle = -\int_\Omega \langle \varphi, \eta + (\partial_t J)\phi \rangle \tag{7.3.7}$$

   *holds for every test function $\varphi \in \mathcal{C}^\infty_0(\Omega, \mathbb{R}^{2n})$, then $\phi \in W^{k+1,p}_{loc}(\Omega, \mathbb{R}^{2n})$ and $\phi$ satisfies*

$$\partial_s\phi + J\partial_t\phi = \eta \,, \quad \phi(\Omega \cap \mathbb{R}) \subset \mathbb{R}^n \times \{0\} \tag{7.3.8}$$

   *almost everywhere.*

2. *If $\phi \in W^{k+1,p}_{loc}(\Omega, \mathbb{R}^{2n})$ satisfies the boundary condition $\phi(\Omega \cap \mathbb{R}) \subset \mathbb{R}^n \times \{0\}$, then*

$$\|\phi\|_{W^{k+1,p}(\Omega')} \leq c\left(\|\partial_s\phi + J\partial_t\phi\|_{W^{k,p}(\Omega)} + \|\phi\|_{W^{k,p}(\Omega)}\right) \,. \tag{7.3.9}$$

*Proof.* The proposition and its proof can be found in [MS04]. There it is proposition B.4.9. ☐

**Remark 7.3.3** I would like to state a short remark on the notations of the above proposition. The subscript "loc" stands for local and indicates the fact that the definig property has to be true on each precompact open subset of $\Omega$. For instance $W^{k,p}_{\text{loc}}(\Omega, \mathbb{R}^{2n})$ is the Sobolov space of maps whose weak derivatives up to order $k$ exist and are in $L^p$ on each precompact open subset of $\Omega$. The inner product $\langle \cdot, \cdot \rangle$ is given by the standard inner product in $\mathbb{R}^{2n}$, whereas $\mathcal{C}^\infty_0(\Omega, \mathbb{R}^{2n})$ denotes the class of all smooth maps from $\Omega$ to $\mathbb{R}^{2n}$ with compact support.

This section treated an important result. It states in particular that the curve part $\Phi$ of a $\mathcal{J}$-holomorphic pair is smooth if the embedding and the almost generalized complex structure is smooth.

## 7.4. Critical Points, Somewhere Injective Curves and Simple Curves

As mentioned at the end of section 7.2, this one is devoted to critical points, simple curves and somewhere injective curves in the generalized complex context. Let $\lambda : TM \to E$ be a fixed smooth isotropic embedding into an exact Courant algebroid $(E, q, [\cdot,\cdot], \pi)$. By theorem 6.4.1 it is true that $\lambda = \Lambda \circ s$ for an orthogonal automorphism $\Lambda$ of $E$ with respect to $q$ and a smooth isotropic embedding $s$. This $\Lambda$ can be absorbed into $\mathcal{J}$. Using proposition 6.4.4 we are allowed to assume without loss of generality that $\lambda = \iota$ and $E = \mathbb{T}M$, i.e. we can restrict ourselves to $\mathcal{J}$-holomorphic curves.

At the beginning we will show that $\mathcal{J}$-holomorphic curves respect the Carleman similarity principle. Thereafter we will define the relevant objects in definition 7.4.2. After that we will show that the natural extensions of the corresponding results in the theory of ordinary pseudoholomorphic curves, which can be found in [MS04], are also true in the generalized setting. In particular we will show that the set of preimages of a critical value is finite and, moreover, $\Phi^{-1}(p)$ is also finite. Then we will show the main theorem of the local theory of $(E, \mathcal{J})$-holomorphic curves. It states that $(E, \mathcal{J})$-holomorphic curves locally look like ordinary $J$-holomorphic curves in a space of doubled dimension. This implies the following immediately. Two $\mathcal{J}$-holomorphic curves which coincide at a sequence whose limit is 0 and at 0 itself are connected by a holomorphic map. The last result concerning critical points will show that two non-constant $\mathcal{J}$-holomorphic curves can only intersect in at most countably many points and those can only accumulate at the critical values. Then we will turn to somewhere injective curves and simple curves. In doing so we will prove that a simple $\mathcal{J}$-holomorphic curve is somewhere injective and that the set of non-injective points of $\Phi$ can only accumulate at its critical points. At the end we will give two corollaries of this fact.

One important tool in the proof of these results has been established in the last section. It is the elliptic regularity theorem. Another one is the Carleman similarity principle,

**Theorem 7.4.1.** *Let $p > 2$, $C \in L^p(B_\epsilon, \mathbb{R}^{2n\times 2n})$ and $J \in W^{1,p}(B_\epsilon, \mathbb{R}^{2n\times 2n})$ such that $J^2 = -\mathbb{1}$. Suppose that $\phi \in W^{1,p}(B_\epsilon, \mathbb{R}^{2n})$ is a solution of*

$$\partial_s \phi(z) + J(z)\partial_t \phi(z) + C(z)\phi(z) = 0\,, \quad z = s + \mathrm{i}t \tag{7.4.1}$$

*such that $\phi(0) = 0$. Then there is a $\delta \in ]0, \epsilon[$, a map $\Xi \in W^{1,p}(B_\delta, \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{R}^{2n}))$ and a holomorphic map $f : B_\delta \to \mathbb{C}^n$ such that $\Xi(z)$ is invertible and*

$$\phi(z) = \Xi(z) f(z)\,, \quad f(0) = 0\,, \quad \Xi(z)^{-1} J(z)\Xi(z) = \mathrm{i} \tag{7.4.2}$$

*for every $z \in B_\delta$.*

*Proof.* This is theorem 2.3.5. in [MS04]. In their notation $\phi = u$, $\Xi = \Phi$ and $f = \sigma$. □

The significance of the Carleman similarity principle for the present work is the following. Let $(\Phi, \lambda)$ be a $\mathcal{J}$-holomorphic pair. As mentioned above we are able to choose without loss of generality $\lambda = \iota$. Then $\Phi$ solves equation (6.3.1). In local coordinates $\Phi$ can be represented by a map $\phi$ and we saw in the proof of theorem 7.3.1 that $\mathcal{J}$ induces an almost generalized complex structure $\mathcal{J}$ on $\mathbb{R}^{4n}$, which is in particular almost complex. Then it is true that $\phi \oplus 0$ fulfills equation (7.4.1) for $n$ even, $J(z) = \mathcal{J}(\phi(z))$ and $C = 0$. Hence, $\mathcal{J}$-holomorphic curves (and thus also pairs) fulfill the Carleman similarity principle and the local analytic behavior of $\Phi$ is dictated by the holomorphic map $f$.

Some notions which are important in the local study of $\mathcal{J}$-holomorphic pairs are those which can be found in

**Definition 7.4.2.** *Let $(\Sigma, j_\Sigma)$ be a compact Riemann surface, $M$ be a smooth $2n$-dimensional manifold, $(E, q, [\cdot,\cdot], \pi)$ be an exact Courant algebroid over $M$, $\mathcal{J}$ be an almost generalized complex structure on $E$, $\lambda : TM \to E$ be a smooth isotropic embedding and $\Phi : \Sigma \to M$ be such that $(\Phi, \lambda)$ is an $(E, \mathcal{J})$-holomorphic pair. Then we call*

1. *$\sigma \in \Sigma$ a critical point of $(\Phi, \lambda)$ iff $T_z\Phi = 0$. The image of a critical point is called a critical value.*
2. *$(\Phi, \lambda)$ multiply covered if there exists a compact Riemann surface $(\Sigma', j')$, a map $\Phi' : \Sigma' \to M$ and a holomorphic branched covering $\chi : \Sigma \to \Sigma'$ such that $(\Phi', \lambda)$ is a $\mathcal{J}$-holomorphic pair and*

   $$\Phi = \Phi' \circ \chi\,, \quad \deg(\chi) > 1\,. \tag{7.4.3}$$

   *The pair is called simple if it is not multiply covered.*
3. *$(\Phi, \lambda)$ somewhere injective if $\Phi : \Sigma \to M$ is somewhere injective, i.e. if there exists a $\sigma \in \Sigma$ such that*

   $$T_\sigma\Phi \neq 0\,, \quad \phi^{-1}(\Phi(\sigma)) = \{\sigma\}\,. \tag{7.4.4}$$

   *A point with this property is named an injective point.*

After having defined critical values, simple and somewhere injective points, we should turn to the examination of these objects. A first question which arises in this context is how many preimages a critical value has. The answer will tell us that there are only finitely many and in particular we will show

**Lemma 7.4.3.** *Let $(\Sigma, j_\Sigma)$ be a compact Riemann surface, $M$ be a smooth $2n$-dimensional manifold, $(E, q, [\cdot,\cdot], \pi)$ be an exact Courant algebroid over $M$, $\mathcal{J}$ be an almost generalized complex structure on $E$, $\lambda : TM \to E$ be a smooth isotropic embedding and $\Phi : \Sigma \to M$ be such that $(\Phi, \lambda)$ is an $(E, \mathcal{J})$-holomorphic pair. Moreover, let $\mathcal{J}$ be $\mathcal{C}^1$ and $\Phi$ be nonconstant. Then the set*

$$X := \Phi^{-1}\left(\{\Phi(\sigma) | \sigma \in \Sigma\,,\, T_z\Phi = 0\}\right) \tag{7.4.5}$$

*of preimages of critical values is finite. Furthermore, $\Phi^{-1}(z)$ is a finite set for every $p \in M$.*

*Proof.* First, we will reduce the assertion from generalized $(E, \mathcal{J})$-holomorphic pairs to the case of generalized pseudoholomorphic curves. Afterwards we will proof the lemma using the Carleman similarity principle.

Let $s$ be a smooth isotropic splitting of $E$. By theorem 6.4.1 it follows that there exits a smooth orthogonal transformation $\Lambda$ such that $\Lambda \in \mathcal{C}^\infty$ and $\lambda = \Lambda \circ s$. Therefore, $\mathcal{J}' := \Lambda^{-1} \circ \mathcal{J} \circ \Lambda$ is an almost generalized complex structure which is as smooth as $\mathcal{J}$. Proposition 6.4.3 shows that $(\Phi, \lambda)$ is an $(E, \mathcal{J})$-holomorphic pair if and only if $\Phi$ is a $(E, \mathcal{J}')$-holomorphic curve. If we denote the almost generalized complex structure on $\mathbb{T}M$ induced by $\mathcal{J}'$ by $\mathcal{J}'$, too, proposition 6.4.4 shows that $\Phi$ is a $\mathcal{J}'$-holomorphic curve. Hence, we are left to prove the lemma for the latter objects.

Since $\Sigma$ is supposed to be a compact Riemann surface, we need to show that critical points are isolated. Thus we may work locally without loss of generality.

The proof of theorem 7.3.1 shows that the coordinate maps $\xi_M$ on $M$ induce a push-forward of the almost generalized complex structure $\mathcal{J}'$ on $\mathbb{R}^{4n}$. This structure is also almost generalized complex, but on $\mathbb{R}^{4n}$. Let us slightly abuse notation and denote this induced structure by $\mathcal{J}' : \mathbb{R}^{2n} \to \mathrm{GL}(4n, \mathbb{R})$.

Assume that $\mathcal{J}'$ is of class $\mathcal{C}^1$, $\Omega \subset \mathbb{C}$ and $\phi : \Omega \to \mathbb{R}^{2n}$ is a locally $\mathcal{J}'$-holomorphic curve. Using theorem 7.3.1 it follows that $\phi \oplus 0 : \Omega \to \mathbb{R}^{4n}$ is of class $W^{2,p}$ for every $p < \infty$. Then $\phi \oplus 0$ satisfies theorem 7.4.1 for $J(z) = \mathcal{J}' \circ \phi(z)$ and $C(z) = 0$. Obviously, $J(z)$ is of class $\mathcal{C}^1$. This implies that $\phi \oplus 0$ is the composition of an invertible map $\Xi$ and a holomorphic map $f$. Since zeros of holomorphic maps are isolated it follows that $\phi^{-1}(p)$ is discrete in $M$ for every $p \in M$. As $M$ is compact, we infer that $\phi^{-1}(p)$ is finite. If we differentiate equation (7.4.1) for $C = 0$ with respect to $s$ and define $\psi := \partial_s\phi \oplus 0 \in W^{1,p}(\Omega, \mathbb{R}^{4n})$ we obtain

$$\partial_s\psi(z) + J(z)\partial_t\psi + (\partial_s J(z))\psi(z) = 0\,. \tag{7.4.6}$$

It is possible to use theorem 7.4.1 again. The same argument as above shows that zeros of $\psi$ are finite. Differentiating (7.4.1) with respect to $t$ yields that the zeros of $\partial_t\phi$ are finite. Hence, it follows that the set of critical points of $\Phi$ is finite. □

Up to now we were able to prove our lemmas, propositions and theorems more or less directly. We saw that the local behavior is exactly the same as we would expect from the theory of ordinary $J$-holomorphic curves. We observed that locally $\mathcal{J}$-holomorphic curves fulfill a generalized nonlinear Cauchy-Riemann equation, which looks like the ordinary one in a space which has doubled dimension and the second half of the coordinates set to a constant. We will call this the local doubling trick. In the ordinary theory of pseudoholomorphic curves one proves the analogue of lemma 7.4.8 by showing that an almost complex structure looks locally, after some diffeomorphism, as the standard complex structure on $\mathbb{C}^n$. We can not expect that in the generalized context. In the best case we are allowed to hope that this would hold up to diffeomorphisms and a choice of a splitting of $E$. But it is possible to resolve this problem using the fact that the convexity radius in a (precompact) finite dimensional manifold is finite [Whi32]. At the same time we get a precise statement of the above mentioned local doubling trick. It is

**Theorem 7.4.4.** *Let $M$ be a $2n$-dimensional manifold, $(E, q, [\cdot,\cdot], \pi)$ be an exact Courant algebroid, $\mathcal{J}$ be a $\mathcal{C}^l$ almost generalized complex structure on $E$, $\lambda : TM \to E$ be a smooth isotropic embedding with respect to $q$ and $l > 1$. Furthermore, let $(\Sigma, j_\Sigma)$ be a Riemann surface and $\Phi : \Sigma \to M$ be such that $(\Phi, \lambda)$ is an $(E, \mathcal{J})$-holomorphic pair. Then for every $\sigma \in \Sigma$ there exist neighborhoods $\Omega \subset \Sigma$ of $\sigma$ and $U \subset M$ of $\Phi(\sigma)$, and an almost complex structure $J$ on $U \times U$ of class $\mathcal{C}^l$ such that $(\Phi, p_0) : \Omega \to U \times U$ is a (local) $J$-holomorphic curve for any fixed $p_0 \in U$.*

*Proof.* Let $M$, $\mathcal{J}$, $\Sigma$, $j_\Sigma$, $\Phi$, $\lambda$ and $\sigma$ be as in the assertion. We will construct $U$ and $J$ explicitly. Theorem 6.4.1 and propositions 6.4.3, 6.4.4 show that $(\Phi, \lambda)$ is an $(E, \mathcal{J})$-holomorphic pair if and only if $\Phi$ is a $\Lambda^{-1} \circ \mathcal{J} \circ \Lambda$-holomorphic curve. Let us denote the generalized complex structure on $\mathbb{T}M$ being induced by $\Lambda^{-1} \circ \mathcal{J} \circ \Lambda$ as $\mathcal{J}'$. The idea of this proof is to use $\mathcal{J}'$ to define an almost complex structure on $T(U \times U)$. It is true that $\mathcal{J}'$ maps $TM \oplus T^*M$ onto and into itself. In order to get an almost complex structure on some subset of $M \times M$, we first have to map $T^*M$ to $TM$. This can be done by a Riemannian metric, which every smooth manifold admits.

Let $g$ be an arbitrary smooth Riemannian metric on $M$. Then it is true that

$$\mathcal{J}'' := \begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix} \mathcal{J}' \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} \tag{7.4.7}$$

is a map $\mathcal{J}'' : TM \oplus TM \to TM \oplus TM$. Recall that $T_{(p_1,p_2)}(M \times M) = T_{p_1}M \oplus T_{p_2}M$. Equation (7.4.7) only yields an almost complex structure on the diagonal of $M \times M$. We will use parallel transport along geodesics in $M$ with respect to $g$ in order to extend $\mathcal{J}''$ to $U \times U$ for some open neighborhood $U$ being specified in a moment.

As we wish to get a well defined map from $T_{p_2}M \to T_{p_1}M$, we have to ensure that there is exactly one geodesic of specific type connecting $p_2$ and $p_1$. This can be achieved by looking at a geodesically convex neighborhood of $\Phi(\sigma)$. Its existence has been shown in [Whi32]. Whitehead calls it a simple convex region. A geodesically convex neighborhood $U$ of $\Phi(\sigma)$ is a neighborhood of $\Phi(\sigma)$ such that every pair of points $p_1$, $p_2 \in U$ can be connected along exactly one length minimizing geodesic $\gamma$ which does not leave $U$. Hence, let us choose length minimizing geodesics as the family of curves used in the above mentioned parallel transport.

Let $U$ be a geodesically convex neighborhood of $\Phi(\sigma)$ and $p_1$, $p_2 \in U$. Furthermore, let $\mathbb{P}_{p_2 \to p_1}$ denote the parallel transport from $T_{p_2}M$ to $T_{p_1}M$ along the uniquely defined length minimizing geodesic starting at $p_2$ and ending at $p_1$. For $X_1 \in T_{p_1}M$ and $X_2 \in T_{p_2}M$ define

$$J \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ 0 & (\mathbb{P}_{p_2 \to p_1})^{-1} \end{pmatrix} \mathcal{J}'' \begin{pmatrix} 1 & 0 \\ 0 & \mathbb{P}_{p_2 \to p_1} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} . \tag{7.4.8}$$

Obviously it is true that $J^2 = -\mathbb{1}$. If $\mathbb{P}_{p_2 \to p_1}$ is smooth in every argument, it follows that $J$ is also of class $\mathcal{C}^l$, like $\mathcal{J}$. Thus we next show that $\mathbb{P}_{p_2 \to p_1}$ is smooth.

We will first argue that length minimizing geodesics connecting $p_2$ and $p_1$ depend smoothly on $p_1$ and $p_2$. Then we will use this to show that parallel transport along length minimizing geodesics is a smooth map.

It is well known that given a metric $g$ the exponential map $\exp_p(X)$ with respect to $\nabla$ is smooth in $p$ and $X$. Hence, $\gamma(t) := \exp_{p_2}(tX)$ is a smooth geodesic starting at $p_2$ which depends smoothly on $p_2$ and $X$. Now let $p_1, p_2 \in U$ and $\gamma$ denote the length minimizing geodesic connecting $p_2$ and $p_1$ in $U$. Then it is true that $\gamma(t) = \exp_{p_2}(t\dot{\gamma}(0))$. Because of the fact that the exponential map is a local diffeomorphism, it follows that $\exp_{p_2}$ maps a neighborhood of $0$ in $T_{p_2}M$ smoothly to a neighborhood of $p_1$ in $M$. Hence, the connecting geodesic from $p_2$ to $p_1$ depends smoothly on $p_2$ and $p_1$. Furthermore, it is well known that parallel transport of a vector along a curve depends smoothly on the initial conditions. Therefore, parallel transport along the curve depends smoothly on $p_2$ and $p_1$, too. Hence, equation (7.4.8) defines an almost complex structure $J$ on $U \times U$ of class $\mathcal{C}^l$.

Finally, we have to show that there is a neighborhood $\Omega$ of $\sigma$ such that $\Phi \times p_0 : \Omega \to U \times U$ is a local $J$-holomorphic curve for every fixed $p_0 \in U$.

It is true that $\Phi(\Sigma)$ is a subset of $M$. As such it can be equipped with the relative topology induced by the given topology of $M$. Then open sets $W$ in $\Phi(\Sigma)$ are exactly those which are intersections with open sets in $M$, i.e. $W \cap V$ for some open set $V \subset M$. Therefore, $U \cap \Phi(\Sigma)$ is open in the relative topology on $\Phi(\Sigma)$.

By theorem 7.3.1 it follows that $\Phi$ is of class $W^{l+1,p}$ for every $p < \infty$. Using the Sobolev embedding theorem this yields $\Phi$ being $\mathcal{C}^l$. Therefore, it is in particular continuous. It is also true that $\Phi : \Sigma \to M$ is continuous if and only if $\Phi : \Sigma \to \Phi(\Sigma)$ is continuous in the relative topology. Hence, $\Omega := \Phi^{-1}(U \cap \Phi(\Sigma))$ is open as the preimage of an open set. Since $\Phi(\sigma) \in U$ and $\Phi(\sigma) \in \Phi(\Sigma)$, we infer that $\Omega$ is a neighborhood of $\sigma$.

Finally, we have to check whether $\Phi \times p$ is a local $J$-holomorphic curve for every $p \in U$ if $\Phi$ is a local $\mathcal{J}$-holomorphic curve. For all $\sigma \in \Omega$ it holds

$$\begin{aligned} J \circ T(\Phi \times p_0) &= \begin{pmatrix} 1 & 0 \\ 0 & \mathbb{P}_{\Phi(\sigma)\to p_0} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix} \mathcal{J}' \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \mathbb{P}_{p_0\to\Phi(\sigma)} \end{pmatrix} \begin{pmatrix} T\Phi \\ 0_{p_0} \end{pmatrix} = \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \mathbb{P}_{\Phi(\sigma)\to p_0} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix} \mathcal{J}' \begin{pmatrix} T\Phi \\ 0^*_{\Phi(\sigma)} \end{pmatrix} = \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \mathbb{P}_{\Phi(\sigma)\to p_0} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} T\Phi \circ j_\Sigma \\ 0^*_{\Phi(\sigma)} \end{pmatrix} = \\ &= \begin{pmatrix} T\Phi \circ j_\Sigma \\ 0_{p_0} \end{pmatrix} = \begin{pmatrix} T\Phi \circ j_\Sigma \\ Tp_0 \circ j_\Sigma \end{pmatrix} = T(\Phi \times p_0) \circ j_\Sigma \,, \end{aligned} \tag{7.4.9}$$

where $0^*_{\Phi(\sigma)}$ denotes the zero co-vector at $\Phi(0)$. Hence, $\Phi \times p_0 : \Omega \to U \times U$ is a local $J$-holomorphic curve.

□

**Remarks 7.4.5**

1. The above construction can be generalized to connected geodesically convex manifolds. There it is globally true that two points can be connected by exactly one length minimizing geodesic. A trivial example is for instance $\mathbb{R}^n$. A counter example is the sphere $S^2$. If one looks at antipodal points $p_1$, $p_2$, there are infinitely many length minimizing geodesics which connect $p_1$ and $p_2$.
2. If the manifold is flat, the above construction can also be applied in some sense. If $M$ is flat, it follows that there is trivial holonomy which in turn shows that parallel transport along a curve is independent of the curve chosen, as long as one ensures, that the curve induces trivial monodromy.
3. If we act with $J$ on $T\Phi \times 0$, the parallel transport maps the zero at an arbitrary point to zero at $\Phi(\sigma)$. Therefore, it seems that it might be possible to use any curve for parallel transport. Then one has to take care of monodromy effects. Furthermore, it only defines an almost complex structure on $U \times p$. But as we wish to use the methods of pseudoholomorphic curves, we need an almost complex structure in a proper neighborhood of $\Phi(\sigma)$. The above construction achieves this.

4. In algebraic topology one uses the existence of a geodesically convex neighborhood of a point in a manifold to show that a compact (finite dimensional) manifold obeys a "good covering",i.e. the coordinate charts are contractible.

An immediate corollary of above theorem and remarks is

**Corollary 7.4.6.** *Let $M$ be a $2n$-dimensional smooth manifold, $(E, q, [\cdot,\cdot], \pi)$ be an exact Courant algebroid, $\mathcal{J}$ be a $\mathcal{C}^l$ almost generalized complex structure on $E$, $\lambda : TM \to E$ be a smooth isotropic embedding with respect to $q$ and $l > 1$. Furthermore, let $(\Sigma, j_\Sigma)$ be a Riemann surface and $\Phi : \Sigma \to M$ be such that $(\Phi, \lambda)$ is an $(E, \mathcal{J})$-holomorphic pair. If in addition $M$ is geodesically convex for some arbitrary smooth metric $g$ on $M$, it follows that there exists an almost complex structure $J$ of class $\mathcal{C}^l$ on $M \times M$ with the property that $\Phi \times p$ for any $p \in M$ is a $J$-holomorphic curve if and only if $(\Phi, \lambda)$ is a $\mathcal{J}$-holomorphic pair.*

*Proof.* The crucial point in the proof is the definition of the almost complex structure $J$. If $M$ is geodesically convex, we can adopt the proof of theorem 7.4.4. Since parallel transport depends smoothly on the initial conditions it follows that $J$ is also of class $\mathcal{C}^l$. □

**Remark 7.4.7** This corollary reduces the theory of $\mathcal{J}$-holomorphic pairs on a geodesically convex manifold to $J$-holomorphic curves on $M \times M$. At least as long as we are not interested in topological invariants. This tells us that the properties of solutions to equation (6.3.1) can be easily extracted from those of $J$-holomorphic curves. If one looks at deformations, one has to be cautious in order to ensure that the second component of $\Phi \times p$ stays constant. It is possible to show that this problem is connected to admissible vector fields along $\Phi$, confer definition 8.1.6.

Another important technical lemma which we need in the local theory of $\mathcal{J}$-holomorphic pairs is

**Lemma 7.4.8.** *Let $M$ be a $2n$-dimensional smooth manifold, $(E, q, [\cdot,\cdot], \pi)$ be an exact Courant algebroid, $\mathcal{J}$ be a $\mathcal{C}^2$ almost generalized complex structure on $E$ and $\lambda : TM \to E$ be a smooth isotropic embedding with respect to $q$. Moreover, let $\Omega \subset \Sigma$ be an open neighborhood of $\sigma \in \Sigma$ and $\Phi, \Psi : \Omega \to M$ be $(E, \mathcal{J})$-holomorphic curves with respect to the same $\lambda$ such that*

$$\Phi(\sigma) = \Psi(\sigma)\,, \quad T_\sigma\Phi \neq 0\,. \tag{7.4.10}$$

*Then it is true that the existence of sequences $z_\nu$, $\zeta_\nu \in \Omega$ with*

$$\Phi(z_\nu) = \Psi(\zeta_\nu)\,, \quad \lim_{\nu\to\infty} z_\nu = \lim_{\nu\to\infty} \zeta_\nu = \sigma\,, \quad \zeta_\nu \neq \sigma\,, \tag{7.4.11}$$

*implies the existence of a holomorphic function* $\chi : B_\epsilon(0) \to \xi_\Sigma(\Omega)$ *defined in some neighborhood of* $0$ *such that* $\chi(0) = \xi_\Sigma(\sigma)$ *and locally*

$$\Psi \circ \xi_\Sigma = \Phi \circ \xi_\Sigma \circ \chi\,. \tag{7.4.12}$$

*Proof.* Let $M$, $\mathcal{J}$, $\Sigma$, $j_\Sigma$, $\Phi$, $\lambda$ and $\sigma$ be as in the assertion. As usual we use theorem 6.4.1 and propositions 6.4.3, 6.4.4 to deduce that $(\Phi, \lambda)$ is an $(E, \mathcal{J})$-holomorphic pair if and only if $\Phi$ is a $\Lambda^{-1} \circ \mathcal{J} \circ \Lambda$-holomorphic curve. Let $\mathcal{J}' := \Lambda^{-1} \circ \mathcal{J} \circ \Lambda$.

Theorem 7.4.4 shows that there is a neighborhood $\Omega$ of $\sigma$, a neighborhood $U$ of $\Phi(\sigma)$ and an almost complex structure $J$ on $U \times U$ such that $\Phi \times p$ is an ordinary pseudo-holomorphic curve in $U \times U$. Suppose without loss of generality that the coordinate function $\xi_\Sigma$ acts on $\Omega$. Otherwise shrink $\Omega$. Furthermore, we can assume without loss of generality $\xi_\Sigma(\sigma) = 0$. Thus the assertion follows from the known case of pseudo-holomorphic curves. We will give the details until the proofs are exactly the same as in the ordinary theory.

First, we will show that there are nice coordinates of $U \times U$ around $\Phi(\sigma) \times p$. After that we will prove the assertion by using Carleman's similarity principle.

As in the proof of theorem 7.4.4 we deduce that $\Phi$ is of class $\mathcal{C}^l$. Let $z = s + \mathrm{i}t \in \xi_\Sigma(\Omega)$ and $w = (w_1, w_2, \ldots, w_{2n}) \in \mathbb{C}^{2n}$ where $w_j = x_j + \mathrm{i}y_j$. Take a complex $\mathcal{C}^{l-1}$ frame $Z_1$, $Z_2$, ..., $Z_{2n}$ of $(\Phi \times p)^* T(U \times U)_\mathbb{C}|_\Omega$ such that

$$Z_1(\sigma), \ldots, Z_{2n}(\sigma) \in T_{\Phi(\sigma)\times p}(U \times U)_\mathbb{C}\,, \qquad Z_1 = \frac{\partial\left((\Phi \times p) \circ \xi_\Sigma^{-1}\right)}{\partial s}\,. \tag{7.4.13}$$

Here we identified sections in the pullback-bundle $(\Phi \times p)^* TM$ with vector fields along $\Phi \times p$, cp. lemma 4.2.1. Define $f : \xi_\Sigma(\Omega) \times \mathbb{C}^{2n-1} \to M$ by

$$\begin{aligned} f(w_1, \ldots, w_n) := \exp_{(\Phi\times p)\circ\xi_\Sigma^{-1}(w_1)} \Bigg( &\sum_{j=2}^{2n} x_j \left(Z_j \circ \xi_\Sigma^{-1}\right)(w_j) + \\ &+ \sum_{j=2}^{2n} y_j J\left((\Phi \times p) \circ \xi_\Sigma^{-1}(w_1)\right)\left(Z_j \circ \xi_\Sigma^{-1}\right)(w_j) \Bigg)\,, \end{aligned} \tag{7.4.14}$$

where the exponential map is taken with respect to the Lie derivative. Since $\xi_\Sigma$ is a diffeomorphism, it follows that $f$ is a $\mathcal{C}^{l-1}$ diffeomorphism of a neighborhood $V \ni 0$ in $\mathbb{C}^{2n}$ onto a neighborhood $U' \ni \Phi(\sigma) \times p$ in $M$. It fulfills $f(z, 0, \ldots, 0) = (\Phi \times p) \circ \xi_\Sigma^{-1}(z)$ and

$$\frac{\partial f}{\partial x_j} + J(f)\frac{\partial f}{\partial y_j} = 0\,, \quad j = 1, \ldots, 2n\,, \tag{7.4.15}$$

at all points of the form $(z_1, 0, \dots, 0)$. Its inverse $\xi_{U\times U} := f^{-1}$ satisfies

$$\xi_{U\times U} \circ (\Phi \times p) \circ \xi_\Sigma^{-1}(z) = (z, 0, \dots, 0) \tag{7.4.16}$$

and

$$T_{(\Phi\times p)\circ\xi_\Sigma^{-1}(z)}\xi_{U\times U} J = J_0 T_{(\Phi\times p)\circ\xi_\Sigma^{-1}(z)}\xi_{U\times U}\,, \tag{7.4.17}$$

where

$$J_0 = \begin{pmatrix} 0 & \mathbb{1}_{2n\times 2n} \\ -\mathbb{1}_{2n\times 2n} & 0\,. \end{pmatrix} \tag{7.4.18}$$

The last property used the inverse function theorem $T_{f(z)}(f^{-1}) = (T_z f)^{-1}$ for $f$ being regular, the equivalence $T_{f(z)}(f^{-1})J = J_0 T_{f(z)}(f^{-1}) \Leftrightarrow -J(T_{f(z)}(f^{-1}))^{-1} = -T_{f(z)}f^{-1}J_0$ and equation (7.4.15).

Since the assertion of lemma 7.4.8 is a local property, the above consideration shows that we are allowed to assume without loss of generality that $U = \mathbb{C}^n$, $J : \mathbb{C}^{2n} \to \mathrm{GL}(4n, \mathbb{R})$ is a $\mathcal{C}^l$ almost complex structure, and

$$u(z) := \xi_{U\times U} \circ (\Phi \times p) \circ \xi_\Sigma^{-1}(z) = (z, 0)\,, \quad J(w_1, 0) = \mathrm{i}\,, \tag{7.4.19}$$

where $w = (w_1, \tilde{w})$ with $\tilde{w} \in \mathbb{C}^{2n-1}$. Theorem 7.4.4 shows that $u$ is a local $J$-holomorphic curve in $U \times U$. The proof of Lemma 2.4.3. in [MS04] implies that $\Psi \times p = (\Phi \times p) \circ \chi = (\Phi \circ \chi) \times p$ for some $\chi$ having the required properties. There they used the Carleman similarity principle and constructed $C$ explicitly by integrating the derivative of $J$ along a line of the form $(v, \tau\tilde{v})$. ☐

If we look at the proofs in the local theory of $J$-holomorphic curves, e.g. concerning critical points and the property of a curve being somewhere injective, it becomes clear that the above statements are enough to apply the known proofs for the following statements word by word. The first one considers the set of intersections of essentially different $\mathcal{J}$-holomorphic pairs.

**Proposition 7.4.9.** *Let $M$ be a $2n$-dimensional smooth manifold, $(E, q, [\cdot,\cdot], \pi)$ be an exact Courant algebroid, $\mathcal{J}$ be a $\mathcal{C}^2$ almost generalized complex structure on $E$ and $\lambda : TM \to E$ be a smooth isotropic embedding with respect to $q$. Furthermore, let $\Sigma_0$, $\Sigma_1$ be compact connected Riemann surfaces without boundary. Suppose $\Phi_0 : \Sigma_0 \to M$ and $\Phi_1 : \Sigma_1 \to M$ are such that $(\Phi_0, \lambda)$ and $(\Phi_1, \lambda)$ are $(E, \mathcal{J})$-holomorphic pairs where $\Phi_0(\Sigma_0) \neq \Phi_1(\Sigma_1)$ and $\Phi_0$ is nonconstant. Then the set ${\Phi_0}^{-1}(\Phi_1(\Sigma_1))$ is at most countable and can accumulate only at the critical points of $\Phi_0$.*

*Proof.* The assertion follows from theorem 6.4.1, propositions 6.4.3, 6.4.4, theorem 7.4.4, lemma 7.4.3 and the proof of proposition 2.4.4. in [MS04]. First, it is clear by lemma 7.4.3 that the assertion is trivial for $\Phi_1$ being constant. Hence, assume $\Phi_1$ is not constant. Let $X_0 \subset \Sigma_0$ and $X_1 \subset \Sigma_1$ denote the sets of critical points of $\Phi_0$ and $\Phi_1$, respectively. It is possible to use lemma 7.4.8 to show that for all $z_0 \in \Sigma_0 \setminus X_0$ the following assertions are equivalent.

1. There exists a neighborhood $U_0 \subset \Sigma_0$ of $z_0$ such that $\Phi_0(U_0) \subset \Phi_1(\Sigma_1)$.
2. There exists a sequence $z_\nu \in \Phi_0^{-1}(\Phi_1(\Sigma)) \setminus \{z_0\}$ that converges to $z_0$.

Proving proposition 7.4.9 is equivalent to showing that the set $W_0 \subset \Sigma_0 \setminus X_0$ of all points $z_0$ which satisfy 1 and 2 is empty. By definition this set is open and relatively closed in $\Sigma_0 \setminus X_0$. Assuming $W_0 \neq \emptyset$ implies $W_0 = \Sigma_0 \setminus X_0$ and, hence, by 1 $\Phi_0(\Sigma_0) \subset \Phi_1(\Sigma_1)$. In complete analogy it follows that $\Phi_1(\Sigma_1) \subset \Phi_0(\Sigma_0)$. Thus $\Phi_0(\Sigma_0) = \Phi_1(\Sigma_1)$ which contradicts the assumptions of proposition 7.4.9. ☐

Let us recall the results of this section up to now. After extending the definition of critical values, somewhere injective curves and simple curves to $(E, \mathcal{J})$-holomorphic pairs, we examined the properties of critical values. We proved that the set of preimages of a point $p \in \Phi(\Sigma)$ is finite and that the set of intersection points between two essentially different $\mathcal{J}$-holomorphic pairs, i.e. $\Phi_0(\Sigma_0) \neq \Phi(\Sigma_1)$ for the same $\lambda$, is at most countable and can only accumulate at the critical values. The reason why the local theory of $\mathcal{J}$-holomorphic pairs is the same as those for $J$-holomorphic curves is theorem 7.4.4. It states roughly that the map part of a $\mathcal{J}$-holomorphic pair is locally an ordinary $J$-holomorphic curve in $U \times U \subset M \times M$ for some almost complex structure $J$ on $U \times U$. Next we will give the generalization of the results for ordinary somewhere injective pseudoholomorphic curves to generalized pseudoholomorphic pairs. As we were able to extend important lemmas in the local theory of $J$-holomorphic curves, the following proofs can be taken word by word from [MS04].

Let us denote the set of noninjective points, i.e. the complement of the set of injective points, as

$$Z(\Phi) := \{\sigma \in \Sigma \,|\, T_\sigma\Phi = 0 \vee \#\Phi^{-1}(\Phi(z)) > 1\}. \tag{7.4.20}$$

Then we are able to formulate

**Proposition 7.4.10.** *Let $M$ be a $2n$-dimensional smooth manifold, $(E, q, [\cdot,\cdot], \pi)$ be an exact Courant algebroid, $\mathcal{J}$ be a $\mathcal{C}^2$ almost generalized complex structure on $E$ and $\lambda : TM \to E$ be a smooth isotropic embedding with respect to $q$, $(\Sigma, j_\Sigma)$ be a compact Riemann surface without boundary and $\Phi : \Sigma \to M$ be such that $(\Phi, \lambda)$ is a simple $(E, \mathcal{J})$-holomorphic pair. Then $(\Phi, \lambda)$*

*is somewhere injective. Moreover, the set $Z(\Phi)$ of noninjetive points is at most countable and can only accumulate at the critical points of $\Phi$.*

*Proof.* The assertion follows from theorem 6.4.1, propositions 6.4.3, 6.4.4, lemma 7.4.8 and the proof of proposition 2.5.1. in [MS04]. □

**Remark 7.4.11** This proposition shows that the set of injective points of $\Phi : \Sigma \to M$ is open and dense for every $\mathcal{J}$-holomorphic pair $(\Phi, \lambda)$.

Next we will state a global version of lemma 7.4.8. It is

**Corollary 7.4.12.** *Let $M$ be a $2n$-dimensional smooth manifold, $(E, q, [\cdot, \cdot], \pi)$ be an exact Courant algebroid, $\mathcal{J}$ be a $\mathcal{C}^2$ almost generalized complex structure on $E$ and $\lambda : TM \to E$ be a smooth isotropic embedding with respect to $q$. Furthermore, let $\Sigma_0$, $\Sigma_1$ be compact Riemann surfaces without boundary and $\Phi_j : \Sigma_j \to M$ be maps such that $(\Phi_j, \lambda)$ are simple $(E, \mathcal{J})$-holomorphic pairs with $\Phi_0(\Sigma_0) = \Phi_1(\Sigma_1)$. Then there exists a holomorphic diffeomorphism $\chi : \Sigma_1 \to \Sigma_0$ such that*

$$\Phi_1 = \Phi_0 \circ \chi\,. \tag{7.4.21}$$

*Proof.* As we proved lemma 7.4.3 and proposition 7.4.10 the proof of corollary 2.5.3 of [MS04] also shows corollary 7.4.12. □

**Corollary 7.4.13.** *Let $M$ be a $2n$-dimensional smooth manifold, $(E, q, [\cdot, \cdot], \pi)$ be an exact Courant algebroid, $\mathcal{J}$ be a $\mathcal{C}^2$ almost generalized complex structure on $E$ and $\lambda : TM \to E$ be a smooth isotropic embedding with respect to $q$. Furthermore, let $\Sigma_0, \ldots, \Sigma_N$ be compact Riemann surfaces without boundary and $\Phi_j : \Sigma_j \to M$, $j = 0, \ldots, N$ be such that $(\Phi_j, \lambda)$ are simple $(E, \mathcal{J})$-holomorphic pairs and $\Phi_0(\Sigma_0) \neq \Phi_j(\Sigma_j)$ for $j > 0$. Then, for every $\sigma_0 \in \Sigma_0$ and every open neighborhood $U_0 \subset \Sigma_0$ of $\sigma_0$, there exists an annulus $A_0 \subset U_0$ centered at $\sigma_0$ such that $\Phi_0 : A_0 \to M$ is an embedding, ${\Phi_0}^{-1}(\Phi_0(A_0)) = A_0$ and $\Phi_0(A_0) \cap \Phi_j(\Sigma_j) = \emptyset$.*

*Proof.* Proposition 7.4.10, proposition 7.4.9 and the proof of corollary 2.5.4. in [MS04]. □

We stop here the examination of local properties of $(E, \mathcal{J})$-holomorphic pairs. The next chapter treats part of their deformation theory. We will not be able to give a complete examination but we will get many insights into the global structure of solutions to (6.3.1).

# 8. Deformations of Generalized Pseudoholomorphic Curves

In the last two chapters we established the foundation of the theory of $\mathcal{J}$-holomorphic pairs. We gave the relevant definitions and developed the local theory. This chapter is devoted to the study of deformations of generalized pseudoholomorphic curves. Equation (7.1.2) will play a special role.

In the ordinary theory one uses its analog to define a section $S(\Phi) := (\Phi, \overline{\partial}_J(\Phi))$ in a Banach bundle. Pseudo-holomorphic curves are then the intersection of $S$ with the zero section. In order to prove that the solutions to (6.1.1) form a finite dimensional manifold one uses transversality arguments. An almost complex structure $J$ is called regular if the vertical part of the linearization of $\overline{\partial}_J(\Phi)$ at $J$-holomorphic curves is surjective. It can be shown (cf. for instance [MS04]) that the universal moduli space $\mathcal{M}^*(A, \Sigma, \mathfrak{J}^l)$ of simple curves representing the homology class $A$ is a separable Banach submanifold of $\mathcal{B}^{k,p} \times \mathfrak{J}^l$. Here $\mathcal{B}^{k,p}$ is the Sobolev completion of the Fréchet manifold of all smooth maps $\Sigma \to M$. This can be used to prove that the space of regular almost complex structures is a subset of the second category of $\mathfrak{J} = \mathfrak{J}(M, \omega)$ or $\mathfrak{J} = \mathfrak{J}_\tau(M, \omega)$, i.e. it contains an intersection of countably many open and dense subsets of $\mathfrak{J}$. Since the vertical part of the linearization of $S$ at a pseudoholomorphic curve is a Fredholm operator for a regular almost complex structure, the implicit function theorem shows that the moduli space of $J$-holomorphic curves is a finite dimensional manifold of dimension

$$\dim \mathcal{M}^*(A, \Sigma; J) = 2n(1-g) + \langle A, \mathrm{c}_1(M) \rangle .$$

Unfortunately, the situation is not so simple in the generalized context. We will argue that the vertical linearization of $S$ is the composition of a Fredholm operator $\mathbb{D}_\Phi$ and a semi-Fredholm operator $s$. The latter is only semi-Fredholm since its co-kernel is infinite dimensional in general. Thus, the decomposition is semi-Fredholm, too. Even if we restrict our calculations to the well known case of ordinary $J$-holomorphic curves, the generalized vertical differential is only semi-Fredholm. Moreover, if we call a generalized complex structure regular if its associated vertical differential is surjective, there are no regular structures. We will resolve this issue by introducing admissible vector fields along $\Phi$. They are vector fields inducing deformations inside a generalized distribution of $TM$ in the sense of [Sus73]. We will not give a rigorous proof of the statement that the resulting object solves all problems in deformation theory of $(E, \mathcal{J})$-holomorphic pairs in this work. Instead we will look at the extreme cases of symplectic and complex manifolds and demonstrate how the new formalism reproduces the known results.

In the usual theory one uses the Levi-Civita connection associated with the metric $g$ built from a symplectic form $\omega$ and an $\omega$-tame almost complex structure $J$ to get an explicit formula for the vertical linearization. In exact Courant algebroids there is no notion of a Levi-Civita connection since there is no tensorial torsion. Instead, we will define a torsion operator $T$ on a Courant algebroid, which will not be a tensor. But it will transform as a tensor if we restrict it to an isotropic sub-bundle of $E$. This will be enough to be able to construct the vertical part of $S$ and to show that it is the above mentioned composition.

## 8.1. The Vertical Differential and Admissible Vector Fields

This section is devoted to the construction of the linearization of the vertical part of $S$. To this end let us start with

**Definition 8.1.1.** *Let $(\pi, E, M, q, [\cdot,\cdot])$ be an exact Courant algebroid and $\nabla : \Omega^0(M, E) \to \Omega^1(M, E)$ be a connection on $E$. Then we define the generalized torsion operator $T$ associated with $\nabla$ as*

$$T(A, B) := \nabla_{\pi(A)} B - \nabla_{\pi(B)} A - [A, B] \,. \tag{8.1.1}$$

**Remarks 8.1.2**

1. Since $\nabla$ is a usual connection on the vector bundle $E$, the difference of the covariant derivatives behaves like a usual Lie-bracket. On the other hand, the Courant bracket $[A, B]$ behaves only like a usual Lie-bracket if we restrict $A, B$ to lie inside some almost Dirac structure, cf. appendix A. This is the reason why $T$ is not a tensor on $E$.
2. There is a natural way to generalize the notion of a Levi-Civita connection to exact Courant algebroids. The resulting object is not a connection anymore, but it is a connection up to exact terms. Let us call it the Levi-Civita operator $\nabla^0$. Restricted to any almost Dirac-structure it is a usual connection on that subbundle. Since we will not need it in the following, we will not go into the details. But for seek of completeness, let us write down its expression. By considering

$$\pi(A)G(B, C) - \pi(C)G(A, B) + \pi(B)G(C, A)$$

and demanding that

$$\pi(A)G(B,C) = \left(\nabla^0_{\pi(A)}G\right)(B,C) + G(\nabla^0_{\pi(A)}B,C) + G(B,\nabla^0_{\pi(A)}C)\,, \tag{8.1.2}$$
$$\nabla^0_X G = 0\,, \quad \text{as well as} \tag{8.1.3}$$
$$\nabla^0_{\pi(A)}B - \nabla^0_{\pi(B)}A - [A,B] = 0\,, \tag{8.1.4}$$

a simple calculation shows that

$$\begin{aligned} G(\nabla^0_{\pi(A)}B,C) = \frac{1}{2}\,(&\pi(A)G(B,C) - \pi(C)G(A,B) + \pi(B)G(C,A) - \\ &-G([B,A],C) - G([A,C],B) - G([B,C],A))\,. \end{aligned} \tag{8.1.5}$$

It is clear that there is no connection on an exact Courant algebroid which is torsion free on $E$. The best we could hope for is a connection which is torsion free in some subbundle of $E$. If we choose a splitting $s$ of $E$, it is natural to demand that $\nabla$ is torsion free in $s(TM)$. Furthermore, in order to be able to parameterize the deformations of $\Phi$ by geodesics pointing in the direction of a vector field along $\Phi$, we have to ensure the existence of geodesics with respect to a connection $\nabla$.

Let us next define this connection. Since we are interested in the moduli space of generalized pseudoholomorphic curves with respect to an almost generalized complex structure $\mathcal{J}_2$ which is tamed by $\mathcal{J}_1$, we are aware of a metric $G$. It is given in equation (6.3.8). This metric can be used to give a metric on $TM$ via $g = s^*G$, i.e.

$$g(X,Y) := G(s(X),s(Y))\,. \tag{8.1.6}$$

Now let $\nabla$ be the Levi-Civita connection associated withfd $g$. Observing $E = s(TM) \oplus \pi^*(T^*M) \cong TM \oplus T^*M$, where the bracket is a $H$-twisted version of the standard Dorfman-bracket, let us define

$$\nabla_X A = \nabla_X(s(Y) + \pi^*(\xi)) := s(\nabla_X Y) + \pi^*(\nabla^*_X \xi) + \frac{1}{2}\pi^*(i_X i_Y H)\,. \tag{8.1.7}$$

We call $\nabla$ the generalized Levi-Civita connection. Recall that

$$H(X,Y,Z) = s^*([s(X),s(Y)])(Z)\,. \tag{8.1.8}$$

Some of the properties of $\nabla$ can be found in

**Proposition 8.1.3.** *Let $(E,q,[\cdot,\cdot],\pi))$ be an exact Courant algebroid and $\nabla$ be the operator defined in equation* (8.1.7). *Then $\nabla$ is a connection on $E$ which is torsion free in $s(TM)$ and $\pi^*(T^*M)$. Furthermore, $\gamma : [0,1] \to M$ is a geodesic with respect to $\nabla$ if and only if*

$$\nabla_{\dot\gamma}s(\dot\gamma) = 0\,. \tag{8.1.9}$$

*Proof.* Let us first show that equation (8.1.7) defines a connection on $E$. The following equations are evident:

$$\nabla_{fX}B = f\nabla_X B\,, \tag{8.1.10}$$
$$\nabla_{X+Y}C = \nabla_X C + \nabla_Y C\,, \tag{8.1.11}$$
$$\nabla_X(B+C) = \nabla_X B + \nabla_X C\,. \tag{8.1.12}$$

Using the fact that $\nabla$ and $\nabla^*$ are connections it follows that

$$\begin{aligned}\nabla_X(fB) = s(\nabla_X fY) + \pi^*(\nabla^*_X f\xi) + \frac{1}{2}(i_X i_{fY}H) = s\left(X[f]Y + f\nabla_X Y\right) + \\ + \pi^*(X[f]\xi + f\nabla^*_X\xi) + \frac{1}{2}\pi^*(f i_X i_Y H) = X[f]B + f\nabla_X B\,.\end{aligned} \tag{8.1.13}$$

Hence, $\nabla$ is a connection on $E$, indeed. Since $E \cong TM \oplus T^*M$ and the bracket is mapped to the $H$-twisted Dorfman-bracket, it follows that

$$\begin{aligned}T(s(X),s(Y)) &= \nabla_{\pi\circ s(X)}s(Y) - \nabla_{\pi\circ s(Y)}s(X) - [s(X),s(Y)]_H = \\ &= \nabla_X s(Y) - \nabla_Y s(X) - \pi^*(i_X i_Y H) - s([X,Y]) = \\ &= s(\nabla_X Y - \nabla_Y X - [X,Y]) = 0\,,\end{aligned} \tag{8.1.14}$$

where we used $\nabla$ being torsion free. Moreover, $\pi \circ \pi^* = 0$ yields

$$T(\pi^*(\eta),\pi^*(\xi)) = \nabla_0\eta - \nabla_0\xi - [0\oplus\eta, 0\oplus\xi]_H = 0\,. \tag{8.1.15}$$

Finally, let $\gamma : I \to M$ be a smooth curve in $M$. As $H$ is antisymmetric, it follows that

$$\nabla_{\dot\gamma}s(\dot\gamma) = s(\nabla_{\dot\gamma}\dot\gamma) + \frac{1}{2}\pi^*(i_{\dot\gamma}i_{\dot\gamma}H) = s(\nabla_{\dot\gamma}\dot\gamma)\,. \tag{8.1.16}$$

Hence, $\nabla_{\dot\gamma}s(\dot\gamma) = 0$ if and only if $\gamma$ is a geodesic with respect to $\nabla$. □

In order to be able to either compute the generalized chiral ring in topological string theory or to be able to construct generalized Gromov-Witten invariants, it is important to know the structure of $\mathcal{J}$-holomorphic curves. In particular the question arises whether they form a finite dimensional manifold and how far their moduli space is compact. We will not be able to answer this question in this work, but we will take a step towards an answer of the first question. To this end we have to construct the generalization of the vertical differential $D_\Phi$ from ordinary $J$-holomorphic curves. It is the linearization of the vertical part of $(\Phi, \overline{\partial}_{\mathcal{J}}(\Phi))$, where the vertical directions are induced by a connection $\nabla$ on $E$.

Let us assume that we found the structure of the "moduli space" of solutions to equation (6.3.1). If we take a look at the neighborhood of some $\mathcal{J}$-holomorphic curve $\Phi$, it is given by all small deformations $\Phi'$ of $\Phi$ such that $\Phi'$ is still a solution of (6.3.1). Therefore, we should first think of how to parameterize deformations of $\Phi$.

Let $\Phi_t$ be a smooth family of curves $\Phi_t : \Sigma \to M$. An infinitesimal deformation is given by

$$\frac{d}{dt}\Phi_t(\sigma) = \xi(\sigma) \in T_{\Phi(\sigma)}M \,, \tag{8.1.17}$$

hence, a vector field $\xi$ along $\Phi$.

Since the exponential map is locally a diffeomorphism, we can use geodesics starting at $\Phi(\sigma)$ to generate all small finite deformations out of infinitesimal ones. This gives us smooth deformations of $\Phi$. A $W^{k,p}$-neighborhood of $\Phi$ in $W^{k,p}(\Sigma, M)$ is diffeomorphic to a $W^{k,p}$-neighborhood of $0$ in $W^{k,p}(\Sigma, \Phi^*TM)$ via the geodesic flow in direction of $\xi$

$$\Phi \mapsto \exp_\Phi(\xi)\,, \exp_\Phi(\xi)(\sigma) := \exp_{\Phi(\sigma)}(\lambda\xi(\sigma))|_{\lambda=1}\,. \tag{8.1.18}$$

Here $\exp_\Phi(\xi)(\sigma)$ is the evaluation at $\lambda = 1$ of the geodesic $\gamma(\lambda)$ starting at $\Phi(\sigma)$ with $\dot\gamma(0) = \xi(\sigma)$. We wish to find deformations which are still $\mathcal{J}$-holomorphic curves. Therefore, we have to search for vector fields $\xi$ along $\Phi$ such that $\overline{\partial}_{\mathcal{J}}(\exp_\Phi(\xi)) = 0$. To be able to use transversality arguments, we have to transport $\overline{\partial}_{\mathcal{J}}(\exp_\Phi(\xi))$, which is an element of $\Omega^{(0,1)}(\Sigma, \exp_\Phi(\xi)^*E)$, back to $\Phi$. This can be achieved by pointwise parallel transport, with respect to some connection $\tilde\nabla$, of $\overline{\partial}_{\mathcal{J}}(\exp_\Phi(\xi))(\zeta)$ along the geodesic $\exp_\Phi(\lambda\xi)(\sigma)$. We have to ensure that this connection preserves the almost generalized complex structure $\mathcal{J}$. To this end let us define a new connection

$$\tilde\nabla_X A := \nabla_X A - \frac{1}{2}\mathcal{J}(\nabla_X\mathcal{J})A\,. \tag{8.1.19}$$

It preserves $\mathcal{J}$,

$$\begin{aligned}(\tilde\nabla_X\mathcal{J})A = \tilde\nabla_X(\mathcal{J}A) - \mathcal{J}\tilde\nabla_X A = (\nabla_X\mathcal{J})A + \mathcal{J}\nabla_X A - \frac{1}{2}(\nabla_X\mathcal{J})A - \mathcal{J}\nabla_X A -\\ - \frac{1}{2}(\nabla_X\mathcal{J})A = 0\,.\end{aligned} \tag{8.1.20}$$

Let us denote the complex bundle isomorphism given by pointwise parallel transport along $\gamma_\sigma(\lambda) := \Phi_\lambda(\sigma) := \exp_{\Phi(\sigma)}(\lambda\xi(\sigma))$ as $\Psi_\Phi(\xi) : \Phi^*E \to \Phi_\lambda^*E$ (we parallel transport using $\tilde\nabla$ and geodesics are with respect to $\nabla$). Then define an operator $\mathcal{F}_\Phi : \Omega^0(\Sigma, \Phi^*TM) \to \Omega^{(0,1)}(\Sigma, \Phi^*E)$ via

$$\mathcal{F}_\Phi(\xi) := \Psi_\Phi(\xi)^{-1} \circ \overline{\partial}_{\mathcal{J}}(\exp_\Phi(\xi))\,. \tag{8.1.21}$$

This map is precisely the vertical part of $\overline{\partial}_{\mathcal{J}}(\Phi)$ induced by $\tilde{\nabla}$. The following argument shows that it is an anti-holomorphic one form, indeed.

The two assertions "$\mathcal{F}_\Phi(\xi)$ is an anti-holomorphic one form" and "$\mathcal{J} \circ \mathcal{F}_\Phi(\xi) \circ j_\Sigma = \mathcal{F}_\Phi(\xi)$" are obviously equivalent. Since $\mathcal{J} \circ \overline{\partial}_{\mathcal{J}}(\Phi) \circ j_\Sigma = \overline{\partial}_{\mathcal{J}}(\Phi)$, we have to show that $\mathcal{J} \circ \Psi_\Phi(\xi) = \Psi_\Phi(\xi) \circ \mathcal{J}$ for all $\xi$, $\Phi$ and $\mathcal{J}$. The latter statement means that parallel transport has to preserve the $+\,$i-eigenbundle. Let

$$\mathcal{P}_\pm := \frac{1}{2}(1 \mp \mathrm{i}\mathcal{J}) \tag{8.1.22}$$

be the projection from $E$ to the $\pm\,$i-eigenbundle of $\mathcal{J}$. Using $\nabla_X \mathcal{J} = 0$ it follows that

$$\tilde{\nabla}_{\dot\gamma} A = \mathcal{P}_\pm \tilde{\nabla}_{\dot\gamma} X \tag{8.1.23}$$

for any curve $\gamma$ in $M$. If $A$ is the parallel transport of $A_0$ along $\gamma$, i.e. a solution of $\nabla_{\dot\gamma} A = 0$ with boundary condition $A(0) = A_0$ and $\mathcal{P}_\pm A_0 = A_0$, we deduce that $\mathcal{P}_\pm A$ is a solution of the same boundary value problem. From the theorem of Picard-Lindelöf we infer that $\mathcal{P}_\pm A = A$ for all $t$ which means that parallel transport with respect to $\tilde{\nabla}$ preserves $\mathcal{J}$. Thus, equation (8.1.21) is true and $\mathcal{F}_\Phi(\xi) \in \Omega^{(0,1)}(\Sigma, \Phi^* E)$.

Now we turn to the linearization of $\mathcal{F}_\Phi$ at $0$. If $\Phi$ is a $\mathcal{J}$-holomorphic curve, it is the linearization $D_\Phi$ of $\overline{\partial}_{\mathcal{J}}(\Phi)$ and we call it the generalized vertical differential. For non $\mathcal{J}$-holomorphic curves we use $D_\Phi(\xi) := T\mathcal{F}_\Phi(0)\xi$ as a definition. An explicit formula of $D_\Phi$ can be found in

**Proposition 8.1.4.** *Let $M$ be a smooth $2n$-dimensional manifold, $(E, q, [\cdot,\cdot], \pi)$ be an exact Courant algebroid over $M$, $s$ be a smooth isotropic splitting of $E$, $\mathcal{J}$ be an almost generalized complex structure on $E$ and $\Phi : \Sigma \to M$ be a smooth map. Define the operator*

$$\mathcal{D}_\Phi : \Omega^0(\Sigma, \Phi^* TM) \to \Omega^{0,1}(\Sigma, \Phi^* E) \tag{8.1.24}$$

*by $\mathcal{D}_\Phi(\xi) := T\mathcal{F}_\Phi(0)\xi$. Then*

$$\mathcal{D}_\Phi \xi = \mathbb{D}_\Phi \circ s(\xi) \quad \forall\, \xi \in \Omega^0(\Sigma, \Phi^* TM)\,, \tag{8.1.25}$$

*where*

$$\mathbb{D}_\Phi \mathfrak{X} = \frac{1}{2}\left((\Phi^*\nabla)\mathfrak{X} + \mathcal{J}(\Phi) \circ (\Phi^*\nabla)\mathfrak{X} \circ j_\Sigma\right) - \frac{1}{2}\mathcal{J}(\Phi)(\nabla_{\pi(\mathfrak{X})}\mathcal{J})(\Phi) \circ \partial_{\mathcal{J}}(\Phi) \tag{8.1.26}$$

*and $\partial_{\mathcal{J}}(\Phi) := \frac{1}{2}(s \circ d\Phi - \mathcal{J} \circ s \circ d\Phi \circ j_\Sigma)$ for all $\mathfrak{X} \in \Omega^0(\Sigma, \Phi^* E)$.*

*Proof.* Let us slightly abuse notation and denote sections in $\Phi^*TM$ and vector fields along $\Phi$ by the same symbol $\xi$. Our first step towards a proof of this proposition is to show that

$$\frac{d}{d\lambda}\mathcal{F}_\Phi(\lambda\xi)\Big|_{\lambda=0} = \tilde{\nabla}_\lambda \circ \Psi_\Phi(\lambda\xi) \circ \mathcal{F}_\Phi(\lambda\xi)\Big|_{\lambda=0}\,, \tag{8.1.27}$$

where $\tilde{\nabla}_\lambda$ denotes the (pointwise) connection along the curve $\lambda \longmapsto \exp_{\Phi(\sigma)}(\lambda\xi(\sigma)) = \gamma_\sigma(\lambda)$. For any connection $\nabla_t$ along a smooth curve $\gamma$ it is true that

$$\nabla_t A(t) = \nabla_{\dot\gamma} A(t) = \lim_{h\to 0}\left(\mathbb{P}^\gamma_{t+h,t}(A\circ\gamma) - A\circ\gamma(t)\right)\,. \tag{8.1.28}$$

Here, $\mathbb{P}^\gamma_{t+h,t}(A\circ\gamma)$ denotes the parallel transport of the vector field $A\circ\gamma$ along the curve $\gamma$ with respect to $\nabla$ starting at $\gamma(t)$ and ending at $\gamma(t+h)$. In the case at hand this yields

$$\begin{aligned}
\tilde{\nabla}_\lambda \left(\Psi_{\Phi(\sigma)}(\lambda\xi(\sigma))\mathcal{F}_{\Phi(\sigma)}(\lambda\xi(\sigma))(\zeta)\right)\Big|_{\lambda=0} &= \lim_{h\to 0}\left(\Psi_{\Phi(\sigma)}(h\xi(\sigma))^{-1}\Psi_{\Phi(\sigma)}(h\xi(\sigma))\right. \\
\times\mathcal{F}_{\Phi(\sigma)}(h\xi(\sigma))(\zeta) - \Psi_{\Phi(\sigma)}(0)\mathcal{F}_{\Phi(\sigma)}(0)(\zeta)\big)\big|_{\lambda=0} &= \\
= \lim_{h\to 0}\left(\mathcal{F}_{\Phi(\sigma)}(h\xi(\sigma)) - \mathcal{F}_{\Phi(\sigma)}(0)\right)(\zeta) &= \frac{d}{d\lambda}\mathcal{F}_{\Phi(\sigma)}(\lambda\xi(\sigma))(\zeta)\Big|_{\lambda=0}\,,
\end{aligned} \tag{8.1.29}$$

where $\zeta$ is a vector field along $\gamma_\sigma$. Hence, we are able to simplify the linearization of $\mathcal{F}_\Phi$ at $0$ and evaluated at $\sigma\in\Sigma$,

$$T\mathcal{F}_{\Phi(\sigma)}(0)\xi = \frac{d}{d\lambda}\mathcal{F}_{\Phi(\sigma)}(\lambda\xi(\sigma))\Big|_{\lambda=0} = \tilde{\nabla}_\lambda \circ \overline{\partial}_{\mathcal{J}}\left(\exp_{\Phi(\sigma)}(\lambda\xi(\sigma))\right) = \tag{8.1.30}$$

$$= \tilde{\nabla}_{\dot\gamma_\sigma} \circ \frac{1}{2}\left(s\circ d\Phi_\lambda(\sigma) + \mathcal{J}\circ s\circ d\Phi_\lambda(\sigma)\circ j_\Sigma\right)\Big|_{\lambda=0}\,. \tag{8.1.31}$$

A straight forward computation which uses the fact that $\tilde{\nabla}_{\dot\gamma_\sigma}\mathcal{J} = 0$ and $\dot\gamma_\sigma|_{\lambda=0} = \xi(\sigma)$ yields

$$\begin{aligned}
\tilde{\nabla}_\lambda \circ \overline{\partial}_{\mathcal{J}}\left(\exp_{\Phi(\sigma)}(\lambda\xi(\sigma))\right) &= \frac{1}{2}\left(\nabla_{\dot\gamma_\sigma}\circ s\circ d\Phi_\lambda(\sigma) + \mathcal{J}\circ\nabla_{\dot\gamma_\sigma}\circ s\circ d\Phi_\lambda(\sigma)\circ j_\Sigma\right)\Big|_{\lambda=0} \\
&- \frac{1}{2}\mathcal{J}(\Phi(\sigma))\circ\left(\nabla_{\xi(\sigma)}\mathcal{J}\right)(\Phi(\sigma))\circ\partial_{\mathcal{J}}(\Phi(\sigma))\,.
\end{aligned} \tag{8.1.32}$$

In order to simplify the expression $\nabla_{\dot\gamma_\sigma}\circ s\circ d\Phi_\lambda(\sigma)|_{\lambda=0}$, us show that $[\dot\gamma,\gamma'] = 0$ for any sufficiently smooth $\gamma : U\subset\mathbb{R}^2\to M$ and in particular for $\gamma(t_1,t_2) = \exp_{\Phi(\sigma(t_2))}(t_1\xi)$. For a coordinate chart $(U,\xi_U)$ on $M$ it is true that

$$[\dot\gamma,\gamma'](f) = \dot\gamma(\gamma'(f)) - \gamma'(\dot\gamma(f)) = \left(\frac{\partial x^\nu}{\partial t_1}\frac{\partial}{\partial x^\nu}\frac{\partial\gamma^\mu}{\partial t_2} - \frac{\partial x^\nu}{\partial t_2}\frac{\partial}{\partial x^\nu}\frac{\partial\gamma^\mu}{\partial t_1}\right)\frac{\partial f}{\partial x^\mu} = \tag{8.1.33}$$

$$= \left(\frac{\partial^2(\xi^\mu_M\circ\gamma)}{\partial t_1\partial t_2} - \frac{\partial^2(\xi^\mu_M\circ\gamma)}{\partial t_2\partial t_1}\right)\frac{\partial f}{\partial x^\mu} = 0\,. \tag{8.1.34}$$

Here, $\xi_M^\mu$ is the $\mu$-th component of the coordinate map. Hence, we deduce that

$$[\dot{\gamma}_\sigma, d\Phi_\lambda(\zeta)] = 0\,. \tag{8.1.35}$$

Since $\nabla$ is torsion free in $s(TM)$, it holds

$$(\nabla_{\dot{\gamma}_\sigma} \circ s \circ d\Phi_\lambda)\big|_{\lambda=0}(\zeta) = \nabla_{d\Phi_\lambda(\zeta)} s \circ \dot{\gamma}_\sigma\big|_{\lambda=0} = \nabla_{d\Phi(\zeta)}(s \circ \xi) = (\Phi^*\nabla)_\zeta s \circ \xi\,. \tag{8.1.36}$$

Combining equations (8.1.32) and (8.1.36) yields the desired result. ☐

**Remark 8.1.5** The operator $\mathbb{D}_\Phi$ is a real linear Cauchy-Riemann operator and, hence, Fredholm. On the other hand, $s$ is only semi-Fredholm as its co-kernel is naturally isomorphic to $\pi^*(T^*M)$, which is infinite dimensional.

It seems natural to call an almost generalized complex structure $\mathcal{J}$ regular, if and only if $\mathbb{D}_\Phi$ is surjective at a $\mathcal{J}$-holomorphic curve $\Phi$. If $\mathcal{J} = \mathcal{J}_J$ is associated with an almost complex structure, it is easy to see that $\mathbb{D}_\Phi \circ s = s \circ D_\Phi$, where $D_\Phi$ is the usual vertical differential known from the theory of ordinary $J$-holomorphic curves. This also shows that even in the almost complex case $\mathbb{D}_\Phi \circ s$ is not Fredholm. But the image of $\Omega_0(\Sigma, \Phi^*TM)$ under $\mathbb{D}_\Phi \circ s$ is isomorphic to $\Omega^{0,1}(\Sigma, \Phi^*TM)$ via the projection to $TM$, i.e. the map induced by $\pi : E \to TM$. That implies that $\pi \circ \mathcal{D} = \pi \circ \mathbb{D}_\Phi \circ s = D_\Phi$. This situation is in a naive sense similar to two curves intersecting transversally in a plane but not being transversal to each other in 3-dimensional space.

One of the main difference between pseudoholomorphic and generalized pseudoholomorphic curves lies in the fact that $I$ does not square to $-\mathrm{id}$ in general and $T\Phi$ has to be a map into $\ker B$. However, on $\ker B$, it is $I^2 = -\mathrm{id}$. If we restrict $\xi$ to vector fields along $\Phi$ such that the differential of the induced deformation $\Phi_t$ via the geodesic flow in direction of $\xi$, i.e. $\Phi_t = \exp_\Phi(t\xi)$, is in $\ker B$ for all $t$, we tackled the problem of $T\Phi(T\Sigma) \subseteq \ker B$. Let us call these vector fields admissible.

**Definition 8.1.6.** *Let $(\Sigma, j_\Sigma)$ be a Riemann surface, $(M, g)$ be a $2n$-dimensional compact Riemannian manifold, $(E, q, [\cdot,\cdot], \pi)$ be an exact Courant algebroid over $M$, $s$ be a smooth isotropic splitting of $E$, $\mathcal{J}$ be an almost generalized complex structure on $E$ and $B := s^* \circ \mathcal{J} \circ s$. Furthermore, let $\Phi : \Sigma \to M$ be a map which obeys $T\Phi \in \ker B$ and $\xi \in \Omega^0(\Sigma, \Phi^*TM)$ be a vector field along $\Phi$. Then we call $\xi$ an admissible vector field along $\Phi$, if*

$$\forall \lambda \in [0,1] \,:\, T\exp_\Phi(\lambda\xi)(T\Sigma) \subseteq \ker B\,, \tag{8.1.37}$$

*The set of all admissible $\xi$ along $\Phi$ is denoted as $\mathcal{V}(s, \Phi; \mathcal{J})$.*

**Remarks 8.1.7** 1. For any $\sigma \in \Sigma$ it is true that $f\xi \in \mathcal{V}(s, \Phi, \mathcal{J})$ for $\xi \in \mathcal{V}(s, \Phi; \mathcal{J})_\sigma$ and $f \in \mathcal{C}^\infty(\Sigma, [0,1])$. In particular, if $\{\rho_i\}$ is a partition of unity with respect to a good covering, it follows that $\rho_i\xi$ is an admissible vector field along $\Phi$.

2. It is an important question whether $\mathcal{V}(s, \Phi; \mathcal{J})$ is complete in the topology being induced by that on $\Omega^0(\Sigma, \Phi^*TM)$. We will not address this question in this work, but we will examine that in the future.

How can we use admissible vector fields along $\Phi$ to tackle the deformation problem of generalized pseudoholomorphic curves? The answer to this question will be given in the following. The kernel of $B$ is a generalized smooth subbundle of $TM$. That means it is locally generated by smooth vector fields but its rank is allowed to vary along $M$. If the rank is constant, we call $\ker B$ regular. In order to be able to use constructions analogous to proposition 8.1.4, we need a connection on $\ker B$. If $\ker B$ is regular, this is a usual connection on the smooth vector bundle $\ker B$. If $\ker B$ is not regular, we call an operator $\nabla^B : \Omega^0(M, \ker B) \to \Omega^1(M, \ker B)$ a connection on $\ker B$ if it obeys the usual properties

$$\nabla^B_{X+Y} s = \nabla^B_X s + \nabla^B_Y s\,, \tag{8.1.38}$$
$$\nabla^B_{fX} s = f \nabla_X s\,, \tag{8.1.39}$$
$$\nabla^B_X (s+t) = \nabla^B_X s + \nabla^B_X t \quad \text{and} \tag{8.1.40}$$

$$\nabla^B_X (fs) = X[f]s + \nabla^B_X s\,, \tag{8.1.41}$$

for all $X\,, Y \in \Gamma(M, TM)$ $s\,, t \in \ker B$ and $f \in \mathcal{C}^\infty(M, \mathbb{R})$. If $\ker B$ is a smooth vector subbundle of $TM$ (of possibly varying rank), we can project smoothly onto $\ker B$. Obviously we can project $T_pM$ onto $\ker B_p$ at any $p \in M$. This can be done smoothly as the following argument shows. Let us look at a open neighborhood $U$ of $p$. Since $\ker B$ is finitely generated, there exist a minimal number of smooth local sections $s_1$, ..., $s_N \in \ker B$ which generate $\ker B|_U$. Observe that $s_1$, ..., $s_N$ cannot supposed to be linearly independent since $\operatorname{rank} \ker B$ may not be constant along $M$. Recall that we are aware of a metric $g$ on $M$ since we examine $(E, \mathcal{J}_2)$-holomorphic curves where $\mathcal{J}_2$ is tamed by a generalized complex structure $\mathcal{J}_1$. Hence, a projection operator $\mathrm{pr}_{\ker B} \in \Omega^1(M, \ker B)$ on $U$ is given by orthogonal projection of $TM$ onto $\ker B$ with respect to $g$:

$$\mathrm{pr}_{\ker B} := \sum_{i=1}^{N} \frac{1}{g(s_i, s_i)}\, i_{s_i} g \otimes s_i \tag{8.1.42}$$
$$\mathrm{pr}_{\ker B}(X) = \sum_{i=1}^{N} \frac{g(s_i, X)}{g(s_i, s_i)} s_i\,. \tag{8.1.43}$$

It is smooth because $g$ is smooth and acts as $\mathrm{id}_{\ker B}$ on $\ker B$. We claim that a connection on $\ker B$ is given by

$$\nabla^B := \mathrm{pr}_{\ker_B} \circ \nabla\,, \tag{8.1.44}$$
$$\nabla^B_X s = \mathrm{pr}_{\ker B}(\nabla_X s)\,. \tag{8.1.45}$$

for $\nabla$ being the Levi-Civita connection associated with $g$. To justify this we have to show that equations (8.1.38) - (8.1.41) hold. Since $\mathrm{pr}_{\ker B}$ is linear, equations (8.1.38) - (8.1.40) are evident. The last property can be inferred from

$$\nabla^B_X(fs) = \mathrm{pr}_{\ker B}(\nabla_X fs) = \mathrm{pr}_{\ker B}\left(X[f]s + f\nabla_X s\right) = X[f]\,\mathrm{pr}_{\ker B}\,(s) + \tag{8.1.46}$$
$$+ f\,\mathrm{pr}_{\ker B}(\nabla_X s) = X[f]s + \nabla^B_X s\,. \tag{8.1.47}$$

Therefore, $\nabla^B$ is a connection on $\ker B$. The fact that $\mathcal{J}$ squares to $-\mathbb{1}$ implies in particular that $BI - I^*B = 0$. Hence, $\ker B$ is stable under $I = \pi \circ \mathcal{J} \circ s$ which squares to $-\,\mathrm{id}_{\ker B}$. Moreover, $\nabla^B$ being a connection on $\ker B$ shows that

$$\left(\nabla^B_X I\right) s = \nabla^B_X(Is) - I\nabla^B_X s \in \ker B\,. \tag{8.1.48}$$

Hence, we infer that $\nabla^B_X I$ is a map from $\ker B$ to $\ker B$. In particular it is true that $(\nabla^B_X I)I + I\nabla^B_X I = 0$. In analogy to equation (8.1.19) we define a new connection

$$\tilde\nabla^B_X s := \nabla^B_X s - \frac{1}{2} I\left(\nabla^B_X I\right) s\,, \tag{8.1.49}$$

which preserves $I$. Let us denote the complex bundle isomorphism given by pointwise parallel transport along $\gamma_\sigma(\lambda) = \exp_{\Phi(\sigma)}(\lambda\xi(\sigma))$ as $\Psi^B_\Phi(\xi) : \Phi^* \ker B \to \Phi^*_\lambda \ker B$ (we parallel transport using $\tilde\nabla^B$ and geodesics are with respect to $\nabla$). Then define an operator $\mathcal{F}^B_\Phi : \mathcal{V}(s, \Phi; \mathcal{J}) \to \Omega^{(0,1)}(\Sigma, \Phi^* \ker B)$ via

$$\mathcal{F}^B_\Phi(\xi) := \Psi^B_\Phi(\xi)^{-1} \circ \pi \circ \overline{\partial}_{\mathcal{J}}(\exp_\Phi(\xi))\,. \tag{8.1.50}$$

This map is the analogous expression to equation (8.1.21). Recall definition 8.1.6 and $\overline{\partial}_{\mathcal{J}}(\Phi) = s \circ \overline{\partial}_I(\Phi)$ for $T\Phi \in \ker B$. Moreover, $\ker B$ being linear and stable under the action of $I$ implies $\overline{\partial}_I(\Phi)(u) \in \Gamma(\Phi(\Sigma), \ker B)$ for $u \in \Gamma(\Sigma, T\Sigma)$. Now we are ready to state

**Proposition 8.1.8.** *Let $M$ be a smooth $2n$-dimensional manifold, $(E, q, [\cdot,\cdot], \pi)$ be an exact Courant algebroid over $M$, $s$ be a smooth isotropic splitting of $E$, $\mathcal{J}$ be an almost generalized complex structure on $E$ and $\Phi : \Sigma \to M$ be a smooth map. Define the operator*

$$\mathcal{D}^B_\Phi : \mathcal{V}(s, \Phi; \mathcal{J}) \to \Omega^{0,1}(\Sigma, \Phi^* \ker B) \tag{8.1.51}$$

*by $\mathcal{D}^B_\Phi(\xi) := T\mathcal{F}^B_\Phi(0)\xi$. We call this the reduced generalized vertical differential. Then*

$$\mathcal{D}^B_\Phi(\xi) = \frac{1}{2}\left(\nabla^B_\xi \circ d\Phi + I(\Phi) \circ \nabla^B_\xi \circ d\Phi \circ j_\Sigma\right) - \frac{1}{2} I(\Phi)(\nabla^B_\xi I)(\Phi) \circ \partial_I(\Phi) \tag{8.1.52}$$

*for all $\xi \in \mathcal{V}(s, \Phi; \mathcal{J})$, where $\partial_I(\Phi) := \frac{1}{2}(d\Phi - I \circ d\Phi \circ j_\Sigma)$.*

*Proof.* If we use the considerations above proposition 8.1.8, the same argumentation as in the proof of proposition 8.1.4 also shows proposition 8.1.8. □

In order to be able to use transversality arguments, we need to find a generalization of regular almost complex structures. The term "regular" is already used in generalized complex geometry. There it is a generalized complex structure such that its associated Poisson structure $\pi \circ \mathcal{J} \circ \pi^*$ is regular, i.e. has constant rank. Thus we name the generalization of a regular almost complex structure a nonsingular almost generalized complex structure. More concretely we state

**Definition 8.1.9.** *Let $M$ be a smooth $2n$-dimensional manifold, $(E, q, [\cdot,\cdot], \pi)$ be an exact Courant algebroid over $M$, $s$ be an isotropic splitting of $E$, $\mathcal{J}$ be an almost generalized complex structure on $E$ and $A \in H_2(M, \mathbb{Z})$. An almost generalized complex structure is called nonsingular for $A$ and $\Sigma$ if $\mathcal{D}_\Phi^B$ is surjective for every $(E, \mathcal{J})$-holomorphic curve $\Phi$ representing the homology class $A$.*

Here we stop the search for a solution of the deformation problem in the general case. But before looking at some simple examples we should take a moment to think about how to proceed the examination of the moduli space of $(E, \mathcal{J})$-holomorphic curves. As a next step we should examine whether $\mathcal{V}(s, \Phi; \mathcal{J})$ is a Banach space or at least whether there is some completion with respect to a Sobolev norm. The question whether $\mathcal{V}(s, \Phi; \mathcal{J})$ is a vector space might be problematic to answer. After answering these questions we should show that $\mathcal{D}_\Phi^B$ is a Fredholm operator and whether the set of nonsingular almost generalized complex structures is of the second category in the set of all $\mathcal{J}_1$-tamed or $\mathcal{J}_1$-compatible almost generalized complex structures. If $\mathcal{D}_\Phi^B$ is Fredholm and $\mathcal{J}$ is nonsingular and $\ker B$ is regular it follows from the implicit function theorem that the moduli space of $(E, \mathcal{J})$-holomorphic curves is a finite dimensional manifold. All these considerations will be done somewhere else.

Instead, we will simplify the situation by some assumptions. This will be done in the following section. It will be the last section of this work.

## 8.2. Examples

As mentioned at the end of the last section we will make some assumptions which will simplify the deformation problem. Their treatment is done in the following subsections. For simplicity we set $E = TM \oplus T^*M$ with $s(X) = X \oplus 0$ and $H = 0$, i.e. we consider untwisted generalized complex geometry.

### 8.2.1. Almost Complex Manifolds

As a first example we should show that ordinary pseudoholomorphic curves are covered by our constructions. To this end let $(M, I)$ be an almost complex manifold and $\mathcal{J}_I$ be its associated almost generalized complex structure. Generalized pseudoholomorphic curves are exactly ordinary $I$-holomorphic curves. From $B = 0$ we infer that $\ker B = TM$ and any vector field along $\Phi$ is admissible. Hence, we arrived at the known theory of pseudoholomorphic curves.

### 8.2.2. Almost Symplectic Manifolds

Let $(M, \omega)$ be an almost symplectic manifold, i.e. $\omega$ is a non-degenerate $2$-form, $(\Sigma, j_\Sigma)$ be a Riemann surface and $\Phi : \Sigma \to M$ be a $\mathcal{J}_\omega$ holomorphic curve. Since $T\Phi u$ has to lie inside $\ker B = \ker \omega$ for all $u \in T\Sigma$, we infer that $\Phi$ has to be a constant map $\Phi : \sigma \longmapsto p_0$ and, hence, $T\Phi = 0$. This implies that $\Phi^* TM = \Sigma \times T_{p_0} M$, $\Phi^* E = \Sigma \times E_{p_0}$, $\partial_{\mathcal{J}}(\Phi) = 0$ and $\Phi^* \nabla = 0$. The linearization of $\mathcal{F}_\Phi(\xi)$ at $0$ then vanishes identically and the vertical differential then decomposes into $\mathcal{D}_\Phi = s \circ 0$ for any $s : TM \to E$. Moreover, we realize that $\mathcal{D}^B_\Phi = 0$. Hence, any almost symplectic structure is nonsingular.

We have shown above that only constant maps are $\mathcal{J}_\omega$-holomorphic. Hence, admissible vector fields along $\Phi$ are constant along $\Phi$ (this corresponds to a unique vector at $p_0$). Thus the space of sections in $\mathcal{V}(s, \Phi; \mathcal{J}_\omega)$ is finite dimensional. Therefore, the vertical differential is a Fredholm operator. Any constant vector field along $\Phi$ is admissible and lies in the kernel of $\mathcal{D}_\Phi$. It follows that deformations of $\Phi$ via $p_0 \longmapsto \exp_{p_0}(\xi)$ are exactly those which are still $\mathcal{J}$-holomorphic curves and the moduli-space of $\mathcal{J}_\omega$-holomorphic curves is $M$ itself.

This is not surprising as only constant maps are $\mathcal{J}_\omega$-holomorphic. But we wanted to demonstrate that we get the intuitive answer using above formalism.

### 8.2.3. Parallelizable Manifolds

Our next example is a target manifold $M$ which is paralellizable. That means that the tangent bundle of $M$ is trivial

$$TM = M \times V \cong M \times \mathbb{R}^n \cong M \times (\mathbb{R} \oplus \cdots \oplus \mathbb{R}) \cong (M \times \mathbb{R}) \oplus \cdots \oplus (M \times \mathbb{R}) \,. \quad (8.2.1)$$

Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be two tamed generalized complex structures such that $\mathcal{J}_1$ is integrable and $\mathcal{J}_2$ is $\mathcal{C}^l$. Then they induce metric structures $G$ on $E$ and $g$ on $TM$. The ordinary exterior differential $d$ is a flat connection on $M \times \mathbb{R}$ and induces a flat connection $\nabla'$

on $TM$. In analogy to the proof of theorem 7.4.4 we can use parallel transport with respect to this connection to extend $\mathcal{J}$ acting on $TM \oplus g(TM)$ to an $\mathcal{C}^l$ almost complex structure $J$ acting on $M \times M$.[1] Now we have the whole apparatus of deformation theory of pseudoholomorphic curves at hand.

The proof of theorem 7.4.4 also shows that a curve $\Phi : \Sigma \to M$ is $\mathcal{J}$-holomorphic if and only if $\Phi' := \Phi \times p : \Sigma \to M \times M$ is $J$-holomorphic for any fixed $p \in M$. Let us assume that $J$ is regular in the usual sense. Then the moduli space of $J$-holomorphic curves $\mathcal{M}(A, \Sigma; J)$ is a smooth manifold of dimension $4n(1-g) + \langle A, \mathrm{c}_1(M)\rangle$. The moduli space of all $\mathcal{J}$-holomorphic curves is then given by the sub-variety $\Phi_2 = p$ for $\Phi = (\Phi_1, \Phi_2) : \Sigma \to M \times M$.

### 8.2.4. Existence of a Foliation into Holomorphic Poisson Manifolds

In the last example we assume that sections in $\ker B$ are involutive. If $X, Y \in \ker B$, this is equivalent to

$$0 = i_{[X,Y]}B = i_X \mathcal{L}_Y B - \mathcal{L}_Y i_X B = i_X i_Y dB - i_X d i_Y B = i_X i_Y dB\,. \tag{8.2.2}$$

We immediately see that $dB = 0$ is sufficient for $\ker B$ being involutive. Since $\ker B$ is involutive, it induces a foliation of $M$ into submanifolds such that sections in $\ker B$ are exactly the vector fields tangential to them. If $X \in \ker B$, the action of $\mathcal{J}$ on $X$ simplifies to

$$\mathcal{J}s(X) = \begin{pmatrix} I & \beta \\ B & -I^* \end{pmatrix} \cdot \begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} IX \\ BX \end{pmatrix} = \begin{pmatrix} IX \\ 0 \end{pmatrix}\,. \tag{8.2.3}$$

Combined with $\mathcal{J}^2 = -\mathbb{1}$ this implies that $\mathcal{J}$ induces an almost complex structure $I$ on each leave of the foliation. As $T\Phi$ has to map $T\Sigma$ into $\ker B$, we deduce that $\Phi$ has to lie inside one leave.

Admissible vector fields are those which generate deformations of $\Phi$ inside one leaf or which make $\Phi$ switch the leaf. In the latter case we have to ensure that during the deformation $\Phi_t$ always lies in exactly one leaf.

**Remark 8.2.1** Observe that in both the almost complex and the almost symplectic case the kernel of $B$ is involutive. Whereas in the almost complex case we get the trivial foliation of $M$ into $M$ itself ($M$ is the only leaf), we get in the almost symplectic case a foliation of $M$ into points. In the latter case the deformations of $\Phi$ are in particular those who switch the leaf in which $\Phi$ lies.

[1] There seems to be a connection to the recently introduced notion of doubled geometry, but we will not go into the details here.

For simplicity let us look at deformations which stay inside one leaf $L$ and assume that $\ker B$ has constant rank. As these leaves are almost complex manifolds on their own, we are able to use the known theory of ordinary pseudoholomorphic curves. This shows that the moduli space of simple pseudoholomorphic curves which lie inside the leaf $L$ and represent the homology class $A \in H_2(L, \mathbb{Z})$ forms a finite dimensional manifold $\mathcal{M}^*_L(A, \Sigma; \mathcal{J})$. The moduli space of all deformations lying inside a particular leaf is then given by the disjoint union over all leaves admitting the homology class $A$,

$$\mathcal{M}^*_0(A, \Sigma; \mathcal{J}) := \coprod_L \mathcal{M}^*_L(A, \Sigma; \mathcal{J}) = \{(L, \Phi) | \Phi \in \mathcal{M}^*_L(A, \Sigma; \mathcal{J})\} \ . \tag{8.2.4}$$

It should be viewed as a family of manifolds. Its dimension is given by

$$\dim \mathcal{M}^*_0(A, \Sigma; \mathcal{J}) = (2 - g) \dim \ker B + \langle A, \mathrm{c}_1(M) \rangle + \operatorname{codim} \ker B \, . \tag{8.2.5}$$

The last term in the dimension formula originates in the dimension of the leaf space of the foliation induced by $\ker B$. This situation should be compared to theorem 3.1.7. in [MS04]. It states that the moduli space of simple $J_\lambda$-holomorphic curves for a regular homotopy $J_\lambda$ is a finite dimensional manifold of dimension $2n(1 - g) + \langle A, \mathrm{c}_1(M) \rangle + 1$. It is an interesting question which geometric structure $\mathcal{M}^*_0(A, \Sigma; \mathcal{J})$ precisely obeys.

If $\ker B$ does not have constant rank, $\mathcal{M}^*_0(A, \Sigma; \mathcal{J})$ cannot be a manifold. Since $\ker B$ is a smooth distribution in the sense of Sussman, each leaf on the boundary where the rank of $\ker B$ jumps has a subfoliation into leaves of the lower dimension. The dimension of the moduli space of leaf preserving deformations depends on which side of the boundary we are. It seems more likely that $\mathcal{M}^*_0(A, \Sigma; \mathcal{J})$ is really a stratifold. It is possible that these problems can be solved by considering deformations which are allowed to switch the leaf, i.e which are generated by admissible vector fields which are not in $\ker B$.

### 8.2.5. Non Constant Isotropic Embedding and the Physical Case

Let us consider the case of integrable almost generalized complex structures. In particular we should do that keeping an eye on the physical interpretation of $B$-transformations. Recall that a $B$-transformation can be interpreted as a canonical transformation on the string super phase space of $M$. What is then the local form of a $\mathcal{J}$-holomorphic curve? By the generalized Darboux theorem it follows that $\mathcal{J}$ is up to a diffeomorphism and a $B$-transformation the direct sum of a symplectic and a complex structure. In physical language that means that there is a local canonical transformation such that $\mathcal{J}$ is the product of a complex and a symplectic structure. Examples 6.3.3 show that

$\Phi$ is then constant in symplectic directions and pseudoholomorphic in complex directions. This is the most general case in the physical situation, i.e. amongst other things $\mathcal{J}$ being integrable.

### 8.2.6. Generalized B-Model on Hyperkähler Manifolds

In section 5.2 we gave a smooth interpolation between topological string theories being defined on a Hyperkähler manifold $M$. We found a family of generalized B-models which connects the A-model being present at $t = 0$ and associated with $(J, \omega_J)$ and the B-model being present at $t = \pi/2$ and associated with $(I, \omega_I)$. Let us examine the instantons for this family. In mathematical terms we will consider generalized pseudoholomorphic curves for a family of generalized complex structures being defined by equation (5.2.3).

Let $(M, I, J, K)$ be a Hyperkähler manifold, $(\Sigma, j_\Sigma)$ be a Riemann surface, $\mathbb{T}M$ the standard Courant algebroid over $M$, $\mathcal{J}_2(t)$ be as in equation (5.2.3) and $\iota : TM \to \mathbb{T}M$ be the canonical smooth isotropic splitting of $\mathbb{T}M$, i.e. $\iota(X) := X \oplus 0$. A map $\Phi$ is $\mathcal{J}_2(t)$-holomorphic if and only if

$$\mathcal{J}_2(t) \circ \iota \circ T\Phi = \iota \circ T\Phi \circ j_\Sigma \,. \tag{8.2.6}$$

Using the splitting $\iota$ this becomes equivalent to

$$\cos(t) J \circ T\Phi = T\Phi \circ j_\Sigma \quad \text{and} \tag{8.2.7}$$

$$\sin(t) \omega_I \circ T\Phi = 0 \,. \tag{8.2.8}$$

Since $\omega_I$ is a symplectic form and in particular non degenerate, equation (8.2.8) forces $\Phi$ to be constant for $t \neq 0$. If $t = 0$, however, the equation (8.2.7) is trivially true and the equation (8.2.7) tells us that $\Phi$ is pseudoholomorphic with respect to $J$.

We observe that the moduli spaces of instantons during this deformation are not isomorphic.

# Part IV.

# Epilogue

*"Es ist nicht das Wissen, sondern das Lernen, nicht das Besitzen, sondern das Erwerben, nicht das Dasein, sondern das Hinkommen, was den größten Genuß gewährt."*

Carl Friedrich Gauß (1777-1855)

# 9. Conclusions, Conjectures and Outlook

We presented the foundation of an extension of symplectic topology towards generalized complex manifolds. In doing so we let us guide by principles originated in topological string theory. To give a self contained exposition, we gave an introductory chapter on supersymmetric quantum mechanics. Afterwards we reviewed the construction of the generalized B-model on generalized Calabi-Yau metric geometries. As topological string theory on ordinary Calabi-Yau manifolds provides us with a motivation of e.g. quantum cohomology, the generalized topological B-model motivates our ansatz for generalized complex topology. Subsequent to the presentation of generalized B-models we examined the transformation behavior of instantons under $B$-transformations. We found instantons not being invariant, but only modulo canonical transformations acting on $\Pi T^*\mathcal{L}M$. This pointed towards the fact that additional degrees of freedom had to be included in order to obtain a rigorous and $B$-field invariant definition of pseudoholomorphic curves taking values in generalized complex manifolds.

Part II also contained an intrinsically geometric formulation of nonlinear sigma-models on a Riemannian manifold $(M, g)$. We expressed the action of nonlinear sigma-models as $S[\Phi] = \langle d\Phi, d\Phi \rangle$, where $d\Phi \in \Omega^1(\Sigma, \Phi^*TM)$ is defined by $d\Phi(u) := T\Phi \circ u$ for $u \in \Gamma(\Sigma, T\Sigma)$ and $\langle \cdot, \cdot \rangle$ is a $L^2$-norm on the space of $\Phi^*TM$-valued $k$-forms. The equations of motion were found in this global picture to be $d^\dagger_{\Phi^*\nabla} d\Phi = 0$. Here $\Phi^*\nabla$ denotes the pullback connection of the Levi-Civita connection $\nabla$ on $(M, g)$. Moreover, we showed that $\nabla$ being torsion free implies $d_{\Phi^*\nabla} d\Phi = 0$. Hence, harmonic maps were found to be $d_{\Phi^*\nabla}$-harmonic 1-forms with values in $\Phi^*TM$. This is in some sense a nonlinear version of non-commutative Hodge theory.

In order to achieve linearity, enhanced solution spaces $\mathcal{H}_\Phi$ were introduced. The enhanced solution space which is associated to $\Phi$ consists of all $\Phi^*TM$-valued 1-forms $\xi$ fulfilling $d^\dagger_{\Phi^*\nabla}\xi = 0$. In particular, linearity implies that the principle of superposition is true here. Therefore, there should exist a set of basis solutions for every $\Phi$. Instead of $d^\dagger_{\Phi^*\nabla}$ we could also use $\Delta_{\Phi^*\nabla} := d_{\Phi^*\nabla} d^\dagger_{\Phi^*\nabla} + d^\dagger_{\Phi^*\nabla} d_{\Phi^*\nabla}$. We showed that if $(M, g)$ is flat and $\Phi_1$ and $\Phi_2$ are homotopic to each other, it follows that $\mathcal{H}_{\Phi_1}$ and $\mathcal{H}_{\Phi_2}$ are isomorphic. Moreover, by exploiting deformation methods which used series expansions, we gave a heuristic motivation why this should also be true if $M$ is not flat. From a physical perspective, the enhancement of the space of harmonic maps by $\Phi^*TM$-valued 1-forms $\xi$ which cannot be expressed as some $d\Phi$ corresponds to the enhancement of physical solutions of the nonlinear sigma-model by non-physical modes. This restores the existence of a mode expansion. Every physical solution corresponds to some mode expansion, but not every mode expansion is physically sensible. Only those which obey the cohomological constraint $\xi = d\Phi$. At the end of chapter 4 we gave a brief summary of a program of how to quantize non-linear sigma models on Riemannian manifolds with non-vanishing curvature. It will be easier to achieve a quantization under the assumption of

**Conjecture 1.** *Let $\Phi_1 : \Sigma \to M$ and $\Phi_2 : \Sigma \to M$ be two differentiable maps which are homotopic to each other. Then it holds $\mathcal{H}^k_{\Phi_1} \cong \mathcal{H}^k_{\Phi_2}$ for all $k$.*

Part III gives an exposition of the theory of $(E, \mathcal{J})$-holomorphic pairs. They were the main objects of interest during this work. In order to give a precise definition, let $M$ be a $2n$-dimensional manifold, $(E, q, [\cdot,\cdot], \pi)$ be an exact Courant algebroid over $M$, $\mathcal{J}$ be an almost generalized complex structure acting on $E$, $(\Sigma, j_\Sigma)$ be a Riemann surface, $\Phi : \Sigma \to M$ be a map and $\lambda : TM \to E$ be an isotropic embedding. Then $(\Phi, \lambda)$ is called an $(E, \mathcal{J})$-holomorphic pair, if and only if

$$\mathcal{J} \circ \lambda \circ T\Phi = \lambda \circ T\Phi \circ j_\Sigma \,. \tag{6.3.1}$$

We proved that this yields a $B$-field invariant notion. Without $B$-field invariance, it would not be possible to give a consistent notion of pseudoholomorphic curves for exact Courant algebroids.

If $(\Phi, \lambda)$ is an $(E, \mathcal{J})$-holomorphic pair and $\lambda$ is an isotropic splitting $\lambda : TM \to E$, we name $\Phi$ an $(E, \mathcal{J})$-holomorphic curve. If additionally to that $(E, q, [\cdot,\cdot], \pi)$ is given by the standard Courant algebroid $(\mathbb{T}M, q, [\cdot,\cdot]_0, \mathrm{pr}_1)$, $(E, \mathcal{J})$-holomorphic curves are simply called $\mathcal{J}$-holomorphic curves. Thereafter, tamed almost generalized complex structures, compatible almost generalized complex structures as well as the generalized energy of a pair $(\Phi, \lambda)$ have been introduced. An important technical result which we proved is theorem 6.4.1. It states the following: it is possible to find for any isotropic embedding $\lambda$ and every isotropic splitting $s$ an orthogonal automorphism $\Lambda$ of $E$ such that $\lambda = \Lambda \circ s$. This enabled us to show that any $(E, \mathcal{J})$-holomorphic pair gives rise to a $\mathcal{J}'$-holomorphic curve. We showed that $(E, \mathcal{J})$-holomorphic pairs admit an energy identity which is very similar to the corresponding expression in usual symplectic topology. Moreover, we proved that there exists an isotropic embedding $\lambda_0$ such that the generalized energy of $(\Phi, \lambda_0)$ is invariant under homotopy if $\mathcal{J}$ is regular and $E$ has vanishing Ševera class $[H]$. This still covers manifolds which do not admit any integrable complex structures or symplectic structures.

Then we covered the local theory of $(E, \mathcal{J})$-holomorphic pairs. To this end we calculated a local expression of equation (6.3.1) and named it the generalized nonlinear Cauchy-Riemann equation. Using a theorem of Aronszajn we proved an identity theorem for generalized pseudoholomorphic pairs. Thereafter we showed that $(E, \mathcal{J})$-holomorphic pairs obey elliptic regularity, i.e. if $\mathcal{J} \in \mathcal{C}^l$ and $\lambda$ is smooth, it follows that $\Phi$ is of class $\mathcal{C}^l$. In analogy to the local theory of ordinary $J$-holomorphic curves, we also introduced the notions of critical points, injective points, simple pairs as well as somewhere injective pairs and examined their properties.

It transpired that $\Phi$, being part of an $(E, \mathcal{J})$-holomorphic pair, locally behaves exactly as a usual pseudoholomorphic curve. We realized that this is true because of theorem 7.4.4. It states that for any $\sigma \in \Sigma$ there exist neighborhoods $\Omega \subseteq \Sigma$ and $U \subseteq M$ and an almost complex structure $J$ being defined on $U \times U$ such that $\Phi \times p$ is a local $J$-holomorphic curve for any $p \in U$.

After the examination of their local behavior we changed our focus towards the deformation theory of $(E, \mathcal{J})$-holomorphic pairs. To this end we calculated the vertical differential $\mathcal{D}_\Phi$ of

$$\overline{\partial}_{\mathcal{J}}(\Phi) := \frac{1}{2}\left(\lambda \circ d\Phi + \mathcal{J} \circ \lambda \circ d\Phi \circ j_\Sigma\right) . \tag{7.1.2}$$

We viewed $\overline{\partial}_{\mathcal{J}}$ as a section in the infinite dimensional vector bundle $\mathcal{E} \to \mathcal{B}$ whose fiber at $\Phi$ is the space $\mathcal{E}_\Phi := \Omega^{0,1}(\Sigma, \Phi^*E)$ of anti-holomorphic 1-forms with values in $\Phi^*E$. In order to be able to give an explicit expression of the vertical differential, we had to introduce the notions of generalized torsion and generalized Levi-Civita connections on exact Courant algebroids. Torsion does only behave like a tensor up to exact terms involving $q$, but, hence, acts tensorial on any isotropic subbundle of $E$. We defined a generalized Levi-Civita connection of $E$ (equipped with a fiber metric $G$) with respect to an isotropic splitting $s$ to be a connection $\nabla$ on $E$ defined by

$$\nabla_X A = \nabla_X(s(Y) + \pi^*(\xi)) := s(\nabla_X Y) + \pi^*(\nabla^*_X \xi) + \frac{1}{2}\pi^*(i_X i_Y H) , \tag{8.1.7}$$

where $\nabla$ is the Levi-Civita connection on $TM$ associated to $g := s^*G$. In particular the restriction of $T$ to $s(TM)$ vanishes. The vertical part is then taken with respect to some connection $\tilde{\nabla}$ on $E$ which arises from the generalized Levi-Civita connection $\nabla$ of $(E, G)$ and leaves $\mathcal{J}$ invariant. It transpired that $\mathcal{D}_\Phi$ is the composition of a real linear Cauchy-Riemann operator $\mathbb{D}_\Phi$ and an upper semi-Fredholm operator $\lambda$. Hence, the vertical differential is only semi-Fredholm and it is not useful to apply the theory of transversal intersections directly.

In order to tackle this problem, we introduced admissible vector fields along $\Phi$. They were defined as infinitesimal deformations of $\Phi$ which generate geodesic flows $\Phi_t$ such that the differential of $\Phi_t$ lies inside $\ker(s^* \circ \mathcal{J} \circ s)$ for all $t$. By restricting small deformations of $\Phi$ to those which are generated by admissible vector fields, the exceed degrees of freedom in the generalized vertical differential separated and we obtained the decomposition $\mathcal{D}_\Phi = s \circ D_\Phi$.[1] Here $D_\Phi$ is formally given by the vertical differential of ordinary pseudoholomorphic curves. In order to apply transversality arguments, it remains to clarify the topological structure of admissible vector fields along a map. If $\xi$ is an admissible vector field along $\Phi$ and $\{\rho_i\}$ is a partition of unity with respect to a good covering into geodesically convex neighborhoods, $\rho_i \xi$ is also an admissible vector field along $\Phi$. Deformations of $\Phi$ via the geodesic flow generated by $\rho_i \xi$ can be computed locally since it is possible to use theorem 7.4.4. Then deformations of $\Phi$ correspond to deformations of $\Phi \times p$ such that $p$ stays a constant map. It is possible that the importance of leaving $p$ a constant map implies the existence of singularities in the moduli space of generalized pseudoholomorphic curves. The exact behavior of this construction will be examined in the future, but this ansatz seems very promising to prove

**Conjecture 2.** *The moduli space of $(E, \mathcal{J})$-holomorphic curves locally obeys the structure of a manifold. More precisely, it is given by a stratifold. Gromov compactness may be incorporated locally.*

If we were able to prove some type of Gromov compactness of the moduli space of $(E, \mathcal{J})$-holomorphic curves, it should be possible to prove

**Conjecture 3.** *Quantum cohomology in generalized complex manifolds is given by a deformation of Lie algebroid cohomology using $(E, \mathcal{J})$-holomorphic curves.*

This directs us to the question of how to apply the theory of $(E, \mathcal{J})$-holomorphic curves to the open sector of mirror symmetry. It should be

**Conjecture 4.** *There exists an extension of Fukaya categories towards generalized complex manifolds. In particular, its objects are given by generalized calibrated generalized Lagrangian submanifolds and its morphisms are an extension of Floer homology towards generalized complex manifolds. Thereby, morphisms interpolate between $Ext$ functors in the complex case and Floer homology in the symplectic case.*

[1] Strictly speaking we used another connection in order to compute an explicit expression, namely $\nabla^B$ defined on $\ker(s^* \circ \mathcal{J} \circ s)$.

One important application of the above construction would be

**Conjecture 5.** *Hyperkähler manifolds $(M, I, J, K)$ are self-mirror. By that we mean that $(M, I)$ is mirror to $(M, J)$.*

This would be the most important and difficult part of the proof of the physical formulation of Langlands correspondence in terms of electric-magnetic duality as it proves that the Hitchin moduli space is self mirror. As a hint towards that result we gave a smooth family of generalized Kähler structures on Hyperkähler manifolds which interpolates between the Bogoliubov transformation of the A-model with respect to some Kähler form $\omega_I$ and the B-model with respect to some complex structure $J$. This enables one to view mirror symmetry in contrast to the literature not as a discrete symmetry, but as a continuous one.

It seems that every concept from symplectic topology can be translated into a concept on generalized complex manifolds. Every of the above conjectures can be summarized in

**Conjecture 6.** *There should be a functorial correspondence between symplectic topology and complex topology.*

This functorial correspondence may be connected to the miraculous duality of complex and symplectic geometry/topology on the same manifold $M$. In physical terms this is S-duality. I hope to be able to work on these topics in the future.

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# Part V.

# Appendix

# A. Generalized Complex Geometry

In this appendix we give a few facts about generalized complex geometry on exact Courant algebroid. We take this appendix from [Gua11] and restrict ourselves to the topics which are needed to be able to follow this work.

## A.1. Linear Geometry of $V \oplus V^*$

Let $V$ be a real vector space of dimension $n$ and let $V^*$ be its dual space. Let us endow $V \oplus V^*$ with a natural non-degenerate symmetric bilinear form $q$ of signature $(n, n)$, given by

$$q(X \oplus \xi, Y \oplus \eta) := \xi(Y) + \eta(X)\,. \tag{A.1.1}$$

In order to examine orthogonal symmetries of this pseudo metric, we look at its Lie algebra. An arbitrary element $g$ in the Lie algebra may be written as a block matrix in the splitting $V \oplus V^*$ via

$$g = \begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix}\,, \tag{A.1.2}$$

where $A : V \to V$, $B : V \to V^*$, $\beta : V^* \to V$ and $B$, $\beta$ are skew. Here $B$ and $\beta$ are viewed as a 2-form and a bivector. By exponentiation, we obtain orthogonal symmetries of $V \oplus V^*$ in the identity component of $\mathrm{SO}(V \oplus V^*)$. They are

$$\exp(B) = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}\,,\ \exp(\beta) = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}\,,\ \exp(A) = \begin{pmatrix} \exp A & 0 \\ 0 & (\exp A^*)^{-1} \end{pmatrix}\,. \tag{A.1.3}$$

The transformations $\exp(B)$ and $\exp(\beta)$ are referred to as $B$-transformations and $\beta$-transformations , respectively. Transformations of the form $\exp(A)$ define a distinguished embedding of $\mathrm{GL}^+(V)$ into the identity component of the orthogonal group.

A maximal isotropic subspace, i.e. an isotropic subspace of maximal dimension, $L$ of $V \oplus V^*$ is also known as a linear Dirac structure. Let $\iota : \Delta \to V$ be a subspace inclusion and let $\epsilon \in \wedge^2 \Delta^*$. Then

$$L(\Delta, \epsilon) := \{X \oplus \xi \in \Delta \oplus V^* \,|\, \iota^* \xi = i_X \epsilon\} \subset V \oplus V^* \tag{A.1.4}$$

is a maximal isotropic subspace of $V \oplus V^*$. Moreover, any linear Dirac structure in $V \oplus V^*$ is of this form. Note that $B$-transformations act on $L(\Delta, \epsilon)$ as

$$\exp(B) \cdot L(\Delta, \epsilon) = L(\Delta, \epsilon + \iota^* B)\,. \tag{A.1.5}$$

In fact, any maximal isotropic can be expressed as a $B$-transformation of $L(\Delta, 0) = \Delta \oplus \mathrm{Ann}(\Delta)$. Another important notion can be found in

**Definition A.1.1.** *The type of a maximal isotropic $L \subset V \oplus V^*$ is the codimension $k$ of its projection onto V.*

Maximal isotropic subspaces can also be described by their associated spinor lines. The action of $V \oplus V^*$ on the exterior algebra $\wedge^\bullet V^*$ (the spinors), given by

$$(X \oplus \xi) \cdot \varphi = I_X \varphi + \xi \wedge \varphi\,, \tag{A.1.6}$$

extends to a spin representation of the Clifford algebra $\mathrm{CL(V \oplus V^*)}$ associated to $q$.

**Definition A.1.2.** *A spinor $\varphi$ is pure when its null space $L_\varphi := \{v \in V \oplus V^* \,|\, v \cdot \varphi = 0\}$ is maximal isotropic.*

Every maximal isotropic subspace $L \subset V \oplus V^*$ is represented by a unique line $K_L \subset \wedge^\bullet V^*$ of pure spinors. By equation (A.1.5), any maximal isotropic $L(\Delta, \epsilon)$ is a $B$-transformation of $L(\Delta, 0)$ for $B$ chosen such that $\iota^* B = -\epsilon$. The pure spinor line which has $L(\Delta, 0)$ as its null space is $\det(\mathrm{Ann}(\Delta)) \subset \wedge^k V^*$, for $k$ being the dimension of $\Delta \subset V$. Moreover, the spinor line $K_L$ determined by a maximal isotropic $L$ constitutes the beginning of a filtration on spinors

$$K_L = F_0 \subset F_1 \subset \cdots \subset F_n = \wedge^\bullet V^* \otimes (\det V)^{1/2}\,. \tag{A.1.7}$$

Here $F_k$ is defined as $\mathrm{CL}^k \cdot K_L$, where $\mathrm{CL}^k$ is spanned by products of at most $k$ generators of the Clifford algebra.

## A.2. Courant Algebroids

Let $M$ be a real $n$-dimensional smooth manifold. The definition of a Courant algebroid can be found in

**Definition A.2.1.** *Let $\pi_E : E \to M$ be a vector bundle over $M$ equipped with a bundle morphism $\pi : E \to TM$, called the anchor, a non-degenerate symmetric bilinear form $q$ and a* *bracket* $[\cdot,\cdot]$. *If* $(E, q, [\cdot,\cdot], \pi)$ *satisfies the conditions*

$$[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]\,, \tag{A.2.1}$$
$$\pi([e_1, e_2]) = [\pi(e_1), \pi(e_2)]\,, \tag{A.2.2}$$
$$[e_1, fe_2] = f[e_1, e_2] + \pi(e_1)[f]e_2\,, \tag{A.2.3}$$
$$\pi(e_1)q(e_2, e_3) = q([e_1, e_2], e_3) + q(e_2, [e_1, e_3])\,, \tag{A.2.4}$$
$$[e_1, e_1] = \pi^* dq(e_1, e_1)\,, \tag{A.2.5}$$

*for all* $e_1$*,* $e_2$*,* $e_3 \in E$ *and* $f \in \mathcal{C}^\infty(M)$*, we call* $(E, q, [\cdot,\cdot], \pi)$ *a Courant algebroid.*

**Remarks A.2.2** 1. Strictly speaking, $\pi^*$ is a map from $T^*M$ into $E^*$, where $\pi^*$ is the dual map of $\pi$ with respect to $q$. In order to get a map from $T^*M$ into E, we have to use $q^{-1} \circ \pi^*$ instead. If $A^* \in E^*$, $q^{-1}(A^*)$ is the unique $A \in E$ such that $q(A, B) = A^*(B)$ for all $B \in E$. In order to simplify formulas, we slightly abuse notation and denote, as common in the literature, $q^{-1} \circ \pi^*$ as $\pi^*$.

2. Any Courant algebroid fits into a short sequence

$$0 \longrightarrow T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \longrightarrow 0\,. \tag{A.2.6}$$

This can be seen from the following argument. Equations (A.2.2) and (A.2.5) imply $\pi \circ \pi^* dq(e_1, e_1) = 0$. It is sufficient that $\pi \circ \pi^* = 0$ on any chart $U$ of $M$. As any 1-form on $U$ can be expressed as $q(e, e)$ it follows that $\pi \circ \pi^* = 0$ on any chart $U$. Hence the claim is true.

3. We use the convention that $[\cdot,\cdot]$ is given by a Dorfman bracket. Its associated Courant bracket is given by the anti-symmetrization of $[\cdot,\cdot]$. Both descriptions are equivalent concerning the applications given in this work.

Another important definition is

**Definition A.2.3.** *Let* $(E, q, [\cdot,\cdot], \pi)$ *be a Courant algebroid over* $M$*. We call* $(E, q, [\cdot,\cdot], \pi)$ *an exact Courant algebroid if* (A.2.6) *is a short exact sequence.*

**Remarks A.2.4** 1. For any exact Courant algebroid there is a canonically defined isotropic embedding $q^{-1} \circ \pi^* : T^*M \to E$. This is true since

$$q\left(q^{-1}\left(\pi^*(\xi)\right), q^{-1}(\pi^*\eta)\right) = \pi^*(\eta)\left(q^{-1}(\pi^*(\xi))\right) = \eta(\pi \circ q^{-1} \circ \pi^*(\xi)) = 0\,. \tag{A.2.7}$$

2. Since $q$ has split signature, there exists a smooth isotropic splitting $s : TM \to E$, i.e. a smooth map $s : TM \to E$ such that $\pi \circ s = \mathrm{id}$, of (A.2.6) if $E$ is exact.

**Example A.2.5** Let $E = \mathbb{T}M := TM \oplus T^*M$. It can be endowed with the same canonical bilinear form $q$ we described on $V \oplus V^*$. As an anchor we use $\pi(X \oplus \xi) := X$ and as a bracket we choose the twisted Dorfman bracket

$$[X \oplus \xi, Y \oplus \eta]_H := [X, Y] \oplus (\mathcal{L}_X \eta - i_Y d\xi + i_X i_Y H) . \tag{A.2.8}$$

A straight forward calculation shows that equations (A.2.1) - (A.2.5) are satisfied. A smooth isotropic splitting is given by $\iota(X) := X \oplus 0$. If $H = 0$ we call $\mathbb{T}M$ with the prescribed data the standard Courant algebroid.

Let $s$ be a smooth isotropic splitting of an exact Courant algebroid $(E, q, [\cdot,\cdot], \pi)$. Then $t = s^* \circ q$ is a left inverse of $q^{-1} \circ \pi^*$. By observing $E = s(TM) \oplus \pi^*(T^*M)$ it is possible to express the non-degenerate inner product as

$$\begin{aligned} q\left(s(X) + q^{-1}(\pi^*(\xi)), s(Y) + q^{-1}(\pi(\eta))\right) &= q\left(s(X), s(Y)\right) + \\ + q\left(q^{-1}(\pi^*(\eta)), s(X)\right) &+ q\left(q^{-1}(\pi^*(\xi)), s(Y)\right) + \\ + q\left(q^{-1}(\pi^*(\xi)), q^{-1}(\pi^*(\eta))\right) &= \pi^*(\eta)(s(X)) + \pi^*(\xi)(s(Y)) = \\ &= \eta(\pi \circ s(X)) + \xi(\pi \circ s(Y)) = \eta(X) + \xi(Y) . \end{aligned} \tag{A.2.9}$$

Moreover, the bracket gets mapped to the $H$-twisted Dorfman bracket, where $H$ is the curvature of the splitting, i.e.

$$i_X i_Y H := s^*[s(X), s(Y)] , \quad X, Y \in \Gamma(TM) . \tag{A.2.10}$$

Hence, any exact Courant algebroid $(E, q, [\cdot,\cdot], \pi)$ is isomorphic to $(\mathbb{T}M, q, [\cdot,\cdot]_H, \mathrm{pr}_1)$.

**Definition A.2.6.** *A $B$-transformation of an exact Courant algebroid $(E, q, [\cdot,\cdot], \pi)$ is defined by a closed 2-form $B$ via*

$$e \longmapsto e + \pi^* i_{\pi(e)} B . \tag{A.2.11}$$

*A diffeomorphism $\chi : M \to M$ lifts to an orthogonal automorphism of $\mathbb{T}M$ given by*

$$\chi_* = \begin{pmatrix} T\chi & 0 \\ 0 & (\chi^*)^{-1} \end{pmatrix} . \tag{A.2.12}$$

*We denote $\chi_*$ also by $\exp(T\chi)$.*

The most general type of automorphism which preserves the twisted Dorfman bracket can be found in

**Proposition A.2.7.** *Let $F$ be an orthogonal automorphism of $\mathbb{T}M$ covering the diffeomorphism $\chi : M \to M$, and preserving the $H$-twisted Dorfman bracket. Then $F = \chi_* \circ \mathrm{e}^B$ for a unique 2-form $B \in \Omega^2(M)$ satisfying $\chi^* H - H = dB$.*

## A.3. Dirac Structures

The Dorfman bracket fails to be a Lie bracket due to exact terms involving the inner product $q$. Therefore, upon restriction to a subbundle $L \subset TM \oplus T^*M$ which is closed under the Dorfman bracket as well as being isotropic, the anomalous terms vanish. Then $(L, [\cdot,\cdot], \pi)$ defines a Lie algebroid, with associated differential graded algebra $(\mathcal{C}^\infty(\wedge^\bullet L^*), d_L)$. Here $d_L$ is the Lie algebroid de Rham differential with respect to $\pi$ and $[\cdot,\cdot]$. In fact, there is a tight a priori constraint on which proper subbundles may be involutive:

**Proposition A.3.1.** *If $L \subset E$ is an involutive subbundle of an exact Courant algebroid, then $L$ must be isotropic, or of the form $\pi^{-1}(\Delta)$, for $\Delta$ an integrable distribution in $TM$.*

Now we are ready to define Dirac structures in an exact Courant algebroid.

**Definition A.3.2.** *A maximal isotropic subbundle $L \subset E$ of an exact Courant algebroid $(E, q, [\cdot,\cdot], \pi)$ is called an almost Dirac structure. If $L$ is involutive with respect to $[\cdot,\cdot]$, then the almost Dirac structure is said to be integrable, or simply a Dirac structure.*

At any point $p$, a Dirac structure has a unique description as a generalized graph $L(\Delta_p, \epsilon_p)$, where $\Delta_p = \pi(L)_p$ and $\epsilon_p \in \wedge^2\Delta_p^*$. Assuming that $L$ is regular near $p$, we have the following description of the integrability condition:

**Proposition A.3.3.** *Let $\Delta \subset TM$ be a subbundle and $\epsilon \in \Gamma(\wedge^2\Delta^*)$. Then the almost Dirac structure $L(\Delta, \epsilon)$ is integrable for the $H$-twisted Dorfman bracket if and only if $\Delta$ integrates to a foliation and $d_\Delta = \iota^* H$, where $d_\Delta$ is the leafwise exterior derivative.*

In neighborhoods where $\Delta$ is not regular, one has the following description of the integrability condition.

**Theorem A.3.4.** *The almost Dirac structure $L \subset TM \oplus T^*M$ is involutive for the $H$-twisted Courant bracket if and only if*

$$d_H(\mathcal{C}^\infty(F_0)) \subset \mathcal{C}^\infty(F_1)\,, \tag{A.3.1}$$

*where $d_H\varphi := d\varphi + H \wedge \varphi$. That is, for any local trivialization $\varphi$ of $K_L$, there exists a section $X \oplus \xi \in \Gamma(\mathbb{T}M)$ such that*

$$d_H\varphi = I_X\varphi + \xi \wedge \varphi\,. \tag{A.3.2}$$

*Furthermore, condition* (A.3.1) *implies that*

$$d_H(\mathcal{C}^\infty(F_k)) \subset \mathcal{C}^\infty(F_{k+1})\,. \tag{A.3.3}$$

## A.4. Generalized Complex Structures and Integrability

It is possible to transport the definition of an almost complex structure to exact Courant algebroids.

**Definition A.4.1.** *An almost generalized complex structure $\mathcal{J}$ on an exact Courant algebroid $(E, q, [\cdot,\cdot], \pi)$ is an almost complex structure $\mathcal{J}$ on $E$ which is orthogonal with respect to $q$, i.e.*

$$\mathcal{J}^2 = -\mathbb{1} \quad \textit{and} \tag{A.4.1}$$

$$q(\mathcal{J}e_1, \mathcal{J}e_2) = q(e_1, e_2)\,. \tag{A.4.2}$$

*If the $+\,\mathrm{i}$-eigenbundle $L$ of $\mathcal{J}$, acting on the complexification of $E$, is involutive with respect to $[\cdot,\cdot]$, we call $\mathcal{J}$ an integrable almost generalized complex structure , or simply a generalized complex structure*

**Remark A.4.2** The second equation in above definition can also be formulated as $\mathcal{J}^* = -\mathcal{J}$. Hence, we observe that an almost generalized complex structure on $E$ is in some sense an almost complex structure and an almost symplectic structure at the same time.

An immediate consequence of $\mathcal{J}$ being orthogonal with respect to $q$ is that the $+\,\mathrm{i}$-eigenbundle of $\mathcal{J}$ is isotropic, i.e.

$$q(e_1, e_2) = q(\mathcal{J}e_1, \mathcal{J}e_2) = \mathrm{i}^2 q(e_1, e_2) = -q(e_1, e_2) = 0 \quad \forall\, e_1\,, e_2 \in L\,. \tag{A.4.3}$$

**Examples A.4.3** We give two simple examples on the standard Courant algebroid which show that complex and symplectic structures can be embedded into this formalism.

1. If $I$ is an almost complex structure, i.e. an endomorphism on $TM$ which squares to $-\mathbb{1}$, it is evident that

$$\mathcal{J}_I = \begin{pmatrix} I & 0 \\ 0 & -I^* \end{pmatrix} \tag{A.4.4}$$

   is an almost generalized complex structure associated to $I$. The $+\,\mathrm{i}$-eigenbundle is given by $L_I = TM^{1,0} \oplus T^*M^{0,1}$. Moreover, it is true that $\mathcal{J}_I$ is integrable if and only if $I$ is integrable as an almost complex structure.

2. If $\omega$ is a non-degenerate $2$-form on $M$, we can associate the almost generalized complex structure

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}\,. \tag{A.4.5}$$

   The $+\,\mathrm{i}$-eigenbundle is given by $L_\omega = \mathrm{e}^{-\,\mathrm{i}\omega} TM = \{X \oplus -\,\mathrm{i}\omega(X) \,|\, X \in TM\}$. It is true that $\mathcal{J}_\omega$ is integrable if and only if $d\omega = 0$, i.e. $\omega$ is a symplectic structure.

The connection between Dirac structures and generalized complex structures is

**Proposition A.4.4.** *A generalized complex structure is equivalent to a complex Dirac structure $L \subset E \otimes \mathbb{C}$ such that $L \cap \overline{L} = \{0\}$, where $L$ is the $+\,\mathrm{i}$-eigenbundle of $\mathcal{J}$.*

As a result, the $+\,\mathrm{i}$-eigenbundle $(L, [\cdot,\cdot], \pi)$ defines the structure of a Lie algebroid and we obtain a differential complex

$$\cdots \xrightarrow{d_L} \mathcal{C}^\infty(\wedge^k L^*) \xrightarrow{d_L} \mathcal{C}^\infty(\wedge^{k+1} L^*) \xrightarrow{d_L} \cdots \quad , \tag{A.4.6}$$

where

$$\begin{aligned} d_L\omega(l_1,\dots,l_{k+1}) :=& \sum_{i=1}^{k+1}(-1)^{i+1}\pi(l_i)\omega(l_1,\dots,\hat{l}_i,\dots,l_{k+1}) \\ &+ \sum_{i<j}(-1)^{i+j}\omega([l_i,l_j],l_1,\dots,\hat{l}_i,\dots,\hat{l}_j,\dots,l_{k+1})\,. \end{aligned} \tag{A.4.7}$$

**Proposition A.4.5.** *The Lie algebroid complex of a generalized complex structure is elliptic.*

This provides us with

**Corollary A.4.6.** *The cohomology of the complex* (A.4.6)*, called the Lie algebroid cohomology $H^\bullet(M,L)$, is a finite dimensional graded ring associated to any compact generalized complex manifold.*

In order to prepare for a local classification of generalized complex structures, we need

**Definition A.4.7.** *The type of the generalized complex structure $\mathcal{J}$ is the upper semi-continuous function*

$$\mathrm{type}(\mathcal{J}) = \frac{1}{2}\dim_{\mathbb{R}} T^*M \cap \mathcal{J}T^*M\,. \tag{A.4.8}$$

Moreover, we need

**Proposition A.4.8.** *Let $\mathcal{J}$ be a generalized complex structure with $+\,\mathrm{i}$-eigenbundle $L \subset E\otimes\mathbb{C}$. Then $P := \pi\circ\mathcal{J}\circ\pi^*$ is a Poisson bivector. The distribution $\Delta = \pi\circ\mathcal{J}\circ\pi^*(T^*M)$ integrates to a generalized foliation by smooth symplectic leaves with codimension $2k$, where $k = \mathrm{type}(\mathcal{J})$.*

A local form for a generalized complex structure is given in

**Theorem A.4.9.** *At any point, a generalized complex structure of type $k$ is equivalent, by a choice of a smooth isotropic splitting of $E$, to the direct sum of a complex structure of complex dimension $k$ and a symplectic structure of real dimension $2n - 2k$.*

To get an expression in a neighborhood we need

**Definition A.4.10.** *A point $p \in M$ in a generalized complex manifold is called regular when the Poisson structure $P$ is regular at $p$, i.e. $\mathrm{type}(\mathcal{J})$ is locally constant at $p$. A neighborhood in which every point is regular is called a regular neighborhood. If any $p \in M$ is regular, we call $\mathcal{J}$ regular.*

By corollary A.4.8, a generalized complex structure defines, in a regular neighborhood $U$, a foliation $\mathcal{F}$ by symplectic leaves of codimension $2k = 2\mathrm{type}(\mathcal{J})$, integrating the distribution $\Delta$. We get a complex structure on the leave space

**Proposition A.4.11.** *The leaf space $U/\mathcal{F}$ of a regular neighborhood of a generalized complex manifold inherits a canonical complex structure.*

This enables us to state

**Theorem A.4.12** (Generalized Darboux theorem)**.** *A regular point of type $k$ in a generalized complex manifold has a neighborhood which is equivalent to the product of an open set in $\mathbb{C}^k$ with an open set in the standard symplectic space $(\mathbb{R}^{2n-2k}, \omega_0)$.*

At the end of this section we should state the definition of a generalized Calabi-Yau structure, a generalized Kähler structure and a generalized Calabi-Yau metric geometry.

**Definition A.4.13.** *The canonical line bundle of a generalized complex structure on $\mathbb{T}M$ is the complex pure spinor line subbundle $K \subset \wedge^\bullet T^* \otimes \mathbb{C}$ annihilated by the $+\,\mathrm{i}$-eigenbundle $L$ of $\mathcal{J}$.*

**Definition A.4.14.** *A generalized Calabi-Yau structure is a generalized complex structure with holomorphically trivial canonical bundle, i.e. admitting a nowhere-vanishing $d_H$-closed section $\rho \in \mathcal{C}^\infty(K)$.*

**Definition A.4.15.** *A generalized Kähler structure is a pair $(\mathcal{J}_1, \mathcal{J}_2)$ of commuting generalized complex structures such that $G = -\mathcal{J}_1\mathcal{J}_2$ is a positive definite metric on $\mathbb{T}M$.*

**Definition A.4.16.** *A generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ is called a generalized Calabi-Yau metric geometry if $\mathcal{J}_1$ and $\mathcal{J}_2$ define a generalized Calabi-Yau structure on their own, with nowhere vanishing $d_H$-closed sections $\rho_i \in \mathcal{C}^\infty(K_i)$, and if both sections are related by a constant, i.e.*

$$(\rho_1, \overline{\rho}_1) = c(\rho_2, \overline{\rho}_1)\,, \tag{A.4.9}$$

*where $c \in \mathbb{R}$ is a constant.*

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# Index

## Eidesstattliche Versicherung

(Siehe Promotionsordnung vom 12.07.11, § 8, Abs. 2 Pkt. .5.)

Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

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Name, Vorname

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Formular 3.2