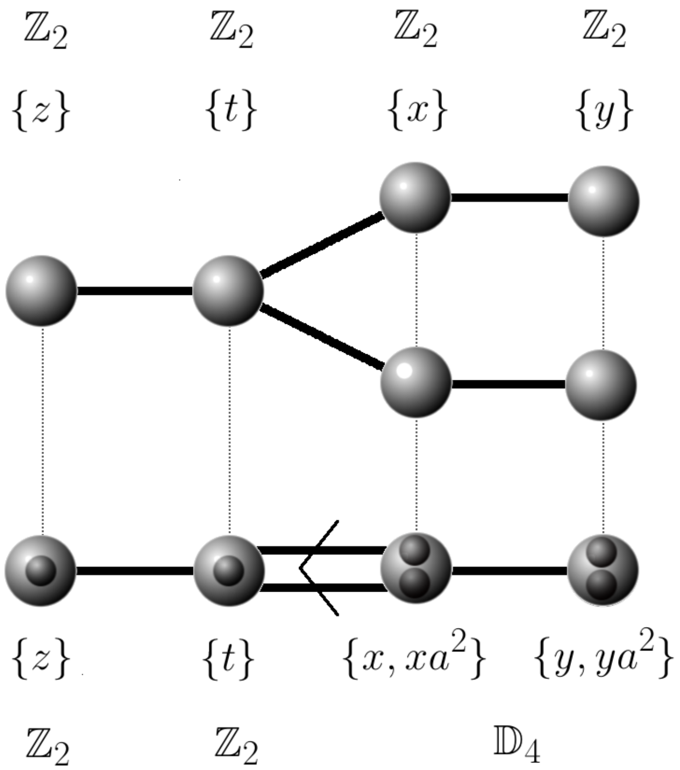
Orbifoldizing Hopf- and Nichols-Algebras



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Reading Directions: The thesis contains 5 rather independent parts.

To encourage the respective non-expert reader, casual introductions to the respective topics are given. Moreover, personal notes are included regarding goals, motivations and outlooks to possible further work.

These *italic sections* may be **skipped**, as the core material is self-contained and equipped with separate technical overviews.

Notations: (for further notions, see the **index** on the last page)

We are working over the base field $\mathbb{k} = \mathbb{C}$ and denote by $\mathbb{k}_n$ the set of primitive n-th roots of unity. The Galois fields are $\mathbb{F}_{p^n}$. All algebras will be finite-dimensional from chapter 2 on, except for part 5.

We denote by $\mathbb{Z}_n, \mathbb{D}_n, \mathbb{Q}_n, \mathbb{A}_n, \mathbb{S}_n$ the usual finite groups, whereas the symbols $A_n, B_n, C_n, D_n, E_n, F_4$ are reserved for the respective Dynkin diagrams (and the associated simple groups of Lie type). Extraspecial groups are denoted as usual $p_{\pm}^{2n+1}$, especially $\mathbb{D}_4, \mathbb{Q}_8 = 2_{\pm}^{2+1}$ ([**Hu83**] p. 349ff).

We add the notation X_n for an arbitrary diagram and $Z_n = A_{n-1}^{(1)}$ for a simply-laced n-cycle.We use the graph theory notation $A_2 \cup B_3$ rather than the geoemtric $A_2 \times B_3$ for disconnected diagrams. Sometimes we speak of the *shape* (triangular, A_2, D_4 etc.) and mean the graph without distinguishing multiple edges or other differing decorations.

The dual group is always denoted as G^*, whereas $\mathbb{k}^\times$ is the multiplicative group. The center is $Z(G)$, the commutator subgroup G' and any centralizer $Cent(g)$. Kernel and image of maps are denoted Ker, Im.

For an action of a group on a set and a given subset, we use the more suggestive terms *normalizer* and *centralizer* for the stabilizer resp. point wise stabilizer of the subset.

Second version with many small corrections, February 2013

Contents

Abstract

The **main goal of this thesis** is to explore a new general construction of orbifoldizing **Hopf-** and **Nichols algebras**, describe the growth of the **automorphism group** and compare the behaviour of certain associated **categories** to Kirillov's orbifoldizing. Together with outlooks towards **vertex algebras** these aspects form the **5-fold subdivision** of this thesis.

The **main applications** of this theory is the construction of new finite-dimensional Nichols algebras with sometimes large rank. In the process, the associated group is centrally extended and the root system is folded, as shown e.g. for $E_6 \to F_4$ on the title page. Thus, in some sense, orbifoldizing constructs new finite-dimensional quantum groups with nonabelian Cartan-algebra.

Orbifoldizing for **me** is the following class of phenomena: Given some proper object H and several "twistings" $A(p)$ thereof, that are forming a group $p \in \Sigma$ with $A(e) = H$. Then the sum of all $A(p)$ is again a proper object, the **orbifold** Ω:

$$\Omega = \bigoplus_{p \in \Sigma} A(p)$$

- The **geometric intuition** behind this (see example below) is the decomposition of functions $\Omega = \mathcal{F}(G)$ on a covering Lie group $G \to \Gamma$ into "twisted" functions $A(p) = \mathcal{F}(\Gamma_p)$ on the quotient i.e. sections in nontrivial line bundles Γ_p over Γ with monodromy prescribed by p. Especially $H = A(e) = \mathcal{F}(\Gamma)$.
- The **algebraic intuition** relies on generalized Schur cover groups [**Hu83**]: For a finite groupring $H = \mathbb{k}[\Gamma]$ and a subgroup $\Sigma \subset H^2(\Gamma, \mathbb{k}^\times)$, the sum (as an algebra) of twisted grouprings $A(p) = \mathbb{k}_p[\Gamma]$ yields the groupring $\Omega = \mathbb{k}[G]$ of a central extension by Σ. The aim has been to reduce projective representation theory for Γ to ordinary ones over G.

The group-interpretation has been the driving force behind the construction and for group-Hopf-algebras it is recovered accordingly.

Zusammenfassung

Das **Hauptziel dieser Arbeit** ist es, eine neue allgemeine Konstruktion von Orbifold **Hopf-** und **Nichols-Algebren** zu untersuchen, sowie das Wachstum der **Automorphismen-Gruppe** zu beschreiben und das Verhalten bestimmter damit assoziierter Kategorien mit der Orbifold-Konstruktion von Kirillov zu vergleichen. Mit einem Ausblick auf **Vertex Algebren** stellen diese Aspekte die **5 Teile** dieser Arbeit dar.

Die **Hauptanwendung** dieser Theorie ist die Konstruktion neuer, endlich-dimensionaler Nichols-Algebren von teils großem Rang. Bei dem Vorgang wird die Gruppe zentral erweitert und das Wurzelsystem gefaltet, siehe z.B. $E_6 \to F_4$ auf der Titelseite. Wir konstruieren also neue, endlich-dim. Quantengruppen mit nichtabelscher Cartan-Algebra.

Unter **Orbifoldizing** verstehe ich persönlich dabei die folgende Klasse von Phänomenen: Gegeben sei ein Objekt H und mehrere "twists" $A(p)$ hiervon, welche eine Gruppe $p \in \Sigma$ bilden, wobei $A(e) = H$. Dann erhält die Summe aller $A(p)$ wieder die Struktur eines Objektes im herkömmlichen Sinne, dem **Orbifold**:

$$\Omega = \bigoplus_{p\in\Sigma} A(p)$$

- Die **geometrische Intuition** hierfür (siehe folgendes Beispiel) ist die Zerlegung von Funktionen $\Omega = \mathcal{F}(G)$ auf einer überlagernden Lie-Gruppe $G \to \Gamma$ in "getwistete" Funktionen $A(p) = \mathcal{F}(\Gamma_p)$ auf dem Quotienten, d.h. Schnitte in nichttrivialen Geradenbündeln Γ_p auf Γ, wobei die Monodromie durch p gegeben wird. Insbesondere ist $H = A(e) = \mathcal{F}(\Gamma)$.
- Die zugrundegelegte **algebraische Intuition** stammt von Darstellungsgruppen [**Hu83**]: Für eine endlich-dimensionalen Gruppenalgebra $H = \mathbb{k}[\Gamma]$ und eine Untergruppe $\Sigma \subset H^2(\Gamma, \mathbb{k}^\times)$ ist die Summe (als Algebren) der getwisteten Gruppenringe $A(p) = \mathbb{k}_p[\Gamma]$ wieder ein Gruppenring $\Omega = \mathbb{k}[G]$ einer zentralen Erweiterung mit Σ. Das Ziel dieser Konstruktion war die Zurückführung von projektiver Darstellungstheorie von Γ auf gewöhnliche Darstellungstheorie von G.

Letztere Interpretation war der Leitfaden dieser neuen Konstruktion und für Gruppen-Hopfalgebren ergeben sich Darstellungsgruppen.

A Geometric Example To Start With

First off all, let us consider an intuitive geometric example, before we proceed to algebraic one's and summarize our methods and results:

Suppose G is a semisimple simply-connected complex Lie group and Σ a finite abelian group with it's dual Σ^* ($\cong \Sigma$) normally contained in G. By standard theory, the quotient $\Gamma := G/\Sigma^*$ is again a Lie group with fundamental group $\pi_1(\Gamma) \cong \Sigma^*$.

Now elements in the algebra of (continuous) $\mathbb{C}$-functions $f \in \mathcal{F}(G)$ (or equivalently every section in the **trivial line bundle over** G) can be uniquely written as a linear combination of Σ**-covariant functions** with respect to some 1-dimensional representation $p \in \Sigma^{**} \cong \Sigma$ of Σ^*:

$$\mathcal{F}(G) \cong \bigoplus_{p\in\Sigma} \mathcal{F}_p(G) \quad via \quad f(x) = \sum_{p\in\Sigma} f_p(x) = \sum_{p\in\Sigma} \left(\frac{1}{|\Sigma|} \sum_{g\in\Sigma^*} p(g) f(g.x) \right)$$

$$f_p \in \mathcal{F}_p(G) := \{ f \in \mathcal{F}(G) \mid \forall_{g\in\Sigma^*,\, x\in G}\ f(g^{-1}.x) = p(g)f(x) \}$$

For $p = e$ trivial (the Σ^*-invariant functions) this leads exactly to the algebra of Γ-functions $\mathcal{F}_{1^*}(G) = \mathcal{F}(\Gamma)$ or again the sections in the respective **trivial line bundle over** Γ. The other $\mathcal{F}_p(G)$ correspond to sections in precisely all **nontrivial line bundles** Γ_p, i.e. are "functions" on Γ with prescribed monodromy $p(g)$ along each cycle $g \in \pi(\Gamma) \cong \Sigma$. Note that for $p \neq e$ the $\mathcal{F}_p(G) = \mathcal{F}(\Gamma_p)$ are no algebras any more, but **modules over the algebra** $\mathcal{F}(\Gamma)$.

$$\mathcal{F}(G) \cong \bigoplus_{p\in\Sigma} \mathcal{F}(\Gamma_p)$$

Be **warned**, that in this thesis the structures appear **dualized**. E.g. $A(p)$ will be comodule algebras, $A(p)A(q) = A(pq)$ becomes the coproduct and the natural algebra map $H = A(e) \subset \Omega$ a quotient $\Omega \to H$.

As well, the reader is **warned**, that in this geometric example, the term "orbifold" is reserved for the smaller space Γ. In contrast, in existing and new cases below, **orbifoldizing** shall describe the entire algebraic process above (twist and sum) and **orbifold** the larger algebra Ω.

My Motivations And Goals

My **initial motivation** to search for a notion "orbifoldizing", such as the two examples described above, in a more general Hopf algebra setting, emerged from my diploma thesis [**Len07**]: I constructed vertex algebras "uniformly" from strong Hopf algebra structures, including so-called lattice algebras. Upon finishing, I came across the much celebrated vertex algebra orbifoldizing yielding the **Moonshine module** (see part 5). Here, the relevant existing orbifoldizing constructions are:

- An **equivariant category** composed of a braided part (untwisted sector) and several "modules-alikes" (twisted sectors) contains a new braided category as its invariant part.
- A **vertex algebra** (axiomatizing CFT operators) and a given (cyclic) group acting on it, leads to the notion of twisted vertex modules, which are proper modules over the fixed vertex subalgebra. The equivariant part of all of them summed up (as vertex modules) can sometimes again be given the structure of a new full vertex algebra.

I decided to also study the effect of orbifoldizing in the algebraic setting first, where I kept close connection to the Schur cover group case: I hoped to then establish similar to my thesis a "messy" once-and-for-all-isomorphy to the series' calculations and be able to perform much of the ad-hoc work in a "cleaner" purely algebraic setting.

Thus, in the following work I want to give a general orbifoldizing construction for **Hopf algebras** and **Nichols algebras** in particular. The latter are tensor algebras of braided vector spaces modulo some relations associated to the braiding. They appear e.g. as quantum Borel part in the classification of pointed Hopf algebras [**AS**], such as the truncated $\mathcal{U}_q(\mathfrak{g})$. Hence I wish to add the following to the lists above:

- Suppose a given **Hopf algebra** and multiple Bigalois objects forming a group. These "twistings" will be explained below and are sometimes viewed as noncommutative principle fibre bundles; compare this to the example above! As well, they are algebras, but no Hopf algebras. Then, the direct sum of all these Bigalois objects as an algebra can again be given the structure of a Hopf algebra. The construction specializes to generalizations of Schur cover groups if applied to grouprings.

- Suppose a **Nichols algebra** over a finite group, some subgroup of group-2-cocycles and an action of this twisting group on the vector space as "twisted symmetries" $H \to H_\sigma$. Then, the direct sum of all twistings contains a new Nichols algebra over a centrally extended group (as a certain Σ-stabilizer, excluding the newly appearing coradical).

The **vertex algebras**, my diploma thesis' construction would assign to both the base Hopf algebra and the orbifold, seem to behave accordingly, but there are several complications. Thus, in this thesis I will neither dare a general construction nor tackle the tremendous calculations necessary on the vertex side to show how it could coincide especially with the ad-hoc constructed Moonshine module, but leave it with a qualitative outlook on both.

The surprising occurrence during my work was on the other hand, that the mere construction on the algebraic side can contribute noteworthy to the present research on finite-dimensional Nichols algebras – objects with a remarkably rich structure continuing root systems of semisimple Lie algebras, that enabled their classification over abelian groups by Heckenberger [**H05**]. Over nonabelian groups, this trend persist ([**HS08**]), but only few examples are known so far.

The more I got fascinated by the algebraically strong and notoriously combinatorially flavoured subject, and the more some use of orbifoldizing became clear, my thesis' goal willingly shifted into an according direction, such that now the backbone, the more general results and the worked-out applications fall solely into this branch of algebra.

Doubtless, a completed path to constructing the moonshine module along these approaches, let alone new cases, would require much future work, many further adaptions and might very well be unsuccessful after all. However, I would like to voice my opinion about the necessity to explore new options for a purely algebraic analysis of the impressing phenomena this field of study has to offer, and to the stimuli it might yet feed back to algebra itself.

Simon Lentner, Munich, May 20th 2012

Summary: Methods And Results Of This Thesis

We give a brief overview over each part, point to central notions and theorems and give credit to valuable personal influences along the way.

Orbifoldizing Hopf Algebras

The first part of this thesis deals with the orbifoldizing procedure itself on the level of Hopf algebras. We establish a rather general categorical setting[1] (twisting groups, see Definition 1.5), in which we can prove the main result of this part: The construction of the orbifold Hopf algebra in Theorem 1.6. We will subsequently (section 1.6) realize this abstract situation by a subgroup of Bigalois objects and additional data.

We also describe the behaviour of some characteristic subsets, as one passes to the orbifold, such as the coradical (Theorem 2.4) and the skew-primitives (Theorem 2.9). Under certain conditions, two usually "desired" properties of a Hopf algebra, namely *pointedness* and *link-indecomposability* will survive the process and also hold in the orbifold.

Finally we apply this to the situation, in which the initial Hopf algebra is composed of a groupring and a Nichols algebra (a sort of quantum Borel part). We will give a construction (Theorem 3.2), that uses far more concrete data, namely a group of group-2-cocyles, that determine the new coradical, and a representation of Σ on H by so-called twisted symmetries (isomorphisms to a Doi twist). This situation of just orbifoldizing Nichols algebras will be focused on in part 2, where these twisted symmetries (though no automorphisms) still preserve the Dynkin diagram and hence can be well identified.

We conclude by addressing the vice-versa question on how to inherently characterize all Hopf algebras, that arise as orbifolds. The answer is the surprisingly general Reconstruction Theorem 3.6, proven by a variant of Masuoka's push-out construction **[M01]**, which has some classificatory value for Nichols algebras (see part 2).

[1] The author thanks Prof. Schneider and Prof. Masuoka for suggesting this course of action, that especially clarified the issue of "coherent choices" of Bigalois isomorphisms $\iota_{p,q}$. See also Remark 1.10.

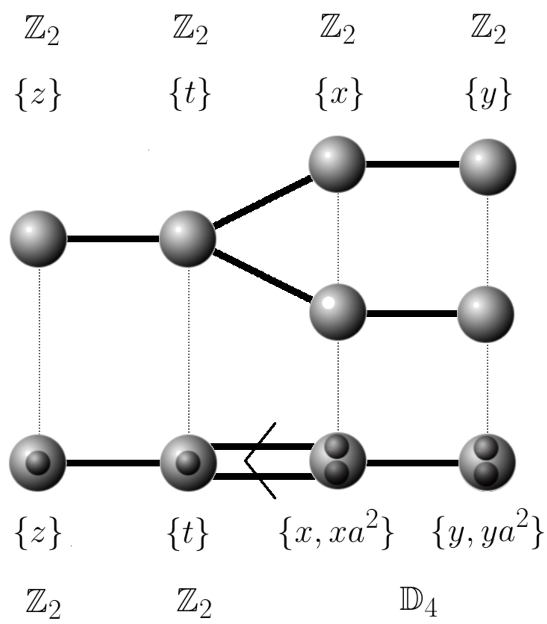
Orbifoldizing Nichols Algebras

This is the main part of the thesis. To keep independent of the first part, we shall start the second part by giving a quick, yet thorough construction of orbifolds purely in the context of Nichols algebras (Theorem 4.4). These ad-hoc constructions have been already prepared in Theorem 3.2: The Bigalois objects (twistings) in the first part are replaced by group 2-cocycles and twisted symmetries of the underlying Yetter-Drinfel'd-module.

Orbifoldizing then constructs new examples of finite–dimensional indecomposable (even faithful) Nichols algebras[2] over a nonabelian group extension by the twisting group. E.g. in sections 3.2 and 4.4 we find:

$$\mathbb{Z}_2^2 \longleftarrow \mathbb{D}_4, \mathbb{Q}_8 \qquad \mathbb{S}_4 \longleftarrow GL_2(\mathbb{Z}_3)$$

The already mentioned **root systems** and their generalized **Dynkin diagrams** reduce in the orbifold to a subsystem/-diagram fixed by the twisted symmetries, a behaviour known for semisimple Lie algebras as **diagram folding** (see e.g. [**Gi06**], p. 47ff), while the dimension of each node (roughly a conjugacy class) increases as shown on the cover:



[2]The author thanks Prof. Schneider for stimulating discussions, especially regarding root systems of Nichols algebras, for pointing out the sources [**H05**][**HS10**], and for addressing the question of constructing faithful Nichols algebras.

Mainly, we shall study, which of the specimen in Heckenberger's list allow an $\mathbb{Z}_p$-orbifoldizing to a nonabelian nilpotent group of class 2 (classification in Theorem 6.1, proof entire chapter 6). Exemplary, section 7.1 uses this result to clarify the existence of such Nichols algebras over most groups of order 16 and 32. We also do some steps towards a classification by deorbifoldizing hypothetical Nichols algebras back to an abelian groupring, where we consult Heckenberger's list. Thereby we can find all such Nichols algebras (examples in sections 5.3 as well as 7.2 and 7.3) or rule out their very existence (examples in section 7.4).

The **key methods** are:

- First of all in section 6.1 the analysis of **which folding are possible** for Dynkin diagrams (generally in Theorem 6.8 and for abelian groups in Theorem 6.9), and checking it against "Heckenberger's list" for abelian groups (tediously in section 6.6).
- For cases not ruled out, we conversely prove in Theorem 6.15 the existence of so-called **symplectic root systems** (Definition 6.14) for the Dynkin diagrams in question: This is a basis of $\mathbb{Z}_2^n$ (viewed as symplectic vector space), which reflects the desired diagram. It is similar to usual root systems, but far weaker (many graphs are possible) and should be rather seen as an additional datum ensuring the twisted symmetry.
- To yield even **faithful Nichols algebras** (orbifolds have always trivial Σ-action), one may Doi twist certain orbifolds over the nonabelian group (see examples in section 7.1).
- Conversely, we derive certain conditions on any possible Σ-action from constraints given in **[HS08]**. This can in several cases restrict their number to such an extend, that the remaining cases can be numerically **exhausted by Doi twists** using Matsumoto's spectral sequence (section 5.2). Thus, by the reconstruction theorem, every such Nichols algebra is a Doi twist of an orbifold. The first example is in section 5.3, others have been described above.

Orbifoldizing Automorphisms

We then turn our attention to the behaviour of the group of Hopf algebra automorphisms as the Hopf algebra undergoes orbifoldization - particularly in the aim of identifying large simple groups[3]. We define certain subgroups $B, N \subset Aut(\Omega)$ (Definition 9.5) related to $Aut(H)$ and the permutation action on certain central idempotents[4] (by Lemma 9.3). These subgroups should be visualized as the group-theoretic generalization of the subgroup of upper triangular resp. monomial matrices in Lie groups over finite fields.

Under very specific conditions (section 9.2) we are able to construct a so-called **Tits building** in Theorem 9.10. This is an abstract simplicial complex with an action of $Aut(\Omega)$. Corollary 9.11 then immediately shows the previously defined $B, N \subset Aut(\Omega)$ to form a so-called **BN-pair** in these cases by standard theory (e.g. [**L05**]).

[3]The author thanks Prof. Humphreys and Prof. Pasechnik for pointing out literature on the stricter notion of a "split" BN-pair in low rank and for laying out the weaker amalgam construction for sporadic simple groups upon my question in MathOverflow (http://mathoverflow.net /questions/93463/weak-bn-pair-tits-system-for-sporadic-groups).

[4]The author thanks Dr. Steinberg for providing an explicit description of the simplicial complex (see below) associated to idempotents of an algebra upon my question in MathOverflow regarding this (http://mathoverflow.net/questions/93862/simplicial-complex-made-of-central-idempotents-of-an-algebra). This direct approach, however, turned out not to be suitable afterwards.

Orbifoldizing Categories

There is an **existing notion** of orbifoldizing equivariant categories by Kirillov [**K04**]. We will connect to this notion[5] by showing, that the category of bicomodule algebras, the Bigalois groupoid as well as the Yetter-Drinfel'd modules, behave accordingly if we pass to the orbifold Hopf algebra.

More specifically, there is a two-step process: The respective categories over H correspond to the **untwisted sectors** of the equivariant category. We will extend them to include twisted sectors consisting of respective **projective representations**, which altogether yields an equivariant category in all cases. Then, Theorems 10.1 resp. 11.2 show the invariant part (Kirillov's orbifoldization) to be categorically equivalent to the respective category over Ω.

Note that again, this is very much inspired by the behaviour of Schur cover groups, the model for our construction: As already mentioned, they have been defined to study projective representations of the smaller groups in terms of ordinary representations of the larger group.

[5]The author thanks Prof. Schweigert for stimulating discussions after a mini-talk the author gave in Oberwolfach 2010, in which he pointed out this notion and asked for a connection, as well as for the invitation to a talk in his Research Seminar (Hamburg 2011) and the discussions afterwards.

Orbifoldizing Quantum fields

Last, we give an outlook on the status of the initial motivation: The construction of the **Monster vertex algebra** purely from Hopf algebra structures. A vertex algebra is an infinite-dimensional, graded structure and certain operator-valued Laurent series given with their product associative only up to δ-functions. It is commonly viewed as axiomatizing quantum field theory operators.

The remarkable vertex algebra in question has as automorphisms the Monster group and as graded dimensions the Fourier coefficients of the modular $j(z)$-function. It was (as a module) constructed by Frenkel, Meurman and Lepowsky in [**FLM84**] and is an important step in Borcherd's proof of the **Moonshine Conjecture**, see for example the extensive survey [**G06**].

We start by an overview of the authors diploma thesis [**Len07**][6], which constructs a vertex algebra from certain rather general Hopf algebra data (Theorem 12.9). We also describe in section 12.3, which Hopf structure leads to the so-called lattice vertex algebras. For the Leech-lattice this is the starting point, which is orbifoldized to the Moonshine Module, that subsequently even supports a rather ad-hoc vertex algebra structure.

The **conjectural aim** is now to perform an orbifoldization **on the Hopf-algebra side** and obtain an infinite-dimensional Nichols algebra still possessing a root system! Then one has to show, that the associated vertex algebra is the desired vertex algebra. Note that already the Moonshine Module construction points to an explicit conjectural twisting 2-cocycle in section 12.2.

Moreover, the BN-pair established above should directly proof the automorphism group to be the monster group - in fact, this assumption gives more valuable hints on the assumed orbifold (see section 12.4).

[6]The author thanks Prof. Schottenloher, supervisor of both thesis', for his long-term support and encouragement even for far-fetched goals, and for the countless hours of stimulating discussions about quantum field theories, vertex algebras, and their connection to various fields of mathematics and theoretical physics.

However, two very severe obstacles appear:

- Section 12.3: The orbifoldization starts with a non-proper "twisting group", yielding only a **quasi-associative Hopf algebra**, with associativity constraint prescribed by the Parker loop.
- Section 12.4: The orbifoldiziation is performed not over a group, but a groupoid Σ of different Doi twist Hopf algebras. Hence we obtain first a **weak Hopf algebra** Ω' (see Remark 1.7) and hope to yield the actual Hopf algebra Ω as an amalgam completion. This should correspond to the well-known BN-pair of the Monster group being non-proper in the sense that $B \cap N$ is not normal in N and the quotient being the Weyl groupoid.

Especially for these two extensions of this thesis, we at present have only vague clues – moreover, up to now, there seems to exist **no theory of Hopf algebra amalgams**.

The author wishes to emphasize again, that this goal is far from being completed and it is very likely, that the aspired approach will not be possible and/or helpful after all! Nevertheless, his supervisor has encouraged the author to (gladly) include these thoughts as an outlook to this thesis.

Part 1

Orbifoldizing Hopf Algebras

Basic Concepts:

Physics, Symmetry And Hopf Algebras

The concept of **symmetry** has been fundamental to physics. Compact Lie groups usually corresponding to **local gauge fields** leading e.g. to the **standard model** of 3 nature forces, namely U_1, SU_2, SU_3 for electromagnetic, weak and strong interaction, unified in a single SU_5. The irreducible representations thereby determine the particle spectrum of the theory and one studies fusion rules of couples of particles by tensoring the representations and again decomposing them into irreducible representations (á la Clebsch Gordan). On the other hand the symmetries of **spacetime** is governed by the noncompact $SL_2(\mathbb{C})$ (covering the Lorentz group $SO_{3,1}(\mathbb{R})$) leading to fields of scalars, spinors, vectors etc and again their respective tensors, such as the field stress.

One may introduce **Hopf algebras** solely by searching for more general algebraic symmetry principles, that still support the fundamental notions of tensoring and dualizing their representations:

Suppose H an algebra of symmetries and a representation/module V or A (with even an algebra structure); the **four main examples** we may want to have in mind are formulated as algebras

- a **discrete group** $H_{Group} = \mathbb{C}[\mathbb{Z}_2]$ (linearly extended)
- a **Lie algebra** $H_{Lie} = U(\mathfrak{sl}_2)$ (multiplicatively extended)

acting typically either on

- a **finite-dimensional representation** V
- the **algebra of functions** on the manifold e.g. (for simplicity) the polynomial ring $A = \mathbb{C}[x, y]$ on $M = \mathbb{C}^2$ with tangent space $V = \langle x, y \rangle_{\mathbb{C}}$.

Widespread examples in physics include the following:

	vector space V	Algebra A
H_{Group}	Anyon models	Reflection g.
		$f(x,y) \overset{g.}{\longmapsto} f(-x,-y)$
H_{Lie}	Particle Multiplets	Lie Derivatives, e.g. $\mathfrak{sl}_2 :\ \mathfrak{L}_E, \mathfrak{L}_F, \mathfrak{L}_H$
	(angular momentum etc.)	$f(x,y) \overset{\mathfrak{L}_E}{\longmapsto} y\frac{\partial}{\partial x} f(x,y), \ldots$

Given two such representations V, V' we form the **tensor representation** $V \otimes_{\mathbb{C}} V$ and the **dual representation** V^* as it is well known:

- **Group elements** g simply act on each tensor factor simultaneously and via g^{-1} on dual elements.
- **Lie algebra elements (primitives)** E act on the tensor factors via Leibniz rule and on dual elements by $-E$.

One should require any additional **algebra structure** $A \otimes A \to A$ to entwine the respective actions defined above (=module homomorphisms). This explains (see above), why group elements act on the algebra of functions naturally as **automorphisms**, while Lie algebra elements act as **derivatives**. Such is called a **module algebra**.

A **Hopf algebra** H in general is now defined to be an algebra with an additional **comultiplication**, **counit** and **antipode**

$$H \xrightarrow{\Delta} H \otimes H$$

$$H \xrightarrow{\epsilon} \mathbb{k}$$

$$H \xrightarrow{S} H$$

As intended, the tensor product, trivial representation ($\mathbb{k}_\epsilon$) and dual representation may be formed via the new action:

$$H \otimes (V \otimes W) \xrightarrow{\Delta} (H \otimes H) \otimes (V \otimes W) \xrightarrow{\rho_V \otimes \rho_W} V \otimes W$$

$$H \otimes \mathbb{k} \xrightarrow{\epsilon} \mathbb{k} \otimes \mathbb{k} \xrightarrow{mult.} \mathbb{k}$$

$$H \otimes V^* \xrightarrow{S} H \otimes V^* \xrightarrow{oh.} V^*$$

The datum (Δ, ϵ, S) of a Hopf algebra comes with certain **compatibility conditions**, that ensure precisely these constructions are well-behaved (let μ be the multiplication of H):

Δ is algebra map	$\Leftrightarrow$	$U \otimes V$ is again representation
Δ is **coassociative** $(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta$	$\Leftrightarrow$	$U \otimes (V \otimes W) \to (U \otimes V) \otimes W$ entwines the H-action
ϵ is algebra map	$\Leftrightarrow$	$\mathbb{k}_\epsilon$ is a representation
Δ, ϵ are **counital** $(id \otimes \epsilon)\Delta = (\epsilon \otimes id)\Delta = \Delta$	$\Leftrightarrow$	$V \otimes \mathbb{k}_\epsilon, \mathbb{k}_\epsilon \otimes V \to V$ entwine both the H-action
S fulfills the **antipode condition** $\mu(S \otimes id)\Delta = \mu(id \otimes S)\Delta = 1_H \cdot \epsilon$	$\Leftrightarrow$	$V^* \otimes V \xrightarrow{eval} \mathbb{k}_\epsilon \xrightarrow{dual} V \otimes V^*$ both entwine the H-action

The classical examples H_{Group}, H_{Lie} fit into this picture by becoming Hopf algebras, if they are endowed with structures exactly matching the rules given above:

$$\Delta(g) = g \otimes g \qquad \epsilon(g) = 1 \qquad S(g) = g^{-1}$$
$$\Delta(E) = 1 \otimes E + E \otimes 1 \qquad \epsilon(E) = 0 \qquad S(E) = -E$$

Note that under certain conditions, there is even an equivalence! There exist two generalizations ("weak quasi-Hopf algebras“ allowing e.g. a nontrivial F-matrix) that exhaust at least all tensor categories with finitely many simple objects [**EO03**].

We conclude by introducing important subsets for a Hopf algebra H::

- By Δ being an algebra map (and S giving an inverse) the set of all **grouplike** elements $g \in H$ with $\Delta(g) = g \otimes g$ of a Hopf algebra H forms a group $G(H) \subset H$.
- More generally, the sum of all simple subcoalgebras, i.e. minimal in being stable under Δ, ϵ, is called **coradical** and is the dual (co-) version of the Jacobson radical in algebra. As each grouplike $g \in G(H)$ for itself is already stable (i.e. a 1-dimensional subcoalgebra) the coradical contains $\mathbb{k}[G(H)]$. If they even coincide, we call the Hopf algebra **pointed**.
- Moreover, elements with $\Delta(X) = g \otimes X + X \otimes h$ for g, h grouplike are called **skew-primitives** , and they correspond to skew-derivational action with respect to some additionally existing automorphisms (i.e. grouplikes) g, h.

In this thesis we mainly concern ourselves with pointed Hopf algebras. Their classification (especially for abelian groups $G(H)$) is addressed in the second part's introduction, leading directly to Nichols algebras.

group

Bigalois Objects As Twisted Hopf Algebras

One can easily define 2-cocycles over an arbitrary Hopf algebra H:

$\sigma:\ H\otimes H \to \Bbbk^\times$ with $\sigma(x^{(1)},y^{(1)})\sigma(x^{(2)}y^{(2)},z) = \sigma(y^{(2)},z^{(2)})\sigma(x,y^{(1)}z^{(1)})$

However, in contrast to the cocommutative case (e.g. a group), they do **not** form a group! Rather, one has to simultaneously consider 2-cocycles over different, slightly deformed Hopf algebras. Their product is only again a 2-cocycle, if they "fit together" as we shall see now:

Definition. *A* **groupoid** Σ *is a category, such that every morphism is an isomorphism. Especially a group is presented as a single object* $\mathfrak{O}$ *with the group being* $Mor(\mathfrak{O},\mathfrak{O})$.

Instead of dealing with the actual 2-cocycles, one usually considers:

Definition. *A* H-L-**Bigalois object** *between Hopf algebras* H, L *is a* **bicomodule** *A between them, with an* **algebra** *structure on A compatible with left-*H*- and right-*L*-coaction:*

$$\delta_L:\ A \to H\otimes A \quad \delta_L(ab)=\delta_L(a)\delta_L(b) \quad \delta_L(1_A)=1_H\otimes 1_A$$

$$\delta_R:\ A \to A\otimes H \quad \delta_R(ab)=\delta_R(a)\delta_R(b) \quad \delta_R(1_A)=1_A\otimes 1_H$$

Thirdly, both sides need to satisfy a **nondegeneracy**, *namely the canonical map* $A\otimes A\to H\otimes A$ *shall be bijective:*

$$can:(a\otimes b)\to a^{(0)}\otimes a^{(1)}b$$

We call the set of isomorphism class of H-L*-Bigalois objects* $BiGal(H,L)$.

We will show how $BiGal(H,L)$ forms a groupoid and how this can be used instead of bare 2-cycles. A comprehensive reference is [**S04**].

Definition. *Two* **fitting** *Bigalois objects, i.e.* $A\in BiGal(H,L)$, $B\in BiGal(L,E)$ *may be* **cotensored** *over* L *to get a new Bigalois object:*

$$\{a\otimes b\in A\otimes B \mid (\delta_R\otimes id)(a\otimes b)=(id\otimes\delta_L)(a\otimes b)\} =: A\Box_L B\in BiGal(H,E)$$

In several instances (e.g. H pointed or finite-dimensional), all Bigalois objects are **cleft**, meaning that each is isomorphic to H as left comodule via a **cleaving map** (if a Bigalois object is even isomorphic to H as bicomodule, it is called **bicleft**).
Any cleft Bigalois object is isomorphic as left comodule algebra to a cocycle-twists ${}^\sigma H$:

- The left H-coaction coincides with the coproduct on H.
- The multiplication is deformed by a 2-cocycle $\sigma \in Z^2(H, \Bbbk^\times)$.

$$a \cdot_{^\sigma H} b := \sigma(a^{(1)}, b^{(1)})a^{(2)}b^{(2)}$$

- Every such one-sided Galois object may be non-uniquely completed to a Bigalois object in $BiGal(H, L)$ for a unique Hopf algebra L (this is generally true). For cocycle-twists, L turns out to be the **Doi twist** H_σ, which H as coalgebra with doubly deformed multiplication

$$a \cdot_{H_\sigma} b := \sigma(a^{(1)}, b^{(1)})a^{(2)}b^{(2)}\sigma^{-1}(a^{(3)}, b^{(3)})$$

  which can be proven to be again a Hopf algebra.

Throughout this work, this Doi twist appears as "mild modification" of a Hopf algebra structure (i.e. to change the Σ-action in section 5.2). Especially, their categories of modules are equivalent. It should not be confused with the "twisted" Bigalois object!

Finally, $H \in BiGal(H, H)$ itself (and all it's Doi twists $L \in BiGal(L, L)$) are respective **units** and for any $A \in BiGal(H, L)$ there is an **inverse** Bigalois object $B \in BiGal(L, H)$ such that

$$A\square_L B \cong H \qquad B\square_H A \cong L$$

Hence taking as objects all Doi twists of some given H and as morphisms all Bigalois objects between them, multiplied via $\square$, we obtain the **Bigalois groupoid** $BiGal(H)$.

Example. *In case $H = \Bbbk[G]$ (or another cocommutative H) there are no nontrivial Doi twists ($L = H$), and we get a Bigalois group:*

$$BiGal(H) = BiGal(H, H) \cong Aut(G) \ltimes H^2(G, \Bbbk^\times)$$

Here, the algebra ${}^\sigma H$ defined above is the well-known **twisted groupring** $\Bbbk_\sigma[G]$*, while the additional automorphism corresponds to different right H-coaction to choose from (we mentioned the "completion" is non-unique).*

More generally, the subgroup of the groupoid $BiGal(H, H) \subset BiGal(H)$ correspond to so-called **lazy 2-cocycles** σ.

Technical Overview On Methods & Results

This part describes the author's abstract concept of orbifoldizing Hopf algebras. It starts with Definition 1.5 of a categorical context "twisting group", that stages the general setting, where our ansatz works.

The **basic idea** is to take a finite, abstract subgroup(oid) Σ of twistings (e.g. inside the Bigalois groupoid $BiGal(H)$) with coherently chosen isomorphisms of the underlying twisted objects' multiplication:

$$A_p \Box A_q \overset{\iota_{p,q}}{\cong} A_{pq} \qquad p,q \in \Sigma$$

This can be cleanly formulated as a bifunctor between two bicategories. We then construct in several steps (sections 1.3 to 1.5) from such context a new Hopf algebra Ω composed as a direct sum of the $|\Sigma|$ differently twisted algebras $A(p),\ p \in \Sigma$ of a smaller given one $H = A(e)$, with a "mixed" coalgebra structure extended from H by the demanded fixed isomorphisms $\iota_{q,r}^{-1}$ for each $p = qr$:

$$A(p) \xrightarrow{\iota_{q,r}^{-1}} A(q) \Box A(r) \subset A(q) \otimes A(r)$$

Main Theorem (1.6)**.** *Given a twisting group Σ of H, the Ω defined above is a Hopf algebra and H-H-bicomodule algebra. We have a Hopf algebra surjection and injection:*

$$i_\Sigma : \Bbbk^\Sigma \to \Omega \qquad \pi_H : \Omega \to H$$

Thus this **basic construction** can be understood as practically forming the dual groupring of Σ

$$\Bbbk^\Sigma = \bigoplus_{p \in \Sigma} e_p k$$

but instead of (1-dimensional) primitive idempotents $e_p \overset{\Delta}{\mapsto} \sum_{p=qr} e_q \otimes e_r$ we use the entire algebras $A(p)$. Ω has therefore $H = A(e)$ as quotient (untwisted sector) and further contains new idempotents $e_p = 1_{A(p)}$ forming the dual groupring of the twisting group $\Bbbk^\Sigma \subset \Omega$.

We then quickly turn in section 1.6 to a concrete realization of the abstract twisting setting as one well known to Hopf algebra theory, namely Bigalois objects of H. This case has the particularly nice property of small coinvariants and thus we find:

Theorem (1.13). *We have an exact sequence of Hopf algebras*

$$\Bbbk \to \Bbbk^\Sigma \xrightarrow{i_\Sigma} \Omega \xrightarrow{\pi_H} H \to \Bbbk$$

The embedding s of H as $A(e)$ is a cleaving/section, hence this central extension is cleft. Then Ω is isomorphic to a bicrossed product

$$(\Bbbk^\Sigma)^{\tau,\rho}\#_{\sigma,1}H$$

Our ansatz can hence be **alternatively understood** as to produce a bicrossed product datum (obeying rather complicated compatibilities) from a suitable group of Bigalois objects with fixed isomorphisms $\iota_{p,q}$.

We proceed in section 1.7 with the first example of H,Ω being grouprings. We recover our initial motivation (Schur cover group) of Ω begin a centrally extended groupring, now even as a Hopf algebra. A **curious occurrence** compared to the classical Schur cover (that uses only cohomology classes) is, that the specific choice of a subgroup of 2-cycles necessary to define the coalgebra structure on the orbifold, already pins down the group elements in the Schur cover groupring. Thereby it determines a specific Schur cover group; note that this is in general not unique (despite the fixed isomorphy class of the algebra structure). For example are $\Bbbk[\mathbb{D}_4]\cong\Bbbk[\mathbb{Q}_8]$ the two Schur covers of $\mathbb{Z}_2\times\mathbb{Z}_2$.

We then should turn our attention to the influence of orbifoldization to a couple of characteristic subsets of the Hopf algebra (chp. 2), namely the **coradical**, the **grouplikes** and the **skew-primitives**. In each case we describe their behaviour (Theorems 2.4 and 2.9) and give precise conditions ensuring that certain aspired properties hold still in the orbifoldization, namely pointedness (Corollary 2.5) and link-indecomposability (Theorem 2.10).

Explicitly we will find altogether:

- Dimension is controlled by $dim(\Omega) = dim(H) \cdot |\Sigma|$.
- Semisimplicity of the algebra H is preserved (Remark 2.1).
- The group of grouplikes undergoes a central extension

$$\Sigma^* \longrightarrow G := G(\Omega) \longrightarrow \Gamma := G(H)$$

prescribed by restricting the twisting 2-cocycle to the group.

$$res: \ \Sigma \to H^2(\Gamma, \Bbbk^\times)$$

- Pointedness survives if among others Σ abelian (see below).
- The vector space $M = H_1/H_0$ of skew-primitives (modulo trivials) in H is preserved, but the homogeneous components decompose into eigenspaces of Σ acting as twisted symmetries.
- Generation only by grouplikes and skew-primitives (-derivations) is preserved under the same conditions as pointedness.
- The number of link-components in H grows moderately, bounded by $Ker(res)$ and thus:
- Link indecomposability of H is preserved in Ω, if G is even a stem-extension $\Sigma \subset G'$ whence $Ker(res) = 0$.

The **proof idea** is to (quite) uniquely pin down (cleft images of) grouplikes resp. skew-primitives in any Bigalois object solely in terms of their coaction, while left-to-right some nontrivial correspondence may apply, which leads to an action of the group of Bigalois objects Σ on $G(H)$ as automorphism resp. on $Prim(H)$ as **twisted symmetries**. All this is technically done in the Lemma 2.2 resp. 2.8 using the Galois property. These observations determine the coradical Ω_0 resp. skew-primitives Ω_1 by proving them to be sub-orbifolds of H_0 resp. H_1 inside Ω.

The preceding study of the groupring's orbifold behaviour (Lemma 1.17) then gives quickly precise conditions for Ω to again be pointed:

Corollary (2.5)**.** *Let H be pointed and finite-dimensional, then an orbifold of Bigalois objects Ω is pointed iff Σ is abelian and the above restriction of the twisting group to the grouplikes $(G(p))_{p\in\Sigma}$ is bicleft.*

We will refer to these conditions as the **usual setting***, under which we will work throughout the rest of this thesis.*

On the other hand, in Theorem 2.9 we find the new skew-primitives by simultaneously diagonalizing the twisted symmetries (Σ now abelian!). Thereby the space of $\bar{g}, 1$-skew-primitives decomposes into eigenspaces to different eigenvalues λ, corresponding to different liftings of $\bar{g} \in \Gamma := G(H)$ to the central extension $g \in G := G(\Omega)$. Especially for stem-extension $\Sigma^* \subset G'$ (and more generally Frattini extensions) we can use, that any such lift choices of generating $(g_i)_{i\in\mathbb{I}} \in G(H)$ generate the extension, hence Ω may also again be link-indecomposable.

To apply our construction, in section 3.1 we restrict ourselves to the later-on most relevant case of H a Radford biproduct $\mathbb{k}[\Gamma]\#\mathcal{B}(M)$ of the group with a Nichols algebra. We solely use group 2-cocyces σ extended trivially to all of H and thereof construct a general twisting group in Theorem 3.2. This orbifoldizing of Nichols algebras will be discussed extensively in the second part of this thesis.

Finally in Theorem 3.6 we prove conversely, that a Hopf algebra is an orbifoldization for a given central $\Sigma^* \subset G$, iff Σ^* is central in all of Ω. The proof uses a variant of Masuoka's push-out construction [**M01**]. We shall exploit it in the second part to reversely disprove existence of finite-dimensional Nichols algebras over some larger G by writing it as an orbifold from the central quotient Γ. Note that this rather trivial behaviour is (in both directions) enhanced by considering also Doi twists of orbifolds!

CHAPTER 1

Categorically Orbifoldizing

We start by describing our construction in an abstract, categorical manner and give an explicit realization by Bigalois objects in section 1.6:

The notion of **bicategories** (defined by Bénabou in [**B67**]) will be used in the following to combine the structure of a groupoid (only "fitting ends" may be multiplied) with an enrichment of the arrows to being objects in a new category, including nontrivial "second-order" morphisms.

The reader should keep in mind e.g. the **Morita category** of rings with bimodules, where we may tensor such bimodules over one ring and get associativity up to bimodule isomorphisms.

1. Bicategories

Definition 1.1 ([**B67**] p. 3-6)**.** *A* **bicategory** $\mathcal{S}$ *consists of a set of* **points** $\mathcal{S}_0$*, where for each pair* $H, L \in \mathcal{S}_0$ *a category* $\mathcal{S}(H, L)$ *is defined. We call its objects* p, q **edges** *and its morphisms* **2-cells** *(or just* **morphisms***). Additionally, the data includes given* **identity edges** $I_H \in \mathcal{S}(H, H)$ *and* **composition functors***:*

$$\mathcal{S}(H, L) \times \mathcal{S}(L, E) \overset{c_{HL,LE}}{\longrightarrow} \mathcal{S}(H, L)$$

Especially for two edges p, q *we thus get a composite, denoted* $p \otimes_{\mathcal{S}} q$*.*

Additionally one demands respective natural transformations:

- **Associativity** *isomorphism* α*:*

  $$c_{HE,EF} \circ (c_{HL,LE} \times id_{\mathcal{S}(E,F)}) \cong c_{HL,LF} \circ (id_{\mathcal{S}(H,L)} \times c_{LE,EF})$$

  such that the pentagonal identity holds.
- **Left/right identities** *isomorphisms:*

  $$c_{HL}(I_H \times id_{\mathcal{S}(H,L)}) \cong Id \cong c_{HL}(id_{\mathcal{S}(H,L)} \times I_L)$$

  such that the triangular identities hold.

Especially there is the **step-down category** $\bar{\mathcal{S}}$*, an ordinary category with objects* $H \in \mathcal{S}_0$ *and morphisms the isomorphy classes* $[\mathcal{S}(H,L)]$ *of edges. For bimodules e.g. this yields the usual Morita category.*

Remark 1.2. *Nowadays, edges are often called horizontal morphisms opposed to the (vertical) morphisms. The composition is often as well denoted as horizontal.*

As bicategories are 2-categories with non-strict associativity, they are sometimes called weak 2-categories.

Definition 1.3 ([**B67**] p. 29f)**.** *A* **bicategory functor** *between bicategories* $\mathcal{S} \to \mathcal{C}$ *consists of the following data:*

- *A map* $A : \mathcal{S}_0 \to \mathcal{C}_0$
- *Functors* $A(H,L) : \mathcal{S}(H,L) \to \mathcal{C}(A(H), A(L))$*. We denote the specialization to an object (edge)* $p \in \mathcal{S}(H,L)$ *by* $A(p)$*.*
- *For each point* $H \in \mathcal{S}_0$ *an identity morphisms* $I_{a(H)} \to A(H,H)I_H$
- *A family of natural transformations*

  $$c^{\mathcal{C}}_{a(H)a(L),a(L)a(E)} \circ (A(H,L) \times A(L,E)) \xrightarrow{\iota} A(H,E) \circ c^{\mathcal{S}}_{HL,LE}$$

  We denote the specialization of this transformation to some objects $(p,q) \in \mathcal{S}(H,L) \times \mathcal{S}(L,E)$ *by*

  $$\iota_{p,q} :\ A(p) \otimes_{\mathcal{C}} A(q) \to A(p \otimes_{\mathcal{S}} q)$$

Furthermore, for each triple $(p,q,r) \in \mathcal{S}(H,L) \times \mathcal{S}(L,E) \times \mathcal{S}(E,F)$ *we have the* **coherence condition***:*

$$A\alpha_{\mathcal{S}}(p,q,r) \circ \iota_{p\otimes_{\mathcal{S}} q,r} \circ (\iota_{p,q} \otimes_{\mathcal{S}} id_{A(r)}) = \iota_{p,q\otimes_{\mathcal{S}} r} \circ (id_{A(p)} \otimes_{\mathcal{S}} \iota_{q,r}) \circ \alpha_{\mathcal{C}}(A(p), A(q), A(r))$$

and a similar coherence for the identity morphisms.

2. Twisting groups

For our purposes, a bicategory functor (see above) $A :\ \mathcal{S} \to \mathcal{C}$ is basically a functor between the step-down-categories of points and isomorphy classes of edges $\bar{A} :\ \bar{\mathcal{S}} \to \bar{\mathcal{C}}$. However, A has to assign to an edge $p \in \mathcal{S}(H,L)$ a **specific representative** $A(p) \in \mathcal{C}(A(H), A(L))$ and for edge concatenation a **specific second-order-morphisms**

$$\iota_{p,q} :\ A(p) \otimes_{\mathcal{C}} A(q) \to A(p \otimes_{\mathcal{S}} q)$$

The following notion should be viewed as some formalized group of "twistings" of a Hopf algebra, i.e. the set of twistings bears a group structure - and its structural maps carry over to maps between the "twisted algebras" (multiplication to ι and inversion to ρ). It is designed solely to enable the next subsections' constructions.

Definition 1.4. *A* **twisting semigroupoid** (Σ, A, ι) *for a finite semigroupoid* Σ *(i.e. an ordinary category) is a bifunctor* (A, ι) *between the following bicategories* $\mathcal{S} \to \mathcal{C}$*:*

- $\mathcal{S}$ *the bicategory with points* $H \in Obj(\Sigma)$*, edges* $p \in Mor(\Sigma)$*, and only the respective identity* I_H *as morphisms on each edge.*
- $\mathcal{C}$ *the bicategory where points* H, L *are Hopf algebras, edges in* $\mathcal{C}(H, L)$ *are* H*-*L*-bicomodule algebras, concatenations* $\otimes^{\mathcal{C}}$ *are the respective cotensor-products, the identity edges* $I_H \in \mathcal{C}(H, H)$ *are* H *with the natural* H*-*H*-bicomodule structure given by* (Δ_H, Δ_H) *and morphisms are bicolinear algebra maps.*
- *such that the* $\iota_{p,q}$ *are bijective and for* $p \in \mathcal{S}(H, L)$ *the maps* $\iota^{-1}_{I_H,p}, \iota^{-1}_{p,I_L}$ *coincide with the left/right comodule maps on* $A(p)$*.*

(we usually identify the points $H \in \mathcal{S}_0$ *with the Hopf algebra* $A(H)$ *and the bicomodule algebra* $I_{A(H)} \cong A(I_H)$*)*

While this will be sufficient to define (possibly weak) orbifold bialgebras, we need an additional datum to obtain an antipode. For Bigalois objects (section 1.6) **this can be derived solely from the data above** (as proven in Lemma 1.11), but in the general case the author does neither see a proof, nor a solid categorical definition.

Definition 1.5. *A* **twisting groupoid** (Σ, A, ι, ρ) *is a twisting semigroupoid for a finite groupoid* Σ*, and for each* Σ*-edge* p *two* $\mathbb{k}$*-linear maps (that will turn out to be actually equal in section 1.5):*

$$\rho_p^{L,R} : A(p^{-1}) \to A(p)$$

such that with $\mu_{A(p)}$ *the algebra multiplication the following holds:*

$$\mu_{A(p)}(\rho_p^L \otimes id_{A(p)})\iota^{-1}_{p^{-1},p} = 1_{A(p)}\epsilon_H = \mu_{A(p)}(id_{A(p)} \otimes \rho_p^R)\iota^{-1}_{p,p^{-1}}$$

(this implies especially $\rho^R_{I_H} = \rho^L_{I_L} = S_H$*)*

If Σ *is a proper group (i.e.* $Obj(\Sigma) = \{\mathfrak{O}\}$*) we briefly call* Σ *a* **twisting group** *of* $H := A(\mathfrak{O})$*. In this case, we abbreviate as usual* $p \in \Sigma(\mathfrak{O}, \mathfrak{O})$ *by* $p \in \Sigma$ *and the unique unit* $I_{\mathfrak{O}}$ *by* e*.*

3. The Bicomodule Algebra

Given a twisting group, we define Ω as H-H-bicomodule algebra to be the direct sum of all "twisted" bicomodule algebras $A(p)$:

$$\Omega(\Sigma) = \bigoplus_{p\in\Sigma} A(p)$$

Clearly the sum of all proper twistings $\bigoplus_{p\neq e} A(p)$ is an ideal and sub-bicomodule, so we have the following bicomodule algebra surjection (splitting multiplicatively and as a bicomodule map, but **not** unit-preserving via the obvious inclusion $s : H = A(e) \to \Omega$):

$$\pi_H : \Omega \to A(e) = H$$

There's also the later most relevant algebra inclusion of the dual groupring of Σ obviously landing in the coinvariants and the center of Ω:

$$i_\Sigma : \Bbbk^\Sigma \ni \phi \mapsto \sum_{p\in\Sigma} \phi(p) 1_{A(p)} \in \Omega(\Sigma)$$

4. The Coalgebra

Since Ω is a direct sum of $A(p)$ we define Δ, ϵ on each of them:

$$\Delta|_{A(p)} : A(p) \to \bigoplus_{p=qr\in\Sigma} A(q) \otimes A(r) \subset \Omega \otimes \Omega$$

We define this map by piecing together the demanded ι^{-1}-morphisms

$$\Delta|_{A(p)} = \bigoplus_{p=qr\in\Sigma} \iota^{-1}_{q,r}$$

We further choose $\epsilon|_{A(e)} = \epsilon_H$ and zero on all other $A(p)$.

Coassociativity: This follows directly after restricting to a summand $A(p)\otimes A(q)\otimes A(r)$ from the pentagonal identity satisfied by the ι's and the associativity of Σ.

Counitality: This holds, because on every summand $A(p)$:

$$(id_\Omega \otimes \epsilon)\Delta = (id_\Omega \otimes \epsilon) \bigoplus_{p=qr\in\Sigma} \iota^{-1}_{q,r} = (id_{A(p)} \otimes \epsilon_H)\iota^{-1}_{p,e} = id_{A(p)}$$

as all summands except $p = pe$ vanish by definition of $\epsilon = \epsilon_\Omega$, while the last equation follows from $\iota^{-1}_{p,e}$ being the H-comodule map. The other way around works identical.

Note further that for any $A(p)$ with $p \neq e$ the coproduct always has left **or** right tensor factors in some $A(q)$ with $q \neq e$. So $\bigoplus_{p\neq e} A(p)$ is also a coideal and π_H a coalgebra map. We calculate easily, that the inclusion $i_\Sigma : \Bbbk^\Sigma \to \Omega$ is also a coalgebra map:

Since the ι are algebra maps, clearly $\iota^{-1}_{p,q}(1_{A(pq)}) = 1_{A(p)} \otimes 1_{A(q)}$. Now

$$\Delta(i_\Sigma(\phi)) = \Delta(\sum_{p\in\Sigma} \phi(p)1_{A(p)}) = \sum_{p=qr\in\Sigma} \phi(qr)1_{A(q)} \otimes 1_{A(r)} =$$

is by construction of the $\Bbbk^\Sigma$-coproduct $\phi(qr) = \phi^{(1)}(q) \otimes \phi^{(2)}(r)$

$$= \sum_{q,r\in\Sigma} \phi^{(1)}(q)1_{A(q)} \otimes \phi^{(2)}(r)1_{A(r)} = i_\Sigma(\Delta_{\Bbbk^\Sigma}(\phi))$$

Furthermore $\epsilon(i_\Sigma(\phi)) = \phi(e) = \epsilon_{\Bbbk^\Sigma}(\phi)$.

5. The Hopf Algebra

Let us first check the bialgebra axioms:

ϵ is an algebra map: This is clear since we defined it induced by the algebra map ϵ_H on a direct summand of the algebra.

Δ is unital: This is the consequence of the group law in Σ:

$$\Delta(1_\Omega) = \sum_{p\in\Sigma} \sum_{p=qr} 1_{A(q)} \otimes 1_{A(r)} = \sum_{q,r\in\Sigma} 1_{A(q)} \otimes 1_{A(r)} = 1_\Omega \otimes 1_\Omega$$

Here we used again that $\iota^{-1}_{q,r}(1_{A(qr)}) = 1_{A(q)} \otimes 1_{A(r)}$.

Δ is multiplicative: We have to distinguish two cases for any $a \in A(p), b \in A(q)$ (which again suffices by linear extension): Let first be $p \neq q$. Then $\Delta(ab) = \Delta(0) = 0$ by construction of the algebra. But $\Delta(a)$ and $\Delta(b)$ consist of elements in the spaces A of the respective decompositions of p, q, and their tensor factors cannot lay in the same A on both sides simultaneously, since the factors determine their product in Σ uniquely. Thus the product of any elementary tensors in $\Delta(a)$ with $\Delta(b)$ also vanishes. Now take $p = q$. By the argument above, the only non-vanishing products of elementary tensors in $\Delta(a), \Delta(b)$ are the ones in the very same decompositions of p, i.e. for every ι^{-1} separately. But these were bicomodule algebras maps, yielding each respective summand of $\Delta(ab)$.

Finally we are coming to the **antipode** $S := \bigoplus_{p\in\Sigma} \rho_p^{L,R}$. Note that we actually get a-priori different left and right antipodes, but as $*$-inverses of id_H, they have to coincide. Let us thus check its defining property

$$\mu(S\otimes id)\Delta = 1_\Omega\epsilon = \mu(id\otimes S)\Delta$$

On all direct summands $A(p), p\neq e$, ϵ vanishes. But for any such $a\in A(p)$ the expression $S(a^{(1)})\otimes a^{(2)}$ takes values in the sum of $A(q^{-1})\otimes A(r)$ over all $p = qr$ and thus the two tensor factors lay in the same direct summand iff $q^{-1} = r$ which is impossible for $p \neq e$. Thus all products in $S(a^{(1)})a^{(2)}$ also vanish. The other way around is totally analogous.

On the unit summand $H = A(e)$ however, for any $h\in A(e)$, the expression $S(a^{(1)})\otimes a^{(2)}$ is a sum of products coming from $A(p)\otimes A(p)$ for all possible $p\in\Sigma$ ($e = p^{-1}p$). So to prove $S(a^{(1)})a^{(2)} = \epsilon(a)1_\Omega$ we can restrict ourselves to any $A(p)$. But there it follows from the defining condition on ρ_p^L. Again the other way around is analogously for ρ^R. Summarizing the preceding section we have proven:

Theorem 1.6. *Given a twisting group of H, then the* **orbifold** *Ω defined step-by-step in the preceding sections is a Hopf algebra and H-H-bicomodule algebra. We have a Hopf algebra injection and surjection:*

$$i_\Sigma : \mathbb{k}^\Sigma \to \Omega \qquad \pi_H : \Omega \to H$$

Remark 1.7. *Note without proof that if Σ were a general twisting groupoid, we expect to obtain* **weak Hopf algebras [EO03]** *with 1 defined as above and ϵ the sum of all $\epsilon_{A(H)} \cong \epsilon_{A(id_H)}$ for all base objects $H\in Obj(\Sigma)$. Especially, Ω contains the weak Hopf algebra dual to $\mathbb{k}^\Sigma$.*

Note that Theorem 1.13 generally shows that $Ker(\pi_H) = Im(i_\Sigma)^+\Omega$ and the embedding $s : H = A(e) \to \Omega$ is a cleaving/section. However, the coinvariants may be considerably larger, if not obtained from Bi-galois objects. Hence these maps generally form no exact sequence in the sense of e.g. [**A96**] p. 7.

Remark 1.8. *Note without further details that by construction Ω is even a* **functor** *from twisting groups to Hopf algebras, where morphisms of twisting groups are natural transformations between the respective underlying bicategory functors A, A'. The maps π_H, i_Σ are special cases thereof for the trivial twisting groups $(\Sigma, A(e)) = (\{e\}, H)$ resp. $(\Sigma, \mathbb{k})$.*

6. Realization Via Bigalois Objects

We now want to give an explicit realization of the categorical data demanded above and interpret the resulting orbifold (as defined above) to be a bicrossed product. The proper generalization of twisted grouprings to arbitrary Hopf algebras H are the Galois objects resp. 2-cocycles, which however fail to form a group. So one usually considers (isomorphy classes of) Bigalois objects, and these form a groupoid $BiGal(H)$ via the cotensor-product $\Box_H$ studied extensively (see **[S04]**). Turning this situation into a twisting group(oid) has been the model to our definition, but a certain technical "choice-" problem arises:

Definition 1.9. *As a* **twisting group(oid) of Bigalois objects** *, we understand a group(oid)-morphism $\bar{A}: \Sigma \to BiGal(H)$ and specifically chosen representatives $A(p)$ (with each $A(I_H) = H$) and specifically chosen isomorphisms ι realizing the $\Box$-multiplication of Bigalois objects*

$$\iota_{p,q} : A(p)\Box_H A(q) \to A(pq)$$

obeying pentagonal identity. The bifunctor and ρ are constructed below!

While the **existence** of the ι is already guaranteed by the structure of the Bigalois groupoid, they are **not unique** and these ambiguities could easily result in the ι-pentagonal identity to fail - there simply may not be a natural all-at-once choice. Hence we can not simply write down twisting groups from $\bar{A}$ without additional knowledge. The main cases where we **can** are the **bicleft/lazy Bigalois objects** in Lemma 1.14, which we will use to determine the coradical of Ω in Theorem 2.4.

Remark 1.10. *To resolve the issue of uniqueness generally, by a helpful comment of A. Masuoka, we fix directly a specific twisting* **2-cocycle** *$\sigma \in Z^2(H)$ in its cohomology class, thereby arriving in an extension of the original Bigalois group by the 2-borders.*

This can be done conceptually well by fixing a so-called cleaving $j_A : H \to A$ yielding immediately a specific 2-cocycle representative. The cleaving on a product $A\Box B$ is thereby defined as $(id_A \otimes j_B) \circ \delta_A^R \circ j_A$.

This is the line of action, we will take in section 3.1, especially Lemma 3.4 (product cleaving) to construct twisting groups, that will lead to the further study of orbifoldizing Nichols algebras in the second part.

It may, however, in more general cases cause a Bigalois object to appear multiple times, corresponding to different cohomologous cocycles respectively cleavings. To see an example for this, see Remark 1.16.

Otherwise, this is the only obstruction and we obtain a twisting group:

Lemma 1.11. *Given a set of choices for the $\iota_{p,q}$ above fulfilling the conditions in Definition 1.5, we can obtain suitable $\rho_p^{R,L}$ from the respective left and right can-maps on each Bigalois object $A(p)$*

Corollary 1.12. *Thus, the data in Definition 1.9 defines a bifunctor A (Definition 1.3) and hence a twisting group (Definition 1.5):*

- *The map on points is clear from $\bar{A}$.*
- *The functors $A(H,L)$ map each edge $p \in \Sigma$ to the chosen representative $A(p)$ and the only trivial morphism accordingly.*
- *By choice of $A(I_H) = H$ the identity morphism is strict. Thus, also the respective coherence condition is trivial.*
- *The associativity constraint $\alpha^{\mathcal{S}}$ in Σ is strict, whereas $\alpha^{\mathcal{C}}$ comes from vector spaces. Hence a natural transformation ι satisfying coherency is given by the $\iota_{p,q}$ satisfying the pentagonal identity.*

PROOF. Viewing $A(p)$ as **right** Galois object yields the well-known

$$\mathrm{can}^{-1}(1_{A(p)} \otimes -) : H \ni h \mapsto h^{[1]} \otimes h^{[2]} \in A(p) \otimes A(p)$$

We omit the first argument in what follows! Define ρ_p^L by:

$$A(p^{-1}) \overset{\iota^{-1}_{e,p^{-1}}}{\to} H\square A(p^{-1}) \overset{\mathrm{can}^{-1}}{\longrightarrow} A(p)\otimes A(p)\square A(p^{-1}) \overset{\iota_{p,p^{-1}}}{\to} A(p)\otimes H \overset{\epsilon_H}{\to} A(p)$$

Note that we omitted the brackets on the (co-)tensor factors, because *can* is right colinear with the comodule structure of the tensor product induced by the right factor (**[S04]** Lemma 2.1.7). We have to verify the defining condition from the pentagonal identity of the ι:

$$(\rho_p^L \otimes id_{A(p)})\iota^{-1}_{p^{-1},p} = (1_{A(p)} \otimes \epsilon_H \iota_{p,p^{-1}} \otimes id_{A(p)})(\mathrm{can}^{-1} \otimes \iota^{-1}_{p^{-1},p})\iota^{-1}_{e,e}$$

Since $\iota^{-1}_{e,e} = \Delta_H$ and again using the above right colinearity of can^{-1} (in the right factor) we furthermore have:

$$= (1_{A(p)} \otimes \epsilon_H \iota_{p,p^{-1}} \otimes id_{A(p)})(id_{A(p)} \otimes id_{A(p)} \otimes \iota^{-1}_{p^{-1},p})(id_{A(p)} \otimes \iota^{-1}_{p,e})\mathrm{can}^{-1}$$
$$= \mathrm{can}^{-1}$$

In the last equation we simplified (again by pentagonal identity) the maps on in the right factors $(\epsilon_H \iota_{p,p^{-1}} \otimes id_{A(p)})(id_{A(p)} \otimes \iota^{-1}_{p^{-1},p})\iota^{-1}_{p,e}$ to

$(\epsilon_H \otimes id_{A(p)})\iota_{e,p}^{-1}$ which is by counitality just $id_{A(p)}$.

This proves the defining condition by **[S04]** lemma 2.1.7:

$$\mu_{A(p)}(\rho_p^L \otimes id_{A(p)})\iota_{p^{-1},p}^{-1} = \mu_{A(p)}\mathrm{can}^{-1} = 1_{A(p)}\epsilon_H$$

Analogously we may consider $A(p)$ as left Galois object to obtain ρ^R. □

We end this section by describing an additional property of Ω, that is true when we obtain the twisting group from Bigalois objects as described above. We already saw that i_Σ lands in the center and the H-coinvariants of Ω. In the Bigalois case, this is already all of Ω^{coinv}:

Theorem 1.13. *We have an exact sequence of Hopf algebras*

$$\Bbbk \to \Bbbk^\Sigma \xrightarrow{i_\Sigma} \Omega \xrightarrow{\pi_H} H \to \Bbbk$$

The embedding s of H as $A(e)$ is a cleaving/section, hence this central extension is cleft. By [**A96**] *p. 17 then Ω is isomorphic to a bicrossed product*

$$(\Bbbk^\Sigma)^{\tau,\rho}\#_{\sigma,1}H$$

Proof. The maps are by construction injective resp. surjective. We first show generally that

$$Ker(\pi_H) = Im(i_\Sigma)^+\Omega = i_\Sigma(Ker(\epsilon_\Sigma))\Omega$$

"$\supset$" follows easily, as the $\phi \in \Bbbk^\Sigma$ with $0 = \epsilon_\Sigma(\phi) = \phi(e)$ are precisely those with $i_\Sigma(\phi)$ vanishing on $A(e)$. By considering the basis of primitive idempotents $e_p \in \Bbbk^\Sigma$ we have $i_\Sigma(e_p)\Omega = A(p)$, hence they generate (for $p \neq e$) already all of $Ker(\pi_H)$.

Secondly, in the case of the $A(p)$ being Bigalois objects we have:

$$\Omega^{coinvH} = \bigoplus_{p\in\Sigma} A(p)^{coinvH} = \bigoplus_{p\in\Sigma} 1_{A(p)}\Bbbk = Im(i_\Sigma)$$

Hence the sequence is exact in the sense of [**A96**] p. 7. Finally we note that the natural embedding $s : H = A(e) \to \Omega$ is a cleaving/section in the sense of [**A96**] p. 18: It is clearly colinear and $*$-invertible via $s \circ S_H$, as s is an algebra map. □

7. Example: Grouprings

Let us discuss a easy situation where we can immediately write down a twisting theory, namely for **lazy** Bigalois objects, see [**BC06**]. This is of special interest for orbifolds of grouprings, as due to their cocommutativity all Bigalois objects are lazy. Note that in this case our construction has already been independently considered in [**Bo97**] to enable projective liftings.

Lazy Bigalois objects are bicleft up to an automorphism and we shall further see, that the automorphisms are trivial (bicleft case) if Ω should again be a groupring (Lemma 1.17). Hence this presents the Hopf algebraic description of the **Schur-group setting in a more general form** (see Theorem 2.7): Some cocycles may appear multiple times and others none, while the coproduct fixes a specific group extension.

Lemma 2.2 will show, that the above situation is contained in every orbifold, which will lead us ultimately to the description of an orbifold's coradical as sub-orbifold in Theorem 2.4.

Lemma 1.14. *Given a group morphism σ from Σ to the group of* **lazy 2-cocycles** *of H. Then for $A(p) := {}^{\sigma(p)}H$ the associated bicleft Bigalois objects, there is a natural choice of the ι via Δ_H that satisfies pentagonal identity. We can also write down $\rho^{R,L}$ from S_H and hence immediately obtain a twisting group.*

Definition 1.15. *We will speak of a* **bicleft twisting group** *and use the symbol $j_p, p \in \Sigma$ with $j_e = id_H$ for the associated bijective bicleaving maps. Note that besides their bicolinearity, also their determination of the special choices for the ι is of most importance!*

Proof. Let ${}^{\sigma}H, {}^{\tau}H, {}^{\tau\sigma}H$ be bicleft, i.e. twistings of H by a lazy 2-cocycle, isomorphic to H as bicomodule algebras. Then the cotensor product ${}^{\sigma}H \square_H {}^{\tau}H \cong {}^{\tau\sigma}H$ as Bigalois objects, we can even have such an isomorphism induced by Δ_H (via the above identifications with H, see e.g. [**BC06**]). Clearly, taking these as ι, the pentagonal identity holds by coassociativity of H. We obtain $\rho^{R,L}$ again by Lemma 1.11. $\square$

Note that this by no means is the only choice. One may obtain non-bicleft Bigalois objects from lazy 2-cocycles by modifying the left comodule structure by an automorphism of H.

Remark 1.16. *In fact, by a result of Schauenburg* $BiGal(\Bbbk[G])$ *is a semidirect product* $Aut(G) \ltimes H^2(G, \Bbbk^\times)$*. If one changes the latter to* $Z^2(G, \Bbbk^\times)$ *we get an extension thereof, corresponding to Bigalois objects with fixed cleavings. Any group morphism to this group can be turned into a twisting theory by an argument similar to the above. This is a model for the general case (see Remark 1.10).*

Also, even bicleft Bigalois objects could be combined with the ι modified by an obstructional 2-cocycle of Σ. However, both these cases will later be excluded as an obstruction to pointedness in Corollary 2.5:

Lemma 1.17. *A twisting group of a finite-dimensional groupring* $\Bbbk[\Gamma]$ *yields as orbifold again a groupring* **iff** *it is bicleft in the sense above and* Σ *is abelian.*

Proof. **Suppose** $\Omega = \Bbbk[G]$ **be a groupring.** Since i_Σ is a Hopf algebra injection of $\Bbbk^\Sigma$, surely Σ has to be abelian and hence $\Bbbk^\Sigma \cong \Bbbk[\Sigma^*]$. Also, π_H has to come from a surjection of groups and hence splits via some j as a coalgebra map. We may use the restrictions as compatible bicleavings:

$$j_p := j|_{A(p)} : H \to A(p)$$

Clearly $j_e = id_H$ and as the H-H-bicomodule structure can be recovered by Δ_Ω and π_H, the j_p are all bicolinear. Having j a coalgebra map also shows they induce the ι via Δ_H:

$$\iota_{p,q} j_{pq} = (j_p \otimes j_q)\Delta_H$$

To finally show $*$-invertibility, note by definition

$$1_{A(p)}\epsilon_H = \mu_{A(p)}(id_{A(p)} \otimes \rho_p^{\Bbbk})\iota_{p,p^{-1}}$$

which me may concatenate with $j_e = id_H$ and use the above formula:

$$= \mu_{A(p)}(j_p \otimes \rho_p^{\Bbbk} j_{p^{-1}})\Delta_H = j_p * (\rho_p^{\Bbbk} j_{p^{-1}})$$

Bijectivity follows by normal basis (Doi/Takeuchi, see [**S04**]).

Conversely: We give an explicit isomorphism of coalgebras using the bijective j_p that induce the ι's

$$f : \Bbbk^\Sigma \otimes \Bbbk[\Gamma] \to \Omega$$
$$f(\phi \otimes g) := \sum_{p \in \Sigma} \phi(p) j_p(g)$$

proven by explicit calculation:

$$\Delta_\Omega(f(\phi \otimes g)) = \sum_{q,r \in \Sigma} \phi(pq) j_q(g) \otimes j_r(g) = f(\phi^{(1)} \otimes g) f(\phi^{(2)} \otimes g)$$

$$\epsilon_\Omega(f(\phi \otimes g)) = \phi(e)\epsilon_H(j_e(g)) = \phi(e) = \epsilon_{\Bbbk^\Sigma}(\phi)$$

□

The isomorphism at the end of the last proof also clarifies how the choice of a specific group of 2-cocycles in Lemma 1.14 determines the specific resulting group extension. This will be needed in section 3.1 to yield a prescribed extended coradical:

Corollary 1.18. *Consider a central extension $\Sigma^* \to G \to \Gamma$. It's well known, that any set-theoretic split $s:\ \Gamma \to G$ yields a $u \in Z^2(\Gamma, \Bbbk^\times)$.*

$$\Sigma \to Z^2(\Gamma, \Bbbk^\times)$$

$$p \mapsto p \circ u$$

then yields a bicleft twisting group structure Σ on $\Bbbk[\Gamma]$ and the corresponding orbifolds is precisely $\Omega \cong \Bbbk[G]$ as Hopf algebras. Note this is stricter than determining the groupring, as it completely fixes G.

Proof. Extend s by left multiplication (and then linearly) to an isomorphism of coalgebras, multiplicatively only in the left factor:

$$\Bbbk^\Sigma \otimes \Bbbk[\Gamma] \cong \Bbbk[\Sigma^*] \otimes \Bbbk[\Gamma] \to \Bbbk[G]$$

The concatenation with the f^{-1} in the proof above yields again a bijective coalgebra map $\Omega \to \Bbbk[G]$, but now also an algebra map as seen on each $A(p)$: Different $A(p), A(q)$ map to different idempotents $e_p, e_q \in \Bbbk^\Sigma$, hence cancel. Elements $j_p(g), j_p(h) \in A(p)$ map to

$$\Bbbk[G] \ni e_p j(g) j(h) = e_p j(gh) c(g,h) = e_p j(gh) c(g,h)(p)$$

which is exactly the multiplication in $A(p) \cong k_{c(g,h)(p)}[\Gamma]$ under the natural correspondence. □

CHAPTER 2

Properties

We now want to discuss the structure of the Hopf algebras defined above in more depth in case of Bigalois objects over some pointed and finite-dimensional H. We calculate the coradical and in case Ω is pointed the space of skew-primitives (especially the link-decomposition) and thus find conditions for pointedness and link indecomposibility.

Remark 2.1. *Since s is a multiplicative splitting of π_H preserving ϵ, left/right* **integral** *Λ_H carry to resp. integrals Λ_Ω. Especially if H is finite-dimensional, the well known criterion of Eilenberg/Sweedler ("Maschke") asserts that Ω is semisimple iff H is.*

1. The Coradical

First, we concern ourselves with the coradical Ω_0, i.e. the sum of all simple subcoalgebras of Ω and clarify pointedness. Denote $\Gamma = G(H)$ and $G = G(\Omega)$ in what follows. We prove now, that we may restrict our study to the case of a groupring H in Lemma 1.17:

Lemma 2.2. *Take a cleft Bigalois object $A(p)$ and Γ finite:*

(1) *For every $g \in \Gamma$ there is a $z \in A(p)$ with $\delta^R(z) = z \otimes g$, unique up to a scalar factor $\Bbbk^\times$.*
(2) *For all z above, there is a unique $h \in \Gamma$ with $\delta^L(z) = h \otimes z$.*
(3) *The subspace $G(p) \subset A(p)$ spanned by the z obtained above for all $g \in \Gamma$ is an underlying $\Bbbk[\Gamma]$-$\Bbbk[\Gamma]$-Bigalois object (namely the image of $\Bbbk[\Gamma]$ under any left- or right cleaving). Especially $G(e) = \Bbbk[\Gamma] \subset H = A(e)$ itself.*
(4) *The restriction of any $\iota_{p,q}$ to $G(pq)$ maps bijectively to $G(p)\Box G(q)$ and the restriction of $\rho^{R,L}$ to $G(p^{-1})$ maps to $G(p)$. Hence we get an underlying twisting group of $\Bbbk[\Gamma]$ and its orbifold is a sub-Hopf algebra of Ω.*

Remark 2.3. *As in the later case of the skew-primitives (Lemma 2.8) one may combine the unique left-right-association $g \leftrightarrow h$ for each $p \in \Sigma$*

to a group homomorphism $\Sigma \to Aut(\Gamma)$. Since we will only be interested in $G(p)$ being bicleft (pointedness!), this will always be trivial and $g = h$ above; so we shall pursue this no further.

PROOF.
1. By using any right-colinear cleaving j, one may obtain such a $z = j(g)$, which by $*$-invertibility of j is even invertible. Now for a second z' fulfilling the condition, the expression $z^{-1}z'$ is coinvariant and hence a scalar.
2. Take such a g, z; since $\delta^{R,L}$ commute, by uniqueness $\delta^L(z)$ is also of the form $h \otimes z$, where $h \in H$. But this already concludes $h \in \Gamma$.
3. Using the above invertibility of each z, we find a split of the *can*-map on both sides, as $\mathrm{can}(z, z^{-1}b) = g \otimes b$. Since Γ is finite, bijectivity follows by dimension (probably finiteness is unnecessary, as our split is colinear).
4. First, the bicolinearity of the ι ensures the above property left/right-sided for the left/right tensor factor(s) of $\iota(z) \in A(p)\Box A(q)$ and we just saw this implies the respective other-sided version, so both tensor factors land in $G(p)$ resp. $G(q)$. We show the same for ρ^L (ρ^R): Note first that, as above, for any $g \in \Gamma$ we have $\iota_{p^{-1},p}(g) = s(z \otimes w)$ with $0 \neq s \in k$ and z, w respective left/right cleaving images of g. Thus the defining condition of ρ^L reads:

$$\rho^L(z)w = 1_{A(p)}$$

Again using invertibility $\rho^L(z) = w^{-1} \in G(p)$. Varying g and extending linearly, this concludes the assertion on all of $G(p^{-1})$. □

Theorem 2.4. *Let H be finite-dimensional, Ω an orbifold of Bigalois objects (Definition 1.9) and denote by $L \subset \Omega$ the sub-Hopf algebra constructed as the groupring orbifold above:*

$$L = \bigoplus_{p \in \Sigma} G(p) \subset \Omega$$

Then $\Bbbk[G(\Omega)] \subset L \subset \Omega_0$. If furthermore H is pointed, the second inclusion is an equality: $L = \Omega_0$.

PROOF. $\Bbbk[\Gamma]$ is semisimple, and so is its orbifold by Remark 2.1. With finite dimension and characteristic zero this also implies cosemisimplicity (Larson/Radford); hence L is contained in the coradical Ω_0.

Suppose further we are given a grouplike

$$G(\Omega) \ni z = \sum_{p \in \Sigma} z_p$$

Because π_H is a Hopf algebra map, $z_e = \pi_H(z)$ is also grouplike in $H = G(e)$. The grouplike condition in Ω reads $\iota_{p,q}^{-1}(z_{pq}) = z_p \otimes z_q$ and hence by part 4 of lemma 2.2 all $z_p \in G(p)$ and thus $z \in L$.

For the second assertion, consider any simple subcoalgebra $C \in \Omega$: Since the H-H-bicomodule structure can be obtained by coproduct and the Hopf algebra map $\pi_H : \Omega \to H$ restricting to $A(e) = H$, C is also a H-H-subbicomodule. Since H is pointed, we can find an 1-dimensional subcomodule $vk \subset C$ with $\delta^L(v) = g \otimes v$ for some $g \in G$. The direct summands v_p of v in each sub-bicomodule algebra $A(p)$ share this property, so by definition $v_p \in G(p)$. Hence this v implies a nontrivial intersection of C with the group orbifold and by assumed simplicity C is already entirely contained in L. $\square$

Since we already discussed the orbifold of a groupring in section 1.7 and especially when it is again a groupring in Lemma 1.17, we can immediately give necessary and sufficient conditions for pointedness:

Corollary 2.5. *Let H be pointed and finite-dimensional, then an orbifold of Bigalois objects Ω is pointed* **iff** *Σ is abelian and the above restriction of the twisting group to the grouplikes $(G(p))_{p\in\Sigma}$ is bicleft.*

We will refer to these conditions as the **usual setting** *, under which we will work throughout the rest of this thesis.*

Also, the characterization of bicleft twisting groups gives us a useful map restricting the twisting 2-cocycles underlying the $A(p)$ to $\Bbbk[\Gamma]$:

$$res : \Sigma \xrightarrow{\sigma} Z^2(\Gamma, \Bbbk^\times) \to H^2(\Gamma, \Bbbk^\times)$$

It controls G, as it will enable us to write G as a double extension of Γ: The first is Schur-group-alike in the sense that it is also made up of different (but not necessarily all) noncohomological group rings, i.e. the image of res. This will turn out to preserve link-indecomposability.

Definition 2.6. *A* **stem extension** *$\Sigma^* \to G \to \Gamma$ is a central extension with $\Sigma^* \subset G'$.*

The second extension is made up of the trivial twistings, and is in contrast solely abelian in the sense that it has trivial intersection with the commutators of G and hence already fully appears as abelianized extension $G_{ab} \to \Gamma_{ab}$. It exhibits a tendency to be decomposable, though not in general (see Theorem 2.10 and the counterexample).

Theorem 2.7. *Every central extension $G/\Sigma^* \cong \Gamma$ can be decomposed into a stem extension $G \to N$ and an extension $N \to \Gamma$, where the kernel has trivial intersection with the image of G' (this is folk).*

In our usual setting, the respective kernels turn out isomorphic to $Im(res)$ and $Ker(res)$. We prove this by characterizing $\Sigma^* \cap G'$ as exactly those characters of Σ factorizing over res. Especially for res injective, G is a stem extension of Γ, and for res bijective, G is a Schur-cover of Γ.

Proof. For the general decomposition set $N := G/(\Sigma^* \cap G')$. Clearly $G \to N$ is a stem extension. The kernel of $N \to \Gamma$ on the contrary is $\Sigma^*/(\Sigma^* \cap G')$ and hence has trivial intersection with $N' = G'/(\Sigma^* \cap G')$.

We prove the actual claim via the given characterization of commutators by the factorizing condition, which immediately shows the first kernel to be $Im(res)^*$ and hence the second kernel to be $\Sigma^*/Im(res)^* \cong Ker(res)^*$ - so by duality of abelian groups we are done.

Any 1-dimensional representation f of $\Bbbk[G]$ has exactly one direct summand $p \in \Sigma$ where its restriction is again a 1-dimensional representation $f|_{G(p)}$ and 0 elsewhere. But properly twisted grouprings have no such representations (e.g. since there is a 1:1-correspondence between these representations of any G and the Schur group $D(G)$ due to $D(G)_{ab} = G_{ab}$). Hence $G(p)$ is always untwisted and $p \in Ker(res)$.

Thus, some $\phi \in \Sigma^* \subset G$ factorizes, iff we have $\phi(p) = 1$ for all such $p \in Ker(res)$, iff $f(\phi) = 1$ for all 1-dimensional representations f. But G' is exactly the set of all grouplikes in the kernel of every 1-dimensional representation. $\square$

2. The Skew Primitives

We want to show that $\Omega_1/\Omega_0 = H_1/H_0$, i.e. the nontrivial skew-primitives in Ω are unique π_H-liftings of the non-trivial skew-primitives in H. However, the space will decompose to skew-primitives over differently lifted grouplikes according to an eigenspace decomposition under the action of "twisted symmetries" θ (the name becomes clear in Theorem 3.2 and part 2). Then we will give a criterion for link-indecomposability solely in terms of the res-map.

As with the grouplikes we need to link left- and right coaction on the primitives (compare Lemma 2.2), yielding the twisted symmetries:

Lemma 2.8. *For a Bigalois object $A(p)$ in the usual setting, suppose we are given a $g \in \Gamma$ and $X \in Prim_{1,g}$ a skew-primitive in H:*

(1) *There exists a $z \in A(p)$ with*
$$\delta^R(z) = z \otimes g + 1_{A(p)} \otimes X$$
unique up to adding $j_p(g)\mathbb{k}$, where $j_p : \mathbb{k}[\Gamma] \to G(p) \subset A(p)$ is the bicleaving of the twisting group of the grouplikes demanded by the usual setting (Corollary 2.5).

(2) *There is a corresponding skew-primitive $Y \in Prim_{1,g}$ with*
$$\delta^L(z) = 1 \otimes z + Y \otimes j_p(g)$$
subsequently unique up to adding $(g - 1_H)\mathbb{k}$.

(3) *This correspondence gives rise to a bijective map*
$$\theta(p) : H_1/H_0 \to H_1/H_0$$
preserving the trivial skew-primitives $H_0 \cap Prim_{1,g}$.

(4) *These maps can be combined to a group morphism:*
$$\theta : \Sigma \to GL(H_1/H_0)$$

PROOF.
1. The existence is clear by an arbitrary cleaving. Now let z, z' both have the property we demanded, then
$$\delta^R(z - z') = (z - z') \otimes g$$
which implies $z - z' \in j_p(g)\mathbb{k}$ by the uniqueness in Lemma 2.2.
2. Write down a completely general expression for the left coaction of

an arbitrary such z:

$$\delta^L(z) = 1 \otimes z + \sum_i a_i \otimes b_i$$

We first use the relation $\delta^L\delta^R = \delta^R\delta^L$:

$$(\delta^L \otimes id_H)\delta^R(z) = (1 \otimes z \otimes g + \sum_i a_i \otimes b_i \otimes g) + 1 \otimes 1_{A(p)} \otimes X$$

$$(id_H \otimes \delta^R)\delta^L(z) = (1 \otimes z \otimes g + 1 \otimes 1_{A(p)} \otimes X) + \sum_i a_i \otimes \delta^R(b_i)$$

hence all b_i are in $j_p(g)\mathbb{k}$ again by uniqueness and we may write:

$$\sum_i a_i \otimes b_i =: Y \otimes j_p(g)$$

To clarify Y further we use that δ^L is a comodule structure:

$$(id_H \otimes \delta^L)\delta^L(z) = (1 \otimes 1 \otimes z + 1 \otimes Y \otimes j_p(g)) + Y \otimes g \otimes j_p(g)$$

$$\stackrel{!}{=} (\Delta_H \otimes id_{A(p)})\delta^L(z) = 1 \otimes 1 \otimes z + \Delta(Y) \otimes j_p(g)$$

which concludes $Y \in Prim_{1,g}$. A different choice $z' = z + j_p(g)t$ with $t \in \mathbb{k}$ just changes Y by $(g - 1_H)t$:

$$\begin{aligned}\delta^L(v + j_p(g)t) &= 1 \otimes v + Y \otimes j_p(g) + g \otimes j_p(g)t \\ &= 1 \otimes (v + j_p(g)t) + (Y + (g - 1_H)t) \otimes j_p(g)\end{aligned}$$

3. We consider $\theta(p) : X \mapsto Y + (g - 1_H)\mathbb{k}$, which is a well defined map $H_1 \to H_1/H_0$. Since again by uniqueness (Lemma 2.2) trivial skew-primitives are sent to trivial skew-primitives, this map factorizes:

$$\theta(p) : H_1/H_0 \to H_1/H_0$$

Again, considering the "left-to-right" situation instead yields an inverse by the uniqueness property shown above.

4. Unitality is clear as $H = A(e)$ is bicleft (with the trivial cleaving), so let us check multiplicativity: For any $p, q \in \Sigma$ and $X \in Prim_{1,g}$, we may choose a $z_X \in A(q)$ and get a $Y \in Prim_{1,g}$ as above. Subsequently we obtain for Y some $z_Y \in A(p)$ and $Z \in Prim_{1,g}$, which reflects the situation $\theta_p\theta_q(\overline{X}) = \overline{Z}$.

To construct a respective element $z_{XY} \in A(pq)$, note first that

$$z := 1_{A(p)} \otimes z_X + z_Y \otimes j_q(g) \in A(p)\Box A(q)$$

as calculated explicitly from the above choices:

$$(\delta^R \otimes id_{A(q)})z = (1_{A(p)} \otimes \delta^L)z$$

$$= 1_{A(p)} \otimes 1 \otimes z_X + 1_{A(p)} \otimes Y \otimes j_q(g) + z_Y \otimes g \otimes j_q(g) =$$

Note further that:

$$\delta^R(z) = 1_{A(p)} \otimes z_X \otimes g + z_Y \otimes j_q(g) \otimes g + 1_{A(p)} \otimes 1_{A(q)} \otimes X = z \otimes g + \iota_{p,q}^{-1}(1_{A(pq)}) \otimes X$$

$$\delta^L(z) = 1 \otimes 1_{A(p)} \otimes z_X + 1 \otimes z_Y \otimes j_q(g) + Z \otimes j_p(g) \otimes j_q(g) = 1 \otimes z + Z \otimes \iota_{p,q}(j_{pq}(g))$$

Bicolinearity preserves these two properties for $z_{XY} := \iota_{p,q}(z) \in A(pq)$ and hence the latter element concludes $\theta_{pq}(\overline{X}) = \overline{Z}$. □

Now we can prove the main theorem of this section, stating how θ controls Ω_1. The main idea is to **simultaneously diagonalize** all $\theta(p)$ (Σ is abelian!). This yields a decomposition of $Prim_{1,g}$ into eigenspaces that π_H-lift to different spaces $Prim_{1,g_\phi}$ where $g_\phi \in G$ are π_H-lifts of g determined by the respective eigenvalue in $\phi \in \Sigma^*$. The fact that θ is only defined up to trivial skew-primitives reflects the fact, that on the other hand *their* number greatly increases with the grouplikes. This adds some some technicality to the proof below:

Theorem 2.9. *Take a θ-eigenbasis $\bar{X}_i$ of H_1/H_0 adapted to $Prim_{1,g}$ with eigenvalues $\phi_i \in \Sigma^*$. For every representing 1-g-skew-primitives $X_i \in H_1$ there exists a π_H-lift $z_i \in \Omega_1$ that is 1-g_ϕ-skew-primitive for*

$$G \ni g_{\phi_i} := \sum_{p \in \Sigma} j_p(g)\phi_i(g)$$

(see Lemma 1.14). This already yields all skew-primitives in Ω_1 up to trivial ones. Especially as vector spaces $\Omega_1/\Omega_0 = H_1/H_0$.

Proof. Note that in the usual setting Σ is abelian and H finite dimensional, so we really can diagonalize all $\theta(p)$ simultaneously on H_1/H_0. Moreover, as θ preserves the $Prim_{1,g}$, the eigenbasis is indeed adapted to this decomposition. It's also clear, that we can choose representatives in the $Prim_{1,g}$ by multiplying a suitable grouplike.

For the **first claim** we restrict our attention to one such θ-eigenvector $X \in Prim_{1,g}$ with eigenvalue ϕ, dropping the index i. By the preceding lemma, we can find for every $q \in \Sigma$ a $z_q \in A(q)$, unique up to $j_q(g)\mathbb{k}$, with the properties given there, especially a left analogon $Y \in H_1$. Since $\overline{Y} = \phi(q)\overline{X}$ by the definition of $\theta(q)$ we have for some scalar t:

$$Y = \phi(q)X + (g - 1_H)t$$

By taking $z_q - j_q(g)t$ instead, we have **unique choices** for z_q with $t = 0$ for each $q \in \Sigma$; especially $z_e = X \in H$.

Exactly as in the preceding lemma's proof 4 (for $z_X = z_q$ and $Y = \phi(q)X, z_Y = \phi(q)z_p$), we can form

$$1_{A(p)} \otimes z_q + \phi(q)z_p \otimes j_q(g) \in A(p)\Box A(q)$$

and this and its $\iota_{p,q}$-image enjoy the same properties with $Z = \phi(p)\phi(q)X$. Hence by uniqueness the latter is already equal to z_{pq} and thus:

$$\iota_{p,q}^{-1}(z_{pq}) = 1_{A(p)} \otimes z_q + \phi(q)z_p \otimes j_q(g)$$

Now finally the following π_H-lifting of X

$$z := \sum_{p\in\Sigma} z_p$$

can easily be calculated to be a g_ϕ-skew-primitive:

$$\Delta(\sum_{r\in\Sigma} z_r) = \sum_{p,q\in\Sigma} \iota_{p,q}^{-1}(z_p q) = \sum_{p,q\in\Sigma} 1_{A(p)} \otimes z_q + \phi(q)z_p \otimes j_q(g)$$

$$= 1_\Omega \otimes (\sum_q z_q) + (\sum_p z_p) \otimes (\sum_q \phi(q)j_q(g)) = 1 \otimes z + z \otimes g_\phi$$

To show the **second claim**, we analyze the kernel K of the map

$$\Omega_1/\Omega_0 \xrightarrow{\pi_H} H_1 \to H_1/H_0$$

where we know the restriction lands in $\pi_H(\Omega_1) \subset H_1$ and the map factorizes over the Ω_0 quotient, both because π_H is a Hopf algebra map. Note we've just proven above, that this is a surjection.

Take an element $z = \sum_{p\in\Sigma} z_p \in \Omega_1$ with $\overline{z} \in K$, meaning we have $z_e = \pi_H(z) \in \Bbbk[\Gamma] = G(e)$. By Lemma 2.2 for every $p \in \Sigma$ the $A(p)\otimes A(p^{-1})$-term $\iota_{p,p^{-1}}^{-1}(z_e)$ is $\Delta(z)$ again lands in $G(p) \otimes G(p^{-1})$. But on the other hand, skew-primitiveness implies this summand to be in

$$G(p) \otimes z_{p^{-1}} + z_p \otimes G(p^{-1})$$

concluding all $z_p \in G(p)$ and thus $z \in \Omega_0$. Hence K is trivial and the above map is a bijection. $\square$

While the nontrivial skew-primitives get uniquely lifted, still something worth noticing happens. Recall the definition of **link-components** for a pointed Hopf algebra. One defines a graph called **quiver**, whose nodes are $G(H)$ and between g and h an edge iff there exist nontrivial g-h-primitives (trivial are the scalar multiples of $g-h$). Now one looks at the **connected components** of the graph. A connected H is called **link-indecomposable** . Another way of putting this is to just consider the edges joining 1, i.e. the 1-g-primitives: The connected component of 1 then is just the subgroup of $G(H)$ generated by all g, for which a nontrivial 1-g-primitive exists.

Note that it is clear from the above result and the Hopf algebra map π_H that if $g, h \in \Gamma \subset H$ are disconnected, so are all liftings $g_\phi, h_\chi \in G \subset \Omega$. On the other hand we have:

Theorem 2.10. *If H is link-indecomposable, there is a subgroup $\overline{M} \subset G_{ab}$ with surjective restriction of $G_{ab} \to \Gamma_{ab}$, such that the entire preimage M of $\overline{M}$ in G is precisely the link-component of* 1.

Especially the number of connected components of Ω_0 is at most $|Ker(res)|$, thus if res is injective, Ω is also link-indecomposable.

Remark 2.11. *The proof below is a slight refinement of the well known fact that for stem extension $G_{ab} = \Gamma_{ab}$ (the case of res injective above), any lifts of Γ-generators already generate all of G.*

Proof. Because H is supposed link-indecomposable, we have a set of nontrivial $1 - g_i$−skew-primitives X_i, such that the g_i generate $\Gamma = G(H)$. Now by the previous result, these can be lifted to $1-(g_i)_{\phi_i}$−skew-primitives for some ϕ_i, i.e. for some liftings of g_i under $G \to \Gamma$. We want to characterize the subgroup M of G generated by these in terms of its image $\overline{M}$ in G_{ab}:

First note that the restriction of $G \to \Gamma$ to M remains surjective, because the images g_i generate Γ (the same is certainly true in the abelianized situation, which proves this condition on $\overline{M}$).

Next we observe, that because $\Sigma^* \to G \to \Gamma$ is a **central** extension, the commutator subgroup M' is already all of G', because in every product of commutators in G we may change the entries by Σ^* (i.e. in their fibers) without changing the actual commutator, until all lay in M.

Finally consider the extension $G \to G_{ab}$, sending M to $\overline{M}$. Because $ker = G' = M' \subset M$, M is the entire preimage of $\overline{M}$, concluding the main assertion.

The numerical bound follows from this, the "surjective restriction"-condition above and Theorem 2.7:

$$[G : M] = [G_{ab} : \overline{M}] \leq [G_{ab} : \Gamma_{ab}] = [N_{ab} : \Gamma_{ab}] = [N : \Gamma] = |Ker(res)|$$

□

The precise description of the link-component of 1 is included in the statement, because the estimation alone may sometimes be to crude: In section 4.4 we shall see an example with

$$G_{ab} \cong \mathbb{Z}_4 \xrightarrow{mod\,2} \mathbb{Z}_2 \cong \Gamma_{ab}$$

so res is not injective, but nevertheless the only possible $\overline{M}$ is obviously all of G_{ab}. Hence this Ω is still link-indecomposable, even though we are not dealing with a stem extension.

Remark 2.12. *Note that incorporating knowledge from the θ one may thoroughly sharpen the "surjective restriction"-condition on $\overline{M}$ above, because there we only used one lifting per $g \in \Gamma$.*

Take as an example $G_{ab} = \mathbb{Z}_6 = \langle g \rangle$ and $\Gamma_{ab} = \mathbb{Z}_2 = \langle h \rangle$: If the lift of any $1-a-$skew-primitive ($\overline{a} = h$) yields a lift $\overline{a_\phi} = g, g^5$, we are fine ($\overline{M} = G_{ab}$), yet we cannot rule out the case $\overline{M} = \langle g^3 \rangle \neq G_{ab}$. But if we know that θ has two different eigenvalues, $\overline{M}$ is again always all of G_{ab} and the orbifold stays indecomposable.

One could thus derive combinatorial lower bounds on the number of different eigenvalues depending on the "new" cyclic factors in $G_{ab} \to \Gamma_{ab}$, that always ensure link-indecomposability – but we do not pursue this any further here.

CHAPTER 3

Orbifoldizing back and forth

1. Constructing Smash-Examples

In this section, we want to give a practical approach to quickly write down a twisting group (Σ, A, ι, ρ) for a Hopf algebra H, that is Radford-products of the coradical $\mathbb{k}[\Gamma]$ with a braided Hopf algebra $\mathcal{B}(M)$. Introducing the notion of **twisted symmetry** we will be able to choose the twisting group such that the orbifold Ω has a prescribed coradical $\mathbb{k}[G]$, where G is a stem-extension of Γ by Σ.

After setting up some machinery, our first example is originally due to Milinski and Schneider [**MS00**] and over $G = \mathbb{D}_4$, which we stem-extend from $\Gamma = \mathbb{Z}_2 \otimes \mathbb{Z}_2$, where we can use the classification results from Heckenberger [**H08**].

This situation will be analyzed in much greater depth in the second part of this thesis – the non-expert reader is referred to the introductions there, especially regarding Nichols algebras on see page 2. The main purpose of this section is to connect the explicit, but rather ad-hoc constructions in part 2 (Theorems 4.4 and 5.1) to the abstract approach taken above.

Assumption 3.1. *Suppose for the remainder of the section Γ a finite group and H a finite-dimensional, indecomposable Hopf algebra of the form $H = \mathbb{k}[\Gamma]\#\mathcal{B}(M)$ with $\mathcal{B}(M)$ the Nichols algebra of some $\mathbb{k}[\Gamma]$-Yetter-Drinfel'd module M. We demand further M to be* **minimally indecomposable***, i.e. no proper indecomposable submodule exists.*

Note that this is not as restrictive as it seems at first glance: If there at all exists a finite-dimensional, pointed, indecomposable Hopf algebra with the prescribed coradical $\mathbb{k}[\Gamma]$, we may take its sub-Hopf algebra H' generated by grouplikes and skew-primitives and consider $gr(H')$ which is of the form demanded. To fulfill the last condition, we choose any minimal set of simple Yetter-Drinfel'd modules, that is still indecomposable and take H the sub-Hopf algebra generated by these.

Theorem 3.2. Suppose *we are given a stem-extension* $\Sigma^* \to G \to \Gamma$ *and a finite-dimensional, indecomposable Hopf algebra* $H = \mathbb{k}[\Gamma]\#\mathcal{B}(M)$ *with coradical* $\mathbb{k}[\Gamma]$*. To reach the specific coradical* $\mathbb{k}[G]$*, we identify* Σ *with a subgroup of* $Z^2(\Gamma, \mathbb{k}^\times)$ *(Corollary 1.18). Note that for stem-extensions by Theorem 2.7, the* $\sigma \in \Sigma$ *belong even to distinct cohomology classes, so the especially the map* $\Sigma \to Z^2(\Gamma, \mathbb{k}^\times)$ *is injective.*

Suppose *further a given linear action* θ *of* Σ *on* H *as* **twisted symmetries***, i.e.* θ_σ *is for each* $\sigma \in \Sigma$ *a Hopf algebra isomorphism* $H_\sigma \cong H$ *to the Doi-twist (with the trivial extension of* σ *to* H*, see proof below). Note that we denote the* Σ*-argument* σ *by a lower index. Suppose further, that* θ_σ *restricted to the coradical is the trivial identification.*

Then *(this is the content of the theorem) we can define a twisting group* Σ *of* H *in accordance with the usual setting, where res (p. 45) bijectively maps every* $\sigma \in \Sigma$ *to its own cohomology class* $[\sigma] \in H^2(\Gamma, \mathbb{k}^\times)$ *and the* θ *defined in Theorem 2.8 coincides with the one above.*

Thus *by Theorems 1.6, 2.10 and 2.4 the orbifold construction yields a finite-dimensional link-indecomposable orbifold Hopf algebra* Ω *with coradical* $\mathbb{k}[G]$*.*

Remark 3.3. *Actually it is already possible to construct the twisted symmetry solely on the Yetter-Drinfel'd modules* M *(see Definition 4.2), which greatly eases their construction by using their structure theory. This is used in part 2, especially Theorem 4.4.*

Proof. Out aim is to construct a twisting theory (especially all $A(\sigma)$ and $\iota_{\tau,\sigma}$) fulfilling the above assertions. Note first that since $H \cong \mathbb{k}[\Gamma]\#\mathcal{B}(M)$ is a Radford biproduct, we may extend each given $\sigma \in \Sigma \subset Z^2(\Gamma, \mathbb{k}^\times)$ trivially to a 2-cocycle over H (e.g. **[CF04]** Prop. 4.2) by:

$$\sigma(g_1\#x_1, g_2\#x_2) := \epsilon(x_1)\epsilon(x_2)\sigma(g_1, g_2)$$

Restricting again to $\mathbb{k}[\Gamma]$ shows that the res-map is as described.

Next the twist ${}^\sigma H$ is a H_σ-H-Galois object. To turn this into an H-H-Bigalois object $A(\sigma)$, we concatenate to the left with the assumed given Hopf algebra isomorphism θ_p. Note that thereby θ_σ restricted to skew-primitives matches the θ_p-maps from Lemma 2.8.

Having defined the Bigalois objects $A(\sigma)$, let us turn to the other structural elements of a twisting group; we need to establish the existence of some ι's. The following lemma generalizes the well-known fact, that lazy 2-cocycles can be multiplied, to the present situation where the cocycles are trivial extensions of lazy ones.

Lemma 3.4. *Each $\theta_\sigma : H_\sigma \to H$ gives rise to an isomorphism of Bigalois objects (where the product 2-cocycle $\tau\sigma$ coincides with the trivial extension of the product of group 2-cocycles to H):*

$$\iota_{\tau,\sigma}^{-1} : {}^{\tau\sigma}H \cong {}^{\tau}H \square {}^{\sigma}H$$
$$h \mapsto \theta_\sigma(h^{(1)}) \otimes h^{(2)}$$

Here and in the proof we use extensively the natural (right colinear) identification (resp. cleavings) of the Bigalois objects with H.

Proof. It is clear from coassociativity and the left comodule algebra structure on ${}^\sigma H$ defined above, that ι^{-1} lands in the cotensor product. Right colinearity is obvious, whereas left colinearity relies on the fact that we chose the isomorphisms such that $\theta_\tau\theta_\sigma = \theta_{\tau\sigma}$.

To check $\iota_{\tau,\sigma}^{-1}$ is an algebra map (unitality is clear), note that θ is generally **not** H-colinear, but $\Bbbk[\Gamma]$-colinear, as detected on $G(\sigma)$, where θ_σ was trivial. This quotient comodule structure is however all we need to calculate the trivially extended 2-cocycles: For any $a, b \in H$

$$\begin{aligned}
&\iota_{\tau,\sigma}^{-1}(a) \cdot_{(A(\tau)\square A(\sigma))} \iota_{\tau,\sigma}^{-1}(b) \\
&= \tau(\theta_\sigma(a^{(1)})^{(1)}, \theta_\sigma(b^{(1)})^{(1)})\theta_\sigma(a^{(1)})^{(2)}\theta_\sigma(b^{(1)})^{(2)} \otimes \sigma(a^{(2)}, b^{(2)})a^{(3)}b^{(3)} \\
&= \tau(a^{(1)}, b^{(1)})\theta_\sigma(a^{(2)})\theta_\sigma(b^{(2)})\sigma(a^{(3)}, b^{(3)}) \otimes a^{(4)}b^{(4)}
\end{aligned}$$

because $\theta : H_\sigma \to H$ is multiplicative with multiplication in H_σ given by $\sigma(a^{(1)}, b^{(1)})a^{(2)}b^{(2)}\sigma^{-1}(a^{(3)}, b^{(3)})$ we finally get:

$$\begin{aligned}
&= \tau(a^{(1)}, b^{(1)})\sigma(a^{(2)}, b^{(2)})\theta_\sigma(a^{(3)}b^{(3)}) \otimes a^{(4)}b^{(4)} \\
&= \theta_\sigma((a \cdot_{A(\tau\sigma)} b)^{(1)}) \otimes (a \cdot_{A(\tau\sigma)} b)^{(2)} = \iota_{\tau,\sigma}^{-1}(a \cdot_{A(\tau\sigma)} b)
\end{aligned}$$

$\square$

The $\iota_{\sigma,\tau}^{-1}$ satisfy (the opposite version of) the pentagonal identity by the group law $\theta_\sigma\theta_{tau} = \theta_{\sigma\tau}$ and the fact, that $H_\sigma \cong H$ as a coalgebra and θ_σ are coalgebra morphisms:

$$\begin{aligned}(\iota_{\sigma,\tau}^{-1} \otimes id)(\iota_{\sigma\tau,\nu}^{-1}(h)) &= (\iota_{\sigma,\tau}^{-1} \otimes id)(\theta_\nu(h^{(1)}) \otimes h^{(2)}) \\ &= \theta_\tau\theta_\nu(h^{(1)}) \otimes \theta_\nu(h^{(2)}) \otimes h^{(3)} \\ &= \theta_{\tau\nu}(h^{(1)}) \otimes \theta_\nu(h^{(2)}) \otimes h^{(3)} \\ &= (id \otimes \iota_{\tau,\nu}^{-1})(\theta_{\tau\nu}(h^{(1)}) \otimes h^{(2)}) \\ &= (id \otimes \iota_{\tau,\nu}^{-1})(\iota_{\sigma,\tau\nu}^{-1}(h))\end{aligned}$$

Thus the inverse maps $\iota_{\sigma,\tau}$ as well satisfy the pentagonal identity. Hence we have gathered all necessary data to find a $\rho^{R,L}$ by Lemma 1.11 and obtain a twisting group.

The orbifold Ω is **pointed** by Corollary 2.5 with coradical as prescribed and **link-indecomposable** by Theorem 2.10, as res is bijective. This concludes the proof of Theorem 3.2. □

2. A Known Example Over $\mathbb{D}_4$

In [**MS00**] Milinski and Schneider gave examples of indecomposable Hopf algebras over the non-abelian Coxeter groups $\mathbb{D}_4, \mathbb{S}_3, \mathbb{S}_4, \mathbb{S}_5$. We want to show how the former can be constructed as an orbifold of the abelian case. In the framework of [**H08**], we may simply try to add suitable Yetter-Drinfel'd modules to force the existence of twisted symmetries and apply Theorem 3.2.

Care has to be taken, that the sum has again finite-dimensional Nichols algebras. While this is the case in the later "unramified" examples in section 6.3 (the present example is a toy-model for this), in other cases we will rather rely on the Yetter-Drinfel'd module to be already inherently twisted symmetric ("ramified" examples in sections 6.4 and 6.5 as well as the ad-hoc orbifolds over $\mathbb{S}_4, \mathbb{S}_5$ in section 4.4).

Now to the construction – we take the 4-dimensional diagonal Yetter-Drinfel'd module defined by the following grouplikes and characters:

$$\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle g \rangle \times \langle h \rangle$$
$$g_x = g_u = g \qquad g_y = g_v = h$$

$$\chi_x = \chi_v = (-1,-1) \qquad \chi_u = (-1,+1) \qquad \chi_y = (+1,-1)$$

where the tuples denote the resp. characters image of g, h and we use as an index set I directly the skew-primitives

$$x, y \qquad u, v$$

with Cartan matrix of $A_2 \cup A_2$. We do not impose nontrivial linking- nor root vector relations, so the resulting finite-dimensional Hopf algebra H (which is clearly link-indecomposable) is a Radford biproduct of the groupring and a Nichols algebra.

Remark 3.5. *For the convenience of the reader, we write down the relations in H as they follow from* [**AS**]*, without ever needing them:*

- *It is generated by the group*

$$\Gamma = \langle g, h \rangle$$

  and skew primitives x, y, u, v and relations
- *The conjugation action of group elements on skew-primitives:*

$$g_i x_j = \chi_j(g_i) x_j g_i \qquad i, j \in \{x, y, u, v\}$$

- *The trivial braided adjoint action (Serre-Relations):*

$$xu + ux = 0 \qquad xv + vx = 0$$

$$yu - uy = 0 \qquad yv + vy = 0$$

- *Two identical Serre Relations for A_2:*

$$xyxy + yxyx = 0 \qquad uvuv + vuvu = 0$$

- *Two identical sets of trivial root relations:*

$$x^2 = y^2 = 0 \qquad u^2 = v^2 = 0$$

The dimension of the Nichols algebra is 64 *and hence* $dim H = 64 \cdot 4$.

Now to the construction: $H^2(\Gamma, \mathbb{k}^\times) = \mathbb{Z}_2 = \{1, [\sigma]\}$ and $\mathbb{D}_4$ is a stem-extension (even a Schur cover) of Γ by its center. We may find a representing 2-cocycle such that $\mathbb{k}[\Gamma] \oplus k_\sigma[\Gamma] \cong \mathbb{k}[\mathbb{D}_4]$ by Corollary 1.18. Namely, $s : \mathbb{D}_4 \to \Gamma$ lifts the elements $1, g, h, gh$ to $1, b, ab, a^3 = bab$

where $a^4 = b^2 = 1$ generate $\mathbb{D}_4$, we get (columns and rows resp. $1, g, h, gh$):

$$\sigma = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

To check the condition in Theorem 3.2 we need to find an involutory twisted symmetry. Using the calculation of the Doi twisting in advance (Theorem 4.4) we calculate the twisted centralizer characters, where the centralizers are all of Γ:

$$\sigma(-,g)\sigma^{-1}(g,-) = (+1,-1) = \chi_x/\chi_u = \chi_u/\chi_x$$
$$\sigma(-,h)\sigma^{-1}(h,-) = (-1,+1) = \chi_y/\chi_v = \chi_v/\chi_y$$

Hence the Doi twisting switches $\chi_x \leftrightarrow \chi_u$ and $\chi_y \leftrightarrow \chi_v$ and we are done by choosing $\theta(\sigma)$ to be switching $x \leftrightarrow u$ and $y \leftrightarrow v$. Thus $\mathbb{Z}_2$-orbifoldizing yields an indecomposable Hopf algebra of dimension $dimH \cdot |\Sigma| = 64 \cdot 4 \cdot 2$ with coradical $\mathbb{D}_4$.

To connect to the results in **[MS00]** we calculate the new skew-primitives using Theorem 2.9. They are the eigenvectors to the trivial eigenvalue $+ = \epsilon_\Sigma$ or the unique nontrivial $- \in \Sigma^*$:

$$x_1 := x + u \in Prim_{1,g_+} = Prim_{j(g)} = Prim_{1,b}$$
$$x_2 := y + v \in Prim_{1,h_+} = Prim_{j(h)} = Prim_{1,ab}$$
$$x_3 := x - u \in Prim_{1,g_-} = Prim_{a^2 j(g)} = Prim_{1,a^2 b}$$
$$x_4 := y - v \in Prim_{1,h_-} = Prim_{a^2 j(h)} = Prim_{1,a^3 b}$$

Note that in *cit. loc.* Schneider and Milinski announced, that introducing the nonhomogeneous x, y, u, v considerably ease the calculation of the relations, reducing them to usual $A_2 \cup A_2$-relations. The present approach may be viewed as a direct explanation for this phenomenon.

3. Reconstructing Twisting Groups

So far we have used orbifoldization to constructed pointed Hopf algebras Ω, where $G = G(\Omega)$ is a central extension by the twisting group Σ^*.

One may conversely ask, which Hopf algebras can be constructed this way. There is a surprisingly simple characterization: By construction

$\mathbb{k}^\Sigma \cong \mathbb{k}[\Sigma^*]$ is central in Ω and it turns out, this is already sufficient.

We will demonstrate in the second part (the exemplary sections 5.3 as well as sections 7.2 to 7.4), how this result can be used to classify pointed Hopf algebras with certain prescribed coradicals. Surely a central subgroup $\Sigma^* \subset G$ needs not to be central in all of Ω - counterexamples can be provided already by a Doi twist. However conversely, for a certain group theoretical situation we can find "enough" 2-cocycles of G to force centrality of Σ in minimal examples via a respective inverse Doi twisting.

Theorem 3.6. *A finite-dimensional pointed Hopf algebra Ω arises as an orbifold by a central subgroup $\Sigma^* \subset G(\Omega) =: G$ of some smaller Hopf algebra H with $\Gamma := G(H) \cong G/\Sigma^*$* **iff** *$\Sigma^*$ is central in all of Ω.*

Remark 3.7. *Note that the construction below reminds on Masuoka's push-out construction* [**M01**] *of Bigalois objects, to which it reduces in the case $\Omega = \mathbb{k}[G]\#A$. In general, however, our ideals are not conjugate, but given directly.*

Proof. Certainly $\mathbb{k}[\Sigma^*]$ is central in every orbifold. Thus suppose conversely a given pointed finite-dimensional Hopf algebra Ω, G with some subgroup $\Sigma^* \subset G$ central in Ω.

As already noted, Corollary 1.18 allows us to consider $\mathbb{k}[G]$ as an orbifold of $\mathbb{k}[\Gamma] := \mathbb{k}[G/\Sigma^*]$ and respectively to identify $p \in \Sigma$ with (possibly equal) group 2-cocycles σ_p. Define thus the following algebra homomorphisms on $\mathbb{k}[G]$ projecting on each Bigalois object (twisted groupring):

$$f_p : \mathbb{k}[G] \to \mathbb{k}_{\sigma_p}[\Gamma] = G(p)$$

with kernels generated by $\phi - \phi(p)\epsilon_\Sigma$ for respectively all $\phi \in \Sigma^* \subset G$.

Since by assumption $\mathbb{k}[\Sigma]$ is central in all of Ω we get two-sided ideals:

$$I_p = ker(f_{\sigma_p})\Omega$$

First consider the case $p = e$, i.e. f_e the quotient map $\mathbb{k}[G] \to \mathbb{k}[\Gamma]$ with kernel generated by $\phi - 1$. As such, I_e thus is a Hopf ideal. Define

$$H := \Omega/I_e$$

and consider Ω as a H-H-bicomodule algebra (via Δ, Δ). Since the $\phi \in \Sigma^*$ are grouplike and mapped to 1_H by the above quotient map, Σ^* and the ideals' generators $\phi - \phi(\sigma)\epsilon_\Sigma$ are coinvariants. Hence all I_σ are sub-bicomodules and the quotients Ω/I_σ are H-H-bicomodule algebras. We show that they are even Bigalois object – consider therefore their can-map:

$$\Omega/I_p \otimes \Omega/I_p \to \Omega/I_e \otimes \Omega/I_p$$

It's surjective, because it is induced by the bijective can_Ω. Then bijectivity follows from dimension.

Some $\iota_{p,q} : \Omega/I_{pq} \cong \Omega/I_p \Box \Omega/I_q$ satisfying the pentagonal identity, may be chosen from the isomorphism $\Delta : \Omega \to \Omega \Box \Omega$ as:

$$\Delta(\phi - \phi(pq)\epsilon_\Sigma) = \phi \otimes \phi - \phi(pq)\epsilon_\Sigma \otimes \epsilon_\Sigma = (\phi - \phi(p)\epsilon_\Sigma) \otimes (\phi + \phi(q)\epsilon_\Sigma) +$$
$$+ (\phi + \phi(p)\epsilon_\Sigma) \otimes (\phi - \phi(q)\epsilon_\Sigma) \subset I_p \otimes \Omega + \Omega \otimes I_q$$

By construction ι is injective and by dimension bijective.

Adding all quotients we get a map of bicomodule algebras:

$$f : \Omega \to \bigoplus_{p \in \Sigma} \Omega/I_p$$

It is injective, since the kernels intersect trivially (for $\phi \neq \eta \in \Sigma^*$ there must be some $p \in \Sigma$ with $\phi(p) \neq \eta(p)$). Bijectivity of f follows again by dimension. By construction of the ι this is also a coalgebra map (considering the right side as orbifold). $\rho^{R,L}$ can immediately be recovered from S_Ω by their defining properties. $\square$

Part 2

Orbifoldizing Nichols Algebras

Basic Concepts:

Nichols Algebras As Borel Part Of Quantum Groups

Instead of attempting a full historical overview, let us start empirically and then sketch a line of development[1] leading to the central ideas and notions, to which this thesis hopes to make a small contribution:

The known "quantum groups", such as $\mathcal{U}_q(\mathfrak{g})$ or their finite-dimensional truncations, are **constructed** as follows – conversely the very same **decomposition** will govern the classification as well (see below)

- The **Cartan algebra**, a groupring $\Bbbk[\Gamma]$ spanned by K_i.
- Two dual **quantum Borel parts** $\mathcal{B}(M), \mathcal{B}(M^*)$ generated by the vector spaces M, M^* spanned[2] by the skew-derivational/skew-primitive simple roots E_i respectively F_i.

The Cartan algebra **acts** on M, M^* as prescribed by the roots and **coacts** by the obvious i-graduation. Moreover, one has to **link** the Borel parts together by additional relations $[E_i, F_i] \subset \Bbbk[\Gamma]$ (nowadays possible by a single 2-cocycle Doi twist σ [**M08**]):

$$\mathcal{U}_q(\mathfrak{g}) \cong (\mathcal{B}(M)\#\Bbbk[\Gamma]\#\mathcal{B}(M^*))_\sigma \cong (\Bbbk[\Gamma]\#\mathcal{B}(M \oplus M^*))_\sigma$$

As in the classical case, these quantum Borel parts $\mathcal{B}(M)$ are the Serre-relation-quotients of the tensor algebra of the vector spaces M, M^*. In the **quantum case** however, M, M^* carry a deforming braiding induced by action and coaction of the Cartan algebra – this turns out to be the reason, that enables $\mathcal{U}_q(\mathfrak{g})$ to have finite-dimensional quotients and produces the atypical representation theory when $q^n = 1$.

[1]The "historical development" is largely derived from facts from an interview in May 2011 by N. Andruskiewitsch (http://www.sciencewatch.com/dr/nhp/2011/11maynhp/11maynhpAndr/) and from the respective publication lists.

[2]To be precise, there may be additional skew-primitives, whenever further truncation was possible (q root of unity). For the notation $\mathcal{B}$ to coincide with the Nichols algebra, one then would have to include these in M as well – see below.

The **minimal quotient** $\mathcal{B}(M)$ afforded by the given braided vector space is the **Nichols algebra** – a thorough introduction will be given in the next section! Apart from W. Nichols himself, they had been considered e.g. by S.L.-Woronowicz as quotient of the tensor algebra by the "quantum symmetrizer" in the context of noncommutative geometry and special cases include Y.I. Manin's "quantum linear space".

In the late 90's, H.-J. Schneider and N. Andruskiewitsch had started to classify finite-dimensional pointed Hopf algebras (as $\mathcal{U}_q$'s truncations).

A very brief sketch of their program [**AS**] can be given as follows (for some notions, see part 1 introduction): Take an arbitrary finite-dimensional pointed Hopf algebra H with coradical a groupring $H_0 = \mathbb{k}[\Gamma]$ (this is pointedness!), then the so-called **coradical filtration**

$$H_n := \Delta^{-1}(H_{n-1} \otimes H + H \otimes H_0) \quad \Rightarrow \quad H_0 \subset H_1 \subset \cdots \quad \bigcup_n H_n = H$$

is even a Hopf algebra filtration and one may take the **graded** object

$$gr(H) := H_0 \oplus H_1/H_0 \oplus H_2/H_1 \oplus \cdots$$

$H_1 = M$ is the Γ-Yetter-Drinfel'd module of all skew-primitives, higher H_i are either products thereof or (a-priori possible) exotic elements. The graded sub-Hopf algebra generated only by H_0, H_1 is a **Radford biproduct** (or smash-cosmash-product or **bosonization**), a semidirect product of the group Hopf algebra with the braided (!) Hopf algebra $\mathcal{B}(M)$ generated by the skew-primitives M:

$$gr(H) \cong \mathbb{k}[\Gamma]\#\mathcal{B}(M)$$

If $\mathcal{B}(M)$ would be a "non-minimal" quotient, additional skew-primitives would appear in higher degree, contrary to the assumption, hence $\mathcal{B}(M)$ is exactly the Nichols algebra!

Completion of the program requires a profound knowledge of these Nichols algebras and especially, when they become finite-dimensional:

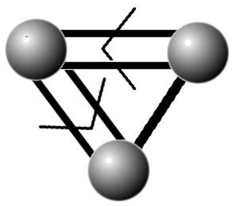
- **Liftings:** Find all H belonging to some $gr(H)$ obtained as above. This includes the linkings as above, but also more exotic ones, nontrivial truncations (root relations) etc.
- **Generation in degree 1:** Show that H_0, H_1 already generates all of H. To the present, no counterexample is known!

As part of their efforts for a general result, they asked the young researcher I. Heckenberger for a **classification** of general finite-dimensional Nichols algebras $\mathcal{B}(M)$ (of diagonal type, corresponding to abelian Γ), where they had so far only used quantum linear spaces (1998) and further ones constructed by-hand in analogy to semisimple Lie algebras (Cartan type, 2000).

Coming from the area of noncommutative geometry, Heckenberger had already concerned himself with the behaviour of quantum groups at roots of unity (diploma thesis 1993), and various questions of noncommutative differential calculus on them in the spirit of Woronowicz (dissertation 1998 and later papers).

In 2004 Heckenberger successfully completed the classification of finite-dimensional Nichols algebras of diagonal type of rank 2 and subsequently for arbitrary rank in 2005 (habilitation thesis [**H05**]).

He deepened the remarkable connection to the theory of semisimple Lie algebras by classifying **arithmetic root systems** of Nichols algebras and found (besides Cartan type) several sporadic non-Cartan examples, appearing only for some small primes in the order of q. In these cases, multiple Dynkin diagrams appear simultaneously and their **Weyl group** becomes a groupoid, interchanging these diagrams. An example for such an exotic Dynkin diagram is the following:



The cooperation of these three scientists has been remarkably fruitful and continues to the present: Schneider and Andruskiewitsch completed their program for abelian groups, where none of the exceptional primes divide the group order [**AS**] and all three continued to investigate the powerful root systems and the Weyl groupoid also for nonabelian groups, e.g. in [**HS08**], [**AHS09**] and [**HS10**]. While Andruskiewitsch continued especially to derive conditions ruling out finite-dimensional Nichols algebras, Heckenberger and Schneider could e.g. identify so-called coideal subalgebras with elements of the Weyl groupoid.

Defining Yetter-Drinfel'd Modules And Nichols Algebras

We continue by a more detailed introduction to the theory of Nichols algebras[3], starting by a generic method to write down braided vector spaces over some finite group G. Note that this is a special case of a much more general construction.

Definition. *A* **Yetter-Drinfel'd module** *M over a group G is a G-graded vector space over $\Bbbk$ denoted by layers*

$$M = \bigoplus_{g \in G} M_g$$

with a G-action on M such that

$$g.M_h = M_{ghg^{-1}}$$

To exclude trivial cases, we call M **indecomposable** *iff the support $\{g \mid M_g \neq 0\}$ generates all G and* **faithful** *iff the action is.*

Note that for **abelian** groups, the compatibility condition means nothing more then stability of the layers M_g.

The notion of a Yetter-Drinfel'd module automatically brings with it a **braiding** τ on M – in fact, each group G defines an entire braided category of G-Yetter-Drinfel'd modules with morphisms graded module homomorphisms.

Lemma. *Consider the following map $M \otimes M \to M \otimes M$:*

$$M_g \otimes M_h \ni \quad v \otimes w \stackrel{\tau}{\longmapsto} g.w \otimes v \quad \in M_{ghg^{-1}} \otimes M_g$$

Then τ fulfills the **Yang-Baxter-equation**

$$(id \otimes \tau)(\tau \otimes id)(id \otimes \tau) = (\tau \otimes id)(id \otimes \tau)(\tau \otimes id)$$

turning M into a **braided vector space***.*

Example. *The widespread concept of a* **superspace** *can be interpreted as a $\mathbb{Z}_2$-Yetter-Drinfel'd module: It is a $\mathbb{Z}_2$-graded $M = M_0 \oplus M_1$ and the nontrivial generator of $\mathbb{Z}_2$ acts as scalar -1. Hence the braiding τ is a trivial flip except on $M_1 \otimes M_1$ where the typical* **fermionic** *negative flip occurs.*

[3]For a thorough introduction see [**HLecture08**]. Note that the following brief introduction largely coincides with the respective Wikipedia-page "Nichols algebras" initially written by the author.

The structure of general Yetter-Drinfel'd modules is well understood:

Lemma. *For $\mathbb{k} = \mathbb{C}$ (as always in this thesis) any finite-dimensional Yetter-Drinfel'd module M over a finite group G decomposes as such into simple Yetter-Drinfel'd modules (the number is called* **rank** *of M):*

$$M = \bigoplus_i M_i$$

Lemma. *Any simple M is isomorphic to some $\mathcal{O}_g^\chi$ for $g \in G$ and $\chi : G \to \mathbb{k}$ character of an irreducible representation V of the centralizer subgroup $Cent(g) = \{h \in G \mid gh = hg\}$; defined as follows:*

- *Define the G-graduated vector space layerwise*

$$\mathcal{O}_g^\chi = \bigoplus_{h \in G} \left(\mathcal{O}_g^\chi\right)_h$$

$$\left(\mathcal{O}_g^\chi\right)_h := \begin{cases} V & \text{for } g\text{-conjugates } h \in [g] \\ \{0\} & \text{else} \end{cases}$$

- *Choose a set $S = \{s_1, \dots s_n\}$ of representatives for the left $Cent(g)$-cosets $G = \bigcup_k s_k Cent(g)$. Then for any g-conjugate $h \in [g]$ there is precisely one s_k with $h = s_k g s_k^{-1}$.*
- *For the action of any $t \in G$ on any $v_h \in \left(\mathcal{O}_g^\chi\right)_h$ for $h \in [g]$ determine the unique s_i, s_j, such that*

$$s_i g s_i^{-1} = h \qquad s_j g s_j^{-1} = tht^{-1}$$

Then $s_j^{-1} t s_i \in Cent(g)$ and using the given $Cent(g)$-action on V we may define

$$t.v_h := (s_j^{-1} t s_i.v)_{tht^{-1}}$$

To summarize the above: There are V-layers for each conjugate of g, acted on as prescribed by $Cent(g)$ (assuming the choice $s_0 = 1$) and solely permuted by the representing s_k. Other elements are decomposed into some $Cent(g)s_k$ and act accordingly.

Example. *For* **abelian** *G (and over $\mathbb{k} = \mathbb{C}$) we have 1-dimensional simple Yetter-Drinfel'd modules $M_i = \mathcal{O}_{g_i}^{\chi_i} = x_i\mathbb{k}$ and hence the braiding is* **diagonal** *with* **braiding matrix** *$q_{ij} := \chi_j(g_i)$*

$$x_i \otimes x_j \stackrel{\tau}{\longmapsto} q_{ij}(x_j \otimes x_i)$$

Definition. *Consider the tensor algebra* $\mathfrak{T}M$*, i.e. for any homogeneous basis* $x_i \in M_{g_i}$ *the algebra of words in all* x_i*. We may uniquely define* **skew derivations** *on this algebra, i.e. maps* $\partial_i : \mathfrak{T}M \to \mathfrak{T}M$ *by*

- $\partial_i(1) = 0$
- $\partial_i(x_j) = \delta_{ij}1$
- $\partial_i(ab) = \partial_i(a)b + (g_i.a)\partial_i(b)$

These derivations can be thought of as a **differential structure** on $\mathfrak{T}M$; intuitively one would expect 1 to be the only "constant" element in the kernel of all ∂_i. However for general braidings this is far from being true and in specific instances, only finitely many dimensions will remain. This is a remarkable phenomenon (and the direct reason for the finite-dimensional truncations of $\mathcal{U}_q(\mathfrak{g})$ for q a root of unity):

Definition. *The* **Nichols algebra** $\mathcal{B}(M)$ *is the quotient of* $\mathfrak{T}M$ *by the largest homogeneous ideal* $\mathfrak{I}$*, that is invariant under all* ∂_i *such that* $M \cap \mathfrak{I} = \{0\}$.

So roughly, it is $\mathfrak{T}M$ modulo all higher-order relations, that the differential structure is "blind" with respect to.

Example. *Take* M *with all* $q_{ij} = 1$ *(e.g. by trivial action), then* $\mathfrak{I}$ *is generated by the relations* $x_i x_j = x_j x_i$*, so* $\mathcal{B}(M)$ *is just the infinite algebra of polynomials* $\mathcal{B}(M) = \Bbbk[x_1, x_2 \ldots, x_{rank}]$ *(bosonic).*

Example. *Take* $G = \mathbb{Z}_2$ *and* $M = M_0 \oplus M_1$ *the superspace with dimension* $0 + 1$ *i.e. fermionic* $q_{11} = -1$*, then* $x^2 \in \mathfrak{I}$ *and hence* $\mathcal{B}(M) = \Bbbk[x]/(x^2)$*. This matches what one would expect from* **Pauli exclusion principle***; especially the Nichols algebra is now* **finite-dimensional**.

More generally a 1-dimensional Yetter-Drinfel'd module with $q_{ii} \in \Bbbk_n$ a primitive n-th root of unity has Nichols algebra $\mathcal{B}(M) = k[x]/(x^n)$.

Example. *Take again* $G = \mathbb{Z}_2$ *and* $M = M_0 \oplus M_1$ *the superspace with dimension* $0 + 2$ *(and basis say* x, y*), then*

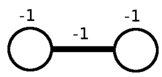
$$\mathcal{B}(M) = \Bbbk\langle x, y\rangle/(x^2, y^2, xy + yx) = \bigwedge M$$

Hence in contrary to the above a **anticommutator** *is divided out and the Nichols algebra is the (fermionic) exterior algebra.*

In the abelian case, Heckenberger (e.g [**H08**]) introduced q-decorated diagrams, with each node corresponding to a simple Yetter-Drinfel'd module decorated by q_{ii}, and each edge decorated by $\tau^2 = q_{ij}q_{ji}$ and edges are drawn if the decoration is $\neq 1$; it turns out that this data is all needed to determine the respective Nichols algebra.

For a symmetric braiding $\tau^2 = q_{ij}q_{ji} = 1$ (all examples so far) the **braided commutator** $[x_i, x_j]_\tau = x_i x_j - q_{ij} x_j x_i$ vanishes in $\mathcal{B}(M)$, which is the reason one does not draw a line in the decorated diagram. Already Kharchenko proved for G abelian, that any $\mathcal{B}(M)$ has a PBW-basis of iterated braided commutators and one draws **Dynkin diagrams** (with nodes again all simple M_i) much like for semisimple Lie algebras, see section 6.1.

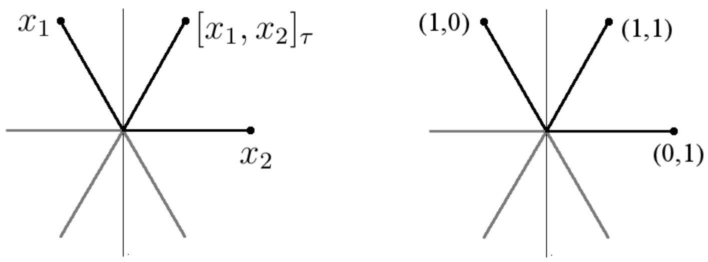
Example. *Assume $q_{11} = q_{22} = q_{12}q_{21} = -1$, then the diagram is:*



Then some calculations show, that $x_3 := [x_1, x_2]_\tau \neq 0$ ($\notin M$!) but

$$[x_2, [x_1, x_2]_\tau]_\tau = [x_1, [x_1, x_2]_\tau]_\tau = 0$$

Hence $\mathcal{B}(M)$ corresponds to the Borel part of $A_2 = \mathfrak{sl}_3$. Compare the PBW-basis in $\{x_1, x_2, x_3 = [x_1, x_2]_\tau\}$ to the A_2-root system



As $q_{11} = q_{22} = -1$ as well as $q_{33} = (q_{11}q_{12})(q_{22}q_{21}) = -1$, all three Nichols algebras $\mathcal{B}(x_i\mathbb{k})$ of rank 1 are fermionic. The Nichols algebra $\mathcal{B}(x_1\mathbb{k} \oplus x_2\mathbb{k})$ itself is 8-dimensional with PBW-Basis $x_1^i x_2^j x_3^k \quad i, j, k \in \{0, 1\}$, i.e. multiplication in $\mathcal{B}(M)$ yields a vector space bijection:

$$\mathcal{B}(M) \cong \mathcal{B}(x_1\mathbb{k}) \otimes \mathcal{B}(x_2\mathbb{k}) \otimes \mathcal{B}(x_3\mathbb{k}) = \mathbb{k}[x_1]/(x_1^2) \otimes \mathbb{k}[x_2]/(x_2^2) \otimes \mathbb{k}[x_3]/(x_3^2)$$

In the same sense, over abelian G for a_{ij} any proper Cartan matrix of a semisimple Lie algebra is realized for braiding matrix $q_{ij}q_{ji} = q_{ii}^{-a_{ij}}$.

However, several additional exotic examples of finite-dimensional Nichols algebras exist, that possess unfamiliar Dynkin diagrams, such as a multiply-laced triangle, and where Weyl reflections may connect different diagrams (yielding a **Weyl groupoid**). Heckenberger completely classified all Nichols algebras over abelian G in [**H08**].

Over **nonabelian** groups however, still much is open. Heckenberger and Schneider studied the Weyl groupoid in this setting as well and established a root system and a PBW-basis for finite-dimensional Nichols algebras in [**HS08**]. Only few finite-dimensional indecomposable examples are known so far, namely $\mathbb{D}_4$ of type A_2 and $\mathbb{S}_3, \mathbb{S}_4, \mathbb{S}_5$ of type A_1 (Schneider et. al. [**MS00**]), higher analogues of $\mathbb{D}_4$ ([**HS10**]), and a couple rank 1 examples over metacyclic groups [**GranaZoo**].

On the other hand, by detecting certain "defect" subconfigurations (so-called type D) most higher symmetric and all alternating groups $\mathbb{S}_{n\geq 6}$, $\mathbb{A}_{n\geq 4}$ and later many especially sporadic groups were totally discarded (Andruskiewitsch et. al. [**AZ07**],[**AFGV10**] etc.).

Technical Overview On Methods & Results

This is the main part of the thesis and solely concerned with orbifoldizing Nichols algebras. As we saw in Theorem 3.2, a twisting group Σ for a Hopf algebra $\mathbb{k}[\Gamma]\#\mathcal{B}(M)$ may be written down in terms of a subgroup of $Z^2(\Gamma,\mathbb{k}^\times)$ and an action of Σ as (Doi-)twisted symmetries $\theta_p : H_{\sigma_p} \xrightarrow{\sim} H$. Orbifoldizing then constructed a pointed Hopf algebra $\mathbb{k}[G]\#\mathcal{B}(\tilde{M})$ (and hence a Nichols algebra) over the extended group $\Sigma^* \to G \to \Gamma$. $\tilde{M}$ has a new basis of homogeneous elements as simultaneous eigenvectors of the twisted symmetries θ_p acting on M (the eigenvalues distinguishing different liftings to G).

To make the approach easier accessible and keep independence of part 1, we start with Theorem 4.4 by giving a short, direct but complete construction of the orbifoldized Nichols algebra solely in terms of Yetter-Drinfel'd modules and twisted symmetries θ_p thereof. Thus we construct an **orbifoldizing map**[4]:

$$
{}^{\Gamma}_{\Gamma}YDMod_{TwistSym(\Sigma)} \longrightarrow {}^{G}_{G}YDMod
$$

$$
(M,\mu,\delta,\theta_{p\in\Sigma}) \longmapsto \tilde{M} = \Big(M, \mu\circ(\pi\otimes id), \tilde{\delta}\Big)
$$

We already observed, that the image of our correspondence consists of modules with **trivial action** of $\Sigma^* \subset G$. Our reconstruction Theorem 3.6 (which will be reestablished directly as well, Theorem 5.1) will once again show this condition to be **iff**. Hence we get a bijection:

$$
{}^{\Gamma}_{\Gamma}YDMod_{TwistSym(\Sigma)} \hookrightarrow\!\!\!\!\rightarrow {}^{G}_{G}YDMod_{\Sigma.\,trivial} \;\cong\; {}^{\Gamma}_{G}YDMod
$$

We will give a first example of a new finite-dimensional Nichols algebra over $\mathbb{Z}_2 \to \mathbb{Q}_8 \to \mathbb{Z}_2^2$ and $\mathbb{Z}_2 \to GL_2(\mathbb{F}_3) \to \mathbb{S}_4$ in sections 4.3 and 4.4.

[4]The author apologizes for not providing thorough categorical definitions of Yetter-Drinfel'd modules with twisted symmetries, cocycle-deformation etc. at this point – otherwise we could also express the **functoriality** of the orbifoldization map. This has already been neglected in the first part, see Remark 1.8

Then we describe an easy way to extend our correspondence to produce Doi twists of Yetter-Drinfel'd modules with nontrivial Σ-action.

$$^{\Gamma}_{\Gamma}YDMod_{Sym(\Sigma)} \times Ker_{G\leftarrow\Sigma}H^2(G,\Bbbk^\times) \longrightarrow {}^G_GYDMod$$

$$(M,\mu,\delta,\theta_{p\in\Sigma},[\sigma]) \longmapsto \tilde{M}_\sigma = \Big(M,(\mu\circ(\pi\otimes id))_\sigma,\tilde{\delta}\Big)$$

We will furthermore get in section 5.2, through Matsumotos extension of the spectral sequence for central extensions [**IM64**], a way of precisely determining the nontrivial (scalar) actions of any $a\in\Sigma^*\subset G$ on any homogeneous component $(\tilde{M}_\sigma)_g$, that is induced by Doi twist:

$$\cdots\to H^2(\Gamma,\Bbbk^\times) \xrightarrow{inf} Ker_{G\leftarrow\Sigma}H^2(G,\Bbbk^\times) \xrightarrow{\gamma} Pairing(\Sigma^*\otimes G) \to\cdots$$

$$\text{“coincidence”:}\quad a.|_{(\tilde{M}_\sigma)_g} = \sigma(a,g)\sigma^{-1}(g,a) = \gamma(\sigma)(a,g)\in\Bbbk^\times$$

This will immediately show the above map to factor to an injection:

$$^{\Gamma}_{\Gamma}YDMod_{Sym(\Sigma)} \times \frac{Ker_{G\leftarrow\Sigma}H^2(G,\Bbbk^\times)}{Im_{\Gamma\leftarrow G}H^2(\Gamma,\Bbbk^\times)} \hookrightarrow {}^G_GYDMod$$

This allows to construct **new faithful examples** from orbifolds, such as the examples in section 7.1 over nonabelian groups of order 16 and 32.

On the other hand, in explicit cases we can narrow down possible Σ^*-actions using the constraints given in [**HS08**] for Yetter-Drinfel'd modules with finite-dimensional Nichols algebras. In several instances, the left-side cohomology term will be large enough to numerically exhaust the remaining. This then proves the map above to be surjective and thus every Nichols algebra to be a Doi twist of an orbifold.

$$^{\Gamma}_{\Gamma}YDMod_{Sym(\Sigma)} \times \frac{Ker_{G\leftarrow\Sigma}H^2(G,\Bbbk^\times)}{Im_{\Gamma\leftarrow G}H^2(\Gamma,\Bbbk^\times)} \hookrightarrow\!\!\!\!\rightarrow {}^G_GYDMod$$

To bound the number of actions and avoid the complication of group-realizations, we shall always restrict ourselves to **minimally** indecomposable Nichols algebras (M contains no proper indecomposable submodule). As such is contained in every indecomposable M, this suffices to study which groups admit finite-dimensional Nichols algebras at all.

The leading example will be the (known) classification of all minimally-indecomposable finite-dimensional Nichols algebras over $\mathbb{D}_4$ (and $\mathbb{Q}_8$) in section 5.3. Later examples are several related groups of order 16 and 32 in sections 7.2 and 7.3, where again only type A_2 appears.

Ultimately in section 7.4 we discard any indecomposable Nichols algebra over nine groups of order 32 (the only ones of this order with rank 3) by proving them to be Doi twists of $\mathbb{Z}_2^2$-orbifolds of an infinite-dimensional Nichols algebra with Dynkin diagram an 8-cycle. Checking all cases is tedious and largely performed by considering all **Weyl equivalents** at once – this table is computed by hand in section 8.1.

The centerpiece of the thesis is Theorem 6.1 which determines **all orbifoldized** minimally indecomposable finite-dimensional connected Nichols algebras over groups with $G' = \mathbb{Z}_p$, starting from the abelian $\Gamma = G/G'$ (with Nichols algebras classified in [**H08**]). The proof will require all of chapter 6. Especially we find necessarily $p = 2$ and the diagram of Cartan type! We construct indecomposable Nichols algebras with all Dynkin diagrams $A_n, D_n, B_n, E_{6,7,8}, F_4$ (and decomposable ones of type C_n, G_2 ($p = 3$!) and several non-Cartans). It's application will be section 7.1, where we give such examples for most groups of order 16 and 32, including many nondiagonal and some even faithful Doi twists.

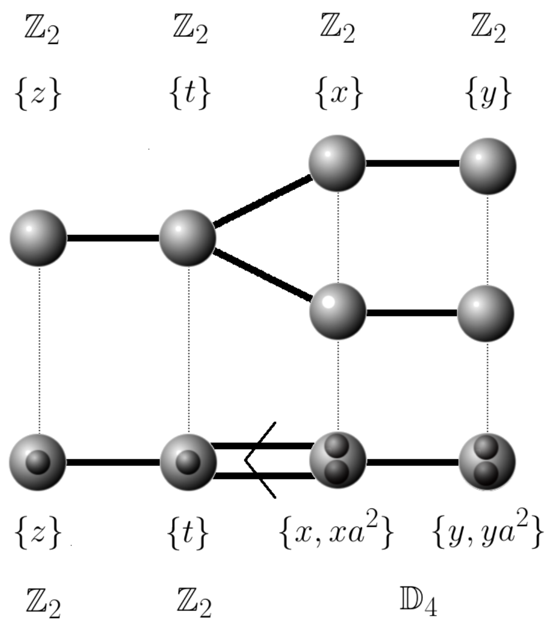
The **main proof idea** for this classification is quite intriguing:

Step I: Clarify orbifoldizing on Dynkin diagram level

Although no algebra automorphisms, we will find twisted symmetries to be automorphisms of the Dynkin diagram of $\mathcal{B}(M)$ (section 6.1) and in Theorem 6.9 derive certain necessary conditions in rank 1 and 2.

Structurally, we show in Theorem 6.8 that the Dynkin diagram of $\mathcal{B}(\tilde{M})$ is **folded** corresponding to the sub-rootsystem fixed by Σ (a known concept for Lie algebras, see e.g. [**Gi06**], p. 47ff), while different nodes (resp. simple Yetter-Drinfel'd modules) in the orbit of the twisted symmetry agglutinate to a single node of higher dimension over G.

The behaviour of nodes and edges will be described using **geometric vocabulary**, such as "splitting" and "ramification". The latter is highly restricted and appears for adjacent simple Yetter-Drinfel'd modulesover G-conjugacy classes of different length – the connecting edge becomes multiply-laced, as shown on the cover of this thesis:



The two leftmost nodes are **inert** *(i.e. invariant under twisted symmetries), while the two rightmost nodes are* **split** *orbits. Accordingly, the left edge is* **inert***, the right is* **split** *and the intermediate is* **ramified***.*

Step II: Search Heckenberger's list for suitable candidates

Actually this is the final step in section 6.6, but of course it points to the yet-to-be constructed examples in step III. Having clarified necessary conditions for an orbifoldization of a diagonal Yetter-Drinfel'd module, we go through Heckenberger's classification [**H08**] and search for all appropriate candidates with finite-dimensional Nichols algebras

- **Step 1** is the observation of a diagram automorphism and excludes totally inert orbifolds as decomposable.
- **Step 2** consist of multiple revisions of Heckenberges list:
  - **Step 2a** searches the list for diagrams eligible to be disconnectedly doubled and orbifoldized unramified to a Dynkin diagram of the same type. We know from step I (split edge) that therefore all edges must be decorated by -1, leaving only all Cartan types for $q = -1$ except B_n.

  - **Step 2b** searches the list for all loopfree[5] diagrams with involutory automorphism, resulting in possible ramified orbifolds from type E_6, A_{2n-1}, D_n and several non-Cartan.
  - **Step 2c** searches the list for possible non-loopfree diagrams with involutory automorphisms. Again, step I heavily restricts this case and leaves only an isolated loop $A_2 \to A_1$ for $q \in \Bbbk_3$.
  - **Step 2d** searches the list for all diagrams with higher-order automorphisms $p \neq 2$, resulting only in $D_4 \to G_2$.
- **Step 3a** excludes multiply laced diagrams C_n, F_4, G_2 from the unramified case by exhibiting a loop between the two copies of any long root, leaving only Cartan type ADE.
- **Step 4** shows that orbifolds with a unique split node only lead to decomposable Nichols algebras. This excludes small and most ramified cases (e.g. $D_{n+1} \to C_n$ and $D_4 \to G_2$) and leaves only the later-on realized unramified cases $A_{n\geq 2}$, $D_{n\geq 4}$, $E_{6,7,8}$ and ramified cases $E_6 \to F_4$ and $A_{2n-1} \to B_{n\geq 3}$.

Step III: Construct the remaining examples (and restrict the possible groups that realize them)

In order to bundle combinatorial considerations, the author introduces the notion of a **symplectic root systems** (Definition 6.14). Be aware, that this is a consequent, but far less powerful extension of ordinary root systems to symplectic vector spaces with possible nullspaces, and many graphs admit them! On the other hand they give precise nontrivial necessary and sufficient conditions on the groups rank and center (i.e. nullspace) to realize the respective diagram as orbifold. E.g. D_n requires more center than others. For all diagrams in question, these symplectic root systems are classified in Theorem 6.15 case-by-case.

With this knowledge, we can construct the actual Nichols algebras, compactly describe dimension, root system etc. and give examples. This is done case-wise:

[5]As for Lie algebra folding (not admitted there), we call a connected orbit **loop**

- In section 6.3 the unramified cases, that are constructed by doubling a diagram of type ADE and orbifoldize it to a diagram of same type. Note that the well-known example over $\mathbb{D}_4$ (section 3.2) was our toy-model for this case: $A_2 \cup A_2 \to A_2$.
- In section 6.4 the exceptional $E_6 \to F_4$ on the cover.
- In section 6.5 furthermore $A_{2n-1} \to B_n$.

The systematic construction so far has only constructed finite-dimensional Nichols algebras for nilpotent groups of class 2. We conclude the part by personal notes that summarize a longer-term effort of the author, that could not have been achieved in this thesis, namely the clarification, which higher-class nilpotent groups admit finite-dimensional Nichols algebras:

Commutators with odd order can be discarded rather easily and new general results in [**HS10**][6] seem to exclude class > 4 with an additional trick (making the authors original argument superfluous, that targeted rather tediously and ad-hoc the nilpotent case using [**HS08**]). However, particular 2-groups of class 3 (most notably the quasidihedral group $\tilde{\mathbb{D}}_8$) are very resilient, and the author has no opinion, whether they lead to new Nichols algebras or can be discarded.

The author has presented his progresses in this direction (including some key results of this thesis), but also this particular serious obstacle, in a mini-Talk "Nichols Algebras over Nilpotent Groups" at the Oberwolfach conference "Deformations in Mathematics and Physics" (October 2010).

[6]The authors thanks Prof. Schneider for pointing out this new research of his.

CHAPTER 4

A Shortcut To Orbifold Construction

In this chapter we give short, direct constructions and proofs for orbifoldizing a Nichols algebras $\mathcal{B}(M)$ over a group Γ to a Nichols algebra $\mathcal{B}(\tilde{M})$ over a central extension $\Sigma^* \to G \to \Gamma$. The rather ad-hoc formulas correspond to applying the abstract machinery in part 1 to a Radford biproduct $H = \Bbbk[G]\#\mathcal{B}(M)$ as derived in Theorem 3.2.

1. Central Group Extensions

Suppose a central extension of finite groups:

$$1 \to \Sigma^* \to G \xrightarrow{\pi} \Gamma \to 1 \qquad \Sigma \subset Z(G)$$

It can be described in terms of a class of 2-cocycles

$$[u] \in H^2(\Gamma, \Sigma^*)$$

For our purposes, we will find it more convenient to rewrite the Hopf algebra $\Bbbk[G]$ in terms of multiple twisted grouprings $\Bbbk_\sigma[\Gamma],\ \sigma \in Z^2(\Gamma, \Bbbk^\times)$:

Lemma 4.1. *We have an algebra isomorphism*

$$\phi :\ \Bbbk[G] \cong \bigoplus_{p \in \Sigma} \Bbbk_{\sigma_p}[\Gamma]$$

where we concatenated some representative u with all 1-dimensional representations $p \in \Sigma^{**} \cong \Sigma$ of the coefficient group Σ^* to yield a subgroup of 2-cocycles $\sigma_p := p \circ u \in Z^2(\Gamma, \Bbbk^\times)$.

Proof. Fix a set-theoretic split $s : \Gamma \to G$ of π with $s(1) = 1$ and $u \in Z^2(\Gamma, \Sigma^*)$ the associated (then normalized) 2-cocycle with $s(a)s(b) = u(a,b)s(ab)$. For $\sigma_p := p \circ u \in Z^2(\Gamma, \Bbbk^\times)$ consider the map:

$$\phi : G \ni g \mapsto \sum_{p \in \Sigma} \pi(g)1_p \cdot p\left(\frac{g}{s(\pi(g))}\right) \in \bigoplus_{p \in \Sigma} \Bbbk_{\sigma_p}[\Gamma]$$

Here the expression $a1_p$ for $a \in \Gamma$ and $p \in \Sigma$ shall denote the image in the twisted groupring $\Bbbk_{\sigma_p}[\Gamma]$. The split-condition $\pi(s(a)) = a$ ensures the fraction to be in the kernel of π and hence in $\Sigma^* \subset G$.

We first show this is an **algebra homomorphism**:

$$\phi(1_G) = \sum_{p\in\Sigma}(1_\Gamma)_p \cdot p(1) = \sum_{p\in\Sigma} 1_{k_{\sigma_p}[\Gamma]} = 1_{\bigoplus k_{\sigma_p}[\Gamma]}$$

$$\begin{aligned}
\phi(g)\phi(h) &= \sum_{p\in\Sigma} \pi(g)\pi(h)\sigma_p(\pi(g),\pi(h))1_p \cdot p\left(\frac{g}{s(\pi(g))}\right) p\left(\frac{h}{s(\pi(h))}\right) \\
&= \sum_{p\in\Sigma} \pi(gh)1_p \cdot \sigma_p(\pi(g),\pi(h))p\left(\frac{gh}{s(\pi(g))s(\pi(h))}\right) \\
&= \sum_{p\in\Sigma} \pi(gh)1_p \cdot \sigma_p(\pi(g),\pi(h))p\left(\frac{gh}{u(\pi(g),\pi(h))s(\pi(gh))}\right) \\
(\sigma_p = p\circ u) &= \sum_{p\in\Sigma} \pi(gh)1_p \cdot p\left(\frac{gh}{s(\pi(gh))}\right) = \phi(gh)
\end{aligned}$$

(note that $(a1_p)(b1_q) = ab1_p\sigma_p(a,b)$ for $p = q$ and 0 else). Certainly $\pi(g) \neq \pi(h)$ yields independent images in the target, moreover elements $g \neq h$ with $\pi(g) = \pi(h)$ differ by a Σ-element detection by some $p \in \Sigma$ over $\mathbb{k} = \mathbb{C}$ by duality of abelian groups. Hence the image's dimension $|G| = |\Gamma| \cdot |\Sigma|$ is also the target dimension, showing **bijectivity**. □

We prove the following **facts**, that are implicitly consequences of constructing the group Hopf algebra $\mathbb{k}[G]$ via ϕ as an **orbifoldization** of $\mathbb{k}[\Gamma]$ (being cocommutative, the easiest case). This means here, that the **coproduct** carries over to the target of ϕ. Let us define the map

$$\Delta_{p,q} : \mathbb{k}_{\sigma_{pq}}[\Gamma] \ni g1_{pq} \mapsto g1_p \otimes g1_q \in \mathbb{k}_{\sigma_p}[\Gamma] \otimes \mathbb{k}_{\sigma_q}[\Gamma]$$

Then we have the correspondence

$$\begin{aligned}
(\phi\otimes\phi)\Delta(g) &= \sum_{p\in\Sigma} \pi(g)1_p \cdot p\left(\frac{g}{s(\pi(g))}\right) \otimes \sum_{q\in\Sigma} \pi(g)1_q \cdot q\left(\frac{g}{s(\pi(g))}\right) \\
&= \sum_{r\in\Sigma} r\left(\frac{g}{s(\pi(g))}\right) \cdot \sum_{pq=r} (\pi(g)1_p \otimes \pi(g)1_q) = \sum_{p,q\in\Sigma} (\Delta_{p,q}\phi)(g)
\end{aligned}$$

Let further π_e denote the algebra projection to the 1_e-summand, then:

$$(\pi_e\phi)(g) = \pi_e\left(\sum_{p\in\Sigma} \pi(g)1_p \cdot p\left(\frac{g}{s(\pi(g))}\right)\right) = \pi(g)$$

Finally this gives us an orbifold expression for the **counit**

$$\epsilon_{\mathbb{k}[G]} = \epsilon_{\mathbb{k}[\Gamma]} \circ \pi = (\epsilon_{\mathbb{k}[\Gamma]} \circ \pi_e) \circ \phi$$

2. Construction Theorem

The setting is as follows: Let M be a finite dimensional Yetter-Drinfel'd module over Γ with Nichols algebra $\mathcal{B}(M)$.

Definition 4.2. *An* **abelian** *group* Σ *together with a group homomorphism* $\sigma:\ \Sigma \to Z^2(\Gamma, \Bbbk^\times)$ *acts as* **twisted symmetries** *on* M *via twisted actions* $\theta_p:\ M \to M$ *for each* $p \in \Sigma$, *iff*

- $\theta_e = id_M$
- $\theta_p \circ \theta_q = \theta_{pq}$
- θ_p *is a Yetter-Drinfel'd module-isomorphism (linear, colinear)* $M_{\sigma_p} \to M$, *where* M_{σ_p} *for a 2-cocycle* $\sigma_p \in H^2(\Gamma, \Bbbk^\times)$ *is defined as* M *with modified* Γ*-action on homogeneous elements:*

$$g._{\sigma_p} v_h = \sigma_p(ghg^{-1}, g)\sigma_p^{-1}(g,h)g.v_h$$

It is well known (e.g. [**M08**] Prop. 5.2) that the modified Γ-action above produces Doi twists $\Bbbk[\Gamma]\#\mathcal{B}(M_{\sigma_p}) \cong (\Bbbk[\Gamma]\#\mathcal{B}(M))_{\sigma_p}$. Hence twisted symmetry means "Doi twist stability" under all σ_p, $p \in \Sigma$ with a coherent choice of isomorphisms, that turns M into a Σ-representation.

Lemma 4.3. *The q-decorated generalized Dynkin diagram (see page 69 or* [**H08**]*) is preserved by the above Doi twists (i.e. by trivially extended group 2-cocycles); as are the root systems* $\tilde{\Delta}$ *(see section 6.1 or* [**HS08**] *Definition 6.1) and thus the Cartan matrices, Dynkin diagrams etc.*

Hence twisted symmetries are no Yetter-Drinfel'd module-automorphisms, but still **Dynkin diagram automorphisms** *in both senses.*

Proof. For the first assertion, we decompose M

$$M = \bigoplus_i \mathcal{O}_{g_i}^{\chi_i}$$

and directly calculate the q-factors after twisting:

$$\begin{aligned} q''_{ii} &= \chi_i^{\sigma_p}(g_i) \\ &= \sigma_p(g_i, g_i)\sigma_p^{-1}(g_i, g_i)\chi_i(g_i) \\ &= \chi_i(g_i) = q'_{ii} \end{aligned}$$

$$\begin{aligned} q''_{ij}q''_{ji} &= \chi_i^{\sigma_p}(g_j)\chi_j^{\sigma_p}(g_i) \\ &= \sigma_p(g_i,g_j)\sigma_p^{-1}(g_j,g_i)\chi_i(g_j)\cdot \\ &\cdot \sigma_p(g_j,g_i)\sigma_p^{-1}(g_i,g_j)\chi_j(g_i) \\ &= \chi_i(g_j)\chi_j(g_i) = q'_{ij}q'_{ji} \end{aligned}$$

For the second assertion, note that the images of some family $(W_l)_{l\in L}$ again satisfy the conditions of *cit. loc.*, as the Doi twist preserves the $\mathbb{N}_0^\theta$-grading and the tensor product in ${}^\Gamma_\Gamma YDM$. Hence the set of degrees is preserved as well. □

Theorem 4.4. *Suppose a central extension $\Sigma^* \to G \to \Gamma$ with some $\pi, s, u, \sigma_p, \phi$ chosen as in Lemma 4.1. Suppose further M a Γ-Yetter-Drinfel'd module with (possibly infinite dimensional) Nichols algebra $\mathcal{B}(M)$ and an action of Σ on M by twisted symmetries θ_p with respect to the σ_p, $p \in \Sigma$ (Definition 4.2).*

We can then define the **orbifoldized** *Yetter-Drinfel'd module $\tilde{M}$ over G as follows: Take $\tilde{M} = M$ as vector space, pull back the action to G via π and define a new G-coaction for a (former!) homogeneous $v_h \in M_h$ by ϕ^{-1}-piecing together all twisted Γ-coactions:*

$$\begin{aligned} \delta_{\tilde{M}}: \quad \tilde{M} &\to \left(\bigoplus_{p\in\Sigma} \mathbb{k}_{\sigma_p}[\Gamma]\right) \otimes \tilde{M} \xrightarrow{\phi^{-1}} \mathbb{k}[G] \otimes \tilde{M} \\ v_h &\mapsto (\phi^{-1}\otimes 1_{\tilde{M}}) \sum_{p\in\Sigma} h1_p \otimes \theta_p(v_h) \end{aligned}$$

Also, $\tilde{M} \cong M$ as a braided vector space and hence $\mathcal{B}(M) \cong \mathcal{B}(\tilde{M})$.

Proof. We have to verify that the pull-back G-action and the above G-coaction $\delta_{\tilde{M}}$ indeed turn $\tilde{M}$ into a G-Yetter-Drinfel'd module:

Clearly, the pull-back action turns $\tilde{M}$ into a G-**module**

$$\mathbb{k}[G] \otimes \tilde{M} \xrightarrow{\pi\otimes id} \mathbb{k}[\Gamma] \otimes M \to M$$

Secondly, the coaction $\delta_{\tilde{M}}$ defines a G-**comodule** via the relations of ϕ with the group Hopf algebra $\mathbb{k}[G]$ to $\Delta_{p,q}, \pi_e$ (see page 78):

$$\begin{aligned}
(\phi \otimes \phi \otimes 1)(1 \otimes \delta_{\tilde{M}})\delta_{\tilde{M}}(v_h) &= (1 \otimes (\phi \otimes 1)\delta_{\tilde{M}}) \sum_{q \in \Sigma} h1_q \otimes \theta_q(v_h) \\
&= \sum_{p,q \in \Sigma} h1_p \otimes h1_q \otimes \theta_p(\theta_q(v_h)) \\
&= \sum_{p,q \in \Sigma} h1_p \otimes h1_q \otimes \theta_{pq}(v_h) \\
&= \sum_{r \in \Sigma} \sum_{pq=r} h1_p \otimes h1_q \otimes \theta_r(v_h) \\
&= \sum_{p,q} (\Delta_{p,q} \otimes 1_{\tilde{M}}) \left(\sum_{r \in \Sigma} h1_r \otimes \theta_r(v_h) \right) \\
&= (\phi \otimes \phi \otimes 1)(\Delta \otimes 1_{\tilde{M}})\delta_{\tilde{M}}(v_h) \\
(\epsilon_{\mathbb{k}[G]} \otimes 1_{\tilde{M}})\delta_{\tilde{M}}(v_h) &= (\epsilon_{\mathbb{k}[G]}\phi^{-1} \otimes 1) \left(\sum_p h1_p \otimes \theta_p(v_h) \right) \\
&= (\epsilon_{\mathbb{k}[\Gamma]}\pi_e \otimes 1) \left(\sum_p h1_p \otimes \theta_p(v_h) \right) \\
&= 1_{\mathbb{k}} \otimes \theta_e(v_h) = 1_k \otimes v_h
\end{aligned}$$

The thirdly prove the **Yetter-Drinfel'd condition** the assumptions of the θ_p as twisted symmetries is required; denote $\pi(g) = \bar{g}$ and again use that $h1_p g1_q = 0$ for different direct summands $p \neq q$:

$$\begin{aligned}
&(\phi \otimes 1)(ad_g \otimes g.)\delta_{\tilde{M}}(v_h) \\
&= (\phi(g) _ \phi(g^{-1})) \otimes g.)(\phi \otimes 1)\delta_{\tilde{M}} v_h \\
&= \left(\sum_p \bar{g}1_p p \left(\frac{g}{s(\pi(g))} \right) _ (\bar{g}^{-1})1_p p \left(\frac{g^{-1}}{s(\pi(g^{-1}))} \right) \otimes g. \right) (\phi \otimes 1)\delta_{\tilde{M}}(v_h) \\
&= \left(\sum_p \sigma_p^{-1}(\bar{g}^{-1}, \bar{g})\bar{g}1_p _ \bar{g}^{-1}1_p \otimes g. \right) (\phi \otimes 1)\delta_{\tilde{M}}(v_h) \\
&= \sum_p \sigma_p^{-1}(\bar{g}^{-1}, \bar{g})\bar{g}1_p \cdot_{\sigma_p} h1_p \cdot_{\sigma_p} \bar{g}_p^{-1} \otimes g.\theta_p(v_h) \\
&= \sum_p \frac{\sigma_p(\bar{g}, h)\sigma_p(\bar{g}h, \bar{g}^{-1})}{\sigma_p(\bar{g}h, 1)\sigma_p(\bar{g}^{-1}, \bar{g})} \left(\bar{g}h\bar{g}^{-1} \right) 1_p \otimes g.\theta_p(v_h)
\end{aligned}$$

$$\begin{aligned}
\textit{(cocycle)} &= \sum_p \frac{\sigma_p(\bar{g},h)\sigma_p(\bar{g}h,\bar{g}^{-1})}{\sigma_p(\bar{g}h,\bar{g}^{-1})\sigma_p(\bar{g}h\bar{g}^{-1},\bar{g})} \left(\bar{g}h\bar{g}^{-1}\right) 1_p \otimes g.\theta_p(v_h) \\
&= \sum_p \frac{\sigma_p(\bar{g},h)}{\sigma_p(\bar{g}h\bar{g}^{-1},\bar{g})} \left(\bar{g}h\bar{g}^{-1}\right) 1_p \otimes g.\theta_p(v_h) \\
\textit{(}\theta_p\textit{ colinear)} &= \sum_p \frac{\sigma_p(\bar{g},h)}{\sigma_p(\bar{g}h\bar{g}^{-1},\bar{g})} \left(\bar{g}h\bar{g}^{-1}\right) 1_p \otimes \theta_p(g._{\sigma_p} v_h) \\
&= \sum_p \left(\bar{g}h\bar{g}^{-1}\right)_p \otimes \theta_p(g.v_h) = \delta_{\tilde{M}}(g.v_h)
\end{aligned}$$

Finally, let us show that $\tilde{M} \cong M$ as **braided vector space**:

$$\begin{aligned}
\tau_{\tilde{M}}(v_g \otimes v_h) &= \phi^{-1}\left(\sum_p \bar{g}1_p \cdot p\left(\frac{g}{s(\pi(g))}\right)\right).v_h \otimes v_g \\
\textit{(action via pullback)} &= \pi\phi^{-1}\left(\sum_p \bar{g}1_p \cdot p\left(\frac{g}{s(\pi(g))}\right)\right).{}_\Gamma v_h \otimes v_g \\
&= \pi_e\left(\sum_p \bar{g}1_p \cdot p\left(\frac{g}{s(\pi(g))}\right)\right).{}_\Gamma v_h \otimes v_g \\
\textit{(only } p=e \textit{ nonvanishing)} &= \bar{g}.{}_\Gamma v_h \otimes v_g = \tau_M(v_g \otimes v_h)
\end{aligned}$$

$\square$

Note that although we have $Prim(\mathcal{B}(M)) = M = Prim(\mathcal{B}(\tilde{M}))$ this does **not** mean, that the v_h are still homogeneous elements (resp. skew-primitives in the bosonization)! Rather, the old $M_h \subset \tilde{M}$ decompose into **different** π**-preimages** of h, i.e. elements in $s(h)\Sigma^* \subset G$:

To achieve this, one has to decompose each M_h into simultaneous θ_p-eigenspaces $M_{h,\lambda}$; this is possible as we chose Σ abelian.

The λ-eigenvectors v then correspond to $\lambda s(h)$-homogeneous elements:

$$\begin{aligned}
\delta_{\tilde{M}} v &= \sum_p h1_p \otimes \theta_p(v) = \sum_p h1_p \otimes \lambda(p)v = \sum_p h1_p \cdot p(\lambda) \otimes v \\
&= \sum_p \pi(s(h)\lambda)1_p \cdot p\left(\frac{s(h)\lambda}{s(h)}\right) \otimes v = \phi(s(h)\lambda) \otimes v
\end{aligned}$$

Hence Σ-**eigenspaces** of Γ-**layer** form the new G-**layer** $M_{h,\lambda} = \tilde{M}_{s(h)\lambda}$.

Especially since there always **are** eigenvectors over $k = \mathbb{C}$ we get homogeneous elements over at least some π-preimage:

Corollary 4.5. *In case of a* **stem extension** *($\Sigma \subset G'$), or more generally a Frattini extension $\Sigma \subset \Phi(G)$, the orbifoldized $\tilde{M}$ is* **indecomposable** *iff M is, because any preimages of a Γ-generating set generates G (see* [**Hu83**]*). This was derived generally in Theorem 2.10.*

3. Example: A New Nichols Algebra Over $\mathbb{Q}_8$

Besides $\mathbb{D}_4$, the second Schur cover of

$$\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$$

is the quaternion group $\mathbb{Q}_8$, again a stem-extension by its center. Let us again calculate the respective 2-cocycle by Lemma 1.18, namely, lift the elements $\mathbb{Q}_8 \to \Gamma \ni 1, g, h, gh$ to $1, i, j, k = ij$ where $i^2 = j^2 = k^2 = -1$ generate $\mathbb{Q}_8$ (columns and rows resp. $1, g, h, gh$):

$$\sigma = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

Note that even though this naturally differs from the cocycle obtained for $\mathbb{D}_4$, their σ-quotients determining the resp. Doi twists are identical:

$$\sigma(-, g)\sigma^{-1}(g, -) = (+1, -1)$$

$$\sigma(-, h)\sigma^{-1}(h, -) = (-1, +1)$$

Hence we may proceed completely equivalent to the preceding example, even use the same H and obtain a pointed, indecomposable Hopf algebra of dimension $dimH \cdot |\Sigma| = 64 \cdot 4 \cdot 2$. Also the description of the skew-primitives is identical to the one above.

Remark 4.6. *This could have been also calculated directly from he Yetter-Drinfel'd modules $O_b^\chi \oplus O_{ab}^\chi$ and $O_i^\chi \oplus O_j^\chi$ over $\mathbb{D}_4$ resp. $\mathbb{Q}_8$, with isomorphic braiding and hence isomorphic Nichols algebras:*

$$c = \begin{pmatrix} -t_1 \otimes t_1 & -t_2 \otimes t_1 & -\bar{t}_2 \otimes t_1 & -\bar{t}_1 \otimes t_1 \\ -t_1 \otimes t_2 & -t_2 \otimes t_2 & -\bar{t}_2 \otimes t_2 & -\bar{t}_1 \otimes t_2 \\ -t_2 \otimes \bar{t}_1 & -t_1 \otimes \bar{t}_1 & -\bar{t}_1 \otimes \bar{t}_1 & -\bar{t}_2 \otimes \bar{t}_1 \\ -t_2 \otimes \bar{t}_2 & -t_1 \otimes \bar{t}_2 & -\bar{t}_1 \otimes \bar{t}_2 & -\bar{t}_2 \otimes \bar{t}_2 \end{pmatrix}$$

Note however, that because $\mathbb{Q}_8$ has trivial second cohomology group, there are **no faithful Doi twists** – in contrast to $\mathbb{D}_4$!

4. Example: A New Nichols Algebra Over $GL_2(\mathbb{F}_3)$

$G := GL(2,3) = GL_2(\mathbb{F}_3)$ is a group with 48 elements and possesses a normal subgroup $\Sigma = \mathbb{F}_3^* \cong \mathbb{Z}_2$ of scalar multiplication. The quotient $\Gamma := PGL_2(\mathbb{F}_3)$ can be shown to act faithfully and 4-transitively on the 4 projective points resp. 1-dimensional vector subspaces of $(\mathbb{F}_3)^2$. Thus $\Gamma \cong \mathbb{S}_4$ and one can show that this quotient makes $GL(2,3)$ a Schur cover of $\mathbb{S}_4$ ([**Hu83**] p.653).

We could find a suitable cocycle again by corollary 1.18, but we will not require its explicit form in what follows.

Remark 4.7. *Note without proof, that the first two indecomposable Hopf algebras over $\Gamma = \mathbb{S}_4$ given already in* [**MS00**] *are* **not** *suitable for orbifoldizing, as a Doi twist interchanges them.*

Take $g \in \mathbb{S}_4$ a 4-cycle with centralizer $\langle g \rangle \cong \mathbb{Z}_4$, a centralizer character $\chi(g) = -1$ and $V = O_g^\chi$. By [**AHS09**], p. 48, $\mathcal{B}(V)$ is the third indecomposable Nichols algebra over $\mathbb{S}_4$ of finite dimension 24^2. We note that the condition in Theorem 4.4 is fulfilled trivially:

Lemma 4.8. *Suppose some $g \in \Gamma$ has the minimal centralizer $\langle g \rangle$, then for any $V = O_g^\chi$ we have $V_\sigma = V$, meaning the Hopf algebra $H\#\mathcal{B}(O_g^\chi)$ is its own Doi twist under the identity map.*

Proof. χ, χ_σ are uniquely determined by their value on g, if it generates all of the centralizer as demanded. But we immediately calculate from the definition:

$$\chi_\sigma(g) = \sigma(g,g)\sigma^{-1}(g,g)\chi(g) = \chi(g)$$

□

Thus we are again finished and obtain a $\mathbb{Z}_2$-orbifold of dimension $dimH \cdot |\Sigma| = 24^2 \cdot 24 \cdot 2$, which is pointed and indecomposable with coradical $GL(2,3)$. This example is apparently new and matches an open possibility (conjugacy class C_4) in [**FGV07**].

We conclude the section by giving hints to the situation $\mathbb{S}_5 \cong PGL(2,5)$. The extension $GL(2,5) \to \mathbb{S}_5$ is **no** stem extension, which one may see from the abelianization $\mathbb{Z}_4 = \mathbb{F}_5^* \to \mathbb{Z}_2$ or from the fact that we extend by $\Sigma = \mathbb{F}_5 = \mathbb{Z}_4$ whereas the group cohomology is only $\mathbb{Z}_2$, so res cannot be injective. However (as already noted there), Theorem 2.10 is still applicable and thus an orbifold would still be indecomposable.

Note that trying this with the known finite-dimensional Nichols algebra of the transposition conjugacy class (see [**MS00**]), we get the same problems as above, because σ-twisting interchanges the two choices. However, there **may exist** a third possibility (see [**AFGV10**]) over the $2+3$-conjugacy class. In this case, observe, that again the centralizer of this conjugacy class is minimal and the above lemma would apply, yielding as $\mathbb{Z}_4$-orbifold a pointed, indecomposable, finite-dimensional Hopf algebra over $GL(2,5)$.

CHAPTER 5

A Shortcut To Orbifold Reconstruction

1. Reconstruction Theorem

We now want to give conversely a short, direct construction and proof for the reconstruction Theorem 3.6 in the case of a Nichols algebra $\mathcal{B}(M)$ of a Yetter-Drinfel'd module over some finite group G. Again, the correspondence consists in the application of the abstract concepts to the Radford biproduct $\Bbbk[G]\#\mathcal{B}(M)$.

Suppose a central extension $\Sigma^* \to G \to \Gamma$ with some $\pi, s, u, \sigma_p, \phi$ chosen as in Lemma 4.1. By construction (Theorem 4.4), any orbifold $\tilde{M}$ has trivial action of $\Sigma^* \subset G$ (pull-back). We show also the converse is true:

Theorem 5.1. *Suppose $\tilde{M}$* **any** *G-Yetter-Drinfel'd module with $\Sigma^* \subset G$ acting trivial. Then there is a Γ-Yetter-Drinfel'd module M with a Σ-action $(\theta_p)_{p\in\Sigma}$ by twisted symmetries (Definition 4.2), such that $\tilde{M}$ is isomorphic to the orbifold of M with respect to the θ_p (Theorem 4.4).*

Proof. We first construct the Γ-Yetter-Drinfel'd module M: Take $M := \tilde{M}$ as **vector space**, the **coaction** shall be the push-forward by π (simple concatenation of $\delta_{\tilde{M}}$), while the condition (Σ^* acting trivial) ensure the **action** factorizes to a Γ-action. This fulfills obviously the Yetter-Drinfel'd condition if $\tilde{M}$ does. Now define **twisted symmetries** for each $p \in \Sigma$ on any homogeneous v_h ($h \in G$) by:

$$\theta_p : \; M_{\sigma_p} \to M \qquad v_h \mapsto p\left(\frac{h}{s(\pi(h))}\right)v_h$$

Here again, the fraction always lands in Σ. We easily verify, that these various linear transformations obey a **group law**:

$$\theta_e(v_h) = e\left(\frac{h}{s(\pi(h))}\right)v_h = v_h$$

$$(\theta_p\circ\theta_q)(v_h) = p\left(\frac{h}{s(\pi(h))}\right)q\left(\frac{h}{s(\pi(h))}\right)v_h = (pq)\left(\frac{h}{s(\pi(h))}\right)v_h = f_{pq}(v_h)$$

We yet have to check, that they respect the Yetter-Drinfel'd structures. The (untwisted) **colinearity** is obvious by construction of the coaction.

The (twisted) **linearity** holds as follows (again denoting $\bar{g} := \pi(g)$)

$$\begin{aligned}
\theta_p(\bar{g}.{}_{\sigma_p} v_{\bar{h}}) &= \sigma_p(\bar{g}\bar{h}\bar{g}^{-1}, \bar{g})\sigma_p^{-1}(\bar{g}, \bar{h})\theta_p(\bar{g}.v_{\bar{h}}) \\
\textit{($\tilde{M}$-YD-condition)} &= \sigma_p(\bar{g}\bar{h}\bar{g}^{-1}, \bar{g})\sigma_p^{-1}(\bar{g}, \bar{h})p\left(\frac{\bar{g}\bar{h}\bar{g}^{-1}}{s(\pi(\bar{g}\bar{h}\bar{g}^{-1}))}\right)(\bar{g}.v_{\bar{h}}) \\
&= \frac{\sigma_p(\bar{g}\bar{h}\bar{g}^{-1}, \bar{g})\sigma_p(\bar{g}, \bar{h})\sigma_p(\bar{g}\bar{h}, \bar{g}^{-1})}{\sigma_p(\bar{g}, \bar{h})} \cdot \\
&\quad \cdot p\left(\frac{\bar{g}}{s(\pi(\bar{g}))}\right)p\left(\frac{\bar{h}}{s(\pi(\bar{h}))}\right)p\left(\frac{\bar{g}^{-1}}{s(\pi(\bar{g}^{-1}))}\right)(\bar{g}.v_{\bar{h}}) \\
&= \frac{\sigma_p(\bar{g}\bar{h}\bar{g}^{-1}, \bar{g})\sigma_p(\bar{g}, \bar{h})\sigma_p(\bar{g}\bar{h}, \bar{g}^{-1})}{\sigma_p(\bar{g}, \bar{h})\sigma_p(\bar{g}^{-1}, \bar{g})} \cdot \\
&\quad \cdot p\left(\frac{\bar{g}\bar{g}^{-1}}{s(\pi(\bar{g}\bar{g}^{-1}))}\right)p\left(\frac{\bar{h}}{s(\pi(\bar{h}))}\right)(\bar{g}.v_{\bar{h}}) \\
&= \frac{\sigma_p(\bar{g}\bar{h}\bar{g}^{-1}, \bar{g})\sigma_p(\bar{g}\bar{h}, \bar{g}^{-1})}{\sigma_p(\bar{g}^{-1}, \bar{g})} \\
\textit{(cocycle)} &= \sigma_p(\bar{g}\bar{h}, 1)p\left(\frac{\bar{h}}{s(\pi(\bar{h}))}\right)(\bar{g}.v_{\bar{h}}) \\
&= \bar{g}.\theta_p(v_{\bar{h}})
\end{aligned}$$

Now it is easy to show, that the **above construction** again applied to an orbifoldization via $M, (\theta_p)_{p\in\Sigma}$ yields back $\tilde{M}$. For actions this is clear and we check now for a homogeneous element $v_h \in \tilde{M}$ with $h \in G$, that this coincides with the coaction, that were obtained by orbifoldizing the M we just found:

$$\begin{aligned}
\delta_{Orb_M}(v_h) &:= (\phi^{-1} \otimes 1)\sum_{p\in\Sigma} \bar{h}1_p \otimes \theta_p(v_h) \\
&:= (\phi^{-1} \otimes 1)\sum_{p\in\Sigma} \bar{h}1_p \otimes p\left(\frac{h}{s(\pi(h))}\right)v_h \\
&= \phi^{-1}\left(\sum_{p\in\Sigma} \bar{h}1_p p\left(\frac{h}{s(\pi(h))}\right)\right) \otimes v_h \\
&:= (\phi^{-1}\phi)(h) \otimes v_h = \delta_{\tilde{M}}(v_h)
\end{aligned}$$

$\square$

2. Matsumotos Exact Sequence

Already by construction $\Sigma^* \subset G$ acts trivial on an orbifold $\tilde{M}, \mathcal{B}(\tilde{M})$. Thus we shall in what follows frequently consider an additional **Doi twist** $\tilde{M}_\sigma$ with an additional 2-cocycle $\sigma \in Z^2(G, \Bbbk^\times)$ over G – note that the twistings above usually use 2-cocycles u, σ_p over the smaller Γ!

We already noted in section 4.2, that the Doi twist of the Hopf- or Nichols algebra produces the following twisted action on the twisted Yetter-Drinfel'd module $\tilde{M}_\sigma$:

$$a._\sigma v_h = \sigma(aga^{-1}, a)\sigma^{-1}(a, g)a.v_h$$

Because Σ^* is central in G, acting formerly trivial, is acts in the twisting on any $\mathcal{O}_h^\chi$ by multiplication of the scalar

$$\gamma(a, g) := \sigma(g, a)\sigma^{-1}(a, g)$$

It is a **lucky "coincidence"**, that this expression appears already in literature on group cohomology, namely in **Matsumoto's extension** [**IM64**] for central group extensions of the general Lyndon-Hochschild-Serre spectral sequence:

$$1 \to \Gamma^* \to G^* \to \Sigma \to H^2(\Gamma, \Bbbk^\times) \to H^2(G, \Bbbk^\times)_\Sigma \xrightarrow{\gamma} \Sigma^* \otimes G$$

Here, $H^2(G, \Bbbk^\times)_\Sigma$ denotes the kernel of the restriction map and the map γ yields as expected a **bimultiplicative pairing** that exactly matches the expression above!

This technique will be used in what follows to quickly determine the result of a Doi twist on the action of Σ^*, but more importantly to enumerate **all** actions, that can be reached by this method.

The next section gives a first example, where already **all admissible** actions (i.e. possibly producing finite-dimensional Nichols algebras) can be exhausted this way and Doi twists of orbifolds already classify all such Nichols algebras.

3. Example: All Minimal Nichols Algebras over $\mathbb{D}_4, \mathbb{Q}_8$

We already saw numerous times, that **most Yetter-Drinfel'd modules cannot be an orbifoldization**, since this is equivalent to trivial Σ-action by Theorem 5.1. We can at most hope for this to be the **only obstruction**, and in cases with "enough twists" and "few irreducible summands" we can prove such a statement. This is an exemplary case:

Theorem 5.2. *All finite-dimensional, minimally indecomposable (Definition 3.1) Nichols algebras $\mathcal{B}(M)$ over $G = \mathbb{D}_4, \mathbb{Q}_8$ are Doi twists of $\Sigma = \mathbb{Z}_2$-orbifoldizations of finite-dimensional Nichols algebras over $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. The **first step** is to use structural results to narrow down possible Σ-actions, that are admitted in finite dimensional specimen.

Theorem 5.3 ([**AZ07**] Lemma 2.2 and [**HS08**] Theorem 8.6)**.** *Suppose a finite-dimensional Nichols algebra: For V over a* **real class** $s^{-1} \in \mathcal{O}_s$ *we have $\chi_V(s) = -1$; for V, W over* **discommuting** $\mathcal{O}_s \neq \mathcal{O}_t$ *we get $U = [V, W] \neq \{0\}$ over $[st]$ with $\chi_U(st) = -\chi_V(s)\chi_W(t)$.*

Let us apply this to the real groups $\mathbb{D}_4, \mathbb{Q}_8$, where squares are in $\Sigma^* = \{1, g\}$. Any minimal generating set of conjugacy classes consists of two distinct discommuting $\mathcal{O}_s, \mathcal{O}_t$. obviously the former condition determines (maybe trivially) some Σ-action on V, W over these classes:

$$1 \overset{!}{=} \chi_V(s^2) = \chi_W(t^2)$$

The latter condition can also be rewritten, as $(st)^2$ is central:

$$1 \overset{!}{=} \chi_{[V,W]}((st)^2) = \chi_V((st)^2)\chi_W((st)^2)$$

Hence, depending on how many nontrivial $s^2, t^2, (st)^2 \overset{?}{=} g$ this reduces the 2^2 possibilities for $\chi_{V,W}(g) = \pm 1$ to $2^2, 2^1, 2^0$ many.

Facing the above restrictions, in the **second step** we need to find enough 2-cocycles of G to exhaust the remaining possible actions; if the Nichols algebra is finite, so are its Doi twist and hence all the obtained twisted actions have to fulfill the above conditions, too.

Although it is almost trivial in this case, we exemplary **use Matsumotos sequence** above (section 5.2) to determine the number of different action $|Im(\gamma)|$ of Σ^* on Doi twists:

$$1 \to \Gamma^* \to G^* \to \Sigma \to H^2(\Gamma, \Bbbk^\times) \to H^2(G, \Bbbk^\times)_\Sigma \xrightarrow{\gamma} \Sigma^* \otimes G$$

Using that the relevant cohomology groups are (see e.g. [**Hu83**])

$$H^2(\mathbb{Q}_8, \mathbb{C}^\times) = \{e\} \qquad H^2(\mathbb{Z}_2^2, \mathbb{C}^\times) = H^2(\mathbb{D}_4, \mathbb{C}^\times) = \mathbb{Z}_2$$

and the cohomology of the cyclic Σ^* is trivial, i.e. $H^2(G, \Bbbk^\times)_\Sigma = H^2(G, \Bbbk^\times)$ we find $|Im(\gamma)| = 2^0, 2^1$ for $G = \mathbb{Q}_8, \mathbb{D}_4$ respectively:

$$1 \to \mathbb{Z}_2^2 \to \mathbb{Z}_2^2 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to \{e\} \xrightarrow{\gamma} \Sigma^* \otimes G$$

$$1 \to \mathbb{Z}_2^2 \to \mathbb{Z}_2^2 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \xrightarrow{\gamma} \Sigma^* \otimes G$$

So we need to check all configurations of conjugacy classes against this and see, whether the Doi twist actions for the given group G exhaust all 2^2 possible Σ^*-actions minus the relations established above for this specific configuration:

- **Case** $\mathbb{Q}_8$**,** $\mathcal{O}_i \oplus \mathcal{O}_j$**:** $|Im(\gamma)| = 2^0$, but as $i^2 = j^2 = g$ we also get 2 independent basis relations:

$$\chi_V(g) = \chi_W(g) = 1$$

- **Case** $\mathbb{D}_4$**,** $\mathcal{O}_a \oplus \mathcal{O}_b$**:** There's now nontrivial Doi twist actions $|Im(\gamma)| = 2^1$ and still one basis relation left:

$$\chi_V(g) = \chi_V(a^2) = 1$$

- **Case** $\mathbb{D}_4$**,** $\mathcal{O}_b \oplus \mathcal{O}_{ab}$**:** There are no more basis relations, but by $(st)^2 = a^2 = g$ now one product relation:

$$\chi_V(g)\chi_W(g) = 1$$

  Note that this case could have been derived from the former by using a **Weyl-reflection** – for larger examples this is a considerable reduction of necessary calculations. We shall use this for the proof in section 7.4 with Weyl equivalence classes worked out in section 8.1.

Having obtained all possible Σ-actions by Doi twists, we may twist back $\tilde{M}$ to have a trivial one, hence our minimally indecomposable Nichols algebra is an orbifoldization by the reconstruction theorem. $\square$

CHAPTER 6

Orbifoldizing Nichols Algebras To $G' \cong \mathbb{Z}_p$

This is the centerpiece of the thesis: Throughout this chapter assume

$$\Sigma^* = \mathbb{Z}_p \to G \to \Gamma$$

to be a stem extension $\Sigma^* \subset G' \cap Z(G)$ of a finite **abelian** group Γ by a cyclic group $\mathbb{Z}_p$ of prime order. Using Heckenberger's classification [**H08**] of finite-dimensional Nichols algebras $\mathcal{B}(M)$ over Γ abelian we now construct and classify all **finite-dimensional minimally indecomposable Nichols algebras $\mathcal{B}(\tilde{M})$ with connected Dynkin diagram**, which appear as orbifoldized Nichols algebra of some $\mathcal{B}(M)$ via the given extension. We denote $1_n := n \; mod \; 2 \in \{0,1\}$.

Theorem 6.1. *For a finite stem extension $\Sigma^* = \mathbb{Z}_2 \to G \to \Gamma$ of a finite abelian group Γ we can construct a finite-dimensional minimally indecomposable Nichols algebra $\mathcal{B}(\tilde{M})$ over G for the following combination of data: Necessary 2-rank Γ (3rd column) and 2-center of G and (4th column) and Dynkin diagram of $\mathcal{B}(\tilde{M})$ (2nd column). Each case is orbifoldized from a suitable $\mathcal{B}(M)$ over Γ (1st column).*

Conversely (section 6.6), this list covers all components of finite-dimensional minimally indecomposable **orbifold** *Nichols algebras over stem extensions $\mathbb{Z}_p \to G \to \Gamma$ (p prime) with connected Dynkin diagram. In particular we find necessarily $p = 2$.*

- **Unramified** *(generic) simply laced components from a disconnected double with a symplectic root system (section 6.3):*

M	$\tilde{M}$	$dim_{\mathbb{F}_2}(\Gamma/\Gamma^2)$	$dim_{\mathbb{F}_2}(Z(G)/G'G^2)$
$A_n \cup A_n$	$A_{n\geq 2}$	n	1_n
$D_n \cup D_n$	$D_{n\geq 4}$	n	$2 - 1_n$
$E_n \cup E_n$	$E_{n=6,7,8}$	n	1_n

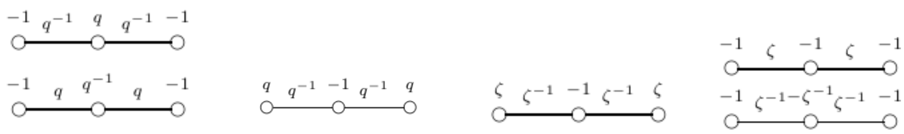
- **Ramified** *components from a single diagram with an order 2 automorphism and a symplectic root system for the split part of the diagram A_2, A_{n-1} (sections 6.4 and 6.5).*

M	$\tilde{M}$	$dim_{\mathbb{F}_2}(\Gamma/\Gamma^2)$	$dim_{\mathbb{F}_2}(Z(G)/G'G^2)$
E_6	F_4	$n=4$	$2+1_2=2$
A_{2n-1}	$B_{n\geq 3}$	n	$1+1_{n-1}=2-1_n$

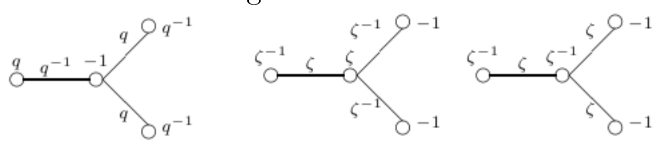
Note that rank and center precisely corresponds to the dimension and nullspace-dimension of all **symplectic root systems** *in Theorem 6.15.*

Note by section 6.6 additional link-**decomposable** Nichols algebras might be orbifoldized with one split node from the following types over Γ. The non-Cartan q-diagrams with involutive diagram automorphisms appear in [**H08**] (with $\zeta \in \Bbbk_3$):

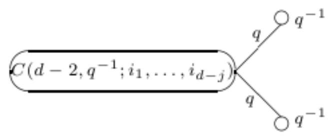
- an isolated loop diagrams $A_2 \to A_1,\ q \in \Bbbk_3$.
- unramified $A_1 \cup A_1 \to A_1$
- ramified $A_3 \to B_2$
- ramified $D_{n+1} \to C_n$
- ramified $D_4 \to G_2$ which is the **only** $\mathbb{Z}_3$-orbifold
- several non-Cartan diagrams of shape alike A_3, folding ramified to another non-Cartan diagram of rank 2.



- three non-Cartan diagrams of shape alike D_4, folding ramified to another non-Cartan diagram of rank 3.



- A family of non-Cartan diagrams of shape alike D_n, folding ramified to C_2 plus an inert non-Cartan part:



Remark 6.2. *In all these cases, the nonabelian (smaller) root lattice corresponds to the Lie-subalgebra fixed by the diagram automorphism of the larger root lattice over the abelian group. This correspondence has been a classical use of* **diagram folding**.

In our context, it can be understood as the **melting** *of simple Γ-Yetter-Drinfel'd modules into a simple one over G (resp. multiple equal Γ-elements to a single G-conjugacy class) by the redefined G-coaction, that uses the entire orbit of the twisted symmetries (resp. by the group's Σ-extension).*

In chapter 7 we will give nondiagonal (and especially some faithful!) Doi twists with rank ≤ 4 over various groups of order 16 and 32 and hence many **new large-rank Nichols algebras over 2-groups**.

Note that by the Reconstruction Theorem 5.1 **every** finite-dimensional indecomposable Nichols algebra over such G with trivial G'-action hence contains a connected component described below. Subsequent **full classifications** without the assumption of a trivial Σ^*-action in special cases are given in sections 7.2 - 7.4.

1. Orbifoldizing Dynkin Diagrams

In what follows we shall describe the effect of $\mathbb{Z}_p$-orbifoldizing to Dynkin diagrams of Yetter-Drinfel'd modules. We will only treat the later-on relevant cases; however the definitions are general context and one may calculate the orbifold's Dynkin diagram in the same manner as below.

Suppose a finite-dimensional semisimple Yetter-Drinfel'd module $M = \bigoplus_{i\in I} M_i$ with M_i simple. Following [**HS08**] Definition 6.4, one defines the generalized **Cartan matrix** $(a_{ij})_{i,j\in I}$ by

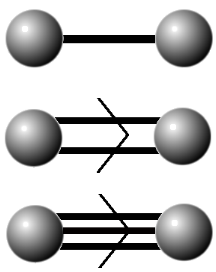
$$a_{ii} = 2 \qquad a_{ij} = -sup_h\{ad(M_i)^h(M_j) \neq 0\} \; for \; i \neq j$$

with the **adjoint action** resp. braided commutator in the Nichols algebra $\mathcal{B}(M)$ with multiplication denoted by μ:

$$ad(x)(y) := \mu(x \otimes y - \tau_M(x \otimes y))$$

One may organize this data into a **Dynkin diagram** by taking I as node-set and connecting i, j whenever $m_{ij} < 0$. More detailedly, one may use standard Lie algebra symbolics for the edge, if the Cartan matrix restricted to i, j matches that of a Lie algebra of rank 2; otherwise one had to decorate th edge with the explicit tuple $(-m_{ij}, -m_{ji})$ as custom for Coxeter groups or affine Lie algebras.

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$
$$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$$
$$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$



Especially we shall call M and $\mathcal{B}(M)$ **(diagram-)connected**, iff the Dynkin diagram is connected, regardless of the type of edge, and **simply laced** if only $m_{ij} = 2, 0, -1$ appear.

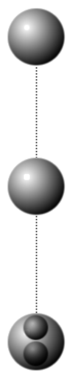
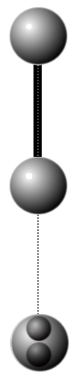
Suppose now we are given a Nichols algebra $\mathcal{B}(M)$ over a group, $\Sigma \to G \to \Gamma$, and Σ acting on M by respective twisted symmetries and hence by **diagram automorphisms** (Lemma 4.3). Especially Σ permutes the simple sub-Yetter-Drinfel'd modules $M_i \subset M$. Construction Theorem 4.4 then orbifoldizes M to a Yetter-Drinfel'd module $\tilde{M}$ over G. By construction of the coaction on $\tilde{M}$, the direct sum $\tilde{M}_i$ of an orbit $\{M_{i_1}, \ldots, M_{i_n}\}$ of simple M_i under this action of Σ is a sub-Yetter-Drinfel'd module of $\tilde{M}$. It is **not necessary simple**, but:

Corollary 6.3. *For a* **stem extension***, the orbifold $\mathcal{B}(\tilde{M})$ may only be* **minimally indecomposable** *(see Definition 3.1), if all $\tilde{M}_i$ are simple. This is again, because any lift of Γ-generators already generate G and hence possible summands in M_i could be omitted. Especially by dimensionality the* **conjugacy class** *$[g] \subset G$ vs. $[\bar{g}] \subset \Gamma$ needs to have grown according to the orbit length!*

We suppose these assumptions in what follows, and use it to visualize the influence of orbifoldizing as follows:

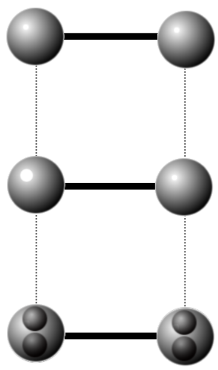
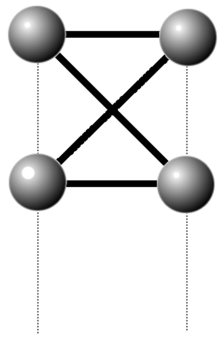
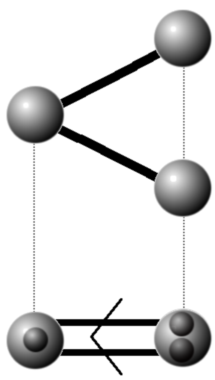
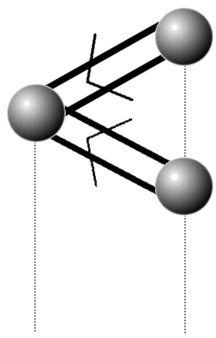
We draw the M-nodes at the top and the $\tilde{M}$-nodes at the bottom, such that each orbit M_{i_k} lays above the **single simple** $\tilde{M}_i$ and "project" by **dotted lines**. We draw **little nodes** for the orbits M_{i_k} inside $\tilde{M}_i$.

Definition 6.4. *For $\Sigma = \mathbb{Z}_p$ we may heuristically classify the* **nodes** *by their Σ-orbits being totally stable or unstable:*

$\{M_i\}$ *is* **inert**, *if it is fixed under Σ. Then $\tilde{M}_i = M_{i_1}$ as Yetter-Drinfel'd modules meaning no orbifoldization is performed and Σ acts trivial. Especially the conjugacy class stays the same.*	*An orbit $\{M_{i_1}, ...M_{i_p}\}$ of lenght p is called* **split** *. By the above for N the length of the Γ-conjugacy class underlying each M_{i_l}, the lenght of the G-conjugacy class underlying $\tilde{M}_i$ is pN.*	*A Γ-edge $(i_1 i_2)$ (i.e. $ad_{M_{i_1}}(M_{i_2}) \neq 0$) is a* **loop**, *if M_i, M_j are in the same split Σ-orbit. This case is heavily restricted for Γ abelian (Theorem 6.9) and later appear only in decomposable Nichols algebras (section 6.6).*
		
$\tilde{M}_i = M_{i_1}$	$\tilde{M}_i = \oplus_{k=1}^{p} M_{i_k}$	$\tilde{M}_i = \oplus_{k=1}^{p} M_{i_k}$

We shall now focus our interest to edges in the orbifold, i.e. let $\tilde{M} = \tilde{M}_i \oplus \tilde{M}_j$ with $\tilde{M}_i, \neq \tilde{M}_j$ simple G-Yetter-Drinfel'd modules. Either, if both nodes are inert, we call the edge **inert**. Otherwise, there are two cases other depending on the nodes' splitting behaviour, **split** and **branched** which we shall subdivide into generic and exotic cases:

Definition 6.5. *For a G-edge (ij) we define (the next Theorem 6.8 clarifies the resulting diagram in all later-on relevant cases!):*

(ij) is called **tamely split**, *iff both nodes are and each i_k connected to precisely one j_l (with equal edge-type by Σ-symmetry). See e.g. $\mathbb{D}_4$ in section 3.2.* 	*(ij) is called* **wildly split**, *if each node i_k connects to multiple i_k (possibly of different edge type!). This type will not appear for Γ abelian (Theorem 6.9).* 
(ij) is called **tamely branched**, *iff one node is inert (say i), the other (j) is split and the Γ-edges (again all equal) of type A_2.* 	*(ij) is called* **wildly branched**, *iff again one node inert and one split, but the Γ-edges not of type A_2. They will appear only in decomposable Nichols algebras (section 6.6).* 

We will introduce a more comfortable notation to diagrams of arbitrary rank, giving credit to increasingly serious non-genericity, as they appear in the later-on discussions:

Definition 6.6. *An orbifoldized diagram is exactly one of the following:*

- **inert** *iff all nodes are inert (and hence all edges)*
- **unramified** *iff no loops occur and all edges are tamely split.*
- **ramified** *iff no loops occur, all edges are either tamely split or tamely branched and at least one branched occurs (hence some nodes are split, some inert).*
- **wild** *iff at least one wildly branched or wildly split edge occurs or at least one node is a loop.*

Remark 6.7. *For* Γ *abelian, we shall prove in theorem 6.9, that*

- *Only a very specific loop may occur for* $p = 2$.
- *No wildly split edges can occur.*
- *Split edges only occur for* $p = 2$ *and a specific* Γ*-braiding matrix and do not contain loop nodes (hence are unramified).*
- *For two given nodes, an unramified edge occurs* **iff** *the two underlying* G*-elements discommute.*

We shall further see by explicitly checking all cases in section 6.6

- *Wildly branched and looped diagrams are both possible, but lead only to decomposable Nichols algebras.*
- *Ramification cannot occur for* $p \neq 2, 3$ *and* $D_4 \to G_2$ *for* $p = 3$ *leads to a decomposable Nichols algebra.*

The result is quite intriguing: The only remaining cases are all $p = 2$*, nonwild, and lead to indecomposable Nichols algebras Cartan type diagrams, as listed in Theorem 6.1.*

We now prove, that the Dynkin diagram of $\tilde{M}$ is indeed of the form given above. Hereby, we will solely use the knowledge of the root system over Γ and derive statements in analogy to classical Lie algebra folding. In contrast however, for Nichols algebras there are several exceptional cases possible

Theorem 6.8. *The orbifold's edge-type* $\tilde{m}_{ij}, \tilde{m}_{ji}$ *for the following cases used later-on is as asserted above:*

(1) *If all nodes* $M_{i_1}, \ldots M_{i_n}$ *over some orbifold node* $\tilde{M}_i$ *are* **disconnected** *to all nodes* $M_{j_1}, \ldots M_{j_n}$ *over some orbifold node*

$\tilde{M}_j$, then so are i,j, i.e.

$$ad(\tilde{M}_i)(\tilde{M}_j) = 0$$

(2) *A* **tamely split edge** *orbifoldizes to a rank 2 G-Nichols algebra of with the same edge type:*

$$ad(\tilde{M}_i)^n(\tilde{M}_j) = \sum_k ad(M_{i_k})^n(M_{j_k})$$

$$ad(\tilde{M}_j)^n(\tilde{M}_i) = \sum_k ad(M_{i_k})^n(M_{j_k})$$

$$\Rightarrow \quad \tilde{m}_{ij} = m_{ij} \qquad \tilde{m}_{ji} = m_{ji}$$

(3) *A* **tamely branched** *edge (say i inert, j split) for $p = 2,3$ orbifolds to a rank 2 Nichols algebra over G of type B_2, G_2 respectively. More precisely i.e.*

$$\begin{array}{cc} ad(\tilde{M}_i)(\tilde{M}_j) \neq 0 & ad(\tilde{M}_j)(\tilde{M}_i) \neq 0 \\ ad^2(\tilde{M}_i)(\tilde{M}_j) = 0 & ad^2(\tilde{M}_j)(\tilde{M}_i) \neq 0 \\ & \cdots \\ & ad^p(\tilde{M}_j)(\tilde{M}_i) \neq 0 \\ & ad^{p+1}(\tilde{M}_j)(\tilde{M}_i) = 0 \end{array}$$

PROOF. By the construction Theorem 4.4 we have $\mathcal{B}(M) \cong \mathcal{B}(\tilde{M})$, hence it suffices to successively derive the adjoint action of the Γ-Yetter-Drinfel'd modules solely from the knowledge of the Dynkin diagram over Γ; note that the elements $M_{i_k} \subset \tilde{M}_i$ are **not** homogeneous over G any longer, so using braided commutators would be tedious!

(1) In this case, simply all ad vanish:

$$ad(\bigoplus_k M_{i_k})(\bigoplus_l M_{i_l}) = \sum_{k,l} ad(M_{i_k})(M_{i_l}) = 0$$

(2) To calculate the necessary ad-spaces

$$ad(\tilde{M}_i)^n(\tilde{M}) \qquad ad(\tilde{M}_i)^n(\tilde{M})$$

we may number the split nodes

$$\tilde{M}_i = \bigoplus_k \tilde{M}_{i_k} \qquad \tilde{M}_i = \bigoplus_k \tilde{M}_{i_k}$$

in such a way, that over Γ precisely the pair M_{i_k}, M_{j_k} is connected , which is possible by the definition of tamely split (also by loopfreeness no M_{i_k}, M_{i_l} are connected!). It was already mentioned, that by the transitive action of $\Sigma = \mathbb{Z}_p$ on both orbits, these edges have equal type:

$$m_{i_k j_k} =: m_{ij} \qquad m_{j_k i_k} =: m_{ji}$$

We now show inductively, that

$$ad(\tilde{M}_i)^n(\tilde{M}) = \sum_k ad(M_{i_k})^n(M_{j_k})$$

(and vice versa for j, i). From this the second assertion on the $\tilde{m}_{ij}, \tilde{m}_{ji}$ follows immediately. The statement is certainly true for $n = 0$, and the induction step uses as assumed $ad(M_{i_k})(M_{j_l}) = 0$ for $k \neq l$ and all $ad(M_{i_k})(M_{i_l}) = 0$:

$$\begin{aligned}
ad(\tilde{M}_i)^{n+1}(\tilde{M}) &= ad(\tilde{M}_i)\left(ad(\tilde{M}_i)^n(\tilde{M})\right) \\
&= ad\left(\bigoplus_k M_{i_k}\right)\left(\sum_l ad(M_{i_l})^n(M_{j_l})\right) \\
&= \sum_{k,l} ad(M_{i_k})\left(ad(M_{i_l})^n(M_{j_l})\right) \\
&= \sum_k ad(M_{i_k})\left(ad(M_{i_k})^n(M_{k_l})\right) \\
&= \sum_l ad(M_{i_l})^{n+1}(M_{j_l})
\end{aligned}$$

(3) Although we would assume this statement to be true for all p, later-on only the cases $p = 2, 3$ will be relevant, that can be realized over abelian groups. In these cases the assertions follow directly from the knowledge of the respective root systems of type A_3, D_4 as in **[Gi06]**, p. 47ff:
  - $p = 2$ with (M_{j_1}, M_i, M_{j_2}) of type A_3 to B_2:

$\tilde{M}$-root	ad-space above	M-roots
$\{i\}$	M_i	$\{i\}$
$\{j\}$	$M_{j_1} \oplus M_{j_2}$	$\{j_1, j_2\}$
$\{i+j\}$	$ad(M_i)(M_{j_1} \oplus M_{j_2})$	$\{i+j_1, i+j_2\}$
	$ad^2(M_i)(M_{j_1} \oplus M_{j_2})$	$\{\}$
$\{i+2j\}$	$ad^2(M_{j_1} \oplus M_{j_2})(M_i)$	$\{i+j_1+j_2\}$
	$ad^3(M_{j_1} \oplus M_{j_2})(M_i)$	$\{\}$

- $p = 3$ with $(M_i, M_{j_1}, M_{j_2}, M_{j_3})$ of type D_4 with M_i in the center of the diagram to G_2:

$\tilde{M}$-root	ad-space above	M-roots
$\{i\}$	M_i	$\{i\}$
$\{j\}$	$M_{j_1} \oplus M_{j_2} \oplus M_{j_3}$	$\{j_1, j_2, j_3\}$
$\{i+j\}$	$ad(M_i)(M_{j_1} \oplus M_{j_2} \oplus M_{j_3})$	$\{i+j_1, i+j_2, i+j_3\}$
	$ad^2(M_i)(M_{j_1} \oplus M_{j_2} \oplus M_{j_3})$	$\{\}$
$\{i+2j\}$	$ad^2(M_{j_1} \oplus M_{j_2})(M_i)$	$\{i+j_1+j_2, i+j_1+j_3, i+j_2+j_3\}$
$\{i+3j\}$	$ad^3(M_{j_1} \oplus M_{j_2} \oplus M_{j_3})(M_i)$	$\{i+j_1+j_2+j_3\}$
	$ad^4(M_{j_1} \oplus M_{j_2} \oplus M_{j_3})(M_i)$	$\{\}$
$\{2i+3j\}$	*iterated, not above*	$\{2i+j_1+j_2+j_3\}$

□

We finally give certain general necessary conditions for such rank $1, 2$ orbifoldizings to be possible over Γ abelian, relying heavily on Heckenberger's list [**H08**]; in the remaining chapter we will then clarify all possible diagrams against this list and construct the remaining:

Theorem 6.9. *For Γ abelian the following necessary conditions apply:*

(1) *A* **loop** *may only appear for $p = 2$ and a decorated subdiagram A_2 in* [**H08**] *with q a primitive third root of unity.*
(2) *No* **wildly split** *edges can occur.*
(3) *A* **tamely split** *edge (ij) appears* **only** *for $p = 2$
$$\tilde{M}_i = M_{i_1} \oplus M_{i_2} \qquad \tilde{M}_j = M_{j_1} \oplus M_{j_2}$$
and* **only** *if the edge decoration of the rank 4 diagram over Γ is of (for a suitable numbering)
$$q_{i_1j_1}q_{j_1i_1} = -1 \qquad q_{i_2j_2}q_{j_2i_2} = -1$$
$$q_{i_1j_2}q_{j_2i_1} = +1 \qquad q_{i_2j_1}q_{j_1i_2} = +1$$
and* **only** *if the conjugacy classes underlying $\tilde{M}_i, \tilde{M}_j$ are mutually discommuting.*
(4) *A* **loop** *node cannot be part of a split edge.*

Remark 6.10. *Note that the discommuting-statement is* **iff***: Such conjugacy classes have both length > 1, hence are split, and by* [**HS08**] *Proposition 8.1 have to be connected!*

Remark 6.11. *Note further, that the prescribed edge-decorations for a split edge may corresponding to several decorated diagrams, most notably $A_2 \cup A_2$ and $C_2 \cup C_2$, for $q = -1$ respectively $q = \sqrt{-1}$; the latter will however exhibit an impossible loop in section 6.6.*

PROOF. Denote with σ a generator of $\Sigma = \mathbb{Z}_p$ and equivalently the 2-cocycle generator and $\theta = \theta_\sigma$ the associated twisted symmetry θ. Denote the simple and diagonal (hence 1-dimensional) Γ-Yetter-Drinfel'd modules

$$M_{i_1} = \mathcal{O}_{\bar{g}}^{\chi} \qquad M_{j_1} = \mathcal{O}_{\bar{h}}^{\eta}$$

then the other can be calculated from the twisted symmetry action, such as e.g. $\theta.M_{i_1} = \mathcal{O}_{\bar{g}}^{\chi_\sigma}$ etc.

(1) First note, that by transitivity action of θ of order p, a loop contains over Γ a p-cycle, which is impossible for $p \geq 4$ by [**H08**] ($p = 3$ will be disregarded at the end).

Let θ map M_{i_1} to some connected M_{i_2}, then

$$\begin{aligned} q_{i_1 i_2} q_{i_2 i_1} &= \chi(g)\chi^\sigma(g) \\ &= \chi(g)\sigma(g,g)\sigma^{-1}(g,g)\chi(g) \\ &= q_{i_1 i_1}^2 \end{aligned}$$

Going through Heckenberger's list for diagonal rank 2 Nichols algebras of finite dimension ([**H08**] table A.1), we find the only cases with symmetric node decoration $q_{i_1 i_1} = q_{i_2 i_2}$ to be Row 1, 2 and 3.

Row	$q_{i_1 i_1} = q_{i_2 i_2}$	$q_{i_1 i_2} q_{i_2 i_1}$	Cartan	Relation above demands
1	$q \neq 1$	1	$A_1 \cup A_1$	$q = -1$
2	$q \neq 1$	q^{-1}	A_2	$q^3 = 1$
3	-1	$q \neq \pm 1$	-	impossible

The first case is no loop, the second is the sporadic case of the statement, the third cannot support the relation given above.

Finally, note in in Heckenberger's list there is no 3-cycle with all nodes, edges decorated with such $q, q^{-1} \in \Bbbk_3$, hence $p = 3$ is impossible.

(2) For a wildly split edge, each Γ-node M_{i_k} has to be connected to at least two distinct M_{j_k}, M_{j_l} and vice versa; this statement could be formulated, as though the (p,p)-bipartite graph contains a $(2,2)$-regular bipartite graph. As obviously no dead ends occur, such a graph contains a cycle. Moreover the bipartitiveness forces the cycle length to be even, the length being $\neq 2$ by $k \neq l$, hence ≥ 4. But such a cycle is outruled again by [**H08**] Lemma 20.

(3) Assume again by proper numbering $i_1 j_1$ to be connected with edge-decoration $q_{i_1 j_1} q_{j_1 i_1} =: q \neq 1$. To be tamely split, all others $i_{k\neq 1} j_1$ have to be disconnected. Choose the further numbering in such a way, that the twisted symmetry θ (associated to the chosen generator) acts as

$$\theta^n.M_{i_1} = M_{i_{k+1}}$$

i.e. $M_{i_{k+1}} = \mathcal{O}_g^{\chi_{\sigma^k}}$. Then with the value

$$r := \sigma(\bar{h}, \bar{g})\sigma^{-1}(\bar{g}, \bar{h})$$

we get for the other edge decorations:

$$\begin{aligned} q_{i_{k+1} j_1} q_{j_1 i_{k+1}} &= \eta(\bar{g})\chi_{\sigma^k}(\bar{h}) \\ &= \eta(\bar{g})\sigma^k(\bar{h}, \bar{g})\sigma^{-k}(\bar{g}, \bar{h})\chi(\bar{h}) \\ &= r^k q \end{aligned}$$

As we assumed only $i_1 j_1$ to be connected via $q \neq 1$, the only possibility to achieve the others to be $rq = r^2 q = \cdots = r^{p-1} = 1$ is $p = 2$ and $q = r = -1$, showing the first and second **"only"**. For the third we proceed with the established

$$r = \sigma(\bar{h}, \bar{g})\sigma^{-1}(\bar{g}, \bar{h}) = -1$$

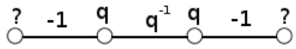
This means, that the underlying 2-cocycle of the stem-extension $u \in Z^2(\Gamma, \mathbb{Z}_2)$ is also nonsymmetric on $\bar{g}, \bar{h}$:

$$u(\bar{h}, \bar{g})u^{-1}(\bar{g}, \bar{h}) \neq 1$$

which concludes g, h to discommute in G.

(4) Suppose in the above statement i to be a loop, then j cannot be, as this would form a 4-cycle. Hence the decorated diagram

of rank 4 is fairly determined from the established decoration of split edges and the loop:



where q has to be a primitive third root of unity. But such a diagram does not appear in Heckenberger's list for rank 4 **[H08]** Table B.

$\square$

2. Symplectic Root Systems

Suppose we are given a finite group with $G' = \mathbb{Z}_p$; it is a stem-extension of it's abelianization $\Gamma = G/G'$. As usual for p-groups we consider the skew-symmetric, isotropic **commutator map** $[,]$ (see **[Hu83]**):

$$G \times G \xrightarrow{[,]} G' = \mathbb{Z}_p$$
$$g, h \mapsto [g, h] = ghg^{-1}h^{-1}$$
$$[h, g] = [g, h]^{-1} \quad [g, g] = 1$$

Because G' is central ("nilpotency class 2"), the map is multiplicative (the other argument's works analogously):

$$\begin{aligned}
[g, h][g', h] &= (ghg^{-1}h^{-1})(g'hg'^{-1}h^{-1}) \\
&= g(g'hg'^{-1}h^{-1})hg^{-1}h^{-1} \\
&= gg'hg'^{-1}g^{-1}h^{-1} \\
&= [gg', h]
\end{aligned}$$

and factors to $c : \Gamma \times \Gamma \to \mathbb{Z}_p$.

Lemma 6.12. *Let $u \in Z^2(\Gamma, \mathbb{Z}_2)$ be the 2-cocycle associated to the stem-extension $\mathbb{Z}_2 \to G \to \Gamma$ and set-theoretic split $s : \Gamma \to G$, then*

$$u(\bar{g}, \bar{h})u^{-1}(\bar{h}, \bar{g}) = [g, h] = c(\bar{g}, \bar{h})$$

Proof. Because $[,]$ is invariant, when central elements, such as $\in G'$, are multiplied to the argument, it is sufficient to check the assertion on the images of s, where we calculate:

$$\begin{aligned}
s(\bar{g})s(\bar{h}) &= u(\bar{g}, \bar{h})s(\bar{g}\bar{h}) \\
s(\bar{h})s(\bar{g}) &= u(\bar{h}, \bar{g})s(\bar{g}\bar{h}) \\
&= u(\bar{h}, \bar{g})s(\bar{h}\bar{g}) \\
\Rightarrow \ [s(\bar{g}), s(\bar{h})] &= u(\bar{g}, \bar{h})u^{-1}(\bar{h}, \bar{g})
\end{aligned}$$

□

Because of multiplicativity $[g^p, h] = [g, h]^p = 1$ and thus the commutator map even factorizes one step further to $V := \Gamma/p\Gamma \cong \mathbb{F}_p^n$

$$V \times V \xrightarrow{\langle,\rangle} \mathbb{F}_p \qquad \textit{denoted additively}$$

Theorem 6.13 (Burnside Basis Theorem)**.** *Every minimal generating set of G (no element may be omitted) consists precisely of $n = dim(V)$ elements $g_1, \ldots g_n$, whose images in V form a basis (this holds much more generally for all p-groups with $V = G/\Phi(G)$).*

Proof. Take a set $\{g_1, \ldots g_n\}$ with their images forming a basis of $V = \Gamma/p\Gamma$, then obviously the $\{\bar{g}_i\}$ also generate Γ; because some g_i, g_j ought to be discommuting (otherwise $G' = 1$), they already generate all of G. Also, no element may be omitted, otherwise the remaining images could not generate all of V. But the images of a generating set of G certainly have to generate the quotient V. Hence, the g_i form a minimally generating set.

Assume conversely some set $\{g_1, \ldots g_k\}$ to minimally generate G: Again, the images in V generate the entire quotient V. Assumed some linear dependency, one may omit an element g_l without compromising the generation of all V and (as shown above) the remaining g_i still generate the entire group. Thus the images form a basis. □

In what follows, we shall consider $V = G/(G'G^p)$ as a **symplectic vector space** $\mathbb{F}_p^n$ with (possibly degenerate!) **symplectic form** $\langle v, w\rangle$. For a sub-vector space $W \subset V$ we define the **orthogonal complement**:

$$W^\perp := \{v \in V \mid \forall_{w \in W}\ \langle v, w\rangle = 0\}$$

Especially $V^\perp = Z(G)/(Z(P) \cap G'G^p) = Z(G)/(G'G^p)$ is the **nullspace** of vectors orthogonal on all of V (note that always $\langle v, v\rangle = 0$). For $V^\perp = \{0\}$ we call V **nondegenerate**. It is well known (see e.g. [**Hu83**]) that there is always a **symplectic basis** $\{x_i, y_i, z_j\}$ consisting of mutually orthogonal **nullvectors** $z_j \in V^\perp$ and **symplectic base pairs** $\langle x_i, y_i\rangle = 1$ generating a maximal nondegenerate subspace. Note especially, that nondegenerate symplectic vector spaces hence always have even dimension! These nondegenerate spaces lead for example to the extraspecials $G = 2_\pm^{dim(V)+1}$, especially for $dim(V) = 2$ to $\mathbb{D}_4$ and $\mathbb{Q}_8$.

By Theorem 6.8 split edges may only appear over discommuting group elements. The following notion is used to construct a generating set of the nonabelian group, such that the noncommutativity matches the edges of a given graph. Note that it is a **considerably weaker** notion than that of an ordinary root system, and rather arbitrary graphs may be realized that way; it should be rather viewed as addition to a proper root system and Dynkin diagram, that allows stem-extension of the underlying group as prescribed by the symplectic form.

Definition 6.14. *Given a symplectic vector space V over $\mathbb{F}_2$ and a graph, a* **symplectic root system** *for this graph is a decoration $\beta : Nodes \to V$, such that the images form a basis of V and nodes $i \neq j$ are connected iff $\langle \beta(i), \beta(j) \rangle = 1_{\mathbb{F}_2}$ (note that always $\langle v, v \rangle = 0$)*

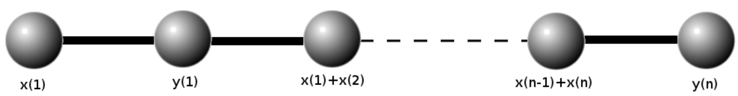
We will use the notion for one directly on simply-laced Dynkin diagrams, but also as tool for others, where only a part of the diagram is split (such as A_2, A_{n-1} for ramified $E_6 \to F_4$ and $A_{2n-1} \to B_n$).

Theorem 6.15. *Any simply laced Dynkin diagram X_n of rank n (viewed as graph) admits a symplectic root system over a symplectic vector space V of dimension n, if and only if the nullspace has minimal dimension ($= 0, 1$ for n even/odd), except D_{2n} requires a 2-dimensional nullspace.*

PROOF. The hardest part will be the case A_{2n}, the other will be derived thereof. By-hand constructions are needed for E_6 and E_8.

Case A_{2n} (if): We first give for $dim(V^\perp) = 0$ (nondegenerate) a realization similar to the ordinary case. For symplectic base pairs x_i, y_i and the obvious numbering of A_n take the following alternating decoration

- $\beta(1) = x_1$
- $\beta(2k+1) = x_k + x_{k+1}$
- $\beta(2k) = y_k$

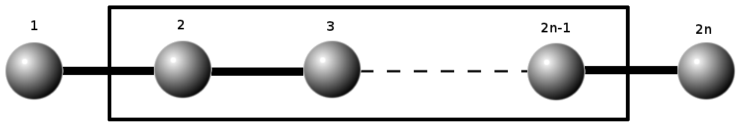


This obviously forms a basis and it is easy to see, that the only non-trivial scalar products are:

$$\begin{aligned}\langle \beta(1), \beta(2)\rangle &= \langle x_1, y_1\rangle = 1\\ \langle \beta(2k), \beta(2k+1)\rangle &= \langle y_k, x_k + x_{k+1}\rangle = 1\\ \langle \beta(2k+1), \beta(2k+2)\rangle &= \langle x_k + x_{k+1}, y_{k+1}\rangle = 1\end{aligned}$$

Case A_{2n} (only if): This is proven inductively: First note for $n = 1$, that V of dimension 2 is either nondegenerate as asserted or $V = V^\perp$ consists only of a nullspace, in which case A_2 cannot have a realization, because all $\langle v, w\rangle = 0$.

Now suppose we had a realization of A_{2n} over some V with nontrivial nullspace $dim(V^\perp) \geq 2$ (1 is impossible by even dimension). We consider the subspace generated by the intermediate node decorations $W = \bigoplus_{k=2}^{2n-1} \beta(k)\mathbb{F}_2$, which realizes the diagram $A_{2(n-1)}$; by induction W has to nondegenerate!



The remaining base elements $\beta(1), \beta(2n)$ have in conjunction with W to generate all of V with its assumed nontrivial nullspace of dimension at least 2, hence there is a basis z, z' of $V^\perp$ with

$$\beta(1) = z + w \in z + W \qquad \beta(2n) = z' + w' \in z + W$$

We now **claim**:

$$w = \sum_{k=1}^{n-1} \beta(2k+1) \qquad w' = \sum_{k=1}^{n-1} \beta(2k)$$

which we prove for w using the nondegeneracy of W on the basis $\beta(l), 2 \leq l \leq 2n-1$ (and analogously for w'):

$$\begin{aligned}
d &:= w - \sum_{k=1}^{n-1} \beta(2k+1) \\
\langle d, \beta(l)\rangle &= \langle w - \sum_{k=1}^{n-1} \beta(2k+1), \beta(l)\rangle \\
&= \langle w, \beta(l)\rangle - \sum_{k=1}^{n-1} \langle \beta(2k+1), \beta(l)\rangle \\
&= \langle \beta(1), \beta(l)\rangle - \sum_{k=1}^{n-1} \langle \beta(2k+1), \beta(l)\rangle
\end{aligned}$$

To show this expression to be **zero** for all l, we use the knowledge of the diagram: For $l = 2$, exactly the first term and the $k = 1$-term is nonzero, for all other even l exactly the two adjacent $2k+1 = l \pm 1$ are nonzero, while for odd l all summands are zero. This proves d to be orthogonal on all basis elements of W (we are over $\mathbb{F}_2$!) and hence by nondegeneracy $d = 0$, which **concludes the claim** (w' follows symmetrically).

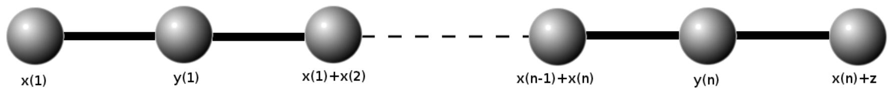
But now in contrast to the assumed diagram, 1 and n have also to be connected, yielding a contradiction:

$$\begin{aligned}
\langle \beta(1), \beta(2n)\rangle &= \langle w + z, w' + z'\rangle = \langle w, w'\rangle \\
&= \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \langle \beta(2i+1), \beta(2j)\rangle \\
&= 2n - 1 = 1
\end{aligned}$$

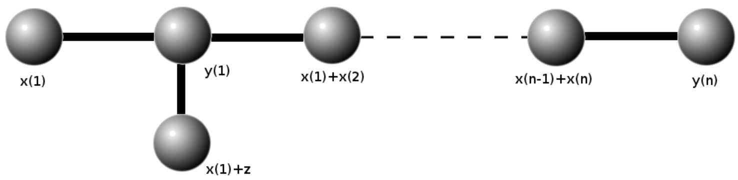
Case A_{2n+1} **(only if):** For $n = 0$, the 1-dimensional V has to be a nullspace; so consider $n > 0$. The basis elements $\beta(1), \ldots \beta(2n)$ generate a subspace $W = \bigoplus_{k=1}^{2n} \beta(k)\mathbb{F}_2$ that realizes A_{2n}. Thus by the above W is nondegenerate and V can have at most 1-dimensional nullspace, and by odd dimension even of exact dimension 1.

Case A_{2n+1} **(if):** For $n = 0$, the 1-dimensional nullspace $V = z\mathbb{F}_2$ supports the A_1 realization ($\beta(1) = z$); so consider $n > 0$. To realize this diagram over V with minimal nullspace $V^\perp = z\mathbb{F}_2$, i.e. $V = W \oplus$

$z\mathbb{F}_2$ with W nondegenerate, realize A_{2n} over W as above and add a node $\beta(2n+1) = z + x_n$. Because y_n only appears in $\beta(2n)$, the only additional nontrivial scalar-product is $\langle \beta(2n), \beta(2n+1) \rangle = 1$

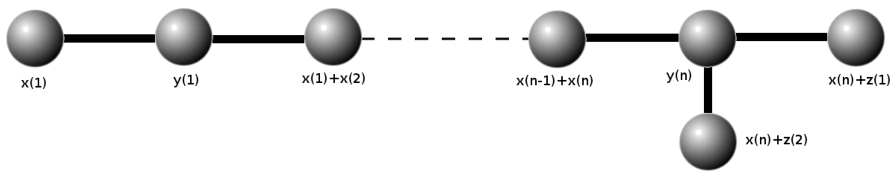


Case D_{2n+1} (if and only if): We proceed as for A_{2n+1}, but add to A_{2n} a different node $\beta(2n+1) = z + x_1$. Conversely, this contained A_{2n} generating a nondegenerate W shows again $dim(V^\perp)$ to be 1-dimensional.



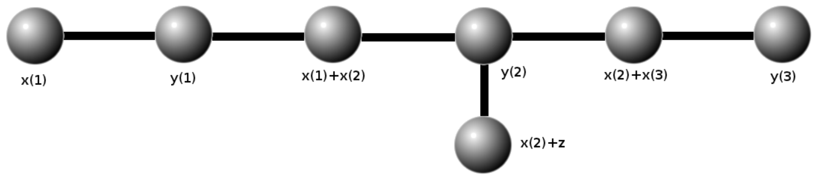
Case D_{2n+2} (if and only if): Note again that the subspace W generated by the contained A_{2n} node decorations is nondegenerate, hence V has nullspace of dimension at most 2, which only leave the cases 2 and 0. We give a construction for the former and a contradiction for the latter.

First we construct D_{2n+2} over V with the atypically large nullspace $V^\perp = z_1\mathbb{F}_2 \oplus z_2\mathbb{F}_2$ from A_{2n} by adding two nodes $\beta(2n+1) = x_n + z_1$ and $\beta(2n+2) = x_n + z_2$:

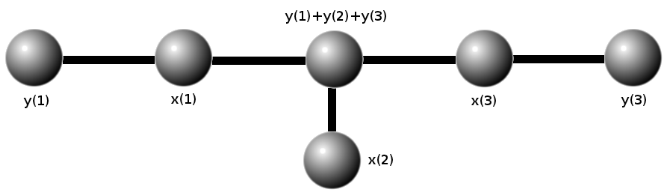


Secondly, suppose we had a nondegenerate V supporting D_{2n+2}. We remove the branching point and consider the subspace $W \subset V$ generated by the remaining node decorations; it has dimension $2n+1$ and a 1-dimensional nullspace. Also, W supports now a symplectic root system for the remaining diagram $A_1 \cup A_1 \cup A_{2n-1}$. But the linear independent node decorations of the two disconnected A_1 have to be orthogonal on all of W, hence nullvectors, contradicting W to have a 1-dimensional nullspace.

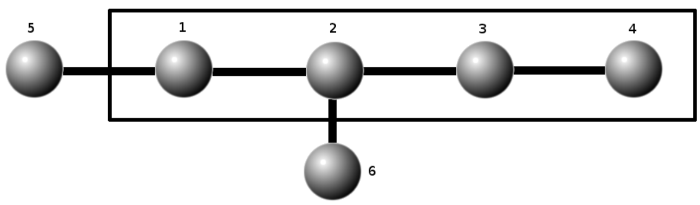
Case E_7 (if and only if): E_7 contains the diagram A_6, which requires a nondegenerate vector space of dimension 6. Hence the only possible choice for V (dimension 7) has a 1-dimensional nullspace $V^\perp = z\mathbb{F}_2$. We construct a realization by adding a node $\beta(7) = x_2 + z$:



Case E_6 (if): It was quite surprising to the author, that (in contrast to E_7) for E_6 no subdiagram seem to significantly ease a construction. We shall thus directly give an exceptional construction for V nondegenerate, and subsequently a by-hand exclusion of further solutions with nontrivial nullspace by reducing to A_4.



Case E_6 (only if): Assume we had a symplectic root system β with nontrivial nullspace $dim(V^\perp) \geq 2$. We reduce to a contained A_4, which only supports a nondegenerate W with basis $\beta(1), \beta(2), \beta(3), \beta(4)$



hence the nullspace were even exactly 2-dimensional and

$$\beta(5) = z + w \in z + W \qquad \beta(6) = z' + w' \in z' + W \qquad V^\perp = z\mathbb{F}_2 \oplus z'\mathbb{F}_2$$

Now in analogy to the A_{2n} induction step we **claim**:

$$w = \beta(2) + \beta(4) \qquad w' = \beta(1)$$

Again, this is proven by using W to be nondegenerate. For w this was done in the A_{2n} proof, and for w' we calculate:

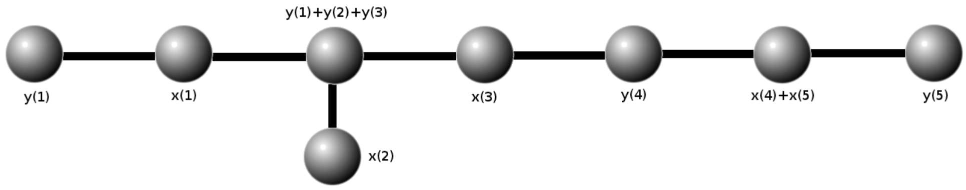
$$\begin{aligned}
d' &= w' - \beta(1) \\
\langle d', \beta(1) &= \langle w' - \beta(1), \beta(1) \rangle \\
&= \langle \beta(6), \beta(1) \rangle - \langle \beta(1), \beta(1) \rangle = 0 \\
\langle d', \beta(2) &= \langle w' - \beta(1), \beta(2) \rangle \\
&= \langle \beta(6), \beta(2) \rangle - \langle \beta(1), \beta(2) \rangle = 0 \\
\langle d', \beta(3) &= \langle w' - \beta(1), \beta(3) \rangle \\
&= \langle \beta(6), \beta(3) \rangle - \langle \beta(1), \beta(3) \rangle = 0 \\
\langle d', \beta(4) &= \langle w' - \beta(1), \beta(4) \rangle \\
&= \langle \beta(6), \beta(4) \rangle - \langle \beta(1), \beta(4) \rangle = 0
\end{aligned}$$

Hence d' is in W orthogonal to all basis elements and hence $d' = 0$ which **concludes to claim**.

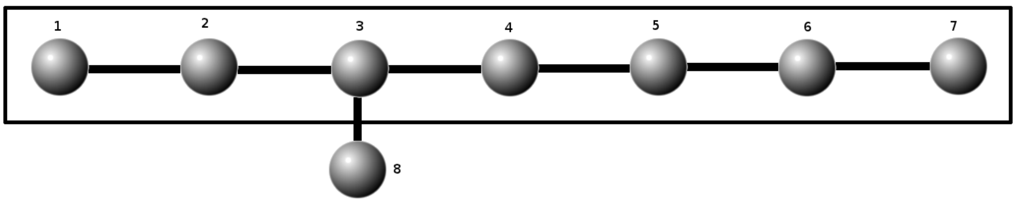
Now we can finally recover a contradicting edge between 5 and 6:

$$\begin{aligned}
\langle \beta(5), \beta(6) \rangle &= \langle z + \beta(2) + \beta(4), z' + \beta(1) \rangle \\
&= \langle \beta(2), \beta(1) \rangle + \langle \beta(4), \beta(1) \rangle \\
&= 1 + 0 = 1
\end{aligned}$$

Case E_8 (if): As E_8 contains E_6, which requires a nondegenerate vector space W, V may only have nullspace dimensions 0 and 2. We construct an example of the former, derived from the exceptional E_6 root system and an additional symplectic base-pair x_5, y_5, and contradict the latter coming from the contained A_7 by a new argument.



Case E_8 (only if): Thus finally assume there were a realization over a vector space V with nullspace of dimension 2. There is a contained A_7 diagram over the vector space W (nullspace dimension 1) generated by the decorations $\beta(1), \ldots \beta(7)$.



As we supposed $dim(V^\perp) = 2$ there is a basis element $z \in V^\perp$ such that $\beta(8) = z + w \in z + W$, which means by the diagram

$$\langle w, \beta(3) \rangle = 1 \qquad \langle w, \beta(k) \rangle = 0 \qquad k = 1, 2, 4, 5, 6, 7$$

In contrast to the previous arguments, we now want to conclude that no such w can exist: It had to be a $\mathbb{F}_2$-linear combination of the basis $\beta(i) \in W$, which corresponds to giving a subset

$$w = \sum_{k \in S} \beta(k) \qquad S \subset \{1, 2, 3, 4, 5, 6, 7\}$$

As $\langle w, \beta(3) \rangle = 1$ we have either $2 \in S$ or $4 \in S$. But:

- $2 \in S$ would concludes $\langle w, \beta(1) \rangle = 1$ regardless of the rest of S, which contradicts the above assumption.
- $4 \in S$ would also require $6 \in S$ in order to keep as assumed $\langle w, \beta(5) \rangle = 0$. But then $\langle w, \beta(7) \rangle = 1$ again contradicts the assumption.

$\square$

3. Unramified Cases $ADE \cup ADE \to ADE$

The most natural and generic way to construct a Yetter-Drinfel'd module with twisted symmetry $\mathbb{Z}_2$ (actually $\mathbb{Z}_p$) has already been demonstrated on the case $\mathbb{D}_4, \mathbb{Q}_8$ in section 3.2; we force twisted symmetry by **doubling** the diagram. We subsequently give explicit examples for $A_4 \cup A_4 \to A_4$ and $D_4 \cup D_4 \to D_4$.

The **technical proof idea** is to take a (suitable, see below) diagonal finite-dimensional Nichols algebra $\mathcal{B}(M')$ realizing the given diagram over Γ. Then we calculate for an arbitrary given cocycle σ of order 2 the twisted $M'' := M'^{\sigma}$ and hence obtain a natural twisted symmetry θ of order 2 on $M = M' \oplus M''^{\sigma}$.

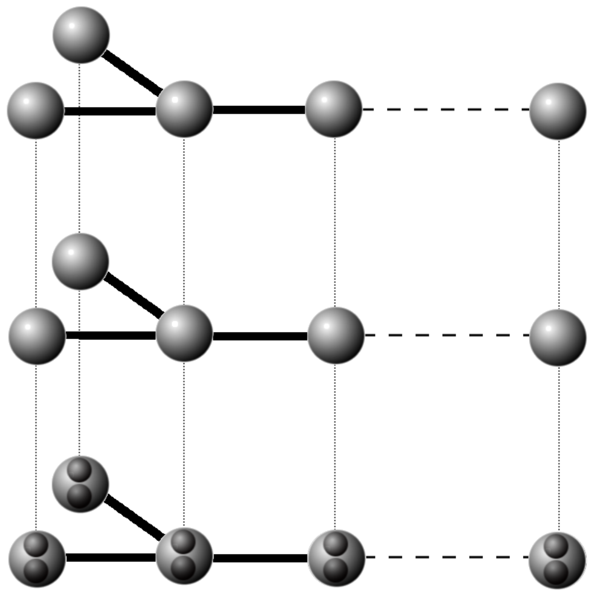
Care has to be taken, not to cause additional edges between the copies, such that $\mathcal{B}$ certainly stays finite:

$$\begin{aligned} dim(\mathcal{B}(M)) &= dim(\mathcal{B}(M' \oplus M''^{\sigma})) \\ &\stackrel{?}{=} dim(\mathcal{B}(M') \otimes \mathcal{B}(M''^{\sigma})) \\ &= dim(\mathcal{B}(M'))dim(\mathcal{B}(M'^{\sigma})) \\ &= dim(\mathcal{B}(M'))^2 \end{aligned}$$

This detailed statement was found by **systematically avoiding** in the necessary conditions Theorem 6.9.

- that the splitting of each edge becomes **wildly** in the third proof part of the theorem, avoiding "diagonal additional edges".
- that **loops** occur, avoiding "vertical additional edges".

The former would requires $p = 2$, all edge decorations in M to be $q_{ij}q_{ji} = -1$ and edges exactly between G-discommuting nodes, as prescribed by σ. This is the crucial role for the **symplectic root systems** established in the last section for this construction. The latter additionally require the node decorations to be $q_{ii} = -1$ and **excludes multiply-laced** M', which is done "the other way around" as part of checking all possible diagonal Nichols algebras in section 6.6. We again use the notation $1_n := n \; mod \; 2 = 0, 1$.



Theorem 6.16. *Suppose a simply-laced Dynkin diagram of rank n and any group G with* $G' = \mathbb{Z}_2$ *and* $\Gamma := G/G'$*, such that*

- $dim_{\mathbb{F}_2}(V) = dim_{\mathbb{F}_2}(\Gamma/\Gamma^2) = n \geq 2$
- $dim_{\mathbb{F}_2}(V^\perp) = dim_{\mathbb{F}_2}(Z(G)/G'G^2) = 1_n$
  respectively $= 2 - 1_n$ *for diagrams* D_n

Then orbifoldizing two disconnected copies of the diagram over Γ *through it's obvious involutory diagram automorphism constructs a G-Yetter-Drinfel'd module* $\tilde{M} = \bigoplus_{i=1}^n \tilde{M}_i$ *of dimension* $2n$ *with:*

- G' *acts trivially on* $\tilde{M}$*, which is hence diagonal, but the quotient* V *acts faithfully.*
- $\tilde{M}$ *is minimally indecomposable, i.e. indecomposable and not properly containing an indecomposable module.*
- $\mathcal{B}(\tilde{M})$ *is finite-dimensional, the dimension being the square of the single diagram's in the abelian case* $\mathbb{Z}_2^n$*.*
- $\tilde{M}$ *has the prescribed Cartan matrix and Dynkin diagram with all nodes* $\tilde{M}_i$ *dimension 2 (i.e. underlying conjugacy class of length 2).*

Several faithful Doi twist and hence nondiagonal Nichols algebras for small rank D_4, A_2, A_3, A_4 *over various G are given in section 7.1.*

Proof. The strategy has been outlined above:

Step 1: We first construct a Yetter-Drinfel'd module M' of dimension n over Γ, such that two nodes decorated with group elements $\bar{g}_i, \bar{g}_j$ are connected iff $c(\bar{g}_i, \bar{g}_j) \neq 0$ (i.e. lifts $g_i, g_j \in G$ discommute) and the braiding matrix only contains ± 1. This is done by using precisely the symplectic root systems constructed in theorem 6.15: $V := \Gamma/\Gamma^2$ is a symplectic vector space as described in the cited section with dimension $dim_{\mathbb{F}_2}(\Gamma/\Gamma^2)$ and nullspace dimension $dim_{\mathbb{F}_2}(Z(G)/G'G^2)$. Hence the assumptions of the present theorem exactly match those of *cit. loc.* and we get a symplectic root system basis $\beta(i)$ $(1 \leq i \leq n)$ of V. Choosing $\bar{g}_i \in \Gamma$ any lift of $\beta(i)$ fulfills $c\langle \bar{g}_i, g_j \rangle \neq 0$ iff i, j are connected.

We have to construct suitable characters $\chi_i : \Gamma \to \Bbbk^\times$ that realize the given diagram with braiding matrix ± 1. Because the $\beta(i)$ were a basis of Γ/Γ^2, there is exactly one χ_i such that $\chi_i(\bar{g}_j) = -1$ if $i = j$ or $i < j$ are connected and $+1$ otherwise. Then $M' := \oplus \mathcal{O}_{\bar{g}_i}^{\chi_i}$ has the braiding matrix $q_{ij}q_{ji} = \pm 1$ depending on whether g_i, g_j discommute. Note by construction, as $\mathbb{F}_2$-matrix $\chi_1, \ldots \chi_n$ is triangular, hence the basis g_k acts faithful, which also proves this part of the statement.

Step 2: The connection to the 2-cocycles is rather generic and similar to previous cases: The central (stem!) extension in question is

$$\Sigma = \mathbb{Z}_2 \to G \to \Gamma$$

Take a section s and $u \in Z^2(\Gamma, \Sigma)$ the respective cocycle. The a symmetry of u describes the commutator map to Σ:

$$u(\bar{a}, \bar{b})u^{-1}(\bar{b}, \bar{a}) = [a, b]$$

Thus the symplectic form describes the demand of the twisted symmetry on a Γ-Yetter-Drinfel'd module $\bar{M}$ – take $p \in \Sigma$ the nontrivial $p(1_{\mathbb{F}_2}) = -1$, then twisted linearity of θ_p on an element $v_{\bar{b}} \in V_{\bar{b}}$ reads as:

$$\begin{aligned}
\bar{a}.\theta_p(v_{\bar{b}}) &= \sigma_p(\bar{a}, \bar{b})\sigma_p^{-1}(\bar{b}, \bar{a})\theta_p(\bar{a}.v_{\bar{b}}) \\
&= p\left(u(\bar{a}, \bar{b})u^{-1}(\bar{b}, \bar{a})\right)\theta_p(\bar{a}.v_{\bar{b}}) \\
&= p\left(\langle \bar{a}, \bar{b} \rangle\right)\theta_p(\bar{a}.v_{\bar{b}})
\end{aligned}$$

Hence any decorating character on some decorating group element $\chi_k(g_l)$ picks up an additional -1 iff $[g_k, g_l] = -1$ iff $\langle \bar{g}_k, \bar{g}_l \rangle \neq 0$.

Step 3: We now construct such a twist-symmetric Γ-Yetter-Drinfel'd module as in the case $\mathbb{D}_4, \mathbb{Q}_8$. We start with the indecomposable $M' = \bigoplus_{i=1}^n M'_i$ given by the Dynkin Diagram over $\Gamma \cong \mathbb{Z}_2^n$. Then we add the necessary twisted part (p again the nontrivial one):

$$M'' := M'^{\sigma_p} = M'^{p \circ u}$$

It consists of simple Yetter-Drinfel'd modules M''_i given by the same group elements $\beta(i)$ but with twisted characters:

$$\chi_i^{\sigma_p}(v_{\bar{b}}) := p\left(\langle \beta(i), \bar{b} \rangle\right) \chi_i(v_{\bar{b}})$$

By construction $M := M' \oplus M''$ now admits an involutory twisted symmetry θ_p interchanging the copies $M'_i \leftrightarrow M''_i$.

Step 4: We yet have to check that M still has a finite Nichols algebra, so we determine its full Dynkin diagram – as intended, we prove now, that it really consists of two disconnected copies of the given one. First be reminded on Lemma 4.3 that twisted symmetries leave Dynkin diagrams and decoration invariant $M' \cong M''$.

Hence the tricky part is, that there are no additional mixed edges between any M'_i, M''_j. This is precisely where we need the specific base choice $\beta(i)$ and the fact that all $q_{ij} = \pm 1$. We have to calculate their mixed braiding factors:

$$\begin{aligned}
q &:= q_{M'_i, M''_j} q_{M''_j, M'_i} \\
&= \chi_i(\beta(j)) \chi_j^{\sigma_p}(\beta(i)) \\
&= \chi_i(\beta(j)) \cdot \sigma_p(\beta(j), \beta(i)) \sigma_p^{-1}(\beta(i), \beta(j)) \chi_j(\beta(i)) \\
&= p\left(\langle \beta(i), \beta(j) \rangle\right) \chi_i(\beta(j)) \chi_j(\beta(i)) \\
&= p\left(\langle \beta(i), \beta(j) \rangle\right) q'_{ij} q'_{ji}
\end{aligned}$$

We have to distinguish two cases that yield $q = 1$ in different ways:

- Suppose i, j disconnected in the original diagram. Then $q'_{ij} q'_{ji} = 1$ and at the same time by construction $\langle \beta(i), \beta(j) \rangle = 0$, hence $q = 1$.
- Suppose i, j connected by a single edge. Then $q'_{ij} q'_{ji} = -1$ and at the same time by construction $\langle \beta(i), \beta(j) \rangle = 1$, hence again $q = 1$.

Step 5: Thus we are done: We constructed a twist-symmetric indecomposable M over Γ with finite-dimensional Nichols algebra of dimension $dim(M) = dim(M')dim(M'') = dim(M')^2$. We may orbifoldize it to an indecomposable G-Yetter-Drinfel'd module M with Nichols algebra of the same dimension, gluing each M_i', M_i'' to a single G-conjugacy class $\tilde{M}_i$ of length 2.

□

We shall give two explicit examples, as they would arise from the general construction given above:

Example 6.17. *We realize A_4 as prescribed over a group G with 2-rank $dim_{\mathbb{F}_2}(\Gamma/\Gamma^2) = 4$ and $dim_{\mathbb{F}_2}(Z(G)/G'G^2) = 0$, such as the extraspecial group $G = 2_+^{4+1} = \mathbb{D}_4 * \mathbb{D}_4$ (the central product identifies the two dihedral centers), which is generated by mutually discommuting involutions x, y and x', y', corresponding to a symplectic basis of $V = \Gamma = \mathbb{F}_2^4$ nondegenerate. We need a Γ-Yetter-Drinfel'd module of type $A_4 \cup A_4$ admitting an involutory twisted symmetry*

$$M = M' \oplus M'' =: (M_1 \oplus M_2 \oplus M_3 \oplus M_4) \oplus (M_5 \oplus M_6 \oplus M_7 \oplus M_8)$$

where each $M_k = \mathcal{O}_{g_k}^{\chi_k}$ is 1-dimensional. The **group elements** *are determined by the respective symplectic root system in Theorem 6.15:*

$$g_1 = g_5 = x \qquad g_2 = g_6 = y \qquad g_3 = g_7 = xx' \qquad g_4 = g_8 = y'$$

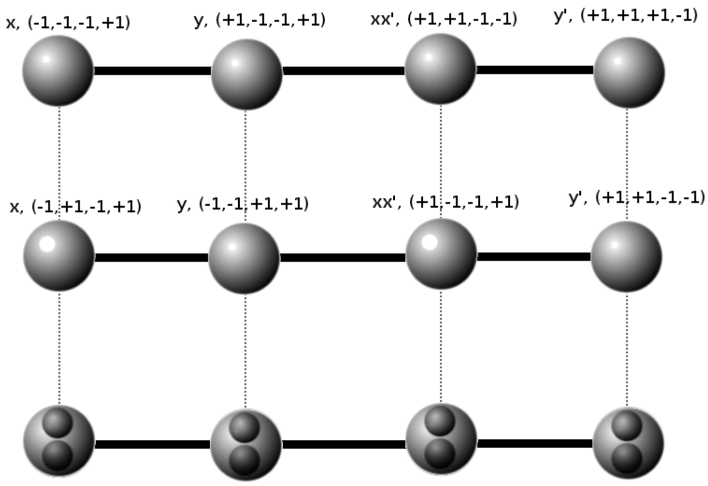
Then the **characters** *χ_k for $k \leq 4$ were defined in such a way that $\chi_k(g_k) = -1$, and $\chi_k(g_l) = -1$ for edges $k < l$ and $+1$ else. This has to be basis-transformed to be expressed as row vector showing the values in the original basis $(\chi(x), \chi(y), \chi(x'), \chi(y'))$:*

$$\begin{aligned}
\chi_1 &= (-1, -1, -1, +1) \\
\chi_2 &= (+1, -1, -1, +1) \\
\chi_3 &= (+1, +1, -1, -1) \\
\chi_4 &= (+1, +1, +1, -1)
\end{aligned}$$

As generally calculated, the **twisted characters** $\chi_{4+k} = \chi_k^\sigma$ *catch an additional* -1 *on every element* G*-discommuting with* g_k *resp. non-orthogonal in* V*:*

$$\chi_5 = (-1, +1, -1, +1)$$
$$\chi_6 = (-1, -1, +1, +1)$$
$$\chi_7 = (+1, -1, -1, +1)$$
$$\chi_8 = (+1, +1, -1, -1)$$

Altogether we orbifoldize the following Γ*-Yetter-Drinfel'd module, which has a faithful Doi twist by section 7.1:*



Example 6.18. *We realize* D_4 *as prescribed over a group* G *with 2-rank* $dim_{\mathbb{F}_2}(\Gamma/\Gamma^2) = 4$ *and atypically large* $dim_{\mathbb{F}_2}(Z(G)/G'G^2) = 2$*, such as the group* $G = \mathbb{Z}_2^2 \times \mathbb{D}_4$*, which is generated by two mutually discommuting involutions* x, y *and two central involutions* z, z' *corresponding to a symplectic basis of* $V = \Gamma = \mathbb{F}_2^4$ *with* $dim(V^\perp) = 2$*. We need a* Γ*-Yetter-Drinfel'd module of type* $D_4 \cup D_4$ *admitting an involutory twisted symmetry*

$$M = M' \oplus M'' =: (M_1 \oplus M_2 \oplus M_3 \oplus M_4) \oplus (M_5 \oplus M_6 \oplus M_7 \oplus M_8)$$

where each $M_k = \mathcal{O}_{g_k}^{\chi_k}$ *is* 1*-dimensional. The* **group elements** *are determined by the respective symplectic root system in Theorem 6.15:*

$$g_1 = g_5 = x \qquad g_2 = g_6 = y \qquad g_3 = g_7 = xz \qquad g_4 = g_8 = xz'$$

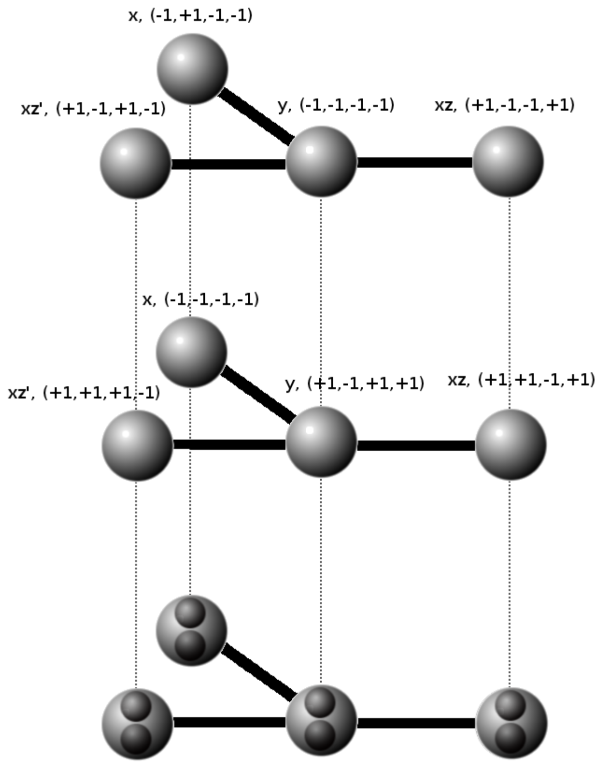
Then the **characters** χ_k *for* $k \leq 4$ *were defined in such a way that* $\chi_k(g_k) = -1$*, and* $\chi_k(g_l) = -1$ *for edges* $k < l$ *and* $+1$ *else. This has to* *be basis-transformed to be expressed as row vector showing the values in the original basis* $(\chi(x), \chi(y), \chi(z), \chi(z'))$*:*

$$\chi_1 = (-1, +1, -1, -1)$$
$$\chi_2 = (-1, -1, -1, -1)$$
$$\chi_3 = (+1, -1, -1, +1)$$
$$\chi_4 = (+1, -1, +1, -1)$$

As generally calculated, the **twisted characters** $\chi_{4+k} = \chi_k^\sigma$ *catch an additional* -1 *on every element* G*-discommuting with* g_k *resp. non-orthogonal in* V*:*

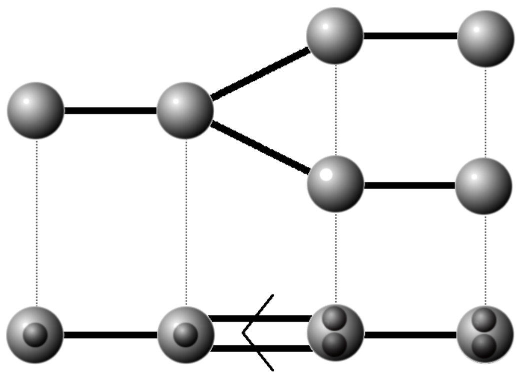
$$\chi_5 = (-1, -1, -1, -1)$$
$$\chi_6 = (+1, -1, +1, +1)$$
$$\chi_7 = (+1, +1, -1, +1)$$
$$\chi_8 = (+1, +1, +1, -1)$$

Altogether we orbifoldize the following Γ*-Yetter-Drinfel'd module, which has a faithful Doi twist by section 7.1:*



4. Ramified Case $E_6 \to F_4$

The examples of the last two sections are "generic" in the sense, that they exploit a disconnected doubling of a rather arbitrary Dynkin diagram, and the very same diagram is reproduced in the nonabelian setting. Especially, every (nonabelian) edge corresponds to the $\mathbb{D}_4$ example above; it is not allowed for the Dynkin diagrams to connect conjugacy classes of different length (e.g. abelian and nonabelian). It turns out, that this "interconnected case" is far more restrictive! We shall now give an example of this type, where the $\mathbb{Z}_2$-automorphism of a single E_6-diagram is orbifoldized to the non-simply laced F_4:



Theorem 6.19. *Suppose a group G with $G' = \mathbb{Z}_2$ and $\Gamma := G/G'$ s.t.*

- $dim_{\mathbb{F}_2}(V) = dim_{\mathbb{F}_2}(\Gamma/\Gamma^2) = 4$
- $dim_{\mathbb{F}_2}(V^\perp) = dim_{\mathbb{F}_2}(Z(G)/G'Z(G)^2) = 2$

Then orbifoldizing a suitable Γ-Yetter-Drinfel'd module of type E_6 through it's involutory diagram automorphisms constructs a G-Yetter-Drinfel'd module $\tilde{M} = \bigoplus_{i=1}^4 \tilde{M}_i$ of dimension 6, such that:

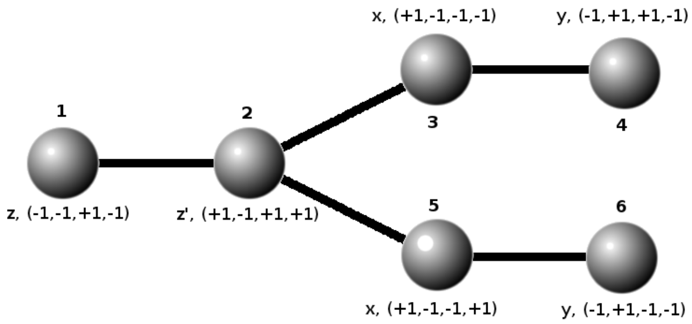
- *G' acts trivially on $\tilde{M}$, which is hence diagonal, but the quotient V acts faithfully.*
- *$\tilde{M}$ is minimally indecomposable, i.e. indecomposable and not properly containing an indecomposable module.*
- *$\mathcal{B}(\tilde{M})$ has dimension 2^{36} (as $E_6^{q=-1}$ in* **[H08]***).*
- *$\tilde{M}$ has the Dynkin diagram F_4, where the long roots corresponds to conjugacy classes of length 2 and the short roots to a central elements (length 1).*

There's also a faithful Doi twist and hence a nondiagonal Nichols algebra over $G = \mathbb{Z}_2^2 \times \mathbb{D}_4, \mathbb{Z}_2^2 \times \mathbb{Q}_8$, see section 7.1.

PROOF. Denote by $\bar{z}, \bar{z}', \bar{x}, \bar{y} \in \Gamma$ some lifts of a basis of the 4-dimensional symplectic vector space $V = \Gamma/\Gamma^2$ with 2-dimensional nullspace, such that $\bar{z}, \bar{z}'$ were nullvectors and $\bar{x}, \bar{y}$ was a symplectic base pair in V, i.e. any lifts $z, z', x, y \in G$ obey:

$$z, z' \in Z(G) \qquad [x, y] \neq 1$$

We directly construct the Γ-Yetter-Drinfel'd module $\bigoplus_{k=1}^{6} \mathcal{O}_{g_k}^{\chi_k}$ of type E_6, but otherwise proceed as in the unramified case. Note that the following could also be derived systematically using the (rather trivial) symplectic root system $\bar{x}, \bar{y}$ for the aspired split part of V and character via some ordering of the nodes, as it is done for the remaining ramified case below; but here we want to keep everything explicit! Further denote any **character** $\chi \in \Gamma^*$ as row-vectors containing the basis images $(\chi(\bar{z}), \chi(\bar{z}'), \chi(\bar{x}), \chi(\bar{y}))$, then M shall be (we've introduced additional signs for the faithfulness-statement):



One can check directly, that $q_{ii} = -1$ and the $q_{ij}q_{ji} \pm 1$ exactly match the given diagram; further already $\chi_1, \chi_2, \chi_3, \chi_4$ is $\mathbb{F}_2$-linearly independent and z, z' have been constructed to act as -1 on x resp. y, hence the faithfulness assertions hold. This defined a proper Nichols algebra $\mathcal{B}(M)$ of dimension 2^{36}, because of **[HS10]** Theorem 4.5 we have the following bijection via multiplication and for $q = -1$ know all simple Nichols algebras (none with $q = 1$ appear!):

$$\mathcal{B}(M) \cong \bigotimes_{l \in L} \mathcal{B}(W_l) \qquad \mathcal{B}(W_l) \cong \mathbb{k}[w_l]/(w_l^2)$$

$$dim(\mathcal{B}(M)) = \prod_{l \in L} 2 = 2^{|L|} = 2^{|\Delta^+|} = 2^{36}$$

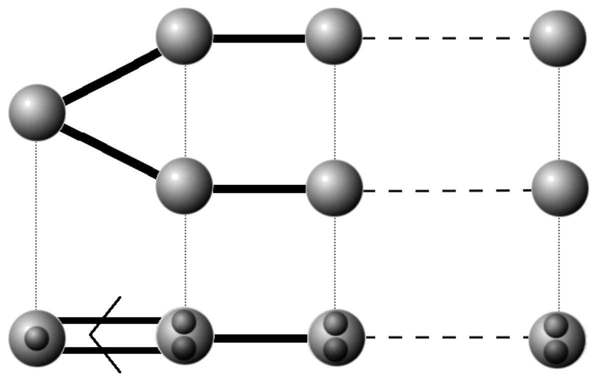
We further check directly, that the obvious involutory diagram automorphisms θ is even a twisted automorphism:

$$\begin{aligned}
\chi_1^\sigma(g_k) &= \sigma(g_k, g_1)\sigma_p^{-1}(g_1, g_k)\chi_1(g_k) = \langle \bar{g}_k, z\rangle \chi_1(g_k) = \chi_1(g_k) \\
\chi_3^\sigma(z) &= \langle z, x\rangle \chi_3(z) = \chi_3(z) = +1 = \chi_5(z) \\
\chi_3^\sigma(z') &= \langle z', x\rangle \chi_3(z') = \chi_3(z') = -1 = \chi_5(z') \\
\chi_3^\sigma(x) &= \langle x, x\rangle \chi_3(x) = \chi_3(x) = -1 = \chi_5(x) \\
\chi_3^\sigma(y) &= \langle y, x\rangle \chi_3(z') = -\chi_3(y) = +1 = \chi_5(y)
\end{aligned}$$

This shows $\chi_1^\sigma = \chi_1$ and $\chi_3^\sigma = \chi_5$. The same calculations prove $\chi_2^\sigma = \chi_2$ and $\chi_4^\sigma = \chi_6$, hence $\theta : M^\sigma \to M$ is an automorphism of Yetter-Drinfel'd modules. Orbifoldizing again constructs a Yetter-Drinfel'd module $\bar{M}$ of the given form by section 6.1. $\square$

5. Ramified Cases $A_{2n-1} \to B_n$

The second ramification will on the other hand be completely reduced to the unramified case $A_{n-1} \cup A_{n-1} \to A_{n-1}$ and an additional inert node causing an additionally tamely branched edge.



Theorem 6.20. *Suppose a group G with $G' = \mathbb{Z}_2$ and $\Gamma := G/G'$, s.t.*

- $dim_{\mathbb{F}_2}(V) = dim_{\mathbb{F}_2}(\Gamma/\Gamma^2) = n \geq 3$
- $dim_{\mathbb{F}_2}(V^\perp) = dim_{\mathbb{F}_2}(Z(G)/G'Z(G)^2) = 1 + 1_{n-1}$

Then orbifoldizing a suitable Γ-Yetter-Drinfel'd module of type A_{2n-1} through it's involutory diagram automorphisms constructs a G-Yetter-Drinfel'd module $\tilde{M}$ of rank n and dimension $2n-1$, such that:

- *G' acts trivially on $\tilde{M}$, which is hence diagonal, but the quotient V acts faithfully.*
- *$\tilde{M}$ is minimally indecomposable, i.e. indecomposable and not properly containing an indecomposable module.*
- *$\mathcal{B}(\tilde{M})$ has dimension $2^{n(2n-1)}$ (as $A_{2n-1}^{q=-1}$ in* **[H08]***).*
- *$\tilde{M}$ has the nonabelian Dynkin diagram B_n where the long roots corresponds to conjugacy classes of length 2 and the unique short root to a central element (length 1).*

Exemplary nondiagonal and even faithful Doi twists of B_3 over various G are given in section 7.1.

Proof. As in the ramified case $E_6 \to F_4$ above, we use the prescribed dimension $1 + 1_{n-1}$ nullspace of $V = \Gamma/\Gamma^2$ to separate $V = \bar{z}\mathbb{F}_2 \oplus W$ with $dim(W^\perp) = 1_{n-1}$ for the split nodes and $z \in Z(G)$ for the inert node.

Our main goal is to construct a Γ-Yetter-Drinfel'd module M of dimension $1+2(n-1)$ and Dynkin diagram A_{2n-1} with the involutory diagram automorphism a twisted symmetry. The starting point is the Yetter-Drinfel'd module constructed in the proof of section 6.3 of dimension $2(n-1)$ and Dynkin diagram $A_{n-1} \cup A_{n-1}$, numbered $2\ldots 1+2(n-1)$, with an involutory twisted symmetry over the subgroup $\Gamma' \subset \Gamma$ generated by any lifts of W. Denote the leftmost nodes $2,3$ of both copies by $\mathcal{O}_g^{\chi'}, \mathcal{O}_g^{\chi'}$. We extend all used characters trivially to Γ except

$$\chi(z) = -1 \qquad \chi(g) = -1 \qquad \chi(g_k) = +1$$

for all other g_k, which is possible because $g = g_1, \ldots g_n$ was a W-basis. Note that the former Yetter-Drinfel'd module had already been proven to be faithful over the Γ-quotient W, with z now acting trivial on all but the new node M_1, hence faithfulness of V again holds.

First we have to check that M indeed has decorated diagram $A_{1+2(n-1)}$ for $q = -1$, especially $dim(\mathcal{B}(M)) = 2^{n(2n-1)} < +\infty$ (determined as in the proof in section 6.4). We've shown that already for the subdiagram $A_{n-1} \cup A_{n-1}$, and the additional node M_1 obeys for $k \geq 4$:

$$\begin{aligned} q_{11} &= \chi(z) = -1 \\ q_{12}q_{21} &= \chi(g)\chi'(z) = (-1)(+1) = -1 \\ q_{13}q_{31} &= \chi(g)\chi''(z) = (-1)(+1) = -1 \\ q_{1k}q_{k1} &= \chi(g_k)\chi_k(z) = (+1)(+1) = +1 \end{aligned}$$

Secondly we have to extend the established twisted symmetry θ of $A_{n-1} \cup A_{n-1}$ by $\theta(M_1) = M_1$, which is possible by z's centrality:

$$\begin{aligned} \chi^\sigma(h) &= \sigma(z,h)\sigma_p^{-1}(h,z)\chi(h) \\ &= \langle \bar{h}, z\rangle \chi(h) = \chi(h) \end{aligned}$$

Finally orbifoldizing constructs $\tilde{M}$ with the asserted properties. $\square$

6. Proof Finish: The List Is Complete

We finally want to prove, that any finite-dimensional Nichols algebra with $\Sigma := G' \cong \mathbb{Z}_p$ acting trivial has solely as **connected** components $\tilde{M}$ the types constructed above. Hence we have to check all possible Dynkin diagrams M (possibly disconnected, see unramified examples above) over abelian groups with twisted symmetries. We excessively use the necessary conditions proven in Theorem 6.8 for rank $1, 2$ and the lists of Nichols algebras in the abelian case from [**H08**] and [**H05**]. The **proof strategy** is organized as follows:

- **Step 1** is the observation of a diagram automorphism and excludes totally inert orbifolds as decomposable.
- **Step 2** consist of multiple revisions of Heckenberges list:
  - **Step 2a** searches the list for diagrams eligible for (one connected copy of) the unramified case by the necessary conditions from cit. loc., i.e. all edges decorated by -1 resulting in all classical Cartan types for $q = -1$ but B_n.
  - **Step 2b** searches the list for all loopfree diagrams with involutory automorphism, resulting in E_6, A_{2n-1}, D_n (ramified) and several non-Cartan (mostly wildly branched) diagrams of shape alike A_3, D_4, D_n.
  - **Step 2c** searches the list for loop diagrams with involutory automorphisms under heavy use of the necessary condition (established cit. loc.), that a loop has a precise decoration and cannot be directly connected to split nodes. The only result is an isolated loop A_2 for $q \in \Bbbk_3$.
  - **Step 2d** searches the list for all diagrams with higher-order automorphisms, resulting only in D_4.
- **Step 3a** excludes multiply laced diagrams C_n, F_4, G_2 from the unramified case by exhibiting a loop between the two copies of any long root, leaving only Cartan type ADE.
- **Step 4** shows that orbifoldizings with a unique split node only lead to decomposable Nichols algebras. This leaves only the later-on realized unramified cases $A_{n\geq 2}, D_{n\geq 4}, E_{6,7,8}$ and ramified cases E_6 and A_{2n-1} for $n \geq 3$.
- **Step 5** applies the necessary condition on G established for the symplectic root systems (Theorem 6.15) and finally states the remaining cases to have been realized in the previous sections.

Step 1: First we shall exploit the fact, that $\Sigma = \langle p \rangle$ needs to act on $\tilde{M}$ via a twisted symmetry θ_p preserving the diagram including its decorations by q_{ii} and $q_{ij}q_{ji}$ (rank $1, 2$). By the group law $\theta_p^p = \theta_{p^p} = id$, thus θ_p acts as a diagram automorphism of order p. Suppose otherwise the diagram totally inert, i.e. $\theta_p = id$, then all G-nodes have conjugacy classes of length 1. They are hence central and may not generate the entire nonabelian G, outruling the orbifold to be indecomposable.

Step 2a: In the unramified case we find all connected M': We've proven (rank 2) to be necessary, that $q_{ij}q_{ji} = \pm 1$, hence this has to be true for **all** M'. Going though Heckenberger's list we find then only possibly:

- Rank 1 of type A_1 for a free $q \neq 1$.
- For M rank 2 (see **[H05]** table A.1) this can only be achieved by Cartan-type edges:

Row	$q \in$	Cartan
(1	$\neq 1$	$A_1 \times A_1$)
2	$\mathbb{k}_2$	A_2
4	$\mathbb{k}_4$	B_2
11	$\mathbb{k}_6$	G_2

- In rank 3 (see **[H05]** table A.2) again only Cartan-type diagrams appear, but B_3 requires a second, single edge decorated by the $\mathbb{k}_4$-element q in contrast to C_3 bearing only $\mathbb{k}_2$ at edges. Going through the entire list indeed shows:

Row	$q \in$	Cartan
1	$\mathbb{k}_2$	A_3
2	$\mathbb{k}_4$	C_3

- Rank 4 (see **[H08]** appendix B) is similar, but type D_n appears:

Row	$q \in$	Cartan
1	$\mathbb{k}_2$	A_4
2	$\mathbb{k}_4$	C_4
5	$\mathbb{k}_2$	D_4

- In rank $n \geq 5$ (see **[H08]** appendix C) the series remain and Cartan type E appears. Again, most diagrams can be discarded because an edge decorated by q resp. q^2 and q of larger order is demanded. Note finally that rows 2 resp. 10 are excluded, because any simple chain $C(l, q; i_1 \ldots)$ has edge-weights $q^{\pm 1}$ and the above two cases do not admit $q = -1$!

Row	$q \in$	Cartan
1	$\Bbbk_2$	A_n
3	$\Bbbk_4$	C_n
8	$\Bbbk_2$	D_n
16	$\Bbbk_2$	E_6
20	$\Bbbk_2$	E_7
22	$\Bbbk_2$	E_8

Step 2b: In the $p = 2$ loopfree connected case we once again consult Heckenberges list and look for diagram symmetries respecting the decoration. Additionally, still the split edges have to be -1-decorated. The complete list is:

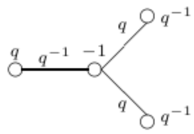
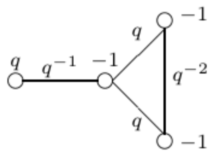
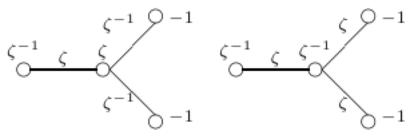
- For M connected of rank 2 this is not possible (loop).
- For rank 3 (see **[H05]** table A.2) we have for one triangles, which have loops at all symmetries. The usual chain diagrams (e.g. A_3) admit an apparent 2-symmetry, which would require the right- and leftmost node decoration to coincide. These criteria leaves (apart from the table below) only the following cases, that have unsymmetrical edge-decorations.
  - Row 5, diagram 3 for $q \in \Bbbk_6$
  - Row 7, diagram 4 for $q \in \Bbbk_8$
  - Row 17, multiple diagrams

  On the other hand we find the following cases, which (except of A_3) are non-Cartan.

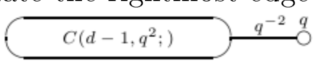
Row	$q \in$	Diagram	Folds to
1	$\Bbbk_2$	A_3	B_2
		$\overset{-1}{\circ}\overset{q^{-1}}{\text{———}}\overset{q}{\circ}\overset{q^{-1}}{\text{———}}\overset{-1}{\circ}$	
8	$\neq 1$	$\overset{-1}{\circ}\overset{q}{\text{———}}\overset{q^{-1}}{\circ}\overset{q}{\text{———}}\overset{-1}{\circ}$	-
10	$\neq 1$	$\overset{q}{\circ}\overset{q^{-1}}{\text{———}}\overset{-1}{\circ}\overset{q^{-1}}{\text{———}}\overset{q}{\circ}$	-
11	$\Bbbk_3$	$\overset{\zeta}{\circ}\overset{\zeta^{-1}}{\text{———}}\overset{-1}{\circ}\overset{\zeta^{-1}}{\text{———}}\overset{\zeta}{\circ}$	-
		$\overset{-1}{\circ}\overset{\zeta}{\text{———}}\overset{-1}{\circ}\overset{\zeta}{\text{———}}\overset{-1}{\circ}$	
15	$\Bbbk_3$	$\overset{-1}{\circ}\overset{\zeta^{-1}}{\text{———}}\overset{-\zeta^{-1}}{\circ}\overset{\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$	-

  Note that all but A_3 yield **wildly ramified** rank 2 edge and we did not determine these foldings in Theorem 6.8, because only a single node is split, they will yield only decomposable Nichols algebras in Step 4.

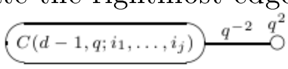
- Rank 4 (see **[H08]** appendix B) can (as any even rank) not be of chain form, otherwise the center edge becomes a loop. Also no symmetries can be expected from the prolonged triangles. So the only diagrams in question have a branch (alike D_4). Note that the fitting case row 20 is discarded by unsymmetrical node/edge decorations:

Row	$q \in$	Cartan	Folds to
5	free	D_3	C_2
12	free		-
13	free		-
18	$\Bbbk_3$		-

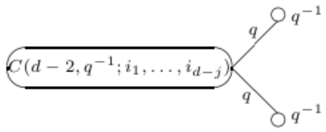
  All but D_3 and the second diagram of row 13 (say M) orbifoldize again to a **wildly ramified** edge, connected to a an **inert** edge of non-Cartan type (except A_1 for row 18). M yields a **tamely ramified** orbifold B_2, connected to an inert non-Cartan edge. Again all of these examples will only determine **decomposable Nichols algebras** in Step 4, as they only have a single split node.
- In rank $n \geq 5$ (see **[H08]** appendix C), again a chain form would require n odd (loop-freeness); hence all but the middle two edges are split and have to be decorated by -1, which again excludes most cases as in Step 3a. Combination with node-decoration symmetry is needed to discard:
  - In row 3 the outmost nodes require ($q = q^2$ or) $q = q^{-2}$, which would violate the rightmost edge to bear -1.

  - In row 10 the outmost nodes require ($q = q^2$ or) $q = q^{-2}$, which would violate the rightmost edge to bear -1.

The prolonged triangles there have always loops in their symmetries (even those prolonged to both sides), and the E_7, E_8-shape no symmetry at all. It remains to check all D_n, E_6-shaped, where we compare the two equally-long-branch-end's node decorations (for row 19 diagram 5 compare two inner nodes!). Altogether we get:

Row	$q \in$	Cartan	Folds to
1	$\mathbb{k}_2$	A_{2n-1}	B_n
8	$\neq 1$	D_{n+1}	C_n
10	$\neq 1$		-
16	$\mathbb{k}_2$	E_6	F_4

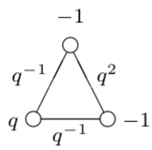
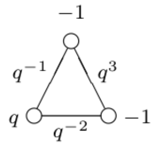
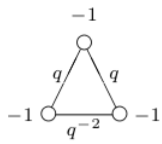
Again, although all are **tamely ramified**, row 18 and 10 have only a single split node and only yield decomposable Nichols algebras in Step 4. Note that D_n is the “prototype for this behaviour in all previous cases.

Step 2c: Now consider the $p = 2$ connected case with at least one loop, we shall show that **only** A_2 for $q \in \mathbb{k}_3$ remains! We showed, that necessarily such a loop needs node decoration q^{-1} and edge decorations q for $q \in \mathbb{k}_3$ **and** cannot be directly connected to a split node. Hence it is **either isolated** and thus of rank 2 with symmetric node decoration (see [**H05**] table A.1):

Row	$q \in$	Cartan	Folds to
2	$\mathbb{k}_3$	A_2	A_1
9	$\mathbb{k}_{12}$	$\overset{-\zeta^2}{\circ}\overset{\zeta}{\text{———}}\overset{-\zeta^2}{\circ}$	-

The latter is discarded by incorrect edge decoration $\notin \mathbb{k}_3$.

The **second possibility** is the loop to be connected to an inert node, with the edge hence branched; this means the loop is part of a triangle with 2-fold symmetry; see **[H05]** table A.2 to find (row 9 is discarded by unsymmetrical edge-decoration):

Row	$q \in$	Cartan	Folds to
6	$\neq \pm 1$		-
7	$\neq \pm 1, \notin \Bbbk_3$		-
10	$\neq \pm 1, \notin \Bbbk_3$		-

All three possibilities have incorrect node decoration $-1 \notin \Bbbk_3$, hence no triangle can be contained (also in higher rank!)

Step 2d: We finally consider connected diagrams with automorphisms of order $p \geq 3$, first by restricting just their shape without decorations.

- Loops have already been discarded for $p \neq 2$ in theorem 6.9, hence especially no triangularly shaped diagrams can appear.
- Note that there is no branch point of order ≥ 3 in any diagram in Heckenberger's list.
- Secondly suppose a $p = 3$ and rank > 4, then by symmetry we would require a 3 branch point with prolongations of all ends, which again does not appear in Heckenberger's list.
- Hence only $p = 3$ and $rank = 4$ with shape D_4 is possible.

Finally we consider rank 4 and shape D_4 in **[H08]** appendix B and search for threefold symmetry. We find that rows 13, 18, 20 have unsymmetrical node decoration and row 12 has unsymmetrical edge decoration, leaving

Row	$q \in$	Cartan	Folds to
5	$\neq 1$	D_4	G_2

Step 3a We now want to exclude those unramified diagrams with multiply-laced diagrams, i.e. a longer root node decorated with $q \neq -1$ (actually $q \in \mathbb{k}_4$ for C_n resp. $q \in \mathbb{k}_6$ for G_2). Note that a node $M_1 = \mathcal{O}_g^\chi$ with decoration $q = \chi(g)$ appears in the unramified diagram together with a copy $M_2 = \mathcal{O}_g^{\chi_\sigma}$, hence

$$\begin{aligned} q_{12}q_{21} &= \chi_\sigma(g)\chi(g) \\ &= \sigma(g,g)\sigma^{-1}(g,g)\chi(g)\chi(g) = q^2 \end{aligned}$$

and $q \neq \pm 1$ would cause a loop contrary to the assumption; however the diagram could (but doesn't) appear in Step 2c.

Step 4: Next we claim, that diagrams with a single split edge may only orbifoldize to decomposable Nichols algebras: Suppose otherwise, then the group elements associated to all G-nodes generate G. Hence at least two of these need to discommute. Then by Theorem 6.9 these nodes have to be connected by a split edge and thus both nodes are splits, which shows the claim.

This additional condition rules out as decomposable orbifolds:

- the isolated loop A_2
- the ramified cases D_4, D_n and further non-Cartan cases above.
- the ramified $D_4 \to G_2$
- small ranks for the remaining: ramified A_2 and unramified A_1.

which leaves only the later-on realized unramified cases $A_{n\geq 2}, D_{n\geq 4}, E_{6,7,8}$ and ramified cases E_6 and A_{2n-1} for $n \geq 3$.

Step 5: We also need to show, that the conditions (matching those in the last sections' constructions) on the group given in Theorem 6.1, which we still prove, are **necessary**:

First, we prove that $dim_{\mathbb{F}_2}(\Gamma/\Gamma^2) = dim_{\mathbb{F}_2}(V)$ has to coincide with the rank of the **minimally** indecomposable orbifold. For this, we invoke the Burnside Basis Theorem 6.13, that states that every minimally generating set corresponds to a V-basis, hence has precise cardinality $dim_{\mathbb{F}_2}(V)$. Further, since conjugacy classes map to single V-element, the same holds for minimally generating sets of conjugacy classes.

Secondly we need to prove the given restrictions on

$$dim_{\mathbb{F}_2}(Z(G)/(G'G^2)) = dim_{\mathbb{F}_2}(V^{\perp})$$

Again, the conjugacy classes underlying the G-nodes form a basis of V. Split nodes are connected **iff** the conjugacy classes discommute (Theorem 6.9) and hence the subspace W generated by the classes' images in V supports these as symplectic root system, which means a nullspace $W^{\perp}$ exactly prescribed by Theorem 6.15. Also note, that inert G-nodes correspond to conjugacy classes of length 1 and hence central elements $\in G$ resp. nullvectors $\in V$.

- For all unramified cases holds $V = W$ and hence $dim_{\mathbb{F}_2}(V^{\perp})$ can directly be read from the symplectic root system (1_n for A_n, E_6, E_7, E_8 resp. $2 - 1_n$ for D_n).
- For ramified $E_6 \to F_4$ we have two inert nodes (i.e. nullvectors), as well as two split nodes with diagram A_2 generating a nondegenerate $W \subset V$ by symplectic root systems. Hence $V^{\perp} = 2$ as asserted.
- For ramified $A_{2n-1} \to B_n$ we have one inert node (i.e. nullvector), as well as split nodes with diagram A_{n-1} generating a $W \subset V$ with nullspace $dim(W^{\perp}) = 1_{n-1}$ by symplectic root systems. Hence $V^{\perp} = 1 + 1_{n-1}$ as asserted.

Finally, the cases above with the given restrictions on G and ranks bounded from below by Step 4 were realized in the previous three sections. This proves Theorem 6.1 and concludes the classification of this chapter.

CHAPTER 7

Applications To Nondiagonal Nichols Algebras

1. Nichols Algebras Over Most Groups Of Order 16 And 32

We shall demonstrate the result of the last sections and point to cases yet to be treated. We denote by the symbol (M_n for any) the orbifoldized Dynkin diagram and by the superscript $[I],[U],[R]$ inert, split or ramified orbifoldizing (M_n any rank-n-module). In each case, we use Matsumotos sequence (section 5.2) and the known cohomologies in (section 8.2) to find Doi twists of the orbifolds with nontrivial action of G', especially **nondiagonal**. Elementary case-by-case considerations are used to assert that the action is even **faithful** in some of these cases (especially for $\Gamma \cong \mathbb{Z}_2^n$). Note that the absence does not generally contradict nondiagonal Nichols algebras. An example were Doi twists of orbifolds exhausts already all was given in section 5.3 and will be the content of the remaining chapter.

Group G [(a)]	Known as..	Nichols algebra	Orbifolds $G = \Sigma.(\Gamma)$	section
$\#1-5$	abelian	(YES, faithful)	$X_n^{[I]},\ n = 1\ldots 4$	**[H08]**
$\#6$	$\mathbb{Z}_2 \times \mathbb{D}_4$	**YES**, **faithful**	$\mathbb{Z}_2.(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2)$	
$\#7$	$\mathbb{Z}_2 \times \mathbb{Q}_8$	**YES**	$\mathbb{Z}_2.(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2)$	
$\#8$	$\mathbb{Z}_4 * \mathbb{D}_4 \cong \mathbb{Z}_4 * \mathbb{Q}_8$	**YES**	$\mathbb{Z}_2.(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2)$	
			$A_3^{[U]},\ X_1^{[I]} \cup A_2^{[U]}$	6.3
			$B_3^{[R]}$	6.5
$\#9$	$(G' \not\subset G^2)$	**YES**, **nondiag.**	$\mathbb{Z}_2.(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$	
$\#10$	$(G' \subset G^2)$	**YES**, **faithful**	$\mathbb{Z}_2.(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$	
$\#11$	$(G' \subset G^4)$	**YES**	$\mathbb{Z}_2.(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$	
			$A_2^{[U]}$	6.3
$\#12-14$	$\mathbb{D}_8, \tilde{\mathbb{D}}_8, \mathbb{Q}_{16}$	**(?)** Class 3	$\mathbb{Z}_4^{Aut}.(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ [(b)]	

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[(a)] From the classification **[Group16]**

[(b)] A noncentral extension. This is case $\mathcal{G}_4$ in **[HS10]**, see outlook 3 of this part

Group G [c]	Known as..	Nichols algebra	Orbifolds $G = \Sigma.(\Gamma)$	section
$\#1-7$	abelian	(YES, faithful)	$X_n^{[I]},\ n = 1\ldots 5$	**[H08]**
$\#8$	$\mathbb{Z}_2^2 \times \mathbb{D}_4$	**YES**, **faithful**	$\mathbb{Z}_2.(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2)$	
$\#9$	$\mathbb{Z}_2^2 \times \mathbb{Q}_8$	**YES**	$\mathbb{Z}_2.(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2)$	
$\#10$	$\mathbb{Z}_2 \times (\mathbb{Z}_4 * \mathbb{D}_4)$	**YES**	$\mathbb{Z}_2.(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2)$	
			$X_2^{[I]} \cup A_2^{[U]},\ D_4^{[U]}$	6.3
			$X_1^{[I]} \cup B_3^{[R]}$	6.5
			$F_4^{[R]}$	6.4
$\#11$	$\mathbb{Z}_2 \times \#_{16}9$	**YES**, **nondiag.**	$\mathbb{Z}_2.(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4)$	
$\#12$	$\mathbb{Z}_2 \times \#_{16}10$	**YES**, **faithful**	$\mathbb{Z}_2.(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4)$	
$\#13$	$\mathbb{Z}_2 \times \#_{16}11$	**YES**	$\mathbb{Z}_2.(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4)$	
$\#14$	$\mathbb{Z}_4 \times \mathbb{D}_4$	**YES**, **nondiag.**	$\mathbb{Z}_2.(\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2)$	
$\#15$	$\mathbb{Z}_4 \times \mathbb{Q}_8$	**YES**	$\mathbb{Z}_2.(\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2)$	
$\#16, 17$		**YES**	$\mathbb{Z}_2.(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4)$	
			$A_3^{[U]},\ X_1^{[I]} \cup A_2^{[U]}$	6.3
			$B_3^{[R]}$	6.5
$\#18$	$(G' \not\subset G^2)$	**YES**, **nondiag.**	$\mathbb{Z}_2.(\mathbb{Z}_4 \oplus \mathbb{Z}_4)$	
$\#19$	$(G' \subset G^2)$	**YES**	$\mathbb{Z}_2.(\mathbb{Z}_4 \oplus \mathbb{Z}_4)$	
$\#20, 21$	$(G' \not\subset G^2)$	**YES**, **nondiag.**	$\mathbb{Z}_2.(\mathbb{Z}_2 \oplus \mathbb{Z}_8)$	
$\#22$	$(G' \subset G^4)$	**YES**	$\mathbb{Z}_2.(\mathbb{Z}_2 \oplus \mathbb{Z}_8)$	
			$A_2^{[U]}$	6.3
$\#23-25$	$\mathbb{Z}_2 \times \mathbb{D}_8, \tilde{\mathbb{D}}_8, \mathbb{Q}_{16}$	**(?)** Class 3	$\mathbb{Z}_4^{Aut}.(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$ [e]	
$\#26-32$		**(?)** Class 3	$\mathbb{Z}_4^{Aut}.(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$ [e]	
$\#33-35$	fibre products	**NO**	$\mathbb{Z}_2^2.(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2)$ [d]	
$\#36-41$		**NO**	$\mathbb{Z}_2^2.(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2)$ [d]	
			$A_7^{(1)} \longmapsto D_5^{(2)}$	7.4
$\#42, 43$	$\mathbb{D}_4 * \mathbb{D}_4,\ \mathbb{D}_4 * \mathbb{Q}_8$	**YES**	$\mathbb{Z}_2.(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2)$	
			$A_2^{[U]} \cup A_2^{[U]},\ A_4^{[U]}$	6.3
$\#44-48$		**(?)** Class 3	$\mathbb{Z}_4^{Aut}.(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$ [e]	
$\#49-51$	$\mathbb{D}_{16}, \tilde{\mathbb{D}}_{16}, \mathbb{Q}_{32}$	**NO** Class 4	[f]	

.

[c] From the classification [**Group16**]

[d] These are discarded by orbifoldizing it back to an 8-cycle.

[e] Noncentral extension. This is case $\mathcal{G}_4$ in [**HS10**], see outlook 3 of this part

[f] Noncentral extension of higher class, discarded by [**HS10**], see outlook 3.

Remark 7.1. *Before we proceed to the proof, here are some comments.*

- *The cases* $A_2^{[U]}$ *extending the construction for* $\mathbb{D}_4$ *have independently been found in* [**HS10**].
- *Some Dynkin diagrams do not appear due to larger rank. E.g.* $E_6^{[U]}$ *is first realized over the extraspecial groups* $G =^{12+1}$.
- $G' = \mathbb{Z}_2^2$ *(such as* $\#_{32}\ 33 - 41$*) by no means generally contradicts the existence of finite-dimensional Nichols algebras. Rather, there might be disconnected ones with supports* $N' = \mathbb{Z}_2$*. The smallest examples include e.g.*
$$\mathbb{D}_4 \times \mathbb{D}_4,\ A_2^{[U]} \cup A_2^{[U]}$$
- *Several disconnected diagrams are "adapted" to G being a direct sum. This is not necessary, as the use of* 2 *symplectic basis-pairs for* $A_2^{[U]} \cup A_2^{[U]}$ *over* $\mathbb{D}_4 * \mathbb{D}_4$ *show, or the central element generating the* $X_1^{[I]}$ *in* $\#_{32}16, 17$.

Proof. For all **connected** Nichols algebras over $G' = \mathbb{Z}_2$ follow from theorem 6.1 and structure constants for each group (note that the ordering used in the classification is adapted to these!):

G	$exp(\Gamma)$	$dim_{\mathbb{F}_2}(\Gamma/\Gamma^2)$	$dim_{\mathbb{F}_2}(Z(G)/G'G^2)$
$\#_{16}6 - \#_{16}8$	2	3	1
$\#_{16}9 - \#_{16}11$	4	2	0
$\#_{32}8 - \#_{32}10$	2	4	2
$\#_{32}11 - \#_{32}17$	4	3	1
$\#_{32}18 - \#_{32}19$	4	2	0
$\#_{32}20 - \#_{32}22$	8	2	0
$\#_{32}42 - \#_{32}43$	2	4	0

Note that a nonabelian group cannot be generated by central elements, hence not all connected components may be inert; also we've proven in section 6.6, that a single split node may never appear in an indecomposable Nichols algebra. Hence the number of split nodes is always at least 2.

In all cases, where **disconnect diagrams** may appear, we need to find (by Theorem 6.9) commuting conjugacy classes with the respective supports and realize each connected components over the respective subgroups:

- $\#_{16}6 - \#_{16}8,\ X_1^{[I]} \cup A_2^{[U]}$: Take the central element generating the summand $\mathbb{Z}_2$ and the symplectic base pair for $A_2^{[U]}$ ($N = \mathbb{D}_4$).
- $\#_{32}8 - \#_{32}10,\ X_2^{[I]} \cup A_2^{[U]}$: As the last case.
- $\#_{32}8 - \#_{32}10,\ X_1^{[I]} \cup B_3^{[R]}$: Take one central generator of a summand $N = \mathbb{Z}_2$ for $X_1^{[I]}$ and realize $B_3^{[R]}$ over the other summand $N = \#_{16}6 - \#_{16}8$.
- $\#_{32}11 - \#_{32}13,\ X_1^{[I]} \cup AB_2^{[U]}$: Take again the direct sum and realize $A_2^{[U]}, B_2[U]$ over $N = \#_{16}9 - \#_{16}11$.
- $\#_{32}14 - \#_{32}15,\ X_1^{[I]} \cup A_2^{[U]}$: As above.
- $\#_{32}14 - \#_{32}15,\ X_1^{[I]} \cup B_2^{[U]}$: Note that the direct-sum approach fails here! Take a, b generators of the extraspecial summand, c the generator of the $\mathbb{Z}_4$-summand and $g = [a, b]$ the central commutator. Then we take as conjugacy classes $[c]$ over $N = \mathbb{Z}_4$ and $[ac] \oplus [bc]$ over $N = \#_{16}9 - 11$.
- $\#_{32}16,\ X_1^{[I]} \cup AB_2^{[U]}$: Take for $X_1^{[I]}$ the central $[G_1G_2]$ with support $N = \mathbb{Z}_4$. Realize $A_2^{[U]}, B_2^{[U]}$ over $[G_2] \oplus [G_3]$ with support $N = \#_{16}9 - \#_{16}11$.
- $\#_{32}17,\ X_1^{[I]} \cup AB_2^{[U]}$: Take for $X_1^{[I]}$ the central $[G_3]$ with support $N = \mathbb{Z}_4$. Realize $A_2^{[U]}, B_2^{[U]}$ over $[G_1] \oplus [G_2]$ with support $N = \#_{16}9 - \#_{16}11$.
- $\#_{32}42 - \#_{32}43,\ A_2^{[U]} \cup A_2^{[U]}$: Take the symplectic base pairs $[x_i] \oplus [y_i]$, each with support $N = \mathbb{D}_4$.

The exclusion of the cases $G' = \mathbb{Z}_2^2$ ($\#_{32}33 - \#_{32}41$) will be the content of section 7.4, while higher classes are discussed in outlook 3.

To check **nondiagonality** of a Doi twist, i.e. $G' \cong \mathbb{Z}_2$ acting nontrivial, we use Matsumotos exact sequence to determine the possible Σ actions $Im(\gamma)$ induced by Doi twists (see section 5.2). We showed, that for cyclic stem extensions $\mathbb{Z}_2 \cong \Sigma \subset G'$

$$log_2|Im(\gamma)| = log_2|H^2(G, \mathbb{k}^\times)| + 1 - log_2|H^2(\Gamma, \mathbb{k}^\times)|$$

which we check against all respective cases $G' \cong \mathbb{Z}_2$ of order $|G| = 16,\ 32$ in section 8.2.

$\#_{16}$	$H^2(\Gamma)$	$\leq log_2\lvert H^2(G,\Bbbk^\times)\rvert$	$\leq log_2\lvert Im(\gamma)\rvert$
#6	$\mathbb{Z}_2^3$	3	1
#7	$\mathbb{Z}_2^3$	2	0
#8	$\mathbb{Z}_2^3$	2	0
#9	$\mathbb{Z}_2^2$	2	2
#10	$\mathbb{Z}_2^2$	1	1
#11	$\mathbb{Z}_2^2$	0	0

$\#_{32}$	$H^2(\Gamma)$	$\leq log_2\lvert H^2(G,\Bbbk^\times)\rvert$	$\leq log_2\lvert Im(\gamma)\rvert$
#8	$\mathbb{Z}_2^6$	6	1
#9	$\mathbb{Z}_2^6$	5	0
#10	$\mathbb{Z}_2^6$	5	0
#11	$\mathbb{Z}_2^3$	4	2
#12	$\mathbb{Z}_2^3$	4	2
#13	$\mathbb{Z}_2^3$	2	0
#14	$\mathbb{Z}_2^3$	3	1
#15	$\mathbb{Z}_2^3$	2	0
#16	$\mathbb{Z}_2^3$	2	0
#17	$\mathbb{Z}_2^3$	2	0
#18	$\mathbb{Z}_4$	3	2
#19	$\mathbb{Z}_4$	1	0
#20	$\mathbb{Z}_2$	2	2
#21	$\mathbb{Z}_2$	2	2
#22	$\mathbb{Z}_2$	0	0
#42	$\mathbb{Z}_2^4$	5	0
#43	$\mathbb{Z}_2^4$	5	0

Note that the last number determines even a $\mathbb{Z}_2$-basis of linearly independent actions of Σ, that can be achieved by Doi twists of the respective orbifolds.

Faithfulness follows in all nondiagonal cases above, where Σ is already the entire center. This is because all non-central elements act surely non-trivial, while the center acts by a nontrivial scalar. Moreover, when the center is cyclic $Z = \langle g\rangle$ with $nG = \Sigma$ for some n, we again achieve faithfulness by a nontrivial scalar center action. All these cases are marked accordingly in the above list. □

2. All Nichols Algebras over $\#_{16}\,9,10$ (rank 2)

As we from now on pursue the **classification** of certain minimally indecomposable, finite-dimensional Nichols algebras, we first shall give two more examples of groups, where all such Nichols Algebras are Doi twists of orbifolds, and which we will require in the next section. The groups and results are very similar to the cases $\mathbb{D}_4, \mathbb{Q}_8$ (section 5.3):

Theorem 7.2. *For G the group $\#9,10$ of order 16 in* [**Group16**]*, every minimally indecomposable finite-dimensional Nichols algebra is a Doi twist of an orbifold, with Dynkin diagram A_2 (unramified).*

Both groups have $G/G' =: \Gamma = \mathbb{Z}_4 \times \mathbb{Z}_2$ and hence by Burnside's basis Theorem 6.13 minimally indecomposable Nichols algebras are of Rank 2:

$$\tilde{M} = \mathcal{O}_a \oplus \mathcal{O}_b$$

Besides from their nontrivial commutator x (both G_4 in [**Group16**]) there is a second central element y (both G_3 in [**Group16**]) and for both groups:

$$Z(G) = G^2 = \langle x, y \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

We shall characterize different choices for a, b by the symbol

$$(u, v) = (a^2, b^2) \in Z(G) \times Z(G)$$

Note that then the product square is then already determined

$$(ab)^2 = aba^{-1}b^{-1}ba^2b = xuv$$

and the relations determine the group completely as central extension:

$$\langle x, y \rangle = \mathbb{Z}_2^2 \to G \to \mathbb{Z}_2^2 = \langle \bar{a}, \bar{b} \rangle \qquad x \in G',\ y \in G^2$$

Now, **instead** of calculating conjugacy classes for each group, we **list all** possible 4^2 such configurations, sorted by symmetry (on row) and some easily recognizable invariants $I_{1,2}$ (number of involutions resp. squaring to x among $\{a, b, ab\}$). Because of the group classification in order 16 ([**Group16**]) we know all possible groups are exactly $\#_{16}9, 10$ and find the above invariants to be decisive.

Note that if $I_1 + I_2 = 3$ than y doesn't appear at all, hence $Z(G) = \mathbb{Z}_2$ and the groups are $\mathbb{D}_4, \mathbb{Q}_8$, which shall be included here to allow the reader easy comparison to section 5.3:

$\#_{16}, I_1, I_2$	$(u,v) = (a^2, b^2)$	$xuv = (ab)^2$	Relations	$\leq log_2\|Im(\gamma)\|$
$(\mathbb{D}_4, 2, 1)$	$(1,1)$	x	$0+1$	1
	$(1,x),\ (x,1)$	1	$1+0$	
$(\mathbb{Q}_8, 0, 3)$	(x,x)	1	$2+0$	0
$\#9, 1, 1$	$(1,y),\ (y,1)$	xy	$0+0$	2
	$(1,xy),\ (xy,1)$	y	$0+0$	
	$(y,xy),\ (xy,y)$	1	$0+0$	
$\#10, 0, 1$	$(x,y),\ (y,x)$	y	$1+0$	1
	(y,y)	x	$0+1$	
	$(x,xy),\ (xy,x)$	xy	$1+0$	
	(xy,xy)	x	$0+1$	

As in *cit. loc.* we give the number of independent relations (base+product) $\chi_i(x) = -1$ resp. $\chi_i(x)\chi_j(x) = -1$ on real classes $g_i^2 = x$ resp. $(g_ig_j)^2 = x$. This is compared in the last column to the bounds on the image of the map $\gamma : H^2(G, \mathbb{k}^\times) \to G' \otimes \Gamma$ in Matsumotos sequence obtained in the previous section. We find again, that in all cases the relations and the remaining Doi twists already exhaust all possible 2^2 actions of G', hence we again may Doi twist every such Nichols algebra back to one with trivial action and hence an orbifold. The form of this Nichols algebra is then obtained from the table in section 7.1.

3. All Nichols Algebras over $\#_{32}$ 18 (rank 2)

Solely for application in the next section, we include yet another group of the type above, namely (as we will see) the **only**

$$\langle x, y, z\rangle = \mathbb{Z}_2^2 \to G \to \mathbb{Z}_2^2 = \langle \bar{a}, \bar{b}\rangle \qquad x \in G'; y, z \in G^2$$

Theorem 7.3. *For G the group #18 of order 32 in* **[Group32]**, *every minimally indecomposable finite-dimensional Nichols algebra is a Doi twist of an orbifold, with Dynkin diagram A_2 (unramified).*

By the table in 7.1 the cohomology of this group fulfills again $Im(\gamma) \geq 2$, so no relations are necessary to exhaust all possible Σ-actions by Doi twist, which again concludes the proof.

We only need to prove the uniqueness statement above by consulting the list [**Group32**]:

- Only $\#_{32}18-22$ are central extensions of abelian Γ with rank $dim_{\mathbb{F}_2}(\Gamma/\Gamma^2)=2$
- Only $\#_{32}18,19$ have $\Gamma=\mathbb{Z}_4^2$ as demanded by

$$\langle y,z\rangle=\mathbb{Z}_2^2\to\Gamma\to\mathbb{Z}_2^2$$

- $\#_{32}19$ is discarded, as there is one generator G_1 with $1\neq G_1^4\in G'$ contrary to the demanded extension.

(note even further, that another additional independent $z'\in G^2$ cannot be supported within $\langle x,y,z,z'\rangle$)

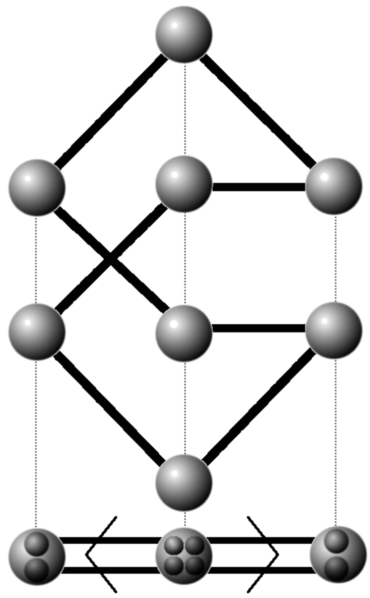
4. No Nichols Algebras over $\#_{32}\ 33-41$ (rank 3)

We shall pursue our classificatory interests beyond $G'\cong\mathbb{Z}_p$ and demonstrate a generic technique discarding all groups of order 32 and $G'=\mathbb{Z}_2^2=\langle x,y\rangle$.

Theorem 7.4. *There is no finite-dimensional indecomposable Nichols algebra over the groups* $\#33-41$ *of order 32 in* [**Group32**].

As in the last section we will show all minimally indecomposable Nichols algebras to be Doi twists of $\mathbb{Z}_2^2$-orbifolds, which is the tedious part. The rank 2 subalgebras have been determined in the preceding sections and the enumeration of possible conjugacy class configurations is done in section 8.1 by using **reflections** to determine the entire **Weyl equivalence class**. Then, we choose a representative with many apparent relations and exhaust the remaining possible Σ^*-actions by known cohomology.

Then, contrary to the above cases, the **two generators** of twisted symmetries $\Sigma=\mathbb{Z}_2^2$ will show the orbifold to be of affine form $A_7^{(1)}\to D_5^{(2)}$. The former cannot support any a minimally indecomposable finite-dimensional Nichols algebras over Γ abelian (again by [**H08**] Lemma 20), hence also not G, which totally discards indecomposable one.



Remark 7.5. *Note that the latter arguments will generally hold* **in the orbifold***, whenever a triple of elements has independent commutators, leaving for G' noncyclic only disconnected diagrams over mutually commuting support and again cyclic commutators. However, there seems no apparent argument discarding all possible nontrivial G'-actions.*

Note for convenience, that these groups are (or resemble) **fibre-products**

$$G\square_{\mathbb{Z}_2}H := \{(g,h) \in G \times H \mid q_1(g) = q_2(h)\}$$

for $G, H = \mathbb{D}_4, \mathbb{Q}_8$ and chosen quotients $q_{1,2}$ to $\mathbb{Z}_2$. While for $\mathbb{Q}_8$ all quotients have kernel $\mathbb{Z}_4$, for $\mathbb{D}_4$ we use the suggestive symbol *long* for kernel $\mathbb{Z}_2^2$ and *short* for kernel $\mathbb{Z}_4$ (projecting on long/short elements):

- $\#33 \cong \mathbb{D}_4^{long}\square_{\mathbb{Z}_2}\mathbb{D}_4^{long}$
- $\#34 \cong \mathbb{D}_4^{short}\square_{\mathbb{Z}_2}\mathbb{D}_4^{short}$
- $\#35 \cong \mathbb{D}_4^{short}\square_{\mathbb{Z}_2}\mathbb{Q}_8 \cong \mathbb{Q}_8\square_{\mathbb{Z}_2}\mathbb{Q}_8$
- $\#36 \cong \mathbb{D}_4^{short}\square_{\mathbb{Z}_2}\mathbb{D}_4^{long}$
- $\#37-41$ are similar, but generators in one factor power to the commutator of the other factor.

Remark 7.6. *Hence, besides $\mathbb{A}_5 \subset \mathbb{S}_5$ these are new examples of groups admitting finite-dimensional Nichols algebras (e.g. $A_2 \cup A_2$ over $\mathbb{D}_4 \times \mathbb{D}_4$) having subgroups which do not admit such.*

We now proceed to the **proof** along the lines of the exemplary section 5.3 (and the preceding sections):

Step 1: Clarify rank and possible Dynkin diagrams

Again, by Bursides Basis Theorem 6.13, every minimal generating set of conjugacy classes has precisely cardinality $3 = dim_{\mathbb{F}_2}(\Gamma/\Gamma^2)$, hence any minimally indecomposable Yetter-Drinfel'd module is of the form:

$$\tilde{M} = \mathcal{O}_a \oplus \mathcal{O}_b \oplus \mathcal{O}_c$$

Again, all configurations of such conjugacy classes are distinguished by the central squares ("fusions") of the conjugacy classes

$$\mathcal{O}_a^2 = \{u\} \qquad \mathcal{O}_b^2 = \{v\} \qquad \mathcal{O}_c^2 = \{w\} \qquad u, v, w \in \Sigma$$

At all by $\Sigma^* \otimes G \cong \mathbb{Z}_2^6$ there are 2^6 different possible actions of Σ^* on these three simple Yetter-Drinfel'd modules. Only one of these, trivial one, corresponds to a possible orbifold!

Lemma 7.7. *Suppose a minimally indecomposable finite-dimensional Nichols algebra over G, then the Dynkin diagram is simply laced, i.e. of type A_3 or Z_3 (3-cycle).*

Proof. We reduce to the rank 2 cases treated above: Consider the the minimally indecomposable $\mathcal{O}_a \oplus \mathcal{O}_b$ over $G_{a,b} = \langle a, b\rangle$ (resp. b, c and a, c), which is depending on the configuration an extension

$$\begin{aligned}
\langle x\rangle = \mathbb{Z}_2^2 \to & G_{a,b} \to \mathbb{Z}_2^2 = \langle \bar{a}, \bar{b}\rangle && x \in G' \\
\langle x, y\rangle = \mathbb{Z}_2^2 \to & G_{a,b} \to \mathbb{Z}_2^2 = \langle \bar{a}, \bar{b}\rangle && x \in G' \; y \in G^2 \\
\langle x, y, z\rangle = \mathbb{Z}_2^2 \to & G_{a,b} \to \mathbb{Z}_2^2 = \langle \bar{a}, \bar{b}\rangle && x \in G' \; y, z \in G^2
\end{aligned}$$

but all such minimally indecomposable finite-dimensional Nichols algebras were shown to be of type A_2 in sections 5.3, 7.2 and 7.3 $\square$

Step 2: Bound the number of possible actions by relations

Next we use again, that not all 2^6 actions are admissible for a finite-dimensional Nichols algebra by using [**AHS09**] and [**HS08**]. To reduce the amount of by-hand case treatment of all different configurations, we use the knowledge of the diagrams in question and perform a **Weyl reflection** to obtain a different configuration, and the former has the same dimension as the latter! Hence in what follows, we only have to find **one** discarding configuration in each Weyl equivalence orbit. This has to be done rather tediously:

Lemma 7.8. *Every group's $G = \#_{32}33 - 41$ configurations only consists of one Weyl-equivalence class and in each we find a configuration with r independent relations:*

$G =$	#33	#34	#35	#36	#37	#38	#39	#40	#41
$r \geq$	*3*	*2*	*3*	*2*	*4*	*3*	*3*	*4*	*4*

PROOF. This is the tedious by-hand part devised to section 8.1, especially as we will need to introduce some technical notation to obtain/present them as efficient as possible. □

Step 3: Exhaust the remaining actions by Doi twists

We use again Matsumoto's sequence (section 5.2 and the knowledge of the respective cohomologies to enumerate the number of actions $|Im(\gamma)|$, that can be achieved by Doi twisting the orbifold:

$$1 \to \Sigma^* \to G \to \Gamma \to 1$$

$$1 \to \Gamma^* \to G^* \to \Sigma \to H^2(\Gamma, \Bbbk^\times) \to H^2(G, \Bbbk^\times)_\Sigma \xrightarrow{\gamma} \Sigma^* \otimes G$$

Let us calculate resp. bound the orders of all sequence terms:

- Generally for stem-extensions ($\Sigma \subset [G, G]$) we have $G^* \cong \Gamma^*$.
- Since Σ is abelian, $|\Sigma^*| = |\Sigma| = 2^2$.
- We know $|H^2(\Gamma, \Bbbk^\times)| = |H^2(\mathbb{Z}_2^3, \Bbbk^\times)| = 2^3$.
- Let some $2^m \leq |H^2(G, \Bbbk^\times)|$ bound the cohomology; as $|H^2(\Sigma)| = 2^1$ the kernel of the restriction is then $2^{m-1} \leq |H^2(G, \Bbbk^\times)_\Sigma|$.
- Hence by the exact sequence $2^{m-2} \leq |Im(\gamma)|$

Step 4: State that the Doi twists exhaust all possibilities

Thus our proof amount to checking $6 - r \leq m - 2$. The left side is stated in Step 2 (and checked in section 8.1) , while the right side is from Step 3 with a bound $2^m \leq |H^2(G, \Bbbk^\times)|$ on the respective cohomology listed in section 8.2. The proof works, because they are in lucky coincidence:

We get $r + m = 8$ except for #35 with even $r + m = 9$. Hence we get enough twists for all admissible actions, and in consequence all our

Nichols algebras can again be reversely Doi twisted to ones with trivial Σ^*-action:

Corollary 7.9. *Every minimally indecomposable finite-dimensional Nichols algebra over one of these groups is an $\mathbb{Z}_2^2$-orbifold of a $\Gamma = \mathbb{Z}_2^3$.*

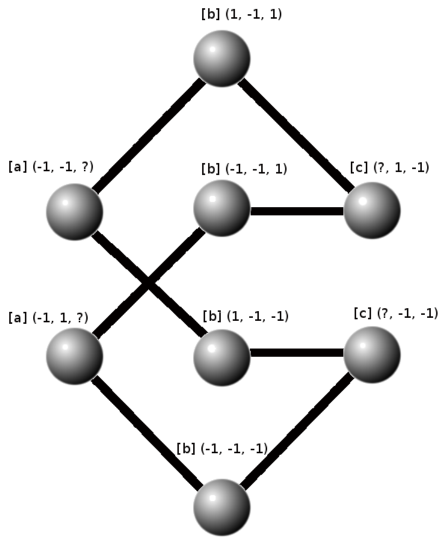
Step 5: Analyze (and here discard) the possible orbifold

To complete the proof of the main theorem, we now discard the corresponding Nichols algebra over Γ with two simultaneous involutory twisted symmetries $\Sigma \cong \mathbb{Z}_2^2 = \langle \sigma, \tau \rangle$ generating an extension of $\mathbb{Z}_2^3$ by $\Sigma^* \cong \mathbb{Z}_2^2 = \langle g_\sigma, g_\tau \rangle$ (as in Corollary 1.18):

Suppose the first summand of M contains some irreducible $[\bar{a}]_{(-1,1,?)}$, where we denoted as 1-dimensional character the image of the basis and used, that for finite dimension always $\chi_g(g) \neq 1$. Then the twisted symmetry θ_σ maps this to $[\bar{a}]_{(-1,-1,?)}$ and vice versa, because $[a,b] = g_\sigma$; an analogous argument holds by $[b,c] = g_\tau$ with θ_τ for $[\bar{c}]_{(?,\pm1,-1)}$. However for b we have both nontrivial action of $\theta_\sigma, \theta_\tau$, yielding all 4 irreducible Yetter Drinfel'd modules $[\bar{b}]_{(\pm1,-1,\pm1)}$. Hence in order to afford the prescribed twisted symmetries, M has dimension ≥ 8 and

$$M \supset [\bar{a}]_{(-1,\pm1,?)} \oplus [\bar{b}]_{(\pm1,-1,\pm1)} \oplus [\bar{c}]_{(?,\pm1,-1)}$$

Altogether, we can draw the associated Dynkin diagram (omitting all unclear edges between a and c) and it happens to contain an 8-cycle, which is impossible by **[H08]** Lemma 20.



CHAPTER 8

Tables

1. Weyl Equivalence Classes for $\#_{32}\, 33 - 41$

These worked out tables prove Lemma 7.8 used in section 7.4.

Assuming all diagrams to be simply laced we check by hand all configuration of squares

$$a^2, b^2, c^2 = u, v, w \in \Sigma^* = \langle x, y\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

of some fixed conjugacy classes with one of the prescribed commutator structures (for A_3, Z_3 respectively)

$$\begin{array}{llll} \mathcal{O}_a, \mathcal{O}_b, \mathcal{O}_c & [a,b] = x & [b,c] = y & [a,c] = 1 \\ \mathcal{O}_a, \mathcal{O}_b, \mathcal{O}_c & [a,b] = x & [b,c] = y & [a,c] = xy \end{array}$$

Not all configurations are independent, but may be **Weyl equivalent**.

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad \Longleftrightarrow \quad \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

In this section we will calculate explicitly all orbits of the **Weyl groupoid** acting on the configurations, and find them to represent all possible configurations resp. Yetter-Drinfel'd modules for each of the groups $\#33 - \#41$.

We again (see section 5.3) obtain on relation for the action of u, v, w on one of the simple summands of the Yetter-Drinfel'd module $M \supset \mathcal{O}_a^\chi, \mathcal{O}_b^\rho, \mathcal{O}_c^\phi$, assuming $\mathcal{B}(M) < +\infty$ and the respective class **real**:

$$\chi(u) = \rho(v) = \phi(w) = 1$$

After calculating each Weyl equivalence class (i.e. orbits of reflections) we may check the representative with the highest number of such relations r. This finally yields the necessary relations for the proof of section 7.4, that are in luckily remarkable coincidence with the Schur multiplier rank n (section 8.2) $r+n=8$ (9 in the exceptional #35).

$G=$	#33	#34	#35	#36	#37	#38	#39	#40	#41
$r\geq$	3	2	3	2	4	3	3	4	4

In the following, we decorate the respective Dynkin diagram's nodes with the conjugacy classes and the edges with the commutator. Below we denote the powers (a^2,b^2,c^2) of the nodes and the prescribed commutators $(x,y)_{A_3}$ resp. $(x,y,xy)_{Z_3}$.

Often, after a reflection the diagram needs to be **reordered** in order to be in this standard presentation. For one, we'll need to permute the nodes (and accordingly permute the node squares), e.g.

$$\begin{array}{lll}(12) & (u,v,w)\mapsto(v,u,w) & (x,y,xy)_{Z_3}\mapsto(y,xy,x)_{Z_3}\\ (23) & (u,v,w)\mapsto(u,w,v) & (x,y,xy)_{Z_3}\mapsto(xy,y,x)_{Z_3}\\ (13) & (u,v,w)\mapsto(w,v,u) & (x,y)_{A_3,Z_3}\mapsto(y,x)_{A_3,Z_3}\end{array}$$

Secondly, still we need to perform a base transformation in $\Sigma^*=\langle x,y\rangle$ and again accordingly transform the node square expressions involving x,y. We denote this by a superscript of the x,y-images, e.g. $(13)^{(y,x)}$.

Now, we start calculating the influence of a **reflection** R_i on the i-th node. Note hereby e.g. for R_1, that from $a^2=u$ always follows

$$R_1\left(\mathcal{O}_a\right)=\mathcal{O}_a^{-1}=\mathcal{O}_{au}$$

with equal node square $(a^{-1})^2=u$ and for (single-) connected

$$R_1\left(\mathcal{O}_a\oplus\mathcal{O}_b\right)=\mathcal{O}_{au}\oplus\mathcal{O}_{ab}$$

from $a^2=u, b^2=v, [a,b]=x$ that the new node square is $(ab)^2=uvx$. The edge (commutator) is not changed $[au,ab]=[a,b]=c$. The influence on the third depends on the shape of the diagram:

- Case A_3: if $\mathcal{O}_c$ was not connected to $\mathcal{O}_a$, it behaves trivially under reflection, hence we get

  $$R_1\left(\mathcal{O}_a\oplus\mathcal{O}_b\oplus\mathcal{O}_c\right)=\mathcal{O}_{au}\oplus\mathcal{O}_{ab}\oplus\mathcal{O}_c$$

  and the third edge is again $[ab,c]=y$ yielding A_3 in standard presentation as well.
- Case Z_3: if $\mathcal{O}_c$ was connected to $\mathcal{O}_a$ as well, the influence is the same as on $\mathcal{O}_b$

  $$R_1\left(\mathcal{O}_a\oplus\mathcal{O}_b\oplus\mathcal{O}_c\right)=\mathcal{O}_{au}\oplus\mathcal{O}_{ab}\oplus\mathcal{O}_{ac}$$

and the third edge is now $[ab, ac] = x \cdot y \cdot xy = 1$ hence the new diagram is of type A_3 with $\mathcal{O}_{au}$ now the central node. It needs to be **reordered** by $(12)^{(x,xy)}$ to be in standard presentation!

- Case R_2A_3: A special variation of the latter is the connection of the reflected node to two mutually disconnected nodes:

$$R_2 \left(\mathcal{O}_a \oplus \mathcal{O}_b \oplus \mathcal{O}_c\right) = \mathcal{O}_{ba} \oplus \mathcal{O}_{bv} \oplus \mathcal{O}_{bc}$$

  And these two are now connected by an edge $[ba, bc] = xy$ and hence now of type Z_3.

Altogether for A_3 with node squares (u, v, w) and edges $(x, y)_{A_3}$:

$$\begin{array}{lll} R_1 & (u,v,w) \mapsto (u, uvx, w) & (x,y)_{A_3} \mapsto (x,y)_{A_3} \\ R_2 & (u,v,w) \mapsto (uvx, v, vwy) & (x,y)_{A_3} \mapsto (x,y,xy)_{Z_3} \\ R_3 & (u,v,w) \mapsto (u, vwy, w) & (x,y)_{A_3} \mapsto (x,y)_{A_3} \end{array}$$

For a Z_3 diagram with node squares (u, v, w) and edges $(x, y, xy)_{Z_3}$:

$$\begin{array}{lll} R_1 & (u,v,w) \mapsto (u, uvx, uwxy) & (x,y,xy)_{Z_3} \mapsto (x,1,xy)_{(12)^{(x,xy)}A_3} \\ R_2 & (u,v,w) \mapsto (uvx, v, vwy) & (x,y,xy)_{Z_3} \mapsto (x,y,1)_{A_3} \\ R_3 & (u,v,w) \mapsto (uwxy, vwy, w) & (x,y,xy)_{Z_3} \mapsto (1,y,xy)_{(23)^{(xy,y)}A_3} \end{array}$$

Now finally note, that a class $\mathcal{O}_a$ in this notation is **real**, iff its (central) power $u = a^2$ is contained in the subgroup of $\Sigma^* = \langle x, y\rangle$ generated the adjacent edges' decorations, because in this case the adjacent node's elements b, b' conjugate a to $a^{-1} = ua$. For Z_3 this is always the case, but for A_3 care has to be taken! We get a nontrivial Σ^*-action relation for each **real classes with nonzero square**. Similarly we recognize nontrivial product relations.

We now start multiple times with some not-appeared configuration $M = (u, v, w)$ with diagram A_3 (where we aim to the most real classes with nontrivial square) and calculate the full **Weyl equivalence class**. [(a)] To **recognize the support groups** for one configuration per class, the author calculated the involutions from the given relations; this gives the conjugacy classes of elementary abelian subgroups and respective centralizers, that can be looked up in the classification list [**Group32**]. Then an explicit isomorphism was constructed, given at the end of each *Configuration* entry below.

[(a)] Actually we calculate the Weyl groupoid modulo the above symmetry transformations. Disconnected Weyl-orbits connected by a symmetry might be fused!

Any Configuration $(a^2, b^2, c^2) \in \Sigma^*$**:** The following is the generic picture if starting with type A_3. In each particular case, at some point the entries will repeat, possibly involving a symmetry above, and we will have thus found the entire respective Weyl orbit.

Reflection	(a^2, b^2, c^2)	$[a,b],[b,c],[a,c]$	$=$ Dynkin diagram
M	(u,v,w)	$x,y,1$	A_3
$R_1(M)$	(u,uvx,w)	$x,y,1$	A_3
$R_2R_1(M)$	$(v,uvx,uvwxy)$	x,y,xy	Z_3
$R_1R_2R_1(M)$	(v,u,uw)	$x,1,xy$	$(12)^{(x,xy)}A_3$
$R_2R_1R_2R_1(M)$	$(uvx,u,uw) = R_1R_2(M)$		
$R_3R_1R_2R_1(M)$	$(uvwxy,u,uw)$ $\cdots$	$x,1,xy$	$(12)^{(x,xy)}A_3$
$R_3R_2R_1(M)$	$(uw,w,uvwxy)$	$1,y,xy$	$(23)^{(xy,y)}A_3$
$R_1R_3R_2R_1(M)$	$(uw,w,v) = R_2R_3R_2(M)$		
$R_2R_3R_2R_1(M)$	(uw,w,uvx) $\cdots$	$1,y,xy$	$(23)^{(xy,y)}A_3$
$R_3R_1(M)$	$(u,uvwxy,w)$	$x,y,1$	A_3
$R_1R_3R_1(M)$	$(u,wvy,w) = R_3(M)$		
$R_2R_3R_1(M)$	$(vwy,uvwxy,uvx)$	x,y,xy	Z_3
$R_1R_2R_3R_1(M)$	(vwy,u,uw) $\cdots$	$x,1,xy$	$(12)^{(x,xy)}A_3$
$R_3R_2R_3R_1(M)$	(uw,w,uvx) $\cdots$	$1,y,xy$	$(23)^{(xy,y)}A_3$
$R_2(M)$	(uvx,v,vwy)	x,y,xy	Z_3
$R_1R_2(M)$	(uvx,u,uw)	$x,1,xy$	$(12)^{(y,xy)}A_3$
$R_2R_1R_2(M)$	$(v,u,uw) = R_1R_2R_1(M)$		
$R_3R_1R_2(M)$	(vwy,u,uw)	$x,1,xy$	$(12)^{(x,xy)}A_3$
$R_1R_3R_1R_2(M)$	$(vwy,uvwxy,uvx)$ $\cdots$	x,y,xy	Z_3
$R_2R_3R_1R_2(M)$	$(uvwxy,u,uw) = R_3R_1R_2R_1(M)$		
$R_3R_2(M)$	(uw,w,vwy)	$1,y,xy$	$(23)^{(xy,y)}A_3$
$R_1R_3R_2(M)$	$(vwy,u,uw) = R_3R_1R_2(M)$		
$R_2R_3R_2(M)$	(uw,w,v)	$1,y,xy$	$(23)^{(xy,y)}A_3$
$R_1R_2R_3R_2(M)$	$(uw,w,uvwxy) = R_3R_2R_1(M)$		
$R_3R_2R_3R_2(M)$	$(uvwxy,vwy,v) = R_2R_3(M)$		
$R_3(M)$	(u,vwy,w)	$x,y,1$	A_3
$R_1R_3(M)$	$(u,uvwxy,w) = R_3R_1(M)$		
$R_2R_3(M)$	$(uvwxy,vwy,v)$	x,y,xy	Z_3
$R_1R_2R_3(M)$	$(uvwxy,u,v)$ $\cdots$	$x,1,xy$	$(12)^{(x,xy)}A_3$
$R_3R_2R_3(M)$	$(uw,w,v) = R_2R_3R_2$		

Configuration $(1, xy, 1)$ **yields** $\#33 \cong \mathbb{D}_4^{long} \square \mathbb{D}_4^{long}$**:**

Reflection	(u, v, w) + Symmetries	Dynkin diagram
M	$(1, xy, 1)$	A_3
$R_1(M)$	$(1, y, 1), (1, x, 1)$	A_3
$R_2R_1(M)$	$(xy, y, 1), (x, 1, y), (1, x, xy)$	Z_3
$R_1R_2R_1(M)$	$(1, y, 1) = R_1(M)$	
$R_3R_2R_1(M)$	$(1, 1, 1) = R_3R_1(M)$	
$R_3R_1(M)$	$(1, 1, 1)$	A_3
$R_2R_3R_1(M)$	$(x, 1, y) \cong R_2R_1(M)$	
$R_2(M)$	(y, xy, x)	Z_3
$R_1R_2(M)$	$(1, xy, 1) = M$	
$R_3R_2(M)$	$(1, xy, 1) = M$	
$R_3(M)$	$(1, x, 1) \cong R_1(M)$	

We get at least 3 relations from $R_2(M)$. The group is isomorphic to $\mathbb{D}_4^{long} \square \mathbb{D}_4^{long}$ (in it's standard presentation) as $\langle M \rangle$.

Configuration $(x, 1, y)$ **yields** $\#34 \cong \mathbb{D}_4^{short} \square \mathbb{D}_4^{short}$**:**

Reflection	(u, v, w) + Symmetries	Dynkin diagram
M	$(x, 1, y)$	A_3
$R_1(M)$	$(x, 1, y) = M$	
$R_2(M)$	$(1, 1, 1)$	Z_3
$R_1R_2(M)$	$(x, 1, y) = M$	
$R_3R_2(M)$	$(x, 1, y) = M$	
$R_3(M)$	$(x, 1, y) = M$	

We get at least 2 relations from M, as $\mathcal{O}_a, \mathcal{O}_c$ are real even for A_3 (see above). The group is isomorphic to $\mathbb{D}_4^{short} \square \mathbb{D}_4^{short}$ as $\langle M \rangle$.

Configuration (x, y, y) **yields** $\#35 \cong \mathbb{D}_4^{short} \square \mathbb{Q}_8 \cong \mathbb{Q}_8 \square \mathbb{Q}_8$**:**

Reflection	(u, v, w) + Symmetries	Dynkin diagram
M	$(x, y, y), (x, x, y)$	A_3
$R_1(M)$	$(x, y, y) = M$	
$R_2(M)$	$(y, y, y), (xy, xy, xy), (x, x, x)$	Z_3
$R_1R_2(M)$	(x, xy, y)	A_3
$R_3R_1R_2(M)$	$(x, xy, y) = R_1R_2(M)$	
$R_3R_2(M)$	$(x, y, y) = M$	
$R_3(M)$	$(x, y, y) = M$	

We get at least 3 relations from $R_2(M)$. The group is isomorphic to $\mathbb{D}_4^{short} \square \mathbb{Q}_8$ as $\langle M \rangle$ and to $\mathbb{Q}_8 \square \mathbb{Q}_8$ as $\langle R_1R_2(M) \rangle$.

Configuration $(x,y,1)$ **yields** $\#36 \cong \mathbb{D}_4^{short}\Box\mathbb{D}_4^{long}$**:**

Reflection	(u,v,w) + Symmetries	Dynkin diagram
M	$(x,y,1),(1,x,y)$	A_3
$R_1(M)$	$(x,y,1)=M$	
$R_2(M)$	$(y,y,1),(xy,xy,1),(y,1,y)$	Z_3
	$(1,x,x),(x,1,x),(1,xy,xy)$	
$R_1R_2(M)$	$(x,xy,x),(y,xy,y)$	A_3
$R_3R_1R_2(M)$	$(x,1,x),(y,1,y)$	A_3
$R_1R_3R_1R_2(M)$	$(1,1,y),(1,1,xy),(1,y,1)$	Z_3
	$(x,1,1),(xy,1,1),(1,x,1)$	
$R_3R_2(M)$	$(xy,1,1),(1,1,xy)$	A_3
$R_2R_3R_2(M)$	$(xy,y,1),(1,x,xy)$	A_3
$R_3(M)$	$(x,1,1),(1,1,y)$	A_3
$R_2R_3(M)$	$(1,1,y)\cong R_3(M)$	

We get at least 2 relations from $R_2(M)$. The group is isomorphic to $\mathbb{D}_4^{short}\Box\mathbb{D}_4^{long}$ as $\langle M\rangle$.

Configuration $(1,xy,y)$ **yields** $\#37 \cong \mathbb{D}_4^{long}\Box\mathbb{Q}_8$**:**

Reflection	(u,v,w) + Symmetries	Dynkin diagram
M	$(1,xy,y),(x,xy,1)$	A_3
$R_1(M)$	$(1,y,y),(x,x,1)$	A_3
$R_2R_1(M)$	$(xy,y,y),(xy,y,xy),(x,y,y)$	Z_3
	$(x,x,xy),(xy,x,xy),(x,x,y)$	
$R_1R_2R_1(M)$	$(1,y,xy),(xy,x,1)$	A_3
$R_3R_1R_2R_1(M)$	$(1,xy,xy)=R_1R_2(M)$	
$R_3R_2R_1(M)$	$(y,y,y),(x,x,x)$	A_3
$R_2R_3R_2R_1(M)$	$(y,y,y)=R_3R_2R_1(M)$	
$R_3R_1(M)$	$(1,y,y)=R_1(M)$	
$R_2(M)$	$(y,xy,xy),(y,xy,y),(y,x,x)$	Z_3
	$(xy,xy,x),(x,xy,x),(y,y,x)$	
$R_1R_2(M)$	$(1,xy,xy),(xy,xy,1)$	A_3
$R_3R_1R_2(M)$	$(1,y,xy)=R_1R_2R_1(M)$	
$R_3R_2(M)$	$(y,x,y),(x,y,x)$	A_3
$R_2R_3R_2(M)$	$(y,x,y)=R_3R_2(M)$	
$R_3(M)$	$(1,xy,y)=M$	

We get at least 3 relations from $R_2(M)$ plus 1 product relation from $(bc)^2=y$. The group is isomorphic to $\mathbb{D}_4^{long}\Box\mathbb{Q}_8$ as $\langle M\rangle$.

Configuration $(y,1,1)$ **yields** $\#38$**:**

Reflection	(u,v,w) + Symmetries	Dynkin diagram
M	$(y,1,1),(1,1,x)$	A_3
$R_1(M)$	$(y,xy,1),(1,xy,x)$	A_3
$R_2R_1(M)$	$(1,xy,x),(y,1,x),(y,xy,1)$	Z_3
$R_1R_2R_1(M)$	$(xy,1,xy)$	A_3
$R_3R_1R_2R_1(M)$	$(xy,x,xy)\cong R_1R_2(M)$	
$R_3R_2R_1(M)$	$(y,xy,1)\cong R_2R_1(M)$	
$R_3R_1(M)$	$(y,x,1),(1,y,x)$	A_3
$R_2R_3R_1(M)$	$(y,x,xy),(x,xy,y),(xy,y,x)$	Z_3
$R_1R_2R_3R_1(M)$	$(xy,xy,xy)=R_3R_1R_2(M)$	
$R_3R_2R_3R_1(M)$	$(y,x,1)=R_3R_1(M)$	
$R_2(M)$	$(xy,1,y),(1,y,xy),(x,y,1)$	Z_3
	$(x,1,xy),(xy,x,1),(1,x,y)$	
$R_1R_2(M)$	$(xy,y,xy),(xy,x,xy)$	A_3
$R_3R_1R_2(M)$	(xy,xy,xy)	A_3
$R_3R_2(M)$	$(y,y,1)=R_3(M)$	
$R_3(M)$	$(y,y,1),(1,x,x)$	A_3
$R_2R_3(M)$	$(x,y,1)\cong R_2(M)$	

We get at least 3 relations from $R_2R_3R_1(M)$. The group is isomorphic to #38 in **[Group32]** $(a,b,c,x,y)=(G_1,G_2,G_3,G_4,G_4)$ by $(12)R_1(M)$.

Configuration $(y,1,x)$ **yields** $\#39$**:**

Reflection	(u,v,w) + Symmetries	Dynkin diagram
M	$(y,1,x)$	A_3
$R_1(M)$	(y,xy,x)	A_3
$R_2R_1(M)$	$(1,xy,1),(y,1,1),(1,1,x)$	Z_3
$R_1R_2R_1(M)$	$(xy,1,y),(x,1,xy)$	A_3
$R_3R_2R_1(M)$	$(x,1,xy)\cong R_1R_2R_1(M)$	
$R_3R_1(M)$	$(y,1,x)=M$	
$R_2(M)$	$(xy,1,xy),(1,y,y),(x,x,1)$	Z_3
$R_1R_2(M)$	$(xy,y,y),(x,x,xy)$	A_3
$R_3R_1R_2(M)$	$(xy,y,y)=R_1R_2(M)$	
$R_3R_2(M)$	$(x,x,xy)\cong R_1R_2(M)$	
$R_3(M)$	$(y,xy,x)=R_1(M)$	

We get at least 2 relations from $R_2(M)$ plus 1 product relation from $(ab)^2=y$. The group is isomorphic to #39 by $(12)R_1R_2(M)$.

Configuration (xy, x, y) **yields** $\#40$**:**

M	$(xy, x, y), (x, y, xy)$	A_3
$R_1(M)$	$(xy, xy, y), (x, xy, xy)$	A_3
$R_2R_1(M)$	$(x, xy, xy) \cong R_2(M)$	
$R_3R_1(M)$	$(xy, xy, y) = R_1(M)$	
$R_2(M)$	$(xy, x, x), (x, y, x), (x, xy, xy)$ $(y, y, xy), (y, x, y), (xy, xy, y)$	Z_3
$R_1R_2(M)$	$(y, y, x), (y, x, x)$	A_3
$R_3R_1R_2(M)$	$(y, x, x) = R_1R_2(M)$	
$R_3R_2(M)$	$(xy, xy, y) = R_1(M)$	
$R_3(M)$	$(xy, x, y) = M$	

We get at least 3 relations from $R_2(M)$ plus 1 product relation from $(ab)^2 = y$. The group is isomorphic to $\#40 = \langle G_1, G_2, G_1G_3, G_4, G_5\rangle$ by $(123)R_1R_2(M)$.

Configuration $(y, 1, xy)$ **yields** $\#41$**:**

Reflection	(u, v, w) + Symmetries	Dynkin diagram
M	$(y, 1, xy), (xy, 1, x)$	A_3
$R_1(M)$	$(y, xy, xy), (xy, xy, x)$	A_3
$R_2R_1(M)$	$(1, xy, y) \cong R_2(M)$	
$R_3R_1(M)$	$(y, y, xy), (xy, x, x)$	A_3
$R_2R_3R_1(M)$	$(x, y, xy), (xy, x, y), (xy, x, y), (x, y, xy)$	Z_3
$R_1R_2R_3R_1(M)$	$(xy, x, x) \cong R_3R_1$	
$R_3R_2R_3R_1(M)$	$(xy, x, x) \cong R_3R_1$	
$R_2(M)$	$(xy, 1, x), (1, y, x), (x, xy, 1)$ $(y, 1, xy), (y, x, 1), (1, xy, y)$	Z_3
$R_1R_2(M)$	$(xy, y, x) \cong R_3(M)$	
$R_3R_2(M)$	$(xy, xy, x) \cong R_1(M)$	
$R_3(M)$	$(y, x, xy), (xy, y, x)$	A_3
$R_2R_3(M)$	$(y, x, 1) \cong R_2(M)$	

We get at least 3 relations from $R_2R_3R_1(M)$ plus 1 product relation from $(ab)^2 = y$. The group is isomorphic to $\#41 \cong \langle G_1, G_2, G_2G_3, G_4, G_5\rangle$ by $R_3R_1(M)$.

2. Groups And Cohomologies

We list all groups G of order 32 and class 2 (and how they are obtained as stem extensions $\Sigma^*.\Gamma$) from the classification [**Group32**]. It especially contains information about the local cohomology ring $H^*(G,\mathbb{Z}_2)$ and we shall extract lower bounds for the Schur multiplier $H^2(G,\Bbbk^\times)$ by using the long exact sequence induced by the short exact sequence of coefficients $\mathbb{Z}_2 \xrightarrow{\pm 1} \Bbbk^\times \xrightarrow{t^2} \Bbbk^\times$

$$1 \to Hom(G,\mathbb{Z}_2) \to Hom(G,\Bbbk^\times) \to Hom(G,\Bbbk^\times) \to H^2(G,\mathbb{Z}_2) \to H^2(G,\Bbbk^\times) \cdots$$

Since $Hom(G,\Bbbk^\times) \cong G/G'$ and smaller in characteristic 2 we get: $G/[G,G] = \Gamma = \mathbb{Z}_2^3$ the first three terms are equal

$$|H^2(G,\mathbb{Z}_2)| \cdot |G/G' \otimes \mathbb{Z}_2|^{-1} \leq |H^2(G,\Bbbk^\times)|$$

Group $\#_{16}$	$\Sigma^* := G'$	$\Gamma := G/G'$	$log_2 \lvert H^2(G,\mathbb{Z}_2)\rvert$	$\leq log_2 \lvert H^2(G,\Bbbk^\times)\rvert$
#6	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	6	3
#7	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	5	2
#8	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	5	2
#9	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	4	2
#10	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	3	1
#11	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	2	0

Group $\#_{32}$	$\Sigma^* := G'$	$\Gamma := G/G'$	$log_2 \lvert H^2(G,\mathbb{Z}_2)\rvert$	$\leq log_2 \lvert H^2(G,\Bbbk^\times)\rvert$
#8	$\mathbb{Z}_2$	$\mathbb{Z}_2^4$	10	6
#9	$\mathbb{Z}_2$	$\mathbb{Z}_2^4$	9	5
#10	$\mathbb{Z}_2$	$\mathbb{Z}_2^4$	9	5
#11	$\mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	7	4
#12	$\mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	7	4
#13	$\mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	5	2
#14	$\mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	6	3
#15	$\mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	5	2
#16	$\mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	5	2
#17	$\mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	5	2

(to be continued...)

(continuing....)

Group $\#_{32}$	$\Sigma^* := G'$	$\Gamma := G/G'$	$log_2\vert H^2(G,\mathbb{Z}_2)\vert$	$\leq log_2\vert H^2(G,\Bbbk^\times)\vert$
#18	$\mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_4$	5	3
#19	$\mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_4$	3	1
#20	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_8$	4	2
#21	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_8$	4	2
#22	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_8$	2	0
#33	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	9	6
#34	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	9	6
#35	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	8	5
#36	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	9	6
#37	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	7	4
#38	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	8	5
#39	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	8	5
#40	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	7	4
#41	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	7	4
#42	$\mathbb{Z}_2$	$\mathbb{Z}_2^4$	9	5
#43	$\mathbb{Z}_2$	$\mathbb{Z}_2^4$	9	5

These information is used together with Matsumotos exact sequence (section 5.2) for one to find faithful Doi twists of orbifolds (section 7.1) or to exhaust all possible actions of Σ by such twists:

- $\#_{16}$ $9, 10$ in section 7.2
- $\#_{32}$ 18 in section 7.3
- $\#33 - \#41$ in section 7.4

Outlook: 3 Conjectural Steps To All Nilpotent Groups

The above techniques construct link-indecomposable finite-dimensional Nichols algebras over many nilpotent groups of class 2. As conclusion of the second part of this thesis, the author would like to point out some open questions, that describe consequent pursues of the above techniques towards a total classification of minimally indecomposable finite-dimensional Nichols algebras over the nilpotent groups and are part of the author's ongoing effort:

1. *Negation Of All Noncommuting Rank 3 Cases*

First we want to discuss an exploit of the rather general argument in the preceding section **beyond exponent 4**. One could aim to prove, that $G' \cong \mathbb{Z}_2^n$ is (mostly) generated by mutually commuting subsets of nodes with support N_i of the type $N_i' \cong \mathbb{Z}_p$; the Dynkin diagram would be disconnected and the Nichols algebra **factorize** (see below)! Since Theorem 6.1 indeed constructs Nichols algebras for such groups, this would **clarify existence over all nilpotent groups of class 2**.

Example. *The group $G = \mathbb{D}_4 \times \mathbb{D}_4$ has $G' \cong \mathbb{Z}_2^2$, but admits Nichols algebras over the mutually commuting subgroups $\mathbb{D}_4$ and the overall Dynkin diagram is $A_2 \cup A_2$.*

Problem 8.1. *As in the above rank 3 examples, consider all stem extensions $\mathbb{Z}_2^2 \to G \to \Gamma$ with Γ abelian of rank 3, but in contrast possibly $\Gamma \ncong \mathbb{Z}_2^n$. We want to prove, that only exceptional examples may appear, that could allow finite-dimensional indecomposable Nichols algebras.*

- *Clarify whether in most cases of Weyl equivalence class of conjugacy configurations as in section 8.1 there exists some Yetter-Drinfel'd module with $r + log_2|Im(\gamma)| \geq 6$ using respective knowledge of the cohomology and give a complete list of all* **exceptionals***, which do not contain such a configuration!*

- *With* [**HS08**] *Theorem 8.6 (double coset statement) then prove, that any triple of mutually discommuting elements is of the form above (only two independent commutators) and hence discarded!*
- *Deduce, that apart from exceptional (low rank?) configuration any Nichols algebra with $G' \cong \mathbb{Z}_p^n$* **factorizes** *into indecomposable Nichols algebras $B(G_i)$, in the sense that*
  - *The Dynkin diagram is* **disconnected**
  - *Hence multiplication in $\mathcal{B}(G)$ induces a bijective map:*

$$\mathcal{B}(G) \cong \bigotimes_i \mathcal{B}(G_i)$$

  - *By theorem 6.8 the conjugacy classes underlying each connected components nodes generate* **mutually commuting** *supports with $G \supset G_i' \cong \mathbb{Z}_2$*

2. *Classifying All Nichols Algebras Over $G' = \mathbb{Z}_2$*

Also, we do not settle with just knowing **whether** a group admits Nichols algebras, but rather aim at **classifying** all minimally indecomposable finite-dimensional Nichols algebra (with possibly G' acting nontrivial!) as Doi twist of orbifold. This suggests an extension of the techniques used for about $\mathbb{D}_4, \mathbb{Q}_8$ (section 5.3) and the groups #9, 10 of order 16 and #18 of order 32 (sections 7.2 and 7.3):

Problem 8.2. *Suppose some Dynkin diagrams of $\mathcal{B}(M)$ with all nodes $\mathcal{O}_{g_k}^{\chi_k}$* **colored white or black** *depending on whether $G' \cong \mathbb{Z}_2$ acts as $+1$ or -1. The unique nontrivial Doi twist hence inverts black and white coloration. Both* **subdiagrams by themselves** *(and after Doi twisting the latter) present each a proper orbifold (G' acting trivial) and hence appear in Theorem 6.1. They especially have as Dynkin diagram*

$$A_{n\geq 3}, D_{n\geq 4}, E_{6,7,8}, F_4, B_{n\geq 3}$$

- *Clarify whether in most cases,* **if** *two black and two white elements are all mutually commuting, forming an independent set in the graph, then the generated abelian subrack over G! (this technique was introduced in* [**AZ07**]*) contains a 4-cycle and hence had infinite dimension!*

- *Deduce that hence in any configurations associated to a finite-dimensional orbifold, only black* A_1 *or* A_2 *(at most one independent node) may be coupled with one of the Dynkin diagrams of the list above in white (or by Doi twist vice-versa in white/black)!*
- *Clarify, whether in most cases* **if** *one black and* $n \geq 4$ *white elements are not mutually commuting forming an* n*-branch in the graph, then the generated abelian subrack (over G!) contains an* n*-branch and hence had infinite dimension!*
- *Clarify, whether in most cases,* **if** *one black and three white elements are not mutually commuting, forming a* D_4 *with white center, then the reflection on this center generates a 3-cycle in* G *and hence cannot occur over 2-groups by Heckenberger's list!*
- *Deduce a general statement (probably including several exceptions from above), linking the graphs to* **simply-laced affine Dynkin diagrams**, *with the black nodes precisely the additonal nodes from affinization.*
- *Try to construct examples or contradict such indecomposable finite-dimensional Nichols algebras associated to affine Dynkin diagrams and with nontrivial action* G' *prescribed as above precisely on the additional affinization nodes.*

3. *Nichols Algebras Over Nilpotent Groups Of Class* ≥ 3

Finally we need to target nilpotent groups of class ≥ 3. While the case > 3 seems to be discarded rather easily by extending a result of Schneider and Heckenberger regarding the rank 2 case in general groups [**HS10**], in class 3 there is a class of more resilient groups corresponding to the case $\mathcal{G}_4$ in *cit. loc.*. **If** there were an orbifold over this group, it had to be from a certain non-minimally indecomposable Nichols algebra over $\mathbb{D}_4$ "found" in the preceding section. The author has no opinion, what the Dynkin diagram might be, or whether this yields new cases or can be negated by a more skillful approach!

The author's study of the case of general nilpotent group G started with the following observations about Nichols algebras $\mathcal{B}(\mathcal{O}_s^\chi \oplus \mathcal{O}_t^\rho)$ of rank 2 of finite dimension, that are strong consequences of [**HS08**] in the nilpotent case and have implications on the structure of G, if an indecomposable finite-dimensional $\mathcal{B}(\bigoplus_i \mathcal{O}_{g_i}^{\chi_i})$ exists:

- The general $stst = tsts$ implies for $g := [s,t]$ that ${}^{st}g = g^{-1}$
- However, a **nilpotent group** is direct product of Sylow-subgroups $G(p)$. Consider $\bar{s}, \bar{t}$ in the largest quotient of odd order $G/G(2)$; since there squaring is invertible, we get from the condition on the other hand $\bar{s}\bar{t} = \bar{t}\bar{s}$.
- Suppose now there exists an indecomposable $\bigoplus_i \mathcal{O}_{g_i}^{\chi_i}$ where any two discommuting $s = g_i$ and $t = g_j$ commute as demanded above, then $\langle\{[\bar{s}_i]\}\rangle$ generate also $G/G(2)$, but there they all commute. Hence $G/G(2)$ is abelian.

The behaviour of the rank 2 Nichols algebras $\mathcal{B}(\mathcal{O}_s^\chi \oplus \mathcal{O}_t^\rho)$ in question of course depends greatly on the order of $g = [s,t]$. Suppose the nilpotency class of G to be just 2 (=commutators are central), then our first observation shows that always $g^2 = 1$ meaning G' is 2-elementary-abelian. In this work, we construct large examples of Nichols over such groups and clarify conversely existence of minimally indecomposable finite-dimensional Nichols algebras (even connected) at least for $G' \cong \mathbb{Z}_2$.

The author had also put considerable effort into clarifying the other cases $g^{2^n} = 1$ with ${}^{(st)}g = g^{-1} \neq g$. Before deciding the existence of an orbifold (in one class lower), one has to exclude possible examples with nontrivial g-actions, that are no Doi twist of trivial action; although the author was able to derive certain conditions, the particular lack of cohomology for some groups has prevented to derive the aimed conclusion.

Example 8.3. *Particularly resisent was the following case of the* **quasidihedral group** $\tilde{\mathbb{D}}_8$, *as the author announced in a mini-talk given at the "Oberwolfach Conference" 2010.*

$$\tilde{\mathbb{D}}_8 := \langle a, b \mid b^2 = a^8 = 1,\ ba = a^3 b\rangle$$
$$\tilde{M} := \mathcal{O}_a \oplus \mathcal{O}_{a^2b} \qquad \Sigma \ni g^2 = [a, a^2b]^2 = a^4$$

In contrast to $\mathbb{Q}_{16}$, *only one* $\mathcal{O}_a$ *powers to* g^2, *which proves* g^2 *to act trivial only there; this are* **few relations** *(compare section 5.3. On the other hand, in contrast to* $\mathbb{D}_8$, *there is* **not enough cohomology** *to generate the remaining case (*$g^2.x_{a^2b} = -x_{a^2b}$*) as a Doi twist.*

Towards the end of this dissertation, Schneider suggested in this context the recent paper [**HS10**]. Heckenberger and himself had proven strong implications for the structure of such groups in the general case $G' = \mathbb{Z}_n$ for $\mathcal{B}(\mathcal{O}_s^\chi \oplus \mathcal{O}_t^\rho)$ finite-dimensional. More specific, they proved that the support $\langle \mathcal{O}_s, \mathcal{O}_t \rangle$ for $\mathcal{O}_s, \mathcal{O}_t$ discommuting ("exceptional pair") has to be a quotient of one of the following three infinite groups:

$$\mathcal{G}_2 := \langle s, t, g \mid {}^s g = g \; {}^t g = g \rangle$$
$$\mathcal{G}_3 := \langle s, t, g \mid {}^s g = g^{-1} \; {}^t g = g \; g^3 = 1 \rangle$$
$$\mathcal{G}_4 := \langle s, t, g \mid {}^s g = g^{-1} \; {}^t g = g \; g^4 = 1 \rangle$$

Note that no nonabelian quotient of $\mathcal{G}_3$ may be nilpotent!

Problem 8.4. Decide Class 3**:** *(such as the above groups of order* 16*). If such a Nichols algebra were an orbifold by* $\Sigma = \langle g^2 \rangle$ *of an example with* $\bar{g}^2 = 1$ *in* $\Gamma = G/\Sigma$*, the smaller Nichols algebra (by some calculations) of a non-minimally indecomposable Nichols algebra with both trivial and non-trivial* Σ^**-action over* $\mathbb{D}_4$*, hence cannot be generally deorbifoldized further to the abelian case. The necessary edges between discommuting nodes severely restricts the possible diagrams.*

A good guess might be $A_3 \cup A_3 \to C_2$ *between conjugacy classes of orders* 2, 4 *(for* $q = -1$ *of dimension* 2^{12}*), which could even lead to a family* $D_n \cup D_n \to C_{n-1}$ *over central products with extraspecial groups.*

A second possibility would be $A_3^{(1)}$ *over conjugacy classes of both order* 4*. We would require the discussions of the preceding section to decide it's existence.*

Problem 8.5. Discard Class > 3**:** *By the result quoted above, there may not be any rank* 2 *Nichols algebra with support already class* > 3*; namely* $\mathcal{G}_2, \mathcal{G}_4$ *have class* $2, 3$*. Hence the only possibility were a situation* $\mathcal{O}_a \oplus \mathcal{O}_b \oplus \mathcal{O}_c$ *with* $\langle a, b \rangle, \langle a, c \rangle, \langle b, c \rangle$ *of class* ≤ 3*, such that e.g.* $g \in [a, b]^2$ *(which is central in* $\langle a, b \rangle$*) is nontrivially conjugated upon by* c*. We now sketch an argument, that requires the knowledge of the Dynkin diagram in the class* 3 *case above: By possibly using a Weyl reflection on* $\mathcal{O}_a$ *we may suppose* $b^2 = [a, b]$*, but then* $\langle b, c \rangle$ *could not have been of class* 3*, because* $b^4 = g$ *is not central in* $\langle b, c \rangle$*.*

Part 3

Orbifoldizing Automorphisms

Basic Concepts:

We start by reviewing very roughly the classification of finite simple groups including the fundamental induction along the centralizers of involutions and the characterization via BN-pairs. The former is pursued in the Monster construction in part 5, which relies on a series of remarkably unique structures. The latter we shall establish in this part for the automorphism group of a Hopf algebra in the aim of identifying large sporadic groups and especially the monster group $\mathbb{M}$.

The Classification Of Simple Groups

A remarkable beauty about the long-term efforts to completely classify all finite simple groups was, that construction and classification converged towards the end, and new simple groups were discovered along the way - some with virtually no other description (especially the pariahs). Often, such a case exhibited already in a very early stage a chain of "coincidences", that strongly suggest a simple group, and hence in many cases group order, character tables etc. was known many years before the construction could be completed. Note that actually constructing them or excluding other cases is very tedious and requires numerous ingenious concepts (such as signalizer method or local analysis) we may not attempt to present here. We will however in the succeeding section give a deeper introduction into the construction of the simple Monster group, that will be relevant to the following discussions.

Of all simple groups, the most generic are perhaps the Lie groups over finite fields. This is also, where the notion of BN-pairs emerges. This is a pair of subgroups with very specific properties, that axiomatize the notions of Borel part, Weyl group, etc. and hence to some extend transfer classical Lie theory into group theory. They even characterize the simple group (this is a hard theorem, see below).

It is the conviction of the author, that proving statements about a simple group (without computer-aid), should walk along these lines to the group – or make use of the symmetry construction principles, that often follow them as well. We want to confront the reader with the entire list before going into details for the groups in question:

Theorem 8.6. [(2)] *Each nonabelian finite simple group G is isomorphic to either a group of* **Lie type** [(3)] *over a finite field $\mathbb{F}_q = \mathbb{F}_{p^n}$*

$A_{n\geq 1}(q)\quad B_{n\geq 2}(q)\quad C_{n\geq 3}(q)\quad D_{n\geq 4}(q)\quad E_6(q), E_7(q), E_8(q), G_2(q), F_4(q)$

or of **twisted Lie type** [(4)]*, i.e. a simple subgroup ${}^kM_n(q) \subset M_n(q^k)$, roughly fixed by a Dynkin diagram- and field automorphism (see below)*

${}^2A_n(q)\quad {}^2D_n(q)\quad {}^2E_6(q)\quad {}^3D_4(q)\quad {}^2B_2(2^{2n+1})\quad {}^2F_4(2^{2n+1})\quad {}^2G_2(3^{2n+1})$

or to an **alternating group** $\mathbb{A}_{n\geq 5}$ *with exceptional isomorphisms to small Lie type groups* [(5)] *or to one of the* 26 **sporadic groups***:*

Groups	Common Name	Construction Principle
$M_{24}, M_{23}, M_{22}, M_{12}, M_{11}$	*Mathieu groups*	**Golay Code** + *stabilizers (see below)*
$Co_1, Co_2, Co_3, McL, HS$	*Conway groups*	**Leech lattice** + *stabilizers* [(5)]
Suz, J_2		+ *centralizer of Suzuki chain* [(6)]
$Fi'_{24}, Fi_{23}, Fi_{22}$	*Fischer groups*	*3-Transposition graph of an involution* [(7)]
$\mathbb{M}, B, Th, HN, He$	**Monster group**	**Griess algebra** + $Cent(2A, 3C, 5A, 7A)$ [(8)]
		or from $2B$**-centralizer** $Co_1 \ltimes 2_+^{24+1}$
$J_1, J_3, Ly, Ru, ON, J_4$	*Pariahs*	*from prescribed 2-centralizers*
		$\mathbb{A}_5,\ \mathbb{A}_5,\ \mathbb{A}_{11},\ {}^2B_2(2^3),\ A_2(2^2),\ M_{22}$

(2) For classification/notation see the ATLAS, details are largely from an ICTS-lecture: http://www.icts.res.in/media/uploads/Old_Talks_Lectures/Document/1285648885MuellerWshopGAC2010.pdf

(3) $A_1(2), A_1(3)$ are solvable, $B_2(2)', G_2(2)'$ are the simple index-2 commutator subgroup of their respective actual Lie type group. Exceptional isomorphisms occur.

(4) ${}^2A_2(2^2), {}^2B_2(2)$ solvable and ${}^2F_4(2)', {}^2G_2(3)'$ have index $2, 3$ (as above)

(5) $\mathbb{A}_5 \cong A_1(2^2), A_1(5) \qquad \mathbb{A}_6 \cong A_1(3), B_2(2)' \qquad \mathbb{A}_8 \cong A_3(2)$

(5) of Leech vectors of length 4 resp. 6 resp. a 2/2/3-triangle resp. a 100-graph

(6) $Cent(3D) \cong 3 \times \mathbb{A}_9 \supset \mathbb{A}_8 \supset \mathbb{A}_7 \supset \mathbb{A}_6 \supset \mathbb{A}_5 \supset \mathbb{A}_4 \supset \mathbb{A}_3 \supset \mathbb{A}_2$ leading to to centralizers $\mathbb{S}_3 \subset \mathbb{S}_4 \subset A_2(2) \subset PSU_3(3) \subset J_2 \subset G_2(4) \subset 3.Suz \subset Co_1$

(7) Possesses a conjugacy class of an involution with products $ord(g_i g_j) \leq 3$

(8) centralizers of M-conjugacy classes in ATLAS-notation

Groups of Lie type [(1)] were historically (apart from the alternating groups) the first groups with established simplicity namely $A_1(p) = PSL_2(p)$ in works of Galois (1832) and $A_n(p)$ by Jordan (1870). The trend continued and soon simplicity of other classical Lie groups over $\mathbb{F}_p$ was established. Dickson took the step to consider arbitrary finite fields (1901) and found the first exceptional case $G_2(q)$ in 1905. It took until 1955 that Chevalley gave an elegant uniform construction of all (untwisted) groups of Lie type, by giving roughly an integral presentation of the semisimple Lie algebra envelopings, which allows considering them over finite fields.

Groups of twisted Lie type [(1)] were subsequently found by Steinberg (1959) in a similar way one constructs unitary groups over the complex numbers. Let σ be a field automorphism $\mathbb{F}_{q^k}/\mathbb{F}_q$ and τ an outer automorphism of a semisimple Lie algebra X_n (and hence the Dynkin diagram), than one considers the fixed subgroup of $\sigma\tau$ inside the Lie-type group ${}^kX_n(q) \subset X_n(q^k)$, that turns again out to be (close to) simple. Suzuki and Ree (1960, 1961) discovered further series', which come from the curious fact, that the diagrams B_2, F_4, G_2 have additional automorphisms over characteristic $2, 2$ resp. 3, where the multiple-edge orientation ("arrow“) can be ignored.

We proceed by reviewing [(2)] the **induction step of the classification** for simple groups. In 1963 Feit and Thompson had proven that every odd order group is solvable, hence every nonabelian simple G has to be of even order and contains thus an involution. Brauer had started to classify simple groups in terms of the **centralizers of such an involution** ($\cong GL_2(\mathbb{F}_q)$ in 1954). As this group is smaller than G it allows an inductive approach, checking any known group as possible such 2-centralizer. This philosophy has been the driving force behind the classification effort and explains the tremendous length of the proof (over 10.000 pages). Luckily it has turned out, that usually a 2-centralizer of a simple group is very close to a smaller simple group.

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(1) The historical remarks are taken from Wikipedia.

(2) A short version of this has been posted by the author to answer the respective MathOverflow-Question http://mathoverflow.net/ questions/17617/why-are-the-sporadic-simple-groups-huge/93459#93459

Example 8.7. *Janko (1965) investigated possible G with 2-centralizer $2 \times A_1(q)$ $q \geq 3$ and found, that either $q = 3^n$ and (later) $G \cong {}^2G_2(3^{2n+1})$, which was his prototype, or $q = 4, 5$ ($A_1(q) \cong \mathbb{A}_5$) and $G \cong J_1$ is the new simple* **Janko group**, *the first in almost a century.*

Example 8.8. *The* **McLaughlin group** *McL (1969) is derived from Co_1 as stabilizer of a $2/2/3$ triangle in the Leech lattice. A 2-centralizer is the Schur cover $2.\mathbb{A}_8$. The subsequent search for any $2.\mathbb{A}_n$ ruled out all other cases except $n = 11$ leading to the* **Lyons group** *Ly (1972).*

Example 8.9. *Quite ironically, the* **most resistant cases** *were 2-centralizers resembling those of Lie type groups, distinguished by characteristic and rank. While odd characteristics were already dealt with by Aschbacher ("Classical involution theorem", 1977), the even characteristic case was proven first for rank $1, 2$ ("Thin groups", Aschbacher, 1978), for rank ≥ 4 ("Trichotomy theorem", Gorenstein & Lyons, 1983) and ultimately for rank 3 in* **"Quasithin groups", Aschbacher & Smith, 2004**. *The last proof took incredible 20 years, 1221 pages and finally concluded the classification theorem.*

While these (and almost all other) examples stop already one induction step beyond the Lie case, the following groups represent a **very remarkable** chain of induction steps. It will be discussed in more depth in the introduction of part 5.

$$A_2(2^2) \longrightarrow M_{24} \longrightarrow Co_1 \longrightarrow \mathbb{M}$$

Simple Groups And Their BN-Pairs

The following observation of a very specific constellation of subgroups B, N places the groups of (twisted) Lie type in a first classificatory context, proves their simplicity and appears (with modifications) in the treatment of sporadic simple groups as the monster (see part 5).

Definition 8.10. *A* **BN-pair** *(J. Tits 1964) or* **Tits system** *of a finite group G is a pair of subgroups $B, N \subset G$ generating G such that*

- *$D = B \cap N$ is normal in N.*
- *$W = N/D$ is a Coxeter group generated by involutions w_i.*
- *No w_i normalizes B.*
- *For any such w_i and $w \in W$ we have $w_iBw \subset Bw_iwB \cup BwB$ (a sort of triangle inequality).*

Example 8.11. *For $A_n(q) \cong PSL_{n+1}(\mathbb{F}_q)$ one takes for B the subgroup of upper triangular matrices (mod scalars $\mathbb{F}_q^\times$) and N the monomial matrices (one entry per column and row). Then D are the diagonal matrices and $W \cong \mathbb{S}_n$. For this reason, we call B* **Borel subgroup**, *D* **Cartan subgroup** *and W* **Weyl group** *with the number of generators w_i the* **rank**.

All groups of (twisted) Lie type admit similarly a BN-pair and under certain conditions (B solvable, $\bigcap_g B^g = \{1\}$, W indecomposable) the simplicity of G follows from G being perfect; the latter is what requires mild modifications in some small examples, so e.g. the "proper" simple commutator subgroup ${}^2F_4(2)'$ has index 2 in ${}^2F_4(2)$.

In 1974 Tits showed, that for rank ≥ 3 in fact **all** groups with BN-pair are of Lie type! He associated to every such pair an abstract simplicial complex with G-symmetry the **Tits building** [**L05**]. We will work with the more involved definition in *cit. loc.* in what follows, but for convenience we include now an elementary definition from Wikipedia:

Definition 8.12. *A* **Tits building** *of rank $n+1$ is an abstract simplicial complex (call the n-simplices* **chambers***) which is the union of certain subcomplexes (***apartments***), such thats*

- *any two chambers lay in exactly one common apartment.*
- *any $n-1$-simplex in some apartments A lays in exactly two adjacent chambers in A and the thereby defined graph of chambers is connected.*

- *for any two chambers C, C' laying both in two apartments A, A' there is a simplicial isomorphisms $A \to A'$ fixing C, C'.*
- *the building is called* **thick**, *if every k-simplex $k \leq n$ lays within at least three chambers.*

A group G acting transitively on pairs $C \in A$ of a building has a BN-pair via $B = Cent(C_0)$ and $N = Cent(A_0)$ for some fixed choice. For finite, thick buildings the converse also holds.

Example 8.13. *The alternating groups $\mathbb{A}_n$ possess BN-pairs as well, coming from the doubly transitive action on $|\Omega| = n$: Let $x \neq y \in \Omega$ and take $B \cong \mathbb{A}_{n-1}$ the stabilizer of x and N the stabilizer of the subset $\{x, y\}$; then $D \cong \mathbb{A}_{n-2}$ is the stabilizer of both x, y and $W \cong \mathbb{Z}_2$ exchanging x, y. Thus in contrast to the above, they are of rank 1.*

For these **thin** buildings [(3)] one introduces **split** BN**-pairs**, meaning additionally $B \cong H \rtimes U$ for U nilpotent. In this case the theorem of Fong-Seitz classifies rank 2 as $A_2, B_2, {}^2A_4, {}^2A_5, G_2, {}^3D_4, {}^3F_4$. A similar result exists for rank 1.

This story also partly continues for alternating and sporadic simple groups and the building geometry and -combinatorics are studied with great profit; there is a great number of generalization targeting these. They also can be used e.g. to compute the cohomology rings (by Quillen's map). Compare finally the following to $\mathbb{A}_n$:

Example 8.14. *The monster group $\mathbb{M}$ will be constructed via a non-proper BN-pair (see part 5), where $B = Cent(z_1)$ and $N = Norm(\langle z_1, z_2 \rangle)$ for z_1, z_2 suitable commuting involutions. We have $\tilde{D} = Cent(\langle z_1, z_2 \rangle)$ normal in N with Weyl group $\mathbb{S}_3$, but the actual $D = B \cap N$ is slightly larger (permuting $z_2, z_2 z_1$) and non-normal.*

The latter two will be prototypes for our recovery of a BN-pair of an orbifolds automorphism group in what follows.

[(3)] The author thanks Prof. Humphreys and Prof. Pasechnik for pointing out literature on the stricter notion of a "split" BN-pair in low rank and for laying out the weaker amalgam construction for sporadic simple groups upon my question in MathOverflow (http://mathoverflow.net /questions/93463/weak-bn-pair-tits-system-for-sporadic-groups).

CHAPTER 9

The Automorphism Group Of An Orbifold

We want to establish the existence of a BN-pair for the group of Hopf algebra automorphisms of an orbifold $Aut(\Omega)$. The approach resides in between the BN-pairs classically defined via the triangular and monomial subgroup of matrix groups, that have arbitrary high rank, and the 2-transitive groups with B, N the 1- resp. 2-point-stabilizer (rank 1, see above!). In some sense, we again replace points in the latter approach by algebras, which allows far more “space” for higher-rank specimen, while on the other hand the necessary rigidity is kept (as in $\mathbb{A}_n$ vs. $\mathbb{M}$):

2 points normalized + 1 point centralized = 2 points centralized

The actual proof is carried out as usual via Tits-buildings [**L05**]. These are simplicial complexes with additional structure and a symmetry action of the group to be studied. The role of the **chamber** system is hereby taken by the orbit of embeddings $H = A(e) \subset \Omega$ under $Aut(\Omega)$, while the set of **apartments** is the orbit of a fixed intermediate $\mathbb{Z}_2$-orbifold embeddings $L \subset \Omega$. We will give an alternative characterization in terms of just the involved central idempotents of Ω.

1. Two Subgroups $B, N \subset Aut(\Omega)$

Let $\mathbb{k}^\Sigma \to \Omega \to H$ be an orbifold as constructed in Theorem 1.6. Depending on a fixed chosen involution $p \in \Sigma$ we shall define a set $\mathcal{C}$ (“chambers”) and a set of subsets thereof $\mathcal{A}$ (“apartments”):

Definition 9.1. *The group $G = Aut(\Omega)$ of Hopf algebra automorphisms Ω acts naturally on the set of multiplicative (NOT necessarily unit-preserving) embeddings $s_H : H \to \Omega$. Define $\mathcal{C}_{Aut}$ to be the orbit of the fixed chosen embedding $s_{H,0} : H = A(e) \subset \Omega$. Then, identify embeddings $[s] = [s']$ with equal images $s(H) = s'(H) \subset \Omega$ to define $\mathcal{C}$.*

Definition 9.2. *Fix now any involution $p \in \Sigma$ and consider the orbifold $L := H \oplus H_p$. Then G acts also naturally on the set of multiplicative embeddings $s_L : L \to \Omega$. Define $\mathcal{A}_{Aut}$ to be the orbit of the embedding $s_{L,0} : L \subset \Omega$ above induced by the chosen p. Again, identify embeddings $[s] = [s']$ with equal images $s(L) = s'(L) \subset \Omega$ to define $\mathcal{A}$.*

Note that $\mathcal{C}_{Aut}, \mathcal{A}_{Aut}$ are extensions of $\mathcal{C}, \mathcal{A}$ by all H, L-automorphisms induced from Ω. We next identify $\mathcal{C}, \mathcal{A}$ with certain **central idempotents** in Ω.

Lemma 9.3. *The evaluation of an embedding at 1_H resp. 1_L induces a bijection of $\mathcal{C}, \mathcal{A}$ with the orbits $\mathcal{C}_{Idem}, \mathcal{A}_{Idem}$ of the central idempotents $s_{H,0}(1_H) = 1_e$ resp. $s_{L,0}(1_L) = 1_e + 1_p$ under $Aut(\Omega)$.*

PROOF. The given elements clearly are central idempotents in Ω, so are all automorphic images. Hence evaluation at $1_H, 1_L$ yields an epimorphisms of permutation representation $\mathcal{C}_{Aut} \to \mathcal{C}_{Idem}$ resp. $\mathcal{A}_{Aut} \to \mathcal{A}_{Idem}$. As the idempotent already defines the entire image $s(H) = s(1_H)\Omega$ resp. $s(H) = s(1_H)\Omega$ the above epimorphisms identify embeddings with equal image in Ω, hence it factors through $\mathcal{C}, \mathcal{A}$ to an isomorphism.

□

Remark 9.4. *We view an element $[s_L] \in \mathcal{A}$ as* **subset** *of $\mathcal{C}$ by defining $[s_H] \in [s_L]$ whenever the image is contained $s_H(H) \subset s_L(L) \subset \Omega$. This corresponds to the obvious view of a central idempotent $x \in \mathcal{C}_{Idem}$ being contained in another one $y \in \mathcal{A}_{Idem}$ whenever $x \in y\Omega$. We freely use the terms $[s], s(1_H)\Omega, s(H)$ to describe an element in $\mathcal{C}$ ($\mathcal{A}$ respectively).*

Given this data, we define general subsets $B, N \subset G$ by taking those automorphisms preserving the respective structure:

Definition 9.5. *Let $B := Norm(H)$ and $N := Norm(L)$ or equivalently via idempotents $B = Cent(1_e)$ and $N = Cent(1_e + 1_p)$.*

Note that they also precisely normalize the orthogonal complements

$$B = Norm(\sum_{q \neq e} H_q) \qquad N = Norm(\sum_{q \neq e,p} H_q)$$

Hence B, N can alternatively be characterized as precisely those automorphisms, that factor through the orbifold quotient π_H, π_L to H, L.

Definition 9.6. *Take $\mathcal{C} \cap L$ the orbit of 1_e under N and define for this permutation representation ρ the image, i.e. the permutation group $N/Ker(\rho)$ as* **Weyl group** *W.*

Note that in contrast to proper BN-pairs, we so far have in general $B \cap N \supset Ker(\rho)$ not even normal! This is topic of the next section.

2. Conditions Establishing A Generic BN-Pair

Next we assume the following conditions, that ultimately prove B, N defined above to be a BN-pair for $G = Aut(\Omega)$. They amount to assuming W a Coxeter group acting on $\mathcal{C} \cap L$ as the regular representation, together with conditions ensuring a "sufficiently dense" covering of Ω by suborbifolds L along with "sufficient space" outside such an L.

(1) *Condition ensuring normality*: The Weyl group W acts on $\mathcal{C} \cap L$ **barely-transitive**, i.e. transitive with trivial one-point-stabilizers. Hence $\mathcal{C} \cap L$ is the regular W-representation .
(2) *Conditions enabling the definition of a building*:
    (a) The Weyl group W is a **Coxeter group** (we call the Coxetersystem) $S \subset W$.
    (b) Every pair of chambers g_1H, g_2H are elements (i.e. algebra subsets) of at least one **common apartment** g_3L.
(3) *Condition ensuring nondegeneracy* and establishing the B, N-pair: The building is **thick**, i.e. for any $w \in S$ not all $(gwg^{-1})H$ for $g \in B$ coincide. Equivalently this means, that no $w \in S$ normalizes B.

Already the first condition implies (the "rigidity" mentioned above):

Lemma 9.7. *$B \cap N$ is normal in N and the quotient is W.*

Proof. Any $g \in B \cap N$ by definitions centralizes $H, L \subset \Omega$. Consider now $\mathcal{C} \cap L$ with $N \ni g$ acting on it. As $g \in B$ centralizes H, by the barely-transitiveness condition above it already centralizes the entire set $\mathcal{C} \cap L$.

Now take some $h \in N$, then it normalizes $\mathcal{C} \cap L$ and hence hgh^{-1} again centralizes the entire set. But this especially means it stabilizes H once again and hence $hgh^{-1} \in B \cap N$. Thus $B \cap N$ is normal in N.

To identify the quotient, note that by the above $B \cap N$ is the kernel of the permutation representation ρ of N on $\mathcal{C} \cap L$. Hence $N/(B \cap N) \cong N/Ker(\rho) =: W$. $\square$

Each element of the Weyl group defines an equivalence relation on $\mathcal{C}$. Especially for each w the Coxeter system S we consider:

Definition 9.8. *Two elements $x, y \in \mathcal{C} \cap L$ are called w-**equivalent**, if some preimage in $w(B \cap N) \subset N$ maps $x \mapsto y$. Every $A \in \mathcal{A}$ is by definition an automorphic image of the standard apartment $A = gL$, thus we may also define the translated equivalence relation on $\mathcal{C} \cap A$. Finally we define w-equivalence on all $\mathcal{C}$ as the transitive closure. The equivalence class of some $x \in \mathcal{C}$ shall be denoted $[x]_w$, the w-**residue**.*

We show two additional rigidity statements:

Lemma 9.9.

- *Let $A, B \in \mathcal{A}$ and $x, y \in \mathcal{C}$ with $x, y \in A \cap B$, then there is an automorphisms $g \in G$ with $gA = B$ and centralizing $\{x, y\}$.*
- *Let $A, B \in \mathcal{A}$ and $x \in \mathcal{C}$ with $x \in A \cap B$ and $[y]_w$ some residue, then there is an automorphisms $g \in G$ with $A = B$, stabilizing x and sending $g(A \cap [y]_w) = B \cap [y]_w$.*

PROOF.

(1) By definition of $\mathcal{A}$ as orbit there are automorphisms $g_A A = L, g_B B = L$. Consider for $x \in A \cap B$ the elements $x_A = g_A x, x_B = g_B x \in L$. Because W acts transitive on $\mathcal{C} \cap L$ we may find a $g_L \in N$ with $g x_A = x_B$ and because it acts barely-transitive, this already fixes analogously $g y_A = y_A$. Hence we obtain $g := g_B^{-1} g_N g_A$ with the asserted properties.

(2) As in the first part of the proof, we easily find $g := g_B^{-1} g_N g_A$ mapping A to B and fixing x, y. It is clear from the definition (N normalizes all $L \cap [y]_w$), that then already the entire residue of y is preserved.

$\square$

This already defines enough structures on the set $\mathcal{C}$ and the subsets $\mathcal{A}$ thereof to endow the pair with the structure of a **building** with **chambers** $\mathcal{C}$ forming a **chamber system** via the equivalence relations from each $w \in S \subset W$:

Theorem 9.10. *Under conditions 1+2 the structure* $\mathcal{B} = (\mathcal{C}, \mathcal{A}, []_w)$ *defines a* **building** *by Theorem 3.6 in* [**L05**]*, because it satisfies all conditions given there:*

- *Each apartment is isometric to the Coxeter complex of* G*: This is clear, as we already noticed, that* $\mathcal{C} \cap L$ *is isomorphic to the regular representation of* W*, which we required to be a Coxeter group with Coxeter system* S *in condition 2.*
- *Each pair of chambers lay in a common apartment: This was the second part of condition 2 above.*
- *Each pair of chambers in the intersection of two apartments* A, B*, allows an isometry* $A \to B$ *fixing* x, y*: Take the automorphisms* g *found in the last theorem.*
- *Each chamber in the intersection of two apartments* AmB *and* w*-residue* R *allows an isometry* w *with* $gA = B, g(A \cap R) = A \cap R, gx = x$*: Take again the automorphism above.*

Note that an **isometry** *means a permutation of* $\mathcal{C}$*, such that the set of apartments and each equivalence-relation/residue are preserved, which is especially true for any automorphisms* $g \in G = Aut(\Omega)$*!*

The building has a **strongly transitive** G**-action** (i.e. transitive on chamber/apartment pairs $x \in A$) as already established above and is **thick** if the additional condition 3 is fulfilled. Hence by theorem 4.5 cit. loc. the B, N above define a proper BN-pair.

Corollary 9.11. *Under the conditions 1+2+3 the subgroups* B, N *of* $G = Aut(\Omega)$ *define a* BN*-pair with Weyl group* W*.*

Remark 9.12. *We conclude by further observations and conjectures:*

- *Note from the proof in cit. loc. the* **Bruhat-decomposition** *into double-cosets* $G = BNB$*.*
- *What does the additional condition* $wBw^{-1} \cap B = 0$ *amount to? If this condition is true,* B *is solvable,* W *connected (as Coxeter graph) and* G *is perfect, then by a standard argument* G *is already* **simple***!*
- *When is* G *already the* **full isometry group** *of the building? The author would assume* $\Sigma \cong \mathbb{Z}_2^n$ *to suffice, as then any direct summand* H_σ *in* Ω *lays in an apartment.*

3. An Artificial Example $Aut(\Omega) \to \mathbb{S}_3 \ltimes 2_+^{12+1} \supset \mathbb{S}_4$

We conclude by giving an **artificial, disconnected example** going from a partial orbifold (i.e. by a subgroup of twisted symmetries), which has already nonabelian coradical and far less automorphisms then the primary Hopf algebra, to the full orbifold and find the expected growth of the automorphism groups. Especially we give an explicit "triality" element t (analogous to the Monster's triality elements?) mixing the secondary orbifoldizing's twisted symmetry with the primary ones, which is at the start of the secondary orbifoldization an ordinary group element (a central commutator).

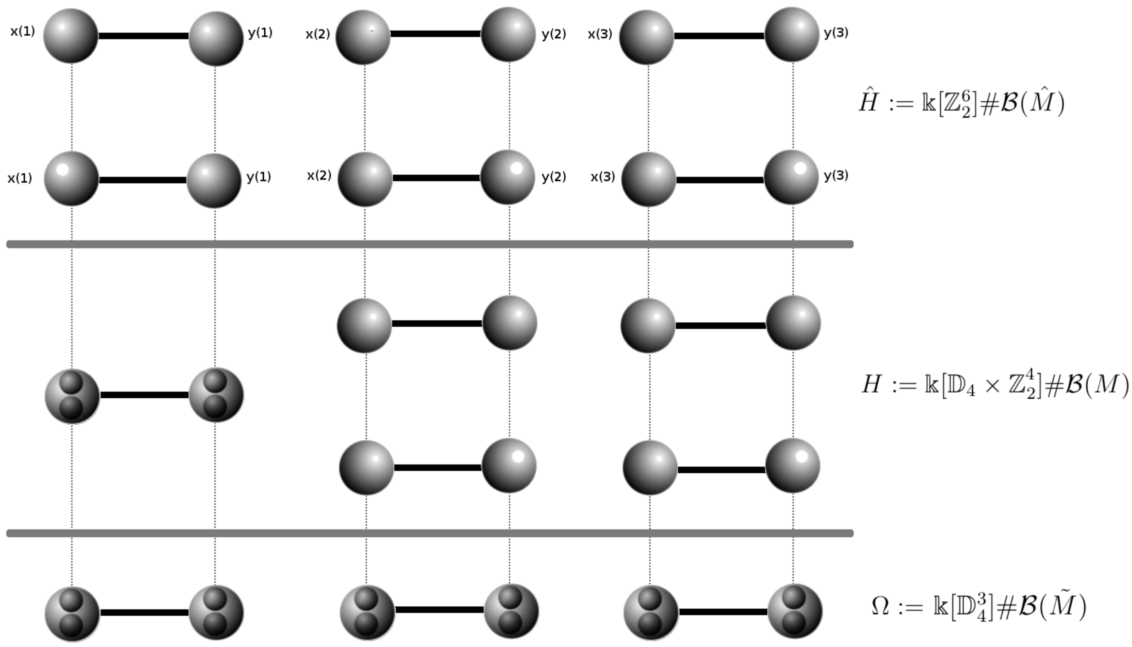
Without working out all details, we will find that with respect to the above construction yields (using $\mathbb{S}_n \wr G := \mathbb{S}_n \ltimes G^n$):

- $Aut(H) = \mathbb{S}_1 \wr (\mathbb{D}_4 \times (\Bbbk^\times)^2) \times \mathbb{D}_4 \wr (\mathbb{Z}_2 \ltimes (\Bbbk^\times)^2)$
- $B = (\mathbb{S}_1 \times \mathbb{S}_2) \wr (\mathbb{D}_4 \ltimes (\Bbbk^\times)^2)$
- $N = (\mathbb{S}_2 \times \mathbb{S}_1) \wr (\mathbb{D}_4 \ltimes (\Bbbk^\times)^2) \cong B$
- $B \cap N = (\mathbb{S}_1 \times \mathbb{S}_1 \times \mathbb{S}_1) \wr (\mathbb{D}_4 \ltimes (\Bbbk^\times)^2) \cong (\mathbb{D}_4 \ltimes (\Bbbk^\times)^2)^3$
- $N/(B \cap N) \cong \mathbb{S}_2$ is a rank-1 Coxeter group
- $Aut(\Omega) = \mathbb{S}_3 \wr (\mathbb{D}_4 \ltimes (\Bbbk^\times)^2) \to \mathbb{S}_3 \ltimes 2_+^{12+1}$
- A Weyl generator $n \in N$ and the product $t := bn$ can be viewed as **duality- and triality-element**, each element mixing twisted and untwisted sectors.

Note that the continuous automorphisms $\Bbbk^\times$ come from rescaling the nilpotent skew-primitives and may reduce to a finite group (e.g. $\mathbb{Z}_2 \subset \Bbbk^\times$) if a nontrivial lifting is present! As all of Σ acts trivial by construction, we may further fuse the $\mathbb{D}_4^3$ to an extraspecial group $\mathbb{D}_4^{*3} =: 2_+^{12+1}$. **The remaining interesting (quotient-)part** of $Aut(\Omega)$ arises from the automorphisms of the 2-groups underlying finite symplectic vector space ($G/(G^2G') \cong \mathbb{F}_2^{12}$), which are heavily restricted by additionally having to preserve the Dynkin diagram (not only) as a graph.

Example 9.13. *Start with the Yetter-Drinfel'd module $\hat{M}$ over $\hat{\Gamma} = \mathbb{Z}_2^6 = \langle x_1, y_2, \ldots, x_3, y_3 \rangle$-of type $A_2^{6\cup}$, two over each conjugacy class pair $\mathcal{O}_{x_i} \oplus \mathcal{O}_{y_i}$), each pair of A_2 interchanged by a twisted symmetry $\mathbb{Z}_2 \cong \Sigma_{1,2,3} \subset \Sigma \cong \mathbb{Z}_2^3$.*

We "partially" orbifoldize $\hat{M}$ by Σ_1 to M of type $A_2^{[U]} \cup A_2^{4\cup}$ over $\Gamma = \mathbb{D}_4 \times \mathbb{Z}_2^4$. On the other hand the subsequent full orbifold $\tilde{M}, \Omega$ by $\Sigma_{2,3} = \mathbb{Z}_2^2$ is over $G = \mathbb{D}_4^3$ and of type $\left(A_2^{[U]}\right)^{3\cup}$.



The Hopf algebra automorphisms of $H = \mathbb{k}[G]\#\mathcal{B}(\hat{M})$ and resp. of Ω and the subgroups B, N have a decomposition series as follows

$$1 \lhd Cent(\mathbb{k}[G]) \lhd Cent(Diagram) \lhd Norm(ConnectedComponents) \lhd Aut(H)$$

The respective quotients are:

- *$Cent(\mathbb{k}[G])/1$ are the automorphisms of the Nichols algebra (coradical fixed). For each connected component (type $A_2, A_2^{[U]}$) this is $(\mathbb{k}^\times)^2$ from rescaling the skew-primitives associated to the nodes. Hence these therms are overall $(\mathbb{k}^\times)^{10}, (\mathbb{k}^\times)^6$ for H, Ω respectively.*
- *$Cent(Diagram)/Cent(\mathbb{k}[G])$ are the group automorphisms fixing the Dynkin diagram. For A_2 this is trivial, but for $A_2^{[U]}$ we get additional $\mathbb{Z}_2 \times \mathbb{Z}_2$ corresponding to conjugating with the respective other involution (inner automorphism).*
- *$Norm(ConnectedComponents)/Cent(Diagram)$ are the group automorphisms flipping an $A_2, A_2^{[U]}$-copy, i.e. $\mathbb{Z}_2$ for each.*

- *So far this means for every connected component* $A_2, A_2^{[U]}$ *of the Dynkin a automorphisms group of* $\mathbb{Z}_2 \wr k^\times$ *respectively* $\mathbb{Z}_2 \wr (\mathbb{Z}_2 \times \Bbbk^\times) = \mathbb{D}_4 \ltimes (\Bbbk^\times)^2$.
- $Aut(H)/Norm(ConnectedComponents)$ *is the permutation action on the connected components. Hence we get*
  - $\mathbb{S}_1 \times \mathbb{D}_4$ *for* $Aut(H)$
  - $\mathbb{S}_3$ *for* $Aut(\Omega)$
  - $\mathbb{S}_1 \times \mathbb{S}_2 = \langle b \rangle$ *for* B
  - $\mathbb{S}_2 \times \mathbb{S}_1 = \langle n \rangle$ *for* N

Note that n *does not fix* H, *so it mixes twisted and untwisted sector (as below one could work the map out explicitly). An explicit additional* Ω*-automorphism* $t = bn$ *of order* 3 *interchanging all three copies of* A_2 *can be worked out from the assumed cyclic permutation action on* a_i^2. *Denoting* $a = a_1 \in \Omega$ *and take the basis of central idempotents* $1_{A(p)}$, *than*

$$(a^2, a^2, a^2, a^2) \xmapsto{t} (1, -1, 1, -1) \xmapsto{t} (1, 1, -1, -1)$$

Altogether (using multiplicativity of t*) we find the following matrix and notice that it mixes twisted and untwisted sectors:*

$$t = \frac{1}{2}\begin{pmatrix} 1+a^2 & 0 & 1-a^2 & 0 \\ 1+a^2 & 0 & -a^2 & 1 \\ 0 & a^2 & 1 & 1-a^2 \\ 0 & 1+a^2 & 0 & 1-a^2 \end{pmatrix}$$

Part 4

Orbifoldizing Categories

In 2010 the author was given a chance to present his ongoing work in a mini-talk at the Oberwolfach Workshop *Deformations in Mathematics and Physics*. At the very same workshop, Prof. Christoph Schweigert (Hamburg) gave a talk *Geometric and algebraic structures for general cross modules* involving also an orbifoldizing procedure for equivariant categories introduced by Kirillov [**K04**] (who attributes the work already to Turaev). It describes e.g. categorically the behaviour of vertex algebra orbifoldizing (untwisted and twisted vertex modules, see *cit. loc.*), which will be topic of part 5. In a subsequent discussion, the question came up, if and how the construction presented in this thesis has a connection to the existing one – the result is the following part.

The category of **comodules** over an orbifold is just a duplication of the respective category over the base Hopf algebra (as coalgebras $\Omega \cong \Bbbk[\Sigma^*] \otimes H$). On the other hand, the category of **modules** corresponds to projective representations of the original Hopf algebra, which was the reason for Schur to treat the group-case long ago and was also the motivation behind [**Bo97**].

Hence we can only expect more nontrivial behaviour for category notions, that depend both on algebra- and coalgebra-structure. Here indeed we find categorical orbifoldizing. In both cases we take the respective category over H as untwisted sector, extend it naturally to an Σ-equivariant category and get as orbifoldization the respective category over the Hopf algebra orbifold Ω:

- The category of **bicomodules**, especially the contained **Bigalois groupoid** ("Hopf algebra cohomology", see part 1) can be duplicated ($\times\Bbbk^\Sigma$) to the Σ-equivariant category and yields coinciding orbifolds. Note that this is a rather trivial equivariant category and it comes **without braiding**, which is the most distinguishing feature of orbifoldizing categories. It is however interesting to calculates the orbifold's Bigalois groupoid!
- The category of **Yetter-Drinfel'd modules** with inverse braiding c^{-1} can be extended by **projective** Yetter-Drinfel'd modules to the Σ-equivariant category and yields coinciding orbifolds (again with inverse braiding). Note that this requires to consider both categories. This is the more involved case.

Basic Concept: Equivariant Category Orbifoldization

The following notion from **[K04]** appears already in Turaev:

Definition 9.14. *For a finite group Σ, a Σ-***equivariant category** *is a category $\mathcal{C}$ with a given formal Σ-decomposition of full subcategories ("untwisted" and "j-twisted" sectors):*

$$\mathcal{C} = \bigoplus_{j\in\Sigma} \mathcal{C}_j$$

*together with a Σ-***action***, i.e. functors $R_p : \mathcal{C} \to \mathcal{C}$ and natural isomorphisms $\alpha_{j,k} : R_j \circ R_k \cong R_{jk}$ such that:*

$$R_1 = id \quad R_j(\mathcal{C}_k) \subset \mathcal{C}_{jkj^{-1}}$$

$$\alpha_{jk,l}\alpha_{j,k} = \alpha_{j,kl}\alpha_{k,l}$$

In what follows additionally assume a strict monoidal structure on $\mathcal{C}$.

Definition 9.15. *Such a Σ-equivariant monoidal category $\mathcal{C}$ is called* **equivariant fusion category***, if the action is monoidal, commutes with the biduality functor and there is a* **pre-braiding** *satisfying the pentagonal identity (for $V \in \mathcal{C}_j$):*

$$c_{V,W} : V \otimes W \to R_j(W) \otimes V$$

Especially the untwisted sector $\mathcal{C}_1$ is a braided category. To yield a braided category, we take the coherently covariant part:

Theorem 9.16. *The* **orbifoldization** *$\mathcal{C}//\Sigma$ of a Σ-equivariant category $\mathcal{C}$ is a braided category: Take objects $(V \in \mathcal{C}, (\phi_p)_{p\in\Sigma})$ with:*

$$\phi_j : \ R_j(V) \cong V \qquad \phi_j \circ R_j(\phi_k) = \phi_{jk}$$

and morphisms between (V, ϕ), (W, ψ) to be usual $\mathcal{C}$-morphism $f : V \to W$ satisfying $\psi_j \circ R_p(f) = f \circ \phi_p$ and as braiding (for $V \in \mathcal{C}_j$):

$$V \otimes W \xrightarrow{c_{V,W}} R_j(W) \otimes V \xrightarrow{\phi_j^W \otimes id} W \otimes V$$

CHAPTER 10

Bicomodules And The Bigalois Groupoid

1. An Equivariant Category (without braiding)

Let Σ be a twisting group [(1)] of Bigalois objects of a Hopf algebra H (Definitions 1.5 and 1.9), then we consider the **category of** H,H-**bicomodules** over H:

$$\mathcal{C} = \bigoplus_{j\in J} \mathcal{C}_i := \bigoplus_{j\in J} Bicomod(H,H) = Bicomod(H,H)^{|\Sigma|}$$

we obviously get a **monoidal** structure:

$$\square_H : \mathcal{C}_j \times \mathcal{C}_k \to \mathcal{C}_{jk}$$

To turn $\mathcal{C}$ into an Σ-**equivariant category**, consider

$$R_j : \mathcal{C}_k \to \mathcal{C}_{jkj^{-1}}$$

$$V \mapsto A(j)\square_H V \square_H A(j^{-1})$$

We easily verify that this is monoidal by taking the natural transformation $R_j(\_)\square_H R_j(\_) \cong R_j(\_\square_H\_)$ as follows:

$$\begin{aligned} R_j(V)\square_H R_j(W) &= A(j)\square_H V\square_H A(j^{-1})\square_H A(j)\square_H W\square_H A(j^{-1}) \\ &\cong A(j)\square_H V\square_H W\square_H A(j^{-1}) \\ &= R_j(V\square_H W) \end{aligned}$$

where we used the isomorphisms $A(j^{-1})\square_H A(j) \cong H$ in **[S04]**.

We get a natural transformation $R_jR_k \cong R_{jk}$ obeying associativity by using the fixed choices $\iota_{j,k}$ for isomorphisms $A(j)\square A(k) \cong A(jk)$, that have been demanded by the definition of a twisting group.

$$\begin{aligned} R_jR_k(V) &= A(j)\square_H A(k)\square_H V\square_H A(k^{-1})\square_H A(j^{-1}) \\ &\overset{\iota_{j,k}\ \iota_{j^{-1},k^{-1}}}{\longrightarrow}\ A(jk)\square_H V\square_H A((jk)^{-1}) \end{aligned}$$

(1) In contrast to the pointed case in most of this thesis, we do not necessarily have Σ abelian here.

2. Orbifoldizations Coincides With Kirillov

We now consider the orbifoldization of the Σ-equivariant category $\mathcal{C}$ defined above

$$\mathcal{C}//\Sigma := \{(V,(\phi_j)_{j\in J}) \mid \phi_j : R_j(V) \cong V, \phi_{jk} = \phi_j \circ R_j(\phi_k)\}$$

and the orbifoldized Hopf algebra (Theorem 1.6)

$$\Omega(H,\Sigma) = \bigoplus_{j\in J} A(j) \quad \Delta = \sum_{i,j\in\Sigma} \iota_{i,j}^{-1}$$

and show that the former describes again the new $Bicomod(\Omega,\Omega)$.

Theorem 10.1. *The following natural transformation gives a* **monoidal category equivalence**

$$\Phi : \mathcal{C}//\Sigma \cong Bicomod(\Omega,\Omega)$$

$$(V,(\phi_i)_i) \mapsto V$$

The non-degeneracy of can is preserved, hence we get

$$\Phi : BiGal(H)//J \cong BiGal(\Omega,\Omega)$$

Proof. Split $V = \bigoplus_{i\in J} V_i$ and take this as **vector space**. Define the left (right) Ω-**comodule structure** using the cleaving maps $c_j : H \to A(j)$ by

$$\delta_L^\Omega(v_i) = \sum_{j\in J}(c_j\otimes\phi_j^{-1})\delta_L(v_i) \subset \bigoplus_j A(j)\otimes V_{jij^{-1}}$$

They commute, because:

$$\begin{aligned}
(1\otimes\delta_R^\Omega)\delta_L^\Omega &= \sum_{j,k}(1\otimes\phi_j^{-1}\otimes c_j)(1\otimes\delta_R)(c_k\otimes\phi_k^{-1})\delta_L(v_i)\\
&= \sum_{j,k}(1\otimes\phi_j^{-1}\otimes c_j)(1\otimes\delta_R)(c_k\otimes\phi_k^{-1})\delta_L(v_i)\\
\phi\in Mor(\mathcal{C}) \qquad &= \sum_{j,k}(c_k\otimes\phi_j^{-1}\phi_k^{-1}\otimes c_j)(1\otimes\delta_R)\delta_L(v_i)
\end{aligned}$$

and analog for the right side. This turns V into a Ω,Ω- **bicomodules**.

Φ is **invertible** by taking for any Ω,Ω-bicomodule the projections (push-forward) π of the comodule structures on $\Omega \to H$, say δ_L,δ_R (as π is a Hopf algebra map, these are again comodule structures), which makes V an H,H-bicomodule. It is clear from the construction of $\pi : \Omega \to H$ as dividing out all twisted sectors $A(j),\ j\neq e$, that the so-defined coactions coincide with the original ones. The additional structural maps ϕ_j^{-1} can be recovered via $\epsilon_H c_j^{-1}\delta_L^\Omega$.

The orbifoldizations of H,H-**Bigalois objects** V_k yield again Ω,Ω-Bigalois objects, as we calculate for the new left (right) can-map:

$$can_L^\Omega(v,w) = \sum_{j\in J}(c_j\otimes\phi_j^{-1}\cdot w_{jkj^{-1}})\delta_L(v_k) = \sum_{j\in J}(c_j\otimes\phi_j^{-1})can_L(v_k,\phi_j(w_{jkj^{-1}}))$$

which is clearly bijective. $\square$

CHAPTER 11

Yetter-Drinfel'd Modules

Note that as there is currently no definition of Yetter-Drinfel'd modules with projective action [(1)] in literature (and this was out of scope of this thesis). Thus we may only define the bare category $\mathcal{C}$ and recover **afterwards** the R_j and the pre-braiding from the known structures on Ω-Yetter-Drinfel'd modules. Although this is technically rigid, the author would hope for an intrinsic description of these structures solely in $\mathcal{C}$, as remarked below.

1. An Equivariant Category

Recall, that a twisting group Σ of Bigalois objects of a Hopf algebra H by Definitions 1.5 and 1.9 determines corresponding 2-cocyles $\sigma_? : \Sigma \to Z^2(H, \mathbb{k}^\times)$ by providing standard cleavings $c_j : \ H \to A(j)$.

Consider now $\mathcal{C}$ the category of **projective H-Yetter-Drinfel'd modules** associated to the above 2-cocycles, i.e.

$$\mathcal{C} = \bigoplus_{j \in \Sigma} \mathcal{C}_j$$

with $\mathcal{C}_j$ Yetter-Drinfel'd modules with an action that is projective with respect to the 2-cocycle σ_j, i.e.

$$V \in \mathcal{C}_j \implies a.(b.v) = \sigma_j(a^{(1)}, b^{(1)}) \left((a^{(2)}a^{(2)}).v\right)$$

(1)This is **not** the same as the projective Yetter-Drinfel'd modules defined over groups in K. Vocke's diploma thesis [**V10**] and used below in outlook section 12.3. This natural notion is equivalent to modules over Dijkgraaf's Drinfel'd double $D(G)^\omega$ for a 3-cocycle ω and are projective layer-wise with respect to different 2-cocycles. Especially their category is a properly braided category! This mistake has been noted by Prof. Schweigert in a discussion following a talk the author gave about Nichols algebras in Hamburg 2011.

Clearly, the usual tensor product in YDM (diagonal action and coaction) extends accordingly, because the cocycles are (convolution-) multiplied and $\sigma_j\sigma_k = \sigma_{jk}$:

$$\otimes : \ \mathcal{C}_j \times \mathcal{C}_k = \mathcal{C}_{jk}$$

As we'll see in the following, $\mathcal{C}$ can even be endowed with the structure of a Σ-equivariant category, including a pre-braiding. The author was so far not able to directly construct these structure from within $\mathcal{C}$. Note however, that the proof below gives valuable hints on how these should be defined generally.

Remark 11.1. *The author assumes $\mathcal{C}_j$ to be the category of relative Yetter-Drinfel'd modules with respect to the action of σ_j-"twisted" bicomodule algebras $A(j)$ over H. This would be a more natural definition.*

2. Orbifoldizations Coincides With Kirillov

Theorem 11.2. *The category $\mathcal{C}$ of projective Yetter-Drinfel'd modules with $\mathcal{C}_e = {}^G_G YDM$ above can be endowed with the structure of a Σ-equivariant category including a pre-braiding, such that the orbifoldization $\mathcal{C}//\Sigma$ is categorically equivalent to the braided category ${}^\Omega_\Omega YDM$ with the* **inverse braiding***!*

Proof. We construct the category equivalence vice-versa and recover the necessary structures in $\mathcal{C}$ as we move along.

Take a Yetter-Drinfel'd module M over Ω. By construction (Theorem 1.6) Ω contains a new dual groupring $\Bbbk^\Sigma$ and hence central idempotents $(1_j)_{j\in J} = 1_{A(j)}$, that are orthogonal $e_j e_k = 0$ (for $j \neq k$) and have sum $\sum_{j\in J} 1_j = 1_\Omega$.

Consider the induced decompositions on Yetter-Drinfel'd modules M as Ω-modules

$$M = \bigoplus_{j\in J} 1_j.M =: \bigoplus_{j\in J} M_j$$

They are submodules with only $A(j) \cong {}^{\sigma_j}H$ acting nontrivial. Hence they are σ_j-projective H-modules, which proves $M_j \in \mathcal{C}_j$ As in the above case, the Hopf algebra projection $\Omega \to H$ immediately yields an H-coaction on M. This yields a functor $f : {}^\Omega_\Omega YDM \to \mathcal{C}$.

In the **next step** we define $R_j:\ \mathcal{C}_k\to\mathcal{C}_{jkj^{-1}}$ and for each M a family of $(\phi_j^M)_{j\in J}$, such that $\left(f(M),\phi^M\right)$ lands in the orbifoldized category

$$C//J := \{(V,(\phi_j)_{j\in J})\mid \phi_j : R_j(V)\cong V,\ \phi_{jk}=\phi_j\circ R_j(\phi_k)\}$$

Let for each M set $R_j(M_k):=M_{jkj^{-1}}$. To define the ϕ_j, we again require the coherent choices of cleavings $c_j:\ H\cong A(j)$ (as coalgebras) demanded by Definition 1.9 to define a coalgebra map $\kappa:\ \Omega\to\mathbb{k}^\Sigma$ to the linear forms on Σ. Then we may use the M-coaction

$$\phi_j:\ v\mapsto\kappa\left(v^{(-1)}\right)(j)\cdot v^{(0)}$$

By the Yetter-Drinfel'd condition on M, ϕ_j maps $M_{jkj^{-1}}$ to M_k:

$$\begin{aligned}
\phi_j(1_{jkj^{-1}}.v) &= \kappa\left((1_{jkj^{-1}}.v)^{(-1)}\right)(j)\cdot(1_{jkj^{-1}}.v)^{(0)}\\
&= \kappa\left(1^{(1)}_{jkj^{-1}}v^{(-1)}S\left(1^{(3)}_{jkj^{-1}}\right)\right)(j)\cdot\left(1^{(2)}_{jkj^{-1}}.v^{(0)}\right)\\
&= \sum_{abc=jkj^{-1}}\kappa\left(1_a v^{(-1)}1_{c^{-1}}\right)(j)\cdot\left(1_b.v^{(0)}\right)\\
a,c^{-1},j\ \mathit{equal}\qquad &= \kappa\left(v^{(-1)}\right)(j)\cdot\left(1_k.v^{(0)}\right)\in M_k
\end{aligned}$$

We finally need to check $\phi_{jk}=\phi_j\circ R_j(\phi_k)$:

$$\begin{aligned}
\left(\phi_j\circ R_j(\phi_k)\right)(v) &= \phi_j\left(\kappa\left(v^{(-1)}\right)(jkj^{-1})\cdot v^{(0)}\right)\\
&= \kappa\left(v^{(-2)}\right)(jkj^{-1})\cdot\kappa\left(v^{(-1)}\right)(j)\cdot v^{(0)}\\
\kappa\ \mathit{coalg.-map}\qquad &= \kappa\left(v^{(-1)}\right)^{(1)}(jkj^{-1})\cdot\kappa\left(v^{(-1)}\right)^{(2)}(j)\cdot v^{(0)}\\
\mathbb{k}^\Sigma-\mathit{coprod.}\qquad &= \kappa\left(v^{(-1)}\right)(jkj^{-1}j)\cdot v^{(0)}=\phi_{jk}(v)
\end{aligned}$$

As **last step** we observe that the inverse standard braiding [(2)] on ${}^\Omega_\Omega YDM$

$$v\otimes w\mapsto w^{(0)}\otimes S^{-1}(w^{(-1)}).v$$

restricts to the desired pre-braiding

$$c_{M_j,M_k}:\ M_j\otimes M_k\to M_{jkj^{(-1)}}\otimes M_j$$

[(2)]Here we assume the antipode of Ω invertible.

by verifying the codomain works out correctly

$$\begin{aligned}
1_j.v \otimes 1_k.w &\mapsto (1_k.w)^{(0)} \otimes S^{-1}((1_k.w)^{(-1)}).1_j.v \\
YD-cond. \qquad &= 1_k^{(2)}.w^{(0)} \otimes S^{-1}(1_k^{(1)} w^{(-1)} S(1_k^{(3)})).1_j.v \\
&= \sum_{abc=k} 1_b.w^{(0)} \otimes 1_c S^{-1}(w^{(-1)}) 1_{a^{-1}} 1_j.v \\
c, a^{-1}, j\ equal \qquad &= 1_{jkj^{-1}}.w^{(0)} \otimes 1_j.S^{-1}(w^{(-1)}).v
\end{aligned}$$

$\square$

Part 5

Orbifoldizing Quantum Fields

Finally we give an outlook on the initial motivation of this thesis - to construct the **monster vertex algebra** solely on the level of Hopf algebras. It plays a crucial role in the proof of Monstrous Moonshine by having the simple Monster group as its automorphism group and it is constructed by orbifoldizing the Leech lattice vertex algebra (a thorough introduction to these topics is given in the following). The author hopes that orbifoldizing Hopf algebras not only reproduces the rather abstract mechanism of constructing the orbifoldized (e.g. conformal field theory) categories in the sense above, but also to keep track of a representing vertex algebra. This is an infinite-dimensional space of Laurent-series'-valued operators, fulfilling a suitable operator-product-expansion (associativity up to δ-functions). It can be seen as an explicit quantization of the conformal quantum field theory.

Note, that this effort is presently far from its conclusion!

In [**Len07**] a general technique was introduced to write down vertex algebras (which obey natural, but long and technical axioms) from a given set of data, corresponding to an infinite-dimensional Hopf algebra and a lifting in the sense of part 1. Once established, the properties of a vertex algebra are proven tediously, but once-and-forall. In the main chapter 12, we give a detailed overview of these theorems, including the examples of lattice algebras. Now the question arises:

Can one orbifoldize the Hopf algebra underlying the Leech lattice vertex algebra, such that the orbifold Hopf algebra produces the Monster vertex algebra? Does the Monster group action show more easily at this level?

Curiously, the latter seems to be easier approachable, as we shall see that Hopf- and vertex-automorphisms coincide and we already established the structure of a BN-pair on the former in part 3. The assumption of this automorphism group already gives valuable hints for a construction, as does the explicit Moonshine module underlying the monster vertex algebra; the main missing links is constructing an orbifoldizing scenario, that obeys a number of very explicit conditions. The main obstacles are the technically involved vertex algebra orbifoldizing and several (non-existing) generalizations needed on the Hopf side of this thesis (e.g. amalgams). They are discussed in an extensive outlook.

Basic Concepts:

Constructing Sporadics And Especially the Monster

We have seen in part 3, that $\mathbb{A}_n$ possesses a BN-pair as well due to the highly transitive action; a fact that in some sense points to and continues for **sporadic simple groups** [(3)] (see below). Already in 1861 (resp. 1873) the five **Mathieu groups** were discovered in the search for highly transitive permutation groups: M_{24} acts 5-transitive on $|S| = 24$ (the others are stabilizers of respectively many points) and apart from $\mathbb{S}_n, \mathbb{A}_n$ they are (except M_{21}) the only 4-transitive groups. The combinatorial structure on S preserved exactly by M_{24} is quite remarkable:

Definition 11.3. *A* **Steiner system** $S(k, m, n)$ *consists of a set* $|S| = n$ *and a system of subsets ("blocks") each of order* m*, such that every subset of order* k *is contained in exactly one block.*

Example 11.4. *For a finite projective plane* $S = \mathbb{F}_q\mathbb{P}^2 = \mathbb{F}_q^3\backslash\{0\}/\sim$ *the lines form a Steiner system* $S(2, q+1, q^2+q+1)$ *as two distinct points determine exactly a line (affine geometries yield subsystems).*

Example 11.5. *There is a* **unique** Steiner system $S(5, 8, 24)$ *(Witt geometry) and its automorphism group is* M_{24}*; respective omission of some elements of* Ω *lead to respective Steiner systems corresponding to the other Mathieu groups (no Steiner systems are known for* $k > 5$*).*

The Witt geometry generates (via symmetric difference) the exceptional **Golay code** $\mathcal{G}$ [(4)] $\subset \Omega$ (with automorphisms M_{24} as well) and ultimately leads to the **Leech lattice** Λ, a highly exceptional 24-dimensional lattice presenting a very dense spherical packing and having as automorphism group the simple **Conway group** Co_1 (1968).

[(3)] Information on historical and technical matters come from a large number of sources, including Wikipedia, but most notably Griess' original- (Aschbacher: "Quasithin Groups") and Conway's revised construction of the monster group.

[(4)] A necessity in the Golay code's construction gives a nice "reason" for 24 being special: It is the largest number with $1^2 + 2^2 + \cdots + 24^2$ a square, namely 70^2

To understand the relationship between M_{24}, Co_1 and the monster $\mathbb{M}$ we recall from part 3 the classification of finite simple groups by inductively clarifying centralizer-of-involution subgroups:

Example 11.6. *The following groups are a chain of induction steps*

$$PSL_3(\mathbb{F}_{2^2}) = A_2(2^2) \longrightarrow M_{24} \longrightarrow Co_1 \longrightarrow \mathbb{M}$$

accompanied by respective (here symbolic) extensions of the combinatorial structure having the respective group as automorphisms (compare to the contrary taking of stabilizers to obtain simple subgroups)

$$\mathbb{F}_{2^2}\mathbb{P}^2 = S(2,5,21) \longrightarrow S(5,8,24) \longrightarrow \Lambda \longrightarrow \mathcal{A}$$

While the Leech lattice has still been studied for its own sake, the **Griess algebra** A is obtained "purposely", relying heavily on the hypothesis of a prescribed 2-centralizer $Co_1 \ltimes 2_+^{24+1} \subset \mathbb{M}$ and has to be seen in strong correspondence to the theory of BN-pairs for $\mathbb{A}_n$ (see part 3). Let us finally review this construction in some more detail:

Assume an involution $z = z_1 \in \mathbb{M}$ in the yet-to-be-constructed group with its supposed centralizer $B = Cent(z) \subset \mathbb{M}$ a split extension of an extraspecial group $Q \cong 2_+^{24+1}$ (viewed as a symplectic 24-dimensional $\mathbb{Z}_2$-vector space, see section 6.2) by the simple Conway group Co_1 acting on Q as on the Leech lattice *mod* 2; the element $z \in Cent(z)$ is the central commutator in Q and preserved by Co_1. Suppose further z_2 a (suitable) second involution in Q and define $N \subset \mathbb{M}$ as the normalizer of the Klein-4-group $V = \langle z_1, z_2 \rangle \cong \mathbb{F}_2^2$.

N can be constructed as a central extension of $M_{24} \times \mathbb{S}_3$ and Conway's approach uses the newly discovered **Parker loop** with great profit. This is a remarkable algebraic structure with 2^{12+1} elements and should be visualized from a Hopf algebra point of view as follows: Identify the 2^{12} Golay code words $\mathcal{G} \ni A \subset \{1, \ldots 24\}$ with the basis of a finite vector space $A \in \mathbb{Z}_2^{12}$. Extend the latter by an additional $\mathbb{Z}_2 = \{\pm 1\}$ to an algebra with nontrivial associativity constraint (given as follows by set cardinalities $|\cdot|$) ultimately yields a "deformed group" (loop):

$$A^2 = E \cdot (-1)^{|A|/4}$$

$$BA = AB \cdot (-1)^{|A \cap B|/2}$$

$$A(BC) = (AB)C \cdot (-1)^{|A \cap B \cap C|}$$

As for proper BN-pairs B, N generate the (future) Monster and the intersection $D = B \cap N = Cent(V)$ is roughly M_{24}. However, while $\tilde{D} = Cent(V)$ is clearly normal in N with Weyl group $N/H = \mathbb{S}_3 \cong A_1(2)$ permuting $\{z_1, z_2, z_1z_2\}$, the actual D is slightly larger (still containing (23)) and non-normal. A representative is the utmost important **triality element** w conjugating the three involutions' centralizers. From the assumed B, N, D the monster $\mathbb{M}$ can be obtained as an amalgam, the explicit construction proceeds as follows:

Start with a monomial B-representation $\mathcal{A}_- \cong \Lambda \otimes 2^{12}$ and a B-representation $\mathcal{A}_S$ from the symmetric tensor square $\Lambda \otimes_{sym} \Lambda$ (viewed as symmetric matrices). Furthermore, in Λ there are 196560 "minimal vectors" $\mathcal{R}$ (presenting the spheres packed around the center) and one chooses an appropriate decomposition as $\mathcal{R}_+ \cup \mathcal{R}_-$, yielding a third B-representation $\mathcal{A}_R$ spanned by symbols $y(r),\ r \in \mathcal{R}_+$

Definition 11.7. *The* **Griess algebra** $\mathcal{A}, \gamma, \tau$ *is the* B*-module* $\mathcal{A} := \mathcal{A}_- \oplus \mathcal{A}_S \oplus \mathcal{A}_R$ *of dimension* $196.884 = 24 \cdot 2^{12} + \frac{24 \cdot 25}{2} + \frac{196560}{2}$*, equipped with a natural* B*-invariant bilinear form* γ *from the construction and several* B*-invariant composition structures:*

- $\tau_S :\ \mathcal{A}_S \otimes \mathcal{A}_S \to \mathcal{A}_S$ *the symmetrized matrix product.*
- $\tau_{SR} : \mathcal{A}_S \otimes \mathcal{A}_R \to \mathcal{A}_R$ *dualized from the* Λ*-coaction* ξ *on* A_R
$$M \otimes y(r) \mapsto \gamma(M, \xi(r) \otimes_{sym} \xi(r)) y(r)$$
- $\tau_R :\ \mathcal{A}_R \otimes \mathcal{A}_R \to \mathcal{A}_R$ *adding "positive" minimal Leech vectors* $y(r) \otimes y(r') \mapsto y(rr')$ *where possible* ($rr' \in \mathcal{R}_+$) *and* 0 *otherwise*
- $\tau_{S-}\ \mathcal{A}_S \otimes \mathcal{A}_- \to \mathcal{A}_-$ *by the matrix* $M \in \mathcal{A}_S$ *acting on* Λ
- $\tau_{R-}\ \mathcal{A}_R \otimes \mathcal{A}_- \to \mathcal{A}_-$ *from reflecting the* Λ*-factor along* $\xi(r)$*.*

Simply adding all these structure maps yields $\tau : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$*, commonly viewed as a (nonassociative) algebra or an invariant* **tri***linear form.*

One than restricts the action on $\mathcal{A}$ to $H = B \cap N$ and subsequently extends to N in such a way, that γ, τ are still invariant (Griess had to guess the extra triality automorphism $w \in N$). Hence $B, N \subset Aut(\mathcal{A}) =: G$ and the proof is concluded by showing finiteness of G (from a finite orbit in $\mathcal{A}$) and $Cent_G(z) = B$, then subsequently simplicity of G follows by a rather general argument working for other groups as well (Q is a **large extraspecial group** in G and hence G is close to simple).

Vertex Algebras And Monstrous Moonshine

Monstrous Moonshine (a comprehensive survey is **[G06]**, for a proof overview see p. 412ff) has first been observed rather accidentally [5] by John McKay in the late 1970's: The monster group $\mathbb{M}$ has not even been constructed, but much evidence pointed to an explicit character table. McKay noticed, that the dimensions of the smallest irreducible representations of this new sporadic group

$$dim(\rho_0), dim(\rho_1), dim(\rho_2), dim(\rho_3), \ldots = \; 1,\; 196883,\; 21296876,\; 842609326\ldots$$

seemed in strange coincidence to the distinguished Fourier series

$$\begin{aligned} j(z) &= (q^{-1} + 744) \qquad\qquad\qquad q = e^{2\pi i z} \\ &+ q(1 + 196883) \\ &+ q^2(1 + 196883 + 21296876) \\ &+ q^3(2 \cdot 1 + 2 \cdot 196883 + 21296876 + 842609326) \\ &+ \;\cdots \end{aligned}$$

of the **modular function** $j(z)$. This function is well-known in complex analysis and analytical number theory for being (in some sense) the only holomorphic function on the upper halfplane (single pole only in $i\infty$), that is **invariant** for integral Möbius transformations $PSL_2(\mathbb{Z})$ acting as fractional linear maps on the complex plane ("modular").

As a "reason", Conway and Norton soon conjectured the existence of an infinite-dimensional graded representation $V^\natural$ of the monster group, the **Moonshine module**, with layers according to the observation

$$V_1^\natural, V_2^\natural, V_3^\natural, V_4^\natural, \ldots = \;\; 0,\; \rho_0 \oplus \rho_1,\; \rho_0 \oplus \rho_1 \oplus \rho_2,\; 2\rho_0 \oplus 2\rho_1 \oplus \rho_2 \oplus \rho_3, \ldots$$

and graded dimension ("Thompson series") [6] the $j(z)$-function

$$j(z) - q^{-1} - 744 \;\stackrel{!}{=}\; \chi_{V^\natural}(z) := \sum_{n \geq 0} dim(V_n^\natural) q^n$$

---

(5) Because this observation seemed too much of a coincidence, but also no one could even imagine a connection between so distinct areas of mathematics, the term "Moonshine" (being illegally distilled liqueur) is ascribed either to J.H. Conway describe the craziness of this idea ("talking moonshine") or the bottle of whiskey offered by A. Ogg for the proof of a related observation involving the monster.

(6) Actually, Monstrous Moonshine even conjectures the graded trace of each element $g \in \mathbb{M}$ acting on $V^\natural$ to be a so-called genus-0 Hauptmodule (-function).

A Moonshine Module $V^\natural$ having $\mathbb{M}$ as automorphism group was found by I. Frenkel, J. Lepowsky and A. Meurman [**FLM84**]. They explicitly orbifoldized a 24-dimensional bosonic string theory compactified over the **Leech lattice** Λ (typical infinite-dimensional structures, where modular function appear elsewhere) by the lattice's involutive automorphism. Note that at this time, physicist have practically used vertex algebras in string theory, but no mathematical definition existed. Today we can speak mathematically of an orbifoldization of the standard lattice vertex algebra V^Λ.

Definition 11.8. *A* **vertex algebra** *(see e.g* [**FB01**]*) is a vector space of* **states** V *with a* **vacuum state** $1 \in V$ *together with a* **translation operator** $T \in End(V)$ *and a* **state-field-correspondence**

$$Y:\ V \to End(V)[[z, z^-1]$$

such that $Y(a)v$ is a Laurent series in z and (briefly)

- **First vacuum axiom:** $Y(1) = vz^0$
- **Second vacuum axiom:** $Y(a)1|_{z=0}$ *defined and* $= a$
- **Translation axiom:** $[T, Y(a)]v = \frac{\partial}{\partial z} Y(a)v$
- **Locality:** $Y(a)Y(b) \sim Y(b)Y(a)$ *up to δ-functions*

Especially **associativity** *follows:* $Y(Y(a)b)v \sim Y(a)Y(b)v$

The **action of the monster group** is obtained from Griess' original construction (see above): $\bar{B} = Co_1 \ltimes \mathbb{F}_2^{24}$ acts on Λ and is (almost the discrete part of) $Aut(V^\Lambda)$. In the orbifoldization, a central extension by $\mathbb{Z}_2 = \langle z \rangle$ appears $B \cong Co_1 \ltimes 2_+^{24+1}$, but it turns out **non-central** in $Aut(V^\natural)$: An explicitly "guessed" **triality automorphisms** w mixes twisted and untwisted sectors of the orbifold and hence does not normalize B. It turns out to be the **triality element** $w \in N$ that generates together with B the entire monster $\mathbb{M}$. The first nontrivial layer of the vertex algebra $V_2^\natural$ is the **Griess algebra** above (compare dimensions!).

Richard Borcherd's, a group-theoretician from Cambridge (who had written his PhD thesis under H.J. Conway's supervision on the Leech lattice) started a long-term effort, that finally lead him to a complete proof for the Moonshine conjecture, and earned the 1998 fields medal.

His proof proceeds in several steps and we shall give a rough overview:

Step 1: He constructed on the Moonshine module $V^\natural$ described above the structure of again a full vertex algebra. In fact, his efforts lead him 1986 to the very definition of this structure, that mathematically rigidifies the approach above.

Step 2: He associated to $V^\natural$ an infinite-dimensional Lie algebra $\mathfrak{m}$ with $\mathbb{M}$ acting as automorphisms. It turns out to possess a root system, although all simple roots except one are imaginary. The structure is termed **generalized Kac-Moody algebra** or Borcherd's-Kac-Moody algebra. It is studied under heavy use of "physical" representation theory of the **Virasoro algebra** ("no-ghost" i.e. a charge 24 module has only positive L_0 eigenvalues)

Step 3: Similar to the usual Weyl character formula type he obtained a **denominator identity** argument for $\mathfrak{m}$, which is used to derive a so-called **replication formula** for $f = \chi_{V^\natural}$. This means there is a polynomial P of degree n expressing the action of certain fractional linear transformations as

$$\sum_{ad=n,\ 0\leq d<b} f\left(\frac{az+b}{d}\right) = P(f(z))$$

Such replication formulae determine the function from the first couple Fourier coefficients. In fact Norton conjectured (still open):

Conjecture 11.9. *Any replicable function with rational coefficients is either a genus-0 Hauptmodul of Moonshine type or one of the so-called* **modular fictions** $q^{-1},\ q^{-1}+q, q^{-1}-q$

Similarly, twisted denominator identities yield replication formulae for the graded trace of the action of each $g \in M$ (on the left side of the above then other such traces for g^a appear.

Step 4: Compare by-hand the coefficients a_1, a_2, a_3, a_4, a_6 of $\chi_V^\natural$ and $j - q^{-1} - 744$ (respectively for the graded traces and Hauptmodules). Since both are replicable functions, this proves them to coincide.

CHAPTER 12

Constructing Vertex Algebras From Hopf Algebras

We shall now concisely review the authors approach **[Len07]** to construct vertex algebras from a Fock-space type Hopf algebra with a comodule algebra and a skew-form providing a **linking** in the sense of Nichols algebras as quantum Borel algebras (see part 2). We shall also review, how the well-known examples of a Heisenberg vertex algebra and a lattice vertex algebra can be obtained from this construction.

1. The Coordinate Ring

The critical point was a smooth algebraic framework, that serves as a *coordinate ring* in the sense of algebraic geometry. The following turns out to suffice and especially reproduce the *classical* vertex algebra notion with complex functions:

Definition 12.1. *A* **coordinate ring datum** (H, R) *consist of*

- **Symmetry operation:** *A Hopf algebra H, e.g. classically $\mathbb{k}[T]$ with* **Translation** $\Delta(T) = T \otimes 1 + 1 \otimes T$.
- **Polar part:** *An* **right** *H-module algebra R, classically $\mathbb{k}[\mathbb{Z}] \cong \mathbb{k}[g, g^{-1}]$ with T acting as $-\frac{\partial}{\partial g}$. This is* **much** *smaller and (algebraically) well-behaved than the space of Laurent series!*

We call the space of linear maps $M := Hom_{\mathbb{k}}(H, R)$ **coordinate ring**. *Do* **not** *mistakenly interpret H as argument of the resulting* **functions** *(see the isomorphy $M \approx \mathbb{k}[[z, z^{-1}]]$ in the classical case below)!*

The "fine structure" imposed by this definition on the to-be-used coordinate ring M supports several crucial structures and definitions:

- The natural convolution **multiplication** on M from R-multiplication and H-comultiplication.
- A natural H-**action** on M from the one on R and the adjoint Hopf algebra action on H itself.
- A subring of **regular functions**

$$M_{reg} = Hom_{\mathbb{k}}(H, k) \subset Hom_{\mathbb{k}}(H, R) = M$$

with a map **evaluation at zero**

$$\zeta : M_{reg} \to \mathbb{k}$$
$$\phi \mapsto \epsilon_{H^*}(\phi) = \phi(1)$$

- A ring of **multivariable functions**

$$M^{(n)} := Hom_{\mathbb{k}}(H^{\otimes n}, R^{\otimes n})$$

  that appear later as n-point functions.
- An equivalence relation $\equiv$ **up to** $\delta(z_1 - z_2)$**-function** for two variables $Hom_{\mathbb{k}}(H_1 \otimes H_2, \mathbb{k}_1 \otimes \mathbb{k}_2)$ generated for all $r \in R$ by:

$$(h_1 \otimes h_2 \mapsto \epsilon(h_2) \otimes r.S^{-1}(h_1)) \equiv (h_1 \otimes h_2 \mapsto r.S^{-1}(h_2) \otimes \epsilon(h_1))$$

  Here and in what follows we use the index $H_i, h_i, \ldots$ to indicate which coordinate (i.e. tensor factor $H_i, \mathbb{k}_i$) the respective h, r corresponds to.

Definition 12.2. *A* **symmetric coordinate ring datum** (H, R, γ) *consists additionally of an algebra map $\gamma : R \to R$, that is subsequently extended by S_H to $M \supset R$. It comes with an additional δ-equivalence relation*

$$(h_1 \otimes h_2 \mapsto \epsilon(h_2) \otimes r.h_1) \equiv (h_1 \otimes h_2 \mapsto \gamma(r).h_2 \otimes \epsilon(h_1)$$

γ should "classically" be chosen $g \mapsto -g$ to produce the correct classical δ-function as well.

Remark 12.3. *Note the difference between 2-variable functions and (finite!) tensors of 1-variable functions*

$$M^{(2)} = Hom_{\mathbb{k}}(H_1 \otimes H_2, \mathbb{k}_1 \otimes \mathbb{k}_2) \supset Hom_{\mathbb{k}}(H, R) \otimes Hom_{\mathbb{k}}(H, R) = M \otimes M$$

In Borcherd's' abstract categorical formalization [**Bor99**]*, the former corresponds to "inequivalent" coordinates $i, j = 1, 2$ and the functions representing objects for the bifunctor of* **singular** *bilinear maps.*

For $(H, R) = (\mathbb{k}[M], \mathbb{k}[\mathbb{Z}])$ we can roughly construct a map $M \to \mathbb{k}[[z]]$ to the ring of Laurent series' by mapping $R \ni g \mapsto z$ and the projector $T^* \mapsto z$, that reduces all the above notions to their meromorphic origin (proof in [**Len07**], section 5.1). Note that while an arbitrary linear map $\phi \in Hom_{\mathbb{k}}(H, R)$ may not yield proper Laurent series', all produced by the vertex operators below indeed do (by an easy general argument). This is certainly not true for more complicated vertex expressions, that may involve δ-functions!

2. Obtaining The Vertex Algebra

In the author's new approach to vertex algebras [**Len07**], we define the vertex algebra A implicitly by providing a **vertex operator** through a map:

$$Y : A \otimes A \otimes H \to A \otimes R$$

While such an expression **cannot** be directly rewritten (as one might attempt) to $Y : A \to End(A) \otimes M$ without infinite sums

$$A \to Hom_{\Bbbk}(A, A) \hat{\otimes} Hom_{\Bbbk}(H, R) = Hom_{\Bbbk}(A, A) \hat{\otimes} M$$

it is very well possible to cleanly (finitely) extract **n-point functions** where all A has been plugged in resp. projected to, such that a matrix element in the coordinate ring $M^{(n)} = Hom_{\Bbbk}(H^{\otimes n}, R^{(\otimes n)})$ is left over. Consider e.g:

$$\langle w | \, Y(a) \, | v \rangle := (h \mapsto w^*(Y(a \otimes v \otimes h))) \in Hom_{\Bbbk}(H, R) = M$$

$$\langle w | \, Y(a)Y(b) \, | v \rangle := (h_1, h_2 \mapsto w^*(Y(a \otimes Y(b \otimes v \otimes h_2) \otimes h_1)) \in M^{(2)}$$

Now we may formulate the **construction and basic results**:

Definition 12.4. *A* **Hopf vertex algebra datum** $(A, \bar{A}, \langle\rangle)$ *for a coordinate ring datum* (H, R) *consists of an right* H*-module Hopf algebra* $\bar{A}$ *(the* **undeformed state space***, an* $\bar{A}$*-comodule algebra* A *in the category of* H*-modules (the* **deformed state space***, possibly* $= A$*, but not in the lattice algebra) and a skew-multiplicative linear form (the* **linking***):*

$$\langle , \rangle : \bar{A} \otimes \bar{A} \to R$$

satisfying translation covariance with respect to the H*-action on* R*:*

$$\langle a, b.h \rangle = \langle a, b \rangle .h \qquad \langle a.h, b \rangle = \langle a, b \rangle .S(h)$$

Definition 12.5. Translation operators series' *on states* $\bar{A}$ *and geometry* R*:*

$$\Gamma : \ \bar{A} \otimes H \xrightarrow{\mu_{\bar{A}}} \bar{A}$$

$$\Gamma_{ij} : \ \Bbbk_j \otimes H_i \xrightarrow{\mu_R} \Bbbk_j$$

We may interpret the multiplication in $\bar{A}$ *and the action of* H *on a different* R *as above of for maps* $H \otimes V \to V \otimes R$ *as matrix elements resp. n-point functions and (using infinite sums)* $End(V)$*-valued operator series'.*

In this view, the former is a End(A)-valued regular function and the latter is a regular function valued by differential operators on another coordinate function (again we index the coordinate in multi-variable expressions by i, j)

Both operators intrinsically exponentiate the action of H by it's double role as regular part of the coordinate ring. In the classical setting they were proven to both reduce to finite translation of states and functions:

$$\Gamma_A = e^{-zM} \qquad \Gamma_{12} f(z_2) = f(z_2 - z_1)$$

They will even in the general case present exactly the additional modifications appearing in the definition of skew-symmetry (former) and associativity (latter).

Definition 12.6. *A* **symmetric Hopf vertex algebra datum** $(A, \bar{A}, \langle\rangle, \beta)$ *for a symmetric coordinate ring datum* (H, R, γ) *consists additionally of an H-linear $\bar{A}$-braiding β such that $\bar{A}$ is β-commutative and $\langle,\rangle$ is β-symmetric*

$$\langle \beta(a \otimes b) \rangle = \gamma(\langle a, b \rangle)$$

(this is why we need to keep the freedom to deform $\bar{A}$ to A accordingly)

The symmetry is actually defined much more generally and produces α-local vertex algebras; this includes naturally super-locality, as it appears in odd lattice algebras (see below).

Remark 12.7. Note, *that in* **[Len07]** *the author has used a different definition, namely braiding with coefficients satisfying the Yang-Baxter equation, product rules and certain (H-) translation compatibilities*

$$\tau : \bar{A} \otimes \bar{A} \to \bar{A} \otimes \bar{A} \otimes R$$

This notion was axiomatized rather artificially, but gives more flexibility. The fairly natural construction of such a τ from a skew-form as above was only in the case of "classical" $\alpha = 1$ (yielding locality) and $H = \mathbb{k}[M]$; but as it applies to the here relevant examples, the author has decided to choose this approach here.

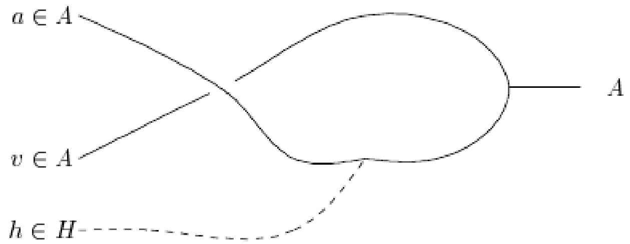
Definition 12.8.

$$\tau:\ a\otimes b\longmapsto b^{(1)}\otimes a^{(1)}\otimes\langle a^{(0)},b^{(0)}\rangle$$

As two $\bar{A}$ are braided, both coact into H, which subsequently is plugged into the skew-form yielding an R-coefficient. We draw τ as braiding and omit R in the following visualization.

For a (symmetric) Hopf vertex algebra datum we define on $\bar{A}$ the **(local) vertex operator** Y by

$$\begin{aligned} Y:\ &\bar{A}\otimes\bar{A}\otimes H\longrightarrow\bar{A}\otimes R\\ &a\otimes v\otimes h\longmapsto\mu_{\bar{A}}(id_{\bar{A}}\otimes .h)\tau(a,v)=v^{(1)}(a^{(1)}.h)\otimes\langle a^{(0)},v^{(0)}\rangle \end{aligned}$$



The Hopf vertex algebra datum and especially its symmetry implies **very strong consequences** on this on the composition structure of this non-multiplicative map Y in either argument. The following general properties are proven by explicit calculation once-and-for-all in section 4.2 [**Len07**]. Viewing Y as a vertex operator, the properties precisely describe the desired vertex-axioms and for the classical choices of the (symmetric) coordinate datum (H,R), Y thus equips $\bar{A}$ with the structure of a (local) vertex algebra with translation operator $T\in H$ and vacuum vector $1_{\bar{A}}$ ([**Len07**] section 5.1).

Theorem 12.9. *For a Hopf vertex algebra datum we obtain the following properties, that define a vertex algebra* [(1)] *in the classical context:*

- **First vacuum "axiom":**

$$\langle w|Y(1)|v\rangle=\langle w|v\rangle 1_M$$

- **Second vacuum "axiom":**

$$\langle w|Y(a)|1\rangle\in M_{reg}\qquad\zeta\langle w|Y(a)|1_A\rangle=\langle w|a\rangle 1_M$$

---

(1) Note that the grading is omitted, as well as conformal vector. Note on the other hand, that we always have an action of $H\ltimes\bar{A}$, hence it is *quasi-conformal*

- **Translation "axiom":** *for* $t \in H$

$$\langle w|(.t^{(2)})Y(a)(.S(t^{(1)}))|v\rangle = (.t)\langle w|Y(a)|v\rangle$$

Especially for "classically" $H = \mathbb{k}[M]$ the derivational (primitive) M is the well-known translation operator and the result above reads as:

$$\langle w|[.M, Y(a)]|v\rangle = -\frac{\partial}{\partial z}\langle w|Y(a)|v\rangle$$

- **Associativity** *up to δ-functions:*

$$\langle w|Y_1(a)Y_2(b)|v\rangle \equiv \langle w|Y_2\left(\Gamma_{21}Y_1(a)|b\rangle\right)|v\rangle$$

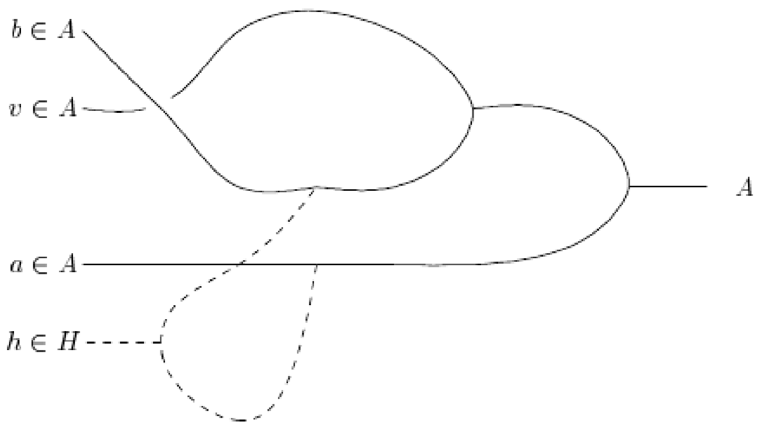
For a **symmetric** *Hopf vertex datum, we additionally prove* **skew-symmetry** *and (by expanding both sides using associativity)* **locality***:*

$$\gamma\langle w|\Gamma_A Y(a)|b\rangle \equiv \langle w|Y(b)|a\rangle \qquad \langle w|Y(a)Y(b)|v\rangle \equiv \langle w|Y(b)Y(a)|v\rangle$$

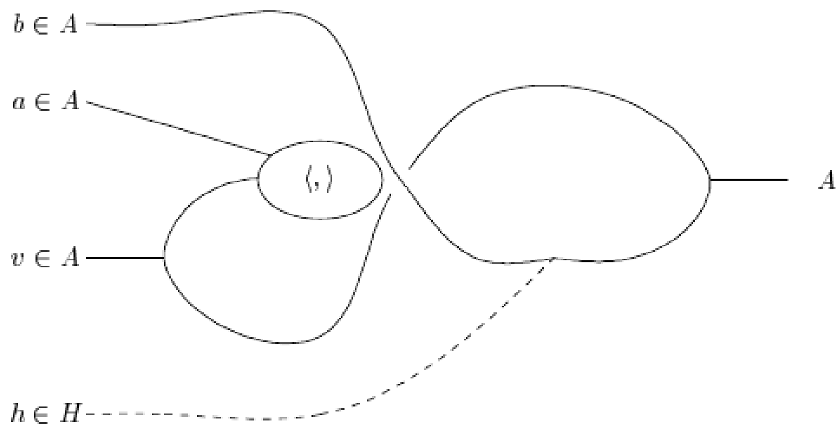
Notice that apart from avoiding infinite sums and especially δ-expressions, the structure of the state space A is the second elegance about the suggested approach: As an algebra, we only consider a "Borel" subalgebra of **creation operators**, set equal to the Fock space. The additional, second copy of **annihilation operators** appears from the **same** elements through the skew-form (linking) as for semisimple Lie algebras.

This gets particularly clear, if some derivational (=primitive) element $\Delta(a) = 1 \otimes a + a \otimes 1$ in the vertex operator's argument is explicitly worked out to $\langle 1, -\rangle a = a$ resp. $\langle a, -\rangle a$ (first resp. second Δ-summand).

This yields a decomposition of $Y(ba)|v\rangle$ into the classical creation term acting **after** $Y(b)$



and the classical annihilation term acting **before** $Y(b)$



Note that by these order issues let a **normally ordered product** appears by itself - it simply reflects the "non-multiplicatively" of the map above in the first argument. This is especially used in the later-on isomorpy proofs to known vertex algebras.

3. Examples: Lattice Algebras

We proceed by providing a first easy examples of a vertex algebra, that have been shown in [**Len07**] to arise from the presented construction.

Example 12.10 (Heisenberg algebra)**.** *Take $V = H = \mathbb{k}[T]$ with basis $p_i := T^i$ as an H-module and $A = \bar{A} = \mathfrak{T}V$ the Fock space with V primitive elements $\Delta_{\bar{A}}(p_i) = 1 \otimes p_i + p_i \otimes 1$.*

The H-action on V extends diagonally (i.e. via Δ_H) to A, turning it into a module algebra. Induce a **linking** *for any $\kappa \in k^\times$ by:*

$$\langle p_0, p_0 \rangle = \kappa z^{-2} \in R$$

This immediately yields a (nonlocal) vertex algebra. By choice of the even z-power, this skew-form is symmetric (in the sense above) already for $A = \bar{A}$. Hence we obtain a local vertex algebra and the author has proven it ([**Len07**]*, section 5.2) to be isomorphic to the Heisenberg vertex algebra π_0^κ.*

Similarly, one associates to larger affine Lie algebra $\mathfrak{g}$ the **affine Kac-Moody vertex algebra**. We expect again that this construction can be recovered from taking $V = \langle \mathcal{R} \rangle_{\mathbb{C}}$ the respective root system with the null-root derivation T and $A = \bar{A}$ the Borel part of the universal enveloping (see [**Len07**], concluding remark in section 5.2).

Example 12.11 (Lattice algebra)**.** *Suppose a lattice Λ with $(-,-)$ the biadditive map, take $V = \Bbbk[\Lambda] \otimes \Bbbk[T]$ with basis $p_i^g := g \otimes T^i$ as H-module and $\bar{A} := \Bbbk[\Lambda] \otimes \mathfrak{T}V$ an extended Fock space with above's action on $\mathfrak{T}V$ and $T.g = gp_0^g$. Induce a* **linking** *by the following:*

$$\langle p_0^g, p_0^h \rangle := (g,h)z^{-2}$$
$$\langle g, h \rangle := z^{(g,h)}$$
$$\langle p_0^g, h \rangle := (g,h)z^{-1} = -\langle h, p_0^g \rangle$$

This immediately yields a (nonlocal) vertex algebra. An easy calculation shows, that taking $A = k_\sigma[\Lambda] \otimes \mathfrak{T}V = \bar{A}^{(\sigma \otimes 1)}$ for the usual 2-cocycle ([**FB01**] *section 4.4) yields a (super-)local vertex algebra for odd/even lattices Λ. Again, the author has provided an isomorphism to the lattice vertex algebra in* [**Len07**] *section 5.3.*

There is a very nice correspondence connecting these two notions, the **boson-fermion correspondence** ([**FB01**] section 4.3). Namely, simply-laced Kac-Moody vertex algebras are isomorphic to the respective lattice algebra for the root lattice. Moreover, odd lattices correspond to super-vertex algebras (α nontrivial as above). Note that this shows adequately from the choice of the deformed states $\bar{A} \neq A$ necessary to obtain locality.

Example 12.12. *The even lattice $\Lambda = \sqrt{2}\mathbb{Z}$ has a vertex algebra isomorphic to $\hat{\mathfrak{sl}}_2 = A_1^{(1)}$ with Dynkin diagram $(m_{12}, m_{21}) = (2,2)$.*

Example 12.13. *The odd lattice $\Lambda = \mathbb{Z}$ has vertex algebra isomorphic to the vertex super-algebra of a single fermion (hence the name of the correspondence).*

The author was be interested to verify this correspondence generally at the level of the $\bar{A}$, but this was out of scope of the diploma thesis:

$$\Bbbk[\Lambda] \otimes \mathfrak{T}\left(\Bbbk[\Lambda] \otimes k[T]\right) \cong \mathcal{B}(\hat{\mathfrak{g}})$$

if Λ is the root lattice of a simply laced $\mathfrak{g}$ and $\mathcal{B}(\hat{\mathfrak{g}})$ the Borel part of the affinization, i.e. $\mathfrak{T}\mathcal{R}/Serre$ for the affine root system $\mathcal{R}$ as above.

Question 12.14. *Is this true? How does the more complicated correspondence for non-simply-laced root systems look like from this point of view? Is there a natural construction of the respective super-vertex algebras from root systems in of a $\mathbb{Z}_2$ braided category?*

Outlook: 5 Conjectural Steps To Moonshine

We finally **sketch a far-fetched** path, that could leading from the constructions so far to an infinite dimensional Nichols algebra with root system over a nonabelian group and subsequently to a vertex algebra with automorphism group $\mathbb{M}$ prescribed by the BN-pair in Theorem 9.10. The author still has no complete picture, but rather a series of explicit clues from different points of view, how such a construction would have to look like, if it existed. Even further away is the technically tedious proof, that this structure (once established) could coincide with the ad-hoc construction due to Borcherd's.

Hence this topic is far from conclusion!, but his supervisor has nevertheless encouraged the author to also include these thoughts as an outlook to this thesis as follows:

Particularly nontrivial are the the following generalizations to the techniques established in this thesis:

- On one side, we can prove vertex algebra automorphism to correspond to Hopf algebra automorphisms and have a fairly solid understanding, how twisted vertex modules are defined. However, tedious calculations would be needed to indeed verify the twisted Jacobi identity (section 12.1). From the definition of the Moonshine module, in section 12.2 we can conjecture a specific twisting 2-cocycle that should underlay the $\mathbb{Z}_2$-sub-orbifolds L, as they appear in Theorem 9.10.
- The appearance of **projective** modules in the construction above strongly suggest an orbifoldization without a coherent choice of cleavings resulting in a quasi-Hopf-algebra-alike structure. More precisely, the associativity constraint should pick up additional signs prescribed by the Parker loop's associator used in the Monster group's construction. This is addressed in section 12.3.

- The involutive extension is **not central**, resulting in a non-proper BN-pair. Hence it seems necessary to perform an orbifoldization with respect to a groupoid Σ with endomorphisms $\mathbb{Z}_2$, leading as described in Remark 1.7 to a **weak Hopf algebra**. The actual Hopf algebra might then be derived thereof by a universal completion much like a group amalgam. This is described in section 12.4.
- Finally in section 12.5 we try to glimpse at the "big picture", combining all of the above. We describe an explicit twisting groupoid Σ, consisting of multiple $\mathbb{Z}_2$, while the triality symmetry is hidden in the groupoid structure and only shows after orbifoldizing and amalgamization. The overall (amalgamed) orbifold of an affine $E_8^{(1)}$ could exhibit a specific Dynkin diagram, while the three contained E_8-root lattices are combined to the Leech lattice. This might yield the right BN-pair to prove this infinite-dimensional Yetter-Drinfel'd module $\tilde{M}$ to have automorphism group $\mathbb{M}$.

1. *Orbifoldizing Vertex Algebras Vs. Hopf Algebras*

We first assume $A = \bar{A}$. When this is no longer true (especially for lattice algebras!) we would yield **projective** versions of the Hopf algebra notions used below, that are yet to be defined.

Definition 12.15. *A* **Hopf vertex module** *is an $\bar{A} = A$-module and A-comodule algebra V as objects in ${}_H Mod$ with*

$$(v.a)^{(0)} \otimes (v.a)^{(1)} = v^{(0)}a^{(0)} \otimes v^{(1)}.a^{(1)}$$

Proof. The main theorem 12.9 for constructing a vertex algebra thereof shows, that associativity, translation and vacuum hold as for usual vertex algebras. Hence *classically* these are vertex modules. $\square$

Consider now the injective extension of abelian groups and -grouprings of the symmetric coordinate ring datum $(H, R) = (\mathbb{k}[T], \mathbb{k}[g, g^{-1}])$:

$$\begin{array}{ccccccc}
1 & \longrightarrow & \mathbb{Z} & \xrightarrow{n\cdot} & \mathbb{Z} & \longrightarrow & \mathbb{Z}_n = \Sigma \\
0 & \longrightarrow & \mathbb{k}[g, g^{-1}] & \longrightarrow & \mathbb{k}[\sqrt[n]{g}, \sqrt[n]{g^{-1}}] & & \\
0 & \longrightarrow & R & \longrightarrow & R' & &
\end{array}$$

where the latter is a $\mathbb{k}[\Sigma]$-comodule for $\Sigma = \langle p \rangle \cong \mathbb{Z}_n$ by quotient and module via $\sqrt[n]{g} \mapsto \zeta_n \sqrt[n]{g}$ with $\zeta_n \in \mathbb{Z}_n \subset k^\times$, forming a Yetter-Drinfel'd module (as $\mathbb{k}[\Sigma]$ is commutative and cocommutative, while action/coaction commute). Redefine with this $H' := H \otimes \Bbbk[\Sigma]$

$$0 \longrightarrow \Sigma \longrightarrow H' \longrightarrow H \longrightarrow 0$$

and call (H', R') the $\Sigma = \mathbb{Z}_n$-**extended coordinate ring datum**.

Remark 12.16. *It appears very natural to consider other Galois extensions E of the field of rational functions $\Bbbk(T)$ with ring of integers $\Bbbk[T] = H$ and localizations $\Bbbk[g, g^{-1}] = R$. Taking then Σ the Galois group, as in the example above, leads to a respective more general notion of twisted vertex module for this context.*

Assume now **first** a given Hopf vertex datum $(A, A, \langle\rangle, \beta)$ over the extended coordinate ring. The following can be thought of as an "intrinsically exponentiation" of the action of Σ on A, and indeed is the respective Γ_A.

Theorem 12.17. *The group Σ acts naturally (as any central grouplike in an arbitrary H) as*

(1) **Hopf algebra automorphisms** *on the undeformed states A.*
(2) **vertex algebra automorphism** *on the associated vertex algebra:*

$$\langle w|Y(a.p)(.p)|v\rangle = \langle w|(.p)Y(a)|v\rangle$$

Proof. The **first statement** follows by construction via Hopf vertex algebra datum, as A is an $H \otimes \Bbbk[\Sigma] =: H'$-module-Hopf-algebra and hence grouplikes p act as Hopf automorphisms.

The **second statement** is tougher - it is consequence of an additional property proven in **[Len07]** p. 29, namely for any central $t \in H$ we have:

$$\langle w|Y(a.t)|v\rangle = \langle w|(.t^{(2)})Y(a)(.S(t^{(1)}))|v\rangle$$

This property expresses a **general symmetry principle** on H-action:

$$Y \circ (.t^{(2)} \otimes .t^{(1)}) = (.t) \circ Y$$

$$\langle w|Y(a.t^{(2)})t^{(1)}|v.\rangle = \langle w|(.t^{(3)})Y(a)(.t^{(1)}S(t^{(2)}))|v\rangle = \langle w|(.t)Y(a)|v\rangle$$

For $t = p$ grouplike $t^{(1)} \otimes t^{(2)} = p \otimes p$ this expresses the definition of a vertex algebra automorphism.

Note, that our symmetry principle can also be applied to the classical translation operator $t = M$, which is derivational (=primitive) $t^{(1)} \otimes t^{(2)} = 1 \otimes M + M \otimes 1$. Thus, the symmetry of M is also derivational, a fact shown e.g. in **[FB01]** Corollary 3.6.1 as a consequence of Goddard's uniqueness theorem: M (and any other central primitive) acts as **infinitesimal** vertex algebra automorphism:

$$\langle w|Y(a.M)|v\rangle + \langle w|Y(a)(.M)|v\rangle = \langle w|(.M)Y(a)|v\rangle$$

$$\langle w|Y(a.M)|v\rangle = \langle w|[.M, Y(a)]|v\rangle$$

$\square$

Then we can describe, how to get back to a new *classical* vertex algebra as stabilizer of the new symmetry $\langle p\rangle \cong \mathbb{Z}_n$. While the substructures are established via the last theorem, most significantly one gets a **reduction** to the classical coordinate ring H, R simply by enforced Σ-invariance through the R'/R-Galois property.

Theorem 12.18. *The stabilizer under the action of p is a*

(1) **sub-***Hopf algebra $A^p \subset A$.*
(2) **restricted** *vertex module $A, Y|_{A^p\otimes A}$ over A^p.*
(3) **sub-***vertex algebra $A^p, Y|_{A^p\otimes A^p}$.*

and the latter are over (H, R), i.e. the action factorizes over $H \leftarrow H'$ and the restricted skew-form lands in R

$$\langle,\rangle|_{A\otimes A^p + A^p\otimes A} \to R \subset R'$$

Proof. Deriving **statements 1-3** from the last theorem are folk in their respective subject.

The **module structure** factorizes by construction, as A^p is defined as the Σ-stabilizer and the respective quotient projects to $H \subset H'$.

For the **skew-form** observe, that by the defining translation covariance of $\langle\rangle$, applied to $p \in \Sigma \subset H'$, we have for $b \in A^p$ (and the other side respectively):

$$\langle a, b\rangle = \langle a, b.p\rangle = \langle a, b\rangle.p$$

This means that for (a or) $b \in A^p$ the resulting R'-element $r := \langle a, b\rangle$ is stable under $\langle p\rangle = \Sigma$, and hence by the Galois property in $R \subset R'$. Namely in our specific case, all Laurent polynomials $P(g) \in$

$\Bbbk[\sqrt[n]{g}, \sqrt[n]{g^{-1}}] = R'$ invariant under $\sqrt[n]{g} \mapsto \zeta_n \sqrt[n]{g}$ lays already in $\Bbbk[g, g^{-1}] = R$. □

Now note, that on the other hand, there is an induced H', R'-vertex module structure for any eigenvalue $\lambda \in \Sigma^{**} \cong \Sigma$ of the Σ-action on A (Σ abelian, hence we may simultaneously diagonalize)

$$A^{p,\lambda} = \{a \in A | a.p = a\lambda(p)\} \subset A$$

Corollary 12.19. *The vertex operators $Y|_{A \otimes A^{p,\lambda}}$ defines an (H', R')-vertex module structure on $A^{p,\lambda}$, that lands in the respective* **twisted sector** $\sqrt[n]{g^m}\Bbbk[g, g^{-1}]$ *(as easily seen from the action of Σ on the definition of Y.*

Conjecture 12.20. *The author strongly assumes, that this defines also classically* **twisted vertex modules***, as they have (H', R')-associativity holding, p acting covariantly via λ and landing in the right function subspace. There seems however no direct "twisted associativity" to generalize, so one had to compute the* **twisted Jacobi identity** *- so far he shied away from the necessary calculations.*

As we described bicomodule algebras and especially Bigalois objects as orbifoldizings, we may just write down the respective cocycle we "know" to obtain the aspired example structure:

Example 12.21. *There exists (for n prime) a Bigalois structure on $A = \mathfrak{T}V$, given by the cocycle*

$$\sigma(p_i, p_j) = Res_{z_0} Res_{z_2} z_0^{-i} z_2 \sum_{r=1}^{n} \left(\frac{z + z_2}{z + z_0}\right)^{r/p} \frac{1 - \frac{r}{p} + \frac{r}{p}\frac{x+x_0}{x+x_2}}{(x_0 - x_2)^2} \langle \alpha_r, \beta \rangle$$

and the map obtained above is the twisted vertex module structure over the Heisenberg vertex algebra.

2. *The sub-Orbifold L Underlying The Moonshine Module*

We have no good clue so far, what exact orbifold should be formed. However, guessing from the "known" structure of the Moonshine module, we assume the following to be a good candidate for the **Bigalois object** underlying the sub-Orbifold $L \subset \Omega$ used throughout part 3:

Consider for G an even lattice the following decomposition of the sublattice $2G = K \cup -K$:

$$K = (-1)^{(a,a)/2} a^2$$

and choose an irreducible $k_\sigma[G/K]$-module T with a H-linear map $T \xrightarrow{j} \mathfrak{T}V$ with $V = \Bbbk[G] \otimes H$, that clarifies the action of $H = \Bbbk[M]$:

$$T \xrightarrow{\cdot M} T \otimes \mathfrak{T}V$$

$$t \mapsto t \otimes t \otimes j(t)$$

(note that in the lattice algebra above we had $T = k_\sigma[G]$ and $j(g) = p_0^g$). For the moonshine case $G = \Lambda$ such a representation T can be constructed of dimension 2^{12} leading to the unique twisted vertex algebra $A = T \otimes \mathfrak{T}V$.

Conjecture 12.22. *There exists a projective Hopf module structure on $\mathfrak{T}V$, such that the cocycle is generated by the series*

$$\sum_{i,j} \sigma(p_i, p_j) x^i y^j = -ln\left(\frac{(1+x)^{1/2} + (1+y)^{1/2}}{2}\right)$$

(see **[FLM84]***) and a map j sending T-elements to* **order 2 elements** *in $\mathfrak{T}V$, such that the map obtained above is the unique twisted vertex module structure over the Leech lattice vertex algebra.*

3. *Projectivity And Quasi Hopf Algebras*

The action of Σ^* is not trivial for the Moonshine module! Rather, it seems we have to start with non-coherent choices of cleavings, yielding a Yetter-Drinfel'd module in a category with nontrivial associativity constraints. As we'll work over the groupring $\Bbbk[\mathbb{Z}_2^{12}]$, it would be natural to try using the Parker loop (see introduction). It should yield a group-3-cocycle ω with the category in question the modules over the deformed Dijkgraaf double $D^\omega(\mathbb{Z}_2^{12})$, such that the R, F-matrices of the Parker loop are recovered accordingly. Note that the author has introduced (from a different motivation) **ω-projective Yetter-Drinfel'd modules** in an attempt to treat such categories as well via the Yetter-Drinfel'd module approach using conjugacy classes and projective irreps of the centralizer. The definition, category equivalence, basic structure theorems and examples have been worked out cleanly as diploma thesis by Karolina Vocke in 2010 **[V10]**, co-supervised by the author.

This corresponds to taking (as already for the lattice algebra) **deformed states** $\bar{A} \neq A$, a comodule algebra over A. To formulate the results of the preceding section, we need to carry over the respective notions:

Step 1: Vertex modules are now **projective Hopf modules**, as the action $\bar{A}$ has been twisted by a cocycle. To the knowledge of the author, this notion has not been defined yet, though he assumes it to be relative $\bar{A}$-A-Hopf modules structure.

Step 2: The group $\Sigma \ni p$ acts naturally (as any central grouplike in an arbitrary H) as a **"twisted" bicolinear algebra automorphisms** on the deformed states $\bar{A}$ with respect to the action of $p \in \Sigma$ on the coacting Hopf algebra A, i.e.

$$\left(a^{(0)}\right).p \otimes \left(a^{(0)}\right).p = (a.p)^{(1)} \otimes (a.p)^{(0)}.p$$

$$(ab).p = (a.p)(b.p)$$

Step 3: The stabilizer under p is a **sub-**bicomodule algebra $\bar{A}^p \subset \bar{A}$.

4. *Amalgams Of Groupoids And Weak Hopf Algebras*

In part 3 we have established a generic BN-pair for the automorphism group $Aut(\Omega)$ of an orbifold. As we've seen in this part's introduction, the monster $\mathbb{M}$ does **not** possess such, because (z_1, z_2 two commuting $2A$-involutions)

$$D = B \cap N = Cent(z_1) \cap Norm(\langle z_1, z_2 \rangle) \neq Cent(\langle z_1, z_2 \rangle) = \tilde{D}$$

is not normal in N. Rather, there are 3 subgroups $D, w.D, w^2.D$ conjugate by the **triality element** w corresponding to $N\cap$ the centralizer of $\{z_1, z_2, z_1z_2\}$ accordingly. This already points to the fact, that the constructions in this thesis are not general enough for this situation.

Recall that we introduced twisting groups in Definition 1.5 for arbitrary groupoids Σ leading presumably to weak Hopf algebras by Remark 1.7. This corresponds to non-isomorphic Doi twists $\Omega \supset H_n = A(\mathfrak{O}_n)$ for all objects $\mathfrak{O}_n \in Obj(\Sigma)$, where a group had a unique $H = A(\mathfrak{O})$ (as above, we identify object and Hopf algebra below). The twisted symmetries in Σ now permute the H_n and the coproduct maps Bigalois objects

$$A\left(Mor(\mathfrak{O}_l,\mathfrak{O}_n)\right) \to \bigcup_{\mathfrak{O}_m\in Obj(\Sigma} A\left(Mor(\mathfrak{O}_l,\mathfrak{O}_m)\right)\square_{H_m} A\left(Mor(\mathfrak{O}_m,\mathfrak{O}_n)\right)$$

Is it possible to complete such a weak Ω to a proper Hopf algebra? Well, certainly one may redefine $H = \bigotimes_n H_n$ and take as new twisting group Σ this single object $\mathfrak{O}$ (again Doi twist stable!) and as morphisms N-fold tensor products of Bigalois objects with left/right each H_n precisely once (N the number of groupoid objects); the author likes to call these H-Bigalois objects **stacks**. If the groupoid was connected, then certainly Σ is a group of order $|Mor(\mathfrak{O},\mathfrak{O})| = N!\cdot Mor(\mathfrak{O}_1,\mathfrak{O}_1)^N$; note that only if there are canonical identifications we get actually $\Sigma \cong \mathbb{S}_N \wr Hom(\mathfrak{O}_1,\mathfrak{O}_1)$.

But one can do better: Suppose we have underlying sub-Hopf-algebras $H' \subset H$, that **are** Doi twist stable (with coherent choices of isomorphism), then we could try to identify at least these H_n' and thus yield the analogon of a groups **universal amalgam completion**: Given (in this context conjugate) subgroups U_n and intersections $U_{ij}, U_{ijk}, \ldots$ find the universal group affording this situation!

5. *Conclusion: An Infinite Monster Nichols algebra*

We directly continue the preceding section and give an explicit example, where the amalgam that would intuitively fit the Moonshine case, especially the observations on the BN-pair:

Note that with so few knowledge so far, the author does not dare to claim this to be the right choice!

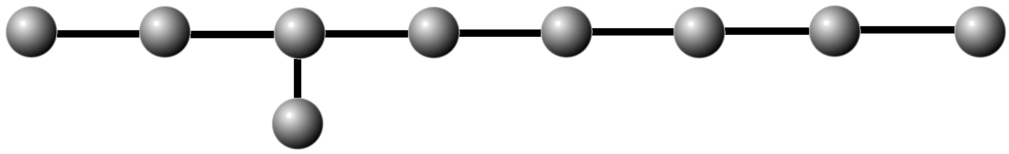
Take as twisting groupoid Σ the the so-called **action groupoid** of $\mathbb{S}_3$ acting on its three involution subgroups, i.e.

- Objects $\mathfrak{O}, w\mathfrak{O}, w^2\mathfrak{O} = \langle(23)\rangle, \langle(13)\rangle, \langle(12)\rangle$, identified with the D, wD, w^2D interchanged by the triality element w.
- Morphisms multiple copies of $\mathbb{S}_3$ elements, namely (for other objects accordingly)

$$Hom(\mathfrak{O},\mathfrak{O}) = \langle(23)\rangle \cong \mathbb{Z}_2$$

$$Hom(\mathfrak{O},w\mathfrak{O}) = \{w = (123), (12)\}$$

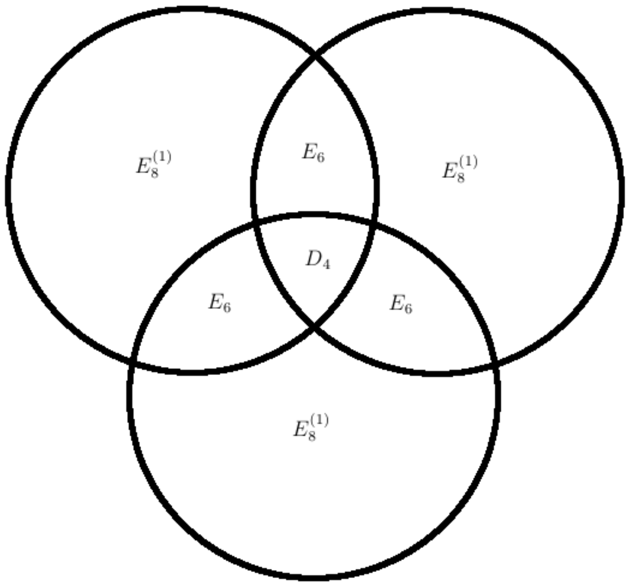
Assume for the moment we had for $\Gamma = \mathbb{Z}_2^{12}$ a projective Yetter-Drinfel'd module M (i.e. a module over $D^\omega(\Gamma)$ with ω producing the Parker loop, see section 12.3) with Dynkin diagram of affine type $E_8^{(1)}$ (possibly using Golay code group elements $\mathcal{G} \subset \Gamma$).



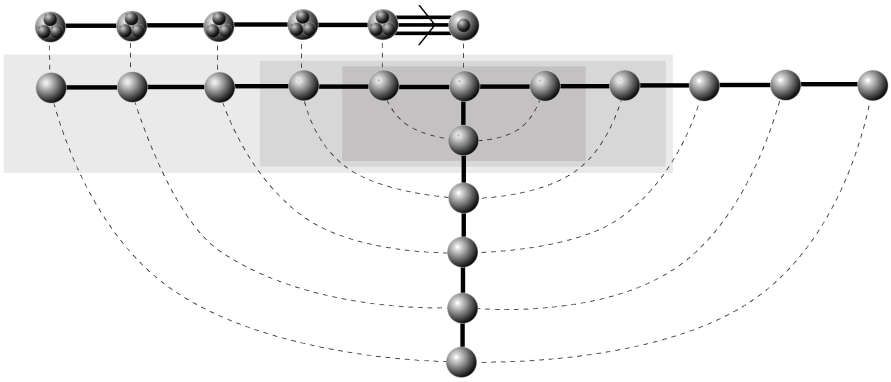
By Lemma 4.3 the Dynkin diagram is invariant under Doi twist, so M_{σ_w} and $M_{\sigma_{w^2}}$ have the same diagram. Then at least diagrammatically the following situation is **possible** (of course the decorations must be chosen appropriately to allow the twisted symmetries on the subdiagrams!):

- The sub-Yetter-Drinfel'd modules and -Nichols-algebras generated by the $E_6 \subset E_8^{(1)}$ afford a involutive twisted symmetry (hence are Doi twist invariant).
- The sub-Yetter-Drinfel'd modules and -Nichols-algebras generated by the $D_4 \subset E_8^{(1)}$ afford a $\mathbb{S}_3$-twisted symmetry (hence are Doi twist invariant).

Then we could attempt the (hypothetical) amalgam completion, that identifies these sub-Diagrams:



If the author's intuitive assumption about an amalgamed diagram is right, we would get for the orbifoldized and amalgamed $\tilde{M}$:



Note that we only marked one sub-Yetter-Drinfel'd module of each type, which we amalgamed along; actually there are 6 of type $E_8^{(1)}$, 3 of type E_6 and a unique of type D_4.

We could faintly hope for the following to finally happen:

- The three contained E_8-root-lattices form together the Leech lattice as in Turyn's construction of the latter.
- The three grouprings $\mathbb{k}[\Gamma] = \mathbb{Z}_2^{12}$ are amalgamed/orbifoldized to a single $G = 2_+^{24+1}$
- Hence the B in part 3 turns out to be $Co_1 \ltimes 2_+^{24+1}$.
- If N is also correct, an amalgamed version of Theorem 9.10 could yield $Aut(\Omega) = \mathbb{M}$.

So what we've done is actually construct a nonabelian $\mathbb{S}_3$-orbifold, but from the construction side only the three $\mathbb{Z}_2$-orbifoldizations corresponding to the $Hom(w^k\mathfrak{D}, w^k\mathfrak{D}) \cong \mathbb{Z}_2$ in the twisting groupoid Σ with 3 objects appear; together they generate an additional triality symmetry $(123) \in \mathbb{S}_3$ in the amalgam.

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All braiding diagrams in this thesis were generated by **Strid**, a tool to generate LaTeX-String diagrams by S. Mimram and N. Tabareau, *http://strid.sourceforge.net/*.

The q-decorated diagrams were copied from the papers of I. Heckenberger [**H05**][**H08**].

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