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Risk-Minimization for Life Insurance Liabilities

Irene Schreiber

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Erstgutachter:	Prof. Dr. Francesca Biagini
Zweitgutachter:	Prof. Dr. Thorsten Rheinländer

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Zusammenfassung

Die folgende Dissertation basiert auf drei wissenschaftlichen Arbeiten zum Thema Pricing und Hedging von Lebensversicherungsrisiken mit Hilfe des Risk-Minimization Ansatzes. Eine der wichtigen Charakteristika dieser Arbeit ist, dass das Hedging der Versicherungsrisiken mit Hilfe von Longevity Bonds möglich ist, die das systematische Mortalitätsrisiko repräsentieren. Im ersten Artikel wird der Fall einer einzelnen versicherten Person in einem sehr allgemeinen Setting bezüglich des zugrundeliegenden Asset-Preises und der Payoff-Struktur untersucht, d.h. es wird außerhalb des Brownschen Settings gearbeitet, insbesondere sind Sprünge des Asset-Preises zulässig. Außerdem werden weder die Unabhängigkeit der zugrundeliegenden Prozesse noch bestimmte technische Annahmen wie die Existenz der Mortalitätsintensität benötigt. Der zweite Artikel ist eine Erweiterung für den Fall eines homogenen Versicherungsportfolios. Hauptneuerungen dieser Arbeit sind, dass das sogenannte Basisrisiko berücksichtigt und konkret modelliert wird. Dieses Risiko entsteht dadurch, dass die Versicherung ihre Risiken nicht perfekt absichern kann, indem sie in ein Hedging Instrument investiert, das auf einem Longevity Index basiert, nicht auf dem Versicherungsportfolio selbst. Die Abhängigkeit zwischen dem Index und dem Versicherungsportfolio wird mit Hilfe eines affinen Diffusionsprozesses mit stochastischem Drift modelliert. Der letzte Artikel analysiert den Fall eines Portfolios, das aus Individuen verschiedener Altersklassen besteht. Die kohortenübergreifende Abhängigkeitsstruktur des Portfolios wird berücksichtigt, indem die Mortalitätsintensitäten als Random Field modelliert werden. Anhand konkreter Beispiele wird die Konsistenz mit historischen Mortalitätsdaten und Korrelationsstrukturen gezeigt.

Abstract

This dissertation is based on three papers on pricing and hedging of life insurance liabilities by means of the risk-minimization approach. One of the important features of this work is that we allow for hedging of the risk inherent in the life insurance liabilities by investing in longevity bonds representing the systematic mortality risk. In the first article we consider the case of one insured person in a very general setting regarding the underlying asset price and the structure of the insurance payment process studied, i.e. we work outside the Brownian setting, in particular the asset price may have jumps. Besides that, we are able to relax certain technical assumptions such as the existence of the mortality intensity and we do not require the independence of the underlying processes. The second paper provides an extension to the case of a homogenous insurance portfolio. Main novelties of this work are that we take into account and explicitly model the basis risk that arises due to the fact that the insurance company cannot perfectly hedge its exposure by investing in a hedging instrument that is based on a longevity index, not on the insurance portfolio itself. We model the dependency between the index and the insurance portfolio by means of an affine mean-reverting diffusion process with stochastic drift. The last article considers an insurance portfolio that consists of individuals of different age cohorts. In order to capture the cross-generational dependency structure of the portfolio we model the mortality intensities as random fields. We also provide specific examples consistent with historical mortality data and correlation structures.

Contents

Chapter 1

Introduction

This dissertation is based on three papers on pricing and hedging of life insurance liabilities by means of the well-known risk-minimization approach (see Biagini and Schreiber [8], Biagini, Rheinländer, and Schreiber [10] and Biagini, Botero, and Schreiber [9]). In the first article we consider the case of one insured person in a very general setting regarding the underlying asset price and the structure of the insurance payment process studied. The second paper provides an extension to the case of a homogenous insurance portfolio, where all individuals belong to the same age cohort. The last article considers an insurance portfolio that consists of individuals of different age cohorts.

Mortality or longevity is a primary source of risk for many insurance and pension products. For example, annuity providers face the risk that the mortality rates of pensioners might fall at a faster rate than expected, whereas life insurers are exposed to the risk of unexpected increases in mortality. The traditional method of dealing with mortality risk is through suitable insurance or reinsurance contracts. However, reinsurers are often reluctant to take on the aggregated bulk risk typical of these transactions, thus leading to securitization as a new form of risk transfer and consequently to the creation of a new life market, see, e.g., Blake et al. [17]. In this context pricing and modeling of mortality-linked securities has been studied extensively in the literature, for an overview on the valuation and securitization of mortality risk we refer to Cairns et al. [19].

Mortality risk can essentially be split into systematic risk represented by the mortality intensity, i.e. the risk that the mortality rate of an age cohort differs from the one expected at inception, and idiosyncratic or unsystematic risk, i.e. the risk that the mortality rate of the individual is different from that of its age cohort. The first kind of risk may be hedged by investing in longevity bonds, see, e.g., Cairns et al. [19]. These bonds pay out the conditional survival probability at maturity as a function of the hazard rate or mortality intensity, which is given by so-called longevity or survivor indices. Survivor indices, provided by various investment banks, consist of publicly available mortality data aggregated by population, hence providing a good proxy for the systematic component of the mortality risk. One of the features of our approach is to allow for hedging of the risk inherent in the life insurance liabilities by investing not only in the stock and money market account, but also in longevity bonds, accounting for the systematic mortality risk. When modeling life insurance liabilities we make use of the similarities between mortality and credit risk and follow the intensity-based or hazard rate approach of reduced-form modeling, see, e.g., Bielecki and Rutkowski [12]. Since it is impossible to completely hedge the financial and mortality risk inherent in the liabilities of the insurance company, even in this setting where we allow for investments in products representing the systematic mortality risk, the market is incomplete and it is thus necessary to select one of the techniques for pricing and hedging in incomplete markets. Here we make use of the popular risk-minimization method first introduced by Föllmer and Sondermann [36]. The idea of this technique is to allow for a wide class of admissible strategies that in general might not necessarily be self-financing, and to find an optimal hedging strategy with "minimal risk" within this class of strategies that perfectly replicates the given claim. For a survey on risk-minimization and other quadratic hedging methods we refer to Schweizer [58].

In Chapter 2, based on results from Biagini and Schreiber [8], we study the problem of pricing and hedging life insurance liabilities for the case of one insured person in a very general setting by means of the risk-minimization approach. First we consider a financial market model with one risky primary asset, e.g. a stock, and the riskless money market account. In this setting we compute the price and hedging strategy for an insurance payment process whose value may depend on the primary assets as well as on the time of death of a single individual, such as a unit-linked life insurance contract. In a second step we extend the financial market by introducing two mortality-linked securities, a longevity bond, incorporating the systematic longevity risk, and a pure endowment contract, representing the unsystematic mortality risk. The main idea then is to hedge the financial and (systematic and unsystematic) mortality risk by investing in both the stock and the bank account, as well as in the two mortality-linked securities. We would like to emphasize, that hedging with these two mortality-linked securities is intrinsic in our modeling context, in a sense that it does not depend on the specific form of the insurance payment process. One may argue, that in the case where large portfolios with independent risks are pooled by insurance companies, the unsystematic risk might be eliminated by law of large number arguments. However, in many cases portfolios with a smaller number of insured lives are of interest. Furthermore in some situations, for instance in the case of catastrophic mortality events, it is not realistic to assume independence between members of the portfolio. Hence hedging of both the systematic and unsystematic mortality risk and thus completely eliminating the cost term may be of great value in many practical applications. There exist a number of studies that focus on applications of the risk-minimization approach in the context of mortality modeling, see, e.g., Barbarin [4], Biagini et al. [11], Møller et al. [26, 27, 53, 54] and Riesner [57]. However, most authors study quadratic hedging for very specific insurance products in a Brownian setting, whereas we allow for more general assumptions regarding the given filtrations and the structure of the insurance liabilities. Also, some authors such as Møller [53, 54] and Riesner [57] assume independence between the financial market and the insurance model. In the context of credit risk modeling Biagini and Cretarola [5, 6, 7] study local risk-minimization for defaultable claims, again in a Brownian setting. Here we allow for mutual dependence between the time of death and the asset prices behavior as in Biagini and Cretarola [6, 7] and Biagini et al. [11], however we extend their results since we allow for a more general structure of the insurance payment process and we do not require the existence of the mortality intensity. Besides that, similarly as in Barbarin [4], we work outside the Brownian setting, in particular we allow for jumps in the asset price. Hence in this chapter we extend earlier work on risk-minimization for insurance products in several directions: we work in a very general setting regarding the underlying asset price and the structure of the payment process studied, we are able to relax certain technical assumptions such as the existence of the mortality intensity and we do not require the independence of the underlying processes. We also allow for investments in the primary assets as well as in two further mortality-linked securities.

In Chapter 3, based on results from Biagini, Rheinländer, and Schreiber [10], we study pricing and hedging of life insurance liabilities for the case of a homogenous insurance portfolio. In a financial market where we allow for hedging by investing in longevity bonds representing the systematic mortality risk we take into account and explicitly model the basis risk that arises due to the fact that the insurance company cannot perfectly hedge its exposure by investing in a hedging instrument that is based on a longevity index, not on the insurance portfolio itself. Because of differences in socioeconomic profiles (with respect to e.g. health, income or lifestyle), the mortality rates of the population typically differ from those of the insurance portfolio. Hence the hedge will be imperfect, leaving a residual amount of risk, know as basis risk. There exist a number of empirical studies concerned with quantifying and modeling mortality basis risk, see, e.g., Cairns et al. [20], Coughlan et al. [23], Dowd et al. [30], Jarner and Kryger [43], Li and Hardy [48] and Li and Lee [49]. Li and Lee [49] are the first to study the mortality rate of closely related populations within a global modeling context. They extend the well-known Lee-Carter model by introducing the concept of a global improvement process together with mean-reverting idiosyncratic variations for each population. Cairns et al. [20], Dowd et al. [30] and Jarner and Kryger [43] model the mortality rates of a small population that is a subpopulation of a larger reference population, where the relationship between the large and small population's mortality rates is determined by a mean-reverting stochastic spread. In this chapter, similarly as in Biffis [13], we model the mortality intensity of the insurance portfolio together with the intensity of the population by means of a multivariate affine square-root diffusion. The dependency between the two populations is captured by the fact that the intensity of the insurance portfolio is fluctuating around a stochastic drift, which is given by the mortality intensity of the reference index. This model is intuitive in its interpretation, as well as analytically tractable through its affine structure. Affine models have become very popular in many areas of applied financial mathematics, such as exotic option pricing, or interest rate and credit risk modeling. An overview of the theory of affine processes can be found in Duffie et al. [32], as well as in Filipović and Mayerhofer [35] for the case of affine diffusions. As mentioned above, there exist a number of studies that focus on applications of the risk-minimization approach in the context of mortality modeling or in related areas such as credit risk. As in Chapter 2, here we work in a general setting where we allow for mutual dependence between the times of death and the financial market, as well as for general payoff structures similarly as in Barbarin [4] and Biagini et al. [6, 7]. Besides that, similarly as in Biagini et al. [11] and Dahl et al. [27], we allow for hedging of the insurance liabilities by investing not only in the primary financial market, but also in an instrument representing the systematic mortality risk. Dahl et al. [27] also model the dependency between two death counting processes, the first one representing an insurance portfolio and the second one the whole population. They allow for dependency between the mortality intensities via correlated diffusion terms. Here we consider an affine mean-reverting diffusion model with stochastic drift and model the portfolio mortality intensity as depending on the evolution of the intensity of the population. This has the great advantage, of capturing the basis risk between the insurance portfolio and the longevity index in a very natural way, thereby offering an intuitive interpretation while remaining analytically tractable due to the affine structure. Also in this way it is not necessary to artificially introduce a second death counting process representing the population. Hence in this chapter we extend earlier work on risk-minimization for insurance products in several directions: we provide explicit computations of risk-minimizing strategies for a portfolio of life insurance liabilities in a complex setting. Thereby we explicitly take into account and model the basis risk between the insurance portfolio and the longevity index and allow for investments in hedging instruments representing the systematic mortality risk. Besides that, we allow for a general structure of the insurance products studied and we do not require certain technical assumptions such as the independence of the financial market and the insurance model.

In Chapter 4, based on results from Biagini, Botero, and Schreiber [9], we consider an insurance portfolio that consists of individuals of different age cohorts. In a financial market where we allow for hedging by investing in longevity bonds representing the systematic mortality risk we capture the cross-generational dependency structure of the portfolio by modeling the mortality intensities as random fields. We also provide specific applications consistent with historical mortality data and correlation structures. In practice, one typically considers homogeneous classes of policyholders and then aggregates market valuations of liabilities at portfolio level without taking dependencies between cohort classes into account. To the best of our knowledge there exist only very few studies concerned with quantifying and modeling inter-age dependencies in stochastic mortality models. Based on a multivariate time series study of yearly mortality rates Loisel and Serrant [50] propose a discrete-time multi-dimensional extension of the well-known Lee-Carter model that takes inter-age correlations into account. Blackburn and Sherris [15] and Jevtic et al. [45] propose affine continuous-time factor models for the mortality surface, allowing for correlation across different generations. Biffis and Millossovich [14] model the mortality intensity surface as a random field and with a view on the insurer's future business consider market valuations of pure endowment contracts with deterministic survival benefit. Random fields have been employed in mathematical finance when modeling the term structure of interest rates (see, e.g., Furrer [38], Goldstein [39], Kennedy [47]) and have proven to be very useful in our context as well. Similarly as Biffis and Millossovich [14], we model the mortality surface as a random field parameterized in time and age at inception of the contract. In a complex setting with a portfolio consisting of different age cohorts we study risk-minimization for life insurance liabilities (unit-linked pure endowment, term insurance and annuity contracts) at an aggregate level. By modeling the mortality intensities as a random surface we are able to look simultaneously in both the time and age direction. This is very important, since there is statistical evidence that typical downward mortality improvement trends are not homogeneous across age cohorts (see, e.g., Andreev [2] and Forfar and Smith [37]). Besides that, this approach enables us to establish a mortality model consistent with historical data that takes inter-age correlations into account in a natural and elegant way. Since the mortality intensity of every age cohort is an affine process, the model is analytically tractable, allowing us to compute hedging strategies for life insurance liabilities in an immediate and parsimonious way. Hence in this chapter we extend earlier work on risk-minimization for insurance products in several directions: we provide explicit computations of risk-minimizing strategies for life insurance liabilities written on an insurance portfolio in a complex setting where we consider different age cohorts simultaneously. Thereby we take into account and explicitly model the dependency structure of the insurance portfolio by introducing analytically tractable affine models for the mortality intensities consistent with historical mortality data based on Gaussian random fields. Besides that, similarly as in Chapter 2 and 3, we allow for hedging by investing in a family of longevity bonds representing the systematic mortality risk and we do not require certain technical assumptions such as the independence of the financial market and the insurance model.

Chapter 2

The Single Life Case

2.1 Introduction

In this chapter, based on Biagini and Schreiber [8], we study the problem of pricing and hedging life insurance liabilities for the case of one insured person in a very general setting by means of the risk-minimization approach. First we consider a financial market model with one risky primary asset, e.g. a stock, and the riskless money market account. In a second step we extend the financial market by introducing two mortality-linked securities, a longevity bond, incorporating the systematic longevity risk, and a pure endowment contract, representing the unsystematic mortality risk. We allow for a very general setting regarding the structure of the payment process studied and the underlying asset prices, in particular we work outside the Brownian setting and the asset prices may have jumps. Besides that we are able to relax certain technical assumptions such as the existence of the mortality intensity and we do not require the independence of the underlying processes. The remainder of this chapter is organized as follows: Section 2.2 introduces the general setup. In Section 2.3, we provide our main result by computing the Galtchouk-Kunita-Watanabe (GKW) decomposition and finding the price and risk-minimizing strategy of the life insurance liabilities. The financial market is extended by two tradable mortality-linked securities representing the systematic and unsystematic mortality risk in Section 2.4, thus completing the market and eliminating the cost process. Section 2.5 then concludes this chapter with a specific example where we consider a unit-linked term insurance contract in a jump-diffusion model for the asset price with affine stochastic mortality intensity.

2.2 The Setting

For a fixed time horizon $T > 0$ we consider a simple financial market model defined on a given probability space $(\Omega, \mathcal{G}, \mathbb{P})$ consisting of one risky asset with discounted

asset price X and discounted bank account X^0, i.e. $X^0_t \equiv 1$, $t \in [0,T]$. On this probability space we assume given a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, such that X is a local $(\mathbb{P},\mathbb{F})$-martingale, i.e. the financial market given by X is arbitrage-free.
We now introduce the *time of death* of an individual, given by a strictly positive random variable $\tau : \Omega \to [0,T] \cup \{\infty\}$, defined on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$ with $\mathbb{P}(\tau = 0) = 0$ and $\mathbb{P}(\tau > t) > 0$ for each $t \in [0,T]$. Note that since the time horizon T is usually fixed as the maturity of the life insurance contract, in order to ensure that $\mathbb{P}(\tau > T) > 0$ (the remaining lifetime τ is not necessarily bounded by T) it is necessary to allow τ to take values larger than T, indicated here by the convention that τ can assume the value infinity. We define the *death process* $H_t = \mathbb{1}_{\{\tau\le t\}}$ and denote by $\mathbb{H} = (\mathcal{H}_t)_{t\in[0,T]}$ the filtration generated by this process. In this setting we consider the extended market $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, such that the information available to all agents in the market at time t is assumed to be $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ and we put $\mathcal{G} = \mathcal{G}_T$. It is clear that τ is an $\mathbb{H}$-stopping time, as well as a $\mathbb{G}$-stopping time, but not necessarily an $\mathbb{F}$-stopping time. In fact here we assume that the random time τ avoids every $\mathbb{F}$-stopping time $\tilde{\tau}$, i.e. $\mathbb{P}(\tau = \tilde{\tau}) = 0$, and under this hypothesis we have that τ is a totally inaccessible $\mathbb{G}$-stopping time and $\Delta U_\tau = 0$ for any $\mathbb{F}$-adapted càdlàg process U (see, e.g., Coculescu et al. [21] or Blanchet-Scalliet and Jeanblanc [18]). All filtrations are assumed to satisfy the usual hypotheses of completeness and right-continuity. We postulate that all $\mathbb{F}$-local martingales are also $\mathbb{G}$-local martingales, and in the sequel we refer to this hypothesis as Hypothesis (H). This hypothesis is well-known in the literature of reduced form approaches for valuating defaultable claims, for a discussion of this hypothesis we refer to Blanchet-Scalliet and Jeanblanc [18]. In this setting we follow the hazard rate or intensity-based approach, well-known from reduced-form modeling of credit derivatives (see, e.g., Bielecki and Rutkowski [12]), which means that as opposed to the structural approach the default time occurs as a surprise for the market participants, since the time of death τ is a totally inaccessible stopping time. Therefore it is not possible to predict τ, and an important role is then played by the conditional distribution function of τ, given by

$$F_t = \mathbb{P}(\tau \le t \,|\, \mathcal{F}_t),$$

and we assume $F_t < 1$ for all $t \in [0,T]$. Then the *hazard process* Γ of τ under $\mathbb{P}$

$$\Gamma_t = -\ln(1 - F_t) = -\ln \mathbb{E}[\mathbb{1}_{\{\tau>t\}} \,|\, \mathcal{F}_t]$$

is well-defined for every $t \in [0,T]$. In particular under the above conditions the hazard process Γ is continuous and increasing (see, e.g., Coculescu et al. [21]) and we additionally assume that Γ_T is bounded. Note that this rather strong assumption is not always required in concrete examples, since it may be possible to directly check the necessary integrability conditions (see also Section 2.5). The process

$$e^{-\Gamma_t} = \mathbb{P}(\tau > t \,|\, \mathcal{F}_t), \quad t \in [0,T],$$

is often called a *survivor index* and according to Cairns et al. [19] can be seen as the basic building block for many other mortality-linked securities. The need for standardization in the life markets has led to the creation of various such indices by investment banks comprising publicly available mortality data for various age cohorts across populations of many different countries. Therefore many market-traded securities have payments linked to a survivor index, e.g. they pay out the survivor index or a function of the survivor index at maturity T. Hence a fundamental role is played by the $\mathbb{F}$-martingale

$$\mathbb{E}[e^{-\Gamma_T} \,|\, \mathcal{G}_t] = \mathbb{E}[e^{-\Gamma_T} \,|\, \mathcal{F}_t] = \mathbb{E}[\mathbb{1}_{\{\tau > T\}} \,|\, \mathcal{F}_t], \quad t \in [0, T], \tag{2.2.1}$$

since it represents the information on the mortality risk contained in the filtration $\mathbb{F}$, i.e. it describes the systematic mortality risk as we will see in Section 2.3 and 2.4. Note that in the first equation of (2.2.1) we have used that Hypothesis (H) is equivalent to the fact that conditioning on $\mathcal{G}_t$ can be replaced by conditioning on $\mathcal{F}_t$ for $\mathcal{F}_T$-measurable random variables (see, e.g., Bielecki and Rutkowski [12]).
Commonly the hazard rate process Γ is represented as an integral over the mortality intensity, which itself is given by a diffusion. Here we work in a more general setting, since we do not require the existence of the mortality intensity. Instead we describe the systematic mortality risk component $\mathbb{E}[\mathbb{1}_{\{\tau > T\}} \,|\, \mathcal{F}_t]$, $t \in [0, T]$, as driven by a local $\mathbb{F}$-martingale Y strongly orthogonal[1] to X, see (2.4.1) in Section 2.4 and also (2.5.4) in Section 2.5, where Y is given by a Brownian motion. In general, financial markets may be affected by consistent or sudden variations of the mortality rate, hence we a priori do not consider X and Y to be independent. However, we suppose that they are strongly orthogonal, since mortality is external to the financial markets, and not hedgeable by investing only in the primary assets.
By Proposition 5.1.3 of Bielecki and Rutkowski [12] we obtain that the compensated process M given by

$$M_t = H_t - \Gamma_{t \wedge \tau}, \quad t \in [0, T], \tag{2.2.2}$$

follows a $\mathbb{G}$-martingale. Since M is a finite variation process and X has no jump in τ, by Proposition 4.52 of Jacod and Shiryaev [42, Chapter I] for the square bracket process we have

$$\begin{aligned} [X, M]_t &= \langle X^C, M^C \rangle_t + \sum_{0 \le s \le t} \Delta X_s \Delta M_s \\ &= \sum_{0 \le s \le t} \Delta X_s \Delta M_s = X_0 M_0 = 0, \end{aligned}$$

$t \in [0, T]$, where X^C and M^C denote the continuous martingale parts of X and M. Hence by Proposition 4.50 of Jacod and Shiryaev [42, Chapter I], XM is a

[1] We recall that two local martingales X, Y are said to be *strongly orthogonal* if the product $(X_t Y_t)_{t \in [0,T]}$ is a local martingale.

local martingale, i.e. X and M are strongly orthogonal. Note that by the same arguments M is in fact strongly orthogonal to any $\mathbb{F}$-adapted local martingale.
In this setting we now introduce a square integrable (discounted) *life insurance payment process* A:

$$A_t = \mathbb{1}_{\{\tau \leq t\}} \bar{A}_\tau + \mathbb{1}_{\{t=T\}} \mathbb{1}_{\{\tau > T\}} \tilde{A}, \tag{2.2.3}$$

where $\bar{A} = (\bar{A}_t)_{t\in[0,T]}$ is an $\mathbb{F}$-predictable process, such that $\mathbb{E}\left[\sup_{t\in[0,T]} \bar{A}_t^2\right] < \infty$ and $\tilde{A}$ is a $\mathcal{G}_T$-measurable random variable, such that $\mathbb{E}[\tilde{A}^2] < \infty$.

Remark 2.2.1. *We would like to comment on the structure of A as defined in* (2.2.3)*. The first part*

$$\mathbb{1}_{\{\tau \leq T\}} \bar{A}_\tau$$

consists of a so-called term insurance contract*, i.e. the contract pays out $\bar{A}_\tau$ at the random time τ in case of death before T. The second part*

$$\mathbb{1}_{\{\tau > T\}} \tilde{A}$$

is a pure endowment contract*, i.e. the contract pays out $\tilde{A}$ in case of survival until T. It is now widely acknowledged (see, e.g. Barbarin [4], Biffis [13] and Møller [53]) that most mortality linked securities of practical relevance are of the form* (2.2.3)*. For example, consider an* annuity contract *with accumulated payments up to the time of death given by $C = (C_t)_{t\in[0,T]}$, where C is an $\mathbb{F}$-adapted, non-negative continuous increasing process such that $C_0 = 0$. Then the accumulated payoff can be decomposed as*

$$\int_0^T \mathbb{1}_{\{\tau > s\}} \mathrm{d}C_s = C_\tau \mathbb{1}_{\{\tau \leq T\}} + C_T \mathbb{1}_{\{\tau > T\}},$$

i.e. the payoff is given by (2.2.3) *with $\bar{A}_t = C_t$ and $\tilde{A} = C_T$. Also note that the form of A is in fact very general as a consequence of Lemma 4.4 in Chapter IV.2 of Jeulin [44], where the general form of a $\mathbb{G}$-predictable process in terms of $\mathbb{F}$-predictable processes is given.*

Recall that the primary financial market is arbitrage-free (but not necessarily complete), and it is a well-known fact that Hypothesis (H) is a sufficient condition for the market given by the larger filtration $\mathbb{G}$ to be arbitrage-free, see, e.g., Blanchet-Scalliet and Jeanblanc [18]. Nevertheless, since it is impossible to hedge a short position in A by investing in a portfolio consisting only of the primary assets our extended market model $\mathbb{G}$ is incomplete even in the case where the reduced market generated by X is complete. In order to find a price and hedge for the life insurance liabilities we therefore make use of a well-known quadratic hedging method for pricing and hedging in incomplete markets, the (local) risk-minimization approach that will be briefly discussed in the following section.

Remark 2.2.2. *In this work our focus is on risk-minimization with an application to life markets in a general setting, i.e. our modeling framework regarding the two primary assets is generic in a sense that we do not specify the dynamics of the bank account, but instead directly consider everything in a discounted world. In fact various choices for the discounting factor are feasible in this context, such as the so-called $\mathbb{P}$-numéraire portfolio*[2]*, under which according to Platen and Heath [55] the discounted asset prices are local martingales if they are described by continuous processes or in a wide class of jump-diffusion models.*

2.3 Risk-Minimization for Life Insurance Liabilities

Under the hypotheses of Section 2.2 we now compute the price and hedging strategy for the life insurance payment process A as introduced in (2.2.3) by applying the results of Appendix A. In order to find a hedging strategy with optimal cost, we compute the GKW decomposition of

$$\mathbb{E}[A_T \mid \mathcal{G}_t] = \underbrace{\mathbb{E}[\mathbb{1}_{\{\tau \le T\}} \bar{A}_\tau \mid \mathcal{G}_t]}_{a)} + \underbrace{\mathbb{E}[\mathbb{1}_{\{\tau > T\}} \tilde{A} \mid \mathcal{G}_t]}_{b)}, \quad t \in [0,T]. \tag{2.3.1}$$

We now separately compute the terms $a)$ and $b)$ in (2.3.1). We start with $a)$.

Lemma 2.3.1. *Let $\bar{A} = (\bar{A}_t)_{t\in[0,T]}$ be given as in* (2.2.3)*. Then for $a)$ in* (2.3.1) *we have the following decomposition:*

$$\mathbb{E}[\mathbb{1}_{\{\tau \le T\}} \bar{A}_\tau \mid \mathcal{G}_t] = \bar{m}_0 + \int_{]0,t]} \mathbb{1}_{\{\tau \ge s\}} e^{\Gamma_s}\, \mathrm{d}\bar{m}_s + \int_{]0,t]} (\bar{A}_s - e^{\Gamma_s} U_s)\, \mathrm{d}M_s, \quad t \in [0,T],$$

where

$$U_t = \bar{m}_t - \int_0^t \bar{A}_s e^{-\Gamma_s}\, \mathrm{d}\Gamma_s \tag{2.3.2}$$

and

$$\bar{m}_t = \mathbb{E}\left[\int_0^T \bar{A}_s e^{-\Gamma_s}\, \mathrm{d}\Gamma_s \,\middle|\, \mathcal{F}_t \right]. \tag{2.3.3}$$

Proof. First note that since Γ is continuous and increasing and M is stopped in τ, by Proposition 5.1.3 of Bielecki and Rutkowski [12] we have that

$$L_t := \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} = 1 - \int_{]0,t]} L_{s-}\, \mathrm{d}M_s = 1 - \int_{]0,t]} e^{\Gamma_s}\, \mathrm{d}M_s, \quad t \in [0,T]. \tag{2.3.4}$$

[2] A strictly positive, finite, self-financing portfolio V^* with initial capital 1 is called $\mathbb{P}$-*numéraire portfolio*, if every nonnegative, finite, self-financing portfolio V with initial capital 1, when denominated in units of V^*, forms a supermartingale, that is for $0 \le s \le t \le T$:

$$\mathbb{E}\left[\frac{V_t}{V_t^*} \,\middle|\, \mathcal{F}_s \right] \le \frac{V_s}{V_s^*}.$$

Then by Corollary 5.1.3 of Bielecki and Rutkowski [12] for $t \in [0,T]$ we have

$$\mathbb{E}[\mathbb{1}_{\{\tau \le T\}} \bar{A}_\tau \,|\, \mathcal{G}_t] = \mathbb{1}_{\{\tau \le t\}} \bar{A}_\tau + L_t U_t,$$

where U is given by (2.3.2) and $\bar{m}$ is given by (2.3.3). By (2.3.4) and an application of Itô's formula we get

$$\begin{aligned} L_t U_t &= \bar{m}_0 + \int_{]0,t]} L_{s-} \,\mathrm{d}U_s + \int_{]0,t]} U_{s-} \,\mathrm{d}L_s \\ &= \bar{m}_0 + \int_{]0,t]} \mathbb{1}_{\{\tau \ge s\}} e^{\Gamma_s} \,\mathrm{d}\bar{m}_s - \int_0^t \mathbb{1}_{\{\tau \ge s\}} \bar{A}_s \,\mathrm{d}\Gamma_s - \int_{]0,t]} e^{\Gamma_s} U_s \,\mathrm{d}M_s, \end{aligned}$$

since U has no jumps in τ. Hence for $t \in [0,T]$ we obtain that

$$\begin{aligned} \mathbb{E}[\mathbb{1}_{\{\tau \le T\}} \bar{A}_\tau \,|\, \mathcal{G}_t] &= H_t \bar{A}_\tau + \bar{m}_0 + \int_{]0,t]} \mathbb{1}_{\{\tau \ge s\}} e^{\Gamma_s} \,\mathrm{d}\bar{m}_s - \int_0^t \mathbb{1}_{\{\tau \ge s\}} \bar{A}_s \,\mathrm{d}\Gamma_s \\ &\quad - \int_{]0,t]} e^{\Gamma_s} U_s \,\mathrm{d}M_s \\ &= \bar{m}_0 + \int_{]0,t]} \mathbb{1}_{\{\tau \ge s\}} e^{\Gamma_s} \,\mathrm{d}\bar{m}_s + \int_{]0,t]} (\bar{A}_s - e^{\Gamma_s} U_s) \,\mathrm{d}M_s, \end{aligned}$$

and the result follows. □

Note that Corollary 5.1.3 of Bielecki and Rutkowski [12] requires $\bar{A}$ to be bounded. However, it can be easily seen that this result also holds if $\mathbb{E}[\sup_{t \in [0,T]} \bar{A}_t^2] < \infty$ and we may therefore apply it in our setting. For the second term $b)$ of (2.3.1) we have the following result.

Lemma 2.3.2. *Let $\tilde{A} \in L^2(\mathcal{G}_T, \mathbb{P})$. Then for $b)$ in (2.3.1) we have the following decomposition:*

$$\mathbb{E}[\mathbb{1}_{\{\tau > T\}} \tilde{A} \,|\, \mathcal{G}_t] = \tilde{m}_0 + \int_{]0,t]} \mathbb{1}_{\{\tau \ge s\}} e^{\Gamma_s} \,\mathrm{d}\tilde{m}_s - \int_{]0,t]} e^{\Gamma_s} \tilde{m}_s \,\mathrm{d}M_s, \quad t \in [0,T],$$

where

$$\tilde{m}_t = \mathbb{E}[\mathbb{1}_{\{\tau > T\}} \tilde{A} \,|\, \mathcal{F}_t]. \tag{2.3.5}$$

Proof. By Corollary 5.1.1 of Bielecki and Rutkowski [12] we have

$$\mathbb{E}[\mathbb{1}_{\{\tau > T\}} \tilde{A} \,|\, \mathcal{G}_t] = L_t \tilde{m}_t, \quad t \in [0,T],$$

where $\tilde{m}$ is given by (2.3.5). By the same arguments as in the proof of Lemma 2.3.1 we get

$$\begin{aligned} L_t \tilde{m}_t &= \tilde{m}_0 + \int_{]0,t]} L_{s-} \,\mathrm{d}\tilde{m}_s + \int_{]0,t]} \tilde{m}_{s-} \,\mathrm{d}L_s \\ &= \tilde{m}_0 + \int_{]0,t]} \mathbb{1}_{\{\tau \ge s\}} e^{\Gamma_s} \,\mathrm{d}\tilde{m}_s - \int_{]0,t]} e^{\Gamma_s} \tilde{m}_s \,\mathrm{d}M_s, \end{aligned}$$

hence the result follows. □

The next theorem states the most important result of this chapter, providing the risk-minimizing strategy of the insurance payment process as defined in (2.2.3).

Theorem 2.3.3. *In the market model outlined in Section 2.2, every insurance payment process admits a risk-minimizing strategy* $\varphi = (\xi, \xi^0)$ *given by*

$$\begin{aligned}\xi_t &= \mathbb{1}_{\{\tau \geq t\}} e^{\Gamma_t} \xi_t^m, \\ \xi_t^0 &= V_t - \xi_t X_t = V_t - \mathbb{1}_{\{\tau \geq t\}} e^{\Gamma_t} \xi_t^m X_t,\end{aligned}$$

with discounted value process

$$\begin{aligned}V_t(\varphi) &= \mathbb{E}[A_T \,|\, \mathcal{G}_t] - A_t \\ &= m_0 + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \xi_s^m \, \mathrm{d}X_s + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \eta_s^m \, \mathrm{d}Y_s \\ &\quad + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \, \mathrm{d}C_s^m + \int_{]0,t]} \psi_s^M \, \mathrm{d}M_s - A_t \qquad (2.3.6)\end{aligned}$$

and optimal cost process

$$C_t(\varphi) = m_0 + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \eta_s^m \, \mathrm{d}Y_s + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \, \mathrm{d}C_s^m + \int_{]0,t]} \psi_s^M \, \mathrm{d}M_s, \quad (2.3.7)$$

for $t \in [0, T]$*, where the processes* M*,* m*,* ψ^M*,* ξ^m*,* η^m *and* C^m *are introduced respectively in* (2.2.2) *and* (2.3.9) - (2.3.11).

Proof. By Lemma 2.3.1 and Lemma 2.3.2 for $t \in [0, T]$ we have that

$$V_t^A := \mathbb{E}[A_T \,|\, \mathcal{G}_t] = m_0 + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \, \mathrm{d}m_s + \int_{]0,t]} \psi_s^M \, \mathrm{d}M_s, \qquad (2.3.8)$$

with

$$m_t = \bar{m}_t + \tilde{m}_t = \mathbb{E}\left[\int_0^T \bar{A}_s e^{-\Gamma_s} \, \mathrm{d}\Gamma_s \,\Big|\, \mathcal{F}_t\right] + \mathbb{E}[\mathbb{1}_{\{\tau > T\}} \tilde{A} \,|\, \mathcal{F}_t] \qquad (2.3.9)$$

and

$$\psi_t^M = \bar{A}_t - e^{\Gamma_t}(U_t + \tilde{m}_t), \qquad (2.3.10)$$

where $\bar{m}$, $\tilde{m}$, and U are defined in (2.3.2), (2.3.3) and (2.3.5).
We now compute the martingale representation for the process m as defined in (2.3.9) in terms of the underlying driving process X and Y, as introduced in Section 2.2. By Lemma 2.1 of Schweizer [58] for all ξ, η $\mathbb{F}$-predictable processes satisfying

$$\mathbb{E}\left[\int_0^T \xi_s^2 \, \mathrm{d}[X]_s\right], \ \mathbb{E}\left[\int_0^T \eta_s^2 \, \mathrm{d}[Y]_s\right] < \infty,$$

we have that the integral processes $\int \xi_s \, \mathrm{d}X_s$, $\int \eta_s \, \mathrm{d}Y_s$ are square integrable $\mathbb{F}$-martingales. Furthermore, since X and Y are strongly orthogonal, by Proposition

4.50 of Jacod and Shiryaev [42, Chapter I] the bracket process $[X, Y]$ is a local martingale, hence

$$\left[\int \xi_s \, \mathrm{d}X_s, \int \eta_s \, \mathrm{d}Y_s\right]_t = \int_0^t \xi_s \eta_s \, \mathrm{d}[X, Y], \quad t \in [0, T],$$

is a local martingale, e.g. by Jacod and Shiryaev [42, Chapter I, 3.23], and since by the Kunita-Watanabe inequality we have

$$\begin{aligned} &\mathbb{E}\left[\sup_{t \in [0,T]} \left| \int_0^t \xi_s \eta_s \, \mathrm{d}[X, Y]_s \right| \right] \\ &\leq \mathbb{E}\left[\int_0^T \xi_s^2 \, \mathrm{d}[X]_s\right]^{1/2} \mathbb{E}\left[\int_0^T \eta_s^2 \, \mathrm{d}[Y]_s\right]^{1/2} < \infty, \end{aligned}$$

it is in fact a (uniformly integrable) martingale, and therefore (again by Proposition 4.50 of Jacod and Shiryaev [42, Chapter I]) the product

$$\int_0^t \xi_s \, \mathrm{d}X_s \cdot \int_0^t \eta_s \, \mathrm{d}Y_s, \quad t \in [0, T],$$

is a (uniformly integrable) martingale, i.e. the two processes are strongly orthogonal. Since by (2.2.3) and Jensen's inequality for any $t \in [0, T]$ we have

$$\mathbb{E}[\bar{m}_t^2] \leq \mathbb{E}\left[\left(\int_0^T \bar{A}_s \, \mathrm{d}(e^{-\Gamma_s})\right)^2\right] \leq \mathbb{E}\left[\sup_{t \in [0,T]} \bar{A}_t^2\right] < \infty,$$

as well as

$$\mathbb{E}[\tilde{m}_t^2] \leq \mathbb{E}[\tilde{A}^2] < \infty,$$

the process m as given in (2.3.9) is a square integrable $\mathbb{F}$-martingale as a sum of square integrable martingales. Hence, e.g. by Protter [56, Chapter IV.3], m admits a decomposition

$$m_t = m_0 + \int_{]0,t]} \xi_s^m \, \mathrm{d}X_s + \int_{]0,t]} \eta_s^m \, \mathrm{d}Y_s + C_t^m, \quad t \in [0, T], \tag{2.3.11}$$

where ξ^m, η^m are $\mathbb{F}$-predictable processes satisfying

$$\mathbb{E}\left[\int_0^T (\xi_s^m)^2 \, \mathrm{d}[X]_s\right], \ \mathbb{E}\left[\int_0^T (\eta_s^m)^2 \, \mathrm{d}[Y]_s\right] < \infty,$$

and C^m is a square integrable martingale strongly orthogonal to $\int \xi_s^m \, \mathrm{d}X_s$ and $\int \eta_s^m \, \mathrm{d}Y_s$, i.e. $\int \xi_s^m \, \mathrm{d}X_s \cdot C^m$, $\int \eta_s^m \, \mathrm{d}Y_s \cdot C^m$ are (uniformly integrable) martingales. Therefore we have that

$$\begin{aligned} V_t^A = \mathbb{E}[A_T \,|\, \mathcal{G}_t] = m_0 &+ \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \xi_s^m \, \mathrm{d}X_s + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \eta_s^m \, \mathrm{d}Y_s \\ &+ \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \, \mathrm{d}C_s^m + \int_{]0,t]} \psi_s^M \, \mathrm{d}M_s, \quad t \in [0, T]. \end{aligned} \tag{2.3.12}$$

We now prove that (2.3.12) is indeed the GKW decomposition of $\mathbb{E}[A_T \,|\, \mathcal{G}_t]$. To this end we need to show that the integral with respect to X in (2.3.12) is square integrable, and that the process

$$L_t^A := \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \eta_s^m \,\mathrm{d}Y_s + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \,\mathrm{d}C_s^m + \int_{]0,t]} \psi_s^M \,\mathrm{d}M_s,$$

$t \in [0,T]$, is square integrable and strongly orthogonal to the space $\mathcal{I}^2(X)$ of stochastic integrals with respect to X. First note that by (2.2.3) we have

$$\mathbb{E}[(V_t^A)^2] \leq \mathbb{E}[A_T^2] < \infty, \quad t \in [0,T],$$

hence V^A is a square integrable martingale and $\mathbb{E}[[V^A]_T] < \infty$ (see, e.g. Corollary 3 of Theorem 27 in Chapter II of Protter [56]). Since

$$\mathbb{E}[[V^A]_T] = \mathbb{E}\left[\int_0^T (\mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s})^2 \,\mathrm{d}[m]_s\right] + \mathbb{E}\left[\int_0^T (\psi_s^M)^2 \,\mathrm{d}[M]_s\right],$$

where we have used (2.3.8) and the fact that m has no jumps in τ, i.e.

$$[m, M]_t = \sum_{0 \leq s \leq t} \Delta m_s \Delta M_s = 0, \quad t \in [0,T],$$

we have that

$$\mathbb{E}\left[\int_0^T (\psi_s^M)^2 \,\mathrm{d}[M]_s\right] < \infty.$$

Besides that since Γ is increasing and Γ_T is bounded, we have that

$$\mathbb{E}\left[\int_0^T (\mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \xi_s^m)^2 \,\mathrm{d}[X]_s\right] < \infty, \tag{2.3.13}$$

and analogously for the second and third term in (2.3.12) we have that

$$\mathbb{E}\left[\int_0^T (\mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \eta_s^m)^2 \,\mathrm{d}[Y]_s\right] < \infty, \tag{2.3.14}$$

as well as

$$\mathbb{E}\left[\int_0^T (\mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s})^2 \,\mathrm{d}[C^m]_s\right] < \infty.$$

Hence all integrals in (2.3.12) are square integrable martingales by Lemma 2.1 of Schweizer [58], and L^A is square integrable as the sum of square integrable martingales. Furthermore for a $\mathbb{G}$-predictable process $\psi \in L^2(X)$, i.e.

$$\mathbb{E}\left[\int_0^T \psi_s^2 \,\mathrm{d}[X]_s\right] < \infty, \tag{2.3.15}$$

we have that

$$\left[\int \psi_s \,\mathrm{d}X_s, \int \mathbb{1}_{\{\tau\geq s\}} e^{\Gamma_s} \eta^m_s \,\mathrm{d}Y_s\right]_t = \int_0^t \mathbb{1}_{\{\tau\geq s\}} e^{\Gamma_s} \psi_s \eta^m_s \,\mathrm{d}[X,Y], \quad t\in[0,T],$$

is a local martingale, e.g. by Jacod and Shiryaev [42, Chapter I, 3.23], and in view of (2.3.14) and (2.3.15) and again by the Kunita-Watanabe inequality we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} \Big| \int_0^t \mathbb{1}_{\{\tau\geq s\}} e^{\Gamma_s} \psi_s \eta^m_s \,\mathrm{d}[X,Y]_s \Big|\right] < \infty,$$

i.e. the bracket process is in fact a (uniformly integrable) martingale, and therefore by Proposition 4.50 of Jacod and Shiryaev [42, Chapter I] the product

$$\int_0^t \psi_s \,\mathrm{d}X_s \cdot \int_0^t \mathbb{1}_{\{\tau\geq s\}} e^{\Gamma_s} \eta^m_s \,\mathrm{d}Y_s, \quad t\in[0,T],$$

is a (uniformly integrable) martingale. With the same arguments it can easily be seen that

$$\int_0^t \psi_s \,\mathrm{d}X_s \cdot \int_0^t \mathbb{1}_{\{\tau\geq s\}} e^{\Gamma_s} \,\mathrm{d}C^m_s, \quad t\in[0,T],$$

is a (uniformly integrable) martingale. Finally

$$\begin{aligned}&\left[\int \psi_s \,\mathrm{d}X_s, \int \psi^M_s \,\mathrm{d}M_s\right]_t \\ &= \int_0^t \psi_s \psi^M_s \,\mathrm{d}[X,M]_s = 0,\end{aligned}$$

for $t\in[0,T]$ since X and M are strongly orthogonal and X has no jumps in τ, i.e. the product

$$\int_0^t \psi_s \,\mathrm{d}X_s \cdot \int_0^t \psi^M_s \,\mathrm{d}M_s, \quad t\in[0,T],$$

is also a (uniformly integrable) martingale. Putting everything together we have obtained that

$$\int_0^t \psi_s \,\mathrm{d}X_s \cdot L^A_t, \quad t\in[0,T],$$

is a (uniformly integrable) martingale for $\psi\in L^2(X)$, i.e. L^A is strongly orthogonal to $\mathcal{I}^2(X)$ and thus (2.3.12) is the GKW decomposition of $\mathbb{E}[A_T \,|\, \mathcal{G}_t]$. By the results of Appendix A it then follows that the risk-minimizing strategy $\varphi=(\xi,\xi^0)$ is given by

$$\begin{aligned}\xi_t &= \mathbb{1}_{\{\tau\geq t\}} e^{\Gamma_t} \xi^m_t, \\ \xi^0_t &= V_t - \xi_t X_t,\end{aligned}$$

for $t\in[0,T]$, with discounted value process $V_t(\varphi)=\mathbb{E}[A_T \,|\, \mathcal{G}_t] - A_t$ and optimal cost process $C_t(\varphi) = m_0 + L^A_t$. □

Note that in (2.3.6) every term is stopped in τ, i.e. the value process is constant after τ. Besides that every integral with respect to the local $\mathbb{F}$-martingales X, Y and C^m contains the ratio

$$\frac{\mathbb{1}_{\{\tau \geq t\}}}{\mathbb{P}(\tau > t \,|\, \mathcal{F}_t)},$$

i.e. the actual survival event divided by the conditional survival probability on $\mathbb{F}$. Also note that the cost process in (2.3.7) is essentially made up of three components, given in terms of orthogonal integrals with respect to the processes Y, C^m and M. While the component associated to C^m in general cannot be eliminated unless the processes X and Y have the predictable representation property (see Corollary 2.3.4), the other two integrals with respect to Y and M represent the systematic and unsystematic component of the mortality risk. As we will see in Section 2.4, these risks may be eliminated by introducing suitable mortality-linked products related to Y and M on the financial market.

Corollary 2.3.4. *Assume X and Y have the predictable representation property with respect to the filtration $\mathbb{F}$ (see, e.g., Protter [56, Chapter IV.3]). Then $C^m \equiv 0$ in decomposition* (2.3.11) *and the square integrable martingale m defined in* (2.3.9) *admits a decomposition*

$$m_t = m_0 + \int_{]0,t]} \tilde{\xi}^m_s \, \mathrm{d}X_s + \int_{]0,t]} \tilde{\eta}^m_s \, \mathrm{d}Y_s, \quad t \in [0,T], \tag{2.3.16}$$

where $\tilde{\xi}^m$ and $\tilde{\eta}^m$ are $\mathbb{F}$-predictable process satisfying

$$\mathbb{E}\left[\int_0^T (\tilde{\xi}^m_s)^2 \, \mathrm{d}[X]_s\right], \ \mathbb{E}\left[\int_0^T (\tilde{\eta}^m_s)^2 \, \mathrm{d}[Y]_s\right] < \infty.$$

In this case the insurance payment process A as defined in Section 2.2 admits a risk-minimizing strategy $\tilde{\varphi} = (\tilde{\xi}, \tilde{\xi}^0)$ given by

$$\begin{aligned} \tilde{\xi}_t &= \mathbb{1}_{\{\tau \geq t\}} e^{\Gamma_t} \tilde{\xi}^m_t, \\ \tilde{\xi}^0_t &= V_t - \tilde{\xi}_t X_t = V_t - \mathbb{1}_{\{\tau \geq t\}} e^{\Gamma_t} \tilde{\xi}^m_t X_t, \end{aligned}$$

with discounted value process

$$\begin{aligned} V_t(\tilde{\varphi}) &= \mathbb{E}[A_T \,|\, \mathcal{G}_t] - A_t \\ &= m_0 + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \tilde{\xi}^m_s \, \mathrm{d}X_s \\ &\quad + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \tilde{\eta}^m_s \, \mathrm{d}Y_s + \int_{]0,t]} \psi^M_s \, \mathrm{d}M_s - A_t \end{aligned} \tag{2.3.17}$$

and optimal cost process

$$C_t(\tilde{\varphi}) = m_0 + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \tilde{\eta}^m_s \, \mathrm{d}Y_s + \int_{]0,t]} \psi^M_s \, \mathrm{d}M_s,$$

for $t \in [0,T]$, where the processes M, m, ψ^M, $\tilde{\xi}^m$ and $\tilde{\eta}^m$ are introduced respectively in (2.2.2), (2.3.9), (2.3.10) *and* (2.3.16).

The predictable representation property is often associated with the completeness of the underlying financial market. However, assuming that the predictable representation property holds does not necessarily imply that the financial market is complete and vice versa, since these properties depend largely on the specific characteristics of the underlying driving price processes as well as the structure of the filtration (see, e.g., Cont and Tankov [22, Remark 9.1]).
Note that in Corollary 2.3.4, (2.3.16) means that the GKW decomposition of the square integrable $\mathbb{F}$-martingale m has a special structure, where the orthogonal part consists only of the integral with respect to Y. In particular we have that $C^m \equiv 0$ in (2.3.11) if (X, Y) have the predictable representation property with respect to the filtration $\mathbb{F}$. This is the case for example if $\mathbb{F}$ is generated by two independent Brownian motions driving X and Y. However in more general settings it often may not be possible to decompose m in this way, in fact this is the case in many jump diffusion models. However we will see in Section 2.5, that if the life insurance payment process has a special structure then it might be possible to find a decomposition of m as in (2.3.16), even if X and Y do *not* have the predictable representation property with respect to $\mathbb{F}$ (see (2.5.7) in Section 2.5).

2.4 Extending the Financial Market

We now turn to a more detailed analysis of the cost process in (2.3.7). If we consider the GKW decomposition as computed in (2.3.6) for a given payment process, we can see that the cost is generated by the following orthogonal components:

- Y, the driving process of the conditional survival probability,
- M, the compensated jump process of the time of death, and
- C^m, the orthogonal part due to the predictable decomposition of the $\mathbb{F}$-martingale m in (2.3.9).

Then a natural question is: Can we introduce mortality-linked products into the financial market, that can be used to hedge the cost parts due to Y and M? For illustration purposes in the following we set $C^m \equiv 0$ and for the short rate we assume $r_t \equiv 0$, $t \in [0, T]$, i.e. the bank account is constant. Following Cairns et al. [19] we now assume there exists another risky asset traded on the market, a zero-coupon *longevity bond* with maturity T, $(P_t^T)_{t\in[0,T]}$, with discounted asset price given by

$$P_t^T = \mathbb{E}[\, e^{-\Gamma_T} \,|\, \mathcal{G}_t] = \mathbb{E}[\, e^{-\Gamma_T} \,|\, \mathcal{F}_t] = \mathbb{E}[\mathbb{1}_{\{\tau > T\}} | \, \mathcal{F}_t], \quad t \in [0, T],$$

i.e. a zero-coupon bond that pays out the survivor index at maturity. As discussed in Section 2.2, since it is given by the conditional survival probability, it may be seen as incorporating the systematic mortality risk. Recall that we have defined Y as driving the martingale P^T, i.e.

$$P_t^T = P_0^T + \int_{]0,t]} \zeta_s P_{s-}^T \, \mathrm{d}Y_s, \quad t \in [0,T], \tag{2.4.1}$$

for an $\mathbb{F}$-predictable process ζ. If $\zeta_t \neq 0$ a.s. for all $t \in [0,T]$, inserting this in (2.3.17) immediately leads to

$$\begin{aligned} V_t(\tilde{\varphi}) &= \mathbb{E}[A_T \,|\, \mathcal{G}_t] - A_t \\ &= m_0 + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \tilde{\xi}_s^m \, \mathrm{d}X_s + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} \frac{e^{\Gamma_s} \tilde{\eta}_s^m}{\zeta_s P_{s-}^T} \, \mathrm{d}P_s^T \\ &\quad + \int_{]0,t]} \psi_s^M \, \mathrm{d}M_s - A_t, \end{aligned} \tag{2.4.2}$$

i.e. we obtain the price of the life insurance payment process in terms of the investments in X and P^T, thereby eliminating the cost part associated to the systematic mortality risk. We now assume that there exists a third risky asset actively traded on the market, directly related to the time of death τ. This will finally enable us to eliminate the cost part due to M. We introduce a *pure endowment* contract $E = (E_t)_{t \in [0,T]}$, i.e. a life insurance contract that pays 1 at maturity T if the individual survived, with present value

$$E_t = \mathbb{E}[\mathbb{1}_{\{\tau > T\}} \,|\, \mathcal{G}_t], \quad t \in [0,T].$$

As the following computations show, E, P^T and M are closely related to each other and we may find a representation of M in terms of P^T and E. By Lemma 5.1.2 of Bielecki and Rutkowski [12] we have that

$$E_t = \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}[\mathbb{1}_{\{\tau > T\}} \,|\, \mathcal{F}_t]}{\mathbb{P}(\tau > t \,|\, \mathcal{F}_t)} = L_t P_t^T, \quad t \in [0,T],$$

where $L_t = (1 - H_t) e^{\Gamma_t}$. By (2.3.4) and since P^T has no jumps in τ, it follows that

$$\begin{aligned} L_t P_t^T &= L_0 P_0^T + \int_{]0,t]} L_{s-} \, \mathrm{d}P_s^T + \int_{]0,t]} P_{s-}^T \, \mathrm{d}L_s \\ &= L_0 P_0^T + \int_{]0,t]} L_{s-} \, \mathrm{d}P_s^T - \int_{]0,t]} P_{s-}^T e^{\Gamma_s} \, \mathrm{d}M_s, \end{aligned}$$

i.e. for $t \in [0,T]$ we have

$$\mathrm{d}E_t = L_{t-} \, \mathrm{d}P_t^T - P_{t-}^T e^{\Gamma_t} \, \mathrm{d}M_t$$

and

$$\mathrm{d}M_t = \frac{\mathbb{1}_{\{\tau \geq t\}}}{P_{t-}^T} \, \mathrm{d}P_t^T - \frac{1}{P_{t-}^T e^{\Gamma_t}} \, \mathrm{d}E_t.$$

Note that $P^T_{t-}, e^{\Gamma_t} \neq 0$ for all $t \in [0,T]$. Hence by inserting this in (2.4.2) we obtain

$$\begin{aligned} V_t(\tilde{\varphi}) &= \mathbb{E}[A_T \,|\, \mathcal{G}_t] - A_t \\ &= m_0 + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \tilde{\xi}^m_s \, \mathrm{d}X_s \\ &\quad + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} \left(\frac{e^{\Gamma_s} \tilde{\eta}^m_s}{\zeta_s P^T_{s-}} + \frac{\psi^M_s}{P^T_{s-}} \right) \mathrm{d}P^T_s - \int_{]0,t]} \frac{\psi^M_s}{P^T_{s-} e^{\Gamma_s}} \, \mathrm{d}E_s - A_t. \end{aligned}$$

In practice insurance companies often trade mortality-linked contracts similar to P^T, where the payoff at maturity is directly linked to a survivor index. Examples of such products include e.g. the EIB/BNP and Swiss Re bonds in 2004 or futures and options on survivor indices (see, e.g., Blake et al. [16]). However, as shown very clearly by the above computations, by themselves these products are not able to offer complete protection against mortality risk, since a remaining source of randomness is directly related to the knowledge of τ, i.e. the unsystematic, individual mortality risk, and requires an additional asset in order to be hedged.

2.5 Example: Unit-linked Life Insurance

In this section we assume given two independent Brownian motions $W = (W_t)_{t\in[0,T]}$, $W^\mu = (W^\mu_t)_{t\in[0,T]}$ and a compound Poisson process $Q = (Q_t)_{t\in[0,T]}$,

$$Q_t = \sum_{i=1}^{N_t} Y_i, \quad t \in [0,T],$$

where $N = (N_t)_{t\in[0,T]}$ is a Poisson process with intensity $\lambda > 0$ and Y_i are i.i.d. random variables independent of N with $Y_i > -1$ a.s., $i \geq 1$, such that $\mathbb{E}[Y_1] = \beta < \infty$ and $\mathbb{E}[Y_1^2] < \infty$. We then assume that the filtration $\mathbb{F}$ is generated by these three processes, i.e. $\mathbb{F} = \mathbb{F}^W \vee \mathbb{F}^{W^\mu} \vee \mathbb{F}^Q$, where $\mathbb{F}^W$, $\mathbb{F}^{W^\mu}$ and $\mathbb{F}^Q$ are the natural filtrations of W, W^μ and Q. For the discounted asset price process we assume a jump diffusion model

$$\mathrm{d}X_t = \sigma_t X_t \, \mathrm{d}W_t + X_{t-} \, \mathrm{d}\widetilde{Q}_t, \quad X_0 = x, \tag{2.5.1}$$

for $t \in [0,T]$, with $\widetilde{Q}_t = Q_t - \beta\lambda t$ and $\sigma = (\sigma_t)_{t\in[0,T]}$ is a bounded, $\mathbb{F}$-adapted process. Then X is a local martingale and the solution to (2.5.1) is given by

$$X_t = x \exp\left\{ \int_0^t \sigma_s \, \mathrm{d}W_s - \left(\beta\lambda t + \frac{1}{2} \int_0^t \sigma_s^2 \, \mathrm{d}s \right) \right\} \prod_{i=1}^{N_t} (Y_i + 1), \quad t \in [0,T].$$

Since σ is bounded and by Doob's maximal inequalities we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} X_t^2 \right] \leq 4\mathbb{E}[X_T^2] \leq c_1 \exp\{(c-1)\lambda T\} < \infty, \tag{2.5.2}$$

where $c_1 \in \mathbb{R}_+$ and $c = \mathbb{E}[(Y_1 + 1)^2]$, hence X is in fact a (uniformly integrable) martingale. We assume that the hazard process admits the following representation:

$$\Gamma_t = \int_0^t \mu_s \, \mathrm{d}s, \quad t \in [0, T], \tag{2.5.3}$$

where the *default intensity* or *mortality rate* μ is a non-negative $\mathbb{F}$-progressively measurable process. Stochastic mortality modeling has been studied extensively in the literature, see, e.g., Biffis [13], Dahl [25] and Milevsky and Promislow [52] for different modeling approaches for the spot force of mortality in continuous time and Cairns et al. [19] for an overview of existing modeling frameworks for stochastic mortality. Here we follow the affine approach of Dahl [25] and assume that μ is given by the Cox-Ingersoll-Ross model

$$\mathrm{d}\mu_t = (a + b\mu_t)\,\mathrm{d}t + c\sqrt{\mu_t}\,\mathrm{d}W_t^\mu, \quad \mu_0 = 0,$$

for $t \in [0, T]$, $b \in \mathbb{R}$ and $a, c \in \mathbb{R}_+$. Note that the process Γ as introduced in (2.5.3) is not bounded, however we will show later, that the results of Section 2.3 remain valid in this setting even without this assumption, in particular equations (2.3.13) and (2.3.14) still hold. Since μ is an affine process, e.g. by Filipović [34] we have that

$$\mathbb{E}[e^{-\Gamma_T} \mid \mathcal{G}_t] = e^{-\Gamma_t}\mathbb{E}[e^{-\int_t^T \mu_s \,\mathrm{d}s} \mid \mathcal{F}_t^{W^\mu}] = e^{-\Gamma_t} e^{\alpha(t) + \beta(t)\mu_t}, \quad t \in [0, T],$$

where the functions $\alpha(t)$ and $\beta(t)$ satisfy the following equations:

$$\partial_t \alpha(t) = -a\beta(t), \quad \alpha(T) = 0,$$
$$\partial_t \beta(t) = 1 - b\beta(t) - \frac{1}{2}c^2\beta^2(t), \quad \beta(T) = 0,$$

$t \in [0, T]$. It is well-known that the explicit solutions are given by

$$\alpha(t) = \frac{2a}{c^2} \ln\left(\frac{2\gamma e^{(\gamma - b)(T-t)/2}}{(\gamma - b)(e^{\gamma(T-t)} - 1) + 2\gamma} \right),$$
$$\beta(t) = -\frac{2(e^{\gamma(T-t)} - 1)}{(\gamma - b)(e^{\gamma(T-t)} - 1) + 2\gamma},$$

$t \in [0, T]$, where $\gamma := \sqrt{b^2 + 2c^2}$. Then by Itô's formula, since the process $e^{-\Gamma_t + \alpha(t) + \beta(t)\mu_t}$, $t \in [0, T]$, is continuous we get

$$\mathrm{d}(e^{-\Gamma_t} e^{\alpha(t) + \beta(t)\mu_t}) = e^{-\Gamma_t + \alpha(t) + \beta(t)\mu_t} c\sqrt{\mu_t}\beta(t)\,\mathrm{d}W_t^\mu, \tag{2.5.4}$$

hence in this setting the local martingale Y, introduced in Section 2.2 and (2.4.1) in Section 2.4 as the driving process of the martingale $\mathbb{E}[\exp\{-\Gamma_T\} \mid \mathcal{F}_t]$, is given by the Brownian motion W^μ. Note that Q and W, W^μ are independent (see, e.g., Chapter 11 of Shreve [60]) and by simple calculations it is easy to see that

W, W^μ are independent if and only if they are strongly orthogonal. Similarly one can also show that $\widetilde{Q}W$ and $\widetilde{Q}W^\mu$ are martingales. Hence in this context we may apply the results of Section 2.3, since the underlying driving processes are strongly orthogonal martingales. We now study the case where the insurance payment process as defined in Section 2.2 is given by a (discounted) unit-linked term insurance contract:

$$A_T = \mathbb{1}_{\{\tau \leq T\}} X_\tau,$$

i.e. a life insurance contract that pays out the discounted asset price in the case of death prior to maturity. Since X has no jumps in τ, we have that

$$\mathbb{1}_{\{\tau \leq T\}} X_\tau = \mathbb{1}_{\{\tau \leq T\}} X_{\tau-},$$

i.e. $\bar{A}_t = X_{t-}$ for $t \in [0,T]$ and $\tilde{A} = 0$ in (2.2.3) and consequently for m as defined in (2.3.9) it follows that

$$\begin{aligned} m_t &= \mathbb{E}\left[\int_0^T X_{s-} e^{-\Gamma_s}\, \mathrm{d}\Gamma_s \,\Big|\, \mathcal{F}_t\right] \\ &= \int_0^t X_s e^{-\Gamma_s}\, \mathrm{d}\Gamma_s + \mathbb{E}\left[\int_t^T X_s e^{-\Gamma_s}\, \mathrm{d}\Gamma_s \,\Big|\, \mathcal{F}_t\right], \quad t \in [0,T]. \end{aligned} \tag{2.5.5}$$

By the independence of W^μ and W, Q we get

$$\begin{aligned} \mathbb{E}\left[\int_t^T X_s e^{-\Gamma_s}\, \mathrm{d}\Gamma_s \,\Big|\, \mathcal{F}_t\right] &= \int_t^T \mathbb{E}\left[X_s e^{-\Gamma_s} \mu_s \,\Big|\, \mathcal{F}_t\right] \mathrm{d}s \\ &= \int_t^T \mathbb{E}\left[X_s \,\Big|\, \mathcal{F}_t^W \vee \mathcal{F}_t^Q\right] \mathbb{E}\left[e^{-\Gamma_s}\mu_s \,\Big|\, \mathcal{F}_t^{W^\mu}\right] \mathrm{d}s \\ &= X_t \mathbb{E}\left[\int_t^T e^{-\Gamma_s}\mu_s\, \mathrm{d}s \,\Big|\, \mathcal{F}_t^{W^\mu}\right] \\ &= X_t \left(e^{-\Gamma_t} - \mathbb{E}\left[e^{-\Gamma_T} \,\Big|\, \mathcal{F}_t^{W^\mu}\right]\right), \quad t \in [0,T]. \end{aligned} \tag{2.5.6}$$

Then by (2.5.5) and (2.5.6) we have that

$$\begin{aligned} m_t &= \int_0^t X_s e^{-\Gamma_s}\, \mathrm{d}\Gamma_s + X_t e^{-\Gamma_t} - X_t \mathbb{E}\left[e^{-\Gamma_T} \,\Big|\, \mathcal{F}_t^{W^\mu}\right] \\ &= \int_0^t e^{-\Gamma_s}\, \mathrm{d}X_s - X_t \mathbb{E}\left[e^{-\Gamma_T} \,\Big|\, \mathcal{F}_t^{W^\mu}\right], \quad t \in [0,T], \end{aligned}$$

and by inserting (2.5.4) and again by the independence of W and W^μ we have

$$\mathrm{d}(X_t e^{-\Gamma_t + \alpha(t) + \beta(t)\mu_t}) = e^{-\Gamma_t + \alpha(t) + \beta(t)\mu_t}[\mathrm{d}X_t + X_t c\sqrt{\mu_t}\beta(t)\mathrm{d}W_t^\mu]$$

for $t \in [0,T]$. Therefore we obtain

$$\begin{aligned} m_t = x(1 - e^{\alpha(0)}) &+ \int_{]0,t]} e^{-\Gamma_s}(1 - e^{\alpha(s) + \beta(s)\mu_s})\, \mathrm{d}X_s \\ &- \int_0^t c\sqrt{\mu_s}\beta(s) X_s e^{-\Gamma_s + \alpha(s) + \beta(s)\mu_s}\, \mathrm{d}W_s^\mu, \quad t \in [0,T]. \end{aligned} \tag{2.5.7}$$

We would now like to comment on the integrability conditions as imposed in Section 2.3 and how they apply in this context. Note that (2.5.7) implies that the processes ξ^m and η^m as introduced in (2.3.11) are well-defined and in this setting given by

$$\xi_t^m = e^{-\Gamma_t}(1 - e^{\alpha(t)+\beta(t)\mu_t}), \quad t \in [0,T],$$

and

$$\eta_t^m = -c\sqrt{\mu_t}\beta(t) X_t e^{-\Gamma_t+\alpha(t)+\beta(t)\mu_t}, \quad t \in [0,T].$$

Hence in (2.3.13) we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T (\mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \xi_s^m)^2 \, \mathrm{d}[X]_s\right] \\
&\leq \mathbb{E}\left[\int_0^T (1 - e^{\alpha(s)+\beta(s)\mu_s})^2 \, \mathrm{d}[X]_s\right] \\
&\leq 4\mathbb{E}[[X]_T] \leq c_2 \mathbb{E}\left[\sup_{t\in[0,T]} X_t^2\right] < \infty,
\end{aligned}$$

$c_2 \in \mathbb{R}_+$, where we have used (2.5.2) and the Burkholder-Davis-Gundy inequalities. Furthermore, in (2.3.14) we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T (\mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \eta_s^m)^2 \, \mathrm{d}[Y]_s\right] \\
&\leq \mathbb{E}\left[\int_0^T (c\sqrt{\mu_s}\beta(s) X_s e^{\alpha(s)+\beta(s)\mu_s})^2 \, \mathrm{d}[Y]_s\right] \\
&\leq c_3 \mathbb{E}\left[\sup_{t\in[0,T]} X_t^2\right] \mathbb{E}\left[\int_0^T \mu_s \, \mathrm{d}s\right] < \infty,
\end{aligned}$$

$c_3 \in \mathbb{R}_+$, since $\beta(\,\cdot\,)$ is bounded on $[0,T]$ and the integral over the square root process μ has finite first moments, see, e.g., Dufresne [33]. Hence the results of Section 2.3 remain valid in the context of this model, even though Γ is not bounded.
By (2.3.2), (2.5.5) and (2.5.6) for U_t we obtain

$$U_t = m_t - \int_0^t X_s e^{-\Gamma_s} \, \mathrm{d}\Gamma_s = X_t \left(e^{-\Gamma_t} - \mathbb{E}\left[e^{-\Gamma_T} \,\middle|\, \mathcal{F}_t^{W^\mu}\right]\right), \quad t \in [0,T],$$

hence for ψ_t^M as defined in (2.3.10) we have

$$\psi_t^M = \bar{A}_t - e^{\Gamma_t} U_t = X_{t-} - X_t(1 - e^{\alpha(t)+\beta(t)\mu_t}), \quad t \in [0,T],$$

and again, since X has no jump in τ, in (2.3.6) we obtain:

$$\begin{aligned} V_t &= \mathbb{E}[\mathbb{1}_{\{\tau \le T\}} X_\tau \mid \mathcal{G}_t] - A_t \\ &= x(1 - e^{\alpha(0)}) + \int_{]0,t]} \mathbb{1}_{\{\tau \ge s\}} (1 - e^{\alpha(s)+\beta(s)\mu_s}) \, \mathrm{d}X_s \\ &\quad - \int_0^t \mathbb{1}_{\{\tau \ge s\}} c\sqrt{\mu_s} \beta(s) X_s e^{\alpha(s)+\beta(s)\mu_s} \, \mathrm{d}W_s^\mu \\ &\quad + \int_{]0,t]} X_s e^{\alpha(s)+\beta(s)\mu_s} \mathrm{d}M_s - A_t, \quad t \in [0,T], \end{aligned}$$

or, in the setting of Section 2.4 with two additional risky assets P^T and E,

$$\begin{aligned} V_t &= x(1 - e^{\alpha(0)}) + \int_{]0,t]} \mathbb{1}_{\{\tau \ge s\}} (1 - e^{\alpha(s)+\beta(s)\mu_s}) \, \mathrm{d}X_s \\ &\quad - \int_0^t \mathbb{1}_{\{\tau \ge s\}} e^{\Gamma_s} X_s \, \mathrm{d}P_s^T + \int_0^t \frac{1}{P_s^T} \left(\mathbb{1}_{\{\tau \ge s\}} X_s e^{\alpha(s)+\beta(s)\mu_s} \right) \mathrm{d}P_s^T \\ &\quad - \int_{]0,t]} \frac{1}{P_s^T e^{\Gamma_s}} \left(X_s e^{\alpha(s)+\beta(s)\mu_s} \right) \mathrm{d}E_s - A_t \\ &= x(1 - e^{\alpha(0)}) + \int_{]0,t]} \mathbb{1}_{\{\tau \ge s\}} (1 - e^{\alpha(s)+\beta(s)\mu_s}) \, \mathrm{d}X_s \\ &\quad - \int_{]0,t]} X_s \, \mathrm{d}E_s - A_t, \quad t \in [0,T]. \end{aligned}$$

Chapter 3

Homogeneous Insurance Portfolio

3.1 Introduction

In this chapter, based on Biagini, Rheinländer, and Schreiber [10], we study the problem of pricing and hedging life insurance liabilities for the case of a homogenous insurance portfolio. Main features of this work are that we take into account and explicitly model the basis risk that arises due to the fact that the insurance company cannot perfectly hedge its exposure by investing in a hedging instrument that is based on a longevity index, not on the insurance portfolio itself. We model the dependency between the index and the insurance portfolio by means of an affine mean-reverting diffusion process with stochastic drift. The remainder of this chapter is organized as follows: Section 3.2 introduces the general setup, including the structure of the insurance portfolio and the financial market. In Section 3.3 we compute the price and the risk-minimizing strategy of the life insurance payment streams. We also provide specific applications in the context of unit-linked life insurance contracts.

3.2 The Model

Let $T > 0$ be a fixed finite time horizon and $(\Omega, \mathcal{G}, \mathbb{P})$ a probability space equipped with a filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ which contains all available information. We define $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, and put $\mathcal{G} = \mathcal{G}_T$, where $\mathbb{H} = (\mathcal{H}_t)_{t\in[0,T]}$ is generated by the death counting processes of the insurance portfolio (see Subsection 3.2.1). The *background filtration* $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ contains all information available except the information regarding the individual survival times. Here we define $\mathcal{F}_t = \sigma\{(W_s, W_s^{\mu}, W_s^{\bar{\mu}}) : 0 \le s \le t\}$, $t \in [0, T]$, where W, W^{μ} and $W^{\bar{\mu}}$ are independent Brownian motions driving the financial market and the mortality intensities (see

Subsections 3.2.1 and 3.2.2). In the following we introduce the three components of the model: the insurance portfolio, the financial market and the combined model.

3.2.1 Insurance Portfolio and Mortality Intensities

We consider a portfolio of n lives all aged x at time 0, with *death counting process*

$$N_t = \sum_{i=1}^{n} \mathbb{1}_{\{\tau^{x,i} \le t\}}, \quad t \in [0,T],$$

where $\tau^{x,i} : \Omega \to [0,T] \cup \{\infty\}$, and for convenience in the following we omit the dependency on x. We assume that $\mathbb{P}(\tau^i = 0) = 0$ and $\mathbb{P}(\tau^i > t) > 0$ for $i = 1, \dots, n$ and $t \in [0,T]$. Note that since the time horizon T is usually fixed as the maturity of the life insurance liabilities, in order to ensure that $\mathbb{P}(\tau^i > T) > 0$ for $i = 1, \dots, n$ (the remaining lifetimes are not necessarily bounded by T), it is necessary to allow τ^i to take values larger than T, indicated here by the convention that τ^i can assume the value infinity. We define $\mathcal{H}_t = \mathcal{H}_t^1 \vee \dots \vee \mathcal{H}_t^n$, with $\mathcal{H}_t^i = \sigma\{H_s^i : 0 \le s \le t\}$ and $H_t^i = \mathbb{1}_{\{\tau^i \le t\}}$. We assume that the times of death τ^i are totally inaccessible $\mathbb{G}$-stopping times, and an important role is then played by the conditional distribution function of τ^i, given by

$$F_t^i = \mathbb{P}(\tau^i \le t \,|\, \mathcal{F}_t), \quad i = 1, \dots, n,$$

and we assume $F_t^i < 1$ for all $t \in [0,T]$. Then the *hazard process* Γ^i of τ^i under $\mathbb{P}$

$$\Gamma_t^i = -\ln(1 - F_t^i) = -\ln \mathbb{E}[\mathbb{1}_{\{\tau^i > t\}} \,|\, \mathcal{F}_t], \tag{3.2.1}$$

is well-defined for every $t \in [0,T]$. Since the insurance portfolio is homogenous in the sense that all individuals belong to the same age cohort, we set $\Gamma^i = \Gamma$, where

$$\Gamma_t = \int_0^t \mu_s \, \mathrm{d}s, \quad t \in [0,T].$$

Similarly as in Biffis [13], we assume that the *mortality intensity* μ is given as the solution of the following set of stochastic differential equations:

$$\mathrm{d}\mu_t = \gamma_1(\bar{\mu}_t - \mu_t)\mathrm{d}t + \sigma_1 \sqrt{\mu_t}\, \mathrm{d}W_t^{\mu}, \tag{3.2.2}$$

$$\mathrm{d}\bar{\mu}_t = \gamma_2(m(t) - \bar{\mu}_t)\mathrm{d}t + \sigma_2 \sqrt{\bar{\mu}_t}\, \mathrm{d}W_t^{\bar{\mu}}, \tag{3.2.3}$$

for $t \in [0,T]$ and $\mu_0 = \bar{\mu}_0 = 0$, where γ_1, γ_2, σ_1, $\sigma_2 > 0$, and $m : [0,T] \to \mathbb{R}_+$ is a continuous deterministic function. The existence and uniqueness of a strong solution $(\mu, \bar{\mu})$ to the set of stochastic differential equations (3.2.2) - (3.2.3) is proved in Appendix E of Biffis [13] by using Proposition 2.13 and 2.18 in Chapter 5 of Karatzas and Shreve [46], as well as results of Deelstra and Delbaen [29]. The process $\bar{\mu}$ represents the mortality intensity of the equivalent age cohort of the population, and can be derived by means of publicly available data of the *survivor index*

$$S_t^{\bar{\mu}} = \exp\left(-\int_0^t \bar{\mu}_s \, \mathrm{d}s \right), \quad t \in [0, T]. \tag{3.2.4}$$

According to Cairns et al. [19] survivor indices can be seen as basic building blocks for many mortality-linked securities. The need for standardization in the life markets has led to the creation of various such indices by investment banks and many market traded securities have payments linked to survivor indices. The dynamics of $\bar{\mu}$ in (3.2.3) are given by a non-negative affine square root diffusion, mean-reverting towards the deterministic drift m, which can be seen as best estimate for $\bar{\mu}$, such as an available mortality table. Hence μ as defined in (3.2.2) is a non-negative affine process, fluctuating around a stochastic drift given by the mortality intensity $\bar{\mu}$ of the respective age cohort of the population. Note that many empirical studies have shown that the mortality of life insurance portfolios is often lower than that of the equivalent age cohort of the population due to socioeconomic factors such as lifestyle, income, etc. This characteristic feature can easily be incorporated in our model e.g., by replacing the stochastic drift $\bar{\mu}$ by $\bar{\mu} - \varepsilon$ and m by $m - \varepsilon$ for a constant $\varepsilon > 0$ in (3.2.2) - (3.2.3).
We also assume that for $i \neq j$, τ^i, τ^j are conditionally independent given $\mathcal{F}_T$, i.e.

$$\mathbb{E}[\mathbb{1}_{\{\tau^i > t\}} \mathbb{1}_{\{\tau^j > s\}} \,|\, \mathcal{F}_T] = \mathbb{E}[\mathbb{1}_{\{\tau^i > t\}} \,|\, \mathcal{F}_T] \, \mathbb{E}[\mathbb{1}_{\{\tau^j > s\}} \,|\, \mathcal{F}_T], \quad 0 \leq s, t \leq T. \tag{3.2.5}$$

This assumption is well-known in the literature of credit risk modeling, see, e.g., Chapter 9 of Bielecki and Rutkowski [12]. All individuals within the insurance portfolio are subject to idiosyncratic risk factors, as well as common risk factors, given by the information represented by the background filtration $\mathbb{F}$. Intuitively, the assumption of conditional independence means that given all common risk factors are known, the idiosyncratic risk factors become independent of each other.

3.2.2 The Financial Market

Since our focus is on modeling the basis risk between the insurance portfolio and the longevity index, for simplicity we consider a rather simple model of a financial market defined on $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ consisting of a bank account or numéraire B with constant short rate $r > 0$, i.e.

$$B_t = \exp\{rt\}, \quad t \in [0, T],$$

as well as two risky assets with asset prices S and P. We assume that S follows the $\mathbb{P}$-dynamics

$$\mathrm{d}S_t = S_t \left(r\mathrm{d}t + \sigma(t, S_t) \, \mathrm{d}W_t \right), \quad t \in [0, T], \tag{3.2.6}$$

with $S_0 = s$ and we assume that σ satisfies certain regularity conditions that ensure the existence and uniqueness of a solution to (3.2.6). We denote by $X = S/B$ the discounted asset price, i.e. the dynamics of X are given by

$$\mathrm{d}X_t = \mathrm{d}\left(\frac{S_t}{B_t}\right) = \sigma(t, S_t) X_t \,\mathrm{d}W_t, \quad t \in [0, T]. \tag{3.2.7}$$

Following Cairns et al. [19] we assume that P is the price process of a *longevity bond* with maturity T representing the systematic mortality risk, i.e. P is defined as a zero-coupon bond that pays out the value of the survivor or longevity index as defined in (3.2.4) at T. This means the discounted value process $Y = P/B$ is given by

$$Y_t = \mathbb{E}\left[\frac{S_T^{\bar{\mu}}}{B_T}\,\middle|\,\mathcal{G}_t\right] = \mathbb{E}\left[\frac{\exp(-\int_0^T \bar{\mu}_s \,\mathrm{d}s)}{B_T}\,\middle|\,\mathcal{G}_t\right], \quad t \in [0, T]. \tag{3.2.8}$$

Thus the discounted asset prices X, Y are continuous (local) $(\mathbb{P}, \mathbb{F})$-martingales, i.e. the financial market is arbitrage-free, since the physical measure $\mathbb{P}$ belongs to the set of equivalent local martingale measures. Note that the asset prices are $\mathbb{F}$-adapted, however the trading strategies are allowed to be $\mathbb{G}$-adapted, i.e. in the following sections we consider (discounted) hedging portfolios

$$V_t(\varphi) = \xi_t^X X_t + \xi_t^Y Y_t + \xi_t^0, \quad t \in [0, T],$$

where $\varphi = (\xi^X, \xi^Y, \xi^0)$ is a $\mathbb{G}$-adapted process (see also Definition A.0.3 in Appendix A). This implies that all agents invest according to information incorporating the common risk factors such as the financial market and the mortality intensities, as well as the individual times of death.

3.2.3 The Combined Model

We consider the extended market $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, such that the information available in the market at time $t \in [0, T]$ is assumed to be $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$. All filtrations are assumed to satisfy the usual hypotheses of completeness and right-continuity. We postulate that all $\mathbb{F}$-local martingales are also $\mathbb{G}$-local martingales, and in the sequel we refer to this hypothesis as Hypothesis (H). This hypothesis is well-known in the literature on enlargements of filtrations, for a discussion of this hypothesis we refer to Blanchet-Scalliet and Jeanblanc [18] and Bielecki and Rutkowski [12, Chapter 6]. In this setting we study life insurance liabilities in form of insurance payment streams as introduced by Møller [54]. It is now widely acknowledged (see, e.g. Barbarin [4], Biffis [13] and Møller [53]) that most payment streams of practical relevance are covered by the three building blocks pure endowment-, term insurance-, and annuity contracts. Following Barbarin [4], the *pure endowment contract* consists of a payoff

$$C^{pe}\,(n - N_T) \tag{3.2.9}$$

at T, where C^{pe} is a non-negative $\mathcal{F}_T$-measurable random variable such that $\mathbb{E}[(C^{pe})^2] < \infty$, i.e. the insurer pays the amount C^{pe} at the term T of the contract to every policyholder of the portfolio who has survived until T. The *term* *insurance contract* is defined as

$$\int_0^T C_s^{ti}\,\mathrm{d}N_s = \sum_{i=1}^n \int_0^T C_s^{ti}\,\mathrm{d}H_s^i = \sum_{i=1}^n \mathbb{1}_{\{\tau^i \leq T\}} C_{\tau^i}^{ti}, \tag{3.2.10}$$

where C^{ti} is assumed to be a non-negative $\mathbb{F}$-predictable process such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} (C_t^{ti})^2\right] < \infty,$$

i.e. the amount $C_{\tau^i}^{ti}$ is payed at the time of death τ^i to every policyholder i, $i = 1, \dots, n$. The *annuity contract* consists of multiple payoffs the insurer has to pay as long as the policyholders are alive. We model these payoffs through their cumulative value C_t^a up to time t, where C^a is assumed to be a right-continuous, non-negative increasing $\mathbb{F}$-adapted process such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} (C_t^{a})^2\right] < \infty.$$

The cumulative payment up to time T is then given by

$$\int_0^T (n - N_s)\,\mathrm{d}C_s^a = \sum_{i=1}^n \int_0^T \left(1 - H_s^i\right)\mathrm{d}C_s^a. \tag{3.2.11}$$

Similarly as in Møller [53] or Riesner [57] we also provide specific examples (see Corollary 3.3.6, Corollary 3.3.8 and Corollary 3.3.10) where in the context of unit-linked life insurance products we set $C^{pe} = f(S_T)$, $C_t^{ti} = f(S_t)$ and $C_t^a = \int_0^t f(S_u)\,\mathrm{d}u$ in (3.2.9) - (3.2.11) for a function f that satisfies sufficient regularity conditions. Recall that (X, Y) is a $(\mathbb{P}, \mathbb{F})$-local martingale, i.e. the market given by $(\mathbb{P}, \mathbb{F})$ is arbitrage-free, and Hypothesis (H) implies that the extended financial market defined by $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ is also arbitrage-free. However, the market is not complete since the times of death occur as a surprise to the market and hence represent a kind of "orthogonal" risk. In particular any derivative relying on information of the individual times of death such as the insurance liabilities introduced in (3.2.9) - (3.2.11) cannot be perfectly hedged by investing in (X, Y). Therefore in the following section in order to find a price and hedging strategy for the insurance payment processes, we make use of a well-known quadratic hedging method for pricing and hedging in incomplete markets, the risk-minimization approach, a brief review of which is given in Appendix A.

Remark 3.2.1. *We would like to briefly comment on the fact that the insurance payment streams introduced in* (3.2.9) - (3.2.11) *can actually also be interpreted as T-claims, i.e. non-negative $\mathcal{G}_T$-measurable random variables, hence the risk-minimizing strategies may equivalently be found by means of the original method by Föllmer and Sondermann [36]. To this end note that the pure endowment contract* *consists of a single payoff at time T, hence it is a European type contingent claim by definition. Furthermore, the discounted term insurance contract has the same payoff as the discounted T-claim*

$$H = B_T^{-1} \sum_{i=1}^{n} \mathbb{1}_{\{\tau^i \leq T\}} C_{\tau^i}^{ti} B_{\tau^i}^{-1} B_T = \sum_{i=1}^{n} \int_0^T C_s^{ti} B_s^{-1} \,\mathrm{d}H_s^i = \int_0^T C_s^{ti} B_s^{-1} \,\mathrm{d}N_s,$$

where the insurer's liabilities C^{ti} are deferred and accumulated using the riskless asset B. By the same arguments the annuity contract can also be interpreted as discounted T-claim. In Remark A.3 of Møller [54] it is shown how in this case the approaches of Föllmer and Sondermann [36] and Møller [54] coincide in the sense that they deliver equivalent risk-minimizing strategies. In particular the investment in the risky assets is equal in both settings. The portfolio value process and investment in the riskless asset differ only in the sense that the portfolio value in the payment stream setting is seen as after *the insurance payments have been settled, whereas the value process in the setting of Föllmer and Sondermann [36] accounts for the insurance liabilities by accumulating them on the bank account as deferred payments.*

3.3 Risk-Minimization for Life Insurance Liabilities

In the setting of Section 3.2 we now compute the price and hedging strategy for the life insurance liabilities by applying the results of Appendix A. We start with some preliminary results.

3.3.1 Preliminary Results

For $i = 1, \ldots, n$ we consider the finite variation process

$$L_t^i = (1 - H_t^i) e^{\Gamma_t}, \quad t \in [0, T],$$

then by Lemma 5.1.7 of Bielecki and Rutkowski [12] we have that L^i is a local $\mathbb{G}^i$-martingale, where $\mathbb{G}^i := (\mathcal{G}_t^i)_{t \in [0,T]}$ and $\mathcal{G}_t^i = \mathcal{F}_t \vee \mathcal{H}_t^i$, $t \in [0, T]$, $i = 1, \ldots, n$. Since the hazard process Γ_t, $t \in [0, T]$, of τ^i exists and is continuous and increasing, by Proposition 5.1.3 of Bielecki and Rutkowski [12] we have that the compensated process M^i given by

$$M_t^i = H_t^i - \Gamma_{t \wedge \tau^i}, \quad t \in [0, T], \tag{3.3.1}$$

follows a local $\mathbb{G}^i$-martingale, such that

$$M_t^i = - \int_{]0,t]} e^{-\Gamma_s} \,\mathrm{d}L_s^i, \quad t \in [0, T], \tag{3.3.2}$$

and

$$L_t^i = 1 - \int_{]0,t]} L_{s-}^i \,\mathrm{d}M_s^i, \quad t \in [0, T].$$

Furthermore, since

$$\mathbb{E}[[M^i]_T] = \mathbb{E}[H^i_T] \leq 1 < \infty, \quad i = 1, \dots, n,$$

e.g., by Protter [56, Corollary 4 after Theorem 27 in Chapter II] M^i is a square integrable martingale. From (3.3.1) we have that the $\mathbb{F}$-hazard process Γ and the $(\mathbb{F}, \mathbb{G}^i)$-martingale hazard process Λ^i of τ^i coincide. By (3.2.5) (see, e.g., Lemma 9.1.1 of Bielecki and Rutkowski [12]) M^i is also a $\mathbb{G}$-martingale, i.e. Γ is also the $(\mathbb{F}, \mathbb{G})$-martingale hazard process of τ^i. Note that it is easily seen that for $j \neq i$, (3.2.5) implies that $L^i L^j$ is a local $\mathbb{G}$-martingale (see also Proposition 6.1 in Chapter 3 of Barbarin [4]), hence L^i and L^j are strongly orthogonal. Then by (3.3.2) we have that M^i and M^j are also strongly orthogonal. Note that since M^i are $\mathbb{G}$-martingales and

$$M_t := \sum_{i=1}^n M^i_t = N_t - \int_0^t (n - N_{s-}) \mu_s \, \mathrm{d}s, \quad t \in [0, T], \tag{3.3.3}$$

is a $\mathbb{G}$-martingale, the process $(\int_0^t (n - N_{s-}) \mu_s \, \mathrm{d}s)_{t \in [0,T]}$ is the $\mathbb{G}$-compensator of N. In the following by making use of the affine structure of $(\mu, \bar{\mu})$ as introduced in (3.2.2) and (3.2.3) we compute the dynamics of different processes related to $(\mu, \bar{\mu})$, such as the longevity bond introduced in (3.2.8), that will be needed for the computations in Sections 3.3.2 - 3.3.4.

Lemma 3.3.1. *For the longevity bond as introduced in* (3.2.8) *we have the dynamics:*

$$Y_t = Y_0 + \int_0^t Y_s e^{-rT} \beta^T(s) \sigma_2 \sqrt{\bar{\mu}_s} \, \mathrm{d}W^{\bar{\mu}}_s, \quad t \in [0, T],$$

where β^T is given by the following differential equation:

$$\partial_t \beta^T(t) = 1 + \gamma_2 \beta^T(t) - \frac{1}{2} \sigma_2^2 (\beta^T(t))^2, \quad \beta^T(T) = 0. \tag{3.3.4}$$

Proof. We rewrite $(\mu, \bar{\mu})$ as introduced in (3.2.2) and (3.2.3) as

$$\mathrm{d} \begin{pmatrix} \mu_t \\ \bar{\mu}_t \end{pmatrix} = \begin{pmatrix} 0 \\ \gamma_2 m(t) \end{pmatrix} + \begin{pmatrix} -\gamma_1 & \gamma_1 \\ 0 & -\gamma_2 \end{pmatrix} \begin{pmatrix} \mu_t \\ \bar{\mu}_t \end{pmatrix} \mathrm{d}t + \begin{pmatrix} \sigma_1 \sqrt{\mu_t} & 0 \\ 0 & \sigma_2 \sqrt{\bar{\mu}_t} \end{pmatrix} \mathrm{d} \begin{pmatrix} W^{\mu}_t \\ W^{\bar{\mu}}_t \end{pmatrix}$$

for $t \in [0, T]$, i.e. $(\mu, \bar{\mu})$ is affine. By equation (B.0.1) in Appendix B we immediately obtain

$$\tilde{Y}_t := \mathbb{E}\left[\exp\left(-\int_t^T \bar{\mu}_s \, \mathrm{d}s \right) \Big| \mathcal{F}_t \right] = \exp(\alpha^T(t) + \beta^T(t) \bar{\mu}_t), \quad t \in [0, T],$$

where

$$\partial_t \beta^T(t) = 1 + \gamma_2 \beta^T(t) - \frac{1}{2} \sigma_2^2 (\beta^T(t))^2, \quad \beta^T(T) = 0,$$

and

$$\partial_t \alpha^T(t) = -\gamma_2 m(t)\beta^T(t), \quad \alpha^T(T) = 0.$$

Then by Itô's formula we obtain

$$\begin{aligned} \mathrm{d}\tilde{Y}_t &= \tilde{Y}_t(\partial_t \alpha^T(t) + \partial_t \beta^T(t)\bar{\mu}_t)\mathrm{d}t + \tilde{Y}_t \beta^T(t)\mathrm{d}\bar{\mu}_t + \frac{1}{2}\tilde{Y}_t(\beta^T(t))^2 \mathrm{d}\langle\bar{\mu}\rangle_t \\ &= \tilde{Y}_t(\bar{\mu}_t \, \mathrm{d}t + \beta^T(t)\sigma_2 \sqrt{\bar{\mu}_t}\, \mathrm{d}W_t^{\bar{\mu}}), \end{aligned}$$

and by (3.2.8) we have that

$$\mathrm{d}Y_t = Y_t e^{-rT}\beta^T(t)\sigma_2\sqrt{\bar{\mu}_t}\,\mathrm{d}W_t^{\bar{\mu}}, \quad t \in [0,T],$$

hence the result follows. □

The following lemma will be needed in the proofs of Corollary 3.3.6 and 3.3.10.

Lemma 3.3.2. *Fix $u \in [0,T]$. For*

$$Z_t^u := \mathbb{E}[\exp(-\Gamma_u) \,|\, \mathcal{F}_t] = \mathbb{E}\left[\exp\left(-\int_0^u \mu_s \, \mathrm{d}s\right) \,\Big|\, \mathcal{F}_t\right], \quad t \in [0,u],$$

we have the following dynamics:

$$Z_t^u = Z_0^u + \int_0^t Z_s^u \beta_1^u(s)\sigma_1\sqrt{\mu_s}\,\mathrm{d}W_s^{\mu} + \int_0^t Z_s^u \beta_2^u(s)\sigma_2\sqrt{\bar{\mu}_s}\,\mathrm{d}W_s^{\bar{\mu}}, \tag{3.3.5}$$

where β_1^u and β_2^u are given by the following differential equations:

$$\partial_t \beta_1^u(t) = 1 + \gamma_1 \beta_1^u(t) - \frac{1}{2}\sigma_1^2(\beta_1^u(t))^2, \quad \beta_1^u(u) = 0, \tag{3.3.6}$$

$$\partial_t \beta_2^u(t) = -\gamma_1 \beta_1^u(t) + \gamma_2 \beta_2^u(t) - \frac{1}{2}\sigma_2^2(\beta_2^u(t))^2, \quad \beta_2^u(u) = 0. \tag{3.3.7}$$

Proof. Fix $u \in [0,T]$. With the same arguments as in the proof of Lemma 3.3.1, by equation (B.0.1) in Appendix B we have

$$\tilde{Z}_t^u := \mathbb{E}\left[\exp\left(-\int_t^u \mu_s \, \mathrm{d}s\right) \,\Big|\, \mathcal{F}_t\right] = \exp(\alpha^u(t) + \beta_1^u(t)\mu_t + \beta_2^u(t)\bar{\mu}_t) \tag{3.3.8}$$

for $t \in [0,u]$, where the functions α^u, β_1^u and β_2^u are given by

$$\begin{aligned} &\partial_t \beta_1^u(t) = 1 + \gamma_1 \beta_1^u(t) - \frac{1}{2}\sigma_1^2(\beta_1^u(t))^2, \quad \beta_1^u(u) = 0, \\ &\partial_t \beta_2^u(t) = -\gamma_1 \beta_1^u(t) + \gamma_2 \beta_2^u(t) - \frac{1}{2}\sigma_2^2(\beta_2^u(t))^2, \quad \beta_2^u(u) = 0, \\ &\partial_t \alpha^u(t) = -\gamma_2 m(t)\beta_2^u(t), \quad \alpha^u(u) = 0. \end{aligned}$$

Then, again by an application of Itô's formula, we obtain that

$$\begin{aligned} \mathrm{d}\tilde{Z}_t^u &= \tilde{Z}_t^u\Big[(\partial_t\alpha^u(t) + \partial_t\beta_1^u(t)\mu_t + \partial_t\beta_2^u(t)\bar{\mu}_t)\,\mathrm{d}t + \beta_1^u(t)\,\mathrm{d}\mu_t + \beta_2^u(t)\,\mathrm{d}\bar{\mu}_t \\ &\qquad + \frac{1}{2}(\beta_1^u(t))^2\,\mathrm{d}\langle\mu\rangle_t + \frac{1}{2}(\beta_2^u(t))^2\,\mathrm{d}\langle\bar{\mu}\rangle_t + \beta_1^u(t)\beta_2^u(t)\,\mathrm{d}\langle\mu,\bar{\mu}\rangle_t\Big] \\ &= \tilde{Z}_t^u(\mu_t\,\mathrm{d}t + \beta_1^u(t)\sigma_1\sqrt{\mu_t}\,\mathrm{d}W_t^\mu + \beta_2^u(t)\sigma_2\sqrt{\bar{\mu}_t}\,\mathrm{d}W_t^{\bar{\mu}}) \end{aligned}$$

for $t \in [0,u]$, hence the result follows. □

The following lemma will be needed in the proof of Corollary 3.3.8.

Lemma 3.3.3. *Fix $u \in [0,T]$. For*

$$Z_t^{\mu,u} := \mathbb{E}\left[\exp\left(-\int_0^u \mu_s\,\mathrm{d}s\right)\mu_u \,\middle|\, \mathcal{F}_t\right], \quad t \in [0,u],$$

we have the following dynamics:

$$\begin{aligned} Z_t^{\mu,u} &= Z_0^{\mu,u} + \int_0^t Z_s^u\left(\hat{\beta}_1^u(s) + \beta_1^u(s)\hat{Z}_s^u\right)\sigma_1\sqrt{\mu_s}\,\mathrm{d}W_s^\mu \\ &\quad + \int_0^t Z_s^u\left(\hat{\beta}_2^u(s) + \beta_2^u(s)\hat{Z}_s^u\right)\sigma_2\sqrt{\bar{\mu}_s}\,\mathrm{d}W_s^{\bar{\mu}}, \end{aligned} \tag{3.3.9}$$

where $\hat{Z}^u$ is given by

$$\hat{Z}_t^u = \hat{\alpha}^u(t) + \hat{\beta}_1^u(t)\mu_t + \hat{\beta}_2^u(t)\bar{\mu}_t, \quad t \in [0,u], \tag{3.3.10}$$

and $\hat{\alpha}^u$, $\hat{\beta}_1^u$ and $\hat{\beta}_2^u$ are given by the following differential equations:

$$\partial_t\hat{\beta}_1^u(t) = \gamma_1\hat{\beta}_1^u(t) - \sigma_1^2\beta_1^u\hat{\beta}_1^u(t), \quad \hat{\beta}_1^u(u) = 1, \tag{3.3.11}$$

$$\partial_t\hat{\beta}_2^u(t) = -\gamma_1\hat{\beta}_1^u(t) + \gamma_2\hat{\beta}_2^u(t) - \sigma_2^2\beta_2^u(t)\hat{\beta}_2^u(t), \quad \hat{\beta}_2^u(u) = 0, \tag{3.3.12}$$

$$\partial_t\hat{\alpha}^u(t) = -\gamma_2 m(t)\hat{\beta}_2^u(t), \quad \hat{\alpha}^u(u) = 0,$$

and β_1^u, β_2^u, and Z_t^u are given in (3.3.5) - (3.3.7).

Proof. Fix $u \in [0,T]$. With equation (B.0.1) in Appendix B we immediately obtain

$$\mathbb{E}\left[\exp\left(-\int_t^u \mu_s\,\mathrm{d}s\right)\mu_u \,\middle|\, \mathcal{F}_t\right] = \tilde{Z}_t^u\hat{Z}_t^u, \quad t \in [0,u],$$

where $\tilde{Z}_t^u$ is given in (3.3.8) and

$$\hat{Z}_t^u = \hat{\alpha}^u(t) + \hat{\beta}_1^u(t)\mu_t + \hat{\beta}_2^u(t)\bar{\mu}_t, \quad t \in [0,u],$$

with

$$\begin{aligned} \partial_t\hat{\beta}_1^u(t) &= \gamma_1\hat{\beta}_1^u(t) - \sigma_1^2\beta_1^u\hat{\beta}_1^u(t), \quad \hat{\beta}_1^u(u) = 1, \\ \partial_t\hat{\beta}_2^u(t) &= -\gamma_1\hat{\beta}_1^u(t) + \gamma_2\hat{\beta}_2^u(t) - \sigma_2^2\beta_2^u(t)\hat{\beta}_2^u(t), \quad \hat{\beta}_2^u(u) = 0, \\ \partial_t\hat{\alpha}^u(t) &= -\gamma_2 m(t)\hat{\beta}_2^u(t), \quad \hat{\alpha}^u(u) = 0. \end{aligned}$$

Then, again by an application of Itô's formula, we obtain

$$\begin{aligned}\mathrm{d}\hat{Z}_t^u &= [\partial_t\hat{\alpha}^u(t) + \partial_t\hat{\beta}_1^u(t)\mu_t + \partial_t\hat{\beta}_2^u(t)\bar{\mu}_t]\,\mathrm{d}t + \hat{\beta}_1^u(t)\,\mathrm{d}\mu_t + \hat{\beta}_2^u(t)\,\mathrm{d}\bar{\mu}_t \\ &= [-\beta_1^u(t)\hat{\beta}_1^u(t)\sigma_1^2\mu_t - \beta_2^u(t)\hat{\beta}_2^u(t)\sigma_2^2\bar{\mu}_t]\,\mathrm{d}t \\ &\quad + \hat{\beta}_1^u(t)\sigma_1\sqrt{\mu_t}\,\mathrm{d}W_t^\mu + \hat{\beta}_2^u(t)\sigma_2\sqrt{\bar{\mu}_t}\,\mathrm{d}W_t^{\bar{\mu}},\end{aligned}$$

and

$$\begin{aligned}\mathrm{d}(\tilde{Z}_t^u\hat{Z}_t^u) &= \tilde{Z}_t^u\,\mathrm{d}\hat{Z}_t^u + \hat{Z}_t^u\,\mathrm{d}\tilde{Z}_t^u + \langle\tilde{Z}^u, \hat{Z}^u\rangle_t \\ &= \tilde{Z}_t^u[\mu_t\hat{Z}_t^u\,\mathrm{d}t + (\hat{\beta}_1^u(t) + \beta_1^u(t)\hat{Z}_t^u)\sigma_1\sqrt{\mu_t}\,\mathrm{d}W_t^\mu \\ &\quad + (\hat{\beta}_2^u(t) + \beta_2^u(t)\hat{Z}_t^u)\sigma_2\sqrt{\bar{\mu}_t}\,\mathrm{d}W_t^{\bar{\mu}}]\end{aligned}$$

for $t \in [0, u]$, hence the result follows. □

The following remark elaborates more in detail on the technical assumptions regarding the model choice for $(\mu, \bar{\mu})$ in (3.2.2) - (3.2.3).

Remark 3.3.4. *As stated in Biffis [13], from a technical point of view for the existence and uniqueness of a solution $(\mu, \bar{\mu})$ of the set of stochastic differential equations* (3.2.2) - (3.2.3) *it is not necessarily required that the Brownian motions W^μ and $W^{\bar{\mu}}$ are independent. In fact from an intuitive point of view it is plausible to actually allow for correlation between the two Brownian motions driving μ and $\bar{\mu}$. However, we would like to remark that relaxing the independence assumption destroys the affine structure of $(\mu, \bar{\mu})$ (see, e.g., Dai and Singleton [28], Duffie et al. [32] or Filipović and Mayerhofer [35]), hence in order to obtain analytical expressions for the conditional expectations in Lemma 3.3.2 and 3.3.3 it is in fact necessary to assume that W^μ and $W^{\bar{\mu}}$ are independent. Also note that in* (3.3.23), (3.3.29) *and* (3.3.35) *we will make use of the fact that the Brownian motions W driving the asset price S as introduced in* (3.2.6) *and $(W^\mu, W^{\bar{\mu}})$ driving $(\mu, \bar{\mu})$ are independent. Of course it is possible to relax this independence, however then in order to evaluate the conditional expectations in* (3.3.23), (3.3.29) *and* (3.3.35) *it is necessary to define $(S, \mu, \bar{\mu})$ as a multi-dimensional affine diffusion (with respect to correlated Brownian motions). This is only possible if the diffusion coefficients are constants for all three processes, in which case μ and $\bar{\mu}$ are no longer non-negative and for the volatility of the asset price we have $\sigma(t, S_t) \equiv \sigma$, $t \in [0, T]$, for a constant $\sigma > 0$.*

In the following we calculate the prices and hedging strategies of insurance payment streams as introduced in (3.2.9) - (3.2.11) by means of the risk-minimization approach (see Appendix A). We start with the pure endowment contract.

3.3.2 Pure Endowment Contract

For the pure endowment contract introduced in (3.2.9) we define the payment process

$$A_t^{pe} = (n - N_t)\,\frac{C^{pe}}{B_t}\mathbb{1}_{\{t=T\}}, \quad t \in [0, T], \tag{3.3.13}$$

where C^{pe} is a non-negative $\mathcal{F}_T$-measurable random variable and $\mathbb{E}[(C^{pe})^2] < \infty$.

Proposition 3.3.5. *In the setting of Section 3.2 the payment process A^{pe} introduced in* (3.3.13) *admits a risk-minimizing strategy $\varphi = (\xi, \xi^0) = (\xi^X, \xi^Y, \xi^0)$ given by*

$$\xi_t = (\xi_t^X, \xi_t^Y) = \left(\frac{(n - N_t)e^{\Gamma_t}\psi_t}{\sigma(t, X_t)X_t}, \frac{(n - N_t)e^{rT+\Gamma_t}\psi_t^{\bar{\mu}}}{Y_t\beta^T(t)\sigma_2\sqrt{\bar{\mu}_t}} \right),$$
$$\xi_t^0 = V_t^{pe}(\varphi) - \xi_t^X X_t - \xi_t^Y Y_t$$

for $t \in [0, T]$, with discounted value process

$$V_t^{pe}(\varphi) = nU_0^{pe} + \int_0^t \xi_s^X \,\mathrm{d}X_s + \int_0^t \xi_s^Y \,\mathrm{d}Y_s + L_t^{pe} - A_t^{pe},$$

where

$$L_t^{pe} = \int_0^t (n - N_s)e^{\Gamma_s}\psi_s^\mu \,\mathrm{d}W_s^\mu - \int_{]0,t]} U_s^{pe} e^{\Gamma_s} \,\mathrm{d}M_s$$

for $t \in [0, T]$, where $\beta^T(t)$ and M_t are defined in (3.3.3) *and* (3.3.4) *and U^{pe}, ψ, ψ^μ, and $\psi^{\bar{\mu}}$ are given by*

$$U_t^{pe} = \mathbb{E}\left[e^{-\Gamma_T}\frac{C^{pe}}{B_T} \,\Big|\, \mathcal{F}_t\right] = U_0^{pe} + \int_0^t \psi_s \,\mathrm{d}W_s + \int_0^t \psi_s^\mu \,\mathrm{d}W_s^\mu + \int_0^t \psi_s^{\bar{\mu}} \,\mathrm{d}W_s^{\bar{\mu}}, \quad (3.3.14)$$

where ψ, ψ^μ and $\psi^{\bar{\mu}}$ are $\mathbb{F}$-predictable processes satisfying

$$\mathbb{E}\left[\int_0^T (\psi_s)^2 \,\mathrm{d}s\right], \; \mathbb{E}\left[\int_0^T (\psi_s^\mu)^2 \,\mathrm{d}s\right], \; \mathbb{E}\left[\int_0^T (\psi_s^{\bar{\mu}})^2 \,\mathrm{d}s\right] < \infty.$$

The optimal cost and risk processes are given by

$$C_t^{pe}(\varphi) = nU_0^{pe} + L_t^{pe},$$
$$R_t^{pe}(\varphi) = \mathbb{E}[(L_T^{pe} - L_t^{pe})^2 \,|\, \mathcal{G}_t],$$

for $t \in [0, T]$.

Proof. Let $t \in [0, T]$. Then we have that

$$\mathbb{E}[A_T^{pe} \,|\, \mathcal{G}_t] = \sum_{i=1}^n \mathbb{E}\left[\mathbb{1}_{\{\tau^i > T\}}\frac{C^{pe}}{B_T} \,\Big|\, \mathcal{G}_t\right],$$

and by Proposition 4.10 and 5.11 of Barbarin [4, Chapter 3], as well as Corollary 5.1.1 of Bielecki and Rutkowski [12] and (3.2.5) we have

$$\begin{aligned} J_t^{pe,i} &:= \mathbb{E}\left[\mathbb{1}_{\{\tau^i > T\}}\frac{C^{pe}}{B_T} \,\Big|\, \mathcal{G}_t\right] \\ &= U_0^{pe} + \int_0^t L_s^i \,\mathrm{d}U_s^{pe} - \int_{]0,t]} U_s^{pe} e^{\Gamma_s} \,\mathrm{d}M_s^i, \end{aligned} \quad (3.3.15)$$

where

$$U_t^{pe} = \mathbb{E}\left[e^{-\Gamma_T}\frac{C^{pe}}{B_T}\,\Big|\,\mathcal{F}_t\right], \quad t \in [0,T],$$

is a square integrable martingale, since $\mathbb{E}[(C^{pe})^2] < \infty$. By the martingale representation theorem for Brownian filtrations (see, e.g., Theorem 43 of Protter [56, Chapter IV.3]) we have that

$$U_t^{pe} = U_0^{pe} + \int_0^t \psi_s \,\mathrm{d}W_s + \int_0^t \psi_s^{\mu} \,\mathrm{d}W_s^{\mu} + \int_0^t \psi_s^{\bar{\mu}} \,\mathrm{d}W_s^{\bar{\mu}}, \quad t \in [0,T],$$

where ψ, ψ^{μ} and $\psi^{\bar{\mu}}$ are $\mathbb{F}$-predictable processes satisfying

$$\mathbb{E}\left[\int_0^T (\psi_s)^2 \,\mathrm{d}s\right], \; \mathbb{E}\left[\int_0^T (\psi_s^{\mu})^2 \,\mathrm{d}s\right], \; \mathbb{E}\left[\int_0^T (\psi_s^{\bar{\mu}})^2 \,\mathrm{d}s\right] < \infty.$$

Hence by (3.2.7) and Lemma 3.3.1 for $t \in [0,T]$ we have that

$$\begin{aligned}\mathbb{E}[A_T^{pe} \,|\, \mathcal{G}_t] = \sum_{i=1}^n \Big(U_0^{pe} &+ \int_0^t \mathbb{1}_{\{\tau^i \geq s\}} e^{\Gamma_s} \psi_s \,\mathrm{d}W_s + \int_0^t \mathbb{1}_{\{\tau^i \geq s\}} e^{\Gamma_s} \psi_s^{\mu} \,\mathrm{d}W_s^{\mu} \\ &+ \int_0^t \mathbb{1}_{\{\tau^i \geq s\}} e^{\Gamma_s} \psi_s^{\bar{\mu}} \,\mathrm{d}W_s^{\bar{\mu}} - \int_{]0,t]} U_s^{pe} e^{\Gamma_s} \,\mathrm{d}M_s^i \Big) \qquad (3.3.16) \\ = nU_0^{pe} &+ \int_0^t \xi_s^X \,\mathrm{d}X_s + \int_0^t \xi_s^Y \,\mathrm{d}Y_s + L_t^{pe}, \qquad (3.3.17)\end{aligned}$$

where

$$\begin{aligned}\xi_t^X &= \frac{(n-N_t)e^{\Gamma_t}\psi_t}{\sigma(t,X_t)X_t}, \\ \xi_t^Y &= \frac{(n-N_t)e^{rT+\Gamma_t}\psi_t^{\bar{\mu}}}{Y_t \beta^T(t)\sigma_2\sqrt{\bar{\mu}_t}},\end{aligned}$$

and

$$L_t^{pe} = \int_0^t (n-N_s)e^{\Gamma_s}\psi_s^{\mu} \,\mathrm{d}W_s^{\mu} - \int_{]0,t]} U_s^{pe} e^{\Gamma_s} \,\mathrm{d}M_s. \qquad (3.3.18)$$

It remains to prove that (3.3.17) is indeed the GKW decomposition of $\mathbb{E}[A_T^{pe} \,|\, \mathcal{G}_t]$, i.e. that all integrals are square integrable and that

$$\left(\int_0^t \tilde{\xi}_s^X \,\mathrm{d}X_s + \int_0^t \tilde{\xi}_s^Y \,\mathrm{d}Y_s\right) \cdot L_t^{pe}, \quad t \in [0,T],$$

is a (uniformly integrable) martingale for all $\mathbb{G}$-predictable processes $\tilde{\xi}^X \in L^2(X)$, $\tilde{\xi}^Y \in L^2(Y)$. To this end note that since $J^{pe,i}$ is a square integrable martingale, we have that $\mathbb{E}[[J^{pe,i}]_T] < \infty$. Then from (3.3.15) we follow that

$$\mathbb{E}[[J^{pe,i}]_T] = \mathbb{E}\left[\int_0^T (L_s^i)^2 \,\mathrm{d}[U^{pe}]_s\right] + \mathbb{E}\left[\int_0^T (U_s^{pe} e^{\Gamma_s})^2 \,\mathrm{d}[M^i]_s\right] < \infty,$$

$i = 1, \dots, n$, since $[U^{pe}, M^i]_t \equiv 0$, $t \in [0,T]$. Hence by Lemma 2.1 of Schweizer [58] we have that

$$\int_0^t L_s^i \, \mathrm{d}U_s^{pe}, \quad t \in [0,T],$$

and

$$\int_0^t (-U_s^{pe} e^{\Gamma_s}) \, \mathrm{d}M_s^i, \quad t \in [0,T],$$

are square integrable martingales. Again by the martingale representation theorem we have that

$$\int_0^t L_s^i \, \mathrm{d}U_s^{pe} = \int_0^t \tilde{\psi}_s^i \, \mathrm{d}W_s + \int_0^t \tilde{\psi}_s^{\mu,i} \, \mathrm{d}W_s^{\mu} + \int_0^t \tilde{\psi}_s^{\bar{\mu},i} \, \mathrm{d}W_s^{\bar{\mu}}, \quad t \in [0,T], \tag{3.3.19}$$

where $\tilde{\psi}^i$, $\tilde{\psi}^{\mu,i}$ and $\tilde{\psi}^{\bar{\mu},i}$ are $\mathbb{F}$-predictable processes satisfying

$$\mathbb{E}\left[\int_0^T (\tilde{\psi}_s^i)^2 \, \mathrm{d}s\right], \; \mathbb{E}\left[\int_0^T (\tilde{\psi}_s^{\mu,i})^2 \, \mathrm{d}s\right], \; \mathbb{E}\left[\int_0^T (\tilde{\psi}_s^{\bar{\mu},i})^2 \, \mathrm{d}s\right] < \infty.$$

Hence by comparing (3.3.16) with (3.3.19) for $i = 1, \dots, n$ and $t \in [0,T]$ we have that

$$\tilde{\psi}_t^i = \mathbb{1}_{\{\tau^i \geq t\}} e^{\Gamma_t} \psi_t, \quad \tilde{\psi}_t^{\mu,i} = \mathbb{1}_{\{\tau^i \geq t\}} e^{\Gamma_t} \psi_t^{\mu}, \quad \tilde{\psi}_t^{\bar{\mu},i} = \mathbb{1}_{\{\tau^i \geq t\}} e^{\Gamma_t} \psi_t^{\bar{\mu}},$$

since W, W^{μ} and $W^{\bar{\mu}}$ are independent. Hence $(\xi^X, \xi^Y) \in L^2(X,Y)$ and L^{pe} as defined in (3.3.18) is a square integrable martingale as the sum of square integrable martingales. It remains to prove that

$$\left(\int_0^t \tilde{\xi}_s^X \, \mathrm{d}X_s + \int_0^t \tilde{\xi}_s^Y \, \mathrm{d}Y_s\right) \cdot L_t^{pe}, \quad t \in [0,T],$$

is a (uniformly integrable) martingale for all $\mathbb{G}$-predictable processes $\tilde{\xi}^X \in L^2(X)$, $\tilde{\xi}^Y \in L^2(Y)$. However, this follows directly from the fact that for $t \in [0,T]$, $[W, M]_t = [W^{\bar{\mu}}, M]_t \equiv 0$ and $[W, W^{\mu}]_t = [W^{\bar{\mu}}, W^{\mu}]_t \equiv 0$ and by using Proposition 4.50 of Jacod and Shiryaev [42, Chapter I]. □

Note that the cost process is the sum of two orthogonal martingales, the first of which is related to the fact that due to the structure of $(\mu, \bar{\mu})$ as defined in (3.2.2) - (3.2.3) the financial market given by the filtration $\mathbb{F}$ is not complete. The second integral is related to the unpredictability of the times of death.
In the following (see Corollary 3.3.6, 3.3.8 and 3.3.10) we now consider special payoff structures in the context of unit-linked life insurance products, where the life insurance liabilities are given in terms of a non-negative Borel measurable function $f(S_t)$ of the asset price S_t, $t \in [0,T]$. Then following Møller [53] for fixed $u \in [0,T]$ the arbitrage-free price process

$$F^u(t, S_t) = \mathbb{E}\left[\exp\left(-r\,(u-t)\right) f(S_u) | \mathcal{F}_t\right], \quad t \in [0,u], \tag{3.3.20}$$

associated with the payoff $f(S_u)$ at time u can be be characterized by the partial differential equation

$$-rF^u(t,s) + F^u_t(t,s) + rsF^u_s(t,s) + \frac{1}{2}\sigma(t,s)^2 s^2 F^u_{ss}(t,s) = 0, \qquad (3.3.21)$$

with boundary value $F^u(u,s) = f(s)$, where we denote by $F^u_t(t,s)$, $F^u_s(t,s)$ and $F^u_{ss}(t,s)$ the partial first and second order derivatives of F^u with respect to t and s.
The next corollary provides an application of Proposition 3.3.5 where we set

$$C^{pe} = f(S_T)$$

in (3.2.9) and (3.3.13), where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borel measurable function such that

$$\mathbb{E}\left[f(S_T)^2\right] < \infty,$$

i.e. we define the payment process

$$A^{pe,f}_t = (n - N_t)\,\frac{f(S_t)}{B_t}\mathbb{1}_{\{t=T\}}, \quad t \in [0,T]. \qquad (3.3.22)$$

Corollary 3.3.6. *In the setting of Section 3.2 the payment process $A^{pe,f}$ introduced in* (3.3.22) *admits a risk-minimizing strategy $\varphi = (\xi, \xi^0) = (\xi^X, \xi^Y, \xi^0)$ given by*

$$\begin{aligned}
\xi^X_t &= (n - N_t)e^{\Gamma_t} Z^T_t F^T_s(t, S_t),\\
\xi^Y_t &= \frac{(n - N_t)e^{\Gamma_t + r(T-t)}\beta^T_2(t) F^T(t,S_t) Z^T_t}{Y_t \beta^T(t)},\\
\xi^0_t &= V^{pe,f}_t(\varphi) - \xi^X_t X_t - \xi^Y_t Y_t
\end{aligned}$$

for $t \in [0,T]$, with discounted value process

$$V^{pe,f}_t(\varphi) = nZ^T_0 F^T(0,S_0) + \int_0^t \xi^X_s \,\mathrm{d}X_s + \int_0^t \xi^Y_s \,\mathrm{d}Y_s + L^{pe,f}_t - A^{pe,f}_t,$$

where

$$\begin{aligned}
L^{pe,f}_t = &\int_0^t (n - N_s)e^{\Gamma_s - rs}\beta^T_1(s)\sigma_1\sqrt{\mu_s}F^T(s,S_s)Z^T_s \,\mathrm{d}W^\mu_s\\
&- \int_{]0,t]} e^{\Gamma_s - rs} F^T(s,S_s) Z^T_s \,\mathrm{d}M_s,
\end{aligned}$$

for $t \in [0,T]$, where $\beta^T(t)$, $\beta^T_1(t)$, $\beta^T_2(t)$, $F^T(t,S_t)$, $F^T_s(t,S_t)$, Z^T_t and M_t are defined in (3.3.3) - (3.3.7), (3.3.20) *and* (3.3.21).

Proof. By the independence of the underlying driving processes, for U^{pe} as defined in (3.3.14) we have

$$U_t^{pe} = \mathbb{E}\left[e^{-\Gamma_T}\frac{f(S_T)}{B_T}\,\Big|\,\mathcal{F}_t\right] = \mathbb{E}[e^{-\Gamma_T}\,|\,\mathcal{F}_t]\mathbb{E}\left[\frac{f(S_T)}{B_T}\,\Big|\,\mathcal{F}_t\right], \tag{3.3.23}$$

for $t \in [0,T]$, and by (3.2.6) - (3.2.7), (3.3.20) - (3.3.21) and Itô's formula the discounted arbitrage-free price process $\frac{F^T(t,S_t)}{B_t}$, $t \in [0,T]$, follows the dynamics

$$\mathrm{d}\left(\frac{F^T(t,S_t)}{B_t}\right) = F_s^T(t,S_t)\sigma(t,S_t)X_t\,\mathrm{d}W_t = F_s^T(t,S_t)\,\mathrm{d}X_t, \quad t \in [0,T], \tag{3.3.24}$$

and by integration by parts and (3.3.5) and (3.3.24) we obtain that

$$\begin{aligned} U_t^{pe} &= Z_0^T F^T(0,S_0) + \int_0^t \sigma(s,X_s)X_sF_s^T(s,S_s)Z_s^T\,\mathrm{d}W_s \\ &\quad + \int_0^t \beta_1^T(s)\sigma_1\sqrt{\mu_s}\frac{F^T(s,S_s)}{B_s}Z_s^T\,\mathrm{d}W_s^\mu + \int_0^t \beta_2^T(s)\sigma_2\sqrt{\bar{\mu}_s}\frac{F^T(s,S_s)}{B_s}Z_s^T\,\mathrm{d}W_s^{\bar{\mu}} \end{aligned}$$

for $t \in [0,T]$, hence the result follows by using Proposition 3.3.5. □

3.3.3 Term Insurance Contract

For the term insurance contract introduced in (3.2.10) we define the payment process

$$A_t^{ti} = \int_0^t \frac{C_s^{ti}}{B_s}\,\mathrm{d}N_s = \sum_{i=1}^n \int_0^t \frac{C_s^{ti}}{B_s}\,\mathrm{d}H_s^i = \sum_{i=1}^n \mathbb{1}_{\{\tau^i \le t\}}\frac{C_{\tau^i}^{ti}}{B_{\tau^i}}, \quad t \in [0,T], \tag{3.3.25}$$

where C^{ti} is assumed to be a non-negative $\mathbb{F}$-predictable process such that

$$\mathbb{E}\left[\sup_{t\in[0,T]}(C_t^{ti})^2\right] < \infty.$$

Proposition 3.3.7. *In the setting of Section 3.2 the payment process A^{ti} introduced in* (3.3.25) *admits a risk-minimizing strategy $\varphi = (\xi,\xi^0) = (\xi^X,\xi^Y,\xi^0)$ given by*

$$\begin{aligned} \xi_t &= (\xi_t^X,\xi_t^Y) = \left(\frac{(n-N_t)e^{\Gamma_t}\psi_t}{\sigma(t,X_t)X_t}, \frac{(n-N_t)e^{\Gamma_t+rT}\psi_t^{\bar{\mu}}}{Y_t\beta^T(t)\sigma_2\sqrt{\bar{\mu}_t}}\right), \\ \xi_t^0 &= V_t^{ti}(\varphi) - \xi_t^X X_t - \xi_t^Y Y_t \end{aligned}$$

for $t \in [0,T]$, with discounted value process

$$V_t^{ti}(\varphi) = nU_0^{ti} + \int_0^t \xi_s^X\,\mathrm{d}X_s + \int_0^t \xi_s^Y\,\mathrm{d}Y_s + L_t^{ti} - A_t^{ti},$$

where

$$L_t^{ti} = \int_0^t (n-N_s)e^{\Gamma_s}\psi_s^\mu \, \mathrm{d}W_s^\mu + \int_{]0,t]} \left(\frac{C_s^{ti}}{B_s} - \mathbb{E}\left[\int_s^T \frac{C_u^{ti}}{B_u} e^{\Gamma_s - \Gamma_u} \, \mathrm{d}\Gamma_u \,\Big|\, \mathcal{F}_s \right] \right) \mathrm{d}M_s,$$

for $t \in [0,T]$, *where* $\beta^T(t)$ *and* M_t *are defined in* (3.3.3) *and* (3.3.4) *and where* U^{ti}, ψ, ψ^μ *and* $\psi^{\bar{\mu}}$ *are given by*

$$\begin{aligned} U_t^{ti} &= \mathbb{E}\left[\int_0^T \frac{C_s^{ti}}{B_s} e^{-\Gamma_s} \, \mathrm{d}\Gamma_s \,\Big|\, \mathcal{F}_t \right] \\ &= U_0^{ti} + \int_0^t \psi_s \, \mathrm{d}W_s + \int_0^t \psi_s^\mu \, \mathrm{d}W_s^\mu + \int_0^t \psi_s^{\bar{\mu}} \, \mathrm{d}W_s^{\bar{\mu}}, \end{aligned} \tag{3.3.26}$$

where ψ, ψ^μ *and* $\psi^{\bar{\mu}}$ *are* $\mathbb{F}$*-predictable processes satisfying*

$$\mathbb{E}\left[\int_0^T (\psi_s)^2 \, \mathrm{d}s \right], \; \mathbb{E}\left[\int_0^T (\psi_s^\mu)^2 \, \mathrm{d}s \right], \; \mathbb{E}\left[\int_0^T (\psi_s^{\bar{\mu}})^2 \, \mathrm{d}s \right] < \infty.$$

The optimal cost and risk processes are given by

$$\begin{aligned} C_t^{ti}(\varphi) &= nU_0^{ti} + L_t^{ti}, \\ R_t^{ti}(\varphi) &= \mathbb{E}[(L_T^{ti} - L_t^{ti})^2 \,|\, \mathcal{G}_t], \end{aligned}$$

for $t \in [0,T]$.

Proof. By Proposition 4.11 and 5.12 of Barbarin [4, Chapter 3], as well as Corollary 5.1.3 of Bielecki and Rutkowski [12] and (3.2.5) we have

$$\begin{aligned} \mathbb{E}[A_T^{ti} \,|\, \mathcal{G}_t] &= nU_0^{ti} + \int_0^t (n-N_s)e^{\Gamma_s} \, \mathrm{d}U_s^{ti} \\ &\quad + \int_{]0,t]} \left(\frac{C_s^{ti}}{B_s} - \mathbb{E}\left[\int_s^T \frac{C_u^{ti}}{B_u} e^{\Gamma_s - \Gamma_u} \, \mathrm{d}\Gamma_u \,\Big|\, \mathcal{F}_s \right] \right) \mathrm{d}M_s, \end{aligned}$$

where

$$U_t^{ti} = \mathbb{E}\left[\int_0^T \frac{C_s^{ti}}{B_s} e^{-\Gamma_s} \, \mathrm{d}\Gamma_s \,\Big|\, \mathcal{F}_t \right], \quad t \in [0,T],$$

is a square integrable martingale, since by Jensen's inequality for any $t \in [0,T]$ we have

$$\mathbb{E}[(U_t^{ti})^2] \le \mathbb{E}\left[\sup_{t\in[0,T]} (C_t^{ti})^2 \left(\int_0^T \mathrm{d}e^{-\Gamma_s} \right)^2 \right] \le \mathbb{E}\left[\sup_{t\in[0,T]} (C_t^{ti})^2 \right],$$

and $\mathbb{E}\left[\sup_{t\in[0,T]}(C_t^{ti})^2\right] < \infty$. By the martingale representation theorem for Brownian filtrations we have that

$$U_t^{ti} = U_0^{ti} + \int_0^t \psi_s \, \mathrm{d}W_s + \int_0^t \psi_s^\mu \, \mathrm{d}W_s^\mu + \int_0^t \psi_s^{\bar{\mu}} \, \mathrm{d}W_s^{\bar{\mu}}, \quad t \in [0,T],$$

where ψ, ψ^{μ} and $\psi^{\bar{\mu}}$ are $\mathbb{F}$-predictable processes satisfying

$$\mathbb{E}\left[\int_0^T (\psi_s)^2 \, \mathrm{d}s\right], \ \mathbb{E}\left[\int_0^T (\psi_s^{\mu})^2 \, \mathrm{d}s\right], \ \mathbb{E}\left[\int_0^T (\psi_s^{\bar{\mu}})^2 \, \mathrm{d}s\right] < \infty.$$

Hence by (3.2.7) and Lemma 3.3.1 we have that

$$\mathbb{E}\left[A_T^{ti} \,\middle|\, \mathcal{G}_t\right] = nU_0^{ti} + \int_0^t \xi_s^X \, \mathrm{d}X_s + \int_0^t \xi_s^Y \, \mathrm{d}Y_s + L_t^{ti}, \tag{3.3.27}$$

where

$$\xi_t^X = \frac{(n - N_t)e^{\Gamma_t}\psi_t^X}{\sigma(t, X_t)X_t},$$

$$\xi_t^Y = \frac{(n - N_t)e^{\Gamma_t + rT}\psi_t^{\bar{\mu}}}{Y_t \beta^T(t)\sigma_2\sqrt{\bar{\mu}_t}},$$

and

$$L_t^{ti} = \int_0^t (n - N_s)e^{\Gamma_s}\psi_s^{\mu} \, \mathrm{d}W_s^{\mu} + \int_{]0,t]} \left(\frac{C_s^{ti}}{B_s} - \mathbb{E}\left[\int_s^T \frac{C_u^{ti}}{B_u} e^{\Gamma_s - \Gamma_u} \, \mathrm{d}\Gamma_u \,\middle|\, \mathcal{F}_s\right] \right) \mathrm{d}M_s.$$

By the same arguments as in the proof of Proposition 3.3.5 we obtain that all integrals in (3.3.27) are square integrable and strongly orthogonal, hence (3.3.27) is indeed the GKW decomposition of $\mathbb{E}[A_T^{ti} \mid \mathcal{G}_t]$. □

Note that Corollary 5.1.3 of Bielecki and Rutkowski [12] requires C^{ti} to be bounded. However, it can be easily seen that this result also holds if $\mathbb{E}[\sup_{t \in [0,T]}(C_t^{ti})^2] < \infty$ and we may therefore apply it in our setting.
The next corollary provides an application of Proposition 3.3.7 where we set

$$C_t^{ti} = f(S_t), \quad t \in [0, T],$$

in (3.2.10) and (3.3.25), where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borel measurable function such that

$$\mathbb{E}\left[\sup_{t \in [0,T]} f(S_t)^2\right] < \infty,$$

i.e. we define the payment process

$$A_t^{ti,f} = \int_0^t \frac{f(S_s)}{B_s} \, \mathrm{d}N_s = \sum_{i=1}^n \mathbb{1}_{\{\tau^i \le t\}} \frac{f(S_{\tau^i})}{B_{\tau^i}}, \quad t \in [0, T]. \tag{3.3.28}$$

Corollary 3.3.8. *In the setting of Section 3.2 the payment process $A^{ti,f}$ introduced in* (3.3.28) *admits a risk-minimizing strategy $\varphi = (\xi, \xi^0) = (\xi^X, \xi^Y, \xi^0)$ given by*

$$
\begin{aligned}
\xi_t^X &= (n - N_t) e^{\Gamma_t} \int_t^T F_s^u(t, S_t) Z_t^{\mu,u} \,\mathrm{d}u \\
\xi_t^Y &= (n - N_t) e^{\Gamma_t + r(T-t)} Y_t^{-1} (\beta^T(t))^{-1} \int_t^T F^u(t, S_t) Z_t^u (\hat{\beta}_2^u(t) + \beta_2^u(t) \hat{Z}_t^u) \,\mathrm{d}u, \\
\xi_t^0 &= V_t^{ti}(\varphi) - \xi_t^X X_t - \xi_t^Y Y_t
\end{aligned}
$$

for $t \in [0, T]$, with discounted value process

$$
V_t^{ti,f}(\varphi) = n \int_0^T Z_0^{\mu,u} F^u(0, S_0) \,\mathrm{d}u + \int_0^t \xi_s^X \,\mathrm{d}X_s + \int_0^t \xi_s^Y \,\mathrm{d}Y_s + L_t^{ti,f} - A_t^{ti,f},
$$

where

$$
\begin{aligned}
L_t^{ti,f} = &\int_0^t (n - N_s) e^{\Gamma_s} \int_s^T \frac{F^u(s, S_s)}{B_s} Z_s^u (\hat{\beta}_1^u(s) + \beta_1^u(s) \hat{Z}_s^u) \sigma_1 \sqrt{\mu_s} \,\mathrm{d}u \,\mathrm{d}W_s^\mu \\
&+ \int_{]0,t]} \left(\frac{f(S_s)}{B_s} - \mathbb{E}\left[\int_s^T \frac{f(S_u)}{B_u} e^{\Gamma_s - \Gamma_u} \,\mathrm{d}\Gamma_u \,|\, \mathcal{F}_s \right] \right) \mathrm{d}M_s
\end{aligned}
$$

for $t \in [0, T]$, where $\beta^T(t)$, $\beta_1^u(t)$, $\beta_2^u(t)$, $\hat{\beta}_1^u(t)$, $\hat{\beta}_2^u(t)$, $F^u(t, S_t)$, $F_s^u(t, S_t)$, Z_t^u, $Z_t^{\mu,u}$, $\hat{Z}_t^u$ and M_t are defined in (3.3.3) - (3.3.7), (3.3.9) - (3.3.12), (3.3.20) *and* (3.3.21).

Proof. For U^{ti} as defined in (3.3.26) we have

$$
\begin{aligned}
U_t^{ti} &= \mathbb{E}\left[\int_0^T \frac{f(S_u)}{B_u} e^{-\Gamma_u} \mu_u \,\mathrm{d}u \,\Big|\, \mathcal{F}_t \right] \\
&= \int_0^T \mathbb{E}\left[\frac{f(S_u)}{B_u} \,\Big|\, \mathcal{F}_t \right] \mathbb{E}\left[e^{-\Gamma_u} \mu_u \,\Big|\, \mathcal{F}_t \right] \mathrm{d}u, \quad t \in [0, T], \qquad (3.3.29)
\end{aligned}
$$

where we have used Fubini's theorem and the independence of the underlying driving processes. By the same arguments as in the proof of Corollary 3.3.6 we have that

$$
\mathbb{E}\left[\frac{f(S_u)}{B_u} \,\Big|\, \mathcal{F}_t \right] = F^u(0, S_0) + \int_0^t F_s^u(s, S_s) \sigma(s, X_s) X_s \mathbb{1}_{\{s \le u\}} \,\mathrm{d}W_s \qquad (3.3.30)
$$

for $0 \le t,\ u \le T$, where $F^u(u, S_u) = f(S_u)$. Furthermore, by (3.3.9) we have

$$
\begin{aligned}
Z_t^{\mu,u} = \mathbb{E}\left[e^{-\Gamma_u} \mu_u \,\Big|\, \mathcal{F}_t \right] &= Z_0^{\mu,u} + \int_0^t Z_s^u \left(\hat{\beta}_1^u(s) + \beta_1^u(s) \hat{Z}_s^u \right) \sigma_1 \sqrt{\mu_s} \mathbb{1}_{\{s \le u\}} \,\mathrm{d}W_s^\mu \\
&+ \int_0^t Z_s^u \left(\hat{\beta}_2^u(s) + \beta_2^u(s) \hat{Z}_s^u \right) \sigma_2 \sqrt{\bar{\mu}_s} \mathbb{1}_{\{s \le u\}} \,\mathrm{d}W_s^{\bar{\mu}}, \quad 0 \le t,\ u \le T,
\end{aligned}
$$

where β_1^u, β_2^u, $\hat{\beta}_1^u$, $\hat{\beta}_2^u$, Z^u and $\hat{Z}^u$ are given in (3.3.5) - (3.3.7) and (3.3.10) - (3.3.12). Then since all integrands are continuous (see Theorem 15 in Chapter IV of Protter [56]), once again by Itô's formula and by the stochastic Fubini theorem (see, e.g., Theorem 65 in Chapter IV of Protter [56]) we obtain

$$\begin{aligned} U_t^{ti} &= \int_0^T Z_0^{\mu,u} F^u(0, S_0)\,\mathrm{d}u + \int_0^t \int_s^T F_s^u(s, S_s) Z_s^{\mu,u} \sigma(s, X_s) X_s \,\mathrm{d}u\,\mathrm{d}W_s \\ &+ \int_0^t \int_s^T \frac{F^u(s, S_s)}{B_s} Z_s^u (\hat{\beta}_1^u(s) + \beta_1^u(s) \hat{Z}_s^u) \sigma_1 \sqrt{\mu_s}\,\mathrm{d}u\,\mathrm{d}W_s^\mu \\ &+ \int_0^t \int_s^T \frac{F^u(s, S_s)}{B_s} Z_s^u (\hat{\beta}_2^u(s) + \beta_2^u(s) \hat{Z}_s^u) \sigma_2 \sqrt{\bar{\mu}_s}\,\mathrm{d}u\,\mathrm{d}W_s^{\bar{\mu}} \end{aligned}$$

for $t \in [0, T]$, hence the result follows by using Proposition 3.3.7. □

3.3.4 Annuity Contract

For the annuity contract introduced in (3.2.11) we define the payment process

$$A_t^a = \int_0^t (n - N_s) \frac{1}{B_s}\,\mathrm{d}C_s^a = \sum_{i=1}^n \int_0^t \mathbb{1}_{\{\tau^i > s\}} \frac{1}{B_s}\,\mathrm{d}C_s^a, \quad t \in [0, T], \tag{3.3.31}$$

where C^a is assumed to be a right-continuous, non-negative increasing $\mathbb{F}$-adapted process such that

$$\mathbb{E}\left[\sup_{t \in [0,T]} (C_t^a)^2\right] < \infty.$$

Proposition 3.3.9. *In the setting of Section 3.2 the payment process A^a introduced in* (3.3.31) *admits a risk-minimizing strategy $\varphi = (\xi, \xi^0) = (\xi^X, \xi^Y, \xi^0)$ given by*

$$\begin{aligned} \xi_t &= (\xi_t^X, \xi_t^Y) = \left(\frac{(n - N_t) e^{\Gamma_t} \psi_t}{\sigma(t, X_t) X_t}, \frac{(n - N_t) e^{\Gamma_t + rT} \psi_t^{\bar{\mu}}}{Y_t \beta^T(t) \sigma_2 \sqrt{\bar{\mu}_t}} \right), \\ \xi_t^0 &= V_t^a(\varphi) - \xi_t^X X_t - \xi_t^Y Y_t \end{aligned}$$

for $t \in [0, T]$, with discounted value process

$$V_t^a(\varphi) = n U_0^a + \int_0^t \xi_s^X\,\mathrm{d}X_s + \int_0^t \xi_s^Y\,\mathrm{d}Y_s + L_t^a - A_t^a,$$

where

$$L_t^a = \int_0^t (n - N_s) e^{\Gamma_s} \psi_s^\mu\,\mathrm{d}W_s^\mu - \int_{]0,t]} \mathbb{E}\left[\int_s^T \frac{e^{\Gamma_s - \Gamma_u}}{B_u}\,\mathrm{d}C_u^a \,\Big|\, \mathcal{F}_s \right] \mathrm{d}M_s,$$

for $t \in [0,T]$*, where* $\beta^T(t)$ *and* M_t *are defined in* (3.3.3) *and* (3.3.4) *and* U^a, ψ, ψ^μ *and* $\psi^{\bar{\mu}}$ *are given by*

$$U_t^a = \mathbb{E}\left[\int_0^T \frac{e^{-\Gamma_s}}{B_s}\,\mathrm{d}C_s^a \,\Big|\, \mathcal{F}_t\right] = U_0^a + \int_0^t \psi_s\,\mathrm{d}W_s + \int_0^t \psi_s^\mu\,\mathrm{d}W_s^\mu + \int_0^t \psi_s^{\bar{\mu}}\,\mathrm{d}W_s^{\bar{\mu}}, \tag{3.3.32}$$

where ψ, ψ^μ *and* $\psi^{\bar{\mu}}$ *are* $\mathbb{F}$*-predictable processes satisfying*

$$\mathbb{E}\left[\int_0^T (\psi_s)^2\,\mathrm{d}s\right],\ \mathbb{E}\left[\int_0^T (\psi_s^\mu)^2\,\mathrm{d}s\right],\ \mathbb{E}\left[\int_0^T (\psi_s^{\bar{\mu}})^2\,\mathrm{d}s\right] < \infty.$$

The optimal cost and risk processes are given by

$$\begin{aligned} C_t^a(\varphi) &= nU_0^a + L_t^a, \\ R_t^a(\varphi) &= \mathbb{E}[(L_T^a - L_t^a)^2 \,|\, \mathcal{G}_t], \end{aligned} \tag{3.3.33}$$

for $t \in [0,T]$*.*

Proof. By Proposition 4.12 and 5.13 of Barbarin [4, Chapter 3], as well as Proposition 5.1.2 of Bielecki and Rutkowski [12] and (3.2.5) we have

$$\begin{aligned} \mathbb{E}[A_T^a \,|\, \mathcal{G}_t] = nU_0^a &+ \int_0^t (n - N_s)e^{\Gamma_s}\,\mathrm{d}U_s^a \\ &- \int_{]0,t]} \mathbb{E}\left[\int_s^T \frac{e^{\Gamma_s - \Gamma_u}}{B_u}\,\mathrm{d}C_u^a \,\Big|\, \mathcal{F}_s\right]\mathrm{d}M_s, \end{aligned}$$

where

$$U_t^a = \mathbb{E}\left[\int_0^T \frac{e^{-\Gamma_s}}{B_s}\,\mathrm{d}C_s^a \,\Big|\, \mathcal{F}_t\right] = U_0^a + \int_0^t \psi_s\,\mathrm{d}W_s + \int_0^t \psi_s^\mu\,\mathrm{d}W_s^\mu + \int_0^t \psi_s^{\bar{\mu}}\,\mathrm{d}W_s^{\bar{\mu}},$$

$t \in [0,T]$, is a square integrable martingale, since $\mathbb{E}\left[\sup_{t\in[0,T]}(C_t^a)^2\right] < \infty$ and where ψ, ψ^μ and $\psi^{\bar{\mu}}$ are $\mathbb{F}$-predictable processes satisfying

$$\mathbb{E}\left[\int_0^T (\psi_s)^2\,\mathrm{d}s\right],\ \mathbb{E}\left[\int_0^T (\psi_s^\mu)^2\,\mathrm{d}s\right],\ \mathbb{E}\left[\int_0^T (\psi_s^{\bar{\mu}})^2\,\mathrm{d}s\right] < \infty.$$

The result follows by the same arguments as in the proofs of Proposition 3.3.5 and 3.3.7. □

Note that Proposition 5.1.2 of Bielecki and Rutkowski [12] requires C^a to be bounded. However, it can be easily seen that this result also holds if

$$\mathbb{E}\left[\sup_{t\in[0,T]}(C_t^a)^2\right] < \infty.$$

The next corollary provides an application of Proposition 3.3.9 where we set

$$C_t^a = \int_0^t f(S_s)\,\mathrm{d}s, \quad t \in [0,T],$$

in (3.2.11) and (3.3.31), where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borel measurable function such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} f(S_t)^2\right] < \infty,$$

i.e. we define the payment process

$$A_t^{a,f} = \int_0^t (n - N_s)\frac{f(S_s)}{B_s}\,\mathrm{d}s = \sum_{i=1}^n \int_0^t \mathbb{1}_{\{\tau^i > s\}}\frac{f(S_s)}{B_s}\,\mathrm{d}s, \quad t \in [0,T]. \tag{3.3.34}$$

Corollary 3.3.10. *In the setting of Section 3.2 the payment process $A^{a,f}$ introduced in* (3.3.34) *admits a risk-minimizing strategy $\varphi = (\xi, \xi^0) = (\xi^X, \xi^Y, \xi^0)$ given by*

$$\begin{aligned}
\xi_t^X &= (n - N_t)e^{\Gamma_t}\int_t^T F_s^u(t, S_t) Z_t^u\,\mathrm{d}u\\
\xi_t^Y &= (n - N_t)e^{\Gamma_t + r(T-t)} Y_t^{-1}(\beta^T(t))^{-1}\int_t^T \beta_2^u(t) F^u(t, S_t) Z_t^u\,\mathrm{d}u,\\
\xi_t^0 &= V_t^a(\varphi) - \xi_t^X X_t - \xi_t^Y Y_t
\end{aligned}$$

for $t \in [0,T]$, with discounted value process

$$V_t^{a,f}(\varphi) = n\int_0^T Z_0^u F^u(0, S_0)\,\mathrm{d}u + \int_0^t \xi_s^X\,\mathrm{d}X_s + \int_0^t \xi_s^Y\,\mathrm{d}Y_s + L_t^{a,f} - A_t^{a,f},$$

where

$$\begin{aligned}
L_t^{a,f} = &\int_0^t (n - N_s)e^{\Gamma_s}\int_s^T \beta_1^u(s)\sigma_1\sqrt{\mu_s}\frac{F^u(s, S_s)}{B_s} Z_s^u\,\mathrm{d}u\,\mathrm{d}W_s^\mu\\
&- \int_{]0,t]} \mathbb{E}\left[\int_s^T \frac{e^{\Gamma_s - \Gamma_u}}{B_u}\,\mathrm{d}C_u^a \,\Big|\, \mathcal{F}_s\right]\mathrm{d}M_s,
\end{aligned}$$

for $t \in [0,T]$, where $\beta^T(t)$, $\beta_1^u(t)$, $\beta_2^u(t)$, $F^u(t, S_t)$, $F_s^u(t, S_t)$, Z_t^u and M_t are defined in (3.3.3) - (3.3.7), (3.3.20) *and* (3.3.21).

Proof. For U^a as defined in (3.3.32) we have

$$\begin{aligned}
U_t^a &= \mathbb{E}\left[\int_0^T e^{-\Gamma_u}\frac{f(S_u)}{B_u}\,\mathrm{d}u \,\Big|\, \mathcal{F}_t\right]\\
&= \int_0^T \mathbb{E}\left[\frac{f(S_u)}{B_u}\Big|\, \mathcal{F}_t\right]\mathbb{E}\left[e^{-\Gamma_u}\,\Big|\, \mathcal{F}_t\right]\mathrm{d}u, \quad t \in [0,T]
\end{aligned} \tag{3.3.35}$$

where we have used Fubini's theorem and the independence of the underlying driving processes. By (3.3.5) we have

$$Z_t^u := \mathbb{E}\left[e^{-\Gamma_u} \,\middle|\, \mathcal{F}_t\right] = Z_0^u + \int_0^t Z_s^u \beta_1^u(s)\sigma_1\sqrt{\mu_s}\mathbb{1}_{\{s\leq u\}}\,\mathrm{d}W_s^\mu$$
$$+ \int_0^t Z_s^u \beta_2^u(s)\sigma_2\sqrt{\bar{\mu}_s}\mathbb{1}_{\{s\leq u\}}\,\mathrm{d}W_s^{\bar{\mu}} \tag{3.3.36}$$

for $0 \leq t,\, u \leq T$, where $\beta_1(t)$ and $\beta_2(t)$ are given in (3.3.6) - (3.3.7), and by the same arguments as in the proof of Corollary 3.3.8 we have that

$$\begin{aligned} U_t^a = & \int_0^T Z_0^u F^u(0, S_0)\,\mathrm{d}u + \int_0^t \int_s^T \sigma(s, X_s)X_s F_s^u(s, S_s) Z_s^u \,\mathrm{d}u\,\mathrm{d}W_s \\ & + \int_0^t \int_s^T \beta_1^u(s)\sigma_1\sqrt{\mu_s}\frac{F^u(s,S_s)}{B_s} Z_s^u \,\mathrm{d}u\,\mathrm{d}W_s^\mu \\ & + \int_0^t \int_s^T \beta_2^u(s)\sigma_2\sqrt{\bar{\mu}_s}\frac{F^u(s,S_s)}{B_s} Z_s^u \,\mathrm{d}u\,\mathrm{d}W_s^{\bar{\mu}}, \quad t \in [0,T], \end{aligned}$$

where we used (3.3.30), (3.3.36), Itô's formula and the stochastic Fubini theorem. Then the result follows by using Proposition 3.3.9. □

We conclude this section with a remark regarding the hedging error of the risk-minimizing strategies as computed in Propositions 3.3.5, 3.3.7 and 3.3.9. Following Barbarin [4], Møller [53] and Riesner [57] we take the initial intrinsic risk $R_0(\varphi)$ as a measure of the total risk associated with the non-hedgeable part of the insurance claims. In the case of the annuity contract, for $R_0^a(\varphi)$ as defined in (3.3.33) we have

$$\begin{aligned} R_0^a(\varphi) = \mathbb{E}[(L_T^a - L_0^a)^2] = & \, \mathbb{E}\left[\left(\int_0^T (n - N_s)e^{\Gamma_s}\psi_s^\mu \,\mathrm{d}W_s^\mu\right)^2\right] + \mathbb{E}\left[\left(\int_0^T \zeta_s \,\mathrm{d}M_s\right)^2\right] \\ & - 2\mathbb{E}\left[\left(\int_0^T (n - N_s)e^{\Gamma_s}\psi_s^\mu \,\mathrm{d}W_s^\mu\right)\left(\int_0^T \zeta_s \,\mathrm{d}M_s\right)\right], \end{aligned}$$

where $\zeta_t = \mathbb{E}[\int_t^T \frac{e^{\Gamma_t} - e^{\Gamma_u}}{B_u}\,\mathrm{d}C_u^a \,|\, \mathcal{F}_t]$, $t \in [0,T]$, and since W^μ and M are strongly orthogonal, the square integrable martingales

$$\left(\int_0^t (n - N_s)e^{\Gamma_s}\psi_s^\mu \,\mathrm{d}W_s^\mu\right), \quad \left(\int_0^t \zeta_s \,\mathrm{d}M_s\right), \quad t \in [0,T]$$

are strongly orthogonal, and e.g., by Proposition 4.50 in Chapter I of Jacod and Shiryaev [42], we have that

$$\mathbb{E}\left[\left(\int_0^T (n - N_s)e^{\Gamma_s}\psi_s^\mu \,\mathrm{d}W_s^\mu\right)\left(\int_0^T \zeta_s \,\mathrm{d}M_s\right)\right] = 0.$$

Furthermore

$$\mathbb{E}\left[\left(\int_0^T (n-N_s)e^{\Gamma_s}\psi_s^\mu \,\mathrm{d}W_s^\mu\right)^2\right] = \sum_{i=1}^n \mathbb{E}\left[\left(\int_0^T \mathbb{1}_{\{\tau^i>s\}}e^{\Gamma_s}\psi_s^\mu \,\mathrm{d}W_s^\mu\right)^2\right]$$
$$+\sum_{i\neq j}\mathbb{E}\left[\left(\int_0^T \mathbb{1}_{\{\tau^i>s\}}e^{\Gamma_s}\psi_s^\mu \,\mathrm{d}W_s^\mu\right)\left(\int_0^T \mathbb{1}_{\{\tau^j>s\}}e^{\Gamma_s}\psi_s^\mu \,\mathrm{d}W_s^\mu\right)\right],$$

and by (3.2.1), (3.2.5) and Fubini's theorem we have that

$$\sum_{i=1}^n \mathbb{E}\left[\left(\int_0^T \mathbb{1}_{\{\tau^i>s\}}e^{\Gamma_s}\psi_s^\mu \,\mathrm{d}W_s^\mu\right)^2\right] = \sum_{i=1}^n \mathbb{E}\left[\int_0^T \mathbb{1}_{\{\tau^i>s\}}e^{2\Gamma_s}(\psi_s^\mu)^2 \,\mathrm{d}s\right]$$
$$=\sum_{i=1}^n \int_0^T \mathbb{E}\left[\mathbb{E}[\mathbb{1}_{\{\tau^i>s\}}\,|\,\mathcal{F}_s]e^{2\Gamma_s}(\psi_s^\mu)^2\right]\mathrm{d}s = n\mathbb{E}\left[\int_0^T e^{\Gamma_s}(\psi_s^\mu)^2 \,\mathrm{d}s\right],$$

as well as

$$\sum_{i\neq j}\mathbb{E}\left[\left(\int_0^T \mathbb{1}_{\{\tau^i>s\}}e^{\Gamma_s}\psi_s^\mu \,\mathrm{d}W_s^\mu\right)\left(\int_0^T \mathbb{1}_{\{\tau^j>s\}}e^{\Gamma_s}\psi_s^\mu \,\mathrm{d}W_s^\mu\right)\right]$$
$$=\sum_{i\neq j}\mathbb{E}\left[\int_0^T \mathbb{1}_{\{\tau^i>s\}}\mathbb{1}_{\{\tau^j>s\}}e^{2\Gamma_s}(\psi_s^\mu)^2 \,\mathrm{d}s\right]$$
$$=\sum_{i\neq j}\int_0^T \mathbb{E}\left[\mathbb{E}[\mathbb{1}_{\{\tau^i>s\}}\mathbb{1}_{\{\tau^j>s\}}\,|\,\mathcal{F}_s]e^{2\Gamma_s}(\psi_s^\mu)^2\right]\mathrm{d}s = (n^2-n)\mathbb{E}\left[\int_0^T (\psi_s^\mu)^2 \,\mathrm{d}s\right].$$

Hence

$$\mathbb{E}\left[\left(\int_0^T (n-N_s)e^{\Gamma_s}\psi_s^\mu \,\mathrm{d}W_s^\mu\right)^2\right] = n\mathbb{E}\left[\int_0^T e^{\Gamma_s}(\psi_s^\mu)^2 \,\mathrm{d}s\right] + (n^2-n)\mathbb{E}\left[\int_0^T (\psi_s^\mu)^2 \,\mathrm{d}s\right].$$

Besides that

$$\mathbb{E}\left[\left(\int_0^T \zeta_s \,\mathrm{d}M_s\right)^2\right] = \sum_{i=1}^n \mathbb{E}\left[\left(\int_0^T \zeta_s \,\mathrm{d}M_s^i\right)^2\right] + \sum_{i\neq j}\mathbb{E}\left[\left(\int_0^T \zeta_s \,\mathrm{d}M_s^i\right)\left(\int_0^T \zeta_s \,\mathrm{d}M_s^j\right)\right],$$

and since M^i and M^j are strongly orthogonal for $i \neq j$, by Proposition 4.15 in Chapter I of Jacod and Shiryaev [42] it follows that

$$\sum_{i\neq j}\mathbb{E}\left[\left(\int_0^T \zeta_s \,\mathrm{d}M_s^i\right)\left(\int_0^T \zeta_s \,\mathrm{d}M_s^j\right)\right] = 0,$$

hence

$$\mathbb{E}\left[\left(\int_0^T \zeta_s \,\mathrm{d}M_s\right)^2\right] = \sum_{i=1}^n \mathbb{E}\left[\int_0^T \zeta_s^2 \,\mathrm{d}\langle M^i\rangle_s\right] = \sum_{i=1}^n \int_0^T \mathbb{E}\left[\zeta_s^2 \mathbb{1}_{\{\tau^i>s\}}\mu_s\right]\mathrm{d}s$$
$$= n\mathbb{E}\left[\int_0^T \zeta_s^2 e^{-\Gamma_s}\mu_s \,\mathrm{d}s\right].$$

Putting the results together we obtain that

$$R_0^a(\varphi) = n\mathbb{E}\left[\int_0^T e^{\Gamma_s}(\psi_s^\mu)^2\,\mathrm{d}s\right] + (n^2-n)\mathbb{E}\left[\int_0^T (\psi_s^\mu)^2\,\mathrm{d}s\right] + n\mathbb{E}\left[\int_0^T \zeta_s^2 e^{-\Gamma_s}\mu_s\,\mathrm{d}s\right],$$

hence

$$\lim_{n\to\infty}\frac{\sqrt{R_0^a(\varphi)}}{n} = \sqrt{\mathbb{E}\left[\int_0^T (\psi_s^\mu)^2\,\mathrm{d}s\right]}. \tag{3.3.37}$$

The analogous results hold for the pure endowment and term insurance contract. Therefore, in contrast to the setting in Møller [53], with increasing portfolio size the hedging error cannot be fully eliminated. As noted already in Barbarin [4] the non-diversifiable term in (3.3.37) is related to the incompleteness of the market given by the filtration $\mathbb{F}$.

Chapter 4

Portfolio with Different Age Cohorts

4.1 Introduction

In this chapter, based on Biagini, Botero, and Schreiber [9], we study the problem of pricing and hedging life insurance liabilities for the case of an insurance portfolio that consists of individuals of different age cohorts. In order to capture the cross-generational dependency structure of the portfolio we model the mortality intensities as random fields. We also provide specific examples consistent with historical mortality data and correlation structures. The remainder of this chapter is organized as follows: Section 4.2 introduces the general setup, including the structure of the insurance portfolio and the financial market. In Section 4.3 we propose two illustrative examples of intensity field models consistent with characteristics of typical mortality data. In Section 4.4 we compute risk-minimizing strategies of the life insurance liabilities at aggregate portfolio level. Section 4.5 then concludes with specific examples where we compare risk-minimizing strategies and the dependency structure for different intensity field models.

4.2 The Setting

Let $T > 0$ be a fixed finite time horizon and $(\Omega, \mathcal{G}, \mathbb{P})$ a probability space equipped with a filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ which contains all available information. We define $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, and put $\mathcal{G} = \mathcal{G}_T$, where $\mathbb{H} = (\mathcal{H}_t)_{t\in[0,T]}$ is generated by the death counting processes of the insurance portfolio (see Subsection 4.2.1). The *background filtration* $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ contains all information available except the information regarding the individual survival times. Here we define $\mathbb{F} = \mathbb{F}^X \vee \mathbb{F}^\mu$, where $\mathbb{F}^X$ is the filtration containing information regarding a risky asset, e.g., a stock (see Subsection 4.2.2), $\mathbb{F}^\mu$ is the filtration containing information regarding the mortality intensities (see Subsection 4.2.1) and we assume that $\mathbb{F}^X$ and $\mathbb{F}^\mu$ are independent. In the subsequent sections we introduce the three components of the model: the insurance portfolio, the financial market and the combined model.

4.2.1 Insurance Portfolio and Mortality Intensities

We consider an insurance portfolio consisting of n individuals belonging to a set of age cohorts $B = \{x_1, \dots, x_m\} \subseteq I$, where the interval $I = [0, x^*]$ is assumed to be a given range of possible ages of individuals at time 0 and $x^* > 0$ is a natural upper bound for the range of ages considered. Note that $m \leq n$, in particular if $m = 1$ all individuals belong to the same age cohort, whereas if $m = n$ all individuals belong to different age cohorts. Similarly as in Biffis and Millossovich [14] we define a function $n^{\cdot} : B \to \mathbb{N}$, such that the quantity n^x represents the number of insureds belonging to the age cohort x, i.e., $\sum_{i=1}^m n^{x_i} = n$. For $x \in B$ and $j = 1, \dots, n^x$, we model the *residual lifetime* of the j-th insured person within the age cohort x as a $\mathbb{G}$-stopping time $\tau^{x,j} : \Omega \to [0, T] \cup \{\infty\}$ and assume that $\mathbb{P}(\tau^{x,j} = 0) = 0$ and $\mathbb{P}(\tau^{x,j} > t) > 0$ for $t \in [0, T]$. Note that since the time horizon T is usually fixed as the maturity of the life insurance liabilities, in order to ensure that $\mathbb{P}(\tau^{x,j} > T) > 0$ for $x \in B$ and $j = 1, \dots, n^x$ (the remaining lifetimes are not necessarily bounded by T), it is necessary to allow $\tau^{x,j}$ to take values larger than T, indicated here by the convention that $\tau^{x,j}$ can assume the value infinity. We define $\mathcal{H}_t = \vee_{x \in B} \mathcal{H}_t^x$ with $\mathcal{H}_t^x = \mathcal{H}_t^{x,1} \vee \dots \vee \mathcal{H}_t^{x,n^x}$, where $\mathcal{H}_t^{x,j} = \sigma\{H_s^{x,j} : 0 \leq s \leq t\}$ and $H_t^{x,j} = \mathbb{1}_{\{\tau^{x,j} \leq t\}}$ for $t \in [0, T]$, $x \in B$ and $j = 1, \dots, n^x$. Furthermore we consider a finite measure ζ on $(B, \mathcal{P}(B))$, where $\mathcal{P}(B)$ denotes the power set of B, allowing us to differently weight the subsets of B. Then

$$\int_B n^x \,\zeta(\mathrm{d}x) = \sum_{i=1}^m n^{x_i} \zeta(x_i)$$

provides us with the weighted dimension of the portfolio B. The death counting process associated with the age cohort $x \in B$ is given by

$$N_t^x = \sum_{j=1}^{n^x} \mathbb{1}_{\{\tau^{x,j} \leq t\}}, \quad t \in [0, T],\ x \in B.$$

Then the weighted random number of insureds alive at time t in the portfolio is given by

$$\int_B (n^x - N_t^x)\,\zeta(\mathrm{d}x) = \sum_{i=1}^m \sum_{j=1}^{n^{x_i}} \mathbb{1}_{\{\tau^{x_i,j} > t\}} \zeta(x_i), \quad t \in [0, T].$$

For $x \in B$ and $j = 1, \dots, n^x$ we assume that the times of death $\tau^{x,j}$ are totally inaccessible $\mathbb{G}$-stopping times, and an important role is then played by the conditional distribution function of $\tau^{x,j}$, given by

$$F_t^{x,j} = \mathbb{P}(\tau^{x,j} \leq t \,|\, \mathcal{F}_t), \quad t \in [0, T],$$

and we assume $F_t^{x,j} < 1$ for all $t \in [0,T]$. Then the *hazard process* $\Gamma^{x,j}$ of $\tau^{x,j}$

$$\Gamma_t^{x,j} = -\ln(1 - F_t^{x,j}) = -\ln \mathbb{E}[\mathbb{1}_{\{\tau^{x,j} > t\}} \,|\, \mathcal{F}_t], \quad t \in [0,T],$$

is well-defined for every $t \in [0,T]$. For $x \in B$ we define $\Gamma^x := \Gamma^{x,j}$ for $j = 1, \ldots, n^x$, i.e., all individuals of the same age cohort have the same hazard process. Moreover, we assume that Γ^x admits a *mortality intensity* μ^x, i.e.

$$\Gamma_t^x = \int_0^t \mu_s^x \, \mathrm{d}s, \quad t \in [0,T], \tag{4.2.1}$$

where $\mu = (\mu_{t,x})_{(t,x) \in [0,T] \times I}$, is a random field generated by a Brownian sheet $W = (W_{t,x})_{(t,x) \in [0,T] \times I}$, as specified in Appendix C. Note that for $t \in [0,T]$ and $x \in I$ we write $\mu_{t,x}$ interchangeably with μ_t^x if we want to emphasize that for fixed $x \in I$ we are integrating in the t-direction (see, e.g., Lemma 4.3.1). For $t \in [0,T]$ and $x \in I$ the natural filtration of the Brownian sheet W is given by $\mathcal{F}_{t,x}^{\mu} := \sigma\{W_{s,v} : 0 \leq s \leq t,\ 0 \leq v \leq x\}$, and we define $\mathbb{F}^{\mu} = (\mathcal{F}_t^{\mu})_{t \in [0,T]}$, where $\mathcal{F}_t^{\mu} := \{\mathcal{F}_{t,x}^{\mu} : 0 \leq x \leq x^*\} = \vee_{x \in I} \mathcal{F}_{t,x}^{\mu}$. For fixed $x \in I$ we assume that the process $(\mu_t^x)_{t \in [0,T]}$ is an affine diffusion process (see also Section 4.3), which facilitates the related computations in Section 4.4. The process μ^x represents the mortality intensity of the age cohort $x \in I$ and can be derived by means of publicly available data of the *survivor index*

$$S_t^{\mu^x} = \exp\left(-\int_0^t \mu_s^x \, \mathrm{d}s\right), \quad t \in [0,T],\ x \in I. \tag{4.2.2}$$

The need for standardization in the life markets has led to the creation of various such indices aggregated for different age cohorts and populations by investment banks. According to Cairns et al. [19] survivor indices can be seen as basic building blocks for many mortality-linked securities, see also the definition of a longevity bond for age cohort $x \in I$ in Subsection 4.2.2. This modeling approach enables us to not only capture the dependency structure in the t-direction, but also in the x-direction and additionally takes into account the cross-generational correlation of the insurance portfolio. In Section 4.3 we provide explicit specifications for μ that are consistent with typical characteristics of historical mortality data (see, e.g., Andreev [2]) in the sense that e.g., for fixed $x \in I$, $(\mu_{t,x})_{t \in [0,T]}$ is decreasing in t (downward mortality trend) and for fixed $t \in [0,T]$, $(\mu_{t,x})_{x \in I}$ is increasing in x.

For $x, y \in B$ and $i = 1, \ldots, n^x$, $j = 1, \ldots, n^y$ with $(x,i) \neq (y,j)$ we also assume

$$\mathbb{E}[\mathbb{1}_{\{\tau^{x,i} > t\}} \mathbb{1}_{\{\tau^{y,j} > s\}} \,|\, \mathcal{F}_T] = \mathbb{E}[\mathbb{1}_{\{\tau^{x,i} > t\}} \,|\, \mathcal{F}_T] \, \mathbb{E}[\mathbb{1}_{\{\tau^{y,j} > s\}} \,|\, \mathcal{F}_T], \tag{4.2.3}$$

for $0 \leq s, t \leq T$, i.e., we assume conditional independence for individuals in different age cohorts as well as for individuals within the same age cohort. This assumption is well-known in the literature of credit risk modeling, see, e.g., Chapter 9 of Bielecki and Rutkowski [12]. All individuals within the insurance portfolio are subject to idiosyncratic risk factors, as well as common risk factors, given by the information within the background filtration $\mathbb{F}$. Intuitively, the assumption of conditional independence means that if the evolution of all common risk factors is known, the idiosyncratic risk factors become independent of each other.

4.2.2 The Financial Market

We consider a financial market defined on $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ consisting of a bank account or numéraire B with constant short rate $r > 0$, i.e.

$$B_t = \exp\{rt\}, \quad t \in [0,T], \tag{4.2.4}$$

as well as a risky asset, e.g., a stock, with asset price S and a family of *longevity bonds* (P^x), $x \in I$. We assume that S follows the $\mathbb{P}$-dynamics

$$\mathrm{d}S_t = S_t \left(r\mathrm{d}t + \sigma(t, S_t)\, \mathrm{d}W_t^X \right), \quad t \in [0,T], \tag{4.2.5}$$

for a Brownian motion $W^X = (W_t^X)_{t\in[0,T]}$ with $S_0 = s$ and we assume that σ satisfies certain regularity conditions that ensure the existence and uniqueness of a solution to (4.2.5). We denote by $X = S/B$ the discounted asset price, i.e., the dynamics of X are given by

$$\mathrm{d}X_t = \mathrm{d}\left(\frac{S_t}{B_t}\right) = \sigma(t, S_t) X_t\, \mathrm{d}W_t^X, \quad t \in [0,T]. \tag{4.2.6}$$

and define $\mathbb{F}^X = \left(\mathcal{F}_t^X\right)_{t\in[0,T]}$, with $\mathcal{F}_t^X := \sigma\{W_s^X : 0 \le s \le t\}$. Following Cairns et al. [19], for $x \in I$ we consider the *longevity bond* P^x with maturity T representing the systematic mortality risk inherent to the life insurance contracts for the age cohort x, i.e., P^x is defined as a zero-coupon bond that pays out the value of the survivor or longevity index as defined in (4.2.2) at T. This means the discounted value process $Y^x = P^x/B$ is given by

$$Y_t^x = \mathbb{E}\left[\frac{S_T^{\mu^x}}{B_T}\,\middle|\, \mathcal{G}_t\right], \quad t \in [0,T],\ x \in I. \tag{4.2.7}$$

Thus the discounted asset prices $X, (Y^x)_{x\in I}$ are (local) $(\mathbb{P}, \mathbb{F})$-martingales, i.e., the financial market is arbitrage-free and the physical measure $\mathbb{P}$ belongs to the set of equivalent local martingale measures.

4.2.3 The Combined Model

We consider the extended market $\mathbb{G} = \mathbb{F} \vee \mathbb{H} = \mathbb{F}^X \vee \mathbb{F}^\mu \vee \mathbb{H}$, such that the information available in the market at time $t \in [0,T]$ is assumed to be $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^\mu \vee \mathcal{H}_t$. All filtrations are assumed to satisfy the usual hypotheses of completeness and right-continuity. We postulate that all $\mathbb{F}$-local martingales are also $\mathbb{G}$-local martingales, and in the sequel we refer to this hypothesis as Hypothesis (H). This hypothesis is well-known in the literature on enlargements of filtrations, for a discussion of this hypothesis we refer to Blanchet-Scalliet and Jeanblanc [18] and Bielecki and Rutkowski [12, Chapter 6]. In this setting we study unit-linked life insurance liabilities in form of insurance payment streams as introduced by Møller [54]. It is now widely acknowledged (see, e.g., Barbarin [4], Biffis [13] and Møller [53]) that most payment streams of practical relevance are covered by the three building blocks pure endowment-, term insurance-, and annuity contracts. Following Barbarin [4] and Møller [53], the *pure endowment contract* consists of a payoff

$$\int_B \sum_{j=1}^{n^x} f(S_T) \mathbb{1}_{\{\tau^{x,j} > T\}} \zeta(\mathrm{d}x) = \sum_{i=1}^{m} \sum_{j=1}^{n^{x_i}} f(S_T) \mathbb{1}_{\{\tau^{x_i,j} > T\}} \zeta(x_i) \tag{4.2.8}$$

at time T, where f is a non-negative function fulfilling certain regularity conditions, i.e., the insurer pays the aggregate amount $f(S_T)$ to every policyholder in the portfolio B who has survived until T, weighted by means of the measure ζ. The *term insurance contract* is defined by the following payoff structure:

$$\int_B \sum_{j=1}^{n^x} f(S_{\tau^{x,j}}) \mathbb{1}_{\{\tau^{x,j} \leq T\}} \zeta(\mathrm{d}x) = \sum_{i=1}^{m} \sum_{j=1}^{n^{x_i}} f(S_{\tau^{x_i,j}}) \mathbb{1}_{\{\tau^{x_i,j} \leq T\}} \zeta(x_i), \tag{4.2.9}$$

i.e., the amount $f(S_{\tau^{x_i,j}})$ is payed at the time of death $\tau^{x_i,j}$ to policyholder j within the age cohort x_i, $i = 1, \ldots, m$, $j = 1, \ldots, n^{x_i}$. The *annuity contract* consists of multiple payoffs as functions of the asset price the insurer has to pay as long as the policyholders are alive, i.e.,

$$\int_B \left(\int_0^T (n^x - N_s^x) \, f(S_s) \, \mathrm{d}s \right) \zeta(\mathrm{d}x) = \sum_{i=1}^{m} \sum_{j=1}^{n^{x_i}} \int_0^T \mathbb{1}_{\{\tau^{x_i,j} > s\}} f(S_s) \, \mathrm{d}s \, \zeta(x_i), \tag{4.2.10}$$

where weighting of the different age cohorts is again enabled through the measure ζ. In the following section we specify examples for the intensity field model introduced in (4.2.1) consistent with characteristics of historical mortality data.

4.3 Intensity Field Model

It is now widely acknowledged (see, e.g., Andreev [2] and Forfar and Smith [37]) that downward mortality trends are not uniform across ages. In this context, modeling mortality intensities by means of random fields as a random surface appears as a natural modeling choice. This approach enables us to look simultaneously at the evolution of death probabilities over time for a given age, death probabilities across all ages at a given time and death probabilities over time for people born in the same year. In this section we provide two specific examples of affine intensity field models for μ as defined in (4.2.1), a Gaussian random field (see Subsection 4.3.1) with nice analytical properties and intuitive interpretation, as well as a χ^2-random field (see Subsection 4.3.2) that has the advantage of restraining the mortality intensities to non-negative values.

4.3.1 Gaussian Intensity Field

We define

$$\mu_{t,x} = \bar{\mu}(t,x) + O_{t,x}, \quad t \in [0,T],\ x \in I, \tag{4.3.1}$$

where $\bar{\mu}$ is a deterministic function, differentiable in x and t, and O is a space-time changed Brownian sheet, i.e.,

$$O_{t,x} = \frac{\sigma}{\sqrt{2\theta\alpha}} e^{-\theta t} e^{-\alpha x} W_{\nu_1(t),\nu_2(x)}, \quad t \in [0,T],\ x \in I, \tag{4.3.2}$$

with

$$\nu_1(t) = e^{2\theta t}, \quad \nu_2(x) = e^{2\alpha x}, \quad t \in [0,T],\ x \in I, \tag{4.3.3}$$

for $\alpha, \theta > 0$. In particular here we assume that $\nu_1 : [0,T] \to [0,T]$ and $\nu_2 : [0,x^*] \to [0,x^*]$. Intuitively, $\mu_{t,x}$ fluctuates around the deterministic mortality level $\bar{\mu}$. We would like to obtain a model for μ that is consistent with typical characteristics of historical mortality data, i.e., for fixed x, $(\mu_{t,x})_{t\in[0,T]}$ is decreasing in t and for fixed t, $(\mu_{t,x})_{x\in I}$ is increasing in x. These properties can be directly imposed on the deterministic function $\bar{\mu}(t,x)$. For example, $\bar{\mu}(t,x)$ could be given by the well-known Lee-Carter model (see, e.g., Loisel and Serrant [50]):

$$\bar{\mu}(t,x) = \exp\left(a(x) + b(x)k(t)\right), \quad t \in [0,T],\ x \in I, \tag{4.3.4}$$

where a is a negative increasing function such that $e^{a(x)}$ represents the general shape of the mortality curve at age x, k is a real-valued decreasing function representing the downward trend in time of the logarithm of the force of mortality. The non-negative function b represents the sensitivity of the logarithm of the force of mortality at age x to variations in k and allows us to model this trend heterogeneously over cohorts (see, e.g., Andreev [2]). For example if b is decreasing for high values of x, it implies that mortality improvement is lower for older ages, as suggested by Forfar and Smith [37]. Note that for $t \in [0,T]$, $x \in I$, we have that

$$\mathbb{E}[\mu_{t,x}] = \bar{\mu}(t,x), \quad t \in [0,T],\ x \in I,$$

and by equation (C.0.1) in Appendix C we have

$$\begin{aligned}
\mathrm{Cov}(\mu_{t,x}, \mu_{s,y}) &= \frac{\sigma^2}{2\alpha\theta} e^{-\theta(t+s)} e^{-\alpha(x+y)} \mathrm{Cov}\left(W_{\nu_1(t),\nu_2(x)}, W_{\nu_1(s),\nu_2(y)}\right) \\
&= \frac{\sigma^2}{2\alpha\theta} e^{-\theta(t+s)} e^{2\theta(t\wedge s)} e^{-\alpha(x+y)} e^{2\alpha(x\wedge y)} \\
&= \frac{\sigma^2}{2\alpha\theta} e^{-\theta|t-s|} e^{-\alpha|x-y|}, \quad s,t \in [0,T],\ x,y \in I,
\end{aligned} \tag{4.3.5}$$

as well as

$$\operatorname{Corr}(\mu_{t,x}, \mu_{s,y}) = e^{-\theta|t-s|}e^{-\alpha|x-y|}, \quad s,t \in [0,T],\ x,y \in I,$$

i.e., correlation is positive, symmetric and exponentially declining. The next lemma provides a stochastic representation of μ in the t-direction.

Lemma 4.3.1. *For μ as defined in* (4.3.1) *and fixed $x \in I$ we have that*

$$\mu_t^x = \bar{\mu}(t,x) + e^{-\theta t}(\mu_0^x - \bar{\mu}(0,x)) + \frac{\sigma}{\sqrt{\alpha}}\int_0^t e^{-\theta(t-s)}\mathrm{d}\tilde{W}_s^{\nu_2(x)}, \quad t \in [0,T],$$

where

$$\tilde{W}_t^{\nu_2(x)} := \frac{W_{t,\nu_2(x)}}{\sqrt{\nu_2(x)}}, \quad t \in [0,T], \tag{4.3.6}$$

is a standard Brownian motion. The set $(\mu^x)_{x\in I}$ is a family of affine diffusion processes, i.e., for fixed $x \in I$ the dynamics of $(\mu_t^x)_{t\in[0,T]}$ are given by

$$\mathrm{d}\mu_t^x = \theta\left[\left(\bar{\mu}(t,x) + \frac{\partial_t\bar{\mu}(t,x)}{\theta}\right) - \mu_t^x\right]\mathrm{d}t + \frac{\sigma}{\sqrt{\alpha}}\mathrm{d}\tilde{W}_t^{\nu_2(x)}, \quad t \in [0,T], \tag{4.3.7}$$

Proof. Fix $x \in I$ and define

$$O_t^x = O_{t,x} = \frac{\sigma}{\sqrt{2\theta\alpha}}e^{-\theta t}\tilde{W}_{\nu_1(t)}^{\nu_2(x)}, \quad t \in [0,T].$$

Note that here as for the random field μ introduced in (4.2.1), for $t \in [0,T]$ and $x \in I$ we write $O_{t,x}$ interchangeably with O_t^x, if we want to emphasize that for fixed $x \in I$ we are integrating in the t-direction. Then $\mathbb{E}[O_t^x] = 0$ and from (4.3.5) we have that $\operatorname{Cov}(O_t^x, O_s^x) = \frac{\sigma^2}{2\theta\alpha}e^{-\theta|t-s|}$. We now show that O^x is a stationary Ornstein-Uhlenbeck (OU) process with dynamics

$$\mathrm{d}O_t^x = -\theta O_t^x\mathrm{d}t + \frac{\sigma}{\sqrt{\alpha}}\mathrm{d}\tilde{W}_t^{\nu_2(x)}, \quad t \in [0,T], \tag{4.3.8}$$

and $O_0^x \sim \mathcal{N}(0, \frac{\sigma^2}{2\theta\alpha})$. To this end we consider a process $\bar{O}^x$ that solves the stochastic differential equation (4.3.8) such that $\bar{O}_0^x \sim \mathcal{N}(0, \frac{\sigma^2}{2\theta\alpha})$. It is easily seen (see, e.g., Example 6.8 in Chapter 5 of Karatzas and Shreve [46]) that $\bar{O}^x$ is given by

$$\bar{O}_t^x = e^{-\theta t}\bar{O}_0^x + \frac{\sigma}{\sqrt{\alpha}}\int_0^t e^{-\theta(t-s)}\mathrm{d}\tilde{W}_s^{\nu_2(x)}, \quad t \in [0,T],$$

i.e., $\mathbb{E}[\bar{O}_t^x] = 0$ and

$$\operatorname{Cov}(\bar{O}_t^x, \bar{O}_s^x) = e^{-\theta(t+s)}\left(\frac{\sigma^2}{2\theta\alpha}\left(e^{2\theta(t\wedge s)} - 1\right) + \operatorname{Var}(\bar{O}_0^x)\right) = \frac{\sigma^2}{2\theta\alpha}e^{-\theta|t-s|}$$

for $s, t \in [0,T]$. Since O^x and $\bar{O}^x$ are Gaussian processes with the same covariance structure, it follows that O^x solves (4.3.8) and

$$O_t^x = e^{-\theta t} O_0^x + \frac{\sigma}{\sqrt{\alpha}} \int_0^t e^{-\theta(t-s)} \mathrm{d}\tilde{W}_s^{\nu_2(x)}, \quad t \in [0,T],$$

with $O_0^x \sim \mathcal{N}(0, \frac{\sigma^2}{2\alpha\theta})$. Then by (4.3.1) it follows that

$$\mu_t^x = \bar{\mu}(t,x) + e^{-\theta t}(\mu_0^x - \bar{\mu}(0,x)) + \frac{\sigma}{\sqrt{\alpha}} \int_0^t e^{-\theta(s-t)} \mathrm{d}\tilde{W}_s^{\nu_2(x)}, \quad t \in [0,T],$$

and by (4.3.1), (4.3.8) and Itô's lemma we get the following dynamics for μ^x:

$$\begin{aligned} \mathrm{d}\mu_t^x &= \partial_t \bar{\mu}(t,x)\mathrm{d}t + \mathrm{d}O_t^x \\ &= \theta \left[\left(\bar{\mu}(t,x) + \frac{\partial_t \bar{\mu}(t,x)}{\theta} \right) - \mu_t^x \right] \mathrm{d}t + \frac{\sigma}{\sqrt{\alpha}} \mathrm{d}\tilde{W}_t^{\nu_2(x)}, \quad t \in [0,T]. \end{aligned}$$

□

In the next lemma we compute the sharp bracket or quadratic covariation process of μ^x and μ^y for x, $y \in I$.

Lemma 4.3.2. *Let μ be given by the Gaussian intensity field model as introduced in* (4.3.1). *Then for fixed x, $y \in I$ the sharp bracket process of $\mu^x = (\mu_t^x)_{t\in[0,T]}$ and $\mu^y = (\mu_t^y)_{t\in[0,T]}$ is given by*

$$\langle \mu^x, \mu^y \rangle_t = \frac{\sigma^2}{\alpha} e^{-\alpha|x-y|} t, \quad t \in [0,T].$$

Proof. Fix $x, y \in I$ and let $\nu_2(\cdot)$ be given as in (4.3.3). For $0 \le s \le t \le T$ we have

$$\begin{aligned} \mathbb{E}\left[W_t^{\nu_2(x)} W_t^{\nu_2(y)} \middle| \mathcal{F}_s^\mu \right] &= \mathbb{E}\Big[\left(W_t^{\nu_2(x)} - W_s^{\nu_2(x)} \right) \left(W_t^{\nu_2(y)} - W_s^{\nu_2(y)} \right) \Big| \mathcal{F}_s^\mu \Big] \\ &\quad + \mathbb{E}\Big[\left(W_t^{\nu_2(x)} - W_s^{\nu_2(x)} \right) W_s^{\nu_2(y)} + W_s^{\nu_2(x)} \left(W_t^{\nu_2(y)} - W_s^{\nu_2(y)} \right) \Big| \mathcal{F}_s^\mu \Big] \\ &\quad + \mathbb{E}\Big[W_s^{\nu_2(x)} W_s^{\nu_2(y)} \Big| \mathcal{F}_s^\mu \Big] \\ &= \mathbb{E}\left[\left(W_t^{\nu_2(x)} - W_s^{\nu_2(x)} \right) \left(W_t^{\nu_2(y)} - W_s^{\nu_2(y)} \right) \right] + W_s^{\nu_2(x)} W_s^{\nu_2(y)} \qquad (4.3.9) \\ &= t(\nu_2(x) \wedge \nu_2(y)) - s(\nu_2(x) \wedge \nu_2(y)) + W_s^{\nu_2(x)} W_s^{\nu_2(y)}, \qquad (4.3.10) \end{aligned}$$

where in (4.3.9) we used the fact that, for fixed $z \in I$, $W_s^{\nu_2(z)}$ is $\mathcal{F}_s^\mu$-measurable and $W_t^{\nu_2(z)} - W_s^{\nu_2(z)}$ is independent of $\mathcal{F}_s^\mu$, and (4.3.10) is a consequence of the covariance structure of the Brownian sheet, see also equation (C.0.1) in Appendix C. Note that here again for $t \in [0,T]$ and $x \in I$ we write $W_{t,x}$ interchangeably with W_t^x. It follows that

$$\mathbb{E}\left[W_t^{\nu_2(x)} W_t^{\nu_2(y)} - t\left(\nu_2(x) \wedge \nu_2(y)\right) \middle| \mathcal{F}_s^\mu \right] = W_s^{\nu_2(x)} W_s^{\nu_2(y)} - s\left(\nu_2(x) \wedge \nu_2(y)\right),$$

i.e., for the two martingales $\tilde{W}^{\nu_2(x)}$ and $\tilde{W}^{\nu_2(y)}$ introduced in (4.3.6) we have that

$$\mathbb{E}\left[\tilde{W}_t^{\nu_2(x)}\tilde{W}_t^{\nu_2(y)} - \frac{\nu_2(x)\wedge\nu_2(y)}{\sqrt{\nu_2(x)\nu_2(y)}}\, t \middle| \mathcal{F}_s^{\mu}\right] = \tilde{W}_s^{\nu_2(x)}\tilde{W}_s^{\nu_2(y)} - \frac{\nu_2(x)\wedge\nu_2(y)}{\sqrt{\nu_2(x)\nu_2(y)}}\, s,$$

hence by Theorem 4.2 in Section 4 of Jacod and Shiryaev [42, Chapter I] we have that

$$\left\langle \tilde{W}^{\nu_2(x)}, \tilde{W}^{\nu_2(y)} \right\rangle_t = \frac{\nu_2(x)\wedge\nu_2(y)}{\sqrt{\nu_2(x)\nu_2(y)}}\, t = e^{-\alpha|x-y|}t, \tag{4.3.11}$$

and by (4.3.7) for fixed x, $y \in I$ we obtain that

$$\begin{aligned}\langle \mu^x, \mu^y \rangle_t &= \frac{\sigma^2}{\alpha}\left\langle \tilde{W}^{\nu_2(x)}, \tilde{W}^{\nu_2(y)} \right\rangle_t \\ &= \frac{\sigma^2}{\alpha}e^{-\alpha|x-y|}t.\end{aligned}$$

□

We would like to conclude this subsection with a short remark on a model drawback of the Gaussian framework. Mortality intensities are by definition non-negative processes. Unfortunately, the Gaussian intensity model, although very convenient due to its simplicity, analytical tractability and intuitive interpretation, allows for negative values with positive probability. However, although one cannot exclude negative mortality rates, in practical applications this probability tends to be very small (see, e.g., Luciano and Vigna [51]), such that the probability of negative values can usually be considered negligible.

4.3.2 χ^2-Field

In order to overcome the drawback of negative mortality intensities, in this subsection we model the mortality intensities as a non-negative χ^2-random field, generated by a Gaussian random field by means of a positive transformation. We define

$$\mu_{t,x} = \left(c(t,x)O_{t,x}\right)^2, \quad t \in [0,T],\ x \in I, \tag{4.3.12}$$

where $c(t,x)$ is a continuously differentiable function in both t and x and O is defined in (4.3.2), $t \in [0,T]$, $x \in I$. From (C.0.2) and (C.0.3) in Appendix C we have that

$$\mathbb{E}\left[\mu_{t,x}\right] = c^2(t,x)\mathrm{Cov}(O_{t,x}, O_{t,x}) = c^2(t,x)\frac{\sigma^2}{2\theta\alpha},$$

for $t \in [0,T]$ and $x \in I$ and

$$\begin{aligned}\mathrm{Cov}(\mu_{t,x}, \mu_{s,y}) &= 2c^2(t,x)c^2(s,y)\mathrm{Cov}(O_{t,x}, O_{s,y})^2 \\ &= \frac{\sigma^4}{2\theta^2\alpha^2}c^2(t,x)c^2(s,y)e^{-2\theta|t-s|}e^{-2\alpha|x-y|},\end{aligned}$$

i.e.,

$$\mathrm{Corr}(\mu_{t,x}, \mu_{s,y}) = e^{-2\theta|t-s|} e^{-2\alpha|x-y|},$$

for $s, t \in [0, T]$ and $x, y \in I$. In particular, the correlation function of the χ^2-field is the square of the correlation function of the Gaussian field, thus featuring the same properties as discussed in Subsection 4.3.1.

Lemma 4.3.3. *For μ as defined in (4.3.12) the set $(\mu^x)_{x\in I}$ is a family of affine diffusion processes, i.e., for fixed $x \in I$ the dynamics of $(\mu^x_t)_{t\in[0,T]}$ are given by*

$$\mathrm{d}\mu^x_t = 2\left(\theta - \frac{\partial_t c(t,x)}{c(t,x)}\right)\left(\frac{\sigma^2}{2\alpha}\bar{c}(t,x) - \mu^x_t\right)\mathrm{d}t + \sqrt{\frac{4}{\alpha}\sigma^2 c^2(t,x)\mu^x_t}\,\mathrm{d}\tilde{W}^{\nu_2(x)}_t, \quad (4.3.13)$$

for $t \in [0, T]$, where $\bar{c}(t, x) = \frac{c^3(t,x)}{(\theta c(t,x) - \partial_t c(t,x))}$.

Proof. Fix $x \in I$. By Itô's formula and (4.3.8) we have that

$$\mathrm{d}(c(t,x)O^x_t) = O^x_t\left(\partial_t c(t,x) - \theta c(t,x)\right)\mathrm{d}t + \frac{\sigma}{\sqrt{\alpha}} c(t,x)\mathrm{d}\tilde{W}^{\nu_2(x)}_t, \quad t \in [0, T],$$

and by (4.3.12) it follows that

$$\begin{aligned}
\mathrm{d}\mu^x_t &= \mathrm{d}\left(c(t,x)O^x_t\right)^2 = 2c(t,x)O^x_t \mathrm{d}(c(t,x)O^x_t) + \frac{\sigma^2}{\alpha}c^2(t,x)\mathrm{d}t \\
&= \left(2c^2(t,x)\left(O^x_t\right)^2\left(\frac{\partial_t c(t,x)}{c(t,x)} - \theta\right) + \frac{\sigma^2}{\alpha}c^2(t,x)\right)\mathrm{d}t \\
&\quad + 2\frac{\sigma}{\sqrt{\alpha}}c^2(t,x)O^x_t \mathrm{d}\tilde{W}^{\nu_2(x)}_t, \quad t \in [0, T].
\end{aligned}$$

The assertion follows by rearranging the terms. □

Lemma 4.3.4. *Let μ be given by the χ^2-intensity field model as introduced in (4.3.12). Then for fixed x, $y \in I$ the sharp bracket process of $\mu^x = (\mu^x_t)_{t\in[0,T]}$ and $\mu^y = (\mu^y_t)_{t\in[0,T]}$ is given by*

$$\langle \mu^x, \mu^y \rangle_t = \frac{4\sigma^2}{\alpha} e^{-\alpha|x-y|} \int_0^t \mu^x_s \mu^y_s c(s,x)c(s,y)\,\mathrm{d}s, \quad t \in [0, T].$$

Proof. Fix $x, y \in I$. By (4.3.11) and (4.3.13) we immediately obtain

$$\begin{aligned}
\mathrm{d}\left\langle \mu^x, \mu^y \right\rangle_t &= \frac{4\sigma^2}{\alpha} c(t,x)c(t,y)\mu^x_t\mu^y_t \mathrm{d}\left\langle \tilde{W}^{\nu_2(x)}, \tilde{W}^{\nu_2(y)} \right\rangle_t \\
&= \frac{4\sigma^2}{\alpha}\mu^x_t\mu^y_t c(t,x)c(t,y)e^{-\alpha|x-y|}\mathrm{d}t, \quad t \in [0, T].
\end{aligned}$$

□

The function c needs to be specified such that μ as defined in (4.3.12) is consistent with typical characteristics of historical mortality data, e.g., we define

$$c(t,x) = \exp\left(\frac{1}{2}[a(x) - kb(x)t]\right)\frac{\sqrt{2\theta\alpha}}{\sigma}, \quad t \in [0,T],\ x \in I, \tag{4.3.14}$$

where a and b are given in (4.3.4) and $k > 0$. Then

$$\mathbb{E}[\mu_t^x] = \exp(a(x) - kb(x)t)$$

and for the mean reversion level we have

$$\frac{\sigma^2}{2\alpha}\bar{c}(t,x) = \frac{\theta \exp(a(x) - kb(x)t)}{\theta + \frac{kb(x)}{2}},$$

for the mean reversion speed

$$2\left(\theta - \frac{\partial_t c(t,x)}{c(t,x)}\right) = 2\left(\theta + \frac{kb(x)}{2}\right)$$

and for the volatility

$$\sqrt{\frac{4}{\alpha}\sigma^2 c^2(t,x)\mu_t^x} = \sqrt{8\theta \exp(a(x) - kb(x)t)\mu_t^x},$$

for $t \in [0,T]$, $x \in I$.

4.3.3 Model Comparison

With the notation of Subsections 4.3.1 and 4.3.2, in Table 4.1 we compare the Gaussian and the χ^2-intensity field models as specified in (4.3.1), (4.3.12) and (4.3.14). When comparing (4.3.8) with (4.3.13), we observe that while in the Gaussian model we have an age- and time dependent mean reversion level with constant mean reversion speed and volatility, in the χ^2-intensity field model all three parameters are age- and time dependent. Both models are affine (see also Example 4.4.1), thus facilitating the computation of conditional survival probabilities, and allow us to model an inhomogeneous downward mortality trend while taking into account a realistic dependency structure. In spite of the drawback of allowing for negative values with positive probability, the Gaussian intensity model is attractive due to its simplicity and intuitive interpretation.

4.4 Risk-Minimization for Life Insurance Liabilities

Recall that the financial market defined in Subsection 4.2.2 is arbitrage-free, however, the market is not complete since the times of death occur as a surprise to the market and hence represent a kind of “orthogonal” risk. Therefore, in this section in order to find a price and hedging strategy for the insurance payment processes we make use of a well-known quadratic hedging method for pricing and hedging in incomplete markets, the risk-minimization approach, a brief review of which is given in Appendix A. We start with some preliminary results.

criteria	Gaussian intensity field	χ^2-intensity field
affine	yes	yes
closed form solution	yes	no
mean reverting	yes	yes
mean	$\bar{\mu}(t,x)$	$\exp(a(x) - kb(x)t)$
mean reversion level	$\bar{\mu}(t,x) + \partial_t\bar{\mu}(t,x)/\theta$	$\theta\exp(a(x) - kb(x)t)/(\theta + kb(x)/2)$
mean reversion speed	θ	$2(\theta + kb(x)/2)$
volatility	$\sigma/\sqrt{\alpha}$	$\sqrt{8\theta\exp(a(x) - kb(x)t)\mu_t^x}$
correlation	$e^{-\theta\|t-s\|}e^{-\alpha\|x-y\|}$	$e^{-2\theta\|t-s\|}e^{-2\alpha\|x-y\|}$
non-negative	no	yes

Table 4.1: Comparison between the intensity field models

4.4.1 Preliminary Results

Recall that for fixed $x \in I$, $\mu^x = (\mu_t^x)_{t\in[0,T]}$ as defined in Section 4.2 is assumed to be an affine diffusion process (see Appendix B), i.e., μ^x follows the dynamics

$$\mathrm{d}\mu_t^x = \delta(t,\mu_t^x)\mathrm{d}t + \sigma(t,\mu_t^x)\mathrm{d}\,\tilde{W}_t^{\nu_2(x)}, \quad t \in [0,T], \tag{4.4.1}$$

with $\delta(t,r) = d_0(t) + d_1(t)r$ and $\sigma^2(t,r) = v_0(t) + v_1(t)r =: (\sigma_t^x)^2$.

Example 4.4.1. *Note that for fixed $x \in I$ both the Gaussian and the χ^2-field models are affine. In particular, for the Gaussian intensity model as defined in* (4.3.1) *by* (4.3.7) *we have that*

$$\begin{aligned} d_0(t) &= \theta\bar{\mu}(t,x) + \partial_t\bar{\mu}(t,x), \quad d_1(t) = -\theta, \\ v_0(t) &= \frac{\sigma^2}{\alpha}, \quad v_1(t) = 0. \end{aligned}$$

for $t \in [0,T]$, $x \in I$. For the χ^2-intensity model as defined in (4.3.12) *by* (4.3.13) *we have that*

$$\begin{aligned} d_0(t) &= \frac{\sigma^2}{\alpha}c^2(t,x), \quad d_1(t) = 2\left(\frac{\partial_t c(t,x)}{c(t,x)} - \theta\right), \\ v_0(t) &= 0, \quad v_1(t) = \frac{4}{\alpha}\sigma^2 c^2(t,x), \end{aligned}$$

for $t \in [0,T]$, $x \in I$.

Lemma 4.4.2. *Fix $u \in [0,T]$ and $x \in I$. If μ^x is an affine diffusion satisfying* (4.4.1)*, then under the hypothesis of Section 4.2, the process*

$$Z_t^{x,u} = \mathbb{E}[\exp(-\Gamma_u^x)\,|\,\mathcal{F}_t] = \mathbb{E}\left[\exp\left(-\int_0^u \mu_s^x\,\mathrm{d}s\right)\,\Big|\,\mathcal{F}_t^\mu\right],$$

$t \in [0, u]$, *has the following dynamics:*

$$Z_t^{x,u} = Z_0^{x,u} + \int_0^t Z_s^{x,u} \sigma_s^x \beta^{x,u}(s) \mathrm{d}\tilde{W}_s^{\nu_2(x)}, \quad t \in [0, u], \tag{4.4.2}$$

where $\beta^{x,u}$ *is given by the following differential equation:*

$$\partial_t \beta^{x,u}(t) = 1 - d_1(t)\beta^{x,u}(t) - \frac{1}{2} v_1(t) \left(\beta^{x,u}(t)\right)^2, \quad \beta^{x,u}(u) = 0. \tag{4.4.3}$$

Proof. Fix $u \in [0, T]$ and $x \in I$. Since μ^x is affine, by equation (B.0.1) in Appendix B we immediately obtain that

$$\tilde{Z}_t^{x,u} := \mathbb{E}\left[e^{-\int_t^u \mu_s^x \mathrm{d}s} \middle| \mathcal{F}_t \right] = \mathbb{E}\left[e^{-\int_t^u \mu_s^x \mathrm{d}s} \middle| \mathcal{F}_t^\mu \right] = e^{\alpha^{x,u}(t) + \beta^{x,u}(t)\mu_t^x}, \tag{4.4.4}$$

for $t \in [0, u]$, where the functions $\alpha^{x,u}$ and $\beta^{x,u}$ are given by

$$\begin{aligned} \partial_t \beta^{x,u}(t) &= 1 - d_1(t)\beta^{x,u}(t) - \frac{1}{2} v_1(t) \left(\beta^{x,u}(t)\right)^2, \quad \beta^{x,u}(u) = 0, \\ \partial_t \alpha^{x,u}(t) &= -d_0(t)\beta^{x,u}(t) - \frac{1}{2} v_0(t) \left(\beta^{x,u}(t)\right)^2, \quad \alpha^{x,u}(u) = 0. \end{aligned} \tag{4.4.5}$$

Then by Itô's formula we have that

$$\begin{aligned} \mathrm{d}\tilde{Z}_t^{x,u} &= \tilde{Z}_t^{x,u} \left(\partial_t \alpha^{x,u}(t) + \partial_t \beta^{x,u}(t) \mu_t^x \right) \mathrm{d}t + \tilde{Z}_t^{x,u} \beta^{x,u}(t) \mathrm{d}\mu_t^x \\ &\quad + \frac{1}{2} \tilde{Z}_t^{x,u} \left(\beta^{x,u}(t)\right)^2 \mathrm{d}\left\langle \mu^x \right\rangle_t \\ &= \tilde{Z}_t^{x,u} \left(\mu_t^x \mathrm{d}t + \beta^{x,u}(t) \sigma_t^x \mathrm{d}\tilde{W}_t^{\nu_2(x)} \right), \end{aligned} \tag{4.4.6}$$

as well as

$$\begin{aligned} \mathrm{d}Z_t^{x,u} &= e^{-\Gamma_t^x} \mathrm{d}\tilde{Z}_t^{x,u} - e^{-\Gamma_t^x} \tilde{Z}_t^{x,u} \mu_t^x \mathrm{d}t \\ &= Z_t^{x,u} \beta^{x,u}(t) \sigma_t^x \mathrm{d}\tilde{W}_t^{\nu_2(x)}, \quad t \in [0, u], \end{aligned}$$

hence the result follows. □

Lemma 4.4.3. *Fix* $u \in [0, T]$ *and* $x \in I$. *If* μ^x *is an affine diffusion satisfying* (4.4.1), *then under the hypothesis of Section 4.2, the process*

$$\bar{Z}_t^{x,u} := \mathbb{E}\left[\exp\left(-\int_0^u \mu_s^x \, \mathrm{d}s \right) \mu_u^x \,\middle|\, \mathcal{F}_t^\mu \right], \quad t \in [0, u],$$

has the following dynamics:

$$\bar{Z}_t^{x,u} = \bar{Z}_0^{x,u} + \int_0^t Z_s^{x,u} \sigma_s^x \left[\beta^{x,u}(s) \hat{Z}_s^{x,u} + \hat{\beta}^{x,u}(s) \right] \mathrm{d}\tilde{W}_s^{\nu_2(x)}, \quad t \in [0, u], \tag{4.4.7}$$

where $\hat{Z}^{x,u}$ is given by

$$\hat{Z}_t^{x,u} = \hat{\alpha}^{x,u}(t) + \hat{\beta}^{x,u}(t)\mu_t^x, \quad t \in [0,u], \tag{4.4.8}$$

and $\hat{\alpha}^{x,u}$ and $\hat{\beta}^{x,u}$ are given by the following differential equations:

$$\begin{aligned} \partial_t \hat{\beta}^{x,u}(t) &= -d_1(t)\hat{\beta}^{x,u}(t) - \beta^{x,u}(t)\hat{\beta}^{x,u}(t)v_1(t), \quad \hat{\beta}^{x,u}(u) = 1, \\ \partial_t \hat{\alpha}^{x,u}(t) &= -d_0(t)\hat{\beta}^{x,u}(t) - \beta^{x,u}(t)\hat{\beta}^{x,u}(t)v_0(t), \quad \hat{\alpha}^{x,u}(u) = 0, \end{aligned} \tag{4.4.9}$$

and $Z^{x,u}$ and $\beta^{x,u}$ are given by (4.4.2) *and* (4.4.3)*.*

Proof. Fix $u \in [0,T]$ and $x \in I$. Since μ^x is affine, by equation (B.0.1) in Appendix B we immediately obtain that

$$\mathbb{E}\left[e^{-\int_t^u \mu_s^x \mathrm{d}s}\mu_u^x \,\middle|\, \mathcal{F}_t\right] = \tilde{Z}_t^{x,u}\hat{Z}_t^{x,u}, \quad t \in [0,u],$$

where $\tilde{Z}_t^{x,u}$ is given in (4.4.4) and

$$\hat{Z}_t^{x,u} = \hat{\alpha}^{x,u}(t) + \hat{\beta}^{x,u}(t)\mu_t^x, \quad t \in [0,u],$$

with

$$\begin{aligned} \partial_t \hat{\beta}^{x,u}(t) &= -d_1(t)\hat{\beta}^{x,u}(t) - \beta^{x,u}(t)\hat{\beta}^{x,u}(t)v_1(t), \quad \hat{\beta}^{x,u}(u) = 1, \\ \partial_t \hat{\alpha}^{x,u}(t) &= -d_0(t)\hat{\beta}^{x,u}(t) - \beta^{x,u}(t)\hat{\beta}^{x,u}(t)v_0(t), \quad \hat{\alpha}^{x,u}(u) = 0. \end{aligned}$$

Then, again by an application of Itô's formula we obtain

$$\begin{aligned} \mathrm{d}\hat{Z}_t^{x,u} &= \left(\partial_t \hat{\alpha}^{x,u}(t) + \partial_t \hat{\beta}^{x,u}(t)\mu_t^x\right)\mathrm{d}t + \hat{\beta}^{x,u}(t)\mathrm{d}\mu_t^x \\ &= \hat{\beta}^{x,u}(t)\sigma_t^x\left(-\beta^{x,u}(t)\sigma_t^x\mathrm{d}t + \mathrm{d}\tilde{W}_t^{\nu_2(x)}\right), \end{aligned} \tag{4.4.10}$$

and by (4.4.6) and (4.4.10) we have

$$\begin{aligned} \mathrm{d}(\tilde{Z}_t^{x,u}\hat{Z}_t^{x,u}) &= \tilde{Z}_t^{x,u}\mathrm{d}\hat{Z}_t^{x,u} + \hat{Z}_t^{x,u}\mathrm{d}\tilde{Z}_t^{x,u} + \mathrm{d}\left\langle \tilde{Z}^{x,u}, \hat{Z}_t^{x,u}\right\rangle_t \\ &= \tilde{Z}_t^{x,u}\hat{\beta}^{x,u}(t)\sigma_t^x\left(-\beta^{x,u}(t)\sigma_t^x\mathrm{d}t + \mathrm{d}\tilde{W}_t^{\nu_2(x)}\right) \\ &\quad + \hat{Z}_t^{x,u}\tilde{Z}_t^{x,u}\left(\mu_t^x\mathrm{d}t + \beta^{x,u}(t)\sigma_t^x\mathrm{d}\tilde{W}_t^{\nu_2(x)}\right) + \tilde{Z}_t^{x,u}\left(\sigma_t^x\right)^2\beta^{x,u}(t)\hat{\beta}^{x,u}(t)\mathrm{d}t \\ &= \tilde{Z}_t^{x,u}\left(\hat{Z}_t^{x,u}\mu_t^x\mathrm{d}t + \sigma_t^x\left(\hat{Z}_t^{x,u}\beta^{x,u}(t) + \hat{\beta}^{x,u}(t)\right)\mathrm{d}\tilde{W}_t^{\nu_2(x)}\right), \end{aligned}$$

hence we obtain that

$$\begin{aligned} \mathrm{d}\bar{Z}_t^{x,u} &= e^{-\Gamma_t^x}\mathrm{d}(\tilde{Z}_t^{x,u}\hat{Z}_t^{x,u}) - e^{-\Gamma_t^x}\tilde{Z}_t^{x,u}\hat{Z}_t^{x,u}\mu_t^x\mathrm{d}t \\ &= Z_t^{x,u}\sigma_t^x\left(\hat{Z}_t^{x,u}\beta^{x,u}(t) + \hat{\beta}^{x,u}(t)\right)\mathrm{d}\tilde{W}_t^{\nu_2(x)}, \quad t \in [0,u], \end{aligned}$$

and the result follows. □

Remark 4.4.4. *For the Gaussian field model as specified in* (4.3.1) *and* (4.3.7), *for fixed* $u \in [0,T]$ *and* $x \in I$ *we can easily compute the functions* $\alpha^{x,u}$, $\beta^{x,u}$, $\hat{\alpha}^{x,u}$ *and* $\hat{\beta}^{x,u}$ *analytically. The closed forms for* $\alpha^{x,u}$ *and* $\beta^{x,u}$ *are given by*

$$\beta^{x,u}(t) = \frac{e^{-\theta(u-t)} - 1}{\theta}, \tag{4.4.11}$$

$$\alpha^{x,u}(t) = \int_t^u \beta^{x,u}(s) \left(\theta\bar{\mu}(s,x) + \partial_s\bar{\mu}(s,x) + \frac{\sigma}{2\sqrt{\alpha}}\beta^{x,u}(s) \right) \mathrm{d}s, \tag{4.4.12}$$

for $t \in [0,u]$, *as it can easily be verified by substitution in* (4.4.3) *and* (4.4.5). *The closed forms for* $\hat{\alpha}^{x,u}$ *and* $\hat{\beta}^{x,u}$ *are given by*

$$\hat{\beta}^{x,u}(t) = e^{-\theta(u-t)},$$

$$\hat{\alpha}^{x,u}(t) = \int_t^u \hat{\beta}^{x,u}(s) \left(\theta\bar{\mu}(s,x) + \partial_s\bar{\mu}(s,x) + \frac{\sigma}{\sqrt{\alpha}}\beta^{x,u}(s) \right) \mathrm{d}s.$$

for $t \in [0,u]$. *For the* χ^2*-field model in* (4.3.12) *if* c *is a function of age only, i.e.,* $c(t,x) \equiv c(x)$, *we obtain the well-known time-homogeneous Cox-Ingersoll-Ross model for* μ^x *for any given fixed* x, *and* $\alpha^{x,u}$, $\beta^{x,u}$, $\hat{\alpha}^{x,u}$ *and* $\hat{\beta}^{x,u}$ *are explicitly computable (see Cox et al. [24]). In general, if the model has time-dependent parameters no closed form solutions are available (see Heath et al. [40] and Hull and White [41]) and the differential equations determining* $\alpha^{x,u}$, $\beta^{x,u}$, $\hat{\alpha}^{x,u}$ *and* $\hat{\beta}^{x,u}$ *have to be solved by using numerical methods.*

In the following we calculate the prices and hedging strategies of the insurance payment streams introduced in (4.2.8) - (4.2.10) by means of the risk-minimization approach (see Appendix A). An important role is played by the $\mathbb{G}$-(local) martingales

$$M_t^{x_i,j} = H_t^{x_i,j} - \Gamma^{x_i}_{t\wedge\tau^{x_i,j}} \quad \text{and} \quad M_t^{x_i} := \sum_{j=1}^{n^{x_i}} M_t^{x_i,j}, \tag{4.4.13}$$

as well as

$$L_t^{x_i,j} = \mathbb{1}_{\{\tau^{x_i,j}>t\}} e^{\Gamma_t^{x_i}} = 1 - \int_{]0,t]} L_{s-}^{x_i,j}\, \mathrm{d}M_s^{x_i,j} = 1 - \int_{]0,t]} e^{\Gamma_s^{x_i}}\, \mathrm{d}M_s^{x_i,j}, \tag{4.4.14}$$

for $t \in [0,T]$, $i = 1,\dots,m$, $j = 1,\dots,n^{x_i}$ (see, e.g., Chapter 5 and Chapter 9 of Bielecki and Rutkowski [12]). Recall that we consider unit-linked life insurance products, i.e., the insurance liabilities defined in (4.2.8) - (4.2.10) are given in terms of a non-negative Borel measurable function $f(S_t)$ of the asset price S_t, $t \in [0,T]$. Then following Møller [53] for fixed $u \in [0,T]$ the arbitrage-free price process

$$F^u(t,S_t) = \mathbb{E}\left[\exp\left(-r\,(u-t)\right) f(S_u) | \mathcal{F}_t^X\right], \quad t \in [0,u], \tag{4.4.15}$$

associated with the payoff $f(S_u)$ at time u can be be characterized by the partial differential equation

$$-rF^u(t,s) + F^u_t(t,s) + rsF^u_s(t,s) + \frac{1}{2}\sigma(t,s)^2 s^2 F^u_{ss}(t,s) = 0, \qquad (4.4.16)$$

with boundary value $F^u(u,s) = f(s)$, where we denote by $F^u_t(t,s)$, $F^u_s(t,s)$ and $F^u_{ss}(t,s)$ the partial first and second order derivatives of F^u with respect to t and s. Also recall that we assume that trading in the (discounted) risky asset X introduced in (4.2.6), as well as in the family of (discounted) longevity bonds Y^x, $x \in I$, defined in (4.2.7) is possible (see Subsection 4.2.2). However, the insurance portfolio introduced in Subsection 4.2.1 only consists of individuals belonging to the age cohorts $\{x_1, \ldots, x_m\} \subset I$. Therefore, the risk-minimizing strategies will be given in terms of investments in X as well as the portfolio of longevity bonds $Y := (Y^{x_1}, \ldots, Y^{x_m})$ corresponding to the age cohorts $x_1, \ldots, x_m$ of the insurance portfolio, see, e.g., (4.4.18) and (4.4.19). In the following, we denote by $\int_0^t \xi_s \,\mathrm{d}Y_s := \sum_{i=1}^m \int_0^t \xi^i_s \,\mathrm{d}Y^{x_i}_s$, for any m-dimensional $\mathbb{G}$-predictable process $\xi = (\xi^1, \ldots, \xi^m)$, as well as $\xi \cdot Y := \sum_{i=1}^m \xi^i Y^{x_i}$.

4.4.2 Pure Endowment Contract

For the pure endowment contract introduced in (4.2.8) we define the payment process

$$A^{pe}_t = \frac{f(S_t)}{B_t} \sum_{i=1}^m \zeta(x_i) \sum_{j=1}^{n^{x_i}} \mathbb{1}_{\{\tau^{x_i,j} > t\}} \mathbb{1}_{\{t=T\}}, \quad t \in [0,T], \qquad (4.4.17)$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borel measurable function such that

$$\mathbb{E}\left[f(S_T)^2\right] < \infty.$$

Proposition 4.4.5. *In the setting of Section 4.2 the payment process A^{pe} introduced in* (4.4.17) *admits a risk-minimizing strategy $\varphi = (\xi, \xi^0) = (\xi^X, \xi^Y, \xi^0)$ with discounted value process*

$$V^{pe}_t(\varphi) = \mathbb{E}[A^{pe}_T \,|\, \mathcal{G}_0] + \int_0^t \xi^X_s \,\mathrm{d}X_s + \int_0^t \xi^Y_s \,\mathrm{d}Y_s + L^{pe}_t - A^{pe}_t,$$

and

$$\xi^0_t = V^{pe}_t(\varphi) - \xi^X_t X_t - \xi^Y_t \cdot Y_t$$

for $t \in [0,T]$, where the investment in the (discounted) risky asset X is given by

$$\xi^X_t = F^T_s(t, S_t) \sum_{i=1}^m \zeta(x_i)(n^{x_i} - N^{x_i}_t) e^{\Gamma^{x_i}_t} Z^{x_i,T}_t, \qquad (4.4.18)$$

and the investment in the family of (discounted) longevity bonds $Y = (Y^{x_1}, \dots, Y^{x_m})$ *is given by* $\xi_t^Y = (\xi_t^{Y^{x_1}}, \dots, \xi_t^{Y^{x_m}})$*, with*

$$\xi_t^{Y^{x_i}} = F^T(t, S_t) e^{r(T-t)} \zeta(x_i)(n^{x_i} - N_t^{x_i}) e^{\Gamma_t^{x_i}}, \tag{4.4.19}$$

and

$$L_t^{pe} = -\sum_{i=1}^m \zeta(x_i) \int_{]0,t]} \frac{F^T(s, S_s)}{B_s} e^{\Gamma_s^{x_i}} Z_s^{x_i,T} \mathrm{d}M_s^{x_i},$$

$t \in [0,T]$*, where* $F^T(t, S_t)$*,* $F_s^T(t, S_t)$*,* $Z_t^{x_i,T}$ *and* $M_t^{x_i}$ *are defined in* (4.4.2)*,* (4.4.13)*,* (4.4.15) *and* (4.4.16)*. The optimal cost and risk processes are given by*

$$C_t^{pe}(\varphi) = \mathbb{E}[A_T^{pe} \,|\, \mathcal{G}_0] + L_t^{pe},$$
$$R_t^{pe}(\varphi) = \mathbb{E}[(L_T^{pe} - L_t^{pe})^2 \,|\, \mathcal{G}_t],$$

for $t \in [0,T]$*.*

Proof. Let $t \in [0,T]$. Then we have that

$$\mathbb{E}[A_T^{pe} \,|\, \mathcal{G}_t] = \sum_{i=1}^m \zeta(x_i) \sum_{j=1}^{n^{x_i}} J_t^{ij},$$

where

$$J_t^{ij} = \mathbb{E}\left[\frac{f(S_T)}{B_T} \mathbb{1}_{\{\tau^{x_i,j} > T\}} \,\Big|\, \mathcal{G}_t\right], \tag{4.4.20}$$

for $t \in [0,T]$, $i = 1, \dots, m$, $j = 1, \dots, n^{x_i}$. By Proposition 4.10 and 5.11 of Barbarin [4, Chapter 3], as well as Corollary 5.1.1 of Bielecki and Rutkowski [12] and (4.2.3) we have

$$J_t^{ij} = U_0^{x_i,pe} + \int_0^t L_s^{x_i,j} \, \mathrm{d}U_s^{x_i,pe} - \int_{]0,t]} U_s^{x_i,pe} e^{\Gamma_s^{x_i}} \, \mathrm{d}M_s^{x_i,j}, \tag{4.4.21}$$

where $M_t^{x_i,j}$ and $L_t^{x_i,j}$ are defined in (4.4.13) and (4.4.14) and

$$U_t^{x_i,pe} := \mathbb{E}\left[\frac{f(S_T)}{B_T} e^{-\Gamma_T^{x_i}} \,\Big|\, \mathcal{F}_t\right],$$

for $t \in [0,T]$, $i = 1, \dots, m$, $j = 1, \dots, n^{x_i}$. By the independence of the underlying driving processes we have

$$U_t^{x_i,pe} = \mathbb{E}\left[\frac{f(S_T)}{B_T} e^{-\Gamma_T^{x_i}} \,\Big|\, \mathcal{F}_t\right] = \mathbb{E}\left[\frac{f(S_T)}{B_T} \Big|\, \mathcal{F}_t^X\right] \mathbb{E}\left[e^{-\Gamma_T^{x_i}} \,\Big|\, \mathcal{F}_t^\mu\right] = \frac{F^T(t, S_t)}{B_t} Z_t^{x_i,T},$$

for $t \in [0,T]$ and $i = 1, \dots, m$. By (4.2.5) - (4.2.6), (4.4.15) - (4.4.16) and Itô's formula the discounted arbitrage-free price process $\frac{F^T(t,S_t)}{B_t}$, $t \in [0,T]$, follows the dynamics

$$\mathrm{d}\left(\frac{F^T(t, S_t)}{B_t}\right) = F_s^T(t, S_t) \sigma(t, S_t) X_t \, \mathrm{d}W_t^X = F_s^T(t, S_t) \, \mathrm{d}X_t, \quad t \in [0,T],$$

and by (4.4.2) and integration by parts we obtain that

$$\begin{aligned} U_t^{x_i,pe} &= U_0^{x_i,pe} + \int_0^t Z_s^{x_i,T}\,\mathrm{d}\left(\frac{F^T(s,S_s)}{B_s}\right) + \int_0^t \frac{F^T(s,S_s)}{B_s}\,\mathrm{d}Z_s^{x_i,T} \\ &= U_0^{x_i,pe} + \int_0^t Z_s^{x_i,T} F_s^T(s,S_s)\sigma(s,S_s)X_s\,\mathrm{d}W_s^X \\ &\quad + \int_0^t \frac{F^T(s,S_s)}{B_s} Z_s^{x_i,T}\sigma_s^{x_i}\beta^{x_i,T}(s)\mathrm{d}\tilde{W}_s^{\nu_2(x_i)} \end{aligned} \tag{4.4.22}$$

for $t \in [0,T]$, $i = 1,\dots,m$. By Lemma 4.4.2 we have that for each $x \in I$ the dynamics of the (discounted) longevity bond with maturity T associated to the age cohort x as defined in (4.2.7) are given by

$$\mathrm{d}Y_t^x = \frac{Z_t^{x,T}}{B_T}\sigma_t^x\beta^{x,T}(t)\mathrm{d}\tilde{W}_t^{\nu_2(x)}, \quad t \in [0,T]. \tag{4.4.23}$$

Hence by (4.2.6) and (4.4.23) we obtain that

$$\begin{aligned} \mathbb{E}[A_T^{pe} \mid \mathcal{G}_t] &= \sum_{i=1}^m \zeta(x_i) \sum_{j=1}^{n^{x_i}} F^T(0,S_0)Z_0^{x_i,T} \\ &\quad + \sum_{i=1}^m \zeta(x_i)\int_0^t (n^{x_i} - N_s^{x_i})e^{\Gamma_s^{x_i}} F_s^T(s,S_s)Z_s^{x_i,T}\sigma(s,S_s)X_s\,\mathrm{d}W_s^X \\ &\quad + \sum_{i=1}^m \zeta(x_i)\int_0^t (n^{x_i} - N_s^{x_i})e^{\Gamma_s^{x_i}} \frac{F^T(s,S_s)}{B_s} Z_s^{x_i,T}\sigma_s^{x_i}\beta^{x_i,T}(s)\mathrm{d}\tilde{W}_s^{\nu_2(x_i)} + L_t^{pe} \\ &= F^T(0,S_0)\sum_{i=1}^m \zeta(x_i)n^{x_i}Z_0^{x_i,T} + \int_0^t \xi_s^X\,\mathrm{d}X_s + \int_0^t \xi_s^Y\,\mathrm{d}Y_s + L_t^{pe}, \end{aligned} \tag{4.4.24}$$

for $t \in [0,T]$, where the investment in the (discounted) risky asset X is given by

$$\xi_t^X = F_s^T(t,S_t)\sum_{i=1}^m \zeta(x_i)(n^{x_i} - N_t^{x_i})e^{\Gamma_t^{x_i}} Z_t^{x_i,T},$$

and the investment in the family of (discounted) longevity bonds $Y = (Y^{x_1},\dots,Y^{x_m})$ is given by $\xi_t^Y = (\xi_t^{Y^{x_1}},\dots,\ \xi_t^{Y^{x_m}})$, with

$$\xi_t^{Y^{x_i}} = F^T(t,S_t)e^{r(T-t)}\zeta(x_i)(n^{x_i} - N_t^{x_i})e^{\Gamma_t^{x_i}},$$

and

$$L_t^{pe} = -\sum_{i=1}^m \zeta(x_i)\int_{]0,t]} \frac{F^T(s,S_s)}{B_s}e^{\Gamma_s^{x_i}} Z_s^{x_i,T}\mathrm{d}M_s^{x_i},$$

$t \in [0,T]$. It remains to prove that (4.4.24) is indeed the GKW decomposition of $\mathbb{E}[A_T^{pe} \mid \mathcal{G}_t]$, $t \in [0,T]$. To this end define $\overline{S} = (X,Y) = (X,Y^{x_1},\dots,Y^{x_m})$ and

$\xi = \left(\xi^X, \xi^Y\right)$. Note that since $\mathbb{E}[f(S_T)^2] < \infty$ and (4.2.4) holds, for $i = 1, \dots, m$ and $j = 1, \dots, n^{x_i}$ we have that J^{ij} introduced in (4.4.20) is a square integrable martingale, hence $\mathbb{E}[[J^{ij}]_T] < \infty$, and from (4.4.21) it follows that

$$\mathbb{E}\left[\int_0^T (L_s^{x_i,j})^2 \, \mathrm{d}[U^{x_i,pe}]_s\right], \quad \mathbb{E}\left[\int_0^T (U_s^{x_i,pe} e^{\Gamma_s^{x_i}})^2 \, \mathrm{d}[M^{x_i,j}]_s\right] < \infty, \tag{4.4.25}$$

because $\mathrm{d}[U^{x_i,pe}, M^{x_i,j}]_t \equiv 0$, $t \in [0,T]$, $i = 1, \dots, m$ and $j = 1, \dots, n^{x_i}$. Since for $i = 1, \dots, m$, $\mathrm{d}[W^X, \tilde{W}^{\nu_2(x_i)}]_t \equiv 0$, $t \in [0,T]$, because W^X and $\tilde{W}^{\nu_2(x_i)}$ are independent, by (4.4.22), (4.4.24) and (4.4.25) and by the Kunita-Watanabe Inequality (see, e.g., Theorem 25 in Chapter II.6 of Protter [56]), we obtain that

$$\mathbb{E}\left[\int_0^T \xi_s' \mathrm{d}[\overline{S}]_s \xi_s\right] < \infty \quad \text{and} \quad \mathbb{E}[[L^{pe}]_T] < \infty,$$

i.e., $\xi \in L^2(\overline{S})$ (see Definition A.0.3 in Appendix A) and L^{pe} is a square integrable martingale. Since L^{pe} is strongly orthogonal to all continuous $\mathbb{F}$-local martingales, it follows that

$$\left(\int_0^t \tilde{\xi}_s \mathrm{d}\overline{S}_s\right) \cdot L_t^{pe}, \quad t \in [0,T],$$

is a (uniformly integrable) martingale for any $\tilde{\xi} \in L^2(\overline{S})$, i.e., (4.4.24) is the GKW decomposition of $\mathbb{E}[A_T^{pe} \,|\, \mathcal{G}_t]$, $t \in [0,T]$ (see equation (A.0.2) in Appendix A). □

4.4.3 Term Insurance Contract

For the term insurance contract introduced in (4.2.9) we define the payment process

$$A_t^{ti} = \sum_{i=1}^m \zeta(x_i) \sum_{j=1}^{n^{x_i}} \frac{f(S_{\tau^{x_i,j}})}{B_{\tau^{x_i,j}}} \mathbb{1}_{\{\tau^{x_i,j} \le t\}}, \quad t \in [0,T], \tag{4.4.26}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borel measurable function such that

$$\mathbb{E}\left[\sup_{t \in [0,T]} f(S_t)^2\right] < \infty.$$

Proposition 4.4.6. *In the setting of Section 4.2 the payment process A^{ti} introduced in (4.4.26) admits a risk-minimizing strategy $\varphi = (\xi, \xi^0) = (\xi^X, \xi^Y, \xi^0)$ with discounted value process*

$$V_t^{ti}(\varphi) = \mathbb{E}[A_T^{ti} \,|\, \mathcal{G}_0] + \int_0^t \xi_s^X \, \mathrm{d}X_s + \int_0^t \xi_s^Y \, \mathrm{d}Y_s + L_t^{ti} - A_t^{ti},$$

and

$$\xi_t^0 = V_t^{ti}(\varphi) - \xi_t^X X_t - \xi_t^Y \cdot Y_t$$

for $t \in [0,T]$, where the investment in the (discounted) risky asset X is given by

$$\xi_t^X = \sum_{i=1}^m \zeta(x_i)(n^{x_i} - N_t^{x_i}) e^{\Gamma_t^{x_i}} \int_t^T \bar{Z}_t^{x_i,u} F_s^u(t, S_t) \, \mathrm{d}u,$$

and the investment in the family of (discounted) longevity bonds $Y = (Y^{x_1}, \dots, Y^{x_m})$ is given by $\xi_t^Y = (\xi_t^{Y^{x_1}}, \dots, \xi_t^{Y^{x_m}})$, with

$$\xi_t^{Y^{x_i}} = \zeta(x_i)(n^{x_i} - N_t^{x_i}) \frac{e^{\Gamma_t^{x_i}} e^{r(T-t)}}{Z_t^{x_i,T} \beta^{x_i,T}(t)} \int_t^T F^u(t, S_t) Z_t^{x_i,u} (\beta^{x_i,u}(t) \hat{Z}_t^{x_i,u} + \hat{\beta}^{x_i,u}(t)) \, \mathrm{d}u,$$

and

$$L_t^{ti} = \sum_{i=1}^m \zeta(x_i) \int_{]0,t]} \left(\frac{f(S_s)}{B_s} - \mathbb{E}\left[\int_s^T \frac{f(S_u)}{B_u} e^{\Gamma_s^{x_i} - \Gamma_u^{x_i}} \, \mathrm{d}\Gamma_u^{x_i} \,\middle|\, \mathcal{F}_s \right] \right) \mathrm{d}M_s^{x_i},$$

$t \in [0,T]$, where $F^u(t, S_t)$, $F_s^u(t, S_t)$, $\beta^{x_i,u}(t)$, $\hat{\beta}^{x_i,u}(t)$, $Z_t^{x_i,u}$, $\bar{Z}_t^{x_i,u}$, $\hat{Z}_t^{x_i,u}$ and $M_t^{x_i}$ are defined in (4.4.2) - (4.4.3), (4.4.7) - (4.4.9), (4.4.13) *and* (4.4.15) - (4.4.16). *The optimal cost and risk processes are given by*

$$C_t^{ti}(\varphi) = \mathbb{E}[A_T^{ti} \,|\, \mathcal{G}_0] + L_t^{ti},$$
$$R_t^{ti}(\varphi) = \mathbb{E}[(L_T^{ti} - L_t^{ti})^2 \,|\, \mathcal{G}_t],$$

for $t \in [0,T]$.

Proof. Let $t \in [0,T]$. Then we have that

$$\mathbb{E}[A_T^{ti} \,|\, \mathcal{G}_t] = \sum_{i=1}^m \zeta(x_i) \sum_{j=1}^{n^{x_i}} \mathbb{E}\left[\frac{f(S_{\tau^{x_i,j}})}{B_{\tau^{x_i,j}}} \mathbb{1}_{\{\tau^{x_i,j} \leq T\}} \,\middle|\, \mathcal{G}_t \right],$$

and by Proposition 4.11 and 5.12 of Barbarin [4, Chapter 3], as well as Corollary 5.1.3 of Bielecki and Rutkowski [12] and (4.2.3) we have

$$\begin{aligned}
\mathbb{E}\left[\frac{f(S_{\tau^{x_i,j}})}{B_{\tau^{x_i,j}}} \mathbb{1}_{\{\tau^{x_i,j} \leq T\}} \,\middle|\, \mathcal{G}_t \right] &= U_0^{x_i,ti} + \int_0^t L_s^{x_i,j} \, \mathrm{d}U_s^{x_i,ti} \\
&+ \int_{]0,t]} \left(\frac{f(S_s)}{B_s} - \mathbb{E}\left[\int_s^T \frac{f(S_u)}{B_u} e^{\Gamma_s^{x_i} - \Gamma_u^{x_i}} \, \mathrm{d}\Gamma_u^{x_i} \,\middle|\, \mathcal{F}_s \right] \right) \mathrm{d}M_s^{x_i,j},
\end{aligned}$$

and

$$\begin{aligned}
U_t^{x_i,ti} &:= \mathbb{E}\left[\int_0^T \frac{f(S_u)}{B_u} e^{-\Gamma_u^{x_i}} \, \mathrm{d}\Gamma_u^{x_i} \,\middle|\, \mathcal{F}_t \right] \\
&= \int_0^T \mathbb{E}\left[\frac{f(S_u)}{B_u} \middle|\, \mathcal{F}_t^X \right] \mathbb{E}\left[e^{-\Gamma_u^{x_i}} \mu_u^{x_i} \,\middle|\, \mathcal{F}_t^\mu \right] \mathrm{d}u,
\end{aligned}$$

for $t \in [0,T]$ and $i = 1,\dots,m$, $j = 1,\dots,n^{x_i}$, where we have used Fubini's theorem and the independence of the underlying driving processes. By the same arguments as in the proof of Proposition 4.4.5 we have that

$$\mathbb{E}\left[\frac{f(S_u)}{B_u}\,\Big|\,\mathcal{F}_t^X\right] = F^u(0,S_0) + \int_0^t F_s^u(s,S_s)\sigma(s,S_s)X_s\mathbb{1}_{\{s\le u\}}\,\mathrm{d}W_s^X$$

for $0 \le t,\, u \le T$, where $F^u(u,S_u) = f(S_u)$. Furthermore by (4.4.7) we have

$$\begin{aligned}\bar{Z}_t^{x_i,u} &= \mathbb{E}\left[e^{-\Gamma_u^{x_i}}\mu_u^{x_i}\,\Big|\,\mathcal{F}_t^\mu\right]\\ &= \bar{Z}_0^{x_i,u} + \int_0^t Z_s^{x_i,u}\sigma_s^{x_i}\left[\beta^{x_i,u}(s)\hat{Z}_s^{x_i,u} + \hat{\beta}^{x_i,u}(s)\right]\mathbb{1}_{\{s\le u\}}\mathrm{d}\tilde{W}_s^{\nu_2(x_i)},\end{aligned}$$

for $0 \le t,\, u \le T$ and $i = 1,\dots,m$, where $\beta^{x_i,u}$, $\hat{\beta}^{x_i,u}$, $Z^{x_i,u}$ and $\hat{Z}^{x_i,u}$ are given in (4.4.2), (4.4.3) and (4.4.8) - (4.4.9). Then for $u \in [0,T]$ by integration by parts we obtain that

$$\begin{aligned}\frac{F^u(t,S_t)}{B_t}\bar{Z}_t^{x_i,u} &= F^u(0,S_0)\bar{Z}_0^{x_i,u} + \int_0^t \bar{Z}_s^{x_i,u}F_s^u(s,S_s)\sigma(s,S_s)X_s\mathbb{1}_{\{s\le u\}}\,\mathrm{d}W_s^X\\ &+ \int_0^t \frac{F^u(s,S_s)}{B_s}Z_s^{x_i,u}\sigma_s^{x_i}\left(\beta^{x_i,u}(s)\hat{Z}_s^{x_i,u} + \hat{\beta}^{x_i,u}(s)\right)\mathbb{1}_{\{s\le u\}}\,\mathrm{d}\tilde{W}_s^{\nu_2(x_i)}, \quad t \in [0,T].\end{aligned}$$

Since all integrands are continuous (see Theorem 15 in Chapter IV of Protter [56]), by the stochastic Fubini theorem (see, e.g., Theorem 65 in Chapter IV of Protter [56]) and integration by parts we obtain

$$\begin{aligned}U_t^{x_i,ti} &= \int_0^T F^u(0,S_0)\bar{Z}_0^{x_i,u}\,\mathrm{d}u + \int_0^t \sigma(s,S_s)X_s\int_s^T F_s^u(s,S_s)\bar{Z}_s^{x_i,u}\,\mathrm{d}u\,\mathrm{d}W_s^X\\ &\quad + \int_0^t \frac{\sigma_s^{x_i}}{B_s}\int_s^T F^u(s,S_s)Z_s^{x_i,u}(\beta^{x_i,u}(s)\hat{Z}_s^{x_i,u} + \hat{\beta}^{x_i,u}(s))\,\mathrm{d}u\,\mathrm{d}\tilde{W}_s^{\nu_2(x_i)}\\ &= \int_0^T F^u(0,S_0)\bar{Z}_0^{x_i,u}\,\mathrm{d}u + \int_0^t\int_s^T F_s^u(s,S_s)\bar{Z}_s^{x_i,u}\,\mathrm{d}u\,\mathrm{d}X_s\\ &\quad + \int_0^t \frac{e^{r(T-s)}}{Z_s^{x_i,T}\beta^{x_i,T}(s)}\int_s^T F^u(s,S_s)Z_s^{x_i,u}(\beta^{x_i,u}(s)\hat{Z}_s^{x_i,u} + \hat{\beta}^{x_i,u}(s))\,\mathrm{d}u\,\mathrm{d}Y_s^{x_i},\end{aligned}$$

$t \in [0,T]$, where in the second equation we have used (4.2.6) and (4.4.23). Finally, we obtain that

$$\mathbb{E}[A_T^{ti}\,|\,\mathcal{G}_t] = \mathbb{E}[A_T^{ti}\,|\,\mathcal{G}_0] + \int_0^t \xi_s^X\,\mathrm{d}X_s + \int_0^t \xi_s^Y\,\mathrm{d}Y_s + L_t^{ti}, \tag{4.4.27}$$

for $t \in [0,T]$, where the investment in the (discounted) risky asset X is given by

$$\xi_t^X = \sum_{i=1}^m \zeta(x_i)(n^{x_i} - N_t^{x_i})e^{\Gamma_t^{x_i}}\int_t^T \bar{Z}_t^{x_i,u}F_s^u(t,S_t)\,\mathrm{d}u,$$

and the investment in the family of (discounted) longevity bonds $Y = (Y^{x_1}, \dots, Y^{x_m})$ is given by $\xi^Y_t = (\xi^{Y^{x_1}}_t, \dots, \xi^{Y^{x_m}}_t)$, with

$$\xi^{Y^{x_i}}_t = \zeta(x_i)(n^{x_i} - N^{x_i}_t)\frac{e^{\Gamma^{x_i}_t} e^{r(T-t)}}{Z^{x_i,T}_t \beta^{x_i,T}(t)} \int_t^T F^u(t, S_t) Z^{x_i,u}_t (\beta^{x_i,u}(t)\hat{Z}^{x_i,u}_t + \hat{\beta}^{x_i,u}(t))\,\mathrm{d}u,$$

and

$$L^{ti}_t = \sum_{i=1}^m \zeta(x_i) \int_{]0,t]} \left(\frac{f(S_s)}{B_s} - \mathbb{E}\left[\int_s^T \frac{f(S_u)}{B_u} e^{\Gamma^{x_i}_s - \Gamma^{x_i}_u}\,\mathrm{d}\Gamma^{x_i}_u \,\middle|\, \mathcal{F}_s \right] \right) \mathrm{d}M^{x_i}_s,$$

$t \in [0, T]$. By the same arguments as in the proof of Proposition 4.4.5 we obtain that the terms in (4.4.27) are square integrable and strongly orthogonal, hence (4.4.27) is indeed the GKW decomposition of $\mathbb{E}[A^{ti}_T \,|\, \mathcal{G}_t]$, $t \in [0, T]$. □

4.4.4 Annuity Contract

For the annuity contract introduced in (4.2.10) we define the payment process

$$A^a_t = \sum_{i=1}^m \zeta(x_i) \sum_{j=1}^{n^{x_i}} \int_0^t \mathbb{1}_{\{\tau^{x_i,j} > s\}} \frac{f(S_s)}{B_s} \mathrm{d}s, \quad t \in [0, T], \tag{4.4.28}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borel measurable function such that

$$\mathbb{E}\left[\sup_{t \in [0,T]} f(S_t)^2 \right] < \infty.$$

Proposition 4.4.7. *In the setting of Section 4.2 the payment process A^a introduced in* (4.4.28) *admits a risk-minimizing strategy $\varphi = (\xi, \xi^0) = (\xi^X, \xi^Y, \xi^0)$ with discounted value process*

$$V^a_t(\varphi) = \mathbb{E}[A^a_T \,|\, \mathcal{G}_0] + \int_0^t \xi^X_s \,\mathrm{d}X_s + \int_0^t \xi^Y_s \,\mathrm{d}Y_s + L^a_t - A^a_t,$$

and

$$\xi^0_t = V^a_t(\varphi) - \xi^X_t X_t - \xi^Y_t \cdot Y_t$$

for $t \in [0, T]$, where the investment in the (discounted) risky asset X is given by

$$\xi^X_t = \sum_{i=1}^m \zeta(x_i)(n^{x_i} - N^{x_i}_t) e^{\Gamma^{x_i}_t} \int_t^T Z^{x_i,u}_t F^u_s(t, S_t)\,\mathrm{d}u,$$

and the investment in the family of (discounted) longevity bonds $Y = (Y^{x_1}, \dots, Y^{x_m})$ is given by $\xi^Y_t = (\xi^{Y^{x_1}}_t, \dots, \xi^{Y^{x_m}}_t)$, with

$$\xi^{Y^{x_i}}_t = \zeta(x_i)(n^{x_i} - N^{x_i}_t)\frac{e^{\Gamma^{x_i}_t} e^{r(T-t)}}{Z^{x_i,T}_t \beta^{x_i,T}(t)} \int_t^T F^u(t, S_t) Z^{x_i,u}_t \beta^{x_i,u}(t)\,\mathrm{d}u,$$

and

$$L_t^a = -\sum_{i=1}^m \zeta(x_i) \int_{]0,t]} \mathbb{E}\left[\int_s^T \frac{f(S_u)}{B_u} e^{\Gamma_s^{x_i} - \Gamma_u^{x_i}} \, \mathrm{d}u \,\Big|\, \mathcal{F}_s\right] \mathrm{d}M_s^{x_i},$$

$t \in [0,T]$, *where* $F^u(t, S_t)$, $F_s^u(t, S_t)$, $\beta^{x_i,u}(t)$, $Z_t^{x_i,u}$ *and* $M_t^{x_i}$ *are defined in* (4.4.2) - (4.4.3), (4.4.13) *and* (4.4.15) - (4.4.16). *The optimal cost and risk processes are given by*

$$C_t^a(\varphi) = \mathbb{E}[A_T^a \,|\, \mathcal{G}_0] + L_t^a,$$
$$R_t^a(\varphi) = \mathbb{E}[(L_T^a - L_t^a)^2 \,|\, \mathcal{G}_t],$$

for $t \in [0,T]$.

Proof. Let $t \in [0,T]$. Then we have that

$$\mathbb{E}[A_T^a \,|\, \mathcal{G}_t] = \sum_{i=1}^m \zeta(x_i) \sum_{j=1}^{n^{x_i}} \mathbb{E}\left[\int_0^T \mathbb{1}_{\{\tau^{x_i,j} > s\}} \frac{f(S_s)}{B_s} \mathrm{d}s \,\Big|\, \mathcal{G}_t\right],$$

and by Proposition 4.12 and 5.13 of Barbarin [4, Chapter 3], as well as Proposition 5.1.2 of Bielecki and Rutkowski [12] and (4.2.3) we have

$$\mathbb{E}\left[\int_0^T \mathbb{1}_{\{\tau^{x_i,j} > s\}} \frac{f(S_s)}{B_s} \mathrm{d}s \,\Big|\, \mathcal{G}_t\right] = U_0^{x_i,a} + \int_0^t L_s^{x_i,j} \, \mathrm{d}U_s^{x_i,a}$$
$$- \int_{]0,t]} \mathbb{E}\left[\int_s^T \frac{f(S_u)}{B_u} e^{\Gamma_s^{x_i} - \Gamma_u^{x_i}} \, \mathrm{d}u \,\Big|\, \mathcal{F}_s\right] \mathrm{d}M_s^{x_i,j},$$

and

$$U_t^{x_i,a} := \mathbb{E}\left[\int_0^T \frac{f(S_u)}{B_u} e^{-\Gamma_u^{x_i}} \, \mathrm{d}u \,\Big|\, \mathcal{F}_t\right]$$
$$= \int_0^T \mathbb{E}\left[\frac{f(S_u)}{B_u} \Big|\, \mathcal{F}_t^X\right] \mathbb{E}\left[e^{-\Gamma_u^{x_i}} \,\Big|\, \mathcal{F}_t^\mu\right] \mathrm{d}u,$$

for $t \in [0,T]$ and $i = 1, \ldots, m$, $j = 1, \ldots, n^{x_i}$, where we have used Fubini's theorem and the independence of the underlying driving processes. We proceed similarly as in the proof of Proposition 4.4.6. By (4.4.2) we have that

$$Z_t^{x_i,u} = \mathbb{E}\left[e^{-\Gamma_u^{x_i}} \,\Big|\, \mathcal{F}_t^\mu\right] = Z_0^{x_i,u} + \int_0^t Z_s^{x_i,u} \sigma_s^{x_i} \beta^{x_i,u}(s) \mathbb{1}_{\{s \le u\}} \mathrm{d}\tilde{W}_s^{\nu_2(x_i)},$$

for $0 \le t$, $u \le T$ and $i = 1, \ldots, m$, where $\beta^{x_i,u}$, is given in (4.4.3). Then by the stochastic Fubini theorem (see, e.g., Theorem 65 in Chapter IV of Protter [56]) and again by integration by parts we obtain

$$\begin{aligned} U_t^{x_i,a} &= \int_0^T F^u(0,S_0)Z_0^{x_i,u}\,\mathrm{d}u + \int_0^t \sigma(s,S_s)X_s \int_s^T F_s^u(s,S_s)Z_s^{x_i,u}\,\mathrm{d}u\,\mathrm{d}W_s^X \\ &\quad + \int_0^t \frac{\sigma_s^{x_i}}{B_s}\int_s^T F^u(s,S_s)Z_s^{x_i,u}\beta^{x_i,u}(s)\,\mathrm{d}u\,\mathrm{d}\tilde{W}_s^{\nu_2(x_i)} \\ &= \int_0^T F^u(0,S_0)Z_0^{x_i,u}\,\mathrm{d}u + \int_0^t\int_s^T F_s^u(s,S_s)Z_s^{x_i,u}\,\mathrm{d}u\,\mathrm{d}X_s \\ &\quad + \int_0^t \frac{e^{r(T-s)}}{Z_s^{x_i,T}\beta^{x_i,T}(s)}\int_s^T F^u(s,S_s)Z_s^{x_i,u}\beta^{x_i,u}(s)\,\mathrm{d}u\,\mathrm{d}Y_s^{x_i}, \end{aligned}$$

where in the second equation we have used (4.2.6) and (4.4.23). Finally, we obtain that

$$\mathbb{E}[A_T^a \,|\, \mathcal{G}_t] = \mathbb{E}[A_T^a \,|\, \mathcal{G}_0] + \int_0^t \xi_s^X\,\mathrm{d}X_s + \int_0^t \xi_s^Y\,\mathrm{d}Y_s + L_t^a, \tag{4.4.29}$$

for $t \in [0,T]$, where the investment in the (discounted) risky asset X is given by

$$\xi_t^X = \sum_{i=1}^m \zeta(x_i)(n^{x_i} - N_t^{x_i})e^{\Gamma_t^{x_i}}\int_t^T Z_t^{x_i,u}F_s^u(t,S_t)\,\mathrm{d}u,$$

and the investment in the family of (discounted) longevity bonds $Y = (Y^{x_1},\dots,Y^{x_m})$ is given by $\xi_t^Y = (\xi_t^{Y^{x_1}},\dots,\ \xi_t^{Y^{x_m}})$, with

$$\xi_t^{Y^{x_i}} = \zeta(x_i)(n^{x_i} - N_t^{x_i})\frac{e^{\Gamma_t^{x_i}}e^{r(T-t)}}{Z_t^{x_i,T}\beta^{x_i,T}(t)}\int_t^T F^u(t,S_t)Z_t^{x_i,u}\beta^{x_i,u}(t)\,\mathrm{d}u,$$

and

$$L_t^a = -\sum_{i=1}^m \zeta(x_i)\int_{]0,t]}\mathbb{E}\left[\int_s^T \frac{f(S_u)}{B_u}e^{\Gamma_s^{x_i}-\Gamma_u^{x_i}}\,\mathrm{d}u \,\Big|\, \mathcal{F}_s\right]\mathrm{d}M_s^{x_i},$$

$t \in [0,T]$. By the same arguments as in the proofs of Propositions 4.4.5 and 4.4.6 we obtain that the terms in (4.4.29) are square integrable and strongly orthogonal, hence (4.4.29) is indeed the GKW decomposition of $\mathbb{E}[A_T^a \,|\, \mathcal{G}_t]$, $t \in [0,T]$. □

Note that Proposition 5.1.2 and Corollary 5.1.3 of Bielecki and Rutkowski [12] requires the process $f(S_t)$, $t \in [0,T]$, to be bounded. However, it can be easily seen that this result also holds if $\mathbb{E}[\sup_{t\in[0,T]} f(S_t)^2] < \infty$ and we may therefore apply it in our setting.

4.5 Examples

We consider the two natural extreme cases of cohort dependency in a portfolio of pure endowment contracts: one portfolio in which all individuals belong to different age cohorts and one where all individuals belong to the same age cohort.

Example 4.5.1. *Consider a portfolio of pure endowment contracts as defined in* (4.2.8) *and let* $m = n$, *i.e., all individuals belong to different cohort classes. We write* $\tau^{x_i} = \tau^{x_i,1}$ *and* $M^{x_i} = M^{x_i,1}$ *for* $i = 1, \dots, n$ *and assume that* $\zeta(x_i) = 1$ *for* $i = 1, \dots, n$, *i.e., all classes are equally weighted. Then from* (4.4.18) *we have that the investment in the (discounted) risky asset* X *is given by*

$$\xi_t^X = F_s(t, S_t) \sum_{i=1}^{n} \mathbb{1}_{\{\tau^{x_i} \geq t\}} e^{\Gamma_t^{x_i}} Z_t^{x_i,T}, \tag{4.5.1}$$

and by (4.4.19) *the investment in the family of (discounted) longevity bonds* $Y = (Y^{x_1}, \dots, Y^{x_m})$ *is given by*

$$\xi_t^Y = (\xi_t^{Y^{x_1}}, \dots, \xi_t^{Y^{x_m}}) \quad \text{with} \quad \xi_t^{Y^{x_i}} = F(t, S_t) e^{r(T-t)} \mathbb{1}_{\{\tau^{x_i} \geq t\}} e^{\Gamma_t^{x_i}}, \tag{4.5.2}$$

for $t \in [0, T]$, $i = 1, \dots, n$, *where* $\Gamma_t^{x_i}$, $Z_t^{x_i,T}$ *and* $F(t, S_t)$ *are given in* (4.2.1), (4.4.2) *and* (4.4.15). *Note that in equations* (4.5.1) *and* (4.5.2) *the processes* $\Gamma_t^{x_i} = \int_0^t \mu_s^{x_i} \, \mathrm{d}s$ *and* $Z_t^{x_i,T}$, $t \in [0, T]$, *depend on the specification of the intensity process* $(\mu_t^{x_i})_{t \in [0,T]}$. *For example, in the case of the Gaussian intensity field defined in* (4.3.1) *we have that*

$$Z_t^{x_i,T} = e^{-\int_0^t \mu_s^{x_i} \mathrm{d}s} e^{\alpha^{x_i,T}(t) + \beta^{x_i,T}(t) O_t^{x_i}}, \quad t \in [0, T],$$

where μ^{x_i}, O^{x_i}, $\alpha^{x_i,T}$ *and* $\beta^{x_i,T}$ *are given in* (4.3.1), (4.3.2), (4.4.12) *and* (4.4.11), *as well as*

$$Corr(\mu_{t,x_i}, \mu_{s,x_j}) = e^{-\theta|t-s|} e^{-\alpha|x_i - x_j|},$$

where $t, s \in [0, T]$ *and* $i, j = 1, \dots, n$. *For the* χ^2*-field defined in* (4.3.12) *we have*

$$Z_t^{x_i,T} = e^{-\int_0^t \mu_s^{x_i} \mathrm{d}s} e^{\alpha^{x_i,T}(t) + \beta^{x_i,T}(t) \mu_t^{x_i}}, \quad t \in [0, T],$$

where μ^{x_i} *is given in* (4.3.12). *Recall that in this case in general the functions* $\alpha^{x_i,T}$ *and* $\beta^{x_i,T}$ *cannot be explicitly computed (see also Remark 4.4.4). Furthermore*

$$Corr(\mu_{t,x_i}, \mu_{s,x_j}) = e^{-2\theta|t-s|} e^{-2\alpha|x_i - x_j|},$$

where $t, s \in [0, T]$ *and* $i, j = 1, \dots, n$.

Example 4.5.2. *Consider a portfolio of pure endowment contracts as defined in* (4.2.8). *Let* $m = 1$ *with* $B = \{x\}$ *and* $n^x = n$, *i.e., all individuals belong to the same age cohort, and assume* $\zeta(x) = 1$. *Then from Proposition 4.4.5 we have that the investment in the (discounted) risky asset* X *is given by*

$$\xi_t^X = F_s(t, S_t)(n^x - N_t^x) e^{\Gamma_t^x} Z_t^{x,T},$$

and the investment in the (discounted) longevity bond Y^x *is given by*

$$\xi_t^{Y^x} = F(t, S_t) e^{r(T-t)} (n^x - N_t^x) e^{\Gamma_t^x},$$

for $t \in [0, T]$. *In the case of the Gaussian intensity field defined in* (4.3.1) *we have*

$$Corr(\mu_{t,x}, \mu_{s,x}) = e^{-\theta|t-s|},$$

for the χ^2*-field defined in* (4.3.12) *we have*

$$Corr(\mu_{t,x}, \mu_{s,x}) = e^{-2\theta|t-s|},$$

for $t, s \in [0, T]$.

Appendices

Appendix A

Risk-Minimization for Payment Processes

The (local) risk-minimization method is a quadratic hedging approach that was first introduced by Föllmer and Sondermann [36] in the case of European type contingent claims and later extended to the case of payment processes by Møller [54] and later Schweizer [59] and Barbarin [4, Chapter 4]. In this section of the appendix for the readers convenience we briefly review all aspects of the theoretical background that are relevant for our purposes. Note that this borrows extensively from Møller [54] and Schweizer [58].

For a finite time horizon $T > 0$ consider a financial market defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ fulfills the usual conditions, consisting of one risk-free asset or numéraire $B = (B_t)_{t\in[0,T]}$, as well as d risky assets $S^i = (S^i_t)_{t\in[0,T]}$, $i = 1, \dots, d$. We denote by $X = (X^1, \dots, X^d)$ the discounted asset prices, where $X^i = S^i/B$, $i = 1, \dots, d$, and we assume that X is a local $\mathbb{P}$-martingale. In particular we assume that the market is arbitrage-free and we are working under a risk-neutral measure, i.e., the measure $\mathbb{P}$ itself belongs to the set of equivalent local martingale measures. In this setting we would like to find a hedging strategy for an $\mathbb{F}$-adapted, square integrable payment process $A = (A_t)_{t\in[0,T]}$, representing cumulative discounted payments up to time t, $t \in [0, T]$. Since the market is not necessarily complete, it is in general not possible to find a self-financing hedging strategy that perfectly replicates the payment process A. In this context the idea of risk-minimization is to relax the self-financing assumption, allowing for a wider class of admissible strategies, and to find an optimal hedging strategy with "minimal risk" within this class of strategies that perfectly replicates A. In the following we now explain how to find the risk-minimizing strategy and explain in what sense this strategy is optimal. We begin with some definitions.

Definition A.0.3. *An* L^2-strategy *is a pair* $\varphi = (\xi, \xi^0) = (\xi^1, \dots, \xi^d, \xi^0)$, *such* *that* $\xi = (\xi^1, \dots, \xi^d)$ *is a d-dimensional process belonging to* $L^2(X)$*, with*

$$L^2(X) := \left\{ \xi \,\middle|\, \xi \ \mathbb{F}\text{-predictable}, \ \left(\mathbb{E}\left[\int_0^T \xi_s' \, \mathrm{d}[X,X]_s \, \xi_s \right] \right)^{1/2} < \infty \right\},$$

and ξ^0 *is a real-valued* $\mathbb{F}$*-adapted process, such that the discounted value process*

$$V_t(\varphi) = \xi_t \cdot X_t + \xi_t^0 := \sum_{i=1}^d \xi_t^i X_t^i + \xi_t^0, \quad t \in [0,T],$$

is right-continuous and square integrable.

For an L^2-strategy φ the *discounted (cumulative) cost process* $C(\varphi)$ is defined as

$$C_t(\varphi) = V_t(\varphi) - \int_0^t \xi_s \, \mathrm{d}X_s + A_t, \quad t \in [0,T],$$

where $\int_0^t \xi_s \, \mathrm{d}X_s := \sum_{i=1}^d \int_0^t \xi_s^i \, \mathrm{d}X_s^i$, describing the accumulated costs of the trading strategy φ during $[0,t]$ including the payments A_t. Note that $V_t(\varphi)$ should therefore be interpreted as the discounted value of the portfolio φ_t held at time t *after* the payments A_t have been made. In particular, $V_T(\varphi)$ is the value of the portfolio upon settlement of all liabilities, and a natural condition is then to restrict to 0-*admissible strategies* satisfying

$$V_T(\varphi) = 0 \quad \mathbb{P}\text{-a.s.}$$

The *risk process* of φ is given by the conditional expected value of the squared future costs

$$R_t(\varphi) = \mathbb{E}[(C_T(\varphi) - C_t(\varphi))^2 \,|\, \mathcal{F}_t], \quad t \in [0,T], \tag{A.0.1}$$

and is taken as a measure of the hedger's remaining risk. We would like to find a trading strategy that minimizes the risk in a sense we define now.

Definition A.0.4. *An* L^2*-strategy* $\varphi = (\xi, \xi^0)$ *is called* risk-minimizing *for the payment stream A, if for any* L^2*-strategy* $\tilde{\varphi} = (\tilde{\xi}, \tilde{\xi}^0)$ *such that* $V_T(\tilde{\varphi}) = V_T(\varphi) = 0$ $\mathbb{P}$*-a.s., we have*

$$R_t(\varphi) \le R_t(\tilde{\varphi}), \quad t \in [0,T],$$

i.e., φ *pointwise minimizes the risk process introduced in* (A.0.1)*.*

The key to finding the strategy with minimal risk is the well-known Galtchouk-Kunita-Watanabe (GKW) decomposition, see Ansel and Stricker [3]. Since A is square integrable, the expected accumulated total payments may be decomposed by use of the GKW decomposition as

$$\mathbb{E}[A_T \,|\, \mathcal{F}_t] = \mathbb{E}[A_T \,|\, \mathcal{F}_0] + \int_{]0,t]} \xi_s^A \, \mathrm{d}X_s + L_t^A, \quad t \in [0,T], \tag{A.0.2}$$

where $\xi^A \in L^2(X)$ and L^A is a square integrable martingale null at 0 that is strongly orthogonal to the space of stochastic integrals with respect to X

$$\mathcal{I}^2(X) := \left\{ \int \psi \, \mathrm{d}X \,\middle|\, \psi \in L^2(X) \right\},$$

i.e., for $\psi \in L^2(X)$, $L^A_t \int_0^t \psi \, \mathrm{d}X$, $t \in [0,T]$, is a (uniformly integrable) martingale.

Theorem A.0.5. *There exists a unique* 0*-admissible risk-minimizing* L^2*-strategy* $\varphi = (\xi, \xi^0)$*, given by*

$$\begin{aligned} \xi_t &:= \xi^A_t, \\ \xi^0_t &:= V_t(\varphi) - \xi_t \cdot X_t, \end{aligned}$$

with discounted value process

$$V_t(\varphi) = \mathbb{E}[A_T \,|\, \mathcal{F}_t] - A_t = \mathbb{E}[A_T \,|\, \mathcal{F}_0] + \int_{]0,t]} \xi_s \, \mathrm{d}X_s + L^A_t - A_t,$$

discounted optimal cost process

$$C_t(\varphi) = \mathbb{E}[A_T \,|\, \mathcal{F}_0] + L^A_t = C_0(\varphi) + L^A_t,$$

and minimal risk process

$$R_t(\varphi) = \mathbb{E}[(L^A_T - L^A_t)^2 \,|\, \mathcal{F}_t],$$

$t \in [0,T]$*, where* ξ^A *and* L^A *are given by* (A.0.2)*.*

Proof. See Schweizer [58] for the single payoff case or Møller [54] and Schweizer [59] for the extension to the case of payment streams. □

Note that the preceding approach relies heavily on the fact that the discounted asset prices are local martingales under the original measure $\mathbb{P}$. In a more general setting, when the discounted asset price is merely required to be a semimartingale under $\mathbb{P}$, one finds the price by following the *local* risk-minimization technique, see Schweizer [59] or Barbarin [4, Chapter 4]. For more information on (local) risk-minimization and other quadratic hedging approaches we would like to refer the interested reader to the survey paper of Schweizer [58].

Appendix B

Affine Diffusion Processes

In this section of the appendix we give a brief review of some aspects of the theory of affine processes that are relevant for this work. Note that this borrows extensively from Section 3 and Appendix A of Biffis [13]. We would also like to refer the interested reader to Duffie et al. [32] and Filipović and Mayerhofer [35]. An affine diffusion process $X = (X_t)_{t\in[0,T]}$ with values in $\mathbb{R}^n$ is a Markov process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ fulfills the usual conditions, solving (in the strong sense) the stochastic differential equation

$$\mathrm{d}X_t = \delta(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}W_t, \quad t \in [0, T],$$

where W is an n-dimensional standard Brownian motion, and $\delta(t, X_t)$ and $\sigma(t, X_t)$ are "affine" in X in the sense that

$$\delta(t, x) = d_0(t) + d_1(t)x,$$

where $d_0 : [0, T] \to \mathbb{R}^n$ and $d_1 : [0, T] \to \mathbb{R}^{n\times n}$ are continuous functions and

$$\left(\sigma(t, x)\sigma(t, x)'\right)_{ij} = (v_0(t))_{ij} + (v_1(t))'_{ij}\, x, \quad i, j = 1, \ldots, n,$$

for continuous functions $v_0 : [0, T] \to \mathbb{R}^{n\times n}$ and $v_1 : [0, T] \to \mathbb{R}^{n\times n\times n}$. Let $c \in \mathbb{C}$, $a, b \in \mathbb{C}^n$ and

$$\Lambda(t, x) = \lambda_0(t) + \lambda_1(t)'x,$$

for $\lambda_0 : [0, T] \to \mathbb{R}$ and $\lambda_1 : [0, T] \to \mathbb{R}^n$ continuous. Under certain technical conditions (see, e.g., Duffie et al. [31]) for $0 \le t \le u \le T$ the following expression holds:

$$\mathbb{E}\left[e^{-\int_t^u \Lambda(s,X_s)\mathrm{d}s} e^{a'X_u} \left(b'X_u + c\right) \,\middle|\, \mathcal{F}_t\right] = e^{\alpha^u(t)+\beta^u(t)'X_t} \left[\hat{\alpha}^u(t) + \hat{\beta}^u(t)'X_t\right] \tag{B.0.1}$$

where α^u and β^u are functions uniquely solving the following ordinary differential equations:

$$\partial_t \beta^u(t) = \lambda_1(t) - d_1(t)'\beta^u(t) - \frac{1}{2}\beta^u(t)'v_1(t)\beta^u(t),$$
$$\partial_t \alpha^u(t) = \lambda_0(t) - d_0(t)'\beta^u(t) - \frac{1}{2}\beta^u(t)'v_0(t)\beta^u(t),$$

and $\hat{\alpha}^u$ and $\hat{\beta}^u$ are functions uniquely solving the following ordinary differential equations:

$$\partial_t \hat{\beta}^u(t) = -d_1(t)'\hat{\beta}^u(t) - \beta^u(t)'v_1(t)\hat{\beta}^u(t),$$
$$\partial_t \hat{\alpha}^u(t) = -d_0(t)'\hat{\beta}^u(t) - \beta^u(t)'v_0(t)\hat{\beta}^u(t),$$

for $t \in [0, u]$ with boundary conditions $\alpha^u(u) = 0$, $\beta^u(u) = a$ and $\hat{\beta}^u(u) = b$, $\hat{\alpha}^u(u) = c$.

Appendix C

Random Fields

In this section of the appendix for the readers convenience we give a brief overview of the theoretical concepts of the theory of random fields that are relevant for our purposes. Note that this borrows extensively from Section 2 of Biffis and Millossovich [14]. A standard reference is Adler [1].

A real-valued *random field* is a collection of random variables $(X_t)_{t\in I}$, with index set $I \subseteq \mathbb{R}^N$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with a collection of distribution functions

$$F_{t_1,\ldots,t_n}(b_1,\ldots,b_n) = \mathbb{P}\left(X_{t_1} \le b_1, \ldots, X_{t_n} \le b_n\right),$$

for $n \in \mathbb{N},\ b_i \in \mathbb{R},\ t_i \in I,\ i = 1,\ldots,n$. Given a square integrable random field $X = (X_t)_{t\in I}$, the *mean function* is defined as $m(t) = \mathbb{E}\left[X_t\right]$, $t \in I$, and the *covariance function* is defined as $c(s,t) = \mathrm{Cov}\left(X_s, X_t\right)$, $s,t \in I$. A square integrable random field X is *homogeneous* or *stationary* if the mean function is independent of t, i.e., $m(t) = m$, $t \in I$, and the covariance function $c(s,t)$ is a function of $t-s$, $s,t \in I$, only. In this case we write $c(h) = c(0,h)$ for $h \in I$.

A *Gaussian random field* is a random field where all finite-dimensional distributions $F_{t_1,\ldots,t_n}$, $n \in \mathbb{N}$ are multivariate normal. Note that a Gaussian random field is completely determined by specifying its mean and covariance functions, and it is well known that given any function $m : I \to \mathbb{R}$ and a symmetric non-negative definite function $c : I \times I \to \mathbb{R}$, it is always possible to construct a Gaussian random field for which m and c are the mean and covariance function, respectively.

A *Brownian sheet* is the natural generalization of a Brownian motion to a multidimensional index set and is defined as the continuous version of a centered Gaussian field $W = (W_t)_{t\in\mathbb{R}_+^N}$ with covariance function

$$c(s,t) = \prod_{i=1}^{N}\left(s_i \wedge t_i\right), \quad t = (t_1,\ldots,t_N),\ s = (s_1,\ldots,s_N) \in \mathbb{R}^N. \tag{C.0.1}$$

In particular we have that for each $i = 1,\ldots,N$ and fixed t_j, $j \ne i$, the process $\left(\prod_{j\ne i} t_j^{-1/2} W_{t_1,\ldots,t_i,\ldots,t_N}\right)_{t_i\in\mathbb{R}_+}$ is a standard Brownian motion.

In many practical applications such as interest rate or credit risk modeling one is often interested in random fields with non-negative values. In this context χ^2-fields as positive transformations of Gaussian random fields have obtained increasing popularity. A χ^2*-field* $Y = (Y_t)_{t\in I}$ with parameter $n \in \mathbb{N}$ is defined as

$$Y_t := (Z_t^1)^2 + \cdots + (Z_t^n)^2, \quad t \in I,$$

where $Z^1, \ldots, Z^n$ are independent, stationary centered Gaussian random fields with common covariance function $c(h)$, $h \in I$, and variance $c(0) = \sigma^2$. For each $t \in I$, the random variable Y_t has χ^2-distribution with n degrees of freedom. It is easily seen that

$$\mathbb{E}\left[Y_t\right] = n\sigma^2, \quad t \in I, \tag{C.0.2}$$

the covariance structure of Y is given by

$$\mathrm{Cov}(Y_s, Y_t) = 2nc^2(s,t) \quad \text{and} \quad \mathrm{Var}(Y_t) = 2n\sigma^4, \tag{C.0.3}$$

for $s, t \in I$, where $c(s,t)$ is the covariance function of the Gaussian fields Z^i, $i = 1, \ldots, n$ (see, e.g., Adler [1]). Note that Y is stationary as a consequence of the stationarity of the Gaussian fields Z^i, $i = 1, \ldots, n$.

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Eidesstattliche Versicherung

(Siehe Promotionsordnung vom 12.07.11, § 8, Abs. 2 Pkt. .5.)

Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

Schreiber, Irene

Name, Vorname

München, 31.10.2012

Ort, Datum

Unterschrift Doktorand/in

Formular 3.2