Topological Set Theories and Hyperuniverses

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Dissertation
an der Fakultät für Mathematik, Informatik und Statistik
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Zusammenfassung

Wir stellen ein neues mengentheoretisches Axiomensystem vor, hinter welchem eine topologische Intuition steht: Die Menge der Teilmengen einer Menge ist eine Topologie auf dieser Menge. Einerseits ist dieses System eine gemeinsame Abschwächung der Zermelo-Fraenkelschen Mengenlehre **ZF**, der positiven Mengenlehre $\mathbf{GPK}^+_\infty$ und der Theorie der Hyperuniversen; andererseits erhält es größtenteils die Ausdruckskraft dieser Theorien und hat dieselbe Konsistenzstärke wie **ZF**. Wir heben das zusätzliche Axiom einer universellen Menge als dasjenige heraus, das die Konsistenzstärke zu der von $\mathbf{GPK}^+_\infty$ erhöht und untersuchen weitere Axiome und Beziehungen zwischen diesen Theorien.

Hyperuniversen sind eine natürliche Klasse von Modellen für Theorien mit einer universellen Menge. Die $\aleph_0$- und $\aleph_1$-dimensionalen Cantorwürfel sind Beispiele von Hyperuniversen mit Additivität $\aleph_0$, da sie homöomorph zu ihrem Exponentialraum sind. Wir beweisen, dass im Bereich der Räume mit überabzählbarer Additivität die entsprechend verallgemeinerten Cantorwürfel diese Eigenschaft nicht haben.

Zum Schluss stellen wir zwei komplementäre Konstruktionen von Hyperuniversen vor, die einige in der Literatur vorkommende Konstruktionen verallgemeinern sowie initiale und terminale Hyperuniversen ergeben.

Summary

We give a new set theoretic system of axioms motivated by a topological intuition: The set of subsets of any set is a topology on that set. On the one hand, this system is a common weakening of Zermelo-Fraenkel set theory **ZF**, the positive set theory $\mathbf{GPK}^+_\infty$ and the theory of hyperuniverses. On the other hand, it retains most of the expressiveness of these theories and has the same consistency strength as **ZF**. We single out the additional axiom of the universal set as the one that increases the consistency strength to that of $\mathbf{GPK}^+_\infty$ and explore several other axioms and interrelations between those theories.

Hyperuniverses are a natural class of models for theories with a universal set. The $\aleph_0$- and $\aleph_1$-dimensional Cantor cubes are examples of hyperuniverses with additivity $\aleph_0$, because they are homeomorphic to their hyperspace. We prove that in the realm of spaces with uncountable additivity, none of the generalized Cantor cubes has that property.

Finally, we give two complementary constructions of hyperuniverses which generalize many of the constructions found in the literature and produce initial and terminal hyperuniverses.

Introduction

The mathematician considers collections of mathematical objects to be objects themselves. A formula $\phi(x)$ with one free variable x divides the mathematical universe into those objects to which it applies and those to which it does not. The collection of all x satisfying $\phi(x)$, the class

$$\{x \mid \phi(x)\}$$

is routinely dealt with as if it were an object itself, a *set*, which can in turn be a member of yet another set. Russell's antinomy is one of several paradoxes which show that this is not possible for all formulas ϕ. Specifically, the class

$$\{x \mid x \notin x\}$$

cannot be a set, or otherwise contradictions arise[1]. An axiomatic set theory can be thought of as an effort to make precise which classes are sets. It simultaneously aims at providing enough freedom of construction for all of classical mathematics and still remain consistent. It therefore must imply that all "reasonable" class comprehensions $\{x \mid \phi(x)\}$ produce sets and explain why $\{x \mid x \notin x\}$ does not.

The answer given by Zermelo-Fraenkel set theory (**ZF**), the most widely used and most deeply studied system of axioms for sets, is the *Limitation of Size Principle*: Only small classes are sets. $x \notin x$ holds true for too many x, and no single set can comprehend all of them. On the other hand, if a is a set, every subclass $\{x \mid \phi(x) \wedge x{\in}a\}$ also is. However, the totality of all mathematical objects, the *universe* $\mathbb{V} = \{x \mid x{=}x\}$, is a proper class in **ZF**. Since $\mathbb{V}$ is of considerable interest for the set theorist, several alternative axiom systems without this perceived shortcoming have been proposed.

W. V. Quine's set theory *New Foundations* (**NF**), probably the most famous one, is built around a different comprehension scheme: The existence of $\{x \mid \phi(x)\}$ is postulated only for *stratified*[2] formulas $\phi(x)$, avoiding circularities like $x \in x$ but still admitting $x = x$. So although the Russell class is a proper class, its superclass $\mathbb{V}$ is a set. An interesting peculiarity of **NF** is that it has been shown by R. B. Jensen in [Jen68] to be consistent relative to **ZF** (and even much weaker theories), but only if one admits *atoms*[3], objects which are not classes. With the condition that every object is a class, there is no known upper bound to its consistency strength.

This thesis is concerned with a third family of set theories originating from yet another possible answer to the paradoxes, or rather from two independent answers:

Firstly, instead of the missing delimitation $x{\in}a$ or the circularity, one might blame the negation in the formula $x \notin x$ for Russell's paradox. The collection of *generalized positive formulas* is recursively defined by several construction steps not including negation. If the existence of $\{x \mid \phi(x)\}$ is stipulated for every generalized positive formula, a beautiful "positive" set theory emerges.

[1] $\{x \mid x{\notin}x\} \in \{x \mid x{\notin}x\}$ if and only if $\{x \mid x{\notin}x\} \notin \{x \mid x{\notin}x\}$, by definition.

[2] A formula is *stratified* if natural numbers $l(x)$ can be assigned to its variables x in such a way that $l(x) = l(y)$ for each subformula $x = y$, and $l(x) + 1 = l(y)$ for every subformula $x \in y$.

[3] also called *urelements*

Secondly, instead of demanding that every class is a set, one might settle for the ability to approximate it by a least superset, a *closure* in a topological sense.

Surprisingly such "topological" set theories tend to prove the comprehension principle for generalized positive formulas, and conversely, in positive set theory, the universe is a topological space. More precisely, the sets are closed with respect to intersections and finite unions, and the universe is a set itself, so the sets represent the closed subclasses of a topology on $\mathbb{V}$. A class is a set if and only if it is topologically closed.

The first model of such a theory was constructed by R. J. Malitz in [Mal76] under the condition of the existence of certain large cardinal numbers. E. Weydert, M. Forti and R. Hinnion were able to show in [Wey89, FH89] that in fact a weakly compact cardinal suffices. In [Ess97] and [Ess99], O. Esser exhaustively answered the question of consistency for a specific positive set theory, $\mathbf{GPK}^+_\infty$ with a choice principle, and showed that it is mutually interpretable with a variant of Kelley-Morse set theory.

All known models of positive set theory are *hyperuniverses*: κ-compact κ-topological Hausdorff spaces homeomorphic to their own hyperspace[4]. These structures have been extensively studied: In [FHL96], M. Forti, F. Honsell and M. Lenisa give several equivalent definitions. Forti and Honsell discovered a much more general construction of hyperuniverses described in [FH96b], yielding among other examples structures with arbitrary given κ-compact subspaces. Finally, O. Esser in [Ess03] identifies the existence of mildly ineffable cardinals as equivalent to the existence of hyperuniverses with a given weight and additivity.

The fact that all known models are hyperuniverses and the concern that axiom schemes given by purely syntactical requirements do not have an immediately clear intuitive meaning, motivate the study of systems of topological axioms rather than theories based on any comprehension scheme, and to axiomatize parts of the notion of a hyperuniverse "from within". In his course "Topologische Mengenlehre" in the summer term 2006 at Ludwig-Maximilians-Universität München, H.-D. Donder gave such a topological set theory, explored its set theoretic and topological consequences and showed that the consistency proofs of positive set theory apply to this system of axioms as well.

Overview

This thesis is divided into two chapters. In the first one we introduce "essential set theory" (**ES**), a theory in which the power set defines a topology on each set, which is not necessarily trivial. A variant of positive set theory and **ZF** both occur as natural extensions of **ES**, and in particular, essential set theory leaves open the question about a universal set, so this can be investigated separately. Also, it allows for atoms and even for the empty class $\emptyset$ being proper – a statement which has topologically connected models. We will show that several basic set theoretic constructions can be carried out in **ES** and in particular the theory of ordinal numbers is still available. We give a criterion for interpretations of **ES** and show that essential set theory with infinity is equiconsistent with several of its extensions, most notably **ZF**. The axiom $\mathbb{V} \in \mathbb{V}$ increases the consistency strength considerably and, together with the

[4] The space whose points are the closed subsets of X, endowed with the Vietoris κ-topology, the coarsest κ-topology such that for each open respectively closed U, the set of all closed subsets of U is open respectively closed.

assumptions that the universe is regular and contains the set of atoms and all unions of sets, proves the comprehension scheme for generalized positive formulas. The first chapter closes with the exploration of the consequences of a very natural choice principle and the beautiful implications of a compactness assumption.

The second chapter takes place in Zermelo-Fraenkel set theory and is concerned with hyperuniverses. As a preparation we give several topological characterizations of mildly λ-ineffable cardinal numbers κ, one of them being that the κ-additive topology on the λ-dimensional Cantor cube is κ-compact. The Cantor space itself as well as its $\aleph_1$-dimensional variant are known to be atomless hyperuniverses. So the question arises whether that is also true for their κ-additive siblings. We define the notion of a space's solidity and use it to show that none of these is homeomorphic to any hyperspace at all. Apart from that, hyperuniverses turn out to be abundant. We give two constructions which lead to plenty of interesting examples. Both of them define functors from very large categories into the category of hyperuniverses, and they complement each other: for any given partially defined hyperuniverse they yield maximal and minimal hyperuniverses completing it.

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Chapter 1

Topological Set Theories

The first chapter takes place within essential, topological and positive set theory and explores the implications of those axiom systems, their interrelations and the ramifications of additional axioms.

1.1 Atoms, sets and classes

We begin by establishing the logical and set theoretic foundations we will use throughout the thesis, in classical as well as alternative set theories. For the sake of clarity we incorporate proper classes into all the theories we consider. This not only enables us to write down many arguments in a more concise yet formally correct way, but it also helps separate the peculiarities of particular theories from the general facts which hold true under very weak common assumptions about atoms, sets and classes. Not repeating these common class axioms explicitly when giving a system of axioms better accentuates the idea specific to that theory.

We use the language of set theory with atoms, whose non-logical symbols are the binary relation symbol $\in$ and the constant symbol $\mathbb{A}$. We say "X is an element of Y" for $X \in Y$. We call X an *atom* if $X \in \mathbb{A}$, and otherwise we call X a *class*. If a class is an element of any other class, it is called a *set*; otherwise it is a *proper class*.

We do not introduce a special symbol to distinguish sets from proper classes. Instead, we denote the objects of our theories – all atoms, sets and classes – by capital letters and adopt the convention to use lowercase letters for sets and atoms only, so:

- $\forall x\ \phi(x)$ means $\forall X.\ (\exists Y\ X{\in}Y) \Rightarrow \phi(X)$ and
- $\exists x\ \phi(x)$ means $\exists X.\ (\exists Y\ X{\in}Y) \wedge \phi(X)$.

For each formula ϕ let ϕ^C be its relativization to the objects which are not proper classes, that is, every quantified variable in ϕ is replaced by a lowercase variable in ϕ^C.

Free variables in formulas that are supposed to be sentences are implicitly universally quantified. For example, we usually omit the outer universal quantifiers in axioms.

Using these definitions and conventions, we can now state the basic axioms concerning atoms, sets and classes. Firstly, we assume that classes are uniquely defined by their extension, that is, two classes are equal iff they have the same elements. Secondly, atoms do not have any elements. Thirdly, there are at least two distinct sets or atoms. And finally, any collection of sets and atoms which can be defined in terms of sets, atoms and finitely many fixed parameters, is a class. Formally:

Extensionality	$(X,Y{\notin}\mathbb{A} \wedge \forall Z.\ Z{\in}X \Leftrightarrow Z{\in}Y) \;\Rightarrow\; X{=}Y$
Atoms	$X \in \mathbb{A} \quad\Rightarrow\quad Y \notin X$
Nontriviality	$\exists x,y \;\; x \neq y$
Comprehension(ψ)	$\exists Z{\notin}\mathbb{A}.\ \forall x.\ x{\in}Z \Leftrightarrow \psi(x,\vec{P})$ for all formulas $\psi = \phi^C$.

We will refer to these axioms as the *class axioms* from now on. Note that the object $\mathbb{A}$ may well be a proper class, or a set. The atoms axiom implies however that $\mathbb{A}$ is not an atom.

We call the axiom scheme given in the fourth line the *weak comprehension* scheme. It can be strengthened by removing the restriction on the formula ψ, instead allowing ψ to be any formula – even quantifying over all classes. Let us call that variant the *strong comprehension* scheme. The axiom of extensionality implies the uniqueness of the class Z. We also write $\{w \mid \psi(w,\vec{P})\}$ for Z, and generally use the customary notation for comprehensions, e.g. $\{x_1,\ldots,x_n\} = \{y \mid y{=}x_1 \vee \ldots \vee y{=}x_n\}$ for the class with finitely many elements $x_1,\ldots,x_n$, $\emptyset = \{w \mid w{\neq}w\}$ for the empty class and $\mathbb{V} = \{w \mid w{=}w\}$ for the universal class. Also let $\mathbb{T} = \{x \mid \exists y.y{\in}x\}$ be the class of nonempty sets. The weak comprehension scheme allows us to define unions, intersections and differences in the usual way.

1.2 Essential Set Theory

Before we can state the axioms of essential set theory, we need to define several topological terms. They all make sense in the presence of only the class axioms, but one has to carefully avoid for now the assumption that any class is a set. Also, the "right" definition of a topology in our context is an unfamiliar one: Instead of the collection of open sets, we consider a topology to be the collection of all nonempty closed sets.

For given classes A and T, we call A T-*closed* if $A = \emptyset$ or $A \in T$. A *topology* on a class X is a class T of nonempty subsets of X, such that:

- X is T-closed.
- $\bigcap B$ is T-closed for every nonempty class $B{\subseteq}T$.
- $a \cup b$ is T-closed for all T-closed sets a and b.

The class X, together with T, is then called a *topological space*. If A is a T-closed class, then its complement $\complement A = X \setminus A$ is T-*open*. A class which is both T-closed and T-open is T-*clopen*. The intersection of all T-closed supersets of a class $A \subseteq X$ is the least T-closed superset and is called the T-*closure* $\mathrm{cl}_T(A)$ of A. Then $\mathrm{int}_T(A) = \complement\mathrm{cl}_T(\complement A)$ is the largest T-open subclass of A and is called the T-*interior* of A. Every A with $x \in \mathrm{int}_T(A)$ is a T-*neighborhood* of the point

x. The explicit reference to T is often omitted and X itself is considered a topological space, if the topology is clear from the context.

If $S \subset T$ and both are topologies, we call S *coarser* and T *finer*. An intersection of several topologies on a set X always is a topology on X itself. Thus for every class B of subsets of X, if there is a coarsest topology $T \supseteq B$, then that is the intersection of all topologies S with $B \subseteq S$. We say that B is a *subbase* for T and that T is *generated* by B.

If $A \subseteq X$, we call a subclass $B \subseteq A$ *relatively closed* in A if there is a T-closed C such that $B = A \cap C$, and similarly for *relatively open* and *relatively clopen*. If every subclass of A is relatively closed in A, we say that A is *discrete*. Thus a T-closed set A is discrete iff all its nonempty subclasses are elements of T. Note that there is an equivalent definition of the discreteness of a class $A \subseteq X$ which can be expressed without quantifying over classes: A is discrete iff it contains none of its accumulation points, where an *accumulation point* is a point $x \in X$ which is an element of every T-closed $B \supseteq A \setminus \{x\}$. Formally, A is discrete iff it has at most one point or:

$$\forall x{\in}A\ \exists b{\in}T.\ \ A \subseteq b{\cup}\{x\}\ \wedge\ x{\notin}b$$

A topological space X is T_1 if for all distinct $x,y \in X$ there exists an open $U \subseteq X$ with $y \notin U \ni x$, or equivalently, if every singleton $\{x\} \subseteq X$ is closed. X is T_2 or *Hausdorff* if for all distinct $x,y \in X$ there exist disjoint open $U,V \subseteq X$ with $x \in U$ and $y \in V$. It is *regular* if for all closed $A \subseteq X$ and all $x \in X \setminus A$ there exist disjoint open $U,V \subseteq X$ with $A \subseteq U$ and $x \in V$. X is T_3 if it is regular and T_1. It is *normal* if for all disjoint, closed $A,B \subseteq X$ there exist disjoint open $U,V \subseteq X$ with $A \subseteq U$ and $B \subseteq V$. X is T_4 if it is normal and T_1.

A map $f : X \to Y$ between topological spaces is *continuous* if all preimages $f^{-1}[A]$ of closed sets $A \subseteq Y$ are closed.

Let $\mathcal{K}$ be any class. We consider a class A to be *$\mathcal{K}$-small* if it is empty or there is a surjection from a member of $\mathcal{K}$ onto A, that is:

$$A = \emptyset \quad \vee \quad \exists x{\in}\mathcal{K}\ \exists F{:}x{\to}A\ F[x]{=}A$$

Otherwise, A is *$\mathcal{K}$-large*. We say *$\mathcal{K}$-few* for "a $\mathcal{K}$-small collection of", and *$\mathcal{K}$-many* for "a $\mathcal{K}$-large collection of". Although we quantified over classes in this definition, we will only use it in situations where there is an equivalent first-order formulation.

If all unions of $\mathcal{K}$-small subclasses of a topology T are T-closed, then T is called *$\mathcal{K}$-additive* or a *$\mathcal{K}$-topology*. If T is a subclass of every $\mathcal{K}$-topology $S \supseteq B$ on X, then T is *$\mathcal{K}$-generated* by B on X and B is a *$\mathcal{K}$-subbase* of T on X. If every element of T is an intersection of elements of B, B is a *base* of T.

A topology T on X is *$\mathcal{K}$-compact* if every T-cocover has a $\mathcal{K}$-small T-subcocover, where a *T-cocover* is a class $B \subseteq T$ with $\bigcap B = \emptyset$. Dually, we use the more familiar term *open cover* for a collection of T-open classes whose union is X, where applicable.

For all classes A and T, let

$$\Box_T A = \{b{\in}T \mid b{\subseteq}A\} \quad \text{and} \quad \Diamond_T A = \{b{\in}T \mid b \cap A \neq \emptyset\}.$$

If T is a topology on X, and if for all $a,b \in T$ the classes $\Box_T a \cap \Diamond_T b$ are sets, then the set $T = \Box_T X = \Diamond_T X$, together with the topology S $\mathcal{K}$-generated by $\{\Box_T a{\cap}\Diamond_T b \mid a,b{\in}T\}$ is called the $\mathcal{K}$-*hyperspace* (or *exponential space*) of X and denoted by $\mathrm{Exp}_{\mathcal{K}}(X,T) = \langle \Box_T X, S\rangle$, or in the short form: $\mathrm{Exp}_{\mathcal{K}}(X)$. Since $\Box_T a = \Box_T a \cap \Diamond_T X$ and $\Diamond_T a = \Box_T X \cap \Diamond_T a$, the classes $\Box_T a$ and $\Diamond_T a$ are also sets and constitute another $\mathcal{K}$-subbase of the exponential $\mathcal{K}$-topology. A notable subspace of $\mathrm{Exp}_{\mathcal{K}}(X)$ is the space $\mathrm{Exp}^c_{\mathcal{K}}(X)$ of $\mathcal{K}$-compact subsets. In fact, this restriction suggests the canonical definition $\mathrm{Exp}^c_{\mathcal{K}}(f)(a) = f[a]$ of a map $\mathrm{Exp}^c_{\mathcal{K}}(f) : \mathrm{Exp}^c_{\mathcal{K}}(X) \to \mathrm{Exp}^c_{\mathcal{K}}(Y)$ for every continuous $f : X \to Y$, because continuous images of $\mathcal{K}$-compact sets are $\mathcal{K}$-compact. Moreover, $\mathrm{Exp}^c_{\mathcal{K}}(f)$ is continuous itself.

Later, $\mathcal{K}$ will usually be a cardinal number, but prior to stating the axioms of essential set theory, the theory of ordinal and cardinal numbers is not available. But to obtain useful ordinal numbers, an axiom stating that the additivity is greater than the cardinality of any discrete set is needed. Fortunately, this can be expressed using the class $\mathcal{D}$ of all discrete sets as the additivity.

Consider the following, in addition to the class axioms:

1st Topology Axiom	$\mathbb{V} \in \mathbb{V}$
2nd Topology Axiom	If $A \subseteq \mathbb{T}$ is nonempty, then $\bigcap A$ is $\mathbb{T}$-closed.
3rd Topology Axiom	If a and b are $\mathbb{T}$-closed, then $a \cup b$ is $\mathbb{T}$-closed.
T_1	$\{a\}$ is $\mathbb{T}$-closed.
Exponential	$\Box_{\mathbb{T}} a \cap \Diamond_{\mathbb{T}} b$ is $\mathbb{T}$-closed.
Discrete Additivity	$\bigcup A$ is $\mathbb{T}$-closed for every $\mathcal{D}$-small class A.

We call this system of axioms *topological set theory*, or in short: **TS**, and the theory **TS** without the 1st topology axiom *essential set theory* or **ES**. During the course of this chapter, we will mostly work in **ES** and explicitly single out the consequences of $\mathbb{V} \in \mathbb{V}$.

Thus in **ES**, the class $\mathbb{T} = \Box_{\mathbb{T}}\mathbb{V} = \Diamond_{\mathbb{T}}\mathbb{V}$ of all nonempty sets satisfies all the axioms of a topology on $\mathbb{V}$, except that it does not need to contain $\mathbb{V}$ itself. Although it is not necessarily a class, we can therefore consider the collection of $\mathbb{V}$ and all nonempty sets a topology on $\mathbb{V}$ and informally attribute topological notions to it. We will call it the *universal topology* and whenever no other topology is explicitly mentioned, we will refer to it. Since no more than one element distinguishes the universal topology from $\mathbb{T}$, any topological statement about it can easily be reformulated as a statement about $\mathbb{T}$ and hence be expressed in our theory. Having said this, we can interpret the third axiom as stating that the universe is a T_1 space.

Alternatively one can understand the axioms without referring to collections outside the theory's scope as follows: Every set a carries a topology $\Box a$, and a union of two sets is a set again. Then the T_1 axiom says that all sets are T_1 spaces (and that all singletons are sets) and the fourth says that every set's hyperspace exists.

If $\mathbb{V}$ is not a set, we cannot interpret the exponential axiom as saying that the universe's hyperspace exists! Since $\Box a = \Box a \cap \Diamond a$, it implies the power set axiom, but it does not imply the sethood of $\Diamond a$ for every set a.

A very handy implication of the 2nd topology axiom and the exponential axiom is that for all sets b, c and every class A,

$$\{x \in c \mid A \subseteq x \subseteq b\} \quad = \quad \begin{cases} c \cap \bigcap_{y \in A}(\Box b \cap \Diamond\{y\}) & \text{if } A \neq \emptyset. \\ (c \cap \Box b) \cup (c \cap \mathbb{A}) & \text{if } A = \emptyset. \end{cases}$$

is closed, given that $c \cap \mathbb{A}$ is closed or A is nonempty.

An important consequence of the T_1 axiom is that for each natural number[1] n, all classes with at most n elements are discrete sets. In particular, pairs are sets and we can define ordered pairs as Kuratowski pairs $\langle x,y\rangle = \{\{x\},\{x,y\}\}$. We adopt the convention that the $n+1$-tuple $\langle x_1,\ldots,x_{n+1}\rangle$ is $\langle\langle x_1,\ldots,x_n\rangle,x_{n+1}\rangle$ and that relations and functions are classes of ordered pairs. With these definitions, all functional formulas ϕ^C on sets correspond to actual functions, although these might be proper classes. We denote the $\in$-relation for sets by $\mathbf{E} = \{\langle x,y\rangle \mid x{\in}y\}$, and the equality relation by $\Delta = \{\langle x,y\rangle \mid x{=}y\}$. Also, we write Δ_A for the equality $\Delta \cap A^2$ on a class A.

We have not yet made any stronger assumption than T_1 about the separation properties of sets. However, many desirable set-theoretic properties, particularly with respect to Cartesian products, apply only to Hausdorff sets, that is, sets whose natural topology is T_2.

We denote by $\square_{<n}A$ the class of all $b \subseteq A$ with less than n elements. Given $t_1,\ldots,t_m \in \{1,\ldots,n\}$. We define:

$$F_{n,t_1,\ldots,t_m} : \mathbb{V}^n \to \mathbb{V}^m, F_{n,t_1,\ldots,t_m}(x_1,\ldots,x_n) = \langle x_{t_1},\ldots,x_{t_m}\rangle$$

With the corresponding choice of $t_1,\ldots,t_m$, all projections and permutations can be expressed in this way.

For a set a, let a' be its *Cantor-Bendixson derivative*, the set of all its accumulation points, and let $a_I = a \setminus a'$ be the class of all its isolated points.

Proposition 1 (ES). Let a and b be Hausdorff sets and $a_1,\ldots,a_n \subseteq a$.

1. $\square_{<n}a$ is a Hausdorff set.
2. The Cartesian product $a_1 \times \ldots \times a_n$ is a Hausdorff set, too, and its universal topology is at least as fine as the product topology.
3. Every continuous function $F : a_1 \to a_2$ is a set.
4. For all $t_1,\ldots,t_m \in \{1,\ldots,n\}$, the function

$$F_{n,t_1,\ldots,t_m} \restriction a^n : a^n \to a^m$$

is a Hausdorff set. It is even closed with respect to the product topology of a^{n+m}.

5. For each $x \in a_I$, let $b_x \subseteq b$. Then for every map $F : a_I \to b$, the class $F \cup (a' \times b)$ is $\mathbb{T}$-closed. Moreover,

$$\prod_{x\in a_I} b_x = \left\{F \cup (a' \times b) \mid F : a_I \to \mathbb{V}, \forall x\, F(x) \in b_x\right\}$$

is $\mathbb{T}$-closed and its natural topology is at least as fine as its product topology.

[1] Until we have defined them in essential set theory, we consider natural numbers to be metamathematical objects.

Proof. (1): To show that it is a set it suffices to prove that it is a closed subset of the set $\Box a$, so assume $b \in \Box a \setminus \Box_{<n} a$. Then there exist distinct $x_1,\dots,x_n \in b$, which by the Hausdorff axiom can be separated by disjoint relatively open $U_1,\dots,U_n \subseteq a$. Then $\Diamond U_1 \cap \dots \cap \Diamond U_n \cap \Box a$ is a relatively open neighborhood of b disjoint from $\Box_{<n} a$.

Now let $b,c \in \Box_{<n} a$ be distinct sets. Wlog assume that there is a point $x \in b \setminus c$. Since c is finite and a satisfies the Hausdorff axiom, there is a relatively open superset U of c and a relatively open $V \ni x$, such that $U \cap V = \emptyset$. Now $\Diamond V \cap \Box_{<n} a$ is a neighborhood of b and $\Box U \cap \Box_{<n} a$ is a neighborhood of c in $\Box_{<n} a$, and they are disjoint. Hence $\Box_{<n} a$ is Hausdorff.

(2): It suffices to prove that $a \times a$ is a set and carries at least the product topology, because then it follows inductively that this is also true for a^n with $n \geqslant 2$. And from this in turn it follows that $a_1 \times \dots \times a_n$ is closed in a^n and carries the subset topology, which implies the claim.

Since a^2 contains exactly the sets of the form $\{\{x\},\{x,y\}\}$ with $x,y \in a$, it is a subclass of the set $s = \Box_{\leqslant 2}\Box_{\leqslant 2} a \cap \Diamond\Box_{\leqslant 1} a$ and we only have to prove that it is closed in s. So let $c \in s \setminus a^2$. Then $c = \{\{x\},\{y,z\}\}$ with $x \notin \{y,z\}$ and $x,y,z \in a$. Since a is Hausdorff, there are disjoint $U \ni x$ and $V \ni y,z$ which are relatively open in a. Then $s \cap \Diamond\Box_{\leqslant 1} U \cap \Diamond\Box_{\leqslant 2} V$ is relatively open in s, and is a neighborhood of c disjoint from a^2.

It remains to prove the claim about the product topology, that is, that for every subset $b \subseteq a$, $b \times a$ and $a \times b$ are closed, too. The first one is easy, because $b \times a = a^2 \cap \Diamond\Box_{\leqslant 1} b$. Similarly, $(b \times a) \cup (a \times b) = a^2 \cap \Diamond\Diamond b$, so in order to show that $a \times b$ is closed, let $c \in (b \times a) \cup (a \times b) \setminus (a \times b)$, that is, $c = \{\{x\},\{x,y\}\}$ with $y \notin b$ and $x \in b$. Since a is Hausdorff, there are relatively open disjoint subsets $U \ni x$ and $V \ni y$ of a. Then $s \cap \Diamond\Box_{\leqslant 1} U \cap \Diamond\Diamond(V \setminus b)$ is a relatively open neighborhood of c disjoint from $a \times b$.

(3): Let $F : a_1 \to a_2$ be continuous and $\langle x,y \rangle \in a_1 \times a_2 \setminus F$, that is, $F(x) \neq y$. Then $F(x)$ and y can be separated by relatively open subsets $U \ni F(x)$ and $V \ni y$ of a_2, and since F is continuous, $F^{-1}[U]$ is relatively open in a_1. $F^{-1}[U] \times V$ is a neighborhood of $\langle x,y \rangle$ and disjoint from F. This concludes the proof that F is relatively closed in $a_1 \times a_2$ and hence a set.

(4): Let $F = F_{n,t_1,\dots,t_m}$. Then $F \subseteq a^n \times a^m \in \mathbb{V}$, so we only have to find for every

$$b = \langle\langle x_1,\dots,x_n\rangle,\langle y_1,\dots,y_m\rangle\rangle, \text{ such that } x_{t_k} \neq y_k \text{ for some } k,$$

a neighborhood disjoint from F. By the Hausdorff property, there are disjoint relatively open $U \ni x_{t_k}$ and $V \ni y_k$. Then

$$\left(a^{t_k - 1} \times U \times a^{n - t_k}\right) \times \left(a^{k-1} \times V \times a^{m-k}\right)$$

is such a neighborhood.

(5): Firstly,

$$F \cup (a' \times b) \quad = \quad \bigcap_{x \in a_I} \left(\{\langle x, F(x)\rangle\} \ \cup \ ((a \setminus \{x\}) \times b)\right)$$

is a set for any such function F.

Secondly, the claim about the product topology follows as soon as we have demonstrated the product to be $\mathbb{T}$-closed, because the product topology is generated by classes of the form $\prod_{x \in a_I} c_x$, where $c_x \subseteq b_x$ and only finitely many c_x differ from b_x.

Since $a \times b$ is $\mathbb{T}$-closed and the product $P = \prod_{x \in a_I} b_x$ is a subset of $\Box(a \times b)$, it suffices to show that P is relatively closed in $\Box(a \times b)$, so let $r \in \Box(a \times b) \setminus P$. There are four cases:

- The domain of r is not a. Then there is an $x \in a$ such that $x \notin \mathrm{dom}(r)$. In that case, $\Box(a \times b) \cap \Diamond(\{x\} \times b)$ is a closed superset of P omitting r.
- $a' \times b \nsubseteq r$. Then some $\langle x, y \rangle \in a' \times b$ is missing and $\Box(a \times b) \cap \Diamond\{\langle x, y \rangle\}$ is a corresponding superset of P.
- $r \upharpoonright a_I$ is not a function. Then there is an $x \in a_I$, such that there exist distinct $\langle x, y_0 \rangle, \langle x, y_1 \rangle \in r$. Since b is Hausdorff, there are closed $u_0, u_1 \subseteq b$, such that $u_0 \cup u_1 = b$, $y_0 \notin u_0$ and $y_1 \notin u_1$. Then P is a subclass of

  $$\Box(\{x\} \times u_0 \ \cup\ (a \backslash \{x\}) \times b) \quad \cup \quad \Box(\{x\} \times u_1 \ \cup\ (a \backslash \{x\}) \times b),$$

  which does not contain r.
- $F = r \upharpoonright a_I$ is a function, but $F(x) \notin b_x$ for some $x \in a_I$. Then

  $$\Box(\{x\} \times b_x \ \cup\ (a \backslash \{x\}) \times b)$$

  is a closed superclass of P omitting r.

Thus for every $r \in \Box(a \times b) \setminus P$, there is a closed superclass of P which does not contain r. Therefore P is closed. ■

The additivity axiom states that the universe is $\mathcal{D}$-additive, that is, that the union of a discrete set's image is $\mathbb{T}$-closed. In other words: For every function F whose domain is a discrete set, the union of the range $\bigcup \mathrm{rng}(F)$ is a set or empty. Had we opted against proper classes, the additivity axiom therefore could have been expressed as an axiom scheme.

Even without a choice principle, we could equivalently have used injective functions into discrete sets instead of surjective functions defined on discrete sets: Point (2) in the following proposition is exactly the additivity axiom.

Proposition 2. In **ES** without the additivity axiom, the following are equivalent:

1. Images of discrete sets are sets, and unions of discrete sets are $\mathbb{T}$-closed.
2. If d is discrete and $F : d \to A$ surjective, then $\bigcup A$ is $\mathbb{T}$-closed.
3. If d is discrete and $F : A \hookrightarrow d$ injective, then $\bigcup A$ is $\mathbb{T}$-closed.

Proof. (1) $\Rightarrow$ (2): If images of discrete sets are sets, then they are discrete, too, because all their subsets are images of subsets of a discrete set. Thus $F[d]$ is discrete, and therefore its union $\bigcup F[d]$ is closed.

(2) $\Rightarrow$ (1): If d is discrete and F is a function, consider the function $G : \mathrm{dom}(F) \to \mathbb{V}$ defined by $G(x) = \{F(x)\}$. Then $F[d] = \bigcup G[d] \in \mathbb{V}$. Applying (2) to the identity proves that $\bigcup d = \bigcup \mathrm{id}[d]$ is closed.

(2) $\Rightarrow$ (3): If $F : A \hookrightarrow d$ is an injection, then $F^{-1} : F[A] \to A$ is a surjection from the discrete subset $F[A] \subseteq d$ onto A, so $\bigcup A$ is closed.

(3) $\Rightarrow$ (2): First we show that $\Box d$ is discrete. We have to show that any given $a \in \Box d$ is not an accumulation point, i.e. that $\Box d \setminus \{a\}$ is closed. Since a is a discrete set, every $d \setminus \{b\}$ for $b \in a$ is closed, as well as $d \setminus a$. But

$$\Box d \setminus \{a\} = \Box d \cap \left(\Diamond(d \setminus a) \cup \bigcup_{b \in a} \Box(d \setminus \{b\}) \right)$$

and this union can be seen to be closed by applying (3) to the map

$$F : \{\Box(d \setminus \{b\}) \mid b \in a\} \hookrightarrow a, \quad F(\Box(d \setminus \{b\})) = b.$$

Now we can prove (2):

Let $G : d \to \mathbb{V}$. Then $F : G[d] \to \Box d, F(x) = G^{-1}[\{x\}]$ is an injective function from $G[d]$ to the discrete set $\Box d$. Therefore, $\bigcup G[d] \in \mathbb{V}$. ∎

Proposition 3 (ES). $\Box d$ is discrete for every discrete set d. Every $\mathcal{D}$-small nonempty class is a discrete set and every nonempty union of $\mathcal{D}$-few discrete sets is a discrete set.

Proof. The first claim has already been shown in the proof of Proposition 2.

Let A be $\mathcal{D}$-small and $B \subseteq A$. Then B and $\widetilde{B} = \{\{b\} \mid b \in B\}$ are also $\mathcal{D}$-small. Therefore $\bigcup \widetilde{B} = B$ is $\mathbb{T}$-closed by the additivity axiom.

Finally, let A be $\mathcal{D}$-small and let every $a \in A$ be a discrete set. We have to show that every nonempty $B \subseteq \bigcup A$ is a set. But if A is $\mathcal{D}$-small, the class C of all nonempty sets of the form $B \cap a$ with $a \in A$ also is. Since $B \neq \emptyset$ and every $a \in A$ is discrete, the union of C is in fact B. ∎

1.3 Ordinal Numbers

We do not assume that the empty class is a set, so there may be no well-founded sets at all, yet of course we want to define the natural numbers and later we will even be looking for an interpretation of a well-founded theory. To this end we need suitable variants of the concepts of well-foundedness and von Neumann ordinal numbers.

Our starting point is finding a substitute for the empty set: A class or atom 0 is called a *zero* if no element of 0 is a superset of 0. Zeros exist in $\mathbb{V}$: By the nontriviality axiom, there are distinct $x, y \in \mathbb{V}$, so we can set $0 = \{\{x\}, \{y\}\}$. But in many interesting cases, there even is a definable zero: Let us set $0 = \emptyset$ if $\emptyset \in \mathbb{V}$, and if $\emptyset \notin \mathbb{V}$ but $\mathbb{V} \in \mathbb{V}$, we set $0 = \{\{\mathbb{V}\}\}$ (its element $\{\mathbb{V}\}$ is not a superset of 0, because by the nontriviality axiom $\mathbb{V}$ is not a singleton). Note that all these examples are sets with at most two elements.

Given a fixed zero 0, we make the following definitions:

$$\begin{array}{rcl}
A^{\oplus} & = & A \setminus 0 \\
A \in_0 B & \text{if} & A \in B^{\oplus} \quad \text{and} \quad 0 \subseteq B. \\
A \text{ is 0-}\textit{transitive} & \text{if} & c \in_0 A \quad \text{for all} \quad c \in_0 b \in_0 A. \\
\text{A 0-transitive } a \text{ is 0-}\textit{pristine} & \text{if} & 0 \subseteq c \notin \mathbb{A} \quad \text{for all} \quad c \in_0 a \cup \{a\}. \\
\alpha \text{ is a 0-}\textit{ordinal number} & \text{if} & \alpha \text{ is 0-transitive, 0-pristine and} \\
 & & \alpha^{\oplus} \text{ is strictly well-ordered by } \in_0,
\end{array}$$

where by a (strict) *well-order* we mean a (strict) linear order such that each nonempty sub*set* has a minimal element. A (strict) order with the property that every sub*class* has a minimal element is called a (strict) *strong well-order*, and we will see shortly that in fact such $\alpha^{\oplus}$ are strictly strongly well-ordered.

We denote the class of 0-ordinals by On_0 and the 0-ordinals themselves by lowercase greek letters. If α and β are 0-ordinals, we also write $\alpha \leqslant_0 \beta$ for $\alpha \subseteq \beta$. A 0-ordinal $\alpha \neq 0$ is a 0-*limit ordinal* if it is not the immediate $\leqslant_0$-successor of another 0-ordinal, and it is a 0-*cardinal number* if there is no surjective map from $\beta^{\oplus}$ onto $\alpha^{\oplus}$ for any $\beta <_0 \alpha$. If there is a least 0-limit ordinal distinct from 0 itself, we call it ω_0, otherwise we define $\omega_0 = On_0$. Its predecessors $n \in_0 \omega_0$ are the 0-*natural numbers*. Obviously 0 is the least 0-ordinal, if $0 \in \mathbb{V}$.

For the remainder of this section, let us assume that our 0 is an atom or a finite set. Unless there is danger of confusion (as in the case of $\in_0$), we omit the prefix and index 0.

Proposition 4 (ES). Let $\alpha \in On$.

1. $\alpha \notin \alpha$, α is discrete and $\alpha = 0 \cup \{\beta \in On \mid \beta \in_0 \alpha\}$.
2. On is strictly strongly well-ordered by $\in_0$ and $<$, and these orders coincide.
3. $\alpha \cup \{\alpha\}$ is the unique immediate successor of α.
4. If A is a nonempty class of ordinals and $\bigcup A \in \mathbb{V}$, then $\bigcup A$ is an ordinal and the least upper bound of A.
5. $\bigcup On = On \cup 0 \notin \mathbb{V}$

Proof. (1): Since $0 \subseteq a$, the equality follows if we can prove that every $x \in_0 \alpha$ is an ordinal. Firstly, let $c \in_0 b \in_0 x$. Then $b \in_0 \alpha$ and $c \in_0 \alpha$ by transitiviy of α. Since $\alpha^{\oplus}$ is strictly linearly ordered by $\in_0$, it follows that $c \in_0 x$, proving that x is transitive. Again by the transitivity of α, we see that $x \subseteq \alpha$, and as a subset of a well-ordered set, $x^{\oplus}$ is well-ordered itself. Also, every $c \in_0 x \cup \{x\}$ is an element of $\alpha^{\oplus}$ and therefore a superset of 0 not in $\mathbb{A}$, so x is pristine.

Since α is a superset of 0, $\alpha \notin 0$. Thus if α were an element of α, it would be in $\alpha^{\oplus}$. But $\alpha \in_0 \alpha$ contradicts the condition that the elements of $\alpha^{\oplus}$ are strictly well-ordered.

Because 0 is a discrete set and $\alpha^{\oplus} = \{x \in \alpha \mid 0 \subseteq x \subseteq \alpha\}$ is closed, it suffices to show that $\alpha^{\oplus}$ is discrete. So let $\gamma \in_0 \alpha$. Since the elements of $\alpha^{\oplus}$ are strictly linearly ordered, every

$\delta \in \alpha^{\oplus} \setminus \{\gamma\}$ is either a predecessor or a successor of γ. Hence

$$\alpha^{\oplus} \setminus \{\gamma\} \quad = \quad \gamma^{\oplus} \cup \{x \in \alpha^{\oplus} \mid \{\gamma\} \subseteq x \subseteq \alpha\}$$

is closed.

(2): If $\alpha \in_0 \beta$, then by transitivity of β, α is a subset of β and because $\alpha \notin_0 \alpha$, it is a proper one. For the converse assume $\alpha < \beta$, that is, $\alpha \subset \beta$. $\beta^{\oplus}$ is discrete and well-ordered, so the nonempty subset $\beta \setminus \alpha$ contains a minimal element δ, which by (1) is an ordinal number. For all $\gamma \in_0 \delta$, it follows from the minimality of δ that $\gamma \in_0 \alpha$. Now let $\gamma \in_0 \alpha$. Then $\gamma \in_0 \beta$ and since β is linearly ordered, γ is comparable with δ. But if $\delta \in_0 \gamma$, then $\delta \in_0 \alpha$ by transitivity, which is false. Hence $\gamma \in_0 \delta$. We have shown that δ and α have the same predecessors, so by (1), they are equal. Thus $\alpha = \delta \in_0 \beta$ and so the orders $\in_0$ and $<$ coincide on the ordinals.

Next we show that ordinals $\alpha, \beta \in On$ are always subsets of each other and hence On is linearly ordered, so assume they are not. Let α_0 be minimal in $\alpha \setminus \beta$ and β_0 in $\beta \setminus \alpha$. Now all predecessors of α_0 must be in $\alpha \cap \beta$. And since α and β are transitive, $\alpha \cap \beta$ is an initial segment and therefore every element of $\alpha \cap \beta$ is also in α_0. The same argument applied to β_0 shows that $\alpha_0 = \alpha \cap \beta = \beta_0$, contradicting our assumption.

Finally, given a nonempty subclass $A \subseteq On$, let $\alpha \in A$ be arbitrary. Then either α has no predecessor in A and thus is minimal itself, or $\alpha \cap A$ is nonempty and has a minimal element δ, because $\alpha^{\oplus}$ is well-ordered and discrete and $\alpha \cap A \subseteq \alpha^{\oplus}$. For every $\gamma \in A \setminus \alpha$, we then have $\delta < \alpha \leqslant \gamma$. Hence δ is in fact minimal in A, concluding the proof that On is strongly well-ordered.

(3): First we verify that $\beta = \alpha \cup \{\alpha\}$ is an ordinal. Since α is transitive, β also is. Since α is pristine and $0 \subseteq \beta \notin \mathbb{A}$, β is pristine itself. And $\beta^{\oplus}$ is a set of ordinal numbers, which by (2) must be well-ordered.

From $\alpha \notin \alpha$ it follows that in fact $\beta \neq \alpha$ and thus $\beta > \alpha$. If $\gamma < \beta$, then $\gamma \in_0 \beta$, so either $\gamma \in_0 \alpha$ or $\gamma = \alpha$, which shows that β is an immediate successor. Since the ordinals are linearly ordered, it is the only one.

(4): As a union of transitive, pristine, well-founded sets, $\bigcup A$ is transitive, pristine and well-founded itself. Since all its predecessors are ordinals, they are strictly well-ordered by (2), so it is an ordinal itself. For each $\beta \in A$, $\beta \subseteq \bigcup A$ and thus $\beta \leqslant \bigcup A$, so it is an upper bound of A. If $\beta < \bigcup A$, there is an element $\gamma \in A$ with $\beta < \gamma$, therefore it is the least upper bound.

(5): By (1), every element x of an ordinal is in $0 \cup On$. Conversely, 0 is an ordinal and by (3), every ordinal is an element of its successor. Therefore, $0 \cup On = \bigcup On$. If $\bigcup On$ were a set, so would $\bigcup On \cup \{\bigcup On\}$ be. But by (4), that would be an ordinal strictly greater than all elements of On, which is a contradiction. ■

These features of On are all quite desirable, and familiar from Zermelo-Fraenkel set theory. Just as in **ZF**, On (or rather $On \cup 0$) resembles an ordinal number itself, except that it is not a set. But in **ZF**, On even has the properties of a regular limit cardinal – a consequence of the replacement axiom. Also, our dependence on the choice of a specific set 0 is rather irritating. This is where the additivity axiom comes in. In the context of ordinal numbers (and discrete sets in general), it is the appropriate analog to the replacement axiom.

By the usual argument, all strongly well-ordered classes whose initial segments are discrete sets are comparable with respect to their length: There is always a unique isomorphism from one of them to an initial segment of the other. In particular, for all finite zeros $0, \widetilde{0} \in \mathbb{V}$, the well-orders of On_0 and $On_{\widetilde{0}}$ are comparable. But if $A \subseteq On_0$ is an initial segment isomorphic to $On_{\widetilde{0}}$, then in fact $A = On_0$, because otherwise A would be a discrete set and by the additivity axiom, $On_{\widetilde{0}} \in \mathbb{V}$, a contradiction. Hence On_0 and $On_{\widetilde{0}}$ are in fact isomorphic and the choice of 0 is not relevant to our theory of ordinal numbers. Also, ω_0 and $\omega_{\widetilde{0}}$ are equally long and we can define a class A to be *finite* if there is a bijection from $n^\oplus$ to A for some natural number n. Otherwise it is *infinite*. It is easy to prove that this definition is equivalent to A being the image of some $n^\oplus$ or embeddable into some $n^\oplus$. Also, it can be stated without quantifying over classes, because such a bijection is defined on a discrete set and therefore a discrete set itself.

Even if there is no limit ordinal, there might still be infinite sets – they just cannot be discrete. So the proper axiom of infinity in the context of essential set theory is the existence of a limit ordinal number:

$$\textit{Infinity} \qquad \omega \in \mathbb{V}$$

We add the axiom of infinity to a theory by indexing it with the symbol ∞.

Using induction on ordinal numbers, one easily proves that for each $\alpha \in On$, the least ordinal $\kappa \in On$ such that there is a surjection from $\kappa^\oplus$ to $\alpha^\oplus$ is a cardinal, and there is a bijection from $\kappa^\oplus$ to $\alpha^\oplus$.

Proposition 5 (ES). *On* is a regular limit, that is:

1. Every function $F : \alpha^\oplus \to On$ is bounded.
2. The class of cardinal numbers is unbounded in *On*.

Proof. (1): By the additivity axiom, $\bigcup F[\alpha^\oplus]$ is a discrete set, so by Proposition 4, it is an ordinal number and an upper bound of $F[\alpha^\oplus]$.

(2): Let us show that for each α there exists a cardinal $\nu > \alpha$. This goes by the usual argument: Every well-order $R \subseteq \alpha^\oplus \times \alpha^\oplus$ on a subset of $\alpha^\oplus$ is a subclass of the discrete set □□$\alpha^\oplus$, so it is a set itself and since $\alpha^\oplus$ is discrete, it is even a strong well-order. Recursively, isomorphisms from initial segments of $\alpha^\oplus$ with respect to R to initial segments of *On* can be defined, and their union is a function from $\alpha^\oplus$ onto some $\beta^\oplus$. We call β the *order type* of R. Now the class A of all well-orders of α is a subclass of □□□α and hence also a discrete set. Mapping every element of A to its order type must therefore define a bounded map $F : A \to On$. Let $\nu = \min(On \setminus \bigcup F[A])$ be the least ordinal which is not an order type of any subset of $\alpha^\oplus$. We show that ν is a cardinal above α. Firstly, α is the order type of a well-order of $\alpha^\oplus$, so $\nu > \alpha$. Secondly, assume that $g : \gamma^\oplus \to \nu^\oplus$ is surjective and $\gamma < \nu$. Then this defines a well-order on $\gamma^\oplus$ of order-type at least ν, and since γ is the order type of a well-order on some subset of $\alpha^\oplus$ by definition, g would define a well-order on a subset of $\alpha^\oplus$ of order-type ν, a contradiction. ■

If $\mathbb{V} \notin \mathbb{V}$, the closure of *On* may well be all of $\mathbb{V}$ and in particular does not have to be a set. But in the case $\mathbb{V} \in \mathbb{V}$, the fact that all $\Diamond a$ are sets determines the closure Ω of $0 \cup On = \bigcup On$ much more precisely. Moreover, *On* then resembles a weakly compact cardinal, which will in fact turn out to be crucial for the consistency strength of the axiom $\mathbb{V} \in \mathbb{V}$.

Proposition 6 (TS).

1. Every sequence $\langle x_\alpha \mid \alpha{\in}On \rangle$ of length *On* has an accumulation point.
2. Every monotonously $\subseteq$-decreasing sequence $\langle x_\alpha \mid \alpha{\in}On \rangle$ of nonempty sets converges to $\bigcap_{\alpha \in On} x_\alpha$. And every monotonously $\subseteq$-increasing one to $\mathrm{cl}\big(\bigcup_{\alpha \in On} x_\alpha\big)$.
3. $\Omega = 0 \cup On \cup \{\Omega\}$
4. $P = \{x \mid 0 \cup \{\Omega\} \subseteq x \subseteq \Omega\}$ is a *perfect set*, that is, $P' = P \neq \emptyset$.
5. *On* has the tree property, that is: If

$$T \quad \subseteq \quad \{f : \alpha^{\oplus} \to \mathbb{V} \mid \alpha \in On\}$$

such that $T_\alpha = \{f{\restriction}\alpha^{\oplus} \mid f{\in}T,\ \alpha^{\oplus}{\subseteq}\mathrm{dom}(f)\}$ is discrete and nonempty for each ordinal $\alpha > 0$, then there is a $G : On \to \mathbb{V}$ such that:

$$G \restriction \alpha^{\oplus} \in T_\alpha \quad \text{for every} \quad \alpha \in On.$$

Proof. (1): Assume that there is no accumulation point. Then every point $y \in \mathbb{V}$ has a neighborhood U such that $\{\alpha \mid x_\alpha{\in}U\}$ is bounded in *On* and therefore discrete. Since the class $\{x_\alpha \mid x_\alpha{\in}U\}$ of members in U is the image of $\{\alpha \mid x_\alpha{\in}U\}$, it is also a discrete set and does not have y as its accumulation point. It follows that firstly, $\{\alpha \mid x_\alpha{=}y\}$ is discrete for each y, and secondly, the image $\{x_\alpha \mid \alpha{\in}On\}$ of the sequence is also discrete. But *On* is the union of the sets $\{\alpha \mid x_\alpha{=}y\}$ for $y \in \{x_\alpha \mid \alpha{\in}On\}$. Since $\mathcal{D}$-small unions of discrete sets are discrete sets, this would imply that *On* is a discrete set, a contradiction.

(2): First let the sequence be decreasing. Then for every $y \in \bigcap_\alpha x_\alpha$, every member of the sequence lies in the closed set $\Diamond\{y\}$, so all its accumulation points do. Now let $y \notin \bigcap_\alpha x_\alpha$. Then there is a $\beta \in On$ such that $y \notin x_\beta$, and hence from x_β on, all members are in $\Box x_\beta$, so all accumulation points are. Thus the only accumulation point is the intersection. (Note that the intersection therefore is nonempty because $\Diamond\mathbb{V}$ is a closed set containing every member of the sequence.)

Now assume that the sequence is ascending and let A be its union. If $y \in A$, then $y \in x_\beta$ for some $\beta \in On$. Then all members from x_β on are in $\Diamond\{y\}$, so each accumulation point also is. Thus all accumulation points are supersets of A. But all members of the sequence are in $\Box\mathrm{cl}(A)$, so each accumulation point is a subset of $\mathrm{cl}(A)$, and therefore equal to $\mathrm{cl}(A)$.

(3): It suffices to prove that Ω is the unique accumulation point of *On*. Since *On* is the image of an increasing sequence, its accumulation point is indeed unique and is the closure of $\bigcup On$ by (2). But $\mathrm{cl}(\bigcup On) = \mathrm{cl}(0 \cup On) = \Omega$.

(4): P is closed, and it is nonempty because $\Omega \in P$. Given $x \in P$, the sequences in P given by

$$y_\alpha \;=\; x \setminus (On \setminus \alpha^{\oplus}) \quad \text{and} \quad z_\alpha \;=\; x \cup (On \setminus \alpha^{\oplus})$$

both converge to x by (2). If $x \cap On$ is unbounded, x is not among the y_α, otherwise it is not among the z_α, so in any case, x is the limit of a nontrivial sequence in P.

(5): Since for every $\alpha \in On$, T_α is nonempty, there is for every α an $f \in T$ with $\alpha^\oplus \subseteq \mathrm{dom}(f)$. Thus the map

$$T \to On, \quad f \mapsto 0 \cup \mathrm{dom}(f)$$

is unbounded in *On* and therefore has a nondiscrete image. Hence T is not discrete and has an accumulation point $g \in \mathbb{V}$. We set $G = g \cap (On \times \mathbb{V})$.

For each $\alpha \in On$, the union $\bigcup_{\beta<\alpha} T_\beta$ is a discrete set, so g is an accumulation point of the difference $T \setminus \bigcup_{\beta<\alpha} T_\beta$, which is the class of all those $f \in T$ whose domain is at least $\alpha^\oplus$. But every such f is by definition the extension of some $h \in T_\alpha$. Thus this difference is the union of the classes $S_h = \{f \in T \mid h \subseteq f\}$ with $h \in T_\alpha$. Since T_α is discrete, $\mathrm{cl}\big(\bigcup_{h\in T_\alpha} S_h\big) = \bigcup_{h\in T_\alpha} \mathrm{cl}(S_h)$, so g must be in the closure of some S_h. But S_h is a subclass of the closed

$$\{x \mid h \subseteq x \subseteq h \cup (\Omega \backslash \alpha \times \mathbb{V})\},$$

so $h \subseteq g \subseteq h \cup (\Omega \backslash \alpha \times \mathbb{V})$, too.

We have shown that for every $\alpha \in On$, the set $g \cap \big(\alpha^\oplus \times \mathbb{V}\big)$ is an element of T_α. This implies that G is a function defined on *On*, and that $G \restriction \alpha^\oplus = f \restriction \alpha^\oplus$ for some $f \in T$, concluding the proof. ∎

In fact, we have just shown that *every* accumulation point g of T gives rise to such a solution G. Hence firstly, $T = \mathrm{cl}(T) \cap \Box(On \times \mathbb{V})$, and secondly, G can always be described as the intersection of a set g with $On \times \mathbb{V}$. In our formulation of the tree property, the two quantifications over classes could thus be replaced by quantifications over sets.

If Ω exists, the hierarchy of well-ordered sets extends well beyond the realm of ordinal numbers. By *linearly ordered set* we shall mean from now on a set together with a linear order $\leqslant$ such that the set's natural topology is at least as fine as the order topology, that is, such that all $\leqslant$-closed intervals are $\mathbb{T}$-closed. And by *well-ordered set* we mean a linearly ordered set whose order is a well-order (or a strong well-order – which in this case is equivalent). Then all well-ordered sets are comparable.

The significance of (4) is that even if $\mathbb{V} \in \mathbb{V}$, the universe cannot be a well-ordered set, because well-ordered sets have no perfect subset. Thus whenever a is a well-ordered set, there is a $p \notin a$ and the set $a \cup \{p\}$ can be well-ordered such that its order-type is the successor of the order-type of a. We use the usual notation for intervals in the context of linearly-ordered sets, and consider ∞ (respectively $-\infty$) as greater (respectively smaller) than all the elements of the set. We will also sloppily write $a + b$ and $a \cdot b$ for order-theoretic sums and products and say that an order-type *exists* if there is a linearly ordered set with that order-type.

Every linearly ordered set a is a Hausdorff set and since its order is closed with respect to the product topology, it is itself a set by Proposition 1. Moreover, the class

$$\bigcap_{b \subseteq a \text{ initial segment}} \Box b \ \cup \ \{c \mid b \subseteq c \subseteq a\}$$

of all its $\mathbb{T}$-closed initial segments is itself a linearly ordered set in which a can be embedded via $x \mapsto (-\infty, x]$. Thus we can limit our investigations to well-ordered sets whose order is given by $\subseteq$ and whose union exists, which makes things considerably easier:

Lemma 7 (ES). If a class $A \subseteq \Box a$ is linearly ordered by $\subseteq$, then $\mathrm{cl}(A)$ is a linearly ordered set ordered by $\subseteq$. If A is well-ordered, then so is $\mathrm{cl}(A)$.

Proof. First we prove that $\mathrm{cl}(A)$ is still linearly ordered. Let $x,y \in \mathrm{cl}(A)$ and assume that $x \nsubseteq y$. Every $z \in A$ is comparable to every other element of A, so A is a subclass of the set $\Box z \cup \{v \mid z \subseteq v \subseteq a\}$ and thus $\mathrm{cl}(A)$ also is. Therefore both x and y are comparable to every element of A and A is a subclass of $\Box x \cup \{v \mid x \subseteq v \subseteq a\}$. Since y is not a superset of x, it must be in the closure of $A \cap \Box x$ and thus a subset of x.

Since $\{v \in \mathrm{cl}(A) \mid x \subseteq v \subseteq y\} = [x,y]$ is closed, $\mathrm{cl}(A)$ in fact carries at least the order topology.

Now assume that A is well-ordered and let $B \subseteq \mathrm{cl}(A)$ be nonempty. Wlog let B be a final segment. If B has only one element, then that element is minimal, so assume it has at least two distinct elements. Since A is dense, it must then intersect B and $A \cap B$ must have a minimal element x. Assume that x is not minimal in B. Then there is a $y \subset x$ in $B \setminus A$, and this y must be minimal, because if there were a $z \subset y$ in B, then (z,x) would be a nonempty open interval in $\mathrm{cl}(A)$ disjoint from A. ■

Thanks to this lemma, to prove that well-ordered sets of a certain length exist, it suffices to give a corresponding subclass of some $\Box a$ well-ordered by $\subseteq$. As the next theorem shows, this enables us to do a great deal of well-order arithmetic in essential set theory.

Proposition 8 (ES). If a and b are Hausdorff sets and $a_x \subseteq a$ is a well-ordered set for every $x \in b_I$, then $\sup_{x \in b_I} a_x$ exists. If in addition, R is a well-order on b_I (not necessarily a set), then $\sum_{x \in b_I} a_x$ exists. In particular, the order-type of R exists, and binary sums and products of well-orders exist.

Proof. Consider families $\langle r_x \mid x \in b_I \rangle$ of initial segments $r_x \subseteq a_x$ with the following property: for all $x,y \in b_I$ such that $r_x \neq a_x$, the length of r_y is the maximum of r_x and a_y. Given such a family, the class

$$b' \times a \quad \cup \quad \bigcup_{x \in b_I} \{x\} \times r_x$$

is a set. And the class of all such sets is a subclass of $\Box(b \times a)$ well-ordered by $\subseteq$ and at least as long as every a_x, because assigning to $y \in a_x$ the set

$$b' \times a \quad \cup \quad \bigcup_{z \in b_I} \{z\} \times r_z,$$

is an order-preserving map, where $r_z = a_z$ whenever a_z is at most as long as a_x, and $r_z = (-\infty, \tilde{y}]$ such that r_z is oder-isomorphic to $(-\infty, y]$ otherwise.

In the well-ordered case, consider for every $\langle x,y \rangle \in b_I \times a$ with $y \in a_x$ the set

$$b' \times a \quad \cup \quad \{x\} \times (-\infty, a_x] \quad \cup \quad (-\infty, x)_R \times a.$$

The class of these sets is again a subclass of $\square(b \times a)$ and well-ordered by $\subseteq$. Its order-type is the sum of the orders a_x.

Setting $a_x = 1^\oplus$ for each x yields a well-ordered set of the length of R. Using a two-point b proves that binary sums exist. And if b is a well-ordered set and $a_x = a$ for each $x \in b_I$, then $(b + 1^\oplus)_I$ has at least the length of b and $a \cdot b$ can be embedded in $\sum_{x \in (b+1^\oplus)_I} a_x$. ■

1.4 Pristine Sets and Inner Models

Pristine sets are not only useful for obtaining ordinal numbers, but also provide a rich class of inner models of essential set theory and prove several relative consistency results. To this end, we need to generalize the notion of a pristine set, such that it also applies to non-transitive sets.

But first we give a general criterion for interpretations of essential set theory. The picture behind the following is this: The elements of the class Z are to be ignored, so Z is interpreted as the empty class. We do this to be able to interpret $\emptyset \in \mathbb{V}$ even if the empty class is proper by choosing a nonempty set $Z \in \mathbb{V}$. Everything that is to be interpreted as a class will be a superclass X of Z, but only the elements of $X \setminus Z$ correspond to actual objects of the interpretation. In particular, $B \supseteq Z$ will be interpreted as the class of atoms and W as the universe. So the extension of an element $x \in W \setminus B$ will be a set X with $Z \subseteq X \subseteq W$, which we denote by $\Phi(x)$. Theorem 9 details the requirements these objects must meet to define an interpretation of **ES**.

Theorem 9 (ES). Let $\mathcal{K} \subseteq \mathcal{D}$ and $Z \subseteq B \subseteq W$ be classes and $\Phi : W \setminus B \to \mathbb{V}$ injective. We use the following notation:

- X is an *inner class* if it is not an atom and $Z \subseteq X \subseteq W$. In that case, let $X^{\oplus} = X \setminus Z$.
- $S = W \setminus B^{\oplus}$ and $T = \Phi[S^{\oplus}]$.
- $\overline{\Phi} = \Phi \cup \mathrm{id}_{B^{\oplus}} : W^{\oplus} \to \mathbb{V}$

Define an interpretation $\mathcal{I}$ as follows:

$$\begin{array}{rcl} X \text{ is in the domain of } \mathcal{I} & \text{if} & X \text{ is an inner class or } X \in B^{\oplus}. \\ X \in^{\mathcal{I}} Y & \text{if} & Y \text{ is an inner class and } X \in \overline{\Phi}[Y^{\oplus}]. \\ \mathbb{A}^{\mathcal{I}} & = & B \end{array}$$

If the following conditions are satisfied, $\mathcal{I}$ interprets essential set theory:

1. $W^{\oplus}$ has more than one element.
2. Every element of T is an inner class, and no element of B is an inner class.
3. $Z \cup \{x\} \in T$ for every $x \in W^{\oplus}$.
4. Any intersection $\bigcap C$ of a nonempty $C \subseteq T$ is Z or an element of T.
5. $x \cup y \in T$ for all $x, y \in T$.
6. If $x \in T$ and $x \setminus \{y\} \in T$ for all $y \in x^{\oplus}$, then $x^{\oplus}$ is $\mathcal{K}$-small.
7. Any union $\bigcup C$ of a nonempty $\mathcal{K}$-small $C \subseteq T$ is an element of T.
8. For all $a, b \in T$, the class $Z \cup \left\{x{\in}S^{\oplus} \mid \Phi(x){\subseteq}a, \Phi(x){\cap}b{\neq}Z\right\}$ is Z or in T.

The length of $On^{\mathcal{I}}$ is the least $\mathcal{K}$-large ordinal κ, or On if no such κ exists (for example in the case $\mathcal{K} = \mathcal{D}$). In particular, $(\omega \in \mathbb{V})^{\mathcal{I}}$ iff ω is $\mathcal{K}$-small.

Proof. Let us first translate some $\mathcal{I}$-interpretations of formulas:

- $(X \notin \mathbb{A})^{\mathcal{I}}$ iff X is an inner class, and $(X \in \mathbb{A})^{\mathcal{I}}$ iff $X \in B^{\oplus}$.
- $(X \in \mathbb{V})^{\mathcal{I}}$ iff $X \in \overline{\Phi}[W^{\oplus}]$, because W is the union of all inner classes, so $\mathbb{V}^{\mathcal{I}} = W$.
- If $(F : X_1 \to X_2)^{\mathcal{I}}$, then there is a function $G : \overline{\Phi}[X_1^{\oplus}] \to \overline{\Phi}[X_2^{\oplus}]$, defined by $G(Y_1) = Y_2$ if $(F(Y_1) = Y_2)^{\mathcal{I}}$, and G is surjective respectively injective iff $(F \text{ is surjective})^{\mathcal{I}}$ respectively $(F \text{ is injective})^{\mathcal{I}}$.

Now we verify the axioms of $\mathbf{ES}^{\mathcal{I}}$:

Extensionality: Assume $(X_1 \neq X_2 \ \wedge \ X_1, X_2 \notin \mathbb{A})^{\mathcal{I}}$. Then X_1 and X_2 are inner classes. But $X_1 \neq X_2$ implies that there exists an element y in $X_1 \setminus X_2 \subseteq W^{\oplus}$ or $X_2 \setminus X_1 \subseteq W^{\oplus}$. $Y = \Phi(y)$ is either in $B^{\oplus}$ or an inner class by (2). Since Φ is injective, this means by definition that $(Y{\in}X_1 \wedge Y{\notin}X_2)^{\mathcal{I}}$ or vice versa.

The *atoms axiom* follows directly from our definition of $\in^{\mathcal{I}}$, because no element of $B^{\oplus}$ is an inner class, and we enforced *Nontriviality* by stating that $W^{\oplus}$ has more than one element.

Comprehension(ψ): If $Y = Z \cup \{x{\in}W^{\oplus} \mid \psi^{\mathcal{I}}(\Phi(x),\vec{P})\}$, then Y witnesses the comprehension axiom for the formula $\psi = \phi^{C}$ with the parameters $\vec{P}$, because $X \in^{\mathcal{I}} Y$ iff

$$X \in \overline{\Phi}[Y^{\oplus}] = \{\Phi(x) \mid x{\in}W^{\oplus} \wedge \psi^{\mathcal{I}}(\Phi(x),\vec{P})\},$$

which translates to $X \in \overline{\Phi}[W^{\oplus}]$ and $\psi^{\mathcal{I}}(X,\vec{P})$.

T_1*:* Let $(X \in \mathbb{V})^{\mathcal{I}}$. Then $X = \overline{\Phi}(x)$ for some $x \in W^{\oplus}$. By (3), $Y = Z \cup \{x\} \in T = \Phi[S^{\oplus}]$, so in particular $(Y \in \mathbb{V})^{\mathcal{I}}$. But X is the unique element such that $X \in^{\mathcal{I}} Y$, so $(Y = \{X\})^{\mathcal{I}}$.

2nd Topology Axiom: Assume (D is a nonempty class of sets$)^{\mathcal{I}}$, because if (D contains an atom$)^{\mathcal{I}}$, the intersection is empty in $\mathcal{I}$ anyway. Then D is an inner class and every $Y \in C = \overline{\Phi}[D^{\oplus}]$ is an inner class, which means $Y \in \Phi[S^{\oplus}]$. So $C \subseteq \Phi[S^{\oplus}]$ and $C \neq \emptyset$. We have $(X \in \bigcap D)^{\mathcal{I}}$ iff $X \in^{\mathcal{I}} Y$ for all $Y \in^{\mathcal{I}} D$, that is:

$$X \in \bigcap_{Y \in C} \overline{\Phi}[Y^{\oplus}] = \overline{\Phi}\left[\left(\bigcap C\right)^{\oplus}\right],$$

because $\overline{\Phi}$ is injective. Hence the inner class $\bigcap C$ equals $(\bigcap D)^{\mathcal{I}}$, and by (4), it is either in T and therefore interpreted as a set, or it is $Z = \emptyset^{\mathcal{I}}$.

Additivity: A similar argument shows that $\bigcup C$ equals $(\bigcup D)^{\mathcal{I}}$. If (D is a discrete set$)^{\mathcal{I}}$, then by (6), $D^{\oplus}$ is $\mathcal{K}$-small and therefore the union of $C = \overline{\Phi}[D^{\oplus}]$ is in T by (7).

3rd Topology Axiom: Let $(X_1,X_2 \in \mathbb{T})^{\mathcal{I}}$. Then $X_1,X_2 \in T$ and $X_1,X_2 \neq Z$. By (5), $Y = X_1 \cup X_2 \in T$, and Y is interpreted as the union of X_1 and X_2.

The *Exponential* axiom follows from (8), because $Y = Z \cup \{x{\in}S^{\oplus} \mid \Phi(x){\subseteq}a, \Phi(x){\cap}b{\neq}Z\}$ equals $(\Box a \cap \Diamond b)^{\mathcal{I}}$. In fact, $X \in^{\mathcal{I}} Y$ iff $X \in T$, $X \subseteq a$ and $X \cap b \neq Z$, and $X \subseteq a$ is equivalent to $(X \subseteq a)^{\mathcal{I}}$, while $X \cap b \neq Z$ is equivalent to $(X \cap b \neq \emptyset)^{\mathcal{I}}$.

The statement about the length of $On^{\mathcal{I}}$ holds true because the discrete sets are interpreted by the classes X with $\mathcal{K}$-small $X^{\oplus}$. ∎

All the conditions of the theorem only concern the image of Φ but not Φ itself, so given such a model one can obtain different models by permuting the images of Φ. Also, if $\Phi[S^{\oplus}]$ is infinite and if $Z \in \mathbb{V}$, one can toggle the truth of the statement $(\emptyset \in \mathbb{V})^{\mathcal{I}}$ by including Z in or removing Z from $\Phi[S^{\oplus}]$.

Proposition 10 (ES). If $Z = \emptyset$, T is a $\mathcal{K}$-compact Hausdorff $\mathcal{K}$-topology on W, W has at least two elements, $B \subseteq W$ is open and does not contain any subsets of W, and $\Phi : W \setminus B \to \mathrm{Exp}_{\mathcal{K}}(W,T)$ is a homeomorphism, then all conditions of Theorem 9 are met and therefore these objects define an interpretation of **ES**. In addition, they interpret the statements $\mathbb{V} \in \mathbb{V}$ and that every set is $\mathcal{D}$-compact Hausdorff.

Proof. All conditions that we did not demand explicitly follow immediately from the fact that W is a $\mathcal{K}$-compact Hausdorff $\mathcal{K}$-topological space and from the definition of the exponential $\mathcal{K}$-topology.

$(\mathbb{V} \in \mathbb{V})^{\mathcal{I}}$ holds true, because $W \in \mathrm{Exp}_{\mathcal{K}}(W,\mathrm{T})$. And since the $\mathcal{K}$-small sets are exactly those interpreted as discrete, the $\mathcal{K}$-compactness and Hausdorff property of W implies that $(\mathbb{V}$ is $\mathcal{D}$-compact Hausdorff.$)^{\mathcal{I}}$. ∎

Such a topological space W, together with a homeomorphism Φ to its hyperspace, is called a $\mathcal{K}$-*hyperuniverse*. We will deal with the construction of hyperuniverses in the second chapter and instead consider a different class of models now given by pristine sets.

Let $Z \subseteq B$ be such that no element of B is a super*set* of Z (they are allowed to be atoms). Again, write $X \in_Z Y$ for:

$$X \in Y^{\oplus} \quad \text{and} \quad Z \subseteq Y.$$

And X is Z-*transitive* if $c \in_Z X$ whenever $c \in_Z b \in_Z X$. We say that X is Z-B-*pristine* if:

- $X \in_Z B$ or:
- $Z \subseteq X \notin \mathbb{A}$, and there is a Z-transitive set $b \supseteq X$, such that for every $c \in_Z b$ either $Z \subseteq c \notin \mathbb{A}$ or $c \in_Z B$.

If a has a Z-transitive superset b, then it has a least Z-transitive superset $\mathrm{trcl}(a) = \bigcap\{b \supseteq a \mid b \text{ Z-transitive}\}$, the Z-*transitive closure* of a. Obviously a set is Z-transitive iff it equals its Z-transitive closure. Also, a is Z-B-pristine iff $\mathrm{trcl}(a)$ exists and is Z-B-pristine. A set a is Z-*well-founded* iff for every $b \ni_Z a$, there exists an $\in_Z$-minimal $c \in_Z b$.

Theorem 11 (ES). Let $Z \in \mathbb{V}$ and $B \supseteq Z$ such that no element of B is a super*set* of Z, and $B^{\oplus}$ is $\mathbb{T}$-closed. Let Φ be the identity on $W \setminus B$ and $\mathcal{K} = \mathcal{D}$. The following classes $W_i^{\oplus}$ meet the requirements of Theorem 9 and therefore define interpretations $\mathcal{I}_i$ of essential set theory:

- the class $W_1^{\oplus}$ of all Z-B-pristine x
- the class $W_2^{\oplus}$ of all Z-B-pristine x with discrete $\mathrm{trcl}(x)^{\oplus}$
- the class $W_3^{\oplus}$ of all Z-well-founded Z-B-pristine x with discrete $\mathrm{trcl}(x)^{\oplus}$

Z is a member of all three classes and thus $(\emptyset \in \mathbb{V})^{\mathcal{I}_i}$ holds true in all three cases. If $i \in \{2,3\}$, then (every set is discrete)$^{\mathcal{I}_i}$, and in the third case, (every set is $\emptyset$-well-founded)$^{\mathcal{I}_3}$.

If $\mathbb{V} \in \mathbb{V}$, then:

1. $(\mathbb{V} \in \mathbb{V})^{\mathcal{I}_1}$
2. (*On* has the tree property)$_i^{\mathcal{I}}$ for all i.
3. If $B^{\oplus}$ is discrete, $\mathcal{I}_3$ satisfies the strong comprehension principle.

Proof. In this proof, we will omit the prefixes Z and B: By "pristine" we always mean Z-B-pristine, "transitive" means Z-transitive and "well-founded" Z-well-founded.

Since $Z^{\oplus}$ is empty and B is pristine and well-founded, $Z \in W_3^{\oplus} \subseteq W_2^{\oplus} \subseteq W_1^{\oplus}$.

Before we go through the requirements of Theorem 9, let us prove that $x^\oplus$ is closed for every $x \in S^\oplus$:

$$x^\oplus \quad = \quad (x \cap B^\oplus) \ \cup \ (\{Z\} \cap x) \ \cup \ \{y \in x \mid Z \subseteq y \notin \mathbb{A}\}$$

Since x is pristine, there is a transitive pristine $c \supseteq x$, and we can rewrite the class $\{y \in x \mid Z \subseteq y \notin \mathbb{A}\}$ as $\{y \in x \cap \Box c \mid Z \subseteq y \subseteq c\}$, which is closed.

Condition (1) of Theorem 9 is satisfied because Z and $Z \cup \{Z\}$ are distinct elements of $W_3^\oplus$.

(2): If $x \in B$, then x is not a superset of Z and therefore not an inner class. Now let $x \in S_1^\oplus$. We have to show that $x = \Phi(x)$ is an inner class. Since $x \notin B$ and x is pristine, $x \notin \mathbb{A}$ and $Z \subseteq x$, so it only remains to prove that $y \in W_1^\oplus$ for every $y \in x^\oplus$. If $y \in_Z B$, y is pristine. If $y \notin_Z B$, then $Z \subseteq y$. Since every transitive superset of x is also a superset of y, y is pristine in that case, too. If in addition, $\mathrm{trcl}(x)^\oplus$ is discrete, y also has that property, by the same argument. And if x is also well-founded, y also is: For any $b \ni_Z y$, $b^\oplus \cup \{x\}$ has a $\in_Z$-minimal element; since $y \in_Z x$ and $y \in_Z b$, this cannot be x, so it must be in $b^\oplus$. This concludes the proof that $y \in W_i^\oplus$ whenever $x \in S_i^\oplus$.

(3): If $x \in W_1^\oplus$, then $Z \cup \{x\}$ is pristine, because if $x \in B^\oplus$, it is already transitive itself, and otherwise if c is a transitive pristine superset of x, then $c \cup \{x\}$ is a transitive pristine superset of $Z \cup \{x\}$. If moreover $c^\oplus$ is discrete, then $c^\oplus \cup \{x\}$ also is, and if x is well-founded, $Z \cup \{x\}$ also is.

(4): Let $C \subseteq S_i^\oplus$ be nonempty. Then $\bigcap C \in S_i^\oplus$, too, because every subset of a pristine set which is a superset of Z is pristine itself, every subset of a discrete set is discrete, and every subset of a well-founded set is well-founded.

(6): Assume that for every $y \in x^\oplus$, we have $x \setminus \{y\} \in S^\oplus$. Then $(x \setminus \{y\})^\oplus = x^\oplus \setminus \{y\}$ is closed, and hence $x^\oplus$ is a discrete set.

(7) (and consequently (5)): Let $C \subseteq S_1^\oplus$ be a nonempty discrete set. Then $\bigcup C \in W_1 \setminus B$, because if c_b is a transitive pristine superset of b for all $b \in C$, then $\bigcup c_b$ is such a superset of the union. If all the c_b are discrete, their union also is, because they are only $\mathcal{D}$-few. And if every element of C is well-founded, $\bigcup C$ also is.

(8): $Y = Z \cup \left\{x{\in}S^\oplus \mid x{\subseteq}a, x{\cap}b{\neq}Z\right\}$ is pristine, because if c is a transitive pristine superset of a, then $z = Z \cup \{Z\} \cup \{x{\in}\Box c \mid Z{\subseteq}x\}$ is a transitive pristine superset of Y. And Y is in fact a set, because $b^\oplus$ is closed, so $Y = Z \cup (z^\oplus \cap \Box a \cap \Diamond b^\oplus)$ also is. If $c^\oplus$ is discrete, $\Box c^\oplus$ is discrete, and so is $z^\oplus \setminus \{Z\} = \{y \cup Z \mid y \in \Box c^\oplus\}$. And if a is well-founded, any set of subsets of a is well-founded, too.

The claims about discreteness and well-foundedness are immediate from the definitions.

Now let us prove the remaining claims under the assumption that $\mathbb{V} \in \mathbb{V}$:

(1): $\mathbb{V} \setminus \mathbb{A}$ is a set, namely $\Diamond\mathbb{V} \cup \{\emptyset\}$ or $\Diamond\mathbb{V}$, depending on whether $\emptyset \in \mathbb{V}$. Let:

$$\begin{aligned} U_0 &= \mathbb{V} \\ U_{n+1} &= B^\oplus \ \cup \ \{x \in \mathbb{V} \setminus \mathbb{A} \mid Z \subseteq x \subseteq Z \cup U_n\} \\ U_\omega &= \bigcap_{n \in \omega} U_n \end{aligned}$$

Then U_ω is a set. Since $W_1^\oplus \subseteq \mathbb{V}$ and $W_1^\oplus \subseteq B^\oplus \cup \{x \in \mathbb{V} \setminus \mathbb{A} \mid Z \subseteq x \subseteq Z \cup W_1^\oplus\}$, it is a subset of U_ω. It remains to show that $U_\omega \subseteq W_1^\oplus$, that is, that every element of U_ω is pristine, because then it follows that W_1 is a pristine set itself and hence $W_1 \in_Z W_1$. In fact, it suffices to prove that $Z \cup U_\omega$ is a transitive pristine set, because then all $x \in_Z U_\omega$ will be pristine, too. So assume $y \in_Z x \in_Z Z \cup U_\omega$. If x were in $B^\oplus$, then $y \notin_Z x$, so x must be in $\mathbb{V} \setminus \mathbb{A}$ and $Z \subseteq x \subseteq Z \cup U_n$ for all n. Thus $x \subseteq Z \cup U_\omega$, which implies that $y \in_Z U_\omega$.

(2) follows from Proposition 6.

(3): It suffices to show that $W_3^\oplus$ does not contain any of its accumulation points, because that implies that every inner class corresponds to a set – it's closure –, so that the weak comprehension principle allows us to quantify over all inner classes. Since $B^\oplus$ is discrete and

$$S_3^\oplus \setminus \{Z\} \quad = \quad W_3^\oplus \setminus (B \cup \{Z\}) \quad \subseteq \quad \{x \in \mathbb{V} \setminus \mathbb{A} \mid Z \subseteq x\} \quad \in \quad \mathbb{V}$$

(recall that no element of B is a superset of Z), B certainly contains no accumulation point of $W_3^\oplus$. So assume now that $x \in W_3^\oplus$ is an accumulation point. Since it is well-founded and $\mathrm{trcl}(x)^\oplus$ is a discrete set, $\mathrm{trcl}(x)^\oplus \cup \{x\}$ has an $\in_Z$-minimal $W_3^\oplus$-accumulation point y. Then $y \in S_3^\oplus$ and y is also an accumulation point of $W_3^\oplus \setminus (B^\oplus \cup \{Z\})$. Since none of the $\mathcal{D}$-few elements of $y^\oplus$ is an $W_3^\oplus$-accumulation point, $W_3^\oplus \setminus (B^\oplus \cup \{Z, y\})$ is a subclass of

$$\Diamond \mathrm{cl}(W_3^\oplus \setminus y) \ \cup \ \bigcup_{z \in_Z y} \Box \mathrm{cl}(W_3^\oplus \setminus \{z\}),$$

which is closed and does not contain y, a contradiction. ■

By the nontriviality axiom, there are distinct $x, y \in \mathbb{V}$. If we set $Z = B = \{\{x\},\{y\}\}$, the requirements of Theorem 11 are satisfied, so $\mathcal{I}_i$ interprets essential set theory with $\emptyset \in \mathbb{V}$ in all three cases. Moreover, since $Z = B$, it interprets $\mathbb{A} = \emptyset$. So $\mathbb{A} = \emptyset \in \mathbb{V}$ is consistent relative to **ES**. In the case $i = 3$, moreover, (every set is $\emptyset$-well-founded and discrete)$^{\mathcal{I}_3}$! And if in addition $\omega \in \mathbb{V}$, then ω is $\mathcal{D}$-small and thus $(\omega \in \mathbb{V})^{\mathcal{I}_3}$ by Theorem 9.

But if in **ES** every set is discrete and $\emptyset$-well-founded, the following statements are implied:

Pair, Union, Power, Empty Set	$\{a, b\}, \bigcup a, \mathfrak{P}(a), \emptyset \in \mathbb{V}$
Replacement	If F is a function and $a \in \mathbb{V}$, then $F[a] \in \mathbb{V}$.
Foundation	Every $x \in \mathbb{T}$ has a member disjoint from itself.

And these are just the axioms of **ZF**[2]! Conversely, all the axioms of **ES** hold true in **ZF**, so **ZF** could equivalently be axiomatized as follows[3]:

- **ES**$_\infty$
- $\mathbb{A} = \emptyset \in \mathbb{V}$
- Every set is discrete and $\emptyset$-well-founded.

[2] With classes, of course. We avoid the name **NBG**, because that is usually associated with a strong axiom of choice.

[3] We will soon introduce a choice principle for **ES**, the uniformization axiom, which applies to all discrete sets. Since in **ZF** every set is discrete, that axiom is equivalent to the axiom of choice.

If in addition $\mathbb{V} \in \mathbb{V}$, then $\mathcal{I}_3$ even interprets the strong comprehension axiom and therefore Kelley-Morse set theory[4] with *On* having the tree property. Conversely, O. Esser showed in [Ess97] and [Ess99] that this theory is equiconsistent with $\mathbf{GPK}^+_\infty$, which in turn is an extension of topological set theory that will be introduced in the next section. In summary, we have the following results:

Corollary 12. $\mathbf{ES}_\infty$ is equiconsistent with **ZF**: The latter implies the former and the former interprets the latter.

$\mathbf{TS}_\infty$ and $\mathbf{GPK}^+_\infty$ both are mutually interpretable with:
Kelley-Morse set theory $+$ *On* has the tree property.

$\mathcal{I}_3$ is a particularly intuitive interpretation if $\emptyset, \mathbb{V} \in \mathbb{V}$, $\mathbb{A} = \emptyset$ and we set $Z = B = \emptyset$. Then every set is ($\emptyset$-$\emptyset$-)pristine and $\in_N$ is just $\in$. Also, $\mathbb{V} \setminus \{\emptyset\} = \Diamond\mathbb{V} \in \mathbb{V}$, so $\emptyset$ is an isolated point. If a set x contains only isolated points, it is discrete, and since $x = \bigcup_{y \in x}\{y\}$ and every $\{y\}$ is open, x is a clopen set. Moreover, x is itself an isolated point, because $\{x\}$ is open:

$$\{x\} \quad = \quad \Box x \cup \bigcup_{y \in x} \Diamond\{y\}$$

Thus it follows that all ($\emptyset$-)well-founded sets are isolated. Define the cumulative hierarchy as usual:

$$\begin{aligned} U_0 &= \emptyset \\ U_{\alpha+1} &= \Box U_\alpha \cup \{\emptyset\} \\ U_\lambda &= \bigcup_{\alpha<\lambda} U_\alpha \quad \text{for limit ordinals } \lambda \end{aligned}$$

Since images of discrete sets in *On* are bounded and since every nonempty class of well-founded sets has an $\in$-minimal element, the union $\bigcup_{\alpha \in On} U_\alpha$ is exactly the class of all well-founded sets, and in fact equals W_3.

1.5 Positive Specification

This section is a short digression from our study of essential set theory. Again starting from only the class axioms we introduce specification schemes for two classes of "positive" formulas as well as O. Esser's theory $\mathbf{GPK}^+$ (cf. [Ess97, Ess99, Ess00, Ess04]), and then turn our attention to their relationship with topological set theory.

The idea of positive set theory is to weaken the inconsistent *naive comprehension scheme* – that every class $\{x \mid \phi(x)\}$ is a set – by permitting only *bounded positive formulas* (BPF), which are defined recursively similarly to the set of all formulas, but omitting the negation step, thus avoiding the Russell paradox. This family of formulas can consistently be widened

[4] The axiom of choice is not necessarily true in that interpretation, but even the existence of a global choice function does not add to the consistency strength, as was shown in [Ess04].

to include all *generalized positive formulas* (GPF), which even allow universal quantification over classes. But to obtain more general results, we will investigate *specification* schemes instead of comprehension schemes, which only state the existence of subclasses $\{x{\in}c \mid \phi(x)\}$ of sets c. If $\mathbb{V}$ is a set, this restriction makes no difference.

We define recursively when a formula ϕ whose variables are among $X_1,X_2,\ldots$ and $Y_1,Y_2,\ldots$ (where these variables are all distinct) is a *generalized positive formula* (GPF) *with parameters* $Y_1,Y_2,\ldots$:

- The atomic formulas $X_i \in X_j$ and $X_i = X_j$ are GPF with parameters $Y_1,Y_2,\ldots$.
- If ϕ and ψ are GPF with parameters $Y_1,Y_2,\ldots$, then so are $\phi \wedge \psi$ and $\phi \vee \psi$.
- If $i \neq j$ and ϕ is a GPF with parameters $Y_1,Y_2,\ldots$, then so are $\forall X_i{\in}X_j\ \phi$ and $\exists X_i{\in}X_j\ \phi$.
- If ϕ is a GPF with parameters $Y_1,Y_2,\ldots$, then so is $\forall X_i{\in}Y_j\ \phi$.

A GPF with parameters $Y_1,Y_2,\ldots$ is a *bounded positive formula* (BPF) if it does not use any variable Y_i, that is, if it can be constructed without making use of the fourth rule. The *specification axiom* for the GPF $\phi(X_1,\ldots,X_m,Y_1,\ldots,Y_n)$ with parameters $Y_1,Y_2,\ldots$, whose free variables are among $X_1,\ldots,X_m$, is:

$$\{x{\in}c \mid \phi(x,b_2,\ldots,b_m,B_1,\ldots,B_n)\} \text{ is } \mathbb{T}\text{-closed for all } c,b_2,\ldots,b_m \in \mathbb{V} \text{ and all classes } B_1,\ldots,B_n.$$

GPF specification is the scheme consisting of the specification axioms for all GPF ϕ, and *BPF specification* incorporates only those for BPF ϕ. Note that we did not include the formula $x \in \mathbb{A}$ or any other formula involving the constant $\mathbb{A}$ in the definition, so $x \in \mathbb{A}$ is not a GPF.

The following theorem shows that BPF specification is in fact finitely axiomatizable, even without classes.[5]

Theorem 13. Assume only the class axioms and that for all $a,b \in \mathbb{V}$, the following are $\mathbb{T}$-closed:

$$\bigcup a, \quad \{a,b\}, \quad a \times b$$

Let Θ be the statement that for all sets $a,b \in \mathbb{V}$, the following are $\mathbb{T}$-closed:

$$\Delta \cap a, \quad \mathbf{E} \cap a, \quad \{\langle x,y\rangle{\in}b \mid \forall z{\in}y\ \langle x,y,z\rangle{\in}a\}, \quad \{\langle y,x,z\rangle \mid \langle x,y,z\rangle{\in}a\}, \quad \{\langle z,x,y\rangle \mid \langle x,y,z\rangle{\in}a\}$$

Then BPF-specification is equivalent to Θ. And GPF specification is equivalent to Θ and the second topology axiom.

Proof. Ordered pairs can be build from unordered ones, and the equality $\langle x,y\rangle = z$ can be expressed as a BPF. Therefore the classes mentioned in Θ can all be defined by applying BPF specification to a given set or product of sets, so BPF specification implies Θ.

[5] A similar axiomatization, but for positive *comprehension*, is given by M. Forti and R. Hinnion in [FH89]. On the other hand, no finite axiomatization exists for *generalized* positive comprehension, as O. Esser has shown in [Ess04].

GPF specification in addition implies the second topology axiom,

$$\forall B{\neq}\emptyset.\quad \emptyset{=}\bigcap B \quad\vee\quad \bigcap B \in \mathbb{V},$$

because $\forall a{\in}B\ \ x{\in}a$ is clearly a GPF with parameter B, and the intersection is a subclass of any $c \in B$.

To prove the converse, assume now that Θ holds. Since it is not yet clear what we can do with sets, we have to be pedantic with respect to Cartesian products. We define

$$A \times_2 B = \{\langle a,b_1,b_2\rangle \mid a{\in}A, \langle b_1,b_2\rangle{\in}B\},$$

which is not the same as $A \times B$ for $B \subseteq \mathbb{V}^2$, because $\langle a,b_1,b_2\rangle = \langle\langle a,b_1\rangle,b_2\rangle$, whereas the elements of $A \times B$ are of the form $\langle a,\langle b_1,b_2\rangle\rangle$. Yet we can construct this and several other set theoretic operations from Θ:

$$\begin{aligned}
a \times_2 b &= \{\langle z,x,y\rangle \mid \langle x,y,z\rangle \in b \times a\}\\
a \cup b &= \bigcup\{a,b\}\\
a \cap b &= \bigcup\bigcup\{\{\{x\}\} \mid x \in a \cap b\} = \bigcup\bigcup(\Delta \cap (a{\times}b))\\
a \cap \mathbb{V}^2 &= a \cap \left(\bigcup\bigcup a\right)^2\\
\{\{x\} \mid \{x\} \in a\} &= a \cap \bigcup\left(\Delta \cap \left(\bigcup a\right)^2\right)\\
\mathrm{dom}(a) &= \bigcup\left\{\{x\} \mid \{x\} \in \bigcup(a \cap \mathbb{V}^2)\right\}\\
a^{-1} &= \mathrm{dom}(\{\langle y,x,z\rangle \mid \langle x,y,z\rangle \in a{\times}\{a\}\})
\end{aligned}$$

We will prove by induction that for all GPF $\phi(X_1,\ldots,X_m,Y_1,\ldots,Y_n)$ with parameters $Y_1,\ldots,Y_n$ and free variables $X_1,\ldots,X_m$, and for all classes $B_1,\ldots,B_n$ and sets $a_1,\ldots,a_m$,

$$A^{\phi}_{a_1,\ldots,a_m} = \{\langle x_1,\ldots,x_m\rangle \in a_1{\times}\ldots{\times}a_m \mid \phi(x_1,\ldots,x_m,B_1,\ldots,B_n)\}$$

is $\mathbb{T}$-closed. This will prove the specification axiom for ϕ, because

$$\{x{\in}c \mid \phi(x,b_2,\ldots,b_m,B_1,\ldots,B_n)\} \quad = \quad \mathrm{dom}\Big(\ldots\mathrm{dom}\Big(A^{\phi}_{c,\{b_2\},\ldots,\{b_m\}}\Big)\ldots\Big),$$

where the domain operation is applied $m-1$ times.

Each induction step will reduce the claim to a subformula or to a formula with fewer quantifiers. Let us assume wlog that no bound variable is among the $X_1,\ldots$ or $Y_1,\ldots$ and just always denote the bound variable in question by Z.

Case 1: Assume ϕ is $\forall Z{\in}Y_i\ \psi$. Then

$$A^{\phi}_{a_1,\ldots,a_m} = \bigcap_{x\in B_i} \mathrm{dom}\left(A^{\psi(Z/X_{m+1})}_{a_1,\ldots,a_m,\{x\}}\right),$$

where $\psi(Z/X_{m+1})$ is the formula ψ, with each free occurrence of Z substituted by X_{m+1}. This is the step which is only needed for GPF formulas. Since it is the only point in the proof where we make use of the closure axiom, we otherwise still obtain BPF specification as claimed in the theorem.

Case 2: Assume ϕ is a bounded quantification. If ϕ is $\exists Z{\in}X_i\ \psi$, then

$$A_{\phi,a_1,\ldots,a_m} = \operatorname{dom}\left(A^{\psi(Z/X_{m+1})\,\wedge\,X_{m+1}\in X_i}_{a_1,\ldots,a_m,b}\right),$$

where $b = \bigcup a_i$. If ϕ is $\forall Z{\in}X_i\ \psi$, then

$$A^{\phi}_{a_1,\ldots,a_m} = \operatorname{dom}\left\{\langle x,y\rangle \in a_1\times\ldots\times a_m\times a_i \mid \forall z{\in}y\ \langle x,y,z\rangle \in A^{\psi(Z/X_{m+2})\,\wedge\,X_{m+1}=X_i}_{a_1,\ldots,a_m,a_i,b}\right\},$$

where again $b = \bigcup a_i$. The class defined here is of the form $\{\langle x,y\rangle{\in}b \mid \forall z{\in}y\ \langle x,y,z\rangle{\in}a\}$ and therefore a set, by our assumption.

Case 3: Assume ϕ is a conjunction or disjunction. If ϕ is $\psi\wedge\chi$ respectively $\psi\vee\chi$, then

$$A^{\phi}_{a_1,\ldots,a_m} = A^{\psi}_{a_1,\ldots,a_m}\cap A^{\chi}_{a_1,\ldots,a_m} \quad\text{respectively}\quad A^{\phi}_{a_1,\ldots,a_m} = A^{\psi}_{a_1,\ldots,a_m}\cup A^{\chi}_{a_1,\ldots,a_m}.$$

Case 4: Assume ϕ is atomic. If X_m does not occur in ϕ, then $A^{\phi}_{a_1,\ldots,a_m} = A^{\phi}_{a_1,\ldots,a_{m-1}}\times a_m$. If ϕ has more than one variable, but X_{m-1} is not among them, then:

$$A^{\phi}_{a_1,\ldots,a_m} = \left\{\langle z,x_{m-1},x_m\rangle \mid \langle z,x_m,x_{m-1}\rangle \in A^{\phi(X_m/X_{m-1})}_{a_1,\ldots,a_{m-2},a_m}\times a_{m-1}\right\}$$

Applying these two facts recursively reduces the problem to the case where either $m = 1$ or where X_m and X_{m-1} both occur in ϕ:

$$\begin{aligned}
A^{X_1=X_1}_{a_1} &= a_1\\
A^{X_1\in X_1}_{a_1} &= \operatorname{dom}\left(\mathbf{E}\cap a_1^2\right)\\
A^{X_{m-1}=X_m}_{a_1,\ldots,a_m} &= a_1\times\ldots\times a_{m-2}\times_2(\Delta\cap(a_{m-1}\times a_m))\\
A^{X_m=X_{m-1}}_{a_1,\ldots,a_m} &= a_1\times\ldots\times a_{m-2}\times_2(\Delta\cap(a_{m-1}\times a_m))\\
A^{X_{m-1}\in X_m}_{a_1,\ldots,a_m} &= a_1\times\ldots\times a_{m-2}\times_2(\mathbf{E}\cap(a_{m-1}\times a_m))\\
A^{X_m\in X_{m-1}} &= a_1\times\ldots\times a_{m-2}\times_2(\mathbf{E}^{-1}\cap(a_{m-1}\times a_m))
\end{aligned}$$

■

As we already indicated, the theory **GPK**$^+$ uses GPF *comprehension*, but if $\mathbb{V}\in\mathbb{V}$, specification entails comprehension. **GPK**$^+$ can be axiomatized as follows:

- $\mathbb{V}\in\mathbb{V}$
- $\mathbb{A} = \emptyset \in \mathbb{V}$
- GPF specification

Proposition 14. **GPK**$^+$ implies **ES** and that unions of sets are sets.

Proof. If $B \subseteq \mathbb{T}$, then $\bigcap B = \{x \mid \forall y{\in}B \;\; x{\in}y\}$ is $\mathbb{T}$-closed, and if $a,b \in \mathbb{T}$, then $a \cup b = \{x \mid x{\in}a \vee x{\in}b\} \in \mathbb{V}$, because these are defined by GPFs, proving the 2nd and 3rd topology axioms. $\{a\} = \{x \mid x{=}a\}$ and $x{=}a$ is bounded positive, so T_1 is also true.

$\Box a \cap \Diamond b = \{c \mid \exists x{\in}b \;\; x{=}x \wedge \forall x{\in}c \;\; x{\in}a \wedge \exists x{\in}b \;\; x{\in}c\}$ is defined by a positive formula as well, yielding the exponential axiom.

$\bigcup a = \{c \mid \exists x{\in}a \;\; c{\in}x\}$ is also $\mathbb{T}$-closed, for the same reason.

The formula $z = \{x,y\}$ can be expressed as $x{\in}z \wedge y{\in}z \wedge \forall w{\in}z \;(w{=}x \vee w{=}y)$, so it is bounded positive. Using that, we see that ordered pairs, Cartesian products, domains and ranges can all be defined by GPFs. This allows us to prove the additivity axiom:

Let $a \in \mathbb{T}$ be discrete and $F : a \to \mathbb{V}$. We first show that $F \in \mathbb{V}$: Firstly, $F \subseteq a \times \mathbb{V}$ and $a \times \mathbb{V}$ is $\mathbb{T}$-closed. Secondly, if $\langle x,y\rangle \in (a \times \mathbb{V}) \setminus F$, then $F(x) \neq y$, so F is a subclass of the $\mathbb{T}$-closed $(a \backslash \{x\} \times \mathbb{V}) \cup \{\langle x, F(x)\rangle\}$, which does not contain $\langle x,y\rangle$. Thus F is a set and hence $\bigcup \mathrm{rng}(F)$ is $\mathbb{T}$-closed. ■

1.6 Regularity and Union

After having seen that essential set theory is provable in **GPK**$^+$, we now aim for a result in the other direction. To this end we need to assume in addition to **ES** the union axiom and that every set is a regular space:

Union	$\bigcup a$ is $\mathbb{T}$-closed for every $a \in \mathbb{V}$.
T_3	$x{\in}a \;\wedge\; b{\in}\Box a \quad\Rightarrow\quad \exists u,v.\;\; u{\cup}v{=}a \;\wedge\; x{\notin}u \;\wedge\; b{\cap}v{=}\emptyset$

In addition to their use in the proof of GPF specification, these two axioms elegantly connect the topological and set-theoretic properties of orders and products. Note that they, too, are theorems of **ZF**, because every discrete set is regular and its union is a set.

Recall that we use the term *ordered set* only for sets with an order $\leqslant$, whose order-topology is at least as fine as their natural topology. By default, we consider the order itself to be the non-strict version.

Proposition 15 (**ES** + Union + T_3). 1. Domains and ranges of sets are sets.

2. Every map in $\mathbb{V}$ is continuous and closed with respect to the natural topology.
3. A linear order $\leqslant$ on a set a is a set iff its order topology is at most as fine as the natural topology of a.
4. The product topology of a^n is equal to the natural topology.
5. If $\mathbb{A}$ is closed, GPF specification holds.

Proof. (1): Let a be a set. Then $c = \bigcup\bigcup a$ is a set, and in fact, $c = \mathrm{dom}(a) \cup \mathrm{rng}(a)$. But $\mathrm{dom}(a) = \bigcup(\Box_{\leqslant 1} c \cap \bigcup a)$, which proves that domains of sets are sets. Now $F_{2,2,1} \restriction c^2 : c^2 \to$

c^2 is a set, and so is $(c^2 \times a) \cap F_{2,2,1}$. But the domain of this set is a^{-1}, and the domain of a^{-1} is $\mathrm{rng}(a)$.

(2): Let $f \in \mathbb{V}$ be a map from a to b, and let $c \subseteq b$ be closed. Then $f \cap (a \times c)$ is a set, too, and so is $f^{-1}[c] = \mathrm{dom}(f \cap (a \times c))$. Thus f is continuous. Similarly, if $c \subseteq a$ is closed, then $f[c] = \mathrm{rng}(f \cap (c \times b))$ is a set and hence f is closed.

(3): Now let a be linearly ordered by $\leqslant$. If $x \in a$, then $[x,\infty) = \mathrm{rng}((\{x\} \times a) \cap \leqslant)$ and $(\infty,x] = \mathrm{dom}((a \times \{x\}) \cap \leqslant)$. Conversely assume that all intervals $[x,y]$ are sets. Then if $\langle x,y\rangle \in a^2 \setminus \leqslant$, that is, $x > y$. If there is a $z \in (y,x)$, then $(z,\infty) \times (-\infty,z)$ is a relatively open neighborhood of $\langle x,y\rangle$ disjoint from $\leqslant$. Otherwise, $(y,\infty) \times (-\infty,x)$ is one.

(4): To show that the topologies on a^n coincide, we only need to consider the case $n = 2$; the rest follows by induction, because products of regular spaces are regular. Since a is Hausdorff, we already know from Proposition 1 that the universal topology is at least as fine as the product topology, and it remains to prove the converse.

Let $b \subseteq a^2$ be a set. We will show that it is closed with respect to the product topology. Let $\langle x,y\rangle \in a^2 \setminus b$. Then $x \notin \mathrm{dom}(b \cap (a \times \{y\}))$, so by regularity, there is a closed neighborhood $u \ni x$ disjoint from that set. Thus $b \cap (a \times \{y\}) \cap (u \times a) = \emptyset$, that is, $y \notin \mathrm{rng}(b \cap (u \times a))$. Again by T_3, there is a closed neighborhood $v \ni y$ disjoint from that. Hence $b \cap (u \times v) = \emptyset$ and $u \times v$ is a neighborhood of $\langle x,y\rangle$ with respect to the product topology.

(5): We only have to prove Θ from Theorem 13: The statements about the permutations of triples are true because the topologies on products coincide. $\Delta \cap a$ is closed in $(\mathrm{dom}(a) \cup \mathrm{rng}(a))^2$, even with respect to the product topology, because every set is Hausdorff. $\mathbf{E} \cap a$ is a set by regularity: If $\langle x,y\rangle \in a \setminus \mathbf{E}$, then $x \notin y$, so x and y can be separated by disjoint $U \ni x$ and $V \supseteq y$ relatively open in $\mathrm{dom}(a) \cup \mathrm{rng}(a)$. $a \cap (U \times V)$ is a neighborhood of $\langle x,y\rangle$ disjoint from $\mathbf{E}$

It remains to show that $B = \{\langle x,y\rangle \in b \mid \forall z \in y\ \langle x,y,z\rangle \in a\}$ is closed for every $a \in \mathbb{V}$. Since

$$B \quad = \quad b \cap \{\langle x,y\rangle \in c^2 \mid \forall z \in y\ \langle x,y,z\rangle \in a \cap c^3\},$$

where $c = \mathrm{dom}(b) \cup \mathrm{rng}(b) \cup \bigcup \mathrm{rng}(b)$, we can wlog assume that $b = c^2$ and $a \subseteq c^3$, and prove that B is a closed subset of c^2. Let $\langle x,y\rangle \in c^2 \setminus B$, that is, let $\exists z \in y\ \langle x,y,z\rangle \notin a$. By (4) there exist relatively open neighborhoods U, V and W of x, y and z in c, such that $U \times V \times W$ is disjoint from a. But then $c \cap \Diamond W$ equals $c \setminus (\mathbb{A} \cup \Box(c \setminus W))$ or $c \setminus (\mathbb{A} \cup \{\emptyset\} \cup \Box(c \setminus W))$, depending on whether $\emptyset \in \mathbb{V}$, so $c \cap \Diamond W$ is relatively open and hence $U \times (V \cap \Diamond W)$ is an open neighborhood of $\langle x,y\rangle$ in c^2 disjoint from B. ■

Together with (5), Proposition 14 thus proves:

Corollary 16. $\mathbf{GPK}^+_{(\infty)} + T_3$ is equivalent to $\mathbf{TS}_{(\infty)} + (\mathbb{A} = \emptyset \in \mathbb{V}) + \mathrm{Union} + T_3$.

1.7 Uniformization

Choice principles in the presence of a universal set are problematic. By Theorem 6, for example, $\mathbb{V} \in \mathbb{V}$ implies that there is a perfect set and in particular that not every set is well-orderable. And in [FH96a, FH98, Ess00], M. Forti, F. Honsell and O. Esser identified plenty of choice principles as inconsistent with positive set theory. On the other hand, many topological arguments rely on some kind of choice. The following *uniformization axiom* turns out to be consistent and yet have plenty of convenient topological implications, in particular with regard to compactness.

A *uniformization* of a relation $R \subseteq \mathbb{V}^2$ is a function $F \subseteq R$ with $\mathrm{dom}(F) = \mathrm{dom}(R)$. The *uniformization axiom* states that we can simultaneously choose elements from a family of classes as long as it is indexed by a discrete set:

Uniformization If $\mathrm{dom}(R)$ is a discrete set, R has a uniformization.

Unless the relation is empty, its uniformization will be a set by the additivity axiom. Therefore the uniformization axiom can be expressed with at most one universal and no existential quantification over classes, and therefore still be equivalently formulated in a first-order way, using axiom schemes. Let us denote by **ESU** respectively **TSU** essential respectively topological set theory with uniformization.

In these theories, at least all discrete sets are well-orderable. The following proof goes back to S. Fujii and T. Nogura ([FN99]). We call $f : \Box a \to a$ a *choice function* if $f(b) \in b$ for every $b \in \Box a$.

Proposition 17 (ESU). A set a is well-orderable iff it is Hausdorff and there exists a continuous choice function $f : \Box a \to a$, such that $b \setminus \{f(b)\}$ is closed for all b.

In particular, every discrete set is well-orderable and in bijection to $\kappa^{\oplus}$ for some cardinal κ.

Proof. If a is well-ordered, we only have to define $F(b) = \min(b)$. In a well-order, the minimal element is always isolated, so $b \setminus F(b)$ is in fact closed. To show that F is a set, let $c \subseteq a$ be closed. Then the preimage of c consists of all nonempty subsets of a whose minimal element is in c. Assume $b \notin F^{-1}[c]$, that is $F(b) \notin c$. Then

$$(\Box a \ \cap \ \Diamond((-\infty, F(b)] \cap c)) \ \cup \ \Box[F(b)+1, \infty)$$

is a closed superset of $F^{-1}[c]$ omitting b, where by $F(b)+1$ we denote the successor of $F(b)$, and if $F(b)$ is the maximal element, we consider the right part of the union to be empty. Hence $F^{-1}[c]$ is in fact closed, proving that F is continuous and a set.

For the converse, assume now that f is a continuous choice function. A set $p \subseteq \Box a$ is an *approximation* if:

- $a \in p$
- p is well-ordered by reverse inclusion $\supseteq$.

- For every nonempty proper initial segment $Q \subset p$, we have $\bigcap Q \in p$.
- For every non-maximal $b \in p$, we have $b \setminus \{f(b)\} \in p$.

We show that two approximations p and q are always initial segments of one another, so they are well-ordered by inclusion: Let Q be the initial segment they have in common. Since both contain $b = \bigcap Q$, that intersection must be in Q and hence the maximal element of Q. If b is not the maximum of either p or q, both contain $b \setminus \{f(b)\}$, which is a contradiction because that is not in Q.

Thus the union P of all approximations is well-ordered. Assume $\bigcap P$ has more than one element. Then $P \cup \{\bigcap P, \bigcap P \setminus f(\bigcap P)\}$ were an approximation strictly larger than P. Thus $\bigcap P$ is empty or a singleton. Since there is no infinite descending chain, and for every bounded ascending chain $Q \subseteq P$, we have $\bigcap Q \in P$, P is closed, so $P \in \mathbb{V}$. Also, $\supseteq$ is a set-well-order on P. Thus a is also set-well-orderable, because $f \restriction P$ is a continuous bijection onto a:

Firstly, it is injective, because after the first b with $f(b) = x$, x is omitted. Secondly, it is surjective, because if $b \in P$ is the first element not containing x, it cannot be the intersection of its predecessors and thus has to be of the form $b = c \setminus \{f(c)\}$. Hence $x = f(c)$. If x is a member of every element of P, then $\bigcap P = \{x\} \in P$ and $x = f(\{x\})$.

Now let a be a discrete set. We only have to prove that a continuous choice function $f : \Box a \to a$ exists. In fact, any choice function will do, since $\Box a$ is discrete and hence every function on $\Box a$ is continuous. And the existence of such a function follows from the uniformization axiom, applied to the relation $R \subseteq \Box d \times d$ defined by: xRy iff $y \in x$.

It follows that every discrete set a is well-orderable. Therefore, it is comparable in length to *On*. If an initial segment of a were in bijection to *On*, then as the image of a discrete set, *On* would be a set. Hence a must be in bijection to a proper initial segment $\alpha^{\oplus}$ of *On*. If κ is the cardinality of α, there is a bijection between $\kappa^{\oplus}$ and $\alpha^{\oplus}$. Composing these bijections proves the claim. ■

It follows that there exists an infinite discrete set iff $\omega \in On$. The uniformization axiom also allows us to define for every infinite cardinal κ a cardinal 2^{κ}, namely the least ordinal in bijection to $\Box\kappa^{\oplus}$. Proposition 17 then shows that, just as in **ZFC**, *On* is not only a weak but even a strong limit.

Like the axiom of choice, the uniformization axiom could be stated in terms of products. Of course, it only speaks of products of $\mathcal{D}$-few factors at first, but surprisingly it even has implications for larger products as long as the factors are indexed by a $\mathcal{D}$-compact well-ordered set. $\mathcal{D}$-compactness for a well-ordered set just means that no subclass of cofinality $\geqslant$ *On* is closed.

Proposition 18 (**ESU** $+ T_3 +$ Union)**.** Let w be a $\mathcal{D}$-compact well-ordered set, $a \in \mathbb{V}$ and $a_x \subseteq a$ nonempty for every $x \in w_I$. Then the product $\prod_{x \in w_I} a_x$ is nonempty.

Proof. Recall that the product is defined as:

$$\prod_{x \in w_I} a_x \quad = \quad \left\{ F \cup (w' \times a) \;\mid\; F : w_I \to \mathbb{V},\ \forall x\ F(x) \in a_x \right\}$$

We do induction on the length of w and we have to distinguish three cases:

Case 1: If w has no greatest element, its cofinality must be $\mathcal{D}$-small or else it would not be $\mathcal{D}$-compact Hausdorff. So let $\langle y_\alpha \mid \alpha < \kappa \rangle$ be a cofinal strictly increasing sequence. Using the induction hypothesis and the uniformization axiom, choose for every $\alpha < \kappa$ an element

$$f_\alpha \quad \in \quad \prod_{x \in]y_\alpha, y_{\alpha+1}]_I} a_x$$

Then the union of the f_α is an element of $\prod_{x \in w_I} a_x$.

Case 2: Assume that w has a greatest element p and that $w \setminus \{p\}$ is a set. Then this is still $\mathcal{D}$-compact Hausdorff and hence the induction hypothesis applies, so there is an element $f : w \setminus \{p\} \to a$ of the product missing the last dimension. For any $y \in a_p$, the set $f \cup \langle p, y \rangle$ is in $\prod_{x \in w_I} a_x$.

Case 3: Finally assume that w has a greatest element p and that $w \setminus \{p\}$ is not a set. By the induction hypothesis,

$$P_y \;=\; \prod_{x \in [-\infty, y]_I} a_x$$

is a nonempty set for every $y < p$. The union $Q = \bigcup_{y<p} P_y$ is not a set, because otherwise its domain $\mathrm{dom}(\bigcup Q) = w \setminus \{p\}$ would also be a set. But since $Q \subseteq \Box(w \times a)$, it does have a closure which is a set, and this closure must have an element g with $p \in \mathrm{dom}(g)$. We will show that $f = g \cup (w' \times a)$ witnesses the claim, that is, $f \in \prod_{x \in w_I} a_x$.

If $z \in w_I$, then g is not in the closure of $\bigcup_{y<z} P_y$, because that is a subclass of the set $\Box((-\infty, z] \times a)$. Thus g is in the closure of $\bigcup_{z \leqslant y < p} P_y$, which is a subclass of:

$$M_z \quad = \quad \Box(w \times a) \;\cap\; \{ r \mid r \cap (\{z\} \times a) \in \Box_{\leqslant 1} a_z \}$$

If we can show that M_z is closed, we can deduce that $g \in M_z$ for every $z \in w_I$ and therefore $g \restriction w_I = f \restriction w_I$ is a function from w_I to a with $f(x) \in a_x$ for all $x \in w_I$. Thus f is indeed an element of the product.

To prove that M_z is closed in $\Box(w \times a) \cap \Diamond(\{z\} \times a_z)$, assume r is an element of the latter but not of the former. Then there are distinct $x_1, x_2 \in a_z$, such that $\langle z, x_1 \rangle, \langle z, x_2 \rangle \in z$. Since a_z is Hausdorff, there are u_1 and u_2, such that $x_1 \notin u_1$, $x_2 \notin u_2$ and $u_1 \cup u_2 = a$, and

$$\Box((w \setminus \{z\}) \times a \;\cup\; \{z\} \times u_1) \quad \cup \quad \Box((w \setminus \{z\}) \times a \;\cup\; \{z\} \times u_2)$$

is a closed superset of M_z omitting r. ∎

Some of the models of topological set theory we will encounter are ultrametrizable, which in the presence of the uniformization axiom is a very strong topological property. A set a is *ultrametrizable* if there is a decreasing sequence $\langle \sim_\alpha \mid \alpha \in On \rangle$ of equivalence relations on a such that $\bigcap_\alpha \sim_\alpha = \Delta_a$ and the *α-balls* $[x]_\alpha = \{ y \mid x \sim_\alpha y \}$ for $x \in a$ and $\alpha \in On$ are a base of the natural topology on a in the sense of open classes, that is, the relatively open classes $U \subseteq a$ are exactly the unions of balls. If that is the case, the α-balls partition a into clopen sets for every α.

Proposition 19 (ESU). Every ultrametrizable set is a $\mathcal{D}$-compact linearly orderable set.

Proof. For every $\alpha \in On$, the class C_α of all α-balls is a subclass of $\Box a$. If $b \in \Box a$ and $x \in b$, then $\Diamond[x]_\alpha$ is a neighborhood of b in $\Box a$ which contains only one element of C_α, namely $[x]_\alpha$. Hence C_α has no accumulation points and is therefore a discrete set. That means there are only $\mathcal{D}$-few α-balls for every $\alpha \in On$.

Now let $A \subseteq \Box a$ and $\bigcap A = \emptyset$. For each α, let B_α be the union of all α-balls which intersect every element of A. Then $\bigcap_\alpha B_\alpha = \emptyset$ and every B_α is closed.

Assume that all B_α are nonempty. Then for every α all but $\mathcal{D}$-few members of the sequence $\langle B_\alpha \mid \alpha{\in}On\rangle$ are elements of the closed set $\Box B_\alpha$, so every accumulation point must be in $\bigcap_{\alpha\in On}\Box B_\alpha$, which is empty. Thus $\{B_\alpha \mid \alpha{\in}On\}$ has no accumulation point and is a discrete subset of $\Box B_0$. Hence it is $\mathcal{D}$-small, which means that the sequence $\langle B_\alpha \mid \alpha{\in}On\rangle$ is eventually constant, a contradiction.

Therefore there is a B_α which is empty, and by definition every α-ball is disjoint from some element of A. Since there are only $\mathcal{D}$-few α-balls, the uniformization axiom allows us to choose for every α-ball $[x]_\alpha$ an element $c_{[x]_\alpha} \in A$ disjoint from $[x]_\alpha$. The set of these $c_{[x]_\alpha}$ is discrete and has an empty intersection. This concludes the proof of the $\mathcal{D}$-compactness.

Since it is discrete, the set C_α can be linearly ordered and there are only $\mathcal{D}$-few such linear orders for every α. If L is a linear order on C_α, let R_L be the partial order relation on a defined by xR_Ly iff $[x]_\alpha L[y]_\alpha$. R_L is a set because it is the union of $\mathcal{D}$-few sets of the form $[x]_\alpha \times [y]_\alpha$. Let S_α be the set of all such R_L. The sequence $\langle S_\alpha \mid \alpha{\in}On\rangle$ can only be eventually constant if a is discrete, in which case it is linearly orderable anyway. If a is not discrete, however, $S = \bigcup_\alpha S_\alpha$ must be $\mathcal{D}$-large and therefore have an accumulation point $\leqslant$ in $\Box a^2$. Because each S_α is $\mathcal{D}$-small, $\leqslant$ is in the closure of every $\bigcup_{\beta>\alpha} S_\beta$. For $x,y \in a$, let

$$t_{\alpha,x,y} \quad = \quad \Box(a^2 \setminus ([y]_\alpha \times [x]_\alpha)) \;\cap\; \Diamond\{\langle x,y\rangle\}.$$

We will show that $\leqslant$ is a linear order on a:

Assume $x \neq y$. Then there is an α such that $x \not\sim_\alpha y$. Every element of $\bigcup_{\beta>\alpha} S_\beta$ assigns an order to $[x]_\alpha$ and $[y]_\alpha$, so it is in exactly one of the disjoint closed sets $t_{\alpha,x,y}$ and $t_{\alpha,y,x}$. Therefore the same must be true of $\leqslant$, so we have $x \leqslant y$ iff not $y \leqslant x$. This proves antisymmetry and totality.

If $x \leqslant y \leqslant z$ and x,y,z are distinct, then there is an α such that $x \not\sim_\alpha y \not\sim_\alpha z \not\sim_\alpha x$. Then $\leqslant$ is in the closure of neither $t_{\alpha,y,x}$ nor $t_{\alpha,z,y}$, and must therefore be in the closure of

$$\bigcup_{\beta>\alpha} S_\beta \;\cap\; t_{\alpha,x,y} \;\cap\; t_{\alpha,y,z},$$

which is a subset of $t_{\alpha,x,z}$, because every element of S is transitive. It follows that $\leqslant$ is also in $t_{\alpha,x,z}$ and thus $x \leqslant z$, proving transitivity.

Finally, $\leqslant$ is reflexive because for every $x \in a$, all of S lies in the set $\Box a^2 \cap \Diamond\{\langle x,x\rangle\}$. ■

Another consequence of the uniformization axiom is the following law of distributivity:

Lemma 20 (ESU). If d is discrete and for each $i \in d$, J_i is a nonempty class, then

$$\bigcup_{i\in d}\bigcap_{j\in J_i} j \quad = \quad \bigcap_{f\in\prod_{i\in d} J_i}\ \bigcup_{i\in I} f(i).$$

Proof. If x is in the set on the left, there exists an $i \in d$ such that x is an element of every $j \in J_i$. Thus for every function f in the product, $x \in f(i)$. Hence x is an element of the right hand side.

Conversely, assume that x is not in the set on the left, that is, for every $i \in d$, there is a $j \in J_i$ such that $x \notin j$. Let f be a uniformization of the relation $R = \{\langle i,j\rangle \mid i \in d,\ x \notin j \in J_i\}$. Then $x \notin \bigcup_{i\in I} f(i)$. ■

It implies that we can work with subbases in the familiar way. Let us call $\mathcal{K}$ *regular* if every union of $\mathcal{K}$-few $\mathcal{K}$-small sets is $\mathcal{K}$-small again. Then in particular $\mathcal{D}$ is regular.

Lemma 21 (ESU). Let $\mathcal{K} \subseteq \mathcal{D}$ and let B be a $\mathcal{K}$-subbase of a topology T such that the union of $\mathcal{K}$-few elements of B always is an intersection of elements of B. Then B is a base of T.

Proof. We only have to prove that the intersections of elements of B are closed with respect to $\mathcal{K}$-small unions and therefore constitute a $\mathcal{K}$-topology. But if I is $\mathcal{K}$-small, and each $\langle b_{i,j} \mid j \in J_i\rangle$ is a family in B, we have

$$\bigcup_{i\in I}\bigcap_{j\in J_i} b_{i,j} \quad = \quad \bigcap_{f\in\prod_{i\in I} J_i}\ \bigcup_{i\in I} b_{i,f(i)}$$

by Lemma 20, and every $\mathcal{K}$-small union $\bigcup_{i\in I} b_{i,f(i)}$ is an element of B again. ■

Thus if $\mathcal{K}$ is regular and S is a $\mathcal{K}$-subbase of T, the class of all $\mathcal{K}$-small unions of elements of S is a base of T. Since $\bigcup_i \Diamond a_i = \Diamond \bigcup_i a_i$, the sets of the following form constitute a base of the exponential $\mathcal{K}$-topology:

$$\Diamond_T a \cup \bigcup_{i\in I} \Box_T b_i,$$

where I is $\mathcal{K}$-small and $a, b_i \in T$ for all $i \in I$. As that is sometimes more intuitive, we also use open classes in our arguments instead of closed sets. By setting $U = \complement a$ and $V_i = \complement b_i$, we obtain that every open class is a union of classes of the following form:

$$\Box_T U \cap \bigcap_{i\in I} \Diamond_T V_i$$

That is, these constitute a base in the sense of *open* classes. Since $\Box U = \Box U \cap \Diamond U$, the class U can always be assumed to be the union of the V_i.

Lemma 21 also implies that given a class B, the weak comprehension principle suffices to prove the existence of the topology $\mathcal{K}$-generated by B: A set c is closed iff for every $x \in \complement a$, there is a discrete family $(b_i)_{i \in I}$ in B, such that $c \subseteq \bigcup_i b_i$ and $x \notin \bigcup_i b_i$. In particular, the $\mathcal{K}$-topology of $\mathrm{Exp}_{\mathcal{K}}(X)$ exists (as a class) whenever the topology of X is a set.

Lemma 22 (ESU). Let $\mathcal{K}$ be regular and X a $\mathcal{K}$-topological T_0-space.

1. If X is T_1, then $\mathrm{Exp}_{\mathcal{K}}(X)$ is T_1 (but not necessarily conversely).
2. X is T_3 iff $\mathrm{Exp}_{\mathcal{K}}(X)$ is T_2.
3. X is T_4 iff $\mathrm{Exp}_{\mathcal{K}}(X)$ is T_3.
4. $D \subseteq X$ is dense in X iff the $\mathcal{K}$-small subsets of D are dense in $\mathrm{Exp}_{\mathcal{K}}(X)$.

Proof. In this proof we use $\Box$ and $\Diamond$ with respect to X, not the universe, so if T is the topology of X, we set $\Box a = \Box_T a$ and $\Diamond a = \Diamond_T a$.

(1): For $a \in \mathrm{Exp}_{\mathcal{K}}(X)$, the singleton $\{a\} = \Box a \cap \bigcap_{x \in a} \Diamond\{x\}$ is closed in $\mathrm{Exp}_{\mathcal{K}}(X)$.

(As a counterexample to the converse consider the case where $\mathcal{K} = \kappa$ is a regular cardinal number and $X = (\kappa+1)^{\oplus}$, with the κ-topology generated by the singletons $\{\alpha\}$ for $\alpha < \kappa$. This is not T_1, because $\{\kappa\}$ is not closed, but it is clearly T_0. We show that its exponential κ-topology is T_1: Let $a \in \mathrm{Exp}_{\mathcal{K}}(X)$. Then either $a \subseteq \kappa$ is small or $a = X$.

In the first case, $\{a\} = \Box a \cap \bigcap_{x \in a} \Diamond\{x\}$ is closed. In the second case, $\{a\} = \{X\} = \bigcap_{x \in \kappa} \Diamond\{x\}$ is also closed.)

(2): ($\Rightarrow$) Let $a, b \in \mathrm{Exp}_{\mathcal{K}}(X)$ be distinct, wlog $x \in b \setminus a$. Then there are disjoint open $U, V \subseteq X$ separating x from a. Hence $\Diamond U$ and $\Box V$ separate b from a.

($\Leftarrow$) Firstly, we have to show that X is T_1. Assume that $\{y\}$ is not closed, so there exists some other $x \in \mathrm{cl}(\{y\})$, and by T_0, y is not in the closure of x, so $\mathrm{cl}(\{x\}) \subset \mathrm{cl}(\{y\})$. The two closures can be separated by open base classes $\Box U \cap \bigcap_i \Diamond U_i$ and $\Box V \cap \bigcap_j \Diamond V_j$ of $\mathrm{Exp}_{\mathcal{K}}(X)$, whose intersection $\Box(U \cap V) \cap \bigcap_i \Diamond U_i \cap \bigcap_j \Diamond V_j$ is emtpy. Hence there either exists a U_i disjoint from V – which is impossible because $\mathrm{cl}(\{x\}) \in \Box V \cap \bigcap_i \Diamond U_i$ –, or there is a V_j disjoint from U: But since $V_j \cap \mathrm{cl}(\{y\}) \neq \emptyset$, we have $y \in V_j$. Hence $y \notin U \ni x$, contradicting the assumption that x is in the closure of y.

Now let $x \notin a$. Then a and $b = \{x\} \cup a$ can be separated by open base classes $\Box U \cap \bigcap_i \Diamond U_i$ and $\Box V \cap \bigcap_j \Diamond V_j$ of $\mathrm{Exp}_{\mathcal{K}}(X)$, whose intersection $\Box(U \cap V) \cap \bigcap_i \Diamond U_i \cap \bigcap_j \Diamond V_j$ is emtpy. Hence there either exists a U_i disjoint from V – which is impossible because $a \in \Box V \cap \bigcap_i \Diamond U_i$ –, or there is a V_j disjoint from U: Then V_j and U separate x from a, because b meets V_j and a does not, so $x \in V_j$.

(3): In both directions, the T_1 property follows from the previous points.

($\Rightarrow$) Let $a \notin c$, $a \subseteq X$ closed and $c \subseteq \mathrm{Exp}_{\mathcal{K}}(X)$ closed. Wlog[6] let c be of the form $\Box b$ or $\Diamond b$ with closed $b \subseteq X$. In the first case, $a \nsubseteq b$, so let U,V separate some $x \in a \setminus b$ from b. Then $\Diamond U, \Box V$ separate $\{a\}, c$. In the second case, $a \cap b = \emptyset$, so let U,V separate them. Then $\Box U, \Diamond V$ separate $\{a\}, c$.

($\Leftarrow$) Now let $\mathrm{Exp}_{\mathcal{K}}(X)$ be T_3 and let $a,b \subseteq X$ be closed, nonempty and disjoint. Then $\{a\}$ and $\Diamond b$ are disjoint and can be separated by disjoint open $U,V \subseteq \mathrm{Exp}_{\mathcal{K}}(X)$. U can be assumed to be an open base class, so $U = \Box W \cap \bigcap_i \Diamond W_i$. We claim that $\mathrm{cl}(W) \cap b = \emptyset$, which proves the normality of X. So assume that there exists $x \in \mathrm{cl}(W) \cap b$. Then $a \cup \{x\} \in \Diamond b$, so one of the open base classes $\Box Z \cap \bigcap_j \Diamond Z_j$ constituting V must contain $a \cup \{x\}$. That means that either one of the Z_j must be disjoint from W – which is impossible because $x \in Z_j$ – or one of the W_i must be disjoint from Z – which also cannot be the case, because all W_i intersect a and $a \subseteq Z$.

(4): If D is not dense in X, there is a nonempty open U disjoint from D. Then $\Box U$ is nonempty, open and disjoint from the class S of all $\mathcal{K}$-small subsets of D.

Conversely, assume D is dense. If the open base class $\Box U \cap \bigcap_{i \in I} \Diamond V_i$ is nonempty, every $U \cap V_i$ is nonempty and thus by the uniformization axiom there is a family of elements $x_i \in U \cap V_i$. The set $\{x_i \mid i{\in}I\}$ is a $\mathcal{K}$-small subset of D and an element of that base class. ∎

A space X is *locally $\mathcal{K}$-compact* if every point in X has a $\mathcal{K}$-compact neighborhood.

Lemma 23. If $\mathcal{K}$ is regular and X is a locally $\mathcal{K}$-compact Hausdorff space, then $\mathrm{Exp}^c_{\mathcal{K}}(X)$ is locally $\mathcal{K}$-compact Hausdorff, too.

Proof. Let $a,b \in \mathrm{Exp}^c_{\mathcal{K}}(X)$ be distinct and wlog assume $x \in a \setminus b$. Since X is Hausdorff, x can be separated from every $y \in b$ by disjoint open $U_y \ni x$ and $V_y \ni y$. Since b is $\mathcal{K}$-compact, there is a $\mathcal{K}$-small $I \subseteq b$ such that the V_y with $y \in I$ cover b. Then $U = \bigcap_{y \in I} U_y$ and $V = \bigcup_{y \in I} V_y$ separate x from b and $\Diamond U$ and $\Box V$ separate a and b in $\mathrm{Exp}^c_{\mathcal{K}}(X)$, proving that it is Hausdorff.

Now let us prove that every $a \in \mathrm{Exp}^c_{\mathcal{K}}(X)$ has a $\mathcal{K}$-compact neighborhood: Every $x \in a$ has a $\mathcal{K}$-compact neighborhood U_x and again, a $\mathcal{K}$-small set $\{U_x \mid x{\in}I\}$ of them covers a. Then the neighborhood $\Box \bigcup_{x \in I} U_x$ of a is $\mathcal{K}$-compact. ∎

1.8 Compactness

Hyperuniverses are $\mathcal{D}$-compact Hausdorff spaces, so $\mathcal{D}$-compactness is another natural axiom to consider. In the case $\mathbb{V} \notin \mathbb{V}$, the corresponding statement would be that every set is $\mathcal{D}$-compact (note that this is another axiom provable in **ZFC**), but if $\mathbb{V} \in \mathbb{V}$, this is equivalent

[6] To verify that a space X is T_3 it suffices to separate each point x from each subbase set b not containing x: Firstly, the $\mathcal{K}$-small unions of subbase sets b are a base, so if x is not in a $\mathcal{K}$-small union $\bigcup_i b_i$, it can be separated with U_i, V_i from every b_i, and $\bigcap U_i, \bigcup V_i$ separate x from the union. This shows that x can then be separated from each base set. Secondly, every closed set is an intersection $\bigcap_i b_i$ of base sets b_i, and if x is not in that intersection, there is an i with $x \notin b_i$ and if U_i, V_i separate x from b_i, they also separate x from $\bigcap_i b_i$.

to $\mathbb{V}$ being $\mathcal{D}$-compact. And in fact, **TSU** with a $\mathcal{D}$-compact Hausdorff $\mathbb{V}$ implies most of the additional axioms we have looked at so far, including the separation properties and the union axiom:

Let $a \subseteq \mathbb{T}$ and $x \notin \bigcup a$. Then for every $y \in a$, there is a b such that $y \subseteq \mathrm{int}(b)$ and $x \notin b$. The sets $\Box\mathrm{int}(b)$ then cover a and by $\mathcal{D}$-compact Hausdorffness, a discrete subfamily also does. But then the union of these b is a superset of $\bigcup a$ not containing x.

Another consequence of global $\mathcal{D}$-compactness is that most naturally occurring topologies coincide: Point (2) of the following theorem not only applies to hyperspaces $\Box a$, but also to products, order topologies and others. If the class of atoms is closed and unions of sets are sets, this even characterizes compactness (note that these two assumptions are only used in (3) $\Rightarrow$ (1)):

Theorem 24 (**ESU** $+ T_2 +$ Union)**.** If $\mathbb{A}$ is $\mathbb{T}$-closed, the following statements are equivalent:

1. Every set is $\mathcal{D}$-compact, that is: If $\bigcap A{=}\emptyset$, there is a discrete $d{\subseteq}A$ with $\bigcap d{=}\emptyset$.
2. Every Hausdorff $\mathcal{D}$-topology $T \in \mathbb{V}$ equals the natural topology: $T = \Box\bigcup T$
3. For every set a, the exponential $\mathcal{D}$-topology on $\Box a$ equals the natural topology.

Proof. (1) $\Rightarrow$ (2): Let $A = \bigcup T$. Since A is T-closed in A, $A \in T$ and thus $A \in \mathbb{V}$. By definition, $T \subseteq \Box A$. For the converse, we have to verify that each $b \in \Box A$ is T-closed, so let $y \in A \setminus b$. Consider the class C of all $u \in T$, such that there is a $v \in T$ with $u \cup v = A$ and $y \notin v$. By the Hausdorff axiom, for every $x \in b$ there is a $u \in C$ omitting x, so $b \cap \bigcap C = \emptyset$. By $\mathcal{D}$-compactness, there is a discrete $d \subseteq C$ with $b \cap \bigcap d = \emptyset$. By definition of C, $y \in \mathrm{int}_T(u)$ for every u, and since d is discrete, the intersection $\bigcap_{u \in d} \mathrm{int}_T(u)$ is open. Therefore, every $y \notin b$ has a T-open neighborhood disjoint from b.

(2) $\Rightarrow$ (3) is trivial, because as a $\mathcal{D}$-compact Hausdorff $\mathcal{D}$-topological space, a is T_3 and hence $\Box a$ is Hausdorff by Lemma 22.

(3) $\Rightarrow$ (1): Lemma 22 also implies that if $\Box\Box a$ is T_2, then $\Box a$ is T_3 and a is T_4, so it follows from the Hausdorff axiom that every set is normal.

Finally, we can prove $\mathcal{D}$-compactness. Let $A \subseteq \Box a$, $\bigcap A = \emptyset$ and let $c = \mathrm{cl}(A)$. Then $\bigcap c = \emptyset$. Since every set is regular and $\mathbb{A}$ is closed, the positive specification principle holds. Therefore

$$B \quad = \quad \left\{ b{\in}\Box c \mid \bigcap b \neq \emptyset \right\} \quad = \quad \{ b{\in}\Box c \mid \exists x\ \forall y{\in}b\ x{\in}y \}$$

is a closed subset of $\Box c$ not containing c. In particular, there is an open base class

$$\Box U \ \cap \ \bigcap_{i \in I} \Diamond V_i$$

of the space $\Box c$ containing c which is disjoint from B. Every $U \cap V_i$ is a relatively open subset of c, so there is an $x_i \in A \cap U \cap V_i$, because A is dense in c. The set $\{x_i \mid i{\in}I\}$ – and here we used the uniformization axiom – then is a discrete subcocover of A. ■

Chapter 2

Models of Topological Set Theories

This chapter deals with hyperuniverses, that is, models of **TS** whose classes are *all* subclasses of a κ-compact Hausdorff space, for some cardinal κ. Since it would be confusing to always have to discern the natural topology from the hyperuniverse's topology, we will work in **ZFC** now, although many of these constructions would be possible in $\mathbf{ES}_\infty$, too. But as we have established that **ZFC** is interpretable in $\mathbf{ES}_\infty$, this is no limitation anyway. And because we have no need for "real" atoms or a universal set, we use the symbols $\mathbb{A}$, $\mathbb{S}$ and $\mathbb{V}$ to denote the atom, set and universe spaces of the models we construct.

2.1 Hyperuniverses

A κ-*hyperuniverse* is a κ-topological Hausdorff space $\mathbb{V}X$ with at least two points, together with a closed subset $\mathbb{S}X \subseteq \mathbb{V}X$ and a homeomorphism $\Sigma^X : \mathbb{S}X \to \operatorname{Exp}_\kappa(\mathbb{V}X)$. We call $\mathbb{A}X = \mathbb{V}X \setminus \mathbb{S}X$ the *atom space* of X, $\mathbb{V}X$ the *universe space* and $\mathbb{S}X$ the *set space*.

We omitted the condition that the set of atoms of the κ-hyperuniverse does not contain any subsets of $\mathbb{V}X$, because in the well-founded realm of **ZFC** it is irrelevant: Just replace $\mathbb{A}X$ by $\mathbb{A}X \times \{\mathbb{V}X\}$, for example, to fulfill this additional requirement. Having done that it follows from Proposition 10 that $\mathcal{I}$ is an interpretation of **TS**, where the domain of $\mathcal{I}$ is $\mathcal{P}(\mathbb{V}X) \cup \mathbb{A}X$, its atoms are $\mathbb{A}^\mathcal{I} = \mathbb{A}X$ and $a \in^\mathcal{I} b$ is defined as

$$b \notin \mathbb{A}X \qquad \wedge \qquad a \;\in\; (b \cap \mathbb{A}X) \;\cup\; \Sigma^X[b \cap \mathbb{S}X].$$

The axiom of infinity holds in $\mathcal{I}$ iff $\kappa > \omega$. In that case, κ must be inaccessible, as Theorem 28 will show. By definition $\mathcal{I}$ interprets the Hausdorff and uniformization axioms and, as will also follow from Theorem 28, $\mathcal{D}$-compactness.

Let us call a κ-hyperuniverse X *clopen* if $\mathbb{S}X$ is clopen in $\mathbb{V}X$, and *atomless* if $\mathbb{S}X = \mathbb{V}X$. Clopen κ-hyperuniverses are of particular interest because by Proposition 15, they are models of GPF comprehension.

At first glance, these interpretations $\mathcal{I}$ never satisfy $\emptyset \in \mathbb{V}$. But if we pick an element $a \in \mathbb{A}X$ and define $\mathbb{A}^\mathcal{I} = \mathbb{A}X \setminus \{a\}$ instead of $\mathbb{A}X$, then $\emptyset^\mathcal{I} = a$. In other words, the empty set is just an atom that has been awarded set status. Therefore a clopen κ-hyperuniverse with exactly one atom provides a model of $\mathbf{GPK}^+$.

Lemma 25. A space X is κ-compact iff there is no continuous[1] strictly descending sequence of nonempty closed sets $\langle C_\alpha \mid \alpha < \lambda \rangle$ with $\bigcap_{\alpha<\lambda} C_\alpha = \emptyset$, whose length is a regular cardinal $\lambda \geqslant \kappa$.

Proof. If such a sequence exists, then its members constitute a cocover of X. Every small subset's indices are bounded by some $\alpha < \lambda$, so its intersection is a superset of $C_\alpha \neq \emptyset$. Thus the sequence disproves κ-compactness.

Assume conversely that X is not κ-compact. Let λ be the least cardinal such that there is a cocover $\{A_\beta \mid \beta < \lambda\}$ with no κ-small subcocover. For each $\alpha < \lambda$, define $C_\alpha = \bigcap_{\beta<\alpha} A_\beta$. Since no κ-small subset of $\{A_\beta \mid \beta < \alpha\}$ is a cocover and λ is minimal, C_α must be nonempty. Every cofinal subsequence of $\langle C_\alpha \mid \alpha < \lambda \rangle$ is also a cocover without a κ-small subcocover, so it must have length λ. Hence λ is regular. In particular, the sequence can be replaced by a strictly descending subsequence. ■

The following argument is based on an idea of J. Keesling ([Kee70]).

Lemma 26. Let X have a dense subset D of size μ and a closed discrete subset C of size ν, such that $2^\mu < 2^\nu$. Then X is not normal.

Proof. If μ were finite, then $X = D$ would have size μ, which is impossible because $C \subseteq X$ has size ν. Hence $\mu, \nu \geqslant \aleph_0$.

Since D is dense, each of the $(2^\omega)^\mu = 2^\mu$ functions from D to $\mathbb{R}$ can be continuously extended in at most one way to all of X. Thus there are at most 2^μ continuous functions from X to $\mathbb{R}$.

If X were normal, by the Tietze extension theorem[2] each of the $(2^\omega)^\nu = 2^\nu$ functions from C to the unit interval could be continuously extended to X, so there would exist at least 2^ν distinct continuous real-valued functions on X, a contradiction. ■

Lemma 27. Let $\lambda \geqslant \kappa$ be a regular cardinal, endowed with a topology at least as fine as its order-κ-topology. Then $\mathrm{Exp}_\kappa(\lambda)$ is not normal.

Proof. First assume that $\lambda = \kappa$, that is, λ is a discrete space of size κ. By Lemma 26, it suffices to find a dense $D \subseteq \mathrm{Exp}_\kappa(\lambda)$ of size κ and a closed discrete $C \subseteq \mathrm{Exp}_\kappa(\lambda)$ of size

[1] In the sense that $C_\alpha = \bigcap_{\beta<\alpha} C_\beta$ for each limit ordinal β.

[2] Every bounded real-valued continuous function defined on a closed subset of a normal space X can be continuously extended to all of X. (cf. [Kel68, Eng89])

2^κ. For D, we can simply take the small subsets of X, because $\kappa = \kappa^{<\kappa}$. To construct C, first partition λ into κ-large X_1, X_2, and let $f_i : X \to X_i$ be bijections. For each $A \subseteq \lambda$ let $F(A) = f_1[A] \cup f_2[\lambda \setminus A]$. Then let $C = \{F(A) \mid A \subseteq \lambda\}$. To prove that C is closed, let $B \notin C$, so $f_1^{-1}[B \cap X_1] \neq \complement f_2^{-1}[B \cap X_2]$:

- Either $f_1^{-1}[B \cap X_1]$ and $f_2^{-1}[B \cap X_2]$ have some point x in common: then $\Diamond\{f_1(x)\} \cap \Diamond\{f_2(x)\}$ is an open neighborhood of B disjoint from C,
- or there is some point x not contained in any of the two sets: then $\Box\complement\{f_1(x), f_2(x)\}$ is such a neighborhood.

To see that it is also discrete, let $F(A) \in C$. Then $\Box F(A)$ is a neighborhood disjoint from the rest of C.

Now assume that $\lambda > \kappa$. Both $Y = \bigcap_{\alpha<\lambda} \Diamond(\lambda \setminus \alpha)$ and $Z = \{\{\alpha\} \mid \alpha < \lambda\}$ are closed in $\mathrm{Exp}_\kappa(\lambda)$, and they are disjoint. Assume that they can be separated by open sets. Then in particular, there is an open $U \supseteq Y$, whose closure is disjoint from Z.

We recursively define κ-small sets $A_\alpha \in U$ for $\alpha < \kappa$: Let $\gamma_\alpha = 1 + \sup\left(\bigcup_{\beta<\alpha} A_\beta\right)$. Then $\lambda \setminus \gamma_\alpha$ is a clopen element of Y and therefore in U. Thus $U \cap \Box(\lambda \setminus \gamma_\alpha)$ is nonempty and, being open, contains a κ-small element A_α. Now the sequence $\langle A_\alpha \mid \alpha < \kappa\rangle$ lies in the κ-compact set $\Box(1+\gamma)$ with $\gamma = \sup_{\alpha<\kappa} \gamma_\alpha$ and hence has an accumulation point B. For every $\alpha < \kappa$, all A_β with $\beta > \alpha$ are members of the closed set $\Box(\lambda \setminus \gamma_\alpha)$, so B must be in their intersection $\Box(\lambda \setminus \gamma)$. Hence $B = \{\gamma\}$, which is in Z, a contradiction. ∎

Theorem 28. Every κ-hyperuniverse is κ-compact. And if there exists a κ-hyperuniverse, then κ is strongly inaccessible or ω.

Proof. By Lemma 25, if $\mathbb{V}X$ is not κ-compact, $\mathrm{Exp}_\kappa(\mathbb{V}X)$ has a closed subset

$$A = \{C_\alpha \mid \alpha < \lambda\} = \Box C_0 \cap \bigcap_{\alpha<\lambda} \left(\Box C_{\alpha+1} \cup \bigcap_{x \in C_\alpha} \Diamond\{x\} \right)$$

from which there exists a continuous bijection $C_\alpha \mapsto \alpha$ to a regular cardinal $\lambda \geqslant \kappa$. By Lemma 27, $\Box A$ is nonnormal. Hence Lemma 22 implies that $\Box\Box A$ is not regular and $\Box\Box\Box A$ is not Hausdorff. But

$$(\Sigma^X \circ \Sigma^X \circ \Sigma^X \circ \Sigma^X)^{-1}[A]$$

is a closed subset of the Hausdorff space $\mathbb{V}X$ homeomorphic to A, a contradiction. Hence $\mathbb{V}X$ is κ-compact.

If $\mathbb{V}X$ were discrete, then $|\mathbb{V}X| + 1 = |\mathrm{Exp}_\kappa(\mathbb{V}X)| + 1 + |\mathbb{A}X| = 2^{|\mathbb{V}X|} + |\mathbb{A}X|$, which is only possible for $|\mathbb{V}X| = 1$. But a κ-hyperuniverse has more than one point and hence $\mathbb{V}X$ is a nondiscrete κ-compact κ-topological space.

If κ is singular, then every set B of closed sets with $|B| = \kappa$ is the union $\bigcup_{\alpha<\gamma} B_\alpha$ of κ-small sets $B_\alpha \subseteq B$ for some $\gamma < \kappa$, and hence $\bigcup B = \bigcup_{\alpha<\gamma} \bigcup B_\alpha$ is closed. Thus every κ-additive space is κ^+-additive. So $\mathbb{V}X$ is a κ-compact κ^+-additive nondiscrete Hausdorff space. In particular there exists a subset $A \subseteq \mathbb{V}X$ of size κ, which by κ^+-additivity is discrete. But then $\{A \setminus \{a\} \mid a \in A\}$ is a cocover with no κ-small subcocover, which contradicts κ-compactness. Hence κ must be regular.

Now let $\gamma < \kappa$ be infinite. Then there is a set $A \subseteq \mathbb{V}X$ of size γ and A is closed and discrete. Hence $\Box A$ is a closed discrete subset of $\mathrm{Exp}_\kappa(\mathbb{V}X)$ of size 2^γ, which is only possible if $2^\gamma < \kappa$. This proves that κ is a strong limit. ■

2.2 Mild Ineffability and Topology

In [Ess03], O. Esser constructs hyperuniverses of a given uniform weight using mildly ineffable cardinals, which form a hierarchy reaching from weakly compact to strongly compact cardinals. In fact, mild ineffability is equivalent to several topological properties related to hyperuniverses which will be important to us. To describe these equivalences, we need to introduce variants of several topological concepts in the context of κ-topologies first. By a *space* we will mean a κ-topological space. Since these results are well-known for ω, we assume in this section that κ is an uncountable regular cardinal.

The *weight* of a space X is the least cardinality of a basis of X. A map $f : X \to Y$ from a space X to a space Y is called κ-*proper* if the preimage of every κ-compact set is κ-compact.

We define the κ-*Alexandroff compactification* ωX of a Hausdorff space X as follows: Let $\omega X = X \cup \{p\}$ with some added point $p \notin X$ (say $p = X$) and $c : X \to \omega X$ the inclusion map. We endow ωX with the following κ-topology: A set a is closed if it either is a κ-compact subset of X or if $a = b \cup \{p\}$ and b is closed in X. Then c is an embedding, ωX is κ-compact and:

Lemma 29. If X is Hausdorff and locally κ-compact, then ωX is Hausdorff.

Let Y be κ-compact Hausdorff, $U \subseteq Y$ open and $f : U \to X$ continuous and κ-proper. Then there is a continuous map $\omega f : Y \to \omega X$ such that $c \circ f = \omega f \restriction U$ and $\omega f[Y \setminus U] \subseteq \{p\}$.

Proof. Let $x, y \in \omega X$ be distinct points. First assume that they both are in X. Then there are open $U, V \subseteq X$ which separate them. By definition, U and V are open in ωX, too. Now assume that one of the points is p, wlog $x = p$. Since X is locally κ-compact, there is an open $U \ni y$ in X such that $\mathrm{cl}(U)$ is κ-compact Hausdorff. Then $(\omega X) \setminus \mathrm{cl}(U)$ and U are disjoint open neighborhoods of x and y in ωX.

We define ωf as follows: If $y \notin U$, let $\omega f(y) = p$, otherwise $\omega f(y) = c(f(y))$. We only have to show that ωf is continuous. Let $a \subseteq \omega X$ be a base set. There are two cases:

Firstly, assume $a = b \cup \{p\}$. Then b is closed in X and $(\omega f)^{-1}[a] = f^{-1}[b] \cup \complement U$. By the continuity of f and openness of U, this is in fact a closed subset of Y.

Secondly, assume a is a κ-compact subset of X. Then $(c \circ f)^{-1}[a] = (\omega f)^{-1}[a]$ is κ-compact by the κ-properness of f and therefore closed in Y. ■

Lemma 30. If $\mathcal{B}$ is a base of a space X, the sets

$$\left(\Diamond\bigcap_{a\in A} a\right) \cup \bigcup_{b\in B} \Box b$$

with κ-small sets $A, B \subseteq \mathcal{B}$ form a base of $\mathrm{Exp}^c_\kappa(X)$. In particular, the weight of $\mathrm{Exp}^c_\kappa(X)$ is at most $|\mathcal{B}|^{<\kappa}$, and if $\mathcal{B}$ is closed with respect to κ-small intersections, the sets $\Diamond a$ and $\Box b$ with $a, b \in \mathcal{B}$ form a κ-subbase of $\mathrm{Exp}^c_\kappa(X)$.

Proof. By definition of the exponential κ-topology, the sets M of the following form constitute a base, where $\widetilde{A}$ and $\widetilde{B}_i$ for $i \in I$ are arbitrary subsets of $\mathcal{B}$ and I is a κ-small index set:

$$M = \left(\Diamond\bigcap_{a\in\widetilde{A}} a\right) \cup \bigcup_{i\in I}\Box\bigcap_{b\in\widetilde{B}_i} b = \left(\Diamond\bigcap_{a\in\widetilde{A}} a\right) \cup \bigcup_{i\in I}\bigcap_{b\in\widetilde{B}_i}\Box b$$

If a point $c \in \mathrm{Exp}^c_\kappa(X)$ is not in $\Diamond\bigcap_{a\in\widetilde{A}} a$, that means that $\widetilde{A}\cup\{c\}$ has an empty intersection and by the κ-compactness of c that there is a κ-small $A \subseteq \widetilde{A}$ such that $A\cup\{c\}$ has an empty intersection and hence $c \notin \Diamond\bigcap_{a\in A} a$. We apply this fact to the left side of the union operator and the distributive law to the right side:

$$\begin{aligned} M &= \left(\bigcap_{A\subseteq\widetilde{A}\ \kappa\text{-small}} \Diamond\bigcap_{a\in A} a\right) \cup \bigcap_{f\in\prod_{i\in I}\widetilde{B}_i}\bigcup_{i\in I}\Box f(i) \\ &= \bigcap_{A\subseteq\widetilde{A}\ \kappa\text{-small}}\ \bigcap_{f\in\prod_{i\in I}\widetilde{B}_i}\left(\left(\Diamond\bigcap_{a\in A} a\right)\cup\bigcup_{i\in I}\Box f(i)\right) \end{aligned}$$

Thus we have expressed M as an intersection of sets of the given form. ∎

Lemma 31. Every regular space of weight $\leqslant \lambda$ has a base of size $\leqslant \lambda^\omega$ consisting of clopen sets.

Proof. We claim that given a base $\mathcal{B}$, the set of all clopen unions of countable subsets of $\mathcal{B}$ is a base itself. To prove this, let U be open and $x \in U$. Then there is a $b_0 \in \mathcal{B}$ such that $x \in \complement b_0 \subseteq U$, because $\mathcal{B}$ is a base. But by regularity, for each $n \in \omega$, x can be separated from b_n by another set $\complement b_{n+1}$ with $x \notin b_{n+1} \in \mathcal{B}$, such that $\mathrm{cl}(\complement b_{n+1}) \subseteq \complement b_n$. The set

$$\complement\left(\bigcup_{n\in\omega} b_n\right) = \bigcap_{n\in\omega}\complement b_n = \bigcap_{n\in\omega}\mathrm{cl}(\complement b_n) \ni x$$

is a clopen neighborhood of x and a subset of U. ∎

By a κ-*algebra* we mean a κ-distributive κ-complete Boolean algebra. A κ-*filter* on a κ-algebra $\mathcal{A}$ is a subset $\mathcal{F} \subseteq \mathcal{A}$ which is closed with respect to supersets and κ-small meets and which does not include 0. $\mathcal{F}$ *decides* a set $\mathcal{B} \subseteq \mathcal{A}$ if either $x \in \mathcal{F}$ or $-x \in \mathcal{F}$ for every $x \in \mathcal{B}$. If $\mathcal{F}$ decides all of $\mathcal{A}$, then $\mathcal{F}$ is a κ-*ultrafilter*.

We denote by $\mathbb{D}$ the discrete space $\{0,1\}$, and by $\mathbb{D}^\lambda_\kappa$ – or just $\mathbb{D}^\lambda$, if κ is clear from the context – the generalized *Cantor cube* $\prod_{\alpha<\lambda} \mathbb{D}$ with the product κ-topology.

A cardinal κ is *mildly λ-ineffable* if for every family $\langle f_x : x \to 2 \,|\, x \in \mathcal{P}_\kappa(\lambda)\rangle$, there exists an $f : \lambda \to 2$ such that for all $x \in \mathcal{P}_\kappa(\lambda)$, there is a $y \supseteq x$ such that $f_y \restriction x = f \restriction x$.

With these preliminaries, we can now state the equivalences, some of which are generalizations of properties of weakly compact (i.e. mildly κ-ineffable) cardinals κ found in [CN74].

Theorem 32. Let κ be inaccessible, $\lambda \geqslant \kappa$ regular and $\lambda^{<\kappa} = \lambda$. The following statements are equivalent:

1. κ is mildly λ-ineffable.
2. Every κ-filter on a κ-algebra can be extended to a κ-filter deciding a given set of size $\leqslant \lambda$.
3. *(Alexander's subbase theorem)* Let $\mathcal{B}$ be a κ-subbase of X of size $\leqslant \lambda$. If every subcocover of $\mathcal{B}$ has a κ-small subcocover, then X is κ-compact.
4. *(Tychonoff's theorem)* Every product of $\leqslant \lambda$ κ-compact spaces with weight $\leqslant \lambda$ is κ-compact.
5. The Cantor cube $\mathbb{D}^\lambda_\kappa$ is κ-compact.
6. $\mathrm{Exp}_\kappa(X)$ is κ-compact for every κ-compact X with weight $\leqslant \lambda$.
7. $\mathrm{Exp}_\kappa(\omega X)$ is κ-compact, where X is a discrete space of size λ.

Proof. **(1 ⇒ 2):** Let $\mathcal{F}$ be a κ-filter on the κ-algebra $\mathcal{A}$ and let $\mathcal{B} \subseteq \mathcal{A}$ have at most λ elements, namely $\mathcal{B} = \{b_\alpha \mid \alpha < \lambda\}$. For every $x \in \mathcal{P}_\kappa(\lambda)$, let $f_x : x \to 2$, that is $f_x \in \mathbb{D}^x$, be such that for all $z \in \mathcal{F}$,

$$z \wedge \bigwedge_{f_x(\alpha)=1} b_\alpha \wedge \bigwedge_{f_x(\alpha)=0} -b_\alpha \quad > \quad 0.$$

Such an f_x does indeed exist for every x: Otherwise there would exist for every $f \in \mathbb{D}^x$ a $z_f \in \mathcal{F}$, such that

$$z_f \wedge \bigwedge_{f(\alpha)=1} b_\alpha \wedge \bigwedge_{f(\alpha)=0} -b_\alpha \quad = \quad 0.$$

Since $2^{|x|} < \kappa$ and $\mathcal{F}$ is κ-complete, $z = \bigwedge_{f\in\mathbb{D}^x} z_f$ is in $\mathcal{F}$. Using κ-distributivity we would obtain a contradiction:

$$\begin{aligned} 0 &= \bigvee_{f\in\mathbb{D}^x} 0 = \bigvee_{f\in\mathbb{D}^x}\left(z \wedge \bigwedge_{f(\alpha)=1} b_\alpha \wedge \bigwedge_{f(\alpha)=0} -b_\alpha\right) \\ &= z \wedge \bigvee_{f\in\mathbb{D}^x}\left(\bigwedge_{f(\alpha)=1} b_\alpha \wedge \bigwedge_{f(\alpha)=0} -b_\alpha\right) = z \wedge 1 = z \in \mathcal{F} \end{aligned}$$

Now let $f \in \mathbb{D}^\lambda$ be the set granted by mild λ-ineffability. By our choice of f_x, the b_α for which $f(\alpha) = 1$, the $-b_\alpha$ for which $f(\alpha) = 0$ and $\mathcal{F}$ generate a κ-filter, because if $z \in \mathcal{F}$ and $x \in \mathcal{P}_\kappa(\lambda)$, there is an f_y with $y \supseteq x$ which agrees with f on x, and thus:

$$\begin{aligned} z \wedge \bigwedge_{f\restriction x(\alpha)=1} b_\alpha \wedge \bigwedge_{f\restriction x(\alpha)=0} -b_\alpha &= z \wedge \bigwedge_{f_y\restriction x(\alpha)=1} b_\alpha \wedge \bigwedge_{f_y\restriction x(\alpha)=0} -b_\alpha \\ &\geqslant z \wedge \bigwedge_{f_y(\alpha)=1} b_\alpha \wedge \bigwedge_{f_y(\alpha)=0} -b_\alpha > 0 \end{aligned}$$

(2 ⇒ 3): Assume X is not κ-compact and let C be a cocover without a κ-small subcocover. Then C generates a κ-filter which by (2) can be extended to a κ-filter $\mathcal{F}$ deciding $\mathcal{B}$. In particular, $\mathcal{B}\cap\mathcal{F}$ does not have a κ-small subcocover, and we claim that $\mathcal{B}\cap\mathcal{F}$ is a cocover itself and thus a counterexample.

Let $x \in X$. Then there is an $a \in C$ omitting x. a is an intersection of κ-small unions of elements of $\mathcal{B}$, so there is such a union $\bigcup B$ with $B \subseteq \mathcal{B}$ omitting x, too. Since $\bigcup B$ is a superset of a, it is an element of $\mathcal{F}$. Therefore by κ-completeness of $\mathcal{F}$ and since all elements of B are decided by $\mathcal{F}$, some element $b \in B$ must be in $\mathcal{F}$. But $x \notin b$, so in particular $x \notin \bigcap(\mathcal{B}\cap\mathcal{F})$.

(3 ⇒ 4): We only need to consider products $X = \prod_{\alpha<\lambda} X_\alpha$ of size λ, because additional one-point space factors do not change the homeomorphism type. For each $\alpha < \lambda$, let $\mathcal{B}_\alpha = \{b_{\alpha,\beta} \mid \beta < \lambda\}$ be a base of the κ-compact space X_α, and let $\pi_\alpha : X \to X_\alpha$ be the projection.

The sets of the form $\pi_\alpha^{-1}[b_{\alpha,\beta}]$ with $\alpha, \beta < \lambda$ constitute a κ-subbase $\mathcal{B}$ of X whose size does not exceed λ. Hence (3) is applicable and we only have to show that a set $C \subseteq \mathcal{B}$ with no κ-small subcocover cannot have an empty intersection.

For a fixed $\alpha < \lambda$ let S_α be the set of all $\beta < \lambda$ for which $\pi_\alpha^{-1}[b_{\alpha,\beta}] \in C$. No κ-small subfamily of $\{b_{\alpha,\beta} \mid \beta\in S_\alpha\}$ can be a cocover of X_α, or else the corresponding preimages would be a cocover of X. Thus by the κ-compactness of X_α, $\{b_{\alpha,\beta} \mid \beta\in S_\alpha\}$ itself has a nonempty intersection and there exists a point $x_\alpha \in \bigcap_{\beta\in S_\alpha} b_{\alpha,\beta}$. Then the function $f \in X$ defined by $f(\alpha) = x_\alpha$ is contained in every element of C.

(4 ⇒ 5) is trivial: $\mathbb{D}$ has a κ-subbase of size 2 and $\mathbb{D}_\kappa^\lambda$ has λ factors.

(5 ⇒ 1): Consider a family $\langle f_x \,|\, x \in \mathcal{P}_\kappa(\lambda)\rangle$. For each $x \in \mathcal{P}_\kappa(\lambda)$, let $C_x \subseteq \mathbb{D}^\lambda$ be the set of all f such that $f \restriction x = f_y \restriction x$ for some $y \supseteq x$. Then the sets C_x are closed and form a cocover of $\mathbb{D}^\lambda$, because $C_x \subseteq C_y$ whenever $y \subseteq x$. Thus by the κ-compactness of $\mathbb{D}^\lambda$, their intersection is nonempty. Let $f \in \bigcap_x C_x$. Then by our choice of C_x, there is a $y \supseteq x$ for every $x \in \mathcal{P}_\kappa(\lambda)$ such that $f_y \restriction x = f \restriction x$.

(3 ⇒ 6): By (3) and Lemma 30, it suffices to consider cocovers consisting of sets of the form $\Box a$ and $\Diamond b$, with $a, b \in \mathcal{B}$ for a given base $\mathcal{B}$ of X of size at most λ.

So let $a_i \in \mathcal{B}$ and $b_j \in \mathcal{B}$ such that $\{\Box a_i \mid i \in \widetilde{I}\} \cup \{\Diamond b_j \mid j \in \widetilde{J}\}$ is a cocover of $\mathrm{Exp}_\kappa(X)$, wlog with $\widetilde{J} \neq \emptyset$. Then

$$\emptyset \quad = \quad \bigcap_{i \in \widetilde{I}} \Box a_i \cap \bigcap_{j \in \widetilde{J}} \Diamond b_j \quad = \quad \Box \bigcap_{i \in \widetilde{I}} a_i \cap \bigcap_{j \in \widetilde{J}} \Diamond b_j,$$

so no closed subset of $\bigcap_{i \in \widetilde{I}} a_i$ intersects every b_j. In particular, $\bigcap_{i \in \widetilde{I}} a_i$ itself does not do so and hence there is a $j_0 \in \widetilde{J}$ such that

$$\emptyset \quad = \quad \bigcap_{i \in \widetilde{I}} a_i \cap b_{j_0}.$$

Since X is κ-compact, there is a κ-small $I \subseteq \widetilde{I}$ for which this equation still holds. But then $\{\Box a_i \mid i \in I\} \cup \{\Diamond b_{j_0}\}$ is a κ-small subcocover.

(6 ⇒ 7): This is just a special case: The κ-Alexandroff compactification has weight at most λ, because the sets $\{x\}$ and $X \setminus \{x\}$ for $x \in X$ are a κ-subbase.

(7 ⇒ 5): Let X be discrete of size λ and let $\langle x_\alpha \mid \alpha < \lambda \rangle$ be an enumeration of its points. We claim that $f : \mathrm{Exp}_\kappa(\omega X) \to \mathbb{D}^\lambda$, where $f(b)(\alpha) = 1$ iff $x_\alpha \in b$, is continuous and surjective and thus since $\mathrm{Exp}_\kappa(\omega X)$ is κ-compact, its image $\mathbb{D}^\lambda$ also is.

It is surjective, because if $g \in \mathbb{D}^\lambda$, then $g^{-1}[1] \cup \{p\}$ is in its preimage. To show that it is continuous, we have to consider the preimages of the subbase sets of the form $U = \{g \in \mathbb{D}^\lambda \mid g(\alpha) = i\}$ for $\alpha \in \lambda$ and $i \in \mathbb{D}$. If $i = 1$, $f^{-1}[U]$ contains all b which contain x_α, so $f^{-1}[U] = \Diamond\{x_\alpha\}$, which is open. If $i = 0$, $f^{-1}[U]$ contains all b which do not contain x_α, so $f^{-1}[U] = \Box(\omega X \setminus \{x_\alpha\})$, which is open, too. ■

Note that the converse of (6) is also true, even without mild ineffability: If $\{a_i \mid a \in \widetilde{I}\}$ is a cocover of X, $\{\Box a_i \mid a \in \widetilde{I}\}$ is a cocover of $\mathrm{Exp}_\kappa(X)$ having a κ-small subcocover $\{\Box a_i \mid a \in I\}$. But then $\{a_i \mid a \in I\}$ is a cocover of X.

Theorem 33 (O. Esser)**.** Let κ be uncountable, $\lambda \geqslant \kappa$ regular. There is a κ-hyperuniverse of weight λ iff κ is mildly λ-ineffable and $\lambda^{<\kappa} = \lambda$.

Proof. Let X be a κ-hyperuniverse and let $\mathbb{V}X$ have weight λ. By Theorem 28, κ must be inaccessible, so we only need to show $\lambda^{<\kappa} = \lambda$ and one of the characterizations of Theorem 32.

Recursively choose clopen subsets b_α for $\alpha < \lambda$ as follows: Given $\alpha < \lambda$, if all pairs of points $x, y \in \mathbb{V}X$ would be separated by some b_β with $\beta < \alpha$, then the κ-topology induced by $\{b_\beta \mid \beta < \alpha\}$ would be κ-compact Hausdorff and therefore equal to the topology of $\mathbb{V}X$. But the set $\{b_\beta \mid \beta < \alpha\}$ cannot be a subbase of $\mathbb{V}X$ because then $\mathbb{V}X$ would have weight $|\alpha| < \lambda$. Hence there exist $x_\alpha, y_\alpha \in \mathbb{V}X$ not separated by any b_β. Let b_α be a clopen set with $x_\alpha \notin b_\alpha \ni y_\alpha$ – such a b_α exists by Lemma 31.

For every $\alpha \leqslant \lambda$, let

$$\tilde{c}_\alpha \quad = \quad \bigcap_{\beta<\alpha} (b_\beta^2 \cup (\complement b_\beta)^2) \quad \subseteq \quad (\mathbb{V}X)^2$$

Then $\langle \tilde{c}_\alpha \mid \alpha < \lambda \rangle$ is a strictly decreasing sequence in $(\mathbb{V}X)^2$ of length λ with nonempty intersection c_λ. Being a κ-hyperuniverse, $\mathbb{V}X$ has a subspace homeomorphic[3] to $(\mathbb{V}X)^2$, so there exists a strictly decreasing sequence of closed sets $c_\alpha \subseteq \mathbb{V}X$ with nonempty intersection c_λ. Wlog we assume that this sequence is continuous, because then $\{c_\alpha \mid \alpha \leqslant \lambda\}$ is closed in $\mathrm{Exp}_\kappa(\mathbb{V}X)$.

The set $D = \{c_{\alpha+1} \mid \alpha < \lambda\}$ does not contain any of its accumulation points, so it is discrete (although not closed). Then $\mathcal{P}_\kappa(D)$ is a discrete subset of $\mathrm{Exp}_\kappa(\mathrm{Exp}_\kappa(\mathbb{V}X))$ of size $\lambda^{<\kappa}$, because for any $a \in \mathcal{P}_\kappa(D)$, if $U_x \ni x$ is open with $U_x \cap D = \{x\}$ for all $x \in a$, then

$$\Box \bigcup_{x\in a} U_x \cap \bigcap_{x \in a} \Diamond U_x$$

is a neighborhood of a disjoint from $\mathcal{P}_\kappa(D) \setminus \{a\}$. Hence the weight of the space must be at least $\lambda^{<\kappa}$ and therefore $\lambda = \lambda^{<\kappa}$.

Let Y be λ with the discrete topology and define $f : \{c_\alpha \mid \alpha \leqslant \lambda\} \to \omega Y$ as follows: $f(c_{\alpha+1}) = \alpha$ for all $\alpha < \lambda$ and $f(c_\alpha) = p$ for limits α. Now f is a continuous map from a closed subspace of $\mathrm{Exp}_\kappa(\mathbb{V}X)$ onto ωY. Thus $\mathrm{Exp}_\kappa(f)$ is a continuous map from a closed subspace of $\mathrm{Exp}_\kappa(\mathrm{Exp}_\kappa(\mathbb{V}X))$ onto $\mathrm{Exp}_\kappa(\omega Y)$. Thus $\mathrm{Exp}_\kappa(\omega Y)$ is κ-compact, which implies that κ is mildly λ-inaccessible.

The converse will follow from Proposition 55. ■

Let us call $c : X \to \tilde{X}$ (or sloppily $\tilde{X}$) a (or "the") κ-*Čech-Stone compactification* of X if $\tilde{X}$ is κ-compact Hausdorff and it has the following universal property: For every κ-compact Hausdorff space Z and every continuous $g : X \to Z$, there is a unique continuous $h : \tilde{X} \to Z$ with $g = h \circ c$. The κ-Čech-Stone compactification – if it exists – therefore is unique up to homeomorphism and we denote $\tilde{X}$ by βX, h by βg and c by ι_X. Note that we do not require X itself to be Hausdorff.

If κ is strongly compact (i.e. mildly λ-ineffable for every λ), the κ-Čech-Stone compactification can be constructed in the familiar way: There is a *set* S of pairs $\langle f, Y\rangle$, where Y is a κ-compact Hausdorff space, $f : X \to Y$ is continuous and $f[X]$ is dense in Y, such that for every κ-compact Hausdorff space Z and every continuous $g : X \to Z$, there is a $\langle f, Y\rangle \in S$ such that $\langle f, Y\rangle$ and $\langle g, g[X]\rangle$ are homeomorphic, that is, such that there is a homeomorphism $h : Y \to g[X]$ with $g = h \circ f$. Then the canonical map

$$\iota_X : X \to \prod_{\langle f,Y\rangle \in S} Y, \quad \iota_X(x)_{\langle f,Y\rangle} = f(x)$$

from X into the product of all those spaces Y, together with the closure of its image, has the universal property. And by (4) of Theorem 32 (Tychonoff's theorem), it is in fact κ-compact Hausdorff.

[3]This is because in the interpretation of **TS** defined by X, $\mathbb{V}^2$ is a set and the product topology coincides with the natural topology.

Lemma 34. Let κ be strongly compact and X a space.

1. If $U \subseteq X$ is open, Hausdorff and locally κ-compact, then $\iota_X[U]$ is open, $c \upharpoonright U$ is a homeomorphism and $\iota_X[\complement U]$ is disjoint from $\iota_X[U]$.
2. If $a, b \subseteq X$ are closed and disjoint and X is normal, then $\iota_X[a]$ and $\iota_X[b]$ have disjoint closures.

Proof. (1): Let $q : X \to \omega U$ be the identity on U and map everything else to the added point p, as in Lemma 29. Then $q \upharpoonright U$ is continuous and injective, and since $\beta q \circ \iota_X = q$, the map $\iota_X \upharpoonright U$ is injective, too. Also, its image is disjoint from $\iota_X[\complement U]$, because the latter is mapped to $\{p\}$. It remains to prove that $\iota_X[V] = (\beta q)^{-1}[V]$ for every open $V \subseteq U$, which will imply that it is in fact open.

So assume $x \in (\beta q)^{-1}[V]$. Let $W \subseteq V$ be a κ-compact neighborhood of $\beta q(x)$. Since $\iota_X[X]$ is dense, x is in the closure of $\iota_X[X] \cap (\beta q)^{-1}[W]$. But that is just equal to $\iota_X[W]$, which is κ-compact and therefore already closed. Hence x is in the image of $W \subseteq V$.

(2): Let $b_0 = b$. If X is normal, there is for every $n \in \omega$ a b_{n+1} such that $b_n \subseteq \operatorname{int}(b_{n+1})$ and $b_{n+1} \cap a = \emptyset$. Then $c = \bigcup_{n\in\omega} b_n = \bigcup_{n\in\omega} \operatorname{int}(b_n)$ is a clopen set with $b \subseteq c$ and $c \cap a = \emptyset$. Thus $f(x) = 1$ iff $x \in c$ defines a continuous $f : X \to \mathbb{D}$ with $f[a] \subseteq \{0\}$ and $f[b] \subseteq 1$. Since $\mathbb{D}$ is κ-compact, there is a map $\beta f : \beta X \to \mathbb{D}$ with $\beta f \circ \iota_X = f$. In particular, $\iota_X[a]$ is a subset of the closed set $(\beta f)^{-1}[0]$ and $\iota_X[b] \subseteq (\beta f)^{-1}[1]$, so they have disjoint closures. ■

2.3 Categories of κ-topological spaces

Before we have a closer look at hyperuniverses, let us quickly review some crucial notions of category theory[4] and elementary topology, with an emphasis on κ-topological spaces for a regular cardinal κ, and establish some general facts which will prove useful in the treatment of those structures.

A *category* consists of a class of *objects*, a class of *morphisms*, for each morphism f an object $\operatorname{dom}(f)$, the *domain* of f, and an object $\operatorname{cod}(f)$, the *codomain* of f, and for all f and g with $\operatorname{cod}(g) = \operatorname{dom}(f)$ a morphism $f \circ g$, the *composite* of g and f, such that:

- $f \circ (g \circ h) = (f \circ g) \circ h$ whenever that is defined.
- $\operatorname{dom}(f \circ g) = \operatorname{dom}(g)$ and $\operatorname{cod}(f \circ g) = \operatorname{cod}(f)$ whenever $f \circ g$ is defined.
- For each X there is a morphism id_X with $\operatorname{cod}(\mathrm{id}_X) = X = \operatorname{dom}(\mathrm{id}_X)$ such that $\mathrm{id}_X \circ g = g$ and $f \circ \mathrm{id}_X = f$ for all f, g with $\operatorname{cod}(g) = X = \operatorname{dom}(f)$.

We write $f : X \to Y$ (read "from X to Y") to indicate that f has domain X and codomain Y, and denote by $\operatorname{Hom}(X,Y)$ the class of all morphisms from X to Y. A morphism $f : X \to Y$ is called an *epimorphism* if it is right cancellable, that is, if for all g and h with domain $\operatorname{cod}(f)$,

[4] For a thorough treatment of the subject we refer the reader to [Mac71] or [Awo10].

$g \circ f = h \circ f$ implies $g = h$. $f : X \to Y$ is called an *isomorphism*, and X and Y are *isomorphic*, if there is a $g : Y \to X$, the *inverse* of f, with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. An *initial object* is an object X such that for every Y there is a unique morphism $f : X \to Y$, and a *terminal object* is one such that for each Y there is exactly one morphism $f : Y \to X$. Since all morphisms between initial objects are necessarily isomorphisms, all initial objects are isomorphic, and the same goes for terminal objects.

Given a fixed object Z, morphisms with domain Z can themselves be considered objects of a category, where the morphisms from $g : Z \to X$ to $h : Z \to Y$ are all $f : X \to Y$ for which $h = f \circ g$. Similarly morphisms with codomain Z form a category, in which $f : X \to Y$ is a morphism from $g : X \to Z$ to $h : Y \to Z$ iff $h \circ f = g$. In these two senses, we will also speak of *initial* and *terminal morphisms*. For example, the κ-Čech-Stone compactification $\iota_X : X \to \beta X$ is initial among the maps from X to κ-compact Hausdorff spaces.

A *functor* F from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a mapping that assigns to each object respectively arrow of $\mathcal{C}$ an object respectively arrow of $\mathcal{D}$ such that whenever $f : X \to Y$, $F(f) : F(X) \to F(Y)$, and such that $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $F(f \circ g) = F(f) \circ F(g)$ for all X, f and g. A *natural transformation* ϕ from a functor F to a functor G, both from $\mathcal{C}$ to $\mathcal{D}$, maps to each object X of $\mathcal{C}$ a morphism $\phi(X) : F(X) \to G(X)$ such that for every $f : X \to Y$ in $\mathcal{C}$, $\phi(Y) \circ F(f) = G(f) \circ \phi(X)$. It is customary to omit the brackets after functors and write Ff for $F(f)$ and FX for $F(X)$.

In the categories we will encounter, the objects are sets with added structure, the morphisms are maps which preserve that structure and $\circ$ is the composition of functions. In particular, $\mathrm{Hom}(X,Y)$ will always be a set. In this section, we will mainly be concerned with the category **Top** whose objects are κ-topological spaces and whose morphisms are continuous maps, as well as with its subcategory of only the κ-compact Hausdorff spaces.

Let I be a *directed set*, that is, a partially ordered set in which two elements always have a common upper bound. A *directed system* in a category $\mathbf{C}$ is a family $\langle X_i, f_{i,j} \rangle_{i \leqslant j \in I}$ of objects X_i and morphisms $f_{i,j} : X_i \to X_j$ for all $i \leqslant j$, indexed by I, such that whenever $i \leqslant j \leqslant k$, $f_{i,k} = f_{j,k} \circ f_{i,j}$, and such that $f_{i,i} = \mathrm{id}_{X_i}$ for all $i \in I$. A *cone* from the family $\langle X_i, f_{i,j} \rangle$ (to Y) is an object Y and a family of morphisms $g_i : X_i \to Y$ such that for all $i \leqslant j$, $g_j \circ f_{i,j} = g_i$. It is a *direct limit* of $\langle X_i, f_{i,j} \rangle$ if it has the following universal property: For every Z and every cone $\langle Z, h_i \rangle_{i \in I}$ from $\langle X_i, f_{i,j} \rangle$ to Z, there is a unique morphism $h : Y \to Z$ such that $h \circ g_i = h_i$ for all $i \in I$. All direct limits of a given family are isomorphic, so we also speak of *the* direct limit and write:

$$\langle Y, g_i \rangle \quad = \quad \varinjlim^{\mathbf{C}} \langle X_i, f_{i,j} \rangle$$

In the category of κ-topological spaces and continuous maps, a direct limit can be constructed as follows: For $x \in X_i$ and $y \in X_{i'}$, let $x \sim y$ iff there is a $j \geqslant i, i'$ such that $f_{i,j}(x) = f_{i',j}(y)$. Let $Y = \bigcup_{i \in I} X_i / \sim$ be the quotient and let $g_i : X_i \to Y$ be the quotients of the embeddings. Then $\langle Y, g_i \rangle$ in fact has the universal property of a direct limit, because whenever $\langle Z, h_i \rangle_{i \in I}$ is a cone from $\langle X_i, f_{i,j} \rangle$ to Z, the union $\bigcup h_i : \bigcup_{i \in I} X_i \to Z$ factors through $\sim$, defining a corresponding map $h : Y \to Z$.

An *inverse system* in $\mathbf{C}$ is a family $\langle X_i, f_{i,j} \rangle_{i \leqslant j \in I}$ of objects X_i and morphisms $f_{i,j} : X_j \to X_i$ (note the reversed direction) for all $i \leqslant j$, indexed by a directed set I, such that whenever $i \leqslant j \leqslant k$, $f_{i,k} = f_{i,j} \circ f_{j,k}$, and $f_{i,i} = \mathrm{id}_{X_i}$ for all $i \in I$. A *cone* to that family (from Y) is an object Y and a family of morphisms $g_i : Y \to X_i$ such that for all $i \leqslant j$, $g_i = f_{i,j} \circ g_j$. It is an *inverse limit* of $\langle X_i, f_{i,j} \rangle$ if it has the following universal property: For every cone $\langle Z, h_i \rangle_{i \in I}$ to $\langle X_i, f_{i,j} \rangle$, there is a unique morphism $h : Z \to Y$ such that $g_i \circ h = h_i$ for all $i \in I$. Again, all inverse limits of a given family are isomorphic, so we also speak of *the* inverse limit and write:

$$\langle Y, g_i \rangle \quad = \quad \varprojlim{}^{\mathbf{C}} \langle X_i, f_{i,j} \rangle$$

For κ-topological spaces, an inverse limit can also be constructed: Let Y be the subspace of $\prod_{i \in I} X_i$ given by those elements $(x_i)_{i \in I}$ such that $f_{i,j}(x_j) = x_i$ for all $j \geqslant i$, and let $g_i : Y \to X_i$ be the projection onto the i-th component. If now $\langle Z, h_i \rangle_{i \in I}$ is a cone to $\langle X_i, f_{i,j} \rangle$, then $h : Z \to Y$, $h(z) = (h_i(z))_{i \in I}$ witnesses the universal property.

Note that if the X_i are Hausdorff spaces, then Y is a closed subspace of the product and also Hausdorff. If in addition the product is κ-compact, Y is, too. In that case, Y is also the inverse limit in the category of κ-compact Hausdorff spaces.

Unfortunately the matter of separation properties is more complicated in the context of direct limits, where even κ-compact Hausdorff spaces can have non-Hausdorff limits[5]. But still, direct limits may exist in the category of κ-compact Hausdorff spaces: If every space has a κ-Čech-Stone compactification – which is the case for $\kappa = \omega$ or strongly compact κ – then the κ-Čech-Stone compactification of the direct limit in the category of κ-topological spaces is the direct limit in the category of κ-compact Hausdorff spaces. This is because the maps from βX to Y correspond bijectively to the maps from X to Y for every κ-compact Hausdorff space Y and every κ-topological space X, which makes the κ-Čech-Stone compactification the left adjoint of the inclusion functor:

Two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ between categories $\mathcal{C}$ and $\mathcal{D}$ are called *adjoint*, where F is the *left adjoint* of G and G is the *right adjoint* of F, if for all objects X of $\mathcal{C}$ and Y of $\mathcal{D}$, there is a bijection

$$\Phi_{X,Y} : \mathrm{Hom}(FX, Y) \to \mathrm{Hom}(X, GY)$$

such that for every $f : X' \to X$, $g : Y \to Y'$ and $h : FX \to Y$,

$$\Phi_{X',Y'}(g \circ h \circ Ff) = Gg \circ \Phi_{X,Y}(h) \circ f.$$

If that is the case, F preserves direct limits and G preserves inverse limits.

We call a subset K of a partial order P ν-*closed* if all ν-small subsets $A \subseteq I$ which have an upper bound in P also have an upper bound in K. A partial order in which ν-few elements always have an upper bound is called ν-*directed*.

[5]For example let $X_0 = \{0, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\} \subseteq \mathbb{R}$ and let X_n be the quotient where the n greatest fractions are identified. Then each X_n is Hausdorff, but the direct limit is the Sierpiński space.

Lemma 35. Let $\mathcal{C}$ be a category in which inverse limits exist. Let $f : X \to \widetilde{X}$ be an isomorphism in $\mathcal{C}$, where

$$\langle X, g_i \rangle_{i \in I} = \varprojlim \langle X_i, g_{ij} \rangle_{i,j \in I} \quad \text{and} \quad \langle \widetilde{X}, \widetilde{g}_i \rangle_{i \in \widetilde{I}} = \varprojlim \langle \widetilde{X}_i, \widetilde{g}_{ij} \rangle_{i,j \in \widetilde{I}}$$

are inverse limits of inverse systems, where all g_i and $\widetilde{g}_i$ are epimorphisms. Assume that for every $i \in I$, there is an $\widetilde{i} \in \widetilde{I}$ and a morphism $s : \widetilde{X}_{\widetilde{i}} \to X_i$ with $s \circ \widetilde{g}_{\widetilde{i}} = g_i \circ f^{-1}$, and for every $\widetilde{i} \in \widetilde{I}$, there is an $i \in I$ and a morphism $\widetilde{s} : X_i \to \widetilde{X}_{\widetilde{i}}$ with $\widetilde{s} \circ g_i = \widetilde{g}_{\widetilde{i}} \circ f$.

$$\begin{tikzcd} X_i & X \arrow[l, "g_i"'] \arrow[d, "f"] \\ \widetilde{X}_{\widetilde{i}} \arrow[u, "s"] & \widetilde{X} \arrow[l, "\widetilde{g}_{\widetilde{i}}"'] \end{tikzcd} \qquad \begin{tikzcd} X_i \arrow[d, "\widetilde{s}"'] & X \arrow[l, "g_i"'] \arrow[d, "f"] \\ \widetilde{X}_{\widetilde{i}} & \widetilde{X} \arrow[l, "\widetilde{g}_{\widetilde{i}}"'] \end{tikzcd}$$

Let $J \subseteq I$ and $\widetilde{J} \subseteq \widetilde{I}$ have cardinality $\leqslant \nu$ for some inaccessible cardinal ν. Then there are ν-closed subsets $K \supseteq J$ of I and $\widetilde{K} \supseteq \widetilde{J}$ of $\widetilde{I}$ of cardinality $\leqslant \nu$, such that the inverse limits

$$\langle Y, e_i \rangle_{i \in K} = \varprojlim \langle X_i, g_{ij} \rangle_{i,j \in K} \quad \text{and} \quad \langle \widetilde{Y}, \widetilde{e}_i \rangle_{i \in \widetilde{K}} = \varprojlim \langle \widetilde{X}_i, \widetilde{g}_{ij} \rangle_{i,j \in \widetilde{K}}$$

are canonically isomorphic.

Proof. Let $\widetilde{h} : I \to \widetilde{I}$ and $h : \widetilde{I} \to I$ be choices of suitable indices, such that for all $i \in I$ and $\widetilde{i} \in \widetilde{I}$ there exist morphisms $s_i : \widetilde{X}_{\widetilde{h}(i)} \to X_i$ and $\widetilde{s}_{\widetilde{i}} : X_{h(\widetilde{i})} \to \widetilde{X}_{\widetilde{i}}$ with $s_i \circ \widetilde{g}_{\widetilde{h}(i)} = g_i \circ f^{-1}$ and $\widetilde{s}_{\widetilde{i}} \circ g_{h(\widetilde{i})} = \widetilde{g}_{\widetilde{i}} \circ f$.

Let $c : \mathcal{P}_\nu(I) \to I$ be a choice of upper bounds in I, that is, for each $A \in \mathcal{P}_\nu(I)$, let $c(A)$ be an upper bound of A if there is one, and an arbitrary element of I otherwise. Similarly, let $\widetilde{c} : \mathcal{P}_\nu(\widetilde{I})$ be such a choice for upper bounds in $\widetilde{I}$. We define $K \subseteq I$ and $\widetilde{K} \subseteq \widetilde{I}$ as the closure of J and $\widetilde{J}$ with respect to h, $\widetilde{h}$, c and $\widetilde{c}$, that is,

$$\begin{array}{lcl} K_0 & = & J \\ K_{\alpha+1} & = & K_\alpha \cup h[\widetilde{K}_\alpha] \cup c[\mathcal{P}_\nu(K_\alpha)] \\ K_\beta & = & \bigcup_{\alpha<\beta} K_\alpha \end{array} \qquad \begin{array}{lcl} K'_0 & = & \widetilde{J} \\ \widetilde{K}_{\alpha+1} & = & \widetilde{K}_\alpha \cup \widetilde{h}[K_\alpha] \cup \widetilde{c}[\mathcal{P}_\nu(\widetilde{K}_\alpha)] \\ \widetilde{K}_\beta & = & \bigcup_{\alpha<\beta} \widetilde{K}_\alpha \end{array}$$

for all ordinals α and all limit ordinals β. Then $K = K_\nu = K_{\nu+1}$ and $\widetilde{K} = \widetilde{K}_\nu = \widetilde{K}_{\nu+1}$ have cardinality $\leqslant \nu$ and are ν-closed and in particular directed.

For all $i < j$ in K, $\widetilde{h}(i), \widetilde{h}(j)$ are elements of $\widetilde{K}$ and there is an index $\widetilde{m} \in \widetilde{K}$ with $\widetilde{m} \geqslant \widetilde{h}(i), \widetilde{h}(j)$.

Since in the diagram

$$\begin{tikzcd}
X_j \arrow[ddd, "g_{ij}"'] & & & \widetilde{X}_{\widetilde{h}(j)} \arrow[lll, "s_j"'] \\
& & & \\
& X \arrow[uul, "g_j"'] \arrow[ddl, "g_i"] \arrow[r, "f"] & \widetilde{X} \arrow[uur, "\widetilde{g}_{\widetilde{h}(j)}"] \arrow[r, "\widetilde{g}_{\widetilde{m}}"] \arrow[ddr, "\widetilde{g}_{\widetilde{h}(i)}"'] & \widetilde{X}_{\widetilde{m}} \arrow[uu, "\widetilde{g}_{\widetilde{h}(j)\widetilde{m}}"'] \arrow[dd, "\widetilde{g}_{\widetilde{h}(i)\widetilde{m}}"] \\
X_i & & & \widetilde{X}_{\widetilde{h}(i)} \arrow[lll, "s_i"']
\end{tikzcd}$$

all cells are commutative, all paths from X to X_i are equivalent, and because f is an isomorphism and $\widetilde{g}_{\widetilde{m}} : \widetilde{X} \to \widetilde{X}_{\widetilde{m}}$ an epimorphism, it follows that $g_{ij} \circ s_j \circ \widetilde{g}_{\widetilde{h}(j)\widetilde{m}} = s_i \circ \widetilde{g}_{\widetilde{h}(i)\widetilde{m}}$ and thus

$$g_{ij} \circ s_j \circ \widetilde{e}_{\widetilde{h}(j)} = g_{ij} \circ s_j \circ \widetilde{g}_{\widetilde{h}(j)\widetilde{m}} \circ \widetilde{e}_m = s_i \circ \widetilde{g}_{\widetilde{h}(i)\widetilde{m}} \circ \widetilde{e}_{\widetilde{m}} = s_i \circ \widetilde{e}_{\widetilde{h}(i)},$$

so the family of morphisms $s_i \circ \widetilde{e}_{\widetilde{h}(i)} : \widetilde{Y} \to X_i$ with $i \in K$ is a cone from $\widetilde{Y}$ to $\langle X_i, g_{ij} \rangle_{i,j \in K}$ and defines via the universal property of the inverse limit a morphism $s : \widetilde{Y} \to Y$. Symmetrically, there is a corresponding morphism $\widetilde{s} : Y \to \widetilde{Y}$. It remains to show that s and $\widetilde{s}$ are inverses of each other and again using symmetry, we will only prove that $s \circ \widetilde{s} = \mathrm{id}_Y$. Again by the universal property of the inverse limit Y, this follows if we can demonstrate $e_i \circ s \circ \widetilde{s} = e_i \circ \mathrm{id}_Y = e_i$ for all $i \in K$.

Given $i \in K$, let $m \in K$ with $m \geqslant i, j$, where $j = h(\widetilde{h}(i))$. Using the fact that $g_m : X \to X_m$ is an epimorphism, we see that the following diagram commutes:

$$\begin{tikzcd}
& X_i & & \\
X_m \arrow[ur, "g_{im}"] \arrow[dr, "g_{jm}"'] & X \arrow[l, "g_m"'] \arrow[u, "g_i"'] \arrow[d, "g_j"] \arrow[r, "f"] & \widetilde{X} \arrow[r, "\widetilde{g}_{\widetilde{h}(i)}"] & \widetilde{X}_{\widetilde{h}(i)} \arrow[ull, "s_i"'] \\
& X_j \arrow[urr, "\widetilde{s}_{\widetilde{h}(i)}"'] & &
\end{tikzcd}$$

Hence $s_i \circ \widetilde{s}_{\widetilde{h}(i)} \circ g_{jm} = g_{im}$. Using that and $g_{im} \circ e_m = e_i$, the commutative diagram

$$\begin{tikzcd}
Y \arrow[d, "e_i"'] & \widetilde{Y} \arrow[l, "s"'] \arrow[d, "\widetilde{e}_{\widetilde{h}(i)}"'] & Y \arrow[l, "\widetilde{s}"'] \arrow[d, "e_j"'] \arrow[dr, "e_m"] & \\
X_i & \widetilde{X}_{\widetilde{h}(i)} \arrow[l, "s_i"'] & X_j \arrow[l, "\widetilde{s}_{\widetilde{h}(i)}"'] & X_m \arrow[l, "g_{jm}"'] \arrow[lll, bend left, "g_{im}"]
\end{tikzcd}$$

shows that $e_i \circ s \circ \widetilde{s} = e_i$. ■

Lemma 36. Let $\langle X_i, g_{ij}\rangle_{i,j\in I}$ and $\langle \widetilde{X}_j, \widetilde{g}_{ij}\rangle_{i,j\in\widetilde{I}}$ be κ-directed systems of κ-small discrete spaces, all g_i and $\widetilde{g}_j$ surjective, and let their inverse limits

$$\langle X, g_i\rangle_{i\in I} = \varprojlim \langle X_i, g_{ij}\rangle_{i,j\in I} \quad \text{and} \quad \langle \widetilde{X}, \widetilde{g}_i\rangle_{i\in I} = \varprojlim \langle \widetilde{X}_i, \widetilde{g}_{ij}\rangle$$

be homeomorphic. Then all subsystems of $\langle X_i, g_{ij}\rangle_{i,j\in I}$ and $\langle \widetilde{X}_i, \widetilde{g}_{ij}\rangle_{i,j\in\widetilde{I}}$ with at most κ objects can be extended to κ-closed subsystems of size $\leqslant \kappa$ whose inverse limits are homeomorphic.

Proof. Let $f : X \to \widetilde{X}$ be a homeomorphism. It suffices to prove the assumptions of Lemma 35. By symmetry, we only have to find $\widetilde{i}$ and $s_i : \widetilde{X}_{\widetilde{i}} \to X_i$ for any given i.

First we note that the preimages of the form $\widetilde{g}_j^{-1}[a]$ are not only a κ-subbase but actually a base of $\widetilde{X}$, because a κ-small union of such sets is of that form itself:

$$\bigcup_{\alpha<\gamma} \widetilde{g}_{j_\alpha}^{-1}[a_\alpha] = \bigcup_{\alpha<\gamma} \widetilde{g}_k^{-1} \circ \widetilde{g}_{j_\alpha k}{}^{-1}[a_\alpha] = \widetilde{g}_k^{-1}\left[\bigcup_{\alpha<\gamma} \widetilde{g}_{j_\alpha k}^{-1}[a_\alpha]\right],$$

where k is an upper bound of the j_α. In particular, every clopen subset c of $\widetilde{X}$ is an intersection of such sets. But since $\complement c$ is κ-compact, c is in fact an intersection of κ-few base sets and therefore a base set itself:

$$c = \bigcap_{\alpha<\gamma} \widetilde{g}_{j_\alpha}^{-1}[a_\alpha] = \bigcap_{\alpha<\gamma} \widetilde{g}_k^{-1} \circ \widetilde{g}_{j_\alpha k}{}^{-1}[a_\alpha] = \widetilde{g}_k^{-1}\left[\bigcap_{\alpha<\gamma} \widetilde{g}_{j_\alpha k}^{-1}[a_\alpha]\right]$$

For every $i \in I$, the sets $f \circ g_i^{-1}[\{x\}]$ with $x \in X_i$ partition $\widetilde{X}$ into κ-few clopen sets $\widetilde{g}_{j_x}^{-1}[a_x]$. If k is an upper bound of $\{j_x \mid x \in X_i\}$ and we set $b_x = \widetilde{g}_{j_x k}^{-1}[a_x]$, we have $f \circ g_i^{-1}[\{x\}] = \widetilde{g}_k^{-1}[b_x]$. Hence we can define $\widetilde{i} = k$ and $s_i(y) = x$ for all $y \in b_x$. ∎

Lemma 37. Let I be a κ-directed set and $\langle g_{ij}, X_i\rangle_{i,j\in I}$ an inverse system in the category of κ-topological spaces and assume that every X_i is κ-compact. Let the spaces $\widetilde{X}_i = \mathrm{Exp}_\kappa(X_i)$ and the maps $\widetilde{g}_{ij} = \mathrm{Exp}_\kappa(g_{ij})$ be its corresponding exponential system. Consider the limits

$$\langle X, g_i\rangle_{i\in I} = \varprojlim \langle X_i, g_{ij}\rangle_{i,j\in I} \quad \text{and} \quad \langle \widetilde{X}, \widetilde{g}_i\rangle_{i\in I} = \varprojlim \langle \widetilde{X}_i, \widetilde{g}_{ij}\rangle.$$

Then $\widetilde{X}$ is canonically homeomorphic to the space $\mathrm{Exp}_\kappa(X)$.

Proof. We can wlog assume that all g_{ij} (and consequently all $\widetilde{g}_{ij}$) are surjective, because an inverse system of spaces always has the same limit as the corresponding system of subspaces $g_i[X] = \bigcap_{j\geqslant i} g_{ij}[X_j] \subseteq X_i$ in which the maps are surjective. Since the X_i are T_3, every $\widetilde{X}_i$ is Hausdorff and so is $\widetilde{X}$.

The maps $\mathrm{Exp}_\kappa(g_i) : \mathrm{Exp}_\kappa(X) \to \widetilde{X}_i$ commute with the maps $\widetilde{g_{ij}}$, so we obtain a continuous map $f : \mathrm{Exp}_\kappa(X) \to \widetilde{X}$ from the inverse limit property of $\widetilde{X}$ and we only have to show that it is bijective.

Firstly, let $a, b \in \mathrm{Exp}_\kappa(X)$ be distinct, wlog let $x \in a \setminus b$. Then for every $y \in b$, there is an $i_y \in I$, such that $g_{i_y}(x) \neq g_{i_y}(y)$. Since X_i is Hausdorff, there are disjoint open $U_y \ni g_{i_y}(x)$ and $V_y \ni g_{i_y}(y)$ separating them. But κ-few sets $g_{i_y}^{-1}[V_y]$ suffice to cover b, and if j is an upper bound of their κ-few indices i_y, then $g_j(x) \notin g_j[b]$, because

$$g_{i_y j} \circ g_j(x) \quad = \quad g_{i_y}(x) \quad \notin \quad V_y \quad = \quad g_{i_y j} \circ g_j \left[g_{i_y}^{-1}[V_y] \right] \qquad \text{for all } y.$$

As a consequence $g_j[a] \neq g_j[b]$. Thus the map $\mathrm{Exp}_\kappa(g_j) : \mathrm{Exp}_\kappa(X) \to \widetilde{X}_j$, which equals $\widetilde{g}_j \circ f$, maps a and b to distinct points, implying that f also does.

Secondly, let $b \in \widetilde{X}$. Then for every i, $\widetilde{g}_i(b)$ is a κ-compact subset of X_i. We claim that the set

$$a \quad = \quad \bigcap_{i \in I} g_i^{-1}[\widetilde{g}_i(b)]$$

is an element of $\mathrm{Exp}_\kappa(X)$ and $f(a) = b$.

Whenever $j > i$,

$$g_j^{-1}[\widetilde{g}_j(b)] \quad \subseteq \quad g_j^{-1}[g_{ij}^{-1}[\widetilde{g}_{ij} \circ \widetilde{g}_j(b)]] \quad = \quad g_i^{-1}[\widetilde{g}_i(b)].$$

So if a κ-small family of sets $\langle g_{i_\alpha}^{-1}[\widetilde{g}_{i_\alpha}(b)] \mid \alpha < \gamma \rangle$ is given, and j is an upper bound of the i_α, every $g_{i_\alpha}^{-1}[\widetilde{g}_{i_\alpha}(b)]$ is a superset of $g_j^{-1}\left[\widetilde{g}_j(b)\right]$, which is nonempty. Therefore these sets have no κ-small subcocover and because X is κ-compact, a is in fact nonempty, so $a \in \mathrm{Exp}_\kappa(X)$

To prove $f(a) = b$, it suffices to verify that $\widetilde{g}_j \circ f(a) = \widetilde{g}_j(b)$ for all j. But $\widetilde{g}_j \circ f = \mathrm{Exp}_\kappa(g_j)$ and it follows from the definition of a that $\mathrm{Exp}_\kappa(g_j)(a) \subseteq \widetilde{g}_j(b)$. To prove the converse, let $x \notin \mathrm{Exp}_\kappa(g_j)(a)$, that is, let $g_j^{-1}[\{x\}]$ be disjoint from a. Since $g_j^{-1}[\{x\}]$ is κ-compact there are κ-few sets $g_i^{-1}[\widetilde{g}_i(b)]$ whose intersection with $g_j^{-1}[\{x\}]$ is empty, and because I is κ-directed, there exists a single $i > j$, such that $g_j^{-1}[\{x\}] \cap g_i^{-1}[\widetilde{g}_i(b)] = \emptyset$. Hence $x \notin g_{ij}[\widetilde{g}_i(b)] = \widetilde{g}_{ij}(\widetilde{g}_i(b)) = \widetilde{g}_j(b)$. ■

2.4 Ultrametric Spaces and Generalized Cantor Cubes

A κ-*ultrametric* on a set X is a map $d : X^2 \to \kappa + 1$ such that:

- $d(x,y) = \kappa$ iff $x = y$
- $d(x,y) = d(y,x)$
- $d(x,z) \geqslant \min\{d(x,y), d(y,z)\}$

Intuitively, greater ordinals $d(x,y)$ denote smaller distances between x and y. The connection with our earlier definition of ultrametrizability becomes clear if we define $x \sim_\alpha y$ as $d(x,y) \geqslant$

α for each $\alpha < \kappa$. Then these are equivalence relations with the following properties:

$$\begin{aligned} \bigcap_{\alpha \in \kappa} \sim_\alpha &= \Delta_X \\ \bigcap_{\alpha \in B} \sim_\alpha &= \sim_\beta \qquad \text{whenever } \beta = \sup(B) \text{ and } B \subseteq \kappa \text{ is bounded.} \end{aligned}$$

In particular $\sim_\beta$ is a subset of $\sim_\alpha$ whenever $\beta \geqslant \alpha$, and $\sim_0 = X^2$. In fact, any family of binary relations on X with these properties defines a κ-ultrametric via $d(x,y) = \sup\{\alpha \mid x \sim_\alpha y\}$ (which is in fact $\max\{\alpha \mid x \sim_\alpha y\}$ unless $x = y$).

We call $[x]_\alpha = \{y \mid x \sim_\alpha y\}$ the *α-ball* of x. The topology *induced by* d is the coarsest κ-topology such that all α-balls for $\alpha < \kappa$ are open. Then for every $\alpha < \kappa$, the α-balls consitute a partition of X of clopen sets, and if X is κ-compact, that partition has only κ-few members. A space is *κ-ultrametrizable* if its topology is induced by a κ-ultrametric.

Lemma 38. Every regular space with weight $\leqslant \kappa$ is κ-ultrametrizable.

Proof. By Lemma 31, there is a base $\langle B_\alpha \mid \alpha < \kappa \rangle$ of clopen sets. Setting

$$x \sim_\alpha y \quad \text{iff} \quad \forall \beta < \alpha. \quad x \in B_\beta \Leftrightarrow y \in B_\beta$$

defines a κ-ultrametric inducing the topology of X, because on the one hand, every $[x]_\alpha$ is the intersection of α clopen sets, and on the other hand, every B_α is a union of $\alpha + 1$-balls. ■

A map $f : X \to Y$ between κ-ultrametric spaces is a *uniformly continuous* if for every β there is an α such that

$$\forall x_0, x_1 \in X. \qquad x_0 \sim_\alpha x_1 \quad \Rightarrow \quad f(x_0) \sim_\beta f(x_1).$$

Given a set $C \subseteq \kappa$, we call f a C-*nonexpansive* map if for all $\alpha \in C$,

$$\forall x_0, x_1 \in X. \qquad x_0 \sim_\alpha x_1 \quad \Rightarrow \quad f(x_0) \sim_\alpha f(x_1),$$

a C-*isometry* if it is bijective and both f and f^{-1} are C-nonexpansive, and simply an *isometry* if it is a κ-isometry.

Lemma 39. Let X and Y be κ-ultrametric spaces and $f : X \to Y$ continuous.

1. If X is κ-compact, f is uniformly continuous.
2. If f is uniformly continuous, there is a closed unbounded $C \subseteq \kappa$ such that f is C-nonexpansive. In fact, the set C of all α such that f is $\{\alpha\}$-nonexpansive is closed unbounded.
3. If f is bijective and both f^{-1} and f are uniformly continuous, there is a closed unbounded $C \subseteq \kappa$ such that f is a C-isometry. In fact, the set C of all α such that f is an $\{\alpha\}$-isometry is closed unbounded. So in particular, for any two κ-ultrametrics on a space X given by $\langle \sim_\alpha \mid \alpha \in \kappa \rangle$ and $\langle \sim'_\alpha \mid \alpha \in \kappa \rangle$, the set of $\alpha \in \kappa$ with $\sim_\alpha = \sim'_\alpha$ is closed unbounded.

Proof. (1): Let $\beta \in \kappa$. Then the β-balls constitute a partition of Y of clopen sets, so the set $P = \{f^{-1}[[y]_\beta] \mid y{\in}Y\}$ of their preimages is a partition of X of clopen sets. The set

$$\{[z]_\gamma \quad | \quad \gamma \in \kappa, \quad z \in X, \quad \text{P contains a superset of } [z]_\gamma\}$$

is an open cover of X, so it has a subcover $\{[z_i]_{\gamma_i} \mid i{\in}I\}$ indexed by a κ-small I. Let

$$\alpha \quad = \quad \sup_{i \in I} \gamma_i.$$

Now whenever $x_0 \sim_\alpha x_1$, x_0 is in some $[z_i]_{\gamma_i}$. Since $\alpha \geqslant \gamma_i$, the same γ_i-ball must contain x_1. But $[z_i]_{\gamma_i}$ is a subset of some element $f^{-1}[[y]_\beta]$ of P. In particular, $f(x_0) \sim_\beta y \sim_\beta f(x_1)$.

(2): First let us show that C is closed, so assume that $\langle \beta_\alpha \mid \alpha{<}\gamma \rangle$ is an increasing sequence in C for some limit ordinal $\gamma < \kappa$, and let $\delta = \sup_{\alpha<\gamma} \beta_\alpha$. Now if $x_0 \sim_\delta x_1$, then in particular $x_0 \sim_{\beta_\alpha} x_1$ for all $\alpha < \gamma$. Since all β_α are in C, this implies $f(x_0) \sim_{\beta_\alpha} f(x_1)$ for all $\alpha < \gamma$. But since $\sim_\delta$ is the intersection of these $\sim_{\beta_\alpha}$, it follows that $f(x_0) \sim_\delta f(x_1)$, so $\delta \in C$.

To verify that C is unbounded, we have to find for every $\alpha \in \kappa$ an ordinal $\beta \geqslant \alpha$ in C. To this end, let $\alpha_0 = \alpha$ and proceed recursively: For every $n \in \omega$, there is an α_{n+1} such that

$$\forall x_0, x_1 \in X. \qquad x_0 \sim_{\alpha_{n+1}} x_1 \quad \Rightarrow \quad f(x_0) \sim_{\alpha_n} f(x_1).$$

If we define $\alpha_\omega = \sup_{n\in\omega} \alpha_n$, then $\alpha \leqslant \alpha_\omega \in C$, because $x_0 \sim_{\alpha_\omega} x_1$ implies $\forall n.\ x_0 \sim_{\alpha_{n+1}} x_1$, which implies $\forall n.\ f(x_0) \sim_{\alpha_n} f(x_1)$, which finally entails $f(x_0) \sim_{\alpha_\omega} f(x_1)$.

(3) follows from (2), because if C_0 is the closed unbounded set given by (2) and C_1 is the corresponding closed unbounded set for f^{-1} instead of f, then $C_0 \cap C_1$ is closed unbounded, too. But that is exactly the set of all $\alpha \in \kappa$ for which f maps the α-balls of X bijectively onto the α-balls of Y, as claimed. ■

Let X be a κ-ultrametric space and let S_X be the set of all limit ordinals $\delta < \kappa$ such that every descending sequence $\langle [x_\alpha]_\alpha \mid \alpha < \delta \rangle$ in X has a nonempty intersection, or equivalently, such that every descending sequence $\langle [x_\alpha]_\alpha \mid \alpha \in C \rangle$ with unbounded $C \subseteq \delta$ has a nonempty intersection. Let $\mathcal{I}_{NS}$ be the nonstationary ideal. Then we call $S_X/\mathcal{I}_{NS} \in \mathcal{P}(\kappa)/\mathcal{I}_{NS}$ the *solidity* of X. A κ-ultrametric space with solidity $\kappa/\mathcal{I}_{NS}$ is called *solid*.

Proposition 40. If X and Y are homeomorphic κ-compact κ-ultrametric spaces, then

$$S_X \equiv S_Y \mod \mathcal{I}_{NS}.$$

In particular, the solidity of a κ-compact κ-ultrametrizable space does not depend on the choice of a specific κ-ultrametric.

Proof. Let $f : X \to Y$ be a homeomorphism and let $C \subseteq \kappa$ be closed unbounded such that f is a C-isometry, as in Lemma 39. We will show that S_X and S_Y agree on the closed unbounded set $\mathrm{Lim}(C)$ of the limit points of C. Since the situation is symmetric, it suffices to prove $S_X \cap \mathrm{Lim}(C) \supseteq S_Y \cap \mathrm{Lim}(C)$, so let $\delta \in S_Y \cap \mathrm{Lim}(C)$. Let $\langle [x_\alpha]_\alpha \mid \alpha < \delta \rangle$ be a descending sequence in X. For every $\alpha \in \delta \cap C$, there is a y_α, such that $f[[x_\alpha]_\alpha] = [y_\alpha]_\alpha$. Then since $\delta \in S_Y$, the sequence $\langle [y_\alpha]_\alpha \,|\, \alpha \in \delta \cap C \rangle$ has a nonempty intersection. Therefore

$$\bigcap_{\alpha<\delta} [x_\alpha]_\alpha = \bigcap_{\alpha\in\delta\cap C} [x_\alpha]_\alpha = \bigcap_{\alpha\in\delta\cap C} f^{-1}[[y_\alpha]_\alpha] = f^{-1}\left[\bigcap_{\alpha\in\delta\cap C} [y_\alpha]_\alpha\right] \neq \emptyset,$$

which proves that $\delta \in S_X$. ■

Theorem 41. Every perfect, κ-compact, solid, κ-ultrametrizable space is homeomorphic to the Cantor cube $\mathbb{D}^\kappa$.
Every perfect, κ-compact, κ-ultrametrizable space is homeomorphic to a closed $a \subseteq \mathbb{D}^\kappa$.

Proof. Let X be perfect, κ-compact, κ-ultrametrizable.

If X is in addition solid and $\langle \sim_\alpha \;|\; \alpha \in \kappa \rangle$ is a κ-ultrametric, there is a closed unbounded $C \supseteq S_X$. Let $\langle \beta_\alpha \,|\, \alpha < \kappa \rangle$ be a monotonously increasing enumeration of the elements of C. Then $\langle \sim_{\beta_\alpha} \;|\; \alpha \in \kappa \rangle$ is a κ-ultrametric on X of which *no* decreasing sequence of balls has an empty intersection. So we can wlog assume that X is κ-ultrametrized in such a way that $S_X = \kappa$.

We recursively define an injective function g from the set of balls to $\mathbb{D}^{<\kappa}$ such that $g(a) \subset g(b)$ iff $a \supset b$, whereby we let $g(X)$ be the empty sequence, and for limit ordinals α, we let $g([x]_\alpha) = \bigcup_{\beta<\alpha} g([x]_\beta)$.

Let $\alpha < \kappa$ and $x \in X$. Let β be minimal such that $[x]_\alpha \neq [x]_\beta$. (Such an ordinal exists because x is not an isolated point.) Choose any enumeration $\langle [y_\gamma]_\beta \,|\, \gamma < \delta{+}2 \rangle$ of the β-balls in $[x]_\alpha$. Since X is κ-compact Hausdorff, $\delta < \kappa$. Define

$$\begin{aligned} g([y_\gamma]_\beta) &= g([x]_\alpha)^\frown 0^\gamma{}^\frown 1 \quad \text{for } \gamma < \delta + 1, \text{ and} \\ g([y_{\delta+1}]_\beta) &= g([x]_\alpha)^\frown 0^\delta{}^\frown 0. \end{aligned}$$

Then rng(g) maps disjoint balls to incomparable sequences. The sequences $g([x]_\alpha)$ with $\alpha < \kappa$ for a given x define a unique cofinal branch in $\mathbb{D}^{<\kappa}$ and therefore a unique point $f(x) = \bigcup_{\alpha<\kappa} g([x]_\alpha) \in \mathbb{D}^\kappa$. We claim that with this definition, $f : X \to \mathbb{D}^\kappa$ is injective and continuous.

First of all, f is injective, because if $x \neq y$, there is an α such that $[x]_\alpha \neq [y]_\alpha$ and thus $g([x]_\alpha)$ and $g([y]_\alpha)$ are incomparable. But $f(x)$ is an extension of the former and $f(y)$ is an extension of the latter.

To show that f is continuous, we have to verify that each preimage $f^{-1}(A_t)$ of an open basis set $A_t = \{s \in \mathbb{D}^\kappa \mid t \subseteq s\}$ is open. $f(x) \in A_t$ means that $g([x]_\alpha)$ is comparable to s for all α, and in particular that some $g([x]_\alpha)$ is an extension of s. So $f^{-1}(A_t)$ is the union of all $[x]_\alpha$ for which $s \subseteq g([x]_\alpha)$, which is open.

To show that f is surjective in the case $S_X = \kappa$, let $s \in \mathbb{D}^\kappa$. Since rng(g) is cofinal and order-reversing, the set of preimages $A = g^{-1}[\{s \restriction \alpha \mid \alpha < \kappa\}]$ for all $\alpha < \kappa$ constitutes a decreasing sequence of balls. If $t \subset s$ and $t \in \text{rng}(g)$, there is an x such that $g([x]_\alpha) = t$. If $\beta > \alpha$ and δ are as in the definition of g, there is either a γ such that $t^\frown 0^\gamma{}^\frown 1 \subset s$ or $t^\frown 0^\delta{}^\frown 0 \subset s$. Then there is a β-ball mapped to that extension of t, and in particular that β-ball is in A. So A has no least element. But if A has κ-few elements then $\bigcap A = [x]_\alpha$ for some $\alpha < \kappa$ and $x \in X$, and $g([x]_\alpha) \subset s$, so $[x]_\alpha \in A$. It follows that A has κ-elements and by compactness $\bigcap A$ is nonempty. Thus it contains exactly one x and $f(x) = s$. ∎

Since $\mathrm{Exp}_\kappa(\mathbb{D}^\kappa)$ is perfect, κ-compact and κ-ultrametrizable, there is a homeomorphism $\Sigma^X : a \to \mathrm{Exp}_\kappa(\mathbb{D}^\kappa)$ for some closed $a \subseteq \mathbb{D}^\kappa$ by Theorem 41. Hence with $\mathbb{V}X = \mathbb{D}^\kappa$ and $\mathbb{S}X = a$, this defines a hyperuniverse. We shall see, however, that $\mathbb{S}X$ cannot be clopen.

Since a similar reasoning also applies to $\kappa = \omega$ and the notion of solidity is not needed for the surjectivity of f, $\mathbb{D}^{\aleph_0}$ is in fact the universe space of an atomless ω-hyperuniverse. S. Sirota proves in [Sir68] that $\mathbb{D}^{\aleph_1}$ is homeomorphic to its hyperspace, too! On the other hand, in [Š76] L. B. Šapiro shows that $\mathrm{Exp}_\omega(\mathbb{D}^{\aleph_2})$ is not the continuous image of any $\mathbb{D}^\lambda$. Since $\mathrm{Exp}_\omega(\mathbb{D}^{\aleph_2})$ is the continuous image of every $\mathrm{Exp}_\omega(\mathbb{D}^\tau)$ for $\tau \geqslant \aleph_2$, this implies that no $\mathbb{D}^\tau$ with $\tau \geqslant \aleph_2$ is an atomless ω-hyperuniverse.

So what about the generalized Cantor cubes with additivity $\kappa > \omega$? For which λ is $\mathbb{D}^\lambda$ an atomless κ-hyperuniverse? Is $\mathbb{D}^\kappa$ an example? Since if κ is weakly compact, $\mathrm{Exp}_\kappa(\mathbb{D}^\kappa)$ is a perfect, κ-compact κ-ultrametrizable space, this amounts to the question of whether it is solid.

The exponential κ-ultrametric on the hyperspace $\mathrm{Exp}_\kappa(X)$ of a κ-ultrametric space X is defined such that $a \sim_\alpha b$ iff a and b intersect the same $< \alpha$-balls, that is:

$$a \sim_\alpha b \qquad \text{iff} \qquad \forall \beta < \alpha \;\; \forall x \in X. \quad a \cap [x]_\beta = \emptyset \quad \Leftrightarrow \quad b \cap [x]_\beta = \emptyset$$

For κ-compact spaces, this is in fact a κ-ultrametric generating the exponential κ-topology: Given $a \in \mathrm{Exp}_\kappa(X)$, the α-ball around a is

$$[a]_\alpha = \bigcap_{\beta<\alpha} \left(\Box \bigcup_{[x]_\beta \cap a \neq \emptyset} [x]_\beta \cap \bigcap_{[x]_\beta \cap a \neq \emptyset} \Diamond [x]_\beta \right),$$

which is open in $\mathrm{Exp}_\kappa(X)$, because the intersections are κ-small. Conversely,

$$\Box[x]_\alpha = [\{x\}]_{\alpha+1} \quad \text{and} \quad \Diamond[x]_\alpha = \bigcup_{[x]_\alpha \in B \subseteq \{[y]_\alpha \mid y \in X\}} \left[\bigcup B\right]_{\alpha+1}$$

are open with respect to the exponential ultrametric, and by Lemma 30, these sets are a subbasis of $\mathrm{Exp}_\kappa(X)$. Hence the topologies coincide.

For every regular $\lambda < \kappa$ let $E^\kappa_\lambda = \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \lambda\}$. Let $W \subseteq \kappa$ be the set of all regular cardinals. Note that each E^κ_λ is almost disjoint from W.

Theorem 42. Let κ be weakly compact and let X be a κ-compact κ-ultrametric space.

1. $S_X \cap E^\kappa_\omega \leqslant S_{\mathrm{Exp}_\kappa(X)} \leqslant S_X \mod \mathcal{I}_{NS}$.
2. If X has a perfect subset, then $S_{\mathrm{Exp}_\kappa(X)} \leqslant W \cup E^\kappa_\omega \mod \mathcal{I}_{NS}$.

In particular, the solidity of every κ-ultrametrizable clopen κ-hyperuniverse X is at most $W \cup E^\kappa_\omega/\mathcal{I}_{NS}$.

Proof. We endow $\mathrm{Exp}_\kappa(X)$ with the exponential κ-ultrametric.

(1): Firstly, assume $\gamma \in S_X \cap E^\kappa_\omega$ and let $\langle [b_\alpha]_{\alpha+1} \mid \alpha < \gamma \rangle$ be a decreasing sequence in $\mathrm{Exp}_\kappa(X)$. Since $b_\alpha \sim_{\alpha+1} \bigcup_{x\in b}[x]_\alpha$, we can wlog assume that b_α is a union of α-balls. Then the b_α constitute a decreasing sequence of subsets of X. We claim that $b = \bigcap_{\alpha<\gamma} b_\alpha$ is in every $[b_\alpha]_{\alpha+1}$. Since $b \subseteq b_\alpha$, it is clear that b intersects at most the α-balls intersected by b_α, so we have to show that it in fact intersects every $[x]_\alpha$ with $x \in b_\alpha$. To prove this, let $\alpha_0 = \alpha$ and let $\langle \alpha_n \mid n \in \omega \rangle$ be an increasing sequence cofinal in γ. Let $x_0 = x$. If $[x_n]_{\alpha_n}$ intersects b_{α_n}, then there must be some α_{n+1}-ball $[x_{n+1}]_{\alpha_{n+1}} \subseteq [x_n]_{\alpha_n}$ intersecting $b_{\alpha_{n+1}}$. Since $\gamma \in S_X$, there exists an $x \in X$ which is a member of every $[x_n]_{\alpha_n}$. And since $[x_n]_{\alpha_n} \subseteq b_{\alpha_n}$, x is a member of b, proving that b in fact intersects $[x]_\alpha$.

Now assume $\gamma \in S_{\mathrm{Exp}_\kappa(X)}$ and let $\langle [x_\alpha]_\alpha \mid \alpha < \gamma \rangle$ a decreasing sequence of balls in X. Then $\langle [\{x_\alpha\}]_{\alpha+1} \mid \alpha < \gamma \rangle$ is a decreasing sequence of balls in $\mathrm{Exp}_\kappa(X)$, which by our assumption has a nonempty intersection. If a is in that intersection and $y \in a$, then $y \sim_\alpha x_\alpha$ for every $\alpha < \gamma$, so y is in the intersection of the $[x_\alpha]_\alpha$, which proves that $\gamma \in S_X$.

(2): We define a closed unbounded $C \subseteq \kappa$ of cardinals such that the closed unbounded set $\mathrm{Lim}(C)$ of its limit points is disjoint from $S_{\mathrm{Exp}_\kappa(X)} \setminus (W \cup E^\kappa_\omega)$. We give C as the range of a normal function $h : \kappa \to \kappa$ with $h(0) = 0$: For each $\alpha < \kappa$, there are κ-few $h(\alpha)$-balls. Since P is perfect, each $h(\alpha)$-ball $[x]_\alpha$ that intersects P has at least κ distinct points in common with P, so there is a least cardinal $\beta_{[x]_\alpha} < \kappa$, such that $[x]_\alpha$ is the union of at least $h(\alpha)$ distinct $\beta_{[x]_\alpha}$-balls intersecting P. Let

$$h(\alpha+1) \quad = \quad \sup_{x\in X} \beta_{[x]_\alpha}.$$

Since there are only κ-few α-balls, $h(\alpha+1) < \kappa$.

Now let $\gamma \in \mathrm{Lim}(C) \setminus (W \cup E^\kappa_\omega)$ and $\tau = \mathrm{cf}(\gamma)$. Then $\omega < \tau < \gamma$ and there is an increasing sequence $\langle \epsilon_\alpha \mid \alpha < \tau \rangle$ in $C \cap [\tau, \gamma)$ converging to γ. Then by our construction of C, each $[x]_{\epsilon_\alpha}$ intersecting P consists of at least τ distinct $\epsilon_{\alpha+1}$-balls intersecting P.

Consider the tree T of all nondecreasing functions $t : \alpha \to \tau$ with $\alpha < \tau$, finite range and $t(\beta) > \beta$ for all $\beta < \alpha$, ordered by inclusion. Since τ is regular and uncountable, this tree has no cofinal branch, but every $t \in T$ has a successor on every level $\alpha < \tau$, and every element has exactly τ distinct immediate successors.

We recursively define a map $g : T \to P$ which maps each level T_α of T injectively to P and in addition has the property that $g(t) = g(t \restriction \alpha)$ whenever there is a $\beta < \alpha$ such that t is constant on $[\beta, \mathrm{dom}(t))$. Firstly, let $g(\emptyset)$ be any point of P. If $g(t)$ has been defined and t is on level α, map the immediate successors of t to any distinct elements of $P \cap [g(t)]_{\epsilon_\alpha}$ which are in different $\epsilon_{\alpha+1}$-balls, such that $g(t^\frown \max(\mathrm{rng}(t))) = g(t)$ whenever $t^\frown \max(\mathrm{rng}(t)) \in T$. By our choice of C, there are indeed enough $\epsilon_{\alpha+1}$-balls intersecting P. If the level $\alpha = \mathrm{dom}(t)$ of t is a limit, then by our construction, $\langle g(t \restriction \beta) \mid \alpha < \beta \rangle$ is eventually constant and we can define $g(t)$ to be that final value, too.

Let $b_\alpha = \{g(t) \mid t \in T_\alpha\}$ for all $\alpha < \tau$. Then the sequence $\langle [b_\alpha]_{\epsilon_\alpha+1} \mid \alpha < \tau \rangle$ in $\mathrm{Exp}_\kappa(X)$ is descending: If $\alpha < \beta < \tau$ and $g(t) \in b_\beta$, then $g(t) \sim_{\epsilon_\alpha} g(t \restriction \alpha) \in b_\alpha$. On the other hand, if $g(t) \in b_\alpha$, then $g(t) \sim_{\epsilon_\alpha} g(t^\frown \beta^{\beta \setminus \alpha}) \in b_\beta$. Thus b_α and b_β in fact intersect the same ϵ_α-balls and hence $b_\alpha \sim_{\epsilon_\alpha+1} b_\beta$, which means $[b_\alpha]_{\epsilon_\alpha+1} = [b_\beta]_{\epsilon_\alpha+1} \supseteq [b_\beta]_{\epsilon_\beta+1}$.

Now assume that $a \in \bigcap_{\alpha<\tau}[b_\alpha]_{\epsilon_\alpha+1}$, and let $x \in a$. Then for every $\alpha < \tau$, there is a $g(t_\alpha) \in b_\alpha$ with $g(t_\alpha) \sim_{\epsilon_\alpha} x$. Thus for all $\alpha < \beta < \tau$, $g(t_\alpha) \sim_{\epsilon_\alpha} g(t_\beta)$, which implies that $t_\alpha \subseteq t_\beta$. Hence $\{t_\alpha \mid \alpha < \tau\}$ is a cofinal branch in T, a contradiction.

We conclude that $\bigcap_{\alpha<\tau}[b_\alpha]_{\epsilon_\alpha+1}$ is a decreasing sequence with an empty intersection and therefore $\gamma \notin S_{\mathrm{Exp}_\kappa(X)}$.

Finally, if X is a clopen κ-ultrametrizable κ-hyperuniverse, then $\mathbb{S}X$ is homeomorphic to $\mathrm{Exp}_\kappa(\mathbb{V}X)$ and has solidity at most $W \cup E^\kappa_\omega/\mathcal{I}_{NS}$. But if a descending sequence of balls has an empty intersection in the clopen subset $\mathbb{S}X$, then it has an empty intersection in $\mathbb{V}X$, so the solidity of $\mathbb{V}X$ cannot be greater: $S_{\mathbb{V}X} \leqslant S_{\mathbb{S}X} \leqslant W \cup E^\kappa_\omega \mod \mathcal{I}_{NS}$. ∎

With the canonical κ-ultrametric defined by $d(f,g) = \min(\{\kappa\} \cup \{\alpha < \kappa \mid f(\alpha) \neq g(\alpha)\})$, the Cantor cube $\mathbb{D}^\kappa$ (as well as the *Pelczynski space* $\mathbb{D}^{\leqslant\kappa}$) is solid, because if $\langle [f_\alpha]_{\alpha+1} \mid \alpha < \delta \rangle$ is a decreasing sequence, $f_\beta \restriction \beta$ must be an extension of $f_\alpha \restriction \alpha$ for all $\alpha < \beta < \delta$, and hence $\bigcup_{\alpha<\delta}(f_\alpha \restriction \alpha)$ is a function each extension of which is an element of $\bigcap_{\alpha<\delta}[f_\alpha]_\alpha$. So it cannot be a clopen κ-hyperuniverse by Theorem 42. From that we can even conclude that none of the spaces $\mathbb{D}^\lambda$ with greater λ is an atomless hyperuniverse:

Theorem 43. Let $\lambda, \tau \geqslant \kappa$ and assume that κ is mildly ν-ineffable for some regular cardinal $\nu \geqslant \lambda, \tau$ with $\nu^{<\kappa} = \nu$. Then $\mathbb{D}^\lambda \not\cong \mathrm{Exp}_\kappa(\mathbb{D}^\tau)$.

Proof. By Theorem 32, these spaces are κ-compact. $\mathbb{D}^\lambda$ is the inverse limit of the system of projections $\pi_{ab} : \mathbb{D}^b \to \mathbb{D}^a$ with $\pi_{ab}(f) = f \restriction a$ for $f \in \mathbb{D}^b$ and $a \subseteq b \in \mathcal{P}_\kappa(\lambda)$, and by Lemma 37, $\mathrm{Exp}_\kappa(\mathbb{D}^\tau)$ is the inverse limit of the system of maps $p_{a,b} : \mathrm{Exp}_\kappa(\mathbb{D}^b) \to \mathrm{Exp}_\kappa(\mathbb{D}^a)$, where $p_{ab} = \mathrm{Exp}_\kappa(\pi_{ab})$ for $a \subseteq b \in \mathcal{P}_\kappa(\tau)$.

By Lemma 36, if they were homeomorphic, then there would exist κ-closed subsets $K \subseteq \mathcal{P}_\kappa(\lambda)$ and $\widetilde{K} \subseteq \mathcal{P}_\kappa(\tau)$ of size κ, such that the inverse limits of the corresponding subsystems are homeomorphic. Now replacing K by its downwards closure in $\mathcal{P}_\kappa(\lambda)$ does neither increase its cardinality nor change the inverse limit nor affect its κ-closedness, so assume that K is downwards closed. Thus there exists a set $a \subseteq \lambda$ of size κ with $K = \mathcal{P}_\kappa(a)$. For the same reason we can assume there is a $b \subseteq \mathcal{P}_\kappa(\tau)$ of size κ with $\widetilde{K} = \mathcal{P}_\kappa(b)$.

But then the inverse limits of these subsystems would be $\mathbb{D}^a$ and $\mathrm{Exp}_\kappa(\mathbb{D}^b)$, and that would imply $\mathbb{D}^\kappa \cong \mathbb{D}^a \cong \mathrm{Exp}_\kappa(\mathbb{D}^b) \cong \mathrm{Exp}_\kappa(\mathbb{D}^\kappa)$, contradicting Theorem 42. ∎

Lemma 44. Let κ be inaccessible, $\lambda \geqslant \kappa$, $\mathbb{D}^\lambda$ κ-compact and $A \subseteq \mathbb{D}^\lambda$ clopen. Then $A \cong \mathbb{D}^\lambda$.

Proof. Let $\pi_a : \mathbb{D}^\lambda \to \mathbb{D}^a$ be the projection, for every κ-small a. Both A and $\complement A$ are unions of open basis sets $\pi_a^{-1}[U]$, and by κ-compactness, these unions can be assumed to be κ-small. Hence there is a single $a \in \mathcal{P}_\kappa(\lambda)$ such that $A = \pi_a^{-1}[U]$, and so A is a union of κ-few clopen sets of the form $\pi_a^{-1}[\{x\}] \cong \mathbb{D}^{\lambda \setminus a} \cong \mathbb{D}^\lambda$.

But if $\gamma < \kappa$, then $(\gamma + 1) \times \mathbb{D}^\lambda \cong \mathbb{D}^\lambda$. For example, mapping $\langle \alpha, s \rangle$ to $0^\alpha \frown 1 \frown s$ for any $\alpha < \gamma$, and $\langle \gamma, s \rangle$ to $0^\gamma \frown s$ defines a homeomorphism. ■

Theorem 45. Let κ be uncountable, $\lambda \geqslant \kappa$ regular. Then $\mathbb{D}^\lambda$ is not the universe space of any clopen κ-hyperuniverse.

Proof. Assume that X is a clopen κ-hyperuniverse and $\mathbb{V}X = \mathbb{D}^\lambda$. By Theorem 33, $\lambda^{<\kappa} = \lambda$ and κ is mildly λ-ineffable. Thus by Theorem 43, $\mathbb{D}^\lambda$ is not homeomorphic to $\mathrm{Exp}_\kappa(\mathbb{V}X)$. But by Lemma 44, $\mathbb{S}X \cong \mathbb{D}^\lambda$, a contradiction. ■

2.5 Direct Limit Models

After these negative results we will now actually construct hyperuniverses. In this section and the next, we give two complementary methods to complete a partially defined hyperuniverse, as well as examples for such hyperuniverse fragments and the resulting models.

Here we assume that κ is strongly compact or ω for the sake of simplicity, although some of the results are also valid for mildly λ-ineffable cardinals, if the spaces in question have weight at most λ and the maps Σ^X are surjective. Unless stated or implied from the context otherwise, all spaces are assumed to be disjoint.

The objects of the *weak extension maps category* $\mathbf{Ex}^*$ are triples $X = \langle \mathbb{V}X, \mathbb{S}X, \mathbb{S}\Sigma^X \rangle$ of

- a κ-topological space $\mathbb{V}X$ called the *universe space* of X,
- a clopen subset $\mathbb{S}X \subseteq \mathbb{V}X$ called the *set space* of X and
- a continuous map $\mathbb{S}\Sigma^X : \mathbb{S}X \to \mathrm{Exp}_\kappa^c(\mathbb{V}X)$.

We call $\mathbb{A}X = \mathbb{V}X \setminus \mathbb{S}X$ the *atom space* of X and define $\mathbb{A}\Sigma^X = \mathrm{id}_{\mathbb{A}X}$ and

$$\mathbb{V}\Sigma^X \quad = \quad \mathbb{A}\Sigma^X \cup \mathbb{S}\Sigma^X \quad : \quad \mathbb{V}X \quad \to \quad \mathbb{A}X \cup \mathrm{Exp}_\kappa^c(\mathbb{V}X),$$

the *extension map* of X. The morphisms $f : X \to Y$ of $\mathbf{Ex}^*$ are continuous maps $\mathbb{V}f : \mathbb{V}X \to \mathbb{V}Y$ such that the image of $\mathbb{A}f = \mathbb{V}f \restriction \mathbb{A}X$ is a subset of $\mathbb{A}Y$, the image of $\mathbb{S}f = \mathbb{V}f \restriction \mathbb{S}X$ is a subset of $\mathbb{S}Y$ and the diagram

$$\begin{tikzcd}
\mathrm{Exp}^c_\kappa(\mathbb{V}X) \arrow[r, "\mathrm{Exp}^c_\kappa(\mathbb{V}f)"] & \mathrm{Exp}^c_\kappa(\mathbb{V}Y) \\
\mathbb{S}X \arrow[u, "\mathbb{S}\Sigma^X"] \arrow[r, "\mathbb{S}f"] & \mathbb{S}Y \arrow[u, "\mathbb{S}\Sigma^Y"']
\end{tikzcd}$$

commutes. We will only use the notation $\mathbb{V}f$ where it is important to explicitly distinguish between the maps $\mathbb{V}f$, $\mathbb{S}f$, $\mathbb{A}f$ and the morphism f, and otherwise simply write f – it is the same object, after all.

We define the *exponential functor* $\Box : \mathbf{Ex}^* \to \mathbf{Ex}^*$, where $\mathbb{V}\Box X = \mathrm{Exp}^c_\kappa(X)$ and:

$$\begin{array}{rclcrcl}
\mathbb{A}\Box X & = & \mathbb{A}X & \qquad & \mathbb{S}\Box X & = & \mathrm{Exp}^c_\kappa(\mathbb{V}X) \\
\mathbb{A}\Box f & = & \mathbb{A}f & & \mathbb{S}\Box f & = & \mathrm{Exp}^c_\kappa(\mathbb{V}f) \\
\mathbb{A}\Sigma^{\Box X} & = & \mathrm{id}_{\mathbb{A}X} & & \mathbb{S}\Sigma^{\Box X} & = & \mathrm{Exp}^c_\kappa(\mathbb{V}\Sigma^X).
\end{array}$$

$\mathbb{V}\Box f$ is in fact continuous whenever $f : X \to Y$ is a morphism, because $\mathbb{V}\Box f \restriction \mathrm{Exp}^c_\kappa(\mathbb{V}X) = \mathbb{S}\Box f$ is the exponential of a continuous map and $\mathbb{V}\Box f \circ \mathbb{V}\Sigma^X = \mathbb{V}\Sigma^Y \circ \mathbb{V}f$, which is also continuous. Σ^X is a morphism $\Sigma^X : X \to \Box X$ and the following diagram commutes, which means that $X \mapsto \Sigma^X$ is a natural transformation from the identity to $\Box$:

$$\begin{tikzcd}
\Box X \arrow[r, "\Box f"] & \Box Y \\
X \arrow[u, "\Sigma^X"] \arrow[r, "f"] & Y \arrow[u, "\Sigma^Y"']
\end{tikzcd}$$

The *extension maps category* **Ex** is the full[6] subcategory given by those objects X whose universe spaces $\mathbb{V}X$ are κ-compact Hausdorff. And the *category of (extension map) hyperuniverses*[7] **EHyp** is the full subcategory of **Ex** given by those objects X where $\mathbb{V}\Sigma^X$ is bijective and thus a homeomorphism. If $\mathbb{V}X$ is κ-compact Hausdorff, then $\mathbb{V}\Box X$ also is, so the restriction $\Box \restriction \mathbf{Ex}$ is a functor from **Ex** to **Ex**.

We define the κ-*Čech-Stone compactification functor* β from $\mathbf{Ex}^*$ to **Ex** by $\mathbb{V}\beta X = \beta\mathbb{V}X$ and $\mathbb{S}\beta X = \mathrm{cl}(\iota_X[\mathbb{S}X]) = \beta\mathbb{S}X$, and using the universal property of the κ-Čech-Stone compactification of a topological space, let $\mathbb{S}\Sigma^{\beta X} : \mathbb{S}\beta X \to \mathrm{Exp}(\mathbb{V}\beta X)$ be the unique map such that the diagram

$$\begin{tikzcd}
\mathrm{Exp}^c_\kappa(\mathbb{V}X) \arrow[r, "\mathrm{Exp}^c_\kappa(\iota_{\mathbb{V}X})"] & \mathrm{Exp}_\kappa(\mathbb{V}\beta X) \\
\mathbb{S}X \arrow[u, "\mathbb{S}\Sigma^X"] \arrow[r, "\iota_{\mathbb{S}X}"] & \mathbb{S}\beta X \arrow[u, "\mathbb{S}\Sigma^{\beta X}"']
\end{tikzcd}$$

commutes, that is, such that $\mathbb{V}\iota_X = \iota_{\mathbb{V}X}$ defines a morphism $\iota_X : X \to \beta X$. For a morphism

[6] That means it contains *all* morphisms $f : X \to Y$ of $\mathbf{Ex}^*$ whenever it contains the objects X and Y.

[7] Note that for technical reasons we include the trivial object with only one point here, although we do not consider it a proper hyperuniverse.

$f : X \to Y$, let $\mathbb{V}\beta f = \beta\mathbb{V}f$. Then all cells in the diagram

$$\begin{tikzcd}
\mathrm{Exp}_\kappa(\mathbb{V}\beta X) \arrow[rrr, "\mathrm{Exp}_\kappa(\mathbb{V}\beta f)"] & & & \mathrm{Exp}_\kappa(\mathbb{V}\beta Y) \\
& \mathrm{Exp}^c_\kappa(\mathbb{V}X) \arrow[ul, "\mathrm{Exp}^c_\kappa(\mathbb{V}\iota_X)"'] \arrow[r, "\mathrm{Exp}^c_\kappa(\mathbb{V}f)"] & \mathrm{Exp}^c_\kappa(\mathbb{V}Y) \arrow[ur, "\mathrm{Exp}^c_\kappa(\mathbb{V}\iota_Y)"] & \\
& \mathbb{S}X \arrow[u, "\mathbb{S}\Sigma^X"] \arrow[r, "\mathbb{S}f"] \arrow[dl, "\mathbb{S}\iota_X"'] & \mathbb{S}Y \arrow[u, "\mathbb{S}\Sigma^Y"] \arrow[dr, "\mathbb{S}\iota_Y"] & \\
\mathbb{S}\beta X \arrow[uuu, "\mathbb{S}\Sigma^{\beta X}"] \arrow[rrr, "\mathbb{S}\beta f"] & & & \mathbb{S}\beta Y \arrow[uuu, "\mathbb{S}\Sigma^{\beta Y}"']
\end{tikzcd}$$

commute, so $\mathbb{S}\Sigma^{\beta Y} \circ \mathbb{S}\beta f \circ \mathbb{S}\iota_X = \mathrm{Exp}_\kappa(\mathbb{V}\beta f) \circ \mathbb{S}\Sigma^{\beta X} \circ \mathbb{S}\iota_X$. Since $\mathbb{S}\iota_X$ is an epimorphism (its image is dense), this implies $\mathbb{S}\Sigma^{\beta Y} \circ \mathbb{S}\beta f = \mathrm{Exp}_\kappa(\mathbb{V}\beta f) \circ \mathbb{S}\Sigma^{\beta X}$ and hence βf is in fact a morphism. The map $X \mapsto \iota_X$ is a natural transformation from the identity to β and β is a left adjoint functor. Its right adjoint is the inclusion from **Ex** to **Ex***. Thus by the following lemma, direct limits in **Ex** exist and they are the κ-Čech-Stone compactifications of the corresponding direct limits in **Ex***:

Lemma 46. Let $\langle X_i, f_{ij}\rangle$ be a directed system in **Ex***. Then the direct limit $\langle X, f_i\rangle$ exists:

$$\langle \mathbb{V}X, \mathbb{V}f_i\rangle = \varinjlim{}^{\mathbf{Top}} \langle \mathbb{V}X_i, \mathbb{V}f_{ij}\rangle \quad , \quad \langle \mathbb{S}X, \mathbb{S}f_i\rangle = \varinjlim{}^{\mathbf{Top}} \langle \mathbb{S}X_i, \mathbb{S}f_{ij}\rangle$$

and $\mathbb{S}\Sigma^X$ is the unique map such that for all i, $\mathbb{S}\Sigma^X \circ \mathbb{S}f_i = \mathrm{Exp}^c_\kappa(\mathbb{V}f_i) \circ \mathbb{S}\Sigma^{X_i}$.

Proof. The definition of $\mathbb{S}\Sigma^X$ does make sense because indeed

$$\mathrm{Exp}^c_\kappa(\mathbb{V}f_j) \circ \mathbb{S}\Sigma^X_j \circ \mathbb{S}f_{ij} \;=\; \mathrm{Exp}^c_\kappa(\mathbb{V}f_j) \circ \mathrm{Exp}^c_\kappa(\mathbb{V}f_{ij}) \circ \mathbb{S}\Sigma^X_i \;=\; \mathrm{Exp}^c_\kappa(\mathbb{V}f_i) \circ \mathbb{S}\Sigma^X_i$$

whenever $i < j$, so the direct limit property of $\mathbb{S}X$ applies.

We show that the object X with the morphisms f_i defined by those topological limits is in fact a direct limit of the given system in **Ex***.

For each i, let $h_i : X_i \to Z$ be a morphism and assume that for all $j \geqslant i$, $h_i = h_j \circ f_{ij}$. We then define $\mathbb{V}h : \mathbb{V}X \to \mathbb{V}Z$ using the direct limit property of $\mathbb{V}X$, which is the unique candidate for a suitable morphism $h : X \to Z$, and we just have to show that it is in fact a morphism. For each i, every path from $\mathbb{S}X_i$ to $\mathrm{Exp}^c_\kappa(\mathbb{V}Z)$ is equivalent in the diagram

$$\begin{tikzcd}
\mathrm{Exp}^c_\kappa(\mathbb{V}X) \arrow[rr, "\mathrm{Exp}^c_\kappa(\mathbb{V}h)"] & & \mathrm{Exp}^c_\kappa(\mathbb{V}Z) \\
& \mathrm{Exp}^c_\kappa(\mathbb{V}X_i) \arrow[ul, "\mathrm{Exp}^c_\kappa(\mathbb{V}f_i)"'] \arrow[ur, "\mathrm{Exp}^c_\kappa(\mathbb{V}h_i)"] & \\
& \mathbb{S}X_i \arrow[u, "\mathbb{S}\Sigma^{X_i}"] \arrow[dl, "\mathbb{S}f_i"'] \arrow[dr, "\mathbb{S}h_i"] & \\
\mathbb{S}X \arrow[uuu, "\mathbb{S}\Sigma^X"] \arrow[rr, "\mathbb{S}h"] & & \mathbb{S}Z \arrow[uuu, "\mathbb{S}\Sigma^Z"']
\end{tikzcd}$$

and in particular $\mathbb{S}\Sigma^Z \circ \mathbb{S}h \circ \mathbb{S}f_i = \mathrm{Exp}^c_\kappa(\mathbb{V}h) \circ \mathbb{S}\Sigma^X \circ \mathbb{S}f_i$. Since $\langle \mathbb{S}X, \mathbb{S}f_i \rangle$ is the direct limit of the $\mathbb{S}X_i$, it follows that $\mathbb{S}\Sigma^Z \circ \mathbb{S}h = \mathrm{Exp}^c_\kappa(\mathbb{V}h) \circ \mathbb{S}\Sigma^X$. ∎

We recursively define functors $\Box^\beta$ from **Ex** to **Ex** and morphisms $\Sigma^X_{\alpha,\beta} : \Box^\alpha X \to \Box^\beta X$ for ordinals $\alpha \leqslant \beta$:

$$\begin{array}{rclcrclcrcl}
\Box^0 X & = & X & & \Sigma^X_{\alpha,\alpha} & = & \mathrm{id}_{\Box^\alpha X} & & \Box^0 f & = & f \\
\Box^{\alpha+1} X & = & \Box\Box^\alpha X & & \Sigma^X_{\alpha,\beta+1} & = & \Sigma^{\Box^\beta X} \circ \Sigma^X_{\alpha,\beta} & & \Box^{\alpha+1} f & = & \Box\Box^\alpha f
\end{array}$$

For limit ordinals γ, let

$$\langle \Box^\gamma X, \Sigma^X_{\alpha,\gamma} \rangle_{\alpha<\gamma} = \varinjlim^{\mathbf{Ex}} \langle \Box^\alpha X, \Sigma^X_{\alpha,\beta} \rangle_{\alpha \leqslant \beta < \gamma}$$

and if $f : X \to Y$ is a morphism, we use the direct limit property of $\Box^\gamma X$: $\Box^\gamma f$ is defined as the unique morphism such that $\Box^\gamma f \circ \Sigma^X_{\alpha,\gamma} = \Sigma^Y_{\alpha,\gamma} \circ \Box^\alpha f$ for all $\alpha < \gamma$. Then $X \mapsto \Sigma^X_{\alpha,\beta}$ is a natural transformation from $\Box^\alpha$ to $\Box^\beta$, that is, for every morphism $f : X \to Y$, the following diagram commutes:

$$\begin{tikzcd}
\Box^\alpha X \arrow[r, "\Sigma^X_{\alpha,\beta}"] \arrow[d, "\Box^\alpha f"] & \Box^\beta X \arrow[d, "\Box^\beta f"] \\
\Box^\alpha Y \arrow[r, "\Sigma^Y_{\alpha,\beta}"] & \Box^\beta Y
\end{tikzcd}$$

Lemma 47. For each object X in **Ex**, $\mathbb{S}\Sigma^X_{\kappa,\kappa+1}$ is surjective and as a consequence, all $\mathbb{S}\Sigma^X_{\beta,\alpha}$ with $\kappa \leqslant \beta \leqslant \alpha$ are.

Proof. The union $D = \bigcup_{\alpha<\kappa} \mathrm{rng}\left(\mathbb{V}\Sigma^X_{\alpha,\kappa}\right)$ is dense in $\mathbb{V}\Box^\kappa X = \beta D$, so by Lemma 22, the set of κ-small subsets of D is dense in $\mathrm{Exp}^c_\kappa(\mathbb{V}\Box^\kappa X) = \mathbb{S}\Box^{\kappa+1} X$. But for each κ-small $S \subseteq D$, there exists an $\alpha < \kappa$ and a κ-small $\widetilde{S} \subseteq \mathbb{V}\Box^\alpha X$, such that $S = \mathbb{V}\Sigma^X_{\alpha,\kappa}[\widetilde{S}]$. Therefore,

$$\bigcup_{\alpha<\kappa} \mathrm{rng}\left(\mathrm{Exp}^c_\kappa(\mathbb{V}\Sigma^X_{\alpha,\kappa})\right) = \bigcup_{\alpha<\kappa} \mathrm{rng}\left(\mathbb{S}\Sigma^X_{\kappa,\kappa+1} \circ \mathbb{S}\Sigma^X_{\alpha+1,\kappa}\right) \subseteq \mathrm{rng}\left(\mathbb{S}\Sigma^X_{\kappa,\kappa+1}\right)$$

is dense in $\mathbb{S}\Box^{\kappa+1} X$. But $\mathbb{S}\Sigma^X_{\kappa,\kappa+1}$ is continuous and its domain is κ-compact, hence its image is closed and therefore $\mathrm{rng}\left(\mathbb{S}\Sigma^X_{\kappa,\kappa+1}\right) = \mathbb{S}\Box^{\kappa+1} X$. ∎

The equivalence relations $E_\alpha \subseteq (\mathbb{V}\Box^\kappa X)^2$ for $\alpha \geqslant \kappa$ defined by

$$xE_\alpha y \qquad \Leftrightarrow \qquad \mathbb{V}\Sigma^X_{\kappa,\alpha}(x) = \mathbb{V}\Sigma^X_{\kappa,\alpha}(y)$$

are monotonously increasing and thus the sequence has to be eventually constant, that is, there is some α such that $\mathbb{V}\Sigma^{\Box_\alpha X}$ is a homeomorphism, so $\Box^\alpha X$ is an object of **EHyp**. Let γ_X be the least such α.

For objects X and Y of **Ex**, morphisms $f : X \to Y$ and $\alpha = \max\{\gamma_X, \gamma_Y\}$, we define:

$$\begin{aligned}
\Box^\infty X &= \Box^{\gamma_X} X \\
\Box^\infty f &= \left(\Sigma^Y_{\gamma_Y,\alpha}\right)^{-1} \circ \Box^\alpha f \circ \Sigma^X_{\gamma_X,\alpha} \\
\Sigma^X_{\beta,\infty} &= \begin{cases} \Sigma^X_{\beta,\gamma_X} & \text{for } \beta \leqslant \gamma_X \\ \left(\Sigma^X_{\gamma_X,\beta}\right)^{-1} & \text{for } \beta > \gamma_X \end{cases}
\end{aligned}$$

$\Box^\infty$ is a functor from **Ex** to **EHyp** and $X \mapsto \Sigma^X_{\beta,\infty}$ is a natural transformation from $\Box^\beta$ to $\Box^\infty$. $\mathbb{V}\Box^\infty X$ is a quotient of $\mathbb{V}\Box^\kappa X$. Next we show that $\Sigma^X_{0,\infty}$ is initial among the morphisms from X to objects of **EHyp**:

Theorem 48. Let X be an object of **Ex**, Y an object of **EHyp** and $f : X \to Y$. Then there is a unique morphism $g : \Box^\infty X \to Y$ such that $f = g \circ \Sigma^X_{0,\infty}$, namely $g = \Box^\infty f$.

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\Sigma^X_{0,\infty}"'] & & Y \\
& \Box^\infty X \arrow[ur, dotted, "g"'] &
\end{tikzcd}$$

The functor $\Box^\infty$: **Ex** $\to$ **EHyp** is a left adjoint of the inclusion functor.

Therefore, for every object X of **Ex*** and Y of **EHyp** and every morphism $f : X \to Y$, there is a unique morphism $g : \Box^\infty \beta X \to Y$ with $f = g \circ \Sigma^{\beta X}_{0,\infty} \circ \iota_X$, namely $g = (\iota_Y)^{-1} \circ \Box^\infty \beta f$.

Proof. Since $\gamma_Y = 0$ and Σ_{0,γ_X} is a natural transformation,

$$\Box^\infty f \circ \Sigma^X_{0,\infty} = \left(\Sigma^Y_{\gamma_Y,\gamma_X}\right)^{-1} \circ \Box^{\gamma_X} f \circ \Sigma^X_{\gamma_X,\gamma_X} \circ \Sigma^X_{0,\gamma_X} = \left(\Sigma^Y_{0,\gamma_X}\right)^{-1} \circ \Box^{\gamma_X} f \circ \Sigma^X_{0,\gamma_X} = f\,,$$

so $\Box^\infty f$ has the required property.

It remains to show that for every morphism g, the equation $f = g \circ \Sigma^X_{0,\infty}$ implies $g = \Box^\infty f$. By induction on $\alpha \leqslant \gamma_X$, we will show that the following diagram commutes:

$$\begin{tikzcd}
\Box^\alpha X \arrow[r, "\Sigma^X_{\alpha,\infty}"] \arrow[d, "\Box^\alpha f"'] & \Box^\infty X \arrow[d, "g"] \\
\Box^\alpha Y & Y \arrow[l, "\Sigma^Y_{0,\alpha}"']
\end{tikzcd}$$

Then the case $\alpha = \gamma_X$ will imply our claim, proving uniqueness:

$$\Box^\infty f \;=\; \left(\Sigma^Y_{0,\gamma_X}\right)^{-1} \circ \Box^{\gamma_X} f \;=\; g \circ \Sigma^X_{\gamma_X,\infty} \;=\; g$$

The case $\alpha = 0$ is just our assumption $f = g \circ \Sigma^X_{0,\infty}$.

Now let $\alpha = \beta + 1$. Applying $\Box$ to the induction hypothesis for β, we obtain the left cell in the diagram

$$\begin{tikzcd}
\Box^\alpha X \arrow[r, "\Sigma^X_{\alpha,\gamma_X+1}"] \arrow[d, "\Box^\alpha f"'] & \Box^{\gamma_X+1} X \arrow[d, "\Box g"] & \Box^{\gamma_X} X \arrow[l, "\Sigma^X_{\gamma_X,\gamma_X+1}"'] \arrow[d, "g"] \\
\Box^\alpha Y & \Box Y \arrow[l, "\Sigma^Y_{1,\alpha}"'] & Y \arrow[l, "\Sigma^Y_{0,1}"']
\end{tikzcd}$$

and the right cell commutes because Σ is a natural transformation. Since $\Sigma^X_{\gamma_X,\gamma_X+1} = \Sigma^{\Box^\infty X}$ is an isomorphism, this proves the case α.

Finally, let α be a limit ordinal and assume the induction hypothesis for all $\beta < \alpha$. Then every cell in the diagram

$$\begin{tikzcd}
\Box^\alpha X \arrow[rr, "\Sigma^X_{\alpha,\infty}"] \arrow[ddd, "\Box^\alpha f"'] & & \Box^\infty X \arrow[ddd, "g"] \\
& \Box^\beta X \arrow[ul, "\Sigma^X_{\beta,\alpha}"] \arrow[ur, "\Sigma^X_{\beta,\infty}"'] \arrow[d, "\Box^\beta f"'] & \\
& \Box^\beta Y \arrow[dl, "\Sigma^Y_{\beta,\alpha}"'] \arrow[dr, "\Sigma^Y_{0,\beta}"] & \\
\Box^\alpha Y & & Y \arrow[ll, "\Sigma^Y_{0,\alpha}"']
\end{tikzcd}$$

commutes and hence

$$\left(\Sigma^Y_{0,\alpha}\right)^{-1} \circ \Box^\alpha f \circ \Sigma^X_{\beta,\alpha} \quad = \quad g \circ \Sigma^X_{\alpha,\infty} \circ \Sigma^X_{\beta,\alpha} \quad : \quad \Box^\beta X \to Y$$

for all $\beta < \alpha$. So by the direct limit property of $\Box^\alpha X$, it follows that $\left(\Sigma^Y_{0,\alpha}\right)^{-1} \circ \Box^\alpha f = g \circ \Sigma^X_{\alpha,\infty}$.

To prove that $\Box^\infty$ is a left adjoint, we prove that

$$\Phi_{X,Y} : \mathrm{hom}(\Box^\infty X, Y) \to \mathrm{hom}(X,Y), \quad f \mapsto f \circ \Sigma^X_{0,\infty}$$

has the required property:

$$\Phi_{\widetilde{X},\widetilde{Y}}(h_1 \circ f \circ \Box^\infty h_0) = h_1 \circ f \circ \Box^\infty h_0 \circ \Sigma^{\widetilde{X}}_{0,\infty} = h_1 \circ f \circ \Sigma^X_{0,\infty} \circ h_0 = h_1 \circ \Phi_{X,Y}(f) \circ h_0,$$

for all morphisms $h_0 : \widetilde{X} \to X$, $h_1 : Y \to \widetilde{Y}$ and $f : X \to Y$. ■

Let X be an object in **Ex*** and $A \subseteq \mathbb{V}X$ open. A morphism $f : X \to Y$ is called an A-*homeomorphism* if

- $f(x) \neq f(a)$ for every $x \in \mathbb{V}X$ and $a \in A$ with $x \neq a$, and
- $f[U]$ is open in $\mathbb{V}Y$ for every open $U \subseteq A$.

Then if $B \subseteq A$ is open, f is also a B-homeomorphism. And whenever g is an $f[A]$-homeomorphism, $g \circ f$ is an A-homeomorphism.

A is called *transitive* if

$$\Sigma^X[A \cap \mathbb{S}X] \quad \subseteq \quad \Box A,$$

and it is called *persistent* if it is transitive, Hausdorff, open, locally κ-compact and Σ^X is an A-homeomorphism. Intuitively, A is transitive iff all the elements of the class A in the model X are subsets of A again, and as we will see the notion of persistence is chosen such that a persistent set is protected from being collapsed in the construction process of the $\Box^\alpha X$, so that it – or rather its isomorphic image – is still present in the hyperuniverse $\Box^\infty X$.

Lemma 49. Let X be an object of $\mathbf{Ex}^*$ and $A \subseteq \mathbb{V}X$.

1. If A is transitive in X, $f[A]$ is transitive in Y for every $f : X \to Y$. $\iota_X[A]$, all $\Sigma^X_{0,\alpha}[A]$ and $\Box A \cup (A \cap \mathbb{A}X)$ are transitive in βX, $\Box^\alpha[X]$ and $\Box X$.
2. If A is persistent and $f : X \to Y$ is an A-homeomorphism, then $\Box f$ is a $\Box A \cup (A \cap \mathbb{A}X)$-homeomorphism. If moreover $f[\mathbb{S}X]$ is dense in $\mathbb{S}Y$, then $f[A]$ is persistent. In particular, $\Sigma^X[A]$ and $\Box A \cup (A \cap \mathbb{A}X)$ are persistent in $\Box X$ and $\iota_X[A]$ is persistent in βX.
3. If $\langle X_\alpha, f_{\alpha,\beta}\rangle_{\alpha,\beta<\lambda}$ is a directed system in $\mathbf{Ex}^*$, each $f_{\alpha,\beta}$ is an $f_{0,\alpha}[A]$-homeomorphism and $f_{0,\alpha}[A]$ is persistent for each $\alpha < \lambda$, then if

$$\langle X_\lambda, f_{\alpha,\lambda}\rangle_{\alpha<\lambda} = \underset{\to}{\lim}^{\mathbf{Ex}^*} \langle X_\alpha, f_{\alpha,\beta}\rangle_{\alpha,\beta<\lambda}$$

is the direct limit, every $f_{\alpha,\lambda}$ is an $f_{0,\alpha}[A]$-homeomorphism and $f_{0,\lambda}[A]$ is persistent.

Proof. (1): These claims are easily verified by direct calculation:

$$\begin{aligned}
\Sigma^Y[f[A] \cap \mathbb{S}Y] &= \Sigma^Y[f[A \cap \mathbb{S}X]] = \mathrm{Exp}^c_\kappa(f)[\Sigma^X[A \cap \mathbb{S}X]] \\
&\subseteq \mathrm{Exp}^c_\kappa(f)[\Box A] = \Box(f[A]) \\
\Sigma^{\Box X}[(\Box A \cup (A \cap \mathbb{A}X)) \cap \mathbb{S}\Box X] &= \Sigma^{\Box X}[\Box A] = \mathrm{Exp}^c_\kappa(\Sigma^X)[\Box A] = \Box(\Sigma^X[A]) \\
&= \Box(\Sigma^X[A \cap \mathbb{S}X] \cup \Sigma^X[A \cap \mathbb{A}X]) \subseteq \Box(\Box A \cup (A \cap \mathbb{A}X))
\end{aligned}$$

(2): Since $\Box f$ is the union of $\mathrm{Exp}^c_\kappa(\mathbb{V}f)$ and $\mathbb{A}f$, its injectivity on $\Box A$ and $A \cap \mathbb{A}A$ follows from the injectivity of f on A. It is also open on $A \cap \mathbb{A}A$, because $\mathbb{A}f$ is. To prove its openness on $\Box A$ it suffices to check the images of subbase sets of the form $\Box V$ and $\Diamond V$ for open $V \subseteq A$:

$$\begin{aligned}
\Sigma^{\Box X}[\Box V] &= \mathrm{Exp}^c_\kappa(\Sigma^X)[\Box V] = \Box(\Sigma^X[V]) \text{ and} \\
\Sigma^{\Box X}[\Diamond V] &= \mathrm{Exp}^c_\kappa(\Sigma^X)[\Diamond V] = \Diamond(\Sigma^X[V]).
\end{aligned}$$

It remains to show that no $b \notin \Box A \cup (A \cap \mathbb{A}X)$ is mapped into the image of that set. But such a b is either an atom – then $\Box f(b) = f(b) \notin f(A \cap \mathbb{A}X) = \Box f(A \cap \mathbb{A}X)$ –, or it contains an element $z \notin A$. In that case, $f(z) \notin f[A]$ and therefore $\Box f(b)$ contains an element not in $f[A]$, so $\Box f(b) \notin \Box(f[A]) = \Box f[\Box A]$.

We have already seen that $f[A]$ is transitive. As the homeomorphic image of a Hausdorff, locally κ-compact set, it also has these properties. Since $\Sigma^X[A] \subseteq \Box A \cup (A \cap \mathbb{A}X)$ and $\Box f$ is an $\Box A \cup (A \cap \mathbb{A}X)$-homeomorphism, $\Box f \circ \Sigma^X = \Sigma^Y \circ f$ is an A-homeomorphism. In particular, Σ^Y maps $f[A]$ homeomorphically onto its image, which is open.

Now assume that $f[\mathbb{S}X]$ is dense in $\mathbb{S}Y$. It remains to prove that $y \in f[A]$ whenever $\Sigma^Y(y) \in \Sigma^Y[f[A]]$. So assume $\Sigma^Y(y) = \Sigma^Y(f(x))$ for some $x \in A$. Then x has a κ-compact neighborhood U in A, and $f[U] \ni f(x)$ as well as $V = \Sigma^Y[f[U]] \ni \Sigma^Y(y)$ are κ-compact neighborhoods. Thus if $y \notin f[A]$, then in particular $y \notin f[A] \cap (\Sigma^Y)^{-1}[V] = f[U]$, and hence y has a neighborhood $W \subseteq (\Sigma^Y)^{-1}[V]$ disjoint from $f[U]$ and therefore disjoint from A. But $f[\mathbb{S}X] \cap W = f[\mathbb{S}X \setminus A] \cap W$ is dense in W. So

$$\Sigma^Y(y) \in \Sigma^Y[\mathrm{cl}(f[\mathbb{S}X \setminus A])] \subseteq \mathrm{cl}(\Sigma^Y[f[\mathbb{S}X \setminus A]]) \subseteq \mathrm{cl}(\Box f[\Sigma^X[\mathbb{S}X \setminus A]]).$$

But since $\Box f \circ \Sigma^X$ is an A-homeomorphism, $\Box f[\Sigma^X[\mathbb{S}X \setminus A]]$ is disjoint from the open set $\Box f[\Sigma^X[A]] = \Sigma^Y[f[A]]$, contradicting $\Sigma^Y(y) \in \Sigma^Y[f[A]]$.

We have already seen that $\Box A \cup (A \cap \mathbb{A}X)$ is transitive and $\Sigma^{\Box X} = \Box\Sigma^X$ is a $\Box A \cup (A \cap \mathbb{A}X)$-homeomorphism. To prove that $\Box A \cup (A \cap \mathbb{A}X)$ is persistent, we only have to verify that it is open, locally κ-compact and Hausdorff. $A \cap \mathbb{A}X$ is an open subset of A, so for that part of the union these properties are immediate. $\Box A$ is open because A is, and Hausdorff and locally κ-compact by Lemma 23.

As a transitive open subset of $\Box A \cup (A \cap \mathbb{A}X)$, $\Sigma^X[A]$ is persistent, too. Finally, ι_X is an A-homeomorphism with a dense image, so $\iota_X[A]$ is persistent.

(3): Let us begin by showing that $f_{0,\lambda}$ is an A-homeomorphism (and analogously every $f_{\alpha,\lambda}$ is an $f_{0,\alpha}[A]$-homeomorphism). If $x \in \mathbb{V}X_0$ and $y \in A$ are distinct, then $f_{0,\alpha}(x) \neq f_{0,\alpha}(y)$ for every α, so $f_{0,\lambda}(x) \neq f_{0,\lambda}(y)$. In particular, $f_{0,\lambda}$ is injective on A. Thus if $U \subseteq A$ is open, then since every $f_{\alpha,\lambda}^{-1}[f_{0,\lambda}[U]] = f_{0,\alpha}[U]$ is open, $f_{0,\lambda}[U]$ is open, too.

As in (2), it follows without any density assumption that $f_{0,\lambda}[A]$ is open, Hausdorff, locally κ-compact and Σ^{X_λ} maps $f_{0,\lambda}[A]$ homeomorphically onto its image, which is open. It remains to show that no $\widetilde{x} \notin f_{0,\lambda}[A]$ is mapped into $\Sigma^{X_\lambda}[f_{0,\lambda}[A]]$.

$\widetilde{x} = f_{\alpha,\lambda}(x)$ for some $x \in \mathbb{V}X_\alpha$. Let $\mathbb{V}Y = f_{\alpha,\lambda}[X_\alpha]$. Since $\Sigma^{X_\lambda} \circ f_{\alpha,\lambda} = \Box f_{\alpha,\lambda} \circ \Sigma^X$, the restriction $\mathbb{S}\Sigma^Y = \mathbb{S}\Sigma^{X_\lambda} \restriction \mathbb{S}Y$ is a map into $\mathrm{Exp}^c_\kappa(f_{\alpha,\lambda}[X_\alpha]) = \mathrm{Exp}^c_\kappa(\mathbb{V}Y)$. Thus $f_{\alpha,\lambda}$ also is an A-homeomorphism from X_α to Y, and then it is surjective, so (2) applies. Hence A is persistent in Y, which proves that $\Sigma^{X_\lambda}(\widetilde{x}) = \Sigma^Y(f_{\alpha,\lambda}(x)) \notin \Sigma^Y[f_{0,\lambda}[A]] = \Sigma^{X_\lambda}[f_{0,\lambda}[A]]$. ∎

Theorem 50. If X is an object of **Ex*** and $A \subseteq \mathbb{V}X$ is persistent, then $\Sigma^{\beta X}_{0,\alpha} \circ \iota_X[A]$ is persistent in $\Box^\alpha\beta X$, and $\Sigma^{\beta X}_{0,\alpha} \circ \iota_X$ is an A-homeomorphism for every ordinal α.

In particular, $\Sigma^{\beta X}_{0,\infty} \circ \iota_X[A]$ is persistent in the hyperuniverse $\Box^\infty\beta X$ and homeomorphic to A.

Proof. The proof is by induction on α and follows immediately from Lemma 49: First of all, ι_X

is an A-homeomorphism and $\iota_X[A]$ is persistent, which proves the case $\alpha = 0$. Now assume $\alpha = \beta + 1$. By the induction hypothesis, $\widetilde{A} = \Sigma^{\beta X}_{0,\beta} \circ \iota_X[A]$ is persistent in $\Box^\beta \beta X$, and $\Sigma^{\beta X}_{0,\beta} \circ \iota_X$ is an A-homeomorphism. Thus $\Sigma^{\beta X}_{\beta,\alpha}$ is a $\widetilde{A}$-homeomorphism, so the composition $\Sigma^{\beta X}_{0,\alpha} \circ \iota_X$ is an A-homeomorphism, and by (2), $\Sigma^{\beta X}_{\beta,\alpha}[\widetilde{A}]$ is also persistent.

Finally, let α be a limit ordinal and let

$$\langle Y, f_\beta \rangle_{\beta<\alpha} \quad = \quad \varinjlim^{\mathbf{Ex}^*} \langle \Box^\beta \beta X, \Sigma^{\beta X}_{\beta,\beta'} \rangle_{\beta \leqslant \beta' < \alpha}.$$

Then by (3), $f_{0,\alpha} \circ \iota_X[A]$ is persistent and $f_{0,\alpha}$ is a $\iota_X[A]$-homeomorphism. Hence by (2) again, the claim follows for $\Box^\alpha \beta X = \beta Y$. ■

This theorem is important because in the case $\mathbb{A}X = \emptyset$, it is the only way to know that $\Box^\infty X$ is a nontrivial hyperuniverse at all, that is, whether $\mathbb{V}\Box^\infty X$ has more than one point. (If $\mathbb{A} \neq \emptyset$, $\Box^\infty X$ can never be trivial.)

An interesting special case is when $\mathbb{S}\Sigma^X[\mathbb{S}X]$ is dense in $\mathrm{Exp}^c_\kappa(\mathbb{V}X)$. Then for all α, $\Sigma^{\beta X}_{0,\alpha}$ is surjective, and thus $\Sigma^{\beta X}_{0,\alpha} \circ \iota_X[\mathbb{V}X]$ is dense in $\mathbb{V}\Box^\alpha X$. If in addition $\mathbb{V}X$ is persistent, then all $\mathbb{V}\Box^\alpha X$ are κ-compactifications of $\mathbb{V}X$, that is, κ-compact Hausdorff spaces in which the image of $\mathbb{V}X$ is dense. In that case, $\Box^\infty X$ is the hyperuniverse whose universe space is the largest possible κ-compactification of $\mathbb{V}X$: Assume Y were another such hyperuniverse and $f : X \to Y$ an inclusion with dense image. Then $\Box^\infty f : \Box^\infty X \to Y$ witnesses the maximality of X.

For example, let $\mathbb{V}V_e = V_\kappa$ with the discrete topology, $\mathbb{A}V_e = \{\emptyset\}$ and $\mathbb{S}\Sigma^{V_e}$ the identity – which makes sense because $\mathrm{Exp}^c_\kappa(\mathbb{V}V_e) = \mathbb{S}V_e$. Then $\overline{V}_e = \Box^\infty \beta V_e$ is a hyperuniverse in which the $e_{V_e}[\mathbb{V}V_e]$ with $e_{V_e} = \Sigma^{\beta V_e}_{0,\infty} \circ \iota_{V_e}$, the set of well-founded sets, is densely embedded. And in fact, $\overline{V}_e$ is the largest such hyperuniverse in the sense of Theorem 48.

As another example, choose any κ-compact Hausdorff atom space $\mathbb{A}X$ and let $\mathbb{S}X = \emptyset$. Then $\Box^\infty X$ is initial among the hyperuniverses with that given atom space.

2.6 Inverse Limit Models

This section is largely dual to the previous one: The arrows are reversed, and with a few adaptations, we obtain analogous results. In fact, some problems which we had to deal with when carrying out the direct limit construction do not even arise here. For example, the construction never takes more than κ steps and inverse limits of at most κ κ-compact spaces are automatically κ-compact, so there is no need for an additional κ-compactification of those limits. Also, it is easier to watch the weight of the spaces involved. So let us now only assume that κ is mildly λ-ineffable for some regular $\lambda = \lambda^{<\kappa} \geqslant \kappa$.

The objects of the *weak comprehension maps category* $\mathbf{Cm}^*$ are triples $X = \langle \mathbb{V}X, \mathbb{S}X, \Sigma^X \rangle$ of

- a locally κ-compact Hausdorff space $\mathbb{V}X$ of weight $\leqslant \lambda$ called the *universe space* of X,
- a clopen subset $\mathbb{S}X \subseteq \mathbb{V}X$ called the *set space* of X, such that $\mathbb{A}X = \mathbb{V}X \setminus \mathbb{S}X$ is κ-compact, and

- a continuous map $\mathbb{S}\Sigma^X : \mathrm{Exp}^c_\kappa(\mathbb{V}X) \to \mathbb{S}X$.

We call $\mathbb{A}X$ the *atom space* of X and define $\mathbb{A}\Sigma^X = \mathrm{id}_{\mathbb{A}X}$ and

$$\mathbb{V}\Sigma^X \quad = \quad \mathbb{A}\Sigma^X \cup \mathbb{S}\Sigma^X : \quad \mathbb{A}X \cup \mathrm{Exp}^c_\kappa(\mathbb{V}_X) \quad \to \quad \mathbb{V}X,$$

the *comprehension map* of X. The morphisms $f : X \to Y$ of $\mathbf{Cm}^*$ are continuous maps $\mathbb{V}f : \mathbb{V}X \to \mathbb{V}Y$ such that the image of $\mathbb{A}f = \mathbb{V}f \upharpoonright \mathbb{A}X$ is a subset of $\mathbb{A}Y$, the image of $\mathbb{S}f = \mathbb{V}f \upharpoonright \mathbb{S}X$ is a subset of $\mathbb{S}Y$ and the diagram

$$\begin{tikzcd}
\mathrm{Exp}^c_\kappa(\mathbb{V}X) \arrow[r, "\mathrm{Exp}^c_\kappa(\mathbb{V}f)"] \arrow[d, "\mathbb{S}\Sigma^X"'] & \mathrm{Exp}^c_\kappa(\mathbb{V}Y) \arrow[d, "\mathbb{S}\Sigma^Y"] \\
\mathbb{S}X \arrow[r, "\mathbb{S}f"] & \mathbb{S}Y
\end{tikzcd}$$

commutes. Note that as the hyperspace of a locally κ-compact Hausdorff space, $\mathrm{Exp}^c_\kappa(\mathbb{V}X)$ is also locally κ-compact Hausdorff by Lemma 23. And by Lemma 30 it has weight $\leqslant \lambda$.

We define the functor $\Box : \mathbf{Cm}^* \to \mathbf{Cm}^*$ as follows:

$$\begin{array}{rclcrcl}
\mathbb{A}\Box X & = & \mathbb{A}X & \qquad & \mathbb{S}\Box X & = & \mathrm{Exp}^c_\kappa(\mathbb{V}X) \\
\mathbb{A}\Box f & = & \mathbb{A}f & & \mathbb{S}\Box f & = & \mathrm{Exp}^c_\kappa(\mathbb{V}f) \\
\mathbb{A}\Sigma^{\Box X} & = & \mathrm{id}_{\mathbb{A}X} & & \mathbb{S}\Sigma^{\Box X} & = & \mathrm{Exp}^c_\kappa(\mathbb{V}\Sigma^X)
\end{array}$$

Then $\Sigma^X : \Box X \to X$ is a morphism and the following diagram commutes, which shows that $X \mapsto \Sigma^X$ is a natural transformation from $\Box$ to the identity:

$$\begin{tikzcd}
\Box X \arrow[r, "\Box f"] \arrow[d, "\Sigma^X"] & \Box Y \arrow[d, "\Sigma^Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

The *comprehension maps category* $\mathbf{Cm}$ is the full subcategory of $\mathbf{Cm}^*$ given by those objects X whose universe space $\mathbb{V}X$ (or equivalently $\mathbb{S}X$) is κ-compact. And the *category of (comprehension map) hyperuniverses* $\mathbf{CHyp}$ is the full subcategory of $\mathbf{Cm}$ given by those objects X where Σ^X is bijective and thus a homeomorphism. The restriction $\Box \upharpoonright \mathbf{Cm}$ is a functor from $\mathbf{Cm}$ to $\mathbf{Cm}$. Finally, let $\mathbf{PCm}^*$ be the subcategory of $\mathbf{Cm}^*$ of only those objects X with κ-proper $\mathbb{S}\Sigma^X$ and only the κ-proper morphisms.

We define the κ-*Alexandroff compactification functor* ω from $\mathbf{PCm}^*$ to $\mathbf{Cm}$ as follows:

$$\begin{array}{rclcrcl}
\mathbb{A}\omega X & = & \mathbb{A}X & \qquad & \mathbb{A}\omega f & = & \mathbb{A}f \\
\mathbb{S}\omega X & = & \omega\mathbb{S}X & & \mathbb{S}\omega f & = & \omega\mathbb{S}f,
\end{array}$$

that is, $\mathbb{S}\omega f(p) = p$, and using the universal property of the κ-Alexandroff compactification, let $\mathbb{S}\Sigma^{\omega X} : \mathrm{Exp}^c_\kappa(\mathbb{V}\omega X) \to \mathbb{S}\omega X$ be the unique map such that the diagram

$$\begin{tikzcd}
\mathrm{Exp}^c_\kappa(\mathbb{V}X) \arrow[r, "\mathrm{Exp}^c_\kappa(\iota_{\mathbb{V}X})"] \arrow[d, "\mathbb{S}\Sigma^X"] & \mathrm{Exp}_\kappa(\mathbb{V}\omega X) \arrow[d, "\mathbb{S}\Sigma^{\omega X}"] \\
\mathbb{S}X \arrow[r, "\iota_{\mathbb{S}X}"] & \mathbb{S}\omega X
\end{tikzcd}$$

commutes and everything not in $\mathrm{rng}(\mathrm{Exp}^c_\kappa(\iota_{\mathbb{V}X}))$ is mapped to p, that is, such that $\mathbb{V}\iota_X = \mathrm{id}_{\mathbb{A}X} \cup \iota_{\mathbb{S}X}$ defines a morphism $\iota_X : X \to \omega X$. This is possible by Lemma 29 because as $\mathbb{S}X$ is open in $\mathbb{S}\omega X$, $\mathrm{Exp}^c_\kappa(\mathbb{V}X)$ is open in $\mathrm{Exp}_\kappa(\mathbb{V}\omega X)$, and $\mathbb{S}\Sigma^X$ is assumed to be κ-proper. For a κ-proper morphism $f : X \to Y$, ωf is in fact a morphism, because every cell in the diagram

$$\begin{tikzcd}
\mathrm{Exp}_\kappa(\mathbb{V}\omega X) \arrow[rrr, "\mathrm{Exp}_\kappa(\mathbb{V}\omega f)"] \arrow[ddd, "\mathbb{S}\Sigma^{\omega X}"'] & & & \mathrm{Exp}_\kappa(\mathbb{V}\omega Y) \arrow[ddd, "\mathbb{S}\Sigma^{\omega Y}"] \\
& \mathrm{Exp}^c_\kappa(\mathbb{V}X) \arrow[r, "\mathrm{Exp}^c_\kappa(\mathbb{V}f)"] \arrow[ul, "\mathrm{Exp}^c_\kappa(\mathbb{V}\iota_X)"'] \arrow[d, "\mathbb{S}\Sigma^X"] & \mathrm{Exp}^c_\kappa(\mathbb{V}Y) \arrow[ur, "\mathrm{Exp}^c_\kappa(\mathbb{V}\iota_Y)"] \arrow[d, "\mathbb{S}\Sigma^Y"] & \\
& \mathbb{S}X \arrow[r, "\mathbb{S}f"] \arrow[dl, "\mathbb{S}\iota_X"'] & \mathbb{S}Y \arrow[dr, "\mathbb{S}\iota_Y"] & \\
\mathbb{S}\omega X \arrow[rrr, "\mathbb{S}\omega f"] & & & \mathbb{S}\omega Y
\end{tikzcd}$$

is commutative, so $\mathbb{S}\Sigma^{\omega Y} \circ \mathrm{Exp}_\kappa(\mathbb{V}\omega f) \circ \mathrm{Exp}^c_\kappa(\mathbb{V}\iota_X) = \mathbb{S}\omega f \circ \mathbb{S}\Sigma^{\omega X} \circ \mathrm{Exp}^c_\kappa(\mathbb{V}\iota_X)$ and $\mathrm{Exp}^c_\kappa(\mathbb{V}\iota_X)$ can be cancelled on the right because if $a \in \mathrm{Exp}_\kappa(\mathbb{V}\omega X)$ is not in the image of $\mathrm{Exp}^c_\kappa(\mathbb{V}\iota_X)$, then it contains p, and in that case, $p \in \mathrm{Exp}_\kappa(\mathbb{V}\omega f)(a)$ and hence

$$\mathbb{S}\Sigma^{\omega Y}(\mathrm{Exp}_\kappa(\mathbb{V}\omega f)(a)) \;=\; p \;=\; \mathbb{S}\omega f(p) \;=\; \mathbb{S}\omega f(\mathbb{S}\Sigma^{\omega X}(a)).$$

The map $X \mapsto \iota_X$ is a natural transformation from the identity on $\mathbf{PCm}^*$ to ω.

Lemma 51. Whenever Y is in **CHyp**, X is in $\mathbf{PCm}^*$, Σ^X is surjective and $f : X \to Y$ homeomorphically embeds $\mathbb{V}X$ as an open subset in $\mathbb{V}Y$, then there is a unique morphism $g : Y \to \omega X$ such that $g \circ f = \iota_X$ and $g(x) = p$ for every $x \in \mathbb{V}Y \setminus \mathrm{rng}(\mathbb{V}f)$.

Proof. Applying Lemma 29 to $(\mathbb{V}f)^{-1}$, we obtain a map $\mathbb{V}g : \mathbb{V}Y \to \mathbb{V}\omega X$ with the required properties, and we only have to show that it defines a homeomorphism. Firstly, let $a = \Box f(b)$. Then by definition

$$\begin{aligned}
\Sigma^{\omega X} \circ \mathrm{Exp}_\kappa(\mathbb{V}g)(a) \;&=\; \Sigma^{\omega X} \circ \mathrm{Exp}^c_\kappa(\mathbb{V}g \circ f)(b) \;=\; \Sigma^{\omega X} \circ \mathrm{Exp}^c_\kappa(\iota_X)(b) \;=\; \iota_X \circ \Sigma^X(b) \\
&=\; \mathbb{V}g \circ f \circ \Sigma^X(b) \;=\; g \circ \Sigma^Y \circ \Box f(b) \;=\; \mathbb{V}g \circ \Sigma^Y(a)
\end{aligned}$$

If, on the other hand, $a \in \mathrm{Exp}_\kappa(\mathbb{V}Y) \setminus \mathrm{rng}(\Box f)$, then a contains an element not in $\mathrm{rng}(f)$ and hence $p \in \mathrm{Exp}_\kappa(\mathbb{V}g)(a)$ and $\Sigma^{\omega X} \circ \mathrm{Exp}_\kappa(\mathbb{V}g)(a) = p$. Therefore it suffices to show that $\Sigma^Y(a) \notin \mathrm{rng}(f)$, because that entails $\mathbb{V}g(\Sigma^Y(a)) = p$, too. So assume $\Sigma^Y(a) = f(x)$. Then $\Sigma^Y(a) = f(\Sigma^X(b))$ for some b, because Σ^X is surjective. It follows that $\Sigma^Y(a) = \Sigma^Y \circ \Box f(b)$ and thus $a = \Box f(b)$, contradicting our assumption. ■

Lemma 52. Inverse limits exist in the category **Cm**. Let $\langle X_i, f_{ij} \rangle$ be an inverse system in **Cm** and let $\langle X, f_i : X \to X_i \rangle$ be its inverse limit in **Cm**. Then:

$$\langle \mathbb{V}X, \mathbb{V}f_i \rangle = \varprojlim^{\mathbf{Top}} \langle \mathbb{V}X_i, \mathbb{V}f_{ij} \rangle \quad \text{and} \quad \langle \mathbb{S}X, \mathbb{S}f_i \rangle = \varprojlim^{\mathbf{Top}} \langle \mathbb{S}X_i, \mathbb{S}f_{ij} \rangle$$

and $\mathbb{S}\Sigma^X$ is the unique map such that for all i, $\mathbb{S}\Sigma^{X_i} \circ \mathrm{Exp}(\mathbb{V}f_i) = \mathbb{S}f_i \circ \mathbb{S}\Sigma^X$.

Proof. This is exactly dual to Lemma 46 about direct limits in **Ex**: We show that the object X with the morphisms f_i defined by those topological limits is in fact an inverse limit of the given system in **Cm**.

For each i, let $h_i : Z \to X_i$ be a morphism and assume that for all $j \geqslant i$, $h_i = f_{ij} \circ h_j$. We then define $\mathbb{V}h$ using the inverse limit property of $\mathbb{V}X$. This is the unique candidate for a suitable morphism $h : Z \to X$, and we just have to show that it is in fact a morphism. For each i, every path from $\mathrm{Exp}^c_\kappa(\mathbb{V}Z)$ to $\mathbb{S}X_i$ is equal in the diagram

$$\begin{tikzcd}
\mathrm{Exp}^c_\kappa(\mathbb{V}X) \arrow[dr, "\mathrm{Exp}^c_\kappa(\mathbb{V}f_i)"] \arrow[dd, "\mathbb{S}\Sigma^X"'] & & \mathrm{Exp}^c_\kappa(\mathbb{V}Z) \arrow[ll, "\mathrm{Exp}^c_\kappa(\mathbb{V}h)"'] \arrow[dl, "\mathrm{Exp}^c_\kappa(\mathbb{V}h_i)"'] \arrow[dd, "\mathbb{S}\Sigma^Z"] \\
& \mathrm{Exp}^c_\kappa(\mathbb{V}X_i) \arrow[d, "\mathbb{S}\Sigma^{X_i}"] & \\
& \mathbb{S}X_i & \\
\mathbb{S}X \arrow[ur, "\mathbb{S}f_i"] \arrow[rr, "\mathbb{S}h"'] & & \mathbb{S}Z \arrow[ul, "\mathbb{S}h_i"']
\end{tikzcd}$$

and thus $\mathbb{S}f_i \circ \mathbb{S}h \circ \mathbb{S}\Sigma^Z = \mathbb{S}f_i \circ \mathbb{S}\Sigma^X \circ \mathrm{Exp}_\kappa(\mathbb{V}h)$. Since $\langle \mathbb{S}X, \mathbb{S}f_i \rangle$ is the inverse limit of the $\mathbb{S}X_i$, it follows that $\mathbb{S}h \circ \mathbb{S}\Sigma^Z = \mathbb{S}\Sigma^X \circ \mathrm{Exp}_\kappa(\mathbb{V}h)$. ∎

Also dually to the situation in the category **Ex**, we recursively define functors $\Box^\beta : \mathbf{Cm} \to \mathbf{Cm}$ and morphisms $\Sigma^X_{\alpha,\beta} : \Box^\beta X \to \Box^\alpha X$ for ordinals $\alpha \leqslant \beta$:

$$\begin{array}{rclcrclcrcl}
\Box^0 X & = & X & & \Sigma^X_{\alpha,\alpha} & = & \mathrm{id}_{\Box^\alpha X} & & \Box^0 f & = & f \\
\Box^{\alpha+1} X & = & \Box\Box^\alpha X & & \Sigma^X_{\alpha,\beta+1} & = & \Sigma^X_{\alpha,\beta} \circ \Sigma^{\Box^\beta X} & & \Box^{\alpha+1} f & = & \Box\Box^\alpha f
\end{array}$$

For limit ordinals γ, we define

$$\langle \Box^\gamma X, \Sigma^X_{\alpha,\gamma} \rangle \quad = \quad \underleftarrow{\lim}^{\mathbf{Cm}} \langle \Box^\alpha X, \Sigma^X_{\alpha,\beta} \rangle_{\alpha \leqslant \beta < \gamma}$$

and if $f : X \to Y$ is a morphism, we use the inverse limit property of $\Box^\gamma X$: $\Box^\gamma f$ is defined as the unique morphism, such that $\Box^\alpha f \circ \Sigma^X_{\alpha,\gamma} = \Sigma^Y_{\alpha,\gamma} \circ \Box^\gamma f$ for all $\alpha < \gamma$. Then $X \mapsto \Sigma^X_{\alpha,\beta}$ is a natural transformation from $\Box^\beta$ to $\Box^\alpha$, that is, for every morphism $f : X \to Y$, the following diagram commutes:

$$\begin{tikzcd}
\Box^\alpha X \arrow[d, "\Box^\alpha f"] & \Box^\beta X \arrow[l, "\Sigma^X_{\alpha,\beta}"'] \arrow[d, "\Box^\beta f"] \\
\Box^\alpha Y & \Box^\beta Y \arrow[l, "\Sigma^Y_{\alpha,\beta}"']
\end{tikzcd}$$

The construction will stop at $\Box^\kappa X$, because $\mathbb{S}\Sigma^X_{\kappa+1,\kappa} = \mathbb{S}\Sigma^{\Box^\kappa X}$ will already be a homeomorphism: Since κ is regular, the systems

$$\langle \mathbb{V}X_\alpha, \mathbb{V}\Sigma^X_{\alpha,\beta} \rangle \quad \text{and} \quad \langle \mathrm{Exp}_\kappa(\mathbb{V}X_\alpha), \mathrm{Exp}_\kappa(\mathbb{V}\Sigma^X_{\alpha,\beta}) \rangle = \langle \mathbb{S}X_{\alpha+1}, \mathbb{S}\Sigma^X_{\alpha+1,\beta+1} \rangle$$

are κ-directed, so by Lemma 37, $\mathbb{S}\Sigma^X_{\kappa,\kappa+1}$ is a homeomorphism from the hyperspace $\mathrm{Exp}_\kappa(\mathbb{V}X_\kappa)$ of the limit of the former to the limit $\mathbb{S}X_\kappa$ of the latter.

We define $\Box^\infty = \Box^\kappa$. Then $\Box^\infty$ is a functor from **Cm** to the category of hyperuniverses **CHyp**. Note that if Σ^X is surjective, $\Sigma^X_{0,\infty}$ also is, so that X is a quotient of the resulting hyperuniverse. Finally, we show that $\Sigma^X_{0,\infty}$ is terminal among the morphisms from objects of **CHyp** to X:

Theorem 53. Let X be an object of **Cm**, Y an object of **CHyp** and $f : Y \to X$. Then there is a unique morphism $g : Y \to \Box^\infty X$ such that $f = \Sigma^X_{0,\infty} \circ g$, namely $g = \Box^\infty f \circ (\Sigma^Y_{0,\infty})^{-1}$.

$$\begin{tikzcd}
X & & Y \arrow[ll, "f"'] \arrow[dl, dotted, "g"] \\
& \Box^\infty X \arrow[ul, "\Sigma^X_{0,\infty}"] &
\end{tikzcd}$$

The functor $\Box^\infty$: **Cm** $\to$ **CHyp** is a right adjoint to the inclusion functor.

Proof. Again, this proof is mostly dual to that of Theorem 48, although the situation is a bit easier because instead of γ_X we can always use κ alias ∞. Since $\Sigma_{0,\kappa}$ is a natural transformation,

$$\Sigma^X_{0,\kappa} \circ g \;=\; \Sigma^X_{0,\kappa} \circ \Box^\kappa f \circ (\Sigma^Y_{0,\kappa})^{-1} \;=\; f\,,$$

so $\Box^\kappa f \circ (\Sigma^Y_{0,\kappa})^{-1}$ has the required property.

It remains to show that for every morhism g, the equation $f = \Sigma^X_{0,\kappa} \circ g$ implies $g = \Box^\kappa f \circ (\Sigma^Y_{0,\kappa})^{-1}$. By induction on $\alpha \leqslant \kappa$, we will show that the following diagram commutes:

$$\begin{tikzcd}
\Box^\alpha X & \Box^\kappa X \arrow[l, "\Sigma^X_{\alpha,\kappa}"] \\
\Box^\alpha Y \arrow[u, "\Box^\alpha f"] \arrow[r, "\Sigma^Y_{0,\alpha}"'] & Y \arrow[u, "g"']
\end{tikzcd}$$

Then the case $\alpha = \kappa$ will imply our claim, proving uniqueness:

$$\Box^\kappa f \;=\; \Sigma^X_{\kappa,\kappa} \circ g \circ \Sigma^Y_{0,\kappa} \;=\; g \circ \Sigma^Y_{0,\kappa}$$

The case $\alpha = 0$ is just our assumption $f = \Sigma^X_{0,\kappa} \circ g$.

Now let $\alpha = \beta + 1$. Applying $\Box$ to the induction hypothesis for β, we obtain the left cell in the diagram

$$\begin{tikzcd}
\Box^\alpha X & \Box^{\kappa+1} X \arrow[l, "\Sigma^X_{\alpha,\kappa+1}"] \arrow[r, "\Sigma^X_{\kappa,\kappa+1}"] & \Box^\kappa X \\
\Box^\alpha Y \arrow[u, "\Box^\alpha f"] \arrow[r, "\Sigma^Y_{1,\alpha}"'] & \Box Y \arrow[u, "\Box g"'] \arrow[r, "\Sigma^Y_{0,1}"'] & Y \arrow[u, "g"']
\end{tikzcd}$$

and the right cell commutes because Σ is a natural transformation. Since $\Sigma^X_{\kappa,\kappa+1} = \Sigma^{\Box^\kappa X}$ is an isomorphism, this proves the case α.

Finally, let α be a limit ordinal and assume the induction hypothesis for all $\beta < \alpha$. Then every cell in the diagram

$$\begin{tikzcd}
\Box^{\alpha}X \arrow[dr, "\Sigma^X_{\beta,\alpha}"] & & \Box^{\kappa}X \arrow[ll, "\Sigma^X_{\alpha,\kappa}"'] \arrow[dl, "\Sigma^X_{\beta,\kappa}"'] \\
& \Box^{\beta}X & \\
& \Box^{\beta}Y \arrow[u, "\Box^{\beta} f"'] & \\
\Box^{\alpha}Y \arrow[uuu, "\Box^{\alpha} f"'] \arrow[ur, "\Sigma^Y_{\beta,\alpha}"] \arrow[rr, "\Sigma^Y_{0,\alpha}"'] & & Y \arrow[ul, "\Sigma^Y_{0,\beta}"'] \arrow[uuu, "g"]
\end{tikzcd}$$

commutes and hence

$$\Sigma^X_{\beta,\alpha} \circ \Box^{\alpha} f \circ \left(\Sigma^Y_{0,\alpha}\right)^{-1} \quad = \quad \Sigma^X_{\beta,\alpha} \circ \Sigma^X_{\alpha,\kappa} \circ g \quad : \quad Y \to \Box^{\beta} X$$

for all $\beta < \alpha$. So by the inverse limit property of $\Box^{\alpha}X$, it follows that

$$\Box^{\alpha} f \circ \left(\Sigma^Y_{0,\alpha}\right)^{-1} \quad = \quad \Sigma^X_{\alpha,\kappa} \circ g.$$

To prove that $\Box^{\kappa}$ is a right adjoint, we show that the bijection

$$\Phi_{Y,X} : \hom(Y,X) \to \hom(Y,\Box^{\kappa}X), \quad f \mapsto \Box^{\kappa} f \circ (\Sigma^Y_{0,\kappa})^{-1}$$

has the required property:

$$\begin{aligned}
\Phi_{\widetilde{Y},\widetilde{X}}(h_1 \circ f \circ h_0) &= \Box^{\kappa}(h_1 \circ f \circ h_0) \circ (\Sigma^{\widetilde{Y}}_{0,\kappa})^{-1} = \Box^{\kappa} h_1 \circ \Box^{\kappa} f \circ \Box^{\kappa} h_0 \circ (\Sigma^{\widetilde{Y}}_{0,\kappa})^{-1} \\
&= \Box^{\kappa} h_1 \circ \Box^{\kappa} f \circ (\Sigma^Y_{0,\kappa})^{-1} \circ h_0 = \Box^{\kappa} h_1 \circ \Phi_{Y,X}(f) \circ h_0
\end{aligned}$$

for all morphisms $h_0 : \widetilde{Y} \to Y$, $h_1 : X \to \widetilde{X}$ and $f : Y \to X$. ■

A difference with the category **Ex** is that proving nontriviality of $\Box^{\infty}X$ is a lot easier here: It suffices that Σ^X is surjective and X is nontrivial. And in fact, every $\Box^{\infty}X$ arises also from an object Y with surjective Σ^Y:

Proposition 54. Let X be an object of **Cm**. Let $\mathbb{V}Y = \operatorname{rng}(\Sigma^X_{0,\infty})$ and $\mathbb{S}Y = \mathbb{S}X \cap \mathbb{V}Y$. Then $\Sigma^Y = \Sigma^X \restriction (\mathbb{A}X \cup \operatorname{Exp}_{\kappa}(\mathbb{V}Y))$ defines an object Y and $\Box^{\infty}X$ is isomorphic to $\Box^{\infty}Y$.

Proof. Given $a \in \operatorname{Exp}_{\kappa}(\mathbb{V}Y)$, let $b = (\Sigma^X_{0,\infty})^{-1}[a] \in \operatorname{Exp}_{\kappa}(\mathbb{V}\Box^{\infty}X)$. Then

$$\Sigma^X(a) \quad = \quad \Sigma^X(\Sigma^X_{0,\infty}[b]) \quad = \quad \Sigma^X_{0,1}(\Sigma^X_{1,\kappa+1}(b)) \quad = \quad \Sigma^X_{0,\infty}(\Sigma^X_{\kappa,\kappa+1}(b)) \quad \in \quad \operatorname{rng}(\Sigma^X_{0,\infty})$$

Therefore Σ^X actually maps $\operatorname{Exp}_{\kappa}(\mathbb{V}Y)$ to $\mathbb{V}Y$ and the restriction defines an object of **Cm**.

$\mathbb{V}\Sigma^X_{0,\infty}$ also defines a surjective morphism $h : \Box^\infty X \to Y$, such that $h \circ f = \Sigma^X_{0,\infty}$ for the canonical inclusion $f : Y \to X$. By Theorem 53, there is a unique $\widetilde{h} : \Box^\infty X \to \Box^\infty Y$ such that $f \circ \Sigma^Y_{0,\infty} \circ \widetilde{h} = \Sigma^X_{0,\infty}$. But then

$$\Sigma^X_{0,\infty} \circ \Box^\infty f \circ \widetilde{h} \;=\; f \circ \Sigma^Y_{0,\infty} \circ \widetilde{h} \;=\; \Sigma^X_{0,\infty} \;=\; \Sigma^X_{0,\infty} \circ \mathrm{id}_{\Box^\infty X}.$$

Since again by Theorem 53, $\mathrm{id}_{\Box^\infty X}$ is the unique morphism $\Box^\infty X \to \Box^\infty X$ with that property, $\Box^\infty f \circ \widetilde{h} = \mathrm{id}_{\Box^\infty X}$. Analogously,

$$f \circ \Sigma^Y_{0,\infty} \circ \widetilde{h} \circ \Box^\infty f \;=\; \Sigma^X_{0,\infty} \circ \Box^\infty f \;=\; f \circ \Sigma^Y_{0,\infty} \;=\; f \circ \Sigma^Y_{0,\infty} \circ \mathrm{id}_{\Box^\infty Y}$$

implies $\widetilde{h} \circ \Box^\infty f = \mathrm{id}_{\Box^\infty Y}$, because f, being injective, cancels on the left, so Theorem 53 can be applied again. ■

Proposition 55. There exists a κ-hyperuniverse of weight $\geqslant \lambda$

Proof. Let $\mathbb{A}X = \emptyset$, $\mathbb{V}X$ discrete with $|\mathbb{V}X| = \lambda$. Then $\mathrm{Exp}^c_\kappa(\mathbb{V}X)$ is discrete, too: Every $a \in \mathrm{Exp}^c_\kappa(\mathbb{V}X)$ is κ-small and

$$\{a\} \;=\; \Box a \,\cap\, \bigcap_{x \in a} \Diamond\{x\}$$

is open. Since $\lambda^{<\kappa} = \lambda$, $\mathrm{Exp}^c_\kappa(\mathbb{V}X)$ also has size λ and there is a bijection $\mathbb{V}\Sigma^X : \mathrm{Exp}^c_\kappa(\mathbb{V}X) \to \mathbb{V}X$ (which in particular is κ-proper). Then X is in **PCm*** and $\Box^\infty \omega X$ is a κ-hyperuniverse. $\Sigma^{\omega X}_{0,\infty}$ is a surjection from $\mathbb{V}\Box^\infty \omega X$ onto $\mathbb{V}\omega X$. But $\mathbb{V}\omega X$ has λ isolated points and their preimages in $\mathbb{V}\Box^\infty \omega X$ must be disjoint open sets, so $\mathbb{V}\Box^\infty \omega X$ has at least weight λ. ■

Another very general example of a surjective map Σ^X is the union map: Take any κ-compact Hausdorff spaces $Z \neq \emptyset$ and $\mathbb{A}X$, set $\mathbb{S}X = \mathrm{Exp}_\kappa(Z)$. Then the map $\mathbb{S}\Sigma^X : \mathrm{Exp}(\mathbb{V}X) \to \mathbb{S}X$ defined as

$$\mathbb{S}\Sigma^X(b) = \begin{cases} \bigcup(b \cap \mathbb{S}X) & \text{if } b \notin \Box\mathbb{A}X \\ Z & \text{if } b \in \Box\mathbb{A}X \end{cases}$$

is continuous, because the preimage of $\Box V$ is $\Diamond\Box V \cup \Box(\Box V \cup \mathbb{A}X)$ and the preimage of $\Diamond V$ is $\Diamond\Diamond V \cup \Box\mathbb{A}X$ for every proper open subset $V \subset Z$. The set of singletons in $\mathbb{S}X$ is homeomorphic to Z, and so is the set of singletons of singletons $\{\{x\}\} \in \mathbb{S}\Box X$ with $x \in Z$, and so on. These subspaces also survive limit steps and it turns out that $\Box^\infty X$ has a closed subset of autosingletons homeomorphic to Z.[8]

A subset $A \subseteq \mathbb{V}X$ is called *transitive* if $(\Sigma^X)^{-1}[A \cap \mathbb{S}X] \subseteq \Box A$. A is *persistent* if it is transitive, open and Σ^X maps $(\Sigma^X)^{-1}[A]$ homeomorphically onto A.

Lemma 56. Let X be an object of **Cm*** and $A \subseteq \mathbb{V}X$.

1. If A is transitive in X, then $f^{-1}[A]$ is transitive in Y for every morphism $f : Y \to X$. Moreover, $\Box A \cup (A \cap \mathbb{A}X)$ is transitive in $\Box X$ and $\iota_X[A]$ is transitive in ωX.

[8] A similar hyperuniverse, constructed in a different way, is described extensively in [FH96b].

2. If A is persistent in X, then $(\Sigma^X)^{-1}[A]$ and $\Box A \cup \mathbb{A}X$ both are persistent in $\Box X$, and $\iota_X[A]$ is persistent in ωX.
3. If $\langle X_\alpha \mid f_{\alpha,\beta} \rangle_{\alpha,\beta<\gamma}$ is an inverse system in **Cm**, $f_{0,\alpha}$ maps $f_{0,\alpha}^{-1}[A]$ homeomorphically onto A for every $\alpha < \gamma$ and $f_{0,\alpha}^{-1}[A]$ is persistent for each $\alpha < \gamma$, then if

$$\langle X_\gamma \mid f_{\alpha,\gamma} \rangle_{\alpha<\gamma} = \varprojlim^{\mathbf{Cm}} \langle X_\alpha \mid f_{\alpha,\beta} \rangle_{\alpha,\beta<\gamma}$$

is the inverse limit, $f_{0,\gamma} \restriction f_{0,\gamma}^{-1}[A]$ is a homeomorphism and $f_{0,\gamma}^{-1}[A]$ is persistent.

Proof. (1): These claims are easily verified by direct calculation:

$$\begin{array}{rcl}
(\Sigma^Y)^{-1}[f^{-1}[A] \cap \mathbb{S}Y] & = & (\Sigma^Y)^{-1}[f^{-1}[\mathbb{S}X \cap A]] = (\Box f)^{-1}[(\Sigma^X)^{-1}[\mathbb{S}X \cap A]] \\
& \subseteq & (\Box f)^{-1}[\Box A] = \Box(f^{-1}[A]) \\
(\Sigma^{\Box X})^{-1}[\Box A] & \subseteq & \Box((\Sigma^X)^{-1}[A]) = \Box((\Sigma^X)^{-1}[A \cap \mathbb{S}X] \cup (A \cap \mathbb{A}X)) \\
& \subseteq & \Box(\Box A \cup (A \cap \mathbb{A}X)) \subseteq \Box(\Box A \cup \mathbb{A}X) \\
(\Sigma^{\omega X})^{-1}[\iota_X[A] \cap \mathbb{S}\omega X] & = & \Box\iota_X[(\Sigma^X)^{-1}[A \cap \mathbb{S}X] \subseteq \Box\iota_X[\Box A] = \Box(\iota_X[A])
\end{array}$$

Since $(\Box A \cup (A \cap \mathbb{A}X)) \cap \mathbb{S}X = \Box A$, the second calculation proves that $\Box A \cup (A \cap \mathbb{A}X)$ is transitive.

(2): In the light of (1) and the fact that $(\Sigma^X)^{-1}[A]$, $\Box A$, $A \cap \mathbb{A}X$ and $\iota_X[A]$ are open, we only have to worry about whether their preimages are mapped homeomorphically onto the sets in question. Since $(\Sigma^X)^{-1}[A]$ is a subset of $\Box A \cup (A \cap \mathbb{A}X)$ and $\mathbb{A}\Sigma^{\Box X}$ is a homeomorphism by definition, it suffices to consider the map $\Sigma^{\Box X} \restriction (\Sigma^{\Box X})^{-1}[\Box A]$. By definition that equals $\operatorname{Exp}_\kappa^c(\Sigma^X \restriction (\Sigma^X)^{-1}[A])$, and the exponential of a homeomorphism is a homeomorphism itself.

For $\iota_X[A]$, the statement is even more trivial, because Σ^X and $\Sigma^{\omega X}$ agree on $\operatorname{Exp}_\kappa^c(\mathbb{V}X)$ and in particular on $\Box A$.

(3): From (1) we know that $f_{0,\gamma}^{-1}[A]$ is transitive. It follows from the construction of the inverse limit in the category of topological spaces that $f_{0,\gamma}^{-1} \restriction A$ is a homeomorphism because every $f_{0,\alpha}^{-1} \restriction A$ is. But then $\operatorname{Exp}_\kappa^c(f_{0,\gamma}^{-1} \restriction \Box A)$ is a homeomorphism, too. Since $(\Sigma^{X_0})^{-1}[A \cap \mathbb{S}X] \subseteq \Box A$, the left, top and bottom arrow in the diagram

$$\begin{tikzcd}
\operatorname{Exp}_\kappa^c(\mathbb{V}X_0) \arrow[d, "\mathbb{S}\Sigma^{X_0}"'] & \operatorname{Exp}_\kappa^c(\mathbb{V}X_\gamma) \arrow[l, "\operatorname{Exp}_\kappa^c(f_{0,\gamma})"'] \arrow[d, "\mathbb{S}\Sigma^{X_\gamma}"] \\
\mathbb{S}X_0 & \mathbb{S}X_\gamma \arrow[l, "\mathbb{S}f_{0,\gamma}"']
\end{tikzcd}$$

map A homeomorphically to its preimage, so the right one must be a homeomorphism between these preimages, too, implying that $f_{0,\gamma}^{-1}[A]$ is persistent. ∎

Theorem 57. If X is an object of $\mathbf{Cm}^*$ and $A \subseteq \mathbb{V}X$ is persistent, then $(\Sigma_{0,\alpha}^{\omega X})^{-1} \circ \iota_X[A]$ is persistent in $\Box^\alpha \omega X$, and $(\Sigma_{0,\alpha}^{\omega X})^{-1} \circ \iota_X \restriction A$ is a homeomorphism for every ordinal α.

In particular, $(\Sigma^{\omega X}_{0,\infty})^{-1} \circ \iota_X[A]$ is persistent in the hyperuniverse $\Box^\infty \omega X$ and mapped homeomorphically onto A.

Proof. The proof is by induction on α and follows immediately from Lemma 56: The κ-compactification step at the beginning is (2) and the limit step is (3). ■

Proposition 58. Let X be an object of $\mathbf{PCm}^*$ with non-κ-compact $\mathbb{V}X$ and let Σ^X be a homeomorphism. Then $e_X = (\Sigma^{\omega X}_{0,\infty})^{-1} \circ \iota_X$ is a morphism which embeds $\mathbb{V}X$ as a dense open subset in $\mathbb{V}\Box^\infty \omega X$.

If Y is in $\mathbf{CHyp}$ and $\tilde{e} : X \to Y$ embeds $\mathbb{V}X$ as a dense open subset in $\mathbb{V}Y$, then there is a unique morphism $g : Y \to \Box^\infty \omega X$ such that $e_X = g \circ \tilde{e}$.

Proof. If Σ^X is a homeomorphism, then $\Sigma^{\omega X}$ maps $\mathrm{rng}(\Box\iota_X) = (\Sigma^{\omega X})^{-1}[\mathrm{rng}(\iota_X)]$ homeomorphically onto $\mathrm{rng}(\iota_X)$, which is dense and open. It follows inductively that every $\Sigma^{\omega X}_{0,\alpha}$ maps the dense and open preimage of $\mathrm{rng}(\iota_X)$ homeomorphically onto $\mathrm{rng}(\iota_X)$. Thus the definition of $\mathbb{V}e_X$ makes sense. $\mathbb{V}e_X$ also defines a morphism: Let $a \in \mathrm{Exp}_\kappa(X)$. Then since $\mathrm{Exp}_\kappa(\Sigma^{\omega X}_{0,\kappa})$ also maps the preimage of $\mathrm{rng}(\mathrm{Exp}^c_\kappa(\iota_X))$ homeomorphically onto $\mathrm{rng}(\mathrm{Exp}^c_\kappa(\iota_X))$,

$$\begin{aligned}
\mathbb{S}e_X \circ \mathbb{S}\Sigma^X &= (\mathbb{S}\Sigma^{\omega X}_{0,\infty})^{-1} \circ \mathbb{S}\iota_X \circ \mathbb{S}\Sigma^X = (\mathbb{S}\Sigma^{\omega X}_{0,\infty})^{-1} \circ \mathbb{S}\Sigma^{\omega X} \circ \mathrm{Exp}^c_\kappa(\iota_X) \\
&= (\mathbb{S}\Sigma^{\omega X}_{0,\kappa})^{-1} \circ \mathbb{S}\Sigma^{\omega X} \circ \mathrm{Exp}_\kappa(\Sigma^{\omega X}_{0,\kappa}) \circ (\mathrm{Exp}_\kappa(\Sigma^{\omega X}_{0,\kappa}))^{-1} \circ \mathrm{Exp}^c_\kappa(\iota_X) \\
&= (\mathbb{S}\Sigma^{\omega X}_{0,\kappa})^{-1} \circ \mathbb{S}\Sigma^{\omega X}_{0,\kappa} \circ \mathbb{S}\Sigma^{\Box^\infty \omega X} \circ (\mathrm{Exp}_\kappa(\Sigma^{\omega X}_{0,\kappa}))^{-1} \circ \mathrm{Exp}^c_\kappa(\iota_X) \\
&= \mathbb{S}\Sigma^{\Box^\infty \omega X} \circ \mathrm{Exp}^c_\kappa(e_X)
\end{aligned}$$

Given $\tilde{e}$, by Lemma 51 there is an $f : Y \to \omega X$ with $f \circ \tilde{e} = \iota_X$, and since the image of $\tilde{e}$ is dense, this f is unique. Theorem 53 yields a unique $g : Y \to \Box^\infty \omega X$ with $f = \Sigma^X_{0,\infty} \circ g$. Thus g is unique such that $\Sigma^X_{0,\infty} \circ g \circ \tilde{e} = \iota_X$. But that means that the image of $g \circ \tilde{e}$ is in $(\Sigma^{\omega X}_{0,\infty})^{-1}[\mathrm{rng}(\iota_X)]$ and by composing with $(\Sigma^{\omega X}_{0,\infty})^{-1}$ on the left, we obtain the equivalent equation $e_X = g \circ \tilde{e}$. ■

As in the case of $\mathbf{Ex}^*$, let $\mathbb{V}V_c = V_\kappa$ with the discrete topology, $\mathbb{A}V_c = \{\emptyset\}$ and $\mathbb{S}\Sigma^{V_c}$ the identity – which makes sense because $\mathrm{Exp}^c_\kappa(\mathbb{V}V_c) = \mathbb{S}V_c$. Then $\overline{V}_c = \Box^\infty \omega V_c$ is a hyperuniverse in which the set $e_{V_c}[\mathbb{V}V_c]$ of well-founded sets is dense and the homeomorphic image of $\mathbb{V}V_c$. By Proposition 58, it is the smallest hyperuniverse with that property. In fact, $\overline{V}_c$ turns out to be κ-ultrametrizable in a canonical way which shows that it is isomorphic to the structure originally described by R. J. Malitz in [Mal76] and E. Weydert in [Wey89]:

Theorem 59. Let $x \sim_0 y$ for all $x, y \in \mathbb{V}\overline{V}_c$. For every α, let $x \sim_{\alpha+1} y$ whenever

$$\forall \tilde{x} \in (\Sigma^{\overline{V}_c})^{-1}(x) \quad \exists \tilde{y} \in (\Sigma^{\overline{V}_c})^{-1}(y) \quad \tilde{x} \sim_\alpha \tilde{y} \quad \text{and vice versa.}$$

At limit steps, take the intersection. Then the sequence $\langle \sim_\alpha \mid \alpha < \kappa \rangle$ defines a κ-ultrametric inducing the topology of $\mathbb{V}\overline{V}_c$.

Proof. Firstly we show that $\sim_\alpha$ actually is a superset of $\sim_\beta$ for all $\alpha < \beta < \kappa$. We do this by induction on β. Assume $x \sim_\beta y$. If β is a limit ordinal, then $x \sim_\alpha y$ for every $\alpha < \beta$ by definition. Now assume $\beta = \gamma + 1$. Since for all $\tilde{x} \in (\Sigma^{\overline{V}_c})^{-1}(x)$ and $\tilde{y} \in (\Sigma^{\overline{V}_c})^{-1}(y)$, $\tilde{x} \sim_\gamma \tilde{y}$ implies $\tilde{x} \sim_\delta \tilde{y}$ for all $\delta < \gamma$ by the induction hypothesis, it follows that $x \sim_{\delta+1} y$. If γ is a limit, this implies $x \sim_\gamma y$, because $\sim_\gamma$ is the intersection of these $\sim_\delta$. Otherwise, it follows from the case where δ is the immediate predecessor of γ.

It also follows inductively that every $\sim_\alpha$ is an equivalence relation: Intersections of equivalences are equivalences, the definition is symmetric, and whenever $x \sim_{\alpha+1} y \sim_{\alpha+1} z$, just choose a $\tilde{y}$ with $\tilde{x} \sim_\alpha \tilde{y}$, and a $\tilde{z}$ with $\tilde{y} \sim_\alpha \tilde{z}$ for every $\tilde{x}$ as in the definition – then by the induction hypothesis $\tilde{x} \sim_\alpha \tilde{z}$, proving that $x \sim_{\alpha+1} z$.

If all $[x]_\alpha$ are open, then by κ-compactness, there are only κ-few of them. Hence every

$$[x]_{\alpha+1} \quad = \quad \Box \bigcup_{\tilde{x} \in x} [\tilde{x}]_\alpha \ \cap \ \bigcap_{\tilde{x} \in x} \Diamond [\tilde{x}]_\alpha$$

is open, too. If α is a limit, every $[x]_\alpha$ equals $\bigcap_{\beta<\alpha} [x]_\beta$. Thus it follows inductively, that there are κ-few $[x]_\alpha$ for every $\alpha < \kappa$, and they are all open.

Next we show that $\sim_\kappa$, which is the intersection of all $\sim_\alpha$ for $\alpha < \kappa$, equals $\sim_{\kappa+1}$, that is, it is not a proper superset of it: Let $x \not\sim_{\kappa+1} y$. Wlog assume that there is an $\tilde{x} \in (\Sigma^{\overline{V}_c})^{-1}(x)$ such that there is no $\tilde{y} \in (\Sigma^{\overline{V}_c})^{-1}(y)$ with $\tilde{x} \sim_\kappa \tilde{y}$. Thus for every $\tilde{y} \in (\Sigma^{\overline{V}_c})^{-1}(y)$, there is an $\alpha_{\tilde{y}}$ such that $\tilde{x}$ is not in $[\tilde{y}]_{\alpha_{\tilde{y}}}$. Since y is κ-compact, there is a family of such $[\tilde{y}_i]_{\alpha_i}$ with a κ-small index set I, which covers y. Let $\beta < \kappa$ be an upper bound to these α_i. This means that $\tilde{x} \sim_\beta \tilde{y}$ holds for *no* $\tilde{y} \in (\Sigma^{\overline{V}_c})^{-1}(y)$. Thus $x \not\sim_{\beta+1} y$ and in particular $x \not\sim_\kappa y$.

Now let $q : \mathbb{V}\overline{V}_c \to \mathbb{V}M$ be the quotient map, where $\mathbb{V}M = \mathbb{V}\overline{V}_c / \sim_\kappa$. Then $q \circ \Sigma^{\overline{V}_c}(a) = q \circ \Sigma^{\overline{V}_c}(b)$ iff $\Sigma^{\overline{V}_c}(a) \sim_\kappa \Sigma^{\overline{V}_c}(b)$, which is equivalent to $\Sigma^{\overline{V}_c}(a) \sim_{\kappa+1} \Sigma^{\overline{V}_c}(b)$, which in turn means that a and b intersect the same equivalence classes $[x]_\kappa$. Thus $\mathrm{Exp}_\kappa(q)(a) = \mathrm{Exp}_\kappa(q)(b)$. We have shown that $\Sigma^{\overline{V}_c}$ factors through $\sim_\kappa$ in the sense that it induces a homeomorphism $\mathbb{V}\Sigma^M$ from $\{\emptyset\} \cup \mathrm{Exp}_\kappa(\mathbb{V}M)$ to $\mathbb{V}M$ and thus a quotient hyperunverse M, with a quotient morphism $q : \overline{V}_c \to M$.

On the other hand, it follows recursively from the definition that $x \not\sim_\alpha y$ whenever $x \neq y$ and $x, y \in e_{V_c}[V_\alpha]$. In particular, $q \circ e_{V_c}$ is still injective and embeds V_c as an open subset in M. Since $\overline{V}_c$ is minimal with that property, q must therefore be an isomorphism. That implies that $\sim_\kappa$ is the diagonal and thus the family $\langle \sim_\alpha \mid \alpha < \kappa \rangle$ is a κ-ultrametric on $\mathbb{V}\overline{V}_c$ itself. Since every $[x]_\alpha$ is open, it induces the topology. ■

The *tree model*, a hyperuniverse presented by E. Weydert in [Wey89], is isomorphic to $\Box^\infty X$, where $\mathbb{A}X$ and $\mathbb{S}X$ are one-point spaces.[9] He conjectured that the isolated points are dense in $\mathbb{V}\Box^\infty X$ (in his terms, that $\mathbb{V}\Box^\infty X$ is *perfect*). Theorem 61 is a more general criterion for this property and proves Weydert's conjecture.

[9] A *sequential tree* T is a tree of sequences closed with respect to restrictions. For sets of sequences A let $A \restriction \alpha = \{x \restriction \alpha \mid x \in A\}$. Define recursively: $B_\alpha = \{x \in \prod_{\xi<\alpha} \mathcal{P}(B_\xi) \mid \forall \zeta < \xi < \alpha \ x_\zeta = x_\xi \restriction \zeta\}$. Then $d(x,y) = \min(\{\kappa\} \cup \mathrm{dom}(x \Delta y))$ defines a κ-ultrametric on $\mathbb{V}Y = B_\kappa$. With $\mathbb{A}Y = \{\emptyset\}$, the map $\Sigma^Y : B_\kappa \to \mathbb{A}Y \cup \mathrm{Exp}_\kappa(B_\kappa), \Sigma^Y(x) = \{y \mid \forall \alpha \ y \restriction \alpha \in x\}$ is a homeomorphism. This is E. Weydert's *tree model*. But in fact $B_{\alpha+1}$ corresponds exactly to the hyperspace of B_α, and for limit ordinals α, B_α is the inverse limit of its predecessors. Hence these sets correspond to the spaces $\mathbb{V}\Box^\alpha X$ in a canonical way.

Given an object X of **Cm**, we call a point $x \in \mathbb{V}\Box^\alpha X$ *simple* if at least one of the following conditions is met:

- α is not a successor's successor.
- $x \in \mathbb{A}\Box^\alpha X$.
- $\alpha = \beta + 2$, $x \in \mathbb{S}\Box^\alpha X$, $\Sigma^X_{\beta,\beta+1} \restriction x$ is injective and every $y \in x$ is simple.

A *simple sequence* is a sequence $\langle x_\gamma, \gamma \in [\alpha, \beta)\rangle$ such that for every γ, $x_\gamma \in \mathbb{V}\Box^\gamma X$ is a simple point and $\Sigma^X_{\gamma_1,\gamma_2}(x_{\gamma_2}) = x_{\gamma_1}$ whenever $\gamma_2 > \gamma_1$.

Lemma 60. Let X be an object of **Cm** such that $\mathbb{V}X$ is discrete and Σ^X is surjective, $\alpha < \beta < \kappa$, and let $\langle x_\gamma, \gamma \in [\alpha, \beta)\rangle$ be a simple sequence. Then the sequence can be extended to $[\alpha, \beta + 1)$ and if $\beta \geqslant \alpha + \omega$, x_β is the unique point such that $\Sigma^X_{\gamma,\beta}(x_\beta) = x_\gamma$ for all $\gamma \in [\alpha, \beta)$.

Proof. First of all, every $\mathbb{V}\Box^\alpha X$ with $\alpha < \kappa$ is discretely small and hence discrete. The proof goes by induction on β.

If β is a limit ordinal, it follows from the definition of the inverse limit that x_β is unique (and it is simple by definition).

Next assume that $\beta = \delta + 1$. Choose any $z \in (\Sigma^X_{\delta,\beta}[\{x_\delta\}])^{-1}$. Then $x_{\gamma+1} = \Sigma^X_{\gamma+1,\beta}(z) = \Sigma^X_{\gamma,\delta}[z]$ for every $\gamma \in [\alpha, \delta)$. Thus for every $y \in z$, the sequence $\langle \Sigma_{\gamma,\delta}(y), \gamma \in [\alpha, \delta)\rangle$ is a simple sequence. By the induction hypothesis, it has a simple extension w_y. Then the set $x_\beta = \{w_y \mid y \in z\}$ is simple itself and since

$$\Sigma^X_{\gamma+1,\beta}(x_\beta) \;=\; \Sigma^X_{\gamma,\delta}[z] \;=\; \Sigma^X_{\gamma+1,\beta}(z)$$

for every $\gamma \in [\alpha, \delta)$, we have $\Sigma^X_{\delta,\beta}(x_\beta) = x_\delta$ (either by the inverse limit property if δ is a limit, or simply because δ is of the form $\gamma + 1$).

If $\beta \geqslant \alpha + \omega$, the sequences $\langle \Sigma_{\gamma,\delta}(y), \gamma \in [\alpha, \delta)\rangle$ by the induction hypothesis have a *unique* extension. Hence $w_y = y$ for all $y \in z$ and thus $z = x_\beta$. ∎

Theorem 61. Let X be an object of **Cm** such that $\mathbb{V}X$ is discrete. Then the isolated points are dense in $\mathbb{V}\Box^\infty X$.

Proof. Wlog assume that Σ^X is surjective. The nonempty sets of the form $(\Sigma^X_{\alpha,\infty})^{-1}[U]$ with limit ordinals α constitute an open base of $\mathbb{V}\Box^\infty X$, so it suffices to show that each contains an isolated point. Given $x_\alpha \in U$, recursively choose simple points x_γ according to Lemma 60, such that $\langle x_\gamma \mid \gamma \in [\alpha, \kappa)\rangle$ is a simple sequence. Then there is exactly one point $x_\kappa \in \mathbb{V}\Box^\infty X$ such that $\Sigma^X_{\gamma,\kappa}(x_\kappa) = x_\gamma$ for each γ. And by Lemma 60, x_κ is in fact the only preimage of the point $x_{\alpha+\omega}$. But $\{x_{\alpha+\omega}\}$ is open, so $\{x_\kappa\}$ is open, too, and hence x_κ is isolated. ∎

2.7 Metric Spaces and the Hilbert Cube

The inverse limit construction can also be considered in the context of (κ-ultra)metric spaces: Since for κ-compact spaces, the Hausdorff (κ-ultra)metric induces the exponential κ-topology, $\mathrm{Exp}_\kappa(X)$ is metrizable whenever X is.

Lemma 62. Let $\gamma \leqslant \kappa$ and let each X_α with $\alpha < \gamma$ be a (κ-ultra)metrizable space. Then the product

$$X = \prod_{\alpha<\gamma} X_\alpha$$

is (κ-ultra)metrizable, too.

Proof. Since all X_α are κ-compact, we can wlog fix a metric d_α for each α such that the diameter of X_α is at least α. Since the case of metrizable (not necessarily ultrametrizable) spaces for $\kappa = \omega$ has to be treated slightly differently, let us denote the spaces by X_n in that case and choose d_n such that the diameter of X_n is at most $\frac{1}{n}$. We define the metric d on the product X as follows:

$$d(x,y) = \inf_{\alpha<\gamma} d_\alpha(x_\alpha,y_\alpha) \quad \text{respectively} \quad d(x,y) = \sup_{n<\gamma} d_n(x_n,y_n)$$

Then all projections are nonexpanding and in particular continuous, so d induces a topology at least as fine as the product topology. Conversely, the open set

$$\prod_{\beta\leqslant\alpha} [x_\beta]_\alpha \times \prod_{\alpha<\beta<\gamma} X_\beta \quad \text{respectively} \quad \prod_{n\leqslant\alpha} [x_n]_{1/m} \times \prod_{m<n<\gamma}$$

is a subset of the ball $[x]_\alpha$ respectively $[x]_{1/m}$, so the two topologies actually coincide. ■

In particular, inverse limits of at most κ (κ-ultra)metrizable spaces are (κ-ultra)metrizable. We will look at some examples in the case $\kappa = \omega$ that produce hyperuniverses with the Hilbert cube $\mathbb{H} = \prod_{n\in\omega}[0,1]$ as their universe space. In [CS78], D. W. Curtis and R. M. Schori proved that hyperspaces of Peano continua (if they have more than one point) are homeomorphic to $\mathbb{H}$, where a *Peano continuum* is a locally connected compact connected metric space. Since the Hilbert cube is a Peano continuum itself, there is a homeomorphism $\Sigma^X : \mathbb{S}X \to \mathrm{Exp}_\kappa(\mathbb{V}X)$ whenever $\mathbb{S}X$ is homeomorphic to the Hilbert cube and $\mathbb{V}X$ is a Peano continuum. Setting, for example, $\mathbb{V}X = [0,1] \times \mathbb{H}$ and $\mathbb{S}X = [0,\frac{1}{2}] \times \mathbb{H}$, we obtain an example of a non-clopen ω-hyperuniverse.

Lemma 63. Let X and Y be compact. Assume that under $f : X \to Y$, all preimages of singletons are connected. Then $\mathrm{Exp}_\omega(f)$ has the same property.

It follows that if X is connected, so is $\mathrm{Exp}_\omega(X)$.

Proof. Let $A \in \operatorname{Exp}_\omega(Y)$, that is, $A \subseteq Y$ closed. Assume $\operatorname{Exp}_\omega(f)^{-1}[\{A\}]$ is not connected, that is, $\operatorname{Exp}_\omega(f)^{-1}[\{A\}] = B \cup C$ is the disjoint union of nonempty closed sets B and C. Since $\operatorname{Exp}_\omega(X)$ is normal, B and C can be separated by disjoint open sets, and since they are compact, those disjoint open sets can be chosen such that they are unions of finitely many sets of the form

$$V_i = \Box(U_{1i} \cup \ldots \cup U_{n_i i}) \cap \Diamond U_{1i} \cap \ldots \cap \Diamond U_{n_i i}$$

with $U_{1i}, \ldots, U_{n_i i}$ relatively open in $B \cup C$ and $i \in \{1, \ldots, m\}$. Then every V_i intersects either B or C, and whenever $V_i \cap B \neq \emptyset$ and $V_j \cap C \neq \emptyset$, then V_i and V_j are disjoint.

We will show that the set $f^{-1}[A]$ is in B. Since the situation is symmetric, it is then also in C, which is impossible.

Let $S_0 \in B$ be arbitrary and recursively define S_l for $l \in \omega$. Namely, let

$$\widetilde{S}_l = f^{-1}[A] \setminus \bigcup \{U_{ki} \mid S_l \cap U_{ki} = \emptyset\}.$$

Then every V_i that did not contain S_l also does not contain $\widetilde{S}_l$, but since $S_l \subseteq \widetilde{S}_l \subseteq f^{-1}[A]$, $\widetilde{S}_l$ is in $\operatorname{Exp}_\omega(f)^{-1}[\{A\}]$. So some V_i that contains S_l must also contain $\widetilde{S}_l$ and thus $\widetilde{S}_l \in B$. If $\widetilde{S}_l = f^{-1}[A]$, we are done, so assume that $f^{-1}[A]$ contains a point $x \notin \widetilde{S}_l$. Now $f^{-1}(f(x))$ intersects $\widetilde{S}_l$ and its open superset $U_{1i} \cup \ldots \cup U_{n_i i}$, and since it is connected, there is a point

$$y \in f^{-1}(f(x)) \cap (U_{1i} \cup \ldots \cup U_{n_i i}) \setminus \widetilde{S}_l.$$

The set $S_{l+1} = \widetilde{S}_l \cup \{y\}$ then is still in V_i and thus in B, but it intersects strictly more of the open sets $U_{k'i'}$ than S_l. Therefore this construction has to come to an end, where some $\widetilde{S}_l = f^{-1}[A]$.

Now if X is connected, the preimages of singletons under $f : X \to \{p\}$ are connected, so the same goes for $\operatorname{Exp}_\omega(f) : \operatorname{Exp}_\omega(X) \to \{\{p\}\}$. Hence $\operatorname{Exp}_\omega(X)$ is connected. ■

Lemma 64. If X is a locally connected metric space, then so is $\operatorname{Exp}_\omega(X)$.

Proof. It suffices to show that whenever $U_1, \ldots, U_n$ are connected, then so is $V = \Box(U_1 \cup \ldots \cup U_n) \cap \Diamond U_1 \cap \ldots \cap \Diamond U_n$, because the sets of that form are a basis. If V were the disjoint union of two nonempty open sets W_1 and W_2, then each of them would have finite elements. Let $m_1, \ldots, m_n$ be such that there are $a_1 \in W_1$ and $a_2 \in W_2$ which intersect each U_i in at most m_i points. Since the map

$$f : U_1^{m_1} \times \ldots \times U_n^{m_n} \to V, \quad f(x) = \{x_i \mid i \leqslant m_1 + \ldots + m_n\}$$

is continuous and the U_i are connected, its image is connected, too. But it contains both a_1 and a_2, a contradiction. ■

Lemma 65. Let $\langle X_n, f_{n,m} \rangle$ be an inverse system of compact, metrizable and connected spaces indexed by ω. Let

$$\langle X_\omega, f_{n,\omega} \rangle = \varprojlim \langle X_n, f_{n,m} \rangle.$$

Then X_ω is compact, metrizable and connected.

Proof. The metrizability follows from Lemma 62, and the inverse limit is compact, because it is a closed subspace of the product $\prod_{n\in\omega} X_n$.

Now assume that all the X_n are connected. Suppose X_ω is the union of two nonempty clopen sets A and B; we have to show that they are not disjoint. As every X_n is connected, $f_{n,m}[A]$ and $f_{n,m}[B]$ always have a nonempty intersection C_n. Then the sets $f^{-1}_{n,\omega}[C_n]$ are a decreasing sequence of nonempty closed sets and hence have some point x in common. Since every $f_{n,\omega}(x)$ is in $f_{n,\omega}[A]$, x is in A, and similarly for B. ■

In [Gsc75], G. R. Gordh and S. Mardešić prove that if in addition all X_n are locally connected and all preimages $f^{-1}_{n,m}[\{x\}]$ of singletons are connected, then X_ω is locally connected, too.

Summing up the results of this section, we now know that whenever X is an object of **Cm** such that $\mathbb{V}X$ is a Peano continuum and all $(\Sigma^X)^{-1}[\{x\}]$ are connected, then $\mathbb{V}\square^\infty X$ is homeomorphic to the Hilbert cube.

As an example, let $\mathbb{V}X = \mathbb{S}X = [0,1]$ and $\mathbb{S}\Sigma^X(A) = \min(A)$. $\mathbb{V}X$ is a Peano continuum, and the sets $\left(\mathbb{S}\Sigma^X\right)^{-1}[\{x\}]$ are even path-connected: For any given element A,

$$F : [0,1] \to \left(\mathbb{S}\Sigma^X\right)^{-1}[\{x\}], \quad F(t) = \{x + t(a - x) \mid a \in A\}$$

is a path that connects A to $\{x\}$. Thus $\mathbb{V}\Sigma^{\square^\omega X}$ is homeomorphic to $\mathbb{H}$. What is the set of autosingletons in this model? $A \in \square\mathbb{V}\square^\omega X$ is an autosingleton iff $A = \{x\}$, where $x = \mathbb{V}\Sigma^{\square^\omega X}(A)$, which means that for all n, $\mathbb{V}\Sigma^X_{n+1,\omega}(x) = \mathbb{V}\Sigma^X_{n,\omega}[A] = \left\{\mathbb{V}\Sigma^X_{n,\omega}(x)\right\}$. This recursive formula shows that the only autosingletons are the objects obtained by repeatedly taking singletons, and thus the set of autosingletons is homeomorphic to $\mathbb{V}X$.

Analogously, we obtain for every natural number n a Hilbert cube model whose set of autosingletons is homeomorphic to $[0,1]^n$, by setting $\mathbb{V}X = \mathbb{S}X = [0,1]^n$ and $\mathbb{S}\Sigma^X(A) = \langle x_1,\dots,x_n\rangle$, where $x_i = \min\{y_i \mid \langle y_1,\dots,y_n\rangle \in A\}$.

Open Questions

A topological characterization of positive set theory

We have seen that $\mathbf{GPK}^+_{(\infty)} + \mathrm{T}_3$ is equivalent to $\mathbf{TS}_{(\infty)} + (\mathbb{A}{=}\emptyset{\in}\mathbb{V}) + \mathrm{T}_3 + \mathrm{Union}$. But we were unable to find such a formulation of $\mathbf{GPK}^+_{(\infty)}$ alone and in fact we neither know whether $\mathbf{GPK}^+_{(\infty)}$ implies the T_3 separation axiom (or even just that $\mathbb{V}$ is a Hausdorff space), nor whether $\mathbf{TS}_{(\infty)} + \mathrm{Union}$ implies the positive comprehension principle. Also, the union axiom is not really a topological statement. Is there a topological axiomatization of $\mathbf{GPK}^+_{(\infty)}$, that is, is GPF comprehension a topological property of $\mathbb{V}$?

Independence of the compactness axiom

In our proof that every κ-hyperuniverse is κ-compact (Theorem 28), we made heavy use of the set theory external to that hyperuniverse, using for example cardinals λ which might be much larger than κ. It is not clear that the theory **TS** implies that the universe is $\mathcal{D}$-compact: Without a choice principle (for classes) much stronger than the uniformization axiom the proof for hyperuniverses cannot be carried out within topological set theory. Does **TS** – or $\mathbf{GPK}^+$ or $\mathbf{TS} + \mathrm{Union} + \mathrm{T}_4$ – imply compactness?

Normality and compactness

It follows from Lemmas 25 and 27 that whenever $\mathrm{Exp}_\kappa(\mathrm{Exp}_\kappa(X))$ is normal, X is κ-compact.

For the case $\kappa = \omega$, N. V. Velichko proved that in fact X is compact whenever $\mathrm{Exp}_\omega(X)$ is normal (cf. [Vel75])! Although not directly related to hyperuniverses and topological set theory, it would be interesting to know for which $\kappa > \omega$ this is true, too.

Cantor cubes

We were able to prove that $\mathbb{D}^\kappa$ is the universe space of a κ-hyperuniverse but no $\mathbb{D}^\lambda$ is the universe space of any *clopen* κ-hyperuniverse. Do the $\mathbb{D}^\lambda$ with $\lambda \geqslant \kappa$ form (non-clopen) κ-hyperuniverses?

The duality of the limit constructions

The categories **EHyp** and **CHyp** of hyperuniverses are canonically isomorphic: the maps Σ^X are just reversed. We can extend this isomorphism to a functor defined on a larger subcategory of $\mathbf{Cm}^*$: For each object X of $\mathbf{Cm}^*$ such that Σ^X is a homeomorphism, let $\mathbf{R}X$ be the corresponding object of $\mathbf{Ex}^*$, that is, $\mathbb{V}\mathbf{R}X = \mathbb{V}X$, $\mathbb{A}\mathbf{R}X = \mathbb{A}X$ and $\Sigma^{\mathbf{R}X} = (\Sigma^X)^{-1}$. If $f : X \to Y$ is a morphism, let $\mathbf{R}f$ be the morphism of $\mathbf{Ex}^*$ defined by the same function $\mathbb{V}f$. In exactly the same way we define $\mathbf{R}X$ if X is an object of $\mathbf{Ex}^*$ such that $\mathbb{V}X$ is Hausdorff and locally κ-compact and $\mathbb{S}\Sigma^X$ is a homeomorphism.

Then for example, $\mathbf{R}V_c = V_e$, and for every object X of **EHyp** and dense embedding $f : V_e \to X$, there are unique horizontal arrows such that the following diagram commutes:

$$\begin{tikzcd}
\overline{V}_e \arrow[r] & X \arrow[r] & \mathbf{R}\overline{V}_c \\
& V_e \arrow[ul, "e_{V_e}"] \arrow[u, "f"] \arrow[ur, "\mathbf{R}e_{V_c}"'] &
\end{tikzcd}$$

Thus $\overline{V}_e$ really is the largest and $\overline{V}_c$ is the smallest hyperuniverse, in which V_e is dense. But we do not know whether they are actually isomorphic, that is, whether the morphism $\overline{V}_e \to \mathbf{R}\overline{V}_c$ in the diagram is given by an injective map. If they are, then there is up to isomorphism only one hyperuniverse in which V_e can be densely embedded.

More generally, the same question poses itself for every suitable object Y instead of V_e.

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