Rational points on quartic hypersurfaces

Markus Andreas Hanselmann

Dissertation
an der Fakultät für Mathematik, Informatik und Statistik
der Ludwig–Maximilians–Universität
München
vorgelegt von
Dipl.-Math. Markus Andreas Hanselmann
aus Schwäbisch Hall

München, den 24. Januar 2012

Rational points on quartic hypersurfaces

Markus Andreas Hanselmann

Dissertation
an der Fakultät für Mathematik, Informatik und Statistik
der Ludwig–Maximilians–Universität
München
vorgelegt von
Dipl.-Math. Markus Andreas Hanselmann
aus Schwäbisch Hall
München, den 24. Januar 2012

Erstgutachter: Prof. Dr. Ulrich Derenthal
Zweitgutachter: Prof. Dr. Jörg Brüdern
Tag der Disputation: 16. Februar 2012

To my family.

Abstract

In this work the existence of $\mathbb{Q}$-rational points on a geometrically integral projective quartic hypersurface $X \in \mathbb{P}^{n-1}$ defined over $\mathbb{Q}$ will be discussed. It will be shown that the set of rational points $X(\mathbb{Q})$ on X is non-empty, provided that there exist a non-singular real and a non-singular p-adic point for each prime p on X and provided that $n - \dim_{\mathbb{Q}} \operatorname{sing} X \geq 41$. A lower bound for the number of $\mathbb{Q}$-rational points of bounded height will be established. In particular, the Hasse-principle for quartic hypersurfaces which are defined by a non-singular form in at least 40 variables will be confirmed. This improves on a result by Browning and Heath-Brown [BHB09] by saving one variable.

Closely related to this is the question under which conditions an absolutely irreducible quartic form $F \in \mathbb{Z}[x_1, \ldots, x_n]$ represents each non-zero rational number when rational values for the variables are allowed. It will be shown that such a result can be established under the following conditions. It will be required that $n - \dim_{\mathbb{Q}} \operatorname{sing} Y \geq 34$, where Y is the quartic hypersurface defined by F, and that there exist a non-singular real and a non-singular p-adic point for each prime p on Y.

In the last part of the work a quartic hypersurface $Z \subset \mathbb{P}^{k+l-1}$ which is defined by a quartic form $F \in \mathbb{Z}[x_1, \ldots, x_k, y_1, \ldots, y_l]$ of the shape

$$F(x_1, \ldots, x_k, y_1, \ldots, y_l) = F_1(x_1, \ldots, x_k) + F_2(y_1, \ldots, y_l)$$

will be examined. The case $l = 1$ is equivalent to the previous problem. In the case $l \geq 2$ it will be shown that $Z(\mathbb{Q})$ is non-empty under the following conditions. It will be assumed that F_1 is absolutely irreducible and that there exist a non-singular real and a non-singular p-adic point for each prime p on Z. Moreover, it will be required that $n - \dim_{\mathbb{Q}} \operatorname{sing} Z \geq 26$ and $n - \dim_{\mathbb{Q}} \operatorname{sing} Z_1 \geq 35$, where Z_1 is the quartic hypersurface defined by Z_1.

Zusammenfassung

In der vorliegenden Arbeit wird die Existenz von $\mathbb{Q}$-rationalen Punkten auf einer geometrisch integeren projektiven quartischen Hyperfläche $X \subset \mathbb{P}^{n-1}$ über $\mathbb{Q}$ untersucht. Es wird gezeigt, dass die Menge der rationalen Punkte $X(\mathbb{Q})$ auf X nicht-leer ist, falls sowohl ein nicht-singulärer reeller als auch ein nicht-singulärer p-adischer Punkt für jede Primzahl p auf X liegt, und für die Dimension des singulären Lokus von X die Beziehung $n - \dim_{\mathbb{Q}} \operatorname{sing} X \geq 41$ gilt. Für die Anzahl $\mathbb{Q}$-rationaler Punkte beschränkter Höhe auf X wird eine untere Schranke etabliert. Insbesondere wird gezeigt, dass das Hasse-Prinzip für quartische Hyperflächen gilt, denen eine nicht-singuläre quartische Form über $\mathbb{Q}$ in mindestens 40 Variablen zu Grunde liegt. Damit wird ein Resultat von Browning und Heath-Brown [BHB09] durch das Einsparen einer Variablen verbessert.

Eng damit verwandt ist die Fragestellung, welchen Bedingungen eine absolut irreduzible quartische Form $F \in \mathbb{Z}[x_1, \ldots, x_n]$ genügen muss, so dass jede rationale Zahl ungleich Null von F dargestellt wird, wenn rationale Werte für die Variablen zugelassen werden. Es wird gezeigt, dass dies der Fall ist, sobald folgende Bedingungen erfüllt sind. Zum einen wird vorausgesetzt, dass $n - \dim_{\mathbb{Q}} \operatorname{sing} Y \geq 34$, wobei Y die quartische Hyperfläche ist, die durch F definiert wird, zum anderen wird angenommen, dass auf Y ein nicht-singulärer reller und für jede Primzahl p ein nicht-singulärer p-adischer Punkt liegt.

Im letzten Teil der Arbeit wird eine quartische Hyperfläche $Z \subset \mathbb{P}^{k+l-1}$ untersucht, der eine quartische Form $F \in \mathbb{Z}[x_1, \ldots, x_k, y_1, \ldots, y_l]$ der Gestalt

$$F(x_1, \ldots, x_k, y_1, \ldots, y_l) = F_1(x_1, \ldots, x_k) + F_2(y_1, \ldots, y_l)$$

zu Grunde liegt. Der Fall $l = 1$ kann als äquivalent zur vorherigen Fragestellung erkannt werden. Im Fall $l \geq 2$ wird gezeigt, dass $Z(\mathbb{Q})$ nicht-leer ist, wenn folgende Annahmen gelten. Es wird gefordert, dass F_1 absolut irreduzibel ist und dass ein nicht-singulärer reller und für jede Primzahl p ein nicht-singulärer p-adischer Punkt auf Z liegt. Zudem wird vorausgesetzt, dass $n - \dim_{\mathbb{Q}} \operatorname{sing} Z \geq 26$ und $n - \dim_{\mathbb{Q}} \operatorname{sing} Z_1 \geq 35$ gilt, wobei Z_1 die quartische Hyperfläche ist, die durch F_1 definiert ist.

Acknowledgments

There are many people without whom this thesis would not have been possible.

First and foremost I would like to offer my sincerest gratitude to Prof. Dr. Jörg Brüdern. It was he who first introduced me to the fascinating world of number theory. His vast expertise and his contagious enthusiasm for his research which I was able to experience in numerous lectures and seminars at the University of Stuttgart have always been an inspiration to me. I appreciate his trust in me to start a Ph.D. under his supervision and I am truly thankful for his continuous support and his thoughtful guidance. Last but not least I want to thank him for encouraging me to do some of my research work abroad in order to broaden my horizons and I appreciate his help in realising my research stay at the University of Oxford.

I am deeply indebted to Prof. Dr. Roger Heath-Brown of the University of Oxford who willingly agreed to act as my mentor during my stay in Oxford from October 2010 to April 2011. I am thankful for his warm welcome, for his constructive comments on my work, for his valuable advice and all the time he spent discussing my work with me. His wealth of knowledge, his ideas and his way to approach mathematical problems have been extremely educational and inspiring for me and my thesis has greatly benefited from all of this.

Let me further express many thanks to Prof. Dr. Ulrich Derenthal who offered me the chance to finish my Ph.D. at the Ludwig-Maximilians-Universität of Munich under his supervision after my stay in Oxford. Since his research interests are at the interface of Algebraic Geometry and Number Theory, he enabled me to open up new perspectives on my own research and I am truly thankful for his support, his good advice and his useful comments on my work.

I would like to thank Prof. Dr. Werner Bley and Prof. Dr. Fabien Morel for willingly joining my board of examiners.

I want to acknowledge the support of the DAAD, the German Academic Exchange Service, and thank them for granting me a scholarship for my stay in Oxford. Moreover, I am very thankful to the Mathematical Institute of the University of Oxford for a very welcoming reception as an academic visitor.

In my daily work at the universities of Stuttgart, Oxford and Munich I have been blessed with many nice colleagues and good friends and I am truly thankful for their support and friendship and the pleasant working atmosphere they created. A special thank goes to Stefan Baur with whom I jointly explored the world of mathematics and number theory from day one at university.

Last but not least I want to thank my whole family from the bottom of my heart, especially my parents, my brother, my grandparents and Cordula for all their love, support and encouragement. It was always very important to my parents to ensure that I would be able to get a good education and they raised me up with a love of science, supported me in all my pursuits and never questioned me.
Finally, my loving thanks go to Cordula – for just everything.

Contents

1 Introduction

In this work we will examine under which circumstances a geometrically integral quartic hypersurface $X \in \mathbb{P}^{n-1}$ defined over $\mathbb{Q}$ has $\mathbb{Q}-$rational points. We will denote the set of singular points on X by $\operatorname{sing}(X)$ and the set of non-singular points by $X_{ns} = X \backslash \operatorname{sing}(X)$. We use the symbols $X(\mathbb{Q})$, $X(\mathbb{R})$ and $X(\mathbb{Q}_p)$ to denote the set of rational, real and $p-$adic points on X for each prime p, whence the set of adèles $X(\mathbb{A}_{\mathbb{Q}})$ is given by $X(\mathbb{A}_{\mathbb{Q}}) = X(\mathbb{R}) \times \prod_p X(\mathbb{Q}_p)$.

Birch [Bir62] not only showed that the set $X(\mathbb{Q})$ is non-empty provided that $X_{ns}(\mathbb{A}_{\mathbb{Q}})$ is non-empty and

$$n - \dim \operatorname{sing}(X) \geq 50, \tag{1.1}$$

but he also established an asymptotic formula for the number of $\mathbb{Q}-$rational points under those conditions in the following sense. Let $x \in X(\mathbb{Q})$ be any rational point on X. Then there exists a representative $\mathbf{x} \in \mathbb{Z}^n$ for x with $\gcd(x_1, \ldots, x_n) = 1$ which is uniquely determined up to a choice of sign. We define the height $H(x)$ of the rational point $x \in X(\mathbb{Q})$ by

$$H(x) = \max\{|x_1|, \ldots, |x_n|\}.$$

We can measure the density of rational points $X(\mathbb{Q})$ by the counting function

$$N_X(P) := \{x \in X(\mathbb{Q}) : H(x) \leq P\}.$$

According to Birch we have

$$N_X(P) = c_X P^{n-4}(1 + \mathrm{o}(1)) \tag{1.2}$$

for some positive constant $c_X > 0$ whenever the conditions (1.1) are met, which confirms the Manin conjecture for these hypersurfaces.

This was the best known result until recently Browning and Heath-Brown [BHB09] were able to extend the admissible range of n to

$$n - \dim \operatorname{sing}(X) \geq 42$$

and thus were able to save eight variables compared to Birch. In particular they were able to establish the Hasse-principle which states that $X(\mathbb{Q})$ is non-empty if and only if $X(\mathbb{A}_{\mathbb{Q}})$ is non-empty for non-singular quartic forms over $\mathbb{Q}$ in at least 41 variables.

Browning and Heath-Brown establish a lower bound for $N_X(P)$ of magnitude (1.2), which is due to the fact that they work with a weighted counting function. Like Birch they use the Hardy-Littlewood circle method to get their lower bound for $N_X(P)$. The main part of their work deals with the treatment of the occurring exponential sums. Compared to Birch, who uses an iterated Weyl differencing process in order to relate the size of the quartic exponential sum to the locus of a system of corresponding trilinear forms, Browning and Heath-Brown use just one discrete differencing step based on van der Corput's method and bound the resulting cubic exponential sums directly.

The first goal of this work is to save one variable compared to the approach of Browning and Heath-Brown. We will be able to prove the following result.

Theorem 1. *Let $X \subset \mathbb{P}_{\mathbb{Q}}^{n-1}$ be a geometrically integral quartic hypersurface with*

$$n - \dim \operatorname{sing}(X) \geq 41.$$

Assume that $X_{ns}(\mathbb{A}_{\mathbb{Q}})$ is non-empty. Then there exist constants $P_0 \geq 1$ and $c > 0$, such that $N_X(P) \geq cP^{n-4}$ for $P \geq P_0$.

The condition on $X_{\mathrm{ns}}(\mathbb{A}_{\mathbb{Q}})$ being non-empty is needed to establish the positivity of the constant c, which basically is a product of local densities.

Our proof will be much in the spirit of the work of Browning and Heath-Brown. The main difference in our approach is that we combine the van der Corput differencing process of the corresponding exponential sum with a mean square average over a short interval. We hereby adapt the approach of Heath-Brown [HB07], who deals with cubic forms, to the setting of quartic forms. The averaging technique allows us a little saving compared to the bound one obtains by applying the van der Corput method pointwise which will be sufficient for the saving of one variable.

Our result has consequences for the representation of a non-zero rational number by an arbitrary absolutely irreducible quartic form $\tilde{G} \in \mathbb{Z}[x_1, \ldots, x_n]$, using rational values for the variables. On multiplying by denominators it is easy to see that it is sufficient to establish that each quartic form of the shape

$$G(x_1, \ldots, x_{n+1}) = F(x_1, \ldots, x_n) + mx_{n+1}^4 \tag{1.3}$$

represents zero non-trivially, where F is an absolutely irreducible quartic form, m is an arbitrary non-zero integer and $x_{n+1} \neq 0$. It can be deduced from the proof of Theorem 1 that this property holds provided that $(n+1) - \dim \operatorname{sing}(X) \geq 41$. But the range for admissible n can be extended. We denote the number of integer solutions $\mathbf{x} \in \mathbb{Z}^{n+1}$ of (1.3) with $\max |x_i| \ll P$ and $x_{n+1} \neq 0$ by $N_F^*(P)$. We get the following result.

Theorem 2. *Let $Y \subset \mathbb{P}^n_{\mathbb{Q}}$ be a quartic hypersurface defined by a form G of the shape* (1.3) *with*

$$(n+1) - \dim \operatorname{sing}(Y) \geq 35.$$

Assume that $Y_{ns}(\mathbb{A}_{\mathbb{Q}})$ is non-empty. Then there exist constants $P_0 \geq 1$ and $c > 0$, such that $N_F^*(P) \geq cP^{(n+1)-4}$ for $P \geq P_0$.

We therefore deduce the following result.

Corollary 1. *Let $F \in \mathbb{Z}[x_1, \ldots, x_n]$ be an absolutely irreducible quartic form with*

$$n - \dim \operatorname{sing}(X) \geq 34,$$

where $X \subset \mathbb{P}^{n-1}_{\mathbb{Q}}$ is the hypersurface defined by F. Assume that $X_{ns}(\mathbb{A}_{\mathbb{Q}})$ is non-empty. Then F represents each non-zero $m \in \mathbb{Q}$.

We will use the circle method to establish the result. On the major arcs we will follow the steps of the proof of Theorem 1. On the minor arcs we will split off the contribution form the term involving x_{n+1} by an application of Hölders inequality. We obtain two integrals which can be bounded separately. The integral involving the form in n variables can be treated with the methods we use for the proof of Theorem 1. The key point of the proof of Theorem 2 is the treatment of the integral involving the variable x_{n+1}. In the cubic case this has been done by Browning [Bro10] by completing the integration range to the unit interval which enables him to interpret the quantity diophantically. In the quartic case one can do better. Instead of completing the integration range to the unit interval we will be able to keep the information that we just have to integrate over comparable short intervals. This finally relates our integral to a diophantic problem which still contains the information that our integration variable has been small in some sense. The number of solutions of the diophantic problem we thereby encounter can be treated by a result of Robert and Sargos [RS06, Theorem 2].

In Theorem 2 we deal with a form that splits off a form in one variable. In our last theorem we will consider the general case of a form F in $n = k + l$ variables splitting into two quartic forms F_1 and F_2, where F_1 is absolutely irreducible and

$$F(x_1, \ldots, x_k, y_1, \ldots, y_l) = F_1(x_1, \ldots, x_k) + F_2(y_1, \ldots, y_l) \tag{1.4}$$

identically in $x_1, \ldots, x_k$ and $y_1, \ldots, y_l$. This problem has been considered in the cubic case by Browning [Bro10] who showed that a cubic form in n variables splitting off a non-zero form has a non-trivial zero provided that $n \geq 13$. In the quartic case we will be able to proof the following result.

Theorem 3. *Let $Z \subset \mathbb{P}^{n-1}_{\mathbb{Q}}$ be a quartic hypersurface defined by an equation of the shape* (1.4)*. Let $Z_1 \subset \mathbb{P}^{k-1}_{\mathbb{Q}}$ be the quartic hypersurface defined by F_1 and $Z_2 \subset \mathbb{P}^{l-1}_{\mathbb{Q}}$ be the quartic hypersurface defined by F_2. Assume that $Z_{ns}(\mathbb{A}_{\mathbb{Q}})$ is non-empty. We have a rational point on Z provided that*

$$n - \dim \operatorname{sing}(Z) \geq 27, \;\; k - \dim \operatorname{sing}(Z_1) \geq 35 \textit{ and } l \geq 2.$$

The condition on the size of the singular locus of Z is needed for the general treatment of the major arcs whereas the one on the singular locus of Z_1 comes into play when we consider the completion of the singular series.

We will establish the theorem by showing, that either the form F_2 has a non-trivial zero, which implies a non-trivial zero of F, or F_2 can be assumed to have a special shape, whence we can establish an asymptotic formula similar to the asymptotic formulas in our other theorems. In the latter case, we will use the circle method to proof the theorem. We will apply Hölders inequality to separate the contribution of the terms involving the polynomials F_1 and F_2 respectively. We will treat the integral involving F_1 with the methods established for Theorem 1. In the case that $l = 2$, the completion of the integration range to the unit interval of the term involving the polynomial F_2 allows us to use a version of Hua's Lemma for binary forms due to Wooley [Woo99]. If $l > 2$ we will treat $l - 2$ variable trivially and use a version of Hua's Lemma for the remaining two variables.

2 Notations

In this section we shall introduce some notation we need. Since our work is based on the one of Browning and Heath-Brown [BHB09] we find it convenient to adopt their notation for the most part.

Let $W \subset \mathbb{P}_{\mathbb{Q}}^{n-1}$ be an arbitrary variety with irreducible components $W_1, \ldots, W_D$. We define the dimension of W by

$$\dim W := \max_{1 \leq i \leq D} \dim W_i,$$

and use analogue definitions for varieties defined over finite fields.

Any non-zero form $G \in \mathbb{Z}[x_1, \ldots, x_n]$ defines a hypersurface $X_G \subset \mathbb{P}^{n-1}$ that can be viewed over any field $\mathbb{F}_v$, where v denotes either a prime p or the symbol ∞. Here we follow the convention that $\mathbb{F}_\infty = \mathbb{Q}$. We denote the singular locus of $X_G \subset \mathbb{P}^{n-1}$ over $\mathbb{F}_v$, a projective subvariety of X_G, by $\mathrm{sing}_{\mathbb{F}_v}(X_G)$ or $\mathrm{sing}_{\mathbb{F}_v}(G)$. We will write $s_v(X_G) = s_v(G)$ for the dimension of this singular locus.

We will work with a certain family of infinitely differentiable weight functions $\omega : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ which have compact support. We denote the smallest value for S such that ω is supported in $[-S, S]^n$ by $S(\omega)$. Furthermore for each $j \in \mathbb{N}_0$ we set

$$S_j(\omega) := \max \left\{ \left| \frac{\partial^{j_1 + \cdots + j_n} \omega(\mathbf{x})}{\partial^{j_1} x_1 \cdots \partial^{j_n} x_n} \right| : \mathbf{x} \in \mathbb{R}^n, j_1 + \cdots + j_n = j \right\}.$$

For given constants c_n and $c_{n,j}$ we say that ω is in the set $\mathcal{W}_n$ provided that $S(\omega) \leq c_n$ and $S_j(\omega) \leq c_{n,j}$ for all $j \geq 0$.

As usual the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural, integer, rational, real and complex numbers. We will write $\mathrm{e}(\alpha) := e^{2\pi i \alpha}$ and $\mathrm{e}_q(\alpha) := e^{2\pi i \alpha / q}$. We shall indicate vectors by bold letters and their components by the same letter in italic font with subscripts. It should always be clear from the context how many components a vector $\mathbf{x}$ is supposed to have. If $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$, the maximum $\max |x_i|$ will be denoted by $|\mathbf{x}|$. We will often perform summations involving vectors $\mathbf{x}, \mathbf{y}$ etc. These will always be restricted to integer vectors. The notation $\sum_{\mathbf{x} \bmod q}$ is used as a shorthand for $\sum_{1 \leq i \leq n} \sum_{1 \leq x_i \leq q}$. We follow the ε-convention: whenever we use the symbol ε in a statement it is asserted that the statement is true for any positive real number ε. Any order constants which we shall write in $\mathrm{O}(\ldots)$ or $\ll$ notation may depend on the actual value of ε, on the form F and the weight function ω. Any other dependencies will be indicated by appropriate subscripts.

3 Circle Method: main steps and treatment of the major arcs

In this section we will recall the main steps of the Hardy-Littlewood circle method and introduce major arcs and minor arcs. Since the basic steps in the treatment of the major arcs are the same in the proof of all our results, we will give a general discussion of them at the end of this section.

Let $X \in \mathbb{P}_{\mathbb{Q}}^{n-1}$ be a quartic hypersurface defined by an absolutely irreducible quartic form $F \in \mathbb{Z}[x_1, \ldots, x_n]$. We may assume that X has a non-singular adelic point and hence we can fix once and for all a non-singular real point $\mathbf{x}_0 \in \mathbb{R}^n$ such that $F(\mathbf{x}_0) = 0$ and $\nabla F(\mathbf{x}_0) \neq 0$. We adopt the notation of [BHB09] and set

$$\sigma := s_\infty(X) = \dim \operatorname{sing}_{\mathbb{Q}}(X). \tag{3.1}$$

We want to count the points lying close to $\mathbf{x}_0$. We set

$$\gamma(x) := \begin{cases} e^{-1/(1-x^2)}, & \text{if } |x| < 1 \\ 0, & \text{if } |x| \geq 1 \end{cases} \tag{3.2}$$

and define for any $\rho \in (0, 1]$ the weight function $\omega_n : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ via

$$\omega_n(\mathbf{x}) = \omega(\mathbf{x}) := \prod_{i=1}^{n} \gamma(\rho^{-1}(x_i - x_{0_i})). \tag{3.3}$$

In due course of the argument we will choose a suitable value for ρ which is small in some sense. It can easily be checked that there exist constants c_n and $c_{n,j}$, which we consider as fixed from now on, such that $\omega_n \in \mathcal{W}_n$.

For the proof of our results we will consider the quantity

$$N_{F;\omega}(P) := \sum_{\substack{\mathbf{x}=(x_1,\ldots,x_n)\in\mathbb{Z}^n \\ F(\mathbf{x})=0}} \omega(\mathbf{x}/P), \tag{3.4}$$

as $P \to \infty$.

We will show that there exist constants $c > 0, \delta > 0$ such that $N_{F;\omega}(P) = cP^{n-4} + \mathrm{O}\left(P^{n-4-\delta}\right)$. The primitive points, which are counted by $N_X(P)$ and $N_Y(P)$ in Theorems 1 and 2, can be singled out by the Möbius function μ (cf. [Bro09, §1.2]): let $V \subset \mathbb{P}^{n-1}$

be a projective variety which is defined by an absolutely irreducible polynomial $g \in \mathbb{Z}[x_1, \ldots, x_n]$. Since $\mathbf{x} = -\mathbf{x}$ in $\mathbb{P}^{n-1}$ we have

$$\begin{aligned}
N_V(P) &= \frac{1}{2}\#\{\mathbf{x} \in \mathbb{Z}^n : g(\mathbf{x}) = 0, \gcd(x_1, \ldots, x_n) = 1, |\mathbf{x}| \le P\} \\
&\ge \sum_{\substack{\mathbf{x}\in\mathbb{Z}^n:|\mathbf{x}|\le P,\\ \gcd(x_1,\ldots,x_n)=1,\\ g(\mathbf{x})=0}} \omega(\mathbf{x}/P) \\
&= \sum_{k=1}^{\infty} \mu(k) \sum_{\substack{\mathbf{x}\in\mathbb{Z}^n:|\mathbf{x}|\le P,\\ k|\mathbf{x},\\ g(\mathbf{x})=0}} \omega(\mathbf{x}/P) \\
&= \sum_{k=1}^{\infty} \mu(k) N_{g;\omega}(k^{-1}P),
\end{aligned}$$

where the results about the respective quantities $N_{g;\omega}$ can be inserted.

We define the generating function $S_F : \mathbb{R} \to \mathbb{C}$ by

$$S_F(\alpha) := \sum_{\mathbf{x}\in\mathbb{Z}^n} \omega(\mathbf{x}/P) \mathrm{e}\left(\alpha F(\mathbf{x})\right), \tag{3.5}$$

whence we have

$$N_{F;\omega}(P) = \int_0^1 S_F\left(\alpha\right) \, d\alpha,$$

by the orthogonality of the exponential function.

One splits the integration range $[0, 1]$ into a set of major arcs and minor arcs, which are both defined modulo 1. We shall take

$$\mathfrak{M}_{a,q}(\Delta) := \left[\frac{a}{q} - P^{-4+\Delta}, \frac{a}{q} + P^{-4+\Delta}\right]$$

as major arcs, for $1 \le a \le q$ such that $\gcd(a, q) = 1$ and $q \le P^{\Delta}$. Here $0 < \Delta < 4/3$ is a small fixed parameter to be defined in due course. It can easily be checked that two distinct intervals $\mathfrak{M}_{a,q}(\Delta)$ are disjoint for these values of Δ. We write $\mathfrak{M}(\Delta)$ for the union of the various $\mathfrak{M}_{a,q}(\Delta)$ and

$$\mathfrak{m}(\Delta) := [0, 1] \setminus \mathfrak{M}(\Delta)$$

for the corresponding set of minor arcs.

For any coprime integers a, q we define the complete exponential sums $S_{a,q}$ by

$$S_{a,q} := \sum_{\mathbf{x} \bmod q} \mathrm{e}_q\left(aF(\mathbf{x})\right),$$

and define the singular series

$$\mathfrak{S} := \lim_{R\to\infty} \mathfrak{S}(R) := \lim_{R\to\infty} \sum_{q\leq R} \frac{1}{q^n} \sum_{\substack{a=1\\ \gcd(a,q)=1}}^{q} S_{a,q}.$$

The corresponding singular integral is given by

$$\mathfrak{I} := \lim_{R\to\infty} \mathfrak{I}(R) := \lim_{R\to\infty} \int_{-R}^{R} \int_{\mathbb{R}^n} \omega(\mathbf{x}) \mathrm{e}\left(zF(\mathbf{x})\right) d\mathbf{x}\, dz.$$

According to Lemma 23 of [BHB09, §10] there exists $\delta > 0$ such that

$$\int_{\mathfrak{M}(\Delta)} S_F\left(\alpha\right) d\alpha = \mathfrak{S}\mathfrak{I}P^{n-4} + \mathrm{O}\left(P^{n-4-\delta}\right)$$

for any Δ in the range $0 < \Delta < 1/5$, provided that $n - \dim \operatorname{sing}(Z) \geq 27$ and if there exists $\phi > 0$ such that

$$\mathfrak{S}(R) = \mathfrak{S} + \mathrm{O}\left(R^{-\phi}\right). \tag{3.6}$$

This Lemma also assures the absolute convergence of the singular integral $\mathfrak{I}$, the absolute convergence of the singular series $\mathfrak{S}$ follows from [BHB09, Theorem 2]. It can be seen by standard arguments (cf. [Bir62, Lemma 7.1]) that $\mathfrak{S} > 0$. By choosing a sufficiently small value of ρ in (3.2) we can ensure that $\mathfrak{I} > 0$ (cf. [BHB09, §8]).

We want to remark that (3.6) is fulfilled for $n - \sigma \geq 38$ because we than have

$$|\mathfrak{S} - \mathfrak{S}(R)| \ll R^{-1/24+\varepsilon}$$

by an argument similar to [BHB09, p.88]. This is sufficient for the treatment of the singular series in Theorem 1. To get respective bounds for the forms considered in the other theorems we will use the fact that they split into two forms F_1 and F_2. Thus, we can apply the simple identity

$$S_{a,q} = \sum_{(\mathbf{x},\mathbf{y}) \bmod q} \mathrm{e}_q\left(a(F_1(\mathbf{x}) + F_2(\mathbf{y}))\right) = \sum_{\mathbf{x} \bmod q} \mathrm{e}_q\left(aF_1(\mathbf{x})\right) \sum_{\mathbf{y} \bmod q} \mathrm{e}_q\left(aF_2(\mathbf{y})\right),$$

which allows us to bound the resulting two sums separately.

4 Exponential sums: estimates

Like Browning and Heath-Brown we want to relate the size of the quartic exponential sum S_F to a certain family of cubic exponential sums. This can be done by a discrete differencing step based on van der Corput's method. Before we proceed by presenting the bounds Browning and Heath-Brown obtain for the exponential sums, we want to acquaint the reader with the main idea of van der Corput's method. We hereby closely follow the steps of [BHB09, p. 41f]. Let $H \leq P$ be a positive integer and write, temporarily,

$$f(\mathbf{x}) = \omega(\mathbf{x}/P)\mathrm{e}\left(\alpha F(\mathbf{x})\right).$$

The starting point of the van der Corput method is the equality

$$H^n S_F(\alpha) = \sum_{\mathbf{h}} \sum_{\mathbf{x}\in\mathbb{Z}^n} f(\mathbf{x}+\mathbf{h}) = \sum_{\mathbf{x}\in\mathbb{Z}^n} \sum_{\mathbf{h}} f(\mathbf{x}+\mathbf{h}), \tag{4.1}$$

where the sum over $\mathbf{h}$ is for vectors in $\mathbb{N}^n$ with $1 \leq h_i \leq H$. By applying Cauchy's inequality we obtain

$$\begin{aligned}
H^{2n} |S_F(\alpha)|^2 &\ll P^n \sum_{\mathbf{x}\in\mathbb{Z}^n} \left| \sum_{\mathbf{h}} f(\mathbf{x}+\mathbf{h}) \right|^2 \\
&= P^n \sum_{\mathbf{h}_1} \sum_{\mathbf{h}_2} \sum_{\mathbf{x}\in\mathbb{Z}^n} f(\mathbf{x}+\mathbf{h}_1)\overline{f(\mathbf{x}+\mathbf{h}_2)} \\
&= P^n \sum_{\mathbf{h}_1} \sum_{\mathbf{h}_2} \sum_{\mathbf{y}\in\mathbb{Z}^n} f(\mathbf{y}+\mathbf{h}_1-\mathbf{h}_2)\overline{f(\mathbf{y})} \\
&= P^n \sum_{\substack{\mathbf{h}\in\mathbb{Z}^n \\ |\mathbf{h}|\leq H}} N(\mathbf{h}) \sum_{\mathbf{y}\in\mathbb{Z}^n} f(\mathbf{y}+\mathbf{h})\overline{f(\mathbf{y})},
\end{aligned}$$

where $N(\mathbf{h})$ is given by

$$N(\mathbf{h}) := \#\{\mathbf{h}_1, \mathbf{h}_2 : \mathbf{h} = \mathbf{h}_1 - \mathbf{h}_2\}. \tag{4.2}$$

Since $N(\mathbf{h}) \ll H^n$, this yields

$$|S_F(\alpha)|^2 \ll P^n H^{-n} \sum_{\mathbf{h}} |T_{\mathbf{h}}(\alpha)| \ll \frac{P^{2n}}{H^n} + \frac{P^n}{H^n} \sum_{\mathbf{h}\neq 0} |T_{\mathbf{h}}(\alpha)|, \tag{4.3}$$

where

$$T_{\mathbf{h}}(\alpha) := \sum_{\mathbf{x}\in\mathbb{Z}^n} \omega_{\mathbf{h}}(\mathbf{x}/P)\mathrm{e}\left(\alpha(F(\mathbf{x}+\mathbf{h}) - F(\mathbf{x}))\right) \tag{4.4}$$

and

$$\omega_{\mathbf{h}}(\mathbf{x}) := \omega(\mathbf{x} + P^{-1}\mathbf{h})\omega(\mathbf{x}). \tag{4.5}$$

We want to point out that by choosing $H = P$ this corresponds to the first step in Birch's differencing process. The advantage of the van der Corput method is the parameter H which value can be chosen appropriately.

In the rest of this section we will summarize the results Browning and Heath-Brown obtain in their treatment of the exponential sums [BHB09, §§4-7] and we find it convenient to adopt their notation for the most part. Let $g \in \mathbb{Z}[x_1, \ldots, x_n]$ be an arbitrary cubic polynomial and let g_0 be the cubic homogeneous part of g. We set

$$\|g\|_P := \left\|P^{-3}g(Px_1, \ldots, Px_n)\right\|,$$

where the height $\|f\|$ of an arbitrary polynomial f is defined to be the maximum over the moduli of the coefficients of f.

Before we can present bounds for the cubic exponential sum

$$\mathcal{T}(\alpha) = \mathcal{T}_n(\alpha; g, \omega, P) := \sum_{\mathbf{x}\in\mathbb{Z}^n} \omega(\mathbf{x}/P)\mathrm{e}\left(\alpha g(\mathbf{x})\right),$$

we have to introduce some more notation. We consider $\alpha = a/q + z$, with $a, q \in \mathbb{Z}$ such that

$$1 \leq a \leq q \leq P^2, \qquad \gcd(a, q) = 1, \tag{4.6}$$

and $z \in \mathbb{R}$ satisfiying

$$|z| \leq q^{-1}P^{-1}.$$

For each $p \mid q$ we set $s_p := s_p(g_0)$ as well as $s_\infty := s_\infty(g_0)$. Furthermore, we will write $q = bc^2d$, where

$$b := \prod_{\substack{p^e \| q \\ e\leq 2}} p^e, \qquad d := \prod_{\substack{p^e \| q \\ e\geq 3, 2\nmid e}} p, \tag{4.7}$$

and thus we have $d \mid c$. For $0 \leq i \leq n$ we define

$$r_i := \prod_{\substack{p^e \| bd \\ s_p = i-1}} p^e.$$

The following result is due to Browning and Heath-Brown [BHB09, Proposition 2].

Lemma 1. *Let $A, \varepsilon > 0$. Let $\omega \in \mathcal{W}_n$ and let $g \in \mathbb{Z}[x_1, \ldots, x_n]$ be a cubic polynomial with $\|g\|_P \le H$, for some H in the range $1 \le H \le P^A$. Let a, q be such that* (4.6) *holds and $q = bc^2 d$ with respect to* (4.7). *Then we have*

$$\mathcal{T}(a/q + z) \ll_A \min_{1+s_\infty \le \eta \le n} q^{-(n-\eta)/2} \left(\prod_{i=\eta}^{n} r_i^{(i-\eta)/2} \right) P^{n+\varepsilon} W^{n-\eta},$$

where W is given by

$$W := V + \min \left\{ (c^2 dH)^{1/3}, c^{1/2} V^{1/2} + c^{5/6} H^{1/6} \right\} \tag{4.9}$$

with

$$V := qP^{-1} \max \left\{ 1, \sqrt{|z| \, HP^3} \right\}.$$

Browning and Heath-Brown obtain bounds for the quartic exponential sum $S_F(\alpha)$ by applying Lemma 1 to the system of cubic exponential sums, which they get by applying a single van der Corput differencing step and summing up over the values of $\mathbf{h}$ afterwards. The following result is due to [BHB09, Proposition 4].

Lemma 2. *Let a, q be coprime integers such that $1 \le a \le q \le P^2$ and $q = bc^2 d$, in the notation of* (4.7). *Let $z \in \mathbb{R}$ such that $|z| \le q^{-1} P^{-1}$. Then we have*

$$S_F(a/q + z) \ll \frac{P^{n+\varepsilon}}{H^{(n-1-\sigma)/2}} \left(1 + \frac{q^{1/2} H}{P} + \sqrt{q \, |z| \, H^3 P} + \frac{H}{q^{1/2}} M \right)^{n/2},$$

where

$$M := \min \left\{ (c^2 dH)^{1/3}, c^{1/2} q^{1/2} (P^{-1/2} + (|z| \, HP)^{1/4}) + c^{5/6} H^{1/6} \right\}.$$

For some values of H, the direct treatment of the cubic exponential sums described above, gives better bounds than an iterated Weyl differencing process as used in [Bir62]. Hence, this is one of the key ingredients in the work of Browning and Heath-Brown. In [BHB09, §6] Birch's approach is slightly modified and leads to the following result.

Lemma 3. *Let a, q, z be such that*

$$1 \leq a \leq q, \qquad \gcd(a,q) = 1, \qquad |z| \leq \frac{1}{q^2}.$$

Then we have

$$S_F\left(a/q + z\right) \ll P^{n+\varepsilon}(q\,|z| + q^{-1}\,|z|^{-1}\,P^{-4})^{(n-\sigma-1)/24}.$$

5 Averaged van der Corput method

In order to obtain satisfactory estimates for the overall contribution of the minor arcs we will relate the size of $S_F\left(\alpha\right)$ to the size of a certain family of cubic exponential sums by using a differencing process based on an averaged version of the van der Corput method. We will adapt the approach of Heath-Brown [HB07], who considers cubic exponential sums, to the setting of quartic forms. We want to study the object

$$\mathcal{M}(\alpha, H) := \int_{\alpha-(HP^3)^{-1}}^{\alpha+(HP^3)^{-1}} |S_F\left(\beta\right)|^2\,d\beta,$$

where S_F is given by (3.5).

Up to formula (5.6), we will basically follow the argument of Heath-Brown [HB07, p. 214f] step by step. Since we work with a quartic exponential sum instead of a cubic one, roughly spoken, the only difference is that we have to insert an extra factor P at some points of the argument. We recall that our weight function is centered about a non-singular point $\mathbf{x}_0$. By reordering the indices, if necessary, we may therefore assume that

$$G := \left|\frac{\partial F(\mathbf{x}_0)}{\partial x_1}\right| > 0.$$

We now perform a discrete differencing step due to van der Corput as we described it in the previous section, starting with (4.1), but with α replaced by β and the summation range for $\mathbf{h}$ altered to $1 \leq h_1 \leq P$ and $1 \leq h_2, \ldots, h_n \leq H$. We therefore obtain

$$P^2 H^{2n-2}\,|S(\beta)|^2 \ll P^n \sum_{\mathbf{h}} N(\mathbf{h}) \sum_{\mathbf{y}\in\mathbb{Z}^n} f(\mathbf{y}+\mathbf{h})\overline{f(\mathbf{y})},$$

with the new summation condition on $\mathbf{h}$. Analogously, we have to mind the new restrictions on $\mathbf{h}$ in the definition of N which apart from that is still given by (4.2). We deduce that

$$\begin{aligned}\mathcal{M}(\alpha, H) &\ll \int_{-\infty}^{\infty} \exp\{-P^6 H^2(\beta-\alpha)^2\} \left|S(\beta)\right|^2 d\beta && (5.1)\\ &\ll P^{n-2}H^{2-2n}\sum_{\mathbf{h}} N(\mathbf{h}) \sum_{\mathbf{y}\in\mathbb{Z}^n} I(\mathbf{h},\mathbf{y}), && (5.2)\end{aligned}$$

where I is given by

$$I(\mathbf{h},\mathbf{y}) := \int_{-\infty}^{\infty} \omega_{\mathbf{h}}(\mathbf{y}/P) \exp\{-(HP^3)^2(\beta-\alpha)^2\}\mathrm{e}\left(\beta F_{\mathbf{h}}(\mathbf{y})\right) d\beta \tag{5.3}$$

and $F_{\mathbf{h}}(\mathbf{y}) = F(\mathbf{y}+\mathbf{h}) - F(\mathbf{y})$. By carrying out the integration we may write

$$I(\mathbf{h},\mathbf{y}) = \omega_{\mathbf{h}}(\mathbf{y}/P)\frac{\sqrt{\pi}}{HP^3}\exp\left(-\pi^2\left(\frac{F_{\mathbf{h}}(\mathbf{y})}{HP^3}\right)^2\right)\mathrm{e}\left(\alpha F_{\mathbf{h}}(\mathbf{y})\right).$$

We set $\mathcal{L} := \log P$ and turn to the terms with

$$|F(\mathbf{y}+\mathbf{h}) - F(\mathbf{y})| \geq HP^3\mathcal{L}. \tag{5.4}$$

We observe that they contribute $\mathrm{O}(1)$ to $\mathcal{M}(\alpha, H)$ since $\omega_{\mathbf{h}} \ll 1$ and $N(\mathbf{h}) \ll PH^{n-1}$. By Taylor-expansion and the conditions on $\mathbf{h}$ we have

$$|F_{\mathbf{h}}(\mathbf{y})| = |F(\mathbf{y}+\mathbf{h}) - F(\mathbf{y})| = \frac{\partial F(\mathbf{y})}{\partial y_1}h_1 + \mathrm{O}\left(HP^3\right) + \mathrm{O}\left(h_1^2P^2\right), \tag{5.5}$$

where the implicit constants depend on the form F only. We now choose ρ in (3.3) sufficiently small, such that

$$\left|\frac{\partial F(\mathbf{y})}{\partial y_1}\right| > \frac{1}{2}P^3G,$$

which we clearly may, since we just have to consider those $\mathbf{y}$ and $\mathbf{h}$ satisfying $\omega_{\mathbf{h}}(\mathbf{y}/P) \neq 0$. If we choose ρ even smaller, if necessary, we can arrange the error term $\mathrm{O}\left(h_1^2P^2\right)$ in (5.5) to be at most $\frac{1}{4}G\left|h_1\right|P^3$. Consequently we get

$$|F_{\mathbf{h}}(\mathbf{y})| \geq \frac{1}{4}G\left|h_1\right|P^3 + \mathrm{O}\left(HP^3\right).$$

We may conclude that the condition (5.4) holds unless $|h_1| \leq 5G^{-1}H\mathcal{L}$, or alternatively $|h_1| \leq H\mathcal{L}^2$. It follows that the terms with $|h_1| \geq H\mathcal{L}^2$ contribute $\mathrm{O}(1)$ in total to (5.2).

Since the contribution of the range $|\beta - \alpha| \geq (HP^3)^{-1}\mathcal{L}$ to (5.1) is obviously $\mathrm{O}(1)$ we finally obtain

$$\begin{aligned} \mathcal{M}(\alpha, H) &\ll 1 + P^{n-2}H^{2-2n} \sum_{|h_1|\leq H\mathcal{L}2} \sum_{|h_2|\leq H} \cdots \sum_{|h_n|\leq H} N(\mathbf{h}) \sum_{\mathbf{y}\in\mathbb{Z}^n} I(\mathbf{h}, \mathbf{y}) \\ &\ll 1 + P^{n-1}H^{1-n} \sum_{|\mathbf{h}|\leq H\mathcal{L}^2} \int_{\alpha-(HP^3)^{-1}\mathcal{L}}^{\alpha+(HP^3)^{-1}\mathcal{L}} |T_{\mathbf{h}}(\beta)|\, d\beta, \end{aligned} \tag{5.6}$$

where $T_{\mathbf{h}}(\beta)$ is given by (4.4). The bound (5.6) should be compared to the bound (4.3) which one obtains by applying the simple version of van der Corput's method. Roughly spoken, the averaging process by the integration allows us to shorten a long summation over values of h_1 to a comparable short one whence by considering the mean-square average of $S_F(\beta)$ over a suitable interval we gain an extra factor HP^{-1}.

In our treatment of the minor arcs we will split up the integration range in dyadic intervals. For given $Q, R, t \in \mathbb{R}$ with $0 < R \leq Q$ we set

$$\mathfrak{m}(R, t) = \bigcup_{R<q\leq 2R} \bigcup_{\substack{1\leq a\leq q \\ \gcd(a,q)=1}} I(a, q, t)$$

with

$$I(a, q, t) = \left\{ \alpha \in \mathbb{R} : t < \left| \alpha - \frac{a}{q} \right| \leq 2t \right\}.$$

For given $1 \leq \theta \leq 2$ we define the integrals

$$\Sigma(R, t, \theta, S_F) := \int_{\mathfrak{m}(R,t)} |S_F(\alpha)|^\theta \, d\alpha. \tag{5.7}$$

We clearly have

$$\Sigma(R, t, \theta, S_F) \ll \sum_{R<q\leq 2R} \sum_{\substack{1\leq a\leq q \\ \gcd(a,q)=1}} \int_{I(a,q,t)} |S_F(\alpha)|^\theta \, d\alpha \tag{5.8}$$

which we will use frequently without further comment.

In order to apply the averaged version of van der Corput's method, we apply Hölder's inequality to (5.8) to deduce that

$$\Sigma(R, t, \theta, S_F) \ll t^{1-\theta/2} \sum_{R<q\leq 2R} \sum_{\substack{1\leq a\leq q \\ \gcd(a,q)=1}} \left(\int_{I(a,q,t)} |S_F(\alpha)|^2 \, d\alpha \right)^{\theta/2}. \tag{5.9}$$

We divide the integration range $[t, 2t]$ into $\mathrm{O}\left(1 + HP^3 t\right)$ intervals of the shape

$$[\tau - (HP^3)^{-1}, \tau + (HP^3)^{-1}]$$

with $t < \tau \leq 2t$. We therefore have

$$\Sigma(R, t, \theta, S_F) \ll t^{1-\theta/2}(1 + HP^3 t)^{\theta/2} \sum_{R<q\leq 2R} \sum_{\substack{1\leq a\leq q \\ \gcd(a,q)=1}} \mathcal{M}\left(\frac{a}{q} + \vartheta, H\right)^{\theta/2}, \qquad (5.10)$$

for some ϑ in the range $t \leq |\vartheta| \leq 2t$. It follows from (5.6) that

$$\mathcal{M}\left(\frac{a}{q} + \vartheta, H\right) \ll 1 + \int_{\vartheta-(HP^3)^{-1}\mathcal{L}}^{\vartheta+(HP^3)^{-1}\mathcal{L}} \frac{P^{2n-1}}{H^{n-1}} + \frac{P^{n-1}}{H^{n-1}} \sum_{\substack{|\mathbf{h}|\leq H\mathcal{L}^2 \\ \mathbf{h}\neq\mathbf{0}}} |T_{\mathbf{h}}(a/q + \beta)| \, d\beta. \quad (5.11)$$

Since $H \leq P$, for $n \geq 4$ the contribution by the integral dominates the term 1.

The integrand, $\mathcal{I}(a/q+\beta; H)$ say, is considered in [BHB09, §7]. A key ingredient for its treatment is an application of Lemma 1 with $A = 1 + \varepsilon$ to the the occurring exponential sums $T_{\mathbf{h}}$, since $|\mathbf{h}| \leq H\mathcal{L}^2 \leq P^{1+\varepsilon}$ and

$$\|F(\mathbf{x} + \mathbf{h}) - F(\mathbf{x})\|_P = \left\|P^{-3}(F(P\mathbf{x} + \mathbf{h}) - F(P\mathbf{x}))\right\| \ll H\mathcal{L}^8.$$

Since there exist constants such that $\omega \in \mathcal{W}_n$ and by the definition (4.5) of $\omega_{\mathbf{h}}$ it can easily be seen that there also exist constants uniform in $\mathbf{h}$ such that $\omega_{\mathbf{h}} \in \mathcal{W}_n$. The homogeneous part of maximal degree of $F_{\mathbf{h}}(\mathbf{x}) = F(\mathbf{x} + \mathbf{h}) - F(\mathbf{x})$ is given by $\mathbf{h}\nabla F(\mathbf{x})$. Furthermore, the equation $\mathbf{h}\nabla F(\mathbf{x}) = 0$ defines a variety $X_{\mathbf{h}}$ in $\mathbb{P}^{n-1}$ over any $\mathbb{F}_v$, where v denotes a prime or the symbol ∞. By setting $s_v = s_v(X_{\mathbf{h}})$ and recalling the corresponding definition (4.8) of the r_i, Lemma 1 gives us

$$T_{\mathbf{h}}(a/q + \beta) \ll \min_{1+s_\infty\leq\eta\leq n} q^{-(n-\eta)/2} \left(\prod_{i=\eta}^{n} r_i^{(i-\eta)/2}\right) P^{n+\varepsilon} W^{n-\eta},$$

since $\mathcal{L} \ll P^\varepsilon$. We may now follow the arguments of [BHB09, §7] to sum this bound over the relevant values of $\mathbf{h}$ and so we obtain

$$\mathcal{I}(a/q + \beta; H) \ll P^{2n-1+\varepsilon} H^{2-n+\sigma} \left(1 + \frac{H^n}{q^{n/2}} W^n\right).$$

We write $q = bc^2d$ in the notation of (4.7). On recalling the definition (4.9) of W and by using the fact that $\beta \ll t + (HP^3)^{-1}\mathcal{L}$ we obtain

$$\mathcal{I}(a/q+\beta;H) \ll P^{2n-1+\varepsilon} H^{2-n+\sigma} \left(1 + \frac{q^{1/2}H}{P} + \sqrt{q\,|t|\,PH^3} + \frac{H}{q^{1/2}}M\right)^n, \qquad (5.12)$$

with

$$M = \min\left\{(c^2dH)^{1/3}, c^{1/2}q^{1/2}(P^{-1/2} + (|t|\,HP)^{1/4}) + c^{5/6}H^{1/6}\right\}. \qquad (5.13)$$

With respect to (5.10), we should compare the square-root of the bound (5.12) to Lemma 2, since for the proof of Theorem 1 we will set $\theta = 1$. Hence, by using the averaged version of van der Corput's method, we gain an extra factor $(H/P)^{1/2}$ in the bound for the quartic exponential sum S_F, which will allow us the extra saving that is crucial to establish Theorem 1.

We set

$$E(H,P,R,t,M) := 1 + \frac{R^{1/2}H}{P} + \sqrt{R\,|t|\,PH^3} + \frac{HM}{R^{1/2}}. \qquad (5.14)$$

Combining (5.10),(5.11) and (5.12) and carrying out the integration, we finally obtain the following result, which should be compared to Lemma 2.

Lemma 4. *Let $1 \leq \theta \leq 2$ and $Q, R, t \in \mathbb{R}_{\geq 0}$ such that $0 < R \leq Q$ and $|t| \leq R^{-1}P^{-1}$. Let H be an integer in the range $1 \leq H \leq P$ and write $q = bc^2d$ with respect to (4.7). We then have*

$$\Sigma(R,t,\theta,S_F) \ll \left(\frac{t^{1-\theta/2}}{P^{2\theta}} + t\left(\frac{H}{P}\right)^{\theta/2}\right) \sum_{R<q\leq 2R} \sum_{\substack{1\leq a\leq q \\ (a;q)=1}} \frac{P^{\theta n+\varepsilon}}{H^{\theta(n-1-\sigma)/2}} E(H,P,R,t,M)^{\theta n/2},$$

where M is given by (5.13) and E is given by (5.14).

For $\theta = 1$ and $H \ll P$, the latter bound is better than the bound one obtains by using Lemma 2, provided that $t > P^{-4}$.

6 The minor arcs

In this section we will discuss the general steps of the treatment of the minor arcs which can be used in the proofs of all our theorems. Let $X \in \mathbb{P}^{n-1}_{\mathbb{Q}}$ be a quartic hypersurface with underlying quartic form $F \in \mathbb{Z}[x_1, \dots, x_n]$. Let $\sigma = \dim \operatorname{sing}_{\mathbb{Q}}(X)$ denote the dimension of the singular locus of X. Our goal is to show that there exists $\delta > 0$ such that

$$\int_{\mathfrak{m}(\Delta)} S_F(\alpha)\, d\alpha = \mathrm{O}\left(P^{n-4-\delta}\right), \tag{6.1}$$

for some Δ in the range $0 < \Delta < 4/3$, where the minor arcs $\mathfrak{m}(\Delta)$ are defined as in §3. Since the basic steps in the treatment of the minor arcs will be the same in the proof of all our theorems we will give a general discussion here. In the following sections we will make explicit use of the special shapes of the forms under consideration as stated in our theorems to get respective estimates of the shape (6.1). The following arguments in this section will be much in spirit with the original argument [BHB09, §9]. The major modification is to include the results of §5, which will finally allow us extra savings.

For any $\alpha \in [0, 1]$ we apply Dirichlet's approximation theorem with the same parameters as in [BHB09]. So we can write $\alpha = a/q + z$ for appropriate coprime $a, q \in \mathbb{N}$ with $1 \le a \le q \le Q$ and $z \in \mathbb{R}$ with $|z| \le 1/(qQ)$. We may conclude that if α lies in the minor arcs $\mathfrak{m}(\Delta)$, then

$$q \le P^{\Delta}, \qquad |z| \le P^{-4+\Delta} \tag{6.2}$$

do not both hold. It will be convenient to work with

$$Q := P^{8/5+\phi}, \tag{6.3}$$

where $0 < \phi < 1/5$ is a small parameter which will be defined in due course.

We want to show that there exists $\delta > 0$ such that

$$\Sigma(R, t, 1, S_F) \ll P^{n-4-\delta}, \tag{6.4}$$

unless

$$2R \le P^{\Delta} \text{ and } 2t \le P^{-4+\Delta}, \tag{6.5}$$

which is satisfactory for establishing (6.1), because summing up over the dyadic decomposition of R, t just gives us an extra factor P^{ε}.

Let us consider the general case

$$\Sigma(R, t, \theta, S_F) \ll \sum_{R < q \le 2R} \sum_{\substack{1 \le a \le q \\ \gcd(a,q)=1}} \int_{I(a,q,t)} |S_F(\alpha)|^{\theta}\, d\alpha.$$

Like Browning and Heath-Brown, we will study the contribution to $\Sigma(R,t,\theta,S_F)$ which arises from those $q = bc^2d$, in the notation of (4.7), whose factors b,c,d are restricted in the following way. For given $\mathbf{R} = (R_0,R_1,R_2) \in \mathbb{R}^3_{\geq 0}$ we consider those $q = bc^2d$ with

$$R_0 < b \leq 2R_0, \qquad R_1 < c \leq 2R_1, \qquad R_2 < d \leq 2R_2, \tag{6.6}$$

and denote their overall contribution to $\Sigma(R,t,\theta,S_F)$ by $\Sigma_{\mathbf{R}}(R,t,\theta,S_F)$. It will be convenient to use the notation

$$\mathcal{Q}_{\mathbf{R}}(R) := \{q = bc^2d \in \mathbb{N} : R < q \leq 2R, \text{ (4.7) and (6.6) hold}\}.$$

Since $d \mid c$, like in [BHB09] the fact that $\Sigma_{\mathbf{R}}(R,t,\theta,S_F) = 0$ unless

$$R_2 \leq 2R_1, \qquad R/16 < R_0R_1^2R_2 \leq 2R, \qquad R_i \geq 1/2 \tag{6.7}$$

for $0 \leq i \leq 2$ will be used frequently. We have

$$\Sigma(R,t,\theta,S_F) \ll P^{\varepsilon} \max_{\mathbf{R}} \Sigma_{\mathbf{R}}(R,t,\theta,S_F),$$

where the maximum is over all those $\mathbf{R} = (R_0,R_1,R_2) \in \mathbb{R}^3_{\geq 0}$ satisfying (6.7).

In the following we will often utilize the next result due to Browning and Heath-Brown [BHB09, Lemma 20].

Lemma 5. *We have*

$$\sum_{\substack{q=bc^2d \\ \text{(6.6) holds}}} 1 \ll R_0R_1^{1/2}R_2^{1/2}.$$

The following bound for $\Sigma_{\mathbf{R}}(R,t,\theta,S_F)$ is a generalisation of [BHB09, Lemma 21].

Lemma 6. *For $t > (RP^2)^{-1}$ we have*

$$\Sigma_{\mathbf{R}}(R,t,\theta,S_F) \ll P^{\theta n+\varepsilon}(Rt)^{1+\theta(n-\sigma-1)/24}R_0R_1^{1/2}R_2^{1/2}$$

while for $t \leq (RP^2)^{-1}$ we have

$$\Sigma_{\mathbf{R}}(R,t,\theta,S_F) \ll P^{\theta(n-(n-\sigma-1)/6)+\varepsilon}(Rt)^{1-\theta(n-\sigma-1)/24}R_0R_1^{1/2}R_2^{1/2}.$$

Proof. An application of Lemma 3 yields

$$\Sigma_{\mathbf{R}}(R,t,\theta,S_F) \ll \sum_{q\in\mathcal{Q}_{\mathbf{R}}(R)} \sum_{\substack{1\leq a\leq q\\ \gcd(a,q)=1}} \int_t^{2t} P^{\theta n+\varepsilon}(q\,|z| + q^{-1}\,|z|^{-1}\,P^{-4})^{\theta(n-\sigma-1)/24}\,dz.$$

We then may use Lemma 5 to complete the proof. □

Our second bound for $\Sigma_{\mathbf{R}}(R,t,\theta,S_F)$ will be the key ingredient for extra savings compared to the original work and can be seen as analogue to [BHB09, Lemma 22]. The proof is similar to the proof of this Lemma. The major difference is that we use the bounds we gained in §5 by using an averaged version of van der Corput's method.

Lemma 7. *Let $1 \leq \theta \leq 2$. For $(RP^2)^{-1} < t \leq (RQ)^{-1}$ we have*

$$\Sigma_{\mathbf{R}}(R,t,\theta,S_F) \ll M_0 + P^{\theta(n-2/5+\phi)+\varepsilon}\frac{R_0R_1^{1/2}R_2^{1/2}}{Q}\mu^{\theta(n-\sigma-1)/2}, \tag{6.8}$$

while for $t \leq (RP^2)^{-1}$ we have

$$\Sigma_{\mathbf{R}}(R,t,\theta,S_F) \ll \widetilde{M_0} + RtP^{\theta n+\varepsilon}R_0R_1^{1/2}R_2^{1/2}\mu^{\theta(n-\sigma-1)/2}, \tag{6.9}$$

where

$$\mu := \min\left\{\frac{R_1^{5/7}}{R^{3/7}} + R_1^{2/5}t^{1/5}P^{1/5} + \frac{R_1^{1/2}}{P^{1/2}}, \frac{R_1^{1/2}R_2^{1/4}}{R^{3/8}}\right\}$$

and

$$\begin{aligned} M_0(\theta) = M_0 &:= P^{\theta(9n/10+(\sigma-3)/10+\phi/4(n-\sigma+3))+\varepsilon}, &(6.10)\\ \widetilde{M_0}(\theta) = \widetilde{M_0} &:= P^{\theta(9n/10+(\sigma+1)/10+\phi/4(n-\sigma-1))-2/5+\phi+\varepsilon}. &(6.11)\end{aligned}$$

Proof. We will deal with the range $(RP^2)^{-1} < t \leq (RQ)^{-1}$ first. Lemma 4 gives us

$$\Sigma_{\mathbf{R}}(R,t,\theta,S_F) \ll \left(\frac{t^{1-\theta/2}}{P^{2\theta}} + t\left(\frac{H}{P}\right)^{\theta/2}\right) \sum_{q\in\mathcal{Q}_{\mathbf{R}}(R)} \sum_{\substack{1\leq a\leq q\\ (a;q)=1}} \frac{P^{\theta n+\varepsilon}}{H^{\theta(n-\sigma-1)/2}} E(H,P,R,t,M)^{\theta n/2}$$

with

$$E(H,P,R,t,M) = 1 + \frac{R^{1/2}H}{P} + \sqrt{|t|\,PRH^3} + \frac{HM}{R^{1/2}},$$

where

$$M=\min\left\{(R_1^2R_2H)^{1/3},R_1^{1/2}R^{1/2}(P^{-1/2}+(|t|\,HP)^{1/4})+R_1^{5/6}H^{1/6}\right\}.$$

We proceed like Browning and Heath-Brown [BHB09, cf. Proof of Lemma 22] by choosing $1\le H\le P$ such that $E(H,P,R,t,M)=\mathrm{O}\left(P^{\varepsilon}\right)$. This can be achieved by the choice

$$H:=1+\min\left\{P^{1/5-\phi/2},\frac{R^{3/7}}{R_1^{5/7}},\frac{1}{R_1^{2/5}t^{1/5}P^{1/5}},\frac{P^{1/2}}{R_1^{1/2}}\right\}+\min\left\{P^{1/5-\phi/2},\frac{R^{3/8}}{R_1^{1/2}R_2^{1/4}}\right\},$$

where compared to Browning and Heath-Brown we replaced the term $P^{9/5}/Q$ by $P^{1/5-\phi/2}$, which we may since we then still have $R^{1/2}H/P+\sqrt{|t|\,PRH^3}=\mathrm{O}\left(1\right)$. Performing the summation over q and using the fact that $H\ll P^{1/5-\phi/2}$ we have that $\Sigma_{\mathbf{R}}(R,t,\theta,S_F)$ is

$$\begin{aligned}
&\ll\ RtR_0R_1^{1/2}R_2^{1/2}P^{\theta n+\varepsilon}\left(t^{-\theta/2}P^{-2\theta}+P^{-\theta(2/5+\phi/4)}\right)\left(P^{-1/5+\phi/2}+\mu\right)^{\theta(n-\sigma-1)/2}\\
&\ll\ \frac{R_0R_1^{1/2}R_2^{1/2}P^{\theta n+\varepsilon}}{Q}\left((RQ)^{\theta/2}P^{-2\theta}+P^{-\theta(2/5+\phi/4)}\right)\left(P^{-1/5+\phi/2}+\mu\right)^{\theta(n-\sigma-1)/2}
\end{aligned}$$

since $t\le(RQ)^{-1}$ and $\theta\le 2$. Using the fact that $R\le Q=P^{8/5+\phi}$ we have

$$(RQ)^{\theta/2}P^{-2\theta}+P^{-\theta(2/5+\phi/4)}\ll P^{\theta(-2/5+\phi)}$$

and hence we obtain the term involving μ on the right hand side of (6.8).

It remains to deal with the terms involving $P^{-1/5+\phi/2}$. By the same arguments as above and using the fact that $t\le(RQ)^{-1}$, $\theta\le 2$ and $R\le Q=P^{8/5+\phi}$ this is

$$\begin{aligned}
&\ll\ \frac{RR_0R_1^{1/2}R_2^{1/2}(RQ)^{\theta/2}}{RQP^{2\theta}}P^{\theta(9n/10+(\sigma+1)/10+\phi/4(n-\sigma-1))+\varepsilon}\\
&\ll\ P^{\theta(9n/10+(\sigma-3)/10+\phi/4(n-\sigma+3))+\varepsilon}.
\end{aligned}$$

This completes the proof of the Lemma for the range $(RP^2)^{-1}<t\le(RQ)^{-1}$.

We now turn to the range $t\le(RP^2)^{-1}$. Instead of using Lemma 4 we use Lemma 2. The same choice for H and following the same steps as above this leads to

$$\Sigma_{\mathbf{R}}(R,t,\theta,S_F)\ll RtR_0R_1^{1/2}R_2^{1/2}P^{\theta n+\varepsilon}\left(P^{-1/5+\phi/2}+\mu\right)^{\theta(n-\sigma-1)/2}.$$

The contribution from the term involving $P^{-1/5+\phi/2}$ is

$$\begin{aligned}
&\ll\ \frac{RR_0R_1^{1/2}R_2^{1/2}}{RP^2}P^{\theta(9n/10+(\sigma+1)/10+\phi/4(n-\sigma-1))+\varepsilon}\\
&\ll\ P^{\theta(9n/10+(\sigma+1)/10+\phi/4(n-\sigma-1))-2/5+\phi+\varepsilon},
\end{aligned}$$

which completes the proof of the Lemma. $\square$

In the following step we will combine Lemma 6 and Lemma 7. We set

$$\xi^{n-\sigma-1} := P^{-2/5+\phi}$$

and define

$$\begin{aligned} M_1 &:= \frac{1}{2}\min\left\{Q^{-1/24}, \xi\frac{R_1^{1/4}R_2^{1/8}}{R^{3/16}}, \xi\frac{R_1^{5/14}}{R^{3/14}}\right\}, \\ M_2 &:= \frac{1}{2}\min\left\{Q^{-1/24}, \xi\frac{R_1^{1/4}R_2^{1/8}}{R^{3/16}}, \xi\frac{R_1^{1/5}P^{1/10}}{(RQ)^{1/10}}\right\}, \\ M_3 &:= \frac{1}{2}\min\left\{Q^{-1/24}, \xi\frac{R_1^{1/4}R_2^{1/8}}{R^{3/16}}, \xi\frac{R_1^{1/4}}{P^{1/4}}\right\}, \end{aligned} \tag{6.12}$$

as well as

$$\begin{aligned} \widetilde{M_1} &:= \frac{1}{2}\min\left\{(RtP^4)^{-1/24}, \frac{R_1^{1/4}R_2^{1/8}}{R^{3/16}}, \frac{R_1^{5/14}}{R^{3/14}}\right\}, \\ \widetilde{M_2} &:= \frac{1}{2}\min\left\{(RtP^4)^{-1/24}, \frac{R_1^{1/4}R_2^{1/8}}{R^{3/16}}, \frac{R_1^{1/5}P^{1/10}}{(RQ)^{1/10}}\right\}, \\ \widetilde{M_3} &:= \frac{1}{2}\min\left\{(RtP^4)^{-1/24}, \frac{R_1^{1/4}R_2^{1/8}}{R^{3/16}}, \frac{R_1^{1/4}}{P^{1/4}}\right\}. \end{aligned}$$

We want to remark that the definitions of the $\widetilde{M_i}$ correspond to the definitons of the M_i in the original work of Browning and Heath-Brown. It is in the definition of our M_i that the extra saving achieved by an application of the averaged van der Corput method comes into play. There we have the extra factor ξ which will allow us some extra saving in the range $(RP^2)^{-1} < t \le (RQ)^{-1}$. This saving will be sufficient for the saving of one variable. In this range by Lemmas 6 and 7 we have

$$\begin{aligned} \Sigma_{\mathbf{R}}(R,t,\theta,S_F) &\ll M_0(\theta) + \frac{R_0R_1^{1/2}R_2^{1/2}}{Q}P^{\theta n+\varepsilon}\min\left\{Q^{-1/24}, \xi\mu^{1/2}\right\}^{\theta(n-\sigma-1)} \\ &\ll M_0(\theta) + \frac{R_0R_1^{1/2}R_2^{1/2}}{Q}P^{\theta n+\varepsilon}(M_1+M_2+M_3)^{\theta(n-\sigma-1)}. \end{aligned} \tag{6.13}$$

Analogously, in the range $t \le (RP^2)^{-1}$ we obtain

$$\Sigma_{\mathbf{R}}(R,t,\theta,S_F) \ll \widetilde{M_0}(\theta) + RtR_0R_1^{1/2}R_2^{1/2}P^{\theta n+\varepsilon}(\widetilde{M_1}+\widetilde{M_2}+\widetilde{M_3})^{\theta(n-\sigma-1)}.$$

We will conclude this section by showing that for $\theta = 1$ and $n-\sigma \geq 32$, which from now on we assume for the rest of this section, in the range $t \leq (RP^2)^{-1}$ we have

$$RtR_0R_1^{1/2}R_2^{1/2}P^{n+\varepsilon}(\widetilde{M_1}+\widetilde{M_2}+\widetilde{M_3})^{n-\sigma-1} \ll P^{n-4-\delta} \tag{6.14}$$

for some small $\delta > 0$. We will be able to use this result in the proof of all our theorems. We will basically follow the route taken in [BHB09, §9] to bound the overall contribution from the terms involving $\widetilde{M_1}, \widetilde{M_2}, \widetilde{M_3}$ to the left hand side of (6.14) and we denote this contribution by $\widetilde{\Sigma_1}, \widetilde{\Sigma_2}, \widetilde{\Sigma_3}$. Browning and Heath-Brown use the inequality

$$\min\{A, B, C\} \ll A^{\alpha}B^{\beta}C^{\gamma}, \tag{6.15}$$

which is valid for any $\alpha, \beta, \gamma \geq 0$ provided that $\alpha+\beta+\gamma = 1$ in order to deal with the terms M_i, where the elements of the respective minimae correspond to A, B, C. The main idea is to choose β and γ such that the term $R_0R_1^{1/2}R_2^{1/2}$ in (6.14) can be cancelled. Therefore, we try to choose values for β and γ such that we get the term $R_0R_1^2R_2$ in the numerator and a factor R in the denominator, although it turns out that this is not always possible and one gets a power of R_2 which is slightly smaller than 1. Nevertheless, the terms involving R_0, R_1, R_2 then can be cancelled by the help of (6.7). One achieves a saving by the term $(RtP^4)^{-\alpha/24}$, which is taken to the $(n-\sigma-1)-$th power via (6.14). The saving turns out to be sufficient for a certain range of $t-$values. In the remaining $t-$range one can use the property that the inequalities (6.5) do not both hold on the minor arcs. Since the inequalities (6.7) yield that $R_1^{1/4}R_2^{1/8}R^{-3/16} \leq 2$ the functions $M_i^{n-\sigma-1}$ are decreasing in n. Thus, it is sufficient to consider just the case $n-\sigma = 32$.

We start with the contribution of the term involving $\widetilde{M_1}$. We choose the values $\alpha = 26/31, \beta = 8/93, \gamma = 7/93$. This gives us

$$\widetilde{M_1}^{n-\sigma-1} \leq \frac{R_1^{3/2}R_2^{1/3}}{P^{13/3}R(Rt)^{13/12}}.$$

By using (6.7) we obtain

$$\widetilde{\Sigma_1} \leq (Rt)^{-1/12}\frac{R_0R_1^2R_2^{5/6}}{R}P^{n-13/3+\varepsilon} \leq (Rt)^{-1/12}P^{n-13/3+\varepsilon},$$

whence $\widetilde{\Sigma_1} = \mathrm{O}\left(P^{n-4-\delta}\right)$ for some small $\delta > 0$ provided that $t > R^{-1}P^{-4+\Delta/2}$. We will deal with the remaining values for t later and turn our focus on the term $\widetilde{M_2}$. We apply (6.15) with $\alpha = 49/62, \beta = 4/31, \gamma = 5/62$. By using the same cancellation effects as above this gives us

$$\widetilde{\Sigma_2} \leq (Rt)^{-1/48} P^{n-127/30-\phi/4+\varepsilon},$$

which is clearly $\mathrm{O}\left(P^{n-4-\delta}\right)$ for $t > R^{-1}P^{-4+\Delta/2}$ and $0 \leq \Delta \leq 4/3$, where $\delta > 0$ is a suitable small positive number. Finally we set $\alpha = 21/31, \beta = 8/31, \gamma = 2/31$ in order to deal with the term involving $\widetilde{M_3}$. We get

$$\widetilde{\Sigma_3} \leq (Rt)^{1/8} P^{n-4+\varepsilon} \leq P^{n-4-1/4+\varepsilon},$$

since $t \leq (RP^2)^{-1}$.

It remains to deal with the case that $t \ll R^{-1}P^{-4+\Delta/2}$. Because of being on the minor arcs and the corresponding inequalities (6.5), we may assume that $2R > P^{\Delta}$. We carry out the same arguments as in [BHB09, §9, p. 83f], whence by Lemma 7 we have

$$\begin{aligned}
\Sigma_{\mathbf{R}}(R,t,1,S_F) &\ll \widetilde{M_0} + \max_{\mathbf{R}} RtR_0R_1^{1/2}R_2^{1/2}P^{n+\varepsilon}\left(\frac{R_1^{1/2}R_2^{1/4}}{R^{3/8}}\right)^{(n-\sigma-1)/2} \\
&\ll \widetilde{M_0} + \max_{\mathbf{R}} R_0R_1^{1/2}R_2^{1/2}P^{n-4+\Delta/2+\varepsilon}\left(\frac{1}{R_0^3R_1^2R_2}\right)^{31/16} \\
&= \widetilde{M_0} + \max_{\mathbf{R}} R_0^{-77/16}R_1^{-27/8}R_2^{-23/16}P^{n-4+\Delta/2+\varepsilon} \\
&= \widetilde{M_0} + P^{n-4-15\Delta/16+\varepsilon}
\end{aligned}$$

for $n-\sigma \geq 32$, where the maximum is over all vectors $\mathbf{R} \in \mathbb{R}^3_{\geq 0}$ such that (6.6) holds. This shows (6.14).

7 Proof of Theorem 1

With the results of the previous sections the proof of Theorem 1 now is straightforward. Let $F \in \mathbb{Z}[x_1, \ldots, x_n]$ be the quartic form defining the quartic hypersurface X with $n - \sigma \geq 41$, where $\sigma = \dim \operatorname{sing}_{\mathbb{Q}}(X)$. We want to compute the integral

$$N_{F;\omega}(P) = \int_0^1 S_F(\alpha)\, d\alpha.$$

We follow the steps of §3 to deal with the major arcs. For $\alpha \in \mathfrak{m}(\Delta)$ we apply Dirichlet's approximation Theorem according to §6 and hence we may write $\alpha = a/q + z$ such that

(6.2) do not both hold. Following the arguments of §6 with $\theta = 1$ further, we deduce that in order to complete the proof of Theorem 1 it just remains to establish a bound of the form (6.4). We will have to deal with the two ranges for t which lead to the estimates (6.13) and (6.14). We will show in both cases that $\Sigma_{\mathbf{R}}(R, t, 1, S_F) = \mathrm{O}\left(P^{n-4-\delta}\right)$ for some small $\delta > 0$ and for the relevant values (6.7) of $\mathbf{R}$ under the assumption that (6.5) do not both hold.

We consider the range $t \leq (RP^2)^{-1}$ first. We have already dealt with the contribution to $\Sigma_{\mathbf{R}}(R, t, 1, S_F)$ from those terms involving $\widetilde{M_1}, \widetilde{M_2}, \widetilde{M_3}$ in §6. It remains to deal with the term $\widetilde{M_0}(1)$. According to (6.11) we have

$$\widetilde{M_0}(1) = P^{9n/10+(\sigma+1)/10+\phi/4(n-\sigma-1)-2/5+\phi+\varepsilon}.$$

For $n - \sigma = k$ and $k \geq 41$ we have $\widetilde{M_0}(1) = \mathrm{O}\left(P^{n-4-\delta}\right)$ for some small $\delta > 0$ provided that $\phi < (2k - 74 - \varepsilon)/(5k + 15)$. For $k \geq 41$ this is valid for $\phi < 2/55$ and ε small enough.

We shift our focus to the range $(RP^2)^{-1} < t \leq (RQ)^{-1}$. According to (6.13) we have to find suitable bounds for

$$\Sigma_{\mathbf{R}}(R, t, 1, S_F) \ll M_0(1) + \frac{R_0 R_1^{1/2} R_2^{1/2}}{Q} P^{n+\varepsilon}(M_1 + M_2 + M_3)^{n-\sigma-1}.$$

By (6.10) and (6.11) we have $\widetilde{M_0}(1) = M_0(1)$, so we can bound the term M_0 in the same way as above. It remains to deal with the overall contribution of the terms involving M_1, M_2, M_3 which we will denote by $\Sigma_1, \Sigma_2, \Sigma_3$. To bound those terms we will follow basically the same steps as in the treatment of the $\widetilde{M_i}$ in the previous section and hence we basically follow the arguments of [BHB09, §9]. Since the term $\xi \ll 1$ the terms $M_i^{n-\sigma-1}$ are decreasing functions in n as well, in fact the term ξ provides the extra saving we need compared to the result of Browning and Heath-Brown. Thus, for the proof of Theorem 1 it is sufficient to consider the case $n - \sigma = 41$ which from now on we assume throughout this section.

Let us start with the term involving M_1. We treat the minimum by using the inequality (6.15) with $\alpha = 7/8, \beta = 1/15, \gamma = 7/120$. This gives us

$$M_1^{n-\sigma-1} = M_1^{40} \ll \frac{R_1^{3/2} R_2^{1/3}}{R P^{1/20-\phi/8} Q^{35/24}},$$

whence

$$\Sigma_1 \ll \frac{P^{n-1/20+\phi/8+\varepsilon}R_0R_1^2R_2^{5/6}}{RQ^{59/24}} \ll P^{n-4+1/60-7\phi/3+\varepsilon}.$$

This is sufficient provided that $\phi > 1/140$ and ε small enough.

Let us consider the term involving M_2, next. This time we set $\alpha = 67/80, \beta = 1/10, \gamma = 1/16$. Thus

$$M_2^{n-\sigma-1} \ll \frac{R_1^{3/2}R_2^{1/2}P^{37/200+13\phi/80}}{RQ^{79/48}}.$$

We use the same cancellation effects as above and get

$$\Sigma_2 \ll \frac{P^{n+37/200+13\phi/80+\varepsilon}R_0R_1^2R_2}{RQ^{127/48}} \ll P^{n-4-29/600-149\phi/60+\varepsilon}.$$

This is satisfactory for all $\phi > 0$ and ε sufficiently small. Finally we turn to the term involving M_3 and use the values $\alpha = 3/4, \beta = 1/5, \gamma = 1/20$ which leads to

$$M_3^{n-\sigma-1} \ll \frac{R_1^{5/2}R_2^1}{R^{3/2}P^{3/5-\phi/4}Q^{5/4}}.$$

We obtain

$$\Sigma_3 \ll \frac{P^{n-3/5+\phi/4+\varepsilon}R_0R_1^3R_2^{3/2}}{R^{3/2}Q^{9/4}} \ll P^{n-4-1/5-2\phi+\varepsilon}.$$

This is clearly satisfactory for all $\phi > 0$. Summarizing all conditions on ϕ, we can establish the needed bounds provided that $1/140 < \phi < 2/55$. Since this interval obviously is non-empty this completes the proof of Theorem 1.

8 Proof of Theorem 2

In this section we will prove Theorem 2. Let $G \in \mathbb{Z}[x_1, \ldots, x_n, y]$ given by

$$G(x_1, \ldots, x_n, y) = F(x_1, \ldots, x_n) + my^4 \tag{8.1}$$

be the form defining the quartic hypersurface Y in the statement. We will use the circle method to proof the result by establishing an asymptotic formula for $N_{G;\omega}(P)$ with respect to (3.4). By an abuse of notation we denote the dimension of the singular locus of Y by σ. We follow the steps of §3. Let the non-singular real point $\mathbf{x}_0 \in \mathbb{R}^{n+1}$ we thereby fix be denoted by $\mathbf{x}_0 = (x_{0_1}, \ldots, x_{0_n}, y_0)$. The starting point of the circle method is the identity

$$N_{G;\omega}(P) = \int_0^1 S_G(\alpha)\, d\alpha = \int_0^1 S_F(\alpha) S_1(\alpha)\, d\alpha, \tag{8.2}$$

where S_F is given by (3.5) and F is defined by (8.1). The one-dimensional sum S_1 is given by

$$S_1(\alpha) := \sum_{y \in \mathbb{Z}} \omega_1(y/P) \mathrm{e}\left(\alpha m y^4\right),$$

where $\omega_1(y) = \gamma(\rho^{-1}(y - y_0))$ in the notation (3.2). We adopt the definition of major arcs and minor arcs from §3 and we can treat the major arcs with the methods described there. It remains to establish an estimate of the form (3.6). Here we have to deal with

$$\mathfrak{S} = \lim_{R \to \infty} \sum_{q \leq R} \frac{1}{q^{n+1}} \sum_{\substack{a=1 \\ \gcd(a,q)=1}}^{q} S_{a,q},$$

where

$$S_{a,q} := \sum_{(x_1, \ldots, x_n, y) \bmod q} \mathrm{e}_q\left(aG((\mathbf{x}, y))\right) = \sum_{\mathbf{x} \bmod q} \mathrm{e}_q\left(aF(\mathbf{x})\right) \sum_{y \bmod q} \mathrm{e}_q\left(amy^4\right) = S_{F;a,q} S_{1;a,q},$$

say. It is a well known fact that $S_{1;a,q} \ll q^{3/4}$ ([Dav05, cf. §7]). We write $q = uv$, where u is the squarefree part of q. According to [BHB09, Lemma 7, (6.12)] we have

$$S_{F;a,uv} \ll u^{(n+\sigma+1)/2+\varepsilon} v^{23n/24+(\sigma+1)/24+\varepsilon}, \tag{8.3}$$

whence by an argument similar to [BHB09, p. 88] we have

$$\begin{aligned}
|\mathfrak{S} - \mathfrak{S}(R)| &\ll \sum_{q=uv>R} u^{3/4-n/2+(\sigma+1)/2+\varepsilon} v^{3/4-n/24+(\sigma+1)/24+\varepsilon} \\
&\ll \sum_{q=uv>R} u^{-63/4+\varepsilon} v^{-5/8+\varepsilon} \\
&\ll R^{-1/8+2\varepsilon} \sum_{q=uv>R} u^{-2} v^{-1/2-\varepsilon} \\
&\ll R^{-1/8+2\varepsilon} \sum_{u,v=1}^{\infty} u^{-2} v^{-1/2-\varepsilon},
\end{aligned}$$

since $(n+1) - \sigma \geq 35$. Both sums are convergent, because the number of square-full integers $v \in (V, 2V]$ is $\mathrm{O}\left(V^{1/2}\right)$. Thus (3.6) can be established for any $\phi \in (0, 1/8)$.

We want to note that the contribution to the major arc integral by those $(\mathbf{x}, y) \in \mathbb{Z}^{n+1}$ with $y = 0$ is satisfactory small. This contribution is given by the integral

$$\int_{\mathfrak{M}(\Delta)} S_F(\alpha)\, d\alpha.$$

We apply the steps of §3 in order to deal with it. By using the fact that the corresponding singular series [BHB09, cf. Theorem 2] and singular integral [BHB09, cf. Lemma 23] are both absolutely convergent we may deduce that $\int_{\mathfrak{M}(\Delta)} S_F(\alpha)\, d\alpha = \mathrm{O}\left(P^{n-4}\right)$. In the following we will show that the overall contribution to the minor arcs is satisfactory small for all $(\mathbf{x}, y) \in \mathbb{Z}^{n+1}$ which will finally give us Theorem 2.

On the minor arcs we want to separate the contributions arising from the two exponential sums appearing in (8.2) by using Hölders inequality. We thereby will take S_1 to the fourth power and we will relate the size of the exponential sum to the number $\mathcal{N}(M, \mu)$ of solutions of an equation of the shape

$$\left|y_1^4 + y_2^4 - y_3^4 - y_4^4\right| \leq \mu M^4,$$

with $M < y_i \leq 2M$ for an integer M and some real constant $\mu > 0$. This quantity can be bounded by the following result due to Robert and Sargos [RS06, Theorem 2].

Lemma 8. *With the above notations we have*

$$\mathcal{N}(M, \mu) \ll M^{2+\varepsilon} + \mu M^{4+\varepsilon}.$$

In order to be able to apply this Lemma later, we proceed by splitting up the summation over y in dyadic intervals. We will consider the term

$$S_{1,Y}(\alpha) := \sum_{y \in \{Y+1, Y+2, \ldots, 2Y\}} \omega_1(y/P) \mathrm{e}\left(\alpha m y^4\right). \tag{8.4}$$

We then clearly have

$$\int_{\mathfrak{m}(\Delta)} S_F(\alpha)\, S_1(\alpha)\, d\alpha \ll P^{\varepsilon} \max_{Y} \int_{\mathfrak{m}(\Delta)} \left|S_F(\alpha)\right| \left|S_{1,Y}(\alpha)\right| d\alpha, \tag{8.5}$$

where the maximum is over all those Y such that there exists $y \in \{Y+1, Y+2, \ldots, 2Y\}$ with $\omega_1(y/P) \neq 0$, which from now on we assume. Since all values of y are taken to the fourth power in $S_{1,Y}$ we can assume without loss of generality that $Y > 0$. Our goal is to show that there exists $\delta > 0$ such that

$$\int_{\mathfrak{m}(\Delta)} S_G(\alpha)\, d\alpha = \mathrm{O}\left(P^{(n+1)-4-\delta}\right).$$

For $\alpha \in \mathfrak{m}(\Delta)$ we apply Dirichlet's approximation theorem and write $\alpha = a/q + z$ such that the same conditions on a, q, z as in §3 are satisfied. Again we use the value (6.3) for Q.

Let us now consider the quantity $\Sigma(R, t, 1, S_G)$ given by (5.7) under the assumption that (6.5) do not both hold. Similarily to §7 it is sufficient to show that for relevant R, t

$$\Sigma(R, t, 1, S_G) = \mathrm{O}\left(P^{(n+1)-4-\delta}\right),$$

since summing up over the dyadic intervals for R, t then completes the proof.

For given $\mathbf{R} = (R_0, R_1, R_2) \in \mathbb{R}^3_{\geq 0}$ we want to study the overall contribution from those $q = bc^2d$ satisfying the conditions (6.6) and (6.7), which we denote by $\Sigma_{\mathbf{R}}(R, t, 1, S_G)$. An application of Hölder's inequality and (8.5) yields

$$\Sigma(R, t, 1, S_G) \ll P^\varepsilon \max_{\mathbf{R}, Y} \left\{\Sigma_{\mathbf{R}}(R, t, 4/3, S_F)^{3/4} \Sigma_{\mathbf{R}}(R, t, 4, S_{1,Y})^{1/4}\right\}, \tag{8.6}$$

where the maximum is over all vectors $\mathbf{R} \in \mathbb{R}^3_{\geq 0}$ satisfying (6.6) and (6.7).

We shall start by considering the term $\Sigma_{\mathbf{R}}(R, t, 4, S_{1,Y})$. We complete the sum over a to the range $1 \leq a \leq q$ and introduce a non-negative integration kernel such that

$$\Sigma_{\mathbf{R}}(R, t, 4, S_{1,Y}) \ll \sum_{q \in \mathcal{Q}_{\mathbf{R}}(R)} \sum_a \int_{-\infty}^{\infty} |S_{1,Y}(a/q + z)|^4 \frac{\sin^2(\pi z/(4t))}{(\pi z/(4t))^2}\, dz.$$

We have

$$|S_{1,Y}(a/q + z)|^4 = \sum_{y_1, \ldots, y_4} \omega_{1,4}(\mathbf{y}) \mathrm{e}\left((a/q + z)(m(y_1^4 + y_2^4 - y_3^4 - y_4^4))\right),$$

where the summation range for the y_i is given by the definition (8.4) of $S_{1,Y}$ and where $\omega_{1,4}(\mathbf{y}) = \prod_{1 \leq i \leq 4} \omega_1(y_i)$. By sorting this according to the value of $\eta_m(\mathbf{y}) := m(y_1^4 + y_2^4 - y_3^4 - y_4^4)$ we can bound $\Sigma_{\mathbf{R}}(R, t, 4, S_{1,Y})$ by

$$\sum_{q \in \mathcal{Q}_{\mathbf{R}}(R)} \sum_{N \in \mathbb{Z}} \sum_{\eta_m(\mathbf{y}) = N} \omega_{1,4}(\mathbf{y}/P) \sum_a \mathrm{e}\left(\tfrac{aN}{q}\right) \int_{-\infty}^{\infty} \mathrm{e}\left(zN\right) \frac{\sin^2(\pi z/(4t))}{(\pi z/(4t))^2}\, dz,$$

where the summation conditions on the y_i are as before. On performing the summation over a we obtain

$$\Sigma_{\mathbf{R}}(R,t,4,S_{1,Y}) \ll R \sum_{q\in\mathcal{Q}_{\mathbf{R}}(R)} \sum_{N\in\mathbb{Z}} \sum_{\substack{\eta_m(\mathbf{y})=N \\ N\equiv 0 \bmod q}} \omega_{1,4}(\mathbf{y}/P) \int_{-\infty}^{\infty} \mathrm{e}\,(zN)\, \frac{\sin^2(\pi z/(4t))}{(\pi z/(4t))^2}\, dz.$$

The contribution arising from the value $N=0$ is

$$\begin{aligned}
&\ll Rt \sum_{q\in\mathcal{Q}_{\mathbf{R}}(R)} \sum_{\substack{Y<y_i\le 2Y \\ y_1^4+y_2^4-y_3^4-y_4^4=0}} \omega_{1,4}(\mathbf{y}/P) \\
&\ll Rt \sum_{q\in\mathcal{Q}_{\mathbf{R}}(R)} \#\left\{Y<y_i\le 2Y : y_1^4+y_2^4-y_3^4-y_4^4=0\right\} \\
&\ll R_0R_1^{1/2}R_2^{1/2}RtP^{2+\varepsilon} \qquad (8.7)
\end{aligned}$$

where we have used Lemma 5 and the fact that $Y \ll P$. For the other values of N we may calculate the well known Fourier integral (cf. [Dav05, chapter 20]). Since $\omega_{1,4}(\mathbf{y}/P) \ll 1$ the overall contribution from those N is

$$\begin{aligned}
&\ll Rt \sum_{q\in\mathcal{Q}_{\mathbf{R}}(R)} \sum_{N\in\mathbb{Z}} \sum_{\substack{\eta_m(\mathbf{y})=N \\ 0\ne N\equiv 0 \bmod q}} \omega_{1,4}(\mathbf{y}/P) \max\{0, 1-|4Nt|\} \\
&\ll Rt \sum_{q\in\mathcal{Q}_{\mathbf{R}}(R)} \mathcal{N}^*(q,Y,t^{-1}),
\end{aligned}$$

with

$$\mathcal{N}^*(k,M,\delta) = \#\left\{ M<y_i\le 2M : \begin{array}{l} 0<\left|y_1^4+y_2^4-y_3^4-y_4^4\right| \ll \delta, \\ m(y_1^4+y_2^4-y_3^4-y_4^4) \equiv 0 \pmod{k} \end{array} \right\}.$$

By standard estimates for the divisor function and Lemma 8 we get

$$\begin{aligned}
Rt \sum_{q\in\mathcal{Q}_{\mathbf{R}}(R)} \mathcal{N}^*(q,Y,t^{-1}) &\ll Rt \sum_{\substack{0<\left|y_1^4+y_2^4-y_3^4-y_4^4\right|\ll t^{-1} \\ Y<y_i\le 2Y}} \#\left\{R<q\le 2R : q \mid \eta_m(\mathbf{y})\right\} \\
&\ll RtP^{\varepsilon}\mathcal{N}(Y,(tY^4)^{-1}) \\
&\ll RtP^{2+\varepsilon} + RP^{\varepsilon}, \qquad (8.8)
\end{aligned}$$

since $Y \ll P$. Combining (8.7) and (8.8) for those t under consideration we have

$$\Sigma_{\mathbf{R}}(R,t,4,S_{1,Y})^{1/4} \ll (Rt)^{1/4}R_0^{1/4}R_1^{1/8}R_2^{1/8}P^{1/2+\varepsilon} + R^{1/4}P^{\varepsilon} =: \Psi_1+\Psi_2, \qquad (8.9)$$

say, since $R_i \ge 1/2$ by (6.7).

We will consider the two ranges for t that arise through an application of Lemma 6 and Lemma 7 separately and start with the range $(RP^2)^{-1} \leq t \leq (RQ)^{-1}$. The following argument is in the spirit of [BHB09, §9]. By the same arguments leading to (6.13) for those t we have

$$\Sigma_{\mathbf{R}}(R,t,4/3,S_F)^{3/4} \ll M_0(4/3)^{3/4} + \frac{R_0^{3/4}R_1^{3/8}R_2^{3/8}}{Q^{3/4}}P^{n+\varepsilon}(M_1+M_2+M_3)^{n-\sigma-1}, \quad (8.10)$$

where M_0, M_1, M_2, M_3 are defined by (6.10) and (6.12). Inserting (8.9) and (8.10) into (8.6), we finally can bound $\Sigma(R,t,1,S_G)$ by

$$\max_{\mathbf{R}} \left(M_0(4/3)^{3/4} + \frac{R_0^{3/4}R_1^{3/8}R_2^{3/8}}{Q^{3/4}}P^{n+\varepsilon}(M_1+M_2+M_3)^{n-\sigma-1} \right)(\Psi_1+\Psi_2). \quad (8.11)$$

First we will consider the contribution from the term involving M_0. We have $\Psi_1+\Psi_2 \ll P^{1/2+\varepsilon}$ since we have $t \leq (RQ)^{-1}$ and (6.7). Thus, by our choice (6.3) for Q and the definition (6.10) for M_0 we get

$$M_0(4/3)^{3/4}P^{1/2+\varepsilon} \ll P^{9n/10+(\sigma+2)/10+\phi(n/4-\sigma/4+3/4)+\varepsilon}. \quad (8.12)$$

For $n-\sigma = k$ and $k \geq 34$ the exponent is strictly less than $(n+1)-4$ provided that $\phi < (2k-64+\varepsilon)/(15+5k)$. For the k under consideration this is fulfilled for all $\phi < 4/185$ and ε sufficiently small.

We denote the total contribution to (8.11) by the term involving M_i and Ψ_j by $\Sigma_{i,j}$ for $i = 1,2,3$ and $j = 1,2$. We will use the same arguments leading to the bounds for the terms Σ_i in §7. Recalling that the $M_i^{n-\sigma-1}$ are decreasing functions in n, it is sufficient to evaluate them just for $(n+1)-\sigma = 35$ which from now on we assume. Like in the proof of Theorem 1 we apply inequality (6.15) to get suitable bounds for the M_i. By similar considerations as in the proof of Theorem 1 we choose optimal values for the exponents. We set

$$(\alpha_{1,1},\beta_{1,1},\gamma_{1,1}) = (28/33, 8/99, 7/99) \text{ for } \Sigma_{1,1}$$
$$(\alpha_{2,1},\beta_{2,1},\gamma_{2,1}) = (53/66, 4/33, 5/66) \text{ for } \Sigma_{2,1}$$
$$(\alpha_{3,1},\beta_{3,1},\gamma_{3,1}) = (23/33, 8/33, 2/33) \text{ for } \Sigma_{3,1}$$

to get

$$\Sigma_{1,1} \ll P^{(n+1)-4-3/110-133\phi/66+\varepsilon}$$
$$\Sigma_{2,1} \ll P^{(n+1)-4-21/220-1139\phi/528+\varepsilon}$$
$$\Sigma_{3,1} \ll P^{(n+1)-4-14/55-437\phi/264+\varepsilon},$$

for $(n+1)-\sigma = 35$ which clearly is satisfactory for any $\phi > 0$.

Let us now consider the terms involving Ψ_2. The goal for the term involving M_1 is to cancel the terms R_0, R_1, R_2 as well as some of the extra factor $R^{1/4}$ we now have in the numerator arising from Ψ_2. It turns out that the choice $(\alpha_{2,1}, \beta_{2,1}, \gamma_{2,1}) = (101/132, 2/11, 7/132)$ for an application of (6.15) is satisfactory. We have

$$\Sigma_{1,2} \ll \frac{R^{1/4}R_0^{3/4}R_1^{5/2}R_2^{9/8}P^{n-31/330+31\phi/132+\varepsilon}}{R^{3/2}Q^{173/96}} \ll P^{(n+1)-4+1/44-1655\phi/1056+\varepsilon}, \quad (8.13)$$

since $R \le Q$. The bound (8.13) is satisfactory for all $\phi > 24/1655$ and ε sufficiently small.

To get a bound for the term $\Sigma_{2,2}$ we set $(\alpha_{2,2}, \beta_{2,2}, \gamma_{2,2}) = (193/264, 7/33, 5/88)$, whence

$$\Sigma_{2,2} \ll \frac{R^{1/4}R_0^{3/4}R_1^{5/2}R_2^{5/4}P^{n+211/2640+71\phi/264+\varepsilon}}{R^{3/2}Q^{373/192}} \ll P^{(n+1)-4-5/176-3535\phi/2112+\varepsilon},$$

which is clearly satisfactory for all $\phi > 0$ and ε sufficiently small.

Finally we set $(\alpha_{3,2}, \beta_{3,2}, \gamma_{3,2}) = (15/22, 8/33, 5/66)$ and obtain

$$\Sigma_{3,2} \ll \frac{R^{1/4}R_0^{3/4}R_1^{3}R_2^{11/8}P^{n-331/440+7\phi/22+\varepsilon}}{R^{3/2}Q^{27/16}} \ll P^{(n+1)-4-23/440-197\phi/176+\varepsilon},$$

which is satisfactory for any $\phi > 0$ provided that ε is sufficiently small.

It remains to deal with the range $t \le (RP^2)^{-1}$ under the condition that the inequalities (6.5) do not both hold. By Hölder's inequality we have

$$\Sigma_{\mathbf{R}}(R,t,1,S_G) \ll P^{\varepsilon}\Sigma_{\mathbf{R}}(R,t,4/3,S_F)^{3/4}\Sigma_{\mathbf{R}}(R,t,4,S_{1,Y})^{1/4}. \quad (8.14)$$

By (6.14), the first factor on the right hand side of (8.14) is bounded by

$$\widetilde{M_0}(4/3)^{3/4} + (Rt)^{3/4}R_0^{3/4}R_1^{3/8}R_2^{3/8}P^{n+\varepsilon}(\widetilde{M_1}+\widetilde{M_2}+\widetilde{M_3})^{n-\sigma-1} \quad (8.15)$$

where $\widetilde{M_0}$ is given by (6.11). We turn to the second factor and use the estimate (8.9). We have $\Psi_1 + \Psi_2 \ll P^{2/5+\phi/4}$ by (6.7) and since $t < (RP)^2$. We get another bound for this factor by using the trivial bound $S_1 \ll P$ and summing over a and q with the help of Lemma 5. We get

$$\Sigma_{\mathbf{R}}(\mathbf{R},t,4,S_{1,Y})^{1/4} \ll \min\left\{P^{2/5+\phi/4+\varepsilon}, (Rt)^{1/4}PR_0^{1/4}R_1^{1/8}R_2^{1/8}\right\}. \quad (8.16)$$

Inserting (8.15) and (8.16) into (8.14), we have that $\Sigma_{\mathbf{R}}(R, t, 1, S_G)$ is bounded by

$$\widetilde{M_0}(4/3)^{3/4}P^{2/5+\phi/4+\varepsilon} + RtR_0R_1^{1/2}R_2^{1/2}P^{(n+1)+\varepsilon}(\widetilde{M_1} + \widetilde{M_2} + \widetilde{M_3})^{n-\sigma-1}.$$

By the arguments at the end of §6 we may deduce that there exists $\delta > 0$ such that

$$\Sigma_{\mathbf{R}}(R, t, 1, S_G) \ll \widetilde{M_0}(4/3)^{3/4}P^{1/2+\varepsilon} + P^{(n+1)-4-\delta+\varepsilon}.$$

Thus, it remains to deal with the term $\widetilde{M_0}(4/3)^{3/4}P^{2/5+\phi/4+\varepsilon}$. By (6.11) this is

$$\widetilde{M_0}(4/3)^{3/4}P^{2/5+\phi/4+\varepsilon} = P^{9n/10+(\sigma+2)/10+\phi/4(n-\sigma+3)+\varepsilon} = M_0(4/3)^{3/4}P^{1/2+\varepsilon},$$

whence the term involving $\widetilde{M_0}$ can be bound by (8.12).

By summarizing the conditions on ϕ, we see that all of them are fulfilled provided that ε is sufficiently small and

$$24/1655 < \phi < 4/185.$$

Since this interval is non-empty this completes the proof of Theorem 2.

9 Proof of Theorem 3

In this section we will consider quartic forms that split off a form in $l \geq 2$ variables. In the case that $l = 2$ we can use a result by Wooley [Woo99], who established a version of Weyl's inequality and Hua's Lemma in the setting of binary forms. Before we can describe his results, we require some notation. Let $\Phi \in \mathbb{Z}[u, v]$ be a binary quartic form. We say that Φ is degenerate if there exist $\lambda, \mu \in \mathbb{C}$ such that $\Phi(v, w) = (\lambda v + \mu w)^4$. One can easily show that each such form Φ can be written in the form $\Phi(v, w) = a(bv + cw)^4$ for some integers a, b, c. We remark that each degenerate binary form therefore has a non-trivial integer zero. We are now ready to present Wooley's version of Hua's Lemma (cf. [Woo99, Theorem 2]) in the the setting of binary quartic forms.

Lemma 9. *Let $\Phi \in \mathbb{Z}[u, v]$ be a non-degenerate binary quartic form. Then we have*

$$\int_0^1 \left| \sum_{0 \leq u, v \leq P} \mathrm{e}\left(\alpha \Phi(v, w)\right) \right|^{2^{j-1}} d\alpha \ll P^{2^j - j + \varepsilon}$$

for any integer $1 \leq j \leq 4$.

As usual we set

$$S_{\Phi}(\alpha) := \sum_{u,v\in\mathbb{Z}} \omega((u,v)/P)\mathrm{e}(\alpha\Phi(u,v)),$$

where $\omega \in \mathcal{W}_2$. From Lemma 9 we immediately get the following Corollary.

Corollary 2. *We have*

$$\int_0^1 |S_{\Phi}(\alpha)|^{2^{j-1}}\, d\alpha \ll P^{2^j - j+\varepsilon}$$

for any integer $1 \le j \le 4$.

Proof. By orthogonality we have

$$\int_0^1 |S_{\Phi}(\alpha)|^{2^{j-1}}\, d\alpha \quad = \quad \sum_{u_i,v_j} \omega((u_1,v_1)/P)\cdot\ldots\cdot\omega((u_{2j},v_{2j})/P) \ll \sum_{u_i,v_j} 1$$

where the sum is over all $u_1,\ldots,u_{2^{j-1}}, v_1,\ldots,v_{2^{j-1}} \ll P$ such that $\Phi(u_1,v_1)+\cdots+\Phi(u_{2^{j-2}},v_{2^{j-2}}) = \Phi(u_{2^{j-2}+1},v_{2^{j-2}+1})+\cdots+\Phi(u_{2^{j-1}},v_{2^{j-1}})$. The right hand side of (9.1) can be rewritten in integral form and estimated by Wooley's result, which completes the proof of the Corollary. □

Since a version of Hua's Lemma for forms in more than two variables is unknown we shall establish a Lemma which allows us to treat $l-2$ variables trivially and get a saving from the remaining two variables. The following result is based on Wooley's work [Woo99].

Lemma 10. *Let* $\Psi(y_1,\ldots,y_l) \in \mathbb{Z}[y_1,\ldots,y_l]$ *be a quartic form. Then either* Ψ *has a non-trivial zero or we have*

$$\mathcal{I}_j^{(l)} := \int_0^1 \left| \sum_{\mathbf{y}\in\mathbb{Z}^l} \omega(\mathbf{y}/P)\mathrm{e}(\alpha\Psi(\mathbf{y})) \right|^{2^{j-1}} d\alpha \ll P^{j(l-1)+\varepsilon} \tag{9.1}$$

for $j=1,2$.

Proof. Let us suppose that

$$\Psi(y_1,\ldots,y_l) = \sum_{i_1,i_2,i_3,i_4=1}^{l} \psi_{i_1,i_2,i_3,i_4} y_{i_1}y_{i_2}y_{i_3}y_{i_4},$$

with integer coefficients ψ_{i_1,i_2,i_3,i_4} which are symmetric in the indices i_1,i_2,i_3,i_4. Let us assume that in each non-vanishing monomial $\psi_{i_1,i_2,i_3,i_4}y_{i_1}y_{i_2}y_{i_3}y_{i_4}$ of Ψ we have that at least three of the indices i_k are pairwise disjoint. Then we clearly have $\Psi(1,0,\dots,0)=0$, which is sufficient for the proof of the Lemma. So from now on, by reordering the indices if necessary, we may assume that we can write

$$\Psi(y_1,\dots,y_l)=\Phi(y_1,y_2)+\varphi(y_1,\dots,y_l), \tag{9.2}$$

such that in each non-vanishing monomial of φ we have at least one factor y_i with $i\notin\{1,2\}$. If there exist $(y_1^*,y_2^*)\in\mathbb{Z}^2\backslash\{\mathbf{0}\}$ with $\Phi(y_1^*,y_2^*)=0$, we have $\Psi(y_1^*,y_2^*,0,\dots,0)=0$, which is satisfactory for the proof of the Lemma. Thus, from now on we assume that Ψ is of the shape (9.2) where $\Phi(y_1,y_2)=0$ is just trivially solvable, whence Φ is a non-degenerate quartic binary form. In this case we will establish the bound (9.1). The case $j=1$ follows from the case $j=2$ through an application of Cauchy's inequality. Let us assume $j=2$. By orthogonality we have

$$\mathcal{I}_2^{(l)} = \sum_{\substack{\mathbf{y}_1,\mathbf{y}_2\in\mathbb{Z}^l\\ \Psi(\mathbf{y}_1)=\Psi(\mathbf{y}_2)}} \omega(\mathbf{y}_1/P)\omega(\mathbf{y}_2/P) \ll \#\left\{(\mathbf{y}_1,\mathbf{y}_2)\ll P:\Psi(\mathbf{y}_1)=\Psi(\mathbf{y}_2)\right\}, \tag{9.3}$$

since our weight function is bounded. For each $h\in\mathbb{Z}$ we set

$$\begin{aligned} r^*(h) &:= \#\left\{(u,v,w_1,\dots,w_{l-2})\ll P:\Psi(u,v,\mathbf{w})=\Phi(u,v)+\varphi(u,v,\mathbf{w})=h\right\}\\ &= \sum_{|\mathbf{w}|\ll P}\#\left\{u,v\ll P:\Psi(u,v,\mathbf{w})=\Phi(u,v)+\varphi(u,v,\mathbf{w})=h\right\}\\ &= \sum_{|\mathbf{w}|\ll P} r^*_{\mathbf{w}}(h), \end{aligned}$$

say. Thus, by (9.3) and an application of Cauchy's inequality we have

$$\mathcal{I}_2^{(l)} \ll \sum_{h\in\mathbb{Z}} r^*(h)^2 \ll \sum_{h\in\mathbb{Z}}\left(\sum_{|\mathbf{w}|\ll P} r^*_{\mathbf{w}}(h)\right)^2 \ll P^{l-2}\sum_{|\mathbf{w}|\ll P}\sum_{h\in\mathbb{Z}} r^*_{\mathbf{w}}(h)^2. \tag{9.4}$$

Before we proceed we introduce further notation. We set

$$r(h):=\#\left\{u,v\in\mathbb{Z},|(u,v)|\ll P:\Phi(u,v)=h\right\}.$$

Wooley [Woo99, cf. proof of Lemma 5.1] showed that for any non-zero h we have $r(h)=\mathrm{O}\left(P^{\varepsilon}\right)$. We will use this result to show that

$$\sum_{h\in\mathbb{Z}} r^*_{\mathbf{w}}(h)^2 \ll P^{2+\varepsilon}$$

uniformly in $\mathbf{w}$, since the statement of the lemma then follows by performing the summation over $\mathbf{w}$ in (9.4). Let us consider the case $h = 0$ first. We have

$$\begin{aligned}
r^*_{\mathbf{w}}(0)^2 &= (\#\{u, v \ll P : \Psi(u, v, \mathbf{w}) = \Phi(u, v) + \varphi(u, v, \mathbf{w}) = 0\})^2 \\
&\ll (\#\{u, v \ll P : \Phi(u, v) = 0, \varphi(u, v, \mathbf{w}) = 0\})^2 \\
&\quad + (\#\{u, v \ll P : \Phi(u, v) = -\varphi(u, v, \mathbf{w}), \Phi(u, v) \neq 0\})^2 \\
&\ll r(0)^2 + \sum_{h \neq 0} r(h)^2 \\
&\ll r(0)^2 + \left(\max_{h \neq 0} r(h)\right) \sum_{h \neq 0} r(h) \\
&\ll 1 + P^{2+\varepsilon},
\end{aligned}$$

due to Wooley's result and since we have assumed that Φ has no non-trivial zero. We now consider the the case $h \neq 0$. We have

$$\begin{aligned}
\sum_{h \neq 0} r^*_{\mathbf{w}}(h)^2 &= \sum_{h \neq 0} (\#\{u, v \ll P : \Psi(u, v, \mathbf{w}) = \Phi(u, v) + \varphi(u, v, \mathbf{w}) = h\})^2 \\
&\ll \sum_{h \neq 0} (\#\{u, v \ll P : \Phi(u, v) = h - \varphi(u, v, \mathbf{w}), h - \varphi(u, v, \mathbf{w}) = 0\})^2 \\
&\quad + \sum_{h \neq 0} (\#\{u, v \ll P : \Phi(u, v) = h - \varphi(u, v, \mathbf{w}), h - \varphi(u, v, \mathbf{w}) \neq 0\})^2 \\
&= \mathcal{P}_1 + \mathcal{P}_2,
\end{aligned}$$

say. By our assumption in $\mathcal{P}_1$ the condition $\Phi(u, v) = 0$ is only met for $(u, v) = (0, 0)$. Since there is at most one value for h such that $h - \varphi(u, v, \mathbf{w}) = 0$ we have $\mathcal{P}_1 = \mathrm{O}(1)$. Furthermore we have

$$\mathcal{P}_2 \ll \sum_{h \neq 0} r(h)^2 \ll \left(\max_{h \neq 0} r(h)\right) \sum_{h \neq 0} r(h) \ll P^{2+\varepsilon},$$

which completes the proof of the Lemma. □

In order to deal with the singular series of our theorem we will state a version of Weyl's inequality which is due to [Woo00].

Lemma 11. *Let $\Phi \in \mathbb{Z}[u, v]$ be a non-degenerate binary quartic form. Let $\varphi(u, v) \in \mathbb{R}[x, y]$ be any polynomial of total degree at most three. Suppose that $q \in \mathbb{Z}$ is sufficiently large in terms of the coefficients of Φ. Then for each $\varepsilon > 0$ one has*

$$\sum_{u, v \bmod q} \mathrm{e}\left(a(\Phi(u, v) + \varphi(u, v))/q\right) \ll q^{7/4+\varepsilon},$$

where the implicit constant depends at most on the coefficients of Φ and ε but is uniform in the coefficients of φ.

We are now ready to prove Theorem 3 for a quartic hypersurface Z defined by a quartic form $F(x_1, \ldots, x_k, y_1, \ldots, y_l) = F_1(\mathbf{x}) + F_2(\mathbf{y})$ in $n = k + l$ variables provided that the conditions as stated in the theorem are met. For our proof we will use Lemma 10 for all values $l \geq 2$. Nevertheless we want to remark that in the case $l = 2$ Corollary 2 with $j = 3$ would allow us better bounds. Since theses bounds turn out to be not good enough to save a further variable, we will deal with all values of l in the described way.

We write $\sigma := \dim \operatorname{sing}_{\mathbb{Q}}(Z_1)$. Following the proof of Lemma 10, by reordering the indices if necessary, we may assume that $F_2(\mathbf{y}) = \Phi(y_1, y_2) + \varphi(\mathbf{y})$ where Φ is non-degenerate, since otherwise there exists a non-trivial zero $\mathbf{y}^*$ of F_2 and $F(0, \ldots, 0, \mathbf{y}^*) = 0$. We follow the steps of §3 to activate the circle method and to deal with the major arcs. It remains to establish an estimate of the shape (3.6). Here we have to deal with

$$\mathfrak{S} = \lim_{R \to \infty} \sum_{q \leq R} \frac{1}{q^{k+l}} \sum_{\substack{a=1 \\ \gcd(a,q)=1}}^{q} S_{a,q},$$

where

$$S_{a,q} = \sum_{(\mathbf{x},\mathbf{y}) \bmod q} \mathrm{e}_q\left(aF((\mathbf{x},\mathbf{y}))\right) = \sum_{\mathbf{x} \bmod q} \mathrm{e}_q\left(aF_1(\mathbf{x})\right) \sum_{\mathbf{y} \bmod q} \mathrm{e}_q\left(aF_2(\mathbf{y})\right) = S_{F_1;a,q} S_{F_2;a,q},$$

say. By Lemma 11 we have

$$\begin{aligned} S_{F_2;a,q} &\ll \sum_{y_3,\ldots,y_l \bmod q} \sum_{y_1,y_2 \bmod q} \mathrm{e}_q\left(a(\Phi(y_1,y_2) + \varphi(\mathbf{y}))\right) \\ &\ll \sum_{y_3,\ldots,y_l} q^{7/4+\varepsilon} \\ &\ll q^{(l-2)+7/4+\varepsilon}. \end{aligned} \tag{9.5}$$

Writing $q = uv$, where u is the squarefree part of q, we get a bound for $S_{F_1;a,q}$ via (8.3). Together with (9.5) we may conclude that

$$\begin{aligned}
|\mathfrak{S} - \mathfrak{S}(R)| &\ll \sum_{q=uv>R} u^{3/4-k/2+(\sigma+1)/2+\varepsilon} v^{3/4-k/24+(\sigma+1)/24+\varepsilon} \\
&\ll \sum_{q=uv>R} u^{-65/4+\varepsilon} v^{-2/3+\varepsilon} \\
&\ll R^{-1/6+2\varepsilon} \sum_{q=uv>R} u^{-2} v^{-1/2-\varepsilon} \\
&\ll R^{-1/6+2\varepsilon} \sum_{u,v=1}^{\infty} u^{-2} v^{-1/2-\varepsilon},
\end{aligned}$$

since $k - \sigma \geq 35$. By the same arguments as in §8 both sums are convergent, and (3.6) can be established for any $\phi \in (0, 1/6)$, which completes the treatment of the major arcs.

The treatment of the minor arcs now is routine, since we basically follow again the arguments of [BHB09, §9]. Following the steps of §6 we have to establish a bound of the form (6.4) to get a satisfactory contribution from the minor arc integral. We therefore have to consider the quantity $\Sigma_{\mathbf{R}}(R, t, 1, S_F)$ for those $\mathbf{R}$ satisfying (6.7). An application of Hölder's inequality yields

$$\Sigma_{\mathbf{R}}(R, t, 1, S_F) \ll (\Sigma_{\mathbf{R}}(R, t, 2, S_{F_1}))^{1/2} (\Sigma_{\mathbf{R}}(R, t, 2, S_{F_2}))^{1/2}, \tag{9.6}$$

and after completing the integration range of the second factor we can apply Lemma 10 with $j = 2$, which turns out to be the optimal choice for j. Thus we have

$$\Sigma_{\mathbf{R}}(R, t, 2, S_{F_2})^{1/2} \ll \left(\int_0^1 |S_{F_2}(\alpha)|^2 \, d\alpha \right)^{1/2} \ll P^{l-1+\varepsilon}. \tag{9.7}$$

We deal with the first factor of (9.6) by considering the two ranges for t that arise through an application of Lemma 6 and Lemma 7. Let us start with the range

$$(RP^2)^{-1} < t \leq (RQ)^{-1}.$$

By the same arguments as in the previous sections, (9.6) and (9.7) we have

$$\begin{aligned}
\Sigma_{\mathbf{R}}(R, t, 1, S_F) &\ll (\Sigma_{\mathbf{R}}(R, t, 2, S_{F_1}))^{1/2} P^{l-1+\varepsilon} \\
&\ll M_0(2)^{1/2} P^{l-1+\varepsilon} + \frac{R_0^{1/2} R_1^{1/4} R_2^{1/4} P^{k+l-1+\varepsilon}}{Q^{1/2}} (M_1 + M_2 + M_3)^{k-\sigma-1},
\end{aligned}$$

where the M_i are defined as in §7 but with n replaced by k. By (6.10) we have that

$$M_0(2)^{1/2}P^{l-1+\varepsilon} \ll P^{(k+l)-4-4/5+19\phi/2+\varepsilon},$$

for $k-\sigma \geq 35$. This is $\mathrm{O}\left(P^{(k+l)-4-\delta}\right)$ for some small $\delta > 0$ provided that $\phi < 8/95$ and ε sufficiently small.

As usual we denote the overall contribution of the terms involving M_i, $i = 1, 2, 3$ by Σ_i. This time we use (6.15) with

$$\begin{aligned}
(\alpha_1, \beta_1, \gamma_1) &= (63/68, 2/51, 7/204) \text{ for } \Sigma_1\\
(\alpha_2, \beta_2, \gamma_2) &= (123/136, 1/17, 5/136) \text{ for } \Sigma_2\\
(\alpha_3, \beta_3, \gamma_3) &= (29/34, 2/17, 1/34) \text{ for } \Sigma_3
\end{aligned}$$

to get satisfactory bounds for the terms M_1, M_2, M_3. Then we have

$$\begin{aligned}
\Sigma_1 &\ll P^{(k+l)-4+6/85-473\phi/272+\varepsilon}\\
\Sigma_2 &\ll P^{(k+l)-4+5/136-985\phi/544+\varepsilon}\\
\Sigma_3 &\ll P^{(k+l)-4-43/1020-637\phi/408+\varepsilon},
\end{aligned}$$

for $k-\sigma \geq 35$ which is satisfactory provided that $\phi > 96/2365$ and ε sufficiently small. The latter condition on ϕ comes from the term involving M_1.

To deal with the range $t \leq (RP^2)^{-1}$ we start with the inequality

$$\Sigma_{\mathbf{R}}(R, t, 1, S_F) \ll \Sigma_{\mathbf{R}}(R, t, 2, S_{F_1})^{1/2}\Sigma_{\mathbf{R}}(R, t, 2, S_{F_2})^{1/2}.$$

By the same arguments leading to (8.15) we can bound the first factor in the following way.

$$\Sigma_{\mathbf{R}}(R, t, 2, S_{F_1})^{1/2} \ll \widetilde{M}_0(2)^{1/2} + (Rt)^{1/2}R_0^{1/2}R_1^{1/4}R_2^{1/4}P^{k+\varepsilon}(\widetilde{M}_1 + \widetilde{M}_2 + \widetilde{M}_3)^{k-\sigma-1}.$$

The second factor can be bound in two ways. We can either use Lemma 10 or we can use the trivial bound $S_{F_2} \ll P^l$ and perform the integration as well as the summation over a, q with the help of Lemma 5. This gives us

$$(\Sigma_{\mathbf{R}}(R, t, 2, S_{F_2}))^{1/2} \ll \min\left\{P^{l-1+\varepsilon}, (Rt)^{1/2}R_0^{1/2}R_1^{1/4}R_2^{1/4}P^l\right\}.$$

Thus, according to section §6 we have

$$\begin{aligned}\Sigma_{\mathbf{R}}(R,t,1,S_F) &\ll \widetilde{M_0}(2)^{1/2}P^{l-1+\varepsilon}+RtR_0R_1^{1/2}R_2^{1/2}P^{k+l+\varepsilon}(\widetilde{M_1}+\widetilde{M_2}+\widetilde{M_3})^{k-\sigma-1}\\ &\ll \widetilde{M_0}(2)^{1/2}P^{l-1+\varepsilon}+P^{(k+l)-4-\delta}\end{aligned}$$

for some $\delta > 0$. We have

$$\widetilde{M_0}(2)^{1/2}P^{l-1+\varepsilon} \ll P^{(k+l)-4-3/5+9\phi+\varepsilon},$$

for $k-\sigma \geq 35$. This is $\mathrm{O}\left(P^{(k+l)-4-\delta}\right)$ for some small $\delta > 0$ provided $\phi < 1/15$.

All conditions on ϕ are met provided that $96/2365 < \phi < 1/15$. Since this interval is non-empty this completes the proof of Theorem 3.

Bibliography

[BHB09] T. D. Browning and D. R. Heath-Brown. Rational points on quartic hypersurfaces. *J. Reine Angew. Math.*, 629:37–88, 2009.

[Bir62] B. J. Birch. Forms in many variables. *Proc. Roy. Soc. Ser. A*, 265:245–263, 1961/1962.

[Bro09] T. D. Browning. *Quantitative arithmetic of projective varieties*, volume 277 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2009.

[Bro10] T. D. Browning. Rational points on cubic hypersurfaces that split off a form. *Compos. Math.*, 146(4):853–885, 2010. With an appendix by J.-L. Colliot-Thélène.

[Dav05] H. Davenport. *Analytic methods for Diophantine equations and Diophantine inequalities*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2005. With a foreword by R. C. Vaughan, D. R. Heath-Brown and D. E. Freeman, Edited and prepared for publication by T. D. Browning.

[HB07] D. R. Heath-Brown. Cubic forms in 14 variables. *Invent. Math.*, 170(1):199–230, 2007.

[RS06] O. Robert and P. Sargos. Three-dimensional exponential sums with monomials. *J. Reine Angew. Math.*, 591:1–20, 2006.

[Woo99] T. D. Wooley. On Weyl's inequality, Hua's lemma, and exponential sums over binary forms. *Duke Math. J.*, 100(3):373–423, 1999.

[Woo00] T. D. Wooley. Weyl's inequality and exponential sums over binary forms.

Funct. Approx. Comment. Math., 28:83–95, 2000. Dedicated to Włodzimierz Staś on the occasion of his 75th birthday.