Scattering amplitudes in open superstring theory

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Für meine Familie

und familiengleiche

Freunde

Zusammenfassung

Die vorliegende Dissertation befasst sich mit dem Themengebiet der Streuamplituden in Theorien offener Superstrings. Insbesondere werden zwei unterschiedliche Formalismen zu Handhabung von Superstrings eingeführt und zur Berechnung von Baumniveau Amplituden verwendet – der Ramond Neveu Schwarz- (RNS-) und der Pure Spinor (PS) Formalismus.

Der RNS Zugang erweist sich als flexibel, um Kompaktifizierungen der anfänglich zehn flachen Raumzeit Dimensionen nach vier Dimensionen zu beschreiben. Wir lösen die technischen Probleme, die sich aus der wechselwirkenden zugrundeliegenden Weltflächentheorie mit konformer Symmetrie ergeben. Dies wird genutzt, um phänomenologisch relevante Streuamplituden von Gluonen und Quarks sowie Produktions- und Zerfallsraten von massiven Oberschwingungen, die schon als virtuelle Austauschteilchen auf masselosem Niveau identifiziert wurden, auszurechnen. Im Falle einer niedrigen String Massenskala im Bereich einiger TeV können die stringspezifischen Signaturen bei Partonkollisionen in naher Zukunft am LHC Experiment am CERN beobachtet und als erster experimenteller Nachweis der Stringtheorie herangeführt werden. Jene Stringeffekte treten universell für eine weite Klasse von String Grundzuständen bzw. internen Geometrien ein und stellen daher einen eleganten Weg dar, das sogenannte Landschaftsproblem der Stringtheorie zu umgehen.

Ein zweiter Themenkomplex in dieser Arbeit basiert auf dem PS Formalismus, welcher eine manifest supersymmetrische Behandlung von Streuamplituden in zehn Raumzeit Dimensionen mit sechzehn Superladungen erlaubt. Wir führen eine Familie von Superfeldern ein, die in masselosen Amplituden des offenen Strings auftreten und natürlicherweise mit Diagrammen aus dreiwertigen Knoten identifiziert werden können. Dadurch erreichen wir nicht nur eine kompakte Superraumdarstellung der n Punkt Feldtheorieamplitude sondern können auch die komplette Superstring n Punkt Amplitude als minimale Linearkombination von Partialamplituden der Feldtheorie sowie hypergeometrischen Funktionen schreiben. Letztere tragen die Stringeffekte und werden von verschiedenen Perspektiven analysiert, vor allem in Hinblick auf die in der Feldtheorie beobachtete Dualität zwischen gruppentheoretischen und kinematischen Beiträgen zu Baumniveau Amplituden.

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Contents

Genius is one percent inspiration,
ninety-nine percent perspiration.

Thomas Alva Edison
(American inventor, scientist, and businessman)

Chapter 1

Introduction

1.1 A brief prehistory of string theory

To begin the introduction to the topic of string theory, let us first of all sketch the developments in theoretical physics which led to its birth. The holy grail in natural science is a "theory of everything" (TOE), an all-encompassing set of rules which manages to describe and predict any process of nature without any additional input. It happened several times in the history of theoretical physics that serveral seemingly disconnected laws of nature were found to have a common origin, i.e. could be identified as special cases of a more general theory. Each instance of such a unification of distinct phenomena pushes the state of the art a little step closer towards the dream of a TOE. This section illustrates why string theory is among the most promising candidates from today's perspective.

1.1.1 History of unification – gravity and electromagnetism

The first two milestones on the road to unification happened before 1900. Two of the most prominent examples are firstly the gravitational law due to Sir Issac Newton and secondly Maxwell's theory of electromagnetism. Both of them managed to explain phenomena in seemingly different contexts of nature within an unexpectedly wider scope.

Newton's contribution to unification can be loosely referred to as bringing heaven and earth together. He realized that the motion of freely falling bodies on terrestrial scales like his (in)famous apple has the same origin as the planetary motion within and outside our solar system. His epoch making book "Philosophiae Naturalis Principia" published in 1687 casts the gravitational law underlying both the terrestrial and celestial physics into a mathematically precise form.

A second historical example of a unifying theory is the work of James Clerk Maxwell. In 1864, he found a set of differential equations which describe the interplay between electricity and magnetism. They encompass a variety of phenomena such as the magnetism of electric currents, the magnetic force and the induction of electricity (discovered by Ørsted, Ampère and Faraday, respectively) which at first appeared rather unrelated.

More recent examples of unifications and the potential of string theory in this context will be adressed later on. Let us first of all follow the historic thread and introduce the striking developments in theoretical physics in the early 20th century.

1.1.2 Challenges of unification – general relativity and quantum field theory

The early 20th century gave birth to two groundbreaking pillars of theoretical physics with deep impact on our understanding of the universe – general relativity (GR) and quantum mechanics. Both of them raised philosophical questions concussing the key concepts in our perception of the world such as time and determinism.

Inspired by the implications of Maxwell's equations on the speed of light, Einstein formulated the theory of special relativity in 1905 which first of all abandons the notion of absolute time. Ten years later, his general theory of relativity [12, 13] arrived at an even more radical insight on the dynamical nature of spacetime. It identifies both gravitational fields and accelerated reference frames as a geometric effect, more precisely as a source of curvature in the spacetime geometry. This discovery deprives spacetime of any absolute meaning since its structure is tied to its matter content. Apart from its philosophical scope, this theory furnishes a refinement of Newton's gravitational law: Among other things, it explains the perihelion precession of Mercury through a spacetime deformation caused by the sun and the planets.

Another construct of ideas which originates from Einstein's annus mirabilis 1905 is quantum mechanics. The energy carried by light was observed to appear in discrete quanta (of minimum energy) instead of following a continuous distribution. Maxwell's theory predicting wave-like propagation of light cannot explain this particle-like behavior. The theory of quantum mechanics degrades "particle" and "wave" to classical concepts which fail to fully describe the behavior of quantum-scale objects like photons, the quanta of light. According to the wave-particle-dualism – one of the key ideas in quantum mechanics – wave-like and particle-like properties always coexist. The a priori unexpected wave nature of elementary particles like the electron implies the uncertainty principle and thereby questions determinism in nature. That is why quantum mechanics opened a philosophical debate of comparable impact to that about the role of spacetime implied by GR.

The marriage of quantum mechanics with the ideas of special relativity leads to the framework of quantum field theory (QFT) [14]. It is the appropriate language to address quantum systems with a fluctuating particle number and infinitely many classical degrees of freedom. In particular, it proved to be an extremely successful language to describe all nongravitational fundamental interactions in nature, see the following subsection on the Standard Model of particle physics. In the perturbative formulation of QFT, forces between particles are mediated by another class of particles, so-called gauge bosons. The following figure 1.1 illustrates how electromagnetic forces are carried by the photon, this viewpoint is referred to as the quantization of Maxwell's theory of electromagnetism.

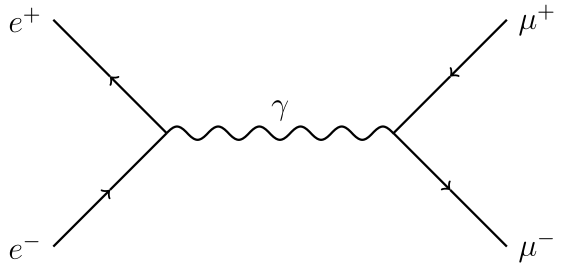


Figure 1.1: The electromagnetic action between electrons e^- and muons μ^- is transmitted through the exchange of a photon γ. In addition to the process shown, additional contributions arise from loop diagrams at higher order in the electromagnetic coupling.

In view of the success of QFT in particle physics and the pursuit of unification, it is desirable to describe all interactions by exchange of messenger bosons, including the gravitational force. But the canonical attempt to introduce a gauge boson mediating gravity – the so-called graviton – fails dramatically. Firstly, the computation of observables is plagued by an uncontrollable set of divergences which cannot be regulated while still keeping the theory predictive. The theory is said to be non-renormalizable (at least perturbatively). Secondly, a perturbative QFT approach to gravity requires the specification of a (possibly curved) background spacetime whose fluctuations are described by the so-called graviton of spin two. The whole framework of QFT must be reformulated in a background independent fashion in order to take the geometric nature of gravity into account.

More generally, the conceptual foundations of GR and QFT appear to be mutually incompatible: The former is a classical, strictly deterministic theory compatible with a dynamical spacetime, the latter theory incorporates quantum fields of intrinsically probabilistic nature but requires a fixed stage in order to define the fields. These conflicts suggest a more revolutionary approach to unification.

In the following subsection, we will put the issue of unification aside and give an overview of the achievements of QFT in the realm of particle physics together with their shortcomings. Even though its fundamental concepts are intrinsically incompatible with the spirit of GR, QFT provides an extremely accurate description of physics at short distances, i.e. on atomic and subatomic scales. The predictions of GR, on the other hand, are most reliable at macroscopic scales. In fact, gravity has hardly been tested at lengths below 10^{-6} m because of its comparative weakness to the other forces, although precision experiments have been recently proposed [15] to mend this shortcoming. Therefore, most of the physical phenomena clearly fall into the validity range of either GR or QFT. However, inevitable clashes of the two frameworks occur in situations with both high energies and strong gravitational fields, such as black holes or the initial Big Bang singularity. Their understanding holds out for a unified description in terms of quantum gravity.

1.1.3 The Standard Model of particle physics

Our present understanding of fundamental particles and their interactions is encompassed by the Standard Model of particle physics (SM) [16,17,18,19]. It is formulated in the language of a gauged quantum field theory, i.e. the fundamental interactions are introduced through the non-abelian gauge symmetry $SU(3)_C \times SU(2)_L \times U(1)_Y$ and mediated through the corresponding vector bosons. Three generations of two leptons and quarks each constitute its fermionic matter content, interacting through the strong, weak and electromagnetic force. The quantized theories of strong and electromagnetic interactions are referred to as quantum chromodynamics (QCD) and quantum electrodynamics (QED), respectively.

Theoretical predictions of the SM have been accurately matched with experimental observations down to length scales of 10^{-16} m (or equivalently up to energy scales of 100 GeV). As the most famous example for its astonishing precision, let us mention the anomalous magnetic moment of the muon where measurements agree with precision calculations [20] to more than 10 significant digits. The predicted masses of the $W^{\pm}$ and Z^0 boson as well as the existence of a third matter generation have been experimentally verified in the late 20th century.

Gauge invariance requires a dynamical mechanism for mass generation, this is accomplished by the postulated Higgs particle – the only Lorentz scalar of the SM. Being a doublet of the $SU(2)_L$ factor group, the Higgs breaks the SM gauge group down to $SU(3)_C \times U(1)_{\mathrm{EM}}$

by acquiring a vacuum expectation value (VEV) at low energies, this mechanism is called electroweak symmetry breaking [21,22,23]. The Higgs VEV provides dynamical masses to both the matter fermions through Yukawa interactions and to three gauge bosons $W^{\pm}$- and Z^0 of the broken symmetry $SU(2)_L \times U(1)_Y$ (while keeping the photon associated with the preserved $U(1)_{\text{EM}}$ symmetry massless).

Only the Higgs can have an explicit mass M_{H} compatible with the SM gauge group. Its value is bounded from below by LEP experiments and from above by inconsistencies with electroweak symmetry breaking. Within the window $114\,\text{GeV} \leq M_{\text{H}} \leq 1.4\,\text{TeV}$, indirect theoretical predictions favor a mass well below the Tevatron exclusion rangle of 158 GeV to 175 GeV, most preferably around 120 GeV [24].

The SM Lagrangian contains as many as 26 unrestricted parameters (including neutrino masses, -mixing and -phase) which must be inferred from experimental observation – a rather unsatisfactory amount of manual input from the theoretical point of view. The fermion masses (or equivalently their Yukawa couplings to the Higgs) vary from 1 eV for the electron neutrino up to 170 GeV for the top quark. This hierarchy over ten orders of magnitude appears rather unnatural and seems to point out our lack of deeper understanding.

An even more severe hierarchy problem concerns the mass of the Higgs particle [25, 26, 27]: The natural scale for quantum corrections to its mass is the cutoff where the SM loses its validity. According to the previous subsection, this most naturally happens when effects of quantum gravity kick in, i.e. at the Planck scale $M_{\text{pl}} = \sqrt{\hbar c/G} \approx 1.22 \cdot 10^{19}\,\text{GeV}/c^2$. (Grand Unified theories modifying the SM at energies higher than $M_{\text{GUT}} \approx 10^{16}\,\text{GeV}$ do not significantly change this hierarchy.) Accommodating these radiative corrections with the electroweak scale $\sim 100\,\text{GeV}$ suggested by experiments requires an unnatural fine tuning in 30 digits of the bare mass. So the question to ask is why the Higgs mass differs from the Planck mass by 17 orders of magnitude or why the weak interaction is ironically about 10^{34} times stronger than gravity.

An even more embarrasing shortcoming of the SM is its misprediction of the cosmological constants by roughly 120 orders of magnitude [28]. The observed accelerated expansion for the universe (due to high precision measurements of distant supernovae and the cosmic microwave background) deviates from a naive QFT computation based on a cutoff at the Plack mass $\sim 10^{19}\,\text{GeV}$ by the truly astronomical factor of 10^{120}. This reflects the need to incorporate quantum gravity effects into the SM.

Explaining the accelerated expansion as well as observed galactic rotational curves within the framework of GR requires two additional forms of energy – dark matter and dark energy.

Both of them are not explained by the SM. Cosmological measurements suggest that dark matter and dark energy make up about 73 % and 23 % of today's universe, respectively. Being applicable to less than 5 % of the universe's energy content relativizes the success of the SM.

To conclude the criticism of the SM, it does not offer any satisfactory explanation for the general structure of its ingredients. Why is the gauge group $SU(3)_C \times SU(2)_W \times U(1)_Y$? Why do four interactions and three families of fermions exist? Why are 26 parameters left for experimental fixing without any theoretical understanding of these values? Finally, why is the flat background spacetime four dimensional?

1.1.4 Supersymmetry

Some of the aforementioned problems can be addressed by a new kind of symmetry relating bosons with fermions – supersymmetry (SUSY) [29, 30, 31]. It adds anticommuting generators to the Poincaré algebra [32] and thereby evades the Coleman Mandula no-go theorem [33] from 1967. Viable extensions to the SM can be built on the basis of SUSY, the simplest of which is called the minimally supersymmetric Standard Model (MSSM) [34].

Unfortunately, the SM does not contain any bose-fermi pair which can be aligned into one SUSY multiplet, so SUSY roughly speaking doubles the particle content. The great virtue of these extra particles is a partial cancellations of bosonic against fermionic contributions to loop corrections of SM observables, in particular, the aforementioned mismatch about the cosmological constant is softened from 10^{120} to 10^{60}.

However, one prediction of unbroken SUSY is a common mass for all members of its multiplets. The SUSY partners to the SM particles would have been observed long ago if SUSY was an exact symmetry at experimentally accessible scales. The non-observation of SUSY partners implies that SUSY must be broken at some energy scale which has not been probed by any accelerator yet. If this SUSY breaking scale is sufficiently low, say in the TeV range, it tames the ultraviolet (UV) divergences from loop diagrams and stays compatible with the non-observation of SUSY. In particular, SUSY offers an appealing solution to the hierarchy problem if broken at the TeV scale because it stabilizes the Higgs mass against radiative corrections and avoids its blowup to the Planck scale.

On the other hand, one has to admit that no natural SUSY breaking mechanism far below the Planck (or GUT-) scale is known. Ad-hoc mechanisms have been considered such as soft SUSY breaking [35] which generically add relevant operators (whose coefficients have a positive mass dimension) to the MSSM Lagrangian and decouple the origin of supersymmetry breaking from its phenomenological consequences. Unfortunately, this introduces a variety of new unfixed input parameters in addition to the 26 of the non-supersymmetric SM. The stabilization of the Higgs mass through SUSY induced cancellations in radiative corrections trades the original hierarchy problem $M_{\mathrm{H}} \ll M_{\mathrm{pl}}$ for a new problem of low-energy SUSY breaking. Recent data from the Large Hadron Collider (LHC) at CERN exclude SUSY partners of SM particles up to masses of $\approx 800\,\mathrm{GeV}$ within representative models of soft SUSY breaking.

Another important feature of TOEs is nicely addressed by SUSY – gauge group unification. The SM has separate coupling constants for each factor of the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$. They evolve with the energy scale according to renormalization group equations, and it turns out that (only) in presence of SUSY, they perfectly meet at some high scale unification scale $M_{\mathrm{GUT}} \approx 10^{16}\,\mathrm{GeV}$. This hints that the SM gauge group in fact originates from a larger group such as $SU(5)$ [36] or $SO(10)$ [37]. However, the construction of such a grand unified theory (GUT) suffers from the need to suppress proton decay (which becomes a serious threat since quarks and leptons end up in the same representation of the GUT gauge group).

If the number $\mathcal{N}$ of spinorial super-Poincaré generators is bigger than one, SUSY is said to be extended. Only the simplest $\mathcal{N} = 1$ SUSY can keep the fermion spectrum chiral as we observe it in the SM, so extended $\mathcal{N} > 1$ SUSY at low energies is phenomenologically ruled out. Nevertheless, maximally supersymmetric theories are a fruitful laboratory for formal progress in QFT as we will explain in the next section on scattering amplitudes.

Also the graviton, the spin two messanger particle of gravity, can be aligned into a SUSY multiplet, it is then accompanied by $\mathcal{N}$ spin 3/2 gravitinos. Gravity is inevitable once SUSY is promoted to a local symmetry, these theories are called supergravity (SUGRA). The maximally supersymmetric $\mathcal{N} = 8$ SUGRA still has the potential to be UV finite to all loop orders [38, 39, 40] because various divergences suggested by naive power counting have been shown to cancel.

1.2 Overview of string theory

One of the most promising attempts to address the aforementioned conceptual and practical problems of the SM is string theory. We have to admit that "string theory" as such is not fully formulated yet, in particular not in a background independent or non-perturbative manner. Nevertheless, a lot of features are necessarily incorporated, irrespective of what its final formulation might be. Most prominently, string theory inevitably includes the graviton and reduces to Einstein's theory at low energies. Moreover, it naturally encompasses Super Yang

Mills (SYM) theory (the supersymmetric generalization of SM gauge theories) in its low energy limit, and even complex chiral representations of fermions can be realized like they are observed in the matter sector of the SM. Finally, the mathematical consistency of string theories with fermions requires spacetime supersymmetry. Knowledge of these general properties turns out to be sufficient to compute scattering amplitudes and to extract their physical implications. Doing so is the main purpose of this work.

There are many nice textbooks on string theory available on the market. The classics are [41, 42, 43, 44, 45], and modern topics are covered in recent textbooks [46, 47, 48]. Moreover, I learned a lot from the brilliant lecture notes [49].

1.2.1 Why string theory?

All the aforementioned features of string theory follow from simple geometrical ideas. The time evolution of a one dimensional string sweeps out a two dimensional surface in spacetime, a so-called worldsheet. It can be thought of as the higher dimensional generalization of the point particle's worldline. Canonical quantization gives rise to a quantum field theory on the worldsheet which turns out to be exactly solvable because of the underlying infinite dimensional (super-)conformal symmetry. The lowest energy excitations of viable string theories have zero mass and shall ultimately be identified with SM particles, the graviton or their SUSY partners. In addition, strings can vibrate in an infinite tower of heavy excited modes.

We will make extensive use of the powerful properties of conformal field theory (CFT) throughout this work. On the basis of CFT methods, it is not difficult to see that string theory predicts the dimensionality of spacetime it lives in[1] – quite in contrast to the god-given four spacetime dimensions of the SM.

As an additional benefit of string theory compared to the SM, it involves no other input parameter than the length scale ℓ_{string} of strings[2] (in contrast to the 26 parameters of the SM). A natural first guess for ℓ_{string} would be a value close to the Planck length $\ell_{\mathrm{pl}} = \sqrt{\hbar G/c^3}$, but this is by no means necessary and we will introduce alternative scenarios within this work. At

[1] There also exist so-called noncritical string theories in a modified number of spacetime dimensons which require a nontrivial linear profile for the dilaton field, a scalar analogue of the graviton. But they violate Lorentz invariance and do not appear to be suitable candidates for a quantum theory of gravity

[2] One might be tempted to argue that the string coupling constant g_{s} governing the perturbative approach should also be counted as a fundamental input parameter. In subsection 5.1.2 we identify it with the VEV of a background field, the so-called dilaton. Therefore, g_{s} parametrizes different backgrounds of one fixed theory rather than a family of string theories.

distances far above ℓ_{string} (or at energies well below the string mass scale), the point particle is still an accurate approximation.

Strings can be either closed or open. The massless excitations in the former topology contain the graviton whereas the lowest open string vibration modes have the properties of gauge bosons of SYM theory. Since open strings can always join to form a closed one, gravity is a *conditio sine qua non* in string theory. One could also think about higher dimensional generalization of strings and try to quantize membranes. But it turns out that the string being a one dimensional object is singled out is two respects: Firstly, only the two dimensional string worldsheet gives rise to an infinite dimensional symmetry group and thereby to an exactly solvable worldvolume QFT. Secondly, membrane theories (with ≥ 3 dimensional worldvolumes) are physically not viable because their quantization yields a continuous spectrum of vibration modes in contrast to the discrete mass gaps between the string excitations.

To summarize this first pleading in favour of string theory – the distinguished role of one dimensional strings as fundamental objects lies in the fact that a QFT on their two dimensional worldvolume is not only exactly solvable but also gives rise to spacetime physics that appears to be compatible with observation: Gravity due to exchange of massless spin two messengers and gauge interactions due to gauge bosons of spin one.

1.2.2 History of string theory

String theory was firstly considered in the late 1960s in the context of the strong nuclear force – the myriad of observed hadrons could be neatly explained as specific oscillator modes of one dimensional objects, in particular the linear relation $M^2 \approx j/\alpha'$ between the spin j of hadrons and mass square M^2 naturally follows from a stringy description [43]. The constant $\alpha' \approx 1\,(\text{GeV})^{-2}$ became known as the Regge slope and gave reliable masses up to hadron spins $j = 11/2$.

However, string theory was temporarily discarded because it contains a massless spin two particle. This is clearly unwanted in a theory of hadrons, so string theory was pushed aside by QCD in the early 1970's. After several years of hibernation, the spin two excitation was identified with the graviton, and string theory turned out to be better suited for a more ambitious challenge, to serve as a theory of quantum gravity. With its goal of unifying QFT and GR, string theory brought the research communities of these subjects together which had developed almost independently till the late 1980's.

In a first superstring revolution in 1984, string theory in ten spacetime dimensions with worldsheet SUSY was shown to be free of quantum anomalies by M. Green and J. Schwarz [50]. Purely bosonic string theory was plagued by a tachyonic state in both the open and closed string sector violating causality and indicating an instability of the vacuum. The virtues of superstring theories established in the early 1980's lie in the absence of tachyons (due to the so-called GSO projection [51]) and their capability to describe fermions which constitute the majority of observed SM particles. In particular, the elimination of the tachyon makes a well-defined field theory limit possible, in which open superstring theory turns out to reproduce SYM theories. Five consistent superstring Theories could be identified in the 1980's – so-called heterotic $E_8 \times E_8$, heterotic $SO(32)$, type I, type IIA and type IIB. The massless closed string spectrum of the latter two boil down to the (two possible) $\mathcal{N} = 1$ supergravity theories in $D = 10$ dimensions.

A second superstring revolution was triggered in 1995 by the discovery of nonperturbative dualities due to E. Witten relating the five seemingly different theories. A web of dualities provides convincing evidence for the uniqueness of an underlying theory called M theory [52]. In addition to the ten dimensions of superstring theories, an extra spatial dimension emerges in the spacetime of M theory at strong coupling, and its low energy limit corresponds to the (unique) $\mathcal{N} = 1$ supergravity theory in eleven dimensions. It would be desirable to improve our currently poor understanding of M theory because it is expected to provide a nonperturbative perspective on quantum gravity and gauge theory [53].

A string duality of particular importance is the $SL(2, \mathbb{Z})$ self-duality of type IIB superstrings. In 1996, it was noticed by C. Vafa that this is precisely the modular group of a two dimensional torus whose modular parameter corresponds to the VEV of a complex field [54]. The elliptic fibration of this torus over the ten dimensional base spacetime yields a twelve dimensional variety, and the theory living on this is called F theory. It allows for a technically more accessible treatment of nonperturbative aspects and was recognized in 2008 as a suitable framework for GUTs with exceptional gauge groups [55].

1.2.3 Extra dimensions and D branes

The ten spacetime dimensions predicted by the CFT on the superstring worldsheet appear questionable in view of the four dimensions we experience in our environment, both in macroscopic and in microscopic experiments. The idea of "invisible" extra dimensions goes back to Kaluza and Klein in the 1920s [56, 57]. They derived the four dimensional Maxwell theory of electromagnetism by compactifying five dimensional GR on a circle.

String theory offers two mechanisms to explain the non-observation of six extra dimensions.

Firstly, they are guaranteed to evade detection if they are compact with spatial extent below the experimentally resolvable length scale (i.e. 10^{-16} m for present days colliders). Secondly, the endpoints of open strings whose massless modes are supposed to reproduce the full non-gravitational SM physics can be confined to lower dimensional regions of spacetime. Then only gravity from the closed string sector can feel the full ten dimensions through a much stronger suppression of the gravitational force with distance than in four dimensions [58]. Since gravitational effects are only tested down to coarse scales of about 10^{-6} m, this scenario imposes less severe constraints on the size of internal spaces.

A consistent treatment of open strings requires boundary conditions for their endpoints of either Dirichlet- or Neumann type. Dirichlet conditions with respect to $9-p$ of the spatial dimensions restrict open string endpoints to $p+1$ dimensional subsets of spacetime which are called Dp branes [59]. They are not quantized as fundamental objects but can rather be thought of nonperturbative excitations with their own dynamics. The simplest explanation why we experience four spacetime dimensions is because we are confined to live on a D3 brane embedded in possibly higher dimensional spacetime. The carefully measured low energy physics from the open string sector feels four dimensional physics while only the gravitational sector due to closed strings (which has been experimenally probed down to 10^{-6} m only) can probe the extra dimensions. Dimensions transverse to the brane are invisible to electromagnetic, weak and strong interactions.

Interpolating scenarios with Dp branes of dimensionality $3 < p < 9$ are more realistic. The $p+1$ dimensional brane worldvolume contains the four macroscopic dimensions and additionally wraps cycles of compact small internal dimensions of size below the experimental limit of 10^{-16} m. Topological properties of the internal manifold and the Dp brane configuration therein determine the particle content and the amount of supersymmetry from the four dimensional point of view. It turns out that the naive compactification on a six-torus preserves $\mathcal{N} = 4$ spacetime supersymmetry in $D = 4$ dimensions and can be ruled out because the observed chiral matter spectrum is compatible with $\mathcal{N} \leq 1$ SUSY only. This phenomenological requirement favors Calabi Yau manifolds for the internal geometry [60] together with further ingredients such as orientifolds [61]. String theory establishes a surprising connection between topologically different Calabi Yau manifolds called mirror symmetry [62]. It is an excellent example for cross fertilization between phenomenology, string theory and mathematics.

However, generic compactifications introduce numerous massless scalars in four dimensions leading to non-acceptable phenomenology. A successful mechanism for generating a scalar potential and thereby rendering them massive is to turn on background fluxes [63]. Because of the quantized nature of these fluxes, admissible string vacua are discretized and can in principle be counted, a commonly quoted estimate of their total number being 10^{500} [64]. Hence, phenomenological constraints are far too weak to determine the string vacua. The failure of string theory to select a unique vacuum which explains the observed SM physics is referred to as the landscape problem. The majority of results in this work remains valid for a huge class of such vacua and thereby evades the landscape problem.

At least, the landscape of string vacua contains scalar potentials which solve the cosmological constant problem: The minimum value of the potential determines the vacuum energy or cosmological constant. In contrast to the SM and MSSM, string theory can accomplish the observed small and positive value for the latter [65, 66].

1.2.4 The AdS/CFT correspondence

A milestone in the implications of string theory for gauge theories is the AdS/CFT correspondence [67, 68, 69] which provides an example of emergent spacetime through the holographic principle, see [70, 71] for reviews. According to this correspondence, the maximally supersymmetric SYM theory is dual to a theory of quantum gravity on Anti-de Sitter space AdS_5, a five dimensional spacetime of constant negative curvature. In this context, the energy scale on the four dimensional SYM side is related to the size of the radial direction of AdS_5, which is therefore referred to as the "holographic" direction.

To understand this correspondence from a string theory point of view, we have to connect two different viewpoints on a collection of N coincident D3 branes in type IIB theory. Firstly, they appear as source terms in the stringy generalization of Einstein's equations and produce a spacetime with horizon, comparable to a black hole. In the vicinity of this horizon, the geometry can be approximated by $AdS_5 \times S_5$. On the other hand, the open strings attached to N coincident D3 branes in type IIB give rise to $\mathcal{N} = 4$ SYM theory with gauge group $SU(N)$ due to their massless excitations.

The AdS/CFT duality connects the strong coupling regime of one side with the weak coupling sector of the other side. Understanding the weakly coupled SYM theory in four dimensions provides insights into the stringy regime of quantum gravity in AdS spacetime. Conversely, weakly coupled strings in $AdS_5 \times S_5$ make the strong coupling regime of the gauge theory accessible [72] which conceals itself from perturbative methods.

Moreover, the AdS/CFT correspondence sheds light on a duality between Wilson loops with lightlike segments and scattering amplitudes involving massless states [72, 73]. This builds the bridge to the main topic of this work – scattering amplitudes in superstring theory.

1.3 Scattering Amplitudes

Scattering amplitudes are perhaps the most basic quantities computed in any QFT, and string amplitudes are in fact older than string theory itself [74,75]. In 1968, G. Veneziano constructed a "dual" four point scattering amplitude with particular crossing symmetries in order to describe hadron scattering. It was motivated by the fact that the high energy behavior of dual models is much softer than that of any field theory, and hadrons were observed to exhibit such a soft high energy behavior. Veneziano's amplitude turned out to emerge from strings, exhibiting the fingerprints of an infinite tower of vibration modes. The generalization of dual models to higher number of legs can be found in [76].

In this work with focus on scattering amplitudes, we take a bit more the particle physicist's point of view on superstring theory. This rephrases the question about high energy physics into that of short distance physics, the length scale being set by the fundamental string length ℓ_{string}.

In the following, we will sketch the structure of superstring scattering amplitudes and give a taste of the beautiful underlying physical and mathematical structures. The interplay between string theory and point particle QFT can be best illustrated by maximally supersymmetric Yang Mills theory, but there is also strong motivation within string theory to get a handle on its S matrix, as we will explain.

1.3.1 Generalities about superstring amplitudes

Scattering amplitude in perturbative QFT are built from summing Feynman diagrams. The perturbative expansion of string theory sums over different worldsheet topologies. Remarkably, there is just one worldsheet contributing to individual orders of perturbation theory in contrast to plethora of Feynman diagram at higher number of legs or loops, see the following figure 1.2.

Each Feynman diagram evaluated separately is more complicated than the complete amplitude, in particular it obscures various kinds of symmetries (e.g. gauge invariance and cyclicity) present in the final answer. Hence, a big advantage of string perturbation theory is absence of these artifacts from the beginning. Moreover, interactions in string theory are uniquely determined by the free worldsheet theory, without the need to specify Feynman rules by a spacetime action. Finally, the joining and splitting of the worldsheet in string scattering does not single

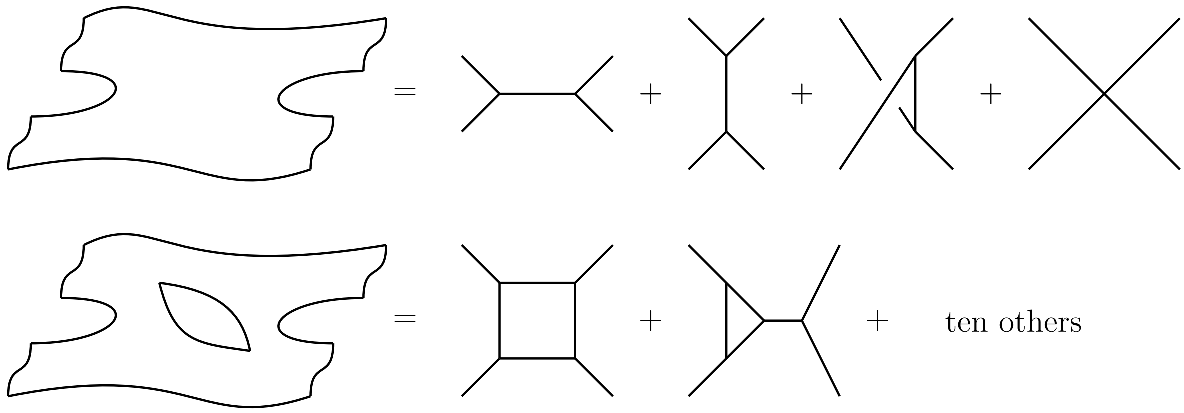


Figure 1.2: At each loop order, a single string worldsheet encompasses several Feynman diagrams.

out an interaction point in spacetime. This keeps string amplitudes free of short distance- or UV divergences.

As we shall see in this work, the leading order of perturbation theory for open string amplitudes still poses serious challenges. All the research results shown are limited to tree level scattering of open strings, the worldsheet of this process having the topology of a disk. We shall convince ourselves that superstring disk scattering of massless states such as gauge bosons reduces to gluon amplitudes of YM theory in the low energy limit, obtained from Feynman diagrams or more modern methods.

1.3.2 The harmony of scattering amplitudes

During the last years, remarkable progress has been accumulated in our understanding and in our ability to compute scattering amplitudes, both for theoretical and phenomenological purposes, see [77,78,79] for a recent account. Striking relations have emerged and simple structures have been discovered leading to a beautiful harmony between seemingly different structures and aspects of gauge and gravity scattering amplitudes [80] which are invisible in a Lagrangian description. As an example, we mention the duality between color and kinematics, which exhibits a new structure in gauge theory [81]. This property allows to rearrange the kinematical factors in the amplitude such that its final form becomes rather simple. Moreover, recently it has been shown [82] that the duality between color and kinematics allows to essentially interchange their roles in the full color decomposition of the amplitude. Many of the nice properties encountered in gauge theory amplitudes take over to graviton scattering.

The properties of scattering amplitudes in both gauge and gravity theories suggest a deeper understanding from string theory, see [39, 40] for a recent review. The color decomposition of gauge theory amplitudes comes almost as a definition from string theory, it isolates the contribution of group theoretic factors and cuts the amplitude into smaller gauge invariant pieces with neat symmetry properties. In fact, many striking field theory relations such as Bern Carrasco Johansson (BCJ) or Kawai Lewellen Tye (KLT) relations can be easily derived from and understood in string theory by tracing these identities back to the monodromy properties of the string worldsheet [83, 84, 85]. We shall demonstrate in this work that the complete result for n point open superstring disk amplitudes for massless states displays properties and symmetries inherent in field theory and reveals structures relevant to field theory. Moreover, we find a beautiful harmony of the string amplitudes with strong interrelations between field theory and string theory.

One of the most important goals in computing amplitudes is to cast their final result into compact and short form. This is achieved by grouping the various ingredients like kinematics and worldsheet integrals into packages. Eventually, it is desirable to express the final result in terms of a minimal basis of both integrals and kinematics.

The computation of scattering amplitudes is a fascinating research area, both in field theory and in superstring theory. On the one hand, it is an experimental science in the sense that facing new classes of unknown amplitudes first of all requires to obtain the result by any means necessary before one can analyze its structure. On the other hand, experience with amplitudes teaches that simplifications do not happen by accident. So the second part of the research work lies in understanding why the result (which was possibly obtained by "brute force") looks like it does and in which ways it simplifies. The interplay of these two steps has led to many examples where the classification of a scattering problem developed from "impossible" to "difficult" to "trivial" [86].

1.3.3 Superstring theory, $\mathcal{N} = 4$ SYM and QCD

Let us take a closer look at the particular field theory to which maximally supersymmetric open superstring theory reduces in the low energy limit and upon dimensional reduction from ten to four dimensions: $\mathcal{N} = 4$ SYM [87]. This field theory was discovered 35 years ago [88] and identified as the four dimensional QFT of highest supersymmetry whose particle content does not exceed spin one. As a gauge theory, $\mathcal{N} = 4$ SYM is a relative of QCD – either a brother or a remote uncle of nth degree, depending on which kind of physicist is asked.

One special feature of $\mathcal{N} = 4$ SYM in four dimensions is superconformal invariance – its beta function turns out to vanish to all loop orders [89]. As a consequence, the theory is UV finite in perturbation theory [90, 91, 92], in agreement with the properties of the underlying string theory.[3] Moreover, the planar limit of $\mathcal{N} = 4$ SYM enjoys a dual superconformal symmetry acting non-locally on scattering amplitudes which is invisible on the level of the Lagrangian [94]. It can be understood from the AdS/CFT correspondence [95]. Together with the aforementioned superconformal invariance, the symmetry of $\mathcal{N} = 4$ SYM is enhanced to the infinite dimensional Yangian [96], the closure of the superconformal algebra and its dual under (anti-)commutators. Because of its rich symmetries, $\mathcal{N} = 4$ SYM appears to be the first solvable QFT, at least in the planar limit. Since last year, the all loop integrand of scattering amplitudes in planar $\mathcal{N} = 4$ SYM is known [97], so the only task missing to its complete solution is to get a systematic handle on the integrals over loop momenta.

The simplicity of the recent results in planar $\mathcal{N} = 4$ SYM rests on the fact that they do not make reference to spacetime and unitarity [98, 99]. The traditional Feynman diagram approach to scattering amplitudes in QFT obscures the simplicity of the answer because it insists on manifest locality and unitarity. There is nothing wrong in talking about local physics without using manifest locality, this is analogous to deriving Newton's deterministic law by minimizing the action for a point particle in a potential. The virtue of the (not manifestly deterministic) principle of least action is its much closer affinity to quantum mechanics, i.e. that it easily emerges as the $\hbar \to 0$ limit of the latter. Similarly, presenting the S matrix of local field theories without manifest locality promises to be helpful for the search of a theory of quantum gravity where the role of locality is questionable. In particular, string theory encompassing an inifinite tower of higher derivative interactions in spacetime fields exhibits non-local properties.

These findings and developments manifest the strong interrelation between maximally supersymmetric superstring theory and its $\mathcal{N} = 4$ field theory limit, in particular on the level of amplitudes. Firstly, the field theory's rich symmetries (on which its solution will heavily rely) can be understood from the underlying string theory. Secondly, starting from the superstring computation to all orders in the string length ℓ_{string}, one can learn a lot about more compact and useful representations of the corresponding field theory amplitudes emerging in the low energy limit $\ell_{\text{string}} \to 0$. In the near future, loop computations in string theory might provide fruitful inspiration for the loop integral technology in $\mathcal{N} = 4$ SYM. Thirdly, novel string computations can be validated by the consistency requirement to reproduce the known SYM amplitude in the field theory limit. Finally, by the AdS/CFT correspondence, solving $\mathcal{N} = 4$ SYM provides information on quantum gravity in AdS space.

More realistic QFTs such as QCD certainly involve additional complications compared to

[3] It turns out that the beta function still vanishes nonperturbatively, but this is harder to prove [93].

$\mathcal{N} = 4$ SYM, but the systematic study of their S matrix definitely profits from the methods used for its supersymmetric brother and lessons drawn from the latter, see e.g. [100] for all tree level amplitudes in massless QCD. Also, most of the string amplitudes discussed in this work are still valid in absence of supersymmetry. Our n gluon disk amplitudes hold independent on the compacitification geometry and reduce to gluon scattering in QCD as $\ell_{\text{string}} \to 0$. And we will argue that also chiral matter can be addressed by superstring theory.

1.3.4 Why superstring amplitudes?

In addition to the aforementioned tight connections between superstring theory and its field theory limit, there is strong intrinsic motivation to study scattering amplitudes for a deeper understanding of superstring theory and its implications.

String scattering can be phenomenologically relevant in case of a low string mass scale in the range of a few TeV/c^2 [1, 101]. As we will argue later on, this is possible if spacetime extends into large extra dimensions [58]. Superstring theory predicts universal resonances in quark- and gluon collisions as they take place at LHC[4]. Disk amplitudes involving mostly gluons are completely insensitive to the compactification model. If the string scale falls into the energy range accessible at LHC, then the stringy signal in four parton cross sections due to exchange of higher spin Regge resonances should deviate from the SM background by the peak shown in figure 1.3 [102]. So far, string resonances have been excluded by the CMS experiment up to 1.67 TeV [103].

Disk amplitudes involving at least four quarks, on the other hand, carry signatures of the low energy dynamics of various compactification geometries. If the model independent peak shown in figure 1.3 is indeed observed, then the next step to carry out is a precision measurement of the internal geometric data on these grounds.

According to [104], multileg processes play a much more dominant role at LHC than at other accelerators due to the high energies available. Decay cascades of SUSY particles are quite likely to result in emission of more than three hadron jets. A careful disentanglement of SUSY- or string signatures from SM backgrounds requires to take scattering amplitudes with many external states into account, on the level of both field- and string theory. The five point superstring amplitudes underlying three jet processes are computed in [1], they allow for a better identification of the spin of the internal states than two jet processes due to their higher angular resolution.

[4] The QCD sector of the SM can be realized in superstring theory by an intersecting D brane configuration.

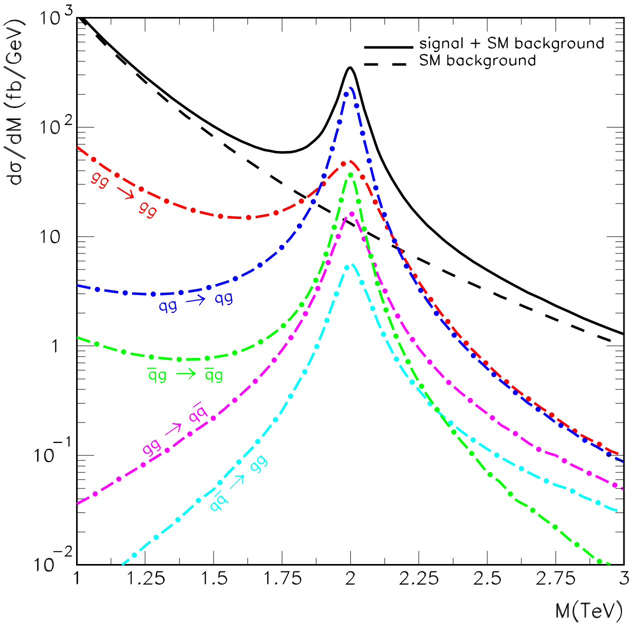


Figure 1.3: Differential cross sections for dijet events in four parton scattering, plotted versus the center of mass energy. The peaked curves show the result of a disk amplitudes involving quarks and gluons in weakly coupled string theory, assuming a string energy scale of 2 TeV.

Apart from their phenomenological motivation, superstring amplitudes allow to test various aspects of duality symmetries relating different string vacua [105]. References [106, 107] for instance address the strong–weak coupling duality between type I and heterotic $SO(32)$ superstring theory conjectured in [108, 109]: Higher genus amplitudes on the heterotic side are related to F^{2n} interactions of the gauge field strength F in type I.

There are plenty of additional benefits from mastering superstring amplitudes. For instance, string theory and its interactions furnish a fruitful laboratory to learn about higher spin gauge theory [110, 111]. We will further elucidate this at various points within the main body of this work. Also, one can use superstring amplitudes (in particular their higher order corrections in the string length) as a generating machine for SUSY invariants. Five- and six loop counterterms for supergravity theories have recently been ruled out on the basis of a low energy expansion of graviton scattering amplitudes in superstring theory [112]. Instead of perpetuating this list with further application of superstring amplitudes, let us turn to the content and novelties of this thesis.

1.4 Achievements in this thesis

The results discussed in this thesis are derived from the broad topic of scattering amplitudes in superstring theory. More specifically, phenomenological aspects of a low string scale [1, 2], the conformal field theory arising in the RNS superstring [3,4,5], higher spin scattering [6] and scattering amplitudes in the pure spinor formalism [7,8,9,10,11] are investigated in detail.

The main result of this thesis is to cast the color ordered n point disk amplitude of the open superstring into the following striking, simple and compact form

$$\mathcal{A}(1,2,\ldots,n-1,n;\alpha') \;=\; \sum_{\sigma \in S_{n-3}} \mathcal{A}^{\mathrm{SYM}}(1,2_\sigma,3_\sigma,\ldots,(n-2)_\sigma,n-1,n)\, F^\sigma(\alpha') \tag{1.4.1}$$

with $i_\sigma \equiv \sigma(i)$ and S_{n-3} denoting the group of permutations in $(2,3,\ldots,n-2)$. The building blocks $\mathcal{A}^{\mathrm{SYM}}$ represent $(n-3)!$ color ordered super Yang Mills (SYM) subamplitudes, while $F^\sigma(\alpha')$ are generalized Euler integrals encoding the full dependence of the string amplitude on the Regge slope α' (i.e. the squares string length or the inverse string tension). Its structure and implications are thoroughly explained in section 12.3 and in [11]. The way to this result was paved by previous work [7,8,9] on superstring- and field theory amplitudes in the pure spinor formalism.

Older work [1,2] focuses on phenomenological aspects of a low string scale. The former article extends string theory's universal prediction on four parton scattering at LHC to the five point level. We compute the full fledged cross sections for five point disk scattering of quarks and gluons and discuss model independent features of multi-parton processes in chapter 8. The latter paper which is summarized in chapter 9 identifies the universal four dimensional particle content of stringy Regge excitations on the first mass level. Excited string states are expected to be produced at the LHC as soon as the string mass threshold is reached in the center of mass energies of the colliding partons. This is why we evaluate amplitudes involving one such massive state and up to three massless ones and express them in a helicity basis. In both cases, the results are cast into a form suitable for the implementation of stringy partonic cross sections in the LHC data analysis.

Three of my publications [3,4,5] are devoted to the old problem that the RNS superstring is only superficially based on a free worldsheet theory, since fermion emission involves subtle ingredients, so-called spin fields. Bosonization techniques make it possible to compute their correlators, but this breaks Lorentz symmetry and requires a precise translation of the results obtained in bosonized language into covariant expressions. This is what we accomplished to do in a variety of cases which appear in fermion amplitudes but have not been systematically addressed in the literature before, see chapter 6. In particular, these results prove to be an essential toolkit in computing the aforementioned multi parton scattering processes relevant at LHC.

In another paper [6], I focus on the leading Regge trajectory in open superstring theories, i.e. on highest spin states at mass level n with spin $s = n+1$ for bosons and $s = n+\frac{1}{2}$ for fermions. It extends the computation of their three point vertices which have already been known for bosonic strings to superstring theory. In addition, four point decay amplitudes of leading Regge trajectory states into partons are computed, providing a higher spin generalization of [2]. For these reasons, these results are also included into chapter 9 on scattering of massive states. Work in progress aims to extract lessons on the field theory of massless higher spin fields [113] and on generating function techniques which govern their scattering amplitudes.

Finally, I started to work with the pure spinor approach to superstring amplitudes when the state of the art in computing tree level scattering of the gauge multiplet with pure spinor methods was at five point level. In [7], we compute the six point disk amplitudes and get a glimpse of the underlying BRST structures. These patterns are generalized to higher numbers of legs which determines the n point field theory amplitude on the basis of BRST cohomology arguments [8]. The n point superstring computation allows for an explicit construction of so-called BCJ numerators in gauge theories [9] which manifest the duality between kinematic and color factors. It was an outstanding problem among field theorists to find better representations for these kinematic building blocks, so string theory serves as a convenient tool here. The end result of my pure spinor projects is [10] with its compact expression (1.4.1) for the n point open superstring disk amplitude. The whole discussion on the pure spinor formalism can be found in chapters 10 to 13.

1.5 Outline

This work is divided into three parts. The first one aims to summarize basic ingredients of the RNS superstring, in particular the CFT machinery necessary for the computation of the scattering amplitudes in the subsequent parts. Chapter 2 introduces the decoupling CFT sectors of worldsheet matter fields X^m, ψ^m, S_A and ghosts b, c, β, γ and sketches the special features of open strings compared to closed ones. The second chapter explains the general rules for constructing the supersymmetric open string spectrum and gives a more detailed account on three examples: massless states, the first mass level and the so-called “leading Regge trajectory” encompassing the highest spin states of each mass level. In the third chapter, we adjust the ten dimensional RNS formalism to $D = 4$ compactifications. After briefly addressing low string scale extensions to the Standard Model, we look at the SCFT implementation of compactifications with variable amount of spacetime supersymmetry and the intersecting D branes. Particular attention is paid to the adjoint and chiral particle content at mass levels zero and one.

The second part II is devoted to disk amplitudes in connection with RNS methods. Chapter 5 thoroughly introduces open superstring tree level amplitudes and points out the parallel structures governing superstring- and field theory amplitudes. The following two chapters 6 and 7 then focus on the two main challenges posed by the prescription for disk amplitudes: The former covers correlation functions in the interacting CFT involving the NS fermions and spin fields which constitute a bottleneck for computing scattering amplitudes involving spacetime fermions. The latter discusses the properties and mathematical background of the worldsheet integrals arising in tree amplitudes. Explicit results on disk scattering appear in the last two chapters of the second part: Scattering amplitudes involving massless particles (mainly members of the SYM vector multiplet and quarks) are the topic of chapter 8, and massive states are included in chapter 9 – both the first mass level in $D = 4$ dimensions and the leading Regge trajectory.

The step to the last part III is a hair pin bend because we switch to a new approach to the worldsheet degrees of freedom of the superstring – the pure spinor formalism. The basic prerequisites for computing disk amplitudes of the massless gauge multiplet are provided in chapter 10. Then, chapter 11 develops methods to determine field theory amplitudes in pure spinor superspace from BRST cohomology arguments rather than by direct computation of the superstring result and its low energy limit. The orthogonal strategy is followed in chapter 12 where the complete string computation is carried out for the n point disk amplitude of the SYM multiplet, resulting in the main result (1.4.1). We will explain the rich structure of and harmony within this equation in section 12.3. As a byproduct of the n point superstring computation, we get a prescription to explicitly compute kinematic numerator factors for gauge theory amplitudes in the low energy limit which satisfy Jacobi identities dual to color factors. This procedure together with explicit examples up to seven point level can be found in the last chapter 13.

The main body is followed by several appendices. In the first appendix A, we will explain our conventions concerning indices and spinor algebra. Appendix B aims to give a lightning introduction into superconformal field theory. The third appendix C provides the background on spinor helicity variables. In four dimensional spacetime, they prove useful to simplify kinematic factors of amplitudes involving states of spin ≥ 1. Appendix D.1 contains supplementing material on hypergeometric worldsheet integrals. Our computations in pure spinor superspace give rise to specific kinematic packages – so-called BRST building blocks and Berends Giele currents. Their explicit definition involves lengthy expressions at higher rank, some examples are gathered in appendix E. Finally, appendix F lists the BCJ numerator for seven point tree amplitudes in gauge theories.

Part I

Basics of the open RNS superstring

Chapter 2

The RNS worldsheet CFTs

As explained in the introduction, the speciality of string theory lies in the emergence of $D = 10$ dimensional spacetime physics from a two dimensional quantum field theory on the worldsheet. The worldsheet theory is governed by the infinite dimensional superconformal symmetry group (which can be understood as a residual gauge symmetry) leading to exact solvability. In this chapter, we want to introduce the degrees of freedom in the superconformal field theory (SCFT) on the worldsheet describing superstrings propagating in flat Minkowski spacetime. It turns out to split into several decoupled CFTs which can be roughly classified as a matter- and ghost sector.

In the RNS approach to superstring theory, the spacetime coordinates X^m describing the string's embedding into spacetime are accompanied by Grassmann odd partners ψ^m under worldsheet supersymmetry (a subset of superconformal transformations). The (X^m, ψ^m) are referred to as the matter degrees of freedom on the worldsheet, they carry a (spacetime) vector index $m = 0, 1, \ldots, D-1$.

The major drawback of the RNS approach to superstring theory lies in the lack of manifest spacetime supersymmetry. The realization of spacetime fermions in the worldsheet SCFT looks completely different from spacetime bosons at first glance. Even though the CFT technique of bosonization makes this gap a bit smaller (see section 2.2), it does not achieve manifest $\mathcal{N} = 1$ spacetime supersymmetry. In $D = 10$ dimensions, this has only been accomplished by the pure spinor formalism, an alternative approach to the superstring which will be introduced in the later part III. It is based on different worldsheet variables compared to the RNS formalism.

The appearence of the residual superconformal gauge symmetry signalizes a redundancy in our description. One way to handle this is a direct elimination of the unphysical degrees of freedom using the so-called light cone gauge on the expense of full spacetime Lorentz symmetry.

We will follow another route throughout part I and II of this work which is compatible with Lorentz invariance in all steps of computations: In section 2.3, ghost fields $\{b, c, \beta, \gamma\}$ are introduced on the worldsheet with a meaning of "negative" degrees of freedom. This will become more transparent from the definition of a BRST charge (due to Becchi-Rouet-Stora-Tyutin) whose cohomology selects the physical spectrum.

The first sections are adapted to the closed string where the worldsheet is periodic in its spatial direction. This effectively doubles all the worldsheet degrees of freedom as we shall explain. The last section 2.4 introduces open strings, their boundary conditions and the SCFT description of their worldsheet.

2.1 Matter fields

The string coordinates X^m are viewed as functions on the worldsheet which describe its embedding into spacetime. The spatial extent of the string is parametrized by a coordinate σ^1, and each string segment sweeps out a worldline with proper time $\tau \equiv \sigma^0$. The σ^1 coordinate is periodic $\sigma^1 \equiv \sigma^1 + 2\pi$ for closed strings and restricted to the interval $\sigma^1 \in (0, \pi)$ for open strings, see the following figure 2.1.

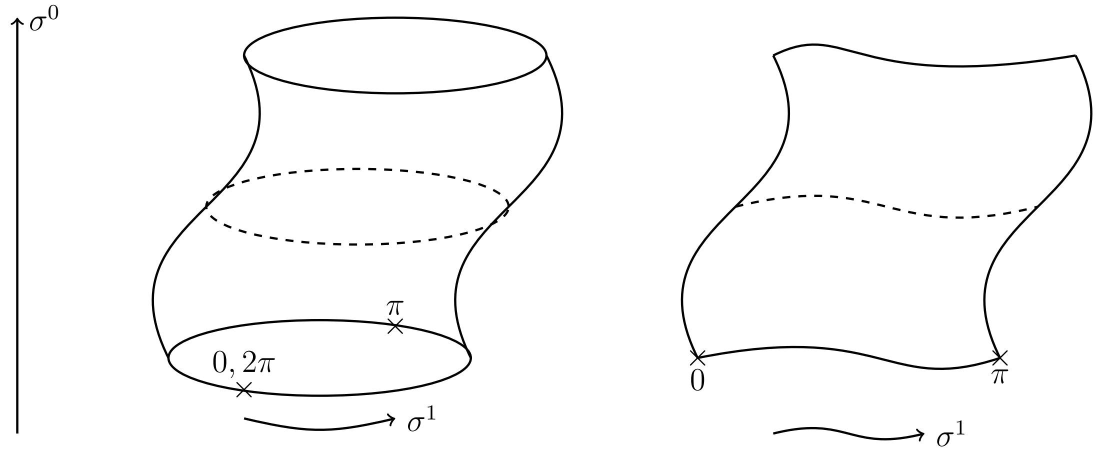


Figure 2.1: Worldsheets of closed and open strings

2.1.1 The bosonic worldsheet action

The embedding coordinates X^m describing the string's propagation in flat Minkowski spacetime are among the most important actors in the worldsheet SCFT. A first promising attempt to describe their dynamics is applying the principle of least action to the worldsheet surface – the natural generalization of the Lagrangian description of the point particle where the worldline length is taken as an action. In this approach, the equations of motion for the X^m are extracted from minimizing the so-called Nambu Goto action[1]

$$\mathcal{S}_{\rm NG}[X] \ = \ -T \int \mathrm{d}^2\sigma \ \sqrt{- \det \gamma} \ , \qquad \gamma_{ab} \ = \ \frac{\partial X^m}{\partial \sigma^a} \, \frac{\partial X^n}{\partial \sigma^b} \, \eta_{mn} \ . \tag{2.1.1}$$

It measures the worldsheet area by means of the induced metric γ_{ab} metric on the worldsheet. The prefactor T can be interpreted as the string tension which is related to the famous Regge slope parameter α' by $T = \frac{1}{2\pi\alpha'}$.

Unfortunately, the canonical quantization procedure is extremely hard to apply to an action with square root dependence on the fields. (The only hope to quantize such an action rests on light cone gauge on the expense of Lorentz symmetry.) This suggests to employ another action for the X^m equivalent to (2.1.1), the so-called Polyakov action. It introduces a worldsheet metric h_{ab} as an a priori independent field:

$$\mathcal{S}[X,h] \ = \ -\frac{T}{2} \int \mathrm{d}^2\sigma \ \sqrt{- \det h} \, h^{ab} \, \partial_a X^m \, \partial_b X^n \, \eta_{mn} \tag{2.1.2}$$

This action first of all describes D worldsheet scalar fields X^m coupled to the two dimensional metric h_{ab}. Equations of motion for the latter imply that h_{ab} is proportional to the induced metric $\gamma_{ab} = \partial_a X^m \partial_b X_m$ from (2.1.1),

$$\frac{\delta \mathcal{S}[X,h]}{\delta h^{ab}} \ = \ 0 \quad \Rightarrow \quad h_{ab} \ = \ \frac{2}{h^{cd} \, \partial_c X^m \, \partial_d X_m} \, \partial_a X^n \, \partial_b X_n \ = \ \frac{2 \, \gamma_{ab}}{h^{cd} \, \partial_c X^m \, \partial_d X_m} \ . \tag{2.1.3}$$

Remarkably, this proportionality constant $2/(h^{cd}\partial_c X^m \partial_d X_m)$ between $h_{ab} \sim \gamma_{ab}$ drops out of the equations of motion for X^m,

$$\frac{\delta \mathcal{S}[X,h]}{\delta X_m} \ = \ 0 \quad \Rightarrow \quad \partial_a(\sqrt{- \det h} \, h^{ab} \, \partial_b X^m) \ = \ \partial_a(\sqrt{- \det \gamma} \, \gamma^{ab} \, \partial_b X^m) \ = \ 0 \tag{2.1.4}$$

since $\sqrt{- \det h}$ scales inversely to h^{ab} in two dimensions. This proves that the Polyakov action (2.1.2) gives rise to the same X^m dynamics as the initial surface action (2.1.1). In the next subsection, we will examine further features of h_{ab} apart from avoiding the square root.

2.1.2 Gauge fixing the action

The Polyakov action (2.1.2) is the more promising starting point to quantize the worldsheet theory of the string's embedding coordinates X^m in Minkowski spacetime. An essential role is played by the action's symmetries which allow to gauge away the local h_{ab} degrees of freedom:

[1] The conventions for worldsheet- and spacetime indices are gathered in appendix A.1.

- Two dimensional diffeomorphisms: the action is written in completely reparametrization invariant manner. Under infinitesimal coordinate changes $\sigma^a \mapsto \tilde{\sigma}^a = \sigma^a - \eta^a(\sigma)$, the worldsheet fields h_{ab} and X^m transform as[2]

$$\delta_\eta h_{ab} \;=\; \nabla_a \eta_b \,+\, \nabla_b \eta_a \,, \qquad \delta_\eta X^m \;=\; \eta^a \, \partial_a X^m \,. \tag{2.1.6}$$

- Weyl invariance: in addition to coordinate changes, the Polyakov action is invariant under local rescaling $h_{ab} \mapsto \mathrm{e}^{2\phi(\sigma)} h_{ab}$ of the metric, so-called Weyl transformations. We have already used Weyl invariance of the X^m equations of motions in (2.1.4). Infinitesimally, Weyl rescalings affect the worldsheet fields as

$$\delta_\phi h_{ab} \;=\; 2\phi \, h_{ab} \,, \qquad \delta_\phi X^m \;=\; 0 \,. \tag{2.1.7}$$

  Note that Weyl transformations should not be thought of as coordinate changes because the X^m are scalars under δ_ϕ.

To summarize the symmetry analysis – using the three worldsheet functions $\eta^a(\sigma)$ and $\phi(\sigma)$, one can locally gauge away all the degrees of freedom in the worldsheet metric h_{ab} and fix it to a Minkowski configuration $\eta_{ab} := \mathrm{diag}(-1, +1)$. This choice is referred to as

$$\text{conformal gauge:} \;\; h_{ab}(\sigma) \;=\; \eta_{ab} \,. \tag{2.1.8}$$

Worldsheets of nonzero genus give rise to global obstructions against conformal gauge. These are briefly addressed in the later subsection 5.1.3.

2.1.3 Residual gauge transformations

To identify the residual gauge symmetry which preserves the gauge choice (2.1.8), we should emphasize again that the gauge parameters are three functions $\eta^{a=1,2}, \phi$ of both worldsheet coordinates σ^0 and σ^1. Functions of only one variable form a subset of measure zero, and it turns out that one can find a family of such reparametrizations which remain a symmetry after fixing conformal gauge.

At this moment, it makes sense to trade the Minkowski signature on the worldsheet for an Euclidean one for the sake of simplicity and elegance. We define Euclidean coordinates (σ^1, σ^2)

[2]The covariant derivative ∇_a strictly speaking involves the two dimensional Christoffel symbols

$$\nabla_a \eta_b \;=\; \partial_a \eta_b \,-\, \Gamma^c{}_{ab} \, \eta_c \;=\; \partial_a \eta_b \,-\, \frac{1}{2} \, h^{cd} \, (\partial_a h_{bd} \,+\, \partial_b h_{ad} \,-\, \partial_d h_{ab}) \, \eta_c \,. \tag{2.1.5}$$

and their complex combinations $(z, \bar{z})$ by

$$(\sigma^1, \sigma^2) \;:=\; (\sigma^1, i\sigma^0)\,, \qquad z \;:=\; \sigma^1 \,-\, i\sigma^2\,, \qquad \bar{z} \;:=\; \sigma^1 \,+\, i\sigma^2\,. \tag{2.1.9}$$

If the σ^1, σ^2 are real, then z and $\bar{z}$ are related by complex conjugation $\bar{z} = z^*$ but we shall think of them as independent coordinates in the following. Since the two dimensional line element factorizes in these complex coordinates, $\mathrm{d}s^2 = \mathrm{d}z\,\mathrm{d}\bar{z} \equiv \eta_{ab}\mathrm{d}x^a\,\mathrm{d}x^b$, (anti-)holomorphic functions of z (or $\bar{z}$ respectively) rescale the metric by an overall prefactor

$$\begin{aligned} z \;&\mapsto\; f(z) \quad\Rightarrow\quad \eta_{ab} \;\mapsto\; \frac{\mathrm{d}f(z)}{\mathrm{d}z}\,\eta_{ab} \;=:\; \partial f(z)\,\eta_{ab} \\ \bar{z} \;&\mapsto\; \bar{g}(\bar{z}) \quad\Rightarrow\quad \eta_{ab} \;\mapsto\; \frac{\mathrm{d}\bar{g}(\bar{z})}{\mathrm{d}\bar{z}}\,\eta_{ab} \;=:\; \bar{\partial}\bar{g}(\bar{z})\,\eta_{ab}\,, \end{aligned} \tag{2.1.10}$$

given by the holomorphic or antiholomorphic derivative $\partial := \frac{1}{2}(\partial_1 + i\partial_2)$, $\bar{\partial} := \frac{1}{2}(\partial_1 - i\partial_2)$ acting on f or $\bar{g}$, respectively. These kinds of h_{ab} transformation can be undone by a compensating Weyl transformation (say $\mathrm{e}^{-2\phi} = \partial f$), so the (anti-)holomorphic maps keep the metric in a configuration equivalent to $h_{ab} = \eta_{ab}$. In two dimensions, the residual gauge symmetry $z \mapsto f(z)$ and $\bar{z} \mapsto \bar{g}(\bar{z})$ coincides with the infinite dimensional group of conformal transformations.

Given the gauge fixing (2.1.8) and the choice of complex coordinates (2.1.9), the Polyakov action now takes the form

$$\mathcal{S}[X, h_{ab} = \eta_{ab}] \;=\; \frac{T}{2}\int \mathrm{d}^2 z\;\partial X^m\;\bar{\partial} X_m \tag{2.1.11}$$

and describes a conformally invariant field theory. Due to the equations of motion $\partial\bar{\partial}X^m = 0$, the first derivatives $\partial X^m(z)$ $\big(\bar{\partial}X^m(\bar{z})\big)$ are purely holomorphic (antiholomorphic). As a consequence, the embedding coordinates decompose into a sum of holomorphic and antiholomorphic parts:

$$\partial\bar{\partial}X^m \;=\; 0 \quad\Rightarrow\quad X^m(z,\bar{z}) \;=\; X^m_L(z) \;+\; X^m_R(\bar{z}) \tag{2.1.12}$$

In other words, any string excitation decomposes into a left- and right mover $X_L(z), X_R(\bar{z})$, depending on one of $\sigma^1 \pm \sigma^0$ only.

2.1.4 Adding worldsheet supersymmetry

The supersymmetric generalization of the reparametrization invariant Polyakov action (2.1.2) must involve two different worldsheet fermions – firstly the partner Ψ^m of the embedding coordinates X^m and secondly a gravitino χ_a pairing up with h_{ab} and gauging the local supersymmetry. The corresponding two dimensional Dirac gamma matrices are denoted by γ^a and the two dimensional spinors require a Dirac conjugate $\bar{\Psi}_m := \Psi^\dagger_m\gamma^0$. The full worldsheet action

$$\mathcal{S}[X, \Psi, h, \chi] \;=\; \frac{1}{8\pi}\int \mathrm{d}^2\sigma\;\sqrt{-\det h}\,\Big\{ -\frac{2}{\alpha'}\;h^{ab}\,\partial_a X^m\,\partial_b X_m \;+\; 2\,\bar{\Psi}^m\,\gamma^a\,\nabla_a\Psi_m$$

$$+ \ (\bar{\chi}_a \, \gamma^b \, \gamma^a \, \Psi^m) \ \left(\sqrt{\frac{2}{\alpha'}} \, \partial_b X_m \ + \ \frac{1}{4} \, \chi_b \, \Psi_m \right) \Bigg\} \qquad (2.1.13)$$

is invariant under local supersymmetry transformations with parameter $\epsilon(\sigma)$ which for instance affects the metric and the gravitino as $\delta_\epsilon h_{ab} \sim \epsilon \gamma_{(a} \chi_{b)}$ and $\delta_\epsilon \chi_a \sim \nabla_a \epsilon$.

The bosonic gauge fixing described in the previous subsection 2.1.2 has supersymmetric analogues. One can first of all use diffeomorphisms and local supersymmetry to bring (h_{ab}, χ_c) into the form $(\mathrm{e}^{2\phi} \eta_{ab}, \gamma_c \zeta)$ and then decouple the conformal factor $\mathrm{e}^{2\phi}$ and the worldsheet spinor ζ by means of super-Weyl transformations. The action (2.1.13) is thereby reduced to the following gauge fixed version, written in complex coordinates (2.1.9)

$$\mathcal{S}[X, \psi, \bar{\psi}] \ = \ \frac{1}{8\pi} \int \mathrm{d}^2 z \left\{ \frac{2}{\alpha'} \, \partial X^m \, \bar{\partial} X_m \ + \ \psi^m \, \bar{\partial} \psi_m \ + \ \bar{\psi}^m \, \partial \bar{\psi}_m \right\} . \qquad (2.1.14)$$

Note that we picked the Majorana Weyl basis for the Dirac spinor where its components read $\Psi^m = (\psi^m, \bar{\psi}^m)$.

The residual gauge invariance now extends to superconformal transformations. Among the general diffeomorphisms $\eta(z, \bar{z})$ and local supersymmetry transformations $\epsilon(z, \bar{z})$, one can still apply the purely holomorphic or antiholomorphic subsets $\eta(z), \epsilon(z)$ and $\bar{\eta}(\bar{z}), \bar{\epsilon}(\bar{z})$ of measure zero and compensate the metric- and gravtitino variation by means of super-Weyl symmetry.

From the fermionic equations of motion $\bar{\partial} \psi^m = \partial \bar{\psi}^m = 0$, we see that the Dirac spinor Ψ has one holomorphic component $\psi(z)$ and an antiholomorphic one $\bar{\psi}(\bar{z})$. This mimics the (anti-)holomorphicity of ∂X^m ($\bar{\partial} X^m$) in the bosonic sector.

2.1.5 Path integrals and two point functions

We will now apply path integral methods to the quadratic action (2.1.14) in order to define correlation functions and compute propagators for the free fields X^m and ψ^m. The correlator $\langle \ldots \rangle_{S^2}$ of some local operators inserted at $\ldots$ is defined by[3]

$$\langle \, \ldots \, \rangle_{S^2} \ := \ \frac{1}{\mathcal{Z}} \int \mathcal{D}X \, \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \, \mathrm{e}^{-\mathcal{S}[X,\psi,\bar{\psi}]} \ \ldots \ , \qquad \mathcal{Z} \ = \ \int \mathcal{D}X \, \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \, \mathrm{e}^{-\mathcal{S}[X,\psi,\bar{\psi}]} \quad (2.1.15)$$

with the standard normalization through the inverse partition function $\mathcal{Z}$. We will use various SCFT methods in later chapters to compute a large class of correlators. For the moment, we restrict out attention to the two point functions or propagators $\langle \phi_i(z, \bar{z}) \phi_j(w, \bar{w}) \rangle_{S^2}$.

[3]The closed string tree level worldsheet has the topology of the Riemann sphere $S^2 \cong \mathbb{C} \cup \{\infty\}$, so we add an extra $_{S^2}$ subscript whenever a closed string correlation function appears. We will most of the time look at disk correlators on open string worldsheets, that is why $\langle \ldots \rangle$ without specification refers to the disk topology by default.

The Dyson-Schwinger equation $\langle\phi_i(z,\bar z)\,\delta\mathcal{S}/\delta\phi_j(w,\bar w)\rangle_{S^2} = \delta_i^j\delta^2(z-w,\bar z-\bar w)$ gives rise to differential equations for the propagators of X^m and $\psi^m,\bar\psi^m$ which are solved by the following expressions[4]:

$$
\begin{aligned}
\langle\, X^m(z,\bar z)\, X^n(w,\bar w)\,\rangle_{S^2} \;&=\; -\frac{\alpha'}{2}\,\eta^{mn}\,\ln|z-w|^2\;, & \langle\, i\partial X^m(z)\, i\partial X^n(w)\,\rangle_{S^2} \;&=\; \frac{\alpha'\,\eta^{mn}}{2\,(z-w)^2} \\
\langle\, \psi^m(z)\,\bar\psi^n(\bar w)\,\rangle_{S^2} \;&=\; 0\;, & \langle\, \bar\psi^m(\bar z)\,\psi^n(w)\,\rangle_{S^2} \;&=\; 0 \\
\langle\, \psi^m(z)\,\psi^n(w)\,\rangle_{S^2} \;&=\; \frac{\eta^{mn}}{z-w}\;, & \langle\, \bar\psi^m(\bar z)\,\bar\psi^n(\bar w)\,\rangle_{S^2} \;&=\; \frac{\eta^{mn}}{\bar z-\bar w}
\end{aligned}
\tag{2.1.17}
$$

This can be directly translated into the following leading OPE singularities, see appendix B.2.2 for a detailed account of OPE techniques:

$$
\begin{aligned}
i\partial X^m(z)\, iX^n(w,\bar w) \;&\sim\; \frac{\alpha'\,\eta^{mn}}{2\,(z-w)} \;+\;\dots\;, & \psi^m(z)\,\psi^n(w) \;&\sim\; \frac{\eta^{mn}}{z-w}\;+\;\dots \\
i\bar\partial X^m(\bar z)\, iX^n(w,\bar w) \;&\sim\; \frac{\alpha'\,\eta^{mn}}{2\,(\bar z-\bar w)} \;+\;\dots\;, & \bar\psi^m(\bar z)\,\bar\psi^n(\bar w) \;&\sim\; \frac{\eta^{mn}}{\bar z-\bar w}\;+\;\dots
\end{aligned}
\tag{2.1.18}
$$

These are the essential SCFT data sufficient for computing any tree level correlation function among the $(X^m,\psi^m,\bar\psi^m)$ fields – see appendix B.4 and chapter 6.

2.1.6 The superconformal algebra

According to the standard methods of quantum field theory (QFT), we can compute the energy momentum tensor and the supercurrent for the $(X^m,\psi^m,\bar\psi^m)$ system by varying the action (2.1.13) with respect to the metric and the gravitino. As required for a conformal theory, the energy momentum tensor is traceless on classical level, $\eta^{ab}T_{ab}=0$ which translates into $T_{z\bar z}=0$ in the complex basis. Moreover, the conservation law $\partial^a T_{ab}=0$ makes sure that the surviving two components $T := T_{zz}$ and $\bar T := T_{\bar z\bar z}$ are holomorphic and antiholomorphic respectively. Explicitly, we have

$$
T(z) \;=\; \frac{1}{\alpha'}\;:\,i\partial X_m(z)\,i\partial X^m(z)\,:\;+\;\frac{1}{2}\;:\,\partial\psi_m(z)\,\psi^m(z)\,:\;. \tag{2.1.19}
$$

If a theory is superconformal (in addition to supersymmetric), then its supercurrent G_a has a vanishing gamma trace, $\gamma^a G_a = 0$ (where the spinor index of G_a is understood to be contracted with γ^a). This is the fermionic analogue of tracelessness $\eta^{ab}T_{ab}=0$. Hence, only two out of four G_a components are nonzero which can be separated into a holomorphic and an antiholomorphic

[4] An important tool in this computation is the Green's function of the $\partial\bar\partial$ operator

$$
\partial\bar\partial\ln|z-w|^2 \;=\; 4\pi\,\delta^2(z-w,\bar z-\bar w)\;. \tag{2.1.16}
$$

one $G_z = \big(G(z),0\big)$ and $G_{\bar z} = \big(0,\bar G(\bar z)\big)$ in the Majorana Weyl spinor basis. In terms of X^m and ψ^m, the holomorphic supercurrent reads

$$G(z) \ = \ \frac{1}{\sqrt{2\alpha'}} \ : i\partial X_m(z)\,\psi^m(z) \ : \ . \tag{2.1.20}$$

The normal ordering colons : . . . : subtract mutual contractions of the encompassed operators such that one point functions vanish, $\langle T(z)\rangle = 0$. Their presence is essential for applying Wick's theorem to the OPE between composite fields such as (T,G) and the elementary ones $(\partial X^m,\psi^m)$. For ease of notation, we will omit the colons for the rest of the thesis. They shall be thought of implicitly present whenever several operators are inserted at the same point.

The OPEs (2.1.18) imply that

$$\begin{aligned} T(z)\,i\partial X^m(w) \ &\sim \ \frac{i\partial X^m(w)}{(z-w)^2} \ + \ \frac{i\partial^2 X^m(w)}{z-w} \ + \ \ldots \\ T(z)\,\psi^m(w) \ &\sim \ \frac{\frac{1}{2}\,\psi^m(w)}{(z-w)^2} \ + \ \frac{\partial\psi^m(w)}{z-w} \ + \ \ldots \\ G(z)\,i\partial X^m(w) \ &\sim \ \sqrt{\frac{\alpha'}{2}}\,\left[\frac{\frac{1}{2}\,\psi^m(w)}{(z-w)^2} \ + \ \frac{\frac{1}{2}\,\partial\psi^m(w)}{z-w}\right] \ + \ \ldots \\ G(z)\,\psi^m(w) \ &\sim \ \sqrt{\frac{2}{\alpha'}}\,\frac{\frac{1}{2}\,i\partial X^m(w)}{z-w} \ + \ \ldots \end{aligned} \tag{2.1.21}$$

Analogous OPEs hold between the antiholomorphic fields $\bar\partial X^m,\bar\psi^m,\bar T$ and $\bar G$ whereas ∂X^m and ψ^m are nonsingular with respect to $\bar T$ and $\bar G$.

By comparison with equation (B.2.17), we can identify $\left(\psi^m, \sqrt{\frac{2}{\alpha'}}i\partial X^m\right)$ as a superconformal primary superfield with conformal weights $h=\frac{1}{2}$ for ψ^m and $h=1$ for $i\partial X^m$. It is certainly no surprise that $i\partial X^m$ transforms like a unit weight primary $i\partial_z X^m = \frac{\partial w}{\partial z}\,i\partial_w X^m$ under change of coordinates $z\mapsto w(z)$, but the $h=\frac{1}{2}$ property of ψ^m is less intuitive to see.

Apart from the conformal weights of the primary fields, there is another important number to characterize a representation of the superconformal algebra, namely the central charge c. It can be read off from the $T(z)T(w)$- and $G(z)G(w)$ OPEs. By computing all the possible contractions of T and G, we exactly reproduce the algebra (B.2.20) with

$$\text{central charge of the } (\partial X^m,\psi^m) \text{ CFT:} \quad c \ = \ \frac{3}{2}\,D \tag{2.1.22}$$

proportional to the number D of spacetime dimensions.

In superstring theory, the analysis can be taken over literally to the right moving or antiholomorphic counterparts $\bar\partial X^m,\bar\psi^m,\bar T$ and $\bar G$.[5] It yields right moving weights $\bar h = \big(\frac{1}{2},1\big)$ for $(\bar\psi^m,\bar\partial X^m)$ and central charge $\bar c = \frac{3D}{2}$.

[5]The situation in different in heterotic string theories [114, 115, 116] where one chiral half of X^m probes 16 additional dimensions.

In this section, we have seen that the embedding string coordinates X^m can be described by a worldsheet SCFT with fermionic partners ψ^m. The superconformal symmetry is a residual gauge symmetry which ultimately leads to a decoupling of the negative norm states in its quantum Hilbert space, see chapter 3. The representations of the superconformal algebra are characterized by conformal weights $h = \left(\frac{1}{2}, 1\right)$ for the primary fields $(\psi^m, \partial X^m)$ and central charge $c = \frac{3D}{2}$. The antiholomorphic counterparts $\bar{\partial} X^m$ and $\bar{\psi}^m$ furnish a decoupled antiholomorphic copy.

2.2 Spin fields and bosonization

Worldsheet fermions ψ^m live on the double cover of the complex plane, i.e. they are defined up to a sign only. Therefore, their boundary conditions under translation $\sigma^1 \mapsto \sigma^1 + 2\pi$ of the closed string can be either periodic or antiperiodic[6]. The states associated with antiperiodic (periodic) boundary conditions are said to fall into the Neveu Schwarz sector (Ramond sector) – in short, the NS- and R sectors, see appendix B.3.2 for more information from the SCFT point of view.

This section introduces spin fields [117,118], conformal primaries which interpolate between the NS- and R sectors. Their basic properties are extracted from the fermions' mode expansion, and their interplay with the ψ^m can be understood by means of the bosonization technique [119]. The whole discussion only takes holomorphic fields into account, the antiholomorphic copy which is present for closed strings follows exactly the same rules.

2.2.1 NS versus R sector

According to the discussion in appendix B.2.4, the mapping from the cylinder (parametrized by $\sigma^1 - i\sigma^2$) to the complex plane (with coordinate $z = \mathrm{e}^{\sigma^2 + i\sigma^1}$) gives rise to the following Laurent mode expansion:

$$\psi^m(z) \ = \ \sum_r \psi^m_r \, z^{-r-1/2} \ , \qquad r \in \begin{cases} \mathbb{Z} + \frac{1}{2} & : \ \text{NS sector} \\ \mathbb{Z} & : \ \text{R sector} \end{cases} \tag{2.2.23}$$

NS sector states are created by acting with half odd integer modes ψ^m_{-r} with $r \in \mathbb{N} - \frac{1}{2}$ on the vacuum. The R sector, on the other hand, is built from integer modes ψ^m_{-r} with $r \in \mathbb{N}_0$. Obviously, the periodicity properties of ψ^m under $z \mapsto \mathrm{e}^{2\pi i} z$ on the plane are opposite to their

[6]Lorentz invariance forces to pick a uniform sign in the boundary conditions for all the ψ^m with $m = 0, 1, ..., 9$

behaviour under $\sigma^1 \mapsto \sigma^1 + 2\pi$ on the cylinder:

$$\psi^m(\mathrm{e}^{2\pi i} z) \ = \ \begin{cases} +\psi^m(z) & : \text{ NS sector} \\ -\psi^m(z) & : \text{ R sector} \end{cases} \tag{2.2.24}$$

In order to arrive at the algebra (B.2.29) for the energy momentum modes with the correct central extension, we have to shift the L_0 generator of the Ramond sector by $\frac{D}{16}$, i.e. one has

$$L_m \ = \ \begin{cases} \frac{1}{2} \sum\limits_{r \in \mathbb{Z} + \frac{1}{2}} r \, \psi^n_{m-r} \, \psi_{n,r} & : \text{ NS sector} \\ \frac{1}{2} \sum\limits_{r \in \mathbb{Z}} r \, \psi^n_{m-r} \, \psi_{n,r} \ + \ \frac{D}{16} \, \delta_{m,0} & : \text{ R sector} \end{cases} . \tag{2.2.25}$$

Hence, lowest energy states of the R sector $|A\rangle_{\mathrm{R}}$ (where the label A is made precise in the next subsection) without any $\psi^m_{-r \leq -1}$ oscillators already have an L_0 eigenvalue of $\frac{D}{16}$. This must be the conformal dimension a primary field S_A which generates R ground states from the NS vacuum via[7]

$$|A\rangle_{\mathrm{R}} \ = \ \lim_{z \mapsto 0} S_A(z) \, |0\rangle_{\mathrm{NS}} \ , \qquad {}_{\mathrm{R}}\langle B| \ = \ \lim_{z \mapsto \infty} {}_{\mathrm{NS}}\langle 0| \, S_B(z) \, z^{D/8} \ , \qquad h(S_A) \ = \ \frac{D}{16} \ . \tag{2.2.26}$$

This saturates a general bound on the conformal weight h of Ramond ground states $|h\rangle_{\mathrm{R}}$ in arbitrary SCFTs: Requiring unitarity yields

$$0 \ \leq \ {}_{\mathrm{R}}\langle h| \, G_0 \, G_0 \, |h\rangle_{\mathrm{R}} \ = \ \left(2h \ - \ \frac{c}{12} \right) {}_{\mathrm{R}}\langle h|h\rangle_{\mathrm{R}} \quad \Rightarrow \quad h \ \geq \ \frac{c}{24} \ = \ \frac{D}{16} \tag{2.2.27}$$

according to the $\{G_r, G_s\}$ anticommutation relation in (B.2.29).

2.2.2 Spin fields as $SO(1,9)$ spinors and covariant OPEs

A peculiar feature of the R sector is the existence of zero modes ψ^m_0. The commutator $[L_0, \psi^m_r] = -r\psi^m_r$ implies that the ψ^m_0 do not change the conformal weight of R states, so they simply reshuffle the Ramond ground states $|A\rangle_{\mathrm{R}}$ (and likewise the descendant states obtained by $\psi^m_{r<0}$ action on $|A\rangle_{\mathrm{R}}$). In fact, the zero modes furnish a representation of the Clifford algebra,

$$\left\{ \psi^m_0 \ , \ \psi^n_0 \right\} \ = \ \eta^{mn} \quad \longleftrightarrow \quad \left\{ \Gamma^m \ , \ \Gamma^n \right\} \ = \ -2 \, \eta^{mn} \ . \tag{2.2.28}$$

Therefore, we can identify the ψ^m_0 with the $2^{D/2} \times 2^{D/2}$ gamma matrices Γ^m and conclude that both the ground states $|A\rangle_{\mathrm{R}}$ and their descendants $\psi^{m_1}_{-r_1} \ldots \psi^{m_k}_{-r_k} |A\rangle_{\mathrm{R}}$ form Dirac spinors of the Lorentz group in D dimensions.

[7] The origin of the $z^{D/8}$ factor in the ${}_{\mathrm{R}}\langle B|$ state is explained in appendix B.3.1.

Given that ψ^m (S_A) fall into vector (spinor) representations of $SO(1, D-1)$, it must be possible to express all their OPEs in manifestly covariant form in terms of Clebsch Gordan coefficients $\eta^{mn}, (\Gamma^m)_A{}^B$ and $\mathcal{C}_{AB}$ of the Lorentz group. Appendix A.1 lists our conventions for Dirac gamma matrices Γ^m and the charge conjugation matrix $\mathcal{C}$. Let us construct covariant OPEs step by step:

- The two point function of spin fields is determined by Poincaré covariance and $h(S_A) = \frac{D}{16}$:

$$\langle\, S_A(z)\, S_B(w)\,\rangle \;=\; \frac{\mathcal{C}_{AB}}{(z-w)^{D/8}} \tag{2.2.29}$$

  This reflects a leading singularity $\mathcal{C}_{AB}(z-w)^{-D/8}$ in the $S_A(z)S_B(w)$ OPE.

- The algebras (2.2.28) imply that the expectation value of ψ_0^m between asymptotic R ground states yields the corresponding matrix element of the product $\Gamma^m\mathcal{C}$,

$$_{\mathrm{R}}\langle A|\, \psi_0^m\, |B\rangle_{\mathrm{R}} \;=\; \frac{1}{\sqrt{2}}\,(\Gamma^m\,\mathcal{C})_{AB} \;=\; \frac{1}{\sqrt{2}}\,(\Gamma^m)_A{}^D\,\mathcal{C}_{DB}\,. \tag{2.2.30}$$

  The left hand side can be evaluated as a three point function which is determined by (B.1.8) in terms of the coefficient $C^m{}_A{}^B$ in $\psi^m(z)S_A(w) \sim (z-w)^{-1/2}C^m{}_A{}^B S_B(w) + ...$

$$\begin{aligned} _{\mathrm{R}}\langle A|\, \psi_0^m\, |B\rangle_{\mathrm{R}} \;&=\; \lim_{z_1\to\infty}\lim_{z_2\to 0}\, {}_{\mathrm{NS}}\langle 0|\, z_1^{D/8}\, S_A(z_1)\, z_2^{1/2}\,\psi^m(z_2)\, S_B(0)\, |0\rangle_{\mathrm{NS}} \\ &=\; \lim_{z_1\to\infty}\lim_{z_2\to 0} \frac{z_1^{D/8}\, z_2^{1/2}\, C^m{}_{AB}}{z_1^{D/8-1/2}\, z_{12}^{1/2}\, z_2^{1/2}} \;=\; C^m{}_A{}^D\,\mathcal{C}_{DB} \end{aligned} \tag{2.2.31}$$

  Comparison with (2.2.30) identifies the desired OPE coeffieient as $C^m{}_A{}^B = \frac{1}{\sqrt{2}}(\Gamma^m)_A{}^B$.

- Now that the complete three point function $\langle \psi^m(z_1)S_A(z_2)S_B(z_3)\rangle = \frac{(\Gamma^m\mathcal{C})_{AB}/\sqrt{2}}{z_{12}^{1/2} z_{13}^{1/2} z_{23}^{D/8-1/2}}$ is available, we can examine its $z_2 \to z_3$ limit. The $z_{23}^{1/2-D/8}$ factor implies that the OPE $S_A(z)S_B(w)$ contains the subleading singularity $(z-w)^{D/8-1/2}(\Gamma^m\mathcal{C})_{AB}\psi_m/\sqrt{2}$.

These findings are summarized as follows:

$$\psi^m(z)\, S_A(w) \;\sim\; \frac{1}{\sqrt{2}\,(z-w)^{1/2}}\,(\Gamma^m)_A{}^B\, S_B(w) \;+\; \dots \tag{2.2.32}$$

$$S_A(z)\, S_B(w) \;\sim\; \frac{\mathcal{C}_{AB}}{(z-w)^{D/8}} \;+\; \frac{(\Gamma^m\,\mathcal{C})_{AB}\,\psi_m(w)}{\sqrt{2}\,(z-w)^{D/8-1/2}} \;+\; \dots \tag{2.2.33}$$

The branch cut singularity in the $\psi^m(z)S_A(w)$ OPE ties in with the role of S_A transforming NS sector into R states and therefore flipping the periodicity of ψ^m. But the successive singularities $(z-w)^{-D/8}$ and $(z-w)^{-D/8+1/2}$ in (2.2.33) differing by half a $z-w$ power only do not fit into the general SCFT framework. This problem can be solved by decomposing the Dirac spinors

S_A into left- and right handed irreducibles S_α and $S^{\dot\alpha}$. Since gamma matrices are block off-diagonal in their chiral decomposition, $\Gamma^m{}_A{}^B = \left(\begin{smallmatrix} 0 & \gamma^m_{\alpha\dot\beta} \\ \bar\gamma^{m\dot\alpha\beta} & 0 \end{smallmatrix}\right)$, each combination of $(S_\alpha(z), S^{\dot\alpha}(\bar z))$ and $(S_\beta(w), S^{\dot\beta}(\bar w))$ only keeps one of the terms in (2.2.33). In $D = 10$ dimensions where $\mathcal{C}_{AB} = \left(\begin{smallmatrix} 0 & C_\alpha{}^{\dot\beta} \\ C^{\dot\alpha}{}_\beta & 0 \end{smallmatrix}\right)$,

$$S_\alpha(z)\, S^{\dot\beta}(w) \quad\sim\quad \frac{C_\alpha{}^{\dot\beta}}{(z-w)^{5/4}} \;+\; \dots \tag{2.2.34}$$

$$S_\alpha(z)\, S_\beta(w) \quad\sim\quad \frac{(\gamma^m\, C)_{\alpha\beta}\, \psi_m(w)}{\sqrt{2}\,(z-w)^{3/4}} \;+\; \dots \tag{2.2.35}$$

The overall physical state correlators (encompassing all decoupling CFT sectors) must not involve any fractional power dependence $z_{ij}^{\pm 1/2}, z_{ij}^{\pm 1/4}$ on worldsheet coordinates. Since the leading singularities in (2.2.34) and (2.2.35) differ in their $z-w$ power by 1/2, a projection to one chiral half of the R ground states is necessary to avoid branch cuts. Details will follow in section 3.1.7 on GSO projection.

2.2.3 Bosonization of worldsheet fermions

It is possible to map the interacting RNS CFT of the ψ^m, S_A fields to a free CFT of chiral bosons. Lorentz covariance gets obscured under this bosonization process but one can restore it later on by expressing correlation functions in terms of $SO(1,9)$ tensors. These covariantizing methods are explained in the later chapter 6.

Two fermions $\psi^{i=1,2}(z)$ subject to $\psi^i(z)\psi^j(w) \sim \delta^{ij}/(z-w)$ give rise to a CFT, equivalent to that of a free chiral boson $H(z)$ with singular behaviour $H(z)H(w) \sim -\ln(z-w)$. The complex combinations $\psi^\pm(z) := \frac{1}{\sqrt{2}}\big(\psi^1(z) \pm i\psi^2(z)\big)$ have the same local properties as the exponentials $\mathrm{e}^{\pm iH(z)}$ due to the following singularity structure[8]:

$$\mathrm{e}^{ipH(z)}\, \mathrm{e}^{iqH(w)} \quad\sim\quad (z-w)^{pq}\, \mathrm{e}^{i(p+q)H(w)} \;+\; \dots \tag{2.2.36}$$

Hence, their tree level correlation functions are guaranteed to match. The equivalence can easily be extended to all local operators such as $\mathrm{e}^{\pm inH(z)}$ with $n \in \mathbb{N}$. Moreover, the boson's derivative can be expressed as a (normal ordered) fermion bilinear, $i\partial H(z) = \psi^+(z)\psi^-(z)$.

Let us now apply this construction to the $D = 10$ component vector ψ^m of fermions. Bosonization requires a grouping into pairs which breaks the $SO(1,9)$ symmetry to a $SU(5)$ subgroup. Then one can check equivalence to a CFT of five chiral bosons H^0 and $H^{i=1,2,3,4}$

$$\psi^m(z)\, \psi^n(w) \quad\sim\quad \frac{\eta^{mn}}{z-w} \;+\; \dots \qquad\longleftrightarrow\qquad H^i(z)\, H^j(w) \quad\sim\quad -\,\delta^{ij}\, \ln(z-w) \;+\; \dots \tag{2.2.37}$$

[8] A careful derivation of this OPE can be found in appendix B.4

where exponentials of H are identified with the Cartan Weyl basis elements of the ten ψ's

$$\begin{aligned} \psi^{\pm 0}(z) &:= \frac{1}{\sqrt{2}} \big(\pm \psi^0(z) \; + \; \psi^1(z) \big) &\equiv \; \mathrm{e}^{\pm i H^0(z)} \\ \psi^{\pm i}(z) &:= \frac{1}{\sqrt{2}} \big(\psi^{2i}(z) \; \pm \; i\psi^{2i+1}(z) \big) &\equiv \; \mathrm{e}^{\pm i H^i(z)} \; . \end{aligned} \tag{2.2.38}$$

In view of a unified presentation of bosonized ψ^m and S_A, it makes sense to introduce five component lattice vectors of $D_5 \equiv SO(1,9)$ here. In slight abuse of notation, we will identify the vector index m with unit vectors $(0,\dots,0,\pm 1,0,\dots,0)$ whose nonzero entry sits at the j'th position (where $j = 0,1,\dots,4$). Then, the fermions in their Cartan Weyl basis can be written as follows,

$$\psi^{\pm j} \;\equiv\; \mathrm{e}^{\pm i \cdot H^j} \quad \Rightarrow \quad \psi^m \;\equiv\; \mathrm{e}^{i m \cdot H} \; . \tag{2.2.39}$$

We should admit that our discussion neglects Jordan-Wigner cocycle factors [120, 121]. These are additional algebraic objects accompanying the exponentials to ensure that $\mathrm{e}^{\pm i H^j}$ and $\mathrm{e}^{\pm i H^{k \neq j}}$ associated with different bosons anticommute. We drop cocycle factors to simplify the notation, it suffices to remember that they are implicitly present and that the bosonized representation of ψ^m still obey fermi statistics.

2.2.4 Bosonization of spin fields

Spin fields also fit into a bosonized picture. Since their defining role is the introduction of a branch cut for $\psi^{\pm j} \equiv \mathrm{e}^{\pm i H^j}$, they must be built as a combination of exponentials $\mathrm{e}^{\pm \frac{i}{2} H^j}$ to enforce the leading power of $(z-w)^{pq} = (z-w)^{\pm 1/2}$ in their OPE (2.2.36) with $\psi^{\pm j}$.

Spinor components of $D_n = SO(1,2n-1)$ can be labelled by their eigenvalues $\pm\frac{1}{2}$ under the n simultaneously diagonalized Lorentz generators $\Gamma^{pq}/2$ which are most conveniently chosen as $\Gamma^{2i,2i+1}$ with $i = 0,\dots,4$ to preserve the $SU(5)$ subgroup of $SO(1,9)$. This suggests to identify spinor indices with lattice vectors $\big(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2}\big)$ from the spinor conjugacy classes of D_5. They have n components with an individual $\pm$ sign each, and the chiral irreducibles can be disentangled by counting the number of minus signs:

$$S_{A = \left(\pm\frac{1}{2},\dots,\pm\frac{1}{2}\right)} \;\equiv\; \begin{cases} \text{left handed spinor } S_\alpha & : \; A \text{ has even number of '$-$' signs} \\ \text{right handed spinor } S^{\dot\alpha} & : \; A \text{ has odd number of '$-$' signs} \end{cases} \tag{2.2.40}$$

Given this dictionary between spinor indices and lattice vectors, we can make the bosonization of spin fields more precise: The S_A are be represented as an exponential of a D_5 lattice vector A from a spinor conjugacy class multiplied with the bosons H^j:

$$S_A(z) \;=\; \mathrm{e}^{i A \cdot H(z)} \, , \qquad A \;\equiv\; \big(\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2}\big) \tag{2.2.41}$$

This is a close parallel to the representation (2.2.39) of the ψ^m (where again, cocycle factors are left implicit).

The Cartan Weyl basis has the remarkable advantage that entries of gamma- and charge conjugation matrices can be written as delta functions for the D_n lattice vectors associated with the vector- and the spinor indices. Up to a complex phase (which could in principle be determined by keeping track of all the cocycles), one has

$$\left.\begin{array}{rcl} (\Gamma^m)_A{}^B & \sim & \sqrt{2}\,\delta(m+A-B) \\ \mathcal{C}_{AB} & \sim & \delta(A+B) \end{array}\right\} \quad \Rightarrow \quad (\Gamma^m\,\mathcal{C})_{AB} \ \sim \ \sqrt{2}\,\delta(m+A+B) \tag{2.2.42}$$

which allows a rederivation of the covariant OPEs (2.2.32) and (2.2.33) in bosonized language:

$$\begin{aligned} \psi^m(z)\,S_A(w) \ &\equiv \ \mathrm{e}^{im\cdot H(z)}\,\mathrm{e}^{iA\cdot H(w)} \ \sim \ \sum_B \frac{\delta(m+A-B)}{(z-w)^{1/2}}\,\mathrm{e}^{i(m+A)\cdot H(w)} \ + \ \dots \\ = \ \sum_B \frac{(\Gamma^m)_A{}^B}{\sqrt{2}\,(z-w)^{1/2}}\,\mathrm{e}^{iB\cdot H(w)} \ &+ \ \dots \ \equiv \ \frac{1}{\sqrt{2}\,(z-w)^{1/2}}\,(\Gamma^m)_A{}^B\,S_B(w) \ + \ \dots \end{aligned} \tag{2.2.43}$$

The leading singularity $(z-w)^{-1/2}$ only occurs if the sign of the m entry is opposite to the corresponding A entry, i.e. if the vector sum $m+A$ does not have a $\pm\frac{3}{2}$ entry and corresponds to another lattice vector B of the spinor conjugacy class. This justifies the occurrence of the $\delta(m+A-B)$ function. The $S_A(z)S_B(w)$ OPE (2.2.34) can be derived by means of a similarly δ function trick.

The computation in (2.2.43) is an example of how Lorentz covariance can be a posteriori restored in results in the bosonized CFT. In the later chapter 6, this prodecure will be applied to correlation functions involving ψ^m and S_A. The H^j system is a CFT of free bosons, and its tree level correlators are shown in appendix B.4 to obey the simple formula

$$\left\langle \prod_{k=1}^n \mathrm{e}^{iq_k\cdot H(z_k)} \right\rangle \ = \ \prod_{k<\ell}^n z_{k\ell}^{q_k\cdot q_\ell}\,\delta\left(\sum_{k=1}^n q_k\right) \tag{2.2.44}$$

in agreement with the singularities (2.2.36). The simplicity of these correlation functions exhibits the power of free field methods, this is the main motivation for the bosonization technique.

2.3 Ghost fields

We have argued in subsection 2.1.3 that conformal transformations are a residual gauge symmetry of the Polyakov action (2.1.2). More precisely, gauge freedom should rather be thought of as a redundancy in describing a system which is urgently needed to decouple negative norm states. Hence, it should definitely remain at quantum level. A main obstacle in preserving conformal symmetry in the quantum theory is the Weyl anomaly

$$\langle T_a{}^a \rangle \;=\; -\frac{c_{\mathrm{tot}}}{12}\, R \,. \tag{2.3.45}$$

in terms of the worldsheet Ricci scalar R and the total central charge c_{tot}. For nonzero c_{tot}, the energy momentum trace $T_a{}^a$ – which should vanish in conformally invariant theories – can receive a nonzero shift under Weyl transformations $h_{ab} \mapsto e^{2\phi} h_{ab}$ which modify the worldsheet Ricci scalar as $R \mapsto e^{-2\phi}(R - 2\nabla_a \nabla^a \phi)$.

In this section, we will carry out a Lorentz covariant gauge fixing along the lines of the standard Fadeev Popov procedure [122]. This introduces ghost systems which furnish an extra SCFT with negative central charge $c_{\mathrm{gh}} = -15$ such that the total central charge $c_{\mathrm{tot}} = c + c_{\mathrm{gh}} = \frac{3D}{2} - 15$ vanishes in $D = 10$ spacetime dimensions. This is one way of determining the critical dimension $D = 10$ in which superstrings can propagate consistently. The BRST quantization of string theory was first accomplished in [123], a broader overview and a more complete list of references can be found in [124].

The superghosts associated with supersymmetry fixing on the worldsheet give rise to some subtleties in selecting the appropriate vacuum of the theory. Again, bosonization will play a key role in simplifying these problems [125, 126, 119].

2.3.1 The b, c ghosts

Let us introduce the first ghost system (b, c) on the level of the bosonic string. Before gauge fixing the worldsheet metric h_{ab}, its partition function is given by

$$\mathcal{Z}_{\mathrm{bos}} \;=\; \frac{1}{\mathrm{vol}} \int \mathcal{D}X \; \mathcal{D}h \; \mathrm{e}^{-\mathcal{S}[X,h]} \,, \tag{2.3.46}$$

in terms of the Polyakov action $\mathcal{S}[X, h]$. The inverse "volume" factor $\frac{1}{\mathrm{vol}}$ refers to the fact that only physically distinct $h_{ab}(z)$ configurations not related by diffeomorphisms or Weyl transformations need to be considered in the $\mathcal{D}h$ integral. According to the Fadeev Popov procedure, the normalized integration $\frac{1}{\mathrm{vol}} \int \mathcal{D}h$ can be removed by setting h_{ab} to some gauge fixed value $\hat{h}_{ab}$ in the Polyakov action (e.g. $\hat{h}_{ab} = \eta_{ab}$) and inserting a $\hat{h}$ dependent (but gauge invariant) Fadeev Popov determinant $\Delta_{\mathrm{FP}}[\hat{h}]$:

$$\mathcal{Z}_{\mathrm{bos}} \;=\; \int \mathcal{D}X \;\, \Delta_{\mathrm{FP}}[\hat{h}] \, \mathrm{e}^{-\mathcal{S}[X,\hat{h}]} \tag{2.3.47}$$

The next step is to compute the Fadeev Popov determinant. Its inverse can be expressed as an integral over the Lie algebra of the diffeomorphisms and Weyl transformations. Given the variations $\delta_\eta \hat{h}_{ab} = \nabla_a \eta_b + \nabla_b \eta_a$ and $\delta_\phi \hat{h}_{ab} = 2\phi \hat{h}_{ab}$ of the gauge fixed metric, we express $\Delta^{-1}_{\rm FP}[\hat{h}]$ in terms of a δ functional[9]

$$\frac{1}{\Delta_{\rm FP}[\hat{h}]} \ = \ \int \mathcal{D}\phi \, \mathcal{D}\eta \ \delta[2\phi \, \hat{h}_{ab} \ + \ \nabla_a \eta_b \ + \ \nabla_b \eta_a] \ . \tag{2.3.48}$$

Starting from (2.3.48), three more steps are necessary to rewrite $\Delta_{\rm FP}$ in terms of a ghost action:

- take the integral representation of the δ functional (with integration variable $t^{ab} = t^{(ab)}$)

$$\frac{1}{\Delta_{\rm FP}[\hat{h}]} \ = \ \int \mathcal{D}\phi \, \mathcal{D}\eta \, \mathcal{D}t \ \exp\left(4\pi i \int {\rm d}^2\sigma \ \sqrt{|\hat{h}|} \, t^{ab} \left[\, \phi \, \hat{h}_{ab} \ + \ \nabla_a \eta_b \, \right] \right) \tag{2.3.49}$$

- integrate out the Weyl parameter ϕ which enforces t to be traceless $t^{ab}\hat{h}_{ab} = 0$

$$\frac{1}{\Delta_{\rm FP}[\hat{h}]} \ = \ \int \mathcal{D}\eta \, \mathcal{D}t \ \exp\left(4\pi i \int {\rm d}^2\sigma \ \sqrt{|\hat{h}|} \ \left[t^{ab} \ - \ \frac{1}{2} \, t^c{}_c \, \hat{h}^{ab} \right] \, \nabla_a \eta_b \right) \tag{2.3.50}$$

- trade the remaining two bosonic integration variables t^{ab}, η_a for fermions b^{ab}, c_a in order to obtain $\Delta_{\rm FP}$ rather than its inverse; also rescale (b^{ab}, c_a) to simplify prefactors

$$\Delta_{\rm FP}[\hat{h}] \ = \ \int \mathcal{D}c \, \mathcal{D}b \ \exp\left(\frac{1}{4\pi} \int {\rm d}^2\sigma \ \sqrt{|\hat{h}|} \, b_{ab} \, \nabla^a c^b \right) \ \equiv \ \int \mathcal{D}c \, \mathcal{D}b \ {\rm e}^{-\mathcal{S}_{\rm gh}[b,c]} \tag{2.3.51}$$

Plugging this back into the partition function,

$$\mathcal{Z}_{\rm bos} \ = \ \int \mathcal{D}X \, \mathcal{D}c \, \mathcal{D}b \ {\rm e}^{-\mathcal{S}[X,\hat{h}] - \mathcal{S}_{\rm gh}[b,c]} \tag{2.3.52}$$

$$\mathcal{S}_{\rm gh}[b,c] \ = \ -\frac{1}{4\pi} \int {\rm d}^2\sigma \ \sqrt{|\hat{h}|} \, b_{ab} \, \nabla^a c^b \tag{2.3.53}$$

we find that the (b, c) ghosts enter the overall action on the same footing as the dynamical fields X^m. Their role is to cancel the unphysical gauge degrees of freedom without giving up manifest Lorentz invariance like in the lightcone quantization procedure.

Many simplifications occur in conformal gauge $\hat{h}_{ab} = {\rm e}^{2\phi}\eta_{ab}$: The covariant derivatives in the ghost action (2.3.53) reduce to ordinary ones, the conformal factor ${\rm e}^{2\phi}$ drops out, and the action factorizes into two free theories of fields $(b, c) := (b_{zz}, c^z)$ and $(\bar{b}, \bar{c}) := (b_{\bar{z}\bar{z}}, c^{\bar{z}})$:

$$\mathcal{S}_{\rm gh}[b,c] \ = \ \frac{1}{4\pi} \int {\rm d}^2 z \ \left(b \, \bar{\partial} c \ + \ \bar{b} \, \partial \bar{c} \right) \tag{2.3.54}$$

This makes use of the tracelessness condition $b_{z\bar{z}} \sim b_a{}^a = 0$. Equations of motion $\bar{\partial} b = \bar{\partial} c = 0$ and $\partial \bar{b} = \partial \bar{c} = 0$ split the ghost components into holomorphic and antiholomorphic pieces

[9]This can be view as the infinite dimensional generalization of integrating over an n dimensional δ function $\int {\rm d}^n x \ \delta^n\big(f(x)\big) = (\det(\partial_i f_j))^{-1}$, composed with some map $f : \mathbb{R}^n \to \mathbb{R}^n$.

$b(z), c(z)$ and $\bar{b}(\bar{z}), \bar{c}(\bar{z})$. Their two point functions and the resulting OPEs can be derived in the spirit of subsection 2.1.5:

$$b(z)\, c(w) \quad \sim \quad \frac{1}{z-w} \ + \ \dots \ , \qquad b(z)\, b(w) \quad = \quad c(z)\, c(w) \quad = \quad \mathcal{O}(z-w) \ . \tag{2.3.55}$$

The energy momentum tensor follows from h_{ab} variation of the action (2.3.53) (where tracelessness of b_{ab} must be implemented by means of Lagrange multipliers)

$$T_{b,c}(z) \quad = \quad 2 \ : \partial c(z)\, b(z) \ : \ + \ : c(z)\, \partial b(z) \ : \tag{2.3.56}$$

which yields the following CFT data for the (b, c) system:

$$h(b) \quad = \quad 2 \ , \qquad h(c) \quad = \quad -1 \ , \qquad c_{b,c} \quad = \quad -\, 26 \tag{2.3.57}$$

This is an elegant way to derive the critical dimension $D = 26$ of the bosonic string: The total central charge receives the contribution D from the matter fields X^m and -26 from the ghosts.

2.3.2 The β, γ superghosts

The anticommuting b, c ghosts discussed in the last subsection remove the bosonic gauge redundancy of the worldsheet action. The supersymmetric completion (2.1.13) of the Polyakov action (see subsection 2.1.4) additionally enjoys local worldsheet supersymmety and super Weyl invariance which require fixing of fermionic gauge freedoms. Fixing these unphysical degrees of freedom therefore introduces a ghost system (β, γ) subject to Bose statistics.

We skip the detailed derivation of the (β, γ) action from a Fadeev Popov procedure. It follows the key ideas from the previous section to perform a path integral over infinitesimal variations of the worldsheet gravitino and to replace both the gauge parameter and the gauge field by a ghost variable of opposite statistics. In superconformal gauge, the fermionic Fadeev Popov determinant gives rise to the superghost action

$$\mathcal{S}_{\rm sgh}[\beta, \gamma] \quad = \quad \frac{1}{4\pi} \int \mathrm{d}^2 z \ \Big(\beta\, \bar{\partial} \gamma \ + \ \bar{\beta}\, \partial \bar{\gamma} \Big) \ . \tag{2.3.58}$$

Equations of motion $\bar{\partial}\beta = \bar{\partial}\gamma = 0$ and $\partial\bar{\beta} = \partial\bar{\gamma} = 0$ again provide chiral fields $\beta(z), \gamma(z)$ and antichiral ones $\bar{\beta}(\bar{z}), \bar{\gamma}(\bar{z})$. Let us give list of CFT data analogous to the end of the previous section:

$$\gamma(z)\, \beta(w) \quad \sim \quad \frac{1}{z-w} \ + \ \dots \ , \qquad \beta(z)\, \beta(w) \quad = \quad \gamma(z)\, \gamma(w) \quad = \quad \mathcal{O}(1) \tag{2.3.59}$$

$$T_{\beta,\gamma}(z) \quad = \quad -\,\frac{3}{2} \ : \partial \gamma(z)\, \beta(z) \ : \ - \ \frac{1}{2} \ : \gamma(z)\, \partial \beta(z) \ : \tag{2.3.60}$$

$$h(\beta) \;=\; \frac{3}{2}\,, \qquad h(\gamma) \;=\; -\frac{1}{2}\,, \qquad c_{\beta,\gamma} \;=\; +11 \tag{2.3.61}$$

The total ghost action allows to construct a supercurrent which intertwines the (b, c) system with its (β, γ) cousin,

$$G_{\rm gh}(z) \;=\; :\frac{1}{2}\,\gamma(z)\,b(z) \;-\; c(z)\,\partial\beta(z) \;-\; \frac{3}{2}\,\partial c(z)\,\beta(z):\;. \tag{2.3.62}$$

Members of two ghost systems can be combined into superconformal primary superfields $B \equiv (\beta, b)$ and $C \equiv (c, \gamma)$ of conformal weights $h(B) = \frac{3}{2}$ and $h(C) = -1$ respectively.

The mode expansions of the $h = 3/2$ and $h = -1/2$ superghosts must ensure that the ghost supercurrent obeys the same boundary conditions as the matter supercurrent:

$$\beta(z) \;=\; \sum_r \beta_r\, z^{-r-3/2}\,, \qquad \gamma(z) \;=\; \sum_r \gamma_r\, z^{-r-1/2}\,, \qquad r \in \begin{cases} \mathbb{Z}+\frac{1}{2} & : \text{ NS sector} \\ \mathbb{Z} & : \text{ R sector} \end{cases} \tag{2.3.63}$$

This will be important for the vacuum structure of the ghost sector, see subsection 2.3.4.

2.3.3 Ghost background charges

The ghost actions (2.3.54) and (2.3.58) possess classical $U(1)$ symmetries generated by the ghost currents

$$j_{b,c}(z) \;=\; -\,b(z)\,c(z)\,, \qquad j_{\beta,\gamma}(z) \;=\; -\,\beta(z)\,\gamma(z) \tag{2.3.64}$$

which associate ghost numbers -1 to (b, β) and $+1$ to (c, γ). However, these currents do not transform as conformal primary fields but instead suffer from anomalous cubic singularities in the OPEs with the corresponding energy momentum tensors:

$$T_{b,c}(z)\,j_{b,c}(w) \;\sim\; \frac{-\,3}{(z-w)^3} \;+\; \frac{j_{b,c}(w)}{(z-w)^2} \;+\; \frac{\partial j_{b,c}(w)}{z-w} \;+\; \dots \tag{2.3.65}$$

$$T_{\beta,\gamma}(z)\,j_{\beta,\gamma}(w) \;\sim\; \frac{+\,2}{(z-w)^3} \;+\; \frac{j_{\beta,\gamma}(w)}{(z-w)^2} \;+\; \frac{\partial j_{\beta,\gamma}(w)}{z-w} \;+\; \dots \tag{2.3.66}$$

The $(z - w)^{-3}$ coefficients are interpreted as background charges $Q_{b,c} = -3$ and $Q_{\beta,\gamma} = +2$ of the ghost systems which appear directly in the anomalous conservation law $\bar\partial j = \frac{1}{4} Q\sqrt{h}R$ for the currents. The anomaly is related to the existence of ghost zero modes, whose number $N_c, N_b, N_\gamma, N_\beta$ is determined by integrating the anomalous conservation law over the worldsheet:

$$N_c \;-\; N_b \;=\; +\,Q_{b,c}\,(g-1) \;=\; 3 \;-\; 3g \tag{2.3.67}$$

$$N_\gamma \;-\; N_\beta \;=\; -\,Q_{\beta,\gamma}\,(g-1) \;=\; 2 \;-\; 2g \tag{2.3.68}$$

The integral over the scalar worldsheet curvature R gives rise to the Euler number $\chi = 2-2g$ of the Riemann surface which we express in terms of the genus g for convenience. More formally, (2.3.67) and (2.3.68) can be viewed as an application of the Riemann-Roch theorem.

Next, the anomalous OPEs (2.3.65) and (2.3.66) of the energy momentum tensors with their ghost currents $j(z)$ imply that the zero modes j_0 have the unusual hermiticity property[10]

$$j_0^\dagger \;\; = \;\; -\,j_0 \; - \; Q \tag{2.3.69}$$

whereas $j_n^\dagger = j_{-n} \; \forall \; n \neq 0$. Let $|q\rangle = \mathcal{O}_q(0)|0\rangle$ denote a ghost number q state, i.e. $|q\rangle$ is generated by an operator $\mathcal{O}_q$ such that $\oint \frac{\mathrm{d}z}{2\pi i} j(z)\mathcal{O}_q(w) = q\mathcal{O}_q(w)$ or $j_0|q\rangle = |q\rangle$. Then its hermitian conjugate must have ghost number $-q-Q$ by virtue of (2.3.69), i.e.

$$|q\rangle^\dagger \;\; = \;\; \langle -q-Q| \; , \qquad \langle -q-Q|q\rangle \;\; = \;\; 1 \; . \tag{2.3.70}$$

We will need this to determine the ghost contribution to the NS ground state.

2.3.4 The (b,c) ghost ground state

We have seen in subsection 2.2.1 that the NS ground state $|0\rangle_{\mathrm{NS}}$ is more fundamental than lowest energy states $|A\rangle_{\mathrm{R}}$ of the R sector in the sense that the latter are created from $|0\rangle_{\mathrm{NS}}$ by action of $h = \frac{D}{16}$ spin fields S_A. Similar phenomena occur in the independent ghost CFTs: The following subsections determine the ghost sector ground states $|1\rangle_{b,c}$ and $|q\rangle_{\beta,\gamma}$ such that

$$|0\rangle_{\mathrm{NS}} \otimes |1\rangle_{b,c} \otimes |q=-1\rangle_{\beta,\gamma} \; , \qquad |A\rangle_{\mathrm{R}} \otimes |1\rangle_{b,c} \otimes \big|q=-\tfrac{1}{2}\big\rangle_{\beta,\gamma} \tag{2.3.71}$$

are lowest energy states with respect to the superghost algebra. In other words, they are annihilated by any "negative frequency" Laurent mode $b_r, c_r, \beta_r, \gamma_r$ with $r > 0$ which would lower the conformal dimension by r units. The rest of the physical superstring spectrum can then be obtained by acting with positive frequency ∂X^m- and ψ^m modes on (2.3.71), details will be explained in chapter 3.

As a consequence of the (c,γ) ghosts' negative conformal dimensions, both ground states in (2.3.71) require some modification of the vacuum state

$$|0\rangle \;\; \equiv \;\; |0\rangle_{\mathrm{NS}} \otimes |0\rangle_{b,c} \otimes |q=0\rangle_{\beta,\gamma} \; , \qquad \left\{ \begin{array}{rcl} L_n\,|0\rangle &=& 0 \quad \forall\, n \geq -1 \\ G_r\,|0\rangle &=& 0 \quad \forall\, r \geq -\frac{1}{2} \end{array} \right. \tag{2.3.72}$$

using operators from the ghost sectors.

[10]The construction (B.3.35) of asymptotic out-states relies on ϕ_h being a primary field, i.e. the non-primary nature of the ghost currents is responsible for the deviation from $j_n^\dagger = j_{-n}$ at $n = 0$.

Let us first of all discuss the vacuum structure of the (b, c) system. From the mode expansion $c(z) = \sum_n c_n z^{-n+1}$ of the $h = -1$ field, we can see that regularity of $c(0)|0\rangle_{b,c}$ requires $c_n|0\rangle_{b,c} = 0$ for $n \geq 2$ only, see subsection B.2.4. There is still one negative frequency mode c_1 which lowers the energy $[L_0, c_1] = -c_1$ but does not annihilate $|0\rangle_{b,c}$, i.e. the latter is not a highest weight state of the ghost algebra.

Since the mode in question squares to zero, $\{c_1, c_1\} = 0$, we can easily construct a state

$$|1\rangle_{b,c} \;\; := \;\; c(0)\,|0\rangle_{b,c} \;\; = \;\; c_1\,|0\rangle_{b,c} \tag{2.3.73}$$

which satisfies the highest weight condition $c_n|1\rangle_{b,c} = 0 \;\forall\, n \geq 1$ and carries conformal dimension $h = -1$ with respect to the overall energy momentum tensor $T + T_{b,c}$. The same is true for its degenerate partner $\partial c(0)c(0)|0\rangle_{b,c} = c_0 c_1|0\rangle_{b,c}$ but this one can be ruled out in view of BRST quantization, see subsection 3.1.2.

As a consequence of (2.3.70) at background charge $Q_{b,c} = -3$, the hermitian conjugate of the ghost number one state $|1\rangle_{b,c}$ has ghost number two,

$$|1\rangle^\dagger_{b,c} \;\; = \;\; \big(c_1\,|0\rangle_{b,c}\big)^\dagger \;\; = \;\; {}_{b,c}\langle 0|\,c_{-1}\,c_0 \quad \longleftrightarrow \quad {}_{b,c}\langle 0|\,c_{-1}\,c_0\,c_1\,|0\rangle_{b,c} \;\; = \;\; 1\;. \tag{2.3.74}$$

This property reflects the existence of three c zero modes on genus $g = 0$ worldsheets which are in one-to-one correspondence with the three globally defined diffeomorphisms on the Riemann sphere.

2.3.5 The NS sector superghost ground state

The problem of L_0 lowering modes also exists on the (β, γ) side: The $h = -1/2$ mode expansion $\gamma(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \gamma_r z^{-r+1/2}$ of the NS sector implies that $|0\rangle_{\beta,\gamma}$ is annihilated by $\gamma_{r\geq 1}$ only, i.e. $\gamma_{1/2}|0\rangle_{\beta,\gamma} \neq 0$. This is quite fatal because $\gamma_{1/2}$ as a bosonic mode does not square to zero like c_1, so it seems as if one could generate states with arbitrarily negative L_0 eigenvalue in view of $[L_0, (\gamma_{1/2})^n] = -\frac{n}{2}(\gamma_{1/2})^n$.

The solution to this problem lies in reexpressing $\beta(z)$ and $\gamma(z)$ as a product of two fermions each and then bosonizing one fermion system:

$$\beta(z) \;\; = \;\; \mathrm{e}^{-\phi(z)}\,\partial\xi(z)\;, \qquad \gamma(z) \;\; = \;\; \mathrm{e}^{\phi(z)}\,\eta(z) \tag{2.3.75}$$

The fields ϕ, η and ξ are defined by having a short distance behaviour which reproduces that of β and γ, see (2.3.59):

$$\phi(z)\,\phi(w) \;\; \sim \;\; -\ln(z-w) \; + \; \dots\;, \qquad \mathrm{e}^{p\phi(z)}\,\mathrm{e}^{q\phi(w)} \;\; \sim \;\; (z-w)^{-pq}\,\mathrm{e}^{(p+q)\phi(w)} \; + \; \dots \tag{2.3.76}$$

$$\eta(z)\,\xi(w) \;\sim\; \frac{1}{z-w} \;+\; \ldots\;, \qquad \eta(z)\,\eta(w) \;\sim\; \xi(z)\,\xi(w) \;\sim\; \mathcal{O}(z-w) \;+\; \ldots \tag{2.3.77}$$

The decoupled CFTs of the chiral boson ϕ and the fermions (η, ξ) are governed by energy momentum tensors[11]

$$T_{\eta,\xi} \;=\; -\,\eta\,\partial\xi\;, \qquad T_\phi \;=\; -\,\frac{1}{2}\,\partial\phi\,\partial\phi \;-\; \partial^2\phi \tag{2.3.78}$$

which give rise to conformal weights

$$h(\mathrm{e}^{q\phi}) \;=\; -\,\frac{q^2}{2} \;-\; q\;, \qquad h(\eta) \;=\; 1\;, \qquad h(\xi) \;=\; 0\;. \tag{2.3.79}$$

Notice that the zero mode of the ξ fermion does not enter the ghost algebra, this will be quite important for the discussion of ghost picture changing in the later subsection 3.1.6.

Given this partial bosonization of $\beta(z)$ and $\gamma(z)$, we claim that

$$|q=-1\rangle_{\beta,\gamma} \;:=\; \mathrm{e}^{-\phi(0)}\,|0\rangle_{\beta,\gamma} \tag{2.3.80}$$

defines a sensible NS superghost ground state. With the OPE (2.3.76), it is easy to check that $|q=-1\rangle_{\beta,\gamma}$ satisfies the essential property $\gamma_{1/2}|q=-1\rangle_{\beta,\gamma} = 0$ necessary for a lower bound in the L_0 spectrum:

$$\begin{aligned}\gamma_{1/2}\,|q=-1\rangle_{\beta,\gamma} \;&=\; \oint \frac{\mathrm{d}z}{2\pi i}\,z^{-1}\,\gamma(z)\,\mathrm{e}^{-\phi(0)}\,|0\rangle_{\beta,\gamma} \;=\; \oint \frac{\mathrm{d}z}{2\pi i}\,z^{-1}\,\eta(z)\,\mathrm{e}^{+\phi(z)}\,\mathrm{e}^{-\phi(0)}\,|0\rangle_{\beta,\gamma}\\ &=\; \oint \frac{\mathrm{d}z}{2\pi i}\,z^{-1}\,\eta(z)\,\big(\,z \;+\; \ldots\big)\,|0\rangle_{\beta,\gamma} \;=\; \big(\,\eta_0 \;+\; \ldots\big)|0\rangle_{\beta,\gamma} \;=\; 0\end{aligned} \tag{2.3.81}$$

We have used that $\eta_{n\geq 0}|0\rangle_{\beta,\gamma} = 0$ and the ellipsis in (2.3.83) due to subleading OPE sigularities contains negative frequency modes $\eta_{n\geq 1}$ only.

2.3.6 The R sector superghost ground state

The (β, γ) ghosts have a different periodicity in the R sector such that their z dependence can be expanded in half odd integer powers $z^{n-1/2}$. Hence, the ghost sector of the R ground states must be created by an operator which opens a branch cut for the (β, γ) system. Moreover, this operator must make sure that the positive frequency mode γ_1 annihilates the ground state.

Both requirements are met by the $h = \frac{3}{8}$ state

$$\big|q=-\tfrac{1}{2}\big\rangle_{\beta,\gamma} \;:=\; \mathrm{e}^{-\phi(0)/2}\,|0\rangle_{\beta,\gamma}\;. \tag{2.3.82}$$

[11] They can be computed from the β, γ energy momentum (2.3.60) by means of the point splitting method [127].

The OPEs $\mathrm{e}^{\pm\phi(z)}\mathrm{e}^{-\phi(0)/2} \sim z^{\pm 1/2}\mathrm{e}^{(\pm 1-\frac{1}{2})\phi(0)}$ give rise to the desired fractional $z^{\pm 1/2}$ power, and the annihilation by γ_1 follows from

$$\begin{aligned}\gamma_1 \left|q=-\tfrac{1}{2}\right\rangle_{\beta,\gamma} \;&=\; \oint \frac{\mathrm{d}z}{2\pi i}\, z^{-1/2}\,\gamma(z)\,\mathrm{e}^{-\phi(0)/2}\,|0\rangle_{\beta,\gamma} \;=\; \oint \frac{\mathrm{d}z}{2\pi i}\, z^{-1/2}\,\eta(z)\,\mathrm{e}^{+\phi(z)}\,\mathrm{e}^{-\phi(0)/2}\,|0\rangle_{\beta,\gamma} \\ &=\; \oint \frac{\mathrm{d}z}{2\pi i}\, z^{-1/2}\,\eta(z)\,\big(\, z^{1/2}\,\mathrm{e}^{\phi(0)/2} \;+\; \dots\big)\,|0\rangle_{\beta,\gamma} \;=\; 0\;. \end{aligned} \tag{2.3.83}$$

Analogous behaviour can be found for $\beta(z)$ with $\beta_{n\geq 0}$ modes annihilating $\left|q=-\frac{1}{2}\right\rangle_{\beta,\gamma}$. One can interpret $\mathrm{e}^{-\phi/2}$ as a superghost spin field.

The presence of the $\mathrm{e}^{-\phi/2}$ operator in the R ground state is essential for locality of physical correlation functions. Mutual OPEs (2.2.34) and (2.2.35) of spin fields involve fractional $z-w$ powers, independent on their relative chirality. In combination with the superghost spin field $\mathrm{e}^{-\phi/2}$, the left handed R ground states $|\alpha\rangle_{\mathrm{R}} \otimes \left|q=-\frac{1}{2}\right\rangle_{\beta,\gamma}$ give rise to local OPEs among themselves

$$S_\alpha(z)\,\mathrm{e}^{-\phi(z)/2}\,S_\beta(w)\,\mathrm{e}^{-\phi(w)/2} \;\sim\; \frac{(\gamma^m\,C)_{\alpha\beta}}{z-w}\,\psi_m(w)\,\mathrm{e}^{-\phi(w)} \;+\; \dots \tag{2.3.84}$$

This follows from the overall conformal weight $\frac{10}{16}+\frac{3}{8}=1$ of the operator $S_\alpha\mathrm{e}^{-\phi/2}$. Right handed spinors become local with respect to $S_\alpha\mathrm{e}^{-\phi/2}$ if their superghost charge gets shifted to $S^{\dot\beta}\mathrm{e}^{+\phi/2}$ or $S^{\dot\beta}\mathrm{e}^{-3\phi/2}$. The significance of these states will become clear in chapter 3.

2.4 Open strings versus closed strings

So far, the presentation of the superstring's worldsheet theory was adapted to the setup of closed strings which sweep out a periodic worldsheet in $\sigma^1 \equiv \sigma^1 + 2\pi$. But this work focuses on scattering amplitudes involving open string states, that is why this section is devoted to the characteristic features of open strings and their two endpoints. In our conventions, the spacelike coordinate σ^1 in the parametrization of an open string worldsheet lives in the interval $\sigma^1 \in (0,\pi)$, see figure 2.1.

We study the CFT description of these endpoints and briefly introduce the notion of D branes, a new class of dynamical objects which emerge from the necessity of boundary conditions. Finally, Chan Paton degrees of freedom are introduced which carry the color degrees of freedom of gauge bosons and other open string states.

2.4.1 Boundary conditions and D branes

In deriving the equations of motion $\partial\bar\partial X = \partial\bar\psi = \bar\partial\psi = 0$ from the gauge fixed worldsheet action (2.1.14) we have discarded boundary terms when integrating worldsheet derivative by parts. This is well justified at the spacelike boundary $\sigma^2 = \pm\infty$, but the timelike worldsheet boundaries furnished by the worldlines of the open string endpoints at $\sigma^1 = 0$ and $\sigma^1 = \pi$ require some extra care. They give rise to potential boundary terms

$$\begin{aligned} \delta\mathcal{S}[X,\psi,\bar\psi] &= \frac{1}{8\pi}\int \mathrm{d}^2 z \, \Big\{ -\frac{4}{\alpha'}\,\delta X^m\,\partial\bar\partial X_m \;+\; 2\,\delta\psi^m\,\bar\partial\psi_m \;+\; 2\,\delta\bar\psi^m\,\partial\bar\psi_m \Big\} \\ &+ \frac{i}{8\pi}\int \mathrm{d}\sigma^2 \, \Big\{ \frac{2}{\alpha'}\, i\delta X^m\, i\partial_1 X_m \;+\; \delta\psi^m\,\psi_m \;-\; \delta\bar\psi^m\,\bar\psi_m \Big\} \Big|_{\sigma^1=0}^{\sigma^1=\pi} \end{aligned} \tag{2.4.85}$$

which only vanish if appropriate boundary conditions at $\sigma^1 = 0, \pi$ are imposed. For each spacetime direction $m = 0, 1, \ldots, 9$ of X^m, there are two consistent choices:

$$\begin{cases} \delta X^m = 0 & : \text{ Dirichlet conditions} \\ \partial_1 X^m = 0 & : \text{ Neumann conditions} \end{cases} \tag{2.4.86}$$

In the Dirichlet case, the endpoints of the string are fixed to constants $X^m(\sigma^1 = 0) = a^m$ and $X^m(\sigma^1 = \pi) = b^m$. Neumann boundary conditions $\partial_1 X^m(\sigma^1 = 0, \pi) = 0$ state that there is no momentum flow through the freely moving endpoints points.

The fermions ψ^m and $\bar\psi^m$ can always be redefined by a relative sign such that $\psi^m(\sigma^1 = 0) = \bar\psi^m(\sigma^1 = 0)$ at the first endpoint. Only the sign between $\psi^m(\sigma^1 = \pi)$ and $\bar\psi^m(\sigma^1 = \pi)$ at the other endpoint $\sigma^1 = \pi$ becomes meaningful:

$$\begin{cases} \psi^m(\sigma^1 = \pi) = +\bar\psi^m(\sigma^1 = \pi) & : \text{ Ramond conditions} \\ \psi^m(\sigma^1 = \pi) = -\bar\psi^m(\sigma^1 = \pi) & : \text{ Neveu Schwarz conditions} \end{cases} \tag{2.4.87}$$

The effect of these boundary conditions can be neatly summarized by the so-called doubling trick: We define a holomorphic field ψ^m on the range $\sigma^1 \in (0, 2\pi)$ and eliminate the antiholomorphic part via $\bar\psi^m(\sigma^1) = \psi^m(2\pi - \sigma^1)$. According to (2.4.87), the piecewise defined field $\psi^m(\sigma^1 \in (0, 2\pi))$ is periodic under $\sigma^1 \mapsto \sigma^1 + 2\pi$ for Ramond conditions and antiperiodic for Neveu Schwarz conditions.

At first glance, one might think that only Neumann boundary conditions for the X^m are compatible with Poincaré invariance because $X^m(\sigma^1 = 0) = a^m$ would single out spacetime points a^m. The only way to accomodate Dirichlet boundary conditions with Poincaré invariance is to think of the spacetime regions where open string endpoints are confined as dynamical objects, i.e. as higher dimensional generalizations of strings. If Dirichlet conditions $X^m(\sigma^1 = 0, \pi) = const$ are imposed in the spacelike directions $m = 1, 2, \ldots, D-p-1$, then the endpoints are confined to a so-called Dp branes. The the D refers to Dirichlet and the specification p reminds of the brane's $p+1$ dimensional worldvolume. The following figure 2.2 gives the

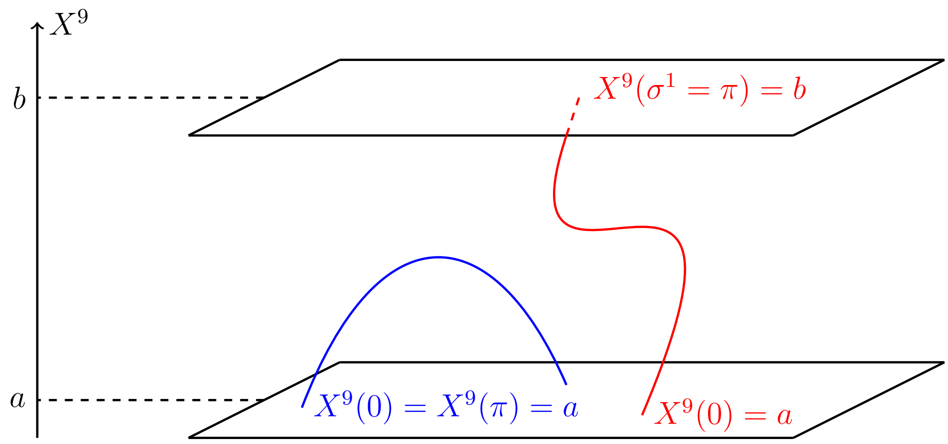


Figure 2.2: Open strings stretching between D8 branes with one transverse direction X^9.

example of a D8 brane with one transverse direction X^9 in ten dimensional spacetime. A D9 brane would be spacetime filling.

D branes are a fruitful area of reasearch for themselves. For good reasons, a full textbook is devoted to this subject [128], and a pedagogical lightning introduction can be found in [129]. Since theories of gravity do not admit rigid objects, D branes necessarily interact with the closed string sector and thereby become dynamical in shape and position. They were originally discovered in [59] as solitonic solutions to supergravity.

We will argue in the later subsection 2.4.4 that D branes give rise to gauge symmetries in spacetime. Section 4.1 shows a way to realize the Standard model gauge group, for instance.

2.4.2 Boundary CFT

The worldsheet of open strings has the topology of an infinite strip with spatial coordinate $\sigma^1 \in (0, \pi)$. In this parametrization, the endpoints of the open string sit at $\sigma^1 = 0$ and $\sigma^1 = \pi$. In contrast to the closed string situation, the complex coordinate $z = \sigma^1 - i\sigma^2$ parametrizes a strip with $0 \leq \mathrm{Re}\{z\} \leq \pi$ rather than a cylinder with a periodic direction. Under the conformal map $w = \mathrm{e}^{iz}$, the strip is mapped to the upper half plane $\mathrm{Im}\{w\} \geq 0$, and the worldlines of the open string endpoints fall onto the real axis $\mathrm{Im}\{w\} = 0$.

Open strings can be described in the framework of boundary SCFT: In presence of a boundary at $\mathrm{Im}\{0\} = 0$, the conformal symmetry group is reduced to those holomorphic maps $f(w)$ which preserve the real axis, $f(\mathbb{R}) \subset \mathbb{R}$. We will therefore need to introduce new worldsheet fields $X^{\mathrm{op}}, T^{\mathrm{op}}$ on the open string worldsheet, adjusted to the smaller symmetry algebra.

Since $T(z)$ and $G(z)$ generate superconformal transformations which are supposed to respect

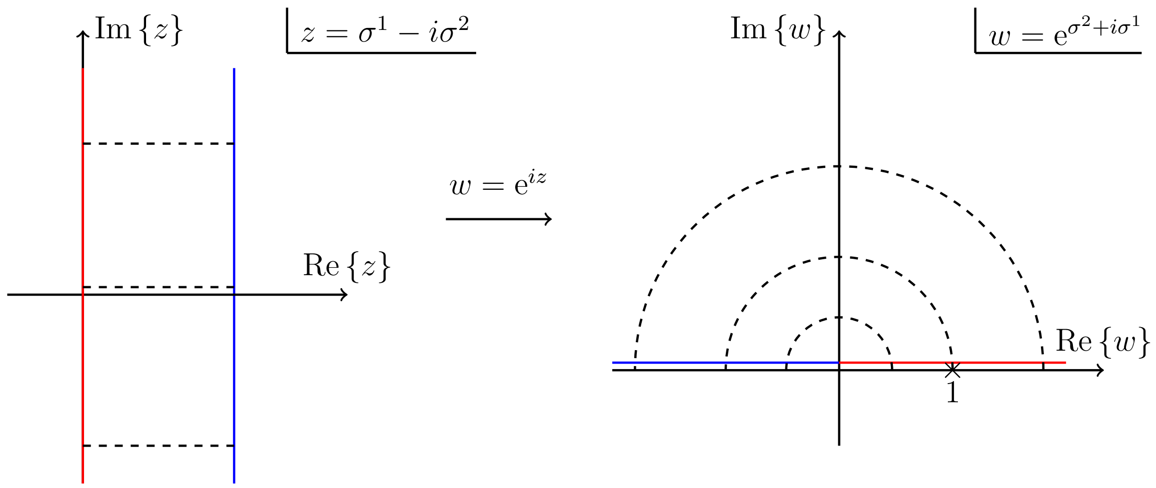


Figure 2.3: Mapping the open string worldsheet from the infinite strip to the upper half plane. In particular, the $\sigma^1 = 0$ ($\sigma^1 = \pi$) boundary drawn in red (blue) is mapped to the positive (negative) real axis.

the boundary, they have to satisfy

$$T(z) \;=\; \bar{T}(\bar{z}) \ , \qquad G(z) \;=\; \bar{G}(\bar{z}) \qquad \forall \ z, \bar{z} \in \mathbb{R} \ . \tag{2.4.88}$$

In the (σ^1, σ^2) coordinates, this first condition translates into $T_{12}(\sigma^1 = 0, \pi) = 0$ and therefore forbids energy momentum flow across the boundary.

The properties (2.4.88) are compatible with both Dirichlet- and Neumann boundary conditions for the worldsheet fields X^m, ψ^m and $\bar{\psi}^m$. On the complex plane[12], they read

$$\begin{array}{rcll} \partial X^m(z) &=& -\bar{\partial} X^m(\bar{z}) & : \ \text{Dirichlet conditions} \\ \partial X^m(z) &=& +\bar{\partial} X^m(\bar{z}) & : \ \text{Neumann conditions} \end{array} \qquad \forall \ z, \bar{z} \in \mathbb{R} \ . \tag{2.4.89}$$

Although the conformal fields $\partial X_m(z), \psi_m(z), T(z)$ and $G(z)$ are restricted to $\mathrm{Im}\,\{z\} \geq 0$, we can still define open string extensions ϕ_h^{op} on the full complex plane $\mathbb{C}$:

$$T^{\mathrm{op}}(z) \ := \ \begin{cases} T(z) & : \ \mathrm{Im}\,\{z\} \geq 0 \\ \bar{T}(\bar{z}) & : \ \mathrm{Im}\,\{z\} < 0 \end{cases} \ , \qquad \partial X_m^{\mathrm{op}}(z) \ := \ \begin{cases} \partial X_m(z) & : \ \mathrm{Im}\,\{z\} \geq 0 \\ \pm\,\bar{\partial} X_m(\bar{z}) & : \ \mathrm{Im}\,\{z\} < 0 \end{cases} \tag{2.4.90}$$

The $\pm$ sign in the definition of $\partial X_m^{\mathrm{op}}$ at $\mathrm{Im}\,\{z\} < 0$ is $+$ for Neumann- and $-$ for Dirichlet boundary conditions. The fermionic counterparts ψ_m^{op} and G^{op} are defined analogously.

The closed string worldsheet SCFT is governed by two energy momentum components T and $\bar{T}$ in the whole plane. In the open string sector, just one copy T^{op} remains in the whole

[12]Note that the relative sign between ∂ and $\bar{\partial}$ on the plane is opposite than on the cylinder.

plane. It contains the same information as both T and $\bar{T}$ in the upper half plane. The analogous construction for the X^m, ψ^m fields ensures that open string theories have fewer states than those of closed strings – on the cylinder, the operators ∂X and $\bar{\partial} X$ give rise to different states; on the strip they give rise to the same state. Generally speaking, states in boundary conformal field theory are in one-to-one correspondence with local operators defined on the boundary. See appendix B.3 for more information on the state operator map in generic CFTs.

The $\phi_h^{\rm op}$ are analytic because of the gluing conditions (2.4.88) and (2.4.89) on the real axis. They can therefore be expanded in Laurent modes such as $T^{\rm op}(z) = \sum_{n\in\mathbb{Z}} L_n^{\rm op} z^{-n-2}$ for the energy momentum tensor. The integral representation of the modes $L_n^{\rm op}$ involve a contour integral with two contributions:

$$L_n^{\rm op} \ = \ \oint_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi i} \, z^{n+1} \, T^{\rm op}(z) \ = \ \int_{\mathcal{C}^+} \frac{\mathrm{d}z}{2\pi i} \, z^{n+1} \, T(z) \ - \ \int_{\mathcal{C}^-} \frac{\mathrm{d}\bar{z}}{2\pi i} \, \bar{z}^{n+1} \, \bar{T}(\bar{z}) \tag{2.4.91}$$

The circle $\mathcal{C}$ around the origin splits into semicircles $z \in \mathcal{C}^+$ and $\bar{z} \in \mathcal{C}^-$ in the upper half plane (where T and $\bar{T}$ are defined) with ends on the real line $\mathbb{R}$, see figure 2.4:

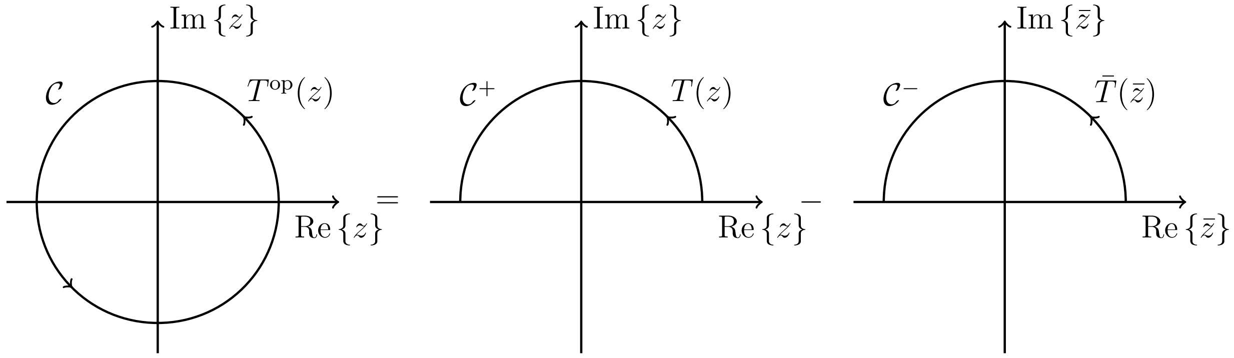


Figure 2.4: The contour integral over fields $T^{\rm op}(z)$ of the boundary CFT involves two integrals of $T(z)$ and $\bar{T}(\bar{z})$ over semicircles in the upper half plane.

The same contours contribute to any other charge built from conformal field $\phi_h^{\rm op}$. Since this work focuses on the open string sector, we will drop the $^{\rm op}$ superscripts from now on.

In the formalism of boundary CFT, D branes can be represented as additional Hilbert space sectors which can be added consistently to a closed string CFT rather than in the target geometric picture [130]. Moreover, the CFT approach leads to D branes without spacetime localization [131].

2.4.3 The free boson at the boundary

Open string scattering amplitudes require correlation functions of the free scalar fields X^m with Neumann boundary conditions on the real axis. As a starting point, let us compute the boundary propagator, starting with the ansatz

$$\langle\, X^m(z,\bar z)\, X^n(w,\bar w)\,\rangle \;\;=\;\; \eta^{mn}\, G(z,\bar z; w,\bar w)\;. \tag{2.4.92}$$

The Dyson-Schwinger equation together with the Neumann boundary conditions imply that the worldsheet function $G(z,\bar z; w,\bar w)$ has to satisfy

$$\partial_z \bar\partial_z G(z,\bar z; w,\bar w) \;=\; -\,2\pi\alpha'\,\delta^2(z-w)\;, \qquad (\partial_z \,-\, \bar\partial_z)\, G(z,\bar z; w,\bar w)\,\Big|_{z,\bar z,w,\bar w\in\mathbb{R}} \;=\; 0\;. \tag{2.4.93}$$

This kind of boundary value problem already arises in electrodynamics. The most convenient way so solve (2.4.93) makes use of an “image charge” in the lower half plane:

$$G(z,\bar z; w,\bar w) \;\;=\;\; -\,\alpha'\,\ln|z-w| \;-\; \alpha'\,\ln|z-\bar w| \tag{2.4.94}$$

If one argument of the two point function (2.4.92) is real (as it will be the case when open string states are inserted on the worldsheet boundary), the two terms in (2.4.94) coincide:

$$\langle\, iX^m(z\in\mathbb{R})\, iX^n(w\in\mathbb{R})\,\rangle \;\;=\;\; 2\alpha'\,\eta^{mn}\,\ln|z-w| \tag{2.4.95}$$

The boundary propagator has an extra factor of two compared to the bulk correlator (2.1.17) at real arguments, this is due to the image charge in (2.4.94). If further derivatives with respect to the real boundary coordinate are taken, we find $\langle i\partial X^m(z\in\mathbb{R}) i\partial X^n(w\in\mathbb{R})\rangle = 2\alpha'\eta^{mn}(z-w)^{-2}$. Hence, the canonically normalized superconformal primary on the open string worldsheet is $(\psi^m, \sqrt{2\alpha'} i\partial X^m)$. The conformal weights $h=\frac{1}{2}$ and $h=1$ are computed with respect to the following energy momentum tensor and supercurrent:

$$T(z) \;\;=\;\; \frac{1}{4\alpha'}\;:\, i\partial X^m\, i\partial X_m(z)\,: \tag{2.4.96}$$

$$G(z) \;\;=\;\; \frac{1}{2\sqrt{2\alpha'}}\;:\, i\partial X^m\,\psi_m(z)\,: \tag{2.4.97}$$

They are obtained from their closed string cousins (2.1.19) and (2.1.20) via $\alpha'\mapsto 4\alpha'$.

2.4.4 Chan Paton charges

Whenever a QFT has distinguished points, it is natural to have additional degrees of freedom residing at these special points. In the boundary SCFT on open string worldsheets, it is consistent with both spacetime Poincaré invariance and superconformal symmetry on the worldsheet to add so-called Chan Paton charges at the endpoints [132]. They do not enter the energy momentum tensor and therefore have trivial worldsheet dynamics.

Each open string state $|\Phi;\alpha_i\alpha_j\rangle$ can have two labels $\alpha_i,\alpha_j = 1,2,\dots,N$ in addition to any other Fock space quantum numbers collectively denoted as Φ. Such a state $|\Phi;\alpha_i\alpha_j\rangle$

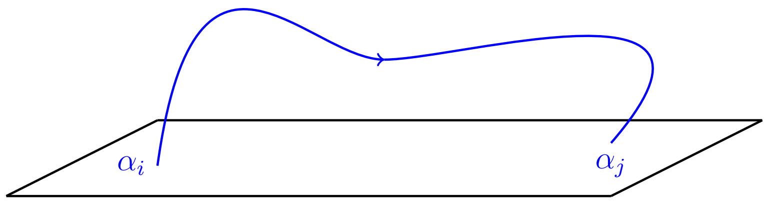


Figure 2.5: Chan Paton charges α_i, α_j associated with the endpoints of an open string.

decomposes into a complete set of $N \times N$ matrices $(T^a)_{\alpha_i}^{\alpha_j}$ via $|\Phi; a\rangle = (T^a)_{\alpha_i}^{\alpha_j} |\Phi; \alpha_i \alpha_j\rangle$. The reality condition for open string states[13] implies that the T^a are hermitian – in other words, they are generators of the gauge group $U(N)$. Since the endpoints transform in inequivalent representations (fundamental and antifundamental) of the gauge group, the open strings must be oriented[14]. The state $|\Phi; a\rangle$ transforms in the adjoint representation of the $U(N)$ symmetry of the Chan Paton degrees of freedom whereas the open string endpoints fall into the fundamental and antifundamental representation, respectively. One can think of this $U(N)$ as a gauge symmetry in spacetime.

An unbroken $U(N)$ symmetry appears whenever N D branes coincide in spacetime. Each endpoint of open strings on this brane stack can be associated with one of N different branes, so there are N^2 possibilities in total. When the branes move apart, the symmetry breaks down to $U(1)^N$, and vibration modes of open string stretching between different branes acquire a mass proportional to the separation of the branes. Therefore, D branes naturally provide a geometric realization of a Higgs mechanism.

[13]This reality condition follows from the fact that open string states are created by conformal fields living on the real axis. They are inserted at the origin, the point of infinite past on the worldsheet boundary.

[14]In type I superstring theories, the worldsheet parity is modded out, i.e. its strings are unoriented. In this case, the Chan Paton matrices are generators of either $SO(N)$ or $USp(N)$.

Chapter 3

The open superstring spectrum in $D = 10$ dimensions

In this chapter, we explain how to build the full superstring spectrum on top of the NS- and R ground states which were systematically constructed in the previous chapter. First of all, physical state conditions are formulated and analyzed from the SCFT point of view in section 3.1. Then, we will apply this to mass levels $n = 0$ and $n = 1$ as well as to the bosonic and fermionic highest spin states of higher mass levels $n > 1$.

All the way through this work, a major task is the organization of the physical states into representations of the Poincaré group, i.e. to identify the scalars, vectors, p forms and higher rank tensors. This chapter focuses on the ten dimensional case, whereas the later chapter 4 discusses dimensional reduction to the Poincaré group in four spacetime dimensions. The mass square $m^2 = -k^2$ is the first Poincaré Casimir for which we establish a one-to-one correspondence to conformal weights very soon. The spin quantum numbers which further specify the occurring representations are governed by the stabilizer subgroup which leaves the spacetime momentum invariant, namely $SO(D-2)$ for massless states (with $k^2 = 0$) and $SO(D-1)$ for massive states in a D dimensional spacetime.

An essential property of superstring theories is their spacetime supersymmetric spectrum. That is why we will introduce the spacetime SUSY charge in a CFT framework and explicitly compute its action on massless physical states.

The closed superstring spectrum can be obtained as a double copy of the open string sector. In other words, the ingredients introduced in this chapter enable to construct closed string states as a straightforward tensor product. Since we are interested in scattering amplitudes of open string states only, their closed string relatives won't be addressed explicitly.

3.1 Physical states

To obtain a mathematically and physically sensible open string spectrum, we need to impose two selection rules. The former is based on a worldsheet BRST operator and guarantees various SCFT mechanisms to conspire consistently. The latter is referred to as "GSO projection" and must be introduced for two reasons: Firstly, it renders RNS correlation functions local (i.e. it completes fractional z_{ij} powers to integers), and secondly, the projection guarantees a supersymmetric mass spectrum free of tachyons.

3.1.1 Vertex operators

As explained in appendix B.3, Hilbert space states in SCFTs are in one-to-one correspondence to conformal fields. Those fields which create a physical state from its value at infinite past are called vertex operators $V(z)$:

$$|\text{phys}\rangle \;\; = \;\; \lim_{z\to 0} V(z)\,|0\rangle \tag{3.1.1}$$

The worldsheet theory of open strings is a boundary SCFT, so the vertex operators for open string states live on the boundary. We will always work in coordinates where it corresponds to the real axis $z \in \mathbb{R}$. According to subsection 2.4.4, open string states carry Chan Paton degrees of freedom subject to trivial worldsheet dynamics. They decouple from all dynamical degrees of freedom in constructing the spectrum and computing scattering amplitudes. That is why we won't display Chan Paton charges in vertex operators, they can always be thought of as implicitly accompanying the $V(z)$ we will introduce in the followings sections.

The action of charges $Q = \oint \frac{\mathrm{d}z}{2\pi i} j(z)$ built from conserved currents $j(z)$ on a state $|\text{phys}\rangle$ follows from the singularities in the $j(z)V(w)$ OPE, see e.g. appendix B.3.1. However, we are dealing with a boundary SCFT in the open string sector, see subsection 2.4.2. The definition of currents j^{op} and vertex operators V^{op} on the full complex plane comes from gluing together a holomorphic and antiholomorphic half $j(z), \bar{j}(\bar{z})$ and $V(z), \bar{V}(\bar{z})$ each of which lives on the upper half plane. Equation (2.4.90) gives an example of this doubling trick. The contour integral involved in the evaluation of $Q|\text{phys}\rangle$ strictly speaking splits into two semicircles $\mathcal{C}^{\pm}$ illustrated in figure 3.1,

$$Q\,|\text{phys}\rangle \;\; = \;\; \oint_0 \frac{\mathrm{d}z}{2\pi i}\, j^{\text{op}}(z)\, V^{\text{op}}(0)\,|0\rangle \;\; = \;\; \left(\int_{\mathcal{C}^+} \frac{\mathrm{d}z}{2\pi i}\, j(z)\, V(0) \; - \; \int_{\mathcal{C}^-} \frac{\mathrm{d}\bar{z}}{2\pi i}\, \bar{j}(\bar{z})\, \bar{V}(0) \right) |0\rangle \tag{3.1.2}$$

Whenever we make reference to contour integrals and -deformations in later chapters, we should be aware of their decomposition (3.1.2) into $\mathcal{C}^{\pm}$ parts with holomorphic and antiholomorphic integrand, respectively. But most of the computations can be performed without detailed

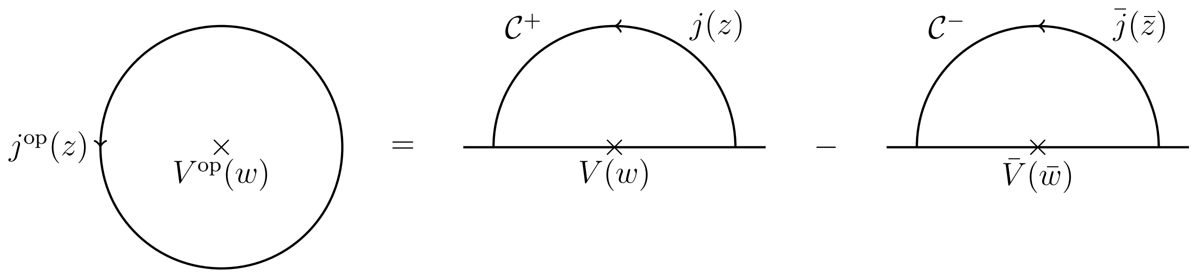


Figure 3.1: The boundary CFT treatment of the contour integral involved in $Q|\mathrm{phys}\rangle$.

knowledge about this, so we won't display the $^{\rm op}$ superscripts along with currents $j(z)$ and open string vertex operators $V(z)$ in the subsequent.

3.1.2 The BRST operator

In contrast to the lightcone quantization approach, the covariant quantization using (b, c, β, γ) ghosts does not explicitly remove the timelike and longitudinal components of the worldsheet fields X^m and ψ^m. It is clear from the lightcone analysis that these components do not contribute to physical states, so we need a mechanism to decouple them and to distinguish between physical and unphysical states in the covariant approach. This job is done by a nilpotent and hermitian BRST operator $Q_{\rm BRST}$ whose cohomology determines the physical spectrum.

In generic gauge theories quantized by BRST methods, the operator $Q_{\rm BRST}$ acts like a fermionic gauge transformation on the matter fields with its gauge parameter replaced by an anticommuting ghost variable. In addition, $Q_{\rm BRST}$ involves quadratic contributions in the ghost variables which represent gauge variations of the ghosts and which are needed to achieve nilpotency $Q_{\rm BRST}^2 = 0$. In superstring theory, requiring nilpotency of the BRST operator is one way of computing the critical dimension $D = 10$.

In the RNS approach to the superstring, the construction of $Q_{\rm BRST}$ is based on the observation that the worldsheet action (after gauge fixing the metric and the gravitino)

$$\begin{aligned} \mathcal{S}_{\rm tot} \ &= \ \frac{1}{8\pi} \int \mathrm{d}^2 z \, \Big\{ \frac{2}{\alpha'} \, \partial X^m \, \bar{\partial} X_m \ + \ \psi^m \, \bar{\partial} \psi_m \ + \ \bar{\psi}^m \, \partial \bar{\psi}_m \\ &\qquad\qquad + \ 2 \left(b \, \bar{\partial} c \ + \ \bar{b} \, \partial \bar{c} \ + \ \beta \, \bar{\partial} \gamma \ + \ \bar{\beta} \, \partial \bar{\gamma} \right) \Big\} \end{aligned} \tag{3.1.3}$$

has a symmetry which mixes matter- and ghost degrees of freedom. It is generated by the conserved current[1]

$$j_{\mathrm{BRST}} \;=\; c\left(T \;+\; \frac{T_{b,c}+T_{\beta,\gamma}}{2}\right) \;-\; \gamma\left(G \;+\; \frac{G_{\mathrm{gh}}}{2}\right) \tag{3.1.4}$$

where T and G denote the matter contributions to the energy momentum tensor and supercurrent whereas $T_{b,c}, T_{\beta,\gamma}, G_{\mathrm{gh}}$ exclusively involve the ghosts, see (2.3.56), (2.3.60) and (2.3.62) for their explicit expressions. In the open string sector, (2.4.96) and (2.4.97) give the correct normalization for the matter fields T and G, respectively.

In subsection 2.3.5, we have introduced a bosonized representation (2.3.75) for the (β,γ) system, in particular we have associated superghost charges q to the exponentials $\mathrm{e}^{q\phi}$. It is convenient to split the conserved charge due to the BRST current (3.1.4) into terms Q_j of different superghost charges $j = 0, 1$ and $j = 2$:

$$\begin{aligned} Q_{\mathrm{BRST}} \;&=\; \oint \frac{\mathrm{d}z}{2\pi i}\, j_{\mathrm{BRST}}(z) \;=\; Q_0 \;+\; Q_1 \;+\; Q_2 && (3.1.5)\\ Q_0 \;&=\; \oint \frac{\mathrm{d}z}{2\pi i}\, \big(c\,(T+T_{\beta,\gamma}) \;+\; b\,c\,\partial c\big) && (3.1.6)\\ Q_1 \;&=\; -\oint \frac{\mathrm{d}z}{2\pi i}\,\gamma\, G \;=\; -\oint \frac{\mathrm{d}z}{2\pi i}\,\mathrm{e}^{\phi}\,\eta\, G && (3.1.7)\\ Q_2 \;&=\; -\frac{1}{4}\oint \frac{\mathrm{d}z}{2\pi i}\, b\,\gamma^2 \;=\; -\frac{1}{4}\oint \frac{\mathrm{d}z}{2\pi i}\, b\,\mathrm{e}^{2\phi}\,\eta\,\partial\eta && (3.1.8) \end{aligned}$$

Physical states $|\mathrm{phys}\rangle$ must lie in the cohomology of the nilpotent and hermitian BRST operator for two reasons:

- Since Q_{BRST} generates gauge transformations, physical states must be BRST closed,

$$Q_{\mathrm{BRST}}\,|\mathrm{phys}\rangle \;=\; 0\;, \tag{3.1.9}$$

  to guarantee their gauge invariance.

- BRST exact states $Q_{\mathrm{BRST}}|\lambda\rangle$ (for arbitrary $|\lambda\rangle$) have zero norm and decouple from BRST closed states due to hermiticity of Q_{BRST}. Hence, physical states cannot be BRST exact,

$$|\mathrm{phys}\rangle \;\neq\; Q_{\mathrm{BRST}}\,|\lambda\rangle\;. \tag{3.1.10}$$

Given the partition (3.1.5) of Q_{BRST} into pieces of different ghost number, physical states $|\mathrm{phys}\rangle$ of uniform ghost number (like the ground states $|q=-1\rangle_{\beta,\gamma}$ and $|q=-\frac{1}{2}\rangle_{\beta,\gamma}$ introduced in subsection 2.3.5 and 2.3.6) are necessarily annihilated by all of Q_0, Q_1 and Q_2 separately.

[1] One can add total derivatives such as $\partial^2 c$ to the present form (3.1.4) of j_{BRST} to turn it into a $h=1$ conformal primary. These additions do not affect Q_{BRST} since total derivatives drop out of contour integrals.

3.1.3 Conformal weights and integrated vertex operators

Let us take a closer look at the ghost charge zero condition $Q_0|\mathrm{phys}\rangle = 0$ and derive the resulting constraint on the associated vertex operator. According to the discussion in subsection 2.3.4, the ground state of the (b,c) sector is generated by $c(0)$, so we can split the vertex operator as $|\mathrm{phys}\rangle = c(0)V_h(0)|0\rangle$ with V_h denoting a weight h field built from matter fields $i\partial X^m, \psi^m$ and (β,γ) superghost operators such as $\mathrm{e}^{q\phi}$. It turns out that Q_0 closedness of $|\mathrm{phys}\rangle$ implies that V_h is a conformal primary field of weight $h = 1$:

$$\begin{aligned} 0 \ \stackrel{!}{=} \ Q_0\,|\mathrm{phys}\rangle \ &= \ \oint \frac{\mathrm{d}z}{2\pi i}\,\big(c\,(T+T_{\beta,\gamma}) \ + \ b\,c\,\partial c\big)(z)\,c(0)\,V_h(0)\,|0\rangle \\ &= \ \oint \frac{\mathrm{d}z}{2\pi i}\,\Big(c(0) \ + \ z\,\partial c(0) \ + \ \dots\Big)\,\Big(\frac{c\,h\,V_h(0)}{z^2} \ + \ \frac{c\,\partial V_h(0)}{z} \ + \ \dots\Big)\,|0\rangle \\ &\quad + \ \oint \frac{\mathrm{d}z}{2\pi i}\,\Big(c\,\partial c(0) \ + \ z\,\partial\big(c\,\partial c\big)(0) \ + \ \dots\Big)\,\Big(\frac{1}{z} \ + \ \dots\Big)\,V_h(0)\,|0\rangle \\ &= \ (h-1)\,\partial c\,c\,V_h(0)\,|0\rangle \end{aligned} \tag{3.1.11}$$

If V_h was non-primary, there would be higher singularities in its OPE with $T(z)$, and $Q_0V_h(0)|0\rangle$ would receive non-cancelling corrections with higher ghost derivatives. We are using the anti-commuting property $c(0)c(0) = 0$ here. Moreover, the conformal dimension of the primary V_h is determined to be $h = 1$ as a necessary condition for BRST closure of $c(0)V_h(0)|0\rangle$.[2] In the subsequent, we will refer to the $h = 1$ conformal primaries $V_{h=1}$ which create physical states from $|1\rangle_{b,c}$ as vertex operators (rather than to the composite $V(z) = c(z)V_h(z)$).

The c ghost has three zero modes at tree level (and less on higher genus Riemann surfaces) which are saturated in a three point correlation function. Therefore, correlation functions of more than three states require a representation of any additional state without further $c(0)$ occurrence. Let us examine the properties of a state $V_h(0)|0\rangle$ without $c(0)$ insertion under Q_0 action:

$$\begin{aligned} Q_0\,V_h(0)\,|0\rangle \ &= \ \oint \frac{\mathrm{d}z}{2\pi i}\,\big(c\,(T+T_{\beta,\gamma}) \ + \ b\,c\,\partial c\big)(z)\,V_h(0)\,|0\rangle \\ &= \ \oint \frac{\mathrm{d}z}{2\pi i}\,\Big(c(0) \ + \ z\,\partial c(0) \ + \ \dots\Big)\,\Big(\frac{h\,V_h(0)}{z^2} \ + \ \frac{\partial V_h(0)}{z} \ + \ \dots\Big)\,|0\rangle \\ &= \ \partial\big(\,c\,V_h\,\big)(0)\,|0\rangle \ + \ (h-1)\,\partial c\,V_h(0)\,|0\rangle \ . \end{aligned} \tag{3.1.12}$$

At conformal weight $h = 1$, the state $Q_0V_{h=1}(0)|0\rangle$ is a total derivative $\partial\big(cV_h\big)(0)|0\rangle$. BRST closedness can be achieved by integrating over insertion points w of $V_h(w)$. Hence, the following "integrated vertex operators" give rise to a potentially BRST invariant state,

$$Q_0 \int \mathrm{d}w \ V_{h=1}(w)\,|0\rangle \ = \ 0 \ . \tag{3.1.13}$$

[2] In the operator approach, this fixes the normal ordering ambiguity in L_0.

The two objects $\int \mathrm{d}w \, V_h(w)|0\rangle$ and $c(0)V_h(0)|0\rangle$ can be thought of as two representations of the same physical state with different (b, c) ghost number. The $h = 1$ primary V_h is obviously more fundamental than the combination $V = cV_h$ which we can view as the c ghost number one representative of the state. This justifies to focus on V_h in the following: As mentioned above, the term "vertex operator" will always refer to V_h rather than $V = cV_h$. We will explore further BRST restrictions on V_h due to Q_1 and Q_2 in the later subsection 3.1.5.

3.1.4 The mass shell condition

In this subsection, we show that conformal weight $h = 1$ of vertex operators V_h determines the mass square of the physical state $|\mathrm{phys}\rangle = c(0)V_h(0)|0\rangle$. The starting point are the Poincaré generators P^m for translations and M^{mn} for Lorentz rotations following from the standard Noether procedure[3]:

$$P_m \;=\; \frac{1}{2\alpha'} \oint \frac{\mathrm{d}z}{2\pi i} \, i\partial X_m \; , \qquad M^{mn} \;=\; \oint \frac{\mathrm{d}z}{2\pi i} \left(\frac{1}{2\alpha'} \, iX^{[m} \, i\partial X^{n]} \;+\; \psi^{[m} \, \psi^{n]} \right) \tag{3.1.16}$$

The spacetime momentum k_m of a physical state is supposed to be its eigenvalue under the momentum operator P_m, so we have to find its eigenstates. According to subsection 2.4.3, the OPEs of the free boson in the open string sector are normalized as

$$iX^m(z) \, iX^n(w) \;\sim\; 2\alpha' \ln|z - w| \;+\; \dots \; , \qquad i\partial X^m(z) \, iX^n(w) \;\sim\; \frac{2\alpha'}{z - w} \;+\; \dots \; . \tag{3.1.17}$$

On these grounds, it is shown in appendix B.4.1 that "plane wave" exponentials $\mathrm{e}^{ik\cdot X}(z)$ obey

$$i\partial X^m(z) \, \mathrm{e}^{ik\cdot X}(w) \;=\; \frac{2\alpha' \, k^m}{z - w} \, \mathrm{e}^{ik\cdot X}(w) \;+\; \dots \tag{3.1.18}$$

such that $\mathrm{e}^{ik\cdot X}(0)|0\rangle$ can be identified as an eigenstate of P^m with eigenvalue k^m. The X^m contribution to the energy momentum tensor $T = \frac{1}{4\alpha'} i\partial X^m i\partial X_m + \dots$ assigns conformal weight

$$h\big(\mathrm{e}^{ik\cdot X}\big) \;=\; \alpha' \, k^2 \;=\; -\,\alpha' \, m^2 \tag{3.1.19}$$

to the plane waves $\mathrm{e}^{ik\cdot X}$, where the mass square is introduced through the mass shell condition $k^2 = -m^2$ for a state of spacetime momentum k^m. Let us separate the $\mathrm{e}^{ik\cdot X}$ part of the vertex operator V_h from the contribution $v_{\hat{h}}$ of β, γ ghosts and $i\partial X^m, \psi^m$ oscillators:

$$V_h(z) \;=\; v_{\hat{h}}(z) \, \mathrm{e}^{ik\cdot X}(z) \; , \qquad |\mathrm{phys}\rangle \;=\; v_{\hat{h}}(0) \, \mathrm{e}^{ik\cdot X}(0) \, c(0) \, |0\rangle \tag{3.1.20}$$

[3] It can be checked by means of the contour integral techniques of B.2.5 that they satisfy the Poincaré algebra

$$\big[\, P^m \, , \, P^n \, \big] \;=\; 0 \; , \qquad \big[\, M^{mn} \, , \, P^p \, \big] \;=\; P^m \, \eta^{np} \;-\; P^n \, \eta^{mp} \tag{3.1.14}$$

$$\big[\, M^{mn} \, , \, M_{pq} \, \big] \;=\; \delta^n_p \, M^m{}_q \;-\; \delta^m_p \, M^n{}_q \;-\; \delta^n_q \, M^m{}_p \;+\; \delta^m_q \, M^n{}_p \; . \tag{3.1.15}$$

Then, m^2 is uniquely determined by the conformal weight $\hat{h}$ of the "remainder" $v_{\hat{h}}$ to ensure that cV_h creates a Q_0 closed state:

$$Q_0 \Big(c(0)\, v_{\hat{h}}(0)\, \mathrm{e}^{ik\cdot X}(0)\, |0\rangle \Big) \ = \ 0 \quad \Rightarrow \quad m^2 \ = \ \frac{\hat{h} \, - \, 1}{\alpha'} \tag{3.1.21}$$

Already the Q_{BRST} component of ghost number zero turns out to fix the mass of a physical state $|\mathrm{phys}\rangle$. It increases with the conformal weight $\hat{h}$ of the $i\partial X^m$-, ψ^m and $\mathrm{e}^{q\phi}$ contribution $v_{\hat{h}}$ to $|\mathrm{phys}\rangle$. Further constraints on polarization wave functions follow from $Q_1|\mathrm{phys}\rangle = 0$, we will illustrate this with some concrete examples in sections 3.2 and 3.3.

3.1.5 Superconformal primaries

In this subsection, we will explore the implications of physical states $|\mathrm{phys}\rangle$ being annihilated by the ghost number one part Q_1 of the BRST charge. For this purpose, the contributions of the (β, γ) system to the remainder field $v_{\hat{h}}$ have to be isolated. This leads to different scenarios in the NS- and R sectors, see subsections 2.3.5 and 2.3.6 for the superghost structure of their ground states:

$$V_{h=1}(z) \ = \ \begin{cases} \Phi^{\mathrm{NS}}_{h=1/2}(z)\, \mathrm{e}^{-\phi(z)} & : \ \text{NS sector} \\ \Phi^{\mathrm{R}}_{h=5/8}(z)\, \mathrm{e}^{-\phi(z)/2} & : \ \text{R sector} \end{cases} \tag{3.1.22}$$

In contrast to the $v_{\hat{h}}$ from the previous subsection, the primary fields Φ^{NS}_h and Φ^{R}_h with respect to the matter CFT encompass the plane wave part $\mathrm{e}^{ik\cdot X}$. For simplicity, we are considering the integrated version of the vertex operator without c ghost insertion. Throughout this work, we will use the notation $V^{(q)} = \Phi_h\, \mathrm{e}^{q\phi}$ for a vertex operator with ghost charge q.

Let us evaluate Q_1 closedness on a physical NS sector state:

$$\begin{aligned} 0 \ &\overset{!}{=} \ \big[\, Q_1 \, , \ \Phi^{\mathrm{NS}}_h(0)\, \mathrm{e}^{-\phi(0)} \,\big] \ = \ - \oint \frac{\mathrm{d}z}{2\pi i}\, G(z)\, \mathrm{e}^{\phi(z)}\, \eta(z)\, \Phi^{\mathrm{NS}}_h(0)\, \mathrm{e}^{-\phi(0)} \\ &= \ - \oint \frac{\mathrm{d}z}{2\pi i} \Big(z\, \eta(0) \ + \ \ldots \Big)\, G(z)\, \Phi^{\mathrm{NS}}_h(0) \end{aligned} \tag{3.1.23}$$

The contour integral in the second line vanishes if the OPE $G(z)\Phi^{\mathrm{NS}}_h(0)$ contains no higher poles than z^{-1}, i.e. if the leading singularity is of the form,

$$G(z)\, \Phi^{\mathrm{NS}}_h(w) \ \sim \ \frac{\frac{1}{2}\, \Phi^{\mathrm{NS}}_{h+1/2}}{z - w} \ + \ \ldots \ . \tag{3.1.24}$$

But this is equivalent to the statement that Φ^{NS}_h is the lowest component of a superconformal primary field $(\Phi^{\mathrm{NS}}_h, \Phi^{\mathrm{NS}}_{h+1/2})$ at $h = \frac{1}{2}$, see (B.2.17).

Closedness of R sector states under Q_1 requires the $(z-w)^{-3/2}, (z-w)^{-5/2}, \ldots$ singularities in the $G(z)\Phi^{\mathrm{R}}_h(w)$ OPE to vanish:

$$0 \ \overset{!}{=} \ \big[\, Q_1 \, , \ \Phi^{\mathrm{R}}_h(0)\, \mathrm{e}^{-\phi(0)/2} \,\big] \ = \ - \oint \frac{\mathrm{d}z}{2\pi i}\, G(z)\, \mathrm{e}^{\phi(z)}\, \eta(z)\, \Phi^{\mathrm{R}}_h(0)\, \mathrm{e}^{-\phi(0)/2}$$

$$= \; - \oint \frac{\mathrm{d}z}{2\pi i} \left(z^{1/2}\,\eta(0)\,\mathrm{e}^{+\phi(0)/2} \; + \; \dots \right) G(z)\,\Phi_h^{\mathrm{R}}(0) \tag{3.1.25}$$

This is equivalent to saying that the state $\Phi_h^{\mathrm{R}}(0)|0\rangle$ is annihilated by all the non-negative modes $G_{r\geq 0}$ and constitutes a highest weight state of the super Virasoro algebra.

Once the conformal weight is fixed to $h = 1$, the constraints $L_{n>0}|\mathrm{phys}\rangle = 0$ due to Q_0 automatically follow from $G_{r>0}|\mathrm{phys}\rangle = 0$, so Q_1 closedness of a vertex operator $V_{h=1}$ implies $[Q_0, V_{h=1}] = 0$.

Closedness under Q_2 does not impose any restriction on vertex operators $V_{h=1} = \Phi_h \mathrm{e}^{q\phi}$ of superghost charge $q < +\frac{1}{2}$:

$$\begin{aligned} \big[\, Q_2 \;,\; \mathrm{e}^{q\phi(w)}\,\Phi_h(w) \,\big] \;&=\; -\frac{1}{4} \oint \frac{\mathrm{d}z}{2\pi i}\, b\,\mathrm{e}^{2\phi}\,\eta\,\partial\eta(z)\,\mathrm{e}^{q\phi(w)}\,\Phi_h(w) \\ &=\; -\frac{1}{4} \oint \frac{\mathrm{d}z}{2\pi i}\, b\,\eta\,\partial\eta(z)\,(z-w)^{-2q}\,\mathrm{e}^{(q+2)\phi(w)}\,\Phi_h(w) \end{aligned} \tag{3.1.26}$$

In the following subsection we will find a context where ghost numbers $q \geq \frac{1}{2}$ can indeed appear.

3.1.6 Superghost pictures

We have pointed out in subsection 3.1.3 that physical states require a representation with c ghost numbers 0 and 1 such that we can saturate the (b, c) zero modes in correlation functions. The same requirement arises for superghosts β and γ whose background charge $Q_{\beta,\gamma} = +2$ imposes the following charge conservation condition:

$$\left\langle \prod_{k=1}^{n} \mathrm{e}^{q_k \phi(z_k)} \right\rangle \;=\; \prod_{k<\ell}^{n} z_{k\ell}^{-q_k q_\ell}\, \delta\left(\sum_{k=1}^{n} q_k + 2 \right) \tag{3.1.27}$$

The notion of superghost picture changing makes sure that physical states have a representative with the appropriate $\mathrm{e}^{q\phi}$ charge to obtan nonzero correlation functions. Also, it gives a meaning to the upper superfield component $\Phi_{h+1/2}^{\mathrm{NS}}$ emerging from (3.1.24).

We have emphasized in subsection 2.3.5 that the bosonization of the β ghost involves the derivative of a $h = 0$ fermion $\xi(z)$ but not its zero mode ξ_0. This allows to construct further representations $V^{(q+1)}$ of a physical state from $\xi_0 V^{(q)} = \xi_0 \Phi_h(z) \mathrm{e}^{q\phi(z)}$: Since the zero mode ξ_0 lies outside the superghost algebra, the state $-2[Q_{\mathrm{BRST}}, \xi \Phi_h \mathrm{e}^{q\phi}]$ is not BRST exact in the strict sense (even though it might look so):

$$V^{(q+1)}(w) \;=\; -\,2\,\big[\, Q_{\mathrm{BRST}} \;,\; \xi(w)\,V^{(q)}(w) \,\big] \tag{3.1.28}$$

This state does not decouple from physical states because of its ξ_0 admixture. But still, the property $Q_{\mathrm{BRST}}^2 = 0$ makes sure that (3.1.29) gives rise to a BRST closed state which is non-exact from the point of view of the "small" superghost algebra excluding the ξ_0 zero mode. The vertex operator $V^{(q+1)}$ is said to create a higher "superghost picture" of its associated physical state, with charge $q+1$ rather than q. By iterating the map (3.1.28), one can generate more representatives of the same physical state with even higher ghost charges.

It turns out that for the purpose of tree amplitudes, we can restrict our attention to ghost pictures $-1 \leq q \leq 1/2$. In this range, only the Q_1 part (3.1.7) of the BRST charge contributes to a higher ghost picture[4]. Let us investigate the Q_1 picture changing on the level of matter fields:

$$\begin{aligned} \Phi_{h+q+3/2}(w)\, \mathrm{e}^{(q+1)\phi(w)} \;\; &:= \;\; -\,2\left[\, Q_1 \,,\; \xi(w)\, \Phi_h(w)\, \mathrm{e}^{q\phi(w)} \,\right] \\ &= \;\; 2 \oint \frac{\mathrm{d}z}{2\pi i}\, G(z)\, \mathrm{e}^{\phi(z)}\, \eta(z)\, \xi(w)\, \Phi_h(w)\, \mathrm{e}^{q\phi(w)} \\ &= \;\; 2 \oint \frac{\mathrm{d}z}{2\pi i} \Big((z-w)^{-q-1} \;+\; \dots \Big)\, G(z)\, \Phi_h(w)\, \mathrm{e}^{(q+1)\phi(w)} \end{aligned} \tag{3.1.29}$$

The canonical ghost picture for the NS sector has charge $q=-1$. Only the Q_1 part of the BRST charge contributes to its $q=0$ ghost picture

$$\Phi^{\mathrm{NS}}_{h+1/2}(w)\, \mathrm{e}^{0\phi(w)} \;\; = \;\; 2 \oint \frac{\mathrm{d}z}{2\pi i} \big(1 \;+\; \dots \big)\, G(z)\, \Phi^{\mathrm{NS}}_h(w) \;\; = \;\; 2\, (G_{-1/2}\, \Phi^{\mathrm{NS}}_h)(w)\;. \tag{3.1.30}$$

The right hand side picks out the upper component $\Phi^{\mathrm{NS}}_{h+1/2} = 2G_{-1/2}\Phi^{\mathrm{NS}}_h$ of a superconformal primary with lower component Φ^{NS}_h. The conformal fields Φ^{NS}_h and $\Phi^{\mathrm{NS}}_{h+1/2}$ from the $q=-1$ and $q=0$ superghost pictures of a NS state, respectively, form a superconformal primary $(\Phi^{\mathrm{NS}}_h, \Phi^{\mathrm{NS}}_{h+1/2})$.

Repeated picture changing further increases the superghost charge of the exponential $\mathrm{e}^{q\phi}$ leading to more and more negative conformal weights $h(\mathrm{e}^{q\phi}) = -\frac{q^2}{2} - q$. Supercurrent action on Φ^{NS} gives rise to descendants $\Phi_{h+q+3/2} = 2G_{-q-3/2}\Phi_h$. To give an overview of the possible ghost pictures in the NS sector:

$$|\mathrm{phys}, \mathrm{NS}\rangle \;\; \longleftrightarrow \;\; \left\{ \begin{array}{rcc} \Phi^{\mathrm{NS}}_h\, \mathrm{e}^{-\phi}\, |0\rangle & \sim & \mathrm{e}^{-\phi}\, |h, \mathrm{NS}\rangle \\ \Phi^{\mathrm{NS}}_{h+1/2}\, \mathrm{e}^{0\phi}\, |0\rangle & \sim & G_{-1/2}\, \mathrm{e}^{0\phi}\, |h, \mathrm{NS}\rangle \\ \Phi^{\mathrm{NS}}_{h+2}\, \mathrm{e}^{+\phi}\, |0\rangle & \sim & G_{-3/2}\, G_{-1/2}\, \mathrm{e}^{1\phi}\, |h, \mathrm{NS}\rangle \\ \vdots & & \vdots \\ \Phi^{\mathrm{NS}}_{h+\frac{1}{2}(n+1)^2}\, \mathrm{e}^{n\phi}\, |0\rangle & \sim & G_{-n-1/2}\, \dots\, G_{-3/2}\, G_{-1/2}\, \mathrm{e}^{n\phi}\, |h, \mathrm{NS}\rangle \end{array} \right. \tag{3.1.31}$$

The canonical ghost picture of R sector states has charge $q=-\frac{1}{2}$. Their matter contributions $\Phi^{\mathrm{R}}_h, \Phi^{\mathrm{R}}_{h+1}, \dots$ in different ghost pictures also fall into representations of the superconformal

[4] As demonstrated in (3.1.12), $[Q_0, \xi\Phi^{\mathrm{NS}}_h \mathrm{e}^{-\phi}]$ yields a total derivative and the contour integrand of $[Q_2, \xi\Phi_h \mathrm{e}^{-\phi}]$ is regular in $z-w$

algebra. Lifting the superghost charge $q = -\frac{1}{2}$ to $q = +\frac{1}{2}$ via Q_1 gives rise to

$$\begin{aligned}\Phi^{\mathrm{R}}_{h+1}(w)\,\mathrm{e}^{\phi(w)/2} &= 2\oint \frac{\mathrm{d}z}{2\pi i}\,\big((z-w)^{1/2} \;+\; \dots\big)\,G(z)\,\Phi^{\mathrm{R}}_{h}(w)\,\mathrm{e}^{\phi(w)/2}\\ &= 2\,(G_{-1}\,\Phi^{\mathrm{R}}_{h})(w)\,\mathrm{e}^{\phi(w)/2}\end{aligned} \tag{3.1.32}$$

such that $\Phi^{\mathrm{R}}_{h+1} = 2G_{-1}\Phi^{\mathrm{R}}_{h}$. This iterates to the following pattern:

$$|\mathrm{phys},\mathrm{R}\rangle \quad\longleftrightarrow\quad \left\{\begin{array}{rcc} \Phi^{\mathrm{R}}_{h}\,\mathrm{e}^{-\phi/2}\,|0\rangle & \sim & \mathrm{e}^{-\phi/2}\,|h,\mathrm{R}\rangle\\ \Phi^{\mathrm{R}}_{h+1}\,\mathrm{e}^{\phi/2}\,|0\rangle & \sim & G_{-1}\,\mathrm{e}^{\phi/2}\,|h,\mathrm{R}\rangle\\ \Phi^{\mathrm{R}}_{h+3}\,\mathrm{e}^{3\phi/2}\,|0\rangle & \sim & G_{-2}\,G_{-1}\,\mathrm{e}^{3\phi/2}\,|h,\mathrm{R}\rangle\\ \vdots & & \vdots\\ \Phi^{\mathrm{R}}_{h+\frac{n}{2}(n+1)}\,\mathrm{e}^{(2n-1)\phi/2}\,|0\rangle & \sim & G_{-n}\,\dots\,G_{-2}\,G_{-1}\,\mathrm{e}^{(2n-1)\phi/2}\,|h,\mathrm{R}\rangle \end{array}\right. \tag{3.1.33}$$

Strictly speaking, also the ghost number two part of Q_{BRST} provides a nonzero contribution to the first non-canonical R sector ghost picture $V^{(1/2)} = -2[Q_{\mathrm{BRST}}, \xi V^{(-1/2)}]$:

$$\begin{aligned}-2\,\big[\,Q_2\,,\;\xi(w)\,\Phi^{\mathrm{R}}_{h}(w)\,\mathrm{e}^{-\phi(w)/2}\,\big] &= \frac{1}{2}\oint\frac{\mathrm{d}z}{2\pi i}\,b(z)\,\mathrm{e}^{2\phi(z)}\,\eta(z)\,\partial\eta(z)\,\xi(w)\,\Phi^{\mathrm{R}}_{h}(w)\,\mathrm{e}^{-\phi(w)/2}\\ &= -\,\frac{1}{2}\,\eta(w)\,b(w)\,\Phi^{\mathrm{R}}_{h}(w)\,\mathrm{e}^{3\phi(w)/2}\end{aligned} \tag{3.1.34}$$

This term cannot affect tree level correlation functions because it has different (b,c)- and (β,γ) ghost numbers than the Q_1 part (3.1.32). It only becomes relevant for checking overall BRST closedness of $V^{(1/2)}$ since Q_2 has a nontrivial action according to (3.1.26). Also, it might contribute to higher ghost pictures via $[Q_0, \xi\eta b\Phi^{\mathrm{R}}_{h}\mathrm{e}^{3\phi/2}]$.

3.1.7 GSO projection

The requirement of BRST invariance puts strong constraints on the mathematical structure of the physical spectrum. In particular, it allows to apply the machinery of SCFT, i.e. to exploit the remarkable properties of superconformal primaries in the computation of correlation functions.

Still, keeping the full BRST cohomology has two main problems of rather physical nature:

- The simplest possible NS sector state $\mathrm{e}^{ik\cdot X}\mathrm{e}^{-\phi(0)}c(0)|0\rangle$ has a negative mass square according to (3.1.21) at $\hat{h} = \frac{1}{2}$. The presence of such a tachyonic state indicates an instability of the theory, that the perturbative spectrum is based on expanding around the wrong vacuum state. The R sector does not contain any tachyons because the inevitable $S_A\mathrm{e}^{-\phi/2}$ fields guarantee $\hat{h} \geq 1$ in the notation of (3.1.21).

- As explained in section 2.2.2, the OPE of two spin fields can have different leading singularities $(z-w)^{-5/4}$ or $(z-w)^{-3/4}$, depending on their relative chirality. Therefore, we can only keep one chiral half of the R sector ground states to avoid fractional z_{ij} powers in correlation functions. Also, the z_{23} power in three point functions $\langle \psi^{m_1} \dots \psi^{m_n}(z_1) S_A(z_2) S_B(z_3) \rangle$ is given by $\frac{n}{2} - \frac{5}{4}$, so the NS-R coupling suffers from branch cut singularities if the number of ψ^{m_j} is even.

The solution to both problems lies in a projection to states of odd worldsheet fermion number F. This procedure was originally invented by Gliozzi, Scherk and Olive [51] and is therefore referred to as GSO projection. Let $|\lambda\rangle$ denote a state in the $Q_{\rm BRST}$ cohomology, then

$$|\lambda\rangle \text{ is physical} \quad \Longleftrightarrow \quad (-1)^{Q_{\rm GSO}} |\lambda\rangle \;=\; +|\lambda\rangle \, . \tag{3.1.35}$$

In the NS sector, the GSO charge $Q_{\rm GSO}$ is defined such that it counts the number of ψ_{-r} oscillators, i.e. $Q_{\rm GSO} = 1 + (\psi \text{ oscillator number})$. Then, the tachyon $\mathrm{e}^{ik\cdot X} c(0)|0\rangle$ and any state with even oscillator number is removed.

To achieve the chiral projection in the R sector, we assign even GSO parity to left handed ground states $|\alpha\rangle$ and odd parity to right handed ground states $|\dot{\beta}\rangle$, i.e. $(-1)^{Q_{\rm GSO}} \sim \Gamma_{11}$ with the chirality matrix Γ_{11} in $D = 10$ dimensions:

$$(-1)^{Q_{\rm GSO}} |A\rangle \;=\; (-1)^{Q_{\rm GSO}} \begin{pmatrix} |\alpha\rangle \\ |\dot{\alpha}\rangle \end{pmatrix} \;=\; \begin{pmatrix} +|\alpha\rangle \\ -|\dot{\alpha}\rangle \end{pmatrix} \;=\; \Gamma_{11} \begin{pmatrix} |\alpha\rangle \\ |\dot{\alpha}\rangle \end{pmatrix} \tag{3.1.36}$$

Additional ψ_{-n} modes with $n \in \mathbb{N}$ again flip the fermion number and require a different chirality in the ground state.

This rather complicated set of rules becomes very compact if we bosonize the worldsheet fermions and spin fields according to the dictionary in subsections 2.2.3 and 2.2.4. Given the OPEs $\partial\phi(z)\mathrm{e}^{q\phi(0)} \sim -\frac{q}{z}\mathrm{e}^{q\phi(0)}$ and $i\partial H^j(z)\mathrm{e}^{ip\cdot H(0)} \sim \frac{p^j}{z}\mathrm{e}^{ip\cdot H(0)}$, we can identify

$$Q_{\rm GSO} \;=\; \oint \frac{\mathrm{d}z}{2\pi i} \left(\sum_{j=0}^{4} i\partial H^j(z) \;-\; \partial\phi(z) \right) \tag{3.1.37}$$

as an appropriate GSO operator with eigenvalue $\sum_{j=0}^4 p^j + q$ on the state $\mathrm{e}^{ip\cdot H}\mathrm{e}^{q\phi}$. The massless NS state $\psi^m \mathrm{e}^{ik\cdot X}\mathrm{e}^{-\phi(0)}c(0)|0\rangle$ has one nonzero $p^j = \pm 1$ and $q = -1$, i.e. even $Q_{\rm GSO}$ eigenvalue 0 or -2. Left handed R ground states $S_\alpha \mathrm{e}^{-\phi/2}$ have $\sum_{j=0}^4 p^j \in \left\{\frac{5}{2}, \frac{1}{2}, -\frac{3}{2}\right\}$ and therefore even GSO eigenvalues.

This reasoning motivates the physical state projection (3.1.35) with bosonized representation (3.1.37) for the GSO operator $Q_{\rm GSO}$. Note that the picture changing operation (3.1.29) preserves the GSO parity, i.e. the projection is independent on the ghost picture representation chosen for $|\lambda\rangle$.

It turns out that the GSO projection gives rise to a supersymmetric spectrum. It is not difficult to compute the GSO projected one loop partition function for the open superstring. Then the so-called "aequatio identica satis abstrusa" due to Jacobi [133] between the bosonic and fermionic contributions implies that any mass level is populated by the same number of bosons and fermions.

3.2 The massless spectrum

This section is devoted to the massless physical states which pass the GSO projection (3.1.35) and live in the cohomology of the BRST operator (3.1.5). We will identify the $8+8$ degrees of freedom from the $\mathcal{N}=1$ Super Yang Mills (SYM) multiplet in $D=10$ dimensions. For massless representations of the ten dimensional Poincaré group, the stabilizer subgroup is $SO(8)$, and the bosons (fermions) transform in its vector (left handed spinor) representation.

3.2.1 NS sector: gluons

Let us recall the splitting $V = v_{\hat{h}} \mathrm{e}^{ik\cdot X}$ of vertex operators into a plane wave $\mathrm{e}^{ik\cdot X}$ (generating the spacetime mometum k^m) of conformal weight $\alpha' k^2$ and a remainder conformal field $v_{\hat{h}}$. Massless states with $k^2 = 0$ clearly have $\hat{h} = 1$. In the canonical $q = -1$ ghost picture, the $\mathrm{e}^{-\phi}$ part covers conformal weight $1/2$, so one is free to add a $h = 1/2$ combination of $(i\partial X^m, \psi^m)$ oscillators. The most general ansatz

$$V^{(-1)}(\xi, k, z) \quad = \quad g_{\mathrm{A}}\, \xi_m\, \psi^m(z)\, \mathrm{e}^{-\phi(z)}\, \mathrm{e}^{ik\cdot X(z)} \tag{3.2.38}$$

for the vertex operators is parametrized by a polarization vector ξ_m and a lightlike spacetime momentum k^m which can be rotated into the form $(E, E, 0, \dots, 0)$ by an appropriate Lorentz transformation. The normalization g_{A} will be fixed in later chapters when scattering amplitudes in superstring theory are matched with results from SYM in the $\alpha' \to 0$ limit.

The BRST condition due to Q_1 is satisfied if the polarization vector ξ is transverse to the spacetime momentum

$$\big[\, Q_1 \;,\; V^{(-1)}(\xi, k, z) \,\big] \quad = \quad 0 \quad \Longrightarrow \quad \xi_m\, k^m \quad = \quad 0\;. \tag{3.2.39}$$

This can be traced back to a double pole with residue $\sim \xi_m k^m$ in the $G(z) V^{(-1)}(\xi, k, w)$ OPE. Conditions like (3.2.39) on the polarization wave functions will be referred to as "on-shell" constraints in the following.

A transverse ten-vector ξ_m has nine independent components, but not all of them are physical: The longitudinal polarization $\xi_m \sim k_m$ can be identified as BRST exact since

$$\big[\, Q_{\mathrm{BRST}}\ ,\ \mathrm{e}^{-2\phi}\, \partial \xi\, \mathrm{e}^{ik\cdot X} \,\big] \quad \sim \quad k_m\, \psi^m\, \mathrm{e}^{-\phi}\, \mathrm{e}^{ik\cdot X} \tag{3.2.40}$$

with no contribution from Q_0 and Q_2 to the right hand side. The vertex operator $V^{(-1)}(\xi = k, k, z)$ is zero in the BRST cohomology and does not create a physical states. The resulting gauge invariance $\xi_m \equiv \xi_m + k_m$ together with the transversality condition $\xi_m k^m = 0$ leaves eight polarization states of positive norm.

The zero ghost picture representative of the vertex operator (3.2.38) can be easily computed using (3.1.30) with lower component $\Phi_h = \xi_m \psi^m \mathrm{e}^{ik\cdot X}$ of the superconformal primary. The result for the upper component is $\Phi_{h+1/2} = \frac{1}{\sqrt{2\alpha'}} \xi_m (i\partial X^m - 2\alpha' k_n \psi^m \psi^n) \mathrm{e}^{ik\cdot X}$, i.e.

$$V^{(0)}(\xi, k, z) \quad = \quad \frac{g_{\mathrm{A}}}{\sqrt{2\alpha'}}\, \xi_m \left(i\partial X^m(z) \ - \ 2\alpha'\, k_n\, \psi^m(z)\, \psi^n(z) \right) \mathrm{e}^{ik\cdot X(z)} \ . \tag{3.2.41}$$

Note that the spurious states with $\xi_m \sim k_m$ are automatically absent in (3.2.41): The second term $\sim k_{(m}\, k_{n)} \psi^{[m} \psi^{n]}$ vanishes on the level of the vertex operator, and the first term yields a total derivative $k_m i\partial X^m \mathrm{e}^{ik\cdot X} = \partial \mathrm{e}^{ik\cdot X}$ which does not contribute in integrated vertex operators. It is a generic phenomenon that spurious states are killed under picture raising operations.

The example of the massless NS sector states is representative to illustrate the interplay between SCFT and physics. Superconformal invariance of the supersymmetric worldsheet action leads to the BRST quantization procedure. Negative norm states $\sim \xi_m = (\xi_0, 0, \dots, 0)$ are a generic problem in gauge theories in Minkowski spacetime, and the requirement (3.2.39) of BRST invariance removes them from the superstring spectrum.

3.2.2 R sector: gluinos

The R ground states exhaust the all the massless spacetime fermions. The left handed spin field S_α can be contracted with a $SO(1,9)$ spinorial wavefunction u^α such that most general vertex operator is given by

$$V^{(-1/2)}(u, k, z) \quad = \quad g_\lambda\, u^\alpha\ S_\alpha(z)\, \mathrm{e}^{-\phi(z)/2}\, \mathrm{e}^{ik\cdot X(z)} \ . \tag{3.2.42}$$

Again, there is a normalization constant g_λ, and we will relate it to the gluon coupling g_{A} in subsection 3.5 on spacetime SUSY.

The OPE $G(z)V^{(-1/2)}(u, k, w)$ has a $(z-w)^{-3/2}$ pole with residue $\sim u^\alpha \gamma^m_{\alpha\dot\beta} k_m$. According to the discussion 3.1.5, this is an obstacle to BRST closedness with respect to Q_1, so it implies the on-shell constraint

$$\big[Q_1 \, , \, V^{(-1/2)}(u,k,z) \big] \;=\; 0 \quad \Longrightarrow \quad u^\alpha \, \gamma^m_{\alpha \dot\beta} \, k_m \;\equiv\; u^\alpha \, \not{k}_{\alpha\dot\beta} \;=\; 0 \tag{3.2.43}$$

which can be identified as a massless Dirac equation[5]. In a Lorentz frame where $k_0 = k_1$, the equation $u^\alpha \not{k}_{\alpha\dot\beta} = 0$ fixes the first entry of the spinor weight $\alpha = \big(\pm\frac{1}{2}, \dots, \pm\frac{1}{2}\big)$ to be $+\frac{1}{2}$, i.e. $\not{k}$ with lightlike k_m has rank eight and the solution space to (3.2.43) is eight dimensional. Since the massless R sector is free of spurious states, the number of physical fermionic degrees of freedom matches the eight bosonic degrees of freedom from the NS sector.

Let us display the state (3.2.42) in two further ghost pictures. Firstly, there exists a "mirror" R sector at ghost charge $q = -\frac{3}{2}$ on each mass level which is necessary to form a two point function of total ghost charge -2. The superghost spin fields $\mathrm{e}^{-3\phi/2}$ and $\mathrm{e}^{-\phi/2}$ happen to carry the same conformal weight 3/8 but their GSO parity is different. Therefore, the mirror R vertex requires a right handed spin field:

$$V^{(-3/2)}(\bar v, k, z) \;=\; \frac{g_\lambda}{\sqrt{\alpha'}} \, \bar v_{\dot\beta} \, S^{\dot\beta}(z) \, \mathrm{e}^{-3\phi(z)/2} \, \mathrm{e}^{ik\cdot X(z)} \tag{3.2.44}$$

Since the $\mathrm{e}^{\phi(z)}\mathrm{e}^{-3\phi(w)/2}$ OPE is less singular than $\mathrm{e}^{\phi(z)}\mathrm{e}^{-\phi(w)/2}$, invariance under Q_1 imposes no constraints on $\bar v$. Picture changing maps $V^{(-3/2)}(\bar v, k, z)$ to $V^{(-1/2)}(u, k, z)$ with $u^\alpha = \bar v_{\dot\beta} \, \not{k}^{\dot\beta\alpha}$ and therefore annihilates those eight $\bar v$ components which can be written as $\bar v_{\dot\beta} = w^\alpha \, \not{k}_{\alpha\dot\beta}$ for some spinor w. These states can be identified as spurious. This is a further example for higher ghost pictures $q = -\frac{1}{2} > -\frac{3}{2}$ carrying less spurious states.

We can further raise the picture of the canonical $q = -\frac{1}{2}$ vertex:

$$V^{(1/2)}(u,k,z) \;=\; \frac{g_\lambda}{2\sqrt{\alpha'}} \, u^\alpha \left(i\partial X_m \;+\; \frac{\alpha'}{2} \, k_n \, \psi^n \, \psi_m \right) \gamma^m_{\alpha\dot\beta} \, S^{\dot\beta} \, \mathrm{e}^{\phi/2} \, \mathrm{e}^{ik\cdot X} \tag{3.2.45}$$

This involves the spin 3/2 projection of the operator $\psi^n \, \not{\psi}_{\alpha\dot\beta} S^{\dot\beta}$. It appears as a subleading singularity in the $\psi^m(z) S_\alpha(w)$ OPE, see (6.1.2).

3.2.3 Outlook

The eight plus eight physical states we have just identified constitute the massless SUSY multiplet in open, uncompactified superstring theory. It precisely provides the particle content of the (unique) ten dimensional $\mathcal{N} = 1$ SYM which has been extensively studied in the literature [134, 135]. We will discuss its superspace formulation in the later section 10.2.

[5] In operator language, it can be obtained from the supercurrent's zero mode $G_0 |\alpha\rangle_{\mathrm{R}} \sim k_m \psi^m_0 |\alpha\rangle_{\mathrm{R}} \stackrel{!}{=} 0$

The masses of all the Regge excitations scale with $m \sim \alpha'^{-1/2}$, so in the formal limit $\alpha' \to 0$, they become infinitely heavy and decouple. The geometric meaning of this limit is to shrink the strings of length $\sqrt{\alpha'}$ to point particles. String theory then reduces to an effective field theory for the massless states. It will be an essential consistency check to verify this on the level of scattering amplitudes, this will be elaborated in more detail in section 5.5 on scattering amplitudes in gauge theories.

Let us furthermore give a formal outlook on how to embed the NS- and R sector operators $\psi^m \mathrm{e}^{-\phi}$ and $S_\alpha \mathrm{e}^{-\phi/2}$ in a more abstract, unifying picture. Upon bosonizing the RNS fields, we can think of both $\psi^m \mathrm{e}^{-\phi}$ and $S_\alpha \mathrm{e}^{-\phi/2}$ as an exponential $\mathrm{e}^{i\vec{w}\cdot(iH,\phi)}$ with a six component vector $\vec{w}$ taking values $(m,-1)$ and $(\alpha,-1/2)$ in the weight lattice of $D_{5,1}$ with Lorentzian signature. Bosonic (fermionic) vertex operators fall into the vacuum (antispinor) conjugacy class.

This is in lines with the other ghost pictures (3.2.41), (3.2.44) and (3.2.45) for the massless vertex operators. They are all built as linear combinations of exponentials $\mathrm{e}^{i\vec{w}\cdot(iH,\phi)}$ whose weight vectors $\vec{w}$ are related to the canonical $(m,-1)$ and $(\alpha,-1/2)$ through a shift by (picture changing) root vectors $(0,\ldots,\pm 1,\ldots,0,\pm 1)$ of $D_{5,1}$. This construction is an example of so-called covariant lattices, they were introduced by [118, 136, 137, 138].

3.3 The first mass level

Let us now proceed to massive states. Their spacetime momentum can be rotated into a rest frame $k_m = (m,0,\ldots,0)$, i.e. the stabilizer group is the $SO(9)$ acting on the nine zero components. Hence, the massive part of the physical spectrum can be decomposed into $SO(9)$ representations. In [139], this is done very explicitly. This exhibits a technical advantage of covariant BRST quantization over light cone quantization: The latter would organize the spectrum into $SO(8)$ representations after decoupling the timelike and longitudinal X^m, ψ^m components. This obscures the $SO(9)$ content, at least one has to find appropriate combinations of $SO(8)$ multiplets to form a representation of the stabilizer group $SO(9)$.

GSO projection removes states at half odd integer mass levels. For instance, any $m^2 = \frac{1}{2\alpha'}$ states would have even fermion oscillator number and are therefore projected out under $(-1)^{Q_{\mathrm{GSO}}}$. Hence, the lightest particles in both sectors of the physical spectrum carry mass $m = \frac{1}{\sqrt{\alpha'}}$. This section reviews the Poincaré representations on the first mass level of the superstring. Its complete ten dimensional content was firstly constructed by [140].

3.3.1 NS sector: spin two tensor and three form

The most general ansatz for the first massive NS sector states involves any possible $h = 3/2$ operator along with the canonical $h = 1/2$ superghost contribution $\mathrm{e}^{-\phi}$ and the $h = -1$ plane wave $\mathrm{e}^{ik\cdot X}$:

$$V^{(-1)}(B,E,H,k,z) \;=\; \hat{g}_A \Big(B_{mn}\, i\partial X^m\, \psi^n \;+\; E_{mnp}\, \psi^m\, \psi^n\, \psi^p \;+\; H_m\, \partial\psi^m \Big)\, \mathrm{e}^{-\phi}\, \mathrm{e}^{ik\cdot X} \quad (3.3.46)$$

The addition of $\xi_m \psi^m \partial\phi \mathrm{e}^{-\phi}$ is neglected because it can be absorbed into a total derivative.

The BRST constraints arising from Q_0 and Q_1 are

$$\begin{aligned} B_{[mn]} \;+\; 3\, k^p\, E_{pmn} \;&=\; 0 \\ 2\alpha'\, k^p\, B_{pm} \;+\; H_m \;&=\; 0 \\ \eta^{mn}\, B_{mn} \;+\; k^p\, H_p \;&=\; 0\,, \end{aligned} \quad (3.3.47)$$

but not all their solutions are physical: Some of them can be generated by Q_{BRST} action on a two-form $\Sigma_{[mn]}$, a vector π_m and a scalar of $SO(9)$:

$$\begin{aligned} \big[\, Q_{\mathrm{BRST}}\,,\, \mathrm{e}^{-2\phi}\, \Sigma_{[mn]}\, \psi^m\, \psi^n\, \partial\xi\, \mathrm{e}^{ik\cdot X} \,\big] \;&\sim\; \Big(2\, \Sigma_{[mn]}\, i\partial X^m\, \psi^n \;+\; \Sigma_{[mn}\, k_{p]}\, \psi^m\, \psi^n\, \psi^p \Big)\, \mathrm{e}^{-\phi}\, \mathrm{e}^{ik\cdot X} \\ \big[\, Q_{\mathrm{BRST}}\,,\, \mathrm{e}^{-2\phi}\, \pi_m\, i\partial X^m\, \partial\xi\, \mathrm{e}^{ik\cdot X} \,\big] \;&\sim\; \Big(\pi_m\, \partial\psi^m \;+\; \pi_m\, k_n\, i\partial X^m\, \psi^n \Big)\, \mathrm{e}^{-\phi}\, \mathrm{e}^{ik\cdot X} \\ \big[\, Q_{\mathrm{BRST}}\,,\, \mathrm{e}^{-2\phi}\, \partial^2\xi\, \mathrm{e}^{ik\cdot X} \,\big] \;&\sim\; \Big(\Big[\frac{\eta_{mn}}{2\alpha'} \;+\; 2\, k_m\, k_n \Big]\, i\partial X^m\, \psi^n \;+\; 3\, k_m\, \partial\psi^m \Big)\, \mathrm{e}^{-\phi}\, \mathrm{e}^{ik\cdot X} \end{aligned} \quad (3.3.48)$$

Therefore, we can identify the antisymmetric part $B_{[mn]}$, the trace $B_m{}^m$ and the full H_m vector as spurious. The on-shell constraints (3.3.47) then require both $B_{(mn)}$ and E_{mnp} to be transverse.

To sum it up, the physical NS degrees of freedom at the first mass level are represented by polarization tensors transverse to k_m (as required by the stabilizer group $SO(9)$), namely a (traceless and symmetric) spin two tensor B_{mn} and a three form E_{mnp}:

$$V^{(-1)}(B,k,z) \;=\; \hat{g}_A\, B_{(mn)}\, i\partial X^m\, \psi^n\, \mathrm{e}^{-\phi}\, \mathrm{e}^{ik\cdot X}\,, \qquad k^m\, B_{mn} \;=\; \eta^{mn}\, B_{mn} \;=\; 0 \quad (3.3.49)$$

$$V^{(-1)}(E,k,z) \;=\; \hat{g}_A\, E_{mnp}\, \psi^m\, \psi^n\, \psi^p\, \mathrm{e}^{-\phi}\, \mathrm{e}^{ik\cdot X}\,, \qquad k^m\, E_{mnp} \;=\; 0 \quad (3.3.50)$$

The number of degrees of freedom is $\frac{9\cdot 10}{2} - 1 = 44$ for B_{mn} and $\frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3} = 84$ for E_{mnp}, i.e. we have $44 + 84 = 128$ bosonic states in total.

For completeness, we also give the zero ghost pictures of the massive NS vertex operators[6]:

$$V^{(0)}(B,k,z) \;=\; \hat{g}_A\, \sqrt{2\alpha'}\, B_{(mn)} \Big(\frac{i\partial X^m\, i\partial X^n}{2\alpha'} \;+\; \partial\psi^m\, \psi^n \;+\; i\partial X^m\, k_p\, \psi^p\, \psi^n \Big)\, \mathrm{e}^{ik\cdot X} \quad (3.3.51)$$

$$V^{(0)}(E,k,z) \;=\; \hat{g}_A\, \sqrt{2\alpha'}\, E_{mnp} \Big(k_q\, \psi^q\, \psi^m\, \psi^n\, \psi^p \;+\; \frac{3}{2\alpha'}\, i\partial X^m\, \psi^n\, \psi^p \Big)\, \mathrm{e}^{ik\cdot X} \quad (3.3.52)$$

[6]The former involves a spin two operator $\partial\psi^{(n}\psi^{p)}$ from the RNS CFT where certain components can be expressed as $\mathrm{e}^{\pm 2iH^j}$ in their bosonized version.

3.3.2 R sector: massive gravitino

Massive vertex operators in the R sector require a conformal field of $h = \frac{13}{8}$ along with the standard ingredients $\mathrm{e}^{-\phi/2}$ and $\mathrm{e}^{ik\cdot X}$. We make the following ansatz for this purpose[7]:

$$V^{(-1/2)}(v,\bar{\rho},k,z) \;=\; \hat{g}_\lambda \left(v^\alpha_m \, i\partial X^m \;+\; 2\alpha' \, \bar{\rho}^m_{\dot\beta} \, \psi_m \, \not{\psi}^{\dot\beta\alpha} \right) S_\alpha \, \mathrm{e}^{-\phi/2} \, \mathrm{e}^{ik\cdot X} \tag{3.3.53}$$

The most general vertex operator is parametrized by two vector spinors $(v^\alpha_m, \bar{\rho}^m_{\dot\beta})$ of opposite chirality which generically contain $SO(1,9)$ irreducibles of spin 1/2 and 3/2. The two chiralities are in lines with the fact that massive fermions are described by Dirac spinors. Their two chiral halves originate from different conformal fields $i\partial X^m S_\alpha$ and $\psi_m \not{\psi}^{\dot\beta\alpha} S_\alpha$.

The BRST conditions involving the R descendant $\psi_m \not{\psi}^{\dot\beta\alpha} S_\alpha$ are quite difficult to evaluate covariantly. Bosonization turned out to be helpful in [140] to find the on-shell conditions. In the end, one can express the right handed wave function $\bar{\rho}^m_{\dot\beta}$ in terms of the left handed one v^α_m and impose an additional constraint on the latter:

$$\bar{\rho}^m_{\dot\beta} \;=\; \frac{1}{36} \, k^n \, v^\alpha_n \, \gamma^m_{\alpha\dot\beta} \;-\; \frac{1}{8} \, v^{m\alpha} \, \not{k}_{\alpha\dot\beta} \tag{3.3.54}$$

$$v^\alpha_m \, \gamma^m_{\alpha\dot\beta} \;=\; 2\alpha' \, k^m \, v^\alpha_m \, \not{k}_{\alpha\dot\beta} \tag{3.3.55}$$

So far, these are nine vector components with 16 Weyl spinor degrees of freedom each. Comparison with the 128 bosonic degrees of freedom suggests that we should identify 16 spurious states within the solutions to (3.3.55). Indeed, one can form the BRST exact state via[8]

$$\begin{aligned} &\big[\, Q_{\mathrm{BRST}} \,,\; \mathrm{e}^{-3\phi/2} \, \bar{u}_{\dot\alpha} \, S^{\dot\alpha} \, \partial\xi \, \mathrm{e}^{ik\cdot X} \,\big] \;\sim\; S_\beta \, \mathrm{e}^{-\phi/2} \, \mathrm{e}^{ik\cdot X} \\ &\times \left(\bar{u}_{\dot\alpha} \, \bar{\gamma}^{\dot\alpha\beta}_m \, i\partial X^m \;+\; 4\alpha' \, \bar{u}_{\dot\alpha} \, \not{k}^{\dot\alpha\beta} \, k_m \, i\partial X^m \;+\; \frac{2\alpha'}{4} \, \bar{u}_{\dot\alpha} \, k_m \, \psi^m \, \not{\psi}^{\dot\alpha\beta} \;-\; \frac{2\alpha'}{24} \, \bar{u}_{\dot\alpha} \, (\not{k} \, \not{\psi} \, \not{\psi})^{\dot\alpha\beta} \right) \end{aligned} \tag{3.3.57}$$

which is related to the finding of [140] via $a^\alpha \sim \bar{u}_{\dot\beta} \, \not{k}^{\dot\beta\alpha}$. We extract the following wavefunctions subject to (3.3.54) and (3.3.55):

$$v^\beta_m \;=\; \bar{u}_{\dot\alpha} \, \bar{\gamma}^{\dot\alpha\beta}_m \;+\; 4\alpha' \, \bar{u}_{\dot\alpha} \, \not{k}^{\dot\alpha\beta} \, k_m \,, \qquad \bar{\rho}^m_{\dot\alpha} \;=\; \frac{1}{4} \, \bar{u}_{\dot\alpha} \, k^m \;-\; \frac{1}{24} \, \bar{u}_{\dot\beta} \, (\not{k}\gamma^m)^{\dot\beta}{}_{\dot\alpha} \tag{3.3.58}$$

[7] The reason why the second ψ must be contracted with a gamma matrix can be better understood in operator language: The second half of (3.3.53) represents the state $\psi^m_{-1}|\dot\beta\rangle_{\mathrm{R}} = \lim_{z\to 0} z^{-1/2} : \psi^m(z) \, S^{\dot\beta}(0) : |0\rangle_{\mathrm{NS}}$ and hence involves the the subleading term of the $\psi^m(z) S^{\dot\beta}(0)$ OPE. According to (6.1.2), this exclusively contains operators of structure $\psi_m \not{\psi}^{\dot\beta\alpha} S_\alpha$.

[8] In intermediate steps of the computation one obtains a term

$$-\,4\alpha' \, \bar{u}_{\dot\alpha} \, \not{k}^{\dot\alpha\beta} \, S_\beta \, \partial \left(\mathrm{e}^{-\phi/2} \right) \mathrm{e}^{ik\cdot X} \;=\; 4\alpha' \, \bar{u}_{\dot\alpha} \, \not{k}^{\dot\alpha\beta} \, \mathrm{e}^{-\phi/2} \, \partial\big(\, S_\beta \, \mathrm{e}^{ik\cdot X} \,\big) \;+ \text{total derivative} \tag{3.3.56}$$

which can be further simplified using $\partial S_\beta = -\frac{1}{36} (\not{\psi} \, \not{\psi})_\beta{}^\gamma S_\gamma$.

d.o.f.	representation	d.o.f.	representation	d.o.f.	representation
156	$\phi^1_{(mnp)}$	126	$\phi^2_{[mnpqr]}$	36	$\phi^3_{[mn]}$
594	$\phi^4_{[mn(p]q)}$	231	$\phi^5_{[m(n]p)}$	9	ϕ^6_m

Table 3.1: Organization of the 1152 bosonic states at second mass level. All the wave functions are understood to be transverse and traceless. The notation $\phi^4_{[mn(p]q)}$ indicates that this rank four tensor ϕ^4 is antisymmetric in its first three indices, but the total antisymmetrization including the fourth index vanishes, $\phi^4_{[mnpq]} = 0$. The two representations $\phi^4_{[mn(p]q)}$ and $\phi^5_{[m(n]p)}$ are associated with a hooked Young tableau.

The essential role of this spurious state is the elimination of the spin 1/2 component from the wavefunctions: Since the vector spinors in (3.3.58) have nonzero gamma matrix trace $(u_{\dot\alpha}\bar\gamma_m^{\dot\alpha\beta} + 4\alpha'\bar u_{\dot\alpha}\, \not k^{\dot\alpha\beta}k_m)\gamma^m_{\beta\dot\beta} \neq 0$, we can absorb the BRST exact states into the physical ones via $\chi^\alpha_m := v^\alpha_m + \lambda\left(u_{\dot\alpha}\bar\gamma_m^{\dot\alpha\beta} + 4\alpha'\bar u_{\dot\alpha}\, \not k^{\dot\alpha\beta}k_m\right)$ and choose $\lambda \in \mathbb{R}$ such that $\chi^\alpha_m\gamma^m_{\alpha\dot\beta} = 0$. Then, (3.3.54) and (3.3.55) guarantee that the wavefunctions are transverse and traceless and furthermore related by a massive Dirac equation:

$$k^m\,\chi^\alpha_m \;=\; \chi^\alpha_m\gamma^m_{\alpha\dot\beta} \;=\; k_m\,\bar\rho^m_{\dot\beta} \;=\; \bar\rho^m_{\dot\beta}\,\gamma_m^{\dot\beta\alpha} \;=\; 0 \tag{3.3.59}$$

$$\bar\rho^m_{\dot\beta} \;=\; -\frac{1}{8}\,\chi^{m\alpha}\,\not k_{\alpha\dot\beta}\,, \qquad \chi^{m\alpha} \;=\; -\,8\alpha'\,\bar\rho^m_{\dot\beta}\,\not k^{\dot\beta\alpha} \tag{3.3.60}$$

Hence, we have identified a massive spin 3/2 Dirac fermion with 128 independent components:

$$V^{(-1/2)}(\chi,k,z) \;=\; \hat g_\lambda\left(\chi^\alpha_m\,i\partial X^m \;-\; \frac{\alpha'}{4}\,\chi^\beta_m\,\not k_{\beta\dot\beta}\,\psi^m\,\not\psi^{\dot\beta\alpha}\right)S_\alpha\,\mathrm{e}^{-\phi/2}\,\mathrm{e}^{ik\cdot X} \tag{3.3.61}$$

The full physical particle content of the first mass level with $m^2 = \frac{1}{\alpha'}$ consists of a spin two tensor B_{mn}, a three form E_{mnp} and massive gravitino χ^α_m with 44+84 bosonic and 128 fermionic degrees of freedom.

3.4 The leading Regge trajectory

At higher mass levels, the number and complexity of the occurring $SO(9)$ representations increases drastically. For instance, table 3.1 shows the polarization tensors for the 1152 bosons at the second mass level (with $m^2 = \frac{2}{\alpha'}$), see [141]. Even though a recent work [139] managed to compute the $SO(9)$ content to any mass level, it is still an open problem to construct the physical vertex operators in full generality.

The only $SO(9)$ representations for which the vertex operators are completely under control for arbitrary mass level n contain highest spin states with $j_{\max} = n+1$ for bosons and $j_{\max} =$

$n+\frac{1}{2}$ for fermions. Since a linear relationship between spins and mass squares was first noticed by Regge for excited mesons, the maximum spin states in string theory are referred to as the “leading Regge trajectory”. The massless field theory encompassing all bosonic integer spins is known as the Vasiliev system [142]. The research areas of string theory and higher spin field theory strongly support and motivate each other:

First of all, the high energy regime of string theory (i.e. the low tension limit $\alpha' \to \infty$ where Regge excitations formally become massless) provides a fruitful laboratory to learn about higher spin gauge theory. Typical no-go theorems on massless higher spin theories are based on subtle assumptions that may well prove too restrictive, such as finiteness of the spectrum or minimal couplings. Results from string theory naturally have the required properties to bypass these no-go theorems.

Secondly, taking all the higher spin modes into account might pave the way beyond the on-shell first quantized picture of string theory and shed light on its true quantum degrees of freedom. Moreover, ideas have been around that masses of Regge excitations arise from some sort of generalized Higgs effect for spontaneous breaking of higher spin gauge symmetries. From this viewpoint, string theory itself might well prove to be a major motivation to study higher spin theories.

Therefore, investigating higher-spin dynamics will help to better understand string theory and, vice versa, a closer look at string theory at high energies in the $\alpha' \to \infty$ limit can provide important clues on higher-spin dynamics. After first explicit investigation of higher spin interactions in [143,144,145,146,147], great progress in finding all order vertices for totally symmetric tensors was made in [142,148], see [149] for a review beyond cubic level.

3.4.1 NS sector: integer spin

In this subsection, we construct higher spin generalizations of the gluon vertex (3.2.38) and the spin two tensor (3.3.49) at mass levels $n=0$ and $n=1$, respectively. It is clear from (3.1.21) that states of mass $m^2 = \frac{n}{\alpha'}$ are generated by $v_{\hat{h}=n+1} \mathrm{e}^{ik\cdot X}$ with v_{n+1} denoting a conformal field of weight $n+1$. The superghost operator $\mathrm{e}^{-\phi}$ of the canonical $q=-1$ picture covers $h=\frac{1}{2}$, so one is free to distribute the remaining weight of $n+\frac{1}{2}$ among α^m_{-n} and ψ^m_{-r} oscillators.

Spin j wave functions ϕ^j are totally symmetric, traceless and transverse rank j tensors, so the leading trajectory state will involve the maximum number of symmetrized $\alpha^{m_i}_{-n_i}, \psi^{m_j}_{-r_j}$ modes with overall weight $\sum_i n_i + \sum_j r_j = n+\frac{1}{2}$. Because of their anticommuting nature, no more than one $\psi^m_{-1/2}$ can contract a symmetric wave function, so the mutually commuting oscillators of minimal conformal weight are α_{-1}^m. These arguments lead to the following vertex operator for NS states of highest spin:

$$V^{(-1)}(\phi^{j=n+1}, k, z) \;=\; g_n \, \phi_{(m_1 \dots m_n m_{n+1})} \, i\partial X^{m_1} \, \dots \, i\partial X^{m_n} \, \psi^{m_{n+1}} \, \mathrm{e}^{-\phi} \, \mathrm{e}^{ik\cdot X} \tag{3.4.62}$$

$$k^{m_i} \, \phi_{m_1 \dots m_{n+1}} \;=\; \eta^{m_i m_j} \, \phi_{m_1 \dots m_{n+1}} \;=\; 0 \, , \qquad k^2 \;=\; -\frac{n}{\alpha'} \tag{3.4.63}$$

Let us also give the zero ghost picture analogue which we will need in the later section (9.2.12) to compute bosonic three point couplings:

$$\begin{aligned} V^{(0)}(\phi^{j=n+1}, k, z) \;&=\; g_n \, \sqrt{2\alpha'} \, \phi_{(m_1 \dots m_{n-1} np)} \, i\partial X^{m_1} \, \dots \, i\partial X^{m_{n-1}} \, \mathrm{e}^{ik\cdot X} \\ &\times \left(n \, \partial\psi^n \, \psi^p \;+\; (k\cdot\psi) \, \psi^n \, i\partial X^p \;+\; \frac{1}{2\alpha'} \, i\partial X^n \, i\partial X^p \right) \end{aligned} \tag{3.4.64}$$

3.4.2 R sector: half odd integer spin

Similar arguments apply to highest spin states in the R sector. Spin $n+\frac{1}{2}$ wavefunctions $\chi^{j=n+\frac{1}{2}}$ are transverse and traceless tensor spinors with n symmetrized vector indices. Moreover, they are $\gamma^m_{\alpha\dot\beta}$ traceless in the spinor index, $\chi^\alpha_{m_1 \dots m_n} \gamma^{m_i}_{\alpha\dot\beta} = 0$.

The α_{-1}^m oscillator is the object of lowest weight to saturate the vector indices of a symmetric wave function, so the first idea to construct a spin $n+\frac{1}{2}$ state would be of schematic form $(\alpha_{-1})^n |\alpha\rangle_{\mathrm{R}}$. BRST invariance requires the addition of a second term with $n-1$ bosonic oscillators and one ψ_{-1}^m such that the physical vertex is given by

$$\begin{aligned} V^{(-1/2)}(\chi^{j=n+1/2}, k, z) \;&=\; g_{n+1/2} \, \chi^\alpha_{(m_1 \dots m_{n-1} n)} \, i\partial X^{m_1} \, \dots \, i\partial X^{m_{n-1}} \\ &\times \left(i\partial X^n \, \delta^\gamma_\alpha \;-\; \frac{\alpha'}{4} \, \not{k}_{\alpha\dot\beta} \, \psi^n \, \not{\psi}^{\dot\beta\gamma} \right) S_\gamma \, \mathrm{e}^{-\phi/2} \, \mathrm{e}^{ik\cdot X} \end{aligned} \tag{3.4.65}$$

$$k^{m_i} \, \chi^\alpha_{m_1 \dots m_n} \;=\; \eta^{m_i m_j} \, \chi^\alpha_{m_1 \dots m_n} \;=\; \chi^\alpha_{m_1 \dots m_n} \, \gamma^{m_i}_{\alpha\dot\beta} \;=\; 0 \tag{3.4.66}$$

where, again, $k^2 = -\frac{n}{\alpha'}$. The second term completes the Dirac spinor required by non-chiral nature of massive fermions.

3.5 Spacetime supersymmetry

The RNS approach to superstring theory which we are applying in part I and II of this work does not provide manifest spacetime supersymmetry. This a drawback compared to the Green Schwarz framework and the pure spinor formalism. Still, one can construct a spacetime SUSY charge $\mathcal{Q}_\alpha$ for the covariantly quantized RNS superstring and explore the action of supersymmetry on individual states using SCFT methods.

A necessary condition for a supersymmetric spectrum is an equal number of bosons and fermion on each mass level. We have explicitly seen this for $n = 0$ and $n = 1$ in the previous sections 3.2 and 3.3, and the bose-fermi populations of higher mass levels $n \geq 2$ can be checked to agree from the superstring one loop partition function.

3.5.1 The SUSY algebra

In this subsection, we will introduce the CFT realization of the ten dimensional $\mathcal{N} = 1$ super Poincaré algebra. All its generators can be represented by a contour integral over conserved $h = 1$ currents from the RNS CFT. The fact that they commute with $Q_{\rm BRST}$ guarantees that the classification of physical states is super Poincaré invariant.

The bosonic generators P^m, M^{mn} of translations and Lorentz rotations, respectively, have already been introduced in subsection 3.1.4. The momentum operator P_m acts with eigenvalue k_m on physical vertex operators, and the rotation generators M^{mn} transform the wave functions according to the $SO(1,9)$ representations they fall into.

Let us now enlarge the Poincaré algebra by the spacetime SUSY charge

$$\mathcal{Q}_\alpha^{(-1/2)} \ := \ \alpha'^{-1/4} \oint \frac{\mathrm{d}z}{2\pi i} \, S_\alpha(z) \, \mathrm{e}^{-\phi(z)/2} \ . \tag{3.5.67}$$

It is proportional to the gaugino vertex (3.2.42) at zero momentum, so BRST invariance of $\mathcal{Q}_\alpha^{(-1/2)}$ immediately follows from that of $V^{(-1/2)}(u,k,z)$. Similarly, we can borrow its higher ghost picture representation from (3.2.45) at $k = 0$:

$$\mathcal{Q}_\alpha^{(1/2)} \ := \ \frac{1}{2\alpha'^{3/4}} \oint \frac{\mathrm{d}z}{2\pi i} \, i\partial X_m(z) \, \gamma^m_{\alpha\dot\beta} \, S^{\dot\beta}(z) \, \mathrm{e}^{+\phi(z)/2} \tag{3.5.68}$$

The latter is needed to compute the anticommutator in an overall ghost neutral fashion,

$$\{ \, \mathcal{Q}_\alpha^{(-1/2)} \, , \, \mathcal{Q}_\beta^{(1/2)} \, \} \ = \ \oint \frac{\mathrm{d}z}{2\pi i} \, i\partial X_m \, \gamma^m_{\beta\dot\beta} \, C_\alpha{}^{\dot\beta} \ = \ - \, (\gamma^m \, C)_{\alpha\beta} \, P_m \ , \tag{3.5.69}$$

it reproduces the momentum operator P_m. The fact that $\mathcal{Q}_\alpha$ transforms as a spinor follows from the commutation relations

$$[\, M^{mn} \, , \, \mathcal{Q}_\alpha^{(q)} \,] \ = \ -\frac{1}{2} \, (\gamma^{mn})_\alpha{}^\beta \, \mathcal{Q}_\beta^{(q)} \tag{3.5.70}$$

valid for both values of $q = \pm\frac{1}{2}$.

This setting does not provide a realization of off-shell SUSY: The anticommutator of SUSY charges $\mathcal{Q}_\alpha^{(-1/2)}$ in their canonical ghost picture yields a picture changed version $\sim \oint \psi^m \mathrm{e}^{-\phi}$ of the momentum operator. But the equivalence between different superghost pictures requires on-shell states as we will explain in subsection (5.2.3). Hence, the SUSY algebra does not close off-shell where $\sim \oint \psi^m \mathrm{e}^{-\phi}$ is inequivalent to the momentum P_m at zero ghost charge (3.1.16).

3.5.2 Applications to light states

This subsection recapitulates the SUSY action on the massless open string states along the lines of [150] and gives an outlook of what happens at the first mass level. For ease of notation, it makes sense to contract the supercharges $\mathcal{Q}_\alpha$ with an anticommuting SUSY spacetime parameter η_α, i.e. to compute the commutators of vertex operators with

$$\mathcal{Q}^{(q)}(\eta) \quad := \quad \eta^\alpha \, \mathcal{Q}^{(q)}_\alpha \; . \tag{3.5.71}$$

At the massless level, spacetime SUSY transformations rotate the gluon- and gluino vertex operators into each other:

$$\big[\, \mathcal{Q}^{(1/2)}(\eta) \; , \; V^{(-1)}(\xi) \, \big] \quad = \quad V^{(-1/2)}\Big(\, u^\beta = \tfrac{1}{\sqrt{2}} \eta^\alpha (\gamma^{mn})_\alpha{}^\beta k_m \xi_n \, \Big) \tag{3.5.72}$$

$$\big[\, \mathcal{Q}^{(-1/2)}(\eta) \; , \; V^{(-1/2)}(u) \, \big] \quad = \quad V^{(-1)}\Big(\, \xi^m = \tfrac{1}{\sqrt{2}} \eta^\alpha (\gamma^m C)_{\alpha\beta} u^\beta \, \Big) \tag{3.5.73}$$

Since the normalization of ξ_m and u^α is fixed by orthonormality of the different helicity components, we can infer a relation between the couplings:

$$g_\lambda \quad = \quad g_{\rm A} \, \alpha'^{1/4} \tag{3.5.74}$$

The SUSY variations (3.5.72) and (3.5.73) can be computed in arbitrary ghost pictures, for instance $\big[\mathcal{Q}^{(+1/2)}(\eta), V^{(-1/2)}(u)\big] = V^{(0)} \left(\xi^m = \frac{1}{\sqrt{2}} \eta^\alpha (\gamma^m C)_{\alpha\beta} u^\beta \right)$. The transformations can therefore be recast on the level of the polarizations:

$$\delta_\eta u^\beta \quad = \quad \frac{1}{\sqrt{2}} \, \eta^\alpha \, (\gamma^{mn})_\alpha{}^\beta \, k_m \, \xi_n \; , \qquad \delta_\eta \xi^m \quad = \quad \frac{1}{\sqrt{2}} \, \eta^\alpha \, (\gamma^m \, C)_{\alpha\beta} \, u^\beta \tag{3.5.75}$$

The on-shell constraints $k_m \xi^m = 0$ and $u^\alpha \, \not{k}_{\alpha\dot\beta} = 0$ are obviously preserved under SUSY. We can check by iterating (3.5.75) that the SUSY algebra closes up to a gauge transformation $\xi^m \rightarrow \xi^m + k^m$ in the gluon polarization vector[9]:

$$\big[\, \delta_{\eta_1} \; , \; \delta_{\eta_2} \, \big] \, u^\alpha \quad = \quad - \, (\eta_1 \, \not{k} \, \eta_2) \, u^\alpha \; , \qquad \big[\, \delta_{\eta_1} \; , \; \delta_{\eta_2} \, \big] \, \xi^p \quad = \quad - \, (\eta_1 \, \not{k} \, \eta_2) \, \xi^p \; + \; (\eta_1 \, \not{\xi} \, \eta_2) \, k^p \tag{3.5.76}$$

Computing the SUSY variation of massive states is more involved because it requires subleading singularities in various OPEs. Let us give just one example

$$\big[\, \mathcal{Q}^{(1/2)}(\eta) \; , \; V^{(-1)}(B,k) \, \big] \quad = \quad V^{(-1/2)}\Big(\, \chi^\gamma_m = \tfrac{1}{\sqrt{2}} \eta^\alpha \, \not{k}_{\alpha\dot\beta} \gamma_n^{\dot\beta\gamma} B_m{}^n \, \Big) \; , \qquad \hat{g}_{\rm A} \, \alpha'^{1/4} \quad = \quad \hat{g}_\lambda \tag{3.5.77}$$

and defer the rest of the multiplet to a later publication [151]. Analogous transformation rules hold for all the bosonic leading Regge trajectory states (3.4.62) and (3.4.65), where $g_n \alpha'^{1/4} = g_{n+1/2}$ and $\delta_\eta \chi^\gamma_{m_1 \ldots m_{n-1}} = \frac{1}{\sqrt{2}} \eta^\alpha \, \not{k}_{\alpha\dot\beta} \gamma_n^{\dot\beta\gamma} \phi^n_{m_1 \ldots m_{n-1}}$.

[9] Evaluating the $\big[\delta_{\eta_1}, \delta_{\eta_2}\big]$ action on u^α requires the famous $D = 10$ gamma matrix identity $(\gamma_m C)_{(\alpha\beta} \, \gamma^m_{\gamma)\dot\delta} = 0$.

Chapter 4

Towards the four dimensional world

This chapter aims to establish connections between superstring theory and the four dimensional world we experience in our everyday life. In particular, it aims to pave the way towards predictions of string theory for LHC experiments.

As we will explain in the following, the string scale $M_{\rm s} = \alpha'^{-1/2}$ can be as low as a few TeV provided that some of the extra dimensions are sufficiently large [58, 101]. In that case, resonances in lepton, quark- and gluon scattering processes due to exchange of virtual Regge excitations can potentially be observed at LHC [152, 153, 154, 155]. Once the mass threshold $M_{\rm s}$ is crossed in the center-of-mass energies of colliding partons, one would also see the string resonance states produced directly [2].

Remarkably, many phenomenological aspects of weakly coupled low mass string theory are universal: Those states of the string spectrum which couple to the gauge bosons of the four dimensional Standard Model (SM) do not depend on the compactification details. In section 4.5, we will work the universal $D = 4$ particle content at first mass level. As a consequence, SM tree level scattering processes involving n gluons and either zero or two chiral fermions are completely model independent – to all orders in the string length $\ell_{\rm string} = \sqrt{\alpha'}$. As we will show in the later section 8.1, they exhibit resonant behavior at the parton center of mass energies equal to the masses of Regge resonances.

An important requirement for making contact between superstring theory and the SM is SUSY breaking. Since only $\mathcal{N} = 0$ or $\mathcal{N} = 1$ spacetime SUSY is compatible with the observed chiral matter spectrum, we will have to explain how to modify the worldsheet SCFT such that it describes a string vacuum with less that $\mathcal{N} = 4$ SUSY. The underlying philosophy is a phenomenology of SCFTs, the requirement that some ground state of string theory obeys the constraints of phenomenology.

For scenarios with $\mathcal{N} = 0, 1$ SUSY, we will construct chiral multiplets from open strings stretching between intersecting D branes. They live in the twisted sector of the internal worldsheet SCFT. This includes (s)quarks and (s)leptons into the superstring spectrum and opens the door to decribing QCD processes as they happen at LHC in a stringy framework.

4.1 Basics of low string scale SM extensions

This section sketches the ideas of Standard Model extensions based on open strings ending on D-branes. In these settings, gauge bosons are due to open strings attached to stacks of D-branes, and chiral matter arises from strings stretching between intersecting D-branes. We will first derive order of magnitude relations between the superstring input data such the string length $\sqrt{\alpha'}$ and the volumes of extra dimensions and the fundamental constants of four dimensional gravity and gauge interactions. Then, in section 4.1.3 we will introduce the most economic brane configurations which can be accomodated with the SM gauge group.

4.1.1 The four dimensional strength of gravity & gauge interactions

Scenarios with large extra dimensions are a very appealing solution to the hierarchy problem [58] because they admit to unify gravitational and gauge interactions at the electroweak scale. The observed weakness of gravity at lower energies is due to the existence of large extra dimensions. The fact that gravitons may scatter into all the directions of the internal space – even those perpendicular to the SM D branes – decreases the gravitational coupling constant to its observed value.

To get a rough estimate of the relevant scales and physical input quantities, let us consider a Dp brane whose worldvolume is parallel to the uncompacified Minkowski spacetime $\mathbb{R}^{1,3}$. Moreover, we assume that the Dp brane wraps a $d_\parallel := p - 3$ cycle of volume $V_\parallel$ within the compact internal manifold of overall volume $V := V_\parallel \cdot V_\perp$. The latter introduces a $9 - p$ dimensional volume $V_\perp$ transverse to the brane. The situation is depicted in the following figure 4.1.

The observed four dimensional coupling constants of gravity and the gauge interactions follow from dimensionally reducing higher dimensional effective actions for massless string excitations. Firstly, the Einstein Hilbert action for the ten dimensional bulk implies that – up to numerical coefficients $\mathcal{O}(1)$ – the four dimensional Planck mass is given by

$$M_{\rm pl}^2 \;\sim\; \frac{1}{g_{\rm s}^2}\, M_{\rm s}^8\, V \;=\; \frac{1}{g_{\rm s}^2}\, M_{\rm s}^8\, V_\parallel\, V_\perp \tag{4.1.1}$$

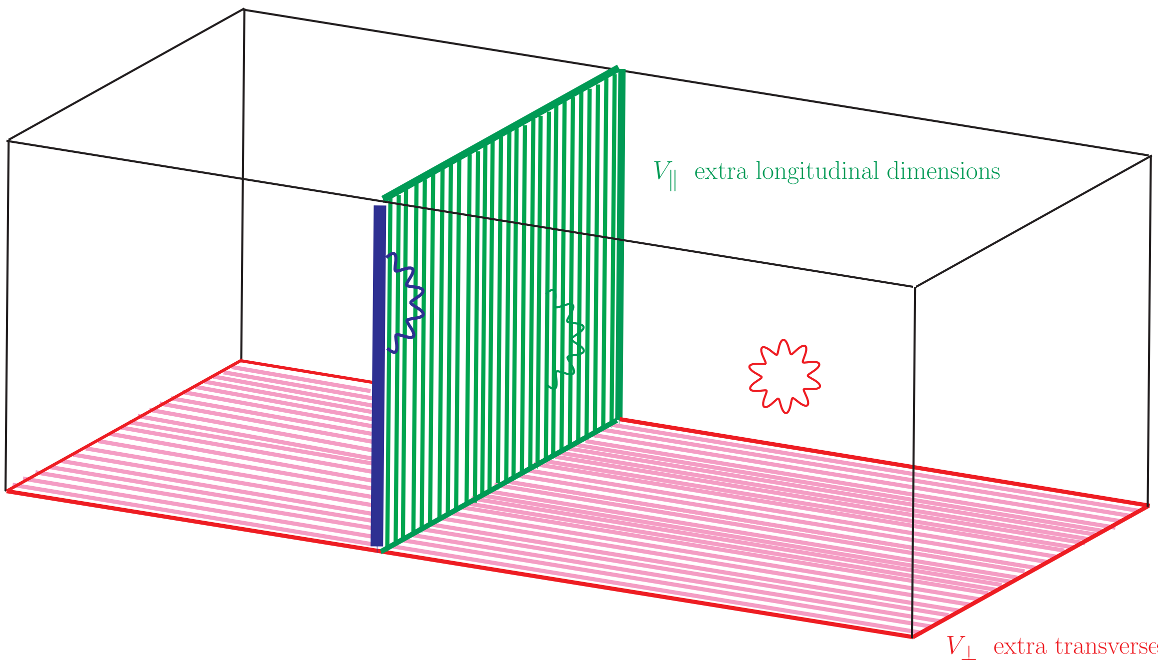


Figure 4.1: The green rectangle represents the extra dimensions parallel to the D brane world volume which can be probed by open string states. The red surface represents transverse extra dimensions probed by the closed string sector only.

where the type II string coupling is related to the dilaton field ϕ_{10} via $g_{\rm s} = {\rm e}^{\phi_{10}}$ [156, 101], see section 5.1 for a brief account on its role in string perturbation theory. Secondly, in type II superstring theory, the gauge theory on the Dp brane worldvolume has the gauge coupling

$$\frac{1}{g_{\rm YM}^2} \sim \frac{1}{g_{\rm s}} M_{\rm s}^{p-3} V_\parallel \tag{4.1.2}$$

where only the internal volume component $V_\parallel$ parallel to the Dp brane contributes. The perpendicular volume $V_\perp$ is inaccessible to open string endpoints. This is the crucial difference to the volume dependence of $M_{\rm pl}$.

4.1.2 Low string scale $M_{\rm s} \sim$ TeV & large extra dimensions

To see the connection between large extra dimensions and a low string scale, it makes sense to consider combinations of $M_{\rm pl}$ and $g_{\rm YM}$ where the string coupling $g_{\rm s}$ drops out, e.g.

$$g_{\rm YM}^2 \, M_{\rm pl} \sim M_{\rm s}^{7-p} \sqrt{\frac{V_\perp}{V_\parallel}} \tag{4.1.3}$$

The left hand side is fixed by experimental observation, so increasing the ratio of volumes $\sqrt{\frac{V_\perp}{V_\parallel}}$ on the right hand side can be compensated by a lower value of the fundamental string mass

$d_\perp := 9-p$	1	2	3	4	5	6
$R_\perp = (V_\perp)^{1/d_\perp}$ [m]	$1,6 \cdot 10^{11}$	$4 \cdot 10^{-4}$	$5,4 \cdot 10^{-9}$	$2 \cdot 10^{-11}$	$7 \cdot 10^{-13}$	$7 \cdot 10^{-14}$

Table 4.1: Admissible sizes of transverse dimensions for Standard Model Dp branes at various values of p. The $d_\parallel := p-3$ extra dimensions parallel to the brane worldvolume are taken to be of the size set by the assumed low string scale $(M_\mathrm{s})^{-1} = (1\ \mathrm{TeV})^{-1} \approx 10^{-18}$ m.

scale M_s. This possibility does not exist in heterotic theories with closed strings only[1].

Low energy SUSY at the TeV scale requires M_s to be at intermediate scales $\sim 10^{11}$ GeV and internal volumes of size $VM_\mathrm{s}^6 \sim 10^{16}$ [157, 158, 159]. If the requirement of TeV scale SUSY is abandoned, on the other hand, then M_s can approach the range of LHC energies $\sim 10^3$ GeV, and string theory can be tested in the near future. The corresponding internal manifolds are then as large as $VM_\mathrm{s}^6 \sim 10^{32}$.

Strong gravity effect such as black hole production [160, 161] are expected to occur at energies $g_\mathrm{s}^2 M_\mathrm{s}$. Hence, it is the value of the string coupling g_s which governs the onset of gravity effects relative to the mass M_s of the lightest Regge excitations. Weakly coupled string theory with $g_\mathrm{s} \ll 1$ allows to detect virtual Regge excitations in parton collisions before black hole production becomes dominant.

Let us next discuss the size of possible extra dimensions. Cavendish type experiments only admit very coarse tests of Newton's law, up to a scale of millimeters. Hence, the typical size $R_\perp := (V_\perp)^{1/(9-p)}$ of extra dimensions transverse to the brane are bounded to lie below the millimeter range. On the other hand, QCD and electroweak scattering experiments give an upper bound on the small extra dimensions' radii $R_\parallel := (V_\parallel)^{1/(p-3)}$ in the range of the electroweak scale $(M_\mathrm{EW})^{-1} \approx 10^{-17}$ m.

If a value of $g_\mathrm{s} = \frac{1}{25}$ is assumend for the string coupling, then an LHC accessible string scale $M_\mathrm{s} = 1$ TeV is compatible with the internal geometries given by table 4.1. The case $d_\perp = 1$ is obviously ruled out by gravity experiments, so branes of D3, D4, ..., D7 type pass the consistency checks due to the order of magnitude analysis of this section.

4.1.3 The SM D brane quiver

In this subsection we will give a brief summary about how the SM is realized by intersecting D branes. More details can be found, e.g., in reference [61, 101]. The following set-up applies for type IIA and type IIB orientifolds with intersecting D6- or D7 branes, respectively. The D

[1] In heterotic superstring theory, the four dimensional Planck scale is given by $M_\mathrm{pl} = g_\mathrm{s}^{-1} M_\mathrm{s}$.

branes are spacetime filling, and they are wrapped around certain p dimensional cycles ($p = 3, 4$) inside the compact space. This setup is in principal also valid for F theory, which provides the non-perturbative uplift of the type IIB orientifolds with intersecting D7 branes [54, 55]. Therefore the considered D brane quiver locally describes a large class of four dimensional string vacua. As it is shown in figure 4.2, the SM particles can be locally realized as massless open string excitations that live on a local quiver of four different stacks of intersecting D branes. Gauge bosons (such as gluons g and electroweak bosons $W^{\pm}$) stem from open strings with both endpoints on the same stack, chiral fermions (quarks q and leptons l) are implemented using open strings stretching between different stacks, see section 4.4 for more details on the latter. The corresponding quiver SM gauge group is given by

$$\mathcal{G} \quad = \quad \mathcal{G}_a \times \mathcal{G}_b \times \mathcal{G}_c \times \mathcal{G}_d \quad = \quad U(3)_a \times U(2)_b \times U(1)_c \times U(1)_d \, . \tag{4.1.4}$$

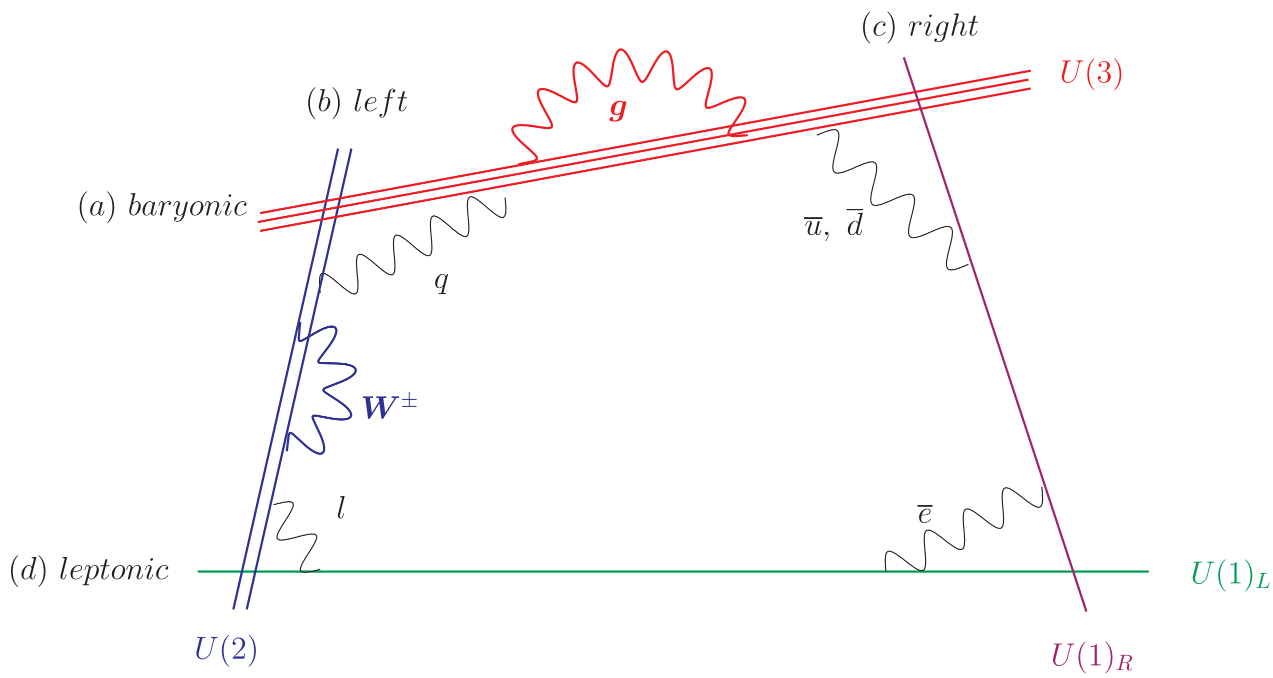


Figure 4.2: D brane quiver realizing the Standard Model.

Note that there are four different $U(1)$ gauge group factors. In a specific compactification, most of them will be anomalous, such that the corresponding gauge bosons become massive by the Green-Schwarz mechanism. However one has to ensure that the gauge boson of the linear combination, which determines the weak hypercharge, stays massive. The $U(1)$ gauge symmetries, which are anomalous or which correspond to massive gauge bosons, nevertheless remain as global symmetries in all perturbative scattering amplitudes. In particular, the symmetry associated to $U(1)_a$ is the baryon number conservation. Hence, all perturbative processes respect baryon number conservation. Of course, there might be other dangerous processes in string perturbation theory, such as flavor changing neutral currents, which are in general not prohibited by the global $U(1)$ symmetries of this D brane quiver. However this is a model dependent issue, which we do not address here.

Specifically, the SM gauge bosons in the adjoint representations of the gauge group $\mathcal{G}$ such as the gluons, the weak gauge bosons and the hypercharge gauge boson (being associated to linear combination of the various $U(1)$ factors), correspond to open strings with two endpoints on the same stack of D branes. On the other hands, the SM matter fields such as quarks and leptons are open strings located at the various intersection points of the four different D branes (and their orientifold images), see section 4.4 for their SCFT implementation. They transform under bifundamental representations of the four gauge group factors, and they can also be in the antisymmetric representation $\mathbf{3}_A$ of $SU(3)_a$, in case the color stack of D branes is intersected by its orientifold image. As it turns out the four stack D brane quiver reproduces all quantum numbers of the SM particles in a straightforward and natural way. In fact, no GUT embedding is necessary to explain the gauge quantum numbers of the SM particles. The family replication is explained by multiple intersections of the D-branes inside the compact space. Moreover it is shown in [162] that one can construct consistent type II string compactifications on the $\mathbb{Z}_6'$ orientifold which reproduce the spectrum of the SM with three generations of quarks and leptons and without chiral exotics.

The four stacks of intersecting D6 branes that give rise to the spectrum of the SM are just a local model that has to be embedded into specific global Calabi Yau manifolds in order to obtain a consistent superstring compactification. Working out the universal tree level scattering amplitudes of SM particles only requires the local information about how the SM is realized on intersecting branes. Fully consistent global orientifold models with all their tadpole and stability conditions satisfied are beyond the scope of this work, partly because these consistency conditions depend on the details of the compactification such as background fluxes. Even model dependent four fermion couplings are argued in [101] to depend only on the local structure of the brane intersections, but not on the global Calabi Yau geometry.

4.1.4 Stringy signatures at LHC

As it is well known for a long time [138], string theory contains a huge number of ground states, a problem which is often referred to as the string landscape problem. This observation raises the question about the predictive power of string theory or, respectively, if string theory is testable. In particular one likes to understand if all or at least some four dimensional string vacua share some common, model independent features, which are true in large region of the string landscape.

Several types of string signatures can be possibly expected at the LHC from a low string scale and from large extra dimensions: Firstly, additional $U(1)$ gauge symmetries from the diagonal part of $U(N)$ give rise to Z like massive bosons [152, 153, 154, 155]. Secondly, mini black holes might hint effects of quantum gravity [160, 161]. We will focus on a third type of signature, namely Regge excitations of SM particles with masses in units of $M_{\rm s}$. They give universal contributions to gluon scattering processes, even in compactification scenarios that completely break spacetime supersymmetry. On the other hand, there are Kaluza Klein and winding excitations along the extra dimensions whose masses depends on geometric data of the internal manifold. Their fingerprint in multi-fermion amplitudes can later on allow a precision measurement of the internal geometry and of the how the D branes are embedded into the compact space.

There also exist KK excitations from the closed string sector of the theory with masses as low as 10^{-3} eV. But their coupling to SM fields only occurs at loop levels of string perturbation theory, hence they are suppressed by powers of $g_{\rm s} \sim g_{\rm YM}^2$ compared to the aforementioned tree level processes on the brane. We need a low string scale, large extra dimensions and also weak string coupling in order for our tree level calculations of chapter 8.1 to be reliable and testable at the LHC.

4.2 Maximal $\mathcal{N} = 4$ supersymmetry

Having given the general framework for SM processes in superstring compactifications, we can now work out the formal aspects of how to implement these scenarios in superconformal field theory language. As a first step, we will consider four dimensional compactifications with all the 16 supercharges preserved. This occurs for D3 branes in $D = 10$ flat Minkowski spacetime or compacification on a generalized T^6 torus.

4.2.1 Dimensional reduction of the SCFT

As a first step of dimensionally reducing the RNS SCFT, this subsection explains how the conformal fields in vector- and spinor representations decompose under $SO(1,9) \to SO(1,3) \times SO(6)$. For this purpose, let us briefly recapitulate the main decomposition rules from appendix A.1:

- vectors decompose into direct sums $X^{m=0,\dots 9} = X^{\mu=0,\dots 3} \oplus X^{i=4,\dots,9}$

- left handed $D = 10$ spinors become $\chi_{\alpha=(\pm,\ldots,\pm)} = \left(\chi_{a=(\pm,\pm)} \otimes \chi^I\right) \oplus \left(\bar{\chi}^{\dot{a}=(\pm,\mp)} \otimes \bar{\chi}_{\bar{J}}\right)$ where $\chi^I(\bar{\chi}_{\bar{J}})$ is a left handed (right handed) spinor with respect to the internal $SO(6)$ with index ranges

$$I \in \left\{(+,+,+),(+,-,-),(-,+,-),(-,-,+)\right\}$$
$$\bar{J} \in \left\{(-,-,-),(-,+,+),(+,-,+),(+,+,-)\right\}$$

- σ^μ and γ^i are the Dirac gamma matrices in four and six dimensions, respectively

In particular, the $SO(1,9)$ covariant conformal fields $i\partial X^m, \psi^m, S_A$ decompose as follows:

$$\begin{aligned} i\partial X^m &= (i\partial X^\mu,\, i\partial Z^i)\ , & S_\alpha &= S_a\,\Sigma^I \,\oplus\, S^{\dot{a}}\,\bar{\Sigma}_{\bar{I}} \\ \psi^m &= (\psi^\mu,\, \Psi^i) \quad\ , & S^{\dot{\alpha}} &= S_a\,\bar{\Sigma}_{\bar{I}} \,\oplus\, S^{\dot{a}}\,\Sigma^I \end{aligned} \tag{4.2.5}$$

This defines $Z^i, \Psi^i, \Sigma^I, \bar{\Sigma}_{\bar{I}}$ to be the internal components of the conformal fields from the ten dimensional theory. The spacetime fields $\psi^\mu, S_a, S^{\dot{b}}$ and the internal fields $\Psi^i, \Sigma^I, \bar{\Sigma}_{\bar{J}}$ belong to different decoupled SCFTs with central charges 6 and 9 respectively. Bosonization of the former requires two chiral fields H^0, H^1, the latter can be represented in terms of three fields H^2, H^3, H^4. We can deduce $SO(1,3)$- and $SO(6)$ covariant OPEs from the ten dimensional ancestors – the most obvious of them are

$$\psi^\mu(z)\,\psi^\nu(w) \;\sim\; \frac{\eta^{\mu\nu}}{z-w} \;+\; \ldots\ , \qquad \Psi^i(z)\,\Psi^j(w) \;\sim\; \frac{\delta^{ij}}{z-w} \;+\; \ldots \tag{4.2.6}$$

whereas $\psi^\mu(z)\Psi^i(w)$ products are regular.

The spacetime spin fields $S_a, S^{\dot{a}}$ for conformal weight $h = \frac{1}{4}$ are governed by the following singularity structure:

$$\begin{aligned} \psi^\mu(z)\,S_a(w) &\sim \frac{1}{\sqrt{2}\,(z-w)^{1/2}}\,\sigma^\mu_{a\dot{b}}\,S^{\dot{b}}(w) \;+\; \ldots \\ S_a(z)\,S_b(w) &\sim \frac{\varepsilon_{ab}}{(z-w)^{1/2}} \;+\; \ldots \\ S^{\dot{a}}(z)\,S^{\dot{b}}(w) &\sim \frac{\varepsilon^{\dot{a}\dot{b}}}{(z-w)^{1/2}} \;+\; \ldots \\ S_a(z)\,S^{\dot{b}}(w) &\sim \frac{1}{\sqrt{2}}\,(\sigma_\mu\,\varepsilon)_a{}^{\dot{b}}\,\psi^\mu(w) \;+\; \ldots \\ S^{\dot{a}}(z)\,S_b(w) &\sim \frac{1}{\sqrt{2}}\,(\bar{\sigma}_\mu\,\varepsilon)^{\dot{a}}{}_b\,\psi^\mu(w) \;+\; \ldots \end{aligned} \tag{4.2.7}$$

Sections 6.2 and 6.3 discuss systematic methods to obtain their correlation functions in $SO(1,3)$ covariant manner and display explicit results.

The internal SCFT with $h = \frac{3}{8}$ spin fields $\Sigma^I, \bar{\Sigma}_{\bar{I}}$, on the other hand, rests on OPEs [163]

$$\Psi_k(z)\,\Sigma^I(w) \;\sim\; \frac{1}{\sqrt{2}\,(z-w)^{1/2}}\,\gamma_k^{I\bar{J}}\,\bar{\Sigma}_{\bar{J}}(w) \;+\; \ldots$$

$$\begin{aligned}
\Sigma^I(z)\,\bar\Sigma_{\bar J}(w) &\sim \frac{C^I{}_{\bar J}}{(z-w)^{3/4}} + \dots \\
\bar\Sigma_{\bar I}(z)\,\Sigma^J(w) &\sim \frac{C_{\bar I}{}^J}{(z-w)^{3/4}} + \dots \\
\Sigma^I(z)\,\Sigma^J(w) &\sim \frac{1}{\sqrt{2}\,(z-w)^{1/4}}\,(\gamma_k\,C)^{IJ}\,\Psi^k(w) + \dots \\
\bar\Sigma_{\bar I}(z)\,\bar\Sigma_{\bar J}(w) &\sim \frac{1}{\sqrt{2}\,(z-w)^{1/4}}\,(\bar\gamma^k\,C)_{\bar I\bar J}\,\Psi_k(w) + \dots\,.
\end{aligned} \tag{4.2.8}$$

A large class of the resulting $SO(6)$ covariant correlators can be found in sections 6.2 and 6.4.

4.2.2 The $\mathcal{N}=4$ SUSY algebra in four dimensions

Dimensionally reducing the 16 supercharges $\mathcal{Q}_\alpha$ yields objects $\mathcal{Q}^I_a$ $(\mathcal{Q}^{\dot a}_{\bar I})$ with a spinor index $a\,(\dot a)$ of the spacetime Lorentz group $SO(1,3)$ and another one $I\,(\bar I)$ of the internal $SO(6)$:

$$\begin{aligned}
\mathcal{Q}^{(-1/2),I}_a &= \alpha'^{-1/4}\oint\frac{\mathrm{d}z}{2\pi i}\,S_a\,\Sigma^I\,\mathrm{e}^{-\phi/2} \\
\bar{\mathcal{Q}}^{(-1/2),\dot a}_{\bar I} &= \alpha'^{-1/4}\oint\frac{\mathrm{d}z}{2\pi i}\,S^{\dot a}\,\bar\Sigma_{\bar I}\,\mathrm{e}^{-\phi/2}
\end{aligned} \tag{4.2.9}$$

The higher ghost picture analogues are given by

$$\begin{aligned}
\mathcal{Q}^{(+1/2),I}_a &= \frac{1}{2\alpha'^{3/4}}\oint\frac{\mathrm{d}z}{2\pi i}\,e^{+\phi/2}\left[i\partial X_\mu\,\sigma^\mu_{a\dot b}\,S^{\dot b}\,\Sigma^I + i\partial Z^k\,S_a\,\gamma_k^{I\bar J}\,\bar\Sigma_{\bar J}\right] \\
\bar{\mathcal{Q}}^{(+1/2),\dot a}_{\bar I} &= \frac{1}{2\alpha'^{3/4}}\oint\frac{\mathrm{d}z}{2\pi i}\,e^{+\phi/2}\left[i\partial X^\mu\,\bar\sigma^{\dot a b}_\mu\,S_b\,\bar\Sigma_{\bar I} + i\partial Z_k\,S^{\dot a}\,\gamma^k_{\bar I J}\,\Sigma^J\right].
\end{aligned} \tag{4.2.10}$$

Due to the correspondence $SO(6)\cong SU(4)$, we can think of the internal spinor index I $(\bar I)$ as a fundamental (antifundamental) $SU(4)$ R symmetry index. That is why $\mathcal{N}=1$ SUSY in $D=10$ dimensions translates into extended $\mathcal{N}=4$ SUSY from the $D=4$ point of view.

The four dimensional reduction of the Poincaré generators P^μ and $M^{\mu\nu}$ is straightforward, the latter give rise to the following commutators with the supercharges

$$\left[M^{\mu\nu}\,,\,\mathcal{Q}^{(q),I}_a\right] = -\frac{1}{2}\,(\sigma^{\mu\nu})_a{}^b\,\mathcal{Q}^{(q),I}_b\,,\qquad \left[M^{\mu\nu}\,,\,\bar{\mathcal{Q}}^{(q),\dot a}_{\bar I}\right] = -\frac{1}{2}\,(\bar\sigma^{\mu\nu})^{\dot a}{}_{\dot b}\,\bar{\mathcal{Q}}^{(q),\dot b}_{\bar I} \tag{4.2.11}$$

In the dimensional reduction of the SUSY algebra (3.5.69), we now have to distinguish the left- and right handed supercharges, this gives rise to the anticommutators

$$\left\{\mathcal{Q}^{(-1/2),I}_a\,,\,\bar{\mathcal{Q}}^{(+1/2),\dot b}_{\bar J}\right\} = -\,(\sigma^\mu\,\varepsilon)_a{}^{\dot b}\,P_\mu\,C^I{}_{\bar J} \tag{4.2.12}$$

$$\left\{\mathcal{Q}^{(-1/2),I}_a\,,\,\mathcal{Q}^{(+1/2),J}_b\right\} = \varepsilon_{ab}\,\mathcal{Z}^{IJ} \tag{4.2.13}$$

$$\left\{\bar{\mathcal{Q}}^{(-1/2),\dot a}_{\bar I}\,,\,\bar{\mathcal{Q}}^{(+1/2),\dot b}_{\bar J}\right\} = \varepsilon^{\dot a\dot b}\,\bar{\mathcal{Z}}_{\bar I\bar J} \tag{4.2.14}$$

which introduce antisymmetric central charges:

$$\mathcal{Z}^{IJ} \;=\; -\frac{1}{2\alpha'} \oint \frac{\mathrm{d}z}{2\pi i}\, i\partial Z^k \, (\gamma_k \, C)^{IJ} \;=\; -\,\mathcal{Z}^{JI} \tag{4.2.15}$$

$$\bar{\mathcal{Z}}_{\bar{I}\bar{J}} \;=\; -\frac{1}{2\alpha'} \oint \frac{\mathrm{d}z}{2\pi i}\, i\partial Z_k \, (\bar{\gamma}^k \, C)_{\bar{I}\bar{J}} \;=\; -\,\bar{\mathcal{Z}}_{\bar{J}\bar{I}} \tag{4.2.16}$$

They can be identified with the internal components of the momentum operator. It is easy to check that they commute with any spacetime Poincaré generator and the supercharges.

4.2.3 The massless $\mathcal{N}=4$ SYM multiplet in four dimensions

This subsection focuses on the massless states in maximally supersymmetric compactifications. The $8+8$ states form the particle content of $\mathcal{N}=4$ SYM theory to which some people lovingly refer to as the "harmonic oscillator" of quantum field theory or the "simplest" QFT in four dimensions [164]. Its S matrix can therefore be probed with the tools of superstring theory, we will say a lot more on that in later chapters.

This work does not discuss states with momenta along the internal directions, i.e. no Kaluza Klein or winding excitations. For any state introduced in a four dimensional setting, the ten momentum has spacetime components only:

$$k^m \;=\; (k^{\mu=0,1,2,3}, 0^{i=4,\dots,9}) \, , \qquad k^\mu \, k_\mu \;=\; -\,m^2 \tag{4.2.17}$$

The four dimensional particle content encompasses one gluon field, four gaugino- and antigaugino species and six scalars. Table 4.2 gives an overview of their relevant quantum numbers. The $D=10$ gluon splits into a four dimensional gauge boson with polarization ξ^μ in spacetime directions only and into six scalars ϕ^i polarized into internal directions, in short: $\xi^m \mapsto (\xi^\mu, \phi^i)$. The gauginos follow the usual $SO(1,9) \to SO(1,3) \times SO(6)$ reduction rules $u^\alpha \mapsto u_I^a \oplus \bar{u}_{\dot{a}}^{\bar{I}}$.

particle	symbol	helicity	$SU(4)$ representation	$SO(6)$ representation	coupling
gluon	g	± 1	scalar	scalar	g_{A}
gaugino	λ^I	$-1/2$	fundamental	left handed spinor	g_λ
antigaugino	$\bar{\lambda}_{\bar{I}}$	$+1/2$	antifundamental	right handed spinor	g_λ
scalar	ϕ_j	0	antisymmetric	vector	g_ϕ

Table 4.2: particle content of the massless four dimensional $\mathcal{N}=4$ SYM multiplet

Let us now introduce the corresponding vertex operators:

- gluon with polarization vector ξ_μ

$$V^{(-1)}(\xi, k, z) \;=\; g_{\mathrm{A}} \, \xi_\mu \, \psi^\mu(z) \, \mathrm{e}^{-\phi(z)} \, \mathrm{e}^{ik\cdot X(z)} \tag{4.2.18}$$

$$V^{(0)}(\xi,k,z) \;=\; \frac{g_{\rm A}}{\sqrt{2\alpha'}}\,\xi_\mu \Big(i\partial X^\mu(z) \;-\; 2\alpha'\,k_\nu\,\psi^\mu(z)\,\psi^\nu(z)\Big)\,\mathrm{e}^{ik\cdot X(z)} \tag{4.2.19}$$

- (anti-) gauginos with spinor wave functions u^a_I ($\bar u^{\bar I}_{\dot a}$)

$$V^{(-1/2)}(u,k,z) \;=\; g_\lambda\, u^a_I\, S_a(z)\,\Sigma^I(z)\,\mathrm{e}^{-\phi(z)/2}\,\mathrm{e}^{ik\cdot X(z)} \tag{4.2.20}$$

$$V^{(-1/2)}(\bar u,k,z) \;=\; g_\lambda\, \bar u^{\bar I}_{\dot a}\, S^{\dot a}(z)\,\bar\Sigma_{\bar I}(z)\,\mathrm{e}^{-\phi(z)/2}\,\mathrm{e}^{ik\cdot X(z)} \tag{4.2.21}$$

$$V^{(+1/2)}(u,k,z) \;=\; \frac{g_\lambda}{2\sqrt{\alpha'}}\, u^a_I\,\mathrm{e}^{\phi(z)/2}\,\mathrm{e}^{ik\cdot X(z)} \tag{4.2.22}$$

$$\times\Big(\big[\, i\partial X_\mu \;+\; 2\alpha'\,k_\nu\,\psi^\nu\,\psi_\mu\,\big]\,\sigma^\mu_{a\dot b}\,S^{\dot b}\,\Sigma^I \;+\; i\partial Z^k\,\gamma_k^{I\bar J}\,S_a\,\bar\Sigma_{\bar J}\Big)(z)$$

$$V^{(+1/2)}(\bar u,k,z) \;=\; \frac{g_\lambda}{2\sqrt{\alpha'}}\, \bar u^{\bar I}_{\dot a}\,\mathrm{e}^{\phi(z)/2}\,\mathrm{e}^{ik\cdot X(z)} \tag{4.2.23}$$

$$\times\Big(\big[\, i\partial X^\mu \;+\; 2\alpha'\,k_\nu\,\psi^\nu\,\psi^\mu\,\big]\,\bar\sigma_\mu^{\dot a b}\,S_b\,\bar\Sigma_{\bar I} \;+\; i\partial Z_k\,\bar\gamma^k_{\bar I J}\,S^{\dot a}\,\Sigma^J\Big)(z)$$

- scalars with internal polarization ϕ_j

$$V^{(-1)}(\phi,k,z) \;=\; g_\phi\,\phi_j\,\Psi^j(z)\,\mathrm{e}^{-\phi(z)}\,\mathrm{e}^{ik\cdot X(z)} \tag{4.2.24}$$

$$V^{(0)}(\phi,k,z) \;=\; \frac{g_\phi}{\sqrt{2\alpha'}}\,\phi_j\Big(i\partial Z^j(z) \;+\; 2\alpha'\,k_\mu\,\psi^\mu(z)\,\Psi^j(z)\Big)\,\mathrm{e}^{ik\cdot X(z)} \tag{4.2.25}$$

The scalars have a natural interpretation as massless transverse fluctuations of the D branes. More precisely, the diagonal element $\phi^i(T^a)_{\alpha_\ell}{}^{\alpha_\ell}$ describes the fluctuation of the ℓ'th brane of the stack where the endpoints of the open string carrying the SYM multiplet are confined.

4.2.4 SUSY transformations within the $\mathcal{N}=4$ multiplet

In this section, we give the SUSY variations of $\mathcal{N}=4$ states by acting with the SUSY charges (4.2.9) and (4.2.10) on the vertex operators. Since Weyl spinors of $SO(1,9)$ give rise to both $SO(1,3)$ chiralities, there are many cases to consider:

$$\big[\,\mathcal{Q}^{(1/2)}(\eta)\,,\,V^{(-1)}(\xi)\,\big] \;=\; V^{(-1/2)}\Big(\,u^b_I=\tfrac{1}{\sqrt2}\eta^a_I(\sigma^{\mu\nu})_a{}^b k_\mu\xi_\nu\Big) \tag{4.2.26}$$

$$\big[\,\bar{\mathcal{Q}}^{(1/2)}(\bar\eta)\,,\,V^{(-1)}(\xi)\,\big] \;=\; V^{(-1/2)}\Big(\,\bar u^{\bar I}_{\dot b}=\tfrac{1}{\sqrt2}\bar\eta^{\bar I}_{\dot a}(\bar\sigma^{\mu\nu})^{\dot a}{}_{\dot b} k_\mu\xi_\nu\Big) \tag{4.2.27}$$

$$\big[\,\mathcal{Q}^{(-1/2)}(\eta)\,,\,V^{(-1/2)}(u)\,\big] \;=\; V^{(-1)}\Big(\,\phi_j=\tfrac{1}{\sqrt2}\eta^a_I\varepsilon_{ab}u^b_J(\gamma_jC)^{IJ}\Big) \tag{4.2.28}$$

$$\big[\,\bar{\mathcal{Q}}^{(-1/2)}(\bar\eta)\,,\,V^{(-1/2)}(\bar u)\,\big] \;=\; V^{(-1)}\Big(\,\phi_j=\tfrac{1}{\sqrt2}\bar\eta^{\bar I}_{\dot a}\varepsilon^{\dot a\dot b}\bar u^{\bar J}_{\dot b}(\bar\gamma_jC)_{\bar I\bar J}\Big) \tag{4.2.29}$$

$$\big[\,\bar{\mathcal{Q}}^{(-1/2)}(\bar\eta)\,,\,V^{(-1/2)}(u)\,\big] \;=\; V^{(-1)}\Big(\,\xi^\mu=\tfrac{1}{\sqrt2}\bar\eta^{\bar J}_{\dot a}(\bar\sigma^\mu\varepsilon)^{\dot a}{}_b C_{\bar J}{}^I u^b_I\Big) \tag{4.2.30}$$

$$\big[\,\mathcal{Q}^{(-1/2)}(\eta)\,,\,V^{(-1/2)}(\bar u)\,\big] \;=\; V^{(-1)}\Big(\,\xi^\mu=\tfrac{1}{\sqrt2}\eta^a_I(\sigma^\mu\varepsilon)_a{}^{\dot b} C^I{}_{\bar J}\bar u^{\bar J}_{\dot b}\Big) \tag{4.2.31}$$

$$\big[\,\mathcal{Q}^{(1/2)}(\eta)\,,\,V^{(-1)}(\phi)\,\big] \;=\; V^{(-1/2)}\Big(\,\bar u^{\bar J}_{\dot b}=\tfrac{1}{\sqrt2}\eta^a_I\,\not{k}_{a\dot b}\gamma_j^{I\bar J}\phi^j\Big) \tag{4.2.32}$$

$$\big[\,\bar{\mathcal{Q}}^{(1/2)}(\bar\eta)\,,\,V^{(-1)}(\phi)\,\big] \;=\; V^{(-1/2)}\Big(\,u^b_J=\tfrac{1}{\sqrt2}\bar\eta^{\bar I}_{\dot a}\,\not{k}^{\dot a b}\gamma^j_{\bar I J}\phi_j\Big) \tag{4.2.33}$$

This relates the normalization constants of the vertex operators via

$$g_{\rm A} \;=\; g_\phi \;=\; \frac{g_\lambda}{\alpha'^{1/4}} \,. \tag{4.2.34}$$

The agreement of $g_{\rm A}$ with g_ϕ is quite natural since ϕ^i are simply the internal polarization states of the $D = 10$ gluon.

The SUSY transformation of the wave functions follow by scanning the right hand side for the relevant particle:

$$\delta_\eta \xi^\mu \;=\; \frac{1}{\sqrt{2}}\, \eta_I^a\, (\sigma^\mu\, \varepsilon)_a{}^{\dot b}\, C^I{}_{\bar J}\, \bar u_{\dot b}^{\bar J} \qquad \delta_{\bar\eta} \xi^\mu \;=\; \frac{1}{\sqrt{2}}\, \bar\eta_{\dot a}^{\bar J}\, (\bar\sigma^\mu\, \varepsilon)^{\dot a}{}_b\, C_{\bar J}{}^I\, u_I^b \tag{4.2.35}$$

$$\delta_\eta u_I^a \;=\; \frac{1}{\sqrt{2}}\, \eta_I^b\, (\sigma^{\mu\nu})_b{}^a\, k_\mu\, \xi_\nu \qquad \delta_{\bar\eta} u_I^a \;=\; \frac{1}{\sqrt{2}}\, \bar\eta_{\dot b}^{\bar J}\, \not{k}^{\dot b a}\, \gamma^j_{\bar J I}\, \phi_j \tag{4.2.36}$$

$$\delta_\eta \bar u_{\dot a}^{\bar I} \;=\; \frac{1}{\sqrt{2}}\, \eta_J^b\, \not{k}_{b\dot a}\, \gamma_j^{J\bar I}\, \phi^j \qquad \delta_{\bar\eta} \bar u_{\dot a}^{\bar I} \;=\; \frac{1}{\sqrt{2}}\, \bar\eta_{\dot b}^{\bar I}\, (\bar\sigma^{\mu\nu})^{\dot b}{}_{\dot a}\, k_\mu\, \xi_\nu \tag{4.2.37}$$

$$\delta_\eta \phi_j \;=\; \frac{1}{\sqrt{2}}\, \eta_I^a\, \varepsilon_{ab}\, u_J^b\, (\gamma_j C)^{IJ} \qquad \delta_{\bar\eta} \phi_j \;=\; \frac{1}{\sqrt{2}}\, \bar\eta_{\dot a}^{\bar I}\, \varepsilon^{\dot a\dot b}\, \bar u_{\dot b}^{\bar J}\, (\bar\gamma_j C)_{\bar I\bar J} \tag{4.2.38}$$

It is easy to check that the algebra (4.2.12) to (4.2.14) closes up to gluon gauge transformations $\xi^\mu \to \xi^\mu + k^\mu$ on each state.

4.3 SUSY breaking in compactifications

Unfortunately, any extended $\mathcal{N} > 1$ SUSY in four dimensions is incompatible with the particle content of the SM because it leads to a non-chiral spectrum. In order to get a little closer to realistic SM like scenarios, we have to break $\mathcal{N} = 4$ supersymmetry in $D = 4$ down to $\mathcal{N} = 1$ (or even to $\mathcal{N} = 0$). This amounts to projecting out certain internal $h = \frac{3}{8}$ fields Σ which govern the supercharge species (4.2.9).

4.3.1 Spacetime- versus worldsheet supersymmetry

It has been shown [163,165,166] that the number of spacetime supersymmetries allows to infer a lot of information on the internal $c = 9$ SCFT, in particular on the amount of worldsheet supersymmetry. The existence of a single $\mathcal{N} = 1$ spacetime supersymmetry with one $\Sigma, \bar\Sigma$ species

$$\mathcal{Q}_a^{(-1/2)} \;=\; \alpha'^{-1/4} \oint \frac{{\rm d}z}{2\pi i}\, S_a\, \Sigma\, {\rm e}^{-\phi/2} \,, \qquad \bar{\mathcal{Q}}^{(-1/2),\dot a} \;=\; \alpha'^{-1/4} \oint \frac{{\rm d}z}{2\pi i}\, S^{\dot a}\, \bar\Sigma\, {\rm e}^{-\phi/2} \tag{4.3.39}$$

already implies that the internal SCFT contains a $U(1)$ current $\mathcal{J}(z)$ (see subsection 9.2.1) which rotates one supercurrent $G \sim G^+ + G^-$ into a second one $G' \sim G^+ - G^-$. This enhances the worldsheet supersymmetry to $\mathcal{N} = 2$.

Similar arguments imply that $\mathcal{N} = 2$ spacetime SUSY gives rise to a $c = 6$ sector in the internal SCFT with $\mathcal{N} = 4$ worldsheet SUSYs and another $c = 3$ sector of $\mathcal{N} = 2$ worldsheet SUSY. The latter part of the internal SCFT can be constructed from two free $h = \frac{1}{2}$ superfields, i.e. the $(\partial Z^k, \Psi^k)$ of two internal directions, say $k = 8, 9$. Moreover, the existence of $\mathcal{N} = 4$ spacetime SUSY implies that the internal SCFT splits into three $c = 3$ systems with a representation in terms of two free $h = \frac{1}{2}$ superfields each. From the emergence six free worldsheet multiplets $(\partial Z^k, \Psi^k)$, we can conclude that any string vacuum of maximal $\mathcal{N} = 4$ spacetime SUSY necessarily corresponds to some generalized toroidal compactification geometry [167]. In other words – the spacetime properties of a set of $D = 4$ string vacua allow to completely classify the associated SCFTs and, in case of extended spacetime SUSY, to even recover two or six internal dimensions.

Conversely, having a $\mathcal{N} = 2$ SCFT of $c = 9$ for the internal manifold with quantized charges for the $U(1)$ current is a sufficient condition for $\mathcal{N} = 1$ spacetime SUSY. Hence, we can describe a large class of $\mathcal{N} = 1$ supersymmetric string vacua by implementing the conditions in the internal $c = 9$ SCFT. The following subsection is devoted to the SCFT realization.

4.3.2 SCFT implementation of SUSY breaking

A convenient way to elegantly reduce the number of spin field species $\Sigma, \bar{\Sigma}$ in the internal SCFT is based on orbifold projections: We identify internal coordinate superfields $(\partial Z^k, \Psi^k)$ with certain rotations thereof and discard any non-invariant degree of freedom in the SCFT. In the Cartan Weyl basis (2.2.38), this amounts to performing phase rotations of the following complex variables:

$$\begin{aligned} Z^{\pm 2} &:= \frac{1}{\sqrt{2}} \left(Z^4 \pm i Z^5 \right) , & \Psi^{\pm 2} &:= \frac{1}{\sqrt{2}} \left(\Psi^4 \pm i \Psi^5 \right) \equiv \mathrm{e}^{\pm i H_2} \\ Z^{\pm 3} &:= \frac{1}{\sqrt{2}} \left(Z^6 \pm i Z^7 \right) , & \Psi^{\pm 3} &:= \frac{1}{\sqrt{2}} \left(\Psi^6 \pm i \Psi^7 \right) \equiv \mathrm{e}^{\pm i H_3} \\ Z^{\pm 4} &:= \frac{1}{\sqrt{2}} \left(Z^8 \pm i Z^9 \right) , & \Psi^{\pm 4} &:= \frac{1}{\sqrt{2}} \left(\Psi^8 \pm i \Psi^9 \right) \equiv \mathrm{e}^{\pm i H_4} \end{aligned} \tag{4.3.40}$$

The Z^{+j} (Z^{-j}) can be though of as living in the fundamental (antifundamental) representation of $SU(3)$ subgroup of $SO(6)$.

SUSY breaking can be implemented by means of the following group action on the internal $(\partial Z^{\pm j}, \Psi^{\pm j})$ and the associated bosons H_2, H_3, H_4:

$$(Z^{\pm j}, \Psi^{\pm j}) \ \mapsto \ \mathrm{e}^{\pm 2\pi i \theta_j} \, (Z^{\pm j}, \Psi^{\pm j}) \, , \qquad H_j \ \mapsto \ H_j \, + \, 2\pi \, \theta_j \tag{4.3.41}$$

The orbifold transformation rules for the internal spin fields Σ^I follow from bosonization

$$\begin{aligned}
\Sigma^{I=(+,+,+)} &\equiv \mathrm{e}^{\frac{i}{2}(+H_2+H_3+H_4)} &\mapsto& \ \mathrm{e}^{i\pi(+\theta_2+\theta_3+\theta_4)}\,\Sigma^{I=(+,+,+)} \\
\Sigma^{I=(+,-,-)} &\equiv \mathrm{e}^{\frac{i}{2}(+H_2-H_3-H_4)} &\mapsto& \ \mathrm{e}^{i\pi(+\theta_2-\theta_3-\theta_4)}\,\Sigma^{I=(+,-,-)} \\
\Sigma^{I=(-,+,-)} &\equiv \mathrm{e}^{\frac{i}{2}(-H_2+H_3-H_4)} &\mapsto& \ \mathrm{e}^{i\pi(-\theta_2+\theta_3-\theta_4)}\,\Sigma^{I=(-,+,-)} \\
\Sigma^{I=(-,-,+)} &\equiv \mathrm{e}^{\frac{i}{2}(-H_2-H_3+H_4)} &\mapsto& \ \mathrm{e}^{i\pi(-\theta_2-\theta_3+\theta_4)}\,\Sigma^{I=(-,-,+)}
\end{aligned} \tag{4.3.42}$$

with analogous statements for $\bar{\Sigma}_{\bar{J}=(-,-,-)} \equiv \mathrm{e}^{\frac{i}{2}(-H_2-H_3-H_4)}$ as well as $\bar{\Sigma}_{\bar{J}=(-,+,+)}$, $\bar{\Sigma}_{\bar{J}=(+,-,+)}$ and $\bar{\Sigma}_{\bar{J}=(+,+,-)}$. For generical values of the rotation angles $\theta_{2,3,4}$, none of the $\Sigma, \bar{\Sigma}$ fields will be invariant and remain in the spectrum of the orbifold theory. In order to achieve $\mathcal{N} = 1$ spacetime SUSY with one rotation invariant Σ species, we have to impose[2]

$$\theta_2 \ + \ \theta_3 \ + \ \theta_4 \ \in \ \mathbb{Z} \quad \Leftrightarrow \quad \mathcal{N} \geq 1 \ \text{ spacetime SUSY} \ . \tag{4.3.43}$$

Unless one if the angles happens to vanish, exactly one out of four spin fields and therefore supercharges stays invariant

$$\begin{pmatrix} \Sigma^{(+,+,+)} \\ \Sigma^{(+,-,-)} \\ \Sigma^{(-,+,-)} \\ \Sigma^{(-,-,+)} \end{pmatrix} \mapsto \begin{pmatrix} \Sigma^{(+,+,+)} \\ \mathrm{e}^{2\pi i\theta_2}\,\Sigma^{(+,-,-)} \\ \mathrm{e}^{2\pi i\theta_3}\,\Sigma^{(-,+,-)} \\ \mathrm{e}^{2\pi i\theta_4}\,\Sigma^{(-,-,+)} \end{pmatrix} , \quad \begin{pmatrix} \bar{\Sigma}_{(-,-,-)} \\ \bar{\Sigma}_{(-,+,+)} \\ \bar{\Sigma}_{(+,-,+)} \\ \bar{\Sigma}_{(+,+,-)} \end{pmatrix} \mapsto \begin{pmatrix} \bar{\Sigma}_{(-,-,-)} \\ \mathrm{e}^{-2\pi i\theta_2}\,\bar{\Sigma}_{(-,+,+)} \\ \mathrm{e}^{-2\pi i\theta_3}\,\bar{\Sigma}_{(+,-,+)} \\ \mathrm{e}^{-2\pi i\theta_4}\,\bar{\Sigma}_{(+,+,-)} \end{pmatrix} \tag{4.3.44}$$

and we can use $\Sigma \equiv \Sigma^{(+,+,+)}$ and $\bar{\Sigma} \equiv \bar{\Sigma}_{(-,-,-)}$ to construct the $\mathcal{N} = 1$ supercharges (4.3.39). The $U(1)$ current of the internal SCFT is then given by $\mathcal{J} = \frac{1}{\sqrt{3}} \sum_{j=2}^{4} i\partial H_j$, it assigns charges ± 1 to the Cartan Weyl fermions $\Psi^{\pm j}$ and emerges from the subleading singularity of the OPE [165]:

$$\Sigma(z)\,\bar{\Sigma}(w) \ \sim \ \frac{1}{(z-w)^{3/4}} \ + \ \frac{\sqrt{3}}{2}\,(z-w)^{1/4}\,\mathcal{J}(w) \ + \ \dots \tag{4.3.45}$$

Extended spacetime SUSY can be realized by imposing further constraints on the θ_j in addition to (4.3.43). The trivial case $\theta_{2,3,4} = 0$ would of course preserve the full $\mathcal{N} = 4$ SUSY, and setting $\theta_4 = 0$ with $\theta_2 = -\theta_3 \neq 0$ gives rise to $\mathcal{N} = 2$ spacetime SUSY. The $(\partial Z^8, \Psi^8)$ and $(\partial Z^9, \Psi^9)$ superfields then furnish the decoupling $c = 3$ sector of the internal SCFT.

4.3.3 The $\mathcal{N} = 1$ algebra and its massless multiplet

Once the internal spin fields are restricted to $\Sigma \equiv \Sigma^{(+,+,+)}$ and $\bar{\Sigma} \equiv \bar{\Sigma}_{(-,-,-)}$, the internal charge conjugation matrix trivializes $C^{(+,+,+)}_{(-,-,-)} = 1$ and central charges vanish due to their

[2]Naively, one would expect $\theta_2 + \theta_3 + \theta_4 \in 2\mathbb{Z}$ from (4.3.42), but the freedom to redefine $\theta_j \mapsto 1 - \theta_j$ allows to convert even integer sums into odd ones.

antisymmetry $\mathcal{Z}^{[IJ]}$ and $\bar{\mathcal{Z}}_{[\bar{I}\bar{J}]}$. The supercharges then satisfy the standard $\mathcal{N}=1$ algebra

$$\left\{ \mathcal{Q}_a^{(-1/2)} \, , \, \bar{\mathcal{Q}}^{(+1/2),\dot{b}} \right\} \;=\; -\,(\sigma^\mu\,\varepsilon)_a{}^{\dot{b}}\,P_\mu \tag{4.3.46}$$

$$\left\{ \mathcal{Q}_a^{(-1/2)} \, , \, \mathcal{Q}_b^{(+1/2)} \right\} \;=\; \left\{ \bar{\mathcal{Q}}^{(-1/2),\dot{a}} \, , \, \bar{\mathcal{Q}}^{(+1/2),\dot{b}} \right\} \;=\; 0\,. \tag{4.3.47}$$

On the level of massless states, the $\mathcal{N}=1$ projection kills all the scalars ϕ_j and three of the four gaugino species. The vertex operators of the unique (anti-)gaugino are simply obtained by suppressing the R symmetry index from their $\mathcal{N}=4$ ancestors,

$$V^{(-1/2)}(u,k,z) \;=\; g_\lambda\,u^a\,S_a(z)\,\Sigma(z)\,\mathrm{e}^{-\phi(z)/2}\,\mathrm{e}^{ik\cdot X(z)} \tag{4.3.48}$$

$$V^{(-1/2)}(\bar{u},k,z) \;=\; g_\lambda\,\bar{u}_{\dot{a}}\,S^{\dot{a}}(z)\,\bar{\Sigma}(z)\,\mathrm{e}^{-\phi(z)/2}\,\mathrm{e}^{ik\cdot X(z)}\,, \tag{4.3.49}$$

whereas the gluon vertex operators (4.2.18) and (4.2.19) remain unchanged. These 2+2 states of helicities $\pm 1, \pm\frac{1}{2}$ transform as

$$\left[\,\mathcal{Q}^{(1/2)}(\eta)\,,\,V^{(-1)}(\xi)\,\right] \;=\; V^{(-1/2)}\Big(\,u^b=\tfrac{1}{\sqrt{2}}\eta^a(\sigma^{\mu\nu})_a{}^b k_\mu\xi_\nu\,\Big) \tag{4.3.50}$$

$$\left[\,\bar{\mathcal{Q}}^{(1/2)}(\bar{\eta})\,,\,V^{(-1)}(\xi)\,\right] \;=\; V^{(-1/2)}\Big(\,\bar{u}_{\dot{b}}=\tfrac{1}{\sqrt{2}}\bar{\eta}_{\dot{a}}(\bar{\sigma}^{\mu\nu})^{\dot{a}}{}_{\dot{b}} k_\mu\xi_\nu\,\Big) \tag{4.3.51}$$

$$\left[\,\bar{\mathcal{Q}}^{(-1/2)}(\bar{\eta})\,,\,V^{(-1/2)}(u)\,\right] \;=\; V^{(-1)}\Big(\,\xi^\mu=\tfrac{1}{\sqrt{2}}\bar{\eta}_{\dot{a}}(\bar{\sigma}^\mu\varepsilon)^{\dot{a}}{}_b u^b\,\Big) \tag{4.3.52}$$

$$\left[\,\mathcal{Q}^{(-1/2)}(\eta)\,,\,V^{(-1/2)}(\bar{u})\,\right] \;=\; V^{(-1)}\Big(\,\xi^\mu=\tfrac{1}{\sqrt{2}}\eta^a(\sigma^\mu\varepsilon)_a{}^{\dot{b}}\bar{u}_{\dot{b}}\,\Big) \tag{4.3.53}$$

under the $\mathcal{N}=1$ SUSY algebra (4.3.39) with spinor parameters $\eta^a, \bar{\eta}_{\dot{b}}$. In analogy to (4.2.35) to (4.2.38), we obtain variations

$$\delta_\eta\xi^\mu \;=\; \frac{1}{\sqrt{2}}\,\eta^a\,(\sigma^\mu\,\varepsilon)_a{}^{\dot{b}}\,\bar{u}_{\dot{b}} \qquad\qquad \delta_{\bar{\eta}}\xi^\mu \;=\; \frac{1}{\sqrt{2}}\,\bar{\eta}_{\dot{a}}\,(\bar{\sigma}^\mu\,\varepsilon)^{\dot{a}}{}_b\,u^b \tag{4.3.54}$$

$$\delta_\eta u^a \;=\; \frac{1}{\sqrt{2}}\,\eta^b\,(\sigma^{\mu\nu})_b{}^a\,k_\mu\,\xi_\nu \qquad\qquad \delta_{\bar{\eta}}u^a \;=\; 0 \tag{4.3.55}$$

$$\delta_\eta\bar{u}_{\dot{a}} \;=\; 0 \qquad\qquad \delta_{\bar{\eta}}\bar{u}_{\dot{a}} \;=\; \frac{1}{\sqrt{2}}\,\bar{\eta}_{\dot{b}}\,(\bar{\sigma}^{\mu\nu})^{\dot{b}}{}_{\dot{a}}\,k_\mu\,\xi_\nu \tag{4.3.56}$$

for the wavefunctions $\xi^\mu, u^a, \bar{u}_{\dot{b}}$.

4.4 Chiral matter at brane intersection

This section is devoted to chiral matter multiplets due to open strings located at D brane intersections, following the lines of [168, 156]. Quarks and leptons of the SM transform in bifundamental representations of the gauge group factors $SU(3)$, $SU(2)$ and $U(1)$, hence they cannot originate from open strings with both endpoints on the same stack of branes whose excitations fall into the adjoint representation. We will build a bridge between the geometric picture of a D brane intersection and the massless vertex operators for chiral multiplets which involve boundary changing operators of the internal SCFT.

4.4.1 The geometric picture

Before going into the formal SCFT discussion of string excitations at brane intersection, let us gain some intuition from the geometric picture. We consider two stacks of D branes whose worldvolume has one dimension in the Z^8, Z^9 plane, intersecting at an angle $\pi\vartheta$ with $\vartheta \in [0,1]$, see the following figure 4.3.

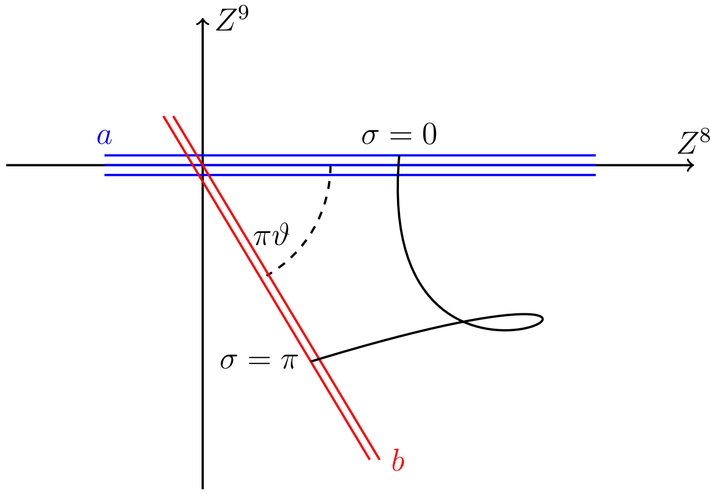


Figure 4.3: Two D brane stacks a and b intersecting in the (Z^8, Z^9) plane.

Open strings with one endpoint ($\sigma = 0$) on brane stack a and the other one ($\sigma = \pi$) on stack b satisfy boundary conditions

$$\begin{aligned} \partial_\sigma Z^8 \Big|_{\sigma=0} &= Z^9 \Big|_{\sigma=0} = 0 \\ \partial_\sigma \Big(\cos(\pi\vartheta)\, Z^8 - \sin(\pi\vartheta)\, Z^9 \Big) \Big|_{\sigma=\pi} &= 0 \\ \Big(\cos(\pi\vartheta)\, Z^9 + \sin(\pi\vartheta)\, Z^8 \Big) \Big|_{\sigma=\pi} &= 0 \, . \end{aligned} \tag{4.4.57}$$

The solutions to the wave equation $\partial\bar{\partial} Z^i = 0$ on the worldsheet subject to these boundary conditions have a shifted mode expansion. More precisely, the complex variables $Z^{\pm 4} = \frac{1}{\sqrt{2}}(Z^8 \pm iZ^9)$ and $\Psi^{\pm 4} = \frac{1}{\sqrt{2}}(\Psi^8 \pm i\Psi^9)$ in the NS sector can be expanded in oscillators

$$Z^{\pm 4} \leftrightarrow \alpha^{+4}_{-\vartheta-n},\, \alpha^{-4}_{\vartheta-1-n} \, , \qquad \Psi^{\pm 4} \leftrightarrow \Psi^{+4}_{-\vartheta-1/2-n},\, \Psi^{-4}_{\vartheta-1/2-n} \, , \qquad n \in \mathbb{Z} \, . \tag{4.4.58}$$

In fact, these strings furnish the twisted sector of the orbifold action (4.3.41) with $2\theta_j = \vartheta_j$. The next subsection will introduce the conformal fields associated with the twisted sector's ground state. The projection of D branes into the remaining internal dimensions which we gathered in complex pairs (4.3.40) can give rise to further intersections: the (Z^4, Z^5) plane with angle ϑ_2, the (Z^6, Z^7) plane with angle ϑ_3 and finally the (Z^8, Z^9) plane with angle ϑ_4.

The Chan Patons degrees of freedom associated with these open strings are different from the standard ones introduced in subsection 2.4.4. The Chan paton matrices do not carry the standard adjoint index T^a but rather bifundamental indices (T^{α}_{β}) with resepect to the gauge groups $\mathcal{G}_a$ and $\mathcal{G}_b$ living on the brane stacks of endpoints a and b. Their matrix elements are given by

$$(T^{\alpha_1}_{\beta_1})_{\alpha_2}{}^{\beta_2} \;=\; \delta^{\alpha_1}_{\alpha_2}\,\delta^{\beta_2}_{\beta_1} \;, \tag{4.4.59}$$

we will revisit them in subsection 8.3.6 for computing color factors in quark amplitudes.

4.4.2 The SCFT implementation

The shifted mode expansions (4.4.58) for the complex fields $\partial Z^{\pm j}$ and $\Psi^{\pm j}$ implies that they depend on fractional powers of the worldsheet coordinate z. These branchings are generated by so-called twist fields $\sigma^{\pm}_{\vartheta_j}$ and $s^{\pm}_{\vartheta_j}$ of conformal dimensions $h(\sigma^{\pm}_{\vartheta_j}) = \frac{1}{2}\vartheta_j(1-\vartheta_j)$ and $h(s^{\pm}_{\vartheta_j}) = \frac{1}{2}(1-\vartheta_j)^2$. Their defining properties are the operator products with ∂Z and Ψ:

$$\partial Z^{\pm j}(z)\,\sigma^{\pm}_{\vartheta_j}(w) \;\sim\; (z-w)^{\vartheta_j-1}\,\tau^{\pm}_{\vartheta_j}(w) \;+\; \dots \tag{4.4.60}$$

$$\partial Z^{\pm j}(z)\,\sigma^{\mp}_{\vartheta_j}(w) \;\sim\; (z-w)^{-\vartheta_j}\,\tilde{\tau}^{\mp}_{\vartheta_j}(w) \;+\; \dots \tag{4.4.61}$$

$$\Psi^{\pm j}(z)\,s^{\pm}_{\vartheta_j}(w) \;\sim\; (z-w)^{1-\vartheta_j}\,\tilde{t}^{\pm}_{\vartheta_j}(w) \;+\; \dots \tag{4.4.62}$$

$$\Psi^{\pm j}(z)\,s^{\mp}_{\vartheta_j}(w) \;\sim\; (z-w)^{\vartheta_j-1}\,t^{\mp}_{\vartheta_j}(w) \;+\; \dots \tag{4.4.63}$$

The right hand side involves excited versions $\tau^{\pm}_{\vartheta_j}, \tilde{\tau}^{\pm}_{\vartheta_j}, t^{\pm}_{\vartheta_j}, \tilde{t}^{\pm}_{\vartheta_j}$ of the twist fields with conformal dimensions $h(\tau^{\pm}_{\vartheta_j}) = \frac{1}{2}\vartheta_j(3-\vartheta_j)$ and $h(\tilde{\tau}^{\pm}_{\vartheta_j}) = \frac{1}{2}(1-\vartheta_j)(2+\vartheta_j)$. The latter OPEs can be reproduced by a bosonized representation

$$\Psi^{\pm j} \;\equiv\; \mathrm{e}^{\pm iH_j} \qquad\qquad s^{\pm}_{\vartheta_j} \;\equiv\; \mathrm{e}^{\pm i(1-\vartheta_j)H_j} \tag{4.4.64}$$

$$t^{\pm}_{\vartheta_j} \;\equiv\; \mathrm{e}^{\mp i\vartheta_j H_j} \qquad\qquad \tilde{t}^{\pm}_{\vartheta_j} \;\equiv\; \mathrm{e}^{\pm i(2-\vartheta_j)H_j} \tag{4.4.65}$$

with obvious conformal weights $h(t^{\pm}_{\vartheta_j}) = \frac{1}{2}\vartheta_j^2$ and $h(\tilde{t}^{\pm}_{\vartheta_j}) = \frac{1}{2}(2-\vartheta_j)^2$.

Worldsheet supersymmetry only admits those combinations of $\sigma^{\pm}_{\vartheta_j}$ and $s^{\pm}_{\vartheta}$ in physical states which are local relative to the internal supercurrent $\sim \sum_j(\partial Z^{+j}\Psi^{-j} + \partial Z^{-j}\Psi^{+j})$, i.e. only the products $\sigma^{\pm}_{\vartheta_j} s^{\mp}_{\vartheta_j}$ can be included into vertex operators. Having a massless ground state in the twisted NS sector requires a twist field of conformal dimension $h = \frac{1}{2}$ in the canonical (-1) superghost picture. This imposes a condition on the angles ϑ_j

$$\vartheta_2 \;+\; \vartheta_3 \;+\; \vartheta_4 \;=\; 2 \quad\Leftrightarrow\quad h\left(\prod_{j=2}^{4} \sigma^{\pm}_{\vartheta_j} s^{\mp}_{\vartheta_j}\right) \;=\; \frac{1}{2} \tag{4.4.66}$$

which we had already found in subsection 4.3.2 as a requirement for spacetime SUSY (recall that $\vartheta_j = 2\theta_j$). If the angles ϑ_j satisfy (4.4.66), then the twisted NS sector contains a (GSO positive) massless scalar with vertex operator

$$V^{(-1)}(C,k,\vartheta_j) \;=\; g_C\, C \prod_{j=2}^{4} \sigma^-_{\vartheta_j}\, s^+_{\vartheta_j}\, \mathrm{e}^{-\phi}\, \mathrm{e}^{ik\cdot X} \tag{4.4.67}$$

$$V^{(-1)}(\bar C,k,\vartheta_j) \;=\; g_C\, \bar C \prod_{j=2}^{4} \sigma^+_{\vartheta_j}\, s^-_{\vartheta_j}\, \mathrm{e}^{-\phi}\, \mathrm{e}^{ik\cdot X}\,. \tag{4.4.68}$$

The higher ghost picture analogue can be computed by means of the Cartan Weyl representation $\sim \sum_j (\partial Z^{+j}\Psi^{-j} + \partial Z^{-j}\Psi^{+j})$ of the supercurrent and th OPEs (4.4.60) to (4.4.63):

$$V^{(0)}(C,k,\vartheta_j) \;=\; \frac{g_C}{\sqrt{2\alpha'}}\, C \left(2\alpha'\, k_\mu\, \psi^\nu \prod_{j=2}^{4} \sigma^-_{\vartheta_j}\, s^+_{\vartheta_j} \;+\; \sum_{j=2}^{4} \tilde\tau^-_{\vartheta_j}\, t^+_{\vartheta_j} \prod_{\ell\neq j}^{4} \sigma^-_{\vartheta_\ell}\, s^+_{\vartheta_\ell} \right) \mathrm{e}^{ik\cdot X} \tag{4.4.69}$$

$$V^{(0)}(\bar C,k,\vartheta_j) \;=\; \frac{g_C}{\sqrt{2\alpha'}}\, \bar C \left(2\alpha'\, k_\mu\, \psi^\nu \prod_{j=2}^{4} \sigma^+_{\vartheta_j}\, s^-_{\vartheta_j} \;+\; \sum_{j=2}^{4} \tilde\tau^+_{\vartheta_j}\, t^-_{\vartheta_j} \prod_{\ell\neq j}^{4} \sigma^+_{\vartheta_\ell}\, s^-_{\vartheta_\ell} \right) \mathrm{e}^{ik\cdot X} \tag{4.4.70}$$

If the angles deviate from the SUSY configuration $\vartheta_2+\vartheta_3+\vartheta_4 = 2$, then this state acquires a mass $m^2 = \frac{1}{2\alpha'}\,(2-\vartheta_2-\vartheta_3-\vartheta_4)$.

4.4.3 The chiral $\mathcal{N}=1$ multiplet

If the intersection angles satisfy the $\mathcal{N}=1$ SUSY condition $\vartheta_2+\vartheta_3+\vartheta_4 = 2$, then one can construct massless partner fermions from the NS scalars (4.4.69), (4.4.70):

$$\left[\, \bar{\mathcal{Q}}^{(-1/2)}(\bar\eta)\,,\; V^{(0)}(C,k,\vartheta_j) \right] \;=\; g_C\, C\, \alpha'^{1/4}\, \bar\eta_{\dot a}\, \not{k}^{\dot a b}\, S_b \prod_{j=2}^{4} \sigma^-_{\vartheta_j}\, \mathrm{e}^{i\left(\frac12-\vartheta_j\right)H_j}\, \mathrm{e}^{-\phi/2}\, \mathrm{e}^{ik\cdot X} \tag{4.4.71}$$

$$\left[\, \mathcal{Q}^{(-1/2)}(\eta)\,,\; V^{(0)}(\bar C,k,\vartheta_j) \right] \;=\; g_C\, \bar C\, \alpha'^{1/4}\, \eta^{a}\, \not{k}_{a\dot b}\, S^{\dot b} \prod_{j=2}^{4} \sigma^+_{\vartheta_j}\, \mathrm{e}^{-i\left(\frac12-\vartheta_j\right)H_j}\, \mathrm{e}^{-\phi/2}\, \mathrm{e}^{ik\cdot X} \tag{4.4.72}$$

$$\left[\, \mathcal{Q}^{(-1/2)}(\eta)\,,\; V^{(0)}(C,k,\vartheta_j) \right] \;=\; \left[\, \bar{\mathcal{Q}}^{(-1/2)}(\bar\eta)\,,\; V^{(0)}(\bar C,k,\vartheta_j) \right] \;=\; 0 \tag{4.4.73}$$

Note that only the first part of the zero ghost picture vertex operators contributes to this variation. It is convenient to collect the twist fields of the right hand side into one boundary condition changing operator

$$\Xi \;:=\; \prod_{j=2}^{4} \sigma^-_{\vartheta_j}\, \mathrm{e}^{i\left(\frac12-\vartheta_j\right)H_j}\,, \qquad \bar\Xi \;:=\; \prod_{j=2}^{4} \sigma^+_{\vartheta_j}\, \mathrm{e}^{-i\left(\frac12-\vartheta_j\right)H_j} \tag{4.4.74}$$

which has conformal dimension $h(\Xi) = \frac38$ independent on the angles. The fermions due to the SUSY transformation (4.4.71) and (4.4.72) remain massless independent on angles ϑ_j, i.e. even if all the spacetime SUSY is broken. Their generic vertex operators are written as

$$V^{(-1/2)}(u,k,\vartheta_j) \;=\; g_\psi\, u^a\, S_a\, \Xi\, \mathrm{e}^{-\phi/2}\, \mathrm{e}^{ik\cdot X} \tag{4.4.75}$$

$$V^{(-1/2)}(\bar u,k,\vartheta_j) \;=\; g_\psi\, \bar u_{\dot a}\, S^{\dot a}\, \bar\Xi\, \mathrm{e}^{-\phi/2}\, \mathrm{e}^{ik\cdot X} \tag{4.4.76}$$

with normalization $g_\psi = \alpha'^{-1/4} g_C$. If $\mathcal{N}=1$ SUSY is preserved then the complex scalar C and the Weyl fermions $u,\bar u$ form a chiral multiplet with $2+2$ on-shell degrees of freedom. Their variations are given as follows:

$$\delta_\eta C \;=\; \eta^a\, u_a \qquad\qquad \delta_{\bar\eta} C \;=\; \bar\eta_{\dot a}\, \bar u^{\dot a} \tag{4.4.77}$$

$$\delta_\eta u^a \;=\; 0 \qquad\qquad \delta_{\bar\eta} u^a \;=\; \bar\eta_{\dot b}\, \not{k}^{\dot b a}\, C \tag{4.4.78}$$

$$\delta_\eta \bar u_{\dot a} \;=\; \eta^b\, \not{k}_{b\dot a}\, C \qquad\qquad \delta_{\bar\eta} \bar u^{\dot a} \;=\; 0 \tag{4.4.79}$$

Note that the vertex operators (4.4.75) and (4.4.76) of the chiral fermions closely resembles the gaugino vertex (4.3.48) and (4.3.49): Apart from their different normalizations $g_\lambda \leftrightarrow g_\psi$ and their different Chan Paton degrees of freedom, they reproduce each other under exchange of the internal $h=\frac{3}{8}$ fields $\Sigma \leftrightarrow \Xi$. The latter give rise to the same two point functions,

$$\langle\, \Sigma(z)\, \bar\Sigma(w)\,\rangle \;=\; \langle\, \Xi(z)\, \bar\Xi(w)\,\rangle \;=\; \frac{1}{(z-w)^{3/4}}\;, \tag{4.4.80}$$

that is why they have the same tree level coupling to gluons which do not see the internal SCFT. Details of this argument are given in subsection 8.2.1, and it remains valid for massive Regge excitations constructed from the spacetime SCFT.

4.5 The first mass level in four dimensions

In this section, we take a look at the first mass level from the four dimensional point of view. The ten dimensional particle content has been identified in section 3.3, it encompasses a spin two tensor B_{mn}, a three form E_{mnp} and chiral gravitino χ^α_m. The fate of these 128+128 states generically depends on the amount of spacetime SUSY which is preserved in the compactification to four dimensions. But still, some states are completely model independent, and we will argue in later chapters that only the universal part of the spectrum will appear as resonances in gluon collisions.

4.5.1 Maximal supersymmetry

In maximally supersymmetric compactifications to $D=4$ dimensions, all the 128+128 states from the first mass level remain in the physical spectrum and fall into some representation of

spin	# of species	bose d.o.f.	spin	# of species	fermi d.o.f.
2	1	$1 \cdot 5 = 5$	3/2	8	$8 \cdot 4 = 32$
1	27	$27 \cdot 3 = 81$	1/2	48	$48 \cdot 2 = 96$
0	42	$42 \cdot 1 = 42$			

Table 4.3: particle content of the massive four dimensional $\mathcal{N} = 4$ spin two multiplet

the $\mathcal{N} = 4$ algebra. Massive SUSY multiplets are generically longer than massless ones, they have $2^{2\mathcal{N}}$ components in four dimensions unless some BPS bound is saturated. The first mass level of the open superstring furnishes precisely one non-BPS multiplet with $2^8 = 256$ states, see [169] for a general reference on massive spin two multiplets. The four dimensional particle content is well-known to consist of the Poincaré representations given in table 4.3.

We can identify all these states on the level of vertex operators by dimensionally reducing their ten dimensional ancestors. A large part of this list is sketched in [170] and will be further discussed in [151]

- spin two tensor $\sim B_{mn} i\partial X^m \psi^n \mathrm{e}^{-\phi} \mathrm{e}^{ik\cdot X}$ in $D = 10$ decomposes into:

$$V^{(-1)}(\alpha, k) = g_\alpha \, \alpha_{\mu\nu} \, i\partial X^\mu \, \psi^\nu \, \mathrm{e}^{-\phi} \, \mathrm{e}^{ik\cdot X} \tag{4.5.81}$$

$$V^{(-1)}(\beta, k) = g_\beta \, \beta^k_\mu \Big(i\partial X^\mu \, \Psi_k \; + \; i\partial Z_k \, \psi^\mu \Big) \, \mathrm{e}^{-\phi} \, \mathrm{e}^{ik\cdot X} \tag{4.5.82}$$

$$V^{(-1)}(\zeta, k) = g_\zeta \, \zeta^{kl} \, i\partial Z_k \, \Psi_l \, \mathrm{e}^{-\phi} \, \mathrm{e}^{ik\cdot X} \tag{4.5.83}$$

$$V^{(-1)}(\varphi, k) = g_\varphi \, \varphi \Big((\eta_{\mu\nu} \; + \; 2\alpha' k_\mu k_\nu) \, i\partial X^\mu \, \psi^\nu \; + \; 2\alpha' \, k_\mu \, \partial\psi^\mu \Big) \, \mathrm{e}^{-\phi} \, \mathrm{e}^{ik\cdot X} \tag{4.5.84}$$

The first vertex operator represents a spin two tensor in $D = 4$ dimensions with transverse and traceless wave function $k^\mu \alpha_{\mu\nu} = \alpha_\mu{}^\mu = 0$, and the second one (4.5.82) describes six vectors $k = 4, ..., 9$ subject to $k^\mu \beta^k_\mu = 0$. The counting of scalar degrees of freedom in the vertex operators (4.5.83) and (4.5.84) is a bit subtle because the only the $D = 10$ trace of the B_{mn} wavefunction vanishes. Here, we have decomposed it into a symmetric and $SO(6)$ traceless part ($\zeta^{kl} = \zeta^{(kl)}$ and $\zeta^{kl}\delta_{kl} = 0$) with 20 components such that another scalar φ comes from the spacetime trace $m, n = 0, 1, 2, 3$.

- three form $\sim E_{mnp} \psi^m \psi^n \psi^p \mathrm{e}^{-\phi} \mathrm{e}^{ik\cdot X}$ in $D = 10$ decomposes into:

$$V^{(-1)}(\theta, k) = g_\theta \, \theta \, 2\alpha' \, \tfrac{1}{6} \, \varepsilon_{\mu\nu\lambda\rho} \, \psi^\mu \, \psi^\nu \, \psi^\lambda \, k^\rho \, \mathrm{e}^{-\phi} \, \mathrm{e}^{ik\cdot X} \tag{4.5.85}$$

$$V^{(-1)}(\omega, k) = g_\omega \, 2\alpha' \, \tfrac{1}{2} \, \omega^\mu_k \, \varepsilon_{\mu\nu\lambda\rho} \, \psi^\nu \, \psi^\lambda \, k^\rho \, \Psi^k \, \mathrm{e}^{-\phi} \, \mathrm{e}^{ik\cdot X} \tag{4.5.86}$$

$$V^{(-1)}(d, k) = g_d \sqrt{2\alpha'} \, d^{kl}_\mu \, \psi^\mu \, \Psi_k \, \Psi_l \, \mathrm{e}^{-\phi} \, \mathrm{e}^{ik\cdot X} \tag{4.5.87}$$

$$V^{(-1)}(\Omega, k) = g_\Omega \sqrt{2\alpha'} \, \Omega^{klm} \, \Psi_k \, \Psi_l \, \Psi_m \, \mathrm{e}^{-\phi} \, \mathrm{e}^{ik\cdot X} \tag{4.5.88}$$

spin	# of species	wavefunctions	spin	# of species	wavefunctions
2	1	$\alpha_{\mu\nu}$	3/2	8	$4\,\chi^a_{\mu,I} \oplus 4\,\bar{\chi}^{\bar{I}}_{\mu,\dot{a}}$
1	27	$6\,\beta^k_\mu \oplus 6\,\omega^k_\mu \oplus 15\,d^{kl}_\mu$	1/2	48	$20\,r^a_{k,I} \oplus 20\,\bar{r}^{\bar{I}}_{k,\bar{a}} \oplus 4\,b^a_I \oplus 4\,\bar{b}^{\bar{I}}_{\dot{a}}$
0	42	$\varphi \oplus \theta \oplus 20\,\zeta^{kl} \oplus 20\,\Omega_{klm}$			

Table 4.4: wavefunctions in the vertex operators for the $\mathcal{N}=4$ spin two multiplet

In (4.5.85) and (4.5.86) we have exploited that a massive p form in $D=4$ dimensions can be Hodge-dualized to a $3-p$ form, this introduces one pseudoscalar θ and six pseudovectors ω^μ_k. With two internal Ψ fields, we obtain another 15 vectors, i.e. the $D=10$ state E_{mnp} gives rise to 21 transversal vectors $k_\mu \omega^\mu_k = k^\mu d^{kl}_\mu = 0$ altogether. Finally, the internal $SO(6)$ three form Ω^{klm} contributes 20 scalars.

- gravitino $\sim (\chi^\alpha_m i\partial X^m + 2\alpha' \bar{\rho}^m_{\dot{\beta}} \psi_m \not{\psi}^{\dot{\beta}\alpha}) S_\alpha \mathrm{e}^{-\phi/2} \mathrm{e}^{ik\cdot X}$ in $D=10$ decomposes into

$$\begin{aligned}
V^{(-1/2)}(\chi,k) &= g_\chi\,\chi^a_{\mu,I}\left(\delta^b_a\, i\partial X^\mu \;-\; \alpha'\,\psi^\mu\,(\not{k}\,\not{\psi})_a{}^b\right) S_b\,\Sigma^I\,\mathrm{e}^{-\phi/2}\,\mathrm{e}^{ik\cdot X} &(4.5.89)\\
V^{(-1/2)}(\bar{\chi},k) &= g_\chi\,\bar{\chi}^{\bar{I}}_{\mu,\dot{a}}\left(\delta^{\dot{a}}_{\dot{b}}\, i\partial X^\mu \;-\; \alpha'\,\psi^\mu\,(\not{k}\,\not{\psi})^{\dot{a}}{}_{\dot{b}}\right) S^{\dot{b}}\,\bar{\Sigma}_{\bar{I}}\,\mathrm{e}^{-\phi/2}\,\mathrm{e}^{ik\cdot X} &(4.5.90)\\
V^{(-1/2)}(r,k) &= g_r\,r^a_{k,I}\left(i\partial Z^k\,S_a\,\Sigma^I \;-\; \frac{\alpha'}{2}\,\not{k}_{a\dot{b}}\,\Psi^k\,\Psi^j\,\gamma^{I\bar{J}}_j\,S^{\dot{b}}\,\bar{\Sigma}_{\bar{J}}\right)\mathrm{e}^{-\phi/2}\,\mathrm{e}^{ik\cdot X} &(4.5.91)\\
V^{(-1/2)}(\bar{r},k) &= g_r\,\bar{r}^{\bar{I}}_{k,\dot{a}}\left(i\partial Z^k\,S^{\dot{a}}\,\bar{\Sigma}_{\bar{I}} \;-\; \frac{\alpha'}{2}\,\not{k}^{\dot{a}b}\,\Psi^k\,\Psi_j\,\bar{\gamma}^j_{\bar{I}J}\,S_b\,\Sigma^J\right)\mathrm{e}^{-\phi/2}\,\mathrm{e}^{ik\cdot X} &(4.5.92)\\
V^{(-1/2)}(b,k) &= g_b\,b^a_I\left(-\tfrac{1}{2}\,(\sigma_\mu\,\not{k})_a{}^b\, i\partial X^\mu \;+\; \tfrac{1}{6}\,(\not{\psi}\,\not{\psi})_a{}^b\right) S_b\,\Sigma^I\,\mathrm{e}^{-\phi/2}\,\mathrm{e}^{ik\cdot X} &(4.5.93)\\
V^{(-1/2)}(\bar{b},k) &= g_b\,\bar{b}^{\bar{I}}_{\dot{a}}\left(-\tfrac{1}{2}\,(\bar{\sigma}_\mu\,\not{k})^{\dot{a}}{}_{\dot{b}}\, i\partial X^\mu \;+\; \tfrac{1}{6}\,(\not{\psi}\,\not{\psi})^{\dot{a}}{}_{\dot{b}}\right) S^{\dot{b}}\,\bar{\Sigma}_{\bar{I}}\,\mathrm{e}^{-\phi/2}\,\mathrm{e}^{ik\cdot X} &(4.5.94)
\end{aligned}$$

First of all, the vertex operators (4.5.89) and (4.5.90) create eight massive spin 3/2 particles (one for each value of the R symmetry indices $I, \bar{I}$) whose wavefunctions are transverse and σ traceless, $\chi^a_\mu \sigma^\mu_{a\dot{b}} = k^\mu \chi^a_\mu = 0$. The next part (4.5.91) and (4.5.92) involving internal vector spinors $r^a_{k,I}$ seems to contain 24+24 Weyl spinors $r^a \oplus \bar{r}_{\dot{a}}$ at first glance. However, it makes sense to impose the internal irreducibility condition $r^a_{k,I}\gamma^{kI\bar{J}} = 0$ which kills one out of six spin 1/2 particles and leaves us with 40 spin 1/2 fermions of "$SO(6)$ spin" 3/2. Finally, (4.5.93) and (4.5.94) represent another eight spin 1/2 fermions which transform in the (anti-)fundamental R symmetry representation.

Table 4.4 summarizes these findings.

4.5.2 Universal states

Most of the vertex operators (4.5.81) to (4.5.94) involve conformal fields $\partial Z^k, \Psi^k, \Sigma^I, \bar{\Sigma}_{\bar{I}}$ from the internal SCFT. They generically disappear after SUSY breaking and therefore become model dependent. This subsection lists the universal states without internal SCFT coupling, they remain in the spectrum even in $\mathcal{N}=0$ compactifications.

Among the NS sector states, only the spin two tensor $\alpha_{\mu\nu}$ and two scalars φ, θ have a vertex operator without internal contributions. It makes sense to combine the latter into a complex scalar whose imaginary part is the parity-odd contribution $\sim \theta\varepsilon_{\mu\nu\lambda\rho}$:

$$V^{(-1)}(\alpha,k) \;=\; g_\alpha\,\alpha_{\mu\nu}\,i\partial X^\mu\,\psi^\nu\,\mathrm{e}^{-\phi}\,\mathrm{e}^{ik\cdot X} \tag{4.5.95}$$

$$\begin{aligned} V^{(-1)}(\varphi^\pm,k) \;&=\; g_\varphi\,\varphi^\pm\,\mathrm{e}^{-\phi}\,\mathrm{e}^{ik\cdot X} \\ &\times\Big(\,(\eta_{\mu\nu}\;+\;2\alpha' k_\mu k_\nu)\,i\partial X^\mu\,\psi^\nu\;+\;2\alpha'\,k_\mu\,\partial\psi^\mu\;\pm\;\frac{i\alpha'}{3}\,\varepsilon_{\mu\nu\lambda\rho}\,\psi^\mu\,\psi^\nu\,\psi^\lambda\,k^\rho\Big) \end{aligned} \tag{4.5.96}$$

We will see in the three point couplings in subsection 9.2.3 that this complex combination is essential to obey certain MHV selection rules.

We have argued in subsection 4.4.3 that chiral fermions (4.4.75) with internal twist field Ξ are present in any $\mathcal{N}=0$ spectrum. The same is true for their Regge excitation with extra conformal fields from the spacetime SCFT, in particular the following spin 3/2- and spin 1/2 states at mass level one are universal:

$$V^{(-1/2)}(\chi,k) \;=\; \tilde g_\chi\,\chi^a_\mu\,\Big(\,\delta^b_a\,i\partial X^\mu\;-\;\alpha'\,\psi^\mu\,(\not k\;\not\psi)_a{}^b\,\Big)\,S_b\,\Xi\,\mathrm{e}^{-\phi/2}\,\mathrm{e}^{ik\cdot X} \tag{4.5.97}$$

$$V^{(-1/2)}(\bar\chi,k) \;=\; \tilde g_\chi\,\bar\chi_{\mu,\dot a}\,\Big(\,\delta^{\dot a}_{\dot b}\,i\partial X^\mu\;-\;\alpha'\,\psi^\mu\,(\not k\;\not\psi)^{\dot a}{}_{\dot b}\,\Big)\,S^{\dot b}\,\bar\Xi\,\mathrm{e}^{-\phi/2}\,\mathrm{e}^{ik\cdot X} \tag{4.5.98}$$

$$V^{(-1/2)}(b,k) \;=\; \tilde g_b\,b^a\,\Big(-\tfrac{1}{2}\,(\sigma_\mu\,\not k)_a{}^b\,i\partial X^\mu\;+\;\tfrac{1}{6}\,(\not\psi\,\not\psi)_a{}^b\,\Big)\,S_b\,\Xi\,\mathrm{e}^{-\phi/2}\,\mathrm{e}^{ik\cdot X} \tag{4.5.99}$$

$$V^{(-1/2)}(\bar b,k) \;=\; \tilde g_b\,\bar b_{\dot a}\,\Big(-\tfrac{1}{2}\,(\bar\sigma_\mu\,\not k)^{\dot a}{}_{\dot b}\,i\partial X^\mu\;+\;\tfrac{1}{6}\,(\not\psi\,\not\psi)^{\dot a}{}_{\dot b}\,\Big)\,S^{\dot b}\,\bar\Xi\,\mathrm{e}^{-\phi/2}\,\mathrm{e}^{ik\cdot X} \tag{4.5.100}$$

Excited quarks certainly deserve another normalization $\tilde g_\chi$ compared to massive gauginos (4.5.89) to (4.5.92), and their Chan Paton degrees of freedom are bifundamentals in the gauge groups of the corresponding D brane stacks.

There also exist universal mass level one fermions with excited twist fields. They are not discussed in [2] and this work. They are of lower phenomenological relevance because a nonvanishing coupling to SM fields requires two of these states.

4.5.3 Completing $\mathcal{N}=1$ multiplets

In the previous subsection, we have identified seven bosonic and twelve fermionic states which universally appear even in $\mathcal{N}=0$ compactifications. Let us now move on to scenarios with $\mathcal{N}=1$ SUSY and identify the extra states which are necessary to fill the corresponding multiplets.

First of all, the $\mathcal{N} = 1$ components $\Sigma \equiv \Sigma^{(+,+,+)}$ and $\bar{\Sigma} \equiv \bar{\Sigma}_{(-,-,-)}$ of the internal spin field each give rise excited gauginos of spin 3/2 and spin 1/2. In contrast to the excited quarks (4.5.97) to (4.5.100), they are associated with adjoint Chan Paton factors.

$$V^{(-1/2)}(\chi, k) \;=\; g_\chi \, \chi^a_\mu \left(\delta^b_a \, i\partial X^\mu \;-\; \alpha' \, \psi^\mu \, (\not k \; \not\psi)_a{}^b \right) S_b \, \Sigma \, \mathrm{e}^{-\phi/2} \, \mathrm{e}^{ik\cdot X} \tag{4.5.101}$$

$$V^{(-1/2)}(\bar{\chi}, k) \;=\; g_\chi \, \bar{\chi}_{\mu,\dot{a}} \left(\delta^{\dot{a}}_{\dot{b}} \, i\partial X^\mu \;-\; \alpha' \, \psi^\mu \, (\not k \; \not\psi)^{\dot{a}}{}_{\dot{b}} \right) S^{\dot{b}} \, \bar{\Sigma} \, \mathrm{e}^{-\phi/2} \, \mathrm{e}^{ik\cdot X} \tag{4.5.102}$$

$$V^{(-1/2)}(b, k) \;=\; g_b \, b^a \left(-\tfrac{1}{2} \, (\sigma_\mu \not k)_a{}^b \, i\partial X^\mu \;+\; \tfrac{1}{6} \, (\not\psi \; \not\psi)_a{}^b \right) S_b \, \Sigma \, \mathrm{e}^{-\phi/2} \, \mathrm{e}^{ik\cdot X} \tag{4.5.103}$$

$$V^{(-1/2)}(\bar{b}, k) \;=\; g_b \, \bar{b}_{\dot{a}} \left(-\tfrac{1}{2} \, (\bar{\sigma}_\mu \not k)^{\dot{a}}{}_{\dot{b}} \, i\partial X^\mu \;+\; \tfrac{1}{6} \, (\not\psi \; \not\psi)^{\dot{a}}{}_{\dot{b}} \right) S^{\dot{b}} \, \bar{\Sigma} \, \mathrm{e}^{-\phi/2} \, \mathrm{e}^{ik\cdot X} \tag{4.5.104}$$

Massive spin two multiplets of $\mathcal{N} = 1$ SUSY combine the spin two states (4.5.95) with the spin 3/2 gaugino-antigaugino pair (4.5.101), (4.5.102). Degree of freedom counting suggests that there must exist a missing vector particle to complete the multiplet. Indeed, the following spin one state is universal to all $\mathcal{N} = 1$ compactifications,

$$V^{(-1)}(d, k) \;=\; g_d \, d_\mu \, \psi^\mu \, \mathcal{J} \, \mathrm{e}^{-\phi} \, \mathrm{e}^{ik\cdot X} \; , \tag{4.5.105}$$

where $\mathcal{J}$ denotes the canonically normalized $U(1)$ current of the internal SCFT which appears as a subleading term in the $\Sigma(z)\bar{\Sigma}(w)$ OPE (4.3.45). It can be regarded as a special combination of the $\mathcal{N} = 4$ vectors (4.5.87) with transverse polarization vector $d^{kl}_\mu \mapsto d_\mu$.

Having accomodated the 8+8 degrees of freedom of $(\alpha_{\mu\nu}, \chi^a_\mu, \bar{\chi}^\mu_{\dot{a}}, d_\mu)$ into spin two multiplet, we shall next take a look at the remaining states (4.5.103) and (4.5.104) of spin 1/2 and the complex scalar (4.5.96). SUSY obviously requires two additional bosonic states, and these are provided by the complex scalar

$$V^{(-1)}(\omega^\pm, k) \;=\; g_\omega \, \omega^\pm \, \mathcal{O}^\pm \, \mathrm{e}^{-\phi} \, \mathrm{e}^{ik\cdot X} \; . \tag{4.5.106}$$

Just like the internal current $\mathcal{J}$, the operator $\mathcal{O}^\pm \equiv \Psi^{\pm 2}\Psi^{\pm 3}\Psi^{\pm 4}$ remains in the SCFT spectrum of any $\mathcal{N} = 1$ supersymmetric geometry. It arises in the spin fields' self-OPE:

$$\Sigma(z) \, \Sigma(w) \;\sim\; (z-w)^{3/4} \, \mathcal{O}^+(w) \,+\, \ldots \; , \qquad \bar{\Sigma}(z) \, \bar{\Sigma}(w) \;\sim\; (z-w)^{3/4} \, \mathcal{O}^-(w) \,+\, \ldots \tag{4.5.107}$$

The states $\omega^\pm$ can be thought of as special components of the internal three form $\Omega_{klm} \mapsto \omega^\pm$. Since each Calabi Yau compactification manifold admits a pulely holomorphic and antiholomorphic three form, we refer to $\omega^\pm$ as the Calabi Yau scalar.

The two real degrees of freedom $\omega^\pm$ complete a massive spin 1/2 multiplet $(b^a, \bar{b}_{\dot{a}}, \Phi^\pm, \omega^\pm)$. Table 4.5 summarizes the two universal $\mathcal{N} = 1$ multiplets of highest spin 2 and 1/2, respectively.

spin	wavefunctions	d.o.f.	spin	wavefunctions	d.o.f.
2	$\alpha_{\mu\nu}$	5 bose	1/2	$b^a \oplus \bar{b}_{\dot{a}}$	(2+2) fermi
3/2	$\chi^a_\mu \oplus \bar{\chi}^\mu_{\dot{a}}$	(4+4) fermi	0	$\Phi^\pm$	2 bose
1	d_μ	3 bose	0	$\omega^\pm$	2 bose

Table 4.5: SUSY multiplets which universally occur in $\mathcal{N} = 1$ compactifications

Part II

Scattering amplitudes in the RNS framework

Chapter 5

Basics of open string tree interactions

After the rather introductory part I, we can now proceed to the main topic of this work – scattering amplitudes of open string states. Although all the full-fledged superstring amplitudes discussed in this thesis are at tree level, we shall start with a general account of string perturbation theory in the first section 5.1. Then, in section 5.2 we develop the definition and computational prescription for open string tree amplitudes, guided by conformal symmetry on the worldsheet. A first look at the role of color- and kinematic degrees of freedom is taken in the third section 5.3.

The later sections 5.4 and 5.5 explain the interplay between tree amplitudes in full-fledged superstring theory and their field theory limit. A lot of deep structures which were recently discovered within field theory amplitudes have a natural explanation from superstring theory. The superstring framework offers powerful worldsheet methods which make a quick and elegant derivation of very general relations available. This fruitful area of research is just at its beginning, that is why a lot of relatively new results are presented in these sections 5.4 and 5.5. Also, they set the stage for the main results of this thesis.

5.1 The perturbative genus expansion

The basic idea in defining transition amplitudes in a quantum theory is to sum over all histories connecting given initial and final states. In string theory, this amounts to performing the Polyakov path integral over gauge inequivalent worldsheet metrics, see subsection 2.3.1 for the disentanglement of gauge inequivalent orbits.

The essential prerequisites for nontrivial scattering amplitudes between physical states are interactions between strings and their excitations. One might feel tempted to enlarge the free worldsheet action (2.1.13) by various cubic and higher order interaction terms in the matter fields. But it turns out that this procedure is incompatible with the residual superconformal gauge transformations which are essential for unitarity and anomaly cancellation.

Instead, it is completely sufficient to consider freely propagating superstrings at local level. Their interactions arises from global properties, in particular from the topology of the worldsheet which could not be captured by extra terms in the action (2.1.13). As shown in figure 5.1, the worldsheet underlying the splitting of one open string into two open ones locally looks like the free case. In fact, this is intimately related to the absence of a distinguished interaction point in spacetime, a feature of point particle QFT which is the source of various UV divergences.

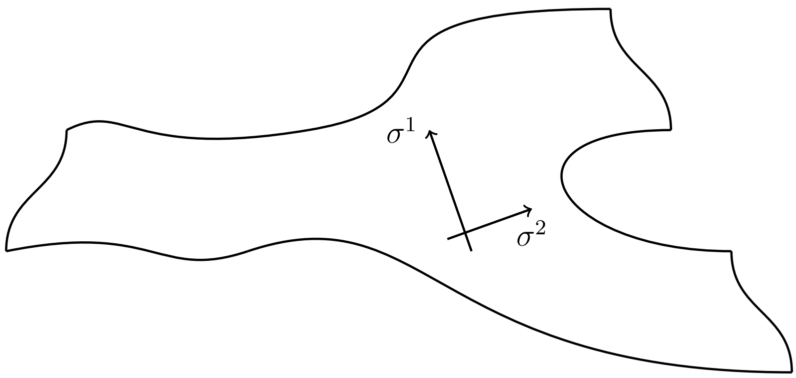


Figure 5.1: One open string splitting into two

5.1.1 The topological worldsheet action

In any field theory with a Lagrangian description, both physical and technical considerations suggest to consider the most general local action which preserves the symmetries of the theory. In case of the worldsheet theory of the bosonic string, there is one extra term beyond the Polyakov action (2.1.2) which respects Weyl- and Poincaré symmetry:

$$\mathcal{S}_{\text{top}} \;=\; \lambda\,\chi\,, \qquad \chi \;=\; \frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{-\det h}\,R \;+\; \frac{1}{2\pi}\int \mathrm{d}s\,k \tag{5.1.1}$$

The subscript 'top' already indicates that this action is of purely topological nature: In two dimensions, the Einstein Hilbert integrand (with R denoting the worldsheet Ricci scalar) is a total derivative. Therefore, adding (5.1.1) to the worldsheet action has no classical effect at all.

For open strings, there is an extra integral of the geodesic curvature k along the worldsheet boundary which can be regarded as a two dimensional Gibbons Hawking boundary term[1]. Altogether, the Einstein Hilbert- and Gibbons Hawking terms give rise to the Euler characteristic

[1]The geodesic curvature of a Lorentzian worldsheet is explicitly given by $k = t^a n_b \nabla_a t^b$ with t^a a unit vector tangent to the boundary and n^a an outward pointing unit normal vector.

χ and the impact of the action (5.1.1) is a weight factor of $e^{-\lambda\chi}$ in the path integral. It controls the relative weighting of different worldsheet topologies such that e^{λ} can be identified with the closed string coupling g_s. The physical meaning of the parameter λ will be clarified in the following subsection.

Worldsheets of open strings require some extra care: Their sources are boundary segments whereas the sources for closed strings are closed boundary loops. The curvature k diverges at corners connecting two different segments giving a contribution of 1/4 to the Euler number. Since the addition of an open string source to the worldsheet increases the number of corners by two, the open string coupling g_{open} can be identified as

$$g_{\text{open}} \quad = \quad e^{\lambda/2} \quad = \quad (g_s)^{1/2} \; . \tag{5.1.2}$$

This will mean that each open string vertex operator must carry such a factor of $e^{\lambda/2}$ in its normalization.

The Euler number is related to the Riemann surface's genus by $\chi = 2-2g$. Adding a handle to the worldsheet increases g by one resulting in weight factor $e^{2\lambda}$ per "hole". The following figure 5.2 shows the genus expansion for worldsheets with a fixed number of open string sources:

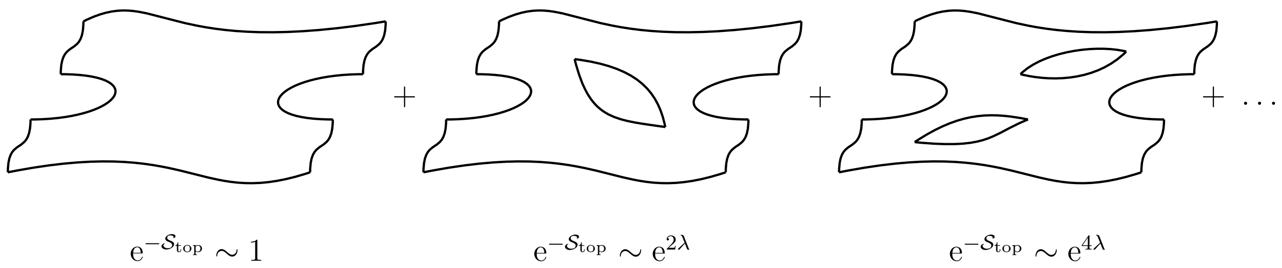


Figure 5.2: The perturbative genus expansion for the contribution of open string worldsheets to a path integral

The basic idea of string perturbation theory is to disentangle the contributions of different worldsheet genera to observables. The path integral over different worldsheets is split according to the loop order g, and each fixed value of g involves only one worldsheet diagram. This is in contrast to the plethora of Feynman diagrams arising in field theory processes with nontrivial multiplicities.

5.1.2 The string coupling from the dilaton field

This subsection is an aside which explains the dynamical origin of the parameter λ in the topological worldsheet action (5.1.1). The end result of the following discussion is that the expectation value ϕ_{10} of the massless dilaton field $\phi_{10}(X)$ from the closed string sector sets the value of the string coupling,

$$\lambda \;=\; \phi_{10} \;=\; \langle\, \phi_{10}(X) \,\rangle \quad\Rightarrow\quad g_{\rm open} \;=\; {\rm e}^{\phi_{10}/2} \; . \tag{5.1.3}$$

To arrive at (5.1.3), we have to consider strings propagating in a curved background spacetime. For simplicity, we restrict the discussion to bosonic string theory, the arguments in superstring theories are straightforward generalizations. To take curved spacetimes into account, we have to modify the Polyakov action (2.1.2) of Minkowski spacetime such that it incorporates a coherent background of massless states $\alpha^m_{-1}\bar\alpha^n_{-1}|0\rangle$ from the closed string sector. The generalization of the Polyakov action with this background follows from adding their vertex operators to (2.1.2)[2]:

$$\mathcal{S}_\sigma \;=\; \frac{1}{4\pi\alpha'} \int {\rm d}^2\sigma \, \sqrt{-\det h}\, \Big(\big[h^{ab}\, G_{mn}(X) \;+\; i\varepsilon^{ab}\, B_{mn}(X)\big]\, \partial_a X^m \, \partial_b X^n \;+\; \alpha'\, R\, \phi_{10}(X) \Big) \tag{5.1.4}$$

The first field G_{mn} denotes the generalized spacetime metric whose traceless part gives rise to the graviton vertex operator. Secondly, the antisymmetric counterpart B_{mn} is referred to as the Kalb Ramond two form field subject to a gauge invariance $B_{mn} \equiv B_{mn} + \partial_m \zeta_n - \partial_n \zeta_m$. Finally, a mixing of the G_{mn} trace and ϕ_{10} gives rise to a scalar degree of freedom, the so-called dilaton. The last term $\sim \alpha' R$ in the action (5.1.4) is required by unitary and conformal invariance or alternatively from the limiting field theory [171, 172, 173, 174].

The inclusion of background fields $G_{mn}, B_{mn}, \phi_{10}$ generically violates Weyl invariance and spoils the theory's consistency. The potential symmetry breaking can be most conveniently parametrized in terms of the energy momentum trace on the worldsheet which necessarily vanishes in conformal theories:

$$T_a{}^a \;=\; -\,\frac{1}{2\alpha'}\, \beta^G_{mn}\, h^{ab}\, \partial_a X^m\, \partial_b X^n \;-\; \frac{i}{2\alpha'}\, \beta^B_{mn}\, \varepsilon^{ab}\, \partial_a X^m\, \partial_b X^n \;-\; \frac{1}{2}\, \beta^{\phi_{10}}\, R \tag{5.1.5}$$

A background is compatible with (super-)conformal symmetry if $T_a{}^a$ and therefore all the β coefficients in (5.1.5) vanish. In terms of the spacetime Ricci tensor $\mathcal{R}_{mn}$ and the Kalb Ramond field strength $H_{mnp} = 3\partial_{[m} B_{np]}$, these are

$$0 \;\stackrel{!}{=}\; \left\{ \begin{array}{lcl} \beta^G_{mn} &=& \alpha'\, \mathcal{R}_{mn} \;+\; 2\alpha'\, \nabla_m \nabla_n \phi_{10} \;-\; \frac{\alpha'}{4}\, H_{mpq}\, H_n{}^{pq} \;+\; \mathcal{O}(\alpha'^2) \\ \beta^B_{mn} &=& -\,\frac{\alpha'}{2}\, \nabla^p H_{pmn} \;+\; \alpha'\, \nabla^p \phi_{10}\, H_{pmn} \;+\; \mathcal{O}(\alpha'^2) \\ \beta^{\phi_{10}} &=& -\,\frac{\alpha'}{2}\, \nabla^2 \phi_{10} \;+\; \alpha'\, \nabla_m \phi_{10}\, \nabla^m \phi_{10} \;-\; \frac{\alpha'}{24}\, H_{mnp}\, H^{mnp} \;+\; \mathcal{O}(\alpha'^2) \end{array} \right. \tag{5.1.6}$$

[2]The worldsheet action (5.1.4) descibes a nonlinear sigma model which is renormalizable because of the dimensionality $d = 2$ of the worldsheet. Higher dimensional worldvolumes would spoil the model's renormalizability.

The first line of (5.1.6) is nothing but the Einstein equation with a ϕ_{10}- and H dependent source term, this is a very direct way to see that string theory contains general relativity in the $\alpha' \rightarrow 0$ regime. But the point of interest in the context of the string coupling is the appearance of ϕ_{10}: Since it always occurs differentiated in (5.1.6), an X^m independent shift of ϕ_{10} preserves Weyl invariance. On the level of the action (5.1.4), such a ϕ_{10} shift changes the weight of the worldsheet Einstein Hilbert action and therefore the suppression factor $\mathrm{e}^{-\lambda\chi}$ of worldsheets with larger Euler number.

The consistency conditions (5.1.6) are solved for the one parameter family

$$G_{mn}(X) = \eta_{mn} , \qquad B_{mn}(X) = 0 , \qquad \phi_{10}(X) = \phi_{10} = \lambda \tag{5.1.7}$$

of flat Minkowski backgrounds. Different values of the constant dilaton and therefore of the string coupling $g_{\mathrm{open}} = \mathrm{e}^{\lambda/2}$ simply correspond to different backgrounds of one single string theory. This should not be confused with a family of different string theories. In principle, the dynamics of string theory determine the background, therefore λ cannot be considered as a free parameter.

5.1.3 Higher genus

Since loop amplitudes are beyond the scope of this work, we only give a brief outlook on some features of higher genus worldsheets. Global aspects of the worldsheet were first addressed by [175,176]. For references on multiloop computations in covariant superstring theory, see [177, 178,179,180,181]. The notion of ellipitic genus was developed in [182,183,184,185,186,187]. The classics on multiloop computations in the RNS framework are [188, 189, 190, 191, 192, 193, 194].

One main complication is that for $g \neq 0$, the gauge redundancy does not completely eliminate the path integral over metrics but leaves behind a finite-dimensional integral over so-called moduli space. The latter is formally defined as the space of metrics for a given topology, modded out by diffeomorphisms and Weyl transformations.

As the simplest open string example, consider a cylindrical worldsheet of genus $g = 1$, parametrized by string coordinate $\sigma^1 \in (0, \pi)$ and periodic time $\sigma^0 \in (0, 2\pi)$. The metric cannot be brought into Minkowski form η_{ab} by means of diffeomorphisms and Weyl transformations which respect the $\sigma^0 \equiv \sigma^0 + 2\pi$ periodicity. The best one can do is to cast it into $\mathrm{d}s^2 = |\mathrm{d}\sigma^1 + it\, \mathrm{d}\sigma^0|^2$ with $t \in \mathbb{R}$. Roughly speaking, this corresponds to cylinders of different lengths at fixed radius – their ratio of length and radius cannot be modified by conformal transformations.

Although one strictly speaking takes the path integral over diffeomorphism- and Weyl inequivalent metrics, one can outsource the integral over moduli space of metrics to an integral over coordinate ranges. In the $g = 1$ case of the cylinder, this means: Instead of a 2π periodicity in time, we identify $\sigma^0 \equiv \sigma^0 + 2\pi i t$ and integrate over $t \in (0, \infty)$ as a finite dimensional remnant of $\int \mathcal{D}h \big|_{g=1}$.

The complications caused by the moduli are taken into account by b ghost insertions into the path integral. The b ghost couples to the metric worldsheet metric h_{ab}, and each metric modulus gives rise to a $\int \mathrm{d}w_i \, b(w_i)$ insertion in the path integral. Since $g = 0$ admits to completely fix $h_{ab} = \eta_{ab}$, the b ghosts do not contribute to tree amplitudes. Like in field theory, ghost fields only affect observables through their running in loops.

As we have argued in subsection 3.1.3, BRST invariance requires that for each fixed coordinate z_i, the position integral $\int \mathrm{d}z$ over the associated physical vertex operator is replaced by a $c(z_i)$ insertion. Hence, the zero mode counting $N_c - N_b = 3 - 3g$ is a relation between the number N_b of moduli and the number N_c of worldsheet positions we can fix at genus g:

$$N_c = \begin{cases} 3 : & g = 0 \\ 1 : & g = 1 \\ 0 : & g \geq 2 \end{cases} , \quad N_b = \begin{cases} 0 : & g = 0 \\ 1 : & g = 1 \\ 3g - 3 : & g \geq 2 \end{cases} \tag{5.1.8}$$

Superstring theories introduce additional moduli due to the Grassmann odd degrees of freedom on the worldsheet, they couple to the β, γ superghost system. They are taken into account by appropriate picture changing, see subsection 3.1.6.

5.2 The string S matrix

In theories which contain gravity and the associated diffeomorphism gauge group, there are no local off-shell gauge invariant observables. Any correlation function $\langle \phi(X_1)...\phi(X_n) \rangle$ for instance is gauge dependent because diffeomorphisms map the points X_i to other positions. The only way to keep such a correlator gauge invariant is to send the spacetime positions to infinity, $X_i \to \infty$. Then, they are unaffected by gauge transformation (which have to die off asymptotically). Amplitudes with all external legs sent to infinity are referred to as elements of the S matrix.

In the context of string theory, this implies that the insertion points z_i of scattering states into the worldsheet SCFT (via vertex operators $V(z_i)$) need to be located at infinity. Then, we can use conformal transformations to bring the z_i to finite distances, details follow in the first subsection 5.2.1. However, Weyl invariance imposes on-shell condition on the $V(z)$ that is why the machinery described in this work does not directly apply to an off-shell S matrix.

5.2.1 From asymptotic states to the punctured disk

In this subsection, we shall explain how asymptotic states can be realized on $g = 0$ open string worldsheet. A scattering process involving n open string states can be described by a worldsheet with n long strips representing the external legs which are pulled off to infinity. Each of the strips can be parametrized by a complex coordinate w, see figure 5.3:

$$0 \ \leq \ \mathrm{Im}(w) \ \leq \ 2\pi\, t \ , \qquad \mathrm{Re}\,(w) \in (0,\pi) \tag{5.2.9}$$

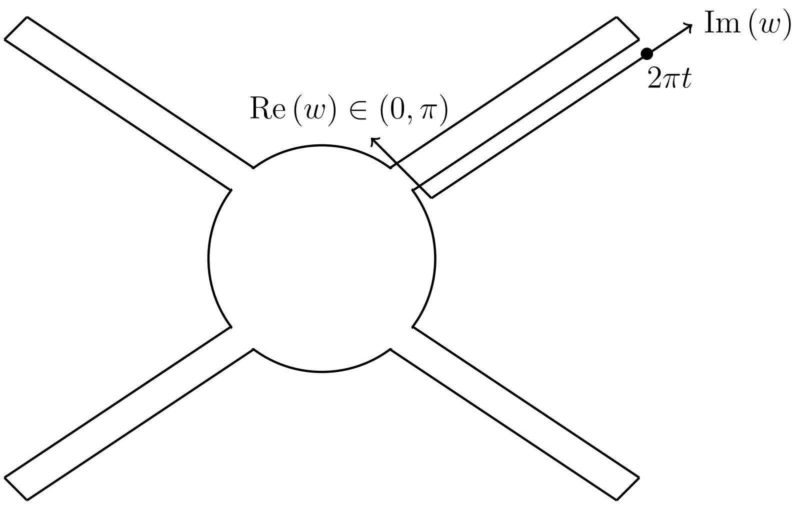


Figure 5.3: Worldsheet with four external open string states, inserted at the end of strips of length $2\pi t$. A complex coordinate w in the range (5.2.9) is convenient to cover one of the strips.

The limit corresponding to an S matrix element is $t \to \infty$ because the lower end $\mathrm{Im}(w) = 2\pi t$ is the position of the external source whereas the upper end $\mathrm{Im}(w) = 0$ fits onto the rest of the worldsheet.

The strip has a conformally equivalent description in terms of the following coordinate z:

$$z \ = \ \mathrm{e}^{iw} \ , \qquad \mathrm{e}^{-2\pi t} \ \leq \ |z| \ \leq \ 1 \ , \qquad \mathrm{Im}(z) \ \geq \ 0 \tag{5.2.10}$$

Under $w \mapsto z = \mathrm{e}^{iw}$, the long strip is mapped into the intersection of the unit disk with the upper half plane, with a tiny semi circle of radius $\mathrm{e}^{-2\pi t}$ cut out at the origin. In the S matrix limit $t \to \infty$, this defect reduces to a point.

The full worldsheet whose w parametrization for the long strips is depicted in figure 5.3 becomes a disk with small dents in the z coordinate, see the following figure 5.4. In other words, the worldsheet underlying an open string tree level scattering process is a disk where the vertex operator positions z_i appear as punctures on the boundary.

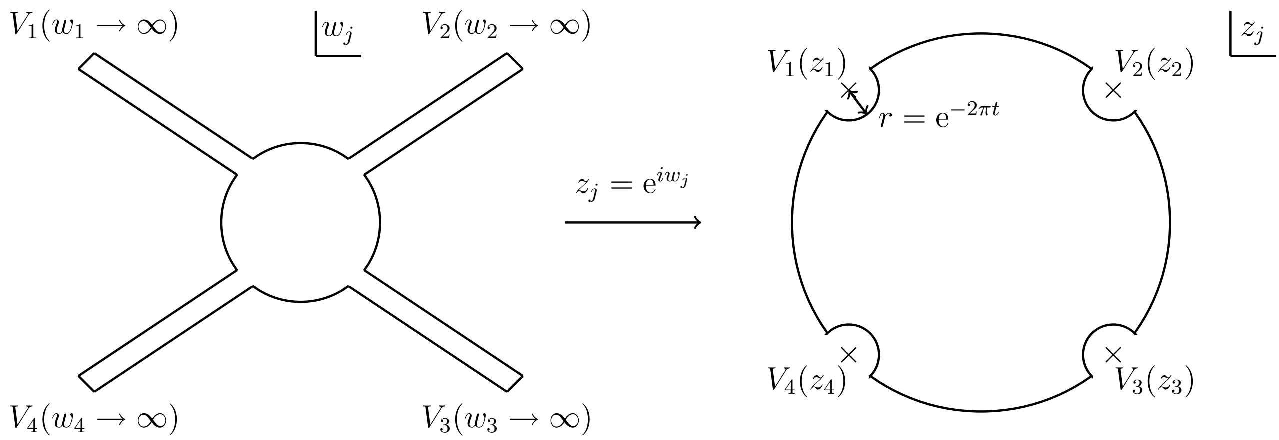


Figure 5.4: Conformally equivalent picture of a worldsheet with four long strips: A disk with small dents of radius $e^{-2\pi t}$

As a last simplifying transformation, let us map the disk boundary to the real axis. This will prove valuable in chapter 7 when the worldsheet integrals are expressed in terms of hypergeometric functions. The conformal reparametrization $z \mapsto x(z)$ which maps the unit circle $|z| = 1$ to the real axis $x \in \mathbb{R}$ is given by

$$z \quad \mapsto \quad x \quad = \quad i \, \frac{1-z}{1+z} \, , \qquad (1, i, -1) \quad \mapsto \quad (0, 1, \infty) \, . \tag{5.2.11}$$

In particular, the interior of the unit circle in the z patch corresponds to the upper half plane in the x patch.

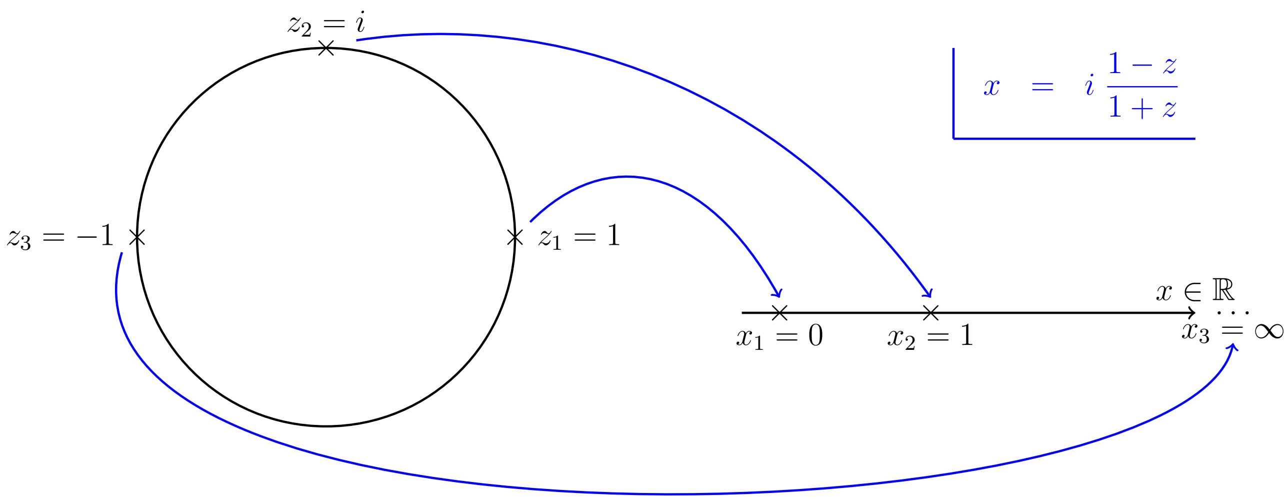


Figure 5.5: Mapping the unit disk $|z| = 1$ to the real axis $x \in \mathbb{R}$ via $x = i\frac{1-z}{1+z}$

5.2.2 The conformal Killing group

According to the discussion of the previous subsection, the starting point for computing open string disk amplitudes $\mathcal{M}(\Phi_1, \Phi_2, \ldots, \Phi_n)$ is inserting the asymptotic states $\Phi_1, ..., \Phi_n$ on the worldsheet boundary. The vertex operators enter as punctures on the disk boundary which is mapped to the real axis for convenience. BRST invariance is guaranteed by integrating over each position, see (3.1.12) and (3.1.13):

$$\mathcal{M}(\Phi_1, \Phi_2, \ldots, \Phi_n) \quad \sim \quad \left\langle \prod_{j=1}^{n} \int_{\mathbb{R}} \mathrm{d}z_j \; V_j(z_j) \right\rangle \tag{5.2.12}$$

Since all the vertex operators V_j are conformal primaries of unit weight, the integrand only depends on $n-3$ cross ratios rather than the full n variables z_i. Three of the n integrations in (5.2.12) badly overcount the conformally inequivalent z_i configurations by an infinite degeneracy factor, so we must compensate this divergence by an appropriate proportionality constant.

We have mentioned in subsection 5.1.3 that each genus admits to fix certain number of vertex positions which agrees with the number of c ghost zero modes. They count the number of so-called conformal Killing vectors, the remnant conformal gauge symmetry on a worldsheet of genus g. The degree of overcounting in (5.2.12) is usually referred to as the "volume" $\mathcal{V}_{\mathrm{CKG}}$ of the conformal Killing group (although it is formally infinite), and this has to be divided out in the (correctly normalized) amplitude $\mathcal{M}$:

$$\mathcal{M}(\Phi_1, \Phi_2, \ldots, \Phi_n) \quad \sim \quad \frac{1}{\mathcal{V}_{\mathrm{CKG}}} \left\langle \prod_{j=1}^{n} \int_{\mathbb{R}} \mathrm{d}z_j \; V_j(z_j) \right\rangle \tag{5.2.13}$$

In order to give a better meaning to the rather symbolic expression $\mathcal{V}_{\mathrm{CKG}}^{-1}$, we have to take a closer look at the conformal Killing group of the disk with $g=0$. In consists of those globally defined conformal transformation which respect the disk boundary (i.e. the real axis). This requirement is met by $z \mapsto z - \eta_{-1}$, $z \mapsto z - \eta_0 z$ and $z \mapsto z - \eta_1 z^2$ for $z \in \mathbb{R}$. These are the only transformation of the form $z \mapsto z - \eta_n z^{n+1}$ with no poles on $\mathbb{C} \cup \{\infty\}$. They generate translations, dilataions and special conformal transformations, respectively, and are referred to as the three conformal Killing vectors.

The exponentiated versions of the infinitesimal maps $\delta z = -\eta_{-1}$, $\delta z = -\eta_0 z$ and $\delta z = -\eta_1 z^2$ have the group structure of $SL(2, \mathbb{R})/\mathbb{Z}_2$:

$$z \quad \rightarrow \quad \frac{az+b}{cz+d} \, , \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \equiv \quad - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \in \; SL(2, \mathbb{R})/\mathbb{Z}_2 \tag{5.2.14}$$

An overall rescaling of a, b, c, d does not affect the map (5.2.14), so we can assume the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$ to have unit determinant. This reduces the four real parameters to three independent ones which ties in with the three infinitesimal parameters $\eta_{\pm 1}$ and η_0. Also, the conformal transformation (5.2.14) is invariant under reflections $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto -\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, so only the discrete $\mathbb{Z}_2$ quotient gives rise to distinct transformations.

The overcounting in the integral of (5.2.13) is cured by fixing three vertex operator positions and dropping the associated integrals – there always exists an appropriate $SL(2,\mathbb{R})/\mathbb{Z}_2$ transformation to map any triplet $(z_1, z_2, z_3) \in \mathbb{R}^3$ to an arbitrary other one $(w_1, w_2, w_3) \in \mathbb{R}^3$. By the BRST arguments of subsection 3.1.3, the position fixing requires the insertion of three c ghosts, and this gives precisely the correct ghost number to cancel the background charge $Q_{b,c} = -3$, see subsection 2.3.3. More formally one could write

$$\mathcal{V}_{\mathrm{CKG}}^{-1} \ = \ c(z_i)\, c(z_j)\, c(z_k)\, \delta(z_i - w_i)\, \delta(z_j - w_j)\, \delta(z_k - w_k) \ , \qquad w_1, w_2, w_3 \ \text{arbitrary} \ . \tag{5.2.15}$$

In other words, the prescription (5.2.15) to divide out the conformal Killing group of the disk kills two birds with one stone – firstly it avoids the overcounting in the n fold integral of (5.2.13) and it secondly achieves ghost charge conservation for the (b, c) system. After evaluating three integrals, the amplitude (5.2.13) becomes

$$\mathcal{M}(\Phi_1, \Phi_2, \ldots, \Phi_n) \ \sim \ \left\langle cV_1(z_1)\, cV_2(z_2)\, cV_3(z_3) \prod_{j=4}^{n} \int_{\mathbb{R}} \mathrm{d}z_j \ V_j(z_j) \right\rangle \tag{5.2.16}$$

where the three unintegrated positions z_1, z_2, z_3 can be fixed to arbitrary positions on the worldsheet boundary. The most convenient choice for the rest of this work is parametrizing the boundary by the real axis and setting $(z_1, z_2, z_3) = (0, 1, \infty)$, see figure 5.5.

The presence of the $\mathcal{V}_{\mathrm{CKG}}^{-1}$ factor in (5.2.16) provides a formal reason for the vanishing of zero-, one- and two point disk amplitudes: The infinite volume of the conformal Killing group cannot be compensated by the overcounting of gauge equivalent configurations of ≤ 2 worldsheet points.

The c ghosts with weight $h = -1$ contribute a three point correlator $\langle c(z_i) c(z_j) c(z_k) \rangle = z_{ij} z_{ik} z_{jk}$ to the amplitude. We can also think of it as a Jacobian $\det \left(\begin{smallmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_1^2 & z_2^2 & z_3^2 \end{smallmatrix}\right)$ from trading the $\int \mathrm{d}z_i \, \mathrm{d}z_j \, \mathrm{d}z_k$ integral for an integration over the $\eta_{-1}, \eta_0, \eta_1$ parameters of the conformal Killing vectors.

Let us conclude this subsection with a few words on higher genus open string worldsheets. The cylinder at $g = 1$ is the last open string worldsheet topology with a conformal Killing vector, namely the translations $\delta z = -\eta_{-1}$. At higher genus $g \geq 2$, we can no longer fix any worldsheet position and have to integrate over all the occurring z's. Ghost charge conservation then requires b ghost insertions, see (5.1.8).

5.2.3 Superghost pictures

So far, we have completely neglected the fermionic degrees of freedom of the superstring. Apart from the (b, c) ghost system which is inherited from bosonic string theory, we will have to include the superpartners (β, γ) into our discussion of superstring amplitudes.

According to subsection 2.3.3, the (β, γ) system is plagued by a background charge $Q_{\beta,\gamma} = +2$ which parametrizes the anomalous conservation law for the superghost number current. Hence, the numbers N_γ, N_β of β, γ zero modes is related to the worldsheet genus by $N_\gamma - N_\beta = 2 - 2g$. In view of the bosonization prescription (2.3.75), this translates into the following superghost charge conservation condition at tree level:

$$\left\langle \prod_{j=1}^{n} \mathrm{e}^{q_j \phi(z_j)} \right\rangle \;=\; \delta \left(\sum_{j=1}^{n} q_j + 2 \right) \prod_{k<l}^{n} z_{kl}^{-q_k q_l} \tag{5.2.17}$$

This constrains the overall superghost charge $\sum_j q_j = -2$ of vertex operators $V_i^{(q_i)}$ in order to provide a nonzero $g = 0$ amplitude:

$$\mathcal{M}(\Phi_1, \Phi_2, \ldots, \Phi_n) \;\sim\; \left\langle c V_1^{(q_1)}(z_1)\, c V_2^{(q_2)}(z_2)\, c V_3^{(q_3)}(z_3) \prod_{j=4}^{n} \int_{\mathbb{R}} \mathrm{d}z_j \; V_j^{(q_j)}(z_j) \Big|_{\sum_{j=1}^n q_j = -2} \right\rangle \tag{5.2.18}$$

The partition of the overall charge -2 is left arbitrary, one can think of various different ghost charge assignments among the scattering states. This reflects a large redundancy in the representation of superstring amplitudes with respect to the β, γ system. We justify in lines of [126] why any partition with $\sum_j q_j = -2$ gives rise to the same amplitude.

It is sufficient to show that exchanging one unit of superghost charge between any two vertex operators does not affect the correlation function, i.e.

$$\langle \ldots \, V_i^{(q_i+1)}(z_i) \, \ldots \, V_j^{(q_j)}(z_j) \, \ldots \rangle \;\stackrel{!}{=}\; \langle \ldots \, V_i^{(q_i)}(z_i) \, \ldots \, V_j^{(q_j+1)}(z_j) \, \ldots \rangle \, . \tag{5.2.19}$$

Vertices in higher ghost pictures are obtained via (3.1.28), so the statement to prove is

$$\begin{aligned} &\left\langle \ldots \oint_{z_i} \frac{\mathrm{d}w}{2\pi i} \, j_{\mathrm{BRST}}(w) \, \xi(z_i) \, V_i^{(q_i)}(z_i) \, V_j^{(q_j)}(z_j) \, \ldots \right\rangle \\ &\qquad \stackrel{!}{=} \; \left\langle \ldots \, V_i^{(q_i)}(z_i) \oint_{z_j} \frac{\mathrm{d}w}{2\pi i} \, j_{\mathrm{BRST}}(w) \, \xi(z_j) \, V_j^{(q_j)}(z_j) \, \ldots \right\rangle \, . \end{aligned} \tag{5.2.20}$$

The $h = 0$ worldsheet fermion ξ only enters the β, γ system via $\beta = \mathrm{e}^{-\phi} \partial \xi$. Physical vertex operators do not depend in its zero mode, so one can include a zero mode integration into the path integral defining the correlation function in the amplitude. Formally, this amounts to inserting $1 = \int \mathrm{d}\xi_0 \, \xi_0$. Since $\xi(z)$ is Grassmann odd, its non-zero modes (and therefore all its z dependence) do not contribute to its path integral such that the position z_i of the $\xi(z)$ insertion does not matter. We can therefore rewrite the $V_i^{(q_i+1)}$ and $V_j^{(q_j)}$ vertices as

$$\langle \dots V_i^{(q_i+1)}(z_i) \, V_j^{(q_j)}(z_j) \dots \rangle \;\; = \;\; -2 \left\langle \dots \oint_{z_i} \frac{\mathrm{d}w}{2\pi i} \, j_{\mathrm{BRST}}(w) \, V_i^{(q_i)}(z_i) \, \xi(z_j) \, V_j^{(q_j)}(z_j) \dots \right\rangle \tag{5.2.21}$$

shifting the ξ argument $z_i \mapsto z_j$ to the position of the $V_j^{(q_j)}$ vertex operator.

The second essential manipulation for proving (5.2.19) is a deformation of the w integration contour according to figure 5.6. A contour around all the vertex operators can be pulled off to infinity, which gives zero contribution due to the z^{-2} falloff of correlation functions involving the $h = 1$ field $j_{\mathrm{BRST}}(z)$. Instead of z_i, the $\oint \mathrm{d}w$ integration contour can encircle any other vertex operator position $z_k \neq z_i$.

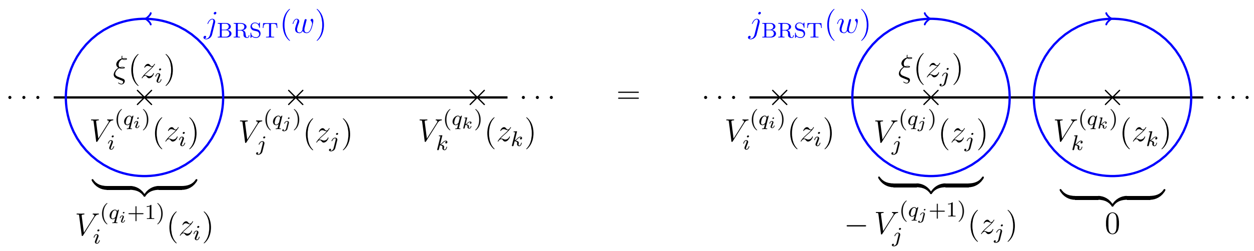


Figure 5.6: Deformation of the $\oint \mathrm{d}w \; j_{\mathrm{BRST}}(w)$ integration contour based on the absence of a residue at infinity.

But only $\xi(z_j) V_j^{(q_j)}(z_j)$ provides a nonzero contribution because the other vertex operator without ξ insertions are BRST closed, $\oint_{z_k} \frac{\mathrm{d}w}{2\pi i} j_{\mathrm{BRST}}(w) V_k^{(q_k)}(z_k) = 0$:

$$\begin{aligned}
&-2 \left\langle \dots \oint_{z_i} \frac{\mathrm{d}w}{2\pi i} \, j_{\mathrm{BRST}}(w) \, V_i^{(q_i)}(z_i) \, \xi(z_j) \, V_j^{(q_j)}(z_j) \dots \right\rangle \\
&\qquad = \; -2 \left\langle \dots \oint_{\{z_k \neq z_i\}} \frac{\mathrm{d}w}{2\pi i} \, j_{\mathrm{BRST}}(w) \, V_i^{(q_i)}(z_i) \, \xi(z_j) \, V_j^{(q_j)}(z_j) \dots \right\rangle \\
&\qquad = \; -2 \left\langle \dots \oint_{z_j} \frac{\mathrm{d}w}{2\pi i} \, j_{\mathrm{BRST}}(w) \, V_i^{(q_i)}(z_i) \, \xi(z_j) \, V_j^{(q_j)}(z_j) \dots \right\rangle \\
&\qquad = \; \left\langle \dots V_i^{(q_i)}(z_i) \, V_j^{(q_j+1)}(z_j) \dots \right\rangle
\end{aligned} \tag{5.2.22}$$

Plugging this into (5.2.21) completes the proof that superstring amplitudes do not depend on the superghost picture assignment. Since $h(\xi) = 0$, there exists a single constrant zero mode ξ_0 on worldsheets of any genus, so the arguments of this subsection carry over to loop amplitudes at $g \geq 1$.

5.2.4 SUSY Ward identities

In this subsection we will examine the impact of spacetime SUSY on superstring amplitudes. We derive relations between disk amplitudes involving different members of a SUSY multiplet, so-called SUSY Ward identities [150]. They are based on the independence of the underlying SCFT correlation functions on the spacetime SUSY charge (3.5.67) or (3.5.68), independent on the superghost picture of $\mathcal{Q}_\alpha$ and the vertex operators involved. We will demonstrate that at tree level, SUSY relations from SYM theories [195, 196, 197, 198, 199] hold to all orders in α'.

We will adapt the following discussion to $D = 10$ uncompactified spacetime dimensions, but it remains valid in compactifications with less supersymmetries. There is always one Ward identity for each $SO(1, D-1)$ spinor of supercharges. In $D = 4$ dimensions, for instance, SUSY transformations can be nicely cast into the spinor helicity language introduced in appendix C.

Since tree amplitudes of $D = 4$ gluons are completely determined by the spacetime sector of the underlying SCFT, one can take advantage of SUSY Ward identities even if all the supersymmetries are broken by the compactification [200]. Disk amplitudes of n gluons can be considerably simplified by their relation to simpler amplitudes involving $n-4$ gluons and four scalars.

SUSY Ward identities are the result of a contour deformation trick similar to that of figure 5.6 in the previous section. Supercharges are contour integrals over $h = 1$ spacetime supercurrents $j^{(q)}_{\mathrm{SUSY}}$ (e.g. $j^{(-1/2)}_{\mathrm{SUSY},\alpha} = \alpha'^{-1/4} S_\alpha \mathrm{e}^{-\phi/2}$ in the canonical ghost picture), so their vanishing integral at infinity can be deformed to small circles around vertex operator positions. The $j^{(q)}_{\mathrm{SUSY}}$ integrals along these circles evaluate to the encircled state's SUSY variation. The situation is depicted in figure 5.7:

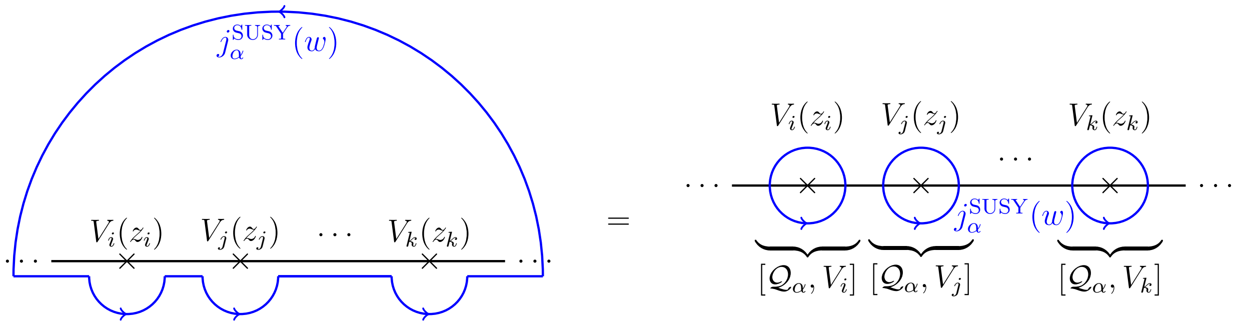


Figure 5.7: Deformation of the $\oint \mathrm{d}w \; j_\alpha^{\mathrm{SUSY}}(w)$ integration contour. The superghost charges are suppressed for ease of notation.

$$0 \;\; = \;\; \left\langle \oint_\infty \frac{\mathrm{d}w}{2\pi i} \, j_\alpha^{\mathrm{SUSY}}(w) \, V_1^{(q_1)}(z_1) \, V_2^{(q_2)}(z_2) \, \dots \, V_n^{(q_n)}(z_n) \right\rangle$$

$$= \sum_{i=1}^{n} \Big\langle V_1^{(q_1)}(z_1) \, \ldots \, V_{i-1}^{(q_{i-1})}(z_{i-1}) \oint_{z_i} \frac{\mathrm{d}w}{2\pi i} \, j_\alpha^{\mathrm{SUSY}}(w) \, V_i^{(q_i)}(z_i) \, V_{i+1}^{(q_{i+1})}(z_{i+1}) \, \ldots \, V_n^{(q_n)}(z_n) \Big\rangle$$
$$= \sum_{i=1}^{n} \Big\langle V_1^{(q_1)}(z_1) \, \ldots \, V_{i-1}^{(q_{i-1})}(z_{i-1}) \big[\, \mathcal{Q}_\alpha^{(q)} \, , \, V_i^{(q_i)}(z_i) \big] \, V_{i+1}^{(q_{i+1})}(z_{i+1}) \, \ldots \, V_n^{(q_n)}(z_n) \Big\rangle \quad (5.2.23)$$

Superghost pictures are understood to add up to the value $q + \sum_{i=1}^n q_i = -2$ required by the background charge. In particular, the n vertex operators have to encompass an odd number of fermionic states in order to get a nontrivial statement from (5.2.23). As mentioned above, the same identities hold for SUSY charges in compactifications, e.g. the $\mathcal{Q}_a^I$ and $\bar{\mathcal{Q}}_{\bar{I}}^{\dot{a}}$ in $D=4$ at maximal $\mathcal{N}=4$ SUSY. One is free to contract (5.2.23) with any reference spinor $\eta^a, \bar{\eta}_{\dot{a}}$ – specific choices give rise to further simplifications in connection with the $D=4$ spinor helicity variables.

On the level of full-fledged scattering amplitudes, including the integral over $n-3$ vertex positions, this implies

$$\sum_{i=1}^{n} \mathcal{M}\big(\Phi_1, \ldots, \Phi_{i-1}, [\mathcal{Q}_\alpha, \Phi_i], \Phi_{i+1}, \ldots, \Phi_n\big) \ = \ 0 \, . \quad (5.2.24)$$

Examples for SUSY variations $[\mathcal{Q}_\alpha, \Phi_i]$ have been given in section 3.5.2, e.g. $\eta^\alpha \mathcal{Q}_\alpha$ transforms a gluon of polarization ξ_m into a gluino of wavefunction $u^\alpha = (\eta \not{k} \not{\xi})^\alpha / \sqrt{2}$.

5.3 Structure of open string disk amplitudes

This section takes a first look at the color- and kinematic structure of superstring amplitudes due to the prescription (5.2.18). Any statement within this section is valid for any number of non-compact dimensions and independent on the internal geometry perpendicular to flat Minkowski spacetime.

5.3.1 Color decomposition

So far, the color degrees of freedom have been completely neglected in the context of scattering amplitudes. Chan Paton charges are conserved under string interactions because they flow across the worldsheet boundary and do not enter Virasoro generators. They therefore decouple from the remaining degrees of freedom which are collectively referred to as “kinematics”.

Since Chan Paton degrees of freedom do not evolve between the vertex operators, the outgoing charge of one state Φ_i agrees with the ingoing charge of its successor Φ_{i+1}. This amounts to performing matrix multiplication of the generators T^{a_i} and $T^{a_{i+1}}$ associated with vertex operators on neighboring positions on the disk boundary. Its S^1 topology requires to take a trace of the overall matrix product (in the fundamental representation of the gauge group), see figure 5.8.

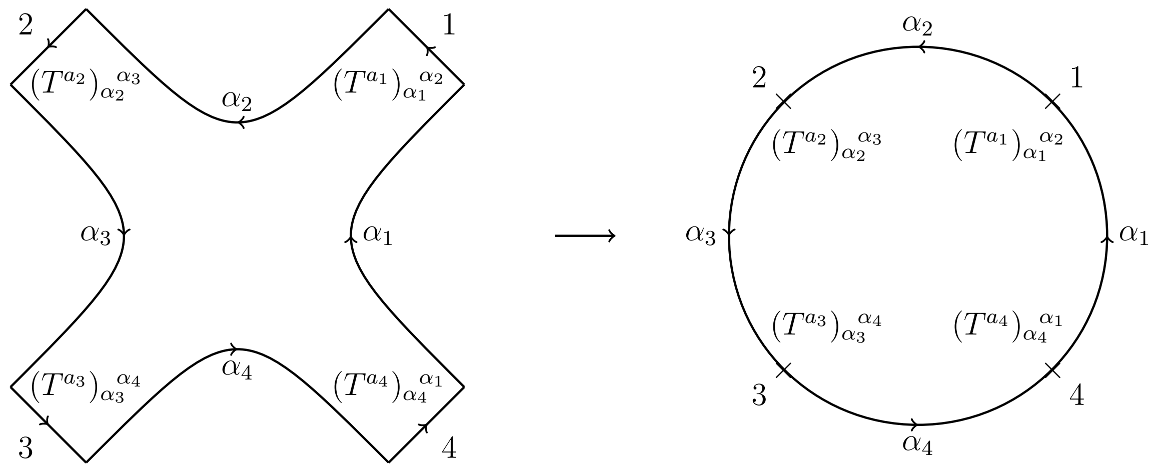


Figure 5.8: The flow of Chan Paton degrees of freedom in a four point disk amplitude leads to the trace $\mathrm{Tr}\{T^{a_1}T^{a_2}T^{a_3}T^{a_4}\}$.

The information on which vertex operators are neighbors lies in the ordering of their worldsheet position $z_i \in \mathbb{R}$, at least up to a jump from $+\infty$ to $-\infty$ which is an artifact of the $S^1 \to \mathbb{R}$ mapping (5.2.11):

$$\mathrm{Tr}\big\{\, T^{a_1}\, T^{a_2} \, \dots \, T^{a_n} \,\big\} \quad \longleftrightarrow \quad z_1 \;\leq\; z_2 \;\leq\; \dots \;\leq z_n \tag{5.3.25}$$

Hence, the integration region $\prod_{j=4}^{n} \int_{\mathbb{R}} \mathrm{d}z_j$ in an n point amplitude must be decomposed into domains of different orderings each of which is accompanied by its individual Chan Paton trace according to (5.3.25).

These arguments lead to the following color decomposition of the color dressed open string tree amplitude:

$$\mathcal{M}\big[(T^{a_1},\Phi_1),\dots,(T^{a_n},\Phi_n)\big] \;\sim\; \sum_{\rho\in S_{n-1}} \mathrm{Tr}\big\{\, T^{a_{\rho(1)}} \,\dots\, T^{a_{\rho(n-1)}}\, T^{a_n} \,\big\}\, \mathcal{A}(\Phi_{\rho(1)},\dots,\Phi_{\rho(n-1)},\Phi_n) \tag{5.3.26}$$

The objects $\mathcal{A}(\Phi_{\rho(1)},\dots,\Phi_{\rho(n-1)},\Phi_n)$ on the right hand side denote color stripped amplitudes (or in short "subamplitudes") which no do not depend on the Chan Paton degrees of freedom. Since the color traces of inequivalent orderings are independent, the $\mathcal{A}$ must vanish for themselves if one of the states Φ is spurious. This in particular implies gauge invariance for gluon states. A further nice property of subampltidues is that they necessarily inherit the cyclic invariance of

Chan Paton traces:

$$\begin{aligned} \mathrm{Tr}\{\, T^{a_1} \, \dots \, T^{a_{n-1}} \, T^{a_n} \,\} \quad &= \quad \mathrm{Tr}\{\, T^{a_n} \, T^{a_1} \, \dots \, T^{a_{n-1}} \,\} \\ \longleftrightarrow \quad \mathcal{A}(\Phi_1, \dots, \Phi_{n-1}, \Phi_n) \quad &= \quad \mathcal{A}(\Phi_n, \Phi_1, \dots, \Phi_{n-1}) \end{aligned} \tag{5.3.27}$$

The order of the arguments Φ_i in $\mathcal{A}$ reflect the worldsheet positions' ordering along the worldsheet boundary $z_{\rho(i)} < z_{\rho(i+1)}$,

$$\begin{aligned} \mathcal{A}(\Phi_{\rho(1)}, \dots, \Phi_{\rho(n-1)}, \Phi_n) \quad \sim \quad \Bigg\langle \, & c V_{\rho(1)}^{(q_{\rho(1)})}(z_{\rho(1)}) \prod_{j=2}^{n-2} \int\limits_{z_{\rho(j-1)}}^{z_{\rho(n-1)}} \mathrm{d}z_{\rho(j)} \; V_{\rho(j)}^{(q_{\rho(j)})}(z_{\rho(j)}) \\ & c V_{\rho(n-1)}^{(q_{\rho(n-1)})}(z_{\rho(n-1)}) \, c V_n^{(q_n)}(\infty) \Big|_{\sum_{j=1}^n q_j = -2} \Bigg\rangle \, . \end{aligned} \tag{5.3.28}$$

We have chosen to fix $z_{\rho(1)}, z_{\rho(n-1)}$ and z_n rather than z_1, z_2, z_3. This will prove convenient in view of the worldsheet integrals accompanying the individual color stripped amplitudes. We will show in the later subsection 5.4 that only $(n-3)!$ color ordered amplitudes are linearly independent, so it suffices to compute $\mathcal{A}(\Phi_1, \Phi_{\sigma(2)}, \dots, \Phi_{\sigma(n-2)}, \Phi_{n-1}, \Phi_n)$ with $\sigma \in S_{n-3}$. The choice $(z_1, z_{n-1}, z_n) = (0, 1, \infty)$ advertised in earlier subsections is consistent with this $(n-3)!$ dimensional basis of subamplitudes.

Throughout the rest of this work, the object of main interest will be the color stripped subamplitude associated with canonical ordering $\rho(j) = j$:

$$\mathcal{A}(\Phi_1, \dots, \Phi_{n-1}, \Phi_n) \quad \sim \quad \Bigg\langle \, c V_1^{(q_1)}(0) \prod_{j=2}^{n-2} \int\limits_{z_{j-1}}^{1} \mathrm{d}z_j \; V_j^{(q_j)}(z_j) \, c V_{n-1}^{(q_{n-1})}(1) \, c V_n^{(q_n)}(\infty) \Big|_{\sum_{j=1}^n q_j = -2} \Bigg\rangle \tag{5.3.29}$$

Evaluation of the Chan Paton traces and performing the color sums required for cross sections will be discussed in the later section 8.3.

The color decomposition at higher genus additionally involves multiple traces, i.e. the structure of (5.3.26) is summplemented by terms like $\mathrm{Tr}\{T^{a_1} \dots T^{a_p}\} \, \mathrm{Tr}\{T^{a_{p+1}} \dots T^{a_n}\}$ together with a so-called non-planar color ordered amplitude. This reflects the fact that $g \neq 0$ open string worldsheets have multiple boundaries where Chan Paton charges flow independently. In the associated SYM theory obtained as $\alpha' \to 0$, non-planar contributions break the dual superconformal symmetry and are therefore much harder to compute.

5.3.2 Kinematic pole structure

It is demonstrated in appendix B.4 that any X^m correlator appearing in open superstring amplitudes at tree level is proportional to the universal plane wave factor $\prod_{i<j}^n |z_{ij}|^{2\alpha' k_i \cdot k_j}$. As

a consequence, the worldsheet integral in color ordered amplitudes takes the form

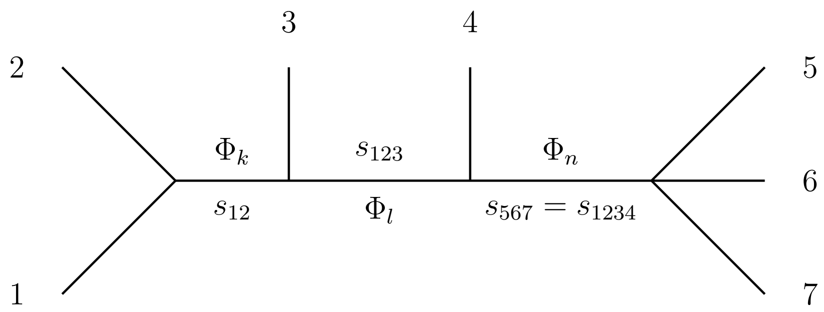
$$\mathcal{A}(\Phi_1, \ldots, \Phi_{n-1}, \Phi_n) \quad \sim \quad \prod_{j=2}^{n-2} \int_{z_{j-1}}^{1} \mathrm{d}z_j \, \prod_{i<j}^{n} |z_{ij}|^{2\alpha' k_i \cdot k_j} \, \sum_I (z_i - z_j)^{n_{ij}^I} \, \mathcal{K}^I \tag{5.3.30}$$

where $\mathcal{K}^I$ denote a collection of kinematic factors and the associated offsets n_{ij}^I in the (z_{ij}) exponents follow from the accompanying RNS and (super-)ghost correlators, see chapter 6 for the former. The GSO projection makes sure that the operator algebra is local such that all the n_{ij}^I are integers, $n_{ij}^I \in \mathbb{Z}$. Their model dependent values do not affect the general statements to be made in the following. The full chapter 7 is devoted to the properties of the integrals in (5.3.30), and we will give a glimpse of their importance for the pole structure of amplitudes right now.

Unitarity requires that intermediate states are exchanged in superstring amplitudes. Also in string theory, Feynman diagrams are very useful to illustrate this. In order to make the propagation of internal states manifest, one has to identify their poles in kinematic invariants (such as momentum products $k_i \cdot k_j$) corresponding to propagators $\frac{1}{p^2+m_k^2}$ or $\frac{1}{\not{p} \pm m_k}$ of internal states Φ_k. In case of color ordered amplitudes, propagating momenta p can only encompass a sequence of r neighboring external momenta, $p = (k_i + k_{i+1} + \ldots + k_{i+r})$. We will use dimensionless versions of Mandelstam variables

$$s_{ij} \quad = \quad \alpha' \, (k_i + k_j)^2 \; , \qquad s_{i_1 \ldots i_r} \quad = \quad \alpha' \, (k_{i_1} + k_{i_2} + \ldots + k_{i_r})^2 \tag{5.3.31}$$

in order to simplify the exponents of (5.3.30). Bosonic propagators are then of the form $(s_{i_1 \ldots i_r}/\alpha' + m_k^2)^{-1}$. The sample diagram shown below represents those parts of a $n = 7$ point amplitude with poles in $(s_{12}/\alpha' + m_k^2)$, $(s_{123}/\alpha' + m_l^2)$ and $(s_{567}/\alpha' + m_n^2)$ where m_k, m_l, m_n are the masses of the virtual particles Φ_k, Φ_l and Φ_n in the internal channels.



In chapter 7 on hypergeometric functions, we will explain in detail how the integrals in (5.3.30) give rise to a pole structure associated with these diagrams, see in particular subsection 7.1.3. Since we are particularly interested in the interplay between superstring- and field theory amplitudes, the main focus of this work lies on massless poles $s_{i_1 \ldots i_r}^{-1}$. They arise from the integration region where the integration variables $z_{i_1}, z_{i_2}, \ldots, z_{i_r}$ approach each other. This gives a rough explanation for the aforementioned selection rule: The canonically ordered subamplitude $\mathcal{A}(\Phi_1, \Phi_2, \ldots, \Phi_n)$ can only have poles in $s_{i,i+1,\ldots,i+r}$ because the ordering $z_1 \leq z_2 \leq \ldots \leq z_n$ only admits $z_i \to z_{i+r}$ if all the intermediate positions $z_{i+1}, z_{i+2}, \ldots, z_{i+r-1}$ collide as well.

5.3.3 The minimal set of Mandelstam invariants

The $k_i \cdot k_j$ appearing in the color ordered amplitudes (5.3.30) can easily be expressed in terms of s_{ij} and the masses m_i:

$$k_i \cdot k_j \; = \; \frac{1}{2\alpha'} \left(s_{ij} \, + \, \alpha' \, m_i^2 \, + \, \alpha' \, m_j^2 \right) \tag{5.3.32}$$

In three point scattering processes, the phase space becomes trivial due to momentum conservation $k_1 + k_2 + k_3 = 0$:

$$\begin{aligned} k_1 \cdot k_2 \; &= \; \frac{1}{2} \left(+ m_1^2 \, + \, m_2^2 \, - \, m_3^2 \right) \\ k_1 \cdot k_3 \; &= \; \frac{1}{2} \left(+ m_1^2 \, - \, m_2^2 \, + \, m_3^2 \right) \\ k_2 \cdot k_3 \; &= \; \frac{1}{2} \left(- m_1^2 \, + \, m_2^2 \, + \, m_3^2 \right) \end{aligned} \tag{5.3.33}$$

Four point kinematics is governed by two independent Mandelstam invariants. They will be referred to as

$$s \; := \; s_{12} \; = \; s_{34} \; , \qquad u \; := \; s_{14} \; = \; s_{23} \; , \tag{5.3.34}$$

and the remaining combinations can be expressed in terms of m_i, s and u

$$t \; := \; s_{13} \; = \; s_{24} \; , \qquad s + t + u \; = \; - \alpha' \, (m_1^2 + m_2^2 + m_3^2 + m_4^2) \; . \tag{5.3.35}$$

Momentum conservation $\sum_{j=1}^n k_j = 0$ in an n particle process allows to reduce any $k_i \cdot k_j$ to a combination of $\frac{1}{2} n(n-3)$ invariants[3]. The most convenient choice of basis is inspired by the color stripped n point amplitude $\mathcal{A}(\Phi_1, \ldots, \Phi_n)$ associated with the canonical ordering $\rho(i) = i$: This basis consists of the cyclic orbit of s_{12} (i.e. $\{s_{12}, s_{23}, \ldots, s_{n-1,n}, s_{n1}\}$) as well as

[3]Subtle exceptions occur if the number n of legs exceeds the number of spacetime dimensions by more than one: Momentum conservation leaves $n-1$ potentially independent D component momentum vectors. If $n - 1 > D$, however, the inevitable linear dependences among $k_1, \ldots, k_{n-1}$ imply the vanishing of Gram determinants $\det s_{ij}$ (with $1 \leq i, j \leq n-1$) which yields polynomial equations between the Mandelstam invariants. In $D = 4$ dimensions, for instance, this effect kicks in at a six point amplitude where $\det s_{ij} = 0$ with $1 \leq i, j \leq 5$ is a fifth order polynomial constraint on the Mandelstam variables. Usually, it is not advisable to eliminate further invariants on these grounds because the involved polynomial equations drastically increase the length of the final expression for the amplitude. We will therefore ignore this class of extra relations between the $\frac{1}{2} n(n-3)$ invariants.

longer chains of adjacent momenta s_{123}, s_{1234}, ... up to the maximum length $s_{12\ldots[n/2]}$, together with their cyclic images. In this context, $[\cdot]$ denotes the Gauss bracket $[x] = \max_{n\in\mathbb{Z}}(n \leq x)$, which picks out the nearest integer smaller than or equal to its argument. If n is even, then momentum conservation implies the last orbit involving $s_{12\ldots n/2} = s_{(n/2+1)\ldots n-1,n}$ contains $n/2$ independent elements rather than n.

A six point amplitude, for instance, is most concisely described by the full cyclic orbit $\{s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{61}\}$ together with $\{s_{123}, s_{234}, s_{345}\}$. Momentum conservation eliminates $s_{456} = s_{123}$ and s_{561}, s_{612}. Any other s_{ik} with non-adjacent labels $i \neq k \pm 1$ can be rewritten in terms of the nine variables above, e.g.

$$s_{13} \;=\; \alpha'\,(k_1 + k_3)^2 \;=\; s_{123} - s_{12} - s_{23} - \alpha'\,(m_1^2 + m_2^2 + m_3^2) \qquad (5.3.36)$$

Let us explicitly check that the aforementioned basis of $s_{i_1\ldots i_r}$ in n point amplitudes with $r \leq n/2$ indeed has $\frac{1}{2}n(n-3)$ elements:

- $n = 2p-1$ is odd: $\exists\ [n/2] - 1 = p-2$ orbits with n elements each, which gives $n(p-2) = \frac{n}{2}(n-3)$ basis elements in total
- n even: $\exists\ n/2 - 2$ orbits with n elements and one half orbit of $s_{i_1\ldots i_{n/2}}$ with $n/2$ elements giving $\frac{n}{2}(n-4) + \frac{n}{2} = \frac{n}{2}(n-3)$ invariants in total

5.3.4 Normalization factors

So far we did not specify the overall normalization of color dressed superstring amplitudes $\mathcal{M}$ or their color stripped ingredients. In principle, the right prefactors could be computed by evaluating all the regularized functional determinants arising in the path integral of the SCFT correlators in (5.2.18). But we will follow a more pragmatic approach in this work and obtain the correct normalization by consistency conditions: Firstly, unitarity requires that $n \geq 4$ point amplitudes factorize into lower point amplitudes on the residue of kinematic poles, and secondly, scattering of massless states has to reduce to the correctly normalized SYM vertices in the $\alpha' \to 0$ limit.

The consistency conditions above are sufficient to determine the set of unknowns on the superstring side. Firstly, we have not specified the normalization of vertex operators in section 3.2, this leaves one unfixed constant for each SUSY multiplet such as g_{A} at the massless level. Secondly, there is a universal prefactor in all disk amplitudes $\mathcal{M}$ coming from the determinants of differential operators in the worldsheet action, independent on the number and types of states involved. In [41], it is referred to as C_{D_2}, the $_{\mathrm{D}_2}$ subscript referring to the disk topology.

However, the presence of chiral matter introduced in section 4.4 requires a modified overall factor $\tilde{C}_{\mathrm{D}_2}$. The worldsheet of open strings streching between branes intersecting at angles introduces a new CFT sector of boundary condition changing fields. Also $\tilde{C}_{\mathrm{D}_2}$ can be determined from factorization and matching with quark gluon amplitudes in the field theory limit. Explicitly, the normalization factors are given by

$$C_{\mathrm{D}_2} \ = \ \frac{1}{g_{\mathrm{YM}}^2\,\alpha'^2}\ , \qquad \tilde{C}_{\mathrm{D}_2} \ = \ \frac{\mathrm{e}^{-\phi_{10}}}{2\,\alpha'^2} \tag{5.3.37}$$

where g_{YM} is the coupling constant of the associated Yang Mills theory and ϕ_{10} denotes the vacuum expectation value of the ten dimensional dilaton field (see subsection 5.1.2). In fact we have $g_{\mathrm{YM}} = g_{\mathrm{open}} = \mathrm{e}^{\phi_{10}/2}$, but we shall keep the notation of the SYM coupling to make contact with field theory amplitudes later on.

According to the discussions above, the normalized n point color ordered disk amplitude in superstring theory is given as follows:

$$\begin{aligned} \mathcal{A}(\Phi_1,\ldots,\Phi_{n-1},\Phi_n) \ &= \ \left\langle cV_1^{(q_1)}(0) \prod_{j=2}^{n-2} \int_{z_{j-1}}^{1} \mathrm{d}z_j\ V_j^{(q_j)}(z_j)\, cV_{n-1}^{(q_{n-1})}(1)\, cV_n^{(q_n)}(\infty) \Big|_{\sum_{j=1}^n q_j=-2} \right\rangle \\ &\quad \times \left\{ \begin{array}{ll} C_{\mathrm{D}_2} & : \ \text{only adjoint matter involved} \\ \tilde{C}_{\mathrm{D}_2} & : \ \text{chiral matter involved} \end{array} \right. \end{aligned} \tag{5.3.38}$$

5.4 Relating massless color ordered disk amplitudes

Having introduced the color decomposition of open string scattering amplitudes, we should next think about how many independent computations need to be performed. Not all of the $(n-1)!$ color stripped amplitudes in (5.3.26) are algebraically independent, and we will explain the methods of [84, 85] to expand any of them in a basis of only $(n-3)!$ subamplitudes.

As first naive step into this direction is based on worldsheet parity. Both the Chan Paton matrices and the oscillator contribution of the vertex operator have a well defined eigenvalue ± 1 under reflection $\sigma^1 \mapsto -\sigma^1$, hence the same is true for the subamplitudes

$$\mathcal{A}(\Phi_1,\Phi_2,\ldots,\Phi_{n-1},\Phi_n) \ = \ \pm\,\mathcal{A}(\Phi_n,\Phi_{n-1},\ldots,\Phi_2,\Phi_1) \tag{5.4.39}$$

with a sign depending on n and the mass levels of the states Φ_j involved. This already reduces the number of potentially independent subamplitudes to $\frac{1}{2}(n-1)!$.

We will restrict our analysis to massless states – firstly they are our main focus during the rest of this work, secondly these are the only states which contribute to the low energy limit of string theory and therefore admit comparison with SYM. More details will follow in subsection 5.5. It also makes sense to lighten notation for the massless state subamplitudes and to denote them by

$$\mathcal{A}(1,2,\ldots,n-1,n) \quad := \quad \mathcal{A}(\Phi_1,\Phi_2,\ldots,\Phi_{n-1},\Phi_n)\Big|_{m(\Phi_i)=0} \tag{5.4.40}$$

from now on.

5.4.1 Worldsheet monodromy relations

In this subsection we apply worldsheet techniques in order to derive algebraic identities between color stripped tree amplitudes relevant for string theory. For this purpose, let us rewrite the canonically ordered subamplitude (5.3.30) in symmetric fashion as

$$\mathcal{A}(1,2,\ldots,n-1,n) \quad \sim \quad \frac{1}{\mathcal{V}_{\text{CKG}}} \prod_{j=1}^{n} \int\limits_{z_1 \leq z_2 \leq \ldots \leq z_n} \mathrm{d}z_j \, \prod_{i<j}^{n} |z_{ij}|^{2\alpha' k_i \cdot k_j} \sum_I (z_i - z_j)^{n_{ij}^I} \, \mathcal{K}^I \ . \tag{5.4.41}$$

Recall that the integers n_{ij}^I specific for a given kinematic factor $\mathcal{K}^I$ are a result of evaluating the RNS correlator and extracting spacetime tensor structures. Moreover, the dimensionless Mandelstam variables take the simple form $2\alpha' k_i \cdot k_j = s_{ij}$ for massless states.

The modulus along with the $|z_{ij}|^{s_{ij}}$ factors is the only obstacle that keeps the worldsheet integrand from being analytic. But we can easily relate it to a holomorphic function in z_1 by taking the phases $(-1)^{s_{ij}} = \mathrm{e}^{i\pi s_{ij}}$ within certain integration regions into account:

$$\prod_{j=2}^{n} z_{j1}^{s_{1j}} \quad = \quad \prod_{j=2}^{n} |z_{1j}|^{s_{1j}} \ \times \left\{ \begin{array}{ccc} 1 & : & -\infty \leq z_1 \leq z_2 \\ \mathrm{e}^{i\pi s_{12}} & : & z_2 \leq z_1 \leq z_3 \\ \mathrm{e}^{i\pi (s_{12}+s_{13})} & : & z_3 \leq z_1 \leq z_4 \\ \vdots & & \vdots \\ \prod_{j=2}^{n-1} \mathrm{e}^{i\pi s_{1j}} & : & z_{n-1} \leq z_1 \leq z_n \end{array} \right. \tag{5.4.42}$$

For a holomorphic integrand in z_1, we know by Cauchy's theorem that a closed contour integral in z_1 vanishes. The contour of interest is the real axis, followed by a semicircle of infinite radius in the upper half plane with zero contribution, see the following figure 5.9:

Let us start with the integrand of (5.4.41) and drop the modulus of $z_{j1}^{s_{ij}}$ factors to make it analytic in z_1. Then we integrate z_1 along the contour from figure 5.9 rather than over $(-\infty, z_2)$ as required for the subamplitude $\mathcal{A}(1,2,\ldots,n)$. The semicircle at infinity does not contribute because the correlation function of conformal $h = 1$ fields behaves like z_1^{-2} as $|z_1| \to \infty$. In this setting, the vanishing of a closed contour integral implies the following relation between

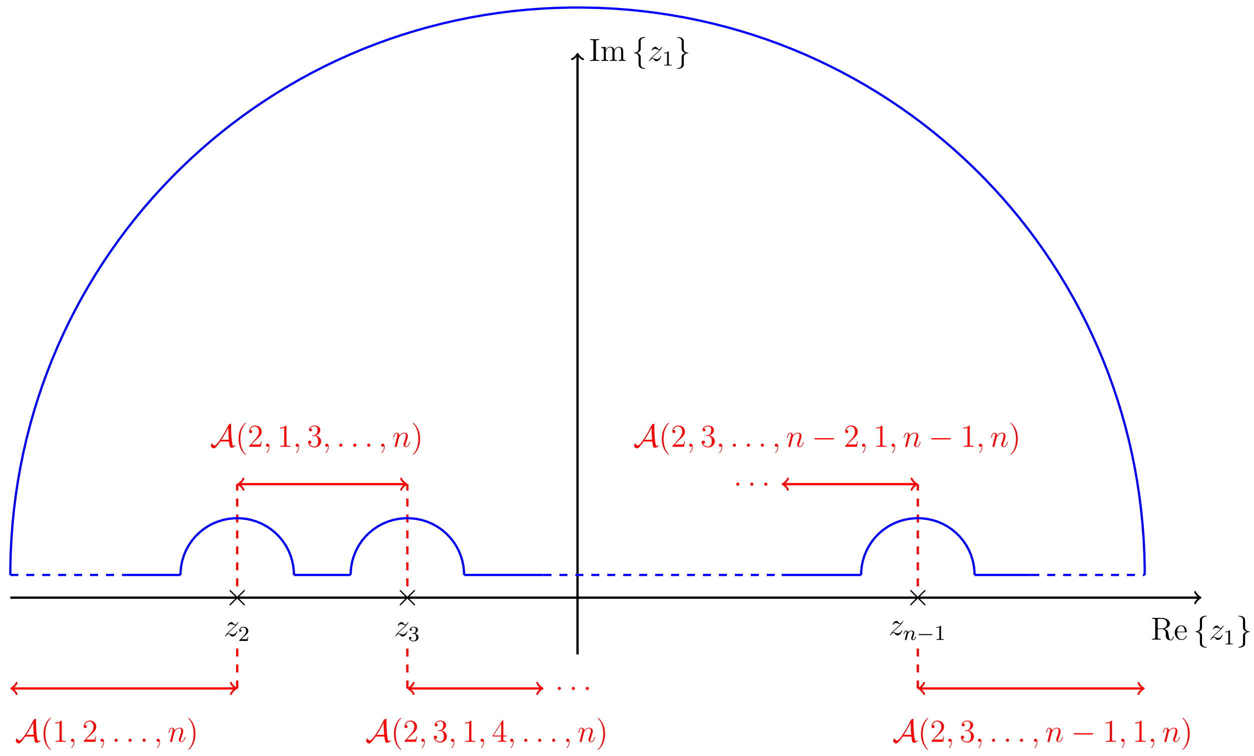


Figure 5.9: The integration contour leading to monodromy relations between subamplitudes

subamplitudes:

$$
\begin{aligned}
0 \;&=\; \int_{\mathbb{R}} \frac{\mathrm{d}z_1}{\mathcal{V}_{\mathrm{CKG}}} \prod_{j=2}^{n} \int_{z_2\le z_3 \le \dots \le z_n} \mathrm{d}z_j \prod_{j=2}^{n} z_{j1}^{s_{1j}} \prod_{\substack{k<l\\ k,l\ge 2}}^{n} |z_{kl}|^{s_{kl}} \sum_I z_{1j}^{n_{1j}^I} z_{kl}^{n_{kl}^I} \mathcal{K}^I \\
&= \int_{-\infty}^{z_2} \frac{\mathrm{d}z_1}{\mathcal{V}_{\mathrm{CKG}}} \prod_{j=2}^{n} \int_{z_2\le z_3 \le \dots \le z_n} \mathrm{d}z_j \prod_{j=2}^{n} |z_{j1}|^{s_{1j}} \prod_{\substack{k<l\\ k,l\ge 2}}^{n} |z_{kl}|^{s_{kl}} \sum_I z_{1j}^{n_{1j}^I} z_{kl}^{n_{kl}^I} \mathcal{K}^I \\
&\quad + \sum_{p=3}^{n} \int_{z_{p-1}}^{z_p} \frac{\mathrm{d}z_1}{\mathcal{V}_{\mathrm{CKG}}} \prod_{j=2}^{n} \int_{z_2\le z_3 \le \dots \le z_n} \mathrm{d}z_j \; \mathrm{e}^{i\pi(s_{12}+\dots+s_{1,p-1})} \prod_{j=2}^{n} |z_{j1}|^{s_{1j}} \\
&\qquad\qquad \times \prod_{\substack{k<l\\ k,l\ge 2}}^{n} |z_{kl}|^{s_{kl}} \sum_I z_{1j}^{n_{1j}^I} z_{kl}^{n_{kl}^I} \mathcal{K}^I \\
&= \sum_{p=2}^{n} \mathrm{e}^{i\pi(s_{12}+\dots+s_{1,p-1})} \mathcal{A}(2,\dots,p-1,1,p,\dots,n)
\end{aligned}
\tag{5.4.43}
$$

On the way to the second line, we have split the z_1 integration region $\mathbb{R}$ into intervals $(-\infty, z_2)$ and (z_{p-1}, z_p) for $p = 3, 4, \dots, n$. The analytic $z_{j1}^{s_{ij}}$ factors are related to the $|z_{j1}|^{s_{ij}}$ which appear in the amplitudes via (5.4.42) – on the expense of catching phase factors $\mathrm{e}^{i\pi(s_{12}+\dots+s_{1,p-1})}$ for the individual $\mathbb{R}$ subsets. In the third line, subamplitudes with Φ_1 at different positions have been identified after pulling out the phases.

The arguments above led to a striking relation between color ordered open superstring amplitudes at tree level

$$\begin{aligned}\mathcal{A}(1,2,\ldots,n) \;+\; \mathrm{e}^{i\pi s_{12}}\,\mathcal{A}(2,1,3,\ldots,n) \;+\; \mathrm{e}^{i\pi(s_{12}+s_{13})}\,\mathcal{A}(2,3,1,4,\ldots,n)\\ +\;\ldots\;+\;\mathrm{e}^{i\pi(s_{12}+s_{13}+\ldots+s_{1,n-1})}\,\mathcal{A}(2,3,\ldots,n-1,1,n) \;=\; 0\,,\end{aligned} \tag{5.4.44}$$

its implication on the number of independent subamplitudes will be explained in the next subsection.

The universality of this result stems from its independence on the integers n_{ij}^{I} in (5.3.30). The latter do not affect the analytic properties of the subamplitudes and hence do not contribute any phases. Also, the kinematics $\mathcal{K}^I$ and their interplay with n_{ij}^{I} are independent on the ordering $\rho \in S_{n-1}$ since the underlying CFT correlators are evaluated before specifying the partial amplitude. It is only the integration region which changes for each subamplitude. Moreover, the proof of the relation (5.4.44) does not rely on the amount of spacetime supersymmetry, the number of spacetime dimensions or whether the massless states belong to the NS- or R sector. Hence, (5.4.44) hold in any spacetime dimension D and for any amount of supersymmetry.

5.4.2 The minimal basis of subamplitudes

The monodromy relation (5.4.44) and its relabellings imply that the basis dimension of independent subamplitudes must be much smaller than the number $(n-1)!/2$ suggested by worldsheet parity and cyclic inequivalence. To get a better handle on the minimal basis, we rewrite the set of monodromy relations in a slightly different way (with α_0 denoting leg 1):

$$\mathcal{A}(1,\alpha_1,\ldots,\alpha_s,n,\beta_1,\ldots,\beta_r) \;=\; (-1)^r \prod_{i<j}^{r} \mathrm{e}^{i\pi s_{\beta_i\beta_j}} \sum_{\sigma\in \mathrm{OP}\{\alpha\}\cup\{\beta^T\}} \left(\prod_{i=0}^{s}\prod_{j=1}^{r} \mathrm{e}^{i\pi s_{\alpha_i\beta_j}} \right) \mathcal{A}(1,\sigma,n) \tag{5.4.45}$$

This follows from a contour integral analysis similar to subsection 5.4.1. By $\{\alpha\}$ and $\{\beta\}$, we denote disjoint ordered subsets of $\{2,\ldots,n-1\}$ such that $\{\alpha\}\cup\{\beta\} = \{2,\ldots,n-1\}$. The summation range $\mathrm{OP}\{\alpha\}\cup\{\beta^T\}$ encompasses the set of all the permutations of $\{\alpha\}\bigcup\{\beta^T\}$ that maintain the order of the individual elements of both sets $\{\alpha\}$ and $\{\beta^T\}$. The notation $\{\beta^T\}$ represents the set $\{\beta\}$ with reversed ordering of its r elements.

The only potentially independent subamplitudes after (5.4.45) have legs 1 and n at neighbouring positions. Their total number is $(n-2)!$ according to the S_{n-2} permutations of the remaining legs $2,3,\ldots,n-1$. In order to make efficient use of (5.4.44) and (5.4.45), we should bear in mind that subamplitudes are real, $\mathcal{A}(1,2,\ldots,n)\in\mathbb{R}$. We can take the real part of

(5.4.45) to perform a reduction to the $(n-2)!$ color orderings $\mathcal{A}(1,\sigma,n)$. The imaginary parts carry additional information and give rise to simpler relations because they involve one terms less than the real parts, e.g.

$$\begin{aligned} &\sin(\pi s_{12})\,\mathcal{A}(2,1,3,\ldots,n) \;+\; \sin\big(\pi(s_{12}+s_{13})\big)\,\mathcal{A}(2,3,1,4,\ldots,n) \\ &\quad +\;\ldots\;+\;\sin\big(\pi(s_{12}+s_{13}+\ldots+s_{1,n-1})\big)\,\mathcal{A}(2,3,\ldots,n-1,1,n) \;=\; 0\,. \end{aligned} \tag{5.4.46}$$

This equation (and its relabellings) are actually sufficient to express any subamplitude in terms of an $(n-3)!$ element basis. This number is identical to the dimension of a minimal basis of generalized Gaussian hypergeometric functions describing the full n point open string amplitude [201, 202, 150], more on that will be said in later chapters.

A sensible basis choice for the $(n-3)!$ independent superstring n point amplitudes (to which we already alluded in subsection 5.3.1) keeps three legs at fixed positions – say 1, $n-1$ and n – and encompasses all permutations of the remaining legs $2,3,\ldots,n-2$:

$$\big\{\,\mathcal{A}(1,\sigma(2),\sigma(3),\ldots,\sigma(n-2),n-1,n)\;,\quad \sigma\in S_{n-3}\,\big\} \tag{5.4.47}$$

All other color ordered amplitudes can be expanded in this basis.

5.4.3 The four- and five point example

Let us demonstrate the power of (5.4.46) relating $n-2$ color stripped n point amplitudes by means of the $n=4$ and $n=5$ examples. Given the $(-1)^n$ worldsheet parity of massless n point subamplitudes, the naive color decomposition would give three independent color orderings at four point:

$$\begin{aligned} \mathcal{M}\big[(T^{a_1},1),\ldots,(T^{a_4},4)\big] \;&=\; \mathrm{Tr}\big\{\,T^{a_1}T^{a_2}T^{a_3}T^{a_4}\;+\;T^{a_4}T^{a_3}T^{a_2}T^{a_1}\,\big\}\,\mathcal{A}(1,2,3,4) \\ &+\;\mathrm{Tr}\big\{\,T^{a_2}T^{a_1}T^{a_3}T^{a_4}\;+\;T^{a_4}T^{a_3}T^{a_1}T^{a_2}\,\big\}\,\mathcal{A}(2,1,3,4) \\ &+\;\mathrm{Tr}\big\{\,T^{a_2}T^{a_3}T^{a_1}T^{a_4}\;+\;T^{a_4}T^{a_1}T^{a_3}T^{a_2}\,\big\}\,\mathcal{A}(2,3,1,4) \end{aligned} \tag{5.4.48}$$

The $n=4$ version of (5.4.46) and its $1\leftrightarrow 2$ relabelling then yield[4]:

$$\sin(\pi s)\,\mathcal{A}(2,1,3,4)\;-\;\sin(\pi u)\,\mathcal{A}(2,3,1,4)\;=\;0 \tag{5.4.49}$$

$$\sin(\pi s)\,\mathcal{A}(1,2,3,4)\;-\;\sin(\pi t)\,\mathcal{A}(1,3,2,4)\;=\;0 \tag{5.4.50}$$

All of $\mathcal{A}(1,2,3,4)$, $\mathcal{A}(2,1,3,4)$ and $\mathcal{A}(2,3,1,4)$ are obviously proportional to each other, the proprotionality constant being a quotient of $\sin(\pi s_{ij})$.

[4] We are using four point Mandelstam variables $s=s_{12}=s_{34},\ u=s_{14}=s_{23}$ and the relation $s+t+u=0$ valid for the massless case.

At five points, the basis (5.4.47) consists of $\mathcal{A}(1,2,3,4,5)$ and $\mathcal{A}(1,3,2,4,5)$. Among the twelve subamplitudes after reflection and cyclic symmetry, ten are nontrivial linear combinations of both basis elements such as

$$\mathcal{A}(1,2,5,4,3) \ = \ \frac{\sin\bigl(\pi(s_{51}-s_{34})\bigr)\,\mathcal{A}(1,2,3,4,5) \ + \ \sin\bigl(\pi\, s_{24}\bigr)\,\mathcal{A}(1,3,2,4,5)}{\sin\bigl(\pi(s_{12}-s_{34}+s_{51})\bigr)} \ , \tag{5.4.51}$$

see subsection 4.3.2 of [84] for the complete list including the remaining nine.

5.5 Field theory versus superstring amplitudes

Open superstring theory reduces to SYM field theory in the limit of vanishing string length $\alpha' \rightarrow 0$. This remains valid after dimensional reduction to four dimensional spacetime, i.e. for $\mathcal{N}=4$ SYM in $D=4$, but also in compactifications with less supersymmetry. After complete SUSY breaking to $\mathcal{N}=0$, gluons and chiral fermions (see subsection 4.4.3 for the latter) being the universal massless states should give rise to QCD physics.

As a consequence, both SYM- and QCD scattering amplitudes must be reproduced by isolating the $\alpha' \rightarrow 0$ limit of massless superstring amplitudes involving (using shorthand $1^{a_1} \equiv (T^{a_1}, \Phi)$ for some massless state Φ):

$$\mathcal{M}^{\rm SYM}\bigl[1^{a_1}, 2^{a_2} \ldots, n^{a_n}\bigr] \ = \ \lim_{\alpha' \rightarrow 0} \mathcal{M}\bigl[1^{a_1}, 2^{a_2} \ldots, n^{a_n}; \alpha'\bigr] \tag{5.5.52}$$

We shall review the structure of gauge theory amplitudes in the first subsections and later on connect them to the string theory technology from the previous section. The following discussion is again universal to the number of spacetime dimensions and supersymmetries, i.e. it applies both to the gluon sector of QCD and to $\mathcal{N}=1$ SYM in ten dimensions. The only simplifying assumption for the color sector is that all external states transform in the adjoint representation of a nonabelian gauge group. Bifundamental fields requires straightforward adjustment in the color factors of their tree amplitudes, see subsection 8.3.6.

Within this section, we use dimensionful Mandelstam invariants

$$\hat{s}_{i_1 i_2 \ldots i_r} \ := \ (k_{i_1} + k_{i_2} + \ldots + k_{i_r})^2 \tag{5.5.53}$$

instead of the dimensionless kinematic invariants $s_{i_1 i_2 \ldots i_r} = \alpha'(k_{i_1} + k_{i_2} + \ldots + k_{i_r})^2$ but drop the hat for ease of notation. In other words – within this section 5.5, $s_{i_1 i_2 \ldots i_r}$ denotes the mass dimension two quantity $(k_{i_1} + k_{i_2} + \ldots + k_{i_r})^2$, but in the remainder of this work, the $s_{i_1 i_2 \ldots i_r}$ are dimensionless through the built in α' in their definition (5.3.31).

5.5.1 Color decomposition in field theory

Scattering amplitudes of gauge theories depend on color- and kinematic degrees of freedom of the external states. The standard Feynman rules due to the Lagrangian description lead to tree amplitudes which we shall parametrize as follows:

$$\mathcal{M}^{\text{SYM}}\big[1^{a_1}, 2^{a_2} \ldots, n^{a_n}\big] \ = \ \sum_i \frac{c_i \, n_i}{\prod_{\alpha_i} s_{\alpha_i}} \tag{5.5.54}$$

The i sum runs over all n point tree diagrams with cubic vertices (or, equivalently, over kinematic pole channels), regardless on the ordering of the external legs. The c_i denote color factors made of $n-2$ structure constants f^{abc} of the gauge group, and the n_i are referred to as the their dual numerators and carry all the information on kinematics and helicities. In other words – the n_i are functions of momenta and polarization tensors. We will leave them unspecified at present because they depend on the external states. Each (n_i, c_i) pair multiplies $n-3$ propagators $s_{\alpha_i}^{-1}$ as required by a cubic n point tree diagram. The contribution of four point vertices in YM fields to (5.5.54) certainly contains less propagators and must be absorbed into the n_i by multiplying with $1 = \frac{s_{\alpha_i}}{s_{\alpha_i}}$ for compatibility with the pole structure. Those parts of the n_i where one of the poles in s_{α_i} can be cancelled are referred to as contact terms.

To make contact with the Chan Paton traces of the string computation, we shall rewrite the color structures f^{abc} from the interaction vertices and δ^{ab} from the propagators in terms of the generators T^a of the gauge group. First of all,

$$\big[T^a \, , \, T^b\big] \ = \ i f^{abc} \, T_c \, , \qquad \text{Tr}\big\{ T^a \, T^b \big\} \ = \ \frac{1}{2} \, \delta^{ab} \tag{5.5.55}$$

fixes our normalization conventions for the generators T^a which might vary between different references. By combining the two equations (5.5.55) we expres the unique color factor f^{abc} of a three point amplitude as

$$\frac{i}{2} \, f^{abc} \ = \ \text{Tr}\big\{ T^a \, T^b \, T^c \ - \ T^b \, T^a \, T^c \big\} \, . \tag{5.5.56}$$

The contractions of structure constants which appear at higher order can be reshuffled in a similar way, e.g. the color factor c_s of the s channel in a four point amplitude (see figure below) contributes to four different color traces:

$$f^{abe} \, f^{ecd} \ = \ -2 \, \text{Tr}\Big\{ \big[T^a \, , \, T^b\big] \, \big[T^c \, , \, T^d\big] \Big\} \tag{5.5.57}$$

By iterating tricks like (5.5.56) and (5.5.57), all the f^{abc} can be replaced by traces of generators T^d. The four point interaction vertex in the gauge field is no obstruction because it

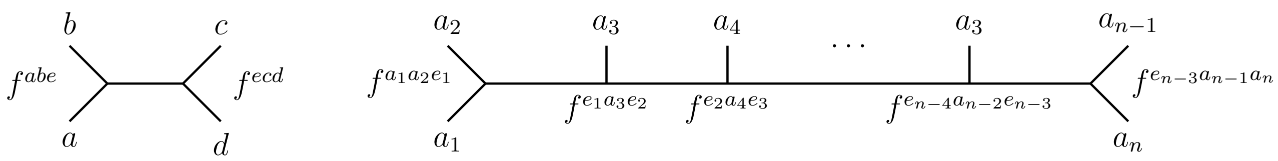


Figure 5.10: The structure constants due to the Feynman rules for three point vertices in the gauge field. The diagram on the left hand side represents the s channel of a four point amplitude with color factor $f^{abe} f^{ecd}$. As shown on the right hand side, a specific channel of an n point amplitude involves $n-2$ structure constants.

provides the same color structure as the product of two cubic vertices connected by a propagator. Hence, the mechanism (5.3.26) of color ordering can be literally transferred to field theory:

$$\mathcal{M}^{\mathrm{SYM}}\big[1^{a_1}, 2^{a_2} \ldots, n^{a_n}\big] \ \sim \ \sum_{\rho \in S_{n-1}} \mathrm{Tr}\big\{ T^{a_{\rho(1)}} \ldots T^{a_{\rho(n-1)}} T^{a_n} \big\} \mathcal{A}^{\mathrm{SYM}}\big(\rho(1), \ldots, \rho(n-1), n\big) \tag{5.5.58}$$

Since the Chan Paton traces are independent, it must be true on the level of color stripped amplitudes that SYM S matrix emerges in the low energy limit (5.5.52) of superstring theory:

$$\mathcal{A}^{\mathrm{SYM}}(1, 2, \ldots, n-1, n) \ = \ \lim_{\alpha' \to 0} \mathcal{A}(1, 2, \ldots, n-1, n; \alpha') \tag{5.5.59}$$

Moreover, each subamplitude $\mathcal{A}^{\mathrm{SYM}}\big(\rho(1), \ldots, \rho(n-1), n\big)$ must be gauge invariant for itself – in constrast to individual Feynman diagrams. The virtue of (5.5.58) lies in the decomposition of the full amplitude $\mathcal{M}^{\mathrm{SYM}}$ into smaller gauge invariant pieces with transparent properties.

As a concrete example for the ideas above, let us reconstruct the color dressed four point amplitude $\mathcal{M}^{\mathrm{SYM}}[1^{a_1}, 2^{a_2}, 3^{a_3}, 4^{a_4}]$ from a decomposition (5.5.58). The color stripped amplitudes $\mathcal{A}^{\mathrm{SYM}}(1,2,3,4)$, $\mathcal{A}^{\mathrm{SYM}}(1,3,2,4)$ and $\mathcal{A}^{\mathrm{SYM}}(1,3,4,2)$ subject to reflection symmetry are parametrized by kinematic numerators n_s, n_t and n_u along with the s-, t- and u channel poles, respectively. Only two out of three channels are compatible with the individual color orderings, see figure 5.11:

$$\begin{aligned}
\mathcal{M}^{\mathrm{SYM}}[1^{a_1}, 2^{a_2}, 3^{a_3}, 4^{a_4}] \ &= \ \mathrm{Tr}\big\{ T^{a_1} T^{a_2} T^{a_3} T^{a_4} \ + \ T^{a_4} T^{a_3} T^{a_2} T^{a_1} \big\} \ \Big(+ \frac{n_s}{s} \ + \ \frac{n_u}{u} \Big) \\
&\quad + \ \mathrm{Tr}\big\{ T^{a_1} T^{a_3} T^{a_2} T^{a_4} \ + \ T^{a_4} T^{a_2} T^{a_3} T^{a_1} \big\} \ \Big(- \frac{n_t}{t} \ - \ \frac{n_u}{u} \Big) \\
&\quad + \ \mathrm{Tr}\big\{ T^{a_1} T^{a_3} T^{a_4} T^{a_2} \ + \ T^{a_2} T^{a_4} T^{a_3} T^{a_1} \big\} \ \Big(- \frac{n_s}{s} \ + \ \frac{n_t}{t} \Big) \\
&= \ \mathrm{Tr}\Big\{ \big[T^{a_1} \, , \, T^{a_2} \big] \big[T^{a_3} \, , \, T^{a_4} \big] \Big\} \, \frac{n_s}{s} \ + \ \mathrm{Tr}\Big\{ \big[T^{a_4} \, , \, T^{a_1} \big] \big[T^{a_2} \, , \, T^{a_3} \big] \Big\} \, \frac{n_u}{u} \\
&\quad + \ \mathrm{Tr}\Big\{ \big[T^{a_1} \, , \, T^{a_3} \big] \big[T^{a_4} \, , \, T^{a_2} \big] \Big\} \, \frac{n_t}{t}
\end{aligned}$$

$$=: \quad \frac{n_s\, c_s}{s} + \frac{n_u\, c_u}{u} + \frac{n_t\, c_t}{t} \tag{5.5.60}$$

From the last equality, we can read off the explicit expressions for the triplet c_s, c_u, c_t of color factors:

$$\begin{aligned} c_s &= \mathrm{Tr}\Big\{ \big[T^{a_1} , T^{a_2} \big] \big[T^{a_3} , T^{a_4} \big] \Big\} &&= -\frac{1}{2} f^{a_1 a_2 e} f^{e a_3 a_4} \\ c_u &= -\frac{1}{2} f^{a_4 a_1 e} f^{e a_2 a_3} , \qquad c_t &&= -\frac{1}{2} f^{a_1 a_3 e} f^{e a_4 a_2} \end{aligned} \tag{5.5.61}$$

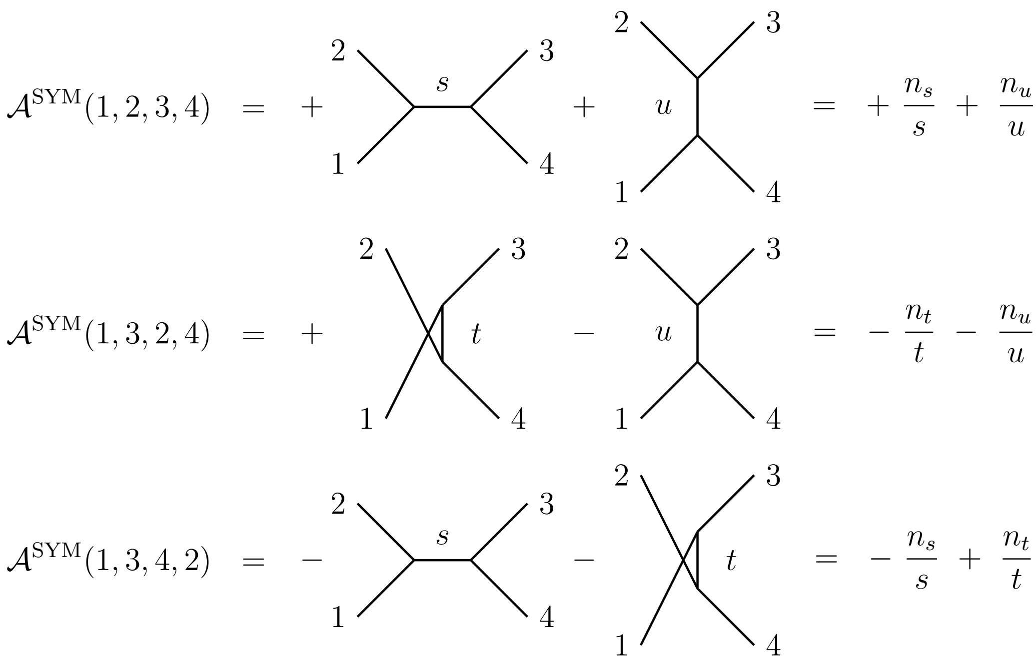


Figure 5.11: The kinematic numerators n_s, n_t and n_u associated with the three channels of the four point amplitude and their appearence in color ordered amplitudes.

5.5.2 Duality between color and kinematics

The kinematics factors n_i in the representation (5.5.54) for $\mathcal{M}^{\rm SYM}$ are building blocks for color stripped amplitudes. It was observed by Bern, Carrasco and Johannson (BCJ) that the n_i can be brought into a parametrization such that they exhibit the same algebraic structures as their color counterparts – although they appear completely unreated at first glance [81]. This duality between color- and kinematic degrees of freedom is an excellent example for hidden simplicity and non-obvious harmony in scattering amplitudes.

More specifically, the duality statement of BCJ concerns the Jacobi identities $f^{b[a_1 a_2} f^{a_3]bc} = 0$ valid for any Lie algebra: Whenever a triplet (c_i, c_j, c_k) of color factors satisfies a Jacobi identity, then one can arrange the associated numerators (n_i, n_j, n_k) such that they obey an analogous relation:

$$c_i + c_j - c_k = 0 \quad \Rightarrow \quad n_i + n_j - n_k = 0 \tag{5.5.62}$$

The Feynman rules of the color sector allow to identify the associated triplet of diagrams, the general case is depicted in figure 5.12. Moreover, a set of numerators n_i compatible with (5.5.62) flips its sign under permutation of external legs whenever the corresponding color factors c_i do. This is due to the antisymmetry of the three point vertex under exchange of two legs – both on the color- and the kinematic side.

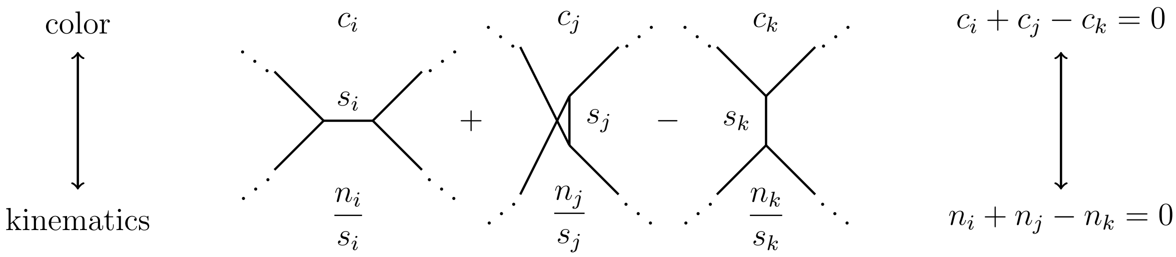
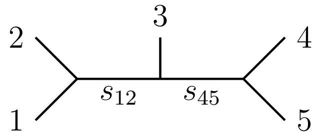


Figure 5.12: A triplet of subdiagrams where the sub over the associated color factors vanishes due to the Jacobi relation $f^{e[ab}f^{c]de} = 0$. The external subdiagrams attached to the dotted lines are arbitrary.

The four point color factors (5.5.61) obviously satisfy $c_s + c_t - c_u = 0$, so the duality predicts the existence of a numerator parametrization such that $n_s + n_t - n_u = 0$. In fact, this four point numerator identity holds independent on the distribution of four gluon contact terms, this is a feature of the simple structure of $\mathcal{M}^{\text{SYM}}[1^{a_1}, 2^{a_2}, 3^{a_3}, 4^{a_4}]$ and its momentum phase space. Concrete examples of n_s, n_t and n_u will be given in section 8.1.

The five point amplitude $\mathcal{M}^{\text{SYM}}[1^{a_1}, 2^{a_2}, 3^{a_3}, 4^{a_4}, 5^{a_5}]$ can be decomposed into fifteen channels with (c_i, n_i), $i = 1, 2, \ldots, 15$. Each color ordered amplitude encompasses five diagrams of the type shown in figure 5.13:

$$\mathcal{A}^{\text{SYM}}(1,2,3,4,5) =$$



$$+ \ \text{cyclic}(12345)$$

$$= \frac{n_1}{s_{12}s_{45}} + \frac{n_2}{s_{23}s_{51}} + \frac{n_3}{s_{12}s_{34}} + \frac{n_4}{s_{23}s_{45}} + \frac{n_5}{s_{34}s_{51}}$$

Figure 5.13: The five cubic diagrams entering the color stripped $\mathcal{A}^{\text{SYM}}(1,2,3,4,5)$ amplitude.

The labelling of the remaining ten channels is uniquely fixed by the subamplitudes

$$\begin{aligned}
\mathcal{A}^{\rm SYM}(1,4,3,2,5) &= \frac{n_6}{s_{14}s_{25}} + \frac{n_5}{s_{34}s_{51}} + \frac{n_7}{s_{23}s_{14}} + \frac{n_8}{s_{34}s_{25}} + \frac{n_2}{s_{23}s_{51}} \\
\mathcal{A}^{\rm SYM}(1,3,4,2,5) &= \frac{n_9}{s_{13}s_{25}} - \frac{n_5}{s_{34}s_{51}} + \frac{n_{10}}{s_{13}s_{24}} - \frac{n_8}{s_{34}s_{25}} + \frac{n_{11}}{s_{51}s_{24}} \\
\mathcal{A}^{\rm SYM}(1,2,4,3,5) &= \frac{n_{12}}{s_{12}s_{35}} + \frac{n_{11}}{s_{24}s_{51}} - \frac{n_3}{s_{12}s_{34}} + \frac{n_{13}}{s_{35}s_{24}} - \frac{n_5}{s_{51}s_{34}} \\
\mathcal{A}^{\rm SYM}(1,4,2,3,5) &= \frac{n_{14}}{s_{14}s_{35}} - \frac{n_{11}}{s_{24}s_{51}} - \frac{n_7}{s_{14}s_{23}} - \frac{n_{13}}{s_{24}s_{35}} - \frac{n_2}{s_{23}s_{51}} \\
\mathcal{A}^{\rm SYM}(1,3,2,4,5) &= \frac{n_{15}}{s_{13}s_{45}} - \frac{n_2}{s_{23}s_{51}} - \frac{n_{10}}{s_{13}s_{24}} - \frac{n_4}{s_{23}s_{45}} - \frac{n_{11}}{s_{24}s_{51}}
\end{aligned} \tag{5.5.63}$$

The pictorial duality dictionary in figure 5.12 implies the following kinematic Jacobi identities

$$\begin{aligned}
0 &= n_3 - n_5 + n_8 = n_3 - n_1 + n_{12} = n_{10} - n_{11} + n_{13} \\
0 &= n_4 - n_2 + n_7 = n_4 - n_1 + n_{15} = n_{10} - n_9 + n_{15} \\
0 &= n_8 - n_6 + n_9 = n_5 - n_2 + n_{11} = n_7 - n_6 + n_{14}
\end{aligned} \tag{5.5.64}$$

which only hold for special assignments of the contact terms. In contrast to the four point analogue $n_s + n_t - n_u = 0$, the five point Jacobi identities (5.5.64) generically fail to hold.

5.5.3 The contact term ambiguity

The quartic self-couplings of gluons in the YM action introduces an ambiguity into the decomposition (5.5.54) of gauge theory amplitudes into diagrams with cubic vertices. The numerators n_i are not uniquely specified because there remains the freedom to add zeros of the form $0 = \left(\frac{s_{\alpha_i}}{s_{\alpha_i}} - \frac{s_{\alpha_j}}{s_{\alpha_j}}\right) \times (\ldots)$ to the amplitudes which amounts to reabsorbing contact terms into a different numerators $n_j \neq n_i$.

The duality (5.5.62) crucially depends on the choice of kinematic packages n_i. A generic assignment of contact terms at $n \geq 5$ points spoils the dual Jacobi identities for the n_i, but the gauge theory setup does not provide any constructive prescription to find such a duality-compatible parametrization. Contact term ambiguities have always been an obstacle in constructing color-dual BCJ numerators directly from the gauge theory. There exist Kawai–Lewellen–Tye (KLT) inspired expressions for n_i in terms of color ordered gauge theory amplitudes [203] but they do not exhibit manifest locality.

The kinematic building blocks $\frac{n_i}{\prod_{\alpha_i} s_{\alpha_i}}$ can essentially be though of as a Feynman like diagram associated with a particular channel. Since individual Feynman diagrams are gauge dependent, we conclude that the same is true for the numerators n_i. The gauge choice can only affect the

n_i by adding terms that cancel one of the s_{α_i} poles. That is why the reshuffling of contact terms is loosely referred to as a generalized gauge transformation.

Remarkably, the pure spinor formalism resolves these ambiguities [11]: The later equation (12.1.13) expresses the n point superstring tree amplitude in terms of $(n-2)!$ basis kinematics. It will be shown in chapter 13 that its low energy limit naturally selects a unique representation of the kinematic numerators n_i which satisfies all the dual Jacobi identities. They are constructed in terms of superfields of $D = 10$ dimensional $\mathcal{N} = 1$ SYM, but it is straightforward to dimensionally reduce the superfield components to $D = 4$, and the bosonic parts describe gluon scattering independently on the existence of supersymmetries.

5.5.4 Kleiss Kuijf relations

The reflection symmetry $\mathcal{A}(1, 2, \ldots, n-1, n) = (-1)^n \mathcal{A}(n, n-1, \ldots, 2, 1)$ of massless state amplitudes is equally valid in string theory and the associated $\alpha' \to 0$ field theory. Only the way of deriving them is different – the string relation is dictated by worldsheet parity whereas the FT reflection property follows from studying the sum of Feynman diagrams which contribute to the subamplitudes in question[5]. In the following, we will give two more examples of field theory technology which finds an alternative derivation from string theory via worldsheet properties.

In field theory, there were two steps of progress in further reducing the number of $\frac{1}{2}(n-1)!$ color orderings. Firstly, Kleiss and Kuijf found a set of relations which boil the number of independent subamplitudes down to $(n-2)!$ [204]:

$$\mathcal{A}^{\rm SYM}(1, \alpha_1, \ldots, \alpha_s, n, \beta_1, \ldots, \beta_r) \ = \ (-1)^r \sum_{\sigma \in {\rm OP}\{\alpha\} \cup \{\beta^T\}} \mathcal{A}^{\rm SYM}(1, \sigma, n) \tag{5.5.65}$$

Just like in (5.4.45), the sum encompasses all permutations of $\{\alpha\} \bigcup \{\beta^T\}$ which preserve the order of the individual elements of both sets $\{\alpha\}$ and $\{\beta^T\}$. The proof of (5.5.65) is carried out in [205] and based on group-theoretic properties only such as the Jacobi identity and the behavior of color traces for a large number of colors.

The information contained in the Kleiss Kuijf (KK) relation (5.5.65) and reflection symmetry is equivalent to writing color dressed amplitude more compactly in terms of structure constants and $(n-2)!$ subamplitudes [206]:

$$\mathcal{M}^{\rm SYM}[1^{a_1}, 2^{a_2}, \ldots, n^{a_n}] \ = \ \frac{i^{n-2}}{2} \sum_{\sigma \in S_{n-2}} f^{a_1 a_{\sigma(2)} e_1} \, f^{e_1 a_{\sigma(3)} e_2} \, \ldots \, f^{e_{n-4} a_{\sigma(n-2)} e_{n-3}} \, f^{e_{n-3} a_{\sigma(n-1)} a_n}$$

[5] More precisely, it is a consequence of the fact that the three vertex picks up a sign under reflections and that an n point tree amplitude has $n-2$ three point vertices.

$$\times \ \mathcal{A}^{\rm SYM}\big(1, \sigma(2,3,\ldots,n-1), n\big) \tag{5.5.66}$$

The S_{n-2} family of color ordered amplitudes $\mathcal{A}^{\rm SYM}\big(1, \sigma(2,3,\ldots,n-1), n\big)$ (which is also referred to as the KK basis or the color ladder) turn out to be sufficient to reconstruct the full color dressed amplitude $\mathcal{M}^{\rm SYM}[1^{a_1}, 2^{a_2}, \ldots, n^{a_n}]$. As a consequence, the number of independent color factors always coincides with the number $(n-2)!$ of KK subamplitudes. If the dual Jacobi identities for the numerators hold, then there are also no more than $(n-2)!$ independent n_i.

Note that the reduction of color factors to structure constants is only possible in field theory. Superstring amplitudes at finite α' generically involve symmetrized traces as well which cannot be reduced to products of f^{abc}.

The complex monodromy relation (5.4.44) can be viewed as the string theory upgrade of field theory relations. Its real part provides an all-orders-in-α'-version of dual Ward identity [31]

$$\begin{aligned} \mathcal{A}^{\rm SYM}(1,2,\ldots,n) \ + \ \mathcal{A}^{\rm SYM}(2,1,3,\ldots,n) \ + \ \mathcal{A}^{\rm SYM}(2,3,1,4,\ldots,n) \\ + \ \ldots \ + \ \mathcal{A}^{\rm SYM}(2,3,\ldots,n-1,1,n) \ = \ 0 \end{aligned} \tag{5.5.67}$$

which is the special case of the KK relations with just one element in the set $\{\beta\}$. It is also known as the photon decoupling identity or the subcyclic property.

5.5.5 Bern Carrasco Johansson relations

The search for a minimal basis of color ordered field theory amplitudes was later on refined by Bern, Carrasco and Johansson (BCJ) [81]. Their so-called BCJ relations further reduce the color ladder of $(n-2)!$ subamplitudes to a minimal basis of $(n-3)!$ only, we will sketch their main ideas in the following.

The concept of generalized gauge transformations explained in subsection 5.5.3 can be extended to non-local transformations. The four point subamplitudes from figure 5.11, for instance, remain unchanged by a transformation $(n_s, n_t, n_u) \mapsto (n_s, n_t, n_u) + \chi(s,t,-u)$ even if the generalized gauge parameter χ is non-local. In particular, we can pick the non-local choice $\chi = \frac{n_u}{u}$ and enforce the transformed n_u to vanish. The Jacobi identity then just leaves one independent numerator $n_s = -n_t$ and yields

$$\mathcal{A}^{\rm SYM}(1,2,3,4) \ = \ \frac{t}{s}\, \mathcal{A}^{\rm SYM}(1,3,2,4) \ = \ \frac{t}{u}\, \mathcal{A}^{\rm SYM}(1,3,4,2) \tag{5.5.68}$$

which is the field theory limit of (5.4.49) and (5.4.50).

This procedure of deriving relations between color ordered amplitudes can be generalized to higher points. For this purpose, the kinematic numerators are treated as being unknown variables subject to the Jacobi equation system (5.5.62). It consists of the following steps:

- express all the numerators in (5.5.54) in terms of $(n-2)!$ independent ones using Jacobi identities (5.5.62)

- solve for $(n-3)!$ basis numerators in terms of a subamplitude to which they contribute

- force the remaining $(n-2)!-(n-3)!$ independent numerators to become zero by means of a non-local generalized gauge transformation

The remaining $(n-2)!-(n-3)!$ KK subamplitudes which did not get involved in the second step are then naturally expressed in terms of a $(n-3)!$ basis from their $\sum_i \frac{n_i}{\prod_{\alpha_i} s_{\alpha_i}}$ parametrization. The authors of [81] have checked that the method above can be successfully applied up to $n=8$ points and conjectured its validity beyond that. An all multiplicity proof in $D=4$ dimensions based on BCFW recursion relations [207, 208] followed in [209].

The resulting system of equation can be brought into the form

$$\begin{aligned} s_{12}\,\mathcal{A}^{\mathrm{SYM}}(2,1,3,\ldots,n) \;&+\; (s_{12}+s_{13})\,\mathcal{A}^{\mathrm{SYM}}(2,3,1,4,\ldots,n) \\ &+\;\ldots\;+\;(s_{12}+s_{13}+\ldots+s_{1,n-1})\,\mathcal{A}^{\mathrm{SYM}}(2,3,\ldots,n-1,1,n) \;=\; 0\,. \end{aligned} \tag{5.5.69}$$

The complex relation (5.4.44) between between superstring amplitudes already reconstructed the photon decoupling identity (5.5.67) from the $\alpha' \to 0$ limit of its real part (such that $\cos(\pi s_{ij}) \to 1$). Its imaginary part, on the other hand, reduces to the from (5.5.69) of the BCJ relations in the low energy limit $\sin(\pi s_{ij}) \to \pi s_{ij}$. However, it should be stressed the full string theory amplitudes generically do not fulfill neither dual Ward nor KK- nor BCJ-relations. Instead, they do fulfill modified relations (5.4.44) or (5.4.45), which boil down to the former in the field theory limit.

5.5.6 BCJ relations versus Jacobi identities

By taking appropriate permutations of (5.5.69) and decomposing the occurring subamplitudes in pole channels, one can derive identities between Jacobi triplets $(n_{i_k}, n_{i_l}, n_{i_m})$ dual to color factors with $c_{i_k}+c_{i_l}+c_{i_m}=0$ of the following form [210, 211]

$$\sum_i \frac{n_{i_k}+n_{i_l}+n_{i_m}}{\prod_{\alpha_i}^{n-4} s_{\alpha_i}} \;=\; 0\,. \tag{5.5.70}$$

The i sum runs over $(n-1)$ point channels of total number $2^{n-3}(2n-7)!!(n-3)/(n-2)!$ and involves the $n-4$ propagators s_{α_i} common among the n_{i_k}, n_{i_l} and n_{i_m} channels. This equation can be viewed as a constraint on the numerators' failure to satisfy the Jacobi identities. Their gauge independent content is that Jacobi triplets $n_{i_k}+n_{i_l}+n_{i_m}$ must vanish at the residue of the $n-4$ poles, independent on the assignment of contact terms to the numerators.

The four point version of (5.5.70) encompasses one term only without any propagators and enforces $n_s+n_t-n_u=0$. But any higher multiplicity $n \geq 5$ gives rise to a relation like

$$\frac{n_4-n_1+n_{15}}{s_{45}} - \frac{n_{10}-n_{11}+n_{13}}{s_{24}} - \frac{n_3-n_1+n_{12}}{s_{12}} - \frac{n_5-n_2+n_{11}}{s_{51}} = 0 \tag{5.5.71}$$

which does not force all the triplets to vanish individually. Satisfying all the Jacobi identities (5.5.64) is not the only solution to (5.5.71), the triplets can still be proportional to a contact term, e.g. $n_4-n_1+n_{15}=s_{45}\chi$ and $n_{10}-n_{11}+n_{13}=s_{24}\chi$ whereas $n_3-n_1+n_{12}=n_5-n_2+n_{11}=0$.

Let us emphasize that deriving (5.5.70) from (5.5.69) is not a circular argument: The BCJ relations hold independent on the choice of numerators n_i. The existence of a generalized gauge where all the Jacobi triplets are zero was just a tool in the approach of [81] to perform the basis reduction to $(n-3)!$ subamplitudes.

5.6 KLT relations

In the context of relating color ordered amplitudes of superstring- and field theory, it makes sense to give a little outlook on the closed string- or gravity sector. In a famous paper [83] by Kawai, Lewellen and Tye (KLT), tree level amplitudes of closed string states were found to factorize into bilinears in open string color stripped amplitudes, the precise formulae are referred to as KLT relations. They are derived by monodromy arguments on the worldsheet, similar to the construction in subsection 5.4.1. This reflects the fact that the Hilbert space of closed string states is simply the tensor product of two open string states.

Let $\mathcal{M}(1,2,\ldots,n)$ denote an n point tree level amplitude of closed string states, then the four- and five point versions decompose as follows into open string subamplitudes of different color orderings:

$$\mathcal{M}(1,2,3,4) \sim \sin(\pi s)\,\mathcal{A}(1,2,3,4) \otimes \mathcal{A}(1,2,4,3) \tag{5.6.72}$$

$$\begin{aligned}\mathcal{M}(1,2,3,4,5) \sim\ & \sin(\pi s_{12})\,\sin(\pi s_{34})\,\mathcal{A}(1,2,3,4,5) \otimes \mathcal{A}(2,1,4,3,5)\\ & + \sin(\pi s_{13})\,\sin(\pi s_{24})\,\mathcal{A}(1,3,2,4,5) \otimes \mathcal{A}(3,1,4,2,5)\end{aligned} \tag{5.6.73}$$

We are not keeping track of the individual coupling constants, they are hidden in the proportionality sign of (5.6.72). The $\otimes$ indicate that the closed string polarization tensors are just tensor products of their open string wavefunctions, e.g. the graviton polarization decomposes via $\zeta_{mn} = \xi_{(m} \otimes \xi_{n)}$ into gluon polarizations. Higher order KLT relations can be found in [212, 203].

These relations are of course inherited by the associated field theories [213] – $\mathcal{N} = 8$ supergravity in case of maximally supersymmetric closed string theory and $\mathcal{N} = 4$ SYM on the open string side [87]:

$$\mathcal{M}^{\text{SUGRA}}(1,2,3,4) \quad \sim \quad s\,\mathcal{A}^{\text{SYM}}(1,2,3,4) \otimes \mathcal{A}^{\text{SYM}}(1,2,4,3) \tag{5.6.74}$$

The five point amplitude $\mathcal{M}^{\text{SUGRA}}(1,2,3,4,5)$ can be obtained from (5.6.73) by replacing $\sin(\pi s_{ij}) \mapsto \pi s_{ij}$, and higher order KLT relations in field theory are given in appendix A of [214] for any number of legs. About 25 years after the discovery of string theory KLT relations based on worldsheet monodromies, a field theory derivation of KLT relations was obtained in [215, 216].

A natural generalization of mapping pure closed string amplitudes to open string constituents is to consider mixed amplitudes of both open and closed string states. It was shown in [84] that also the mixed cases can be mapped to open string subamplitudes where the closed string degrees of freedom are represented by two open string state insertions.

Since we have found that tree amplitudes of open superstring theories decompose into SYM amplitudes via (1.4.1), it is natural to expect that the n point gravity amplitude is somehow related its field theory limit $\mathcal{M}^{\text{SUGRA}}(1,2,\ldots,n)$, its most compact representation is under current research [217].

Chapter 6

Correlation functions of RNS primaries

The computation of superstring amplitudes (5.3.38) requires SCFT correlation functions of vertex operators which create the scattering states. Vertex operators in superstring theory not only contain free fields such as X^m or $\mathrm{e}^{q\phi}$ but also interacting fields ψ^m, S_A of the RNS CFT, see chapters 3 and 4. This chapter summarizes the results of [3, 4, 5] in solving the technical challenges of the interacting SCFT, i.e. in computing higher point correlation functions with ψ^m and S_A fields involved. The applicability of these SCFT results ranges from superstring theories to the heterotic string theories allowing for a CFT description. Working on the aforementioned publications has filled about a third of my research capacity as a PhD student, in collaboration with Daniel Härtl and Stephan Stieberger.

Even though the bosonization methods of subsections 2.2.3 and 2.2.4 admit to reduce ψ^m and S_A to exponentials of free fields, it is a nontrivial problem to cast the result into $SO(1, D-1)$ covariant form. Bosonization breaks Lorentz symmetry down to $U(D/2)$ and only admits to compute individual components of correlators, i.e. one must pick choices for the Lorentz indices m and A before bosonization becomes applicable. The progress of [3, 4, 5] compared to [218, 219, 220] lies in the systematic covariantization of the answers obtained from bosonized computations.

A part of the discussion can be carried out in arbitrary even spacetime dimensions $D = 2n$: We will firstly introduce two algorithms for computing RNS correlation functions and secondly give expressions for the class of correlators $\langle \psi^{m_1} ... \psi^{m_n} S_A S_B \rangle$ with no more than two spin fields. These provide the most difficult CFT input for scattering amplitudes of two massless spacetime fermions and any number of bosons. Although the dimension is kept general, we will use the $D = 10$ indices m for vectors and A $(\alpha, \dot{\alpha})$ for Dirac (Weyl) spinors.

As soon as four or more spin fields get involved, representation theoretic technicalities force us to specify the number of spacetime dimensions. Roughly speaking, this is due to the structure of Fierz identities between tensors with four or more spinor indices which varies a lot with $D = 2n$. We will therefore discuss the cases $D = 4, 6, 8$ and $D = 10$ separately at the level of ≥ 4 spin fields.

Although this thesis is devoted to tree level amplitudes, we will compute most of the correlators on Riemann surfaces of arbitrary genus. The extra effort of computing spin structure dependent factors at $g \geq 1$ is usually quite manageable once the $g = 0$ result is known. There are just a few exceptions where the algorithms do not provide an extension to loops with reasonable effort.

The shortcoming of this chapter is the lack of a systematic study of composite operators such as $\psi^m \not{\psi}_A{}^B S_B$ or $\partial\psi^m\psi^n$. They can carry spins larger than one, i.e. contain exponentials like $\mathrm{e}^{\pm isH_j}$ with Ramond charge $s > 1$ in their bosonized representation. Some lower point correlation functions of the former operators are listed in section 6.5.2, they have been worked out in [2, 6] to compute amplitudes with massive states (see chapter 9).

6.1 The strategy in $D = 2n$ dimensions

This section explains the preliminaries and techniques involved in the computation of RNS correlators in arbitrary even dimensions $D = 2n$. Correlation functions $\langle\psi^{m_1}(z_1)\ldots\psi^{m_n}(z_n)\rangle$ without spin field insertions are determined by (B.4.60) using Wick contractions [221], so we are interested in the nontrivial cases where two or more spin fields appear.

6.1.1 Recapitulating the singularity structures

We have already given the leading OPE singularities of the ψ^m, S_A fields in section 2.2, see in particular (2.2.32) and (2.2.33). As explained in subsection 2.2.4 and explicitly shown in (2.2.43), bosonization provides a neat way to determine OPE coefficients. In any number of dimensions, $\psi^m, S_\alpha, S^{\dot\beta}$ and the identity operator are the only four primary fields with respect to the $SO(1, D-1)$ Kac-Moody algebra at level one generated by the currents $\psi^m\psi^n$ [222, 118, 126, 119]. Leading singularities of those primary fields already determine all tree level correlation functions. This will be exploited in the recursive algorithm to be introduced in the following subsection 6.1.3.

Nevertheless, it is useful to also know some subleading singularities of the OPEs, e.g. to obtain higher ghost picture or SUSY variations of vertex operators. In our case of interest, these are the following:[1]

$$
\begin{aligned}
\psi^m(z)\,\psi^n(w) \;&\sim\; \frac{\eta^{mn}}{z-w} \;+\; \psi^m\,\psi^n(w) \;+\; (z-w)\,\partial\psi^m\,\psi^n(w) \;+\; \dots && (6.1.1)\\
\psi^m(z)\,S_A(w) \;&\sim\; \frac{1}{\sqrt{2}\,(z-w)^{1/2}}\,(\Gamma^m)_A{}^B\,S_B(w)\\
&+\; \frac{2\,(z-w)^{1/2}}{\sqrt{2}\,(D-2)}\,\Big[\,\psi^m\,\psi_n\,(\Gamma^n)_A{}^B \;+\; \tfrac{1}{2(D-1)}\,(\Gamma^m)_A{}^C\,\psi^n\,\psi^p\,(\Gamma_{np})_C{}^B\,\Big]\,S_B \;+\; \dots && (6.1.2)\\
S_A(z)\,S_B(w) \;&\sim\; \frac{\mathcal{C}_{AB}}{(z-w)^{D/8}} \;+\; \frac{(\Gamma^m\,\mathcal{C})_{AB}\,\psi_m(w)}{\sqrt{2}\,(z-w)^{D/8-1/2}} \;-\; \frac{(\Gamma^{mn}\,\mathcal{C})_{AB}\,\psi_m\,\psi_n(w)}{4\,(z-w)^{D/8-1}}\\
&+\; \frac{1}{\sqrt{2}\,(z-w)^{D/8-3/2}}\,\Big[\,\tfrac{1}{2}\,(\Gamma^m\,\mathcal{C})_{AB}\,\partial\psi_m(w) \;-\; \tfrac{1}{12}\,(\Gamma^{mnp}\,\mathcal{C})_{AB}\,\psi_m\,\psi_n\,\psi_p\,\Big] \;+\; \dots && (6.1.3)
\end{aligned}
$$

The structure of the subleading terms can be derived from bosonization[2]:

$$
\begin{aligned}
\mathrm{e}^{iq_1H(z)}\,\mathrm{e}^{iq_2H(w)} \;&=\; (z-w)^{q_1q_2}\,\Big[\,\mathrm{e}^{i(q_1+q_2)\,H(z)} \;+\; q_1\,(z-w)\,i\partial H\,\mathrm{e}^{i(q_1+q_2)\,H(z)}\\
&\quad +\; \tfrac{1}{2}\,(z-w)^2\,\big[\,q_1\,i\partial^2H \;+\; q_1^2\,(i\partial H)^2\,\big]\,\mathrm{e}^{i(q_1+q_2)\,H(z)} \;+\; \mathcal{O}\big((w-z)^3\big)\,\Big]
\end{aligned}
\tag{6.1.5}
$$

The coefficients (in particular the signs) of (6.1.2) and (6.1.3) are most conveniently checked by Taylor expanding some of the correlation functions given in the next sections. Note that this is no circular argument because $g=0$ correlators with any ψ^m, S_A field insertion are completely specified by the *leading* singularities and by their properties under the $SO(1,D-1)$ current algebra.

For conformal primaries such as ψ^m, S_A, correlation functions up to three points are fully determined by conformal invariance, see (B.1.8). The non-vanishing three point function of this type is

$$
\langle\,\psi^m(z_1)\,S_A(z_2)\,S_B(z_3)\,\rangle \;=\; \frac{(\Gamma^m\mathcal{C})_{AB}}{\sqrt{2}\,z_{12}^{1/2}\,z_{13}^{1/2}\,z_{23}^{D/8-1/2}}\;. \tag{6.1.6}
$$

Any higher point correlator in general depends on $SL(2,\mathbb{C})$ invariant cross ratios such as $\frac{z_{ij}z_{kl}}{z_{ik}z_{jl}}$, so the task of solving the CFT amounts to determining the functional dependence on the cross ratios.

---

[1]This work uses the conventions of [5] where $S_AS_B \sim +\mathcal{C}_{AB}(z-w)^{-D/8}$. In older work [1,3,4], there was a relative sign $S_aS_b \sim -\varepsilon_{ab}(z-w)^{-1/2}$ in $D=4$ spin field bilinears. Any result on four dimension spin fields displayed within this chapter is adjusted to the $S_aS_b \sim +\varepsilon_{ab}(z-w)^{-1/2}$ convention.

[2]The analogous OPE for the superghost bosonization reads

$$
\begin{aligned}
\mathrm{e}^{q_1\phi(z)}\,\mathrm{e}^{q_2\phi(w)} \;&=\; (z-w)^{-q_1q_2}\,\Big[\,\mathrm{e}^{(q_1+q_2)\,\phi(z)} \;+\; q_1\,(z-w)\,\partial\phi\,\mathrm{e}^{(q_1+q_2)\,\phi(z)}\\
&\quad +\; \tfrac{1}{2}\,(z-w)^2\,\big[\,q_1\,\partial^2\phi \;+\; q_1^2\,(\partial\phi)^2\,\big]\,\mathrm{e}^{(q_1+q_2)\,\phi(z)} \;+\; \mathcal{O}\big((w-z)^3\big)\,\Big]
\end{aligned}
\tag{6.1.4}
$$

6.1.2 Representation theoretic background

We are interested in $SO(1,D-1)$ covariant expressions for RNS correlation functions, i.e. they need to be expressed in terms of the invariant tensors $\eta^{mn}, \Gamma^m{}_A{}^B$, and $\mathcal{C}_{AB}$. Regardless of which algorithm we use for computing correlators, it is essential to know the number of independent such tensors with a given set of indices, e.g. the Dirac algebra $(\Gamma^m\Gamma^n + \Gamma^n\Gamma^m)_A{}^B = -2\eta^{mn}\delta_A^B$ is necessary to compute $\langle \psi^m(z_1)\psi^n(z_2)S_A(z_3)S_B(z_4)\rangle$.

Starting point for a representation theoretic analysis is the identification of irreducible $SO(1,D-1)$ representations, so we decompose the spinorial expressions into their chiral halves, e.g.

$$S_A \;=\; \begin{pmatrix} S_\alpha \\ S^{\dot\alpha} \end{pmatrix} \, , \qquad (\Gamma^m)_A{}^B \;=\; \begin{pmatrix} 0 & \gamma^m_{\alpha\dot\beta} \\ \bar\gamma^{m\dot\alpha\beta} & 0 \end{pmatrix} \, . \tag{6.1.7}$$

The chiral structure of the charge conjugation matrix $\mathcal{C}$ depends on the number of dimensions:

$$\mathcal{C}_{AB} \;=\; \begin{cases} \begin{pmatrix} C_{\alpha\beta} & 0 \\ 0 & C^{\dot\alpha\dot\beta} \end{pmatrix} & : \quad D = 4 \bmod 4 \\ \begin{pmatrix} 0 & C_\alpha{}^{\dot\beta} \\ C^{\dot\alpha}{}_\beta & 0 \end{pmatrix} & : \quad D = 2 \bmod 4 \end{cases} \tag{6.1.8}$$

Any correlation function $\langle \psi^{m_1}...\psi^{m_n} S_{\alpha_1}...S_{\alpha_p} S^{\dot\beta_1}...S^{\dot\beta_q}\rangle$ (in a notation that distinguishes the chiral irreducibles) can be expressed in terms of Clebsch Gordan coefficients taking the tensor product of n vectors, p left handed spinors and q right handed spinors into scalar representations of $SO(1,D-1)$. The number of linearly independent such tensors $T^{m_1...m_n}{}_{\alpha_1...\alpha_p}{}^{\dot\beta_1...\dot\beta_q}$ is equal to the number of scalar representations in the corresponding tensor product. Table 6.1 summarizes the dimensions of Clebsch Gordan bases for a large class of correlators which we will compute in the following.

6.1.3 The recursive algorithm

A first algorithm to compute the nontrivial correlation functions with more than three points has a recursive nature. It exploits the leading singularities when two $SO(1,D-1)$ primary fields ψ^m, S_α or $S^{\dot\beta}$ approach each other, given by the first term of the OPEs (6.1.1) to (6.1.3). Given the general structure of the OPEs $\phi_i(z)\phi_j(w) \sim C_{ij}{}^k (z-w)^{-h_i-h_j+h_k}\phi_k(w) + \ldots$, the identity

$$\langle \ldots \, \phi_i(z)\, \phi_j(w) \, \ldots \rangle \;=\; \frac{C_{ij}{}^k}{(z-w)^{h_i+h_j-h_k}} \, \langle \ldots \, \phi_k(w) \, \ldots \rangle \;+\; \mathcal{O}\big((z-w)^{-h_i-h_j+h_k+1}\big) \tag{6.1.9}$$

$\otimes$	$D=4$	$D=8$	$\otimes$	$D=6$	$D=10$
$\langle S_\alpha S_\beta \rangle$	1	1	$\langle S_\alpha S^{\dot\beta} \rangle$	1	1
$\langle \psi^\mu S_\alpha S^{\dot\beta} \rangle$	1	1	$\langle \psi^\mu S_\alpha S_\beta \rangle$	1	1
$\langle \psi^\mu \psi^\nu S_\alpha S_\beta \rangle$	2	2	$\langle \psi^\mu \psi^\nu S_\alpha S^{\dot\beta} \rangle$	2	2
$\langle \psi^\mu \psi^\nu \psi^\lambda S_\alpha S^{\dot\beta} \rangle$	4	4	$\langle \psi^\mu \psi^\nu \psi^\lambda S_\alpha S_\beta \rangle$	4	4
$\langle \psi^\mu \psi^\nu \psi^\lambda \psi^\rho S_\alpha S_\beta \rangle$	10	10	$\langle \psi^\mu \psi^\nu \psi^\lambda \psi^\rho S_\alpha S^{\dot\beta} \rangle$	10	10
$\langle \psi^\mu \psi^\nu \psi^\lambda \psi^\rho \psi^\tau S_\alpha S^{\dot\beta} \rangle$	25	26	$\langle \psi^\mu \psi^\nu \psi^\lambda \psi^\rho \psi^\tau S_\alpha S_\beta \rangle$	26	26
$\langle \psi^\mu \psi^\nu \psi^\lambda \psi^\rho \psi^\tau \psi^\xi S_\alpha S_\beta \rangle$	70	76	$\langle \psi^\mu \psi^\nu \psi^\lambda \psi^\rho \psi^\tau \psi^\xi S_\alpha S^{\dot\beta} \rangle$	76	76
$\langle S_\alpha S_\beta S^{\dot\gamma} S^{\dot\delta} \rangle$	1	2	$\langle S_\alpha S_\beta S_\gamma S_\delta \rangle$	1	2
$\langle S_\alpha S_\beta S_\gamma S_\delta \rangle$	2	3	$\langle S_\alpha S_\beta S^{\dot\gamma} S^{\dot\delta} \rangle$	2	3
$\langle \psi^\mu S_\alpha S_\beta S_\gamma S^{\dot\delta} \rangle$	2	4	$\langle \psi^\mu S_\alpha S_\beta S_\gamma S^{\dot\delta} \rangle$	3	5
$\langle \psi^\mu \psi^\nu S_\alpha S_\beta S^{\dot\gamma} S^{\dot\delta} \rangle$	4	9	$\langle \psi^\mu \psi^\nu S_\alpha S_\beta S_\gamma S_\delta \rangle$	6	11
$\langle \psi^\mu \psi^\nu S_\alpha S_\beta S_\gamma S_\delta \rangle$	5	10	$\langle \psi^\mu \psi^\nu S_\alpha S_\beta S^{\dot\gamma} S^{\dot\delta} \rangle$	7	12
$\langle \psi^\mu \psi^\nu \psi^\lambda S_\alpha S_\beta S_\gamma S^{\dot\delta} \rangle$	10	24	$\langle \psi^\mu \psi^\nu \psi^\lambda S_\alpha S_\beta S_\gamma S^{\dot\delta} \rangle$	17	31
$\langle \psi^\mu \psi^\nu \psi^\lambda \psi^\rho S_\alpha S_\beta S^{\dot\gamma} S^{\dot\delta} \rangle$	25	68	$\langle \psi^\mu \psi^\nu \psi^\lambda \psi^\rho S_\alpha S_\beta S_\gamma S_\delta \rangle$	45	88
$\langle \psi^\mu \psi^\nu \psi^\lambda \psi^\rho S_\alpha S_\beta S_\gamma S_\delta \rangle$	28	71	$\langle \psi^\mu \psi^\nu \psi^\lambda \psi^\rho S_\alpha S_\beta S^{\dot\gamma} S^{\dot\delta} \rangle$	48	91
$\langle S_\alpha S_\beta S_\gamma S_\delta S^{\dot\epsilon} S^{\dot\iota} \rangle$	2	10	$\langle S_\alpha S_\beta S_\gamma S_\delta S_\epsilon S^{\dot\iota} \rangle$	4	16
$\langle S_\alpha S_\beta S_\gamma S_\delta S_\epsilon S_\iota \rangle$	5	15	$\langle S_\alpha S_\beta S_\gamma S^{\dot\delta} S^{\dot\epsilon} S^{\dot\iota} \rangle$	6	19
$\langle \psi^\mu S_\alpha S_\beta S_\gamma S^{\dot\delta} S^{\dot\epsilon} S^{\dot\iota} \rangle$	4	24	$\langle \psi^\mu S_\alpha S_\beta S_\gamma S_\delta S_\epsilon S_\iota \rangle$	9	40
$\langle \psi^\mu S_\alpha S_\beta S_\gamma S_\delta S_\epsilon S^{\dot\iota} \rangle$	5	26	$\langle \psi^\mu S_\alpha S_\beta S_\gamma S_\delta S^{\dot\epsilon} S^{\dot\iota} \rangle$	12	45
$\langle \psi^\mu \psi^\nu S_\alpha S_\beta S_\gamma S_\delta S^{\dot\epsilon} S^{\dot\iota} \rangle$	10	68	$\langle \psi^\mu \psi^\nu S_\alpha S_\beta S_\gamma S_\delta S_\epsilon S^{\dot\iota} \rangle$	29	125
$\langle \psi^\mu \psi^\nu S_\alpha S_\beta S_\gamma S_\delta S_\epsilon S_\iota \rangle$	14	76	$\langle \psi^\mu \psi^\nu S_\alpha S_\beta S_\gamma S^{\dot\delta} S^{\dot\epsilon} S^{\dot\iota} \rangle$	32	130
$\langle S_\alpha S_\beta S_\gamma S_\delta S^{\dot\epsilon} S^{\dot\iota} S^{\dot\kappa} S^{\dot\lambda} \rangle$	4	71	$\langle S_\alpha S_\beta S_\gamma S_\delta S_\epsilon S_\iota S_\kappa S_\lambda \rangle$	14	175
$\langle S_\alpha S_\beta S_\gamma S_\delta S_\epsilon S_\iota S^{\dot\kappa} S^{\dot\lambda} \rangle$	5	76	$\langle S_\alpha S_\beta S_\gamma S_\delta S_\epsilon S_\iota S^{\dot\kappa} S^{\dot\lambda} \rangle$	19	196
$\langle S_\alpha S_\beta S_\gamma S_\delta S_\epsilon S_\iota S_\kappa S_\lambda \rangle$	14	106	$\langle S_\alpha S_\beta S_\gamma S_\delta S^{\dot\epsilon} S^{\dot\iota} S^{\dot\kappa} S^{\dot\lambda} \rangle$	24	210

Table 6.1: Number of linearly independent Clebsch Gordan coefficients in various tensor products. Since the chiral structure differs for $D = 4, 8$ and $D = 6, 10$, these two cases are separated into different sets of columns

allows to reduce the singular behaviour of an n point function to smaller $n-1$ point functions.

The reason why the knowledge of leading z_{ij} singularities suffices to determine the full correlator is the following: Suppose we subtract all the singular pieces (i.e. the right hand side of (6.1.9)) from an unknown correlator (the left hand side of (6.1.9)), then the difference has no chance to diverge at finite values of z and w. On the other hand, correlation functions decay at infinite separations, $\lim_{z-w\to\infty}\langle \dots \phi_i(z)\phi_j(w)\dots\rangle = 0$, such that the aforementioned difference between $\langle \dots \phi_i(z)\phi_j(w)\dots\rangle$ and its leading singularities is also bounded at infinity.

Now we can apply Liouville's theorem from complex analysis: A holomorphic function which is bounded on the whole of $\mathbb{C}$ must be a constant. The difference between any $\psi^m, S_\alpha, S^{\dot\beta}$ correlator and its leading singularities was just argued to be bounded both at finite points $z_i \to z_j$ and at infinity $z_i \to \infty$, so this difference must be constant. Considering the behaviour at infinity determine this constant to be zero[3].

As an easy application of this algorithm, let us consider a four point function:

$$\langle\, \psi^m(z_1)\, \psi^n(z_2)\, S_A(z_3)\, S_B(z_4)\,\rangle \quad\to\quad \begin{cases} \dfrac{\eta^{mn}}{z_{12}}\,\langle\, S_A(z_3)\, S_B(z_4)\,\rangle & : \quad z_1 \to z_2 \\ \dfrac{\Gamma^m{}_A{}^C}{\sqrt{2}\, z_{13}^{1/2}}\,\langle\, \psi^n(z_2)\, S_C(z_3)\, S_B(z_4)\,\rangle & : \quad z_1 \to z_3 \\ \dfrac{\Gamma^m{}_B{}^C}{\sqrt{2}\, z_{14}^{1/2}}\,\langle\, \psi^n(z_2)\, S_A(z_3)\, S_C(z_4)\,\rangle & : \quad z_1 \to z_4 \end{cases} \tag{6.1.10}$$

The two- and three point correlators on the right hand are completely fixed by conformal invariance, e.g. $\langle S_A(z_3)\, S_B(z_4)\rangle = \frac{\mathcal{C}_{AB}}{z_{34}^{D/8}}$ and $\langle \psi^n(z_2) S_C(z_3) S_B(z_4)\rangle = \frac{(\Gamma^n \mathcal{C})_{CB}}{\sqrt{2}(z_{23}z_{24})^{1/2} z_{34}^{D/8-1/2}}$. Hence, (6.1.10) contains the complete information on z_1 singularities.

In order to match the three different regimes, we shall fix a basis for the tensor structures. A convenient ansatz is

$$\langle\, \psi^m(z_1)\, \psi^n(z_2)\, S_A(z_3)\, S_B(z_4)\,\rangle \;=\; \eta^{mn}\, \mathcal{C}_{AB}\, f(z_i) \;+\; \frac{1}{2}\, (\Gamma^m\, \Gamma^n\, \mathcal{C})_{AB}\, g(z_i)\;, \tag{6.1.11}$$

where $\eta^{mn}\mathcal{C}_{AB}$ $((\Gamma^m\,\Gamma^n\,\mathcal{C})_{AB})$ are directly generated as $z_1 \to z_2$ $(z_1 \to z_3)$. The last limit $z_1 \to z_4$, on the other hand, introduces a third tensor whose basis decomposition $(\Gamma^m\Gamma^n\mathcal{C})_{BA} = -(\Gamma^n\Gamma^m\mathcal{C})_{AB} = (\Gamma^m\Gamma^n\mathcal{C})_{AB} + 2\eta^{mn}\mathcal{C}_{AB}$ follows from the Dirac algebra. Hence, we have gathered

[3] There is a small loophole in this argument: In $D = 10$ dimensions, the subleading term in the OPE $S_\alpha(z) S^{\dot\beta}(w) \sim (z-w)^{-5/4} C_\alpha{}^{\dot\beta} - \frac{1}{4}(z-w)^{-1/4}(\gamma^{mn}C)_\alpha{}^{\dot\beta}\psi_m\psi_n + \dots$ is still singular. At first glance, it seems to be necessary to subtract as well. In all explicit computations performed, however, we did not run into any ambiguity upon neglecting the subleading singularity. So one could say that the leading term $\sim (z-w)^{-5/4} C_\alpha{}^{\dot\beta}$ was experimentally checked to suffice.

the following information on the functions f, g of the ansatz (6.1.11):

$$f(z_i) \quad \rightarrow \quad \begin{cases} z_{12}^{-1}\, z_{34}^{-D/8} & : \; z_1 \rightarrow z_2 \\ \text{regular} & : \; z_1 \rightarrow z_3 \\ (z_{14}\, z_{23}\, z_{24})^{-1/2}\, z_{34}^{1/2-D/8} & : \; z_1 \rightarrow z_4 \end{cases} \tag{6.1.12}$$

$$g(z_i) \quad \rightarrow \quad \begin{cases} \text{regular} & : \; z_1 \rightarrow z_2 \\ (z_{13}\, z_{23}\, z_{24})^{-1/2}\, z_{34}^{1/2-D/8} & : \; z_1 \rightarrow z_3 \\ (z_{14}\, z_{23}\, z_{24})^{-1/2}\, z_{34}^{1/2-D/8} & : \; z_1 \rightarrow z_4 \end{cases} \tag{6.1.13}$$

This is enough to uniquely determine

$$f(z_i) \;=\; \frac{z_{13}\, z_{24}}{z_{12}\, (z_{13}\, z_{14}\, z_{23}\, z_{24})^{1/2}\, z_{34}^{D/8}} \,, \qquad g(z_i) \;=\; \frac{1}{(z_{13}\, z_{14}\, z_{23}\, z_{24})^{1/2}\, z_{34}^{D/8-1}} \tag{6.1.14}$$

which leads to the final result

$$\langle\, \psi^m(z_1)\, \psi^n(z_2)\, S_A(z_3)\, S_B(z_4)\, \rangle \;=\; \frac{z_{34}^{1-D/8}}{(z_{13}\, z_{14}\, z_{23}\, z_{24})^{1/2}} \left(\frac{z_{13}\, z_{24}}{z_{12}\, z_{34}}\, \eta^{mn}\, \mathcal{C}_{AB} \;+\; \tfrac{1}{2}\, (\Gamma^m\, \Gamma^n\, \mathcal{C})_{AB} \right) . \tag{6.1.15}$$

Knowledge of this four point correlator allows to proceed to higher multiplicity and to compute various new limiting regimes of the five point function $\langle \psi^m(z_1) \psi^n(z_2) \psi^p(z_3) S_A(z_4) S_B(z_5) \rangle$. This show the recursive nature of this OPE based algorithm – computation of an n point function requires a set of $n-1$ (or $n-2$) point functions.

In general situations, one makes an ansatz

$$\begin{aligned} &\langle\, \psi^{m_1}(z_1) \ldots \psi^{m_n}(z_n)\, S_{\alpha_1}(x_1) \ldots S_{\alpha_p}(x_p)\, S^{\dot\beta_1}(y_1) \ldots S^{\dot\beta_q}(y_q)\, \rangle \\ &\qquad = \sum_{j=1}^{s(n,p,q)} f_j(z_i, x_i, y_i)\, T_j^{m_1 \ldots m_n}{}_{\alpha_1 \ldots \alpha_p}{}^{\dot\beta_1 \ldots \dot\beta_q} \end{aligned} \tag{6.1.16}$$

where $s(n,p,q)$ is the number of scalar representations in the tensor product of n vectors, p left handed spinors and q right handed spinors and $T_j^{m_1 \ldots m_n}{}_{\alpha_1 \ldots \alpha_p}{}^{\dot\beta_1 \ldots \dot\beta_q}$ is a basis of Clebsch Gordan coefficients. To determine the associated worldsheet functions $f_j(z_i, x_i, y_i)$ it is sufficient to pick one position, say z_1, and consider all the limits $z_1 \rightarrow z_{j \neq 1}, x_j, y_j$ like we did in the four point example above.

6.1.4 Theta functions and prime forms

The algorithm introduced in the previous subsection is most efficient to determine tree level correlation functions. Since this chapter also gathers g loop results – usually obtained with small extra effort – we shall introduce the relevant classes of worldsheet functions here which can take the periodicities of a genus g Riemann surface into account – generalized (modular) theta functions. Equipped with these periodic functions, we can develop an alternative algorithm which is equally well-suited to compute tree level- and loop correlators.

Generalized theta functions [223, 224, 225] are the natural objects to express correlation functions at nonzero genus. They assure the required periodicity along the homology cycles of the g loop string worldsheet. They can be derived from

$$\Theta(\vec{x}\,|\,\Omega) \;\;:=\;\; \sum_{\vec{n}\in\mathbb{Z}^g} \exp\left[2\pi i\,\left(\tfrac{1}{2}\,\vec{n}\,\Omega\,\vec{n} \;+\; \vec{n}\,\vec{x}\right)\right] \tag{6.1.17}$$

by shifting the first argument according to some spin structure $(\vec{a},\vec{b})$:

$$\begin{aligned}\Theta^{\vec{a}}_{\vec{b}}(\vec{x}\,|\,\Omega) \;\;&:=\;\; \exp\left[2\pi i\,\left(\tfrac{1}{8}\,\vec{a}\,\Omega\,\vec{a} \;+\; \tfrac{1}{2}\,\vec{a}\,\vec{x} \;+\; \tfrac{1}{4}\,\vec{a}\,\vec{b}\right)\right]\,\Theta\left(\vec{x}\;+\;\tfrac{\vec{b}}{2}\;+\;\tfrac{\Omega\,\vec{a}}{2}\,|\,\Omega\right)\\ &=\;\; \sum_{\vec{n}\in\mathbb{Z}^g} \exp\left[\pi i\,\left(\vec{n}\;+\;\tfrac{\vec{a}}{2}\right)\,\Omega\,\left(\vec{n}\;+\;\tfrac{\vec{a}}{2}\right)\;+\;2\pi i\,\left(\vec{n}\;+\;\tfrac{\vec{a}}{2}\right)\,\left(\vec{x}\;+\;\tfrac{\vec{b}}{2}\right)\right]\end{aligned} \tag{6.1.18}$$

In our situations, the g dimensional vectors $\vec{a},\vec{b}$ with entries zero or one characterize the periodicity of the fermion fields along the $2g$ homology cycles of the Riemann surface. The second argument of Θ is the $g\times g$ period matrix Ω which contains the N_b moduli at genus g, see subsection 5.1.3.

We parametrize the two-dimensional string world-sheet by a complex coordinate z. The Abel map $z\mapsto\int_p^z\vec{\omega}$ lifts z to the Jacobian variety of the world-sheet $\mathbb{C}^g/(\mathbb{Z}^g+\Omega\mathbb{Z}^g)$. These vectors of integrals are then natural arguments for the theta function. The periodicity properties of the theta function under transport of z around a homology cycle are summarized in appendix A of [4].

An important expression constructed out of the generalized theta functions is the prime form E,

$$E(z,w) \;\;:=\;\; \frac{\Theta^{\vec{a}_0}_{\vec{b}_0}\left(\int_w^z\vec{\omega}\,|\,\Omega\right)}{h^{\vec{a}_0}_{\vec{b}_0}(z)\,h^{\vec{a}_0}_{\vec{b}_0}(w)}\;, \tag{6.1.19}$$

where $(\vec{a_0},\vec{b_0})$ is an arbitrary odd spin structure (i.e. $\vec{a}_0\cdot\vec{b}_0$ is odd) such that $E(z,w)=-E(w,z)$. The half differentials $h^{\vec{a}_0}_{\vec{b}_0}$ in the denominator are defined by

$$h^{\vec{a}_0}_{\vec{b}_0}(z) \;\;:=\;\; \sqrt{\sum_{j=1}^{g}\omega_j(z)\,\partial_j\Theta^{\vec{a}_0}_{\vec{b}_0}\left(\vec{0}\,|\,\Omega\right)}\;. \tag{6.1.20}$$

They assure that E is independent of the choice of $(\vec{a}_0,\vec{b}_0)$ as long as it is odd. Given the leading behaviour $E(z,w)\sim z-w\;+\;\mathcal{O}\big((z-w)^3\big)$, singularities in correlation functions are caused by appropriate powers of prime forms. It turns out that, starting from a tree level correlator of ψ^m and S_A fields, all the z_{ij} have to be replaced by $E_{ij} := E(z_i, z_j)$ to capture the singular behaviour of the corresponding genus g correlation function. Only the spin structure dependent parts $\Theta^{\vec{a}}_{\vec{b}}$ are then left to be determined.

Considerable simplifications occur at $g = 1$, i.e. on the torus or the cylinder. The period matrix Ω then reduces to the modular parameter τ of the torus (or $\tau = it$ with $t \in \mathbb{R}$ in case of the cylinder), and the theta functions become the standard ones:

$$\theta_1 \;\equiv\; \Theta^1_1 \, , \qquad \theta_2 \;\equiv\; \Theta^1_0 \, , \qquad \theta_3 \;\equiv\; \Theta^0_0 \, , \qquad \theta_4 \;\equiv\; \Theta^0_1 \tag{6.1.21}$$

In particular, the prime form is proportional to the unique odd theta function

$$E_{ij} \Big|_{g=1} \;=\; \frac{\theta_1(z_{ij})}{\theta_1'(0)} \, . \tag{6.1.22}$$

We should point out one technicality about the interplay of the prime forms E_{ij} and theta functions $\Theta^{\vec{a}}_{\vec{b}}$: As a generalization of the crossing identity $z_{ij} z_{kl} = z_{ik} z_{jl} + z_{il} z_{kj}$ at tree level, the so-called Fay trisecant identities hold at higher genus. Their most general form is

$$\begin{aligned} &\Theta^{\vec{a}}_{\vec{b}} \left(\sum_{k=1}^{N} \int_{y_k}^{x_k} \vec{\omega} \,-\, \vec{e} \right) \, \left[\Theta^{\vec{a}}_{\vec{b}}(\vec{e}) \right]^{N-1} \, \frac{\prod_{i<j}^{N} E(x_i, x_j) \, E(y_i, y_j)}{\prod_{i,j=1}^{N} E(x_i, y_j)} \\ &\quad = \; (-1)^{N(N-1)/2} \, \det_{i,j} \left\{ E(x_i, y_j)^{-1} \, \Theta^{\vec{a}}_{\vec{b}} \left(\int_{y_j}^{x_i} \vec{\omega} \,-\, \vec{e} \right) \right\} \end{aligned} \tag{6.1.23}$$

with some arbitrary vector $\vec{e} \in \mathbb{C}^g$. We will mostly need the $N = 2$ case in the specialization $\vec{e} = \frac{1}{2} \sum_{k=1}^{2} \int_{y_k}^{x_k} \vec{\omega} - \vec{\Delta}$:

$$\begin{aligned} &E_{13} \, E_{24} \, \Theta^{\vec{a}}_{\vec{b}} \left(\tfrac{1}{2} \int_{z_2}^{z_1} \vec{\omega} \,+\, \tfrac{1}{2} \int_{z_4}^{z_3} \vec{\omega} \,+\, \vec{\Delta} \right) \, \Theta^{\vec{a}}_{\vec{b}} \left(\tfrac{1}{2} \int_{z_2}^{z_1} \vec{\omega} \,+\, \tfrac{1}{2} \int_{z_4}^{z_3} \vec{\omega} \,-\, \vec{\Delta} \right) \\ &= \; E_{12} \, E_{34} \, \Theta^{\vec{a}}_{\vec{b}} \left(\tfrac{1}{2} \int_{z_3}^{z_1} \vec{\omega} \,+\, \tfrac{1}{2} \int_{z_4}^{z_2} \vec{\omega} \,+\, \vec{\Delta} \right) \, \Theta^{\vec{a}}_{\vec{b}} \left(\tfrac{1}{2} \int_{z_3}^{z_1} \vec{\omega} \,+\, \tfrac{1}{2} \int_{z_4}^{z_2} \vec{\omega} \,-\, \vec{\Delta} \right) \\ &\quad + \, E_{14} \, E_{23} \, \Theta^{\vec{a}}_{\vec{b}} \left(\tfrac{1}{2} \int_{z_2}^{z_1} \vec{\omega} \,+\, \tfrac{1}{2} \int_{z_3}^{z_4} \vec{\omega} \,+\, \vec{\Delta} \right) \, \Theta^{\vec{a}}_{\vec{b}} \left(\tfrac{1}{2} \int_{z_2}^{z_1} \vec{\omega} \,+\, \tfrac{1}{2} \int_{z_3}^{z_4} \vec{\omega} \,-\, \vec{\Delta} \right) \end{aligned} \tag{6.1.24}$$

This identity is essential to combine several additive contributions to the functions $f_j(z_i)$ multiplying the basis tensors $T_j^{m_1 \ldots m_n}{}_{\alpha_1 \ldots \alpha_p}{}^{\dot{\beta}_1 \ldots \dot{\beta}_q}$ in (6.1.16).

6.1.5 Computing correlators in components

Bosonization becomes more subtle beyond tree level [226,227,228]: The sectors of different spin structures within the partition function of fermions are obtained from projecting the associated bosonic partition function onto sectors of certain soliton- or winding numbers. Only the sum over all the fermionic spin structures can establish equivalence to a bosonic theory with any winding numbers $(\vec{m},\vec{n}) \in \mathbb{Z}^g \times \mathbb{Z}^g$ around the $2g$ cycles of the maximal torus allowed.

For the purpose of loop computations, we will identify the covariant RNS fields in $D = 2n$ dimensions with n copies of an $SO(2)$ spin system $(\psi^{\pm}_j, s^{\pm}_j)$ according to

$$\psi^{\pm}_0 \;=\; \frac{1}{\sqrt{2}}\left(\pm\psi^0 \;+\; \psi^1\right), \qquad \psi^{\pm}_{j>0} \;=\; \frac{1}{\sqrt{2}}\left(\psi^{2j} \;\pm\; i\psi^{2j+1}\right) \tag{6.1.25}$$

$$S_{A=\left(\pm\frac{1}{2},\pm\frac{1}{2},\dots,\pm\frac{1}{2}\right)} \;=\; s^{\pm}_0\, s^{\pm}_1 \dots s^{\pm}_{n-1} \tag{6.1.26}$$

without making reference to the exponentials $\mathrm{e}^{\pm iH}$ and $\mathrm{e}^{\pm iH/2}$. The most general genus g correlation function of such a spin system has been given in [219]

$$\begin{aligned}
\left\langle \prod_{i=1}^{N_1} s^+(y_i) \prod_{j=1}^{N_2} s^-(z_j) \prod_{k=1}^{N_3} \psi^-(u_k) \prod_{l=1}^{N_4} \psi^+(v_l) \right\rangle^{\vec{a}}_{\vec{b}} \;&=\; \left(\frac{\prod_{r<s}^{N_1} E(y_r,y_s) \prod_{r<s}^{N_2} E(z_r,z_s)}{\prod_{i=1}^{N_1}\prod_{j=1}^{N_2} E(z_j,y_i)} \right)^{1/4} \\
\times \left(\frac{\prod_{r<s}^{N_3} E(u_r,u_s) \prod_{r<s}^{N_4} E(v_r,v_s)}{\prod_{k=1}^{N_3}\prod_{l=1}^{N_4} E(v_l,u_k)} \right) &\left(\frac{\prod_{j=1}^{N_2}\prod_{k=1}^{N_3} E(u_k,z_j) \prod_{i=1}^{N_1}\prod_{l=1}^{N_4} E(v_l,y_i)}{\prod_{i=1}^{N_1}\prod_{k=1}^{N_3} E(u_k,y_i) \prod_{j=1}^{N_2}\prod_{l=1}^{N_4} E(v_l,z_j)} \right)^{1/2} \\
\times\; \Theta^{\vec{a}}_{\vec{b}} \left(\frac{1}{2}\sum_{i=1}^{N_1} \int_p^{y_i} \vec{\omega} \;-\; \frac{1}{2}\sum_{j=1}^{N_2} \int_p^{z_j} \vec{\omega} \;-\; \sum_{k=1}^{N_3} \int_p^{u_k} \vec{\omega} \;+\; \sum_{l=1}^{N_4} \int_p^{v_l} \vec{\omega} \right) &\frac{\delta\left(\frac{1}{2}(N_1-N_2) - N_3 + N_4\right)}{\Theta^{\vec{a}}_{\vec{b}}(\vec{0})} \; .
\end{aligned} \tag{6.1.27}$$

Due to Ramond charge conservation $\frac{1}{2}(N_1 - N_2) - N_3 + N_4 = 0$, the arbitrary reference point p appearing in the Abel map drops out. In the following, we make use of the following shorthands for the spin structure dependent part:

$$\Theta^{\vec{a}}_{\vec{b}}\left(\frac{1}{2}\left[\int_{z_l}^{z_i} \vec{\omega} \;+\; \int_{z_m}^{z_j} \vec{\omega} \;+ \dots +\; \int_{z_n}^{z_k} \vec{\omega} \right] \right) \;=:\; \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} i & j & \dots & k \\ l & m & \dots & n \end{smallmatrix}\right] \tag{6.1.28}$$

Note in particular that the factor $\frac{1}{2}$ in the argument of $\Theta^{\vec{a}}_{\vec{b}}$ – which is ubiquitous in presence of spin fields – will always be implicit. Its origin lies in the fact that spin fields are responsible for changing the fermions' spin structure by opening or closing branch cuts. The fermion fields flip their sign when transported around these branch points, this can for instance be seen from the OPE (6.1.2). Translating a spin field once around a cycle extends the branch cut all the way across and thus changes the fermion spin structure $(\vec{a},\vec{b})$. The identity

$$\Theta^{\vec{a}}_{\vec{b}}(\vec{x} \,+\, \vec{t} + \Omega\,\vec{s}) \;=\; \exp\Big[-i\pi\left(\vec{s}\,\Omega\,\vec{s} \;+\; \vec{s}\,(2\,\vec{x} \,+\, \vec{b} \,+\, 2\,\vec{t})\right)\Big]\,\Theta^{\vec{a}+2\vec{s}}_{\vec{b}+2\vec{t}}(\vec{x}) \tag{6.1.29}$$

guarantees that the factor of $\frac{1}{2}$ in (6.1.28) leads to the required pseudoperiodicity property of spin fields.

6.1.6 The explicit algorithm

Equipped with the prerequisites on generalized theta functions from the previous subsections, we can now present a second algorithm which admits to compute higher loop correlators of ψ^m

and S_A in $D = 2n$ explicitly rather than recursively. It has already been used in [219, 220] for a limited number of examples.

The idea is to calculate the correlation function $\langle \psi^{m_1}...\psi^{m_n} S_{\alpha_1}...S_{\alpha_p} S^{\dot\beta_1}...S^{\dot\beta_q}\rangle^{\vec a}_{\vec b}$ for specific choices of m_i, α_i and $\dot\beta_i$ by organizing the RNS fields into their spin system content via (6.1.25) and (6.1.26). Evaluating the n individual $SO(2)$ correlators is simply a matter of (6.1.27). The final goal is to express the results in a covariant form, i.e. in terms of Clebsch Gordan coefficients built from gamma matrices and the charge conjugation matrix. As we have explained in section 2.2, they can be viewed as $SO(1, 2n-1)$ covariant Ramond charge conserving delta functions, schematically $\mathcal{C}_{AB} \sim \delta(A+B)$ and $(\Gamma^m \mathcal{C})_{AB} \sim \delta(m+A+B)$ where m, A, B are treated as n component Ramond charge vectors such as $m \equiv (0, \pm 1, 0, \dots, 0)$ and $A \equiv (\pm\frac{1}{2}, \dots, \pm\frac{1}{2})$.

Once again, we start from the ansatz (6.1.16) for the correlation function with a minimal set of Clebsch Gordan coefficients $T_j^{m_1 \dots m_n}{}_{\alpha_1\dots\alpha_p}{}^{\dot\beta_1\dots\dot\beta_q}$. Each of these tensors is accompanied by a z dependent coefficient f_j consisting of prime forms E and theta functions $\Theta^{\vec a}_{\vec b}$. The results obtained for special choices of m_i, α_i and $\dot\beta_i$ have to be matched with this ansatz. It is most economic to first look at configurations $(m_i, \alpha_i, \dot\beta_i)$ where only one tensor is non-zero. Then the loop-level result (6.1.27) directly yields the coefficient f_j for the respective T_j.

In some cases, however, it is not possible to isolate one Clebsch Gordan coefficient and to make the others vanish. If sums of more than one f_j are computed from (6.1.27) for every choice of $(m_i, \alpha_i, \dot\beta_i)$, it can be helpful to switch to different Lorenz tensors which are (anti-)symmetric in some vector- or spinor indices, see appendix B of [5]. In other cases, Fay's trisecant identities (6.1.23) and (6.1.24) have to be used to determine the unknown coefficients [4]. Sign issues can be resolved by matching certain limits $z_i \to z_j$ at tree-level with the RNS OPEs (6.1.1) to (6.1.3).

Let us illustrate this procedure with an easy example, the five point correlation function $\langle \psi^m \psi^n \psi^p S_\alpha S_\beta \rangle^{\vec a}_{\vec b}$ in $D = 6$ dimensions. A convenient ansatz in terms of four Clebsch Gordan coefficients is

$$\begin{aligned} \langle\, \psi^m(z_1)\, \psi^n(z_2)\, \psi^p(z_3)\, S_\alpha(z_4)\, S_\beta(z_5)\,\rangle^{\vec a}_{\vec b} \;&=\; F_1(z)\,(\gamma^{mnp}\, C)_{\alpha\beta} \\ &+\; F_2(z)\,\eta^{mn}\,(\gamma^p\, C)_{\alpha\beta} \;+\; F_3(z)\,\eta^{mp}\,(\gamma^n\, C)_{\alpha\beta} \;+\; F_4(z)\,\eta^{np}\,(\gamma^m\, C)_{\alpha\beta}\,. \end{aligned} \tag{6.1.30}$$

The task is now to determine F_1, F_2, F_3 and F_4 by making several choices for m, n, p, α, β.

The coefficient F_1 immediately follows from $m = 0$, $n = 2$, $p = 4$ because the metric η is diagonal and therefore all tensors except for γ^{mnp} vanish in this configuration. By means of (6.1.25) and (6.1.26), the NS fermions become $(\psi^m, \psi^n, \psi^p) = \frac{1}{\sqrt{2}}\,(\psi_0^+ - \psi_0^-, \psi_1^+ + \psi_1^-, \psi_2^+ + \psi_2^-)$, and we choose $S_\alpha = S_\beta = (s_0^+, s_1^-, s_2^-)$ for the spin fields. The left hand side of (6.1.30) becomes a product of three independent spin system correlators:

$$\frac{1}{\sqrt{2}^3} \, \langle \, (\psi_0^+ - \psi_0^-)(z_1) \, s_0^+(z_4) \, s_0^+(z_5) \, \rangle \, \langle \, (\psi_1^+ + \psi_1^-)(z_2) \, s_1^-(z_4) \, s_1^-(z_5) \, \rangle \, \langle \, (\psi_2^+ + \psi_2^-)(z_3) \, s_2^-(z_4) \, s_2^-(z_5) \, \rangle \tag{6.1.31}$$

Ramond charge conservation in the individual spin systems makes $\psi_0^+(z_1)$, $\psi_1^-(z_2)$ and $\psi_2^-(z_3)$ drop out and (6.1.27) yields the coefficient F_1 up to a sign

$$F_1 \;\; = \;\; \pm \, \frac{\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1 & 1\\ 4 & 5\end{smallmatrix}\right] \, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2 & 2\\ 4 & 5\end{smallmatrix}\right] \, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3 & 3\\ 4 & 5\end{smallmatrix}\right] \, E_{45}^{3/4}}{2\sqrt{2}\, \left[\Theta^{\vec a}_{\vec b}(\vec 0)\right]^3 \, (E_{14}\, E_{15}\, E_{24}\, E_{25}\, E_{34}\, E_{35})^{1/2}} \; . \tag{6.1.32}$$

The coefficient F_2 can be determined in a similar way by setting $m = n = 0$, $p = 2$ and $S_\alpha = (s_0^+, s_1^+, s_2^+)$, $S_\beta = (s_0^-, s_1^+, s_2^-)$. No other tensor than $\eta^{mn}(\gamma^p C)_{\alpha\beta}$ contributes as $\eta^{02} = 0$ and γ^{mnp} is totally antisymmetric. One finds that the results consists of two terms due to the two inequivalent fermion configurations $\psi_0^+(z_1)\,\psi_0^-(z_2)$ and $\psi_0^-(z_1)\,\psi_0^+(z_2)$ in the first spin system:

$$F_2 \;\; = \;\; \pm \, \frac{\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3 & 3\\ 4 & 5\end{smallmatrix}\right] \, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}4\\ 5\end{smallmatrix}\right] \, \left(E_{14}\, E_{25}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1 & 1 & 4\\ 2 & 2 & 5\end{smallmatrix}\right] \; + \; E_{15}\, E_{24}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1 & 1 & 5\\ 2 & 2 & 4\end{smallmatrix}\right]\right)}{2\sqrt{2}\, \left[\Theta^{\vec a}_{\vec b}(\vec 0)\right]^3 \, E_{12}\, (E_{14}\, E_{15}\, E_{24}\, E_{25}\, E_{34}\, E_{35})^{1/2} \, E_{45}^{1/4}} \tag{6.1.33}$$

The remaining z_i functions F_3 and F_4 follow from F_2 by permutation in the vector indices and the $(1,2,3)$ labels.

The signs of the individual coefficients are easily fixed by requiring $\frac{\eta^{mn}}{z_{12}} \langle \psi^p(z_3) S_\alpha(z_4) S_\beta(z_5) \rangle^{\vec a}_{\vec b}$ to emerge in the $z_1 \to z_2$ limit or $\frac{(\gamma^q C)_{\alpha\beta}}{z_{45}^{1/4}} \langle \psi^m(z_1) \psi^n(z_2)\, \psi^p(z_3) \psi_q(z_5) \rangle^{\vec a}_{\vec b}$ in the $z_4 \to z_5$ limit for instance.

6.1.7 Comparing the two algorithms

Both of the two algorithms presented in subsections 6.1.3 and 6.1.6 have their individual advantages and drawbacks. The explicit method is of course quicker to obtain a specific higher point correlator without systematic study of its lower order relatives. Another strength of this algorithm is an easy determination of the spin structure dependent parts $\Theta^{\vec a}_{\vec b}(...)$ – these are quite awkward to find in the recursive approach.

However, it is a notorious trouble spot about RNS correlators to fix the relative signs of the individual terms. The explicit approach can say little about these signs because the formula (6.1.27) for the individual spin system contributions can only be trusted up to a complex phase. It is therefore advisable to fix signs by means of the recursive method: The limits $z_1 \to z_2$ and $z_{n-1} \to z_n$ for neighbouring fields $\langle \phi_1 \phi_2 \ldots \rangle$ ($\langle \ldots \phi_{n-1} \phi_n \rangle$) on the left (right) end of the correlator give reliable absolute signs, and further relative signs can be fixed by any other $z_i \to z_j$ limit. The reason to single out $z_1 \to z_2$ and $z_{n-1} \to z_n$ lies in the uncontrolable complex phases which emerge from moving ψ^m- and S_A fields across each other.

In practical computations of [3, 4, 5], we have often started with the recursive algorithm to obtain a tree level correlator taking good care of all its signs. Then, the explicit method became particularly helpful to determine the missing $\Theta^{\vec{a}}_{\vec{b}}$ functions at $g \geq 1$. The E_{ij} factors arising from (6.1.27) provide consistency checks for to the previously computed tree level limit.

One challenge is common to both algorithms: They both require a basis of Clebsch Gordan coefficients for the tensor structure of interest. Finding a convenient basis is a representation theoretic task which can become quite cumbersome for a large number of indices. It is hard to say in advance which choice of basis is better suited to simplify the computation.

6.2 Correlators with two spin fields in $D = 2n$ dimensions

This section lists the results for correlators $\langle \prod_{j=1}^k \psi^{m_j} S_A S_B \rangle$ with two spin fields and any number k of fermions. They were obtained in [3,4,5] by means of the algorithms of the previous section. They are of phenomenological relevance because g loop scattering of n gluons with a quark-antiquark pair or two gauginos requires correlators $\langle \psi^{2g+2n-1} S_A S_B \rangle^{\vec{a}}_{\vec{b}}$ in the spacetime SCFT. Still, their group structure admits to write them down universally for any even number of spacetime dimensions.

6.2.1 Examples with finite number of fields

Let us first of all display some lower order correlators $\langle \prod_{j=1}^k \psi^{m_j} S_A S_B \rangle$ with two up to $k = 4$ fermions in order to get a feeling of the structure of this correlator class. The loop completion of the two- and three point tree level correlation functions reads as follows:

$$\langle S_A(z_1)\, S_B(z_2) \rangle^{\vec{a}}_{\vec{b}} = \frac{\Theta^{\vec{a}}_{\vec{b}} \left[{}^1_2 \right]^{D/2}}{\left(\Theta^{\vec{a}}_{\vec{b}}(\vec{0}) \right)^{D/2} E_{12}^{D/8}} \tag{6.2.34}$$

$$\langle \psi^m(z_1)\, S_A(z_2)\, S_B(z_3) \rangle^{\vec{a}}_{\vec{b}} = \frac{\Theta^{\vec{a}}_{\vec{b}} \left[{}^2_3 \right]^{D/2-1} (\Gamma^m \mathcal{C})_{AB}\, \Theta^{\vec{a}}_{\vec{b}} \left[{}^{2\ 3}_{1\ 1} \right]}{\sqrt{2} \left(\Theta^{\vec{a}}_{\vec{b}}(\vec{0}) \right)^{D/2} (E_{12}\, E_{13})^{1/2}\, E_{23}^{D/8-1/2}} \tag{6.2.35}$$

The four point function is given in (6.1.15) at tree level. At higher genus, it generalizes to

$$\langle \psi^m(z_1)\, \psi^n(z_2)\, S_A(z_2)\, S_B(z_3) \rangle^{\vec{a}}_{\vec{b}} = \frac{\Theta^{\vec{a}}_{\vec{b}} \left[{}^3_4 \right]^{D/2-1}}{\left(\Theta^{\vec{a}}_{\vec{b}}(\vec{0}) \right)^{D/2} (E_{13}\, E_{14}\, E_{23}\, E_{24})^{1/2}\, E_{34}^{D/8}}$$

$$\times \left\{ \frac{E_{13}\, E_{24}}{E_{12}}\, \eta^{mn}\, \mathcal{C}_{AB}\, \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 1 & 1 & 3 \\ 2 & 2 & 4 \end{smallmatrix} \right] \;+\; \frac{E_{34}}{2\, \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \right]}\, (\Gamma^m\, \Gamma^n\, \mathcal{C})_{AB}\, \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 3 & 4 \\ 1 & 1 \end{smallmatrix} \right] \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 3 & 4 \\ 2 & 2 \end{smallmatrix} \right] \right\} . \tag{6.2.36}$$

Applying the gamma matrix identity $\Gamma^{mnp} = \Gamma^m \Gamma^n \Gamma^p + \Gamma^m \eta^{np} - \Gamma^n \eta^{mp} + \Gamma^p \eta^{mn}$ to the example in subsection 6.1.6 gives the more compact result

$$\begin{aligned} \langle\, \psi^m(z_1)\, \psi^n(z_2)\, \psi^p(z_3)\, S_A(z_4)\, S_B(z_5)\, \rangle^{\vec a}_{\vec b} \;&=\; \frac{\Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \right]^{D/2-2} \left(\Theta^{\vec a}_{\vec b}(\vec 0) \right)^{-D/2}}{\sqrt{2}\, (E_{14}\, E_{15}\, E_{24}\, E_{25}\, E_{34}\, E_{35})^{1/2}\, E_{45}^{D/8-1/2}} \\ \times \Bigg\{ \frac{E_{14}\, E_{25}}{E_{12}}\, \eta^{mn}\, (\Gamma^p\, \mathcal{C})_{AB}\, \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 1 & 1 & 4 \\ 2 & 2 & 5 \end{smallmatrix} \right] \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 4 & 5 \\ 3 & 3 \end{smallmatrix} \right] &- \frac{E_{14}\, E_{35}}{E_{13}}\, \eta^{mp}\, (\Gamma^n\, \mathcal{C})_{AB}\, \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 1 & 1 & 4 \\ 3 & 3 & 5 \end{smallmatrix} \right] \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 4 & 5 \\ 2 & 2 \end{smallmatrix} \right] \\ + \frac{E_{24}\, E_{35}}{E_{23}}\, \eta^{np}\, (\Gamma^m\, \mathcal{C})_{AB}\, \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 2 & 2 & 4 \\ 3 & 3 & 5 \end{smallmatrix} \right] \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 4 & 5 \\ 1 & 1 \end{smallmatrix} \right] &+ \frac{E_{45}\, (\Gamma^m\, \Gamma^n\, \Gamma^p\, \mathcal{C})_{AB}}{2\, \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \right]}\, \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 4 & 5 \\ 1 & 1 \end{smallmatrix} \right] \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 4 & 5 \\ 2 & 2 \end{smallmatrix} \right] \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 4 & 5 \\ 3 & 3 \end{smallmatrix} \right] \Bigg\} . \end{aligned} \tag{6.2.37}$$

As a final example, we state the six point function:

$$\begin{aligned} \langle\, \psi^m(z_1)\, \psi^n(z_2)\, \psi^p(z_3)\, \psi^q(z_4)\, S_A(z_5)\, S_B(z_6)\, \rangle^{\vec a}_{\vec b} \;&=\; \frac{\Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 5 \\ 6 \end{smallmatrix} \right]^{D/2-2} \left(\Theta^{\vec a}_{\vec b}(\vec 0) \right)^{-D/2}}{(E_{15}\, E_{16}\, E_{25}\, E_{26}\, E_{35}\, E_{36}\, E_{45}\, E_{46})^{1/2}\, E_{56}^{D/8}} \\ \times \Bigg\{ &\frac{E_{15}\, E_{26}\, E_{35}\, E_{46}}{E_{12}\, E_{34}}\, \eta^{mn}\, \eta^{pq}\, \mathcal{C}_{AB}\, \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 1 & 1 & 5 \\ 2 & 2 & 6 \end{smallmatrix} \right] \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 3 & 3 & 5 \\ 4 & 4 & 6 \end{smallmatrix} \right] \\ &- \frac{E_{15}\, E_{36}\, E_{25}\, E_{46}}{E_{13}\, E_{24}}\, \eta^{mp}\, \eta^{nq}\, \mathcal{C}_{AB}\, \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 1 & 1 & 5 \\ 3 & 3 & 6 \end{smallmatrix} \right] \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 2 & 2 & 5 \\ 4 & 4 & 6 \end{smallmatrix} \right] \\ &+ \frac{E_{15}\, E_{46}\, E_{25}\, E_{36}}{E_{14}\, E_{23}}\, \eta^{mq}\, \eta^{np}\, \mathcal{C}_{AB}\, \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 1 & 1 & 5 \\ 4 & 4 & 6 \end{smallmatrix} \right] \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 2 & 2 & 5 \\ 3 & 3 & 6 \end{smallmatrix} \right] \\ &+ \frac{E_{56}}{2\, \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 5 \\ 6 \end{smallmatrix} \right]} \Bigg[\frac{E_{15}\, E_{26}}{E_{12}}\, \eta^{mn} (\Gamma^p\, \Gamma^q\, \mathcal{C})_{AB}\, \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 1 & 1 & 5 \\ 2 & 2 & 6 \end{smallmatrix} \right] \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 5 & 6 \\ 3 & 3 \end{smallmatrix} \right] \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 5 & 6 \\ 4 & 4 \end{smallmatrix} \right] \\ &\qquad + \frac{E_{35}\, E_{46}}{E_{34}}\, \eta^{pq} (\Gamma^m\, \Gamma^n\, \mathcal{C})_{AB}\, \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 3 & 3 & 5 \\ 4 & 4 & 6 \end{smallmatrix} \right] \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 5 & 6 \\ 1 & 1 \end{smallmatrix} \right] \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 5 & 6 \\ 2 & 2 \end{smallmatrix} \right] \\ &\qquad - \frac{E_{15}\, E_{36}}{E_{13}}\, \eta^{mp} (\Gamma^n\, \Gamma^q\, \mathcal{C})_{AB}\, \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 1 & 1 & 5 \\ 3 & 3 & 6 \end{smallmatrix} \right] \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 5 & 6 \\ 2 & 2 \end{smallmatrix} \right] \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 5 & 6 \\ 4 & 4 \end{smallmatrix} \right] \\ &\qquad - \frac{E_{25}\, E_{46}}{E_{24}}\, \eta^{nq} (\Gamma^m\, \Gamma^p\, \mathcal{C})_{AB}\, \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 2 & 2 & 5 \\ 4 & 4 & 6 \end{smallmatrix} \right] \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 5 & 6 \\ 1 & 1 \end{smallmatrix} \right] \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 5 & 6 \\ 3 & 3 \end{smallmatrix} \right] \\ &\qquad + \frac{E_{15}\, E_{46}}{E_{14}}\, \eta^{mq} (\Gamma^n\, \Gamma^p\, \mathcal{C})_{AB}\, \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 1 & 1 & 5 \\ 4 & 4 & 6 \end{smallmatrix} \right] \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 5 & 6 \\ 2 & 2 \end{smallmatrix} \right] \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 5 & 6 \\ 3 & 3 \end{smallmatrix} \right] \\ &\qquad + \frac{E_{25}\, E_{36}}{E_{23}}\, \eta^{np} (\Gamma^m\, \Gamma^q\, \mathcal{C})_{AB}\, \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 2 & 2 & 5 \\ 3 & 3 & 6 \end{smallmatrix} \right] \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 5 & 6 \\ 1 & 1 \end{smallmatrix} \right] \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 5 & 6 \\ 4 & 4 \end{smallmatrix} \right] \Bigg] \\ &+ \left(\frac{E_{56}}{2\, \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 5 \\ 6 \end{smallmatrix} \right]} \right)^2 (\Gamma^m\, \Gamma^n\, \Gamma^p\, \Gamma^q\, \mathcal{C})_{AB}\, \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 5 & 6 \\ 1 & 1 \end{smallmatrix} \right] \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 5 & 6 \\ 2 & 2 \end{smallmatrix} \right] \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 5 & 6 \\ 3 & 3 \end{smallmatrix} \right] \Theta^{\vec a}_{\vec b} \left[\begin{smallmatrix} 5 & 6 \\ 4 & 4 \end{smallmatrix} \right] \Bigg\} \end{aligned} \tag{6.2.38}$$

The corresponding seven point function can be found in [4] for $D = 4$.

6.2.2 The n point formula

The cleanest way to present the generalization of the lower point results (6.2.34) to (6.2.38) to arbitrary numbers of ψ^m insertions distinguishes between even and odd fermion numbers:

$$\begin{aligned}
\Omega_{(n,D)}^{m_1 \dots m_{2n-1}}{}_{KL}(z_i) \quad &:= \quad \langle\, \psi^{m_1}(z_1)\, \psi^{m_2}(z_2) \dots \psi^{m_{2n-1}}(z_{2n-1})\, S_K(z_A)\, S_L(z_B)\, \rangle_{\vec{b}}^{\vec{a}} \\
= \quad & \frac{\left[\Theta_{\vec{b}}^{\vec{a}} \left(\frac{1}{2} \int_{z_B}^{z_A} \vec{\omega} \right) \right]^{D/2-n}}{\sqrt{2}\, \left[\, \Theta_{\vec{b}}^{\vec{a}}(\vec{0})\, \right]^{D/2} E_{AB}^{D/8-1/2}\, \prod_{i=1}^{2n-1} (E_{iA}\, E_{iB})^{1/2}} \sum_{\ell=0}^{n-1} \left(\frac{E_{AB}}{2\, \Theta_{\vec{b}}^{\vec{a}} \left(\frac{1}{2} \int_{z_B}^{z_A} \vec{\omega} \right)} \right)^{\ell} \\
\times & \sum_{\rho \in S_{2n-1}/\mathcal{P}_{n,\ell}} \operatorname{sgn}(\rho) \left(\Gamma^{m_{\rho(1)}}\, \Gamma^{m_{\rho(2)}} \dots \Gamma^{m_{\rho(2\ell)}}\, \Gamma^{m_{\rho(2\ell+1)}}\, \mathcal{C} \right)_{KL} \prod_{k=1}^{2\ell+1} \Theta_{\vec{b}}^{\vec{a}} \left(\tfrac{1}{2} \int_{z_{\rho(k)}}^{z_A} \vec{\omega} \, + \, \tfrac{1}{2} \int_{z_{\rho(k)}}^{z_B} \vec{\omega} \right) \\
\times & \prod_{j=1}^{n-\ell-1} \frac{\eta^{m_{\rho(2\ell+2j)} m_{\rho(2\ell+2j+1)}}}{E_{\rho(2\ell+2j),\rho(2\ell+2j+1)}}\, E_{\rho(2\ell+2j),A}\, E_{\rho(2\ell+2j+1),B}\, \Theta_{\vec{b}}^{\vec{a}} \left(\int_{z_{\rho(2\ell+2j+1)}}^{z_{\rho(2\ell+2j)}} \vec{\omega} \, + \, \tfrac{1}{2} \int_{z_B}^{z_A} \vec{\omega} \right) , \qquad (6.2.39)
\end{aligned}$$

$$\begin{aligned}
\omega_{(n,D)}^{m_1 \dots m_{2n-2}}{}_{KL}(z_i) \quad &:= \quad \langle\, \psi^{m_1}(z_1)\, \psi^{m_2}(z_2) \dots \psi^{m_{2n-2}}(z_{2n-2})\, S_K(z_A)\, S_L(z_B)\, \rangle_{\vec{b}}^{\vec{a}} \\
= \quad & \frac{\left[\Theta_{\vec{b}}^{\vec{a}} \left(\frac{1}{2} \int_{z_B}^{z_A} \vec{\omega} \right) \right]^{D/2+1-n}}{\left[\, \Theta_{\vec{b}}^{\vec{a}}(\vec{0})\, \right]^{D/2} E_{AB}^{D/8}\, \prod_{i=1}^{2n-2} (E_{iA}\, E_{iB})^{1/2}} \sum_{\ell=0}^{n-1} \left(\frac{E_{AB}}{2\, \Theta_{\vec{b}}^{\vec{a}} \left(\frac{1}{2} \int_{z_B}^{z_A} \vec{\omega} \right)} \right)^{\ell} \\
\times & \sum_{\rho \in S_{2n-2}/\mathcal{Q}_{n,\ell}} \operatorname{sgn}(\rho) \left(\Gamma^{m_{\rho(1)}}\, \Gamma^{m_{\rho(2)}} \dots \Gamma^{m_{\rho(2\ell)}}\, \mathcal{C} \right)_{KL} \prod_{k=1}^{2\ell} \Theta_{\vec{b}}^{\vec{a}} \left(\tfrac{1}{2} \int_{z_{\rho(k)}}^{z_A} \vec{\omega} \, + \, \tfrac{1}{2} \int_{z_{\rho(k)}}^{z_B} \vec{\omega} \right) \\
\times & \prod_{j=1}^{n-\ell-1} \frac{\eta^{m_{\rho(2\ell+2j-1)} m_{\rho(2\ell+2j)}}}{E_{\rho(2\ell+2j-1),\rho(2\ell+2j)}}\, E_{\rho(2\ell+2j-1),A}\, E_{\rho(2\ell+2j),B}\, \Theta_{\vec{b}}^{\vec{a}} \left(\int_{z_{\rho(2\ell+2j)}}^{z_{\rho(2\ell+2j-1)}} \vec{\omega} \, + \, \tfrac{1}{2} \int_{z_B}^{z_A} \vec{\omega} \right) \qquad (6.2.40)
\end{aligned}$$

The summation ranges $\rho \in S_{2n-1}/\mathcal{P}_{n,\ell}$ and $\rho \in S_{2n-2}/\mathcal{Q}_{n,\ell}$ certainly require some explanation. The conventions are taken from [3] where a more exhaustive presentation can be found. Formally, we define

$$\begin{aligned}
S_{2n-1}/\mathcal{P}_{n,\ell} \;\equiv\; \Big\{ \rho \in S_{2n-1} :\; & \rho(1) < \rho(2) < \dots < \rho(2\ell+1) \;, \\
& \rho(2\ell+2j) < \rho(2\ell+2j+1) \;\forall\; j = 1, 2, \dots, n-\ell-1 \;, \\
& \rho(2\ell+3) < \rho(2\ell+5) < \dots < \rho(2n-1) \Big\} \;, \qquad (6.2.41) \\
S_{2n-2}/\mathcal{Q}_{n,\ell} \;\equiv\; \Big\{ \rho \in S_{2n-2} :\; & \rho(1) < \rho(2) < \dots < \rho(2\ell) \;, \\
& \rho(2\ell+2j-1) < \rho(2\ell+2j) \;\forall\; j = 1, 2, \dots, n-\ell-1 \;, \\
& \rho(2\ell+2) < \rho(2\ell+4) < \dots < \rho(2n-2) \Big\} \;. \qquad (6.2.42)
\end{aligned}$$

In other words, the sums over $S_{2n-1}/\mathcal{P}_{n,\ell}$- and $S_{2n-2}/\mathcal{Q}_{n,\ell}$ in (6.2.39) and (6.2.40) run over those permutations ρ of $(1, 2, \dots, 2n-1)$ or $(1, 2, \dots, 2n-2)$ which satisfy the following constraints:

- Only ordered Γ products are summed over: The indices $m_{\rho(i)}$ attached to a chain of Γ

matrices are increasingly ordered, e.g. whenever the product $\Gamma^{m_{\rho(i)}}\Gamma^{m_{\rho(j)}}\Gamma^{m_{\rho(k)}}$ appears, the sub-indices satisfy $\rho(i) < \rho(j) < \rho(k)$.

- On each metric $\eta^{m_{\rho(i)}m_{\rho(j)}}$ the first index is the "lower" one, i.e. $\rho(i) < \rho(j)$.
- Products of several η's are not double counted. So once we get $\eta^{m_{\rho(i)}m_{\rho(j)}}\eta^{m_{\rho(k)}m_{\rho(l)}}$, the term $\eta^{m_{\rho(k)}m_{\rho(l)}}\eta^{m_{\rho(i)}m_{\rho(j)}}$ does not appear.

These restrictions on the occurring S_{2n-1} (or S_{2n-2}) elements are abbreviated by a quotient $\mathcal{P}_{n,\ell}$ and $\mathcal{Q}_{n,\ell}$. The subgroups removed from S_{2n-1} (S_{2n-2}) are $S_{2\ell+1} \times S_{n-\ell-1} \times (S_2)^{n-\ell-1}$ and $S_{2\ell} \times S_{n-\ell-1} \times (S_2)^{n-\ell-1}$ respectively, therefore the number of terms in (6.2.39) and (6.2.40) at fixed (n, ℓ) is given by

$$\begin{aligned} \big|S_{2n-1}/\mathcal{P}_{n,\ell}\big| \ &= \ \frac{(2n\,-\,1)!}{(2\ell\,+\,1)!\,(n\,-\,\ell\,-\,1)!\,2^{n-\ell-1}}\ , &(6.2.43)\\ \big|S_{2n-2}/\mathcal{Q}_{n,\ell}\big| \ &= \ \frac{(2n\,-\,2)!}{(2\ell)!\,(n\,-\,\ell\,-\,1)!\,2^{n-\ell-1}}\ . &(6.2.44)\end{aligned}$$

To have some easy examples, let us explicitly evaluate the sums over $S_{2n-1}/\mathcal{P}_{n,\ell}$ and $S_{2n-2}/\mathcal{Q}_{n,\ell}$ occurring in the five- and six-point functions $\langle\psi^m\psi^n\psi^p S_A S_B\rangle$ and $\langle\psi^m\psi^n\psi^p\psi^q S_A S_B\rangle$. The formula (6.2.39), applied to $n = 2$, schematically tell us that (up to a z_i dependent pre-factor)

$$\begin{aligned} \langle \psi^{m_1}\,\psi^{m_2}\,\psi^{m_3}\,S_A\,S_B\rangle \ &\sim \ \sum_{\ell=0}^{1}\ \sum_{\rho\in S_3/\mathcal{P}_{2,\ell}} \mathrm{sgn}(\rho)\,\big(\Gamma^{m_{\rho(1)}}\dots\Gamma^{m_{\rho(2\ell+1)}}\,\mathcal{C}\big)_{AB}\,f^{\rho}_{\ell}(z_i)\\ &\sim \ \underbrace{(\Gamma^{m_3}\,\mathcal{C})_{AB}\,\eta^{m_1m_2}\,f^{(312)}_{\ell=0} \ - \ (\Gamma^{m_2}\,\mathcal{C})_{AB}\,\eta^{m_1m_3}\,f^{(213)}_{\ell=0} \ + \ (\Gamma^{m_1}\,\mathcal{C})_{AB}\,\eta^{m_2m_3}\,f^{(123)}_{\ell=0}}_{\rho\in S_3/\mathcal{P}_{2,0}}\\ &\quad + \ \underbrace{(\Gamma^{m_1}\,\Gamma^{m_2}\,\Gamma^{m_3}\,\mathcal{C})_{AB}\,f^{(123)}_{\ell=1}}_{\rho\in S_3/\mathcal{P}_{2,1}}\,, \end{aligned} \tag{6.2.45}$$

with z_{ij} dependencies

$$\begin{aligned} f^{(312)}_{\ell=0} \ &= \ \frac{E_{14}\,E_{25}}{E_{12}}\,\Theta^{\vec a}_{\vec b}\big[\begin{smallmatrix}4&5\\3&3\end{smallmatrix}\big]\,\Theta^{\vec a}_{\vec b}\big[\begin{smallmatrix}1&1&4\\2&2&5\end{smallmatrix}\big]\ , \qquad & f^{(213)}_{\ell=0} \ &= \ \frac{E_{14}\,E_{35}}{E_{13}}\,\Theta^{\vec a}_{\vec b}\big[\begin{smallmatrix}4&5\\2&2\end{smallmatrix}\big]\,\Theta^{\vec a}_{\vec b}\big[\begin{smallmatrix}1&1&4\\3&3&5\end{smallmatrix}\big] \\ f^{(123)}_{\ell=0} \ &= \ \frac{E_{24}\,E_{35}}{E_{23}}\,\Theta^{\vec a}_{\vec b}\big[\begin{smallmatrix}4&5\\1&1\end{smallmatrix}\big]\,\Theta^{\vec a}_{\vec b}\big[\begin{smallmatrix}2&2&4\\3&3&5\end{smallmatrix}\big]\ , \qquad & f^{(123)}_{\ell=1} \ &= \ \frac{E_{45}\,\Theta^{\vec a}_{\vec b}\big[\begin{smallmatrix}4&5\\1&1\end{smallmatrix}\big]\,\Theta^{\vec a}_{\vec b}\big[\begin{smallmatrix}4&5\\2&2\end{smallmatrix}\big]\,\Theta^{\vec a}_{\vec b}\big[\begin{smallmatrix}4&5\\3&3\end{smallmatrix}\big]}{2\,\Theta^{\vec a}_{\vec b}\big[\begin{smallmatrix}4\\5\end{smallmatrix}\big]}\ . \end{aligned} \tag{6.2.46}$$

Equation (6.2.40) for $n = 3$ is expanded as

$$\begin{aligned} \langle\psi^{m_1}\,\psi^{m_2}\,\psi^{m_3}\,\psi^{m_4}\,S_A\,S_B\rangle \ &\sim \ \sum_{\ell=0}^{2}\ \sum_{\rho\in S_4/\mathcal{Q}_{3,\ell}} \mathrm{sgn}(\rho)\,\big(\Gamma^{m_{\rho(1)}}\,\Gamma^{m_{\rho(2)}}\dots\Gamma^{m_{\rho(2\ell)}}\,\mathcal{C}\big)_{AB}\,g^{\rho}_{\ell}\\ \sim \ \mathcal{C}_{AB}\,\eta^{m_1m_2}\,\eta^{m_3m_4}\,g^{(1234)}_{\ell=0} \ &- \ \mathcal{C}_{AB}\,\eta^{m_1m_3}\,\eta^{m_2m_4}\,g^{(1324)}_{\ell=0} \ + \ \mathcal{C}_{AB}\,\eta^{m_2m_3}\,\eta^{m_1m_4}\,g^{(2314)}_{\ell=0}\\ + \ (\Gamma^{m_1}\,\Gamma^{m_2}\,\mathcal{C})_{AB}\,\eta^{m_3m_4}\,g^{(1234)}_{\ell=1} \ &- \ (\Gamma^{m_1}\,\Gamma^{m_3}\,\mathcal{C})_{AB}\,\eta^{m_2m_4}\,g^{(1324)}_{\ell=1} \ + \ (\Gamma^{m_1}\,\Gamma^{m_4}\,\mathcal{C})_{AB}\,\eta^{m_2m_3}\,g^{(1423)}_{\ell=1} \end{aligned}$$

$$+ \ (\Gamma^{m_2}\,\Gamma^{m_3}\,\mathcal{C})_{AB}\,\eta^{m_1 m_4}\,g^{(2314)}_{\ell=1} \ - \ (\Gamma^{m_2}\,\Gamma^{m_4}\,\mathcal{C})_{AB}\,\eta^{m_1 m_3}\,g^{(2413)}_{\ell=1} \ + \ (\Gamma^{m_3}\,\Gamma^{m_4}\,\mathcal{C})_{AB}\,\eta^{m_1 m_2}\,g^{(3412)}_{\ell=1}$$
$$+ \ (\Gamma^{m_1}\,\Gamma^{m_2}\,\Gamma^{m_3}\,\Gamma^{m_4}\,\mathcal{C})_{AB}\,g^{(1234)}_{\ell=2} \,. \tag{6.2.47}$$

The associated world-sheet functions g^{ρ}_{ℓ} can be found looked up in (6.2.38).

The proof of (6.2.39) and (6.2.40) in D dimensions can be carried over almost literally from the four-dimensional case in [4]. The only explicit D dependence lies in the prefactors

$$\Omega_{(n,D)} \ \sim \ \frac{\Big[\Theta^{\vec{a}}_{\vec{b}}\left(\frac{1}{2}\int_{z_B}^{z_A}\vec{\omega}\right)\Big]^{D/2-n}}{\big[\,\Theta^{\vec{a}}_{\vec{b}}(\vec{0})\,\big]^{D/2}\,E_{AB}^{D/8-1/2}}\,, \qquad \omega_{(n,D)} \ \sim \ \frac{\Big[\Theta^{\vec{a}}_{\vec{b}}\left(\frac{1}{2}\int_{z_B}^{z_A}\vec{\omega}\right)\Big]^{D/2+1-n}}{\big[\,\Theta^{\vec{a}}_{\vec{b}}(\vec{0})\,\big]^{D/2}\,E_{AB}^{D/8}} \tag{6.2.48}$$

which are designed to match the leading $z_A \to z_B$ behavior in the OPE (6.1.3) of $S_K(z_A)S_L(z_B)$.

6.3 Correlators in four dimensions

As mentioned at the beginning of this chapter, RNS correlators can only be given for general dimensions $D = 2n$ if the number of spin fields does not exceed two. In the following, we will discuss more general cases with ≥ 4 spin fields for which each dimensionality must be treated separately. This section is devoted to four dimensions which are firstly of phenomenological relevance and secondly of exceptional simplicity as we will explain below. The tree level strategy is taken from [3], and parts of the computational tricks can still be carried out on higher genus [4].

6.3.1 Decomposing tree level correlators

In this subsection, we will demonstrate that the most general RNS tree level correlator in $D = 4$ dimensions with any number of ψ^μ, S_a and $S^{\dot{b}}$ field insertions can be boilt down to correlators $\langle S_{a_1}(x_1)...S_{a_{2n}}(x_{2n})\rangle$ with spin fields of uniform chirality. The first ingredient for this decomposition is a factorization of NS fermions into spin fields: Among the four dimensional RNS OPEs (4.2.7) we have one case with leading $z - w$ power zero,

$$S_a(z)\,S^{\dot{b}}(w) \ = \ \frac{1}{\sqrt{2}}\,(\sigma^\mu\,\varepsilon)_a{}^{\dot{b}}\,(z-w)^0\,\psi_\mu(w) \ + \ \mathcal{O}(z-w)\ . \tag{6.3.49}$$

By setting $z = w$ and using $\sigma^\mu_{a\dot{b}}\,\bar{\sigma}^{\dot{b}a}_\nu = -2\delta^\mu_\nu$, we can invert this relation and express the fermion ψ^μ as a bilinear of $D = 4$ spin fields:

$$\psi^\mu(z) \ = \ -\,\frac{1}{\sqrt{2}}\,S_a(z)\,(\varepsilon\,\sigma^\mu)^a{}_{\dot{b}}\,S^{\dot{b}}(z) \tag{6.3.50}$$

Hence, it is possible to replace all NS fermions in the following correlator:

$$\begin{aligned}\langle\, \psi^{\mu_1}(z_1)\ldots\psi^{\mu_n}(z_n)\, S_{a_1}(x_1)\ldots S_{a_r}(x_r)\, S^{\dot b_1}(y_1)\ldots S^{\dot b_s}(y_s)\,\rangle \;&=\; \prod_{i=1}^{n}\left(-\,\frac{(\varepsilon\,\sigma^{\mu_i})^{c_i}{}_{\dot d_i}}{\sqrt{2}}\right)\\ \times\;\langle\, S_{c_1}(z_1)\ldots S_{c_n}(z_n)\, S_{a_1}(x_1)\ldots S_{a_r}(x_r)\, S^{\dot d_1}(z_1)\ldots S^{\dot d_n}(z_n)\, &S^{\dot b_1}(y_1)\ldots S^{\dot b_s}(y_s)\,\rangle\;. \qquad (6.3.51)\end{aligned}$$

Any correlation function can be written as a spin field correlator contracted by some σ matrices.

The next major simplification exploits the fact that spin field tree correlators in four dimensions factorize into a left handed and a right handed part. The easiest example for this statement is the following four point function:

$$\langle\, S_a(z_1)\, S^{\dot b}(z_2)\, S_c(z_3)\, S^{\dot d}(z_4)\,\rangle \;=\; \frac{\varepsilon_{ac}\,\varepsilon^{\dot b\dot d}}{(z_{13}\,z_{24})^{1/2}} \;=\; \langle\, S_a(z_1)\, S_c(z_3)\,\rangle\,\langle\, S^{\dot b}(z_2)\, S^{\dot d}(z_4)\,\rangle \qquad (6.3.52)$$

In order to see that this factorization property holds for an arbitrary number of spin fields, it is most convenient to work with their bosonized representation [119],

$$\begin{aligned} S_{a=\left(\pm\frac{1}{2},\pm\frac{1}{2}\right)}(z) \;&\sim\; \mathrm{e}^{\pm\frac{i}{2}[H_1(z)+H_2(z)]} \;=:\; \mathrm{e}^{ia\cdot H(z)}\,,\\ S^{\dot b=\left(\pm\frac{1}{2},\mp\frac{1}{2}\right)}(z) \;&\sim\; \mathrm{e}^{\pm\frac{i}{2}[H_1(z)-H_2(z)]} \;=:\; \mathrm{e}^{i\dot b\cdot H(z)}\,, \end{aligned} \qquad (6.3.53)$$

with a two component boson vector $H(z) = \big(H_0(z), H_1(z)\big)$ and spinor weight vectors $a = \left(\pm\frac{1}{2},\pm\frac{1}{2}\right)$, $\dot b = \left(\pm\frac{1}{2},\mp\frac{1}{2}\right)$. In $D=4$, exceptional simplificiations occur because weight vectors of distinct chiralities are orthogonal, $a\cdot\dot b = 0$. Hence, the correlation function of r left handed and s right handed spin fields becomes

$$\begin{aligned} \langle\, S_{a_1}(z_1)\ldots S_{a_r}(z_r)\, S^{\dot b_1}(w_1)\ldots S^{\dot b_s}(w_s)\,\rangle \;&=\; \left\langle \prod_{k=1}^{r}\mathrm{e}^{ia_k\cdot H(z_k)}\;\prod_{l=1}^{s}\mathrm{e}^{i\dot b_l\cdot H(w_l)}\right\rangle\\ &=\; \delta\left(\sum_{k=1}^{r}a_k+\sum_{l=1}^{s}\dot b_l\right)\prod_{\substack{i,j=1\\ i<j}}^{r} z_{ij}^{a_i\cdot a_j}\;\prod_{\substack{\bar\imath,\bar\jmath=1\\ \bar\imath<\bar\jmath}}^{s} w_{\bar\imath\bar\jmath}^{\dot b_{\bar\imath}\cdot\dot b_{\bar\jmath}}\;\underbrace{\prod_{m=1}^{r}\prod_{n=1}^{s}(z_m-w_n)^{a_m\cdot\dot b_n}}_{=\;1\;\text{since}\;a_m\cdot\dot b_n=0}\\ &=\; \delta\left(\sum_{k=1}^{r}a_k\right)\prod_{\substack{i,j=1\\ i<j}}^{r} z_{ij}^{a_i\cdot a_j}\;\delta\left(\sum_{l=1}^{s}\dot b_l\right)\prod_{\substack{\bar\imath,\bar\jmath=1\\ \bar\imath<\bar\jmath}}^{s} w_{\bar\imath\bar\jmath}^{\dot b_{\bar\imath}\cdot\dot b_{\bar\jmath}}\\ &=\; \left\langle\prod_{k=1}^{r}\mathrm{e}^{ia_k\cdot H(z_k)}\right\rangle\left\langle\prod_{l=1}^{s}\mathrm{e}^{i\dot b_l\cdot H(w_l)}\right\rangle\\ &=\; \langle\, S_{a_1}(z_1)\ldots S_{a_r}(z_r)\,\rangle\;\;\langle\, S^{\dot b_1}(w_1)\ldots S^{\dot b_s}(w_s)\,\rangle\;. \end{aligned} \qquad (6.3.54)$$

From the second to the third line we have used that $a_m\cdot\dot b_n = 0$, and the δ-function has been split into the linearly independent a- and $\dot b$ contributions. This proves that a general spin field correlation function in four dimensions splits into separate correlators involving left- and right handed spin fields only.

We should stress that this result does not generalize to arbitrary dimensions. In $D = 2n$ with $n > 2$, the crucial property $\alpha \cdot \dot{\beta} = 0$ for weight vectors $\alpha, \dot{\beta}$ of opposite chirality does not hold any longer. Therefore, the $z_m - w_n$ factors which dropped out of (6.3.54) will generically be present in higher dimensions, and left handed spin fields couple non-trivially to the right handed ones.

Using the factorization property (6.3.54), our previous result (6.3.51) for $D = 4$ tree level correlators can be further reduced to

$$\begin{aligned}\langle\, \psi^{\mu_1}(z_1) \dots \psi^{\mu_n}(z_n)\, S_{a_1}(x_1) \dots S_{a_r}(x_r)\, S^{\dot{b}_1}(y_1) \dots S^{\dot{b}_s}(y_s) \,\rangle \quad &= \quad \prod_{i=1}^{n} \left(- \frac{(\varepsilon\, \sigma^{\mu_i})^{c_i}{}_{\dot{d}_i}}{\sqrt{2}} \right) \\ \times \ \langle\, S_{c_1}(z_1) \dots S_{c_n}(z_n)\, S_{a_1}(x_1) \dots S_{a_r}(x_r) \,\rangle \ \ \langle\, S^{\dot{d}_1}(z_1) \dots S^{\dot{d}_n}(z_n)\, S^{\dot{b}_1}(y_1) \dots S^{\dot{b}_s}(y_s) \,\rangle\ . \quad &(6.3.55)\end{aligned}$$

This formula shows how correlation functions involving NS fermions factorize into a product of correlators with only left- or right handed spin fields. Hence, if the latter are known for an arbitrary number of spin fields, it is possible to calculate in principle any correlator $\langle \psi^{\mu_1} \dots \psi^{\mu_n}\, S_{a_1} \dots S_{a_r}\, S^{\dot{b}_1} \dots S^{\dot{b}_s} \rangle$.

6.3.2 The building block: $2n$ alike spin fields

This section contains an expression for $2n$ point correlators $\langle S_{a_1} \dots S_{a_{2n}} \rangle$ which have just been identified as the basic building block of any tree level correlation functions of the $D = 4$ RNS CFT. As a motivation for the $2n$ point formula, let us first of all give some lower order examples at $n = 2, 3$.

In the four point function, we eliminate one out of three possible tensor structures using the Fierz identity $\varepsilon_{ac}\varepsilon_{bd} = \varepsilon_{ab}\varepsilon_{cd} - \varepsilon_{ad}\varepsilon_{cb}$,

$$\begin{aligned}\langle\, S_a(z_1)\, S_b(z_2)\, S_c(z_3)\, S_d(z_4) \,\rangle \quad &= \quad \frac{1}{(z_{12}\, z_{13}\, z_{14}\, z_{23}\, z_{24}\, z_{34})^{1/2}} \Big(\varepsilon_{ab}\, \varepsilon_{cd}\, z_{14}\, z_{23} \ - \ \varepsilon_{ad}\, \varepsilon_{cb}\, z_{12}\, z_{43} \Big) \\ &= \quad \left(\frac{z_{12}\, z_{14}\, z_{23}\, z_{34}}{z_{13}\, z_{24}} \right)^{1/2} \left(\frac{\varepsilon_{ab}\, \varepsilon_{cd}}{z_{12}\, z_{34}} \ - \ \frac{\varepsilon_{ad}\, \varepsilon_{cb}}{z_{14}\, z_{32}} \right) \qquad (6.3.56)\end{aligned}$$

For the six point correlator, one can think of $5!! = 15$ combinations $\varepsilon_{a_i a_j}\varepsilon_{a_k a_l}\varepsilon_{a_m a_n}$ five of which are linearly independent. However, the result assumes a more symmetric form if we use six tensors, namely:

$$\langle\, S_a(z_1)\, S_b(z_2)\, S_c(z_3)\, S_d(z_4)\, S_e(z_5)\, S_f(z_6) \,\rangle \quad = \quad \left(\frac{z_{12}\, z_{14}\, z_{16}\, z_{23}\, z_{25}\, z_{34}\, z_{36}\, z_{45}\, z_{56}}{z_{13}\, z_{15}\, z_{24}\, z_{26}\, z_{35}\, z_{46}} \right)^{1/2}$$

$$\times \left(\frac{\varepsilon_{ab}\,\varepsilon_{cd}\,\varepsilon_{ef}}{z_{12}\,z_{34}\,z_{56}} - \frac{\varepsilon_{ab}\,\varepsilon_{cf}\,\varepsilon_{ed}}{z_{12}\,z_{36}\,z_{54}} + \frac{\varepsilon_{ad}\,\varepsilon_{cf}\,\varepsilon_{eb}}{z_{14}\,z_{36}\,z_{52}} - \frac{\varepsilon_{ad}\,\varepsilon_{cb}\,\varepsilon_{ef}}{z_{14}\,z_{32}\,z_{56}} + \frac{\varepsilon_{af}\,\varepsilon_{cb}\,\varepsilon_{ed}}{z_{16}\,z_{32}\,z_{54}} - \frac{\varepsilon_{af}\,\varepsilon_{cd}\,\varepsilon_{eb}}{z_{16}\,z_{34}\,z_{52}} \right) \tag{6.3.57}$$

In both cases, the prefactor schematically consists of all possible terms $(z_{\text{odd even}}\ z_{\text{even odd}})^{1/2}$ in the numerator and $(z_{\text{odd odd}}\ z_{\text{even even}})^{1/2}$ in the denominator. Furthermore, the first index at every ε-tensor belongs to a spin field with argument z_{odd} whereas the second index stems from a spin field with argument z_{even}, e.g. odd $\leftrightarrow a, c, e$ and even $\leftrightarrow b, d, f$ in (6.3.57). Finally, every ε-tensor comes with the corresponding factor $(z_{\text{odd}} - z_{\text{even}})^{-1}$, e.g. $\varepsilon_{ad} \leftrightarrow z_{14}^{-1}$. The relative signs between the individual $(\varepsilon/z_{ij})^n$ terms can be understood as the sign of the respective permutation of the even spinor indices.

These observations about (6.3.56) and (6.3.57) suggest the following expression for the $2n$ point function of left-handed spin fields:

$$\begin{aligned} \langle\, S_{a_1}(z_1)\, S_{a_2}(z_2) \ldots S_{a_{2n-1}}(z_{2n-1})\, S_{a_{2n}}(z_{2n})\,\rangle \;&=\; \left(\prod_{i\leq j}^{n} z_{2i-1,2j} \prod_{\bar\imath<\bar\jmath}^{n} z_{2\bar\imath,2\bar\jmath-1} \right)^{1/2} \\ \times \left(\prod_{k<l}^{n} z_{2k-1,2l-1}\, z_{2k,2l} \right)^{-1/2} &\sum_{\rho\in S_n} \operatorname{sgn}(\rho) \prod_{m=1}^{n} \frac{\varepsilon_{a_{2m-1}a_{\rho(2m)}}}{z_{2m-1,\rho(2m)}} \end{aligned} \tag{6.3.58}$$

We prove this claim by induction. The basic cases $n = 2, 3$ reproduce (6.3.56) and (6.3.57). The inductive step makes use of the fact that the $2n-2$ correlator should appear as a limiting case of the $2n$ correlator when two spin fields are contracted via OPE $S_{a_i}(z_i)S_{a_j}(z_j) \sim \varepsilon_{a_i a_j} z_{ij}^{-1/2}$. As every spin field can be permuted to the very right in the correlator, it is sufficient to study the case $z_{2n-1} \to z_{2n}$:

$$\begin{aligned} &\langle\, S_{a_1}(z_1)\ \ldots\ S_{a_{2n-2}}(z_{2n-2})\, S_{a_{2n-1}}(z_{2n-1})\, S_{a_{2n}}(z_{2n})\,\rangle \Big|_{z_{2n-1}\to z_{2n}} \\ &\quad= \frac{\varepsilon_{a_{2n-1}a_{2n}}}{z_{2n-1,2n}^{1/2}} \,\langle\, S_{a_1}(z_1)\ \ldots\ S_{a_{2n-2}}(z_{2n-2})\,\rangle \,+\, \mathcal{O}(z_{2n-1,2n}^{1/2}) \\ &\quad= \frac{\varepsilon_{a_{2n-1}a_{2n}}}{z_{2n-1,2n}}\, z_{2n-1,2n}^{1/2} \left(\prod_{i\leq j}^{n-1} z_{2i-1,2j} \prod_{\bar\imath<\bar\jmath}^{n-1} z_{2\bar\imath,2\bar\jmath-1} \right)^{1/2} \left(\prod_{k<l}^{n-1} z_{2k-1,2l-1}\, z_{2k,2l} \right)^{-1/2} \\ &\qquad\times \underbrace{\left(\frac{\prod_{p=1}^{n-1} z_{2p-1,2n}\, z_{2p,2n-1}}{\prod_{q=1}^{n-1} z_{2q-1,2n-1}\, z_{2q,2n}} \right)^{1/2}}_{=1+\mathcal{O}(z_{2n-1,2n})} \sum_{\rho\in S_{n-1}} \operatorname{sgn}(\rho) \prod_{m=1}^{n-1} \frac{\varepsilon_{a_{2m-1}a_{\rho(2m)}}}{z_{2m-1,\rho(2m)}} \,+\, \mathcal{O}(z_{2n-1,2n}^{1/2}) \\ &\quad= \left(\prod_{i\leq j}^{n} z_{2i-1,2j} \prod_{\bar\imath<\bar\jmath}^{n} z_{2\bar\imath,2\bar\jmath-1} \right)^{1/2} \left(\prod_{k<l}^{n} z_{2k-1,2l-1}\, z_{2k,2l} \right)^{-1/2} \\ &\qquad\times \sum_{\rho\in S_n} \operatorname{sgn}(\rho) \prod_{m=1}^{n} \delta_{\rho(2n),2n}\, \frac{\varepsilon_{\alpha_{2m-1}\alpha_{\rho(2m)}}}{z_{2m-1,\rho(2m)}} \,+\, \mathcal{O}(z_{2n-1,2n}^{1/2}) \end{aligned} \tag{6.3.59}$$

The most singular piece of (6.3.58) in $z_{2n-1,2n}$ is contained in the subset of S_n permutations ρ with fixed point $\rho(2n) = 2n$. This is precisely what we have found in (6.3.59) by applying the OPE of $S_{a_{2n-1}}(z_{2n-1})S_{a_{2n}}(z_{2n})$ and evaluating the remainder $\langle S_{a_1}(z_1) \, \dots \, S_{a_{2n-2}}(z_{2n-2})\rangle$ via (6.3.58) at $n \mapsto n-1$. This completes the proof by induction.

By plugging (6.3.58) and its right handed analogue

$$\begin{aligned} \langle \, S^{\dot b_1}(z_1) \, S^{\dot b_2}(z_2) \dots S^{\dot b_{2n-1}}(z_{2n-1}) \, S^{\dot b_{2n}}(z_{2n}) \, \rangle \; &= \; \left(\prod_{i \leq j}^{n} z_{2i-1,2j} \prod_{\bar\imath < \bar\jmath}^{n} z_{2\bar\imath,2\bar\jmath-1} \right)^{1/2} \\ \times \; \left(\prod_{k<l}^{n} z_{2k-1,2l-1} \, z_{2k,2l} \right)^{-1/2} & \sum_{\rho \in S_n} \mathrm{sgn}(\rho) \prod_{m=1}^{n} \frac{\varepsilon^{\dot b_{2m-1} \dot b_{\rho(2m)}}}{z_{2m-1,\rho(2m)}} \; . \end{aligned} \tag{6.3.60}$$

into (6.3.55), we are now able to calculate any tree level correlator involving fermions ψ^μ and spin fields $S_a, S^{\dot b}$.

6.3.3 The loop generalization

The natural question is whether the tree level strategies can still be applied to loop correlators in four dimensions. The first step trading ψ^μ for a $S_a S^{\dot b}$ product is based on OPEs, i.e. local properties of the conformal fields. Therefore, the factorization prescription (6.3.50) remains valid at $g \geq 1$.

However, the separation of $D = 4$ spin field correlators into left- and right handed halves does not occur beyond tree level. The arguments in subsection 6.3.2 rest on bosonization which becomes more subtle at higher genus, see subsection 6.1.5. The boson's winding numbers have to be projected according to the fermion's spin structure, therefore one has to expect obstacles against factorizing $\langle S_{a_1} ... S_{a_p} S^{\dot b_1} ... S^{\dot b_q} \rangle_{\vec b}^{\vec a}$ into $\langle S_{a_1} ... S_{a_p} \rangle_{\vec b}^{\vec a}$ and $\langle S^{\dot b_1} ... S^{\dot b_q} \rangle_{\vec b}^{\vec a}$ within the spin structure dependent $\Theta_{\vec b}^{\vec a}$ function. This is confirmed by the loop generalization of the toy example (6.3.52):

$$\begin{aligned} \langle \, S_a(z_1) \, S^{\dot b}(z_2) \, S_c(z_3) \, S^{\dot d}(z_4) \, \rangle_{\vec b}^{\vec a} \; &= \; \varepsilon_{ac} \, \varepsilon^{\dot b \dot d} \, \frac{\Theta_{\vec b}^{\vec a} \left(\frac{1}{2} \int_{z_3}^{z_1} \vec\omega \, + \, \frac{1}{2} \int_{z_4}^{z_2} \vec\omega \right) \, \Theta_{\vec b}^{\vec a} \left(\frac{1}{2} \int_{z_3}^{z_1} \vec\omega \, - \, \frac{1}{2} \int_{z_4}^{z_2} \vec\omega \right)}{\left[\Theta_{\vec b}^{\vec a}(\vec 0) \right]^2 \, (E_{13} \, E_{24})^{1/2}} \\ &\neq \; \langle \, S_a(z_1) \, S_c(z_3) \, \rangle_{\vec b}^{\vec a} \, \langle \, S^{\dot b}(z_2) \, S^{\dot d}(z_4) \, \rangle_{\vec b}^{\vec a} \end{aligned} \tag{6.3.61}$$

Hence, determining the $\Theta_{\vec b}^{\vec a}$ arguments within $\langle S_{a_1} ... S_{a_p} S^{\dot b_1} ... S^{\dot b_q} \rangle_{\vec b}^{\vec a}$ is the bottleneck in the computation of $D = 4$ loop correlators. The problem could not be solved in full generality, i.e. not for arbitrary values of p, q, but let us give the partial results of [4]:

Correlation functions of alike spin fields are relatively straightforward generalizations of their tree level limit (6.3.58):

$$\begin{aligned}
\langle\, S_{a_1}(z_1)\, S_{a_2}(z_2) \dots S_{a_{2n-1}}(z_{2n-1})\, S_{a_{2n}}(z_{2n})\,\rangle^{\vec a}_{\vec b} \;&=\; \frac{1}{\big[\Theta^{\vec a}_{\vec b}(\vec 0)\big]^2}\; \left[\Theta^{\vec a}_{\vec b}\left(\frac{1}{2}\sum_{i=1}^{n}\int_{z_{2i}}^{z_{2i-1}}\vec\omega\right)\right]^{2-n} \\
&\times\; \left(\prod_{i\leq j}^{n} E_{2i-1,2j}\prod_{\bar\imath<\bar\jmath}^{n} E_{2\bar\imath,2\bar\jmath-1}\right)^{1/2}\left(\prod_{k<l}^{n} E_{2k-1,2l-1}\,E_{2k,2l}\right)^{-1/2} \\
&\times\; \sum_{\rho\in S_n}\mathrm{sgn}(\rho)\prod_{m=1}^{n}\frac{\varepsilon_{a_{2m-1}a_{\rho(2m)}}}{E_{2m-1,\rho(2m)}}\,\Theta^{\vec a}_{\vec b}\left(\frac{1}{2}\sum_{i=1}^{n}\int_{z_{2i}}^{z_{2i-1}}\vec\omega\;-\int_{z_{\rho(2m)}}^{z_{2m-1}}\vec\omega\right)
\end{aligned} \tag{6.3.62}$$

The negative power of $\Theta^{\vec a}_{\vec b}\big(\frac{1}{2}\sum_{i=1}^{n}\int_{z_{2i}}^{z_{2i-1}}\vec\omega\big)$ functions at $n\geq 3$ can be viewed as a punishment for expressing the correlator in terms of a nonminimal set of $n!$ Lorentz tensors[4] $\prod_{m=1}^{n}\varepsilon_{a_{2m-1}a_{\rho(2m)}},\ \rho\in S_n$.

We also managed to find explicit results for $2n$ left handed and either two or four right handed spin fields:

$$\begin{aligned}
&\langle\, S_{a_1}(z_1)\, S_{a_2}(z_2) \dots S_{a_{2n-1}}(z_{2n-1})\, S_{a_{2n}}(z_{2n})\, S^{\dot c}(z_C)\, S^{\dot d}(z_D)\,\rangle^{\vec a}_{\vec b} \\
&=\; \frac{1}{\big[\Theta^{\vec a}_{\vec b}(\vec 0)\big]^2}\;\left[\Theta^{\vec a}_{\vec b}\left(\frac{1}{2}\sum_{i=1}^{n}\int_{z_{2i}}^{z_{2i-1}}\vec\omega\;\pm\;\frac{1}{2}\int_{z_D}^{z_C}\vec\omega\right)\right]^{2-n}\frac{\varepsilon_{\dot c\dot d}}{E_{CD}^{1/2}} \\
&\times\; \left(\prod_{i\leq j}^{n} E_{2i-1,2j}\prod_{\bar\imath<\bar\jmath}^{n} E_{2\bar\imath,2\bar\jmath-1}\right)^{1/2}\left(\prod_{k<l}^{n} E_{2k-1,2l-1}\,E_{2k,2l}\right)^{-1/2} \\
&\times\; \sum_{\rho\in S_n}\mathrm{sgn}(\rho)\prod_{m=1}^{n}\frac{\varepsilon_{a_{2m-1}a_{\rho(2m)}}}{E_{2m-1,\rho(2m)}}\,\Theta^{\vec a}_{\vec b}\left(\frac{1}{2}\sum_{i=1}^{n}\int_{z_{2i}}^{z_{2i-1}}\vec\omega\;-\int_{z_{\rho(2m)}}^{z_{2m-1}}\vec\omega\;\pm\;\frac{1}{2}\int_{z_D}^{z_C}\vec\omega\right)
\end{aligned} \tag{6.3.63}$$

$$\begin{aligned}
&\langle\, S_{a_1}(z_1)\, S_{a_2}(z_2) \dots S_{a_{2n}}(z_{2n})\, S^{\dot c}(z_C)\, S^{\dot d}(z_D)\, S^{\dot e}(z_E)\, S^{\dot f}(z_F)\,\rangle^{\vec a}_{\vec b} \;=\; \frac{1}{\big[\Theta^{\vec a}_{\vec b}(\vec 0)\big]^2} \\
&\times\; \left(\frac{E_{CD}\,E_{CF}\,E_{DE}\,E_{EF}}{E_{CE}\,E_{DF}}\right)^{1/2}\left(\prod_{i\leq j}^{n} E_{2i-1,2j}\prod_{\bar\imath<\bar\jmath}^{n} E_{2\bar\imath,2\bar\jmath-1}\right)^{1/2}\left(\prod_{k<l}^{n} E_{2k-1,2l-1}\,E_{2k,2l}\right)^{-1/2} \\
&\times\; \left\{\frac{\varepsilon^{\dot c\dot d}\,\varepsilon^{\dot e\dot f}}{E_{CD}\,E_{EF}}\;\left[\Theta^{\vec a}_{\vec b}\left(\frac{1}{2}\sum_{i=1}^{n}\int_{z_{2i}}^{z_{2i-1}}\vec\omega\;\pm\;\frac{1}{2}\int_{z_D}^{z_C}\vec\omega\;\mp\;\frac{1}{2}\int_{z_F}^{z_E}\vec\omega\right)\right]^{2-n}\sum_{\rho\in S_n}\mathrm{sgn}(\rho)\right. \\
&\qquad\prod_{m=1}^{n}\frac{\varepsilon_{a_{2m-1}a_{\rho(2m)}}}{E_{2m-1,\rho(2m)}}\,\Theta^{\vec a}_{\vec b}\left(\frac{1}{2}\sum_{i=1}^{n}\int_{z_{2i}}^{z_{2i-1}}\vec\omega\;-\int_{z_{\rho(2m)}}^{z_{2m-1}}\vec\omega\;\pm\;\frac{1}{2}\int_{z_D}^{z_C}\vec\omega\;\mp\;\frac{1}{2}\int_{z_F}^{z_E}\vec\omega\right) \\
&-\;\frac{\varepsilon^{\dot c\dot f}\,\varepsilon^{\dot e\dot d}}{E_{CF}\,E_{ED}}\;\left[\Theta^{\vec a}_{\vec b}\left(\frac{1}{2}\sum_{i=1}^{n}\int_{z_{2i}}^{z_{2i-1}}\vec\omega\;\pm\;\frac{1}{2}\int_{z_F}^{z_C}\vec\omega\;\mp\;\frac{1}{2}\int_{z_D}^{z_E}\vec\omega\right)\right]^{2-n}\sum_{\rho\in S_n}\mathrm{sgn}(\rho) \\
&\qquad\left.\prod_{m=1}^{n}\frac{\varepsilon_{a_{2m-1}a_{\rho(2m)}}}{E_{2m-1,\rho(2m)}}\,\Theta^{\vec a}_{\vec b}\left(\frac{1}{2}\sum_{i=1}^{n}\int_{z_{2i}}^{z_{2i-1}}\vec\omega\;-\int_{z_{\rho(2m)}}^{z_{2m-1}}\vec\omega\;\pm\;\frac{1}{2}\int_{z_F}^{z_C}\vec\omega\;\mp\;\frac{1}{2}\int_{z_D}^{z_E}\vec\omega\right)\right\}
\end{aligned} \tag{6.3.64}$$

[4]Finding the number of scalars in the $2n$ fold tensor product of left handed $SO(1,3)$ spinors is equivalent to a random walk problem on the positive real axis [229]. It is given by the n'th Catalan number $C(n)=\frac{(2n)!}{n!(n+1)!}\leq n!$.

The sign ambiguity in coupling the right handed fields' positions to the left handed ones vanishes at $n \leq 2$. The cases with $n \geq 3$ offer an increasing number of possibilities to add zeros in the form $0 = \sum_{\rho \in S_m} \mathrm{sgn}(\rho) \prod_{k=1}^{m} \varepsilon_{a_{2k-1} a_{\rho(2k)}}$ for any $3 \leq m \leq n$ which then allow to check equivalence of both sign choices.

The proof for the loop generalizations (6.3.62), (6.3.63) and (6.3.64) is based on similar mechanisms as in (6.3.59) and can be found in appendix B.1 of [4].

With these correlation functions at hand, it is in principle possible to derive a cornucopia of correlators with NS fields included via ψ^μ factorization (6.3.50):

- starting from $\langle S_{a_1} \dots S_{a_{2M}} \, S^{\dot c} \, S^{\dot d} \rangle_{\vec b}^{\vec a}$

$$\Rightarrow \quad \langle \psi^\mu \, S_{a_1} \dots S_{a_{2M-1}} \, S^{\dot c} \rangle_{\vec b}^{\vec a} \ , \quad \langle \psi^\mu \, \psi^\nu \, S_{a_1} \dots S_{a_{2M-2}} \rangle_{\vec b}^{\vec a}$$

- starting from $\langle S_{a_1} \dots S_{a_{2M}} \, S^{\dot c} \, S^{\dot d} \, S^{\dot e} \, S^{\dot f} \rangle_{\vec b}^{\vec a}$

$$\Rightarrow \quad \left\{ \begin{array}{l} \langle \psi^\mu \, S_{a_1} \dots S_{a_{2M-1}} \, S^{\dot c} \, S^{\dot d} \, S^{\dot e} \rangle_{\vec b}^{\vec a} \ , \quad \langle \psi^\mu \, \psi^\nu \, S_{a_1} \dots S_{a_{2M-2}} \, S^{\dot c} \, S^{\dot d} \rangle_{\vec b}^{\vec a} \\ \langle \psi^\mu \, \psi^\nu \, \psi^\lambda \, S_{a_1} \dots S_{a_{2M-3}} \, S^{\dot c} \rangle_{\vec b}^{\vec a} \ , \quad \langle \psi^\mu \, \psi^\nu \, \psi^\lambda \, \psi^\rho \, S_{a_1} \dots S_{a_{2M-4}} \rangle_{\vec b}^{\vec a} \end{array} \right.$$

6.3.4 Examples with four spin fields

After this rather general discussion of RNS correlators in four dimensions, we shall give some explicit examples with four spin fields here which are for instance required for one loop amplitudes with four massless fermions.

Firstly, we shall explicitly display the four point function of left handed spin fields

$$\begin{aligned} \langle \, S_a(z_1) \, S_b(z_2) \, S_c(z_3) \, S_d(z_4) \, \rangle_{\vec b}^{\vec a} \ &= \ \frac{(E_{12} \, E_{14} \, E_{23} \, E_{34})^{1/2}}{\big[\Theta_{\vec b}^{\vec a}(\vec 0) \big]^2 \, (E_{13} \, E_{24})^{1/2}} \\ &\times \ \left\{ \frac{\varepsilon_{ab} \, \varepsilon_{cd}}{E_{12} \, E_{34}} \, \Theta_{\vec b}^{\vec a} \big[\begin{smallmatrix} 1 & 4 \\ 2 & 3 \end{smallmatrix} \big]^2 \ - \ \frac{\varepsilon_{ad} \, \varepsilon_{cb}}{E_{14} \, E_{32}} \, \Theta_{\vec b}^{\vec a} \big[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix} \big]^2 \right\} \end{aligned} \tag{6.3.65}$$

which is the $n = 2$ case of (6.3.62). The nonvanishing five point function with four spin fields reads

$$\begin{aligned} \langle \, \psi^\mu(z_1) \, S_a(z_2) \, S_b(z_3) \, S_c(z_4) \, S^{\dot d}(z_5) \, \rangle_{\vec b}^{\vec a} \ &= \ \frac{1}{\sqrt{2} \, \big[\Theta_{\vec b}^{\vec a}(\vec 0) \big]^2 \, (E_{12} \, E_{13} \, E_{14} \, E_{15} \, E_{23} \, E_{24} \, E_{34})^{1/2}} \\ \times \ \Big\{ (\sigma^\mu \, \varepsilon)_c{}^{\dot d} \, \varepsilon_{ab} \, E_{12} \, E_{34} \, \Theta_{\vec b}^{\vec a} \big[\begin{smallmatrix} 1 & 1 & 2 \\ 3 & 4 & 5 \end{smallmatrix} \big] \, \Theta_{\vec b}^{\vec a} \big[\begin{smallmatrix} 2 & 5 \\ 4 & 3 \end{smallmatrix} \big] \ &+ \ (\sigma^\mu \, \varepsilon)_a{}^{\dot d} \, \varepsilon_{cb} \, E_{14} \, E_{23} \, \Theta_{\vec b}^{\vec a} \big[\begin{smallmatrix} 1 & 1 & 4 \\ 2 & 3 & 5 \end{smallmatrix} \big] \, \Theta_{\vec b}^{\vec a} \big[\begin{smallmatrix} 4 & 5 \\ 2 & 3 \end{smallmatrix} \big] \Big\} \ . \end{aligned} \tag{6.3.66}$$

Among the six point functions $\langle \psi^2 S^4 \rangle$, there are two non-vanishing chirality configurations – either the spin fields are all left handed,

$$\langle \, \psi^\mu(z_1) \, \psi^\nu(z_2) \, S_a(z_3) \, S_b(z_4) \, S^{\dot c}(z_5) \, S^{\dot d}(z_6) \, \rangle_{\vec b}^{\vec a}$$

$$
\begin{aligned}
&= \frac{1}{2\left[\Theta^{\vec a}_{\vec b}(\vec 0)\right]^2 E_{12}\,(E_{13}\,E_{14}\,E_{15}\,E_{16}\,E_{23}\,E_{24}\,E_{25}\,E_{26}\,E_{34}\,E_{56})^{1/2}} \\
&\quad \times \Big\{ (\sigma^\mu\,\varepsilon)_b{}^{\dot d}\,(\sigma^\nu\,\varepsilon)_a{}^{\dot c}\,E_{13}\,E_{15}\,E_{24}\,E_{26}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&3&5\\2&2&4&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&6\\4&5\end{smallmatrix}\right] \\
&\qquad + (\sigma^\mu\,\varepsilon)_a{}^{\dot c}\,(\sigma^\nu\,\varepsilon)_b{}^{\dot d}\,E_{14}\,E_{16}\,E_{23}\,E_{25}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&4&6\\2&2&3&5\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&6\\4&5\end{smallmatrix}\right] \\
&\qquad - (\sigma^\mu\,\varepsilon)_a{}^{\dot d}\,(\sigma^\nu\,\varepsilon)_b{}^{\dot c}\,E_{14}\,E_{15}\,E_{23}\,E_{26}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&4&5\\2&2&3&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&5\\4&6\end{smallmatrix}\right] \\
&\qquad - (\sigma^\mu\,\varepsilon)_b{}^{\dot c}\,(\sigma^\nu\,\varepsilon)_a{}^{\dot d}\,E_{13}\,E_{16}\,E_{24}\,E_{25}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&3&6\\2&2&4&5\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&5\\4&6\end{smallmatrix}\right]\Big\}\ , \qquad (6.3.67)
\end{aligned}
$$

or they are of mixed chirality:

$$
\begin{aligned}
&\langle\,\psi^\mu(z_1)\,\psi^\nu(z_2)\,S_a(z_3)\,S_b(z_4)\,S_c(z_5)\,S_d(z_6)\,\rangle^{\vec a}_{\vec b} \;=\; \frac{1}{\left[\Theta^{\vec a}_{\vec b}(\vec 0)\right]^2} \\
&\times\ \frac{1}{(E_{13}\,E_{14}\,E_{15}\,E_{16}\,E_{23}\,E_{24}\,E_{25}\,E_{26}\,E_{34}\,E_{35}\,E_{36}\,E_{45}\,E_{46}\,E_{56})^{1/2}} \\
&\times\ \Big\{ \frac{\eta^{\mu\nu}}{E_{12}}\,\varepsilon_{ab}\,\varepsilon_{cd}\,E_{36}\,E_{45}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&6\\4&5\end{smallmatrix}\right]\Big(E_{13}\,E_{24}\,E_{25}\,E_{16}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&3&6\\2&2&4&5\end{smallmatrix}\right] + E_{23}\,E_{14}\,E_{15}\,E_{26}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&4&5\\2&2&3&6\end{smallmatrix}\right]\Big) \\
&\quad + \frac{\eta^{\mu\nu}}{E_{12}}\,\varepsilon_{ad}\,\varepsilon_{cb}\,E_{34}\,E_{56}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&4\\5&6\end{smallmatrix}\right]\Big(E_{13}\,E_{14}\,E_{25}\,E_{26}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&3&4\\2&2&5&6\end{smallmatrix}\right] + E_{23}\,E_{24}\,E_{15}\,E_{16}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&5&6\\2&2&3&4\end{smallmatrix}\right]\Big) \\
&\quad + \left[(\sigma^{\mu\nu}\,\varepsilon)_{cd}\,\frac{E_{56}}{2}\right]\varepsilon_{ab}\,\frac{E_{36}\,E_{45}\,E_{13}\,E_{24}}{\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&5\\4&6\end{smallmatrix}\right]}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&6\\4&5\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&3\\4&5&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&2&4\\3&5&6\end{smallmatrix}\right] \\
&\quad + \left[(\sigma^{\mu\nu}\,\varepsilon)_{cb}\,\frac{E_{54}}{2}\right]\varepsilon_{ad}\,\frac{E_{34}\,E_{56}\,E_{13}\,E_{26}}{\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&5\\4&6\end{smallmatrix}\right]}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&4\\5&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&3\\4&5&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&2&6\\3&4&5\end{smallmatrix}\right] \\
&\quad + \left[(\sigma^{\mu\nu}\,\varepsilon)_{ab}\,\frac{E_{34}}{2}\right]\varepsilon_{cd}\,\frac{E_{36}\,E_{45}\,E_{15}\,E_{26}}{\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&5\\4&6\end{smallmatrix}\right]}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&6\\4&5\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&5\\3&4&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&2&6\\3&4&5\end{smallmatrix}\right] \\
&\quad + \left[(\sigma^{\mu\nu}\,\varepsilon)_{ad}\,\frac{E_{36}}{2}\right]\varepsilon_{cb}\,\frac{E_{34}\,E_{56}\,E_{15}\,E_{24}}{\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&5\\4&6\end{smallmatrix}\right]}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&4\\5&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&5\\3&4&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&2&4\\3&5&6\end{smallmatrix}\right]\Big\} \qquad (6.3.68)
\end{aligned}
$$

The $\Theta^{\vec a}_{\vec b}$ function in the denominator is due to the redundancy of the tensors used in (6.3.68),

$$
(\sigma^{\mu\nu}\,\varepsilon)_{ab}\,\varepsilon_{cd} \;-\; (\sigma^{\mu\nu}\,\varepsilon)_{ad}\,\varepsilon_{cb} \;+\; (\sigma^{\mu\nu}\,\varepsilon)_{cd}\,\varepsilon_{ab} \;-\; (\sigma^{\mu\nu}\,\varepsilon)_{cb}\,\varepsilon_{ad} \;=\; 0\ , \qquad (6.3.69)
$$

see the remarks below (6.3.62).

6.4 Correlators in six dimensions

The internal SCFT of four dimensional superstring compactifications with maximal spacetime supersymmetry can be described $SO(6)$ covariantly by six free worldsheet fermions Ψ^k and their associated spin fields Σ^I and $\bar\Sigma_{\bar I}$. This motivates the study of RNS correlation functions in $D = 6$ dimensions. Apart from some lower point results, we will give a general formula for correlators with n insertions of Σ and $\bar\Sigma$ each.

6.4.1 Lower point results

The simplest correlators in six dimensions involving four spin fields only are

$$\langle\, \Sigma^I(z_1)\, \Sigma^J(z_2)\, \Sigma^K(z_3)\, \Sigma^L(z_4)\,\rangle^{\vec a}_{\vec b} \;=\; \frac{\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2\\3&4\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3\\2&4\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&4\\2&3\end{smallmatrix}\right]}{2\,\left[\Theta^{\vec a}_{\vec b}(\vec 0)\right]^3}\,\frac{(\gamma^k\, C)^{IJ}(\gamma_k\, C)^{KL}}{(E_{12}\, E_{13}\, E_{14}\, E_{23}\, E_{24}\, E_{34})^{1/4}}\,, \tag{6.4.70}$$

$$\begin{aligned}\langle\, \Sigma^I(z_1)\, \Sigma^J(z_2)\, \bar\Sigma_{\bar K}(z_3)\, \bar\Sigma_{\bar L}(z_4)\,\rangle^{\vec a}_{\vec b} \;&=\; \frac{\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2\\3&4\end{smallmatrix}\right]}{\left[\Theta^{\vec a}_{\vec b}(\vec 0)\right]^3}\left(\frac{E_{13}\, E_{14}\, E_{23}\, E_{24}}{E_{12}\, E_{34}}\right)^{1/4}\\ &\times\left\{\frac{C^I{}_{\bar K}\, C^J{}_{\bar L}}{E_{13}\, E_{24}}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&4\\2&3\end{smallmatrix}\right]^2 \;-\; \frac{C^I{}_{\bar L}\, C^J{}_{\bar K}}{E_{14}\, E_{23}}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3\\2&4\end{smallmatrix}\right]^2\right\}\end{aligned} \tag{6.4.71}$$

where γ_k are the gamma matrices and $C^I{}_{\bar J}$ is the charge conjugation matrix of $SO(6)$.

The only non-vanishing five-point correlator with four spin fields and one fermion involves three alike chiralities:

$$\begin{aligned}\langle\, \Psi^k(z_1)\, \Sigma^I(z_2)\, \Sigma^J(z_3)\, \Sigma^K(z_4)\, \bar\Sigma_{\bar L}(z_5)\,\rangle^{\vec a}_{\vec b} \;&=\; \frac{1}{\sqrt2\,\left[\Theta^{\vec a}_{\vec b}(\vec 0)\right]^3}\,\frac{(E_{25}\, E_{35}\, E_{45})^{1/4}}{(E_{12}\, E_{13}\, E_{14}\, E_{15})^{1/2}\,(E_{23}\, E_{24}\, E_{34})^{1/4}}\\ &\times\Bigg\{(\gamma^k\, C)^{IJ}\,\frac{C^K{}_{\bar L}}{E_{45}}\,E_{14}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&4\\2&3&5\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&4\\3&5\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&5\\3&4\end{smallmatrix}\right]\\ &\quad-(\gamma^k\, C)^{IK}\,\frac{C^J{}_{\bar L}}{E_{35}}\,E_{13}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&3\\2&4&5\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&3\\4&5\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&5\\3&4\end{smallmatrix}\right]\\ &\quad+(\gamma^k\, C)^{JK}\,\frac{C^I{}_{\bar L}}{E_{25}}\,E_{12}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&2\\3&4&5\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&3\\4&5\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&4\\3&5\end{smallmatrix}\right]\Bigg\}\end{aligned} \tag{6.4.72}$$

There are two six-point correlators involving only spin fields. The first one consists of five left- and one right-handed spin-field:

$$\begin{aligned}\langle\, \Sigma^I(z_1)\, \Sigma^J(z_2)\, \Sigma^K(z_3)\, \Sigma^L(z_4)\, \Sigma^M(z_5)\, \bar\Sigma_{\bar N}(z_6)\,\rangle^{\vec a}_{\vec b} \;&=\; \frac{(E_{16}\, E_{26}\, E_{36}\, E_{46}\, E_{56})^{1/4}}{(E_{12}\, E_{13}\, E_{14}\, E_{15}\, E_{23}\, E_{24}\, E_{25}\, E_{34}\, E_{35}\, E_{45})^{1/4}}\\ \times\frac{1}{2\,\left[\Theta^{\vec a}_{\vec b}(\vec 0)\right]^3}\,\Bigg\{&(\gamma^k\, C)^{IJ}\,(\gamma_k\, C)^{KM}\,\frac{C^L{}_{\bar N}}{E_{46}}\,\frac{E_{45}}{E_{56}}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&6\\3&4&5\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3&6\\2&4&5\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&4&5\\2&3&6\end{smallmatrix}\right]\\ &+(\gamma^k\, C)^{IJ}\,(\gamma_k\, C)^{ML}\,\frac{C^K{}_{\bar N}}{E_{36}}\,\frac{E_{35}}{E_{56}}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&6\\3&4&5\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3&5\\2&4&6\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&4&6\\2&3&5\end{smallmatrix}\right]\\ &+(\gamma^k\, C)^{IM}\,(\gamma_k\, C)^{KL}\,\frac{C^J{}_{\bar N}}{E_{26}}\,\frac{E_{25}}{E_{56}}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&5\\3&4&6\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3&6\\2&4&5\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&4&6\\2&3&5\end{smallmatrix}\right]\\ &+(\gamma^k\, C)^{MJ}\,(\gamma_k\, C)^{KL}\,\frac{C^I{}_{\bar N}}{E_{16}}\,\frac{E_{15}}{E_{56}}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&5\\3&4&6\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3&5\\2&4&6\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&4&5\\2&3&6\end{smallmatrix}\right]\Bigg\}\end{aligned} \tag{6.4.73}$$

In addition we have the correlator with three left- and right-handed spin fields each,

$$\begin{aligned}\langle\, \Sigma^I(z_1)\, \Sigma^J(z_2)\, \Sigma^K(z_3)\, \bar\Sigma_{\bar L}(z_4)\, \bar\Sigma_{\bar M}(z_5)\, \bar\Sigma_{\bar N}(z_6)\,\rangle^{\vec a}_{\vec b} \;&=\; \frac{(E_{14}\, E_{15}\, E_{16}\, E_{24}\, E_{25}\, E_{26}\, E_{34}\, E_{35}\, E_{36})^{1/4}}{\left[\Theta^{\vec a}_{\vec b}(\vec 0)\right]^3\,(E_{12}\, E_{13}\, E_{23}\, E_{45}\, E_{46}\, E_{56})^{1/4}}\\ &\times\Bigg\{\frac{C^I{}_{\bar L}\, C^J{}_{\bar M}\, C^K{}_{\bar N}}{E_{14}\, E_{25}\, E_{36}}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&6\\3&4&5\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3&5\\2&4&6\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&5&6\\2&3&4\end{smallmatrix}\right]\end{aligned}$$

$$
\begin{aligned}
& - \frac{C^I{}_{\bar{L}}\, C^J{}_{\bar{N}}\, C^K{}_{\bar{M}}}{E_{14}\, E_{26}\, E_{35}}\, \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 3 & 6 \\ 2 & 4 & 5 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 5 & 6 \\ 2 & 3 & 4 \end{smallmatrix}\right] \\
& + \frac{C^I{}_{\bar{M}}\, C^J{}_{\bar{N}}\, C^K{}_{\bar{L}}}{E_{15}\, E_{26}\, E_{34}}\, \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 3 & 6 \\ 2 & 4 & 5 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 4 & 6 \\ 2 & 3 & 5 \end{smallmatrix}\right] \\
& - \frac{C^I{}_{\bar{M}}\, C^J{}_{\bar{L}}\, C^K{}_{\bar{N}}}{E_{15}\, E_{24}\, E_{36}}\, \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 2 & 6 \\ 3 & 4 & 5 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 4 & 6 \\ 2 & 3 & 5 \end{smallmatrix}\right] \\
& + \frac{C^I{}_{\bar{N}}\, C^J{}_{\bar{L}}\, C^K{}_{\bar{M}}}{E_{16}\, E_{24}\, E_{35}}\, \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 4 & 5 \\ 2 & 3 & 6 \end{smallmatrix}\right] \\
& - \frac{C^I{}_{\bar{N}}\, C^J{}_{\bar{M}}\, C^K{}_{\bar{L}}}{E_{16}\, E_{25}\, E_{34}}\, \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 4 & 5 \\ 2 & 3 & 6 \end{smallmatrix}\right] \Bigg\} \,. \qquad (6.4.74)
\end{aligned}
$$

Furthermore, two NS fermions can be accompanied by four spin fields, either with uniform chirality

$$
\begin{aligned}
\langle\, \Psi_k(z_1)\, \Psi_l(z_2)\, \Sigma^I(z_3)\, \Sigma^J(z_4)\, \Sigma^K(z_5)\, \Sigma^L(z_6)\,\rangle^{\vec{a}}_{\vec{b}} \;&=\; \frac{-\,(E_{13}\, E_{14}\, E_{15}\, E_{16}\, E_{23}\, E_{24}\, E_{25}\, E_{26})^{-1/2}}{2\left[\Theta^{\vec{a}}_{\vec{b}}(\vec{0})\right]^3 E_{12}\,(E_{34}\, E_{35}\, E_{36}\, E_{45}\, E_{46}\, E_{56}\,)^{1/4}} \\
& \Bigg\{ (\gamma_k\, C)^{IJ}\, (\gamma_l\, C)^{KL}\, E_{15}\, E_{16}\, E_{23}\, E_{24}\, \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 1 & 5 & 6 \\ 2 & 2 & 3 & 4 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 3 & 5 \\ 4 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 3 & 6 \\ 4 & 5 \end{smallmatrix}\right] \\
& + (\gamma_k\, C)^{KL}\, (\gamma_l\, C)^{IJ}\, E_{13}\, E_{14}\, E_{25}\, E_{26}\, \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 1 & 3 & 4 \\ 2 & 2 & 5 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 3 & 5 \\ 4 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 3 & 6 \\ 4 & 5 \end{smallmatrix}\right] \\
& - (\gamma_k\, C)^{IK}\, (\gamma_l\, C)^{JL}\, E_{14}\, E_{16}\, E_{23}\, E_{25}\, \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 1 & 4 & 6 \\ 2 & 2 & 3 & 5 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 3 & 4 \\ 5 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 3 & 6 \\ 4 & 5 \end{smallmatrix}\right] \\
& - (\gamma_k\, C)^{JL}\, (\gamma_l\, C)^{IK}\, E_{13}\, E_{15}\, E_{24}\, E_{26}\, \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 1 & 3 & 5 \\ 2 & 2 & 4 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 3 & 4 \\ 5 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 3 & 6 \\ 4 & 5 \end{smallmatrix}\right] \\
& + (\gamma_k\, C)^{IL}\, (\gamma_l\, C)^{JK}\, E_{14}\, E_{15}\, E_{23}\, E_{26}\, \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 1 & 4 & 5 \\ 2 & 2 & 3 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 3 & 4 \\ 5 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 3 & 5 \\ 4 & 6 \end{smallmatrix}\right] \\
& + (\gamma_k\, C)^{JK}\, (\gamma_l\, C)^{IL}\, E_{13}\, E_{16}\, E_{24}\, E_{25}\, \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 1 & 3 & 6 \\ 2 & 2 & 4 & 5 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 3 & 4 \\ 5 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 3 & 5 \\ 4 & 6 \end{smallmatrix}\right] \Bigg\} \,, \qquad (6.4.75)
\end{aligned}
$$

or with mixed chiralities:

$$
\begin{aligned}
\langle\, \Psi_k(z_1)\, \Psi_l(z_2)\, \Sigma^I(z_3)\, \Sigma^J(z_4)\, \bar{\Sigma}_{\bar{K}}(z_5)\, \bar{\Sigma}_{\bar{L}}(z_6)\,\rangle^{\vec{a}}_{\vec{b}} \;&=\; \frac{(E_{35}\, E_{36}\, E_{45}\, E_{46})^{1/4}\, (E_{34}\, E_{56})^{-1/4}}{\left[\Theta^{\vec{a}}_{\vec{b}}(\vec{0})\right]^3 (E_{13}\, E_{14}\, E_{15}\, E_{16}\, E_{23}\, E_{24}\, E_{25}\, E_{26})^{1/2}} \\
& \Bigg\{ \frac{\delta_{kl}\, C^I{}_{\bar{K}}\, C^J{}_{\bar{L}}}{E_{12}\, E_{35}\, E_{46}}\, E_{13}\, E_{14}\, E_{25}\, E_{26}\, \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 1 & 3 & 4 \\ 2 & 2 & 5 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 3 & 6 \\ 4 & 5 \end{smallmatrix}\right]^2 \\
& - \frac{\delta_{kl}\, C^I{}_{\bar{L}}\, C^J{}_{\bar{K}}}{E_{12}\, E_{36}\, E_{45}}\, E_{13}\, E_{14}\, E_{25}\, E_{26}\, \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 1 & 3 & 4 \\ 2 & 2 & 5 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 3 & 5 \\ 4 & 6 \end{smallmatrix}\right]^2 \\
& + \frac{1}{2}\, (\gamma_k\, C)^{IJ} (\bar{\gamma}_l\, C)_{\bar{K}\bar{L}}\, E_{12}\, \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 1 & 2 & 2 \\ 3 & 4 & 5 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 3 & 5 \\ 4 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 3 & 6 \\ 4 & 5 \end{smallmatrix}\right] \\
& + \frac{1}{2}\, (\gamma_k\, \bar{\gamma}_l\, C)^I{}_{\bar{K}}\, \frac{C^J{}_{\bar{L}}}{E_{46}}\, E_{14}\, E_{26}\, \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 1 & 4 \\ 3 & 5 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 2 & 2 & 6 \\ 3 & 4 & 5 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 3 & 6 \\ 4 & 5 \end{smallmatrix}\right] \\
& - \frac{1}{2}\, (\gamma_k\, \bar{\gamma}_l\, C)^I{}_{\bar{L}}\, \frac{C^J{}_{\bar{K}}}{E_{45}}\, E_{14}\, E_{25}\, \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 1 & 4 \\ 3 & 5 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 2 & 2 & 5 \\ 3 & 4 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 3 & 5 \\ 4 & 6 \end{smallmatrix}\right] \\
& - \frac{1}{2}\, (\gamma_k\, \bar{\gamma}_l\, C)^J{}_{\bar{K}}\, \frac{C^I{}_{\bar{L}}}{E_{36}}\, E_{13}\, E_{26}\, \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 1 & 3 \\ 4 & 5 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 2 & 2 & 6 \\ 3 & 4 & 5 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 3 & 5 \\ 4 & 6 \end{smallmatrix}\right] \\
& + \frac{1}{2}\, (\gamma_k\, \bar{\gamma}_l\, C)^J{}_{\bar{L}}\, \frac{C^I{}_{\bar{K}}}{E_{35}}\, E_{13}\, E_{25}\, \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 1 & 1 & 3 \\ 4 & 5 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 2 & 2 & 5 \\ 3 & 4 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}}\left[\begin{smallmatrix} 3 & 6 \\ 4 & 5 \end{smallmatrix}\right] \Bigg\} \qquad (6.4.76)
\end{aligned}
$$

A seven point function $\langle \Psi_k \Sigma^I \Sigma^J \Sigma^K \Sigma^L \bar{\Sigma}_{\bar{M}} \bar{\Sigma}_{\bar{N}} \rangle^{\vec{a}}_{\vec{b}}$ can be found in [5].

6.4.2 Generalizations to higher point

In six space-time dimensions, the structure of Fierz identities is still sufficiently simple that the $2n$-point correlation function with n left- and right handed spin fields each can be expressed in terms of charge conjugation matrices $C^{K_i}{}_{\bar{L}_j}$ only. Inspired by the lower order results (6.4.71) and (6.4.74), we claim the following generalization:

$$\left\langle \prod_{i=1}^{n} \Sigma^{K_i}(z_{2i-1})\,\bar{\Sigma}_{\bar{L}_i}(z_{2i}) \right\rangle_{\vec{b}}^{\vec{a}} = \frac{\left[\Theta_{\vec{b}}^{\vec{a}}(\sum_{i=1}^{n} \frac{1}{2}\int_{z_{2i-1}}^{z_{2i}} \vec{\omega})\right]^{3-n}}{\left[\Theta_{\vec{b}}^{\vec{a}}(\vec{0})\right]^{3}} \left(\frac{\prod_{i,j=1}^{n} E_{2i-1,2j}}{\prod_{i<j}^{n} E_{2i-1,2j-1}\,E_{2i,2j}} \right)^{1/4}$$
$$\times \sum_{\rho \in S_n} \mathrm{sgn}(\rho) \prod_{m=1}^{n} \frac{C^{K_{2m-1}}{}_{\bar{L}_{\rho(2m)}}}{E_{2m-1,\rho(2m)}}\, \Theta_{\vec{b}}^{\vec{a}}\left(\sum_{i=1}^{n} \frac{1}{2}\int_{z_{2i-1}}^{z_{2i}} \vec{\omega} - \int_{z_{\rho(2m)}}^{2m-1} \vec{\omega} \right) \tag{6.4.77}$$

For higher n, Fay's trisecant identities might be needed to make specific $K_i, \bar{L}_j$ choices compatible with the result above. Due to the chirality structure of the charge conjugation matrix in $D = 6$ the correlator above is the direct relative of $\langle S_{a_1}(z_1)\ldots S_{a_{2n}}(z_{2n})\rangle$ in $D = 4$. Therefore the proof of (6.4.77) proceeds in the same way as in [4].

Having the explicit formula (6.4.77) for this class of correlators is a great benefit in view of the factorization prescription for NS fermions. In $D = 6$ dimensions, one can combine two spin fields of alike chirality to a NS fermion via

$$\Psi_k(w) = -\frac{1}{2\sqrt{2}} \lim_{z\to w} (z-w)^{1/4}\,(C^{-1}\,\bar{\gamma}_k)_{JI}\,\Sigma^I(z)\,\Sigma^J(w) \tag{6.4.78}$$

and thereby derive the following further classes of correlation functions:

$$\langle\, \Psi_{k_1}(z_1)\ldots\Psi_{k_{p+q}}(z_{p+q})\,\Sigma^{I_1}(x_1)\ldots\Sigma^{I_{n-2p}}(x_{n-2p})\,\bar{\Sigma}_{\bar{J}_1}(y_1)\ldots\bar{\Sigma}_{\bar{J}_{n-2q}}(y_{n-2q})\,\rangle_{\vec{b}}^{\vec{a}} \tag{6.4.79}$$

For example the last two correlators calculated in the previous subsection can be derived from the $n = 4$ version of (6.4.77).

6.5 Correlators in higher dimensions

The general results (6.3.62) to (6.3.64) and (6.4.77) for $D = 4, 6$ loop correlators of arbitrary size are not at all transferable to higher dimensions. This is due to the group structure of $SO(1,7)$ and $SO(1,9)$, in particular due to the more involved nature of the corresponding Fierz identities. This is why we can only give a limited number of higher genus examples in $D = 8$ and $D = 10$ with finite number of field insertions.

In this section, we will collect some selected results in $D = 8$ and $D = 10$ dimensions which were presented in more detail in [5]. We give an outlook on how $SO(8)$ triality becomes useful to get a better handle on the RNS CFT in $D = 8$ dimensions in subsection 6.5.1. Then, a second subsection 6.5.2 explicitly displays a few $D = 10$ correlators which have been recognized as useful in the literature.

6.5.1 Tree level triality in $D = 8$

The S_3 permutation symmetry of the Mercedes star–shaped $SO(8)$ Dynkin diagram in figure 6.1 – also referred to as triality – plays an important role for the eight dimensional RNS CFT.

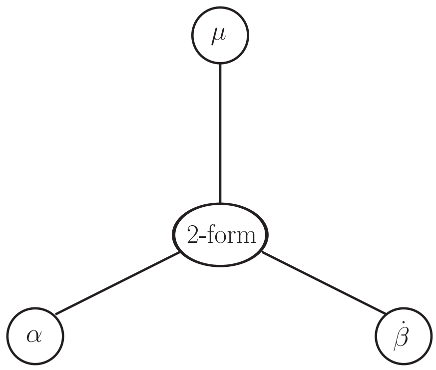


Figure 6.1: Dynkin diagram for $SO(8)$. We will use vector indices $\mu, \nu, \lambda, \ldots$ and spinor indices $\alpha, \beta, \dot{\alpha}, \beta, \ldots$ in this subsection on $D = 8$.

In $D = 8$, NS fermions and spin fields have equal conformal dimension $h = \frac{D}{16} = \frac{1}{2}$. Therefore, the OPEs (6.1.1) to (6.1.3) become particularly symmetric and we can make use of $SO(8)$ triality to rewrite them in unified fashion:

$$P^i(z)\, P^j(w) \;\sim\; \frac{g^{ij}}{z-w} \;+\; \ldots \;, \tag{6.5.80}$$

$$P^i(z)\, Q^j(w) \;\sim\; \frac{G^{ijk}}{(z-w)^{1/2}}\, g_{kl}\, R^l(w) \;+\; \ldots \tag{6.5.81}$$

These OPEs involves two new elements of notation: generalized $h = \frac{1}{2}$ fields P^i, Q^j, R^k

$$(P^i, Q^j, R^k) \;=\; \big(\rho(\psi^\mu), \rho(S_\alpha), \rho(S^{\dot{\beta}})\big) \;, \qquad \rho \in S_3 \;, \tag{6.5.82}$$

and triality covariant Clebsch Gordan coefficients with generalized indices $i, j, k \in \{\mu, \alpha, \dot{\beta}\}$:

$$g^{ij} \;:=\; \begin{cases} \eta^{\mu\nu} & : \; (i,j) = (\mu,\nu) \\ C_{\alpha\beta} & : \; (i,j) = (\alpha,\beta) \\ C^{\dot{\alpha}\dot{\beta}} & : \; (i,j) = (\dot{\alpha},\dot{\beta}) \\ 0 & : \; \text{otherwise} \end{cases} \;, \qquad g_{ij} \;:=\; \begin{cases} \eta_{\mu\nu} & : \; (i,j) = (\mu,\nu) \\ C^{\alpha\beta} & : \; (i,j) = (\alpha,\beta) \\ C_{\dot{\alpha}\dot{\beta}} & : \; (i,j) = (\dot{\alpha},\dot{\beta}) \\ 0 & : \; \text{otherwise} \end{cases} \;, \tag{6.5.83}$$

$$G^{ijk} \;:=\; \begin{cases} \frac{1}{\sqrt{2}}\, (\gamma^\mu\, C)_\alpha{}^{\dot{\beta}} & : \; (i,j,k) = \big(\rho(\mu), \rho(\alpha), \rho(\dot{\beta})\big) \text{ for some } \rho \in S_3 \\ 0 & : \; \text{otherwise} \end{cases} \tag{6.5.84}$$

In this notation, both the Dirac algebra $\eta^{\mu\nu}C_{\alpha\beta} = -(\gamma^{(\mu}\bar{\gamma}^{\nu)}C)_{\alpha\beta}$ and the $D=8$ Fierz identity $C_{\alpha\beta}C^{\dot{\gamma}\dot{\delta}} = \frac{1}{2}(\gamma^\mu C)_\alpha{}^{\dot{\gamma}}(\gamma_\mu C)_\beta{}^{\dot{\delta}} + \frac{1}{2}(\gamma^\mu C)_\beta{}^{\dot{\gamma}}(\gamma_\mu C)_\alpha{}^{\dot{\delta}}$ are special cases of the triality covariant tensor equation

$$g^{i_1 i_2}\, g^{j_1 j_2} \;\; = \;\; G^{i_1 j_1 k_1}\, G^{i_2 j_2 k_2}\, g_{k_1 k_2} \; + \; G^{i_1 j_2 k_1}\, G^{i_2 j_1 k_2}\, g_{k_1 k_2} \; . \tag{6.5.85}$$

The OPEs (6.5.80) and (6.5.81) provide all the input necessary to derive the tree-level correlation function $\langle P^{i_1}(x_1)...P^{i_p}(x_p)\, Q^{j_1}(y_1)...Q^{j_q}(y_q) R^{k_1}(z_1)...R^{k_r}(z_r)\rangle$ in triality covariant form – every assignment $\rho \in S_3$ for $(P^i, Q^j, R^k) = \big(\rho(\psi^\mu), \rho(S_\alpha), \rho(S^{\dot{\beta}})\big)$ is possible.

This circumstance can be used as a strong tool to derive new correlators: Suppose we know $\langle \psi^{\mu_1}...\psi^{\mu_\ell} S_{\alpha_1}...S_{\alpha_m} S^{\dot{\beta}_1}...S^{\dot{\beta}_n}\rangle$ for some $\ell, m, n \in \mathbb{N}_0$, then one can rewrite this result in a covariant way as $\langle P^{i_1}...P^{i_\ell} Q^{j_1}...Q^{j_m} R^{k_1}...R^{k_n}\rangle$ via $(\psi^\mu, S_\alpha, S^{\dot{\beta}}) \equiv (P^i, Q^j, R^k)$ as well as $\eta^{\mu\nu}, C_{\alpha\beta}, C^{\dot{\alpha}\dot{\beta}} \mapsto g^{ij}$ and $(\gamma^\mu C)_\alpha{}^{\dot{\beta}} \mapsto \sqrt{2}G^{ijk}$. One is then free to pick a different assignment, e.g. $(P^i, Q^j, R^k) \equiv (S_\alpha, S^{\dot{\beta}}, \psi^\mu)$ which yields the correlator $\langle \psi^{\mu_1}...\psi^{\mu_n} S_{\alpha_1}...S_{\alpha_\ell} S^{\dot{\beta}_1}...S^{\dot{\beta}_m}\rangle$ with (ℓ, m, n) traded for (n, ℓ, m).

The most interesting application of this procedure is to relate the $2n$- and $(2n+1)$-point correlation functions from section 6.2 to so far unknown correlators with a large number of spin field insertions:

$$\langle\, \psi^{2\ell}\, S^2 \,\rangle \;\; \leftrightarrow \;\; \langle\, \psi^2\, S^{2\ell} \,\rangle \, , \qquad \langle\, \psi^{2\ell}\, S^2 \,\rangle \;\; \leftrightarrow \;\; \langle\, S^{2\ell}\, \dot{S}^2 \,\rangle \, , \qquad \langle\, \psi^{2\ell-1}\, S\, \dot{S} \,\rangle \;\; \leftrightarrow \;\; \langle\, \psi\, S^{2\ell-1}\, \dot{S} \,\rangle$$

This requires rewriting the $D=8$ tree level correlators (6.2.39) and (6.2.40) in $SO(8)$ triality covariant fashion[5]:

$$\begin{aligned}
\Omega^{i_1 \ldots i_{2n-1} jk}_{(n,D=8)}(z_i) \;\; &:= \;\; \langle\, P^{i_1}(z_1)\, P^{i_2}(z_2) \ldots P^{i_{2n-1}}(z_{2n-1})\, Q^j(z_A)\, R^k(z_B) \,\rangle \\
= \;\; &\frac{1}{z_{AB}^{1/2}\, \prod_{i=1}^{2n-1}(z_{iA}\, z_{iB})^{1/2}} \sum_{\ell=0}^{n-1} \big(-\, z_{AB}\big)^\ell \sum_{\rho \in S_{2n-1}/\mathcal{P}_{n,\ell}} G^{i_{\rho(1)} j r_1}\, G^{i_{\rho(2)} q_1}{}_{r_1}\, G^{i_{\rho(3)}}{}_{q_1}{}^{r_2}\, G^{i_{\rho(4)} q_2}{}_{r_2} \ldots G^{i_{\rho(2\ell+1)}}{}_{q_\ell}{}^{k} \\
&\times \operatorname{sgn}(\rho) \prod_{j=1}^{n-\ell-1} \frac{g^{i_{\rho(2\ell+2j)} i_{\rho(2\ell+2j+1)}}}{z_{\rho(2\ell+2j),\rho(2\ell+2j+1)}}\, z_{\rho(2\ell+2j),A}\, z_{\rho(2\ell+2j+1),B} \, ,
\end{aligned} \tag{6.5.86}$$

$$\begin{aligned}
\omega^{i_1 \ldots i_{2n-2} j_1 j_2}_{(n,D=8)}(z_i) \;\; &:= \;\; \langle\, P^{i_1}(z_1)\, P^{i_2}(z_2) \ldots P^{i_{2n-2}}(z_{2n-2})\, Q^{j_1}(z_A)\, Q^{j_2}(z_B) \,\rangle \\
= \;\; &\frac{1}{z_{AB}\, \prod_{i=1}^{2n-2}(z_{iA}\, z_{iB})^{1/2}} \sum_{\ell=0}^{n-1} \big(-\, z_{AB}\big)^\ell \sum_{\rho \in S_{2n-2}/\mathcal{Q}_{n,\ell}} G^{i_{\rho(1)} j_1 r_1}\, G^{i_{\rho(2)} q_1}{}_{r_1}\, G^{i_{\rho(3)}}{}_{q_1}{}^{r_2} \ldots G^{i_{\rho(2\ell)} j_2}{}_{r_\ell} \\
&\times \;\operatorname{sgn}(\rho) \prod_{j=1}^{n-\ell-1} \frac{g^{i_{\rho(2\ell+2j-1)} i_{\rho(2\ell+2j)}}}{z_{\rho(2\ell+2j-1),\rho(2\ell+2j)}}\, z_{\rho(2\ell+2j-1),A}\, z_{\rho(2\ell+2j),B}
\end{aligned} \tag{6.5.87}$$

[5]The $(z_{AB}/2)^\ell$ factors from (6.2.39) and (6.2.40) are converted to $(-z_{AB})^\ell$ by means of $(\gamma^\mu C)_\alpha{}^{\dot{\beta}} \equiv \sqrt{2}G^{ijk}$ as well as $(\bar{\gamma}^\mu C)^{\dot{\beta}}{}_\alpha \equiv -\sqrt{2}G^{ijk}$.

They give rise to new tree level results under $(P^{i_\ell}, Q^j, R^k) \equiv (S_{\alpha_\ell}, \psi^\mu, S^{\dot\beta})$, $(P^{i_\ell}, Q^{j_1}, Q^{j_2}) \equiv (S_{\alpha_\ell}, \psi^\mu, \psi^\nu)$ or $(P^{i_\ell}, Q^{j_1}, Q^{j_2}) \equiv (S_{\alpha_\ell}, S^{\dot\beta}, S^{\dot\gamma})$. However, not all correlation functions can be derived from (6.2.39) and (6.2.40) via triality. Already the six-point function $\langle \psi^\mu \psi^\nu S_\alpha S_\beta S^{\dot\gamma} S^{\dot\delta}\rangle$, for instance, must be derived by the standard algorithms from section 6.1.

6.5.2 Correlators in ten dimensions

The ten dimensional RNS CFT does not seem to admit any shortcuts towards tree level- or loop correlators. We could neither find closed expressions for correlators with a large number of spin fields nor any representation theoretic trick comparable to $SO(8)$ triality. All we can do in this subsection is collecting some lower point examples of loop correlators which were computed by applying the standard algorithms.

Ten dimensional correlators with four spin fields only are given by

$$\begin{aligned}
\langle\, S_\alpha(z_1)\, S_\beta(z_2)\, S_\gamma(z_3)\, S_\delta(z_4)\,\rangle^{\vec a}_{\vec b} \;&=\; \frac{1}{2\,(E_{12}\,E_{13}\,E_{14}\,E_{23}\,E_{24}\,E_{34})^{3/4}}\,\frac{\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2\\3&4\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3\\2&4\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&4\\2&3\end{smallmatrix}\right]}{\left[\Theta^{\vec a}_{\vec b}(\vec 0)\right]^5}\\
&\times\Big\{\,(\gamma^m\,C)_{\alpha\beta}\,(\gamma_m\,C)_{\gamma\delta}\,E_{14}\,E_{23}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&4\\2&3\end{smallmatrix}\right]^2\;-\;(\gamma^m\,C)_{\alpha\delta}\,(\gamma_m\,C)_{\gamma\beta}\,E_{12}\,E_{34}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2\\3&4\end{smallmatrix}\right]^2\,\Big\}\;,\\
\langle\, S_\alpha(z_1)\, S_\beta(z_2)\, S^{\dot\gamma}(z_3)\, S^{\dot\delta}(z_4)\,\rangle^{\vec a}_{\vec b} \;&=\; \left(\frac{E_{12}\,E_{34}}{E_{13}\,E_{14}\,E_{23}\,E_{24}}\right)^{1/4}\,\frac{\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2\\3&4\end{smallmatrix}\right]}{\left[\Theta^{\vec a}_{\vec b}(\vec 0)\right]^5}\\
&\times\Bigg\{\,\frac{C_\alpha{}^{\dot\delta}\,C_\beta{}^{\dot\gamma}}{E_{14}\,E_{23}}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2\\3&4\end{smallmatrix}\right]^2\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3\\2&4\end{smallmatrix}\right]^2\;-\;\frac{C_\alpha{}^{\dot\gamma}\,C_\beta{}^{\dot\delta}}{E_{13}\,E_{24}}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2\\3&4\end{smallmatrix}\right]^2\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&4\\2&3\end{smallmatrix}\right]^2\\
&\qquad+\;\frac{1}{2}\,\frac{(\gamma^m\,C)_{\alpha\beta}\,(\bar\gamma_m\,C)^{\dot\gamma\dot\delta}}{E_{12}\,E_{34}}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3\\2&4\end{smallmatrix}\right]^2\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&4\\2&3\end{smallmatrix}\right]^2\,\Bigg\}\;.
\end{aligned}\tag{6.5.88}$$

In presence of one NS fermion, we find

$$\begin{aligned}
\langle\,\psi^m(z_1)\,S_\alpha(z_2)\,S_\beta(z_3)\,S_\gamma(z_4)\,S^{\dot\delta}(z_5)\,\rangle^{\vec a}_{\vec b}\;&=\;\frac{(E_{23}\,E_{24}\,E_{34})^{-3/4}}{\sqrt2\,\left[\Theta^{\vec a}_{\vec b}(\vec 0)\right]^5\,(E_{12}\,E_{13}\,E_{14}\,E_{15})^{1/2}\,(E_{25}\,E_{35}\,E_{45})^{1/4}}\\
\times\Bigg\{\,&\frac{C_\gamma{}^{\dot\delta}}{E_{45}}\,(\gamma^m\,C)_{\alpha\beta}\,E_{15}\,E_{24}\,E_{34}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&5\\3&4\end{smallmatrix}\right]^2\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&4\\3&5\end{smallmatrix}\right]^2\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&5\\2&3&4\end{smallmatrix}\right]\\
+\;&\frac{C_\alpha{}^{\dot\delta}}{E_{25}}\,(\gamma^m\,C)_{\beta\gamma}\,E_{15}\,E_{23}\,E_{24}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&3\\4&5\end{smallmatrix}\right]^2\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&4\\3&5\end{smallmatrix}\right]^2\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&5\\2&3&4\end{smallmatrix}\right]\\
-\;&\frac{C_\beta{}^{\dot\delta}}{E_{35}}\,(\gamma^m\,C)_{\alpha\gamma}\,E_{15}\,E_{23}\,E_{34}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&5\\2&3&4\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&3\\4&5\end{smallmatrix}\right]^2\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&5\\3&4\end{smallmatrix}\right]^2\\
-\;&\frac{1}{2}\,(\gamma^n\,\bar\gamma^m\,C)_\gamma{}^{\dot\delta}\,(\gamma_n\,C)_{\alpha\beta}\,E_{12}\,E_{34}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&2\\3&4&5\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&3\\4&5\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&4\\3&5\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&5\\3&4\end{smallmatrix}\right]^2\\
+\;&\frac{1}{2}\,(\gamma^n\,\bar\gamma^m\,C)_\alpha{}^{\dot\delta}\,(\gamma_n\,C)_{\beta\gamma}\,E_{14}\,E_{23}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&4\\2&3&5\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&4\\3&5\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&5\\3&4\end{smallmatrix}\right]\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&3\\4&5\end{smallmatrix}\right]^2\,\Bigg\}\;.
\end{aligned}\tag{6.5.89}$$

The following correlator has been partially computed in [220] for the purpose of four fermion scattering at $g = 1$ loop. Let us give the complete g loop result here:

$$
\begin{aligned}
\langle\, \psi^m(z_1)\, \psi^n(z_2)\, S_\alpha(z_3)\, S_\beta(z_4)\, S_\gamma(z_5)\, S_\delta(z_6)\,\rangle^{\vec a}_{\vec b} \;&=\; \frac{(E_{34}\, E_{35}\, E_{36}\, E_{45}\, E_{46}\, E_{56})^{-3/4}}{2\left[\Theta^{\vec a}_{\vec b}(\vec 0)\right]^5 (E_{13}\, E_{14}\, E_{15}\, E_{16}\, E_{23}\, E_{24}\, E_{25}\, E_{26})^{1/2}} \\
&\Bigg\{ \frac{\eta^{mn}}{E_{12}}\, (\gamma^p\, C)_{\alpha\beta}\, (\gamma_p\, C)_{\gamma\delta}\, E_{36}\, E_{45}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&4\\5&6\end{smallmatrix}\right] \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&5\\4&6\end{smallmatrix}\right] \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&6\\4&5\end{smallmatrix}\right]^2 \\
&\qquad \Big(E_{13}\, E_{16}\, E_{24}\, E_{25}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&3&6\\2&2&4&5\end{smallmatrix}\right] \;+\; E_{14}\, E_{15}\, E_{23}\, E_{26}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&4&5\\2&2&3&6\end{smallmatrix}\right] \Big) \\
&- \frac{\eta^{mn}}{E_{12}}\, (\gamma^p\, C)_{\alpha\delta}\, (\gamma_p\, C)_{\gamma\beta}\, E_{34}\, E_{56}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&6\\5&4\end{smallmatrix}\right] \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&5\\6&4\end{smallmatrix}\right] \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&4\\6&5\end{smallmatrix}\right]^2 \\
&\qquad \Big(E_{13}\, E_{14}\, E_{25}\, E_{26}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&3&4\\2&2&6&5\end{smallmatrix}\right] \;+\; E_{15}\, E_{16}\, E_{23}\, E_{24}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&6&5\\2&2&3&4\end{smallmatrix}\right] \Big) \\
&+ \frac{1}{2}\, (\gamma^m\, C)_{\gamma\beta}\, (\gamma^n\, C)_{\alpha\delta}\, E_{34}\, E_{56}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&4\\5&6\end{smallmatrix}\right]^2 \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&5\\4&6\end{smallmatrix}\right] \\
&\qquad \Big(E_{13}\, E_{25}\, E_{46}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&3\\4&5&6\end{smallmatrix}\right] \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&2&5\\3&4&6\end{smallmatrix}\right] \;-\; E_{16}\, E_{24}\, E_{35}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&6\\3&4&5\end{smallmatrix}\right] \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&2&4\\3&5&6\end{smallmatrix}\right] \Big) \\
&+ \frac{1}{2}\, (\gamma^m\, C)_{\alpha\delta}\, (\gamma^n\, C)_{\gamma\beta}\, E_{34}\, E_{56}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&4\\5&6\end{smallmatrix}\right]^2 \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&5\\4&6\end{smallmatrix}\right] \\
&\qquad \Big(E_{14}\, E_{26}\, E_{35}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&4\\3&5&6\end{smallmatrix}\right] \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&2&6\\3&4&5\end{smallmatrix}\right] \;-\; E_{15}\, E_{23}\, E_{46}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&5\\3&4&6\end{smallmatrix}\right] \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&2&3\\4&5&6\end{smallmatrix}\right] \Big) \\
&- \frac{1}{2}\, (\gamma^m\, C)_{\alpha\beta}\, (\gamma^n\, C)_{\gamma\delta}\, E_{36}\, E_{45}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&6\\4&5\end{smallmatrix}\right]^2 \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&5\\4&6\end{smallmatrix}\right] \\
&\qquad \Big(E_{16}\, E_{24}\, E_{35}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&6\\3&4&5\end{smallmatrix}\right] \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&2&4\\3&5&6\end{smallmatrix}\right] \;+\; E_{15}\, E_{23}\, E_{46}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&5\\3&4&6\end{smallmatrix}\right] \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&2&3\\4&5&6\end{smallmatrix}\right] \Big) \\
&+ \frac{1}{2}\, (\gamma^m\, C)_{\gamma\delta}\, (\gamma^n\, C)_{\alpha\beta}\, E_{36}\, E_{45}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&6\\4&5\end{smallmatrix}\right]^2 \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&5\\4&6\end{smallmatrix}\right] \\
&\qquad \Big(E_{13}\, E_{25}\, E_{46}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&3\\4&5&6\end{smallmatrix}\right] \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&2&5\\3&4&6\end{smallmatrix}\right] \;+\; E_{14}\, E_{26}\, E_{35}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&4\\3&5&6\end{smallmatrix}\right] \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&2&6\\3&4&5\end{smallmatrix}\right] \Big) \\
&+ \frac{1}{4}\, (\gamma^{mnp}\, C)_{\alpha\beta}\, (\gamma_p\, C)_{\gamma\delta}\; E_{34}\, E_{36}\, E_{45}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&4\\5&6\end{smallmatrix}\right] \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&6\\4&5\end{smallmatrix}\right]^2 \\
&\qquad \Big(E_{15}\, E_{26}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&5\\3&4&6\end{smallmatrix}\right] \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&2&6\\3&4&5\end{smallmatrix}\right] \;+\; E_{16}\, E_{25}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&6\\3&4&5\end{smallmatrix}\right] \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&2&5\\3&4&6\end{smallmatrix}\right] \Big) \\
&+ \frac{1}{4}\, (\gamma^{mnp}\, C)_{\gamma\delta}\, (\gamma_p\, C)_{\alpha\beta}\; E_{36}\, E_{45}\, E_{56}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&4\\5&6\end{smallmatrix}\right] \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&6\\4&5\end{smallmatrix}\right]^2 \\
&\qquad \Big(E_{13}\, E_{24}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&3\\4&5&6\end{smallmatrix}\right] \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&2&4\\3&5&6\end{smallmatrix}\right] \;+\; E_{14}\, E_{23}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&4\\3&5&6\end{smallmatrix}\right] \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&2&3\\4&5&6\end{smallmatrix}\right] \Big) \\
&- \frac{1}{4}\, (\gamma^{mnp}\, C)_{\alpha\delta}\, (\gamma_p\, C)_{\gamma\beta}\; E_{34}\, E_{36}\, E_{56}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&4\\5&6\end{smallmatrix}\right]^2 \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&6\\4&5\end{smallmatrix}\right] \\
&\qquad \Big(E_{15}\, E_{24}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&5\\3&4&6\end{smallmatrix}\right] \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&2&4\\3&5&6\end{smallmatrix}\right] \;+\; E_{14}\, E_{25}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&4\\3&5&6\end{smallmatrix}\right] \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&2&5\\3&4&6\end{smallmatrix}\right] \Big) \\
&+ \frac{1}{4}\, (\gamma^{mnp}\, C)_{\gamma\beta}\, (\gamma_p\, C)_{\alpha\delta}\; E_{34}\, E_{45}\, E_{56}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&4\\5&6\end{smallmatrix}\right]^2 \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&6\\4&5\end{smallmatrix}\right] \\
&\qquad \Big(E_{13}\, E_{26}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&3\\4&5&6\end{smallmatrix}\right] \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&2&6\\3&4&5\end{smallmatrix}\right] \;+\; E_{16}\, E_{23}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&6\\3&4&5\end{smallmatrix}\right] \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}2&2&3\\4&5&6\end{smallmatrix}\right] \Big) \\
&- (\gamma^{(m}\, C)_{\alpha\gamma}\, (\gamma^{n)}\, C)_{\beta\delta}\, E_{12}\, E_{34}\, E_{36}\, E_{45}\, E_{56}\, \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&1&2&2\\3&4&5&6\end{smallmatrix}\right] \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&4\\5&6\end{smallmatrix}\right]^2 \Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}3&6\\4&5\end{smallmatrix}\right]^2 \Bigg\}
\end{aligned}
\tag{6.5.90}
$$

This representation in terms of antisymmetric products γ^{mnp} rather than $\gamma^m \bar\gamma^n \gamma^p$ was chosen in order to make antisymmetry under the exchange of $\psi^m(z_1) \leftrightarrow \psi^n(z_2)$ and $S_{\alpha_i}(z_i) \leftrightarrow S_{\alpha_j}(z_j)$

manifest (up to a prefactor $E_{ij}^{1/4}$ in the latter case).

The correlator with five left handed spin fields and one righthanded spin field has appeared in the literature before, namely in [230] at tree level for the purpose of six fermion scattering. Let us give its loop generalization here:

$$
\begin{aligned}
\langle\, S_\alpha(z_1)\, S_\beta(z_2)\, S_\gamma(z_3)\, S_\delta(z_4)\, S_\epsilon(z_5)\, S^i(z_6)\,\rangle^{\vec a}_{\vec b} \;&=\; \frac{(E_{12}\,E_{13}\,E_{14}\,E_{15}\,E_{23}\,E_{24}\,E_{25}\,E_{34}\,E_{35}\,E_{45})^{-3/4}}{2\left[\Theta^{\vec a}_{\vec b}(\vec 0)\right]^5 (E_{16}\,E_{26}\,E_{36}\,E_{46}\,E_{56})^{1/4}} \\
\times \Bigg\{ (\gamma^m C)_{\alpha\beta}(\gamma_m C)_{\gamma\delta}\,\frac{C_\epsilon{}^i}{E_{56}}\, & E_{14}\,E_{15}\,E_{23}\,E_{25}\,E_{35}\,E_{46}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&5\\3&4&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3&5\\2&4&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&4&5\\2&3&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&4&6\\2&3&5\end{smallmatrix}\right]^2 \\
-(\gamma^m C)_{\alpha\delta}(\gamma_m C)_{\beta\gamma}\,\frac{C_\epsilon{}^i}{E_{56}}\, & E_{12}\,E_{15}\,E_{25}\,E_{34}\,E_{35}\,E_{46}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&6\\3&4&5\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3&5\\2&4&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&4&6\\2&3&5\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&5\\3&4&6\end{smallmatrix}\right]^2 \\
+(\gamma^m C)_{\alpha\beta}(\gamma_m C)_{\gamma\epsilon}\,\frac{C_\delta{}^i}{E_{46}}\, & E_{14}\,E_{15}\,E_{23}\,E_{24}\,E_{34}\,E_{56}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&4\\3&5&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3&4\\2&5&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&4&5\\2&3&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&5&6\\2&3&4\end{smallmatrix}\right]^2 \\
-(\gamma^m C)_{\alpha\epsilon}(\gamma_m C)_{\beta\gamma}\,\frac{C_\delta{}^i}{E_{46}}\, & E_{12}\,E_{14}\,E_{24}\,E_{34}\,E_{35}\,E_{56}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&6\\3&4&5\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3&4\\2&5&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&5&6\\2&3&4\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&4\\3&5&6\end{smallmatrix}\right]^2 \\
+(\gamma^m C)_{\alpha\beta}(\gamma_m C)_{\delta\epsilon}\,\frac{C_\gamma{}^i}{E_{36}}\, & E_{13}\,E_{15}\,E_{24}\,E_{26}\,E_{34}\,E_{35}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&6\\3&4&5\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3&4\\2&5&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&5&6\\2&3&4\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3&5\\2&4&6\end{smallmatrix}\right]^2 \\
-(\gamma^m C)_{\alpha\epsilon}(\gamma_m C)_{\beta\delta}\,\frac{C_\gamma{}^i}{E_{36}}\, & E_{12}\,E_{13}\,E_{26}\,E_{34}\,E_{35}\,E_{45}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&3\\4&5&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3&4\\2&5&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3&5\\2&4&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&6\\3&4&5\end{smallmatrix}\right]^2 \\
+(\gamma^m C)_{\alpha\gamma}(\gamma_m C)_{\delta\epsilon}\,\frac{C_\beta{}^i}{E_{26}}\, & E_{12}\,E_{15}\,E_{24}\,E_{25}\,E_{34}\,E_{36}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&4\\3&5&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3&6\\2&4&5\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&5&6\\2&3&4\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&5\\3&4&6\end{smallmatrix}\right]^2 \\
-(\gamma^m C)_{\alpha\epsilon}(\gamma_m C)_{\gamma\delta}\,\frac{C_\beta{}^i}{E_{26}}\, & E_{12}\,E_{13}\,E_{24}\,E_{25}\,E_{36}\,E_{45}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&3\\4&5&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&4\\3&5&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&5\\3&4&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3&6\\2&4&5\end{smallmatrix}\right]^2 \\
+(\gamma^m C)_{\beta\epsilon}(\gamma_m C)_{\gamma\delta}\,\frac{C_\alpha{}^i}{E_{16}}\, & E_{12}\,E_{14}\,E_{15}\,E_{23}\,E_{36}\,E_{45}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&3\\4&5&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&4\\3&5&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&5\\3&4&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&4&5\\2&3&6\end{smallmatrix}\right]^2 \\
-(\gamma^m C)_{\beta\gamma}(\gamma_m C)_{\delta\epsilon}\,\frac{C_\alpha{}^i}{E_{16}}\, & E_{12}\,E_{14}\,E_{15}\,E_{25}\,E_{34}\,E_{36}\,\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&4\\3&5&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3&4\\2&5&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&4&5\\2&3&6\end{smallmatrix}\right]\;\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&5\\3&4&6\end{smallmatrix}\right]^2 \\
-\frac{1}{2}\,(\gamma^m\,\bar\gamma^n\,C)_\beta{}^i(\gamma_m C)_{\alpha\epsilon}(\gamma_n C)_{\gamma\delta}\, & E_{12}\,E_{14}\,E_{24}\,E_{35}\,E_{35} \\
& \times\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&6\\3&4&5\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3&5\\2&4&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&4&6\\2&3&5\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&4\\3&5&6\end{smallmatrix}\right]^2 \\
+\frac{1}{2}\,(\gamma^m\,\bar\gamma^n\,C)_\alpha{}^i(\gamma_m C)_{\beta\delta}(\gamma_n C)_{\gamma\epsilon}\, & E_{12}\,E_{15}\,E_{25}\,E_{34}\,E_{34} \\
& \times\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&6\\3&4&5\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3&4\\2&5&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&5&6\\2&3&4\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&5\\3&4&6\end{smallmatrix}\right]^2 \\
+\frac{1}{2}\,(\gamma^m\,\bar\gamma^n\,C)_\epsilon{}^i(\gamma_m C)_{\alpha\beta}(\gamma_n C)_{\gamma\delta}\, & E_{14}\,E_{15}\,E_{23}\,E_{24}\,E_{35} \\
& \times\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&4\\3&5&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3&5\\2&4&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&4&5\\2&3&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&4&6\\2&3&5\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&5&6\\2&3&4\end{smallmatrix}\right] \\
+\frac{1}{2}\,(\gamma^m\,\bar\gamma^n\,C)_\delta{}^i(\gamma_m C)_{\alpha\beta}(\gamma_n C)_{\gamma\epsilon}\, & E_{13}\,E_{15}\,E_{24}\,E_{25}\,E_{34} \\
& \times\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&5\\3&4&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3&4\\2&5&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3&5\\2&4&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3&6\\2&4&5\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&5&6\\2&3&4\end{smallmatrix}\right] \\
-\frac{1}{2}\,(\gamma^m\,\bar\gamma^n\,C)_\gamma{}^i(\gamma_m C)_{\alpha\delta}(\gamma_n C)_{\beta\epsilon}\, & E_{12}\,E_{15}\,E_{24}\,E_{34}\,E_{35} \\
& \times\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&4\\3&5&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&5\\3&4&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&2&6\\3&4&5\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&3&5\\2&4&6\end{smallmatrix}\right]\Theta^{\vec a}_{\vec b}\left[\begin{smallmatrix}1&5&6\\2&3&4\end{smallmatrix}\right]
\end{aligned}
$$

$$- \frac{1}{2} (\gamma^m \bar{\gamma}^n C)_\gamma{}^i (\gamma_m C)_{\alpha\epsilon} (\gamma_n C)_{\beta\delta} E_{12} E_{14} E_{25} E_{34} E_{35} \\ \times \Theta^{\vec{a}}_{\vec{b}} \left[\begin{smallmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}} \left[\begin{smallmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}} \left[\begin{smallmatrix} 1 & 2 & 6 \\ 3 & 4 & 5 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}} \left[\begin{smallmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \end{smallmatrix}\right] \Theta^{\vec{a}}_{\vec{b}} \left[\begin{smallmatrix} 1 & 4 & 6 \\ 2 & 3 & 5 \end{smallmatrix}\right] \Bigg\} \tag{6.5.91}$$

6.6 Correlators with higher spin operators

In this section, we list a choice of correlation functions containing spin fields S_α and the composite operators

$$K^{\dot{\alpha}}_\mu \quad := \quad \psi_\mu \, \psi^\nu \, \bar{\gamma}_\nu^{\dot{\alpha}\beta} \, S_\beta \tag{6.6.92}$$

of conformal dimension $h = 1 + \frac{D}{16}$ which appear in the massive fermion vertex (3.3.53) in its canonical ghost picture or in the massless fermion vertex (3.2.45) in its $+1/2$ picture. Correlations of two spinorial fields will be given for general spacetime dimension D (although dimensions $D = 2 \bmod 4$ strictly speaking require a different relative chirality of the spinors than $D = 4 \bmod 4$), see appendix A.1.

Correlation functions with $K^{\dot{\alpha}}_\mu$ are very involved in general. But in the context of leading Regge trajectory fermions, they are contracted by a γ traceless wavefunction $\bar{\rho}^\mu_{\dot{\alpha}}$ such that only their spin 3/2 irreducible contributes. This spin 3/2 projection simplifies the correlators enormously, so we will always give their $\bar{\rho}$ contractions in the following.

The spin 3/2 component of $K^{\dot{\alpha}}_\mu$ is governed by the following OPEs:

$$\bar{\rho}^\mu_{\dot{\beta}} \, S_\alpha(z) \, K^{\dot{\beta}}_\mu(w) \quad \sim \quad \frac{(D-2) \, C_\alpha{}^{\dot{\beta}}}{\sqrt{2} \, (z-w)^{1/2+D/8}} \, \bar{\rho}^\mu_{\dot{\beta}} \, \psi_\mu(w) \; + \; \ldots \tag{6.6.93}$$

$$\bar{\rho}^\nu_{\dot{\alpha}} \, \psi^\mu(z) \, K^{\dot{\alpha}}_\nu(w) \quad \sim \quad \frac{(D-2)}{\sqrt{2} \, (z-w)^{3/2}} \, \bar{\rho}^\mu_{\dot{\alpha}} \, S^{\dot{\alpha}}(w) \; + \; \ldots \tag{6.6.94}$$

$$\bar{\rho}^\mu_{\dot{\alpha}} \, \bar{\rho}^\nu_{\dot{\beta}} \, K^{\dot{\alpha}}_\mu(z) \, K^{\dot{\beta}}_\nu(w) \quad \sim \quad \frac{(D-2)^2}{2\sqrt{2} \, (z-w)^{3/2+D/8}} \, \bar{\rho}^\mu_{\dot{\alpha}} \, (\bar{\gamma}_\lambda \, C)^{\dot{\alpha}\dot{\beta}} \, \bar{\rho}_{\mu\dot{\beta}} \, \psi^\lambda(w) \; + \; \ldots \tag{6.6.95}$$

Given these prerequisites, one can verify the correct singular behaviour of the three point functions

$$\bar{\rho}^\nu_{\dot{\beta}} \, \langle \, \psi^\mu(z_1) \, S_\alpha(z_2) \, K^{\dot{\beta}}_\nu(z_3) \, \rangle \quad = \quad \frac{(D-2) \, z_{12}^{1/2}}{\sqrt{2} \, z_{13}^{3/2} \, z_{23}^{D/8+1/2}} \, \bar{\rho}^\mu_{\dot{\beta}} \, C_\alpha{}^{\dot{\beta}} \tag{6.6.96}$$

$$\bar{\rho}^\nu_{2\dot{\alpha}} \, \bar{\rho}^\lambda_{3\dot{\beta}} \, \langle \, \psi^\mu(z_1) \, K^{\dot{\alpha}}_\nu(z_2) \, K^{\dot{\beta}}_\lambda(z_3) \, \rangle \quad = \quad \frac{(D-2)^2}{2\sqrt{2} \, (z_{12} \, z_{13})^{1/2} \, z_{23}^{D/8+3/2}} \, \bar{\rho}^\nu_{2\dot{\alpha}} \, (\bar{\gamma}^\mu \, C)^{\dot{\alpha}\dot{\beta}} \, \bar{\rho}_{3\nu\dot{\beta}} \tag{6.6.97}$$

as well as the (no longer D universal) four point function

$$\bar{\rho}^\mu_{\dot{\delta}} \, \langle \, S_\alpha(z_1) \, S_\beta(z_2) \, S_\gamma(z_3) \, K^{\dot{\delta}}_\mu(z_4) \, \rangle \Big|_{D=10} \quad = \quad \frac{4 \, (z_{12} \, z_{13} \, z_{14} \, z_{23} \, z_{24} \, z_{34})^{1/4} \, \bar{\rho}^\mu_{\dot{\delta}}}{z_{14} \, z_{24} \, z_{34}}$$

$$\times \left\{ \frac{(\gamma^\mu C)_{\alpha\beta} \, C_\gamma{}^{\dot\delta}}{z_{12}\, z_{34}} \; - \; \frac{(\gamma^\mu C)_{\alpha\gamma} \, C_\beta{}^{\dot\delta}}{z_{13}\, z_{24}} \; + \; \frac{(\gamma^\mu C)_{\beta\gamma} \, C_\alpha{}^{\dot\delta}}{z_{14}\, z_{23}} \right\} . \tag{6.6.98}$$

Also, we need the following five point correlator for the scattering of two gluons with a massless and a massive fermion:

$$\begin{aligned} \bar\rho^\tau_{\dot\beta} \, \langle \, \psi^\mu(z_1) \, \psi^\nu(z_2) \, \psi^\lambda(z_2) \, S_\alpha(z_3) \, K^{\dot\beta}_\tau(z_4) \, \rangle \;\; &= \;\; \frac{(D-2) \, \bar\rho^\tau_{\dot\beta}}{\sqrt{2} \, (z_{13}\, z_{14})^{1/2} \, z_{23} \, z_{24} \, z_{34}^{3/4}} \\ \left\{ \frac{(\eta^{\mu\nu} \, \delta^\lambda_\tau \; - \; \eta^{\mu\lambda} \, \delta^\nu_\tau) \, C_\alpha{}^{\dot\beta}}{z_{12}\, z_{34}} \, z_{23} \, z_{13} \right. &+ \; \frac{\delta^\mu_\tau \, (\gamma^{\nu\lambda} \, C)_\alpha{}^{\dot\beta}}{2 \, z_{14}} \, z_{13} \\ + \; \frac{\delta^\lambda_\tau \, (\gamma^\mu \, \bar\gamma^\nu \, C)_\alpha{}^{\dot\beta} \; - \; \delta^\nu_\tau \, (\gamma^\mu \, \bar\gamma^\lambda \, C)_\alpha{}^{\dot\beta}}{2 \, z_{24}} \, z_{23} &\left. \vphantom{\frac{1}{2}} \right\} \end{aligned} \tag{6.6.99}$$

In all scattering amplitudes given in this work, the $\bar\rho$ wavefunctions were eliminated by means of the massive Dirac equation $(D-2)\bar\rho_{\dot\alpha} = -v^\alpha \, \not{k}_{\alpha\dot\alpha}$ (suppressing any free vector index).

The four dimensional first mass level also contains massive spin 1/2 fermions (4.5.93) and (4.5.94) whose vertex operators are built from the σ trace of $K^{\dot\alpha}_\mu$. The latter can be identified with the derivative of the standard spin fields in any spacetime dimension D,

$$\partial S_A \;\; = \;\; - \, \frac{1}{4\,(D-1)} \, (\not{\psi} \, \not{\psi})_A{}^B \, S_B \; , \tag{6.6.100}$$

so computing its correlations is straightforward with the results from the previous sections at hand.

Chapter 7

Worldsheet integrals

In section 5.4, we could extract a lot of information on the structure of open string amplitudes from the analyticity properties of the worldsheet integrand, without actually touching the associated integrals. This chapter aims to go one step further and to discuss formal and physical properties of these integrals. As we have already mentioned, they are responsible for the pole structure of color ordered open string amplitudes.

Scattering amplitudes of $n \geq 5$ states were only considered at the massless level so far. That is why we will limit our discussion of the associated integrals to the massless situation where $2\alpha' k_i \cdot k_j = s_{ij}$ and $\sum_{j \neq i} s_{ij} = 0$.

7.1 First look at hypergeometric integrals

This section takes a look at the structure of the worldsheet integrals occurring in open string tree level scattering. The integration regions are mapped to the unit cube, and appropriate manipulations of the integrands allow to relate the integrals in a multileg amplitude to multiple Gaussian hypergeometric series, more precisely to generalized Kampé de Fériet functions [201, 231]. A brief outlook is given on the organization principles which ultimately allow to express any integral in an n point amplitude in terms of an $(n-3)!$ dimensional basis. Also, the structure of the momentum- or α' expansion of the integrals is sketched.

7.1.1 Mapping integration simplices to unit cubes

Our prescription (5.3.30) towards color ordered n point amplitudes with $SL(2,\mathbb{R})$ fixing

$$(z_1, z_{n-1}, z_n) \quad = \quad (0, 1, \infty) \tag{7.1.1}$$

requires to integrate the remaining worldsheet positions $z_j,\ j=2,3,\ldots,n-2$ over a generalized simplex: Only the first integration variable probes the full unit interval $z_2 \in (0,1)$ whereas all the others are restricted as $z_3 \in (z_2,1),\ z_4 \in (z_3,1),\ldots$ to preserve the ordering of the subamplitude in question.

The appropriate coordinate transformation to map the $n-3$ dimensional simplex to the more managable unit cube expresses each $z_j,\ j=2,3,\ldots,n-2$ as a product of variables $x_i,\ i=1,2,\ldots,n-3$ which are integrated over the full $(0,1)$ interval:

$$\left. \begin{array}{rclrcl} z_{n-2} &=& x_{n-3}\ , & z_{n-3} &=& x_{n-4}\, x_{n-3}\ , \\ \ldots\ , & & & z_2 &=& x_1\, x_2\, \ldots\, x_{n-3} \end{array} \right\} \quad \Rightarrow \quad z_j \ = \ \prod_{i=j-1}^{n-3} x_i \tag{7.1.2}$$

This change of variables introduces a Jacobian

$$\prod_{j=2}^{n-2} \int_{z_{j-1}}^1 \mathrm{d}z_j \ = \ \prod_{i=1}^{n-3} \int_0^1 \mathrm{d}x_i \ \left| \det\left(\frac{\partial z_j}{\partial x_i}\right) \right| \ , \qquad \det\left(\frac{\partial z_j}{\partial x_i}\right) \ = \ \prod_{k=1}^{n-3} (x_k)^{k-1} \tag{7.1.3}$$

which reflects the order of integration variables.

To see what happens on the level of the integrand, let us first of all consider the lower order cases $n=4,5$ and $n=6$. The CFT correlator gives rise to z_i dependences of structure $\prod_{i<j} |z_{ij}|^{s_{ij}} z_{ij}^{\tilde{n}_{ij}}$. The exponents contain Mandelstam invariants s_{ij}, s_{ijk} (which are reduced to the basis introduced in subsection 5.3.2 using momentum conservation) and additionally some integers $\tilde{n}_{ij}$ which control the physical poles of the amplitudes:

$$\int_0^1 \mathrm{d}z_2 \prod_{i<j} |z_{ij}|^{s_{ij}}\, z_{ij}^{\tilde{n}_{ij}} \ = \ \int_0^1 \mathrm{d}x_1\ x_1^{s_{12}+\tilde{n}_{12}}\, (1-x_1)^{s_{23}+\tilde{n}_{23}} \tag{7.1.4}$$

$$\begin{aligned} \int_{z_{j-1}}^1 \mathrm{d}z_2\, \mathrm{d}z_3 \prod_{i<j} |z_{ij}|^{s_{ij}}\, z_{ij}^{\tilde{n}_{ij}} \ &= \ \int_0^1 \mathrm{d}x_1\, \mathrm{d}x_2\ x_1^{s_{12}+\tilde{n}_{12}}\, x_2^{1+s_{45}+\tilde{n}_{12}+\tilde{n}_{13}+\tilde{n}_{23}} \\ &\quad \times\ (1-x_1)^{s_{23}+\tilde{n}_{23}}\, (1-x_2)^{s_{34}+\tilde{n}_{34}}\, (1-x_1x_2)^{s_{51}-s_{23}-s_{34}+\tilde{n}_{24}} \end{aligned} \tag{7.1.5}$$

$$\begin{aligned} \int_{z_{j-1}}^1 \mathrm{d}z_2\, \mathrm{d}z_3\, \mathrm{d}z_4 \prod_{i<j} |z_{ij}|^{s_{ij}}\, z_{ij}^{\tilde{n}_{ij}} \ &= \ \int_0^1 \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3\ x_1^{s_{12}+\tilde{n}_{12}}\, x_2^{1+s_{123}+\tilde{n}_{12}+\tilde{n}_{13}+\tilde{n}_{23}} \\ &\quad \times\ x_3^{2+s_{56}+\tilde{n}_{12}+\tilde{n}_{13}+\tilde{n}_{14}+\tilde{n}_{23}+\tilde{n}_{24}+\tilde{n}_{34}}\, (1-x_1)^{s_{23}+\tilde{n}_{23}}\, (1-x_2)^{s_{34}+\tilde{n}_{34}} \\ &\quad \times\ (1-x_3)^{s_{45}+\tilde{n}_{45}}\, (1-x_1x_2)^{s_{234}-s_{23}-s_{34}+\tilde{n}_{24}}\, (1-x_2x_3)^{s_{345}-s_{34}-s_{45}+\tilde{n}_{35}} \\ &\quad \times\ (1-x_1x_2x_3)^{s_{34}+s_{61}-s_{234}-s_{345}+\tilde{n}_{25}} \end{aligned} \tag{7.1.6}$$

The integrals in $n \geq 4$ point amplitudes involve $\frac{1}{2}n(n-3)$ Laurent polynomials in the $n-3$ new integrations variables – either the x_i themselves or $\left(1-\prod_{j=a}^b x_j\right)$. Let us emphasize that in the latter case, only products of successive x_j's occur, i.e. the $n=6$ integrand contains $(1-x_1x_2)$ and $(1-x_2x_3)$ but not $(1-x_1x_3)$.

7.1.2 Relation to hypergeometric functions

Now that the $(n-3)$ fold integral ocurring in an $n = 4, 5$ and $n = 6$ point amplitude is mapped to the unit cube $(0, 1)^{n-3}$, we can identify it with some multiple Gaussian hypergeometric series. For this purpose, the binomial series and the definition of the Pochhammer symbol $(\cdot, m)$ are quite useful:

$$(1-x)^p \;=\; \sum_{m=0}^{\infty} \frac{(-p,m)}{m!}\, x^m \;, \qquad (-p,m) \;=\; \frac{\Gamma(m-p)}{\Gamma(-p)} \tag{7.1.7}$$

With their help, one can identify hypergeometric functions ${}_pF_q$ in the worldsheet integrals at $n = 4, 5, 6$, they are defined by

$${}_pF_q \left[\begin{matrix} a_1,\, \ldots,\, a_p \\ b_1,\, \ldots,\, b_q \end{matrix} \;;\; 1 \right] \;=\; \sum_{s=0}^{\infty} \frac{1}{s!}\, \frac{(a_1,s)\, \ldots\, (a_p,s)}{(b_1,s)\, \ldots\, (b_q,s)} \;. \tag{7.1.8}$$

At $n = 4$, the integral (7.1.4) has two arguments $a := s_{12} + n_{12}$ and $b := s_{23} + n_{23}$. We obtain the Euler Beta function in a and b which coincides with the hypergeometric ${}_2F_1$ function:

$$\int_0^1 \mathrm{d}x\; x^a\,(1-x)^b \;=\; \frac{\Gamma(1+a)\,\Gamma(1+b)}{\Gamma(2+a+b)} \;=\; \frac{1}{1+a}\; {}_2F_1 \left[\begin{matrix} 1+a,\; -b \\ 2+a \end{matrix} \;;\; 1 \right] \tag{7.1.9}$$

Also the $n = 5$ point amplitude can be expressed in terms of a hypergeometric ${}_pF_{p-1}$ function, this time in five arguments:

$$\begin{aligned} &\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\; x^a\, y^b\, (1-x)^c\, (1-y)^d\, (1-xy)^e \\ &\quad = \frac{\Gamma(1+a)\,\Gamma(1+b)\,\Gamma(1+c)\,\Gamma(1+d)}{\Gamma(2+a+c)\,\Gamma(2+b+d)}\; {}_3F_2 \left[\begin{matrix} 1+a,\; 1+b,\; -e \\ 2+a+c,\; 2+b+d \end{matrix} \;;\; 1 \right] \end{aligned} \tag{7.1.10}$$

From $n = 6$ on, however, a single ${}_pF_q$ function is no longer sufficient to capture the $n-3$ fold worldsheet integral. Six point integrals require either an infinite series of hypergeometric functions ${}_4F_3$,

$$\begin{aligned} &\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\; x^a\, y^b\, z^c\, (1-x)^d\, (1-y)^e\, (1-z)^f\, (1-xy)^g\, (1-yz)^h\, (1-xyz)^j \\ &\quad = \Gamma(1+d)\,\Gamma(1+e)\,\Gamma(1+f) \sum_{m,n=0}^{\infty} \frac{\Gamma(m-g)\,\Gamma(n-h)}{\Gamma(-g)\,\Gamma(-h)\, m!\, n!} \\ &\qquad \frac{\Gamma(1+m+a)\,\Gamma(1+m+n+b)\,\Gamma(1+n+c)}{\Gamma(2+m+a+d)\,\Gamma(2+m+n+b+e)\,\Gamma(2+n+c+f)} \\ &\qquad {}_4F_3 \left[\begin{matrix} 1+m+n+b,\; 1+m+a,\; 1+n+c,\; -j \\ 2+m+n+b+e,\; 2+m+a+d,\; 2+n+c+f \end{matrix} \;;\; 1 \right] \;, \end{aligned} \tag{7.1.11}$$

or a triple hypergeometric function $F^{(3)}$ [202, 231].

7.1.3 The n point integrand

The starting point for finding the n point generalization of the x_i integrands (7.1.4) to (7.1.6) is the following form of the n point integrand

$$B_n\left[\tilde{n}_{ij}\right] \quad = \quad \prod_{j=2}^{n-2} \int_{z_{j-1}}^{1} \mathrm{d}z_j \prod_{1\leq i<j\leq n-1} |z_{ij}|^{s_{ij}}\, z_{ij}^{\tilde{n}_{ij}}\ , \tag{7.1.12}$$

with some integers $\tilde{n}_{ij} \in \mathbb{Z}$. We think of (7.1.12) as the result of evaluating the vertex operator correlation function $\langle \prod_{j=1}^{n} V^{(q_j)}_{h_j=1}(z_j)\rangle$ *without* taking the c ghost correlator $\langle c(z_1)c(z_{n-1})c(z_n)\rangle = z_{1,n-1}z_{1,n}z_{n-1,n}$ into account. Since these z_{ij} originate from a correlation function of $h_j = 1$ primary fields, the integers $\tilde{n}_{ij}$ must fulfill the conditions:

$$\sum_{i<j}^{n} \tilde{n}_{ij} \ + \ \sum_{i>j}^{n} \tilde{n}_{ji} \quad = \quad -\,2\ , \qquad j = 1,\ldots,n \tag{7.1.13}$$

After fixing three of the vertex positions as $(z_1, z_{n-1}, z_n) = (0, 1, \infty)$ and parameterizing the integration region $z_2 < \ldots < z_{n-2}$ as $z_k = \prod_{l=k-1}^{n-3} x_l,\ k = 2,\ldots,n-2$ with $0 < x_i < 1$, the integrand in (7.1.14) takes the generic form:

$$B_n\left[n_i, n_{ij}\right] \quad = \quad \left(\prod_{i=1}^{n-3}\int_0^1 \mathrm{d}x_i\right) \prod_{j=1}^{n-3} x_j^{s_{12\ldots j+1}+n_j} \prod_{l=j}^{n-3}\left(1 \ - \ \prod_{k=j}^{l} x_k\right)^{s_{j+1,l+2}+n_{jl}}\ , \tag{7.1.14}$$

with the set of $\frac{1}{2}n(n-3)$ integers $n_j, n_{jl} \in \mathbb{Z}$ and $s_{i,j} \equiv s_{ij}$:

$$\begin{aligned} n_{jl} \quad &= \quad \tilde{n}_{j+1,l+2}\ , \qquad j \leq l \\ n_j \quad &= \quad j\ -\ 1 + \sum_{i<j}^{j+1} \tilde{n}_{il}\ , \qquad 1 \leq j \leq n-3 \end{aligned} \tag{7.1.15}$$

The integrals represent generalized Euler integrals and integrate to multiple Gaussian hypergeometric functions [201], see the previous subsection 7.1.2 for $n = 4, 5$ and $n = 6$ examples.

Factoring out the s_{ij} dependent part of the integrand in (7.1.14) leaves us with a rational function

$$R(x_i) \quad := \quad \prod_{j=1}^{n-3} x_j^{n_j} \prod_{l=j}^{n-3}\left(1 \ - \ \prod_{k=j}^{l} x_k\right)^{n_{jl}} \tag{7.1.16}$$

in the $n-3$ variables x_i. With the conditions (7.1.13), it is in one-to-one correspondence to another rational function $\tilde{R}$ in the original integration variables,

$$\tilde{R}(z_{ij}) \quad := \quad \prod_{1\leq i<j\leq n-1} z_{ij}^{\tilde{n}_{ij}} \tag{7.1.17}$$

depending on the $n-1$ variables z_i and multiplying the integrand of (7.1.12). In the following we write this correspondence as

$$R(x_i) \quad \simeq \quad \tilde{R}(z_{ij})\ . \tag{7.1.18}$$

7.1.4 Finding a basis

There are many relations among integrals (7.1.12) with different sets $\tilde{n}$ of integers as a result of partial fraction decomposition

$$\frac{1}{z_{ij}\,z_{jk}} + \frac{1}{z_{ik}\,z_{kj}} = \frac{1}{z_{ij}\,z_{ik}} \tag{7.1.19}$$

and partial integration with respect to worldsheet variables

$$\begin{aligned} 0 &= \int dz_2 \ldots \int dz_{n-2}\, \frac{\partial}{\partial z_k} \prod_{i<j} |z_{ij}|^{s_{ij}}\, z_{ij}^{\tilde{n}_{ij}} \\ &= \int dz_2 \ldots \int dz_{n-2} \prod_{i<j} |z_{ij}|^{s_{ij}}\, z_{ij}^{\tilde{n}_{ij}} \left(\sum_{m=1}^{k-1} \frac{s_{km}+\tilde{n}_{km}}{z_{mk}} + \sum_{m=k+1}^{n} \frac{s_{km}+\tilde{n}_{km}}{z_{km}} \right) . \end{aligned} \tag{7.1.20}$$

Note, that this way any integral (7.1.12) with powers $\tilde{n}_{ij} < -1$ can always be expressed by a chain of integrals with $\tilde{n}_{ij} \geq -1$. Hence, in the following it is sufficient to concentrate on the cases $\tilde{n}_{ij} \geq -1$.

A quantitative handiness on finding a minimal set of functions can be obtained by performing

- (i) a classification of the integrals (7.1.12) according to their pole structure in the kinematic invariants s_{ij}
- (ii) a Gröbner basis analysis for those integrals (7.1.12) without poles.

Any partial fraction decomposition of an Euler integral with poles can be arranged according to its pole structure (modulo finite or subleading pieces), and the classification (i) yields a basis for them. This is achieved by performing a partial fraction expansion of the leading singularity in the kinematic invariants s_{ij}. On the other hand, the Gröbner basis analysis (ii) (see section 7.4) provides an independent set of rational functions or monomials in the Euler integrals and any integral (7.1.12) without poles can be expanded in terms of this set. Combining (i) and (ii) yields an independent set of integrals (7.1.12), and any partial fraction decomposition can be expressed in terms of the basis obtained this way. In the later sections 7.2 and 7.4 we explicitly construct this "partial fraction basis" for the cases $n = 4, 5$ and $n = 6$ and verify its dimension $(n-2)!$.

7.1.5 The structure of α' expansions

The first classification (i) of the integrals (7.1.12) is done with respect to their pole structure in the kinematic invariants s_{ij}. The maximum number of possible simultaneous poles of an n

point amplitude is $n-3$. Integrals of this type play an important role[1], since they capture the field theory limit of the full amplitude. They assume the following low energy- or power series expansion in α':

$$\begin{aligned} B_n[\tilde{n}] &= \alpha'^{3-n}\, p_{3-n}[\tilde{n}] \;+\; \alpha'^{5-n} \sum_{m=0}^{\infty} \alpha'^m \sideset{}{'}\sum_{\substack{s_r\in\mathbb{N},s_1>1\\ s_1+\ldots+s_d=m+2}} p^{\mathbf{s}}_{5-n+m}[\tilde{n}]\; \zeta(s_1,\ldots,s_d) \\ &= \alpha'^{3-n}\, p_{3-n}[\tilde{n}] \;+\; \alpha'^{5-n}\, p_{5-n}[\tilde{n}]\,\zeta(2) \;+\; \alpha'^{6-N}\, p_{6-n}[\tilde{n}]\,\zeta(3) \;+\; \ldots \end{aligned} \tag{7.1.21}$$

Above the rational functions or monomials $p^{\mathbf{s}}_{5-n+m}[\tilde{n}]$ are of degree $5-n+m$ in the kinematic invariants $(k_i+k_{i+1}+\ldots+k_j)^2$ and depend on the integer set $\tilde{n}$. Furthermore, we have introduced the multi zeta values (MZVs)

$$\begin{aligned} \zeta(s_1,\ldots s_d) &= \prod_{j=1}^{d} \frac{(-1)^{s_j-1}}{\Gamma(s_j)} \int_0^1 \mathrm{d}x_j\; (x_j)^{d-j}\, (\ln x)^{s_j-1} \left(1 \;-\; \prod_{i=1}^{j} x_i\right)^{-1} \\ &= \sum_{n_1>n_2>\ldots>n_d>0} \prod_{j=1}^{d} n_j^{-s_j} \;=\; \sum_{n_1,\ldots,n_d=1}^{\infty} \prod_{j=1}^{d} \left(\sum_{i=j}^{d} n_i\right)^{-s_j} \end{aligned} \tag{7.1.22}$$

of transcendentality degree $\sum_{r=1}^{d} s_r = m+2$ and depth d (where $s_j\in\mathbb{N}$ and $s_1>1$). The prime at the sum (7.1.21) means that the latter runs only over a basis of independent MZVs of weight $m+2$. In (7.1.21) at each order $5-n+m$ in α', a set of MZV of a fixed transcendentality degree $m+2$ appears. We call such a power series expansion transcendental, cf. section 7.3 for a detailed disscussion. In the following section 7.2 we present a method of how to extract the first term of (7.1.21) from integrals (7.1.12) with $n-3$ simultaneous poles. In fact, this method additionally allows to extract any lowest order poles from integrals (7.1.21) with fewer simultaneous poles. However, as we shall demonstrate, this type of integrals generically does not assume the transcendental power series expansion (7.1.21). At any rate, the method proposed in section 7.2 allows to determine the lowest order poles of the integral (7.1.12).

The second classification (ii) of the integrals (7.1.12) is appropriate if the latter have no poles, i.e. their power series expansion in α' starts with some zeta constants. In section 7.4 we introduce a Gröbner basis analysis, which allows to find an independent set of finite integrals (7.1.12), which serves as basis. Any other finite integral (7.1.12) is a $\mathbb{R}$ linear combination of this basis.

Note that the individual integrals entering the functions (12.3.45) and (12.3.48) are of both types – some of them have $n-3$ simultaneous poles and their α' expansion assumes the form (7.1.21), others have no poles and start with some zeta constants. In either case our methods (i) or (ii) can be applied to further reduce them.

[1] In fact, many *individual* integrals entering the functions F^σ in (1.4.1) are of this type.

7.2 Structure of multiple resonance exchanges

Generically, an n point scattering process has multiple resonance exchanges. As a result, the power series expansion in α' of the integrals (7.1.14) may have multiple poles in the Mandelstam variables. These poles come from regions of the integrand for which $x_i \to 0$ or $x_i \to 1$ for some of the variables x_i. To obtain information on the pole structure of the integrals (7.1.14), it is useful to transform the integrand to a different form, in which the poles can be easily extracted.

7.2.1 The general setup of multiparticle dual models

For an n point scattering process there are $\frac{1}{2}n(n-3)$ planar channels $(i,j) \in \mathcal{P}$ represented by delimiters in the set

$$\mathcal{P} \quad := \quad \{\ (1,j)\ |\ 2 \leq j \leq n-2\ \} \cup \{\ (p,q)\ |\ 2 \leq p < q \leq n-1\ \} \tag{7.2.23}$$

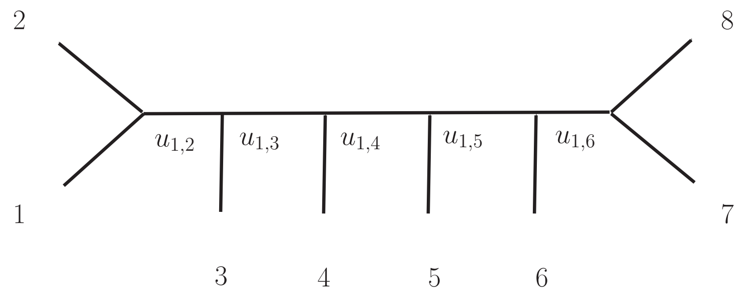
for the color ordering $(1,2,\ldots,n)$. We shall use the following notation for the associated Mandelstam variable:

$$S_{i,j} \quad := \quad \alpha'\,(k_i + k_{i+1} + \ldots + k_j)^2 \tag{7.2.24}$$

The channels (i,j) with states from $i,\ldots,j$ and $(j{+}1,i{-}1)$ with states from $j{+}1,\ldots,n,1,\ldots,i-1$ are identical because of momentum conservation. The set of $n{-}3$ kinematical invariants which can simultaneously appear in the denominator of the α' expansion of the n point amplitude describe the allowed (planar) channels of the underlying field theory diagram involving cubic vertices. Not all combinations of channels are allowed. For instance, adjacent channels as $(i,i+1)$ and $(i+1,i+2)$ cannot appear simultaneously in denominators, they are called dual or incompatible channels. On the other hand, for non-dual channels coincident poles are possible. A geometric way to find all compatible channels is to draw a convex n polygon of n sides representing momentum conservation. The number of ways of cutting this polygon into $n-2$ triangles with $n-3$ non-intersecting straight lines gives the number of distinct sets of allowed channels. According to Eulers polygon division problem this number is given by the Catalan number $C_{n-2} = \frac{2^{n-2}\ (2n-5)!!}{(n-1)!}$ (more generally, $C_k = \frac{1}{k+1}\binom{2k}{k}$). The $n-3$ diagonals of this polygon represent the momenta of possible intermediate states. To each of the $\frac{1}{2}n(n-3)$ channels (i,j) a variable $u_{i,j} \in (0,1)$ may be ascribed, with $u_{i,j} \equiv u_{j+1,i-1}$. For an account and references on the multiparticle dual model see [232].

For a given channel (i,j) with $u_{i,j} = 0$ all incompatible channels (p,q) are required to have $u_{p,q} = 1$. This property is described by the $\frac{1}{2}n(n-3)$ duality constraint equations

$$u_{i,j} \quad = \quad 1 \; - \prod_{\substack{1 \leq p < i \\ i \leq q < j}} u_{p,q} \prod_{\substack{i < r \leq j \\ j < s \leq n-1}} u_{r,s}\ , \qquad 1 \leq i < j \leq n\ , \tag{7.2.25}$$



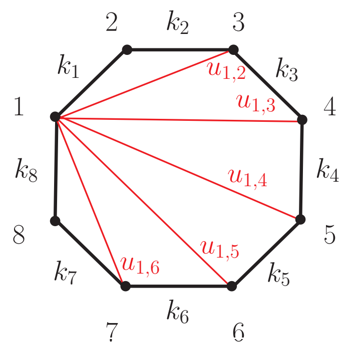


Figure 7.1: Multiperipheral configuration and corresponding dual diagram for $n = 8$.

which are sufficient for excluding simultaneous poles in incompatible channels. We define $u_{i,i} = 0,\ u_{1,n-1} = 1$ and have $u_{k,n} = u_{1,k-1},\ k \geq 3$. Only $\frac{1}{2}(n-2)(n-3)$ of these equations (7.2.25) are independent leaving $n-3$ variables $u_{i,j}$ out of the set of $\frac{1}{2}n(n-3)$ variables free. The set of $n-3$ independent variables $u_{i,j}$ can be associated to the inner lines of one of the C_{n-2} sliced n polygon. In particular, as a canonical choice we may define

$$u_{1,j+1} \quad = \quad x_j\ , \qquad j = 1, \ldots, n-3 \tag{7.2.26}$$

as a set of $n-3$ independent variables corresponding to figure 7.1. Hence, each of the internal lines of the polygon corresponds to an independent variable x_j in the integral (7.1.14). Choosing the inner lines of another sliced n polygon results in a different integral representation (7.1.14). As a consequence of (7.2.25) and (7.2.26) we have:

$$1 \ - \ x_j \quad = \quad \prod_{\substack{0<r\leq j\\ j<s\leq n-2}} u_{r+1,s+1}\ , \qquad j = 1, \ldots, n-3 \tag{7.2.27}$$

$$1 \ - \ \prod_{k=i}^{j} x_k \quad = \quad \prod_{\substack{1\leq p\leq i\\ j+1\leq q\leq n-2}} u_{p+1,q+1}\ , \qquad 1 \leq i \leq j \leq n-3 \tag{7.2.28}$$

With (7.2.27) and the Jacobian $\prod\limits_{2\leq i<j\leq n-1} u_{i,j}^{j-i-1}$, the integral (7.1.14) translates into an integral over all $\frac{1}{2}n(n-3)$ variables $u_{\mathcal{P}}$ related to the partitions $\mathcal{P}$ given in (7.2.23):

$$B_n[n_{i,j}] \quad = \quad \prod_{(i,j)\in\mathcal{P}} \int_0^1 du_{i,j}\ u_{i,j}^{S_{i,j}+n_{i,j}} \prod_{\mathcal{P}'\notin(1,j)} \delta\left(u_{\mathcal{P}'} - 1 + \prod_{\tilde{\mathcal{P}}} u_{\tilde{\mathcal{P}}}\right) \tag{7.2.29}$$

The integers $n_{i,j}$ are determined by the variables n_i and n_{ij} from (7.1.15). The dictionary is given in the later subsections on examples up to $n = 8$ point integrals.

In (7.2.29) the integration is constrained by the duality conditions (7.2.25) resulting in a product of $\frac{1}{2}(n-2)(n-3)$ independent δ functions[2] . In this $B_n[n_{i,j}]$ form (7.2.29) many properties of the integrals (7.1.14) like the pole structure or cyclicity become manifest. This will be elucidated at the following examples.

We can introduce a fundamental set of C_{n-2} integrals B_n

$$\bigcup_{(i_l,j_l)\in\mathcal{P}} \left\{ \prod_{(i,j)\in\mathcal{P}} \int_0^1 du_{i,j}\ u_{i,j}^{S_{i,j}} \left(\prod_{l=1}^{n-3} u_{i_l,j_l}\right)^{-1} \prod_{\mathcal{P}'\notin(1,j)} \delta\left(u_{\mathcal{P}'} - 1 + \prod_{\tilde{\mathcal{P}}} u_{\tilde{\mathcal{P}}}\right)\right\}\ , \tag{7.2.31}$$

with (i_l, j_l) running over all C_{n-2} allowed channels[3]. The α' expansion of each of the elements (7.2.31) assumes the form (7.1.21) with $\prod_{l=1}^{n-3} S_{i_l,j_l}^{-1}$ as its lowest order term. Any other integral (7.1.14) with $n-3$ simultaneous poles can be expressed as $\mathbb{R}$ linear combination of the basis (7.2.31) modulo less singular terms. In case of a sum of $n-3$ simultaneous poles this is achieved by partial fraction decomposition of the polynomials according to their leading singular term and associating the latter with the basis (7.2.31).

A special role is played by the integral

$$B_n[n_{i,j}=-1] \quad = \quad \prod_{(i,j)\in\mathcal{P}} \int_0^1 du_{i,j}\ u_{i,j}^{S_{i,j}-1} \prod_{\mathcal{P}'\notin(1,j)} \delta\left(u_{\mathcal{P}'} - 1 + \prod_{\tilde{\mathcal{P}}} u_{\tilde{\mathcal{P}}}\right) \tag{7.2.32}$$

By construction, it is manifestly invariant under cyclic transformations $S_{i,j} \to S_{i+1,j+1}$, with cyclic identification $i \equiv i+n,\ j \equiv j+n$. Furthermore, it furnishes all C_{n-2} sets of allowed channels at the lowest order, i.e.

$$B_n[n_{i,j}=-1] \quad = \quad \sum_{(i_l,j_l)\in\mathcal{P}} \prod_{l=1}^{n-3} S_{i_l,j_l}^{-1} \ + \ \ldots\ , \tag{7.2.33}$$

with the sum running over all C_{n-2} allowed channels. In terms of (7.1.14), equation (7.2.32) takes the form:

$$B_n\begin{bmatrix} n_i=-1 \\ n_{ii}=-1\end{bmatrix} \quad = \quad \left(\prod_{i=1}^{n-3}\int_0^1 dx_i\right) \prod_{j=1}^{n-3} x_j^{s_{12\ldots j+1}-1}\,(1-x_j)^{s_{j+1,j+2}-1} \prod_{l=j+1}^{n-3}\left(1 \ - \ \prod_{k=j}^{l} x_k\right)^{s_{j+1,l+2}} \tag{7.2.34}$$

[2] The solution to the duality constraint (7.2.25) may be found as ($p=2,3,\ldots,n-2; q=3,4,\ldots,n-1$ and $p<q$):

$$u_{p,q} \quad = \quad \begin{cases} \dfrac{\left(1 \ - \ \prod_{m=p}^{q-1} u_{1,m}\right)\left(1 \ - \ \prod_{n=p-1}^{q} u_{1,n}\right)}{\left(1 \ - \ \prod_{r=p-1}^{q-1} u_{1,r}\right)\left(1 \ - \ \prod_{s=p}^{q} u_{1,s}\right)} & :\ q\neq n-1 \\[2ex] \dfrac{\left(1 \ - \ \prod_{m=p}^{q-1} u_{1,m}\right)}{\left(1 \ - \ \prod_{r=p-1}^{q-1} u_{1,r}\right)} & :\ q=n-1 \end{cases} \tag{7.2.30}$$

[3] As pointed out before, these integrals appear as constituents of some of the functions F^σ discussed in section 12.3. The poles in these B_n combinations are cancelled by corresponding s_{ij} factors in the numerator of the F^σ such that they end up being local.

7.2.2 $n = 4$ point integrals

In the case of $n = 4$, we have the two planar channels $(1, 2)$ and $(2, 3) \equiv (1, 4)$ related to the two variables $u_{1,2}$ and $u_{2,3}$, respectively. After choosing the independent variable $u_{1,2} = x_1 := x$ and following the steps (7.2.27), the integral (7.1.14)

$$B_4[n_i, n_{ij}] \quad = \quad \int_0^1 \mathrm{d}x \; x^{s_{12}+n_1} \; (1-x)^{s_{23}+n_{11}} \tag{7.2.35}$$

takes the form (7.2.29)

$$B_4[n_{i,j}] \quad = \quad \int_0^1 \mathrm{d}u_{1,2} \int_0^1 \mathrm{d}u_{2,3} \; u_{1,2}^{s_{12}+n_{1,2}} \; u_{2,3}^{s_{23}+n_{2,3}} \; \delta(u_{1,2}+u_{2,3}-1) \; , \tag{7.2.36}$$

with $n_{1,2} = n_1$ and $n_{2,3} = n_{11}$. The cyclically invariant integral (7.2.32) is given by

$$B_4 \begin{bmatrix} n_1 = -1 \\ n_{11} = -1 \end{bmatrix} \quad = \quad \int_0^1 \mathrm{d}x \; x^{s_{12}-1} \, (1-x)^{s_{23}-1} \quad = \quad B(s_{12}, s_{23}) \quad = \quad \frac{1}{s_{12}} + \frac{1}{s_{23}} + \ldots \, . \tag{7.2.37}$$

For later purposes, it is convenient to define a regular function capturing the string effects in four point disk amplitudes. This is accomplished by

$$V_t \quad := \quad \frac{s_{12}\, s_{23}}{s_{12}+s_{23}} \, B(s_{12}, s_{23}) \quad = \quad \frac{\Gamma(1+s_{12})\,\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})} \quad = \quad 1 \; - \; \frac{\pi^2}{6} \, s_{12}\, s_{23} \; + \; \ldots \tag{7.2.38}$$

which can be regarded as a stringy formfactor. It is ubiquitous in four point scattering of both massless and massive states, numerous examples follow in chapter 8 and 9. Moreover, we will shed light on its far-reaching physical significance in section 8.1.

7.2.3 $n = 5$ point integrals

At $n = 5$ extermal legs we have five planar channels $(1, 2)$, $(2, 3)$, $(3, 4)$, $(1, 3) \equiv (4, 5)$ and $(2, 4) \equiv (1, 5)$ related to the five variables $u_{1,2}, u_{2,3}, u_{3,4}, u_{4,5} = u_{1,3}$ and $u_{5,1} = u_{2,4}$, respectively. The five point integral (7.1.14) becomes

$$B_5[n_i, n_{ij}] \quad = \quad \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \; x_1^{s_1+n_1} \; x_2^{s_4+n_2} \; (1-x_1)^{s_2+n_{11}} \; (1-x_2)^{s_3+n_{22}} \; (1-x_1 x_2)^{s_{24}+n_{12}} \; , \tag{7.2.39}$$

with $s_i = \alpha'(k_i + k_{i+1})^2$, $i = 1, \ldots, 5$ subject to the cyclic identification $i + 5 \equiv i$. To transform (7.2.39) into the form (7.2.29) according to (7.2.26) we choose the two independent variables $u_{1,2} = x_1$ and $u_{1,3} = u_{4,5} = x_2$. Then, with (7.2.27) the integral (7.2.39) takes the form

$$\begin{aligned} B_5[n_{i,j}] \quad = \quad & \int_0^1 \mathrm{d}u_{1,2} \int_0^1 \mathrm{d}u_{2,3} \int_0^1 \mathrm{d}u_{3,4} \int_0^1 \mathrm{d}u_{4,5} \int_0^1 \mathrm{d}u_{1,5} \; u_{1,2}^{s_1+n_{1,2}} \; u_{2,3}^{s_2+n_{2,3}} \; u_{3,4}^{s_3+n_{3,4}} \; u_{4,5}^{s_4+n_{4,5}} \\ & \times u_{1,5}^{s_5+n_{1,5}} \; \delta(u_{2,3}+u_{1,2}u_{3,4}-1) \; \delta(u_{2,4}+u_{1,2}u_{4,5}-1) \; \delta(u_{3,4}+u_{2,3}u_{4,5}-1) \; , \end{aligned} \tag{7.2.40}$$

with

$$n_{1,2} \;=\; n_1\ , \qquad n_{2,3} \;=\; n_{11}\ , \qquad n_{3,4} \;=\; n_{22}\ , \qquad n_{4,5} \;=\; n_2\ , \qquad n_{1,5} \;=\; 1+n_{11}+n_{22}+n_{12}\ . \tag{7.2.41}$$

In what follows it is convenient to introduce

$$I_5(x,y) \;\;:=\;\; x^{s_4}\ y^{s_1}\ (1-x)^{s_3}\ (1-y)^{s_2}\ (1-xy)^{s_{24}} \tag{7.2.42}$$

arising from (7.2.39) with the identifications $x_1 := y$ and $x_2 := x$. Furthermore, we stick to the following shorter notation for the dual variables $u_{i,j}$

$$X_i \;\;:=\;\; u_{i,i+1}\ , \qquad i=1,\ldots,5 \quad , \quad i+5\equiv i \tag{7.2.43}$$

and define

$$J_5(X) \;\;:=\;\; \left(\prod_{i=1}^{5} X_i^{s_i}\right)\ \delta(X_2+X_1X_3-1)\ \delta(X_3+X_2X_4-1)\ \delta(X_5+X_1X_4-1)\ . \tag{7.2.44}$$

Let us now discuss a few examples. The pole structure of the integral

$$\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\ \frac{I_5(x,y)}{(1-y)\ (1-xy)} \tag{7.2.45}$$

can be easily deduced after transforming it into the form (7.2.40)

$$\left(\prod_{i=1}^{5}\int_0^1 \mathrm{d}X_i\right)\ J_5(X)\ \frac{1}{X_2X_5} \;\;=\;\; \frac{1}{s_2\ s_5}\ +\ \ldots\ . \tag{7.2.46}$$

Hence, the only simultaneous pole is at $X_2, X_5\to 0$ with the product of δ functions yielding the constraints for the three variables $X_1, X_3, X_4 \to 1$. Subsequently, we list a few nontrivial examples:

rational function in equation (7.1.12)	rational function in equation (7.2.39)	rational function in equation (7.2.40)	lowest order poles
$\frac{z_{15}}{z_{12}z_{13}z_{14}z_{25}z_{35}z_{45}}$	$\frac{1}{x\ y}$	$\frac{X_5}{X_1X_4}$	$\frac{1}{s_1s_4}$
$\frac{1}{z_{12}z_{13}z_{24}z_{35}z_{45}}$	$\frac{1}{x\ y\ (1-xy)}$	$\frac{1}{X_1X_4}$	$\frac{1}{s_1s_4}$
$\frac{1}{z_{13}z_{14}z_{23}z_{25}z_{45}}$	$\frac{1}{x\ (1-y)}$	$\frac{1}{X_2X_4}$	$\frac{1}{s_2s_4}$
$\frac{1}{z_{14}z_{15}z_{23}z_{25}z_{34}}$	$\frac{1}{(1-x)\ (1-y)}$	$\frac{1}{X_2X_3X_5}$	$\frac{1}{s_2s_5}+\frac{1}{s_3s_5}$
$\frac{1}{z_{12}z_{15}z_{24}z_{34}z_{35}}$	$\frac{1}{(1-x)\ y\ (1-xy)}$	$\frac{1}{X_1X_3X_5}$	$\frac{1}{s_1s_3}+\frac{1}{s_3s_5}$

(7.2.47)

Let us point out once again that the rational functions of z_{ij} in the first column do not yet encompass the c ghost correlator $\langle c(z_1)c(z_{n-1})c(z_n)\rangle = z_{1,n-1}z_{1,n}z_{n-1,n}$.

The fundamental integrands are $J_5(X)$, multiplied by one of the five rational functions

$$\frac{1}{X_1X_3}\ ,\ \frac{1}{X_2X_4}\ ,\ \frac{1}{X_3X_5}\ ,\ \frac{1}{X_1X_4}\ ,\frac{1}{X_2X_5}\ , \tag{7.2.48}$$

which furnish the $C_3 = 5$ poles

$$\frac{1}{s_1\, s_3}\ ,\ \frac{1}{s_2\, s_4}\ ,\ \frac{1}{s_3\, s_5}\ ,\ \frac{1}{s_1\, s_4}\ ,\frac{1}{s_2\, s_5}\ . \tag{7.2.49}$$

In the basis (7.2.39) the rational functions become

$$\frac{1}{(1-x)y}\ ,\ \frac{1}{x(1-y)}\ ,\ \frac{1}{(1-x)(1-xy)}\ ,\ \frac{1}{xy(1-xy)}\ ,\frac{1}{(1-y)(1-xy)}\ , \tag{7.2.50}$$

respectively. The cyclically invariant integral (7.2.32) is given by

$$\begin{aligned} B_5\begin{bmatrix} n_i=-1 \\ n_{ii}=-1\end{bmatrix} &= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\ \frac{I_5(x,y)}{x\ (1-x)\ y\ (1-y)} \\ &= \frac{1}{s_1\, s_3} + \frac{1}{s_2\, s_4} + \frac{1}{s_3\, s_5} + \frac{1}{s_1\, s_4} + \frac{1}{s_2\, s_5} + \ \ldots \end{aligned} \tag{7.2.51}$$

and exhibits all five poles (7.2.49) in its power series expansion.

Finally, as we shall see in the next section 7.4, there is one rational function without poles and its series expansion starts at $\zeta(2)$:

rational function in equation (7.1.12)	rational function in equation (7.2.39)	rational function in equation (7.2.40)	lowest order
$\frac{1}{z_{13}z_{14}z_{24}z_{25}z_{35}}$	$\frac{1}{(1-xy)}$	1	$\zeta(2)$

(7.2.52)

The function (7.2.52) may be added to (7.2.48) to give rise to another fundamental set

$$\frac{X_2}{X_1X_3}\ ,\ \frac{X_3}{X_2X_4}\ ,\ \frac{X_4}{X_3X_5}\ ,\ \frac{X_5}{X_1X_4}\ ,\frac{X_1}{X_2X_5}\ , \tag{7.2.53}$$

subject to the constraints (7.2.44) and with the same poles (7.2.49), respectivley. In the basis (7.1.14) the latter rational functions correspond to

$$\frac{1-y}{(1-x)y(1-xy)}\ ,\ \frac{1-x}{x(1-y)(1-xy)}\ ,\ \frac{x}{(1-x)(1-xy)}\ ,\ \frac{1}{xy}\ ,\frac{y}{(1-y)(1-xy)}\ , \tag{7.2.54}$$

respectively. Since we have

$$\begin{aligned} \frac{X_3}{X_1X_4} &\simeq \frac{1}{xy}\frac{1-x}{(1-xy)^2} \simeq \frac{z_{25}z_{34}}{z_{12}z_{13}z_{24}^2z_{35}^2z_{45}} \\ \frac{X_2}{X_1X_4} &\simeq \frac{1}{xy}\frac{1-y}{(1-xy)^2} \simeq \frac{z_{14}z_{23}}{z_{12}z_{13}^2z_{24}^2z_{35}z_{45}} \\ \frac{X_3X_5}{X_1X_4} &\simeq \frac{1}{xy}\frac{1-x}{(1-xy)} \simeq \frac{z_{15}z_{34}}{z_{12}z_{13}z_{14}z_{24}z_{35}^2z_{45}} \\ \frac{X_2X_5}{X_1X_4} &\simeq \frac{1}{xy}\frac{1-y}{(1-xy)} \simeq \frac{z_{15}z_{23}}{z_{12}z_{13}^2z_{24}z_{25}z_{35}z_{45}}\ , \end{aligned} \tag{7.2.55}$$

the two rational functions $\frac{1}{X_1X_4}$ and $\frac{X_5}{X_1X_4}$ are the only possibilities to realize the poles $\frac{1}{s_1s_4}$ without double poles in the denominator of (7.1.12). Cyclicity makes sure that these arguments take over to the other four poles (7.2.49) and their rational functions (7.2.48) and (7.2.53). Generally, rational functions other than the latter give rise to double powers in the denominator of (7.1.12), e.g.:

$$\begin{aligned} \frac{1}{X_1} &\simeq \frac{1}{y(1-xy)} \simeq \frac{1}{z_{12}z_{14}z_{24}z_{35}^2} \\ \frac{X_1}{X_2} &\simeq \frac{y}{1-y} \simeq \frac{z_{12}}{z_{13}z_{14}^2z_{23}z_{25}^2} \end{aligned} \tag{7.2.56}$$

We will explain in the following section 7.3 why the presence of double poles obscure the transcendentality properties of the full physical amplitude.

7.2.4 $n=6$ point integrals

Six particle kinematics give rise to nine planar channels $(1,2),\ (1,3),\ (1,4)\equiv(5,6),\ (2,3),\ (2,4),$ $(2,5)\equiv(1,6),\ (3,4),\ (3,5)$ and $(4,5)$ related to the nine variables $u_{1,2},u_{1,3},u_{1,4},u_{2,3},u_{2,4},$ $u_{2,5},u_{3,4},\ u_{3,5}$ and $u_{4,5}$, respectively. The six point integral (7.1.14) becomes

$$\begin{aligned} B_6[n_i,n_{ij}] &= \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3\ x_1^{s_1+n_1}\ x_2^{t_1+n_2}\ x_3^{s_5+n_3}\ (1-x_1)^{s_2+n_{11}}\ (1-x_2)^{s_3+n_{22}} \\ &\times\ (1-x_3)^{s_4+n_{33}}\ (1-x_1x_2)^{s_{24}+n_{12}}\ (1-x_2x_3)^{s_{35}+n_{23}}\ (1-x_1x_2x_3)^{s_{25}+n_{13}}\ , \end{aligned} \tag{7.2.57}$$

with $s_i=\alpha'(k_i+k_{i+1})^2,\ i=1,\ldots,6$ subject to the cyclic identification $i+6\equiv i$ and $t_j=\alpha'(k_j+k_{j+1}+k_{j+2})^2,\ j=1,\ldots,3.$

To bring (7.2.57) into the $B_n[n_{i,j}]$ form (7.2.29) we choose the three independent variables $u_{1,2}=x_1, u_{1,3}=x_2$ and $u_{1,4}=x_3$. Then, with (7.2.27) the integral (7.2.57) takes the form

$$\begin{aligned} B_6[n_{i,j}] &= \int_0^1 du_{1,2}\int_0^1 du_{1,3}\int_0^1 du_{1,4}\int_0^1 du_{2,3}\int_0^1 du_{2,4}\int_0^1 du_{2,5}\int_0^1 du_{3,4}\int_0^1 du_{3,5}\int_0^1 du_{4,5} \\ &\times\ u_{1,2}^{s_1+n_{1,2}}\ u_{2,3}^{s_2+n_{2,3}}\ u_{3,4}^{s_3+n_{3,4}}\ u_{4,5}^{s_4+n_{4,5}}\ u_{1,4}^{s_5+n_{1,4}}\ u_{2,5}^{s_6+n_{2,5}}\ u_{1,3}^{t_1+n_{1,3}}\ u_{2,4}^{t_2+n_{2,4}}\ u_{3,5}^{t_3+n_{3,5}} \\ &\times\ \delta(u_{2,3}+u_{1,2}u_{3,4}u_{3,5}-1)\ \delta(u_{2,4}+u_{1,2}u_{1,3}u_{3,5}u_{4,5}-1)\ \delta(u_{2,5}+u_{1,2}u_{1,3}u_{1,4}-1) \\ &\times\ \delta(u_{3,4}+u_{1,3}u_{2,3}u_{4,5}-1)\ \delta(u_{3,5}+u_{1,3}u_{1,4}u_{2,3}u_{2,4}-1)\ \delta(u_{4,5}+u_{1,4}u_{2,4}u_{3,4}-1) \end{aligned} \tag{7.2.58}$$

with:

$$\begin{aligned} n_{1,2} &= n_1\ , \qquad n_{1,3} = n_2\ , \qquad n_{1,4}=n_3 \\ n_{2,3} &= n_{11}\ , \qquad n_{3,4} = n_{22}\ , \qquad n_{4,5} = n_{33} \\ n_{2,4} &= 1+n_{11}+n_{22}+n_{12}\ , \qquad n_{3,5} = 1+n_{22}+n_{33}+n_{23} \\ n_{2,5} &= 2+n_{11}+n_{22}+n_{33}+n_{12}+n_{13}+n_{23} \end{aligned} \tag{7.2.59}$$

In what follows it is convenient to introduce

$$I_6(x,y,z) \quad := \quad x^{s_5}\, y^{t_1}\, z^{s_1}\, (1-x)^{s_4}\, (1-y)^{s_3}\, (1-z)^{s_2}\, (1-xy)^{s_{35}}\, (1-yz)^{s_{24}}\, (1-xyz)^{s_{25}} \quad (7.2.60)$$

arising from (7.2.57) with the identifications $x_1 := z, x_2 := y$ and $x_3 := x$. Furthermore, we stick to the following shorter notation for the dual variables $u_{i,j}$

$$X_i \quad := \quad u_{i,i+1}\ , \qquad i=1,\ldots,6\ , \qquad i+6\equiv i\ , \qquad Y_j \quad := \quad u_{j,j+2}\ , \qquad j=1,2,3 \quad (7.2.61)$$

and define

$$\begin{aligned} J_6(X,Y) \quad := \quad & \left(\prod_{i=1}^{6} X_i^{s_i}\right) \left(\prod_{j=1}^{3} Y_j^{t_j}\right) \delta(X_2+X_1X_3Y_3-1)\ \delta(Y_2+X_1X_4Y_1Y_3-1) \\ \times\ & \delta(X_6+X_1X_5Y_1-1)\ \delta(X_3+X_2X_4Y_1-1)\ \delta(Y_3+X_2X_5Y_1Y_2-1)\ \delta(X_4+X_3X_5Y_2-1)\ . \end{aligned} \quad (7.2.62)$$

Let us now discuss a few examples. The pole structure of the integral

$$\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\ \frac{I_6(x,y,z)}{(1-x)\ (1-xy)\ (1-xyz)} \quad (7.2.63)$$

can be easily deduced after transforming it into the $B_6[n_{i,j}]$ form (7.2.58)

$$\left(\prod_{i=1}^{6}\int_0^1 \mathrm{d}X_i\right) \left(\prod_{j=1}^{3}\int_0^1 \mathrm{d}Y_j\right) J_6(X,Y)\ \frac{Y_2}{X_4X_6Y_3} \quad = \quad \frac{1}{s_4s_6t_3} + \ldots\ . \quad (7.2.64)$$

Hence, the only simultaneous pole is at $X_4, X_6, Y_3 \to 0$ with the product of δ functions yielding the constraints for the six variables $X_1, X_2, X_3, X_5, Y_1, Y_2 \to 1$. Note that by construction a set of three poles in (7.2.58) does not necessarily yield a compatible set of channels, e.g. the integral

$$\begin{aligned} \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\ \frac{I_6(x,y,z)}{(1-x)\ (1-y)} \quad &= \quad \left(\prod_{i=1}^{6}\int_0^1 \mathrm{d}X_i\right) \left(\prod_{j=1}^{3}\int_0^1 \mathrm{d}Y_j\right) \frac{J_6(X,Y)}{X_3X_4Y_3} \\ &= \quad \frac{1}{s_3t_3} + \frac{1}{s_4t_3} + \frac{1}{s_3} + \frac{1}{s_4} - \frac{s_1}{s_3t_3} - \frac{s_1}{s_4t_3} - \frac{s_6}{s_3t_3} - \frac{s_6}{s_4t_3} + \ldots \end{aligned} \quad (7.2.65)$$

does not yield any triple pole as $\{(3,4),(4,5),(3,5)\}$ are incompatible channels. Similarly, for

$$\begin{aligned} \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\ \frac{I_6(x,y,z)}{z(1-z)(1-xy)(1-xyz)} \quad &= \quad \left(\prod_{i=1}^{6}\int_0^1 \mathrm{d}X_i\right) \left(\prod_{j=1}^{3}\int_0^1 \mathrm{d}Y_j\right) \frac{J_6(X,Y)}{X_1X_2X_6} \\ &= \quad \frac{1}{s_2s_6} + \frac{1}{s_2} + \frac{1}{s_6} - \frac{s_4}{s_2s_6} - \frac{t_2}{s_2s_6} + \frac{\zeta(2)}{s_1} + \ldots \end{aligned} \quad (7.2.66)$$

the channels $(1,2),(2,3)$ and $(6,1)$ are not compatible. In the sequel we list a few non-trivial examples:

rational function in equation (7.1.12)	rational function in equation (7.2.39)	rational function in equation (7.2.40)	lowest order poles
$\frac{z_{16}^2}{z_{12}z_{13}z_{14}z_{15}z_{26}z_{36}z_{46}z_{56}}$	$\frac{1}{x\ y\ z}$	$\frac{X_6^2Y_2Y_3}{X_1X_5Y_1}$	$\frac{1}{s_1s_5t_1}$
$\frac{z_{16}}{z_{12}z_{13}z_{15}z_{26}z_{36}z_{45}z_{46}}$	$\frac{1}{(1-x)\ y\ z}$	$\frac{X_6Y_3}{X_1X_4Y_1}$	$-\frac{1}{s_1s_4t_1}$
$\frac{1}{z_{13}z_{15}z_{23}z_{26}z_{45}z_{46}}$	$\frac{1}{(1-x)\ y\ (1-z)}$	$\frac{1}{X_2X_4Y_1}$	$\frac{1}{s_2s_4t_1}$
$\frac{1}{z_{12}z_{14}z_{25}z_{34}z_{36}z_{56}}$	$\frac{1}{x\ (1-y)\ z\ (1-xyz)}$	$\frac{1}{X_1X_3X_5}$	$\frac{1}{s_1s_3s_5}$
$\frac{z_{13}z_{45}}{z_{12}z_{14}^2z_{25}z_{34}z_{35}z_{36}z_{56}}$	$\frac{y\ (1-x)}{x\ (1-y)\ z\ (1-xy)\ (1-xyz)}$	$\frac{X_4Y_1}{X_1X_3X_5}$	$\frac{1}{s_1s_3s_5}$
$\frac{1}{z_{14}z_{15}z_{23}z_{26}z_{34}z_{56}}$	$\frac{1}{x(1-y)(1-z)}$	$\frac{1}{X_2X_3X_5Y_2}$	$\frac{1}{s_2s_5t_2}+\frac{1}{s_3s_5t_2}$
$\frac{1}{z_{12}z_{15}z_{26}z_{34}z_{36}z_{45}}$	$\frac{1}{z\ (1-x)\ (1-y)}$	$\frac{1}{X_1X_3X_4Y_3}$	$\frac{1}{s_1s_3t_3}+\frac{1}{s_1s_4t_3}$
$\frac{1}{z_{15}z_{16}z_{24}z_{26}z_{34}z_{35}}$	$\frac{y}{(1-y)\ (1-xy)\ (1-yz)}$	$\frac{Y_1}{X_3X_6Y_2Y_3}$	$\frac{1}{s_3s_6t_2}+\frac{1}{s_3s_6t_3}$
$\frac{1}{z_{15}z_{16}z_{23}z_{26}z_{34}z_{45}}$	$\frac{1}{(1-x)\ (1-y)\ (1-z)}$	$\frac{1}{X_2X_3X_4X_6Y_2Y_3}$	$-\frac{1}{s_2s_4s_6}-\frac{1}{s_2s_6t_2}-\frac{1}{s_3s_6t_2}-\frac{1}{s_3s_6t_3}-\frac{1}{s_4s_6t_3}$

(7.2.67)

The fundamental objects correspond to the 14 rational functions

$$\frac{1}{X_1X_3X_5}\,,\ \frac{1}{X_2X_4X_6}\,,\ \frac{1}{X_1X_4Y_1}\,,\ \frac{1}{X_2X_5Y_2}\,,\ \frac{1}{X_3X_6Y_3}\,,\ \frac{1}{X_2X_5Y_1}\,,\ \frac{1}{X_3X_6Y_2}\,,\ \frac{1}{X_1X_4Y_3}\,,$$
$$\frac{1}{X_2X_4Y_1}\,,\ \frac{1}{X_3X_5Y_2}\,,\ \frac{1}{X_4X_6Y_3}\,,\ \frac{1}{X_1X_5Y_1}\,,\ \frac{1}{X_2X_6Y_2}\,,\ \frac{1}{X_1X_3Y_3}\,, \tag{7.2.68}$$

which furnish the $C_4 = 14$ poles

$$\frac{1}{s_1s_3s_5}\,,\ \frac{1}{s_2s_4s_6}\,,\ \frac{1}{s_1s_4t_1}\,,\ \frac{1}{s_2s_5t_2}\,,\ \frac{1}{s_3s_6t_3}\,,\ \frac{1}{s_2s_5t_1}\,,\ \frac{1}{s_3s_6t_2}\,,\ \frac{1}{s_1s_4t_3}\,,$$
$$\frac{1}{s_2s_4t_1}\,,\ \frac{1}{s_3s_5t_2}\,,\ \frac{1}{s_4s_6t_3}\,,\ \frac{1}{s_1s_5t_1}\,,\ \frac{1}{s_2s_6t_2}\,,\ \frac{1}{s_1s_3t_3}, \tag{7.2.69}$$

as single poles in the denominator of (7.1.12), respectively. The cyclically invariant integral (7.2.32) is given by

$$\begin{aligned} B_6\begin{bmatrix} n_i=-1 \\ n_{ii}=-1 \end{bmatrix} &= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\ \frac{I_6(x,y,z)}{x\ (1-x)\ y\ (1-y)\ z\ (1-z)} \\ &= \frac{1}{s_1s_3s_5}+\frac{1}{s_2s_4s_6}+\frac{1}{s_1s_4t_1}+\frac{1}{s_2s_5t_2}+\frac{1}{s_3s_6t_3}+\frac{1}{s_2s_5t_1}+\frac{1}{s_3s_6t_2}+\frac{1}{s_1s_4t_3} \\ &\quad+\frac{1}{s_2s_4t_1}+\frac{1}{s_3s_5t_2}+\frac{1}{s_4s_6t_3}+\frac{1}{s_1s_5t_1}+\frac{1}{s_2s_6t_2}+\frac{1}{s_1s_3t_3}+\ldots\,, \end{aligned} \tag{7.2.70}$$

and exhibits all fourteen poles (7.2.69) in its power series expansion. After triple poles, for an $n=6$ integral the next leading order to start with are single poles. They always come with a $\zeta(2)$. In analogy with (7.2.68), for the latter we may introduce a fundamental set of rational functions[4] furnishing the six single poles $\zeta(2)/s_i,\ i=1,2,\ldots,6$:

rational function in equation (7.1.12)	rational function in equation (7.2.39)	rational function in equation (7.2.40)	lowest order poles
$\frac{1}{z_{12}z_{15}z_{24}z_{35}z_{36}z_{46}}$	$\frac{1}{(1-xy)\ z\ (1-yz)}$	$\frac{1}{X_1}$	$\frac{\zeta(2)}{s_1}$
$\frac{1}{z_{14}z_{15}z_{23}z_{26}z_{35}z_{46}}$	$\frac{1}{(1-z)\ (1-xy)}$	$\frac{1}{X_2}$	$\frac{\zeta(2)}{s_2}$
$\frac{1}{z_{13}z_{15}z_{25}z_{26}z_{34}z_{46}}$	$\frac{1}{(1-y)\ (1-xyz)}$	$\frac{1}{X_3}$	$\frac{\zeta(2)}{s_3}$
$\frac{1}{z_{13}z_{15}z_{24}z_{26}z_{36}z_{45}}$	$\frac{1}{(1-x)\ (1-yz)}$	$\frac{1}{X_4}$	$\frac{\zeta(2)}{s_4}$
$\frac{1}{z_{13}z_{14}z_{24}z_{26}z_{35}z_{56}}$	$\frac{1}{x\ (1-xy)\ (1-yz)}$	$\frac{1}{X_5}$	$\frac{\zeta(2)}{s_5}$
$\frac{1}{z_{13}z_{16}z_{24}z_{25}z_{35}z_{46}}$	$\frac{1}{(1-xy)\ (1-yz)\ (1-xyz)}$	$\frac{1}{X_6}$	$\frac{\zeta(2)}{s_6}$

(7.2.71)

All (transcendental) integrals with single poles can be decomposed with respect to the basis (7.2.71) modulo finite pieces to be discussed in a moment. Subject to (7.2.25), we have e.g.:

$$\begin{aligned}
\frac{1}{z\,(1-xy)} &\simeq \frac{X_6Y_2}{X_1} = \frac{1}{X_1} - Y_1\\
\frac{y}{(1-y)\,(1-xyz)} &\simeq \frac{Y_1}{X_3} = \frac{1}{X_3} - X_6Y_2Y_3\\
\frac{x}{(1-x)\,(1-xyz)} &\simeq \frac{X_5Y_2}{X_4} = \frac{1}{X_4} - Y_3\\
\frac{1}{x\,(1-yz)} &\simeq \frac{X_6Y_3}{X_5} = \frac{1}{X_5} - Y_1
\end{aligned} \tag{7.2.72}$$

After single poles, for an $n=6$ integral the next leading order to start with are constants. They always come with a $\zeta(2)$ or $\zeta(3)$, e.g.:

rational function in equation (7.1.12)	rational function in equation (7.2.39)	rational function in equation (7.2.40)	lowest order
$\frac{1}{z_{14}z_{15}z_{24}z_{26}z_{35}z_{36}}$	$\frac{y}{(1-xy)\ (1-yz)}$	Y_1	$2\ \zeta(3)$
$\frac{1}{z_{13}z_{14}z_{25}z_{26}z_{35}z_{46}}$	$\frac{1}{(1-xy)\ (1-xyz)}$	Y_2	$2\ \zeta(3)$
$\frac{1}{z_{13}z_{15}z_{24}z_{25}z_{36}z_{46}}$	$\frac{1}{(1-yz)\ (1-xyz)}$	Y_3	$2\ \zeta(3)$
$\frac{z_{16}}{z_{13}z_{14}z_{15}z_{25}z_{26}z_{36}z_{46}}$	$\frac{1}{1-xyz}$	$X_6Y_2Y_3$	$\zeta(3)$
$\frac{z_{56}}{z_{14}z_{15}z_{25}z_{26}z_{35}z_{36}z_{46}}$	$\frac{xy}{(1-xy)\ (1-xyz)}$	$X_5Y_1Y_2$	$\zeta(3)$

(7.2.73)

Again, we may add the functions (7.2.73) to (7.2.68) to obtain other fundamental sets subject to the constraints in (7.2.62) and with the same poles as (7.2.69), see section 7.4 for details.

7.2.5 $n=7$ point integrals

In this case we have the 14 planar channels $(1,2),(1,3),(1,4)\equiv(5,7),(1,5)\equiv(6,7),(2,3),(2,4),(2,5),(2,6)\equiv(1,7),(3,4),(3,5),(3,6),(4,5),(4,6)$ and $(5,6)$ related to the 14 variables $u_{1,2},u_{1,3}$,

[4]Note that the rational functions $1/Y_i$ giving rise to single poles in t_i have double poles in their z_{ij} representation, i.e. $\tilde{n}_{ij}=-2$ for some i,j.

$u_{1,4}, u_{1,5}, u_{2,3}, u_{2,4}\ u_{2,5}, u_{2,6}, u_{3,4}, u_{3,5}, u_{3,6}, u_{4,5}, u_{4,6}$ and $u_{5,6}$, respectively. The seven point integral (7.1.14) becomes

$$\begin{aligned} B_7[n_i, n_{ij}] &= \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \int_0^1 \mathrm{d}x_3 \int_0^1 \mathrm{d}x_4\ x_1^{s_1+n_1}\ x_2^{t_1+n_2}\ x_3^{t_5+n_3}\ x_4^{s_6+n_4} (1-x_1)^{s_2+n_{11}} \\ &\times\ (1-x_2)^{s_3+n_{22}} (1-x_3)^{s_4+n_{33}} (1-x_4)^{s_5+n_{44}} (1-x_1x_2)^{s_{24}+n_{12}} (1-x_2x_3)^{s_{35}+n_{23}} \\ &\times\ (1-x_1x_2x_3)^{s_{25}+n_{13}} (1-x_3x_4)^{s_{46}+n_{34}} (1-x_2x_3x_4)^{s_{36}+n_{24}} (1-x_1x_2x_3x_4)^{s_{26}+n_{14}} \end{aligned} \tag{7.2.74}$$

with $s_i = \alpha'(k_i + k_{i+1})^2,\ t_j = \alpha'(k_j + k_{j+1} + k_{j+2})^2,\ i,j = 1,\ldots,7$ subject to the cyclic identifications $i+7 \equiv i$ and $j+7 \equiv j$, respectively.

To bring (7.2.74) into the $B_n[n_{i,j}]$ form (7.2.29) we choose the four independent variables $u_{1,2} = x_1, u_{1,3} = x_2, u_{1,4} = x_3$ and $u_{1,5} = x_4$. Then, with (7.2.27) the integral (7.2.74) assumes the form (7.2.29)

$$\begin{aligned} B_7[n_{i,j}] &= \int_0^1 \mathrm{d}u_{i,j}\ u_{1,2}^{s_1+n_{1,2}}\ u_{2,3}^{s_2+n_{2,3}}\ u_{3,4}^{s_3+n_{3,4}}\ u_{4,5}^{s_4+n_{4,5}}\ u_{5,6}^{s_5+n_{5,6}}\ u_{1,5}^{s_6+n_{1,5}}\ u_{2,6}^{s_7+n_{2,6}}\ u_{1,3}^{t_1+n_{1,3}} \\ &\times\ u_{2,4}^{t_2+n_{2,4}}\ u_{3,5}^{t_3+n_{3,5}}\ u_{4,6}^{t_4+n_{4,6}}\ u_{1,4}^{t_5+n_{1,4}}\ u_{2,5}^{t_6+n_{2,5}}\ u_{3,6}^{t_7+n_{3,6}} \\ &\times\ \delta(u_{2,3} + u_{1,2}u_{3,4}u_{3,5}u_{3,6} - 1)\ \delta(u_{2,4} + u_{1,2}u_{1,3}u_{3,5}u_{3,6}u_{4,5}u_{4,6} - 1) \\ &\times\ \delta(u_{2,5} + u_{1,2}u_{1,3}u_{1,4}u_{3,6}u_{4,6}u_{5,6} - 1)\ \ \delta(u_{2,6} + u_{1,2}u_{1,3}u_{1,4}u_{1,5} - 1) \\ &\times\ \delta(u_{3,4} + u_{1,3}u_{2,3}u_{4,5}u_{4,6} - 1)\ \ \delta(u_{3,5} + u_{1,3}u_{1,4}u_{2,3}u_{2,4}u_{4,6}u_{5,6} - 1) \\ &\times\ \delta(u_{3,6} + u_{1,3}u_{1,4}u_{1,5}u_{2,3}u_{2,4}u_{2,5} - 1)\ \delta(u_{4,5} + u_{1,4}u_{2,4}u_{3,4}u_{5,6} - 1) \\ &\times\ \delta(u_{4,6} + u_{1,4}u_{1,5}u_{2,4}u_{2,5}u_{3,4}u_{3,5} - 1)\ \delta(u_{5,6} + u_{1,5}u_{2,5}u_{3,5}u_{4,5} - 1)\ , \end{aligned} \tag{7.2.75}$$

with

$$\begin{aligned} n_{1,2} &= n_1\ , \qquad n_{1,3} = n_2\ , \qquad n_{1,4} = n_3\ , \qquad n_{1,5} = n_4 \\ n_{2,3} &= n_{11}\ , \qquad n_{3,4} = n_{22}\ , \qquad n_{4,5} = n_{33}\ , \qquad n_{5,6} = n_{44} \\ n_{2,4} &= 1 + \sum_{1\leq i\leq j}^{2} n_{ij}\ , \qquad n_{2,5} = 2 + \sum_{1\leq i\leq j}^{3} n_{ij}\ , \qquad n_{2,6} = 3 + \sum_{1\leq i\leq j}^{4} n_{ij} \\ n_{3,5} &= 1 + \sum_{2\leq i\leq j}^{3} n_{ij}\ , \qquad n_{3,6} = 2 + \sum_{2\leq i\leq j}^{4} n_{ij}\ , \qquad n_{4,6} = 1 + \sum_{3\leq i\leq j}^{4} n_{ij}\ . \end{aligned} \tag{7.2.76}$$

In what follows it is convenient to introduce

$$\begin{aligned} I_7(x,y,z,w) &:= x^{s_6}\ y^{t_5}\ z^{t_1}\ w^{s_1}\ (1-x)^{s_5}\ (1-y)^{s_4}\ (1-z)^{s_3}\ (1-w)^{s_2}\ (1-xy)^{s_{46}} \\ &\times\ (1-wz)^{s_{24}}\ (1-yz)^{s_{35}}\ (1-xyz)^{s_{36}}\ (1-yzw)^{s_{25}}\ (1-xyzw)^{s_{26}}\ , \end{aligned} \tag{7.2.77}$$

arising from (7.2.74) with the identifications $x_1 := w,\ x_2 := z,\ x_3 := y$ and $x_4 := x$. Furthermore, we stick to the following shorter notation for the dual variables $u_{i,j}$

$$X_i := u_{i,i+1}\ , \qquad Y_j := u_{j,j+2}\ , \qquad i,j = 1,\ldots,7\ , \qquad i+7 \equiv i \tag{7.2.78}$$

and define

$$
\begin{aligned}
J_7(X,Y) \quad := \quad & \left(\prod_{i=1}^{7} X_i^{s_i}\right) \left(\prod_{j=1}^{7} Y_j^{t_j}\right) \delta(X_2 + X_1X_3Y_3Y_7 - 1)\ \delta(Y_2 + X_1X_4Y_1Y_3Y_4Y_7 - 1) \\
& \times\ \delta(Y_6 + X_1X_5Y_1Y_4Y_5Y_7 - 1)\ \delta(X_7 + X_1X_6Y_1Y_5 - 1)\ \delta(X_3 + X_2X_4Y_1Y_4 - 1) \\
& \times\ \delta(X_4 + X_3X_5Y_2Y_5 - 1)\ \delta(Y_4 + X_3X_6Y_2Y_3Y_5Y_6 - 1)\ \delta(X_5 + X_4X_6Y_3Y_6 - 1) \\
& \times\ \delta(Y_3 + X_2X_5Y_1Y_2Y_4Y_5 - 1)\ \delta(Y_7 + X_2X_6Y_1Y_2Y_5Y_6 - 1)\ .
\end{aligned}
\tag{7.2.79}
$$

Let us now discuss a few examples. The pole structure of the integral

$$
\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w\ \frac{I_7(x,y,z,w)}{x\ (1-y)\ (1-wz)\ (1-yz)}
\tag{7.2.80}
$$

can be easily deduced after transforming it into the form (7.2.58)

$$
\left(\prod_{i=1}^{7} \int_0^1 \mathrm{d}X_i\right) \left(\prod_{j=1}^{7} \int_0^1 \mathrm{d}Y_j\right) \frac{J_7(X,Y)}{X_4X_6Y_3Y_6} \quad = \quad \frac{1}{s_4s_6t_3t_6} \ +\ \ldots\ .
\tag{7.2.81}
$$

Hence, the only simultaneous pole is at $X_4, X_6, Y_3, Y_6 \to 0$ with the product of δ functions yielding the constraints for the ten variables $X_1, X_2, X_3, X_5, X_7, Y_1, Y_2, Y_4, Y_5, Y_7 \to 1$. Note that by construction a set of four poles in (7.2.75) does not necessarily yield a compatible set of channels, e.g. the integral

$$
\begin{aligned}
& \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w\ \frac{I_7(x,y,z,w)}{w\ (1-x)\ (1-z)\ (1-wyz)\ (1-wxyz)} \\
& \quad = \quad \left(\prod_{i=1}^{7} \int_0^1 \mathrm{d}X_i\right) \left(\prod_{j=1}^{7} \int_0^1 \mathrm{d}Y_j\right) \frac{J_7(X,Y)}{X_1X_3X_5X_7} \quad = \quad \frac{1}{s_1s_3s_5} \ +\ \ldots
\end{aligned}
\tag{7.2.82}
$$

does not give rise to a quadruple pole because $(1,2), (3,4), (5,6)$ and $(7,1)$ are not compatible channels. Subsequently, we list a few nontrivial examples:

rational function in eq. (7.1.12)	rational function in eq. (7.2.39)	rational function in eq. (7.2.40)	lowest order poles
$\frac{z_{17}^3}{z_{12}z_{13}z_{14}z_{15}z_{16}z_{27}z_{37}z_{47}z_{57}z_{67}}$	$\frac{1}{x\ y\ z\ w}$	$\frac{X_7^3Y_2Y_3Y_4Y_6^2Y_7^2}{X_1X_6Y_1Y_5}$	$\frac{1}{s_1s_6t_1t_5}$
$\frac{z_{17}^2}{z_{12}z_{14}z_{15}z_{16}z_{27}z_{34}z_{37}z_{57}z_{67}}$	$\frac{1}{x\ y\ (1-z)\ w}$	$\frac{X_7^2Y_4Y_6Y_7}{X_1X_3X_6Y_5}$	$\frac{1}{s_1s_3s_6t_5}$
$\frac{z_{17}^2}{z_{12}z_{13}z_{14}z_{16}z_{27}z_{37}z_{47}z_{56}z_{57}}$	$\frac{1}{(1-x)\ y\ z\ w}$	$\frac{X_7^2Y_2Y_3Y_6^2Y_7}{X_1X_5Y_1Y_5}$	$\frac{1}{s_1s_5t_1t_5}$
$\frac{z_{17}z_{67}}{z_{12}z_{13}z_{16}z_{27}z_{37}z_{46}z_{47}z_{56}z_{57}}$	$\frac{x}{w\ z\ (1-x)\ (1-xy)}$	$\frac{X_6X_7Y_2Y_3Y_6^2}{X_1X_5Y_1Y_4}$	$\frac{1}{s_1s_5t_1t_4}$
$\frac{z_{17}}{z_{12}z_{14}z_{16}z_{27}z_{34}z_{37}z_{56}z_{57}}$	$\frac{1}{y\ w\ (1-x)\ (1-z)}$	$\frac{X_7Y_6}{X_1X_3X_5Y_5}$	$\frac{1}{s_1s_3s_5t_5}$
$\frac{z_{17}}{z_{12}z_{13}z_{16}z_{27}z_{37}z_{45}z_{47}z_{56}}$	$\frac{1}{z\ w\ (1-x)\ (1-y)}$	$\frac{X_7Y_3Y_6}{X_1X_4X_5Y_1Y_4}$	$\frac{1}{s_1s_4t_1t_4} + \frac{1}{s_1s_5t_1t_4}$
$\frac{z_{17}}{z_{14}z_{15}z_{16}z_{23}z_{27}z_{34}z_{57}z_{67}}$	$\frac{1}{x\ y\ (1-z)\ (1-w)}$	$\frac{X_7Y_4Y_7}{X_2X_3X_6Y_2Y_5}$	$\frac{1}{s_2s_6t_2t_5} + \frac{1}{s_3s_6t_2t_5}$
$\frac{z_{67}}{z_{12}z_{16}z_{27}z_{36}z_{37}z_{45}z_{47}z_{56}}$	$\frac{x\ y}{w(1-x)(1-y)(1-xyz)}$	$\frac{X_6Y_2Y_5Y_6}{X_1X_4X_5Y_4Y_7}$	$\frac{1}{s_1s_4t_4t_7} + \frac{1}{s_1s_5t_4t_7}$
$\frac{1}{z_{12}z_{16}z_{27}z_{34}z_{37}z_{45}z_{56}}$	$\frac{1}{w(1-x)(1-y)(1-z)}$	$\frac{1}{X_1X_3X_4X_5Y_3Y_4Y_7}$	$\frac{1}{s_1s_3s_5t_7} + \frac{1}{s_1s_3t_3t_7} + \frac{1}{s_1s_4t_3t_7} + \frac{1}{s_1s_4t_4t_7} + \frac{1}{s_1s_5t_4t_7}$

(7.2.83)

After quadruple poles, for an $n = 7$ integral the next leading order to start with are double poles. They always come with a $\zeta(2)$, e.g.:

rational function in eq. (7.1.12)	rational function in eq. (7.2.39)	rational function in eq. (7.2.40)	lowest order poles
$\frac{1}{z_{12}z_{15}z_{24}z_{35}z_{37}z_{46}z_{67}}$	$\frac{1}{wx(1-xy)(1-wz)(1-yz)}$	$\frac{1}{X_1X_6}$	$\frac{\zeta(2)}{s_1s_6}$
$\frac{z_{14}}{z_{12}z_{13}z_{16}z_{24}z_{35}z_{46}z_{47}z_{57}}$	$\frac{1}{w(1-xy)z(1-wz)(1-yz)}$	$\frac{1}{X_1Y_1}$	$\frac{\zeta(2)}{s_1t_1}$
$\frac{1}{z_{15}z_{16}z_{26}z_{27}z_{34}z_{37}z_{45}}$	$\frac{yz}{(1-y)(1-z)(1-wxyz)}$	$\frac{Y_1Y_5}{X_3X_4Y_3}$	$\frac{\zeta(2)}{s_3t_3}+\frac{\zeta(2)}{s_4t_3}$

(7.2.84)

After double poles, for an $n = 7$ integral the next leading order to start with are single poles. They are always accompanied by $\zeta(2)$ or $\zeta(3)$ factors:

rational function in eq. (7.1.12)	rational function in eq. (7.2.39)	rational function in eq. (7.2.40)	lowest order poles
$\frac{1}{z_{12}z_{16}z_{24}z_{35}z_{37}z_{46}z_{57}}$	$\frac{1}{w(1-xy)(1-wz)(1-yz)}$	$\frac{1}{X_1}$	$\frac{2\zeta(2)}{s_1}$
$\frac{z_{15}}{z_{12}z_{14}z_{16}z_{25}z_{35}z_{37}z_{46}z_{57}}$	$\frac{1}{w(1-xy)(1-yz)(1-wyz)}$	$\frac{Y_2}{X_1}$	$\frac{2\zeta(2)}{s_1}$
$\frac{1}{z_{12}z_{16}z_{24}z_{35}z_{36}z_{47}z_{57}}$	$\frac{1}{w(1-wz)(1-yz)(1-xyz)}$	$\frac{Y_4}{X_1}$	$\frac{2\zeta(3)}{s_1}$
$\frac{1}{z_{12}z_{15}z_{16}z_{24}z_{35}z_{37}z_{46}z_{47}}$	$\frac{y}{w(1-xy)(1-wz)(1-yz)}$	$\frac{Y_5}{X_1}$	$\frac{2\zeta(3)}{s_1}$
$\frac{z_{15}z_{23}}{z_{12}z_{13}z_{16}z_{24}z_{25}z_{35}z_{37}z_{46}z_{57}}$	$\frac{1-w}{w(1-xy)(1-wz)(1-yz)(1-wyz)}$	$\frac{X_2Y_2}{X_1}$	$\frac{2\zeta(2)}{s_1}$

(7.2.85)

After single poles, for an $n = 7$ integral the next leading order to start with are the zeta constants $\zeta(2), \zeta(3)$ or $\zeta(4)$. First, we display examples without poles and their series expansion starts at $\zeta(2)$ or $\zeta(3)$:

rational function in eq. (7.1.12)	rational function in eq. (7.2.39)	rational function in eq. (7.2.40)	lowest order poles
$\frac{z_{47}}{z_{14}z_{16}z_{24}z_{27}z_{35}z_{37}z_{46}z_{57}}$	$\frac{z}{(1-xy)\ (1-yz)\ (1-wz)}$	Y_1	$2\ \zeta(2)+2\ \zeta(3)$
$\frac{z_{14}z_{37}}{z_{13}z_{15}z_{16}z_{24}z_{27}z_{35}z_{36}z_{47}^2}$	$\frac{y}{(1-yz)\ (1-wz)\ (1-xyz)}$	Y_4Y_5	$\frac{3}{2}\ \zeta(2)+\frac{3}{2}\ \zeta(3)$
$\frac{z_{15}z_{37}}{z_{13}z_{14}z_{16}z_{25}z_{27}z_{35}z_{36}z_{47}z_{57}}$	$\frac{1}{(1-yz)\ (1-xyz)\ (1-wyz)}$	Y_2Y_4	$\frac{5}{2}\ \zeta(4)+4\ \zeta(3)-2\ \zeta(3)$
$\frac{1}{z_{13}z_{14}z_{25}z_{27}z_{36}z_{46}z_{57}}$	$\frac{1}{(1-xy)\ (1-wyz)\ (1-xyz)}$	$Y_2Y_3Y_6$	$3\ \zeta(3)$

(7.2.86)

Finally, we give examples without poles and their series expansion starts at $\zeta(4)$:

rational function in eq. (7.1.12)	rational function in eq. (7.2.39)	rational function in eq. (7.2.40)	lowest order poles
$\frac{1}{z_{13}z_{16}z_{24}z_{27}z_{35}z_{46}z_{57}}$	$\frac{1}{(1-xy)\ (1-yz)\ (1-wz)}$	1	$\frac{27}{4}\ \zeta(4)$
$\frac{1}{z_{14}z_{16}z_{24}z_{27}z_{35}z_{36}z_{57}}$	$\frac{z}{(1-yz)\ (1-wz)\ (1-xyz)}$	Y_1Y_4	$\frac{17}{4}\ \zeta(4)$
$\frac{z_{37}}{z_{13}z_{14}z_{26}z_{27}z_{35}z_{36}z_{47}z_{57}}$	$\frac{1}{(1-yz)\ (1-xyz)\ (1-wxyz)}$	$Y_2Y_4Y_6$	$3\ \zeta(4)$
$\frac{1}{z_{13}z_{14}z_{25}z_{26}z_{37}z_{46}z_{57}}$	$\frac{1}{(1-xy)\ (1-wyz)\ (1-wxyz)}$	$Y_2Y_3Y_6Y_7$	$\frac{5}{2}\ \zeta(4)$
$\frac{z_{16}}{z_{13}z_{14}z_{15}z_{26}z_{27}z_{36}z_{46}z_{57}}$	$\frac{1}{(1-xy)\ (1-xyz)\ (1-wxyz)}$	$Y_6^2Y_2Y_3$	$3\ \zeta(4)$

(7.2.87)

Again, we may add the functions (7.2.87) to the 42 fundamental quadruple poles to obtain other fundamental sets subject to the constraints in (7.2.79), cf. section 7.4 for more details.

7.2.6 $n = 8$ point integrals

In this case we have the 20 planar channels $(1,2)$, $(1,3) \equiv (4,8)$, $(1,4) \equiv (5,8)$, $(1,5) \equiv (6,8)$, $(1,6) \equiv (7,8)$, $(2,3)$, $(2,4)$, $(2,5)$, $(2,6)$, $(2,7) \equiv (1,8)$, $(3,4)$, $(3,5)$, $(3,6)$, $(3,7)$, $(4,5)$, $(4,6)$, $(4,7)$, $(5,6)$, $(5,7)$ and $(6,7)$ related to the 20 variables $u_{1,2}, u_{2,3}, u_{3,4}, u_{4,5}, u_{5,6}, u_{6,7}, u_{1,6} = u_{7,8}, u_{2,7}, u_{1,3}, u_{2,4}, u_{3,5}, u_{4,6}, u_{5,7}, u_{6,8}, u_{2,6}, u_{3,7}, u_{1,4}, u_{2,5}, u_{3,6}, u_{4,8}$. The eight point integral (7.1.14) becomes

$$\begin{aligned}
B_8[n_i, n_{ij}] \;=\; & \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \int_0^1 \mathrm{d}x_3 \int_0^1 \mathrm{d}x_4 \int_0^1 \mathrm{d}x_5\; x_1^{s_1+n_1}\; x_2^{t_1+n_2}\; x_3^{u_1+n_3}\; x_4^{t_6+n_4}\; x_5^{s_7+n_5} \\
\times\; & (1-x_1)^{s_2+n_{11}}\, (1-x_2)^{s_3+n_{22}}\, (1-x_3)^{s_4+n_{33}}\, (1-x_4)^{s_5+n_{44}}\, (1-x_5)^{s_6+n_{55}} \\
\times\; & (1-x_1x_2)^{s_{24}+n_{12}}\, (1-x_2x_3)^{s_{35}+n_{23}}\, (1-x_3x_4)^{s_{46}+n_{34}}\, (1-x_4x_5)^{s_{57}+n_{45}} \\
\times\; & (1-x_1x_2x_3)^{s_{25}+n_{13}}\, (1-x_2x_3x_4)^{s_{36}+n_{24}}\, (1-x_3x_4x_5)^{s_{47}+n_{35}} \\
\times\; & (1-x_1x_2x_3x_4)^{s_{26}+n_{14}}\, (1-x_2x_3x_4x_5)^{s_{37}+n_{25}}\, (1-x_1x_2x_3x_4x_5)^{s_{27}+n_{15}}\;, \qquad (7.2.88)
\end{aligned}$$

with $s_i = \alpha'(k_i + k_{i+1})^2$, $t_j = \alpha'(k_j + k_{j+1} + k_{j+2})^2$, $i, j = 1, \ldots, 8$ subject to the cyclic identifications $i+8 \equiv i$, $j+8 \equiv j$, respectively, and $u_l = \alpha'(k_l+k_{l+1}+k_{l+2}+k_{l+3})^2$, $l = 1, \ldots, 4$.

To bring (7.2.88) into the form (7.2.29) according to (7.2.26) we choose the five independent variables $u_{1,2} = x_1, u_{1,3} = x_2, u_{1,4} = x_3, u_{1,5} = x_4$ and $u_{1,6} = x_5$. Then, with (7.2.27) the integral (7.2.88) assumes the form (7.2.29)

$$\begin{aligned}
B_8[n_{i,j}] \;=\; & \int_0^1 \mathrm{d}u_{i,j}\; u_{1,2}^{s_1+n_{1,2}}\; u_{2,3}^{s_2+n_{2,3}}\; u_{3,4}^{s_3+n_{3,4}}\; u_{4,5}^{s_4+n_{4,5}}\; u_{5,6}^{s_5+n_{5,6}}\; u_{6,7}^{s_6+n_{6,7}}\; u_{1,6}^{s_7+n_{1,6}}\; u_{2,7}^{s_8+n_{2,7}} \\
\times\; & u_{1,3}^{t_1+n_{1,3}}\; u_{2,4}^{t_2+n_{2,4}}\; u_{3,5}^{t_3+n_{3,5}}\; u_{4,6}^{t_4+n_{4,6}}\; u_{5,7}^{t_5+n_{5,7}}\; u_{1,5}^{t_6+n_{1,5}}\; u_{2,6}^{t_7+n_{2,6}}\; u_{3,7}^{t_8+n_{3,7}} \\
\times\; & u_{1,4}^{u_1+n_{1,4}}\; u_{2,5}^{u_2+n_{2,5}}\; u_{3,6}^{u_3+n_{3,6}}\; u_{4,7}^{u_4+n_{4,7}} \\
\times\; & \delta(u_{2,3} + u_{1,2}u_{3,4}u_{3,5}u_{3,6}u_{3,7} - 1)\; \delta(u_{2,4} + u_{1,2}u_{1,3}u_{3,5}u_{3,6}u_{3,7}u_{4,5}u_{4,6}u_{4,7} - 1) \\
\times\; & \delta(u_{2,5} + u_{1,2}u_{1,3}u_{1,4}u_{3,6}u_{3,7}u_{4,6}u_{4,7}u_{5,6}u_{5,7} - 1)\; \delta(u_{2,7} + u_{1,2}u_{1,3}u_{1,4}u_{1,5}u_{1,6} - 1) \\
\times\; & \delta(u_{2,6} + u_{1,2}u_{1,3}u_{1,4}u_{1,5}u_{3,7}u_{4,7}u_{5,7}u_{6,7} - 1)\; \delta(u_{3,4} + u_{1,3}u_{2,3}u_{4,5}u_{4,6}u_{4,7} - 1) \\
\times\; & \delta(u_{3,5} + u_{1,3}u_{1,4}u_{2,3}u_{2,4}u_{4,6}u_{4,7}u_{5,6}u_{5,7} - 1)\; \delta(u_{6,7} + u_{1,6}u_{2,6}u_{3,6}u_{4,6}u_{5,6} - 1) \\
\times\; & \delta(u_{3,7} + u_{1,3}u_{1,4}u_{1,5}u_{1,6}u_{2,3}u_{2,4}u_{2,5}u_{2,6} - 1)\; \delta(u_{4,5} + u_{1,4}u_{2,4}u_{3,4}u_{5,6}u_{5,7} - 1) \\
\times\; & \delta(u_{4,6} + u_{1,4}u_{1,5}u_{2,4}u_{2,5}u_{3,4}u_{3,5}u_{5,7}u_{6,7} - 1)\; \delta(u_{5,6} + u_{1,5}u_{2,5}u_{3,5}u_{4,5}u_{6,7} - 1) \\
\times\; & \delta(u_{4,7} + u_{1,4}u_{1,5}u_{1,6}u_{2,4}u_{2,5}u_{2,6}u_{3,4}u_{3,5}u_{3,6} - 1) \\
\times\; & \delta(u_{5,7} + u_{1,5}u_{1,6}u_{2,5}u_{2,6}u_{3,5}u_{3,6}u_{4,5}u_{4,6} - 1)
\end{aligned}$$

$$\times\ \delta(u_{3,6}+u_{1,3}u_{1,4}u_{1,5}u_{2,3}u_{2,4}u_{2,5}u_{4,7}u_{5,7}u_{6,7}-1) \tag{7.2.89}$$

with

$$\begin{aligned}
n_{1,2} &= n_1\ , \quad n_{1,3} = n_2\ , \quad n_{1,4} = n_3\ , \quad n_{1,5} = n_4 \\
n_{1,6} &= n_5\ , \quad n_{2,3} = n_{11}\ , \quad n_{3,4} = n_{22}\ , \quad n_{4,5} = n_{33} \\
n_{5,6} &= n_{44} n_{6,7} = n_{55}\ , \quad n_{2,4} = 1+\sum_{1\le i\le j}^{2} n_{ij} \\
n_{2,5} &= 2+\sum_{1\le i\le j}^{3} n_{ij}\ , \quad n_{2,6} = 3+\sum_{1\le i\le j}^{4} n_{ij}\ , \quad n_{2,7} = 4+\sum_{1\le i\le j}^{5} n_{ij} \\
n_{3,5} &= 1+\sum_{2\le i\le j}^{3} n_{ij}\ , \quad n_{3,6} = 2+\sum_{2\le i\le j}^{4} n_{ij}\ , \quad n_{3,7} = 3+\sum_{2\le i\le j}^{5} n_{ij} \\
n_{4,6} &= 1+\sum_{3\le i\le j}^{4} n_{ij}\ , \quad n_{4,7} = 2+\sum_{3\le i\le j}^{5} n_{ij}\ , \quad n_{5,7} = 1+\sum_{4\le i\le j}^{5} n_{ij}\ .
\end{aligned} \tag{7.2.90}$$

In what follows it is convenient to introduce

$$\begin{aligned}
I_8(x,y,z,w,v) \ :=\ & x^{s_7}\ y^{t_6}\ z^{u_1}\ w^{t_1}\ v^{s_1}\ (1-x)^{s_6}\ (1-y)^{s_5}\ (1-z)^{s_4}\ (1-w)^{s_3}\ (1-v)^{s_2} \\
& \times\ (1-xy)^{s_{57}}\ (1-yz)^{s_{46}}\ (1-wz)^{s_{35}}\ (1-vw)^{s_{24}}\ (1-xyz)^{s_{47}}\ (1-wyz)^{s_{36}} \\
& \times\ (1-vwz)^{s_{25}}\ (1-wxyz)^{s_{37}}\ (1-vwyz)^{s_{26}}\ (1-vwxyz)^{s_{27}}
\end{aligned} \tag{7.2.91}$$

arising from (7.2.88) with the identifications $x_1 := v,\ x_2 := w,\ x_3 := z,\ x_4 = y$ and $x_5 := x$. Furthermore, we stick to the following shorter notation for the dual variables $u_{i,j}$

$$\begin{aligned}
X_i &:= u_{i,i+1}\ , \quad Y_j = u_{j,j+2}\ , \quad i,j = 1,\ldots,8\ , \quad i+8 \equiv i \\
Z_k &:= u_{k,k+3}\ , \quad k = 1,\ldots,4
\end{aligned} \tag{7.2.92}$$

and define

$$\begin{aligned}
J_8(X,Y,Z) \ :=\ & \left(\prod_{i=1}^{8} X_i^{s_i}\right) \left(\prod_{j=1}^{8} Y_j^{t_j}\right) \left(\prod_{k=1}^{4} Z_k^{u_k}\right)\ \delta(X_2+X_1X_3Y_3Y_8Z_3-1) \\
& \times\ \delta(Y_2+X_1X_4Y_1Y_3Y_4Y_8Z_3Z_4-1)\ \delta(Z_2+X_1X_5Y_1Y_4Y_5Y_8Z_1Z_3Z_4-1) \\
& \times\ \delta(Y_7+X_1X_6Y_1Y_5Y_6Y_8Z_1Z_4-1)\ \delta(X_8+X_1X_7Y_1Y_6Z_1-1) \\
& \times\ \delta(X_3+X_2X_4Y_1Y_4Z_4-1)\ \delta(Y_3+X_2X_5Y_1Y_2Y_4Y_5Z_1Z_4-1) \\
& \times\ \delta(Z_3+X_2X_6Y_1Y_2Y_5Y_6Z_1Z_2Z_4-1)\ \delta(Y_8+X_2X_7Y_1Y_2Y_6Y_7Z_1Z_2-1) \\
& \times\ \delta(X_4+X_3X_5Y_2Y_5Z_1-1)\ \delta(Y_4+X_3X_6Y_2Y_3Y_5Y_6Z_1Z_2-1) \\
& \times\ \delta(Z_4+X_3X_7Y_2Y_3Y_6Y_7Z_1Z_2Z_3-1)\ \delta(X_5+X_4X_6Y_3Y_6Z_2-1)
\end{aligned}$$

$$\times \ \delta(Y_5 + X_4X_7Y_3Y_4Y_6Y_7Z_2Z_3 - 1)\ \delta(X_6 + X_5X_7Y_4Y_7Z_3 - 1)\ . \tag{7.2.93}$$

Let us now discuss a few examples. The pole structure of the integral

$$\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \int_0^1 \mathrm{d}v\ \frac{I_8(x,y,z,w,v)}{w\ (1-v)\ (1-z)\ (1-xy)\ (1-yz)} \tag{7.2.94}$$

can be easily deduced after transforming it into the form (7.2.89)

$$\left(\prod_{i=1}^{8}\int_0^1 \mathrm{d}X_i\right)\left(\prod_{j=1}^{8}\int_0^1 \mathrm{d}Y_j\right)\left(\prod_{k=1}^{4}\int_0^1 \mathrm{d}Z_k\right)\frac{J_8(X,Y,Z)}{X_2X_4Y_1Y_4Z_4} \ = \ \frac{1}{s_2s_4t_1t_4u_4} + \ldots\ . \tag{7.2.95}$$

Hence, the only simultaneous pole is at $X_2, X_4, Y_1, Y_4, Z_4 \to 0$. At these values, the product of δ functions yield the constraints for the 15 variables $X_1, X_3, X_5, X_6, X_7, X_8, Y_2, Y_3, Y_5, Y_6, Y_7, Y_8, Z_1, Z_2, Z_3 \to 1$. Subsequently, we list a few non–trivial examples:

rational function in eq. (7.1.12)	rational function in eq. (7.2.39)	rational function in eq. (7.2.40)	lowest order poles
$\frac{z_{18}^4}{z_{12}z_{13}z_{14}z_{15}z_{16}z_{17}z_{28}z_{38}z_{48}z_{58}z_{68}z_{78}}$	$\frac{1}{x\ y\ z\ w\ v}$	$\frac{X_8^4Y_2Y_3Y_4Y_5Y_7^3Y_8^3Z_2^2Z_3^2Z_4^2}{X_1X_7Y_1Y_6Z_1}$	$\frac{1}{s_1s_7t_1t_6u_1}$
$\frac{z_{18}^3}{z_{12}z_{13}z_{14}z_{16}z_{17}z_{28}z_{38}z_{45}z_{58}z_{68}z_{78}}$	$\frac{1}{x\ y\ (1-z)\ w\ v}$	$\frac{X_8^3Y_2Y_5Y_7^2Y_8^2Z_2Z_3^2Z_4}{X_1X_4X_7Y_1Y_6}$	$\frac{1}{s_1s_4s_7t_1t_6}$
$\frac{1}{z_{17}z_{18}z_{23}z_{24}z_{35}z_{46}z_{57}z_{68}}$	$\frac{1}{(1-v)\ (1-xy)\ (1-wz)\ (1-yz)\ (1-vw)}$	$\frac{1}{X_2X_8Y_2Y_7Z_2}$	$\frac{1}{s_2s_8t_2t_7u_2}$
$\frac{1}{z_{12}z_{17}z_{28}z_{34}z_{36}z_{47}z_{56}z_{58}}$	$\frac{y}{v\ (1-y)\ (1-w)\ (1-xyz)\ (1-wyz)}$	$\frac{Y_6Z_2}{X_1X_3X_5Y_8Z_3}$	$\frac{1}{s_1s_3s_5t_8u_3}$
$\frac{1}{z_{17}z_{18}z_{24}z_{26}z_{35}z_{37}z_{45}z_{68}}$	$\frac{w\ z}{(1-z)\ (1-vw)\ (1-wz)\ (1-vwyz)\ (1-wxyz)}$	$\frac{Y_1Y_5Z_1Z_4}{X_4X_8Y_3Y_7Z_2}$	$\frac{1}{s_4s_8t_3t_7u_2}$
$\frac{z_{18}^2}{z_{12}z_{15}z_{16}z_{17}z_{28}z_{34}z_{38}z_{45}z_{68}z_{78}}$	$\frac{1}{x\ y\ v\ (1-z)\ (1-w)}$	$\frac{X_8^2Y_5Y_7Y_8Z_4}{X_1X_3X_4X_7Y_3Y_6}$	$\frac{1}{s_1s_3s_7t_3t_6} + \frac{1}{s_1s_4s_7t_3t_6}$
$\frac{1}{z_{12}z_{17}z_{24}z_{34}z_{38}z_{56}z_{57}z_{68}}$	$\frac{1}{v\ z\ (1-y)\ (1-w)\ (1-xy)\ (1-vw)}$	$\frac{1}{X_1X_3X_5Y_2Y_5Z_1}$	$\frac{1}{s_1s_3s_5t_5u_1} + \frac{1}{s_3s_5t_2t_5u_1}$
$\frac{1}{z_{13}z_{17}z_{23}z_{25}z_{45}z_{46}z_{68}z_{78}}$	$\frac{1}{x\ y\ w\ (1-z)\ (1-v)\ (1-yz)\ (1-vwz)}$	$\frac{Y_5Y_8}{X_3X_4X_7Y_1Y_4Y_6Y_7Z_2}$	$\frac{1}{s_2s_4s_7}\left(\frac{1}{t_1t_4} + \frac{1}{t_1t_6} + \frac{1}{t_4t_7} + \frac{1}{t_6u_2} + \frac{1}{7u_2}\right)$

(7.2.96)

7.3 Degree of transcendentality in the α' expansion

The α' dependence enters through the kinematic invariants $s_{i_1\ldots i_k}$ into the integrals (7.1.12) or (7.1.14). Hence, in their (integer) power series expansions in α', which (up to overall normalization factors) may start at least at the order α'^{3-n}, each power α'^m is accompanied by some rational function or polynomial of degree m in the kinematic invariants $s_{i_1\ldots i_k}$. The latter have rational coefficients multiplied by MZVs (7.1.22) of certain weights $\sum_{i=1}^k s_i$. The maximal weight thereof appearing at a given order α'^m is related to the power m.

7.3.1 Basic definitions and examples

One important question is, whether the set of MZVs showing up at a given order m in α' is of a fixed weight. In this case we call the power series expansion *transcendental* (we may also call the integral transcendental). E.g. for $n=6$ we may have the following integral and its power series expansion in α':

$$\begin{aligned}\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\ \frac{I_6(x,y,z)}{xyz} \;&=\; \frac{1}{s_1 s_5 t_1} \;-\; \zeta(2)\left(\frac{s_3}{s_1 s_5} \;+\; \frac{s_4}{s_1 t_1} \;+\; \frac{s_2}{s_5 t_1}\right)\\ +\;\zeta(3)&\left(\frac{s_3+s_4-t_3}{s_1} \;+\; \frac{s_2+s_3-t_2}{s_5} \;+\; \frac{s_3^2+s_3 t_1}{s_1 s_5} \;+\; \frac{s_4^2+s_4 s_5}{s_1 t_1} \;+\; \frac{s_2^2+s_1 s_2}{s_5 t_1}\right) \;+\; \mathcal{O}(\alpha')\end{aligned} \tag{7.3.97}$$

In (7.3.97) to each α'^m power a Riemann zeta constant of fixed weight $m+3$ appears. Hence, (7.3.97) represents a transcendental power series expansion. On the other hand, the following two integrals

$$\begin{aligned}\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\ \frac{I_6(x,y,z)}{(1-xyz)^2} \;&=\; \zeta(2) \;+\; \zeta(2)\,(s_3+s_6-t_2-t_3)\\ &-\;\zeta(3)\,(s_1+s_2+2s_3+s_4+s_5+2s_6+t_1-t_2-t_3) \;+\; \mathcal{O}(\alpha^2)\end{aligned} \tag{7.3.98}$$

$$\begin{aligned}\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\ \frac{I_6(x,y,z)}{(1-xy)(1-yz)} \;&=\; 2\,\zeta(2) \;+\; \big[2\,\zeta(2) \;-\; 4\,\zeta(3)\big]\,(t_1+t_2+t_3)\\ &-\;\big[2\,\zeta(2) \;-\; \zeta(3)\big]\,(s_1+s_2+s_3+s_4+s_5+s_6) \;+\; \mathcal{O}(\alpha^2)\end{aligned} \tag{7.3.99}$$

yield examples of non-transcendental power series.

It would be useful to have a criterion at hand, which allows to infer the transcendentality properties of an integral by inspecting its integrand before power series expanding the whole integral. In this section we present a criterion, which allows to deduce from the structure of the integrand, whether we should expect a transcendental power series expansion in α'. Although this is a mathematical question, it will turn out that superstring theory provides a satisfying answer to this.

Transforming the integrals from the representation (7.1.14) into the form (7.1.12) subject to (7.1.13) will prove to be useful in the following. Integrals (7.1.12), whose integrands are rational functions involving double or higher powers of z_{ij} in their denominators, always give rise to non–transcendental power series. This can be seen by performing a partial integration within the integrals, e.g. for a double power we have

$$\begin{aligned}\int \mathrm{d}z_i\ z_{ij}^{s_{ij}-2}\ r(z_{kl}) &= \frac{1}{s_{ij}-1}\int \mathrm{d}z_i\ r(z_{kl})\ \partial_{z_i} z_{ij}^{s_{ij}-1} \\ &= -\ \frac{1}{s_{ij}-1}\int \mathrm{d}z_i\ z_{ij}^{s_{ij}-1}\ \partial_{z_i} r(z_{kl})\ .\end{aligned} \tag{7.3.100}$$

Regardless of the transcendentality structure of the integral $\int z_{ij}^{s_{ij}-1}\ \partial_{z_i} r(z_{kl})$ the geometric series expansion of the prefactor $\frac{1}{s_{ij}-1} = 1 + s_{ij} + s_{ij}^2 + \ldots$ always destroys any transcendentality. This explains, why the integral (7.3.98) with the corresponding rational functions (see e.g. (7.1.18))

$$\frac{1}{(1-xyz)^2} \quad \simeq \quad \frac{1}{z_{13}z_{14}z_{25}^2 z_{36}z_{46}} \tag{7.3.101}$$

yields a non-transcendental power series expansion. On the other hand, the non-transcendentality of the integral (7.1.14) with the rational function $[(1-x)(1-y)(1-z)(1-xyz)]^{-1}$ can only be seen after transforming it into the representation (7.1.12), in which a rational function with a double power in the denominator appears, i.e.

$$\frac{1}{(1-x)(1-y)(1-z)(1-xyz)} \quad \simeq \quad \frac{1}{z_{16}^2 z_{23}z_{25}z_{34}z_{45}}\ . \tag{7.3.102}$$

Let us now discuss the integrals (7.3.97) and (7.3.99). With respect to the two representations (7.1.14) and (7.1.12) we have the following correspondences:

$$\begin{aligned}\frac{1}{xyz} &\simeq \frac{z_{16}^2}{z_{12}z_{13}z_{14}z_{15}z_{26}z_{36}z_{46}z_{56}} \rightarrow \frac{1}{z_{12}z_{13}z_{14}} \\ \frac{1}{(1-xy)(1-yz)} &\simeq \frac{1}{z_{13}z_{15}z_{24}z_{26}z_{35}z_{46}} \rightarrow \frac{1}{z_{13}z_{24}z_{35}}\end{aligned} \tag{7.3.103}$$

The last correspondence follows from the choice (7.1.1), setting $z_6 = \infty$ and taking into account the c ghost factor $\langle c(z_1)c(z_5)c(z_6 = z_\infty)\rangle = z_{15}z_\infty^2$. These operations are denoted by the $\rightarrow$ arrow in (7.3.103). We may regard the rational functions (7.3.103) as originating from a CFT computation of a six gluon amplitude. This fact will be exploited in the following to anticipate the transcendentality properties on an integral from the z_{ij} representation of the integrand.

7.3.2 A transcendentaliy criterion from computing gluon amplitudes

Gluon disk amplitudes in superstring theory have been checked to involve transcendental α' functions only up to seven point level. By imposing transcendentality of the end result, we can derive transcendentality criteria for the individual constituents as a necessary condition. Recall that, given the gluon vertex operators (4.2.18) and (4.2.19) in the (-1) and (0) ghost picture, respectively, the color stripped n gluon amplitude is computed via

$$\mathcal{A}(g_1,\ldots,g_n) \sim \langle\, c(z_1)\, c(z_{n-1})\, c(z_n)\,\rangle \left(\prod_{j=2}^{n-2}\int_{z_{j-1}}^{1} \mathrm{d}z_j\right) \Big\langle\, \xi^i_{\mu_i}\,\psi^{\mu_i}(z_i)\,\mathrm{e}^{-\phi(z_i)}\,\xi^j_{\mu_j}\,\psi^{\mu_j}(z_j)\,\mathrm{e}^{-\phi(z_j)}$$
$$\times \left(\prod_{l\neq i,j}\xi^l_{\mu_l}\left(i\partial X^{\mu_l}(z_l) \;+\; 2\alpha'\,(k^l\cdot\psi)\,\psi^{\mu_l}(z_l)\,\right)\right)\prod_{m=1}^{n}\mathrm{e}^{ik_m\cdot X(z_m)}\Big\rangle \tag{7.3.104}$$

The assignment of superghost charges is left unspecified here. Among the zero ghost picture vertices, the interplay between the ∂X^μ and the $(k\cdot\psi)\psi^\mu$ contribution plays an essential role in the following[5].

In a six gluon amplitude, the integral (7.3.97) describes the kinematic contraction $(\xi_1\xi_6)$ $(\xi_2k_1)(\xi_3k_1)(\xi_4k_1)(\xi_5k_1)$, while the integral (7.3.99) characterizes the contraction $(\xi_2\xi_6)(\xi_1k_3)$ $(\xi_3k_5)(\xi_4k_2)(\xi_5k_1)$. The crucial difference between the two encountered contractions is, that in RNS superstring theory the first contraction can only be realized by contracting

$$\xi_1^{\mu_1}\xi_2^{\mu_2}\xi_3^{\mu_3}\xi_4^{\mu_4}\xi_5^{\mu_5}\xi_6^{\mu_6}k_1^\lambda k_1^\sigma k_1^\rho k_1^\tau\;\langle\psi_1^{\mu_1}\psi_6^{\mu_6}\rangle\langle\partial X_2^{\mu_2}X_1^\lambda\rangle\langle\partial X_3^{\mu_3}X_1^\sigma\rangle\langle\partial X_4^{\mu_4}X_1^\rho\rangle\langle\partial X_5^{\mu_5}X_1^\tau\rangle\ ,$$

with the first and sixth gluon vertex operator in the (-1) ghost picture. Therefore, the integral (7.3.97) gives rise to a nonvanishing piece in the full amplitude. Since the full amplitude is only comprised by transcendental functions multiplying kinematical factors the contribution (7.3.97) must be a transcendental function. On the other hand, the second contraction can be obtained from

$$\xi_1^{\mu_1}\xi_2^{\mu_2}\xi_3^{\mu_3}\xi_4^{\mu_4}\xi_5^{\mu_5}\xi_6^{\mu_6}k_1^{\lambda_1}k_2^{\lambda_2}k_3^{\lambda_3}k_5^{\lambda_5}\;\langle\psi_2^{\mu_2}\psi_6^{\mu_6}\rangle\langle\partial X_1^{\mu_1}X_3^{\lambda_3}\rangle\langle\partial X_3^{\mu_3}X_5^{\lambda_5}\rangle\langle\partial X_4^{\mu_4}X_2^{\lambda_2}\rangle\langle\partial X_5^{\mu_5}X_1^{\lambda_1}\rangle$$

with the second and sixth gluon vertex operator in the (-1) ghost picture. Furthermore, we may also obtain the second contraction from the pure fermionic contraction:

$$\xi_1^{\mu_1}\xi_2^{\mu_2}\xi_3^{\mu_3}\xi_4^{\mu_4}\xi_5^{\mu_5}\xi_6^{\mu_6}k_1^{\lambda_1}k_2^{\lambda_2}k_3^{\lambda_3}k_5^{\lambda_5}\;\langle\psi_2^{\mu_2}\psi_6^{\mu_6}\rangle\langle\psi_1^{\mu_1}\psi_3^{\lambda_3}\rangle\langle\psi_3^{\mu_3}\psi_5^{\lambda_5}\rangle\langle\partial X_4^{\mu_4}X_2^{\lambda_2}\rangle\langle\psi_5^{\mu_5}\psi_1^{\lambda_1}\rangle$$

In fact, after taking into account the anticommuting nature of fermions the two contractions sum up to zero in the full amplitude. We may symbolically write

$$\langle\partial X_1X_3\rangle\langle\partial X_3X_5\rangle\langle\partial X_5X_1\rangle \;-\; \langle\psi_1\psi_3\rangle\langle\psi_3\psi_5\rangle\langle\psi_5\psi_1\rangle \;\;=\;\; 0\ .$$

Otherwise, the latter would give rise to non–transcendent contributions to the full amplitude.

To summarize: in order to investigate the transcendentality properties of an Euler integral (7.1.14) we transform it into the form (7.1.12). This is uniquely possible because of the

[5] The number of spacetime dimensions does not play any role here, see the next chapter 8. The choice of four dimensional vector indices $\mu,\nu,\lambda,\ldots$ is completely arbitrary and irrelevant.

CFT condition (7.1.13). If the rational function $\tilde{R}$ of this integrand involves powers higher than one in the denominator the corresponding integral yields a non–transcendental power series. Otherwise, the rational function (more precisely its limit $z_n \to \infty$ with taking into account the c ghost factor with the choice (7.1.1)) is mapped to a gluon contraction of the form $(\xi_r\xi_n)(\xi_i k_j)\ldots(\xi_l k_m)$ (with no more than one $(\xi\xi)$ product) arising from an n gluon superstring computation with the r th and n th gluon vertex operator in the (-1) ghost picture. If the contraction under consideration can only be realized by the correlator $\langle\psi_r\psi_n\rangle\langle\partial X_i X_j\rangle\ldots\langle\partial X_l X_m\rangle$ the corresponding integral is transcendental. If on the other hand, the contraction under consideration can also be realized by correlators involving more fermionic contractions, the underlying integral is non-transcendental and the two contributions must conspire in some way, e.g. add up to zero. Hence, in the n gluon amplitude computation non-transcendental contributions referring to a given kinematics $(\xi_r\xi_n)(\xi_i k_j)\ldots(\xi_l k_m)$ are always accompanied by contributions involving a circle of fermionic contractions such that all contributions add up to zero. Stated differently, integrals describing a kinematics[6] $(\xi_r\xi_n)(\xi_i k_j)\ldots(\xi_l k_m)$, which can be realized by several field contractions, describe non-transcendental functions.

In fact, this criterion rules out the double poles (7.3.100) to join into a transcendental integral. The latter can be realized by both bosonic and fermionic contractions. E.g. the power $1/z_{ij}^2$ describes the kinematical factor $(\xi_i k_j)(\xi_j k_i)$, which may stem from either $\xi_i^{\mu_i}\xi_j^{\mu_j}k_i^{\lambda_i}k_j^{\lambda_j}$ $\langle\partial X_i^{\mu_i}X_j^{\lambda_j}\rangle\langle\partial X_j^{\mu_j}X_i^{\lambda_i}\rangle$ or from $\xi_i^{\mu_i}\xi_j^{\mu_j}k_i^{\lambda_i}k_j^{\lambda_j}\langle\psi_i^{\mu_i}\psi_j^{\lambda_j}\rangle\langle\psi_j^{\mu_j}\psi_i^{\lambda_i}\rangle$, which add up to zero:

$$\langle\partial X_i X_j\rangle\langle\partial X_j X_i\rangle \;-\; \langle\psi_i\psi_j\rangle\langle\psi_j\psi_i\rangle \;=\; 0$$

Note that kinematics like $(\xi_i\xi_j)$ are realized by both $\xi_i^{\mu_i}\xi_j^{\mu_j}\langle\partial X_i^{\mu_i}\partial X_j^{\mu_j}\rangle$ and $\xi_i^{\mu_i}\xi_j^{\mu_j}\langle\psi_i^{\mu_i}\psi_j^{\mu_j}\rangle\times$ $k_i^{\lambda_i}k_j^{\lambda_j}\langle\psi_i^{\lambda_i}\psi_j^{\lambda_j}\rangle$ giving rise to $(1-2\alpha' k_i k_j)(\xi_i\xi_j)z_{ij}^{-2}$ in the end. The non-transcendentality of the double pole integral according to (7.3.100) is then compensated by the $1-s_{ij}$ factor in the numerator.

Therefore, kinematics involving more than two pairs of $(\xi_i\xi_j)$ scalar products always involve double powers in the denominator. This is why kinematics with more than two pairs of $(\xi_i\xi_j)$ scalar products cannot provide information on the transcendentality property of the underlying integral. On the other hand, when mapping an integral to the kinematics $(\xi_r\xi_n)(\xi_i k_j)\ldots(\xi_l k_m)$ we put the r'th and n'th gluon vertex operator in the (-1) ghost picture such that the double pole from the contraction $(\xi_r\xi_n)$ drops.

Let us mention that the two integrals (7.2.65) and (7.2.66) have non-transcendental power series. Indeed our criterion confirms this: In the representation (7.1.12) the integral (7.2.65)

[6]Note, that this statement assumes the r'th and n'th gluon vertex operator in the (-1)–ghost picture to get rid of the double pole from the correlator $\langle e^{-\phi(z_r)}e^{-\phi(z_n)}\rangle\langle\psi_r\psi_n\rangle$.

gives rise to the rational function $\frac{1}{z_{13}z_{15}z_{26}^2z_{34}z_{45}}$ involving a double pole. As a consequence of the latter the α'-expansion in (7.2.65) is not transcendental. On the other hand, the integral (7.2.66) leads to the rational function

$$\frac{z_{13}z_{26}}{z_{12}z_{14}z_{16}z_{23}z_{25}z_{35}z_{36}z_{46}} \quad\rightarrow\quad \frac{z_{13}z_{15}}{z_{12}z_{14}z_{23}z_{25}z_{35}} \quad=\quad \frac{z_{15}}{z_{12}z_{14}z_{25}z_{35}} + \frac{z_{15}}{z_{14}z_{23}z_{25}z_{35}}\ .$$

According to the previous statements, the last two fractions correspond to the six gluon kinematics $(\xi_1\xi_6)(\xi_2k_1)(\xi_3k_5)(\xi_4k_1)(\xi_5k_2)$ and $(\xi_1\xi_6)(\xi_4k_1)\underline{(\xi_2k_3)(\xi_3k_5)(\xi_5k_2)}$, respectively. The underlined part of the last kinematics may also be realized by contracting fermions along a circle. Hence the power series in (7.2.66) is non-transcendental.

7.3.3 Seven point examples for the transcendentality criterion

Let us now apply our criterion for some $n = 7$ integral examples. The following integrals can be associated to only one kinematical factor. Therefore, they represent integrals with transcendental power series expansions.

rational function in eq. (7.1.14)	rational function in eq. (7.1.12)	kinematics	transcend. power series
$\frac{1}{(1-xy)(1-wz)(1-yz)}$	$\frac{1}{z_{13}z_{24}z_{35}z_{46}}$	$(\xi_1\xi_7)(\xi_2k_4)(\xi_3k_1)(\xi_4k_6)(\xi_5k_3)(\xi_6k_1)$	yes
$\frac{z}{(1-wz)(1-yz)(1-xyz)}$	$\frac{1}{z_{14}z_{24}z_{35}z_{36}}$	$(\xi_1\xi_7)(\xi_2k_4)(\xi_3k_6)(\xi_4k_1)(\xi_5k_3)(\xi_6k_1)$	yes
$\frac{y}{(1-xy)(1-yz)(1-wyz)}$	$\frac{1}{z_{13}z_{25}z_{35}z_{46}}$	$(\xi_1\xi_7)(\xi_2k_5)(\xi_3k_1)(\xi_4k_6)(\xi_5k_3)(\xi_6k_1)$	yes
$\frac{1}{(1-yz)(1-xyz)(1-wxyz)}$	$\frac{z_{16}}{z_{13}z_{14}z_{26}z_{35}z_{36}}$	$(\xi_1\xi_7)(\xi_2k_6)(\xi_3k_1)(\xi_4k_1)(\xi_5k_3)(\xi_6k_3)$	yes
$\frac{z}{(1-wz)(1-wyz)(1-xyz)}$	$\frac{1}{z_{14}z_{24}z_{25}z_{36}}$	$(\xi_1\xi_7)(\xi_2k_4)(\xi_3k_6)(\xi_4k_1)(\xi_5k_2)(\xi_6k_1)$	yes
$\frac{yz}{(1-yz)(1-wxyz)}$	$\frac{1}{z_{14}z_{15}z_{26}z_{35}}$	$(\xi_1\xi_7)(\xi_2k_6)(\xi_3k_5)(\xi_4k_1)(\xi_5k_1)(\xi_6k_1)$	yes
$\frac{yz}{(1-wyz)(1-xyz)}$	$\frac{1}{z_{14}z_{15}z_{25}z_{36}}$	$(\xi_1\xi_7)(\xi_2k_5)(\xi_3k_6)(\xi_4k_1)(\xi_5k_1)(\xi_6k_1)$	yes

(7.3.105)

The rational z_{ij} functions in the second column contain the c ghost factor $\langle c(z_1)c(z_6)c(z_7)\rangle = z_{16}z_{17}z_{67}$.

Sometimes a partial fraction decomposition may be useful before analyzing the integrands. For instance, according to (7.1.18) we have (using partial fraction at the last step):

$$\begin{aligned}\frac{1}{(1-xy)(1-xyz)(1-wz)(1-wxyz)} &\simeq \frac{z_{16}}{z_{13}z_{14}z_{15}z_{26}z_{27}z_{36}z_{46}z_{57}}\\ \rightarrow\quad \frac{z_{16}^2}{z_{13}z_{14}z_{15}z_{26}z_{36}z_{46}} &= \frac{z_{16}}{z_{13}z_{14}z_{15}z_{26}z_{46}} + \frac{z_{16}}{z_{14}z_{15}z_{26}z_{36}z_{46}}\end{aligned} \tag{7.3.106}$$

The two rational functions on the right hand side correspond to the two kinematical factors $(\xi_1\xi_7)(\xi_2k_6)(\xi_3k_1)(\xi_4k_1)(\xi_5k_1)(\xi_6k_4)$ and $(\xi_1\xi_7)(\xi_2k_6)(\xi_3k_6)(\xi_4k_1)(\xi_5k_1)(\xi_6k_4)$, respectively. Both of them do not allow for additional fermionic contractions. Hence, the integral under consideration yields a transcendental series.

Furthermore, let us discuss some integrals with non-transcendental power series expansions. The rational functions of the following integrals describe kinematics, which can be realized in two ways. The second possibility involves contractions of several pairs of fermions. The latter are contracted along a circle and give rise to the underlined subset of the kinematics.

rational function in eq. (7.1.14)	rational function in eq. (7.1.12)	kinematics	transcend. power series
$\frac{z}{(1-xy)(1-wz)(1-yz)}$	$\frac{1}{z_{14}z_{24}z_{35}z_{46}}$	$(\xi_5\xi_7)\underline{(\xi_1k_6)(\xi_4k_1)(\xi_6k_4)}\ (\xi_2k_4)(\xi_3k_5)$	no
$\frac{1}{(1-xy)(1-wz)(1-wxyz)}$	$\frac{z_{16}}{z_{13}z_{15}z_{24}z_{26}z_{46}}$	$(\xi_1\xi_7)\underline{(\xi_2k_6)(\xi_6k_4)(\xi_4k_2)}\ (\xi_3k_1)(\xi_5k_1)$	no
$\frac{xyz}{(1-xy)(1-wyz)(1-xyz)}$	$\frac{1}{z_{14}z_{25}z_{36}z_{46}}$	$(\xi_2\xi_7)\underline{(\xi_1k_6)(\xi_6k_4)(\xi_4k_1)}\ (\xi_3k_6)(\xi_5k_2)$	no

(7.3.107)

Sometimes, before analyzing the integrands a partial fraction decomposition may be useful. E.g. according to (7.1.18) we have

$$\begin{aligned} \frac{z(1-xyz)}{(1-xy)(1-wz)(1-wyz)(1-xyz)} &\simeq \frac{z_{26}z_{47}}{z_{14}z_{16}z_{24}z_{25}z_{27}z_{36}z_{37}z_{46}z_{57}} \\ \to \frac{z_{26}}{z_{14}z_{24}z_{25}z_{36}z_{46}} &= \frac{1}{z_{14}z_{24}z_{25}z_{36}} + \frac{1}{z_{14}z_{25}z_{36}z_{46}} \,. \end{aligned} \tag{7.3.108}$$

The second term on the right hand side corresponds to one of the rational functions discussed in (7.3.107). Hence, the integral under consideration does not give rise to a transcendental series. Another example is:

$$\begin{aligned} \frac{1}{(1-yz)(1-wyz)(1-xyz)} &\simeq \frac{z_{15}z_{37}}{z_{13}z_{14}z_{16}z_{25}z_{27}z_{35}z_{36}z_{47}z_{57}} \\ \to \frac{z_{15}}{z_{13}z_{14}z_{25}z_{35}z_{36}} &= \frac{1}{z_{13}z_{14}z_{25}z_{36}} + \frac{1}{z_{14}z_{25}z_{35}z_{36}} \end{aligned} \tag{7.3.109}$$

The two rational functions on the right hand side correspond to the two kinematical factors $(\xi_2\xi_7)\underline{(\xi_1k_6)(\xi_3k_1)(\xi_6k_3)}(\xi_4k_1)(\xi_5k_2)$ and $(\xi_2\xi_7)(\xi_1k_6)(\xi_3k_5)(\xi_4k_1)(\xi_5k_2)(\xi_6k_3)$, respectively. The first kinematics can also be realized by a fermionic contraction along a circle, which is underlined. Hence, the integral under consideration does not give rise to a transcendental series. Finally, the third integral with the integrand

$$\frac{y}{(1-wz)(1-yz)(1-xyz)} \simeq \frac{z_{14}z_{37}}{z_{13}z_{15}z_{16}z_{24}z_{27}z_{35}z_{36}z_{47}^2}$$

yields a non-transcendental power series due to the double pole.

The results (7.3.107) can be anticipated by explicitly computing the integrals:

$$\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \; \frac{z\, I_7(x,y,z,w)}{(1-xy)(1-wz)(1-yz)} = 2\,\zeta(2) + 2\,\zeta(3) + \ldots$$

$$\begin{aligned}
\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \, \frac{I_7(x,y,z,w)}{(1-xy)(1-wz)(1-wxyz)} \quad &= \quad 3\,\zeta(3) \\
&+ \left(\frac{19}{4}\,\zeta(4) \; - \; 3\,\zeta(3) \right) s_7 \; + \; \frac{4}{5}\,\zeta(2)^2\,(s_1 + s_6 + t_1 + t_5) \; + \; \ldots \qquad (7.3.110) \\
\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \, \frac{xyz\; I_7(x,y,z,w)}{(1-xy)(1-wyz)(1-xyz)} \quad &= \quad -\,2\,\zeta(2) \; + \; 4\,\zeta(3) \; + \; \ldots \\
\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \, \frac{I_7(x,y,z,w)}{(1-yz)(1-wyz)(1-xyz)} \quad &= \quad \frac{5}{2}\,\zeta(4) \; + \; 4\,\zeta(3) \; - \; 2\,\zeta(2) \; + \; \ldots \\
\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \, \frac{y\; I_7(x,y,z,w)}{(1-wz)(1-yz)(1-xyz)} \quad &= \quad \frac{3}{2}\,\zeta(2) \; + \; \frac{3}{2}\,\zeta(3) \; + \; \ldots
\end{aligned}$$

7.4 Polynomial relations and Gröbner basis reduction

For $n_{i,j} \geq 0$, the representation (7.2.29) in the dual variables $u_{i,j}$ gives rise to a polynomial ring $\mathbb{R}[u_{\mathcal{P}}]$ describing polynomials in $u_{i,j}$, $(i,j) \in \mathcal{P}$ with coefficients in $\mathbb{R}$. This ring is suited to perform a Gröbner basis analysis to find a minimal basis for the polynomials in the integrand. Due to the constraints (7.2.25), which give rise to the δ functions in (7.2.29), many polynomials in the variables $u_{i,j}$ referring to different choices of the integers $n_{i,j}$ yield to the same integral B_n. The constraints (7.2.25) define a monomial ideal I in the polynomial ring $\mathbb{R}[u_{\mathcal{P}}]$. Hence, we consider the quotient space $\mathbb{R}[u_{\mathcal{P}}]/I$ and the Gröbner basis method is well appropriate to choose a basis in the ideal I and generate independent sets of polynomials in the quotient ring $\mathbb{R}[u_{\mathcal{P}}]/I$. We are interested in simple representatives of equivalence classes for congruence modulo I. The properties of an ideal are reflected in the form of the elements of the Gröbner basis [233, 234].

7.4.1 Definition of a Gröbner basis

Given a monomial ordering[7] in the ring a Gröbner basis $G = \{g_1, \ldots, g_d\}$ comprises a finite subset of the ideal I such that the leading term[8] of any element of the ideal I is divisible by a

[7] As monomial ordering we may choose lexicographic order or graded lexicographic order. Then, a monomial ordering of two polynomials $f = \sum_\alpha a_\alpha x^\alpha$ and $g = \sum_\beta b_\alpha x^\beta$ can be defined as follows:

(i) *lexicographic order*: $\alpha >_{lex} \beta$, if in the vector difference $\alpha - \beta \in \mathbb{Z}^n$ the leftmost nonzero entry is positive ($x^\alpha >_{lex} x^\beta$ if $\alpha >_{lex} \beta$).

(ii) *graded lexicographic order*: $\alpha >_{grlex} \beta$, if $|\alpha| = \sum_{i=1}^{n} \alpha_i > |\beta|$ and $\alpha >_{lex} \beta$ ($x^\alpha >_{grlex} x^\beta$, if $\alpha >_{grlex} \beta$).

[8] The leading term $LT(f)$ of a polynomial f is defined as follows [233]: For $f = \sum_\alpha a_\alpha x^\alpha$ a nonzero polynomial in $\mathbb{R}[x_1, \ldots, x_n]$ and $>$ a specific monomial order

(i) the *multidegree* of f is $multideg(f) := \mathrm{Max}\{\alpha \in \mathbb{Z}^n_{\geq 0} \mid a_\alpha \neq 0\ \}$,

(ii) the *leading coefficient* of f is: $LC(f) := a_{multideg(f)} \in \mathbb{R}$,

(iii) the *leading monomial* of f is $LM(f) = x^{multideg(f)}$, with coefficient 1, and leading term $LT(g_i)$ of an element of the subset. Alternatively, a finite subset G of an ideal I in a polynomial ring represents a Gröbner basis, if $\langle LT(g_1),\ldots,LT(g_d)\rangle = \langle LT(I)\rangle$ [233,234]. Buchberger's algorithm generates the unique *reduced* Gröbner basis G, in which no monomial in a polynomial $p \in G$ of this basis is divisible by a leading term of the other polynomials in the basis and $LC(p) = 1$.

The main idea is, that after dividing a polynomial $p \in \mathbb{R}[x_1,\ldots,x_n]$ by a Gröbner basis for the ideal $I \subset \mathbb{R}[x_1,\ldots,x_n]$ the remainder $\bar{p}^G$ is uniquely fixed by the polynomial p, cf. Chapter 5, §3 of [233]. More precisely according to the Proposition 1 therein we have: For a given monomial ordering on $\mathbb{R}[x_1,\ldots,x_n]$ and an ideal $I \subset \mathbb{R}[x_1,\ldots,x_n]$,

- (i) Every $f \in \mathbb{R}[x_1,\ldots,x_n]$ is congruent modulo I to a unique polynomial r, which is a $\mathbb{R}$–linear combination of the monomials in the complement of $\langle LT(I)\rangle$.
- (ii) The elements $\{x^\alpha \mid x^\alpha \notin \langle LT(I)\rangle\}$ are linearly independent modulo I, i.e. if $\sum_\alpha c_\alpha x^\alpha = 0 \mod I$, where the x^α are all in the complement of $\langle LT(I)\rangle$, then $c_\alpha = 0$ for all α. As a consequence, for any given $f \in \mathbb{R}[x_1,\ldots,x_n]$ the remainder $\bar{f}^G$ is a $\mathbb{R}$–linear combination of the monomials contained in the complement of $LT(I)$, i.e. $\bar{f}^G \in Span\,(x^\alpha \mid x^\alpha \notin \langle LT(I)\rangle)$.

In the following we want to apply the Gröbner basis method to construct a basis for those polynomials which are independent on the constraints (7.2.25). This basis is determined by the complement of $\langle LT(I)\rangle$ w.r.t. a Gröbner basis G. Note that the representation of this basis (and also of $\langle LT(I)\rangle$ and the remainders) may depend on the chosen monomial ordering. At any rate, there is always the *same* number of monomials in the complement of $\langle LT(I)\rangle$. In addition, we impose a condition on the degree of the basis monomials to ensure that in the denominator of the integrand of (7.1.12) the z_{ij} only appear with powers of at most one. We illustrate the method by the following examples.

7.4.2 Gröbner basis at $n = 4$

We work with the two coordinates $X_1 = u_{1,2}$ and $X_2 = u_{2,3}$ and consider the polynomial ring $\mathbb{R}[X_1, X_2]$. From (7.2.36) we can read off the constraints (7.2.25) giving rise to the monomial

(iv) the *leading term* of f is

$$LT(f) = LC(f)\ LM(f)\ .$$

As an example we consider $f = xyz + 2xy^2z^2 + 3z^3 - 7x^5y + 3x^2z^2$ with $>$ the lexicographic order. Then we have: $multideg(f) = (5,1,0),\ LC(f) = -7, LM(f) = x^5y$ and $LT(f) = -7x^5y$.

ideal

$$I \quad = \quad \langle\, X_1 + X_2 - 1\, \rangle \subset \mathbb{R}[X_1, X_2]\ . \tag{7.4.111}$$

With respect to lexicographic order we find for the Gröbner basis of (7.4.111):

$$G \quad = \quad \{g_1\} \quad = \quad \{\ X_1 + X_2 - 1\ \} \tag{7.4.112}$$

Hence, with respect to lexicographic order the leading term of this monomial gives rise to

$$LT(I) \quad = \quad X_1\ . \tag{7.4.113}$$

Therefore, the set of possible remainders modulo I is the set of all $\mathbb{R}$ linear combinations of the following monomials:

$$\{\, 1,\ X_2,\ X_2^2,\ X_2^3, \ldots\, \} \tag{7.4.114}$$

For some examples let us determine their remainders on dividing them by the Gröbner basis (7.4.112):

$$\begin{aligned}
X_1 &= g_1\ +\ 1\ - X_2 \ \simeq\ 1\ -\ X_2 \\
X_2 &= 0\, g_1\ +\ X_2\ \simeq\ X_2 \\
X_1\, X_2 &= X_2\, g_1\ +\ X_2\ -\ X_2^2\ \simeq\ X_2\ -\ X_2^2 \\
X_1^2 &= (1 + X_1 - X_2)\, g_1\ +\ 1\ -\ 2\, X_2\ +\ X_2^2\ \simeq\ 1\ -\ 2\, X_2\ +\ X_2^2 \\
X_1^2\, X_2 &= X_2\, (1 + X_1 - X_2)\, g_1\ +\ X_2\ -\ 2\, X_2^2\ +\ X_2^3\ \simeq\ X_2\ -\ 2\, X_2^2\ +\ X_2^3
\end{aligned} \tag{7.4.115}$$

Indeed, the remainders (displayed after the $\simeq$ sign) are generated by the basis (7.4.114).

In (7.2.36) the monomials $X_2^{n_{11}},\ n_{11} = 0, 1, \ldots$ of (7.4.114) give rise to the following integrals (7.2.35):

$$B_4[n_{11}] \quad = \quad \int_0^1 \mathrm{d}x\ x^{s_{12}}\ (1 - x)^{s_{23} + n_{11}} \tag{7.4.116}$$

The integrals (7.2.35) without poles in their field theory expansions are given by the integers $n_1, n_{11} \in \mathbb{N}_0$. According to our construction, all these integrals (7.2.35) can be generated from (7.4.116) by linear combinations. However according to (7.1.18) we have

$$(1 - x)^{n_{11}} \quad \simeq \quad \frac{z_{14}^{n_{11}}\ z_{23}^{n_{11}}}{z_{13}^{2+n_{11}}\, z_{24}^{2+n_{11}}}\ , \tag{7.4.117}$$

i.e. all finite integrals (7.4.116) in (7.1.12) imply some powers $\tilde{n}_{ij}$ with $\tilde{n}_{ij} < -1$. As a consequence the set of integrals (7.4.116) cannot serve as a basis and $u_{1,2}^{-1}, u_{2,3}^{-1}$ are the only elements of the partial fraction basis. Note, that this basis is two–dimensional, i.e. $(n-2)! = 2$ for $n = 4$.

7.4.3 Gröbner basis at $n = 5$

We work with the five coordinates (7.2.43) and consider the polynomial ring $\mathbb{R}[X_1, \ldots, X_5]$. From (7.2.44) we can read off the constraints (7.2.25) giving rise to the monomial ideal:

$$I = \langle X_2 + X_1 X_3 - 1,\ X_3 + X_2 X_4 - 1,\ X_5 + X_1 X_4 - 1 \rangle \subset \mathbb{R}[X_1, \ldots, X_5] \tag{7.4.118}$$

With respect to the lexicographic order we find for the (reduced) Gröbner basis of (7.4.118) the three elements:

$$G = \{g_1, g_2, g_3\} = \{\ X_1 + X_2 X_5 - 1,\ X_3 + X_2 X_4 - 1,\ X_4 + X_3 X_5 - 1\ \} \tag{7.4.119}$$

Hence with respect to lexicographic order the leading terms of these monomials give rise to

$$LT(I) = \{\ X_1,\ X_2\, X_4,\ X_3\, X_5\ \}\ . \tag{7.4.120}$$

The set of possible remainders modulo I is thus the set of all $\mathbb{R}$ linear combinations of the following monomials:

$$\bigcup_{m,n=0}^{\infty} \{\ X_2^m\, X_3^n,\ X_2^m\, X_5^n,\ X_3^m\, X_4^n,\ X_4^m\, X_5^n\ \} \tag{7.4.121}$$

For some examples let us determine their remainders on dividing them by the Gröbner basis (7.4.119):

$$\begin{aligned}
X_1 &= g_3 + 1 - X_2\, X_5 \simeq 1 - X_2\, X_5 \\
X_1\, X_4 &= g_1 - X_5\, g_2 + X_4\, g_3 + 1 - X_5 \simeq 1 - X_5 \\
X_3\, X_5 &= g_1 + 1 - X_4 \simeq 1 - X_4 \\
X_3\, X_5^2 &= X_5\, g_1 + X_5 - X_4\, X_5 \simeq X_5 - X_4\, X_5 \\
X_1\, X_2 &= X_2\, g_3 + X_2 - X_2^2\, X_5 \simeq X_2 - X_2^2\, X_5 \\
X_2\, X_3\, X_5 &= X_2\, g_1 - g_2 - 1 + X_2 + X_3 \simeq -1 + X_2 + X_3
\end{aligned} \tag{7.4.122}$$

Indeed, the remainders (displayed after the $\simeq$ sign) are generated by the basis (7.4.121).

We have the following dictionary

monomial in in equation (7.2.40)	rational function in equation (7.2.39)	rational function in equation (7.1.12)
1	$\frac{1}{1-xy}$	$\frac{1}{z_{13}z_{14}z_{24}z_{25}z_{35}}$
X_2	$\frac{1-y}{(1-xy)^2}$	$\frac{z_{23}}{z_{13}^2 z_{24}^2 z_{25}z_{35}}$
X_3	$\frac{1-x}{(1-xy)^2}$	$\frac{z_{34}}{z_{13}z_{14}z_{24}^2 z_{35}^2}$
X_4	$\frac{x}{(1-xy)}$	$\frac{z_{45}}{z_{14}^2 z_{24}z_{25}z_{35}^2}$
X_5	1	$\frac{z_{15}}{z_{13}z_{14}^2 z_{25}^2 z_{35}}$
$X_2\,X_3$	$\frac{(1-x)(1-y)}{(1-xy)^3}$	$\frac{z_{23}z_{34}}{z_{13}^2 z_{24}^3 z_{35}^2}$
$X_2\,X_5$	$\frac{1-y}{1-xy}$	$\frac{z_{15}z_{23}}{z_{13}^2 z_{14}z_{24}z_{25}^2 z_{35}}$
$X_3\,X_4$	$\frac{x(1-x)}{(1-xy)^2}$	$\frac{z_{34}z_{45}}{z_{14}^2 z_{24}^2 z_{35}^3}$
$X_4\,X_5$	x	$\frac{z_{15}z_{45}}{z_{14}^3 z_{25}^2 z_{35}^2}$

(7.4.123)

between monomials in the integral (7.2.40), the polynomial in (7.2.39), and the representation (7.1.12). According to the list (7.4.123), only the element 1 among the generators (7.4.121) of the complement $\overline{\langle LT(I)\rangle}$ does not give rise to higher powers of z_{ij} in the denominator of the integrand (7.1.12). Therefore, we dismiss all other basis elements and the integral

$$\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\ \frac{I_5(x,y)}{1-xy} \quad = \quad \zeta(2)\ +\ \ldots \tag{7.4.124}$$

is left as the only basis element without poles. The integral (7.4.124) yields a transcendental power series in α'. Together with the fundamental set (7.2.48) we obtain a six dimensional partial fraction basis, i.e. $(n-2)! = 6$ for $n=5$.

7.4.4 Gröbner basis at $n = 6$

Using the coordinates (7.2.61) we consider the polynomial ring $\mathbb{R}[X_1,\ldots,X_6,Y_1,\ldots,Y_3]$. From (7.2.62) we can read off the constraints (7.2.25) giving rise to the monomial ideal:

$$\begin{aligned} I \ &= \ \langle\ X_2 + X_1X_3Y_3 - 1\ ,\ X_3 + X_2X_4Y_1 - 1\ ,\ X_4 + X_3X_5Y_2 - 1 \\ &\qquad X_6 + X_1X_5Y_1 - 1\ ,\ Y_2 + X_1X_4Y_1Y_3 - 1\ ,\ Y_3 + X_2X_5Y_1Y_2 - 1\ \rangle \end{aligned} \tag{7.4.125}$$

With respect to lexicographic order we find for the (reduced) Gröbner basis of (7.4.125) the thirteen elements:

$$\begin{aligned} G \ &= \ \{\ 1 - Y_1 + X_6Y_1 - X_6Y_2 - X_6Y_3 + X_6^2Y_2Y_3,\ -1 + X_5Y_1 + X_6Y_3,\ -1 + X_1 + X_2X_6Y_2 \\ &\qquad 1 - X_5 - X_6 + X_5X_6Y_2,\ -1 + X_4Y_3 + X_5Y_2,\ -1 + X_4Y_1 + X_3Y_2, \end{aligned}$$

$$
\begin{aligned}
&X_4 - X_6 - X_4Y_1 + X_4X_6Y_1 + X_6Y_2 - X_4X_6Y_2,\ -1 + X_2Y_1 + X_3Y_3,\\
&X_3 - X_4 + X_6 - X_3Y_1 + X_4Y_1 - X_6Y_1 + X_3X_6Y_1 - X_3X_6Y_3 - X_6Y_2 + X_4X_6Y_2,\\
&-X_3 + X_3X_5 - X_6 + X_3X_6 + X_4X_6,\ 1 - X_2 - X_3Y_3 - X_6Y_3 + X_2X_6Y_3 + X_3X_6Y_3,\\
&-1 + X_2 + X_5 - X_2Y_3 + X_3Y_3 - X_5Y_3 + X_2X_5Y_3 + X_6Y_3 - X_3X_6Y_3 - X_2X_5Y_2,\\
&-X_2 + X_3 + X_2X_4 - X_5 + X_2X_5 + X_6 - X_3X_6 - X_4X_6\ \} \qquad (7.4.126)
\end{aligned}
$$

Hence, with respect to lexicographic order the leading terms of these 13 monomials give rise to

$$
\begin{aligned}
LT(I) \ = \ \{\ &X_6^2Y_2Y_3,\ X_5Y_1,\ X_5X_6Y_2,\ X_4Y_3,\ X_4X_6Y_1,\ X_3Y_2,\ X_3X_6Y_1,\\
&X_3X_5,\ X_2Y_1,\ X_2X_6Y_3,\ X_2X_5Y_3,\ X_2X_4,\ X_1\ \}\ . \qquad (7.4.127)
\end{aligned}
$$

We would like to mention that the Gröbner basis consists of 18 elements in the case of degree lexicographic order.

From the set (7.4.127) the monomials generating the complement $\overline{\langle LT(I)\rangle}$ can be determined. Most of these monomials yield to higher powers of z_{ij} in the denominator of the integrand (7.1.12), i.e. $\tilde{n}_{ij} = -2$ for some z_{ij}. In fact, only the following five monomials give rise to single powers in their denominators, i.e. $\tilde{n}_{ij} \geq -1$:

monomial in in equation (7.2.40)	rational function in equation (7.2.39)	rational function in equation (7.1.12)
1	$\frac{1}{(1-xy)\ (1-yz)}$	$\frac{1}{z_{13}z_{15}z_{24}z_{26}z_{35}z_{46}}$
Y_1	$\frac{y}{(1-xy)\ (1-yz)}$	$\frac{1}{z_{14}z_{15}z_{24}z_{26}z_{35}z_{36}}$
Y_2	$\frac{1}{(1-xy)\ (1-xyz)}$	$\frac{1}{z_{13}z_{14}z_{25}z_{26}z_{35}z_{46}}$
Y_3	$\frac{1}{(1-yz)\ (1-xyz)}$	$\frac{1}{z_{13}z_{15}z_{24}z_{25}z_{36}z_{46}}$
$X_6\,Y_2\,Y_3$	$\frac{1}{(1-xyz)}$	$\frac{z_{16}}{z_{13}z_{14}z_{15}z_{25}z_{26}z_{36}z_{46}}$

(7.4.128)

Therefore, we dismiss all other basis elements of $\overline{\langle LT(I)\rangle}$. All (finite) integrals (7.1.12) with only single powers of z_{ij} in their denominators, i.e. $\tilde{n}_{ij} \geq -1$, are spanned by the following five integrals:

$$
\begin{aligned}
G_0 &= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\ \frac{I_6(x,y,z)}{(1-xy)\ (1-yz)} \ = \ 2\,\zeta(2)\ +\ \ldots\\
G_1 &= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\ \frac{y\ I_6(x,y,z)}{(1-xy)\ (1-yz)} \ = \ 2\,\zeta(3)\ +\ \ldots\\
G_2 &= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\ \frac{I_6(x,y,z)}{(1-xy)\ (1-xyz)} \ = \ 2\,\zeta(3)\ +\ \ldots\\
G_3 &= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\ \frac{I_6(x,y,z)}{(1-yz)\ (1-xyz)} \ = \ 2\,\zeta(3)\ +\ \ldots
\end{aligned}
$$

$$G_4 \quad = \quad \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \; \frac{I_6(x,y,z)}{1-xyz} \quad = \quad \zeta(3) \; + \; \ldots \tag{7.4.129}$$

E.g. we have

$$\begin{aligned}
\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \; \frac{yz \; I_6(x,y,z)}{(1-yz)\;(1-xyz)} \quad &= \quad G_3 \; - \; G_4 \\
\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \; \frac{y\;(1-z)\; I_6(x,y,z)}{(1-xy)\;(1-yz)\;(1-xyz)} \quad &= \quad G_1 \; - \; G_3 \; + \; G_4 \\
\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \; \frac{(1-y)\; I_6(x,y,z)}{(1-xy)\;(1-yz)\;(1-xyz)} \quad &= \quad -G_1 \; + \; G_2 \; + \; G_3 \; - \; G_4 \\
\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \; \frac{(1-x)\; y\; I_6(x,y,z)}{(1-xy)\;(1-yz)\;(1-xyz)} \quad &= \quad G_1 \; - \; G_2 \; + \; G_4 \\
\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \; \frac{xy \; I_6(x,y,z)}{(1-xy)\;(1-xyz)} \quad &= \quad G_2 \; - \; G_4
\end{aligned} \tag{7.4.130}$$

as results from the identities between their corresponding monomials on dividing them by the Gröbner basis (7.4.126):

$$\begin{aligned}
X_1\,Y_1\,Y_3 \quad &= \quad Y_3 \; - \; X_6\,Y_2\,Y_3 \\
X_2\,Y_1\,Y_2 \quad &= \quad Y_1 \; - \; Y_3 \; + \; X_6\,Y_2\,Y_3 \\
X_3\,Y_2\,Y_3 \quad &= \quad -Y_1 \; + \; Y_2 \; + \; Y_3 \; - \; X_6\,Y_2\,Y_3 \\
X_4\,Y_1\,Y_3 \quad &= \quad Y_1 \; - \; Y_2 \; + \; X_6\,Y_2\,Y_3 \\
X_5\,Y_1\,Y_2 \quad &= \quad Y_2 \; - \; X_6\,Y_2\,Y_3
\end{aligned} \tag{7.4.131}$$

Except for the first integral G_0, the other four integrals (7.4.129) yield a transcendental power series in α', cf. section 7.3. Any partial fraction decomposition, which involves G_0 must refer to a non–transcendental integral (7.1.12) and only partial fraction expansions involving the basis $G_1, \ldots, G_4$ comprise into a transcendental integral. In the previous section we have found a set of six transcendental integrals (**??**) with single poles. Together with the fundamental set (7.2.68) we obtain a partial fraction basis (of transcendental integrals (7.1.12)) with $4+6+14=24$ elements, i.e. $(n-2)!=24$ for $n=6$.

7.4.5 Gröbner basis at $n=7$

Using the coordinates (7.2.78) we consider the polynomial ring $\mathbb{R}[X_1,\ldots,X_7,Y_1,\ldots,Y_7]$. From (7.2.79) we can read off the constraints (7.2.25) giving rise to the monomial ideal:

$$\begin{aligned}
I \quad = \quad \langle\; & X_2 + X_1X_3Y_3Y_7 - 1,\; X_3 + X_2X_4Y_1Y_4 - 1,\; X_4 + X_3X_5Y_2Y_5 - 1,\; X_5 + X_4X_6Y_3Y_6 - 1, \\
& Y_4 + X_3X_6Y_3Y_2Y_5Y_6 - 1,\; Y_6 + X_1X_5Y_1Y_4Y_5Y_7 - 1,\; Y_7 + X_2X_6Y_1Y_2Y_5Y_6 - 1,
\end{aligned}$$

$$X_7 + X_1 X_6 Y_1 Y_5 - 1,\ Y_2 + X_1 X_4 Y_1 Y_3 Y_4 Y_7 - 1,\ Y_3 + X_2 X_5 Y_1 Y_2 Y_4 Y_5 - 1\ \rangle \tag{7.4.132}$$

With respect to lexicographic order we find 84 elements in the (reduced) Gröbner basis of (7.4.132). On the other hand, with respect to degree lexicographic order we have 184 basis elements. In the following, we determine the monomials generating the complement $\overline{\langle LT(I)\rangle}$ with respect to degree lexicographic order as this ordering directly yields a cyclic invariant basis. Most of the monomials in the complement $\overline{\langle LT(I)\rangle}$ yield higher powers of z_{ij} in the denominator of the integrand (7.1.12), i.e. $\tilde{n}_{ij} = -2$ for some z_{ij}. After disregarding those only the following six monomials and their cyclic transformations give rise to single powers in their denominators, i.e. $\tilde{n}_{ij} \geq -1$:

monomial in in equation (7.2.40)	rational function in equation (7.2.39)	rational function in equation (7.1.12)
1	$\frac{1}{(1-xy)\ (1-yz)\ (1-wz)}$	$\frac{1}{z_{13}z_{16}z_{24}z_{27}z_{35}z_{46}z_{57}}$
$Y_1\,Y_4$	$\frac{z}{(1-yz)\ (1-wz)\ (1-xyz)}$	$\frac{1}{z_{14}z_{16}z_{24}z_{27}z_{35}z_{36}z_{57}}$
$Y_1\,Y_3\,Y_6$	$\frac{z}{(1-xy)\ (1-wz)\ (1-xyz)}$	$\frac{z_{47}}{z_{14}z_{15}z_{24}z_{27}z_{36}z_{37}z_{46}z_{57}}$
$Y_1\,Y_2\,Y_5$	$\frac{yz}{(1-xy)\ (1-yz)\ (1-wyz)}$	$\frac{1}{z_{14}z_{16}z_{25}z_{27}z_{35}z_{37}z_{46}}$
$Y_2\,Y_4$	$\frac{1}{(1-yz)\ (1-wyz)\ (1-xyz)}$	$\frac{z_{15}z_{37}}{z_{13}z_{14}z_{16}z_{25}z_{27}z_{35}z_{36}z_{47}z_{57}}$
Y_1	$\frac{z}{(1-xy)\ (1-wz)\ (1-yz)}$	$\frac{z_{47}}{z_{14}z_{16}z_{24}z_{27}z_{35}z_{37}z_{46}z_{57}}$

(7.4.133)

Therefore, in total we have a basis of 36 elements and all (finite) integrals (7.1.12) with only single powers in their denominators z_{ij}, i.e. $\tilde{n}_{ij} \geq -1$, are spanned by the following six integrals

$$\begin{aligned}
G_0 &= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w\ \frac{I_7(x,y,z,w)}{(1-xy)\ (1-yz)\ (1-wz)} &&= \frac{27}{4}\,\zeta(4)\ +\ \ldots\\
G_{1a} &= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w\ \frac{z\ I_7(x,y,z,w)}{(1-yz)\ (1-wz)\ (1-xyz)} &&= \frac{17}{4}\,\zeta(4)\ +\ \ldots\\
G_{2b} &= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w\ \frac{z\ I_7(x,y,z,w)}{(1-xy)\ (1-wz)\ (1-xyz)} &&= 3\,\zeta(4)\ +\ \ldots\\
G_{3a} &= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w\ \frac{yz\ I_7(x,y,z,w)}{(1-xy)\ (1-yz)\ (1-wyz)} &&= 3\,\zeta(3)\ +\ \ldots\\
G_{5a} &= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w\ \frac{z\ I_7(x,y,z,w)}{(1-xy)\ (1-wz)\ (1-yz)} &&= 2\,\zeta(3)\ +\ 2\,\zeta(2)\ +\ \ldots\\
G_{4b} &= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w\ \frac{I_7(x,y,z,w)}{(1-yz)\ (1-wyz)\ (1-xyz)} &&= \frac{5}{2}\,\zeta(4)\ +\ 4\,\zeta(3)\\
& &&\quad -\ 2\,\zeta(2)\ +\ \ldots\ ,
\end{aligned} \tag{7.4.134}$$

and their cyclic transformations. For instance, we have

$$\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w\ \frac{I_7(x,y,z,w)}{(1-xy)\ (1-wz)} \quad = \quad G_0 - G_{1b}$$

$$
\begin{aligned}
\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \, \frac{yz^2 \; I_7(x,y,z,w)}{(1-yz)\,(1-wz)\,(1-xyz)} &= -G_0 + G_{1a} + G_{1b} + G_{1d} - G_{2b} \\
\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \, \frac{yz \; I_7(x,y,z,w)}{(1-yz)\,(1-wyz)\,(1-xyz)} &= G_{5b} - G_{3c} - G_{3f} + G_{4b} \\
\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \, \frac{z \; I_7(x,y,z,w)}{(1-wz)\,(1-xyz)} &= G_0 - G_{1b} - G_{1d} + G_{2b} \\
\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \, \frac{y \; I_7(x,y,z,w)}{(1-xy)\,(1-wyz)} &= G_0 - G_{1b} - G_{1f} + G_{2f} \\
\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \, \frac{yz \; I_7(x,y,z,w)}{(1-yz)\,(1-wxyz)} &= -G_{1b} + G_{1g} - G_{3d} - G_{3e} \\
&\quad - 2G_{3g} + G_{4d} + G_{4g} + G_{5f} \\
\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \, \frac{yz \; I_7(x,y,z,w)}{(1-wyz)\,(1-xyz)} &= 2G_0 - 2G_{1b} - G_{1d} - G_{1f} + G_{2a} \\
+G_{2b} - G_{4c} - G_{5c} - G_{5d} + G_{2f} + G_{3a} + G_{3b} + G_{3c} + G_{3f} - G_{4a} & \qquad (7.4.135)
\end{aligned}
$$

as results from the identities between their corresponding monomials on dividing them by the Gröbner basis of (7.4.132):

$$
\begin{aligned}
X_7Y_3Y_6Y_7 &= 1 - Y_1Y_5 \\
Y_1^2Y_4Y_5 &= -1 + Y_1Y_4 + Y_1Y_5 + Y_3Y_6 - Y_1Y_3Y_6 \\
Y_1Y_2Y_4Y_5 &= Y_3 + Y_2Y_4 - Y_2Y_3Y_6 - Y_3Y_4Y_7 \\
X_7Y_1Y_3Y_4Y_6Y_7 &= 1 - Y_1Y_5 - Y_3Y_6 + Y_1Y_3Y_6 \\
X_7Y_2Y_3Y_5Y_6Y_7 &= 1 - Y_1Y_5 - Y_3Y_7 + Y_3Y_5Y_7 \\
X_7Y_1Y_2Y_4Y_5Y_6Y_7 &= -Y_1Y_5 + Y_6 + Y_1Y_6 - Y_2Y_5Y_6 + Y_7 \\
&\quad - Y_1Y_4Y_7 + Y_5Y_7 - 2\,Y_3Y_6Y_7 \\
X_7Y_1Y_2Y_3Y_4Y_5Y_6Y_7 &= 2 - Y_1Y_3 - Y_2 - Y_4 - 2\,Y_1Y_5 - Y_3Y_5 + Y_1Y_3Y_5 + Y_1Y_2Y_5 \\
&\quad + Y_1Y_4Y_5 - Y_3Y_6 + Y_1Y_3Y_6 + Y_2Y_3Y_6 - Y_3Y_7 + Y_3Y_4Y_7 + Y_3Y_5Y_7 \qquad (7.4.136)
\end{aligned}
$$

The set of integrals G_0, G_1, G_2 of four integrals (7.4.134) yields a transcendental power series in α', cf. section 7.3.

To conclude: Any finite integral (7.1.12) with $\tilde{n}_{ij} \geq -1$ can be expressed as $\mathbb{R}$ linear combination of the basis (7.4.134) as a result of partial fraction decomposition of their integrands.

Chapter 8

Tree level scattering of massless states

This chapter contains the explicit results on tree amplitudes of massless states with RNS methods – both in uncompactified $D = 10$ dimensions and in superstring compactifications to four spacetime dimensions. Both adjoint states (i.e. gluons and gluinos) and chiral fermions are addressed. In this sense, we work out stringy predictions for parton scattering processes at LHC. As we will highlight, a lot of these disk amplitudes are in fact independent on the spacetime dimensionality and the compactification geometry.

Multileg scattering of the massless gauge multiplet has been extensively studied in the framework of supersymmetric QFTs. Modern tools such as string inspired color ordering, the resulting relations between subamplitudes (see section 5.5), four dimensional helicity methods and elements of twistor theory [235] have revealed a striking simplicity in the S matrix of field theories. In four dimensional spinor helicity variables, tree amplitudes of n gluons and superpartners in MHV (maximally helicity violating) helicity configurations assume an extremely compact form. This is the famous Parke Taylor formula [236], proven later on by Berends and Giele [237]. By now, any other helicity amplitudes at tree level have been worked out [100], and tremendous progress is taking place on the loop front [97, 238].

We present the superstring completion of the aforementioned tree level results in field theory with corrections to all orders in the Regge slope α'. Firstly, the α' corrections can be analyzed in view of the low energy effective action. For instance, the α'^2 correction to four point gluon scattering admit to identify a quartic interaction in the gluon field strength, and an infinite tower of higher order vertices $\sim D^{2m}\mathcal{F}^n$ can in principle be identified and related to a non-abelian DBI action [239]. Secondly, the pole structure of string amplitudes reflects the exchange of massive Regge excitations and sheds light on their factorization properties.

The aforementioned features of gluon interactions in superstring theory are valid in any compactification, regardless on the amount of supersymmetry preserved. We will argue that the same universality holds for disk amplitudes involving one quark antiquark pair and any number of gluons. That is why they are of (model independent) phenomenological relevance: The resonant behaviour of parton amplitudes at the string mass scale $\alpha'^{-1/2}$ (i.e. at the mass scale of lightest Regge excitations) could become measurable at LHC provided that the string scale is sufficiently low. We have explained in section 4.1 that low string scale scenarios become plausible in the context of large extra dimensions.

We compute full fledged cross sections including color sums and cast the results into a form suitable for the implementation of stringy partonic cross sections in the LHC data analysis. The possibility to detect indirect signals of heavy string vibration modes also motivates to study direct production rates of massive states as carried out in chapter 9.

8.0.1 Conventions in presenting RNS amplitudes

It is essential to have a clean notation to compactly present scattering amplitudes. The starting points for any computation in this chapter are equation (5.3.38) for the (correctly normalized) color ordered n point disk amplitude $\mathcal{A}(\Phi_1, \ldots, \Phi_n)$ and equation (5.3.26) for its color dressed version $\mathcal{M}\big[(T^{a_1}, \Phi_1), \ldots, (T^{a_n}, \Phi_n)\big]$. The color degrees of freedom T^a of the scattering states (T^a, Φ) are clearly separated from the kinematic degrees of freedom to which we collectively refer as Φ. The latter contains the information on spacetime momentum and polarization tensors, e.g.

$$\Phi \ \in \ \begin{cases} g \ \equiv \ (\xi_m, k_m) & : \ \text{gluon with polarization vector } \xi_m \text{ and momentum } k_m \\ \lambda \ \equiv \ (u^\alpha, k_m) & : \ \text{gluino with wave function } u^\alpha \text{ and momentum } k_m \end{cases} \tag{8.0.1}$$

In four dimensional compactifications, the vector index range for momenta and polarization tensors have to be adjusted like $k_m \mapsto k_\mu$.

The notation $\mathcal{A}(1, 2, \ldots, n)$ from section 5.4 and 5.5 refers to disk amplitudes of unspecified members of the massless SUSY multiplet. Giving explicit RNS results requires to specify the SUSY components, i.e. $\Phi \in \{g, \lambda\}$ in $D = 10$ dimensions or $\Phi \in \{g, \lambda, \bar{\lambda}, \phi\}$ in $D = 4$ dimensions. Only the pure spinor formalism gives rise to manifestly supersymmetric field theory- and superstring amplitudes in $D = 10$ dimensions, see part III. Whenever four dimensional spinor helicity methods are applied (like in section 8.2 for instance), the helicity appears as a superscript at the state's label, e.g. $g^\pm$ for gluons with polarization vector $\xi^\pm_\mu$, see (C.1.7).

For massive states, we will represent the kinematical degrees of freedom Φ with the same letter as the polarization wavefunction. At the first mass level, these are $\Phi \in \{B \equiv (B_{mn}, k_m),\ E \equiv$

(E_{mnp}, k_m), $\chi \equiv (\chi_m^\alpha, k_m)\}$ in $D = 10$ dimensions and a zoo of different states in four dimensions which is presented in section 4.5. For $\mathcal{N} = 1$ SUSY in $D = 4$, for instance, we have identified a spin two multiplet $\Phi \in \{\alpha \equiv (\alpha_{\mu\nu}, k_\mu)$, $\chi \equiv (\chi_\mu^a, k_\mu)$, $\bar{\chi} \equiv (\bar{\chi}_{\dot{a}}^\mu, k_\mu)$, $d \equiv (d_\mu, k_\mu)\}$ and a spin 1/2 multiplet $\Phi \in \{\varphi^\pm \equiv (\varphi^\pm, k_\mu)$, $b \equiv (b^a, k_\mu)$, $\bar{b} \equiv (\bar{b}_{\dot{a}}, k_\mu)$, $\omega^\pm \equiv (\omega^\pm, k_\mu)\}$. The even bigger list of states for maximal $\mathcal{N} = 4$ SUSY can be found in subsection 4.5.1.

8.1 Tree amplitudes in ten dimensions

A ten dimensional spacetime is the natural starting point to compute superstring amplitudes. The associated low energy SYM theory encompasses no other states than the eight physical gluon- and gluino polarizations each. Fortunately, the results on multigluon scattering remain fully valid in any lower dimension, the detailed CFT argument will be given in the next section 8.2. The same is true for amplitudes with two gauginos and any number of bosons, independent on the compactifications and on the number of spacetime supersymmetry preserved.

Multigluon amplitudes in ten dimensions have been studied in [240, 201, 202, 241, 242, 243]. Covariant computation of fermion amplitudes in the RNS model has been advanced at tree level in $D = 10$ to the four point level [115, 116, 126, 118, 244] while fermionic amplitudes in $D = 10$ up to six point level are pioneered in [230, 119].

The vertex operators for the ten dimensional gauge multiplet are given in subsection 3.2. Recall that the kinematic degrees of freedom are labelled by $g \equiv (\xi_\mu, k_\mu)$ and $\lambda \equiv (u^\alpha, k_\mu)$.

8.1.1 Three point amplitudes and normalization

The worldsheet positions in color stripped three point amplitudes drop out because the c ghost correlator $\langle c(z_1)c(z_2)c(z_3)\rangle = z_{12}z_{13}z_{23}$ cancels the z dependence of the (ψ, X, S, ϕ) fields:

$$\mathcal{A}(g_1, g_2, g_3) = C_{\mathrm{D}_2} \sqrt{2\alpha'}\, g_{\mathrm{A}}^3 \Big\{ (\xi_1\, \xi_2)\, (\xi_3\, k_1) + (\xi_2\, \xi_3)\, (\xi_1\, k_2) + (\xi_1\, \xi_3)\, (\xi_2\, k_3) \Big\} \tag{8.1.2}$$

$$\mathcal{A}(g_1, \lambda_2, \lambda_3) = \frac{1}{\sqrt{2}}\, C_{\mathrm{D}_2}\, g_{\mathrm{A}}\, g_\lambda^2\, (u_2\, \not{\xi}\, u_3) \tag{8.1.3}$$

Notice the manifest cyclic symmetry of the former. The two amplitudes are consistent with the SUSY Ward identity for $\langle V^{(-1)}(\xi_1, k_1)V^{(0)}(\xi_2, k_2)V^{(-1/2)}(u_3, k_3)\rangle$, see subsection 5.2.4.

The color dressed amplitudes at massless three point level are obtained by combining the cyclically inequivalent orderings $(1, 2, 3)$ and $(3, 2, 1)$. Taking the negative worldsheet parity of massless states into account, one has to subtract the associated Chan Paton traces to get

$\mathrm{Tr}\big\{T^{a_1}T^{a_2}T^{a_3} - T^{a_2}T^{a_1}T^{a_3}\big\} = \frac{i}{2}f^{a_1a_2a_3}$ with vanishing abelian limit:

$$\mathcal{M}\big[(T^{a_1},\Phi_1),(T^{a_2},\Phi_2),(T^{a_3},\Phi_3)\big] \;=\; \frac{i}{2}\,f^{a_1a_2a_3}\,\mathcal{A}(\Phi_1,\Phi_2,\Phi_3) \tag{8.1.4}$$

The three gluon amplitude matches the cubic vertex $\sim i f^{a_1a_2a_3}g_{\mathrm{YM}}$ of SYM if the normalization g_{A} of the gluon vertex operator and the universal disk factor C_{D_2} satisfy

$$C_{\mathrm{D}_2}\,\sqrt{2\alpha'}\,g_{\mathrm{A}}^3\,\frac{i}{2}\,f^{a_1a_2a_3} \;=\; 2i\,f^{a_1a_2a_3}\,g_{\mathrm{YM}}\ . \tag{8.1.5}$$

This is one out of two steps towards reliable normalization of superstring amplitudes. A second independent equation like (8.1.5) can be found from four point amplitudes of the gauge multiplet by requiring it to reproduce the field theory result as $\alpha' \to 0$. Since we have already stated the final answer $C_{\mathrm{D}_2} = (g_{\mathrm{YM}}^2\alpha'^2)^{-1}$ in subsection 5.3.4, let us take the input from four point ahead (which will be rigorously computed in the following subsection). The right normalization constants g_{A} and g_λ for gluon- and gluino vertex operators, respectively, are given by

$$g_{\mathrm{A}} \;=\; \sqrt{2\alpha'}\,g_{\mathrm{YM}}\ , \qquad g_\lambda \;=\; \alpha'^{1/4}\,g_{\mathrm{A}} \;=\; \sqrt{2}\,\alpha'^{3/4}\,g_{\mathrm{YM}}\ . \tag{8.1.6}$$

Hence, the final form of the superstring amplitudes is the following:

$$\begin{aligned}\mathcal{M}\big[(T^{a_1},g_1),(T^{a_2},g_2),(T^{a_3},g_3)\big] \;&=\; 2i\,f^{a_1a_2a_3}\,g_{\mathrm{YM}}\\ &\quad\times\Big\{(\xi_1\,\xi_2)\,(\xi_3\,k_1) \,+\, (\xi_2\,\xi_3)\,(\xi_1\,k_2) \,+\, (\xi_1\,\xi_3)\,(\xi_2\,k_3)\Big\}\end{aligned} \tag{8.1.7}$$

$$\mathcal{M}\big[(T^{a_1},g_1),(T^{a_2},\lambda_2),(T^{a_3},\lambda_3)\big] \;=\; i\,f^{a_1a_2a_3}\,g_{\mathrm{YM}}\,(u_2\,\not{\xi}\,u_3) \tag{8.1.8}$$

As shown in subsection 4.2.4, the scalars in the four dimensional $\mathcal{N}=4$ SYM multiplets have the coupling as the gluons $g_\phi = g_{\mathrm{A}}$.

8.1.2 Four point amplitudes

Four point open superstring amplitudes involve a single worldsheet integral over the unfixed vertex operator position which can be identified with the Euler Beta function of the dimensionless Mandelstam variables $s = 2\alpha' k_1\cdot k_2$ and $u = 2\alpha' k_1 \cdot k_4$:

$$\int_0^1 \mathrm{d}x\ x^{s+n_1}\,(1-x)^{u+n_2} \;=\; B(s+n_1+1,u+n_2+1) \tag{8.1.9}$$

In subsection 7.2.2 we have mentioned that its most convenient representation in the color ordering $1,2,3,4$ is the V_t formfactor

$$V_t \;=\; \frac{\Gamma(s+1)\,\Gamma(u+1)}{\Gamma(s+u+1)} \;=\; \frac{s\,u}{s+u}\,B(s,u) \tag{8.1.10}$$

related to the Euler Beta function $B(\cdot,\cdot)$. It is symmetric in s and u, reduces to 1 in the $\alpha' \to 0$ limit and has well-understood α' expansion (8.1.22). We will say more on its physical interpretation in the later subsections 8.1.3 and 8.1.4.

Given the definition (8.1.10) of V_t, the four point subamplitudes of the $D = 10$ gauge multiplet take the form

$$\begin{aligned}
\mathcal{A}(g_1,g_2,g_3,g_4) \;&=\; -\,C_{\mathrm{D}_2}\, g_{\mathrm{A}}^4\, 2\alpha'\, V_t \left\{ \frac{1}{2\alpha'}\,(\xi_1\,\xi_3)\,(\xi_2\,\xi_4) \right.\\
&+ \frac{1}{s}\Big[\frac{t}{2\alpha'}\,(\xi_1\,\xi_2)\,(\xi_3\,\xi_4) \;+\; (\xi_1\,\xi_2)\,(\xi_3\,k_4)\,(\xi_4\,k_1) \;-\; (\xi_1\,\xi_2)\,(\xi_3\,k_1)\,(\xi_4\,k_3)\\
&\quad +\; (\xi_3\,\xi_4)\,(\xi_1\,k_4)\,(\xi_2\,k_1) \;-\; (\xi_3\,\xi_4)\,(\xi_1\,k_2)\,(\xi_2\,k_4) \;+\; (\xi_1\,\xi_3)\,(\xi_2\,k_1)\,(\xi_4\,k_3)\\
&\quad +\; (\xi_2\,\xi_4)\,(\xi_1\,k_2)\,(\xi_3\,k_4) \;-\; (\xi_1\,\xi_4)\,(\xi_2\,k_1)\,(\xi_3\,k_4) \;-\; (\xi_2\,\xi_3)\,(\xi_1\,k_2)\,(\xi_4\,k_3)\Big]\\
&+ \frac{1}{u}\Big[\frac{t}{2\alpha'}\,(\xi_1\,\xi_4)\,(\xi_2\,\xi_3) \;+\; (\xi_1\,\xi_4)\,(\xi_2\,k_1)\,(\xi_3\,k_2) \;-\; (\xi_1\,\xi_4)\,(\xi_2\,k_3)\,(\xi_3\,k_1)\\
&\quad +\; (\xi_2\,\xi_3)\,(\xi_1\,k_2)\,(\xi_4\,k_1) \;-\; (\xi_2\,\xi_3)\,(\xi_1\,k_4)\,(\xi_4\,k_2) \;+\; (\xi_1\,\xi_3)\,(\xi_2\,k_3)\,(\xi_4\,k_1)\\
&\quad \left. +\; (\xi_2\,\xi_4)\,(\xi_1\,k_4)\,(\xi_3\,k_2) \;-\; (\xi_1\,\xi_2)\,(\xi_3\,k_2)\,(\xi_4\,k_1) \;-\; (\xi_3\,\xi_4)\,(\xi_1\,k_4)\,(\xi_2\,k_3)\Big] \right\}\\
&=\; C_{\mathrm{D}_2}\, g_{\mathrm{A}}^4\, V_t\, \frac{4\alpha'^2}{su}\; t_8^{m_1m_2n_1n_2p_1p_2q_1q_2}\, k^1_{m_1}\,\xi^1_{m_2}\,k^2_{n_1}\,\xi^2_{n_2}\,k^3_{p_1}\,\xi^3_{p_2}\,k^4_{q_1}\,\xi^4_{q_2} \qquad (8.1.11)\\
\mathcal{A}(g_1,g_2,\lambda_3,\lambda_4) \;&=\; C_{\mathrm{D}_2}\, g_{\mathrm{A}}^2\, g_{\lambda}^2 \sqrt{\frac{2\alpha'}{2}}\; V_t \left\{ \frac{1}{u}\Big[\frac{1}{2}\,(u_3\,\not{\xi}_2\,\not{k}_1\,\not{\xi}_1\,u_4) \;-\; (\xi_1\,k_4)\,(u_3\,\not{\xi}_2\,u_4)\Big] \right.\\
&+ \left.\frac{1}{s}\Big[(\xi_1\,\xi_2)\,(u_3\,\not{k}_1\,u_4) \;-\; (\xi_2\,k_1)\,(u_3\,\not{\xi}_1\,u_4) \;+\; (\xi_1\,k_2)\,(u_3\,\not{\xi}_2\,u_4)\Big] \right\} \qquad (8.1.12)\\
\mathcal{A}(\lambda_1,\lambda_2,\lambda_3,\lambda_4) \;&=\; \frac{1}{2}\, C_{\mathrm{D}_2}\, g_{\lambda}^4\, V_t \left\{ \frac{1}{u}\,(u_1\gamma^m u_4)\,(u_2\gamma_m u_3) \;-\; \frac{1}{s}\,(u_1\gamma^m u_2)\,(u_3\gamma_m u_4) \right\}. \\
&\qquad (8.1.13)
\end{aligned}$$

The four gluon amplitude is usually written in terms of the famous t_8 tensor whose full expansion in terms of η symbols encompasses 60 terms. Its contraction with the antisymmetric field strength $F^i_{mn} = 2k^i_{[m}\xi^i_{n]}$ simplifies to

$$\begin{aligned}
t_8(k^1,\xi^1,k^2,\xi^2,k^3,\xi^3,k^4,\xi^4) \;&=\; t_8^{m_1m_2n_1n_2p_1p_2q_1q_2}\, k^1_{m_1}\,\xi^1_{m_2}\,k^2_{n_1}\,\xi^2_{n_2}\,k^3_{p_1}\,\xi^3_{p_2}\,k^4_{q_1}\,\xi^4_{q_2}\\
= \frac{1}{8}\Big[4\,F^{1\,n}_{m}\,F^{2p}_{n}\,F^{3q}_{p}\,F^{4m}_{q} \;&+\; 4\,F^{1\,n}_{m}\,F^{3p}_{n}\,F^{2q}_{p}\,F^{4m}_{q} \;+\; 4\,F^{1\,n}_{m}\,F^{3p}_{n}\,F^{4q}_{p}\,F^{2m}_{q} \qquad (8.1.14)\\
-\; F^{1\,n}_{m}\,F^{2m}_{n}\,F^{3q}_{p}\,F^{4p}_{q} \;&-\; F^{1\,n}_{m}\,F^{3m}_{n}\,F^{2q}_{p}\,F^{4p}_{q} \;-\; F^{1\,n}_{m}\,F^{4m}_{n}\,F^{2q}_{p}\,F^{3p}_{q}\Big]
\end{aligned}$$

The color ordered four point amplitudes (8.1.11), (8.1.12) and (8.1.13) provide explicit examples of kinematic numerators n_i introduced in section 5.5. The identification can be made on the basis of the parametrization $\mathcal{A}(1,2,3,4) = \frac{n_s}{s} + \frac{n_u}{u}$. The four gluon amplitude (8.1.11) obviously contains a contact term $\sim (\xi_1\xi_3)(\xi_2\xi_4)$ which can be distributed in any way between n_s and n_u, the freedom of assignment can be parametrized by a real number $\lambda \in \mathbb{R}$:

$$\begin{aligned} n_s &= (k_1\,k_3)\,(\xi_1\,\xi_2)\,(\xi_3\,\xi_4) \ + \ (\xi_1\,\xi_2)\,(\xi_3\,k_4)\,(\xi_4\,k_1) \ - \ (\xi_1\,\xi_2)\,(\xi_3\,k_1)\,(\xi_4\,k_3) \\ &\quad + \ (\xi_3\,\xi_4)\,(\xi_1\,k_4)\,(\xi_2\,k_1) \ - \ (\xi_3\,\xi_4)\,(\xi_1\,k_2)\,(\xi_2\,k_4) \ + \ (\xi_1\,\xi_3)\,(\xi_2\,k_1)\,(\xi_4\,k_3) \\ &\quad + \ (\xi_2\,\xi_4)\,(\xi_1\,k_2)\,(\xi_3\,k_4) \ - \ (\xi_1\,\xi_4)\,(\xi_2\,k_1)\,(\xi_3\,k_4) \ - \ (\xi_2\,\xi_3)\,(\xi_1\,k_2)\,(\xi_4\,k_3) \\ &\quad + \ \lambda\,(k_1\,k_2)\,(\xi_1\,\xi_3)\,(\xi_2\,\xi_4) \qquad (8.1.15) \\ n_u &= (k_1\,k_3)\,(\xi_1\,\xi_4)\,(\xi_2\,\xi_3) \ + \ (\xi_1\,\xi_4)\,(\xi_2\,k_1)\,(\xi_3\,k_2) \ - \ (\xi_1\,\xi_4)\,(\xi_2\,k_3)\,(\xi_3\,k_1) \\ &\quad + \ (\xi_2\,\xi_3)\,(\xi_1\,k_2)\,(\xi_4\,k_1) \ - \ (\xi_2\,\xi_3)\,(\xi_1\,k_4)\,(\xi_4\,k_2) \ + \ (\xi_1\,\xi_3)\,(\xi_2\,k_3)\,(\xi_4\,k_1) \\ &\quad + \ (\xi_2\,\xi_4)\,(\xi_1\,k_4)\,(\xi_3\,k_2) \ - \ (\xi_1\,\xi_2)\,(\xi_3\,k_2)\,(\xi_4\,k_1) \ - \ (\xi_3\,\xi_4)\,(\xi_1\,k_4)\,(\xi_2\,k_3) \\ &\quad + \ (1-\lambda)\,(k_1\,k_4)\,(\xi_1\,\xi_3)\,(\xi_2\,\xi_4) \end{aligned}$$

The other disk orderings $\mathcal{A}(g_1, g_3, g_2, g_4)$ and $\mathcal{A}(g_1, g_2, g_4, g_3)$ involve different contact terms $\sim (\xi_1\xi_2)(\xi_3\xi_4)$ or $(\xi_1\xi_4)(\xi_3\xi_2)$ and a slightly different represetation of the kinematic numerators. A particularly symmetric choice of n_s, n_u is inspired by the pure spinor formalism, see section 13.2.

To compactly present color dressed amplitudes, let us introduce the shorthand

$$c_+^{a_1a_2a_3a_4} \ := \ \mathrm{Tr}\big\{\, T^{a_1}\,T^{a_2}\,T^{a_3}\,T^{a_4} \ + \ T^{a_4}\,T^{a_3}\,T^{a_2}\,T^{a_1}\,\big\} \qquad (8.1.16)$$

for the color factors which emerges from the even worldsheet parity of massless four point amplitudes. Generalities about color factors will be discussed in section 8.3.

$$\begin{aligned} \mathcal{M}\big[(T^{a_1},g_1),(T^{a_2},g_2),(T^{a_3},g_3),(T^{a_4},g_4)\big] \ &= \ 16\alpha'^2\,g_{\mathrm{YM}}^2\,t_8(k^1,\xi^1,k^2,\xi^2,k^3,\xi^3,k^4,\xi^4) \\ &\quad \left(\frac{V_t}{su}\,c_+^{a_1a_2a_3a_4} \ + \ \frac{V_s}{tu}\,c_+^{a_1a_3a_2a_4} \ + \ \frac{V_u}{st}\,c_+^{a_1a_2a_4a_3}\right) \qquad (8.1.17) \\ \mathcal{M}\big[(T^{a_1},g_1),(T^{a_2},g_2),(T^{a_3},\lambda_3),(T^{a_4},\lambda_4)\big] \ &= \ 4\alpha'\,g_{\mathrm{YM}}^2\,\bigg\{\frac{s}{2}\,(u_3\,\not{\xi}_2\,\not{k}_1\,\not{\xi}_1\,u_4) \\ - \ s\,(\xi_1\,k_4)\,(u_3\,\not{\xi}_2\,u_4) \ + \ u\,\Big[\,(\xi_1\,\xi_2)\,(u_3\,\not{k}_1\,u_4) \ &- \ (\xi_2\,k_1)\,(u_3\,\not{\xi}_1\,u_4) \ + \ (\xi_1\,k_2)\,(u_3\,\not{\xi}_2\,u_4)\Big]\bigg\} \\ &\quad \left(\frac{V_t}{su}\,c_+^{a_1a_2a_3a_4} \ + \ \frac{V_s}{tu}\,c_+^{a_1a_3a_2a_4} \ + \ \frac{V_u}{st}\,c_+^{a_1a_2a_4a_3}\right) \qquad (8.1.18) \\ \mathcal{M}\big[(T^{a_1},\lambda_1),(T^{a_2},\lambda_2),(T^{a_3},\lambda_3),(T^{a_4},\lambda_4)\big] \ &= \ 2\alpha'\,g_{\mathrm{YM}}^2 \\ &\quad \Big\{\,s\,(u_1\gamma^m u_4)\,(u_2\gamma_m u_3) \ - \ u\,(u_1\gamma^m u_2)\,(u_3\gamma_m u_4)\,\Big\} \\ &\quad \left(\frac{V_t}{su}\,c_+^{a_1a_2a_3a_4} \ + \ \frac{V_s}{tu}\,c_+^{a_1a_3a_2a_4} \ + \ \frac{V_u}{st}\,c_+^{a_1a_2a_4a_3}\right) \qquad (8.1.19) \end{aligned}$$

To see that α' drops out in the field theory limit $\alpha' \to 0$, recall that our Mandelstam variables are defined to be dimensionless $s_{ij} = \alpha'(k_i + k_j)^2$, hence each $\frac{1}{s}$-, $\frac{1}{u}$- or $\frac{1}{t}$ pole contains a hidden inverse power α'^{-1}.

8.1.3 Momentum expansion and low energy effective action

The signatures of string specific physics in these four point amplitudes (8.1.17) to (8.1.19) are exclusively captured by the hypergeometric function V_t, V_s and V_u. They contain all the α' dependence and become unity in the low energy limit $\alpha' \to 0$ leaving the associated field theory amplitude behind. In this subsection, we want to learn from the stringy formfactor V_t how the four point amplitude probes α' corrections in the low energy effective action.

The V_t is an analytic function of α', so it can be written as a power series. Since α' always appears in dimensionless combinations $s = 2\alpha' k_1 \cdot k_2$ and $u = 2\alpha' k_1 \cdot k_4$ this amounts to performing a momentum expansion[1]:

$$\begin{aligned} V_t &= \exp\left(\sum_{k=2}^{\infty} \frac{\zeta(k)}{k} \, (-1)^k \left(s^k \, + \, u^k \, - \, (s+u)^k \right) \right) \\ &= 1 \, - \, \zeta(2) \, su \, + \, \zeta(3) \, su \, (s+u) \, - \, \frac{\big(\zeta(2)\big)^2}{10} \, su \, (4s^2 + su + 4u^2) \, + \, \mathcal{O}(\alpha'^5) \end{aligned} \tag{8.1.22}$$

We find the nice correspondence $\zeta(n) \leftrightarrow \alpha'^n$ between zeta values and momentum powers which was discussed in section 7.3.

All the α' corrections in (8.1.22) have a factor of su in common, so the combination $\frac{V_t}{su}$ appearing in the amplitudes (8.1.17), (8.1.18) and (8.1.19) does not give rise to any kinematic poles. Consequently, the stringy contributions to the low energy effective action are pure contact terms. This is consistent with the fact that the three point amplitudes (8.1.7) and (8.1.8) into which $\mathcal{A}(1,2,3,4)$ factorizes on the residue of its poles do not receive any α' correction.

The α' corrections can be translated into a string correction to the low energy effective action: An effective Lagrangian is defined by the requirement of reproducing the amplitude (8.1.17) up to some order α'^k by means of Feynman rules [202, 245, 246, 247, 83, 248]. The simplest example is the α'^2 correction to the four gluon amplitude, reproduced by a Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{eff}} \Big|_{\mathcal{F}^4} \;&=\; \mathrm{Tr}\Big\{ \mathcal{F}_{mn} \, \mathcal{F}^{mn} \; + \; \zeta(2) \, \alpha'^2 \, (\, \mathcal{F}_{mn} \, \mathcal{F}_{pq} \, \mathcal{F}^{nm} \, \mathcal{F}^{qp} \; + \; 2 \, \mathcal{F}_{mn} \, \mathcal{F}^{nm} \, \mathcal{F}_{pq} \, \mathcal{F}^{qp} \\ & \quad - \; 4 \, \mathcal{F}_m{}^n \, \mathcal{F}_n{}^p \, \mathcal{F}_p{}^q \, \mathcal{F}_q{}^m \; - \; 8 \, \mathcal{F}_m{}^n \, \mathcal{F}_n{}^q \, \mathcal{F}_p{}^m \, \mathcal{F}_q{}^p \,) \; + \; \mathcal{O}(\alpha'^3) \Big\} \end{aligned} \tag{8.1.23}$$

[1] This identity is based on the expansion

$$\ln \Gamma(1+z) \;=\; -\gamma \, z \; + \; \sum_{k=2}^{\infty} \frac{\zeta(k)}{k} \, (-1)^k \, z^k \; , \qquad z \in (-1,+1) \tag{8.1.20}$$

of the Gamma function where γ denotes the Euler Mascheroni constant

$$\gamma \;=\; \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} \; - \; \ln(n) \right) \;\approx\; 0.577 \; . \tag{8.1.21}$$

in terms of the gluon field strength $\mathcal{F}_{mn}$ (in position space). Higher order α' corrections give rise to covariant derivatives, i.e. to contributions of schematic form $\mathcal{D}^{2m}\mathcal{F}^4$.

The general effective action for higher point gluon interactions can be written as a formal power series

$$\mathcal{L}_{\text{eff}} \ \sim \ \sum_{n=2}^{\infty} \sum_{m=0}^{\infty} \alpha'^{m+n-2} \, \mathrm{Tr}\big\{ \, \mathcal{D}^{2m} \, \mathcal{F}^n \, \big\} \ . \tag{8.1.24}$$

The way of writing the Lagrangian is not at all unique – there are several sources of ambiguities such as $[\mathcal{D}_m, \mathcal{D}_n]\mathcal{F}_{pq} \sim [\mathcal{F}_{mn}, \mathcal{F}_{pq}]$, field redefinitions, Bianchi identity and integration by parts. If the sum in (8.1.24) is restricted to independent terms, then each contribution accounts for a gluon contact interaction represented by an irreducible Feynman diagram of the underlying (higher derivative) gauge theory. Moreover, spacetime supersymmetry of the superstring admits to read off supersymmetric invariants on these grounds. This reasoning was recently applied to the gravity sector for the purpose of classifying five- and six loop counterterms [112].

The lowest order α' correction (8.1.23) is consistent with the nonabelian DBI action of [239]. But it already departs from the results of computing scattering amplitudes at α'^3 level where $\mathcal{D}^2\mathcal{F}^4$ and $\mathcal{F}^5$ interaction occur. The latter can only be probed from five gluon amplitudes [242].

8.1.4 Pole expansion of four point amplitudes

This subsection opens a second viewpoint on the physical interpretation of the four point formfactor V_t. Instead of a power series expansion in α', we can perform a pole expansion:

$$V_t \ = \ s \sum_{n=0}^{\infty} \frac{\Gamma(n-u)}{n! \, \Gamma(-u) \, (s+n)} \ =: \ s \sum_{n=0}^{\infty} \frac{\gamma(u,n)}{(s+n)} \tag{8.1.25}$$

It is obtained from the binomial series $(1-x)^u = \sum_{n=0}^{\infty} \gamma(u,n) x^n$ for the worldsheet integrand in (8.1.9). The four point amplitude exhibits poles whenever the center of mass energy of the incoming states $(k_1+k_2)^2 = s/\alpha'$ hits the mass of a Regge excitation:

$$\text{poles at } s \ = \ -n \quad \longleftrightarrow \quad n\text{'th mass level } m_n^2 \ = \ \frac{n}{\alpha'} \text{ of Regge excitations} \tag{8.1.26}$$

The residue $\gamma(u,n) = \prod_{J=0}^{n-1}(-u+J)/n!$ at the $s=-n$ pole is a degree n polynomial in the dual Mandelstam variable u, this is an indirect method to identify the maximum spin $j_{\max} = n+1$ at mass level n. The representation (8.1.26) of V_t tells us that heavy states $|n;j\rangle$ with masses $m_n^2 = n/\alpha'$ and spin j up to $n+1$ propagate in the internal channels of four gluon amplitude, see the following figure 8.1. The residue $\gamma(u,n)$ determines the three point coupling of the intermediate string to the external massless states. The spin content of the any mass level can be predicted on these grounds [249, 250].

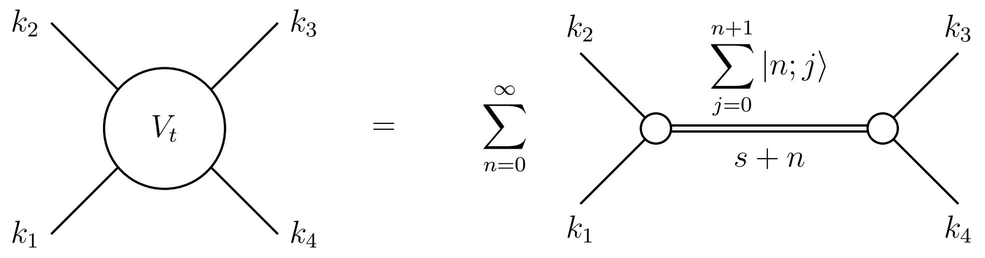


Figure 8.1: Exchange of massive Regge resonances in massless four point amplitudes.

Unitarity implies that on the residue $s = -n$, the four point amplitude factorizes into a product of three point amplitudes with external one mass level n state. This can be used to determine the normalization of massive vertex operators, see the later subsection 9.2.2.

Since V_t is symmetric in s and u, the same type of expansion can be done in the dual channel with poles at $u = -n$ and residues $\sim s^n$. The possibility to write the string amplitude as a sum over only s channel poles *or* as a sum over only u channel poles is in a sharp contrast to the field theory situation. The latter generically requires a sum over both s- and u channel poles.

8.1.5 Higher point amplitudes

The length of the four gluon amplitude (8.1.11) when expanded in terms of polarization vectors and momenta is rather discouraging. Presenting the complete color ordered five point amplitude in terms of $(\xi_i\,\xi_j)(\xi_k\,\xi_l)(\xi_m\,k_p)$ and $(\xi_i\,\xi_j)(\xi_k\,k_p)(\xi_l\,k_q)(\xi_m\,k_r)$ kinematics requires around 900 terms (rather than the 19 terms of $\mathcal{A}(g_1, g_2, g_3, g_4)$), so we won't display it here. In some sense, we can regard this as the punishment of the formalism for our attempt to write amplitudes in terms of gauge dependent variables ξ_m. The gauge redundancy in each gluon's polarization artificially blows up the complexity of the result.

There are two ways around this mess of polarization vectors and momenta. The first one is the use of the $\mathcal{N} = 1$ superfield formulation of $D = 10$ SYM as suggested by the pure spinor formalism. Chapters 11 and 12 show the remarkable compactness of higher point field theory- and superstring amplitudes in terms of appropriate superfield building blocks motivated by the BRST cohomology. A second one is applicable in four dimensions only and based on spinor helicity variables introduced in appendix C. The simplicity and compactness of spinor helicity results heavily depends on the configurations of helicities involved. Details will be explained in subsections 8.3.1 and 8.3.2.

As a first taste of higher point amplitudes, let us review the result of first full-fledged computation of a five gluon superstring amplitude [251]: It is organized into the field theory amplitude $\mathcal{A}^{\rm SYM}(g_1,\ldots,g_5)$ and another kinematic package $\mathcal{A}_{\mathcal{F}^4}(g_1,\ldots,g_5)$ generated by the interaction term (8.1.23):

$$\begin{aligned}\mathcal{A}(g_1,g_2,g_3,g_4,g_5) \ &= \ \big[s_{23}\,s_{51}\,f_1 \ + \ (s_{23}\,s_{34} \ + \ s_{45}\,s_{51})\,f_2 \big]\,\mathcal{A}^{\rm SYM}(g_1,g_2,g_3,g_4,g_5) \\ &\quad + \ f_2\,\mathcal{A}_{\mathcal{F}^4}(g_1,g_2,g_3,g_4,g_5) \end{aligned} \tag{8.1.27}$$

The five point disk amplitude can be expressed in terms of two independent hypergeometric integrals f_1, f_2 carrying the α' dependence

$$f_1 \ = \ \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \ x^{s_{23}-1}\,y^{s_{51}-1}\,(1-x)^{s_{34}}\,(1-y)^{s_{45}}\,(1-xy)^{s_{35}} \tag{8.1.28}$$

$$f_2 \ = \ \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \ x^{s_{23}}\,y^{s_{51}}\,(1-x)^{s_{34}}\,(1-y)^{s_{45}}\,(1-xy)^{s_{35}-1} \tag{8.1.29}$$

and therefore taking the role of V_t. We give an interpretation in terms of Regge exchange in the following subsection. The Mandelstam variables are the usual dimensionless $s_{ij} = 2\alpha' k_i \cdot k_j$.

In section 12.3, we present and discuss the general decomposition of the n point amplitude of massless states in terms of $\mathcal{A}^{\rm SYM}$ building blocks. It is the result of a computation in the pure spinor formalism which can be found in section 12.2. The $\mathcal{A}_{\mathcal{F}^4}(g_1,\ldots,g_5)$ in (8.1.27) can a posteriori be idenfitied with a linear combination of field theory amplitudes of two different color orderings $\mathcal{A}^{\rm SYM}(g_1,g_2,g_3,g_4,g_5)$ and $\mathcal{A}^{\rm SYM}(g_1,g_3,g_2,g_4,g_5)$.

8.1.6 Pole expansion of five point amplitudes

Also the hypergeometric functions (8.1.28) and (8.1.29) appearing in five point amplitudes may be understood as an infinite sum over poles with intermediate string states. The expansion $(1-xy)^{s_{35}} = \sum_{n=0}^{\infty} \gamma(s_{35},n)(xy)^n$ of the integrand allows to reduce the $f_{1,2}$ to an infinite sum of Euler beta functions which were found to capture the α' dependence of four point amplitudes:

$$f_1 \ = \ \sum_{n=0}^{\infty} \gamma(s_{35},n)\,B(s_{23}+n,s_{34}+1)\,B(s_{45}+1,s_{51}+n) \tag{8.1.30}$$

$$f_2 \ = \ \sum_{n=1}^{\infty} \gamma(s_{35}-1,n-1)\,B(s_{23}+n,s_{34}+1)\,B(s_{45}+1,s_{51}+n) \tag{8.1.31}$$

The coefficients $\gamma(s_{35},n) = \prod_{J=0}^{n-1}(-s_{35}+J)/n!$ take the same form as the residues $\gamma(u,n)$ of the pole expansion (8.1.25) of V_t, they can be interpreted as three point couplings. In the following, we present two rewritings of the functions $f_{1,2}$ both of which manifest the exchange of string Regge excitations.

Firstly, (8.1.30) and (8.1.31) may be cast into in infinite double sum over s_{23}- and s_{51} channel poles with intermediate string states $|n,j\rangle$ and $|n',j\rangle$ exchanged:

$$f_1 = \sum_{n=0}^{\infty}\sum_{n'=0}^{\infty}\sum_{j=0}^{\min(n,n')} \gamma(s_{35},j)\ \frac{\gamma(s_{34},n-j)}{s_{23}+n}\ \frac{\gamma(s_{45},n'-j)}{s_{51}+n'} \tag{8.1.32}$$

$$f_2 = \sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\sum_{j=1}^{\min(n,n')} \gamma(s_{35}-1,j-1)\ \frac{\gamma(s_{34},n-j)}{s_{23}+n}\ \frac{\gamma(s_{45},n'-j)}{s_{51}+n'} \tag{8.1.33}$$

This translates into the diagram of figure 8.2:

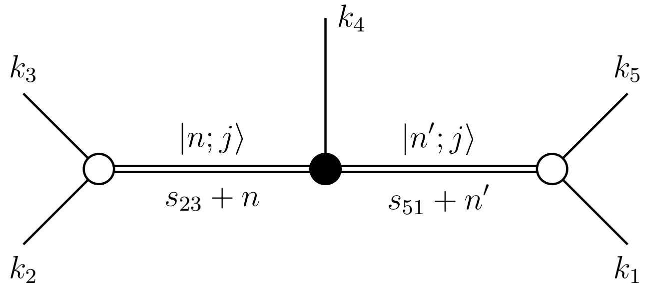


Figure 8.2: Exchange of massive Regge resonances in massless five point amplitudes.

Secondly, the functions $f_{1,2}$ may be written as one infinite sum over s_{23} channel poles with the intermediate string state $|n,j\rangle$ exchanged. More precisely, (8.1.30) and (8.1.31) can be expressed as

$$f_1 = \sum_{n=0}^{\infty} \gamma(s_{35},n)\ \frac{\gamma(s_{34},n-j)}{s_{23}+n}\ B(s_{45}+1,s_{51}+n) \tag{8.1.34}$$

$$f_2 = \sum_{n=1}^{\infty} \gamma(s_{35}-1,n-1)\ \frac{\gamma(s_{34},n-j)}{s_{23}+n}\ B(s_{45}+1,s_{51}+n)\ , \tag{8.1.35}$$

see the following figure 8.3.

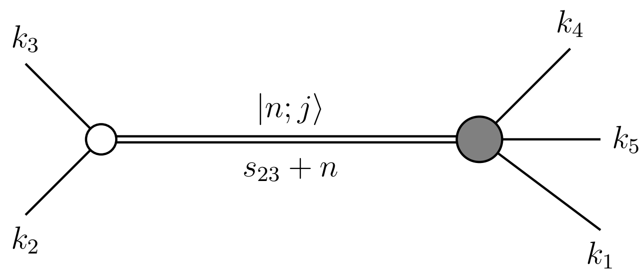


Figure 8.3: Exchange of massive Regge resonances in massless five point amplitudes.

Both viewpoints originate from the same functions f_1, f_2 and must therefore be equivalent. The power series expansion of $f_{1,2}$ yields an $\mathcal{F}^5$ contact interaction at order $\zeta(3)\alpha'^3$.

8.2 Universality in lower dimensions

This section builds the bridge between ten and four spacetime dimensions. We will first of all argue that the results of the previous two sections can be almost literally taken over to $D = 4$ dimensional compactifications. Moreover, we show that on the level of two fermions, quarks and leptons from the chiral multiplet give rise to the same disk coupling to gluons as the adjoint gauginos.

8.2.1 Universal two fermion amplitudes in lower dimensions

Although we have adapted our whole setup to full-fledged ten dimensional superstring theory with spacetime filling D9 branes, it turns out that all the results presented in the previous sections except for $\mathcal{A}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ can be taken over to lower dimensional Dp brane worldvolumes and compactification geometries.

Dimensional reduction of the spacetime gluon vertex operator simply replaces $SO(1, 9)$ vectors ξ_m, k_n, ψ^p, X^q by $SO(1, 3)$ vectors $\xi_\mu, k_\nu, \psi^\lambda, X^\rho$ (since we neglect Kaluza Klein excitations with internal momenta). Correlation functions involving exclusively the spacetime fields ψ^μ and X^ν do not depend on the range of their indices, see appendix B.4. The massless vertex operators do not involve any Kronecker deltas which could imprint a dependence on the number D of uncompactified spacetime dimensions via $\delta^\mu_\mu = D$. Hence, the n gluon amplitude cannot depend on D.

Dimensional reduction of spinorial fields is a bit more subtle. But we will still argue in the following why scattering amplitudes with two fermions and $n-2$ bosons do not depend on the number D of noncompact Minkowski spacetime dimensions.

In chapter 4, we have seen examples of how ten dimensional spin fields S_α of the RNS CFT factorize as $S_a \otimes s^i_{\text{int}}$ or $S^{\dot{b}} \otimes \bar{s}^j_{\text{int}}$ into internal parts $s^i_{\text{int}}, \bar{s}^j_{\text{int}}$ and spinors $S_a, S^{\dot{b}}$ of $SO(1, D-1)$ under dimensional reduction: Four dimensional gauginos involve the $SO(6)$ covariant spin fields $(s^i_{\text{int}}, \bar{s}^j_{\text{int}}) \equiv (\Sigma^I, \bar{\Sigma}_{\bar{J}})$ introduced in (4.2.5), and chiral matter located at D brane intersections require boundary changing operators $(s^i_{\text{int}}, \bar{s}^j_{\text{int}}) \equiv (\Xi, \bar{\Xi})$, see (4.4.74) for their definition and (4.4.75), (4.4.76) for the vertex operators of quarks.

More generally, internal spin fields in a maximally supersymmetric compactification can be represented by exponentials of $m = \frac{10-D}{2}$ free bosons $s^{(\pm,\dots,\pm)}_{\text{int}} = e^{\frac{i}{2}(\pm H_1,\dots,\pm H_m)}$, multiplied by an m component weight vector of $SO(10 - D)$. If some supersymmetries are broken, certain $\pm$ choices and therefore gaugino species are projected out. To describe chiral matter in orbifolds or intersecting brane models, on the other hand, the s^i_{int} can be attributed to the twisted CFT sector and carry details of the compactification geometry such as brane intersection angles. In this case, an appropriate generalization c^{ij} of an internal charge conjugation matrix enters the two point functions (8.2.36). The conformal dimension $\frac{5}{8}$ of S_α splits into $h_1 = \frac{D}{16}$ for the spacetime part S_a and $h_2 = \frac{10-D}{16}$ for the internal spin fields $s_{\rm int}$. Two point functions in both sectors are completely determined by their conformal weights,

$$\langle\, s^i_{\rm int}(z)\, \bar s^j_{\rm int}(w)\,\rangle \;\;=\;\; \frac{c^{ij}}{(z-w)^{(10-D)/8}}\ , \qquad c^{ij} \in \{0,1\}\ . \tag{8.2.36}$$

In massless tree amplitudes $\mathcal{A}(g_1,\ldots,g_{n-2},\lambda_{n-1},\bar\lambda_n)$ and $\mathcal{A}(g_1,\ldots,g_{n-2},q_{n-1},\bar q_n)$ involving two fermions and any number of gluons, (8.2.36) is the only signature of the internal dimensions' CFT. (Internal components of the ψ^m which might interact with $s^i_{\rm int}$ do not occur in the gluon vertex.) At nonzero c^{ij}, the two point correlator $\langle s^i_{\rm int}(z)\bar s^j_{\rm int}(w)\rangle$ cannot distinguish between adjoint fermions from the untwisted sector, say gauginos $\lambda,\bar\lambda$, and chiral matter with a boundary changing vertex operator (e.g. quarks $q,\bar q$ and leptons). The fractional z_{ij} powers combine to integers thanks to the correlations of superghosts and spacetime fields ψ^μ and S_a, the latter are made up of the D dimensional gamma matrices γ^μ_{ab} as we have seen in chapter 6.

To give an easy and still illuminating example of how two fermion amplitudes might become independent on D, let us consider the three point function of the $SO(1,D-1)$ spin fields

$$\langle\, \psi^\mu(z_1)\, S_a(z_2)\, S_b(z_3)\,\rangle \;\;=\;\; \frac{\gamma^\mu_{ab}}{\sqrt{2}\,(z_{12}\,z_{13})^{1/2}\, z_{23}^{D/8-1/2}}\ . \tag{8.2.37}$$

The complementary factor of $z_{23}^{D/8}$ within (8.2.36) makes sure that the overall D dependence drops out of ten dimensional spin field correlation

$$\begin{aligned} \langle\, \psi^{m=\mu}(z_1)\, S_\alpha(z_2)\, S_\beta(z_3)\,\rangle\Big|_{D<10} \;\;&=\;\; \langle\, \psi^\mu(z_1)\, S_a\, s^i_{\rm int}(z_2)\, S_b\, s^j_{\rm int}(z_3)\,\rangle \\ &=\;\; \frac{\gamma^\mu_{ab}\, c^{ij}}{\sqrt{2}\,(z_{12}\,z_{13})^{1/2}\, z_{23}^{3/4}}\ . \end{aligned} \tag{8.2.38}$$

The same mechanism applies to CFT correlators $\Omega_{(n,D)}$ and $\omega_{(n,D)}$ with more $\psi^{m=\mu}$ insertions and two spin fields $S_a(z_A)$ and one of $S_b(z_B), S^{\dot b}(z_B)$, see (6.2.39) and (6.2.40) for their most general form in D dimensions. The only D dependence at tree level enters through the overall z_{AB} powers $z_{AB}^{1/2-D/8}$ and $z_{AB}^{-D/8}$, respectively, which are compensated by the internal contribution (8.2.36). For any D, the product $\Omega_{(n,D)}(z_i)\times\langle s^i_{\rm int}(z)\bar s^j_{\rm int}(w)\rangle$ does no longer depend on the spacetime dimensionality D after compactification, regardless on whether the $s_{\rm int}$ originate from adjoint gauginos or chiral quarks.

One comment on the $SO(1,D-1)$ chirality of S_a spin fields is in order: Because of the different chirality structure in the charge conjugation matrix for dimensions (2 mod 4) and

(4 mod 4), the correlator (8.2.37) vanishes for alike (S_a, S_b) chiralities if $D = 4$ mod 4 and for opposite chiralities in $D = 2$ mod 4 dimensional compactifications. The relative D dimensional chirality of two fermions must therefore be alike if $D = 2$ mod 4 and opposite if $D = 4$ mod 4 for nonvanishing scattering with gluons.

Hence, two gaugino amplitude (8.1.18) presented in the previous section turns out to be independent on D up to the two fermions' relative chirality:

$$\mathcal{A}(g_1, \dots, g_{n-2}, \lambda_{n-1}, \lambda_n)\Big|_{D=10} = \mathcal{A}(g_1, \dots, g_{n-2}, \lambda_{n-1}, \bar{\lambda}_n)\Big|_{D=4,8} \tag{8.2.39}$$

8.2.2 Non-universal four fermion amplitudes

Four fermion amplitudes require four point functions such as $\langle s^i_{\text{int}}(z_1) s^j_{\text{int}}(z_2) s^k_{\text{int}}(z_3) s^l_{\text{int}}(z_4)\rangle$ and $\langle s^i_{\text{int}}(z_1) \bar{s}^j_{\text{int}}(z_2) s^k_{\text{int}}(z_3) \bar{s}^l_{\text{int}}(z_4)\rangle$ as an internal CFT input which cannot be discussed in a model independent fashion. They can have highly non-trivial dependence on cross ratios of the worldsheet positions, parametrized by data of the compactification geometry.

Scattering amplitudes involving four quarks are determined by the four point function of boundary changing operators Ξ carrying the internal geometric data like intersection angles θ^j of the brane configuration:

$$\langle\, \Xi^{a\cap b}(z_1)\, \bar{\Xi}^{b\cap d}(z_2)\, \Xi^{d\cap c}(z_3)\, \bar{\Xi}^{c\cap a}(z_4)\,\rangle = \left(\frac{z_{13}\, z_{24}}{z_{12}\, z_{14}\, z_{23}\, z_{34}} \right)^{3/4} I_\rho(\{z_i\};\theta^j) \tag{8.2.40}$$

This internal part of the chiral fermion vertex operators gives rise to nontrivial mapping from the disk worldsheet into the internal target space. As a result, the four point correlator (8.2.40) of these fields receives a correction factor $I_\rho(\{z_i\};\theta^j) = I_1(\{z_i\};\theta^j)\; I_{2\rho}(\{z_i\};\theta^j)$ describing disk instantons and the quantum correlator of these fields. The quantum four point correlator $I_1(\{z_i\};\theta^j)$ has been already computed in [252, 253, 254, 255, 256], while the effect $I_{2\rho}(\{z_i\};\theta^j)$ of world–sheet disk instantons has been derived in [257, 258, 259] for intersecting D6 branes. Note that the additional gluon insertion has no effect on the instanton part, since the gluon vertex operator has vanishing OPE with the internal field Ξ. The factor I_ρ encoding these effects depends on the specific D–brane setup under consideration and is thoroughly discussed in [101].

But even if we stick to the simplest realization of fermions, for instance with only the two internal spin fields $s^{\pm}_{\text{int}} = \mathrm{e}^{\pm \frac{i}{2}(H_1,\dots,H_m)} \equiv \Sigma, \bar{\Sigma}$ available for the unique gaugino species in $\mathcal{N} = 1$ supersymmetry, one cannot write down D independent expressions for quartic fermion couplings for representation theoretic reasons: The structure of Fierz identities and therefore the appropriate bases of Lorentz tensors vary a lot with D. In $D = 4$ and $D = 6$, there exist three linearly independent tensors with four free spinor indices whereas $D = 8$ and $D = 10$ admit five independent such tensors[2].

Four gaugino amplitudes in $D = 4$ dimensions are given by

$$\mathcal{A}(\lambda_1, \bar{\lambda}_2, \lambda_3, \bar{\lambda}_4) \;=\; C_{\mathrm{D}_2}\, g_\lambda^4\, V_t\, (u_1^I\, u_3^K)\, (\bar{u}_{\bar{J}}^2\, \bar{u}_{\bar{L}}^4) \left\{ \frac{1}{s}\, C^I{}_{\bar{J}}\, C^K{}_{\bar{L}} \;+\; \frac{1}{u}\, C^I{}_{\bar{J}}\, C^K{}_{\bar{L}} \right\} \tag{8.2.42}$$

and

$$\begin{aligned} \mathcal{M}\big[(T^{a_1}, \lambda_1), (T^{a_2}, \bar{\lambda}_2), (T^{a_3}, \lambda_3), (T^{a_4}, \bar{\lambda}_4)\big] \;&=\; 4\alpha'\, g_{\mathrm{YM}}^2\, (u_1^I\, u_3^K)\, (\bar{u}_{\bar{J}}^2\, \bar{u}_{\bar{L}}^4) \\ \Big\{ u\, C^I{}_{\bar{J}}\, C^K{}_{\bar{L}} \;+\; s\, C^I{}_{\bar{J}}\, C^K{}_{\bar{L}} \Big\} & \left(\frac{V_t}{su}\, c_+^{a_1 a_2 a_3 a_4} \;+\; \frac{V_s}{tu}\, c_+^{a_1 a_3 a_2 a_4} \;+\; \frac{V_u}{st}\, c_+^{a_1 a_2 a_4 a_3} \right) \end{aligned} \tag{8.2.43}$$

instead of (8.1.13).

8.2.3 Universal properties of parton amplitudes

Scattering amplitudes of quarks and gluons – to which we will collectively refer as partons – are important for collider phenomenology since multijet production is dominated by tree level QCD scattering. Therefore those parton amplitudes that are generic to any string compactification are especially important, as they may give rise to universal string signals independent on any compactification details. According to our reasoning from the previous subsection, amplitudes involving an arbitrary number of gluons g or gauginos λ but only two quarks q squarks C are of this kind. In section 4.4, we have given vertex operators (4.4.67) and (4.4.68) for the latter. More precisely, the following n point amplitudes

$$\left\{ \begin{array}{c} \mathcal{A}(g_1, g_2, g_3, \ldots, g_n) \\ \mathcal{A}(\lambda_1, \bar{\lambda}_2, g_3, \ldots, g_n) \end{array} \right. , \qquad \left\{ \begin{array}{c} \mathcal{A}(q_1, \bar{q}_2, g_3, \ldots, g_n) \\ \mathcal{A}(C_1, \bar{C}_2, g_3, \ldots, g_n) \end{array} \right. \tag{8.2.44}$$

are completely universal to any string compactification. We are using labels $q \equiv (u^a, k_\mu)$, $\bar{q} \equiv (\bar{u}_{\dot{a}}, k_\mu)$ and $C \equiv (C, k_\mu)$, $\bar{C} \equiv (\bar{C}, k_\mu)$ for the chiral multiplet. By virtue of spacetime supersymmetry, we may also replace gluons g by gauginos λ. Figure 8.4 gives a diagrammatic way of understanding the universality of the above amplitudes.

[2]Explicitly, these are:

$$\begin{array}{cl} C_{\alpha\beta}\, C_{\gamma\delta}\;, \quad C_{\alpha\delta}\, C_{\gamma\beta}\;, \quad C_{\alpha\beta}\, C^{\dot{\gamma}\dot{\delta}} & : \; D = 4 \\ \frac{1}{2}\, (\gamma^\mu\, C)_{\alpha\beta}\, (\gamma_\mu\, C)_{\gamma\delta} \;=\; \varepsilon_{\alpha\beta\delta\gamma}\;, \quad C_\alpha{}^{\dot{\gamma}}\, C_\beta{}^{\dot{\delta}}\;, \quad C_\alpha{}^{\dot{\delta}}\, C_\beta{}^{\dot{\gamma}} & : \; D = 6 \\ C_{\alpha\beta}\, C_{\gamma\delta}\;, \quad C_{\alpha\delta}\, C_{\gamma\beta}\;, \quad C_{\alpha\gamma}\, C_{\beta\delta}\;, \quad (\gamma^\mu\, C)_\alpha{}^{\dot{\gamma}}\, (\gamma_\mu\, C)_\beta{}^{\dot{\delta}}\;, \quad (\gamma^\mu\, C)_\alpha{}^{\dot{\delta}}\, (\gamma_\mu\, C)_\beta{}^{\dot{\gamma}} & : \; D = 8 \\ (\gamma^\mu\, C)_{\alpha\beta}\, (\gamma_\mu\, C)_{\gamma\delta}\;, \quad (\gamma^\mu\, C)_{\alpha\delta}\, (\gamma_\mu\, C)_{\gamma\beta}\;, \quad (\gamma^\mu\, C)_{\alpha\beta}\, (\bar{\gamma}_\mu\, C)^{\dot{\gamma}\dot{\delta}}\;, \quad C_\alpha{}^{\dot{\gamma}}\, C_\beta{}^{\dot{\delta}}\;, \quad C_\alpha{}^{\dot{\delta}}\, C_\beta{}^{\dot{\gamma}} & : \; D = 10 \end{array} \tag{8.2.41}$$

Any higher order γ matrix contraction can be reduced to the shown tensors by means of Fierz identities.

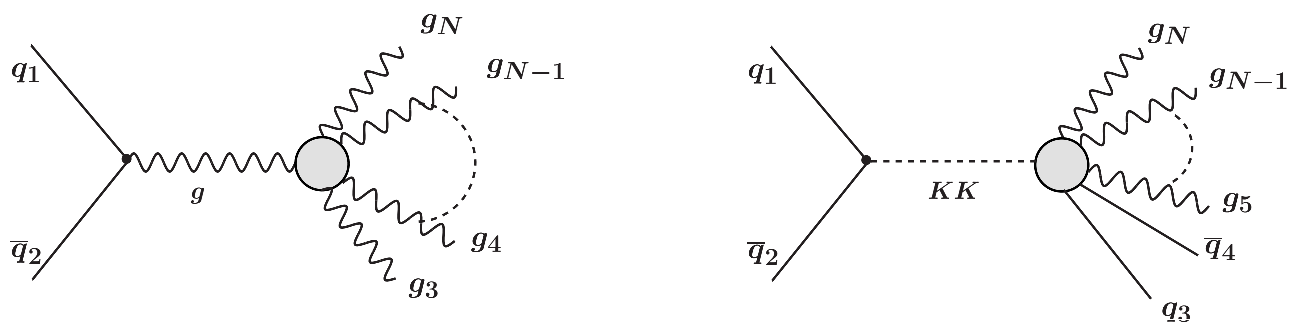


Figure 8.4: Exchange of gluons and Kaluza Klein states in n parton amplitudes.

A possible model dependence would originate from an exchange of a Kaluza Klein- or winding state between the two quarks or squarks and the remaining $n-2$ gluons (or gauginos). However, in contrast to gluons or gauginos, the Kaluza Klein- or winding state carries an internal charge. Such an exchange would violate charge conservation, unless there is at least one quark antiquark pair on the right hand side of the diagram. Hence, amplitudes involving four and more quarks are model dependent due to possible Kaluza Klein- or winding exchange, but the amplitudes (8.2.44) involving no more than one quark antiquark pair share universal properties insensitive to the compactification.

The two amplitudes in the first column of (8.2.44) are related through supersymmetric Ward identities [150]. Similarly, the last two amplitudes of the second column of (2.16) are related through supersymmetry. Since the first set of amplitudes involves only members of a vector multiplet, while the second set also involves chiral multiplets one does not a priori expect a relation between those two sets.

The amplitudes involving more than one quark antiquark pair can be also expressed in terms of $(n-3)!$ functions, but these functions are sensitive to the spectrum of Kaluza Klein excitations, thus to the geometry of extra dimensions, see subsection 5.5 of [101] and sections 6 and 7 of [1].

The fact that both purely gluonic $\mathcal{A}(g_1, g_2, \ldots, g_n)$ and two quark- n gluon amplitudes $\mathcal{A}(q_1, \bar{q}_2, g_3, \ldots, g_n)$, contain the same string formfactors can be explained in the following way. At the disk level, the scattering of gluons and gluinos takes place on a single D–brane stack, say a. On the other hand, quarks are localized on the intersection of stack a with another one, say b, with stack b intersecting stack a at a certain angle ϑ, see section 4.4. One can consider the formal limit $\vartheta \to 0$ which puts stack b on top of stack a, with quarks appearing now as gauginos of the enhanced group associated to the generators $T^a \oplus T^b$. In this limit, the amplitudes involving quarks and gluons fall into the universal class of gluino–gluon amplitudes.

Taking the limit $\vartheta \to 0$ requires some care, however, it becomes particularly simple if only one quark-antiquark pair is present because the corresponding amplitudes are ϑ-independent. This essentially follows from the trivial form $\langle \Xi(z_1)\bar{\Xi}(z_2)\rangle = z_{12}^{-3/4}$ of the two point correlation function of the boundary changing operators.

Thus $\mathcal{A}(q_1, \bar{q}_2, g_3, \dots, g_n)$ amplitudes are related to the universal $\mathcal{A}(g_1, g_2, \dots, g_n)$ amplitudes by a combination of the trivial $\vartheta \to 0$ limit with spacetime supersymmetry. The precise mapping betweeen color stripped amplitudes $\mathcal{A}(\lambda_1, \bar{\lambda}_2, g_3, \dots, g_n)$ and $\mathcal{A}(q_1, \bar{q}_2, g_3, \dots, g_n)$ is a mere adjustment of prefactors $C_{\mathrm{D}_2} g_\lambda^2 \mapsto \tilde{C}_{\mathrm{D}_2} g_\psi^2 = C_{\mathrm{D}_2} g_\lambda^2/2$, i.e. an extra normalization factor 1/2 in the two quark subamplitude relative to its gaugino relative. The universal two quark amplitudes at three- and four point level therefore read

$$\mathcal{A}(g_1, q_2, \bar{q}_3) = g_{\mathrm{YM}} \, (u_2 \, \not{\xi} \, \bar{u}_3) \tag{8.2.45}$$

$$\begin{aligned} \mathcal{A}(g_1, g_2, q_3, \bar{q}_4) &= 2\alpha' \, g_{\mathrm{YM}}^2 \, V_t \left\{ \frac{s}{2} \, (u_3 \, \not{\xi}_2 \, \not{k}_1 \, \not{\xi}_1 \, u_4) \; - \; s \, (\xi_1 \, k_4) \, (u_3 \, \not{\xi}_2 \, u_4) \right. \\ & \left. + \; u \left[(\xi_1 \, \xi_2) \, (u_3 \, \not{k}_1 \, u_4) \; - \; (\xi_2 \, k_1) \, (u_3 \, \not{\xi}_1 \, u_4) \; + \; (\xi_1 \, k_2) \, (u_3 \, \not{\xi}_2 \, u_4) \right] \right\} . \end{aligned} \tag{8.2.46}$$

Moreover, also amplitudes involving more than two chiral fermions boil down to universal gauge amplitudes in the limit $\vartheta \to 0$. In that case, the amplitudes do depend on the intersection angle and the limit $\vartheta \to 0$ has to be taken with some care. At the more technical level, before undertaking this limit some Poisson resummations have to be performed on the instanton part such that the amplitudes are rendered finite in the limit $\vartheta \to 0$. E.g. for quarks from the same intersecting stack the four quark amplitude becomes the four gaugino amplitude in the limit $\vartheta \to 0$

$$\mathcal{A}(q_1, \bar{q}_2, q_3, \bar{q}_4) \quad \stackrel{\vartheta \to 0}{\longrightarrow} \quad \mathcal{A}(\lambda_1, \bar{\lambda}_2, \lambda_3, \bar{\lambda}_4) \; . \tag{8.2.47}$$

The model dependent four quark amplitude $\mathcal{A}(q_1, \bar{q}_2, q_3, \bar{q}_4)$ can be found in [101], and the five parton amplitudes $\mathcal{A}(g_1, q_2, \bar{q}_3, q_4, \bar{q}_5)$ and $\mathcal{A}(g_1, g_2, g_3, q_4, \bar{q}_5)$ have been computed in [1].

8.3 From amplitudes to cross sections

In this section, we present the techniques to obtain squared moduli of disk amplitudes of massless states, summed over helicities and colors of the scattering states. The aim is to cast the result of parton scattering in superstring theory into a form which can be compared with experimental data at LHC, i.e. we will work out full-fledged cross sections.

The sum over polarizations is most conveniently performed using spinor helicity methods (introduced in appendix C). In subsections 8.3.1 and 8.3.2, we give examples of how these $D = 4$

specific variables simplify scattering amplitudes. Particular emphasis is put on the so-called MHV helicity configurations which make the kinematic factors involved collapse into one term.

The color structure of string amplitudes turns out to be richer than in their field theory limit because the Chan Paton traces generically do not reduce to structure constants only, see subsection 8.3.3 or [260] for a discussion of individual α'^k corrections. That is why squaring the five gluon amplitude already requires several novel computations in the color sector which we performed in [1] and present in subsection 8.3.5. Chiral matter transforming in bifundamental representations of the gauge group requires a slightly modified color technology, see subsection 8.3.6.

The $U(N)$ gauge groups on stacks of N D branes give rise to $N^2 - 1$ generators of $SU(N)$ and one additional $U(1)$ generator $T^a \equiv Q\mathbb{1}$. We will negelect these extra gauge bosons in this thesis, their color factors and cross sections are discussed in [101, 1].

8.3.1 Spinor helicity amplitudes

Four dimensions allow to decompose lightlike momenta of massless particles into spinor helicity variables, see appendix C. The main virtue of their bispinor representation is the trivialization of the mass-shell condition $k_i^2 = 0$ which has to be kept as an extra constraint in vector notation. That is why we can expect a lot of simplification by applying the spinor helicity formalism.

For scattering amplitudes involving gluons, the efficiency of the spinor helicity method depends on a clever choice of reference momenta [261, 262]. The assignment should be made such that as many terms as possible vanish in the amplitude. The key identities for this purpose follow directly from (C.1.7)

$$\begin{aligned} q^\mu \, \xi^\pm_\mu(k, q) \;&=\; 0 \\ \xi^\pm(k_1, q) \cdot \xi^\pm(k_2, q) \;&=\; 0 \\ \xi^\pm(k_1, k_2) \cdot \xi^\mp(k_2, q) \;&=\; 0 \; , \end{aligned} \tag{8.3.48}$$

they suggest to use the same reference momentum to all gluon polarizations with the same helicity and that this reference momentum should be the momentum of a gluon with opposite helicity. It is easy to show on grounds of (8.3.48) that n gluon amplitudes of uniform helicity and also those with $n-1$ coinciding helicities vanish:

$$\mathcal{A}(g_1^\pm, g_2^\pm, \ldots, g_n^\pm) \;=\; \mathcal{A}(g_1^\mp, g_2^\pm, \ldots, g_n^\pm) \;=\; 0 \tag{8.3.49}$$

The first nonvanishing class of helicity components $\mathcal{A}(g_1^{\mp}, g_2^{\mp}, g_3^{\pm}, \ldots, g_n^{\pm})$ is called maximally helicity violating (MHV)[3]. One usually reserves the term MHV for amplitudes $\mathcal{A}(g_1^-, g_2^-, g_3^+, \ldots, g_n^+)$ of "mostly +" gluons, whereas their complex conjugates $\mathcal{A}(g_1^+, g_2^+, g_3^-, \ldots, g_n^-)$ with "mostly −" gluons are called $\overline{\text{MHV}}$.

At four point level, the only nonzero helicity amplitude $\mathcal{A}(g_1^-, g_2^-, g_3^+, g_4^+)$ can be most efficiently obtained from (8.1.11) by choosing reference momenta $(3, 3, 2, 2)$. Then, the only nonzero $(\xi_i{\cdot}\xi_j)$ product is $(\xi_1{\cdot}\xi_4)$. Moreover, $q^\mu \xi^\pm_\mu(k, q) = 0$ makes sure that $(\xi_2{\cdot}k_3) = (\xi_3{\cdot}k_2) = 0$ such that only the $(\xi_1 \cdot \xi_4)(\xi_2 \cdot k_1)(\xi_3 \cdot k_4)$ kinematics contributes and the remaining 18 vanish. The end result is

$$\mathcal{A}(g_1^-, g_2^-, g_3^+, g_4^+) \;=\; \frac{4\, g_{\rm YM}^2\, V_t\, \langle 1\,2\rangle^4}{\langle 1\,2\rangle\, \langle 2\,3\rangle\, \langle 3\,4\rangle\, \langle 4\,1\rangle}\;. \tag{8.3.50}$$

A closed formula for the n gluon MHV amplitude in field theory was written down by Parke and Taylor in 1986 [236] and proven by Berends and Giele in 1988 [237],

$$\mathcal{A}^{\rm SYM}(g_1^-, g_2^-, g_3^+, \ldots, g_{n-1}^+, g_n^+) \;=\; \frac{\sqrt{2}^{\,n}\, g_{\rm YM}^{n-2}\, \langle 1\,2\rangle^4}{\langle 1\,2\rangle\, \langle 2\,3\rangle\, \ldots\, \langle n-1\, n\rangle\, \langle n\,1\rangle}\;, \tag{8.3.51}$$

in agreement with the low energy limit $V_t \to 0$ of (8.3.50). Similar expressions exist for two fermion amplitudes with $n-2$ gluons,

$$\mathcal{A}^{\rm SYM}(g_1^-, g_2^+, \ldots, g_{n-2}^+, \lambda_{n-1}, \bar\lambda_n) \;=\; \frac{\sqrt{2}^{\,n}\, g_{\rm YM}^{n-2}\, \langle 1\, n-1\rangle^3\, \langle 1\, n\rangle}{\langle 1\,2\rangle\, \langle 2\,3\rangle\, \ldots\, \langle n-1\, n\rangle\, \langle n\,1\rangle} \tag{8.3.52}$$

they follow for instance from SUSY Ward identities. As explained in the above section 8.2, the corresponding quark subamplitude (both in full superstring theory and in its $\alpha' \to 0$ field theory limit) follows from multiplication with $\frac{1}{2} = \frac{\tilde C_{\rm D_2} g_\psi^2}{C_{\rm D_2} g_\lambda^2}$:

$$\mathcal{A}^{\rm SYM}(g_1^-, g_2^+, \ldots, g_{n-2}^+, q_{n-1}, \bar q_n) \;=\; \frac{(\sqrt{2}\, g_{\rm YM})^{n-2}\, \langle 1\, n-1\rangle^3\, \langle 1\, n\rangle}{\langle 1\,2\rangle\, \langle 2\,3\rangle\, \ldots\, \langle n-1\, n\rangle\, \langle n\,1\rangle} \tag{8.3.53}$$

The corresponding $\overline{\text{MHV}}$ amplitudes are obtained via $\langle ij\rangle \mapsto [ij]$.

Momentum spinors k_a and $k_{\dot a}$ carry charges ± 1 under the stabilizer group $SO(2) \cong U(1)$ of lightlike momenta in four dimensions. The wavefunctions of helicity h accordingly have charge $2h$. The MHV amplitudes (8.3.51), (8.3.52) and (8.3.53) manifest their charges $2h_i$ in the k^i spinor – the cyclic chain $\langle 12\rangle\langle 23\rangle \ldots \langle n1\rangle$ in the denominator contributes $+2$ to each charge, and the numerator factors $\langle 1n-1\rangle^3\langle 1n\rangle$ and $\langle 12\rangle^4$ account for the helicities $+1, +\frac{1}{2}$ and $-\frac{1}{2}$ of g^-, λ and $\bar\lambda$, respectively.

[3]This name can be understood by viewing $[n/2]$ particles in an n point amplitude as outgoing rather than ingoing which reverses the momentum direction and helicity. Helicity conserving amplitudes then contain equal number of g^+ and g^- gluons. The maximal deviation from this helicity conserving configuration (which does not make the amplitude vanish due to (8.3.49)) can be found in $\mathcal{A}(g_1^{\mp}, g_2^{\mp}, g_3^{\pm}, \ldots, g_n^{\pm})$.

The five gluon superstring amplitude (8.1.27) is built from two kinematic factors which evaluate as follows in the helicity basis [201, 1]:

$$\mathcal{A}^{\rm SYM}(g_1^-, g_2^-, g_3^+, g_4^+, g_5^+) \;=\; \frac{\sqrt{2}^5\, g_{\rm YM}^3\, \langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle} \tag{8.3.54}$$

$$\begin{aligned}\mathcal{A}_{\mathcal{F}^4}(g_1^-, g_2^-, g_3^+, g_4^+, g_5^+) \;&=\; \frac{\sqrt{2}^5\, g_{\rm YM}^3\, \langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle} \\ &\times\; \Big(-\frac{1}{2}\,\big(s_{12}s_{23} + \text{cyclic}(12345)\big) \;-\; 2i\,\alpha^2\,\varepsilon_{\mu\nu\lambda\rho}\,k_1^\mu\,k_2^\nu\,k_3^\lambda\,k_4^\rho \Big)\end{aligned} \tag{8.3.55}$$

One can factor out the field theory MHV amplitude. Equivalently, one can obtain the superstring amplitude from (1.4.1) using the two MHV amplitudes $\mathcal{A}^{\rm SYM}(g_1^-, g_2^-, g_3^+ g_4^+, g_5^+)$ and $\mathcal{A}^{\rm SYM}(g_1^-, g_3^+, g_2^-, g_4^+, g_5^+)$. More generally, each superstring MHV amplitude can be expressed in terms of SYM MHV amplitudes using the central result (1.4.1).

8.3.2 MHV versus NMHV amplitudes

The only nonvanishing five point helicity amplitudes arise in either MHV- or $\overline{\text{MHV}}$ configurations. This is very convenient for the computation of cross sections, i.e. for the helicity sum over squared amplitudes. The first instance of nonzero helicity amplitudes beyond MHV and $\overline{\text{MHV}}$ occurs at six point, it falls into the class of so-called next-to-MHV (NMHV) amplitudes [207]:

$$\begin{aligned}\mathcal{A}^{\rm SYM}(g_1^-, g_2^-, g_3^-, g_4^+, g_5^+, g_6^+) \;&=\; \frac{8\,g_{\rm YM}^3}{[2|\,3+4\,|5\rangle}\left\{ \frac{[4|\,5+6\,|1\rangle^3}{[23]\,[34]\,\langle 56\rangle\,\langle 61\rangle\,s_{234}} + \frac{[6|\,1+2\,|3\rangle^3}{[61]\,[12]\,\langle 34\rangle\,\langle 45\rangle\,s_{345}} \right\} \\ =\; 8\,g_{\rm YM}^3 &\left\{ \frac{\langle 23\rangle^3\,[56]^3}{\langle 34\rangle\,[61]\,[1|\,2+3\,|4\rangle\,[5|\,6+1\,|2\rangle\,s_{234}} + \frac{\langle 12\rangle^3\,[45]^3}{\langle 61\rangle\,[34]\,[3|\,4+5\,|6\rangle\,[5|\,6+1\,|2\rangle\,s_{345}} \right. \\ &\left. +\; \frac{s_{123}^2}{[12]\,[23]\,\langle 45\rangle\,\langle 56\rangle\,[1|\,2+3\,|4\rangle\,[3|\,4+5\,|6\rangle} \right\}\end{aligned} \tag{8.3.56}$$

These are two equivalent representations of the six gluon NMHV amplitudes, computed by BCFW recursion relations [263, 208]. The two sides contain different poles of type $[i|j+k|l\rangle$, and their equivalence implies that these poles must be spurious. This is essential in view of locality because $[i|j+k|l\rangle^{-1}$ cannot be reproduced by means of Feynman diagrams due to a local Yang Mills action. An account on the role of locality in scattering amplitudes of $\mathcal{N}=4$ SYM can be found in [99].

Note that there are two inequivalent helicity orderings $\mathcal{A}^{\rm SYM}(g_1^-, g_2^-, g_3^+, g_4^-, g_5^+, g_6^+)$ and $\mathcal{A}^{\rm SYM}(g_1^-, g_2^+, g_3^-, g_4^+, g_5^-, g_6^+)$ which cannot be obtained from (8.3.56) by relabelling or complex conjugation. The first full-fledged six gluon cross section in SYM theories was computed in [262], squaring all nonzero MHV and NMHV contributions.

The complete six gluon amplitude in superstring theory including all NMHV configurations was firstly computed in [200] using SUSY Ward identites. Our recent result (1.4.1) suggests to reduce it to the field theory NMHV building blocks.

8.3.3 Organizing color traces

According to the discussion in subsection 5.3.1, any dependence of disk amplitudes on color degrees of freedom originates from Chan Paton traces $\mathrm{Tr}\{T^{a_1}T^{a_2}\dots T^{a_n}\}$. These traces are not at all optimized for computing the color sum over squared amplitudes for the purpose of cross sections. They have to be rewritten in terms of the structure constants $f^{a_1a_2a_3}$ of the gauge group and symmetrized traces

$$d^{a_1\dots a_p} \;=\; \frac{1}{(p-1)!}\sum_{\rho\in S_{p-1}} \mathrm{Tr}\big\{\, T^{a_{\rho(1)}}\, T^{a_{\rho(2)}} \dots T^{a_{\rho(n-1)}}\, T^{a_n} \,\big\}\,. \tag{8.3.57}$$

Our normalization conventions are fixed by[4]

$$\mathrm{Tr}\big\{\, T^a\, T^b \,\big\} \;=\; \frac{\delta^{ab}}{2}\,, \qquad \big[\, T^a \,,\, T^b \,\big] \;=\; i\, f^{abc}\, T^c\,, \tag{8.3.58}$$

and the two equations (8.3.57) and (8.3.58) in fact comprise all the input needed to decompose any $\mathrm{Tr}\{T^{a_1}T^{a_2}\dots T^{a_n}\}$ into f^{abc} and $d^{b_1\dots b_k}$ (with various $k\le n$). At three- and four point level, for instance, this strategy leads to the rewriting

$$\mathrm{Tr}\big\{T^{a_1}\, T^{a_2}\, T^{a_3}\,\big\} \;=\; d^{a_1a_2a_3} \;+\; \frac{i}{4}\, f^{abc} \tag{8.3.59}$$

$$\begin{aligned}\mathrm{Tr}\big\{T^{a_1}\, T^{a_2}\, T^{a_3}\, T^{a_4}\big\} \;&=\; d^{a_1a_2a_3a_4} \;+\; \frac{i}{2}\,\big(f^{a_2a_3n}\, d^{a_1a_4n} \;-\; f^{a_1a_4n}\, d^{a_2a_3n}\big)\\ &\quad+\;\frac{1}{12}\,\big(f^{a_2a_3n}\, f^{a_1a_4n} \;-\; f^{a_1a_2n}\, f^{a_3a_4n}\big)\,. \end{aligned}\tag{8.3.60}$$

The presentation of the four trace is ambiguous due to Jacobi identities

$$0 \;=\; d^{abe}\, f^{cde} \;+\; d^{bce}\, f^{ade} \;+\; d^{cae}\, f^{bde} \tag{8.3.61}$$

$$0 \;=\; f^{abe}\, f^{cde} \;+\; f^{bce}\, f^{ade} \;+\; f^{cae}\, f^{bde}\,. \tag{8.3.62}$$

The reflection symmetry of subamplitudes makes sure that color factor always combine in pairs

$$c_{\pm}^{a_1a_2\dots a_n} \;:=\; \mathrm{Tr}\big\{\, T^{a_1}\, T^{a_2} \dots T^{a_n} \;\pm\; T^{a_n}\, T^{a_{n-1}} \dots T^{a_1} \,\big\} \tag{8.3.63}$$

[4] Another common convention in the literature is $[T^a, T^b] = \delta^{ab}$ which effectively rescales each generator by $\sqrt{2}$. We needed to take this conversion into account when comparing the results of [1] with the field theory limits in the literature [264].

where the relative sign depends on the number n of legs and the mass levels involved. Both cases project onto certain classes of f^{abc} and $d^{a_1 \dots a_k}$, e.g.

$$
\begin{aligned}
c_+^{a_1 a_2 a_3 a_4} &= 2\, d^{a_1 a_2 a_3 a_4} + \frac{1}{6} \left(f^{a_2 a_3 n}\, f^{a_1 a_4 n} - f^{a_1 a_2 n}\, f^{a_3 a_4 n} \right) && (8.3.64) \\
c_-^{a_1 a_2 a_3 a_4} &= i \left(f^{a_2 a_3 n}\, d^{a_1 a_4 n} - f^{a_1 a_4 n}\, d^{a_2 a_3 n} \right) . && (8.3.65)
\end{aligned}
$$

We will focus on massless subamplitudes with $(-1)^n$ parity, i.e. even (odd) multiplicities n give rise to $c_+^{a_1 \dots a_n}$ ($c_-^{a_1 \dots a_n}$). This was already used in the three- and four point amplitudes of subsections 8.1.1 and 8.1.2.

8.3.4 The group theoretic background

To obtain the color summed modulus square of an n point amplitude, we will meet various invariants due to contractions of $c_\pm^{a_1 a_2 \dots a_n}$. These factors depend on data from the gauge group $\mathcal{G}$, and we will give them as general as possible in the lines of [265, 266].

The normalization (8.3.58) fixes the value of the second index of the fundamental representation $\mathrm{Tr}_{\mathrm{F}} \{ T^a T^b \} = I_2(\mathrm{F}) \delta^{ab}$ to the value $I_2(\mathrm{F}) = \frac{1}{2}$. Further group theoretic quantities of interest are the dimension N_{A} of the adjoint representation and the Casimir operators C_{F} (C_{A}) of the fundamental (adjoint) representation:

$$
\delta^{aa} = N_{\mathrm{A}} \, , \qquad (T^a\, T^a)_{\alpha_1}{}^{\alpha_2} = C_{\mathrm{F}}\, \delta_{\alpha_2}^{\alpha_1} \, , \qquad f^{a_1 bc}\, f^{a_2 bc} = C_{\mathrm{A}}\, \delta^{a_1 a_2} \tag{8.3.66}
$$

It makes sense to decompose the $d^{a_1 a_2 \dots a_n}$ into a traceless part $d_\perp^{a_1 a_2 a_3 a_4} \delta^{a_1 a_2} = 0$ and a trace $\delta^{(a_1 a_2} \delta^{a_3 a_4)}$ as follows:

$$
d^{a_1 a_2 a_3 a_4} = d_\perp^{a_1 a_2 a_3 a_4} + \frac{3\, I_2(\mathrm{F})}{N_{\mathrm{A}} + 2} \left(C_{\mathrm{F}} - \tfrac{C_{\mathrm{A}}}{6} \right) \delta^{(a_1 a_2} \delta^{a_3 a_4)} \tag{8.3.67}
$$

In any representation R of the gauge group $\mathcal{G}$, a symmetrized four-trace splits into a linear combination of the tensors $d_\perp^{a_1 a_2 a_3 a_4}$ and $\delta^{(a_1 a_2} \delta^{a_3 a_4)}$. The coefficients are called the forth indices I_4, $I_{2,2}$ of the corresponding representation,

$$
\mathrm{STr}_{\mathrm{R}} \left\{ T^{a_1}\, T^{a_2}\, T^{a_3}\, T^{a_4} \right\} = I_4(\mathrm{R})\, d_\perp^{a_1 a_2 a_3 a_4} + I_{2,2}(\mathrm{R})\, \delta^{(a_1 a_2} \delta^{a_3 a_4)} \, . \tag{8.3.68}
$$

Comparing (8.3.68) with the fundamental trace (8.3.57) and its decomposition (8.3.67) reveals our normalization convention $I_4(\mathrm{F}) = 1$ for the $d^{a_1 a_2 a_3 a_4}$. Setting an index $I_n(\mathrm{F})$ to another value simply rescales $d^{a_1 a_2 \dots a_n}$. Also, one can read off $I_{2,2}(\mathrm{F}) = \frac{3\, I_2(\mathrm{F})}{N_{\mathrm{A}} + 2} \left(C_{\mathrm{F}} - \frac{C_{\mathrm{A}}}{6} \right)$ from (8.3.67).

Four point cross sections require identities like

$$
f^{a_1 bc}\, f^{a_2 cd}\, f^{a_3 db} = \frac{C_{\mathrm{A}}}{2}\, f^{a_1 a_2 a_3} \tag{8.3.69}
$$

$$d^{abcd}\, d^{abcd} \;=\; d_\perp^{abcd}\, d_\perp^{abcd} \;+\; \frac{3\,N_\mathrm{A}}{4\,(N_\mathrm{A}+2)}\, \Big(C_\mathrm{F} \,-\, \tfrac{C_\mathrm{A}}{6}\Big)^2 \,, \tag{8.3.70}$$

where $d_\perp^{abcd}\, d_\perp^{abcd}$ vanishes for exceptional groups. They imply

$$c_+^{a_1 a_2 a_3 a_4}\, c_+^{a_1 a_2 a_3 a_4} \;=\; 4\, d_\perp^{abcd}\, d_\perp^{abcd} \;+\; \frac{3\,N_\mathrm{A}}{N_\mathrm{A}+2}\, \Big(C_\mathrm{F} \,-\, \tfrac{C_\mathrm{A}}{6}\Big)^2 \;+\; \frac{C_\mathrm{A}^2\, N_\mathrm{A}}{12} \tag{8.3.71}$$

$$c_+^{a_1 a_2 a_3 a_4}\, c_+^{a_1 a_2 a_3 a_4} \;=\; 4\, d_\perp^{abcd}\, d_\perp^{abcd} \;+\; \frac{3\,N_\mathrm{A}}{N_\mathrm{A}+2}\, \Big(C_\mathrm{F} \,-\, \tfrac{C_\mathrm{A}}{6}\Big)^2 \;-\; \frac{C_\mathrm{A}^2\, N_\mathrm{A}}{24} \,, \tag{8.3.72}$$

their special cases for gauge group $\mathcal{G} = SU(N)$ can be found in [101]:

$$c_+^{a_1 a_2 a_3 a_4}\, c_+^{a_1 a_2 a_3 a_4} \Big|_{SU(N)} \;=\; \frac{(N^2-1)\,(N^4-2N^2+6)}{8\,N^2} \tag{8.3.73}$$

$$c_+^{a_1 a_2 a_3 a_4}\, c_+^{a_1 a_2 a_4 a_3} \Big|_{SU(N)} \;=\; \frac{(N^2-1)\,(3-N^2)}{4\,N^2} \tag{8.3.74}$$

Note that the leading N behaviour of the crossterm (8.3.74) is suppressed by a factor of N^{-2} compared to the diagonal product (8.3.73). The suppression of crossterms is a general property of Chan Paton traces, known as color coherence.

8.3.5 Five gluon color factors

As emphasized at the beginning of this section, the color factors for a (massless) five point cross section for adjoint states in superstring theory was computed in [1] for the first time. That is why we present the computation in detail here.

As a starting point, we decompose the relevant Chan Paton traces as

$$\begin{aligned} c_-^{a_1 a_2 a_3 a_4 a_5} \;&=\; \mathrm{Tr}\big\{ T^{a_1}\, T^{a_2}\, T^{a_3}\, T^{a_4}\, T^{a_5} \;-\; T^{a_5}\, T^{a_4}\, T^{a_3}\, T^{a_2}\, T^{a_1} \big\} \\ &=\; i\, f^{a_1 a_2 n} \left(d^{a_3 a_4 a_5 n} \;-\; \frac{1}{12}\, f^{a_3 a_4 m}\, f^{a_5 n m} \right) \;+\; i\, f^{a_1 a_3 n} \left(d^{a_2 a_4 a_5 n} \;-\; \frac{1}{12}\, f^{a_2 a_4 m}\, f^{a_5 n m} \right) \\ &+\; i\, f^{a_2 a_3 n} \left(d^{a_1 a_4 a_5 n} \;-\; \frac{1}{12}\, f^{a_1 a_5 m}\, f^{a_4 n m} \right) \;+\; i\, f^{a_4 a_5 n} \left(d^{a_1 a_2 a_3 n} \;+\; \frac{1}{12}\, f^{a_2 a_3 m}\, f^{a_1 n m} \right) \,. \end{aligned} \tag{8.3.75}$$

Mutual contractions of $c_-^{a_1 a_2 a_3 a_4 a_5} c_-^{a_{\rho(1)} a_{\rho(2)} a_{\rho(3)} a_{\rho(4)} a_{\rho(5)}}$ (with $\rho \in S_5$) receive three different types of contributions each of which requires a specific set of identities:

- $f^3 \times f^3$:

$$\big(f^{a_1 a_2 b}\, f^{a_1 a_2 c}\big) \cdot \big(f^{a_3 a_4 d}\, f^{a_3 a_4 e}\big) \cdot \big(f^{gbd}\, f^{gce}\big) \;=\; C_\mathrm{A}^3\, N_\mathrm{A} \tag{8.3.76}$$

$$\big(f^{a_1 a_2 b}\, f^{a_1 a_2 c}\big) \cdot \big(f^{a_3 de}\, f^{a_4 eg}\, f^{bgd}\big) \cdot f^{a_3 a_4 c} \;=\; \frac{1}{2}\, C_\mathrm{A}^3\, N_\mathrm{A} \tag{8.3.77}$$

$$\big(f^{a_1 bc}\, f^{a_2 cd}\, f^{a_3 db}\big) \cdot \big(f^{a_1 eg}\, f^{a_2 eh}\, f^{a_3 hg}\big) \;=\; \frac{1}{4}\, C_\mathrm{A}^3\, N_\mathrm{A} \tag{8.3.78}$$

$$(f^{a_1a_2a_3}\, f^{b_1b_2b_3}\, f^{c_1c_2c_3}) \cdot (f^{a_1b_1c_1}\, f^{a_2b_2c_2}\, f^{a_3b_3c_3}) \;=\; 0 \tag{8.3.79}$$

The second and third one are due to $f^{a_1bc}f^{a_2cd}f^{a_3db} = \frac{C_\mathrm{A}}{2} f^{a_1a_2a_3}$.

- $fd \times fd$:

$$f^{a_1bc}\, f^{a_2bc}\, d^{a_1def}\, d^{a_2def} \;=\; C_\mathrm{A}\, d^{abcd}\, d^{abcd} \tag{8.3.80}$$

$$f^{a_1a_2b}\, f^{a_3a_4b}\, d^{a_1a_3cd}\, d^{a_2a_4cd} \;=\; \frac{C_\mathrm{A}}{3}\, d^{abcd}\, d^{abcd}\;, \tag{8.3.81}$$

see (8.3.70) for the $d^{abcd}d^{abcd} \leftrightarrow d_\perp^{abcd}d_\perp^{abcd}$ conversion.

- $fd \times f^3$: Since the structure constants f^{abc} are nothing but the matrix elements of T^a in the adjoint representaton, we identify an f^4 chain as a four trace in the adjoint representation of the gauge group $f^{a_1bc}f^{a_2cd}f^{a_3de}f^{a_4eb} = \mathrm{Tr}_\mathrm{A}\{T^{a_1}T^{a_2}T^{a_3}T^{a_4}\}$. Further contraction with a totally symmetric tensor then leads to

$$f^{a_1bc}\, f^{a_2cd}\, f^{a_3de}\, f^{a_4eb}\, d^{a_1a_2a_3a_4} \;=\; d_\mathrm{A}^{a_1a_2a_3a_4}\, d^{a_1a_2a_3a_4}\;. \tag{8.3.82}$$

To reduce this to the fundamental trace contractions $d^{abcd}d^{abcd}$ from (8.3.80), (8.3.81), we need the fourth adjoint indices $I_4(\mathrm{A})$ and $I_{2,2}(\mathrm{A})$ defined by the specialization $d_\mathrm{A}^{abcd} = I_4(\mathrm{A})d_\perp^{abcd} + I_{2,2}(\mathrm{A})\delta^{(ab}\delta^{cd)}$ of (8.3.68) to $\mathrm{R} \equiv \mathrm{A}$. The value $I_{2,2}(\mathrm{A}) = \frac{5C_\mathrm{A}^2}{2(N_\mathrm{A}+2)}$ immediately follows from contraction with δ^{cd}, whereas $I_4(\mathrm{A})$ is tabulated below.

$$d_\mathrm{A}^{a_1a_2a_3a_4}\, d^{a_1a_2a_3a_4} \;=\; I_4(\mathrm{A})\, d_\perp^{abcd}\, d_\perp^{abcd} \;+\; \frac{5\,N_\mathrm{A}\,C_\mathrm{A}^2}{4\,(N_\mathrm{A}+2)}\,\left(C_\mathrm{F} \;-\; \frac{C_\mathrm{A}}{6}\right) \tag{8.3.83}$$

Taking a closer look at the 24 cyclically inequivalent five point subamplitudes (or rather the 12 pairs of opposite ordering), one can identify four distinct invariants in the squaring process,

$$D \;:=\; c_-^{a_1a_2a_3a_4a_5}\, c_-^{a_1a_2a_3a_4a_5} \;=\; \frac{10\,C_\mathrm{A}}{3}\, d^{abcd}\, d^{abcd} \;-\; \frac{1}{3}\, d_\mathrm{A}^{abcd}\, d^{abcd} \;+\; \frac{5\,N_\mathrm{A}\,C_\mathrm{A}^3}{144} \tag{8.3.84}$$

$$X \;:=\; c_-^{a_1a_2a_3a_4a_5}\, c_-^{a_1a_2a_4a_3a_5} \;=\; \frac{4\,C_\mathrm{A}}{3}\, d^{abcd}\, d^{abcd} \;-\; \frac{1}{3}\, d_\mathrm{A}^{abcd}\, d^{abcd} \;-\; \frac{N_\mathrm{A}\,C_\mathrm{A}^3}{144} \tag{8.3.85}$$

$$Y \;:=\; c_-^{a_1a_2a_3a_4a_5}\, c_-^{a_1a_3a_4a_2a_5} \;=\; \frac{2\,C_\mathrm{A}}{3}\, d^{abcd}\, d^{abcd} \;+\; \frac{1}{3}\, d_\mathrm{A}^{abcd}\, d^{abcd} \;-\; \frac{N_\mathrm{A}\,C_\mathrm{A}^3}{72} \tag{8.3.86}$$

$$Z \;:=\; c_-^{a_1a_2a_3a_4a_5}\, c_-^{a_1a_3a_5a_2a_4} \;=\; 0\;, \tag{8.3.87}$$

all the other crossterms can be shown to reduce to these. They take a nice form in terms of the traceless tensors $d_\perp^{abcd}$, see (8.3.67):

$$D \;=\; \frac{10\,C_\mathrm{A} \;-\; I_4(\mathrm{A})}{3}\, d_\perp^{abcd}\, d_\perp^{abcd} \;+\; \frac{5\,N_\mathrm{A}\,C_\mathrm{A}}{4\,(N_\mathrm{A}+2)}\,\left(2\,C_\mathrm{F}^2 \;-\; C_\mathrm{F}\,C_\mathrm{A} \;+\; \frac{(6+N_\mathrm{A})\,C_\mathrm{A}^2}{36}\right) \tag{8.3.88}$$

$$X = \frac{4\,C_{\rm A} \;-\; I_4({\rm A})}{3}\, d_\perp^{abcd}\, d_\perp^{abcd} \;+\; \frac{N_{\rm A}\, C_{\rm A}}{4\,(N_{\rm A}+2)} \left(4\,C_{\rm F}^2 \;-\; 3\,C_{\rm F}\, C_{\rm A} \;+\; \frac{(12-N_{\rm A})\, C_{\rm A}^2}{36} \right) \tag{8.3.89}$$

$$Y = \frac{2\,C_{\rm A} \;+\; I_4({\rm A})}{3}\, d_\perp^{abcd}\, d_\perp^{abcd} \;+\; \frac{N_{\rm A}\, C_{\rm A}}{4\,(N_{\rm A}+2)} \left(2\,C_{\rm F}^2 \;+\; C_{\rm F}\, C_{\rm A} \;-\; \frac{(6+N_{\rm A})\, C_{\rm A}^2}{18} \right) \tag{8.3.90}$$

In the following table 8.1, we list the input data $C_{\rm A}, C_{\rm F}, N_{\rm A}, d_\perp^{abcd} d_\perp^{abcd}$ and $I_4({\rm A})$ as well as the resulting color sums D, X and Y for classical gauge groups $\mathcal{G} \in \{SU(N), SO(N), Sp(N)\}$ at $I_2({\rm F}) = \frac{1}{2}$:

$\mathcal{G}$	$C_{\rm A}$	$C_{\rm F}$	$N_{\rm A}$	$d_\perp^{ijkl}\, d_\perp^{ijkl}$	$I_4({\rm A})$		$D/N_{\rm A}$	$X/N_{\rm A}$	$Y/N_{\rm A}$
$SU(N)$	N	$\frac{N^2-1}{2N}$	N^2-1	$\frac{N_{\rm A}(N_{\rm A}-3)(N_{\rm A}-8)}{96\,(N_{\rm A}+2)}$	$2N$		$\frac{N^4-4N^2+10}{16N}$	$\frac{2-N^2}{8N}$	$\frac{1}{8N}$
$SO(N)$	$\frac{N-2}{2}$	$\frac{N-1}{4}$	$\frac{N(N-1)}{2}$	$\frac{N_{\rm A}(N_{\rm A}-1)(N_{\rm A}-3)}{192\,(N_{\rm A}+2)}$	$N-8$		$\frac{(N-2)(N^2-2N+2)}{128}$	$\frac{(N-2)^2}{128}$	$\frac{N-2}{64}$
$Sp(N)$	$\frac{N+2}{2}$	$\frac{N+1}{4}$	$\frac{N(N+1)}{2}$	$\frac{N_{\rm A}(N_{\rm A}-1)(N_{\rm A}-3)}{192\,(N_{\rm A}+2)}$	$N+8$		$\frac{(N+2)(N^2+2N+2)}{128}$	$-\frac{(N+2)^2}{128}$	$\frac{N+2}{64}$

Table 8.1: Group theoretical factors in the five gluon cross section, evaluated for semi simple gauge groups generated by stacks of D branes

A hierarchy $D \gg X \gg Y$ emerges in the large N limit. With $\mathcal{G} = SO(N), Sp(N)$, their relation is $Y^{SO(N),Sp(N)} \sim \frac{1}{N} X^{SO(N),Sp(N)} \sim \frac{1}{N^2} D^{SO(N),Sp(N)}$ as $N \to \infty$ whereas $SU(N)$ even introduces relative factors of $\frac{1}{N^2}$, namely $Y^{SU(N)} \sim \frac{1}{N^2} X^{SU(N)} \sim \frac{1}{N^4} D^{SU(N)}$.

8.3.6 Color factors involving chiral matter

We have already pointed out in subsection 4.4.1 that open string states located at brane intersections carry different Chan Patons degrees of freedom. The Chan Paton matrices for chial matter carry bifundamental indices (T^α_β) rather than the standard adjoint ones T^a, and their entries are given by $(T^{\alpha_1}_{\beta_1})_{\alpha_2}{}^{\beta_2} = \delta^{\alpha_1}_{\alpha_2} \delta^{\beta_2}_{\beta_1}$. The α and β indices label the fundamental color degrees of freedom associated with the gauge groups living the two intersecting stacks a and b of D branes.

As a consequence, the color factors of universal amplitudes $\mathcal{A}(q_1, \bar q_2, g_3, \ldots, g_n)$ involving two chiral fermions are sensitive on the distribution of the gluons over the individual stacks a and b, respectively

$${\rm Tr}\big\{\, T^{a_1} \, \ldots \, T^{a_k}\, T^{\alpha_1}_{\beta_1}\, T^{b_1} \, \ldots \, T^{b_l}\, T^{\beta_2}_{\alpha_2} \,\big\} \;=\; (T^{a_1} \, \ldots \, T^{a_k})_{\alpha_2}{}^{\alpha_1}\, (T^{b_1} \, \ldots \, T^{b_l})_{\beta_1}{}^{\beta_2} \; . \tag{8.3.91}$$

Not all Chan Paton matrix products give rise to a compatible color index structure: Products of the type $\dots T^a T^\alpha_\beta T^b \dots$ yield sensible results like (8.3.91) whereas color traces involving $\dots T^b T^\alpha_\beta \dots$ or $\dots T^\alpha_\beta T^a \dots$ identically vanish. In other words – the boundary changing matrix T^α_β can only be multiplied by T^a or T^γ_α factors from the left (and by T^b or T^β_γ from the right). Although the worldsheet integrals in their associated color stripped amplitudes are well defined and nonvanishing, they are suppressed by a zero Chan Paton trace. That is why the ordering $(1,3,2)$ does not contribute to the following color dressed three point amplitude:

$$\mathcal{M}\big[(T^{a_1},g_1),(T^{\alpha_2}_{\beta_2},q_2),(T^{\beta_3}_{\alpha_3},\bar{q}_3)\big] \;=\; (T^{a_1})_{\alpha_3}{}^{\alpha_2}\,\delta^{\beta_3}_{\beta_2}\,\mathcal{A}(g_1,q_2,\bar{q}_3) \tag{8.3.92}$$

Four point amplitudes involving two gluons and two quarks can have two different color structures: If both gluons are living on the same a stack of D branes, then two subamplitudes contribute to the color dressed amplitude:

$$\begin{aligned}\mathcal{M}\big[(T^{a_1},g_1),(T^{a_2},g_2),(T^{\alpha_3}_{\beta_3},q_3),(T^{\beta_4}_{\alpha_4},\bar{q}_4)\big] \;&=\; (T^{a_1}\,T^{a_2})_{\alpha_4}{}^{\alpha_3}\,\delta^{\beta_4}_{\beta_3}\,\mathcal{A}(g_1,g_2,q_3,\bar{q}_4)\\ &+\;(T^{a_2}\,T^{a_1})_{\alpha_4}{}^{\alpha_3}\,\delta^{\beta_4}_{\beta_3}\,\mathcal{A}(g_1,g_2,q_3,\bar{q}_4)\end{aligned} \tag{8.3.93}$$

If the two gluons originate from different stacks, then there is just one color ordering with nonzero Chan Paton trace:

$$\mathcal{M}\big[(T^{a},g_1),(T^{b},g_2),(T^{\alpha_3}_{\beta_3},q_3),(T^{\beta_4}_{\alpha_4},\bar{q}_4)\big] \;=\; (T^{a})_{\alpha_4}{}^{\alpha_3}\,(T^{b})_{\beta_3}{}^{\beta_4}\,\mathcal{A}(g_1,q_3,g_2,\bar{q}_4) \tag{8.3.94}$$

The following figure 8.5 illustrates the flow of Chan Paton degrees of freedom along the open strings streching between intersecting branes:

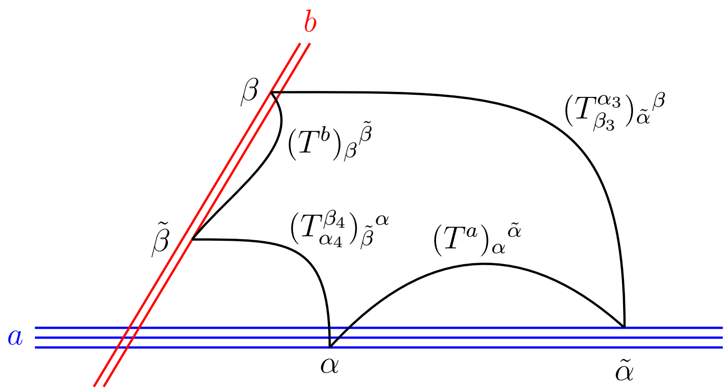


Figure 8.5: Flow of Chan Paton charges in a two quark- two gluon amplitude (with one gluon located at stack a and b each) resulting in $(T^a)_\alpha{}^{\tilde{\alpha}}(T^{\alpha_3}_{\beta_3})_{\tilde{\alpha}}{}^{\beta}(T^b)_\beta{}^{\tilde{\beta}}(T^{\beta_4}_{\alpha_4})_{\tilde{\beta}}{}^{\alpha} = (T^a)_{\alpha_4}{}^{\alpha_3}(T^b)_{\beta_3}{}^{\beta_4}$

The simplest color sums are evaluated as

$$(T^{a_1})_{\alpha_3}{}^{\alpha_2}\,\delta^{\beta_3}_{\beta_2}\,\big[(T^{a_1})_{\alpha_3}{}^{\alpha_2}\,\delta^{\beta_3}_{\beta_2}\big]^* \;=\; \big[N\,C_{\rm F}\big]_a\,[N]_b$$

$$\begin{aligned}
(T^{a_1}\,T^{a_2})_{\alpha_4}{}^{\alpha_3}\,\delta^{\beta_4}_{\beta_3}\,\big[(T^{a_1}\,T^{a_2})_{\alpha_4}{}^{\alpha_3}\,\delta^{\beta_4}_{\beta_3}\big]^* &= \big[N\,C_{\rm F}^2\big]_a\,[N]_b \\
(T^{a_1}\,T^{a_2})_{\alpha_4}{}^{\alpha_3}\,\delta^{\beta_4}_{\beta_3}\,\big[(T^{a_2}\,T^{a_1})_{\alpha_4}{}^{\alpha_3}\,\delta^{\beta_4}_{\beta_3}\big]^* &= \big[N\,C_{\rm F}\,\big(C_{\rm F}\,-\,\tfrac{1}{2}\,C_{\rm A}\big)\big]_a\,[N]_b \\
(T^{a})_{\alpha_4}{}^{\alpha_3}\,(T^{b})_{\beta_3}{}^{\beta_4}\,\big[(T^{a})_{\alpha_4}{}^{\alpha_3}\,(T^{b})_{\beta_3}{}^{\beta_4}\big]^* &= \big[N\,C_{\rm F}\big]_a\,\big[N\,C_{\rm F}\big]_b
\end{aligned} \tag{8.3.95}$$

in terms of the Casimirs $[C_{\rm A}, C_{\rm F}]_j$ and the number of colors $[N]_j$ of the gauge group on stack $j = a, b$. The group constants can be read out from table 8.1, the $SU(N)$ results are given by

$$\begin{aligned}
(T^{a_1})_{\alpha_3}{}^{\alpha_2}\,\delta^{\beta_3}_{\beta_2}\,\big[(T^{a_1})_{\alpha_3}{}^{\alpha_2}\,\delta^{\beta_3}_{\beta_2}\big]^*\Big|_{SU(N)} &= \frac{1}{2}\,(N_a^2-1)\,N_b \\
(T^{a_1}\,T^{a_2})_{\alpha_4}{}^{\alpha_3}\,\delta^{\beta_4}_{\beta_3}\,\big[(T^{a_1}\,T^{a_2})_{\alpha_4}{}^{\alpha_3}\,\delta^{\beta_4}_{\beta_3}\big]^*\Big|_{SU(N)} &= \frac{1}{4\,N_a}\,(N_a^2-1)^2\,N_b \\
(T^{a_1}\,T^{a_2})_{\alpha_4}{}^{\alpha_3}\,\delta^{\beta_4}_{\beta_3}\,\big[(T^{a_2}\,T^{a_1})_{\alpha_4}{}^{\alpha_3}\,\delta^{\beta_4}_{\beta_3}\big]^*\Big|_{SU(N)} &= -\,\frac{1}{4\,N_a}\,(N_a^2-1)\,N_b \\
(T^{a})_{\alpha_4}{}^{\alpha_3}\,(T^{b})_{\beta_3}{}^{\beta_4}\,\big[(T^{a})_{\alpha_4}{}^{\alpha_3}\,(T^{b})_{\beta_3}{}^{\beta_4}\big]^*\Big|_{SU(N)} &= \frac{1}{4}\,(N_a^2-1)\,(N_b^2-1)\ .
\end{aligned} \tag{8.3.96}$$

The model independent five parton amplitude $\mathcal{A}(g_1, g_2, g_3, q_4, \bar{q}_5)$ is accompanied by Chan Paton traces $(T^{a_{\rho(1)}}T^{a_{\rho(2)}}T^{a_{\rho(3)}})_{\alpha_5}{}^{\alpha_4}\delta^{\beta_5}_{\beta_4}$ with $\rho \in S_3$ or $(T^{a_{\rho(1)}}T^{a_{\rho(2)}})_{\alpha_5}{}^{\alpha_4}(T^b)_{\beta_4}{}^{\beta_5}$ with $\rho \in S_2$. Its cross section being among the universal stringy predictions for LHC observables motivates to compute the following color sums:

$$\begin{aligned}
D_q \;&:=\; (T^{a_1}\,T^{a_2}\,T^{a_3})_{\alpha_5}{}^{\alpha_4}\,\big[(T^{a_1}\,T^{a_2}\,T^{a_3})_{\alpha_5}{}^{\alpha_4}\big]^* \;=\; \big[N\,C_{\rm F}^3\big]_a \\
X_q \;&:=\; (T^{a_1}\,T^{a_2}\,T^{a_3})_{\alpha_5}{}^{\alpha_4}\,\big[(T^{a_1}\,T^{a_3}\,T^{a_2})_{\alpha_5}{}^{\alpha_4}\big]^* \;=\; \big[N\,C_{\rm F}^2\,\big(C_{\rm F}\,-\,\tfrac{1}{2}\,C_{\rm A}\big)\big]_a \\
Y_q \;&:=\; (T^{a_1}\,T^{a_2}\,T^{a_3})_{\alpha_5}{}^{\alpha_4}\,\big[(T^{a_2}\,T^{a_3}\,T^{a_1})_{\alpha_5}{}^{\alpha_4}\big]^* \;=\; \Big[N\,C_{\rm F}\,\big(C_{\rm F}\,-\,\tfrac{1}{2}\,C_{\rm A}\big)^2\Big]_a \\
Z_q \;&:=\; (T^{a_1}\,T^{a_2}\,T^{a_3})_{\alpha_5}{}^{\alpha_4}\,\big[(T^{a_3}\,T^{a_2}\,T^{a_1})_{\alpha_5}{}^{\alpha_4}\big]^* \;=\; \big[N\,C_{\rm F}\,\big(C_{\rm F}\,-\,\tfrac{1}{2}\,C_{\rm A}\big)\,(C_{\rm F}\,-\,C_{\rm A})\big]_a
\end{aligned} \tag{8.3.97}$$

A second class of color factors is due to gauge bosons on different stacks of D branes:

$$\begin{aligned}
(T^{a_1}\,T^{a_2})_{\alpha_5}{}^{\alpha_4}(T^b)_{\beta_4}{}^{\beta_5}\,\big[(T^{a_1}\,T^{a_2})_{\alpha_5}{}^{\alpha_4}(T^b)_{\beta_4}{}^{\beta_5}\big]^* &= \big[N\,C_{\rm F}^2\big]_a\,\big[N\,C_{\rm F}\big]_b \\
(T^{a_1}\,T^{a_2})_{\alpha_5}{}^{\alpha_4}(T^b)_{\beta_4}{}^{\beta_5}\,\big[(T^{a_2}\,T^{a_1})_{\alpha_5}{}^{\alpha_4}(T^b)_{\beta_4}{}^{\beta_5}\big]^* &= \big[N\,C_{\rm F}\,\big(C_{\rm F}\,-\,\tfrac{1}{2}\,C_{\rm A}\big)\big]_a\,\big[N\,C_{\rm F}\big]_b
\end{aligned} \tag{8.3.98}$$

8.4 Four- and five parton cross sections

In this section, we list cross sections for the model independent four- and five parton processes $\mathcal{A}(g_1, g_2, g_3, g_4)$, $\mathcal{A}(g_1, g_2, q_3, \bar{q}_4)$, $\mathcal{A}(g_1, g_2, g_3, g_4, g_5)$ and $\mathcal{A}(g_1, g_2, g_3, q_4, \bar{q}_5)$, summed over helicities h_i and colors a_i of the external particles. We will use the notation

$$\big|\,\mathcal{M}[\Phi_1, \Phi_2, \ldots, \Phi_n]\,\big|^2 \;:=\; \sum_{a_i, h_i}\big|\,\mathcal{M}\big[(T^{a_1}, \Phi_1), (T^{a_2}, \Phi_2), \ldots, (T^{a_n}, \Phi_n)\big]\,\big|^2\ . \tag{8.4.99}$$

As mentioned above, the $U(1)$ gauge bosons from the $U(N)$ gauge groups on stacks of N coinciding branes are neglected, i.e. we assume $\mathcal{G}_j \equiv SU(N_j)$ for $j = a, b$. The gauge bosons scattering with a quark antiquark pair can be associated to two different stacks of D branes with gauge groups $\mathcal{G}_a$ and $\mathcal{G}_b$.

8.4.1 Four parton cross sections

As the basic ingredients for four parton cross sections, we need the squares of the following helicity amplitudes:

$$\begin{aligned} \mathcal{M}\big[(T^{a_1}, g_1^-), (T^{a_2}, g_2^-), (T^{a_3}, g_3^+), (T^{a_4}, g_4^+)\big] \;&=\; 4\, g_{\rm YM}^2 \,\langle 1\,2\rangle^4 \\ &\left\{ \frac{V_t\, c_+^{a_1a_2a_3a_4}}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle} \,+\, \frac{V_s\, c_+^{a_1a_3a_2a_4}}{\langle 1\,3\rangle\,\langle 3\,2\rangle\,\langle 2\,4\rangle\,\langle 4\,1\rangle} \,+\, \frac{V_u\, c_+^{a_1a_3a_4a_2}}{\langle 1\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,2\rangle\,\langle 2\,1\rangle} \right\} \end{aligned} \tag{8.4.100}$$

$$\begin{aligned} \mathcal{M}\big[(T^{a_1}, g_1^-), (T^{a_2}, g_2^+), (T^{\alpha_3}_{\beta_3}, q_3), (T^{\beta_4}_{\alpha_4}, \bar q_4)\big] \;&=\; 2\, g_{\rm YM}^2 \,\langle 1\,3\rangle^3\,\langle 1\,4\rangle \\ &\left\{ \frac{V_t\, (T^{a_1}\,T^{a_2})_{\alpha_4}{}^{\alpha_3}\,\delta^{\beta_4}_{\beta_3}}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle} \,+\, \frac{V_u\, (T^{a_2}\,T^{a_1})_{\alpha_4}{}^{\alpha_3}\,\delta^{\beta_4}_{\beta_3}}{\langle 1\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,2\rangle\,\langle 2\,1\rangle} \right\} \end{aligned} \tag{8.4.101}$$

$$\mathcal{M}\big[(T^{a}, g_1^-), (T^{b}, g_2^+), (T^{\alpha_3}_{\beta_3}, q_3), (T^{\beta_4}_{\alpha_4}, \bar q_4)\big] \;=\; 2\, g^a_{\rm YM}\, g^b_{\rm YM} \,\langle 1\,3\rangle^3\,\langle 1\,4\rangle\, \frac{V_s\, (T^a)_{\alpha_4}{}^{\alpha_3}\,(T^b)_{\beta_3}{}^{\beta_4}}{\langle 1\,3\rangle\,\langle 3\,2\rangle\,\langle 2\,4\rangle\,\langle 4\,1\rangle} \tag{8.4.102}$$

The remaining helicity configurations follow from performing crossing operations with the prefactors $\langle 1\,2\rangle^4$ or $\langle 1\,3\rangle^3\,\langle 1\,4\rangle$, e.g. $\mathcal{M}[g_1^-, g_2^+, g_3^-, g_4^+] = \frac{\langle 13\rangle^4}{\langle 12\rangle^4}\mathcal{M}[g_1^-, g_2^-, g_3^+, g_4^+]$.

The color identities (8.3.73) and (8.3.74) together with $\big|\langle ij\rangle\big|^2 = 2k_i\cdot k_j = s_{ij}/\alpha'$ yield a four gluon cross section

$$\begin{aligned} \big|\,\mathcal{M}[g_1, g_2, g_3, g_4]\,\big|^2 \;&=\; 8\,(N^2-1)\,g_{\rm YM}^4\,\left(\frac{1}{s^2} + \frac{1}{t^2} + \frac{1}{u^2}\right) \\ &\times\,\left[N^2\,(s^2\,V_s^2 \,+\, t^2\,V_t^2 \,+\, u^2\,V_u^2) \,+\, \left(\frac{6}{N^2} \,-\, 2\right)(s\,V_s \,+\, t\,V_t \,+\, u\,V_u)^2 \right]\,. \end{aligned} \tag{8.4.103}$$

When squaring the amplitudes for two gluons and a quark antiquark pair, we only sum over the gluon helicities. As a result of the color factors (8.3.96), we get the following cross section for the process with two gluons on stack a (with gauge coupling $g^a_{\rm YM}$):

$$\begin{aligned} \big|\,\mathcal{M}[g^a_1, g^a_2, q_3, \bar q_4]\,\big|^2 \;&=\; \frac{(g^a_{\rm YM})^4\,N_b\,(N_a^2-1)}{N_a}\,\frac{t^2+u^2}{s^2} \\ &\times\,\left[\frac{N_a^2-1}{ut}\,(t\,V_t \,+\, u\,V_u)^2 \,-\, 2\,N_a^2\,(N_a^2-1)\,V_t\,V_u \right] \end{aligned} \tag{8.4.104}$$

The entire process takes place on the stack a while stack b is a mere spectator providing the overall factor N_b. The sum over quark helicities (which is not carried out in (8.4.104)) requires some attention because left- and right-handed quarks originate from different stacks. This can be handled by adding contributions from both stacks, with the net result of doubling the chiral amplitude (8.4.104) and replacing N_b by the number of quark flavors.

The cross section for two gluons live on different stacks a and b is given by

$$\big|\,\mathcal{M}[g_1^a, g_2^b, q_3, \bar{q}_4]\,\big|^2 \;=\; (g^a_{\rm YM})^2\,(g^b_{\rm YM})^2\,(N_a^2-1)\,(N_b^2-1)V_s^2\,\frac{t^2+u^2}{tu}\;. \qquad (8.4.105)$$

8.4.2 Five gluon cross sections

We have argued in subsection 8.3.1 that the five gluon MHV amplitude in superstring theory can be written as its field theory limit (8.3.54), multiplied by a five point formfactor carrying the α' dependence. It comprises two hypergeometric functions f_1, f_2 which were defined in subsection 8.1.5:

$$\mathcal{A}(g_1^-, g_2^-, g_3^+, g_4^+, g_5^+) \;=\; \Big(s_{23}\,s_{51}\,f_1 \;+\; \alpha'^2\,[1\,2]\,\langle 2\,3\rangle\,[3\,5]\,\langle 5\,1\rangle\,f_2\Big)\,\mathcal{A}^{\rm SYM}(g_1^-, g_2^-, g_3^+, g_4^+, g_5^+) \qquad (8.4.106)$$

The result (8.4.106) has a nice factorized form, with the string effects succinctly extracted in a single factor multiplying the SYM amplitude. In order to discuss other orderings, it is convenient to introduce the kinematic function

$$\mathcal{C}(k_1,k_2,k_3,k_4,k_5) \;:=\; \frac{s_{23}\,s_{51}\,f_1 \;+\; \alpha'^2\,[1\,2]\,\langle 2\,3\rangle\,[3\,5]\,\langle 5\,1\rangle\,f_2}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle} \qquad (8.4.107)$$

such that

$$\mathcal{A}(g_1^-, g_2^-, g_3^+, g_4^+, g_5^+) \;=\; 4\sqrt{2}\;g_{\rm YM}^3\,\langle 1\,2\rangle^4\,\mathcal{C}(k_1,k_2,k_3,k_4,k_5)\;. \qquad (8.4.108)$$

Note that the function $\mathcal{C}$ is even under cyclic permutations of the momenta and odd under mirror reflections $(1,2,3,4,5)\to(5,4,3,2,1)$. The amplitudes associated to other orderings (for the same helicity configuration) can be obtained from (8.4.108) by permuting momenta inside the $\mathcal{C}$-function, $(k_1,k_2,k_3,k_4,k_5)\to(k_{1_\rho},k_{2_\rho},k_{3_\rho},k_{4_\rho},k_{5_\rho})$, where $\rho\in S_5$ and $i_\rho\equiv\rho(i)$. Note that the "helicity factor" $\langle 12\rangle^4$ remains intact. Due to antisymmetry of $\mathcal{C}$-functions under mirror reflections, the color dressed amplitude can be written as a sum over twelve terms

$$\begin{aligned}\mathcal{M}\big[&(T^{a_1},g_1^-),(T^{a_2},g_2^-),(T^{a_3},g_3^+),(T^{a_4},g_4^+),(T^{a_5},g_5^+)\big]\\ &=\;4\sqrt{2}\,g_{\rm YM}^3\,\langle 1\,2\rangle^4\sum_{\rho\in\Pi_5}\mathcal{C}(k_{1_\rho},k_{2_\rho},k_{3_\rho},k_{4_\rho},k_{5_\rho})\,c_-^{a_{1_\rho}a_{2_\rho}a_{3_\rho}a_{4_\rho}a_{5_\rho}}\;,\end{aligned} \qquad (8.4.109)$$

where $c_-^{a_1a_2a_3a_4a_5}$ is defined by (8.3.75) and $\Pi_5\equiv S_5/\mathbb{Z}_2$:

$$\Pi_5 \;\equiv\; \Big\{(1,2,3,4,5),\;(1,2,4,3,5),\;(1,3,4,2,5),\;(1,3,2,4,5),\;(1,4,2,3,5),\;(1,4,3,2,5),$$

$$(2,1,3,4,5),\ (2,1,4,3,5),\ (2,3,1,4,5),\ (2,4,1,3,5),\ (3,1,2,4,5),\ (3,2,1,4,5)\Big\} \quad (8.4.110)$$

All other "mostly plus" amplitudes, with the two negative helicity gluons labeled by arbitrary i and j instead of 1 and 2, can be obtained from (8.4.109) by simply replacing the helicity factor $\langle 12\rangle^4 \to \langle i^+ j^+\rangle^4$. "Mostly minus" amplitudes are obtained by complex conjugation.

Let us now take the modulus squared of (8.4.109), sum over all helicity configurations and color (adjoint) indices of five gluons:

$$\big|\,\mathcal{M}[g_1,g_2,g_3,g_4,g_5]\,\big|^2 \;=\; 64\, g_{\mathrm{YM}}^6\, \alpha'^{-4} \left(\sum_{i<j} s_{ij}^4\right) \sum_{\lambda,\lambda'\in\Pi_5} \mathcal{C}_\lambda\, \mathcal{S}_{\lambda\lambda'}\, \mathcal{C}^*_{\lambda'} \tag{8.4.111}$$

We have introduced shorthands for the twelve-vector of $\mathcal{C}$ permutations

$$\mathcal{C}_\lambda \;\equiv\; \mathcal{C}(k_{1_\lambda},k_{2_\lambda},k_{3_\lambda},k_{4_\lambda},k_{5_\lambda}) \tag{8.4.112}$$

and the color matrix

$$\mathcal{S}_{\lambda\lambda'} \;=\; \sum_{a_1,\dots,a_5} c_-^{a_{1_\lambda}a_{2_\lambda}a_{3_\lambda}a_{4_\lambda}a_{5_\lambda}} \left(c_-^{a_{1_{\lambda'}}a_{2_{\lambda'}}a_{3_{\lambda'}}a_{4_{\lambda'}}a_{5_{\lambda'}}}\right)^* . \tag{8.4.113}$$

The matrix elements can be evaluated by using the methods of subsections 8.3.5.

The squared amplitude (8.4.111) can be further simplified by expressing all $\mathcal{C}$-functions in terms of a two element basis. They satisfy the same monodromy relations as the full amplitude, see (5.4.46) and [84].

8.4.3 Three gluon- two quark cross sections

Also the three gluon two quark MHV amplitude has a simple relation to its field theory limit,

$$\mathcal{A}(g_1^-,g_2^+,g_3^+,q_4,\bar q_5) \;=\; \big(\, s_{23}\, s_{51}\, f_1 \;+\; \alpha'^2\, [1\,2]\,\langle 2\,3\rangle\,[3\,5]\,\langle 5\,1\rangle\, f_2\,\big)\, \mathcal{A}^{\mathrm{SYM}}(g_1^-,g_2^+,g_3^+,q_4,\bar q_5)\ , \tag{8.4.114}$$

and the prefactor is given by exactly the same function as it appears in the five-gluon amplitude (8.4.106). By using the $\mathcal{C}$-functions defined in equation (8.4.107), the amplitude (8.4.114) can be written as

$$\mathcal{A}(g_1^-,g_2^+,g_3^+,q_4,\bar q_5) \;=\; 2\sqrt{2}\ g_{\mathrm{YM}}^3\, \langle 1\,4\rangle^3\, \langle 1\,5\rangle\ \mathcal{C}(k_1,k_2,k_3,k_4,k_5)\ . \tag{8.4.115}$$

In order to obtain the full amplitude describing a given helicity configuration, we need the amplitudes associated to all other Chan-Paton factors, with the orderings $\rho = (1_\rho,2_\rho,3_\rho,4_\rho,5_\rho)$, where ρ are the relevant permutations of $(1,2,3,4,5)$ and $i_\rho := \rho(i)$. For all three gluons associated to a single D-brane, these permutations are (1,3,2,4,5), (2,1,3,4,5), (2,3,1,4,5), (3,1,2,4,5)

	(1234)	(1243)	(1342)	(1324)	(1423)	(1432)	(2134)	(2143)	(2314)	(2413)	(3124)	(3214)
(1234)	D	$+X$	$+Y$	$+X$	$+Y$	$-X$	$+X$	$-Y$	$+Y$	0	$+Y$	$-X$
(1243)	$+X$	D	$-X$	$+Y$	$+X$	$+Y$	$-Y$	$+X$	0	$+Y$	$+X$	$-Y$
(1342)	$+Y$	$-X$	D	$+X$	$+Y$	$+X$	$+X$	$-Y$	$-Y$	$-X$	$-Y$	0
(1324)	$+X$	$+Y$	$+X$	D	$-X$	$+Y$	$+Y$	0	$-X$	$+Y$	$+X$	$+Y$
(1423)	$+Y$	$+X$	$+Y$	$-X$	D	$+X$	0	$+Y$	$+Y$	$-X$	$-Y$	$-X$
(1432)	$-X$	$+Y$	$+X$	$+Y$	$+X$	D	$-Y$	$+X$	$-X$	$-Y$	0	$+Y$
(2134)	$+X$	$-Y$	$+X$	$+Y$	0	$-Y$	D	$+X$	$+X$	$+Y$	$-X$	$+Y$
(2143)	$-Y$	$+X$	$-Y$	0	$+Y$	$+X$	$+X$	D	$+Y$	$+X$	$-Y$	$+X$
(2314)	$+Y$	0	$-Y$	$-X$	$+Y$	$-X$	$+X$	$+Y$	D	$-X$	$+Y$	$+X$
(2413)	0	$+Y$	$-X$	$+Y$	$-X$	$-Y$	$+Y$	$+X$	$-X$	D	$-X$	$-Y$
(3124)	$+Y$	$+X$	$-Y$	$+X$	$-Y$	0	$-X$	$-Y$	$+Y$	$-X$	D	$+X$
(3214)	$-X$	$-Y$	0	$+Y$	$-X$	$+Y$	$+Y$	$+X$	$+X$	$-Y$	$+X$	D

Table 8.2: Matrix elements $\mathcal{S}_{\lambda\lambda'}$. Since the elements of the permutation set Π_5 defined by (8.4.110) have the common last number equal 5, the rows and columns are labeled by first four numbers. The entries D, X and Y depend on the gauge group and are listed in table 8.1. Note that the matrix is symmetric.

and (3,2,1,4,5). By explicitly evaluating the amplitudes associated with various orderings of the vertex operators, we find a striking result that they can be obtained from the original amplitude (8.4.115) by simply permuting the arguments of the function $\mathcal{C}$, with the same permutation applied inside the Chan-Paton factor. Hence the full amplitude, obtained by summing all orderings, is given by

$$\begin{aligned}\mathcal{M}\big[&(T^{a_1},g_1^-),(T^{a_2},g_2^+),(T^{a_3},g_3^+),(T^{\alpha_4}_{\beta_4},q_4),(T^{\beta_5}_{\alpha_5},\bar{q}_5)\big]\\ &= 2\sqrt{2}\,g^3_{\rm YM}\,\langle 1\,4\rangle^3\,\langle 1\,5\rangle \sum_{\rho\in\Pi_q}\mathcal{C}(k_{1_\rho},k_{2_\rho},k_{3_\rho},k_{4_\rho},k_{5_\rho})\,(T^{a_{1\rho}}\,T^{a_{2\rho}}\,T^{a_{3\rho}})_{\alpha_5}{}^{\alpha_4}\,\delta_{\beta_4}{}^{\beta_5}\end{aligned} \tag{8.4.116}$$

where

$$\Pi_q \;\equiv\; \Big\{(1,2,3,4,5),\;(1,3,2,4,5),\;(2,1,3,4,5),\;(2,3,1,4,5),\;(3,1,2,4,5),\;(3,2,1,4,5)\Big\}\,. \tag{8.4.117}$$

If one of gauge bosons originates from a different stack, say the boson numbered by '3' is from stack b, then the Chan-Paton factor becomes $(T^{a_1}T^{a_2})_{\alpha_5}{}^{\alpha_4}(T^b)_{\beta_4}{}^{\beta_5}$ and the corresponding amplitude reads

$$\mathcal{M}\big[(T^{a_1},g_1^-),(T^{a_2},g_2^+),(T^{b},g_3^+),(T^{\alpha_4}_{\beta_4},q_4),(T^{\beta_5}_{\alpha_5},\bar{q}_5)\big] \;=\; 2\sqrt{2}\,(g^a_{\rm YM})^2\,g^b_{\rm YM}\,\langle 1\,4\rangle^3\,\langle 1\,5\rangle$$

$$\Big\{ \mathcal{C}(k_1,k_2,k_4,k_3,k_5)\,(T^{a_1}T^{a_2})_{\alpha_5}{}^{\alpha_4}\,(T^b)_{\beta_4}{}^{\beta_5} \;+\; \mathcal{C}(k_2,k_1,k_4,k_3,k_5)\,(T^{a_2}T^{a_1})_{\alpha_5}{}^{\alpha_4}\,(T^b)_{\beta_4}{}^{\beta_5} \Big\}\,. \tag{8.4.118}$$

As in the case of five gluon amplitudes, the dependence of the full amplitudes (8.4.116) and (8.4.118) on the helicity configuration is limited to the $\langle 14\rangle^3\langle 15\rangle$ factor. All other "mostly plus" amplitudes, with the single negative helicity gluon labeled by arbitrary i instead of 1, can be obtained from (8.4.116) by simply replacing the helicity factor $\langle 14\rangle^3\langle 15\rangle \to \langle i4\rangle^3\langle i5\rangle$. "Mostly minus" amplitudes are obtained by complex conjugation.

Let us now take the modulus squared of (8.4.116) and (8.4.118), summing over all helicity configurations and over color indices of quarks and gluons. If all the three gluons originate from the a stack, then

$$\big|\,\mathcal{M}[g_1^a,g_2^a,g_3^a,q_4,\bar{q}_5]\,\big|^2 \;=\; 16\,g_{\rm YM}^6\,\alpha'^{-4}\sum_{i=1,2,3}\big(s_{i4}^3\,s_{i5}\,+\,s_{i4}\,s_{i5}^3\big)\sum_{\lambda,\lambda'\in\Pi_q}\mathcal{C}_\lambda\,\mathcal{P}_{\lambda\lambda'}\,\mathcal{C}^*_{\lambda'}\;, \tag{8.4.119}$$

where the entries of the color matrix $\mathcal{P}$ can be extracted from table 8.3.

	(123)	(132)	(213)	(231)	(312)	(321)
(123)	D_q	X_q	X_q	Y_q	Y_q	Z_q
(132)	X_q	D_q	Y_q	Z_q	X_q	Y_q
(213)	X_q	Y_q	D_q	X_q	Z_q	Y_q
(231)	Y_q	Z_q	X_q	D_q	Y_q	X_q
(312)	Y_q	X_q	Z_q	Y_q	D_q	X_q
(321)	Z_q	Y_q	Y_q	X_q	X_q	D_q

Table 8.3: Matrix elements $\mathcal{P}_{\lambda\lambda'}$. Since the elements of the permutation set Π_q, see (8.4.117), have common last two numbers (4,5), the rows and columns are labeled by first three numbers. The entries D_q, X_q, Y_q and Z_q depend on the gauge group and are given in equation (8.3.97). Note that the matrix is symmetric.

Finally, we consider two gluons from stack a and one from stack b, with the amplitude given by equation (8.4.118). After taking its modulus squared, summing over all helicity configurations and over color indices of quarks and gluons, we obtain

$$\begin{aligned}\big|\,\mathcal{M}[g_1^a,g_2^a,g_3^b,q_4,\bar{q}_5]\,\big|^2 \;&=\; 16\,(g_{\rm YM}^a)^4\,(g_{\rm YM}^b)^2\,[N\,C_F]_b\,[N\,C_F]_a\,\alpha'^{-4}\sum_{i=1,2,3}\big(s_{i4}^3\,s_{i5}\,+\,s_{i4}\,s_{i5}^3\big)\\ &\times\Big\{[C_F]_a\Big(\big|\mathcal{C}_{(1243)}\big|^2+\big|\mathcal{C}_{(2143)}\big|^2\Big)+\Big[C_F-\frac{C_A}{2}\Big]_a\big(\mathcal{C}_{(1243)}\mathcal{C}^*_{(2143)}+\mathcal{C}^*_{(1243)}\mathcal{C}_{(2143)}\big)\Big\}\,.\end{aligned} \tag{8.4.120}$$

8.4.4 Concluding remarks

We have given the universal part of the tree level cross sections for external SM particles at the four- and five parton level, the four fermi completion is addressed in [101, 1]. References [102, 267, 268] discuss possible signatures of these string corrected cross sections in di- and trijet events. One obtains model independent tree level answers irrespective of the details of the type II landscape. This makes universal, stringy predictions possible, and the problem of the string landscape is nullified at the LHC. On the other hand, amplitudes with two quark antiquark pairs do depend on the details of the internal geometry, for instance through the masses of Kaluza Klein- and winding modes exchanged. Measuring their effects at the LHC would allow investigating some properties of the internal compact space.

Chapter 9

Tree level scattering of massive states

In this chapter, we will discuss scattering amplitudes involving massive Regge excitations. Firstly, we will take a closer look at the first mass level in four dimensions and its universal $\mathcal{N} = 1$ SUSY multiplets. For these massive states, any three- and four point production- or decay process involving massless partons otherwise is computed in the following, motivated by its phenomenological relevance at LHC in case of a low string scale: Once the mass threshold $\alpha'^{-1/2}$ is crossed in the center of mass energies of the colliding partons, one would see free Regge states produced directly, in association with jets, photons and other particles.

Secondly, we investigate the leading Regge trajectory. We compute scattering amplitudes of highest spin states at mass level n with spin $s = n + 1$ for bosons and $s = n + 1/2$ for fermions, see section 3.4 for their particularly simple vertex operators in $D = 10$ dimensions. The cubic couplings of bosons and fermions on the leading Regge trajectory are worked out for any triplet of mass levels n_i. The same can be achieved for higher point amplitudes, and we focus on four point level with one heavy maximum spin state and three massless states in any bose-fermi combination, putting particular emphasis on manifest cyclic symmetry. This gives the higher spin generalization of individual results from section 9.2 on the first mass level.

Except for the four fermion coupling, all our results remain valid in any $D < 10$ dimensional compactification scenario, so they might become relevant at LHC in case of an experimentally accessible low string scale. But even if not directly observable, superstring amplitudes involving massive states provide important clues on higher spin dynamics and their consistent interactions in field theory.

The material on the first mass level was developed last year in the context of [2]. The ten dimensional particle content had been well-known for quite some time already [140], and four point amplitudes with one such massive states were computed in [269]. Reference [2], however, performs the first direct construction of massive vertex operators and their scattering amplitudes in four dimensions. Some information had already been obtained indirectly from factorizing massless amplitudes [249, 250].

The results on the leading Regge trajectory are taken from my recent article [6]. Its purpose was to extend the analysis of [110,111] on higher spin interactions in bosonic string theory to the open superstring. The authors of the latter references investigated cubic and quartic couplings of highest spin states at mass $m^2 = \frac{n}{\alpha'}$ on the bosonic string. They extracted their massless limits and explained implications on higher spin gauge symmetry. Investigating the massless limit of the leading Regge trajectory in superstring theory is beyond the scope of [6] and this thesis and left for future work [113]. More detailed physical motivation for studying these highest spin states is given at the beginning of section 9.3.

9.1 General properties of massive amplitudes

Before delving into explicit results, let us first of all explain some general properties of three- and four point disk amplitudes involving massive states Φ_i. Their color- and pole structure turns out to differ from the massless case. We shall keep the mass levels n_i of the external states general in this first section.

9.1.1 Color structure of massive amplitudes

Color stripped three point amplitudes of mass level n_1, n_2, n_3 states have worldsheet parity $(-1)^{1+n_1+n_2+n_3}$, this becomes the relative factor of the two Chan Paton traces associated with the cyclically inequivalent orderings $(1,2,3)$ and $(1,3,2)$. They yield either the structure constants $f^{a_1a_2a_3}$ of the gauge group (defined by commutators $\big[T^a, T^b\big] = i f^{abc} T_c$) or the symmetrized three trace $d^{a_1a_2a_3} := \mathrm{Tr}\big\{T^{(a_1}\, T^{a_2}\, T^{a_3)}\big\}$ as the resulting color factor:

$$\mathcal{M}\big[(T^{a_1}, \Phi_1), (T^{a_2}, \Phi_2), (T^{a_3}, \Phi_3)\big] \;=\; \begin{cases} \frac{i}{2}\, f^{a_1a_2a_3}\, \mathcal{A}(\Phi_1, \Phi_2, \Phi_3) & : \; n_1 + n_2 + n_3 \text{ even} \\ 2\, d^{a_1a_2a_3}\, \mathcal{A}(\Phi_1, \Phi_2, \Phi_3) & : \; n_1 + n_2 + n_3 \text{ odd} \end{cases} \tag{9.1.1}$$

At four point level, the six cyclically inequivalent orderings of total mass level $N = \sum_i n_i$ states can be grouped into three $\mathbb{Z}_2$ orbits of worldsheet parity

$$\mathcal{A}(\Phi_1, \Phi_2, \Phi_3, \Phi_4) \;=\; (-1)^N\, \mathcal{A}(\Phi_4, \Phi_3, \Phi_2, \Phi_1) \;=:\; \varepsilon_N\, \mathcal{A}(\Phi_4, \Phi_3, \Phi_2, \Phi_1)\;. \tag{9.1.2}$$

After placing the $(-1)^N = \varepsilon_N$ sign into the Chan Paton traces, we can organize the full amplitude into three contribution with

$$\begin{aligned} \mathcal{A}_t(\Phi_i) &:= \mathcal{A}(\Phi_1, \Phi_2, \Phi_3, \Phi_4) \\ \mathcal{A}_s(\Phi_i) &:= \mathcal{A}(\Phi_2, \Phi_3, \Phi_1, \Phi_4) \\ \mathcal{A}_u(\Phi_i) &:= \mathcal{A}(\Phi_3, \Phi_1, \Phi_2, \Phi_4) \end{aligned} \tag{9.1.3}$$

which single out one of the s, t or u-channel each:

$$\mathcal{M}\big[(T^{a_1}, \Phi_1), \dots, (T^{a_4}, \Phi_4)\big] \;=\; c_{\varepsilon_N}^{a_1a_2a_3a_4}\, \mathcal{A}_t(\Phi_i) \;+\; c_{\varepsilon_N}^{a_2a_3a_1a_4}\, \mathcal{A}_s(\Phi_i) \;+\; c_{\varepsilon_N}^{a_3a_1a_2a_4}\, \mathcal{A}_u(\Phi_i) \tag{9.1.4}$$

The color factors $c_{\pm}^{a_1a_2a_3a_4} := \mathrm{Tr}\{T^{a_1}T^{a_2}T^{a_3}T^{a_4} \pm T^{a_4}T^{a_3}T^{a_2}T^{a_1}\}$ are defined and further simplified in subsection 8.3.3.

The three color stripped amplitudes in (9.1.4) are still not independent, the worldsheet monodromy analysis of section 5.4 yields relations

$$\sin(\pi s)\, \mathcal{A}_u \;=\; \sin(\pi u)\, \mathcal{A}_s \;, \qquad \sin(\pi u)\, \mathcal{A}_t \;=\; \sin(\pi t)\, \mathcal{A}_u \;, \qquad \sin(\pi t)\, \mathcal{A}_s \;=\; \sin(\pi s)\, \mathcal{A}_t \tag{9.1.5}$$

between the three subamplitudes $\mathcal{A}_t, \mathcal{A}_s$ and $\mathcal{A}_u$. Effectively, there is only one color ordered amplitude left to determine from the scratch.

9.1.2 Kinematic structure of massive four point amplitudes

Momentum conservation $s+t+u \equiv -\alpha' \sum_i m_i^2 = -N$ admits to rewrite the t channel formfactor as $V_t = \frac{\Gamma(s+1)\Gamma(u+1)}{\Gamma(1-t-N)}$, then relations (9.1.5) together with the Euler reflection formula $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$ implies that subamplitudes can be factorized into a channel dependent piece and a universal one $\mathcal{A}_0$ which is common to all of $\mathcal{A}_t, \mathcal{A}_s$ and $\mathcal{A}_u$:

$$\mathcal{A}_t \;=\; \frac{V_t}{\prod_{k=1}^{N-1}(t+k)}\, \mathcal{A}_0 \;, \qquad \mathcal{A}_s \;=\; \frac{V_s}{\prod_{k=1}^{N-1}(s+k)}\, \mathcal{A}_0 \;, \qquad \mathcal{A}_u \;=\; \frac{V_u}{\prod_{k=1}^{N-1}(u+k)}\, \mathcal{A}_0 \tag{9.1.6}$$

The massless case $N = 0$ is somehow exceptional with

$$\mathcal{A}_t \Big|_{N=0} \;=\; \frac{V_t}{su}\, \mathcal{A}_0 \Big|_{N=0} \;, \qquad \mathcal{A}_s \Big|_{N=0} \;=\; \frac{V_s}{tu}\, \mathcal{A}_0 \Big|_{N=0} \;, \qquad \mathcal{A}_u \Big|_{N=0} \;=\; \frac{V_u}{st}\, \mathcal{A}_0 \Big|_{N=0} \;. \tag{9.1.7}$$

in lines of the findings of subsection 8.1.2. In case of identical external states $\Phi_i = \Phi_j$, the universal part $\mathcal{A}_0$ must certainly be either symmetric or antisymmetric under exchange of labels $i \leftrightarrow j$ (symmetric for bosons at even overall mass level N and for fermions at odd N). As a guiding principle in presenting massive four point amplitudes we will make this symmetry manifest.

The structure (9.1.6) and (9.1.7) of four point subamplitudes has a surprising implication on their massless poles $\frac{1}{s}, \frac{1}{t}$ and $\frac{1}{u}$. Only in the massless case are they correlated with the color ordering (such as $\mathcal{A}(1,3,2,4) \sim \frac{1}{tu}$) in agreement with the Feynman rules of the associated SYM theory. Massive amplitudes also receive contributions from exchange of massless states, but this obviously does not come from the channel dependent prefactors like $V_t \prod_{k=1}^{N-1}(t+k)^{-1}$. Hence, the massless poles must orginiate from the universal kinematic factor $\mathcal{A}_0$ which is totally (anti-)symmetric under exchange of labels. Generically, any color ordered four point amplitude $\mathcal{A}(\Phi_1, \Phi_2, \Phi_3, \Phi_4)$ with at least one massive state is expected to exhibit all the three massless pole channels $\frac{1}{s}, \frac{1}{t}$ and $\frac{1}{u}$. This is in sharp contrast to the organization of massless amplitudes $\mathcal{A}(1,2,3,4) = \frac{n_s}{s} + \frac{n_u}{u}$ into color ordered cubic diagrams, see section 5.5.

The bottom line of these arguments is the following simple formula for color dressed four point amplitudes

$$\begin{aligned} \mathcal{M}\big[(T^{a_1}, \Phi_1), (T^{a_2}, \Phi_2), (T^{a_3}, \Phi_3), (T^{a_4}, \Phi_4)\big] \ &= \ \mathcal{A}_0(\Phi_1, \Phi_2, \Phi_3, \Phi_4) \\ \Bigg\{ c_{\varepsilon_N}^{a_1 a_2 a_3 a_4} \, V_t \prod_{k=1}^{N-1} \frac{1}{t+k} \ + \ c_{\varepsilon_N}^{a_3 a_1 a_2 a_4} \, V_u \prod_{k=1}^{N-1} \frac{1}{u+k} \ &+ \ c_{\varepsilon_N}^{a_2 a_3 a_1 a_4} \, V_s \prod_{k=1}^{N-1} \frac{1}{s+k} \Bigg\} \end{aligned} \tag{9.1.8}$$

where the kinematic factor $\mathcal{A}_0(\Phi_1, \Phi_2, \Phi_3, \Phi_4)$ treats all color orderings on equal footing and generically contains all massless poles $\frac{1}{s}$, $\frac{1}{t}$ and $\frac{1}{u}$ unless $N = 0$.

9.2 Amplitudes for $\mathcal{N} = 1$ multiplets at first mass level

We have shown in subsections 4.5.2 and 4.5.3 that the first mass level contains two universal $\mathcal{N} = 1$ multiplets which appear in the spectrum of any compactification with nonzero supersymmetry. In this section, their three- and four point couplings to partons and gauginos are computed.

9.2.1 CFT preliminaries

As argued in subsection 4.3.1, the existence of $\mathcal{N} = 1$ spacetime SUSY charges in $D = 4$ implies that the internal SCFT enjoys an enhanced $\mathcal{N} = 2$ worldsheet supersymmetry. This argument is based on the $U(1)$ current $\mathcal{J}$ which is generated in the OPE of the $h = 3/8$ internal spin fields Σ and $\bar{\Sigma}$:

$$\Sigma(z)\, \bar{\Sigma}(w) \ \sim \ \frac{1}{(z-w)^{3/4}} \ + \ \frac{\sqrt{3}}{2}\, (z-w)^{1/4}\, \mathcal{J}(w) \ + \ \dots \tag{9.2.9}$$

Intersecting branes additionally give rise to $h=3/8$ boundary condition changing operators Ξ and $\bar{\Xi}$. The latter depend on brane intersection angles $\vartheta_{j=2,3,4}$ which add up to $\vartheta_2+\vartheta_3+\vartheta_4\in 2\mathbb{Z}$ if $\mathcal{N}=1$ supersymmetry is preserved, see section 4.4.

Apart from universal states built from the spacetime SCFT, we have identified additional bosonic states at the first mass level in subsection 4.5.3 whose vertex operators contain conformal fields from the internal SCFT: The massive vector (4.5.105) is created by the $h=1$ current $\mathcal{J}$, and the vertex operator (4.5.106) of the Calabi Yau scalar involves a $h=3/2$ internal three form $\mathcal{O}^{\pm}$:

$$\Sigma(z)\,\Sigma(w) \;\sim\; (z-w)^{3/4}\,\mathcal{O}^+(w)\;+\;\dots\,, \qquad \bar{\Sigma}(z)\,\bar{\Sigma}(w)\;\sim\;(z-w)^{3/4}\,\mathcal{O}^-(w)\;+\;\dots \tag{9.2.10}$$

We can check through their bosonized representations that all operators give rise to a canonically normalized two point function[1]:

$$\begin{aligned}
\Sigma &= \mathrm{e}^{+\frac{i}{2}(H^2+H^3+H^4)}\,, & \Xi &= \prod_{j=2}^{4}\sigma^-_{\vartheta_j}\,\mathrm{e}^{i\left(\frac{1}{2}-\vartheta_j\right)H_j}\\
\bar{\Sigma} &= \mathrm{e}^{-\frac{i}{2}(H^2+H^3+H^4)}\,, & \bar{\Xi} &= \prod_{j=2}^{4}\sigma^+_{\vartheta_j}\,\mathrm{e}^{-i\left(\frac{1}{2}-\vartheta_j\right)H_j}\\
\mathcal{J} &= \frac{1}{\sqrt{3}}\sum_{j=2}^{4} i\partial H^j\,, & \mathcal{O}^{\pm} &= \mathrm{e}^{\pm i(H^2+H^3+H^4)}
\end{aligned} \tag{9.2.11}$$

In absence of twist fields, one can use a more economic bosonization scheme,

$$\Sigma = \mathrm{e}^{+\frac{i\sqrt{3}}{2}H}\,, \qquad \mathcal{J} = i\partial H \tag{9.2.12}$$

$$\bar{\Sigma} = \mathrm{e}^{-\frac{i\sqrt{3}}{2}H}\,, \qquad \mathcal{O}^{\pm} = \mathrm{e}^{\pm i\sqrt{3}H}\,. \tag{9.2.13}$$

The following three point functions are needed to derive the amplitudes of interest:

$$\begin{aligned}
\langle\,\mathcal{J}(z_1)\,\Sigma(z_2)\,\bar{\Sigma}(z_3)\,\rangle &= \langle\,\mathcal{J}(z_1)\,\Xi(z_2)\,\bar{\Xi}(z_3)\,\rangle = \frac{\sqrt{3}\,z_{23}^{1/4}}{2\,z_{12}\,z_{13}}\\
\langle\,\mathcal{O}^-(z_1)\,\Sigma(z_2)\,\Sigma(z_3)\,\rangle &= \langle\,\mathcal{O}^+(z_1)\,\bar{\Sigma}(z_2)\,\bar{\Sigma}(z_3)\,\rangle = \frac{z_{23}^{3/4}}{(z_{12}\,z_{13})^{3/2}}\\
\langle\,\mathcal{O}^+(z_1)\,\Sigma(z_2)\,\Sigma(z_3)\,\rangle &= \langle\,\mathcal{O}^-(z_1)\,\bar{\Sigma}(z_2)\,\bar{\Sigma}(z_3)\,\rangle = 0
\end{aligned} \tag{9.2.14}$$

Remarkably, the brane intersection angles drop out of $\langle\mathcal{J}(z_1)\Xi^{a\cap b}(z_2)\bar{\Xi}^{a\cap b}(z_3)\rangle$ because of the SUSY condition $\sum_{j=2}^4\vartheta_j\in 2\mathbb{Z}$, and we arrive at the same result as for $\langle\mathcal{J}(z_1)\Sigma(z_2)\bar{\Sigma}(z_3)\rangle$. As a consequence, the massive vector couples universally to quarks and gauginos, see (9.2.21).

[1]This work follows a more natural normalization for the ingredients of the massive vector compared to [2]: In the reference, the $U(1)$ current $\mathcal{J}$ is defined with an additional factor of $\sqrt{3}$ and the polarization vector d_μ is twice as large as required for $d_\mu(h_1)d^\mu(h_2)=-\delta_{h_1+h_2}$ in the helicity basis (see appendix C.2.2). As a result, our normalization g_d has an extra factor of $\sqrt{12}=2\sqrt{3}$ compared to [2].

9.2.2 Three point amplitudes in vector notation

Let us start with the three point couplings of massive states to partons and gauginos. The nonvanishing results for three bosons are

$$\begin{aligned} \mathcal{A}(g_1, g_2, \alpha_3) &= -\, C_{\mathrm{D}_2}\, g_{\mathrm{A}}^2\, g_\alpha\, \sqrt{2\alpha'}^3\, F^1_{\mu\lambda}\, \alpha^{\mu\nu}\, F^{2\lambda}_{\nu} && (9.2.15)\\ \mathcal{A}(g_1, g_2, \varphi_3^{\pm}) &= -\, C_{\mathrm{D}_2}\, g_{\mathrm{A}}^2\, g_\varphi\, \sqrt{2\alpha'}^3\, F^1_{\mu\nu}\, F^{\mu\nu}_{2\pm}\ . && (9.2.16) \end{aligned}$$

The cubic vertex involving the complex spacetime scalar $\varphi^{\pm}$ involves the imaginary (anti-) self dual gluon field strength $F_i^{\mu\nu} = k_i^\mu \xi_i^\nu - \xi_i^\mu k_i^\nu$:

$$F^{\mu\nu}_{i\pm} \ := \ \frac{1}{2}\left(F_i^{\mu\nu} \ \pm \ i\tilde{F}_i^{\mu\nu} \right) , \qquad \tilde{F}_i^{\mu\nu} \ = \ \frac{1}{2}\, \varepsilon^{\mu\nu\lambda\rho}\, F^i_{\lambda\rho} \tag{9.2.17}$$

The d- and $\omega^{\pm}$ states do not couple to spacetime gluons at disk level because the gluon vertex operator does not involve any internal SCFT operator:

$$\mathcal{A}(g_1, g_2, d_3) \ = \ \mathcal{A}(g_1, g_2, \omega_3^{\pm}) \ = \ 0 \tag{9.2.18}$$

This is in lines with the universality of four gluon amplitudes (8.1.11): Gluons cannot exchange any state at disk level whose existence is tied to spacetime supersymmetry.

Let us next explore the interactions between massive bosons and massless fermions. The universal vector d and the complex Calabi Yau scalar $\omega^{\pm}$ require the internal correlators (9.2.14):

$$\begin{aligned} \mathcal{A}\left(\alpha_1, \left\{ {}^{\lambda_2, \bar{\lambda}_3}_{q_2, \bar{q}_3} \right.\right) &= \sqrt{2}\, \alpha'\, g_\alpha\, k_2^\mu\, \alpha_{\mu\nu}\, (u_2 \sigma^\nu \bar{u}_3) \left\{ {}^{C_{\mathrm{D}_2}\, g_\lambda^2}_{\tilde{C}_{\mathrm{D}_2}\, g_\psi^2} \right. && (9.2.19)\\ \mathcal{A}\left(\varphi_1^{\pm}, \left\{ {}^{\lambda_2, \bar{\lambda}_3}_{q_2, \bar{q}_3} \right.\right) &= 0 && (9.2.20)\\ \mathcal{A}\left(d_1, \left\{ {}^{\lambda_2, \bar{\lambda}_3}_{q_2, \bar{q}_3} \right.\right) &= \frac{\sqrt{3}}{2\sqrt{2}}\, g_d\, (u_2\, \not{d}\, \bar{u}_3) \left\{ {}^{C_{\mathrm{D}_2}\, g_\lambda^2}_{\tilde{C}_{\mathrm{D}_2}\, g_\psi^2} \right. && (9.2.21)\\ \mathcal{A}\left(\left\{ {}^{\omega_1^-, \lambda_2, \lambda_3}_{\omega_1^+, \bar{\lambda}_2, \bar{\lambda}_3} \right.\right) &= C_{\mathrm{D}_2}\, g_\omega\, g_\lambda^2 \left\{ {}^{(u_2\, u_3)}_{(\bar{u}_2\, \bar{u}_3)} \right. && (9.2.22)\\ \mathcal{A}\left(\left\{ {}^{\omega_1^+, \lambda_2, \lambda_3}_{\omega_1^-, \bar{\lambda}_2, \bar{\lambda}_3} \right.\right) &= \mathcal{A}\left(\omega_1^{\pm}, \left\{ {}^{q_2, q_3}_{\bar{q}_2, \bar{q}_3} \right.\right) \ = \ 0 && (9.2.23) \end{aligned}$$

In most cases, gauginos and quarks give rise to the same color stripped disk amplitudes (up to the normalization $C_{\mathrm{D}_2} g_\lambda^2 = 2\tilde{C}_{\mathrm{D}_2} g_\psi^2 = 2/\sqrt{\alpha'}$). This remains true for their interaction with the massive vector d. Only the Calabi Yau scalar $\omega^{\pm}$ feels a difference – it only couples to gauginos of uniform helicity (λ, λ) and $(\bar{\lambda}, \bar{\lambda})$ but not to quarks at non-degenerate values for their intersection angles. This is compatible with the model independent agreement of the two fermion-, two boson amplitudes $\mathcal{A}(g_1, g_2, \lambda_3, \bar{\lambda}_4)$ and $\mathcal{A}(g_1, g_2, q_3, \bar{q}_4)$: There is no s channel exchange of massive states d and $\omega^{\pm}$ which rely on $\mathcal{N} = 1$ SUSY because they do not couple to gluons. They only affect four gaugino amplitudes $\mathcal{A}(\lambda_1, \bar{\lambda}_2, \lambda_3, \bar{\lambda}_4)$ which were argued in section 8.2 to differ heavily from four quark amplitudes $\mathcal{A}(q_1, \bar{q}_2, q_3, \bar{q}_4)$ due to Kaluza Klein mode exchange in the latter.

Among the three point amplitudes involving massive fermions, we only consider excited quarks rather than excited gauginos. The latter can easily be restored via $\tilde{C}_{\mathrm{D}_2} g_\psi \tilde{g}_{\chi,b} \mapsto C_{\mathrm{D}_2} g_\lambda g_{\chi,b}$:

$$\mathcal{A}(g_1, q_2, \bar{\chi}_3) = \tilde{C}_{\mathrm{D}_2} \sqrt{2}\, \alpha'\, g_{\mathrm{A}}\, g_\psi\, \tilde{g}_\chi\, F^1_{\mu\nu}\, (u_2\, \sigma^\mu \bar{\chi}^\nu_3) \tag{9.2.24}$$

$$\mathcal{A}(g_1, q_2, \bar{b}_3) = \tilde{C}_{\mathrm{D}_2} \sqrt{2}\, \alpha'\, g_{\mathrm{A}}\, g_\psi\, \tilde{g}_b\, k_3^\mu\, F^1_{\mu\nu}\, (u_2 \sigma^\nu \bar{b}_3) \tag{9.2.25}$$

The vertex operators of massive fermions contains composite fields $\psi^\mu\ \psi^{\dot{a}b} S_b$ from their spacetime CFT. A few of their correlation functions are given in section 6.6.

The color dressed amplitude of three adjoint states (Φ_1, Φ_2, Φ_3) with odd overall mass level involves the symmetrized trace $\mathrm{Tr}\{T^{a_1} T^{a_2} T^{a_3} + T^{a_2} T^{a_1} T^{a_3}\} = 2 d^{a_1 a_2 a_3}$:

$$\mathcal{M}\big[(T^{a_1}, \Phi_1), (T^{a_2}, \Phi_2), (T^{a_3}, \Phi_3)\big]\Big|_{N=1} = 2\, d^{a_1 a_2 a_3}\, \mathcal{A}(\Phi_1, \Phi_2, \Phi_3) \tag{9.2.26}$$

With two chiral matter states, on the other hand, the color factor is independent on the mass levels:

$$\mathcal{M}\big[(T^{a_1}, \Phi_1), (T^{\alpha_2}_{\beta_2}, \Phi_2), (T^{\beta_3}_{\alpha_3}, \Phi_3)\big] = (T^{a_1})^{\alpha_2}_{\alpha_3}\, \delta^{\beta_3}_{\beta_2}\, \mathcal{A}(\Phi_1, \Phi_2, \Phi_3) \tag{9.2.27}$$

Given these three point vertices, we can determine the normalization factors of the massive vertex operators (displayed in subsections 4.5.2 and 4.5.3) by factorizing four point amplitudes[2] $\mathcal{M}[1^{a_1}, 2^{a_2}, 3^{a_3}, 4^{a_4}]$ of massless states on the residue of the first massive pole $V_t \approx \frac{u}{s+1}$:

$$g_\alpha = g_{\mathrm{YM}}\ , \qquad g_\varphi = \frac{g_{\mathrm{YM}}}{2}\ , \qquad g_d = g_\omega = \sqrt{2\alpha'}\, g_{\mathrm{YM}} \tag{9.2.29}$$

$$g_\chi = \alpha'^{1/4}\, g_{\mathrm{YM}}\ , \qquad g_b = \frac{\alpha'^{3/4}}{\sqrt{2}}\, g_{\mathrm{YM}} \tag{9.2.30}$$

The couplings $\tilde{g}_\chi, \tilde{g}_b$ of excited quarks can be obtained from those of excited gauginos (g_χ, g_b) by replacing $g_{\mathrm{YM}} \mapsto \mathrm{e}^{\phi_{10}/2}$.

9.2.3 Three point helicity amplitudes

In this subsection we evaluate the three point amplitudes computed above in the helicity basis. The polarization tensors of massless states are the usual ones $u^a(k) \equiv k^a$ and $\xi^+_\mu(k,r) \equiv$

[2]The contribution of color degrees of freedom to the factorization is as follows:

$$\sum_e \mathrm{Tr}\{\, T^a\, T^b\, T^e\,\}\, \mathrm{Tr}\{\, T^e\, T^c\, T^d\,\} = \frac{1}{2}\, \mathrm{Tr}\{\, T^a\, T^b\, T^c\, T^d\,\} \tag{9.2.28}$$

$\bar{r}_{\dot{a}}\bar{\sigma}_\mu^{\dot{a}b}k_b/(\sqrt{2}\langle kr\rangle)$ with reference spinor r for gluons, cf. section 8.2. For massive states, on the other hand, timelike momenta require two bispinors $p_a\bar{p}_{\dot{a}}$ and $q_a\bar{q}_{\dot{a}}$ for a lightlike decomposition $k^\mu = p^\mu + q^\mu$, more details about this method can be found in appendix C.2. Wavefunctions of spin j have $2j+1$ physical polarization states with spin component $j_z = -j, \dots, +j$ in the direction of the reference vector p^μ. We will quote the j_z eigenvalue in helicity basis amplitudes along with the wavefunction, e.g. $\alpha(j_z = +2)$, $\alpha(j_z = +1)$, $\dots$, $\alpha(j_z = -2)$ for the spin two state.

We will insert the normalizations (9.2.29) and (9.2.30) for the vertex operator into the results of the previous subsection and isolate constant prefactors before sending the kinematic factor (denoted by $\mathcal{K}[\dots]$ in the following) into the helicity basis. The sum of the moduli squares $\sum_{j_z} |\mathcal{K}[\dots,\Phi(j_z)]|^2$ over all polarizations of the massive state Φ cannot depend on the individual reference vectors p^μ and q^μ, i.e. the choice of quantization axis p^μ drops out of unpolarized cross sections. This is an important consistency check on the subsequent results. Moreover, we will also find selection rules on the helicities of the massless states, i.e. some gluon- and quark helicity configurations make the massive amplitude vanish for all polarizations of the heavy state. This ties in with the factorization properties of massless four point amplitudes on the pole $V_t \approx \frac{u}{s+1}$.

The decay of the spin two particle α into two gluons is described by the amplitude

$$\mathcal{A}(g_1, g_2, \alpha_3) \;=\; 2g_{\rm YM}\,\sqrt{2\alpha'}\,F^1_{\mu\lambda}\,\alpha^{\mu\nu}\,F^{2\lambda}_{\nu} \;=:\; 2g_{\rm YM}\,\sqrt{2\alpha'}^3\,\mathcal{K}[g_1, g_2, \alpha_3] \tag{9.2.31}$$

By substituting the helicity wave functions, we find the selection rule that α only couples to gluons of opposite helicity (independent on its polarization state):

$$\mathcal{K}[g_1^\pm, g_2^\pm, \alpha_3] \;=\; 0 \tag{9.2.32}$$

The nonzero components of this kinematic factor are

$$\begin{aligned}
\mathcal{K}\big[g_1^+, g_2^-, \alpha_3(+2)\big] &= -\frac{1}{4}\,\langle p\,2\rangle^2\,[q\,1]^2 \\
\mathcal{K}\big[g_1^+, g_2^-, \alpha_3(+1)\big] &= -\frac{1}{2}\,\langle p\,2\rangle^2\,[q\,1]\,[p\,1] \\
\mathcal{K}\big[g_1^+, g_2^-, \alpha_3(0)\big] &= -\frac{\sqrt{6}}{4}\,\langle p\,2\rangle^2\,[p\,1]^2 \\
\mathcal{K}\big[g_1^+, g_2^-, \alpha_3(-1)\big] &= +\frac{1}{2}\,\langle p\,2\rangle\,\langle q\,2\rangle\,[p\,1]^2 \\
\mathcal{K}\big[g_1^+, g_2^-, \alpha_3(-2)\big] &= -\frac{1}{4}\,\langle q\,2\rangle^2\,[p\,1]^2\,.
\end{aligned} \tag{9.2.33}$$

The appearance of $\tilde{F}^{\mu\nu}$ in the three point interaction $\mathcal{A}(g_1, g_2, \varphi_3^\pm)$ is essential in view of the helicity basis: The vertex being proportional to $F^1_{\mu\nu}\,F^{\mu\nu}_{2\pm}$ implies the selection rule

$$\mathcal{A}(g_1^\pm, g_2^\mp, \varphi_3^\pm) \;=\; \mathcal{A}(g_1^\pm, g_2^\mp, \varphi_3^\mp) \;=\; \mathcal{A}(g_1^\pm, g_2^\pm, \varphi_3^\mp) \;=\; 0 \tag{9.2.34}$$

such that the only nonzero configurations are

$$\mathcal{A}(g_1^+, g_2^+, \varphi_3^+) \;=\; 2g_{\rm YM}\sqrt{2\alpha'}\,[1\,2]^2\;, \qquad \mathcal{A}(g_1^-, g_2^-, \varphi_3^-) \;=\; 2g_{\rm YM}\sqrt{2\alpha'}\,\langle 1\,2\rangle^2\;. \tag{9.2.35}$$

Such couplings arise naturally from $\mathcal{N}=1$ supersymmetric F-terms $\sim \int {\rm d}^2\theta\,\Phi W^a W_a$ in the effective action where Φ denotes the chiral superfield including the complex scalar $\varphi^\pm$ and W^a is the gluon's gauge field strength superfield.

If φ^+ coupled to both (g^+,g^+) and (g^-,g^-), one would find a nonvanishing factorization channel in the $\mathcal{A}(g_1^+,g_2^+,g_3^+,g_4^+)$ amplitude on its $V_t \approx \frac{u}{s+1}$ pole. Clearly, this is forbidden by MHV selection rules, see subsection 8.3.1. Since we are dealing with a complex scalar which propagates off-diagonally $\varphi^\pm \leftrightarrow \varphi^\mp$, the coupling (9.2.35) can only contribute to an MHV four gluon amplitude $\mathcal{A}(g_1^+,g_2^+,g_3^-,g_4^-)$, see the following figure 9.1.

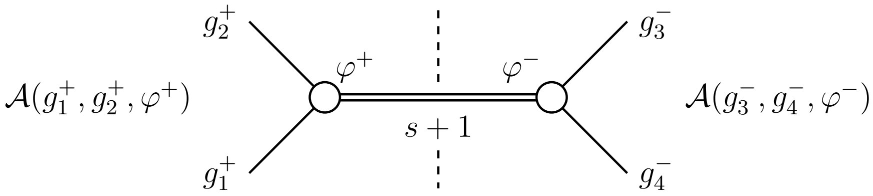


Figure 9.1: Factorization of a four gluon amplitude on the first Regge pole $V_t \approx \frac{u}{s+1}$ in the channel of identical helicities. The off-diagonal propagation $\varphi^\pm \leftrightarrow \varphi^\mp$ of the complex scalar avoids a contribution to $\mathcal{A}(g_1^+,g_2^+,g_3^+,g_4^+)$.

In contrast to the complex scalars, the spin two- and spin one states can decay into both quarks and gauginos. We will focus on quarks:

$$\mathcal{A}\,(\alpha_1, q_2, \bar q_3) \;=\; g_{\rm YM}\sqrt{2\alpha'}\,k_2^\mu\,\alpha_{\mu\nu}\,(u_2\sigma^\nu\bar u_3) \;=:\; g_{\rm YM}\sqrt{2\alpha'}^{\,3}\,\mathcal{K}\,[\alpha_1,q_2,\bar q_3] \tag{9.2.36}$$

In the helicity basis, the kinematic factors become

$$\begin{aligned}
\mathcal{K}\big[\alpha_1(+2),q_2,\bar q_3\big] &= \frac{1}{2}\,\langle p\,2\rangle\,\langle p\,3\rangle\,[q\,2]^2\\
\mathcal{K}\big[\alpha_1(+1),q_2,\bar q_3\big] &= \frac{1}{4}\,\langle p\,3\rangle\,[q\,2]\,\big(\langle q\,2\rangle\,[2\,q] \;-\; 3\,\langle p\,2\rangle\,[2\,p]\big)\\
\mathcal{K}\big[\alpha_1(0),q_2,\bar q_3\big] &= \frac{\sqrt{6}}{4}\,\langle p\,3\rangle\,[p\,2]\big(\langle q\,2\rangle\,[2\,q] \;-\; \langle p\,2\rangle\,[2\,p]\big)\\
\mathcal{K}\big[\alpha_1(-1),q_2,\bar q_3\big] &= \frac{1}{4}\,\langle q\,3\rangle\,[p\,2]\,\big(\langle p\,2\rangle\,[2\,p] \;-\; 3\,\langle q\,2\rangle\,[2\,q]\big)\\
\mathcal{K}\big[\alpha_1(-2),q_2,\bar q_3\big] &= \frac{1}{2}\,\langle q\,2\rangle\,\langle q\,3\rangle\,[p\,2]^2\;.
\end{aligned} \tag{9.2.37}$$

The massive vectors couples as follows

$$\mathcal{A}\,(d_1,q_2,\bar q_3) \;=\; \frac{\sqrt{3}}{2}\,g_{\rm YM}\,(u_2\,\not{d}\,\bar u_3) \;=:\; \sqrt{\frac{3\alpha'}{2}}\,g_{\rm YM}\,\mathcal{K}\,[d_1,q_2,\bar q_3] \tag{9.2.38}$$

with helicity basis components

$$\begin{aligned} \mathcal{K}\big[d_1(+1), q_2, \bar{q}_3\big] &= \langle p\,3\rangle\,[q\,2] \\ \mathcal{K}\big[d_1(0), q_2, \bar{q}_3\big] &= \sqrt{2}\,\langle p\,3\rangle\,[p\,2] \\ \mathcal{K}\big[d_1(-1), q_2, \bar{q}_3\big] &= \langle q\,3\rangle\,[p\,2]\ . \end{aligned} \tag{9.2.39}$$

Finally, the Calabi Yau scalar only couples to gauginos of uniform helicity (but not to quarks):

$$\mathcal{A}(\omega_1^-, \lambda_2, \lambda_3) \;=\; 2\sqrt{2}\,g_{\rm YM}\,[2\,3] \tag{9.2.40}$$

Both spin components among the massive fermions have nonvanishing three point interactions with one massless fermion and one gluon:

$$\mathcal{A}(g_1, q_2, \bar{\chi}_3) \;=\; g_{\rm YM}\,\sqrt{2\alpha'}\,F^1_{\mu\nu}\,(u_2\,\sigma^\mu \bar{\chi}_3^\nu) \;=\; \sqrt{2}\,\alpha'\,g_{\rm YM}\,\mathcal{K}[g_1, q_2, \bar{\chi}_3] \tag{9.2.41}$$

$$\mathcal{A}(g_1, q_2, \bar{b}_3) \;=\; g_{\rm YM}\,\alpha'\,k_3^\mu\,F^1_{\mu\nu}\,(u_2 \sigma^\nu \bar{b}_3) \;=\; \sqrt{2}\,\alpha'\,g_{\rm YM}\,\mathcal{K}[g_1, q_2, \bar{b}_3] \tag{9.2.42}$$

The selection rule is such that each helicity state of the gluon only couples to one of the states $\bar{\chi}$ $(\bar{b})$ of spin 3/2 and spin 1/2, respectively,

$$\mathcal{A}(g_1^+, q_2, \bar{\chi}_3) \;=\; \mathcal{A}(g_1^-, q_2, \bar{b}_3) \;=\; 0\ . \tag{9.2.43}$$

The nonvanishing helicity amplitudes are governed by the following kinematic factors:

$$\begin{aligned} \mathcal{K}\big[g_1^-, q_2, \bar{\chi}_3(+\tfrac{3}{2})\big] &= \langle p\,1\rangle^2\,[2\,q] \\ \mathcal{K}\big[g_1^-, q_2, \bar{\chi}_3(+\tfrac{1}{2})\big] &= \sqrt{3}\,\langle p\,1\rangle\,\langle q\,1\rangle\,[q\,2]\ , \qquad & \mathcal{K}\big[g_1^+, q_2, \bar{b}_3(+\tfrac{1}{2})\big] &= \langle p\,2\rangle\,[12]^2 \\ \mathcal{K}\big[g_1^-, q_2, \bar{\chi}_3(-\tfrac{1}{2})\big] &= \sqrt{3}\,\langle p\,1\rangle\,\langle q\,1\rangle\,[p\,2]\ , \qquad & \mathcal{K}\big[g_1^+, q_2, \bar{b}_3(-\tfrac{1}{2})\big] &= \langle q\,2\rangle\,[12]^2 \\ \mathcal{K}\big[g_1^-, q_2, \bar{\chi}_3(-\tfrac{3}{2})\big] &= \langle q\,1\rangle^2\,[2\,p] \end{aligned} \tag{9.2.44}$$

9.2.4 Four point amplitudes in vector notation

Four point amplitudes with massive states will be presented in the same steps as their three point cousins: This subsection lists the results in vector notation, and the next one translates them into the helicity basis. According to section 9.1, four point subamplitudes with three massless states and one state at mass level one can be written as $\mathcal{A}(\Phi_1, \Phi_2, \Phi_3, \Phi_4) = V_t \mathcal{A}_0(\Phi_1, \Phi_2, \Phi_3, \Phi_4)$ where $\mathcal{A}_0$ is a universal kinematic factor common to all color orderings which we will organize according to its massless poles.

Only the SUSY independent massive bosons α and $\varphi^\pm$ can decay into three gluons:

$$\mathcal{A}(g_1,g_2, g_3, \alpha_4) \;=\; C_{\rm D_2}\,4\alpha'^2\,g_{\rm A}^3\,g_\alpha\,V_t$$

$$
\begin{aligned}
\Bigg\{ \frac{1}{s} \Big[&(\xi_2\,\xi_3)\,(\xi_1\,k_2)\,(k_3^\mu\,\alpha_{\mu\nu}\,k_3^\nu) \;-\; (\xi_1\,\xi_3)\,(\xi_2\,k_1)\,(k_3^\mu\,\alpha_{\mu\nu}\,k_3^\nu) \;+\; (\xi_1\,\xi_2)\,(\xi_3\,k_2)\,(k_1^\mu\,\alpha_{\mu\nu}\,k_3^\nu) \\
&- (\xi_1\,\xi_2)\,(\xi_3\,k_1)\,(k_2^\mu\,\alpha_{\mu\nu}\,k_3^\nu) \;+\; (\xi_1\,k_3)\,(\xi_2\,k_1)\,(k_3^\mu\,\alpha_{\mu\nu}\,\xi_3^\nu) \;-\; (\xi_2\,k_3)\,(\xi_1\,k_2)\,(k_3^\mu\,\alpha_{\mu\nu}\,\xi_3^\nu) \\
&+ (\xi_2\,k_1)\,(\xi_3\,k_4)\,(k_3^\mu\,\alpha_{\mu\nu}\,\xi_1^\nu) \;-\; (\xi_1\,k_2)\,(\xi_3\,k_4)\,(k_3^\mu\,\alpha_{\mu\nu}\,\xi_2^\nu) \;+\; \frac{1}{2\alpha'}\,(\xi_1\,k_2)\,(\xi_2^\mu\,\alpha_{\mu\nu}\,\xi_3^\nu) \\
&- \frac{1}{2\alpha'}\,(\xi_2\,k_1)\,(\xi_1^\mu\,\alpha_{\mu\nu}\,\xi_3^\nu) \;+\; \frac{t}{2\alpha'}\,(\xi_1\,\xi_2)\,(k_2^\mu\,\alpha_{\mu\nu}\,\xi_3^\nu) \;-\; \frac{u}{2\alpha'}\,(\xi_1\,\xi_2)\,(k_1^\mu\,\alpha_{\mu\nu}\,\xi_3^\nu) \Big] \\
+ \frac{1}{u} \Big[&(\xi_1\,\xi_3)\,(\xi_2\,k_3)\,(k_1^\mu\,\alpha_{\mu\nu}\,k_1^\nu) \;-\; (\xi_1\,\xi_2)\,(\xi_3\,k_2)\,(k_1^\mu\,\alpha_{\mu\nu}\,k_1^\nu) \;+\; (\xi_2\,\xi_3)\,(\xi_1\,k_3)\,(k_1^\mu\,\alpha_{\mu\nu}\,k_2^\nu) \\
&- (\xi_2\,\xi_3)\,(\xi_1\,k_2)\,(k_1^\mu\,\alpha_{\mu\nu}\,k_3^\nu) \;+\; (\xi_2\,k_1)\,(\xi_3\,k_2)\,(k_1^\mu\,\alpha_{\mu\nu}\,\xi_1^\nu) \;-\; (\xi_3\,k_1)\,(\xi_2\,k_3)\,(k_1^\mu\,\alpha_{\mu\nu}\,\xi_1^\nu) \\
&+ (\xi_3\,k_2)\,(\xi_1\,k_4)\,(k_1^\mu\,\alpha_{\mu\nu}\,\xi_2^\nu) \;-\; (\xi_2\,k_3)\,(\xi_1\,k_4)\,(k_1^\mu\,\alpha_{\mu\nu}\,\xi_3^\nu) \;+\; \frac{1}{2\alpha'}\,(\xi_2\,k_3)\,(\xi_1^\mu\,\alpha_{\mu\nu}\,\xi_3^\nu) \\
&- \frac{1}{2\alpha'}\,(\xi_3\,k_2)\,(\xi_1^\mu\,\alpha_{\mu\nu}\,\xi_2^\nu) \;+\; \frac{s}{2\alpha'}\,(\xi_2\,\xi_3)\,(k_3^\mu\,\alpha_{\mu\nu}\,\xi_1^\nu) \;-\; \frac{t}{2\alpha'}\,(\xi_2\,\xi_3)\,(k_2^\mu\,\alpha_{\mu\nu}\,\xi_1^\nu) \Big] \\
+ \frac{1}{t} \Big[&(\xi_1\,\xi_2)\,(\xi_3\,k_1)\,(k_2^\mu\,\alpha_{\mu\nu}\,k_2^\nu) \;-\; (\xi_2\,\xi_3)\,(\xi_1\,k_3)\,(k_2^\mu\,\alpha_{\mu\nu}\,k_2^\nu) \;+\; (\xi_1\,\xi_3)\,(\xi_2\,k_1)\,(k_2^\mu\,\alpha_{\mu\nu}\,k_3^\nu) \\
&- (\xi_1\,\xi_3)\,(\xi_2\,k_3)\,(k_1^\mu\,\alpha_{\mu\nu}\,k_2^\nu) \;+\; (\xi_3\,k_2)\,(\xi_1\,k_3)\,(k_2^\mu\,\alpha_{\mu\nu}\,\xi_2^\nu) \;-\; (\xi_1\,k_2)\,(\xi_3\,k_1)\,(k_2^\mu\,\alpha_{\mu\nu}\,\xi_2^\nu) \\
&+ (\xi_1\,k_3)\,(\xi_2\,k_4)\,(k_2^\mu\,\alpha_{\mu\nu}\,\xi_3^\nu) \;-\; (\xi_3\,k_1)\,(\xi_2\,k_4)\,(k_2^\mu\,\alpha_{\mu\nu}\,\xi_1^\nu) \;+\; \frac{1}{2\alpha'}\,(\xi_3\,k_1)\,(\xi_1^\mu\,\alpha_{\mu\nu}\,\xi_2^\nu) \\
&- \frac{1}{2\alpha'}\,(\xi_1\,k_3)\,(\xi_2^\mu\,\alpha_{\mu\nu}\,\xi_3^\nu) \;+\; \frac{u}{2\alpha'}\,(\xi_1\,\xi_3)\,(k_1^\mu\,\alpha_{\mu\nu}\,\xi_2^\nu) \;-\; \frac{s}{2\alpha'}\,(\xi_1\,\xi_3)\,(k_3^\mu\,\alpha_{\mu\nu}\,\xi_2^\nu) \Big] \\
+ \frac{1}{2\alpha'} \Big[&(\xi_1\,\xi_2)\,\xi_3^\mu\,\alpha_{\mu\nu}\,(k_2^\nu - k_1^\nu) \;+\; (\xi_2\,\xi_3)\,\xi_1^\mu\,\alpha_{\mu\nu}\,(k_3^\nu - k_2^\nu) \;+\; (\xi_1\,\xi_3)\,\xi_2^\mu\,\alpha_{\mu\nu}\,(k_1^\nu - k_3^\nu) \Big] \Bigg\}
\end{aligned}
\tag{9.2.45}
$$

Up to V_t, the result is antisymmetric under exchange of gluon labels $(1,2,3)$. Since the associated color factors have the same property, this is in lines with the gluons' bose statistics. The same phenomenon occurs for $\varphi^\pm$:

$$
\begin{aligned}
\mathcal{A}(g_1, g_2, g_3, \varphi_\pm) \;&=\; C_{\mathrm{D}_2}\, 8\,\alpha'^2\, g_{\mathrm{A}}^3\, g_\varphi\, V_t \\
&\times \Bigg\{ \frac{1}{s} \Big[(\xi_2\,k_1)\,F_{3\pm}^{\mu\nu}\,\xi_{1\mu}\,k_{4\nu} \;-\; (\xi_1\,k_2)\,F_{3\pm}^{\mu\nu}\,\xi_{2\mu}\,k_{4\nu} \;+\; (\xi_1\,\xi_2)\,F_{3\pm}^{\mu\nu}\,k_{1\mu}\,k_{2\nu} \Big] \\
&\quad + \frac{1}{t} \Big[(\xi_1\,k_3)\,F_{2\pm}^{\mu\nu}\,\xi_{3\mu}\,k_{4\nu} \;-\; (\xi_3\,k_1)\,F_{2\pm}^{\mu\nu}\,\xi_{1\mu}\,k_{4\nu} \;+\; (\xi_1\,\xi_3)\;F_{2\pm}^{\mu\nu}\,k_{3\mu}\,k_{1\nu} \Big] \\
&\quad + \frac{1}{u} \Big[(\xi_3\,k_2)\,F_{1\pm}^{\mu\nu}\,\xi_{2\mu}\,k_{4\nu} \;-\; (\xi_2\,k_3)\,F_{1\pm}^{\mu\nu}\,\xi_{3\mu}\,k_{4\nu} \;+\; (\xi_2\,\xi_3)\,F_{1\pm}^{\mu\nu}\,k_{2\mu}\,k_{3\nu} \Big] \\
&\quad + \frac{1}{2\alpha'} \Big[F_{3\pm}^{\mu\nu}\,\xi_{1\mu}\,\xi_{2\nu} \;+\; F_{2\pm}^{\mu\nu}\,\xi_{3\mu}\,\xi_{1\nu} \;+\; F_{1\pm}^{\mu\nu}\,\xi_{2\mu}\,\xi_{3\nu} \Big] \Bigg\}
\end{aligned}
\tag{9.2.46}
$$

The $\mathcal{N}=1$ specific states d and $\omega^\pm$ do not couple to gluons,

$$
\mathcal{A}(d_1, g_2, g_3, g_4) \;=\; \mathcal{A}(\omega_1^\pm, g_2, g_3, g_4) \;=\; 0\,. \tag{9.2.47}
$$

All the bosons $\alpha, \varphi^\pm, d$ and $\omega^\pm$ at the first mass level have a four point decay channel into a gluon and two quarks or gauginos:

$$\begin{aligned}
\mathcal{A}(\alpha_1, g_2, q_3, \bar{q}_4) &= \tilde{C}_{\mathrm{D}_2}\, 2\alpha'^{3/2}\, g_{\mathrm{A}}\, g_\alpha\, g_\psi^2\, V_t \\
\times \Bigg\{ \frac{1}{s} \Big[& k_2^\mu\, \alpha_{\mu\nu}\, \xi^\nu\, (u_3\, \not{k}_2\, \bar{u}_4) \;-\; k_2^\mu\, \alpha_{\mu\nu}\, k_2^\nu\, (u_3\, \not{\xi}\, \bar{u}_4) \\
& + k_2^\mu\, \alpha_{\mu\nu}\, (u_3\, \sigma^\nu\, \bar{u}_4)\, (\xi\, k_1) \;-\; \frac{1}{2\alpha'}\, \xi^\mu\, \alpha_{\mu\nu}\, (u_3\, \sigma^\nu\, \bar{u}_4) \Big] \\
+ \frac{1}{t} \Big[& k_3^\mu\, \alpha_{\mu\nu}\, u_3\, \sigma^\nu\, \bar{u}_4\, (\xi\, k_4) \;+\; \frac{1}{2}\, k_3^\mu\, \alpha_{\mu\nu}\, (u_3\, \sigma^\nu\, \not{\xi}\, \not{k}_2\, \bar{u}_4) \Big] \\
+ \frac{1}{u} \Big[& k_4^\mu\, \alpha_{\mu\nu}\, u_3\, \sigma^\mu\, \bar{u}_4\, (\xi\, k_3) \;+\; \frac{1}{2}\, k_4^\mu\, \alpha_{\mu\nu}\, (u_3\, \not{k}_2\, \not{\xi}\, \sigma^\nu\, \bar{u}_4) \Big] \Bigg\}
\end{aligned} \tag{9.2.48}$$

The complex scalar $\varphi^\pm$ couples to two fermions and a gluon through the combination $\varphi \pm i\Gamma_5\theta$ of its real- and imaginary part (where Γ_5 denotes the four dimensional chirality matrix and φ (θ) are the real (imaginary) part of $\varphi^\pm$):

$$\begin{aligned}
\begin{pmatrix} \mathcal{A}(\varphi_1^+, g_2, q_3, \bar{q}_4) & \mathcal{A}\left(\varphi_1^+, g_2, \bar{q}_3, q_4\right) \\ \mathcal{A}(\varphi_1^-, g_2, q_3, \bar{q}_4) & \mathcal{A}\left(\varphi_1^-, g_2, \bar{q}_3, q_4\right) \end{pmatrix} &= \tilde{C}_{\mathrm{D}_2}\, 4\alpha'^{3/2}\, g_{\mathrm{A}}\, g_\varphi\, g_\psi^2\, V_t\, \frac{1}{s}\, F_2^{\mu\nu}\, k_{1\nu} \\
& \times \begin{pmatrix} 0 & (\bar{u}_3\, \bar{\sigma}_\mu\, u_4) \\ -\,(u_3\, \sigma_\mu\, \bar{u}_4) & 0 \end{pmatrix}
\end{aligned} \tag{9.2.49}$$

The absence of massless t- and u channel poles is explained by factorization into $\mathcal{A}(\varphi_1^\pm, q_2, \bar{q}_3) = 0$. Similarly, four point amplitudes involving d and $\omega^\pm$ do not exhibit a massless s channel pole because of their vanishing three point coupling to gluons:

$$\begin{aligned}
\mathcal{A}(d_1, g_2, q_3, \bar{q}_4) &= \frac{3\, \tilde{C}_{\mathrm{D}_2}}{2}\, \sqrt{\alpha'}\, g_d\, g_{\mathrm{A}}\, g_\psi^2\, V_t \\
\times \Bigg\{ \frac{1}{t} \Big[(\xi\, k_4)\, (u_3\, \not{d}\, \bar{u}_4) &+ \frac{1}{2}\, (u_3\, \not{d}\, \not{\xi}\, \not{k}_2\, \bar{u}_4) \Big] \;-\; \frac{1}{u} \Big[(\xi\, k_3)\, (u_3\, \not{d}\, \bar{u}_4) \;+\; \frac{1}{2}\, (u_3\, \not{k}_2\, \not{\xi}\, \not{d}\, \bar{u}_4) \Big] \Bigg\}
\end{aligned} \tag{9.2.50}$$

$$\begin{aligned}
\mathcal{A}(\omega_1^-, g_2, \lambda_3, \lambda_4) &= C_{\mathrm{D}_2}\, \sqrt{2\alpha'}\, g_\omega\, g_{\mathrm{A}}\, g_\lambda^2\, V_t \\
\times \Bigg\{ \frac{1}{t} \Big[(u_3\, u_4)\, (\xi\, k_4) &+ \frac{1}{2}\, (u_3\, \not{\xi}_2\, \not{k}_2\, u_4) \Big] \;-\; \frac{1}{u} \Big[(u_3\, u_4)\, (\xi\, k_3) \;+\; \frac{1}{2}\, (u_3\, \not{k}_2\, \not{\xi}_2\, u_4) \Big] \Bigg\}
\end{aligned} \tag{9.2.51}$$

$$\mathcal{A}(\omega_1^+, g_2, \lambda_3, \lambda_4) = 0 \tag{9.2.52}$$

Massive fermions χ and b of spin 3/2 and spin 1/2, respectively, give rise to the following four point scattering:

$$\begin{aligned}
\mathcal{A}(g_1, g_2, q_3, \bar{\chi}_4) &= \tilde{C}_{\mathrm{D}_2}\, 2\alpha'^{3/2}\, g_{\mathrm{A}}^2\, g_\psi\, \tilde{g}_\chi\, V_t \\
\times \Bigg\{ \frac{1}{t} \Big[(\xi_1\, k_3)\, (u\, \not{\xi}_2\, \bar{\chi}_\mu)\, k_2^\mu &+ (\xi_1\, k_3)\, (u\, \not{k}_4\, \bar{\chi}_\mu)\, \xi_2^\mu \;+\; \frac{1}{2}\, (u\, \not{k}_1\, \not{\xi}_1\, \not{k}_4\, \bar{\chi}_\mu)\, \xi_2^\mu \;+\; \frac{1}{2}\, (u\, \not{k}_1\, \not{\xi}_1\, \not{\xi}_2\, \bar{\chi}_\mu)\, k_2^\mu \Big]
\end{aligned}$$

$$- \frac{1}{u} \Big[(\xi_2\, k_3)\, (u\, \not{\xi}_1\, \bar{\chi}_\mu)\, k_1^\mu \; + \; (\xi_2\, k_3)\, (u\, \not{k}_4\, \bar{\chi}_\mu)\, \xi_1^\mu \; + \; \frac{1}{2}\, (u\, \not{k}_2\, \not{\xi}_2\, \not{k}_4\, \bar{\chi}_\mu)\, \xi_1^\mu \; + \; \frac{1}{2}\, (u\, \not{k}_2\, \not{\xi}_2\, \not{\xi}_1\, \bar{\chi}_\mu)\, k_1^\mu \Big]$$
$$+ \frac{1}{s} \Big[(\xi_1\, k_2)\, (u\, \not{\xi}_2\, \bar{\chi}_\mu)\, k_3^\mu \; - \; (\xi_2\, k_1)\, (u\, \not{\xi}_1\, \bar{\chi}_\mu)\, k_3^\mu \; + \; (\xi_1\, \xi_2)\, (u\, \not{k}_2\, \bar{\chi}_\mu)\, k_1^\mu$$
$$+ \; (\xi_2\, k_1)\, (u\, \not{k}_4\, \bar{\chi}_\mu)\, \xi_1^\mu \; - \; (\xi_1\, k_2)\, (u\, \not{k}_4\, \bar{\chi}_\mu)\, \xi_2^\mu \; - \; (\xi_1\, \xi_2)\, (u\, \not{k}_1\, \bar{\chi}_\mu)\, k_2^\mu \Big] \Big\} \tag{9.2.53}$$

Note that $\bar{\chi}_\mu$ and $\not{k}_4 \bar{\chi}_\mu$ form a Dirac spinor as required for a massive fermion, the same is true for $\bar{b}$ and $\not{k}_4 \bar{b}$:

$$\begin{aligned}
\mathcal{A}(g_1, g_2, q_3, \bar{b}_4) \;\; &= \;\; \tilde{C}_{\mathrm{D}_2}\, 2\alpha'^{3/2}\, g_{\mathrm{A}}^2\, g_\psi\, \tilde{g}_b\, V_t \, \times \, \Big\{ \frac{1}{4\alpha'}\, (u\, \not{\xi}_2\, \not{\xi}_1\, \not{k}_4\, \bar{b}) \\
&+ \frac{1}{s} \Big[(\xi_1\, k_3)\, (\xi_2\, k_1)\, \big(u\, \not{k}_4\, \bar{b}\big) \; - \; \frac{t}{2\alpha'}\, (\xi_1\, \xi_2)\, (u\, \not{k}_4\, \bar{b}) \; + \; \frac{1}{2\alpha'}\, (\xi_1\, k_2)\, (u\, \not{\xi}_2\, \bar{b}) \\
&\quad - (\xi_1\, k_2)\, (\xi_2\, k_3)\, \big(u\, \not{k}_4\, \bar{b}\big) \; + \; \frac{1}{2\alpha'}\, (\xi_1\, \xi_2)\, (u\, \not{k}_1\, \bar{b}) \; - \; \frac{1}{2\alpha'}\, (\xi_2\, k_1)\, (u\, \not{\xi}_1\, \bar{b}) \Big] \\
&- \frac{1}{u} \Big[\frac{1}{4\alpha'}\, (u\, \not{\xi}_1\, \not{k}_2\, \not{\xi}_2\, \bar{b}) \; + \; \frac{1}{2}\, (\xi_1\, k_4)\, (u\, \not{k}_1\, \not{\xi}_2\, \not{k}_2\, \bar{b}) \\
&\quad - \frac{1}{2\alpha'}\, (\xi_2\, k_3)\, (u\, \not{\xi}_1\, \bar{b}) \; + \; (\xi_1\, k_4)\, (\xi_2\, k_3)\, (u\, \not{k}_1\, \bar{b}) \Big] \\
&+ \frac{1}{t} \Big[\frac{1}{4\alpha'}\, (u\, \not{\xi}_2\, \not{k}_1\, \not{\xi}_1\, \bar{b}) \; + \; \frac{1}{2}\, (\xi_2\, k_4)\, (u\, \not{k}_2\, \not{\xi}_1\, \not{k}_1\, \bar{b}) \\
&\quad - \frac{1}{2\alpha'}\, (\xi_1\, k_3)\, (u\, \not{\xi}_2\, \bar{b}) \; + \; (\xi_1\, k_3)\, (\xi_2\, k_4)\, (u\, \not{k}_2\, \bar{b}) \Big] \Big\}
\end{aligned} \tag{9.2.54}$$

Just like in the massless case, four fermion amplitudes with an excited fermion look completely different whether they involve massive gauginos or quarks. Let us focus on the simpler gaugino case here, the quarks are discussed in [2]:

$$\begin{aligned}
\mathcal{A}(\lambda_1, \bar{\lambda}_2, \lambda_3, \bar{\chi}_4) \;\; &= \;\; C_{\mathrm{D}_2}\, 2\alpha'\, g_\lambda^3\, g_\chi\, V_t \Big\{ \frac{1}{t}\, (u_{1I} u_{3K})\, (\bar{u}_2^{\bar{J}} \bar{\chi}_\mu^{\bar{L}})\, k_2^\mu \, \big(C^I{}_{\bar{J}}\, C^K{}_{\bar{L}} \; - \; C^I{}_{\bar{L}}\, C^K{}_{\bar{J}} \big) \\
&+ \frac{C^I{}_{\bar{J}}\, C^K{}_{\bar{L}}}{s} \Big[\tfrac{1}{4}\, (u_{1I}\, \not{k}_4\, \bar{\chi}_\mu^{\bar{L}})\, (u_{3K} \sigma^\mu \bar{u}_2^{\bar{J}}) \; + \; \tfrac{1}{4}\, (u_{3K}\, \not{k}_4\, \bar{\chi}_\mu^{\bar{L}})\, (u_{1I} \sigma^\mu \bar{u}_2^{\bar{J}}) \; - \; (u_{1I} u_{3K})\, (\bar{u}_2^{\bar{J}} \bar{\chi}_\mu^{\bar{L}})\, k_3^\mu \Big] \\
&+ \frac{C^I{}_{\bar{L}}\, C^K{}_{\bar{J}}}{u} \Big[\tfrac{1}{4}\, (u_{1I}\, \not{k}_4\, \bar{\chi}_\mu^{\bar{L}})\, (u_{3K} \sigma^\mu \bar{u}_2^{\bar{J}}) \; + \; \tfrac{1}{4}\, (u_{3K}\, \not{k}_4\, \bar{\chi}_\mu^{\bar{L}})\, (u_{1I} \sigma^\mu \bar{u}_2^{\bar{J}}) \; + \; (u_{1I} u_{3K})\, (\bar{u}_2^{\bar{J}} \bar{\chi}_\mu^{\bar{L}})\, k_1^\mu \Big] \Big\}
\end{aligned} \tag{9.2.55}$$

$$\begin{aligned}
\mathcal{A}(\lambda_1, \bar{\lambda}_2, \lambda_3, \bar{b}_4) \;\; &= \;\; C_{\mathrm{D}_2}\, 2\alpha'\, g_\lambda^3\, g_b\, V_t \, \times \, \Big\{ (u_{1I} u_{3L})\, (\bar{u}_2^{\bar{J}} \bar{a}^{\bar{L}})\, \big(C^I{}_{\bar{J}}\, C^K{}_{\bar{L}} \; - \; C^I{}_{\bar{L}}\, C^K{}_{\bar{J}} \big) \\
&\quad - \frac{C^I{}_{\bar{J}}\, C^K{}_{\bar{L}}}{2\, s}\, (u_{1I} u_{3K})\, (\bar{u}_2^{\bar{J}}\, \not{k}_1\, \not{k}_3\, \bar{b}^{\bar{L}}) \; + \; \frac{C^I{}_{\bar{L}}\, C^K{}_{\bar{J}}}{2\, u}\, (u_{1I} u_{3K})\, (\bar{u}_2^{\bar{J}}\, \not{k}_3\, \not{k}_1\, \bar{b}^{\bar{L}}) \Big\}
\end{aligned} \tag{9.2.56}$$

The notation is adapted to maximal $\mathcal{N}=4$ SUSY with four gaugino species labelled by the R symmetry index $I = 1, 2, 3, 4$. In order to extract the amplitudes for $\mathcal{N}=1$ SUSY, all the internal charge conjugation matrices $C^I{}_{\bar{J}}$ must be set to unity. In this case, the t channel pole drops out of (9.2.55).

The color structure for scattering four adjoint states can be found in (9.1.4) at $\varepsilon_{N=1} = 1$, i.e. the parity odd traces $c_-^{a_1a_2a_3a_4} = \mathrm{Tr}\{T^{a_1}T^{a_2}T^{a_3}T^{a_4} - T^{a_4}T^{a_3}T^{a_2}T^{a_1}\}$ appear. Color factors in presence of chiral matter are exactly the same as in the massless case, see subsection 8.3.6.

9.2.5 Four point helicity amplitudes

The four point amplitudes presented above are now evaluated in the helicity basis. Like in the three point case, we find various selection rules for the helicities of massless states coupling to first mass level excitations.

The four boson amplitudes are written as

$$\begin{aligned} \mathcal{A}(g_1, g_2, g_3, \alpha_4) &= 8\, g_{\mathrm{YM}}^2 \sqrt{2\alpha'}\, V_t\, \mathcal{K}[g_1, g_2, g_3, \alpha_4] && (9.2.57)\\ \mathcal{A}(g_1, g_2, g_3, \varphi_4^+) &= 4\, g_{\mathrm{YM}}^2 \sqrt{\alpha'}\, V_t\, \mathcal{K}[g_1, g_2, g_3, \varphi_4^+]\ . && (9.2.58) \end{aligned}$$

Helicity selection rules forbid coupling of the spin two state α to gluons of uniform helicity, and φ^+ preferably couples to gluons of $+$ helicity:

$$\mathcal{A}(g_1^\pm, g_2^\pm, g_3^\pm, \alpha) = \mathcal{A}(g_1^-, g_2^-, g_3^-, \varphi^+) = \mathcal{A}(g_1^+, g_2^-, g_3^-, \varphi^+) = 0 \qquad (9.2.59)$$

The nonvanishing entries are

$$\begin{aligned} \mathcal{K}[g_1^+, g_2^+, g_3^-, \alpha_4(+2)] &= \frac{\langle p\,3\rangle^4}{2\sqrt{2}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,1\rangle} \\ \mathcal{K}[g_1^+, g_2^+, g_3^-, \alpha_4(+1)] &= \frac{\langle p\,3\rangle^3\,\langle 3\,q\rangle}{\sqrt{2}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,1\rangle} \\ \mathcal{K}[g_1^+, g_2^+, g_3^-, \alpha_4(0)] &= \frac{\sqrt{3}\,\langle p\,3\rangle^2\,\langle 3\,q\rangle^2}{2\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,1\rangle} && (9.2.60)\\ \mathcal{K}[g_1^+, g_2^+, g_3^-, \alpha_4(-1)] &= \frac{\langle q\,3\rangle^3\,\langle 3\,p\rangle}{\sqrt{2}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,1\rangle} \\ \mathcal{K}[g_1^+, g_2^+, g_3^-, \alpha_4(-2)] &= \frac{\langle q\,3\rangle^4}{2\sqrt{2}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,1\rangle} \end{aligned}$$

as well as

$$\mathcal{K}[g_1^+, g_2^+, g_3^+, \varphi^+] = \frac{\alpha'^{-2}}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,1\rangle}\ , \qquad \mathcal{K}[g_1^+, g_2^+, g_3^-, \varphi^+] = \frac{[1\,2]^4}{[1\,2]\,[2\,3]\,[3\,1]}\ , \qquad (9.2.61)$$

the rest can be obtained by complex conjugation.

If two gluons are replaced by a quark antiquark pair, then

$$\mathcal{A}(\alpha_1, g_2, q_3, \bar{q}_4) = 2\, g_{\mathrm{YM}}^2 \sqrt{2\alpha'}\, V_t\, \mathcal{K}[\alpha_1, g_2, q_3, \bar{q}_4] \qquad (9.2.62)$$

with kinematic factor

$$\begin{aligned}
\mathcal{K}[\alpha_1(+2), g_2^+, q_3, \bar{q}_4] &= \frac{\langle p\,3\rangle \, \langle p\,4\rangle^3}{\sqrt{2}\,\langle 2\,3\rangle \, \langle 3\,4\rangle \, \langle 4\,2\rangle} \\
\mathcal{K}[\alpha_1(+1), g_2^+, q_3, \bar{q}_4] &= \frac{\langle p\,4\rangle^2}{2\sqrt{2}\,\langle 2\,3\rangle \, \langle 3\,4\rangle \, \langle 4\,2\rangle} \Big(\langle q\,3\rangle \, \langle p\,4\rangle \; + \; 3\,\langle p\,3\rangle \, \langle q\,4\rangle \Big) \\
\mathcal{K}[\alpha_1(0), g_2^+, q_3, \bar{q}_4] &= \frac{\sqrt{3}\,\langle p\,4\rangle \, \langle q\,4\rangle}{2\,\langle 2\,3\rangle \, \langle 3\,4\rangle \, \langle 4\,2\rangle} \Big(\langle q\,3\rangle \, \langle p\,4\rangle \; + \; \langle p\,3\rangle \, \langle q\,4\rangle \Big) \\
\mathcal{K}[\alpha_1(-1), g_2^+, q_3, \bar{q}_4] &= \frac{\langle q\,4\rangle^2}{2\sqrt{2}\,\langle 2\,3\rangle \, \langle 3\,4\rangle \, \langle 4\,2\rangle} \Big(\langle p\,3\rangle \, \langle q\,4\rangle \; + \; 3\,\langle q\,3\rangle \, \langle p\,4\rangle \Big) \\
\mathcal{K}[\alpha_1(-2), g_2^+, q_3, \bar{q}_4] &= \frac{\langle q\,3\rangle \, \langle q\,4\rangle^3}{\sqrt{2}\,\langle 2\,3\rangle \, \langle 3\,4\rangle \, \langle 4\,2\rangle} \; .
\end{aligned} \tag{9.2.63}$$

The spin two decay $\mathcal{A}(\alpha_1, g_2, q_3, \bar{q}_4)$ into a gluon and two fermions does not impose any selection rule. The process with a gluon of negative helicity follows from

$$\mathcal{K}[\alpha_1, g_2^-, q_3, \bar{q}_4] \;=\; \Big(\mathcal{K}[\alpha_1, g_2^+, q_3, \bar{q}_4] \Big)^* \Big|_{(p\leftrightarrow q),(3\leftrightarrow 4)} \; . \tag{9.2.64}$$

For the complex scalar φ^+, we have

$$\begin{aligned}
\mathcal{A}(\varphi_1^+, g_2^+, q_3, \bar{q}_4) &= 2\, g_{\rm YM}^2 \, \sqrt{\alpha'} \, V_t \, \frac{[1\,3]^2}{[1\,2]} \\
\mathcal{A}(\varphi_1^+, g_2^-, q_3, \bar{q}_4) &= 0 \; .
\end{aligned} \tag{9.2.65}$$

Let us next display the four point helicity amplitudes of $\mathcal{N}=1$ states d and $\omega^\pm$:

$$\mathcal{A}(d_1, g_2, q_3, \bar{q}_4) \;=\; \sqrt{3}\, g_{\rm YM}^2 \, \mathcal{K}[d_1, g_2, q_3, \bar{q}_4] \tag{9.2.66}$$

$$\mathcal{A}(\omega_1^-, g_2, \lambda_3, \lambda_4) \;=\; 4\, g_{\rm YM}^2 \, \mathcal{K}[\omega_1^-, g_2, \lambda_3, \lambda_4] \tag{9.2.67}$$

These processes do not obey any selection rules:

$$\begin{aligned}
\mathcal{K}\big[d_1(+1), g_2^+, q_3, \bar{q}_4\big] &= +\frac{\langle p\,4\rangle^2}{\langle 2\,3\rangle \, \langle 2\,4\rangle} \, , \qquad \mathcal{K}[\omega_1^-, g_2^+, \lambda_3, \lambda_4] \;=\; \frac{1}{\alpha' \, \langle 2\,3\rangle \, \langle 2\,4\rangle} \\
\mathcal{K}\big[d_1(0), g_2^+, q_3, \bar{q}_4\big] &= \frac{\sqrt{2}\,\langle p\,4\rangle \, \langle q\,4\rangle}{\langle 2\,3\rangle \, \langle 2\,4\rangle} \\
\mathcal{K}\big[d_1(-1), g_2^+, q_3, \bar{q}_4\big] &= -\frac{\langle q\,4\rangle^2}{\langle 2\,3\rangle \, \langle 2\,4\rangle} \, , \qquad \mathcal{K}[\omega_1^-, g_2^-, \lambda_3, \lambda_4] \;=\; \frac{[3\,4]^2}{[2\,3]\,[2\,4]}
\end{aligned} \tag{9.2.68}$$

The relevant crossing operations to obtain the remaining entries are

$$\begin{aligned}
\mathcal{K}[d_1, g_2^-, q_3, \bar{q}_4] &= \Big(\mathcal{K}[d_1, g_2^+, q_3, \bar{q}_4] \Big)^* \Big|_{(p\leftrightarrow q),(3\leftrightarrow 4)} \\
\mathcal{K}[\omega_1^+, g_2^\pm, \bar{\lambda}_3, \bar{\lambda}_4] &= -\Big(\mathcal{K}[\omega_1^-, g_2^\mp, \bar{\lambda}_3, \bar{\lambda}_4] \Big)^* \; .
\end{aligned} \tag{9.2.69}$$

Decay amplitudes of massive fermions into a light fermion and two bosons are written as

$$\mathcal{A}(g_1, g_2, q_3, \bar{\chi}_4) \;=\; 2\, g_{\rm YM}^2 \, V_t \, \mathcal{K}[g_1, g_2, q_3, \bar{\chi}_4] \tag{9.2.70}$$

$$\mathcal{A}(g_1, g_2, q_3, \bar{b}_4) \quad = \quad g_{\rm YM}^2 \, \alpha'^{-1} \, V_t \, \mathcal{K}[g_1, g_2, q_3, \bar{b}_4] \ . \tag{9.2.71}$$

The selection rules for spin 3/2 ($\bar{\chi}$) and spin 1/2 ($\bar{b}$) exclude either $(+,+)$ or $(-,-)$ gluons,

$$\mathcal{K}[g_1^+, g_2^+, q_3, \bar{\chi}_4] \quad = \quad \mathcal{K}[g_1^-, g_2^-, q_3, \bar{b}_4] \quad = \quad 0 \ . \tag{9.2.72}$$

Spin 3/2 gives rise to the following nontrivial kinematic factors:

$$\begin{aligned}
\mathcal{K}\big[g_1^-, g_2^-, q_3, \bar{\chi}_4(+\tfrac{3}{2})\big] &= \frac{[3\,q]^3}{[1\,2]\,[2\,3]\,[3\,1]} \ , & \mathcal{K}\big[g_1^+, g_2^-, q_3, \bar{\chi}_4(+\tfrac{3}{2})\big] &= \frac{\sqrt{\alpha'}\,\langle p\,2\rangle^3}{\langle 1\,2\rangle\,\langle 1\,3\rangle} \\
\mathcal{K}\big[g_1^-, g_2^-, q_3, \bar{\chi}_4(+\tfrac{1}{2})\big] &= \frac{\sqrt{3}\,[3\,q]^2\,[p\,3]}{[1\,2]\,[2\,3]\,[3\,1]} \ , & \mathcal{K}\big[g_1^+, g_2^-, q_3, \bar{\chi}_4(+\tfrac{1}{2})\big] &= \frac{\sqrt{3\alpha'}\,\langle p\,2\rangle^2\,\langle q\,2\rangle}{\langle 1\,2\rangle\,\langle 1\,3\rangle}
\end{aligned} \tag{9.2.73}$$

$$\begin{aligned}
\mathcal{K}\big[g_1^-, g_2^-, q_3, \bar{\chi}_4(-\tfrac{1}{2})\big] &= \frac{\sqrt{3}\,[q\,3]\,[3\,p]^2}{[1\,2]\,[2\,3]\,[3\,1]} \ , & \mathcal{K}\big[g_1^+, g_2^-, q_3, \bar{\chi}_4(-\tfrac{1}{2})\big] &= -\,\frac{\sqrt{3\alpha'}\,\langle p\,2\rangle\,\langle q\,2\rangle^2}{\langle 1\,2\rangle\,\langle 1\,3\rangle} \\
\mathcal{K}\big[g_1^-, g_2^-, q_3, \bar{\chi}_4(-\tfrac{3}{2})\big] &= \frac{[3\,p]^3}{[1\,2]\,[2\,3]\,[3\,1]} \ , & \mathcal{K}\big[g_1^+, g_2^-, q_3, \bar{\chi}_4(-\tfrac{3}{2})\big] &= -\,\frac{\sqrt{\alpha'}\,\langle q\,2\rangle^3}{\langle 1\,2\rangle\,\langle 1\,3\rangle}
\end{aligned}$$

The spin 1/2 analogue is:

$$\begin{aligned}
\mathcal{K}\big[g_1^+, g_2^+, q_3, \bar{b}_4(+\tfrac{1}{2})\big] &= \frac{\langle p\,3\rangle}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,1\rangle} \ , & \mathcal{K}\big[g_1^+, g_2^-, q_3, \bar{b}_4(+\tfrac{1}{2})\big] &= +\,\frac{\sqrt{\alpha'}^3\,[q\,1]\,[1\,3]^2}{[1\,2]\,[2\,3]}
\end{aligned} \tag{9.2.74}$$

$$\begin{aligned}
\mathcal{K}\big[g_1^+, g_2^+, q_3, \bar{b}_4(-\tfrac{1}{2})\big] &= \frac{\langle q\,3\rangle}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,1\rangle} \ , & \mathcal{K}\big[g_1^+, g_2^-, q_3, \bar{b}_4(-\tfrac{1}{2})\big] &= -\,\frac{\sqrt{\alpha'}^3\,[p\,1]\,[1\,3]^2}{[1\,2]\,[2\,3]}
\end{aligned}$$

Similar expressions with $(1 \leftrightarrow 2)$ hold for gluons of flipped helicity.

Four fermion amplitudes with excited gauginos are given by

$$\begin{aligned}
\mathcal{A}(\lambda_1, \bar{\lambda}_2, \lambda_3, \bar{\chi}_4) &= 2\, g_{\rm YM}^2 \sqrt{\alpha'}\, \mathcal{K}[\lambda_1, \bar{\lambda}_2, \lambda_3, \bar{\chi}_4] & (9.2.75)\\
\mathcal{A}(\lambda_1, \bar{\lambda}_2, \lambda_3, \bar{b}_4) &= 2\, g_{\rm YM}^2 \sqrt{\alpha'}\, \mathcal{K}[\lambda_1, \bar{\lambda}_2, \lambda_3, \bar{b}_4] \ . & (9.2.76)
\end{aligned}$$

In $\mathcal{N} = 1$ SUSY with one gaugino species, helicity components read

$$\begin{aligned}
\mathcal{K}\big[\lambda_1, \bar{\lambda}_2, \lambda_3, \bar{\chi}_4(+\tfrac{3}{2})\big] &= \frac{\langle 2\,p\rangle^3}{\langle 1\,2\rangle\,\langle 2\,3\rangle} \\
\mathcal{K}\big[\lambda_1, \bar{\lambda}_2, \lambda_3, \bar{\chi}_4(+\tfrac{1}{2})\big] &= \frac{\sqrt{3}\,\langle 2\,p\rangle^2\,\langle 2\,q\rangle}{\langle 1\,2\rangle\,\langle 2\,3\rangle} \ , & \mathcal{K}\big[\lambda_1, \bar{\lambda}_2, \lambda_3, \bar{b}_4(+\tfrac{1}{2})\big] &= \frac{[2\,q]\,[1\,3]^2}{[1\,2]\,[2\,3]} \\
\mathcal{K}\big[\lambda_1, \bar{\lambda}_2, \lambda_3, \bar{\chi}_4(-\tfrac{1}{2})\big] &= \frac{\sqrt{3}\,\langle p\,2\rangle\,\langle 2\,q\rangle^2}{\langle 1\,2\rangle\,\langle 2\,3\rangle} \ , & \mathcal{K}\big[\lambda_1, \bar{\lambda}_2, \lambda_3, \bar{b}_4(-\tfrac{1}{2})\big] &= \frac{[2\,p]\,[1\,3]^2}{[1\,2]\,[2\,3]} \\
\mathcal{K}\big[\lambda_1, \bar{\lambda}_2, \lambda_3, \bar{\chi}_4(-\tfrac{3}{2})\big] &= \frac{\langle q\,2\rangle^3}{\langle 1\,2\rangle\,\langle 2\,3\rangle} \ .
\end{aligned} \tag{9.2.77}$$

Cross sections for jet associated Regge production are tabulated in section VI of [2], listed in order of the initial two particle channels gg, gq and $q\bar{q}$.

9.3 The leading Regge trajectory

This section is devoted to scattering amplitudes of higher spin states on the leading Regge trajectory of the superstring. It extends the analysis of [110, 111] on higher spin interactions in bosonic string theory to the open superstring. Recently, similar calculations have been carried out in heterotic string theories [141] where stable higher spin BPS states were shown to exist and a large class of three- and four point scattering amplitudes was computed for states with higher spin content on the bosonic side of the heterotic string. Further interesting results on scattering of exotic massless higher spin states from a decoupled sector of superstring theory can be found in recent work [270, 271].

A first motivation for studying higher spin interactions in string theory is to extract lessons on higher spin gauge theory. Consistent four point interactions in massless higher spin field theories are still not fully understood [149], and the high energy limit $\alpha' \to \infty$ of string theory can be an important tool for their investigation.

Secondly, knowledge of highest spin three- and four-point interactions can prove valuable for factorization properties of massless multileg superstring disk amplitudes. As we have discussed in chapter 7, the complexity of formfactors in multi gluon scattering grows enormously with increasing number of external legs, see (8.1.25) and subsection 8.1.6 for the explicit four- and five-point results. The expansion (8.1.32) and (8.1.33) of the hypergeometric five point integrals f_1, f_2 encourages to disentangle the contributions of individual spins. More precisely, we conjecture that the summation variable j in these equations can be identified with the spin of the massive state in the internal propagator. The simplifications on the basis of this spin separation might in the end pave the way for new recursion relations.

Thirdly, scattering amplitudes of higher spin states may be relevant for phenomenology. As pointed out in section 4.1, the string scale can be as low as a few TeV provided that some of the extra dimensions are sufficiently large. In that case, the signature of massive string vibrations should become visible at LHC due to the resonant enhancement of parton amplitudes. As explained in the previous section 9.2, Regge excitations can be produced directly, and the amplitudes at higher mass level presented in this section can be regarded as a generalization of the mass level one discussion. In fact, gluon fusion and quark antiquark annihilation can produce any leading trajectory boson if the mass threshold is reached in the center of mass energies of the colliding partons, and similarly, higher spin fermions appear in quark gluon scattering at sufficiently high energy. But one has to admit that also the states from subleading Regge trajectories contribute to these effects and should be included into the analysis on the long hand. Handling the technical challenges posed by the subleading trajectories is an outstanding problem left for future work.

The label used for subamplitudes of highest spin states are $\phi^{j=n+1} \equiv (\phi_{\mu_1 \ldots \mu_j}, k_\mu)$ for bosons and $\chi^{j=n+1/2} \equiv (\chi^{\alpha}_{\mu_1 \ldots \mu_n}, k_\mu)$ for fermions.

9.3.1 Universality in spacetime dimensions

Dimensional reduction of bosonic states and their couplings is straightforward: Suppose $\phi_{m_1 \ldots m_s}$ with $m_i = 0, ..., 9$ is the initially $SO(1,9)$ covariant wave function of the spin $s = n+1$ state on the leading trajectory, then compactification of $10 - D$ spacetime dimensions breaks Lorentz symmetry to $SO(1, D-1)$ and hence requires to specify for each index m_i whether it belongs to the D uncompactified dimensions or to the internal ones. Let us denote the $SO(1, D-1)$ spacetime indices by μ_i, then only the $\phi_{\mu_1 \ldots \mu_s}$ components of the ten dimensional polarization tensor describes a maximum spin state from the D dimensional point of view. As usual, we neglect Kaluza Klein excitations and truncate spacetime momenta to $k_m = (k_\mu, 0)$. The only modification of the spin $s = n+1$ vertex operators (3.4.62) and (3.4.64) upon compactifying $10 - D$ dimensions is the replacement of $SO(1,9)$ indices m_i by $SO(1, D-1)$ indices μ_i:

$$V^{(-1)}(\phi^{j=n+1}, k, z) \;=\; g_n \, \phi_{(\mu_1 \ldots \mu_n \mu_{n+1})} \, i\partial X^{\mu_1} \, \ldots \, i\partial X^{\mu_n} \, \psi^{\mu_{n+1}} \, \mathrm{e}^{-\phi} \, \mathrm{e}^{ik \cdot X} \tag{9.3.78}$$

$$\begin{aligned} V^{(0)}(\phi^{j=n+1}, k, z) \;&=\; g_n \, \sqrt{2\alpha'} \, \phi_{(\mu_1 \ldots \mu_{n-1} \nu \lambda)} \, i\partial X^{\mu_1} \, \ldots \, i\partial X^{\mu_{n-1}} \, \mathrm{e}^{ik \cdot X} \\ &\quad \times \left(n \, \partial \psi^\nu \, \psi^\lambda \;+\; (k \cdot \psi) \, \psi^\nu \, i\partial X^\lambda \;+\; \frac{1}{2\alpha'} \, i\partial X^\nu \, i\partial X^\lambda \right) \end{aligned} \tag{9.3.79}$$

$$k^{\mu_i} \, \phi_{\mu_1 \ldots \mu_{n+1}} \;=\; \eta^{\mu_i \mu_j} \, \phi_{\mu_1 \ldots \mu_{n+1}} \;=\; 0 \; , \qquad k^2 \;=\; - \, \frac{n}{\alpha'} \tag{9.3.80}$$

Note that the spin two operator $\partial \psi^{(\nu} \psi^{\lambda)}$ drops out at the massless level $n = 0$.

Similar to the argument in subsection 8.2.1, the free CFT of the ψ^μ- and X^ν fields does not introduce any D dependence into correlation functions of highest spin states, so their scattering amplitudes take a universal form in any number of spacetime dimensions. The normalizations are set to $g_n = \sqrt{2\alpha'}^{\,-n}$ and $g_{n+1/2} = \sqrt{2\alpha'}^{\,-n}$ in the following.

Highest spin fermions in lower dimensions $D < 10$ have an $h = (10 - D)/16$ internal spin field s_{int} in their vertex operators which is a standard R spin field for excited gauginos or a boundary condition changing operator Ξ for excited quarks (see subsection 8.2.1 for a more detailed definition and discussion of the s_{int}):

$$\begin{aligned} V^{(-1/2)}(\chi^{j=n+1/2}, k, z) \;&=\; g_{n+1/2} \, \chi^a_{(\mu_1 \ldots \mu_{n-1} \nu)} \, i\partial X^{\mu_1} \, \ldots \, i\partial X^{\mu_{n-1}} \\ &\quad \times \left(i\partial X^\nu \, \delta^c_a \;-\; \frac{2\alpha'}{D-2} \, \not{k}_{a\dot{b}} \, \psi^\nu \, \not{\psi}^{\dot{b}c} \right) S_c \, s_{\mathrm{int}} \, \mathrm{e}^{-\phi/2} \, \mathrm{e}^{ik \cdot X} \end{aligned} \tag{9.3.81}$$

$$\begin{aligned} V^{(-1/2)}(\bar{\chi}^{j=n+1/2}, k, z) \ &= \ g_{n+1/2}\, \bar{\chi}_{\dot{a}}^{(\mu_1 \ldots \mu_{n-1} \nu)}\, i\partial X_{\mu_1} \, \ldots \, i\partial X_{\mu_{n-1}} \\ &\times \, \Big(i\partial X_\nu \, \delta^{\dot{a}}_{\dot{c}} \ - \ \frac{2\alpha'}{D-2} \, \not{k}^{\dot{a}b} \, \psi^\nu \, \not{\psi}_{b\dot{c}} \Big) \, S^{\dot{c}} \, \bar{s}_{\rm int} \, \mathrm{e}^{-\phi/2} \, \mathrm{e}^{ik\cdot X} \end{aligned} \tag{9.3.82}$$

$$k^{\mu_i} \, \chi^\alpha_{\mu_1 \ldots \mu_n} \ = \ \eta^{\mu_i \mu_j} \, \chi^\alpha_{\mu_1 \ldots \mu_n} \ = \ \chi^\alpha_{\mu_1 \ldots \mu_n} \, \gamma^{\mu_i}_{\alpha\dot{\beta}} \ = \ 0 \tag{9.3.83}$$

The spin 3/2 component of the spacetime SCFT operator $\psi^\nu \not{\psi}^{\dot{b}c} S_c/(D-2)$ has similar universality properties as the ordinary spin fields S_a: Any correlation function involving no more than two insertions $\in \{S_a s_{\rm int}, \ \psi^\nu \not{\psi}^{\dot{b}c} S_c s_{\rm int}/(D-2)\}$ is independent on D, see section 6.6 for examples. The CFT mechanisms cancelling the D dependence in $\langle s^i_{\rm int}(z_1) \bar{s}^j_{\rm int}(z_2) \rangle = c^{ij} z_{12}^{(D-10)/8}$ against that from the spacetime part are the same ones explained in subsection 8.2.1. Hence, scattering amplitudes involving two highest spin fermions and otherwise bosons take the same form in any number of spacetime dimensions and do not distinguish between Regge excitations of adjoint gauginos and chiral matter.

Again, the arguments for universality fail at the level of four fermions: Firstly, internal correlators $\langle s^i_{\rm int}(z_1) s^j_{\rm int}(z_2) s^k_{\rm int}(z_3) s^l_{\rm int}(z_4) \rangle$ cannot be treated in a model independent fashion, and secondly, the group structure in four point correlators of S_a and $\psi^\nu \not{\psi}^{\dot{b}c} S_c$ becomes more involved.

All the three- and four point amplitudes of highest spin states which we will give in the following subsections 9.3.2 to 9.3.6 are of universal type. Both the lessons on consistent higher spin interactions and the phenomenological implications for LHC experiments remain valid in all lower dimensional superstring compactifications which allow for a CFT description. We will adapt the indices and chiralities to the four dimensional situation which is the most relevant in view of the aforementioned motivations. The only exception occurs for the four fermion amplitude in subsection 9.3.7 which is given in $D = 10$ dimensions and has a different form after dimensional reduction to $D = 4$.

9.3.2 The three boson vertex

Cyclic symmetry of the three boson couplings in superstring theory is initially obscured by the asymmetric assignment of superghost charges. But the three terms in the zero ghost picture vertex (3.4.64) of higher spin bosons conspire such that the labels $1, 2, 3$ enter on the same footing in the end. By virtue of the correlator (B.4.62) with many $i\partial X$ field insertions, one ends up with

$$\mathcal{A}(\phi_1^{s_1}, \phi_2^{s_2}, \phi_3^{s_3}) \ = \ \sqrt{2\alpha'}^{\,s_1+s_2+s_3} \, n_1! \, n_2! \, n_3! \sum_{i,j,k \in \mathcal{I}_b} \frac{(2\alpha')^{-i-j-k} \, (is_3 \, + \, js_2 \, + \, ks_1 \, - \, ij \, - \, ik \, - \, jk)}{i! \, j! \, k! \, (s_1 - i - j)! \, (s_2 - i - k)! \, (s_3 - j - k)!}$$

$$\times\ \phi^1_{\mu_1\ldots\mu_j}{}^{\nu_1\ldots\nu_i}{}_{\rho_1\ldots\rho_{s_1-i-j}}\, k_2^{\rho_1}\ldots k_2^{\rho_{s_1-i-j}}\ \phi^2_{\nu_1\ldots\nu_i}{}^{\lambda_1\ldots\lambda_k}{}_{\tau_1\ldots\tau_{s_2-i-k}}\, k_3^{\tau_1}\ldots k_3^{\tau_{s_2-i-k}}$$
$$\times\ \phi^3_{\lambda_1\ldots\lambda_k}{}^{\mu_1\ldots\mu_j}{}_{\sigma_1\ldots\sigma_{s_3-j-k}}\, k_1^{\sigma_1}\ldots k_1^{\sigma_{s_3-j-k}} \tag{9.3.84}$$

where $s_i = n_i + 1$ and the summation range $\mathcal{I}_b$ is defined exactly like in [110, 111]:

$$\mathcal{I}_b \ := \ \Big\{ i,j,k \in \mathbb{N}_0 \ : \ s_1 - i - j \geq 0 \ , \quad s_2 - i - k \geq 0 \ , \quad s_3 - j - k \geq 0 \Big\} \tag{9.3.85}$$

Let us introduce some shorthands to lighten the notation. A k fold contraction of tensors ϕ^i and ϕ^j is denoted by $\phi^i \phi^j \delta^k_{ij} = \phi^i_{\mu_1\ldots\mu_k}\phi^{j\,\mu_1\ldots\mu_k}$, and the momenta which are dotted into the remaining indices of the ϕ's are written with their power as a collective index, e.g. $\phi^1\cdot k_2^p := \phi^1_{\rho_1\ldots\rho_p}\, k_2^{\rho_1}\ldots k_2^{\rho_p}$. This convention admits to rewrite the three point function (9.3.84) more compactly as

$$\mathcal{A}(\phi_1^{s_1}, \phi_2^{s_2}, \phi_3^{s_3}) \ = \ \sqrt{2\alpha'}^{\,s_1+s_2+s_3}\, n_1!\, n_2!\, n_3! \sum_{i,j,k\in\mathcal{I}_b} \frac{(2\alpha')^{-i-j-k}\,(is_3 \,+\, js_2 \,+\, ks_1 \,-\, ij \,-\, ik \,-\, jk)}{i!\, j!\, k!\, (s_1-i-j)!\, (s_2-i-k)!\, (s_3-j-k)!}$$
$$\times\ (\phi^1\cdot k_2^{s_1-i-j})\,(\phi^2\cdot k_3^{s_2-i-k})\,(\phi^3\cdot k_1^{s_3-j-k})\,\delta^i_{12}\,\delta^j_{13}\,\delta^k_{23}\ . \tag{9.3.86}$$

The essential difference to the analogous formula of bosonic string theory (equation (4.1) of [111]) lies in the factor $is_3 \,+\, js_2 \,+\, ks_1 \,-\, ij \,-\, ik \,-\, jk$ (where the bosonic string has the universal factor $s_1s_2s_3$ instead). As a result, the $i=j=k=0$ term with highest α' power is absent in superstring theory. For three gauge bosons $s_i = 1$, for instance, the summation range $\mathcal{I}_b$ encompasses $(i,j,k) = (0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and suppression of the first term ensures the absence of $\mathcal{F}^3$ interactions occurring on the bosonic string.

One of the most interesting three point vertices is the electromagnetic coupling of spins s_2 and s_3, i.e. the special case $n_1 = 0$ where the first state is set to massless spin $s_1 = 1$ with polarization vector $\phi^1_\mu \equiv \xi_\mu$. For identical spins $s_2 = s_3 \equiv s$ at level $n = s-1$ we find

$$\mathcal{A}(g_1, \phi_2^s, \phi_3^s) \ = \ (2\alpha')^{s+\frac{1}{2}}\,(n!)^2 \sum_{k=0}^{s} \frac{(2\alpha')^{-k}\,\delta^k_{23}}{k!\,\big[\,(s-k)!\,\big]^2} \Big\{ k\,(\xi\cdot k_2)\,(\phi^2\cdot k_3^{s-k})\,(\phi^3\cdot k_1^{s-k})$$
$$+\ \frac{s\,(s-k)}{2\alpha'}\,(\xi^\mu\, k_1^\nu \,-\, \xi^\nu\, k_1^\mu)\,(\phi^2_\mu\cdot k_3^{s-k-1})\,(\phi^3_\nu\cdot k_1^{s-k-1}) \Big\}\ , \tag{9.3.87}$$

and different spins s_2, s_3 give rise to

$$\mathcal{A}(g_1,\phi_2^{s_2}, \phi_3^{s_3}) \ = \ \sqrt{2\alpha'}^{\,1+s_2+s_3}\, n_2!\, n_3! \sum_{k=0}^{\min(s_2,s_3)} \frac{(2\alpha')^{-k}\,\delta^k_{23}}{k!\,(s_2-k)!\,(s_3-k)!}$$
$$\times\ \Big\{ k\,(\xi\cdot k_2)\,(\phi^2\cdot k_3^{s_2-k})\,(\phi^3\cdot k_1^{s_3-k}) \ +\ \frac{s_3\,(s_2-k)}{2\alpha'}\,\xi^\mu\,(\phi^2_\mu\cdot k_3^{s_2-k-1})\,(\phi^3\cdot k_1^{s_3-k})$$
$$+\ \frac{s_2\,(s_3-k)}{2\alpha'}\,(\phi^2\cdot k_3^{s_2-k})\,\xi^\mu\,(\phi^3_\mu\cdot k_1^{s_3-k-1}) \Big\}\ . \tag{9.3.88}$$

As a further specialization, the coupling of two massless gauge bosons with polarization vectors ξ_2, ξ_3 to massive higher spin states ϕ is found to be

$$\mathcal{A}(\phi_1^s, g_2, g_3) \;=\; \sqrt{2\alpha'}^{n+1} \left\{ \frac{n}{2\alpha'} \, \xi_2^\mu \, \xi_3^\nu \, (\phi_{\mu\nu} \cdot k_2^{n-1}) \;+\; (\xi_3 \, k_1) \, \xi_2^\mu \, (\phi_\mu \cdot k_2^n) \right. \\ \left. +\; (\xi_2 \, k_3) \, \xi_3^\mu \, (\phi_\mu \cdot k_2^n) \;+\; (\xi_2 \, \xi_3) \, (\phi \cdot k_2^{n+1}) \right\} . \tag{9.3.89}$$

Note that in contrast to heterotic string theories, higher spin fields always decay into lower spin fields. In the heterotic case at least the BPS higher spin fields are protected from that decay [141].

9.3.3 Two fermion, one boson coupling

A second class of nonvanishing three point couplings among leading Regge trajectory states involves two fermions and one boson. This time, all the vertex operators can appear in their canonical ghost picture. After substituting the $\psi^\mu \not{\psi}_{a\dot{b}} S^{\dot{b}}$ correlators from section 6.6 and appropriately labelling summation indices i, j, k, we find

$$\mathcal{A}(\phi_1^{s_1}, \chi_2^{n_2+1/2}, \bar{\chi}_3^{n_3+1/2}) \;=\; \frac{\sqrt{2\alpha'}^{n_1+n_2+n_3}}{\sqrt{2}} \sum_{i,j,k \in \mathcal{I}_f} \frac{n_1! \, (n_2-1)! \, (n_3-1)! \, (2\alpha')^{-i-j-k} \, \delta_{12}^i \, \delta_{13}^j \, \delta_{23}^k}{i! \, j! \, k! \, (s_1-i-j)! \, (n_2-i-k)! \, (n_3-j-k)!} \\ \times \left\{ n_2 \, n_3 \, (s_1-i-j) \, (\phi_\mu \cdot k_2^{n_1-i-j}) \, (k_3^{n_2-i-k} \cdot \chi_2^a) \, (\sigma^\mu \, \varepsilon)_a{}^{\dot{b}} \, (\bar{\chi}_{3\dot{b}} \cdot k_1^{n_3-j-k}) \right. \\ +\, j \, n_2 \, 2\alpha' \, (\phi \cdot k_2^{n_1+1-i-j}) \, (k_3^{n_2-i-k} \cdot \chi_2^a) (\not{k}_3)_a{}^{\dot{b}} \, (\bar{\chi}_{3\dot{b}} \cdot k_1^{n_3-j-k}) \\ -\, i \, n_3 \, 2\alpha' \, (\phi \cdot k_2^{n_1+1-i-j}) \, (k_3^{n_2-i-k} \cdot \chi_2^a) (\not{k}_2)_a{}^{\dot{b}} \, (\bar{\chi}_{3\dot{b}} \cdot k_1^{n_3-j-k}) \\ \left. -\, \alpha' \, k \, (s_1-i-j) \, (\phi_\mu \cdot k_2^{n_1-i-j}) \, (k_3^{n_2-i-k} \cdot \chi_2^a) \, (\not{k}_2 \, \bar{\sigma}^\mu \, \not{k}_3)_a{}^{\dot{b}} \, (\bar{\chi}_{3\dot{b}} \cdot k_1^{n_3-j-k}) \right\} \tag{9.3.90}$$

with shorthand $(\not{k}_i)_a{}^{\dot{b}} \equiv k_i^\mu (\sigma_\mu \varepsilon)_a{}^{\dot{b}}$ and summation range

$$\mathcal{I}_f \;:=\; \Big\{ i, j, k \in \mathbb{N}_0 \;:\; s_1 - i - j \geq 0 \;, \quad n_2 - i - k \geq 0 \;, \quad n_3 - j - k \geq 0 \Big\} . \tag{9.3.91}$$

Let us explicitly display the electromagnetic coupling with $\xi_\mu \equiv \phi_\mu$ to higher spin fermions, firstly at coinciding mass level $n_2 = n_3 \equiv n$:

$$\mathcal{A}(g_1, \chi_2^{n+1/2}, \bar{\chi}_3^{n+1/2}) \;=\; \frac{(2\alpha')^n}{\sqrt{2}} \, \big[(n-1)! \big]^2 \sum_{k=0}^n \frac{(2\alpha')^{-k} \, \delta_{23}^k}{k! \, \big[(n-k)! \big]^2} \\ \times \left\{ n^2 \, (k_3^{n-k} \cdot \chi_2^a) \, \not{\xi}_a{}^{\dot{b}} \, (\bar{\chi}_{3\dot{b}} \cdot k_1^{n-k}) \;-\; \alpha' \, k \, (k_3^{n-k} \cdot \chi_2^a) \, (\not{k}_2 \, \not{\xi} \, \not{k}_3)_a{}^{\dot{b}} \, (\bar{\chi}_{3\dot{b}} \cdot k_1^{n-k}) \right. \\ \left. +\, n \, (n-k) \, \xi^\mu \Big[(k_3^{n-k} \cdot \chi_2^a) \, (\not{k}_3)_a{}^{\dot{b}} \, (\bar{\chi}_{3\dot{b}} \cdot k_1^{n-k-1}) \;-\; (k_3^{n-k-1} \cdot \chi_{2\mu}^a) \, (\not{k}_2)_a{}^{\dot{b}} \, (\bar{\chi}_{3\dot{b}} \cdot k_1^{n-k}) \Big] \right\} \tag{9.3.92}$$

For different fermion spins $s_2 < s_3$, on the other hand,

$$\begin{aligned} \mathcal{A}(g_1, \chi_2^{n_2+1/2}, \bar{\chi}_3^{n_3+1/2}) &= \frac{\sqrt{2\alpha'}^{n_2+n_3}}{\sqrt{2}} (n_2-1)!\,(n_3-1)! \sum_{k=0}^{n_2} \frac{(2\alpha')^{-k}\,\delta_{23}^k}{k!\,(n_2-k)!\,(n_3-k)!} \\ \times & \Big\{ n_2\,n_3\,(k_3^{n_2-k}\cdot\chi_2^a)\ \not{\xi}_a{}^{\dot b}\,(\bar\chi_{3\dot b}\cdot k_1^{n_3-k}) \ - \ \alpha'\,k\,(k_3^{n_2-k}\cdot\chi_2^a)(\not{k}_2\ \not{\xi}\ \not{k}_3)_a{}^{\dot b}\,(\bar\chi_{3\dot b}\cdot k_1^{n_3-k}) \\ &+ n_2\,(n_3-k)\,(k_3^{n_2-k}\cdot\chi_2^a)\ \not{k}_{3a}{}^{\dot b}\,(\bar\chi_{3\mu\dot b}\cdot k_1^{n_3-k-1})\,\xi^\mu \\ &- n_3\,(n_2-k)\,\xi^\mu\,(k_3^{n_2-k-1}\cdot\chi_{2\mu}^a)\ \not{k}_{2a}{}^{\dot b}\,(\bar\chi_{3\dot b}\cdot k_1^{n_3-k}) \Big\}\,. \end{aligned} \tag{9.3.93}$$

Another interesting special case is one massless Weyl fermion of spin $s_2 = \frac{1}{2}$ whose wave function u^a satisfies the massless Dirac equation $u^a\,\not{k}_{a\dot b} = 0$ with respect to its momentum:

$$\begin{aligned} \mathcal{A}(\phi_1^{s_1}, \lambda, \bar\chi_3^{n_3+1/2}) &= \frac{\sqrt{2\alpha'}^{n_1+n_3}}{\sqrt{2}}\, n_1!\,(n_3-1)! \sum_{j=0}^{\min(n_1+1,n_3)} \frac{(2\alpha')^{-j}\,\delta_{13}^j}{j!\,(s_1-j)!\,(n_3-j)!} \\ \times & \Big\{ n_3\,(s_1-j)\,(\phi_\mu\cdot k_2^{n_1-j})\,u^a\,(\sigma^\mu\,\varepsilon)_a{}^{\dot b}\,(\bar\chi_{3\dot b}\cdot k_1^{n_3-j}) \ + \ j\,2\alpha'\,(\phi\cdot k_2^{n_1+1-j})\,u^a\,\not{k}_{3a}{}^{\dot b}\,(\bar\chi_{3\dot b}\cdot k_1^{n_3-j}) \Big\} \end{aligned} \tag{9.3.94}$$

As explained in subsection 9.3.2, two fermion amplitudes involving leading Regge trajectory states are independent on the superstring compactification and the number of spacetime dimensions, just like parton amplitudes involving two quarks.

9.3.4 Three massless bosons, one higher spin boson

The four boson amplitude with one higher spin state from the leading Regge trajectory requires an equally simple input from the RNS CFT of the ψ^μ fields, regardsless on the spin. The nontrivial contributions come from the $i\partial X^\mu$ correlators (B.4.66), the result for $n \geq 1$ is

$$\begin{aligned} \mathcal{A}(g_1, g_2, g_3, \phi_4^{n+1}) &= \sqrt{2\alpha'}^{n+2}\,(-1)^{n-1} \\ & \Big\{ \frac{n\,(n-1)}{4\alpha'^2}\,\xi_\mu^1\,\xi_\nu^2\,\xi_\lambda^3 \sum_{p=0}^{n-2} \binom{n-2}{p} \left[\phi^{\mu\nu\lambda}\cdot k_1^p\,(-k_3)^{n-2-p}\right] B(s+1+p, u+n-1-p) \\ &+ \frac{1-t}{2\alpha'}\,(\xi^1\,\xi^3)\,\xi_2^\mu \sum_{p=0}^{n} \binom{n}{p} \left[\phi^{\mu\nu\lambda}\cdot k_1^p\,(-k_3)^{n-p}\right] B(s+1+p, u+n+1-p) \\ &+ \Big[(\xi^1\,k_2)\,(\xi^3\,k_1)\,\xi_\mu^2\,k_\nu^2 \ - \ (\xi^1\,\xi^2)\,(\xi^3\,k_1)\,k_\mu^2\,k_\nu^2 \ + \ (\xi^3\,\xi^2)\,(\xi^1\,k_3)\,k_\mu^2\,k_\nu^2 \ - \ (\xi^3\,k_2)\,(\xi^1\,k_3)\,\xi_\mu^2\,k_\nu^2 \\ &\quad + \ (\xi^2\,k_4)\,(\xi^3\,k_1)\,\xi_\mu^1\,k_\nu^2 \ - \ (\xi^2\,k_4)\,(\xi^1\,k_3)\,\xi_\mu^3\,k_\nu^2 \ + \ (\xi^1\,\xi^3)\,(\xi^2\,k_3)\,k_\mu^1\,k_\nu^2 \ - \ (\xi^1\,\xi^3)\,(\xi^2\,k_1)\,k_\mu^3\,k_\nu^2 \\ &\quad + \ \frac{t}{2\alpha'}\,(\xi^2\,\xi^3)\,\xi_\mu^1\,k_\nu^2 \ - \ \frac{t}{2\alpha'}\,(\xi^1\,\xi^2)\,\xi_\mu^3\,k_\nu^2 \ + \ \frac{n}{2\alpha'}\,(\xi^1\,k_3)\,\xi_\mu^2\,\xi_\nu^3 \ - \ \frac{n}{2\alpha'}\,(\xi^3\,k_1)\,\xi_\mu^1\,\xi_\nu^2 \Big] \end{aligned}$$

$$\times \sum_{p=0}^{n-1} \binom{n-1}{p} \left[\phi^{\mu\nu} \cdot k_1^p \, (-k_3)^{n-1-p}\right] B(s+1+p, u+n-p)$$

$$+ \Big[(\xi^1 k_3)(\xi^2 k_1) \xi^3_\mu k^3_\nu - (\xi^1 \xi^3)(\xi^2 k_1) k^3_\mu k^3_\nu + (\xi^2 \xi^3)(\xi^1 k_2) k^3_\mu k^3_\nu - (\xi^2 k_3)(\xi^1 k_2) \xi^3_\mu k^3_\nu$$

$$+ (\xi^3 k_4)(\xi^2 k_1) \xi^1_\mu k^3_\nu - (\xi^3 k_4)(\xi^1 k_2) \xi^2_\mu k^3_\nu + (\xi^1 \xi^2)(\xi^3 k_2) k^1_\mu k^3_\nu - (\xi^1 \xi^2)(\xi^3 k_1) k^2_\mu k^3_\nu$$

$$+ \frac{n}{2\alpha'} (\xi^1 k_2) \xi^2_\mu \xi^3_\nu - \frac{n}{2\alpha'} (\xi^2 k_1) \xi^1_\mu \xi^3_\nu + \frac{n}{2\alpha'} (\xi^1 \xi^2) \xi^3_\mu k^1_\nu - \frac{t}{2\alpha'} (\xi^1 \xi^2) \xi^3_\mu k^3_\nu \Big]$$

$$\times \sum_{p=0}^{n-1} \binom{n-1}{p} \left[\phi^{\mu\nu} \cdot k_2^p \, k_1^{n-1-p}\right] B(s, u+1+p)$$

$$+ \Big[(\xi^2 k_1)(\xi^3 k_2) \xi^1_\mu k^1_\nu - (\xi^1 \xi^2)(\xi^3 k_2) k^1_\mu k^1_\nu + (\xi^1 \xi^3)(\xi^2 k_3) k^1_\mu k^1_\nu - (\xi^3 k_1)(\xi^2 k_3) \xi^1_\mu k^1_\nu$$

$$+ (\xi^1 k_4)(\xi^3 k_2) \xi^2_\mu k^1_\nu - (\xi^1 k_4)(\xi^2 k_3) \xi^3_\mu k^1_\nu + (\xi^2 \xi^3)(\xi^1 k_3) k^1_\mu k^2_\nu - (\xi^2 \xi^3)(\xi^1 k_2) k^1_\mu k^3_\nu$$

$$+ \frac{n}{2\alpha'} (\xi^2 k_3) \xi^1_\mu \xi^3_\nu - \frac{n}{2\alpha'} (\xi^3 k_2) \xi^1_\mu \xi^2_\nu - \frac{n}{2\alpha'} (\xi^2 \xi^3) \xi^1_\mu k^3_\nu + \frac{t}{2\alpha'} (\xi^2 \xi^3) \xi^1_\mu k^1_\nu \Big]$$

$$\times \sum_{p=0}^{n-1} \binom{n-1}{p} \left[\phi^{\mu\nu} \cdot (-k_3)^p \, (-k_2)^{n-1-p}\right] B(s+n-p, u) \Bigg\} . \tag{9.3.95}$$

This pattern of Beta functions has already been observed in [272] in the context of tachyon scattering with one massive leading trajectory state on the bosonic string. The organization scheme is guided by the $n = 1$ result (9.2.45).

Any beta function $B(s+p, u+q)$ with integers $p, q \in \mathbb{Z}$ can be reduced to V_t by factoring out quotients of gamma functions

$$\gamma(x, n) := \frac{\Gamma(n-x)}{n!\,\Gamma(-x)} = \frac{1}{n!} \prod_{j=0}^{n-1} (-x+j) \tag{9.3.96}$$

which already appeared as residue of V_t at massive poles in (8.1.25). In agreement with (9.1.6), the four point amplitude at total mass level n contains a prefactor $V_t \prod_{k=1}^{n-1} (t+k)^{-1}$ which is specific to the ordering $(1,2,3,4)$ of the external legs and a totally antisymmetric kinematic factor $\mathcal{A}_0$ under rearrangement $(1,2,3) \mapsto (2,3,1)$ of the three gluons (although this symmetry is not immediately obvious in the term $\sim \xi^1_\mu \xi^2_\nu \xi^3_\lambda \phi^{\mu\nu\lambda}$). Moreover, exchange of two massless states $\xi^i \leftrightarrow \xi^j$ yields the n dependent sign $(-)^n$ in this color stripped amplitude to compensate for the $(-)^n$ in the color factor $\mathrm{Tr}\{T^{a_1} T^{a_2} T^{a_3} T^{a_4} + (-1)^n T^{a_4} T^{a_3} T^{a_2} T^{a_1}\}$ and preserve the bosonic statistics of gluons.

$$\mathcal{A}(g_1, g_2, g_3, \phi_4^{n+1}) = \frac{\sqrt{2\alpha'}^{\,n+2} (n-1)!\, V_t}{\prod_{k=1}^{n-1} (t+k)}$$

$$\Bigg\{ \frac{n}{4\alpha'^2} \sum_{p=0}^{n-2} \gamma(-s-1, p)\, \gamma(-u-1, n-2-p)\, \xi^1_\mu \xi^2_\nu \xi^3_\lambda \left[\phi^{\mu\nu\lambda} \cdot k_1^p \, (-k_3)^{n-2-p}\right]$$

$$
\begin{aligned}
&- \frac{n}{2\alpha' s} \sum_{p=0}^{n} \gamma(-u-1,p)\,\gamma(-t-1,n-p)\,(\xi^1\,\xi^2)\,\xi^3_\mu \left[\phi^\mu \cdot k_2^p\,(-k_1)^{n-p}\right] \\
&- \frac{n}{2\alpha' t} \sum_{p=0}^{n} \gamma(-s-1,p)\,\gamma(-u-1,n-p)\,(\xi^1\,\xi^3)\,\xi^2_\mu \left[\phi^\mu \cdot k_1^p\,(-k_3)^{n-p}\right] \\
&- \frac{n}{2\alpha' u} \sum_{p=0}^{n} \gamma(-t-1,p)\,\gamma(-s-1,n-p)\,(\xi^2\,\xi^3)\,\xi^1_\mu \left[\phi^\mu \cdot k_3^p\,(-k_2)^{n-p}\right] \\
&+ \frac{1}{s} \sum_{p=0}^{n-1} \gamma(-u-1,p)\,\gamma(-t-1,n-1-p) \left[\phi^{\mu\nu} \cdot k_2^p\,(-k_1)^{n-1-p}\right] \Big[\frac{n}{2\alpha'}\,(\xi^1\,k_2)\,\xi^2_\mu\,\xi^3_\nu \\
&\qquad - \frac{n}{2\alpha'}\,(\xi^2\,k_1)\,\xi^1_\mu\,\xi^3_\nu \;+\; (\xi^1\,k_3)\,(\xi^2\,k_1)\,\xi^3_\mu\,k^3_\nu \;-\; (\xi^1\,\xi^3)\,(\xi^2\,k_1)\,k^3_\mu\,k^3_\nu \\
&\qquad + (\xi^2\,\xi^3)\,(\xi^1\,k_2)\,k^3_\mu\,k^3_\nu \;-\; (\xi^2\,k_3)\,(\xi^1\,k_2)\,\xi^3_\mu\,k^3_\nu \;+\; (\xi^3\,k_4)\,(\xi^2\,k_1)\,\xi^1_\mu\,k^3_\nu \\
&\qquad - (\xi^3\,k_4)\,(\xi^1\,k_2)\,\xi^2_\mu\,k^3_\nu \;+\; (\xi^1\,\xi^2)\,(\xi^3\,k_2)\,k^1_\mu\,k^3_\nu \;-\; (\xi^1\,\xi^2)\,(\xi^3\,k_1)\,k^2_\mu\,k^3_\nu \Big] \\
&+ \frac{1}{t} \sum_{p=0}^{n-1} \gamma(-s-1,p)\,\gamma(-u-1,n-1-p) \left[\phi^{\mu\nu} \cdot k_1^p\,(-k_3)^{n-1-p}\right] \Big[\frac{n}{2\alpha'}\,(\xi^3\,k_1)\,\xi^1_\mu\,\xi^2_\nu \\
&\qquad - \frac{n}{2\alpha'}\,(\xi^1\,k_3)\,\xi^2_\mu\,\xi^3_\nu \;+\; (\xi^3\,k_2)\,(\xi^1\,k_3)\,\xi^2_\mu\,k^2_\nu \;-\; (\xi^3\,\xi^2)\,(\xi^1\,k_3)\,k^2_\mu\,k^2_\nu \\
&\qquad + (\xi^1\,\xi^2)\,(\xi^3\,k_1)\,k^2_\mu\,k^2_\nu \;-\; (\xi^1\,k_2)\,(\xi^3\,k_1)\,\xi^2_\mu\,k^2_\nu \;+\; (\xi^2\,k_4)\,(\xi^1\,k_3)\,\xi^3_\mu\,k^2_\nu \\
&\qquad - (\xi^2\,k_4)\,(\xi^3\,k_1)\,\xi^1_\mu\,k^2_\nu \;+\; (\xi^1\,\xi^3)\,(\xi^2\,k_1)\,k^3_\mu\,k^2_\nu \;-\; (\xi^1\,\xi^3)\,(\xi^2\,k_3)\,k^1_\mu\,k^2_\nu \Big] \\
&+ \frac{1}{u} \sum_{p=0}^{n-1} \gamma(-t-1,p)\,\gamma(-s-1,n-1-p) \left[\phi^{\mu\nu} \cdot k_3^p\,(-k_2)^{n-1-p}\right] \Big[\frac{n}{2\alpha'}\,(\xi^2\,k_3)\,\xi^1_\mu\,\xi^3_\nu \\
&\qquad - \frac{n}{2\alpha'}\,(\xi^3\,k_2)\,\xi^1_\mu\,\xi^2_\nu \;+\; (\xi^2\,k_1)\,(\xi^3\,k_2)\,\xi^1_\mu\,k^1_\nu \;-\; (\xi^1\,\xi^2)\,(\xi^3\,k_2)\,k^1_\mu\,k^1_\nu \\
&\qquad + (\xi^1\,\xi^3)\,(\xi^2\,k_3)\,k^1_\mu\,k^1_\nu \;-\; (\xi^3\,k_1)\,(\xi^2\,k_3)\,\xi^1_\mu\,k^1_\nu \;+\; (\xi^1\,k_4)\,(\xi^3\,k_2)\,\xi^2_\mu\,k^1_\nu \\
&\qquad - (\xi^1\,k_4)\,(\xi^2\,k_3)\,\xi^3_\mu\,k^1_\nu \;+\; (\xi^2\,\xi^3)\,(\xi^1\,k_3)\,k^1_\mu\,k^2_\nu \;-\; (\xi^2\,\xi^3)\,(\xi^1\,k_2)\,k^1_\mu\,k^3_\nu \Big] \Big\} \qquad (9.3.97)
\end{aligned}
$$

Note that at $n = 1$, this reproduces (9.2.45) for decay of spin two into three gluons.

9.3.5 Two massless bosons, one massless and one higher spin fermion

The next four point amplitude of interest involves two fermions of spin 1/2 and $n + 1/2$, respectively. Its evaluation requires the correlation function (6.6.99) of the $\psi^\mu \not\psi_{a\dot b} S^{\dot b}$ operator in the massive fermion's vertex operator (9.3.82):

$$
\mathcal{A}(g_1, g_2, \lambda_3, \bar\chi_4^{n+1/2}) \;=\; \frac{\sqrt{2\alpha'}^{\,n+1}}{\sqrt{2}}\,(-1)^{n-1}
$$

$$\Big\{ \frac{n-1}{2\alpha'} \, \xi_1^\mu \, \xi_2^\nu \, (u \, \not k_4)^{\dot a} \sum_{p=0}^{n-2} \binom{n-2}{p} \left[\bar\chi_{\mu\nu\dot a} \cdot k_1^p \, k_2^{n-2-p} \right] B(s+1, u+n-p-1)$$

$$+ \Big[(\xi^2 \, k_1) \, (u \, \not\xi^1)^{\dot a} \, k_3^\mu \; - \; (\xi^1 \, k_2) \, (u \, \not\xi^2)^{\dot a} \, k_3^\mu \; + \; (\xi^1 \, k_2) \, (u \, \not k_4)^{\dot a} \, \xi_2^\mu \; - \; (\xi^2 \, k_1) \, (u \, \not k_4)^{\dot a} \, \xi_1^\mu$$

$$+ \, (\xi^1 \, \xi^2) \, (u \, \not k_1)^{\dot a} \, k_2^\mu \; - \; (\xi^1 \, \xi^2) \, (u \, \not k_2)^{\dot a} \, k_1^\mu \Big] \sum_{p=0}^{n-1} \binom{n-1}{p} \left[\bar\chi_{\mu\dot a} \cdot k_1^p \, k_2^{n-1-p} \right] B(s, u+n-p)$$

$$+ \Big[\frac{1}{2} \, (u \, \not k_2 \, \not\xi^2 \, \not\xi^1)^{\dot a} \, k_1^\mu \; - \; \frac{1}{2} \, (u \, \not k_2 \, \not\xi^2 \, \not k_1)^{\dot a} \, \xi_1^\mu \; + \; (\xi^2 \, k_3) \, (u \, \not\xi^1)^{\dot a} \, k_1^\mu \; - \; (\xi^2 \, k_3) \, (u \, \not k_1)^{\dot a} \, \xi_1^\mu$$

$$- \, \frac{u}{2\alpha'} \, (u \, \not\xi^2)^{\dot a} \, \xi_1^\mu \Big] \sum_{p=0}^{n-1} \binom{n-1}{p} \left[\bar\chi_{\mu\dot a} \cdot (-k_3)^p \, (-k_2)^{n-1-p} \right] B(s+n-p, u)$$

$$+ \Big[\frac{1}{2} \, (u \, \not k_1 \, \not\xi^1 \, \not\xi^2)^{\dot a} \, k_2^\mu \; - \; \frac{1}{2} \, (u \, \not k_1 \, \not\xi^1 \, \not k_2)^{\dot a} \, \xi_2^\mu \; + \; (\xi^1 \, k_3) \, (u \, \not\xi^2)^{\dot a} \, k_2^\mu \; - \; (\xi^1 \, k_3) \, (u \, \not k_2)^{\dot a} \, \xi_2^\mu$$

$$- \, \frac{t}{2\alpha'} \, (u \, \not\xi^1)^{\dot a} \, \xi_2^\mu \Big] \sum_{p=0}^{n-1} \binom{n-1}{p} \left[\bar\chi_{\mu\dot a} \cdot k_1^p \, (-k_3)^{n-1-p} \right] B(s+1+p, u+n-p) \Big\} \tag{9.3.98}$$

Expressing the Beta functions in terms of V_t admits to extract the universal part of this subamplitude:

$$\mathcal{A}(g_1, g_2, \lambda_3, \bar\chi_4^{n+1/2}) \;=\; \frac{\sqrt{2\alpha'}^{n+1} \, (n-1)! \, V_t}{\sqrt{2} \, \prod_{k=1}^{n-1} (t+k)}$$

$$\Big\{ \frac{1}{2\alpha'} \sum_{p=0}^{n-2} \gamma(-t-1, p) \, \gamma(-u-1, n-2-p) \, \xi_1^\mu \, \xi_2^\nu \, (u \, \not k_4)^{\dot a} \left[\bar\chi_{\mu\nu\dot a} \cdot (-k_1)^p \, k_2^{n-2-p} \right]$$

$$+ \; \frac{1}{u} \sum_{p=0}^{n-1} \gamma(-s-1, p) \, \gamma(-t-1, n-1-p) \left[\bar\chi_{\mu\dot a} \cdot (-k_2)^p \, k_3^{n-1-p} \right] \Big[- \, \frac{u}{2\alpha'} \, (u \, \not\xi^2)^{\dot a} \, \xi_1^\mu$$

$$+ \, (\xi^2 \, k_3) \, (u \, \not\xi^1)^{\dot a} \, k_1^\mu \; - \; (\xi^2 \, k_3) \, (u \, \not k_1)^{\dot a} \, \xi_1^\mu \; + \; \frac{1}{2} \, (u \, \not k_2 \, \not\xi^2 \, \not\xi^1)^{\dot a} \, k_1^\mu \; - \; \frac{1}{2} \, (u \, \not k_2 \, \not\xi^2 \, \not k_1)^{\dot a} \, \xi_1^\mu \Big]$$

$$- \; \frac{1}{t} \sum_{p=0}^{n-1} \gamma(-s-1, p) \, \gamma(-u-1, n-1-p) \left[\bar\chi_{\mu\dot a} \cdot k_1^p \, (-k_3)^{n-1-p} \right] \Big[- \, \frac{t}{2\alpha'} \, (u \, \not\xi^1)^{\dot a} \, \xi_2^\mu$$

$$+ \, (\xi^1 \, k_3) \, (u \, \not\xi^2)^{\dot a} \, k_2^\mu \; - \; (\xi^1 \, k_3) \, (u \, \not k_2)^{\dot a} \, \xi_2^\mu \; + \; \frac{1}{2} \, (u \, \not k_1 \, \not\xi^1 \, \not\xi^2)^{\dot a} \, k_2^\mu \; - \; \frac{1}{2} \, (u \, \not k_1 \, \not\xi^1 \, \not k_2)^{\dot a} \, \xi_2^\mu \Big]$$

$$+ \; \frac{1}{s} \sum_{p=0}^{n-1} \gamma(-t-1, p) \, \gamma(-u-1, n-1-p) \left[\bar\chi_{\mu\dot a} \cdot (-k_1)^p \, k_2^{n-1-p} \right]$$

$$\Big[(\xi^2 \, k_1) \, (u \, \not\xi^1)^{\dot a} \, k_3^\mu \; - \; (\xi^1 \, k_2) \, (u \, \not\xi^2)^{\dot a} \, k_3^\mu \; + \; (\xi^1 \, k_2) \, (u \, \not k_4)^{\dot a} \, \xi_2^\mu$$

$$- \; (\xi^2 \, k_1) \, (u \, \not k_4)^{\dot a} \, \xi_1^\mu \; + \; (\xi^1 \, \xi^2) \, (u \, \not k_1)^{\dot a} \, k_2^\mu \; - \; (\xi^1 \, \xi^2) \, (u \, \not k_2)^{\dot a} \, k_1^\mu \Big] \Big\} \tag{9.3.99}$$

For $n = 1$, this reduces to the spin $s = \frac{3}{2}$ coupling (9.2.53) to a massless fermion and two gluons. This amplitude shares the $(-)^n$ eigenvalue under exchange of the massless bosons $\xi^1 \leftrightarrow \xi^2$ with the previous example.

9.3.6 One massless and one higher spin boson, two massless fermions

This subsection presents another two fermion, two boson amplitude, this time we include a mass level n boson and keep both fermions massless. The correlators are straightforward, and some σ matrix algebra helps to reduce the number of distinct kinematics to the following:

$$\begin{aligned}
\mathcal{A}(\lambda_1, g_2, \bar{\lambda}_3, \phi_4^{n+1}) \;&=\; \frac{\sqrt{2\alpha'}^{n+1}}{\sqrt{2}} \, (-1)^{n-1} \\
\Big\{ \Big[\frac{n}{2\alpha'} \, (u_1 \, \sigma^\mu \, \bar{u}_3) \, \xi^\nu \;&-\; (u_1 \, \sigma^\mu \, \bar{u}_3) \, k_2^\nu \, (\xi \, k_4) \;+\; (k_2^\mu \, \xi^\lambda \;-\; k_2^\lambda \, \xi^\mu) \, (u_1 \, \sigma_\lambda \, \bar{u}_3) \, k_2^\nu \Big] \\
&\sum_{p=0}^{n-1} \tbinom{n-1}{p} \, \big[\phi_{\mu\nu} \cdot k_1^p \, (-k_3)^{n-1-p} \big] \, B(s+1+p, u+n-p) \\
+ \Big[(u_1 \, \sigma^\mu \, \bar{u}_3) \, k_1^\nu \, (\xi \, k_3) \;+\; \frac{1}{2} \, (u_1 \, \sigma^\mu \, \not{\xi} \, \not{k}_2 \, \bar{u}_3) \, k_1^\nu \Big] &\sum_{p=0}^{n-1} \tbinom{n-1}{p} \, \big[\phi_{\mu\nu} \cdot (-k_2)^p \, (-k_3)^{n-1-p} \big] \, B(s+1+p, u) \\
+ \Big[(u_1 \, \sigma^\mu \, \bar{u}_3) \, k_3^\nu \, (\xi \, k_1) \;+\; \frac{1}{2} \, (u_1 \, \not{k}_2 \, \not{\xi} \, \sigma^\mu \, \bar{u}_3) \, k_3^\nu \Big] &\sum_{p=0}^{n-1} \tbinom{n-1}{p} \, \big[\phi_{\mu\nu} \cdot k_2^p \, k_1^{n-1-p} \big] \, B(s, u+1+p) \Big\}
\end{aligned} \tag{9.3.100}$$

The usual rearrangement of the Beta function leads to

$$\begin{aligned}
\mathcal{A}(\lambda_1, g_2, \bar{\lambda}_3, \phi_4^{n+1}) \;&=\; \frac{\sqrt{2\alpha'}^{n+1} \, (n-1)! \, V_t}{\sqrt{2} \, \prod_{k=1}^{n-1}} \\
\Big\{ \frac{1}{t} \sum_{p=0}^{n-1} \gamma(-s-1, p) \, \gamma(-u-1, n-p-1) \, &\big[\phi_{\mu\nu} \cdot k_1^p \, (-k_3)^{n-1-p} \big] \\
&\Big[+ \, (u_1 \, \sigma^\mu \, \bar{u}_3) \, k_2^\nu \, (\xi \, k_4) \;-\; \frac{n}{2\alpha'} \, (u_1 \, \sigma^\mu \, \bar{u}_3) \, \xi^\nu \;+\; (k_2^\lambda \, \xi^\mu \;-\; k_2^\mu \, \xi^\lambda) \, (u_1 \, \sigma_\lambda \, \bar{u}_3) \, k_2^\nu \Big] \\
+ \; \frac{1}{u} \sum_{p=0}^{n-1} \gamma(-t-1, p) \, \gamma(-s-1, n-1-p) \, &\big[\phi_{\mu\nu} \cdot k_3^p \, (-k_2)^{n-1-p} \big] \\
&\Big[(u_1 \, \sigma^\mu \, \bar{u}_3) \, k_1^\nu \, (\xi \, k_3) \;+\; \frac{1}{2} \, (u_1 \, \sigma^\mu \, \not{\xi} \, \not{k}_2 \, \bar{u}_3) \, k_1^\nu \Big] \\
+ \; \frac{1}{s} \sum_{p=0}^{n-1} \gamma(-t-1, p) \, \gamma(-u-1, n-1-p) \, &\big[\phi_{\mu\nu} \cdot k_2^{n-1-p} \, (-k_1)^p \big] \\
&\Big[(u_1 \, \sigma^\mu \, \bar{u}_3) \, k_3^\nu \, (\xi \, k_1) \;+\; \frac{1}{2} \, (u_1 \, \not{k}_2 \, \not{\xi} \, \sigma^\mu \, \bar{u}_3) \, k_3^\nu \Big] \Big\}
\end{aligned} \tag{9.3.101}$$

representing the higher spin extension of (9.2.48).

9.3.7 Three massless fermions, one higher spin fermion in $D = 10$

The last four point coupling of interest involves four fermions one of which is massive with spin $s = n + \frac{1}{2}$ on the leading Regge trajectory. As explained in subsection 9.3.1, four fermi amplitudes require specification of the spacetime dimension and the type of fermion, so let us discuss the $D=10$ dimensional case with four adjoint gauginos here:

$$\begin{aligned}
\mathcal{A}(\lambda_1, \lambda_2, \lambda_3, \chi_4^{n+1/2})^{D=10} &= \frac{1}{2}\,\sqrt{2\alpha'}^{\,n}\,(-1)^{n-1} \\
&\Big\{ - \Big[(u_1\,\gamma^r\,u_2)\,(u_3\,\gamma_r)_\alpha\,k_3^m \;-\; (u_1\,\gamma^m\,u_2)\,(u_3\,\not{k}_4)_\alpha \Big] \\
&\qquad \sum_{p=0}^{n-1} \binom{n-1}{p}\,\big[\chi_m^\alpha \cdot k_1^p\,k_2^{n-1-p}\big]\,B(s, u+n-p) \\
&\quad - \Big[(u_3\,\gamma^r\,u_2)\,(u_1\gamma_r)_\alpha\,k_1^m \;-\; (u_3\,\gamma^m\,u_2)\,(u_1\,\not{k}_4)_\alpha \Big] \\
&\qquad \sum_{p=0}^{n-1} \binom{n-1}{p}\,\big[\chi_m^\alpha \cdot (-k_3)^p\,(-k_2)^{n-1-p}\big]\,B(s+n-p, u) \\
&\quad + \Big[(u_1\,\gamma^r\,u_3)\,(u_2\,\gamma_r)_\alpha\,k_2^m \;-\; (u_1\,\gamma^m\,u_3)\,(u_2\,\not{k}_4)_\alpha \Big] \\
&\qquad \sum_{p=0}^{n-1} \binom{n-1}{p}\,\big[\chi_m^\alpha \cdot k_1^p\,(-k_3)^{n-1-p}\big]\,B(s+1+p, u+n-p) \Big\}
\end{aligned} \tag{9.3.102}$$

Like in the previous examples, we reduce the beta functions to V_t and find manifest cyclic symmetry in the labels $(1,2,3)$ of the massless Weyl fermions and $(-)^{n-1}$ parity under exchange of $1 \leftrightarrow 2$:

$$\begin{aligned}
\mathcal{A}(\lambda_1, \lambda_2, \lambda_3, \chi_4^{n+1/2})^{D=10} &= \frac{\sqrt{2\alpha'}^{\,n}\,(n-1)!\,V_t}{2\,\prod_{k=1}^{n-1}(t+k)} \\
\times \Big\{ \frac{1}{s} \sum_{p=0}^{n-1} &\gamma(-u-1,p)\,\gamma(-t-1,n-1-p) \\
&\Big[(u_1\,\gamma^m\,u_2)\,(u_3\,\not{k}_4)_\alpha \;-\; (u_1\,\gamma^r\,u_2)\,(u_3\,\gamma_r)_\alpha\,k_3^m \Big]\,\big[\chi_m^\alpha \cdot k_2^p\,(-k_1)^{n-p-1}\big] \\
+ \frac{1}{t} \sum_{p=0}^{n-1} &\gamma(-s-1,p)\,\gamma(-u-1,n-1-p) \\
&\Big[(u_1\,\gamma^m\,u_3)\,(u_2\,\not{k}_4)_\alpha \;-\; (u_1\,\gamma^r u_3)\,(u_2\,\gamma_r)_\alpha\,k_2^m \Big]\,\big[\chi_m^\alpha \cdot k_1^p\,(-k_3)^{n-p-1}\big] \\
+ \frac{1}{u} \sum_{p=0}^{n-1} &\gamma(-t-1,p)\,\gamma(-s-1,n-1-p) \\
&\Big[(u_3\,\gamma^m\,u_2)\,(u_1\,\not{k}_4)_\alpha \;-\; (u_3\,\gamma^r\,u_2)\,(u_1\gamma_r)_\alpha\,k_1^m \Big]\,\big[\chi_m^\alpha \cdot k_3^p\,(-k_2)^{n-p-1}\big] \Big\}
\end{aligned} \tag{9.3.103}$$

Let us display the spin 3/2 case explicitly because subsection 9.2.4 only gives the $D=4$ result:

$$\mathcal{A}(\lambda_1, \lambda_2, \lambda_3, \chi_4^{3/2})^{D=10} \;=\; \sqrt{\frac{\alpha'}{2}}\;V_t \Big\{ \frac{1}{s}\,(u_1\,\gamma^m\,u_2)\,\Big[(u_3\,\not{k}_4\,\chi_m) \;-\; (u_3\,\gamma_m\,\chi^p)\,k_p^3 \Big]$$

$$+ \ \frac{1}{t} \, (u_3 \, \gamma^m \, u_1) \Big[(u_2 \, \not{k}_4 \, \chi_m) \ - \ (u_2 \, \gamma_m \, \chi^p) \, k_p^2 \Big] \ + \ \frac{1}{u} \, (u_2 \, \gamma^m \, u_3) \Big[(u_1 \, \not{k}_4 \, \chi_m) \ - \ (u_1 \, \gamma_m \, \chi^p) \, k_p^1 \Big] \Bigg\} \tag{9.3.104}$$

Four point amplitudes with several massive states are under current research [113].

Part III

Massless tree amplitudes in the pure spinor formalism

Chapter 10

Basics of the Pure Spinor formalism

In this chapter, we will introduce the pure spinor approach to superstring theory, an alternative organization scheme of the worldsheet degrees of freedom. In the 90's of the last century, two different prescriptions for the computation of scattering amplitudes have been available, encompassed in the RNS and the Green Schwarz [273, 274, 275] formalisms. Beyond these traditional approaches, the so-called pure spinor formalism has been invented at the beginning of the new millennium [276], see [277, 278] for reviews. It was inspired by a $U(5)$ covariant quantization of the superstring which is related to the RNS formalism by field redefinitions and manifestly preserves six out of sixteen spacetime supercharges [279].

As a big advantage over its RNS- and GS relatives, the pure spinor formalism gives rise to manifestly Lorentz covariant and supersymmetric scattering amplitudes. Also, it allows to compute superstring amplitudes in string theories with non-trivial background e.g. warped setups or Ramond-Ramond fluxes like in $AdS_5 \times S^5$ [280, 281].

Many superstring amplitudes computed so far in the pure spinor formalism [282] turned out to be easier to obtain than with RNS method. A sounding example of this are the massless four-point amplitudes at two-loops: compare the RNS computation in [188,189,190,191,192,193,194] versus the ten-pages-long computation using pure spinors [283, 284]. Results were shown to be equivalent for any loop amplitude computed so far, see e.g. [285,286,287] and most notably [288] for the five point one loop amplitude. The general tree level proof can be found in [289].

Scattering of the massless gauge multiplet is described in the language of ten dimensional super Yang Mills theory [134, 135], see section 10.2. When I started to apply pure spinor methods to the massless SYM multiplet, tree and one-loop amplitudes had been computed to five leg order [288, 290]. About one third of my research activity was devoted to extending the tree level results to a higher number of external legs and to a better understanding in terms of field theory diagrams (as pioneered in [291]). The results of my publications [7, 8, 9, 11] on these topics will be presented in the following chapters 11, 12 and 13.

Massive Regge excitations and their couplings have hardly been explored in the pure spinor framework [292, 293, 294], a lot of work has to be done to construct higher spin multiplets and to obtain massive state amplitudes in superfield language. So far, the equivalence to the RNS formalism was shown for massless states only, and it would be very interesting to compare pure spinor results on massive scattering with RNS computations such as [2, 269]. These issues are beyond the scope of this work.

In its present formulation, the pure spinor formalism is applicable in ten spacetime dimensions with $\mathcal{N} = 1$ spacetime supersymmetry only. Of course, the final result for scattering amplitudes can be translated into those for $\mathcal{N} = 4$ SYM in $D = 4$ by dimensional reduction. Therefore the main results presented in this work can still be recycled for compactifications of any dimension, as explained in chapter 8.

Four dimensional superstring theories which preserve at least $\mathcal{N} = 1$ spacetime supersymmetry can be approached in a manifestly super Poincaré invariant way by means of the so-called hybrid formalism [295, 296, 297]. The latter is based on some non-trivial field redefinitions, which replace the interacting RNS fields $\psi^\mu, S_a, S^{\dot{b}}$ by a new set of free worldsheet fields which closely resemble the pure spinor worldsheet degrees of freedom. It is related to the twistor-like GS description by gauge-fixing six of the eight fermionic worldsheet invariances.

The present chapter gives a very brief introduction to the basic concepts of the pure spinor formalism. It does not aim to provide an equally detailed account as the first chapters 2 and 3 on the RNS framework. We just work out the prerequisites necessary to perform tree level computations for scattering of massless states. More extensive explanations can be found in [277, 282].

Note that a modified set of conventions is used in the following chapters, see appendix A.2.

10.1 Worldsheet CFT of the pure spinor formalism

In this section, we introduce the worldsheet degrees of freedom for the pure spinor formalism. We will argue that the matter- and ghost sectors give rise to the same central charge and to the same level of the Lorentz current algebra as we had it in the RNS CFT.

10.1.1 Matter fields of the pure spinor CFT

The worldsheet variables of the pure spinor formalism are very close to the GS approach: The embedding coordinates X^m are accompanied by a $SO(1,9)$ spacetime spinor variable θ^α rather than a worldsheet spinor ψ^m. More precisely, θ^α is a right handed Majorana Weyl spinor with 16 real components. As proposed by Warren Siegel [298], the conjugate momenta p_α for θ^α are treated as additional free variables. This idea was invented to cure the problems in quantizing the GS superstring. The matter degrees of freedom in the pure spinor formalism are described by Siegel's modification to the GS action:

$$\mathcal{S}[X,p,\theta] \;=\; \frac{1}{4\pi} \int \mathrm{d}^2 z \; \left\{ \frac{1}{2} \, \partial X^m \, \bar\partial X_m \;+\; p_\alpha \, \bar\partial \theta^\alpha \;+\; \bar p_{\hat\beta} \, \partial \bar\theta^{\hat\beta} \right\} \tag{10.1.1}$$

Depending on the $SO(1,9)$ chirality of the right moving spinors $(\bar p_{\hat\beta}, \bar\theta^{\hat\beta}) \in \big\{ (\bar p_\beta, \bar\theta^\beta), (\bar p^\beta, \bar\theta_\beta) \big\}$, (10.1.1) gives rise to type IIA or type IIB closed superstring theory. Since we are interested in open string scattering in this work, we won't follow this issue any further and ignore the right movers from now on.

The action (10.1.1) is invariant under spacetime supersymmetry transformations

$$\delta_\eta X^m \;=\; \frac{1}{2} \, (\eta \, \gamma^m \, \theta) \; , \qquad \delta_\eta \theta^\alpha \;=\; \eta^\alpha \tag{10.1.2}$$

$$\delta_\eta p_\alpha \;=\; -\frac{1}{2} \, \partial X_m \, (\eta \, \gamma^m)_\alpha \;+\; \frac{1}{8} \, (\eta \, \gamma_m \, \theta) \, (\partial\theta \, \gamma^m)_\alpha \tag{10.1.3}$$

genenerated by the charge

$$\mathcal{Q}_\alpha \;=\; \oint \frac{\mathrm{d}z}{2\pi i} \, \left\{ p_\alpha \;+\; \frac{1}{2} \, \gamma^m_{\alpha\beta} \, \theta^\beta \, \partial X_m \;+\; \frac{1}{24} \, (\gamma^m \, \theta)_\alpha \, (\theta \, \gamma_m \, \partial\theta) \right\} \; . \tag{10.1.4}$$

contracted with a spinorial parameter η^α. Hence it makes sense to introduce supersymmetric versions Π^m and d_α of the original variables ∂X^m and p_α:

$$\Pi^m \;:=\; \partial X^m \;+\; \frac{1}{2} \, (\theta \, \gamma^m \, \partial\theta) \tag{10.1.5}$$

$$d_\alpha \;:=\; p_\alpha \;-\; \frac{1}{2} \, \left(\partial X^m \;+\; \frac{1}{4} \, (\theta \, \gamma^m \, \partial\theta) \right) \, (\gamma_m \, \theta)_\alpha \tag{10.1.6}$$

The latter also appears in the Green Schwarz formalism as a fermionic constraint $d_\alpha = 0$.

The action (10.1.1) defines a CFT governed by the energy momentum tensor

$$T \;=\; -\frac{1}{2} \, \Pi^m \, \Pi_m \;-\; d_\alpha \, \partial\theta^\alpha \; . \tag{10.1.7}$$

It identifies $\partial X^m, \Pi^m, p_\alpha$ and d_α as $h = 1$ primary fields subject to the following OPEs:

$$\Pi^m(z) \, \Pi^n(w) \;\sim\; \frac{-\,\eta^{mn}}{(z-w)^2} \;+\; \ldots \; , \qquad \Pi^m(z) \, \mathrm{e}^{k \cdot X}(w) \;\sim\; \frac{-\,k^m}{z-w} \, \mathrm{e}^{k\cdot X}(w) \;+\; \ldots$$

$$\begin{aligned} d_\alpha(z)\,\theta^\beta(w) &\sim \frac{\delta_\alpha^\beta}{z-w} + \ldots\,, & d_\alpha(z)\,\partial\theta^\beta(w) &\sim \frac{\delta_\alpha^\beta}{(z-w)^2} + \ldots \\ d_\alpha(z)\,d_\beta(w) &\sim -\frac{\gamma^m_{\alpha\beta}\,\Pi_m}{z-w} + \ldots\,, & d_\alpha(z)\,\Pi^m(w) &\sim \frac{(\gamma^m\,\partial\theta)_\alpha}{z-w} + \ldots \end{aligned} \tag{10.1.8}$$

These OPEs determine the central charge of each Π^m component and p_α, θ^α pair to be $c(\Pi) = 1$ and $c(p,\theta) = -2$, respectively. In ten spacetime dimensions, the overall central charge of the matter sector is then found to be

$$c_{\rm mat} \ = \ 10 \cdot c(\Pi) \ + \ 16 \cdot c(p,\theta) \ = \ 10 \ - \ 32 \ = \ -22\,. \tag{10.1.9}$$

One can construct $SO(1,9)$ Lorentz currents by means of the standard Noether procedure,

$$\Sigma^{mn}_{p,\theta} \ := \ \frac{1}{2}\,(p\,\gamma^{mn}\,\theta)\,, \tag{10.1.10}$$

but the OPEs (10.1.8) imply that the level of the associated current algebra is $k_{p,\theta} = 4$ rather than the RNS value $k_{\rm RNS} = 1$ following from the currents $\Sigma^{mn}_{\rm RNS} := \psi^m\psi^n$:

$$\Sigma^{mn}_{p,\theta}(z)\,\Sigma^{rs}_{p,\theta}(w) \ \sim \ \frac{2\,\eta^{r[n}\,\Sigma^{m]s}_{p,\theta}(w) \ - \ 2\,\eta^{s[n}\,\Sigma^{m]r}_{p,\theta}(w)}{z-w} \ + \ 4\,\frac{\eta^{ms}\,\eta^{nr} \ - \ \eta^{mr}\,\eta^{ns}}{(z-w)^2} \ + \ \ldots \tag{10.1.11}$$

$$\Sigma^{mn}_{\rm RNS}(z)\,\Sigma^{rs}_{\rm RNS}(w) \ \sim \ \frac{2\,\eta^{r[n}\,\Sigma^{m]s}_{\rm RNS}(w) \ - \ 2\,\eta^{s[n}\,\Sigma^{m]r}_{\rm RNS}(w)}{z-w} \ + \ \frac{\eta^{ms}\,\eta^{nr} \ - \ \eta^{mr}\,\eta^{ns}}{(z-w)^2} \ + \ \ldots \tag{10.1.12}$$

A necessary requirement for equivalent scattering amplitudes in the RNS- and pure spinor formalisms is having identical singularity structures among the symmetry generators. On these grounds, we expect additional contributions to the level of the pure spinor current algebra which compensate for the mismatch $k_{\rm RNS} \neq k_{p,\theta}$.

10.1.2 Ghost fields of the pure spinor CFT

One of the essential ingredients of the pure spinor formalism is its BRST charge

$$Q \ := \ \oint \frac{{\rm d}z}{2\pi i}\,\lambda^\alpha(z)\,d_\alpha(z) \tag{10.1.13}$$

with λ^α denoting a bosonic $SO(1,9)$ Weyl spinor. It must be a ghost variable because it obeys the wrong statistics for a spinor. So far, the BRST operator (10.1.13) could not be derived by gauge fixing a more symmetric theory, see [299] for recent work. One can regard Q as an additional input to the formalism.

According to the $d_\alpha(z)d_\beta(w)$ OPE (10.1.8), nilpotency of Q imposes an algebraic constraint on the spinor variable λ^α:

$$Q^2 \ = \ 0 \quad \Rightarrow \quad \lambda^\alpha \, \gamma^m_{\alpha\beta} \, \lambda^\beta \ = \ 0 \tag{10.1.14}$$

This equation is responsible for the name "pure spinor" formalism: A pure spinor in D dimensions is defined to be a Weyl spinor λ^α which satisfies (10.1.14) for $m = 0, 1, \ldots, D-1$.

The most convenient way of counting the number of independent pure spinor solutions breaks the $SO(1,9)$ Lorentz symmetry down to its $SU(5)$ subgroup [282]. It turns out that (10.1.14) contains $D/2 = 5$ independent constraints rather than $D = 10$ as suggestes by naive component counting $m = 0, 1, \ldots, 9$. Hence, the bosonic ghost λ^α has eleven degrees of freedom which respect the pure spinor constraint.

In contrast to its RNS counterpart, the pure spinor formalism introduces ghost fields in nontrivial representations of $SO(1,9)$. Hence, they also give rise to a Lorentz current N^{mn} characterized by the following OPEs:

$$N^{mn}(z) \, \lambda^\alpha(w) \ \sim \ \frac{(\gamma^{mn})^\alpha{}_\beta \, \lambda^\beta(w)}{2\,(z-w)} \ + \ \ldots \tag{10.1.15}$$

$$N^{mn}(z) \, N^{rs}(w) \ \sim \ \frac{2\,\eta^{r[n} \, N^{m]s}(w) \ - \ 2\,\eta^{s[n} \, N^{m]r}(w)}{z-w} \ - \ 3 \, \frac{\eta^{ms}\,\eta^{nr} \ - \ \eta^{mr}\,\eta^{ns}}{(z-w)^2} \ + \ \ldots \tag{10.1.16}$$

The level $k_{\rm gh} = -3$ of the ghost current algebra cannot be computed in $SO(1,9)$ covariant fashion. The maximally covariant way makes use of $SU(5)$ variables. The double pole in the ghost current OPE combines with that of the matter current to give overall level

$$k_{\rm tot} \ = \ k_{\rm gh} \ + \ k_{p,\theta} \ = \ -3 \ + \ 4 \ = \ +1 \tag{10.1.17}$$

to the total Lorentz currents $\Sigma^{mn}_{p,\theta} + N^{mn}$. This ties in with the level $k_{\rm RNS} = 1$ of the RNS currents $\psi^m \psi^n$.

The construction of the ghost currents and the derivation of their algebra is only possible in the $SU(5)$ decomposition of the λ^α degrees of freedom. The same is true for the energy momentum tensor. A careful $U(5)$ analysis shows that each ghost degree of freedom contributes $c(\lambda) = +2$ to the central charge of the ghost sector such that we are left with a vanishing total central charge:

$$c_{\rm gh} \ = \ 11 \cdot c(\lambda) \ = \ +22 \ = \ -c_{\rm mat} \quad \Rightarrow \quad c_{\rm tot} \ = \ c_{\rm mat} \ + \ c_{\rm gh} \ = \ 0 \tag{10.1.18}$$

The full pure spinor CFT with both matter- and ghost degrees of freedom has therefore passed two important consistency tests: The level of the overall $SO(1,9)$ current algebra adds up to the RNS value $k_{\rm tot} = k_{\rm RNS} = 1$, and the central charges of the matter- and ghost sectors are just opposite such that the conformal anomaly vanishes.

10.2 Superfield formulation of $\mathcal{N} = 1$ SYM in $D = 10$

The aim of this section is the introduction of a supersymmetric vertex operator for the massless open string states, i.e. the $\mathcal{N} = 1$ SYM multiplet in ten spacetime dimensions. The gluon- and gluino degrees of freedom will be packaged into superfields $A(X, \theta)$, functions living on the superspace which is spanned by the ten spacetime coordinates X^m and associated Grassmann-odd spinor variables θ^α. The latter transform as a right handed Majorana Weyl spinor of $SO(1, 9)$. A more general introduction to superspace methods can be found in [300, 301, 302, 303, 304].

The original references for ten dimensional SYM are [134, 135]. It was discovered in later work [305] that pure spinors are useful for implementing the on-shell constraints.

10.2.1 The superfield content

The usual starting point in defining a gauge theory is the introduction of covariant derivatives. Since the gauge theory in question is supposed to be defined on superspace, there exist derivative both in the spacetime directions (i.e. with respect to X^m) and in the direction of the Grassmann odd coordinates θ^α. The canonical differential operator for the latter is slightly modified to ensure the right superalgebra

$$D_\alpha \ := \ \frac{\partial}{\partial \theta^\alpha} \ + \ \frac{1}{2} \, (\gamma^m \, \theta)_\alpha \, \partial_m \quad \Rightarrow \quad \big\{ \, D_\alpha \ , \ D_\beta \, \big\} \ = \ \gamma^m_{\alpha\beta} \, \partial_m \ . \tag{10.2.19}$$

The covariant derivatives in X^m and θ^α directions then involve gauge (super-)fields (A^m, A_α):

$$\nabla_m \ := \ \partial_m \ + \ A_m(X, \theta) \ , \qquad \nabla_\alpha \ := \ D_\alpha \ + \ A_\alpha(X, \theta) \tag{10.2.20}$$

Any physical observable of $\mathcal{N} = 1$ SYM has to be invariant under the gauge transformation

$$A_m \ \equiv \ A_m \ + \ \partial_m \Omega \ , \qquad A_\alpha \ \equiv \ A_\alpha \ + \ D_\alpha \Omega \tag{10.2.21}$$

with some superfield parameter $\Omega(X, \theta)$.

The linearized equations of motion for free gauge fields are derived by the requirement that the supercovariant derivatives ∇_m, ∇_α satisfy the same superalgebra as ∂_m, D_α derivatives,

$$\big\{ \, \nabla_\alpha \ , \ \nabla_\beta \, \big\} \ \stackrel{!}{=} \ \gamma^m_{\alpha\beta} \, \nabla_m \quad \Rightarrow \quad D_\alpha A_\beta \ + \ D_\beta A_\alpha \ = \ \gamma^m_{\alpha\beta} \, A_m \ , \tag{10.2.22}$$

discarding the nonlinear piece $\{A_\alpha, A_\beta\}$ since we are interested in the free theory and introduce the interactions by perturbative computations. In $D=10$, a symmetric bispinor $D_{(\alpha}A_{\beta)}$ has a one-form- and a five-form component, and (10.2.22) can be rephrased as the vanishing of the five form $\gamma^{\alpha\beta}_{mnpqr}D_\alpha A_\beta = 0$. If this construction was carried out in lower dimensions $D<10$, it would not eliminate any p form component from $D_{(\alpha}A_{\beta)}$, i.e. it would not put the fields on shell.

The on-shell constraint admits to express the vector superfield A_m in terms of its spinorial relative A_α and to thereby identify it as an auxiliary superfield:

$$A_m \;=\; \frac{1}{5}\,\gamma_m^{\alpha\beta}\,D_\alpha A_\beta \tag{10.2.23}$$

More formally, the unwanted degrees of freedom due to A_m are eliminated by a set of gauge invariant constraints on the superspace curvature.

The simplest gauge invariant quantities one can construct in this setting are field strengths, i.e. generalized curls in superspace. By antisymmetrizing either two spacetime components or one spacetime- and one Grassmann odd component of the (A_m, A_α) Jacobi matrix, we obtain

$$W^\alpha \;:=\; \frac{1}{10}\,\gamma_m^{\alpha\beta}\,(D_\beta A^m \,-\, \partial^m A_\beta) \tag{10.2.24}$$

$$\mathcal{F}_{mn} \;:=\; \partial_m A_n \,-\, \partial_n A_m\;. \tag{10.2.25}$$

Bianchi identities for the ∇_m and ∇_α operators yield expressions for the fermionic derivatives of all the superfields:

$$D_\alpha A_m \;=\; (\gamma_m\,W)_\alpha \,+\, \partial_m A_\alpha \tag{10.2.26}$$

$$D_\alpha W^\beta \;=\; \frac{1}{4}\,(\gamma^{mn})_\alpha{}^\beta\,\mathcal{F}_{mn} \tag{10.2.27}$$

$$D_\alpha \mathcal{F}_{mn} \;=\; 2\,k_{[m}\,(\gamma_{n]}\,W)_\alpha \tag{10.2.28}$$

They will be important on-shell constraints for the construction of vertex operators in the BRST cohomology.

In order to reproduce the equations of motions for the gluon- and gluino components of the superfields, let us act on (10.2.27) with D_γ and symmetrize in the (α,γ) indices. Taking the δ^γ_β trace then yields a massless Dirac equation for the W^α field strength,

$$\gamma^m_{\alpha\beta}\,\partial_m W^\beta \;=\; 0\;. \tag{10.2.29}$$

The equations of motion for the vector superfield A_m follow from $\gamma_n^{\alpha\gamma}D_\gamma$ action on (10.2.29) together with (10.2.27),

$$\partial^m \mathcal{F}_{mn} \;=\; 0\;. \tag{10.2.30}$$

This already suggests to identify the gluon and gluino with the lowest theta components of A_m and W^α, respectively.

10.2.2 Theta expansions

The Grassmann odd nature of the spinorial superspace coordinates θ^α makes any power series expansion in the latter terminate at the finite order $\mathcal{O}(\theta^{16})$. The expansion coefficient at each order $\mathcal{O}(\theta^k)$ is a spacetime function $\Phi_k(X)$.

For the purpose of θ expanding the $\mathcal{N}=1$ SYM superfields $(A_\alpha, A_m, W^\alpha, \mathcal{F}_{mn})$, the equations of motion (10.2.22) and (10.2.27) give recursion relations between the coefficient $\Phi_k(X)$ at neighboring θ orders. They become particularly simple if we enforce $\theta^\alpha A_\alpha(X,\theta)=0$ by an appropriate choice of the gauge parameter Ω in (10.2.21). This gauge can be viewed as the supersymmetric analogue of choosing normal coordinates.

The solutions to the equations of motion can be parametrized by the gluon polarization vector ξ_m and a gluino wave function u^α. Once the X dependence is organized into plane waves with momentum k_m, the first terms of the superfields' θ expansion are determined as follows [306, 307]:

$$
\begin{aligned}
A_\alpha(X,\theta) \;&=\; \mathrm{e}^{k\cdot X}\biggl\{ \frac{\xi^m}{2}\,(\gamma^m\theta)_\alpha \;-\; \frac{1}{3}\,(u\,\gamma_m\,\theta)\,(\gamma^m\theta)_\alpha \;-\; \frac{1}{16}\,(\gamma_p\,\theta)_\alpha\,(\theta\,\gamma^{mnp}\,\theta)\,k_{[m}\,\xi_{n]} \\
&+ \frac{1}{60}\,(\gamma_m\,\theta)_\alpha\,k_n\,(u\,\gamma_p\,\theta)\,(\theta\,\gamma^{mnp}\,\theta) \;+\; \frac{1}{576}\,(\gamma^m\,\theta)_\alpha\,k^r\,(\theta\,\gamma_{mrs}\,\theta)\,(\theta\,\gamma^{spq}\,\theta)\,k_{[p}\,\xi_{q]} \;+\; \mathcal{O}(\theta^6)\biggr\} \\
A_m(X,\theta) \;&=\; \mathrm{e}^{k\cdot X}\biggl\{ \xi_m \;-\; (u\,\gamma_m\,\theta) \;-\; \frac{1}{4}\,k_p\,(\theta\,\gamma_m{}^{pq}\,\theta)\,\xi_q \;+\; \frac{1}{12}\,k_p\,(\theta\,\gamma_m{}^{pq}\,\theta)\,(u\,\gamma_q\,\theta) \\
&+ \frac{1}{96}\,k_n\,(\theta\,\gamma_m{}^{np}\,\theta)\,k_q\,(\theta\,\gamma_p{}^{qr}\,\theta)\,\xi_r \;-\; \frac{1}{480}\,k_n\,(\theta\,\gamma_m{}^{np}\,\theta)\,k_q\,(\theta\,\gamma_p{}^{qr}\,\theta)\,(\theta\,\gamma_r\,u) \;+\; \mathcal{O}(\theta^6)\biggr\} \\
W^\alpha(X,\theta) \;&=\; \mathrm{e}^{k\cdot X}\biggl\{ u^\alpha \;-\; \frac{1}{2}\,k_{[m}\,\xi_{n]}\,(\gamma^{mn}\,\theta)^\alpha \;+\; \frac{1}{4}\,k_m\,(\gamma^{mn}\,\theta)^\alpha\,(u\,\gamma_n\,\theta) \\
&+ \frac{1}{24}\,k_m\,(\gamma^{mn}\,\theta)^\alpha\,(\theta\,\gamma_n{}^{pq}\,\theta)\,k_{[p}\,\xi_{q]} \;-\; \frac{1}{96}\,(\gamma^{mn}\,\theta)^\alpha\,k_m\,(\theta\,\gamma_n{}^{pq}\,\theta)\,k_p\,(u\,\gamma_q\,\theta) \\
&- \frac{1}{768}\,(\gamma^{mn}\,\theta)^\alpha\,k_m\,(\theta\,\gamma_n{}^{pq}\,\theta)\,k_p\,(\theta\,\gamma_q{}^{rs}\,\theta)\,k_r\,\xi_s \;+\; \mathcal{O}(\theta^6)\biggr\} \\
\mathcal{F}_{mn}(X,\theta) \;&=\; \mathrm{e}^{k\cdot X}\biggl\{ 2\,k_{[m}\,\xi_{n]} \;-\; 2\,k_{[m}\,(u\,\gamma_{n]}\,\theta) \;-\; \frac{1}{2}\,k_{[p}\,\xi_{q]}\,k_{[m}\,(\theta\,\gamma_{n]}{}^{pq}\,\theta) \\
&+ \frac{1}{6}\,k_{[m}\,(\theta\,\gamma_{n]}{}^{pq}\,\theta)\,k_p\,(u\,\gamma_q\,\theta) \;+\; \frac{1}{48}\,k_{[m}\,(\theta\,\gamma_{n]}{}^{tp}\,\theta)\,k_t\,k_q\,(\theta\,\gamma_p{}^{qr}\,\theta)\,\xi_r \\
&- \frac{1}{240}\,k_{[m}\,(\theta\,\gamma_{n]}{}^{tp}\,\theta)\,k_t\,(\theta\,\gamma_p{}^{qr}\,\theta)\,k_q\,(\theta\,\gamma_r\,u) \;+\; \mathcal{O}(\theta^6)\biggr\}
\end{aligned}
\tag{10.2.31}
$$

We will argue in the later subsection 10.3.3 that expanding up to order $\mathcal{O}(\theta^5)$ is sufficient for extracting superfield components from supersymmetric tree amplitudes. More compact expressions for (10.2.31) can be found in [308] in terms of formal power series[1], e.g.

$$\begin{aligned} A_m &= e^{k\cdot X} \left\{ \big(\cosh\sqrt{\mathcal{O}}\big)_m{}^q \, \xi_q \, + \, \Big(\tfrac{\sinh\sqrt{\mathcal{O}}}{\sqrt{\mathcal{O}}}\Big)_m{}^q \, (\theta\,\gamma_q\,u) \right\} && (10.2.32) \\ \mathcal{O}_m{}^q &= -\frac{1}{2}\, k_n \, (\theta\,\gamma_m{}^{np}\,\theta) \, . && (10.2.33) \end{aligned}$$

Note that the θ expansions (10.2.31) have an alternating structure with respect to the gluon and the gluino: The Grassmann odd superfields A_α and W^α involve the gluon polarization along with odd powers of θ and the gluino wave functions with even θ powers. Bosonic superfields $A_m, \mathcal{F}_{mn}$ follow the opposite pattern.

10.2.3 Vertex operators for the SYM multiplet

We have introduced superfields describing the degrees of freedom of the massless gauge multiplet. This paves the way to writing down supersymmetric vertex operators. Just like in the RNS framework, each physical state must have a representative of conformal weight $h=0$ and one of $h=1$ in the worldsheet CFT of the pure spinor formalism.

In the following, we will think of the SYM superfields (10.2.31) as worldsheet functions, depending on z through the superspace variables, e.g. $A_\alpha(z) \equiv A_\alpha\big(X^m(z),\theta(z)\big)$. From the OPEs $d_\alpha(z)\theta^\beta(w) \sim \delta_\alpha^\beta(z-w)^{-1}$ and $\Pi^m(z)e^{k\cdot X}(w) \sim -k^m e^{k\cdot X}(w)(z-w)^{-1}$, we can infer the action of Π^m and d_α on a generic superfield $\mathcal{V} \in \{A_\alpha, A_m, W^\alpha, \mathcal{F}_{mn}\}$:

$$\Pi^m(z)\,\mathcal{V}(w) \ \sim \ \frac{-\,k^m}{z-w}\,\mathcal{V}(w)\ , \qquad d_\alpha(z)\,\mathcal{V}(w) \ \sim \ \frac{1}{z-w}\,D_\alpha\mathcal{V}(w) \qquad (10.2.34)$$

We have argued in subsection 3.1.3 that the $h=1$ RNS open string vertex operators at (b,c) ghost number zero must be integrated over the worldsheet boundary and can be mapped to their position fixed $h=0$ analogue by adjoining a c ghost of conformal weight -1. This suggests to construct the massless pure spinor vertex operator from the λ^α ghost and to combine it with an $h=0$ superfield such that the overall vertex lies in the cohomology of the BRST operator $Q=\oint \frac{\mathrm{d}z}{2\pi i}\lambda^\alpha d_\alpha$. The only possible choice involves the spinorial SYM superfield A_α:

$$V \ = \ \lambda^\alpha\, A_\alpha(X,\theta) \qquad (10.2.35)$$

Recall that all of λ^α, X^m and θ^α depend on the worldsheet position z. By combining the SYM equation of motion (10.2.22) with the d_α action (10.2.34) on superfields, we can check that the

[1]Functions of an operator $\mathcal{O}$ should be understood in terms of their Taylor expansion, i.e. $\cosh\sqrt{\mathcal{O}} = 1+\mathcal{O}/2+\mathcal{O}^2/24+\ldots$ and $\sinh\sqrt{\mathcal{O}}/\sqrt{\mathcal{O}} = 1+\mathcal{O}/6+\mathcal{O}^2/120+\ldots$

vertex operator $\lambda^\alpha A_\alpha$ is BRST closed:

$$Q\,V(w) \;\;=\;\; \oint \frac{\mathrm{d}z}{2\pi i}\,\lambda^{(\alpha}\,\lambda^{\beta)}\,\frac{D_\alpha A_\beta(w)}{z-w} \;\;=\;\; \frac{1}{2}\,\lambda^\alpha\,\gamma^m_{\alpha\beta}\,\lambda^\beta\,A_m \;\;=\;\; 0 \tag{10.2.36}$$

The last step makes use of the pure spinor constraint $(\lambda\gamma^m\lambda) = 0$ for the λ^α ghost.

A bit more work is required to obtain the integrated $h = 1$ version U of this vertex operator. The pure spinor CFT offers four conformal primaries $\partial\theta^\alpha, \Pi^m, d_\alpha, N^{mn}$ of unit weight which might enter the integrated vertex U. Determining the superfield admixtures is guided by the requirement that $QU = \partial V$ such that the integral over $U(z)$ is in the BRST cohomology. An analogous relation $Q_{\mathrm{BRST}}V = \partial(cV)$ holds in the RNS formalism, see (3.1.12). The unique solution to these requirements is

$$U \;\;=\;\; \partial\theta^\alpha\,A_\alpha \;+\; \Pi^m\,A_m \;+\; d_\alpha\,W^\alpha \;+\; \frac{1}{2}\,N^{mn}\,\mathcal{F}_{mn}\;. \tag{10.2.37}$$

Checking $QU = \partial V$ first of all involves various OPEs,

$$\begin{aligned}
Q\,(\partial\theta^\alpha\,A_\alpha) \;&=\; \partial\lambda^\alpha\,A_\alpha \;-\; \partial\theta^\alpha\,\lambda^\beta\,D_\beta A_\alpha \\
Q\,(\Pi^m\,A_m) \;&=\; (\lambda\,\gamma^m\,\partial\theta)\,A_m \;+\; \Pi^m\,\lambda^\alpha\,D_\alpha A_m \\
Q\,(d_\alpha\,W^\alpha) \;&=\; -\,(\lambda\,\gamma^m\,W)\,\Pi_m \;-\; d_\beta\,\lambda^\alpha\,D_\alpha W^\beta \\
Q\,(N^{mn}\,\mathcal{F}_{mn}) \;&=\; \frac{1}{2}\,(\gamma^{mn}\,\lambda)^\alpha\,d_\alpha\,\mathcal{F}_{mn} \;+\; N^{mn}\,\lambda^\alpha\,D_\alpha\mathcal{F}_{mn}\;,
\end{aligned} \tag{10.2.38}$$

then the SYM equations of motion $D_\alpha\{A_\beta, A_m, W^\beta, \mathcal{F}_{mn}\}$ complete the proof:

$$Q\,U \;\;=\;\; \partial\lambda^\alpha\,A_\alpha \;+\; \lambda^\alpha\,\big(\partial\theta^\beta\,D_\beta A_\alpha \;+\; \Pi^m\,\partial_m A_\alpha\big) \;\;=\;\; \partial(\lambda^\alpha\,A_\alpha) \tag{10.2.39}$$

10.3 Tree level amplitudes in the pure spinor formalism

The prescription to compute tree amplitudes in the pure spinor formalism corresponds to the RNS formula (5.3.29) – with the additional simplification that we don't have to take superghosts into account now. The pure spinor part of this work focuses on massless states, so the vertex operators V and U can already be specified to (10.2.35) and (10.2.37) creating states of the $\mathcal{N} = 1$ SYM multiplet:

$$\mathcal{A}(1,2,\dots,n) \;\;\sim\;\; \left\langle V_1(0)\,\prod_{j=2}^{n-2}\int_{z_{j-1}}^{1}\mathrm{d}z_j\;U_j(z_j)\,V_{n-1}(1)\,V_n(\infty)\right\rangle \tag{10.3.40}$$

As explained in subsection 5.2.2, the disk topology of a tree level open string worldsheet gives rise to the conformal Killing group $SL(2,\mathbb{R})$. Therefore, we can fix three vertex positions

$(z_1, z_{n-1}, z_n) = (0, 1, \infty)$ on the real axis and insert unintegrated vertices V_i at these points. The remaining positions $z_2, \ldots, z_{n-2}$ are integrated over segments of the real axis such that the ordering $z_1 \leq z_2 \leq \ldots \leq z_{n-2} \leq z_{n-1}$ of the color stripped amplitude in question is preserved.

In this section, we will explain the general methods for evaluating the CFT correlator in (10.3.40), paying particular attention to the zero modes. Later subsections 10.3.4 and 10.3.5 provide consistency checks for our setup: We prove that the amplitude prescription (10.3.40) is independent on the assignment of V_i and U_j vertices and that its superfield components reproduce the RNS results.

10.3.1 Integrating out nonzero modes

The conformal $h = 1$ primaries $\partial\theta^\alpha, \Pi^m, d_\alpha, N^{mn}$ occurring in the correlator (10.3.40) have no zero modes at tree level. This enormously simplifies the computation of their tree level correlation functions. We can apply Wick's theorem just like in appendix B.4 and replace each primary by the sum over its singularities with the other fields present in the correlation function. A first example for this is the $\Pi^m(z)$ field with $\frac{-k^m}{z-w}$ action on a plane wave $\mathrm{e}^{k\cdot X}(w)$ and hence any superfield $\mathcal{V} \in \{A_\alpha, A_m, W^\alpha, \mathcal{F}_{mn}\}$:

$$\left\langle \Pi^m(z) \prod_{j=1}^{n} \mathcal{V}_j(w_j) \right\rangle \ = \ - \sum_{\ell=1}^{n} \frac{k_\ell^m}{z - w_\ell} \left\langle \prod_{j=1}^{n} \mathcal{V}_j(w_j) \right\rangle \tag{10.3.41}$$

Similarly, we have found d_α to act as a covariant derivative in (10.2.34):

$$\left\langle d_\alpha(z) \prod_{j=1}^{n} \mathcal{V}_j(w_j) \right\rangle \ = \ \sum_{\ell=1}^{n} \frac{1}{z - w_\ell} \left\langle D_\alpha \mathcal{V}_\ell(w_\ell) \prod_{\substack{j=1 \\ j\neq\ell}}^{n} \mathcal{V}_j(w_j) \right\rangle \tag{10.3.42}$$

The ghost current N^{mn} gives rise to a single pole when approaching λ^α, so it has a nontrivial action on the (anticommuting) unintegrated vertex

$$\left\langle N^{pq}(z) \prod_{j=1}^{n} \lambda^{\alpha_j} A^j_{\alpha_j}(w_j) \right\rangle \ = \ \frac{1}{2} \sum_{\ell=1}^{n} \frac{1}{z - w_\ell} \left\langle (\lambda\, \gamma^{pq})^{\alpha_\ell} A^\ell_{\alpha_\ell}(w_\ell) \prod_{\substack{j=1 \\ j\neq\ell}}^{n} (-1)^j \lambda^{\alpha_j} A^j_{\alpha_j}(w_j) \right\rangle . \tag{10.3.43}$$

If multiple $h = 1$ fields are present in correlators, then these mutual singularities have to be summed analogously to (10.3.41) and (10.3.42). According to the OPEs (10.1.8), the Π^m and d_α become singular when they approach the Π^m, d_α and $\partial\theta^\alpha$ primaries. The $\partial\theta^\alpha$ term of the integrated vertex (10.2.37) is nonsingular with respect to $h = 0$ superfields $\mathcal{V}$, but it can feel the presence of other d_α. Finally, the N^{mn} produce the singularities of the ghost current OPE (10.1.16) when approaching the $\frac{1}{2}N^{pq}\mathcal{F}_{pq}$ part of another integrated vertex.

These situations require at least two integrated vertices, so they do not occur in the computation of tree level amplitudes with $n < 5$ legs. In fact, we will argue in the later section 12.1 that even at $n \geq 5$, one can circumvent the explicit computation of $U_i(z_i)U_j(z_j)$ singularities.

The simplicity of the correlation functions (10.3.41) to (10.3.43) needs to be contrasted with RNS computations. The worldsheet interactions of NS fermions ψ^m with their spin fields S_α pose a nontrivial challenge to compute their correlation functions, see chapter 6.

As a sample calculation, let us simplify the correlation function $\langle V_1 U_2 V_3 V_4 \rangle$ entering the four point amplitude $\mathcal{A}(1,2,3,4)$ due to (10.3.40). The basic building block in evaluating this correlator is the single pole in the $V_1(z_1)U_2(z_2)$ OPE:

$$\begin{aligned}
V_1(z_1)\,\Pi^m\,A^2_m(z_2) \quad &\sim \quad \frac{1}{z_{21}}\,(\,-k_1^m)\,\lambda^\alpha\,A^1_\alpha\,A^2_m(z_1) \quad = \quad -\frac{1}{z_{21}}\,(k_1\cdot A_2)\,V_1(z_1)\\
V_1(z_1)\,d_\alpha\,W_2^\alpha(z_2) \quad &\sim \quad -\frac{1}{z_{21}}\,\lambda^\alpha\,D_\beta A^1_\alpha\,W_2^\beta(z_1) \quad = \quad -\frac{1}{z_{21}}\,\lambda^\alpha\,(\,-D_\alpha A^1_\beta \;+\; \gamma^m_{\alpha\beta}\,A^1_m\,)\,W_2^\beta(z_1)\\
&= \quad \frac{1}{z_{21}}\Big[(Q\,A^1_\alpha)\,W_2^\alpha \;-\; A^1_m\,(\lambda\,\gamma^m\,W_2)\Big] \qquad\qquad (10.3.44)\\
V_1(z_1)\,\frac{N^{mn}}{2}\,\mathcal{F}^2_{mn}(z_2) \quad &\sim \quad -\frac{1}{4\,z_{21}}\,\lambda^\alpha\,\gamma^{mn}{}_\alpha{}^\beta\,A^1_\beta\,\mathcal{F}^2_{mn}(z_1) \quad = \quad -\frac{1}{z_{21}}\,A^1_\alpha\,(Q\,W_2^\alpha)(z_1)
\end{aligned}$$

The terms proportional to $(QA^1_\alpha)W_2^\alpha$ and $-A^1_\alpha(QW_2^\alpha)$ add up to a BRST exact combination:

$$V_1(z_1)\,U_2(z_2) \quad \sim \quad \frac{1}{z_{21}}\Big[-(k_1\cdot A_2)\,V_1 \;-\; A^1_m\,(\lambda\,\gamma^m\,W_2) \;+\; Q\,(A_1\,W_2)\Big](z_1) \;+\; \ldots \qquad (10.3.45)$$

Hence, the four point correlator in question is given by

$$\begin{aligned}
&\langle\, V_1(0)\,U_2(z_2)\,V_3(1)\,V_4(\infty)\,\rangle\\
&= \; -\frac{1}{z_2}\Big\langle\Big[-(k_1\cdot A_2)\,V_1 \;-\; A^1_m\,(\lambda\,\gamma^m\,W_2) \;+\; Q\,(A_1\,W_2)\Big](0)\,V_3(1)\,V_4(\infty)\Big\rangle\\
&\quad +\frac{1}{1-z_2}\Big\langle V_1(0)\Big[-(k_3\cdot A_2)\,V_3 \;-\; A^3_m\,(\lambda\,\gamma^m\,W_2) \;+\; Q\,(A_3\,W_2)\Big](1)\,V_4(\infty)\Big\rangle\;. \qquad (10.3.46)
\end{aligned}$$

Note that the contribution $\sim z_{24}$ vanishes because the last position z_4 is fixed to infinity. The next subsection is devoted to the evaluation of the zero mode correlators in (10.3.46).

10.3.2 Zero mode integration

The conformal weight zero variables λ^α and θ^α contain zero modes at tree level which require particular care in evaluating their correlation functions. Since $SL(2,\mathbb{R})$ invariance on the disk requires three unintegrated vertex insertions $V_i = \lambda^\alpha A^i_\alpha$ into tree level correlators, the ghost number is fixed to be three.

Integrating out the conformal fields $\partial\theta^\alpha, \Pi^m, d_\alpha$ and N^{pq} leaves us with a power series $f_{\alpha\beta\gamma}(\theta, z_j)$ in θ, contracted with the λ^α ghosts:

$$\left\langle V_1 \prod_{j=2}^{n-2} \int \mathrm{d}z_j \; U_j(z_j) \, V_{n-1} \, V_n \right\rangle \;=\; \prod_{j=2}^{n-2} \int \mathrm{d}z_j \; \langle \, \lambda^\alpha \, \lambda^\beta \, \lambda^\gamma \, f_{\alpha\beta\gamma}(\theta; z_j) \, \rangle \tag{10.3.47}$$

The specific form of $f_{\alpha\beta\gamma}$ in terms of SYM superfield follows from the OPE contractions discussed above. Since the amplitude is made of BRST closed ingredients V and $\int \mathrm{d}z \, U$, also the final answer must be BRST closed. This constrains $f_{\alpha\beta\gamma}$ to satisfy

$$\prod_{j=2}^{n-2} \int \mathrm{d}z_j \; \lambda^\alpha \, \lambda^\beta \, \lambda^\gamma \, \lambda^\delta \, D_\delta f_{\alpha\beta\gamma}(\theta; z_i) \;=\; 0 \; . \tag{10.3.48}$$

It turns out that among all the occurring θ^k powers $k = 0, 1, \ldots, 16$, only θ^5 yields a consistent nonzero correlation function:

$$\langle \, \lambda^\alpha \, \lambda^\beta \, \lambda^\gamma \, f_{\alpha\beta\gamma}(\theta) \, \rangle \;=\; \left\langle \lambda^\alpha \, \lambda^\beta \, \lambda^\gamma \, f_{\alpha\beta\gamma}(\theta) \Big|_{\theta^5} \right\rangle \tag{10.3.49}$$

More precisely, the $\langle \ldots \rangle$ brackets have to be evaluated via

$$\langle \, (\lambda \, \gamma^m \, \theta) \, (\lambda \, \gamma^n \, \theta) \, (\lambda \, \gamma^p \, \theta) \, (\theta \, \gamma_{mnp} \, \theta) \, \rangle \;=\; 1 \; . \tag{10.3.50}$$

This is justified by the fact that the unique element in the BRST cohomology at ghost number three is proportional to θ^5. Let us prove that the expression within the $\langle \ldots \rangle$ bracket indeed belongs to the BRST cohomology:

- BRST closedness follows from the pure spinor constraint $(\lambda\gamma^m\lambda) = 0$ and its particular form $(\lambda\gamma^m)_\alpha(\lambda\gamma_m)_\beta = 0$.

- Expressions of the form $\lambda^3 \, \theta^5$ cannot be BRST exact $\sim Q(\lambda^2 \, \theta^6)$ because one cannot build a Lorentz scalar from two λ^α and six θ^β: The bispinor $\lambda^\alpha\lambda^\beta = \frac{1}{3840}(\lambda\gamma^{mnpqr}\lambda)\gamma^{\alpha\beta}_{mnpqr}$ only has a five-form component and it can be checked using the LiE program [309] that its tensor product with an antisymmetric six-spinor $\theta^{[\alpha_1} \ldots \theta^{\alpha_6]}$ does not contain any Lorentz scalar[2].

- Uniqueness follows from the fact that the tensor product of three λ^α and five θ^β contains one scalar.

[2] It is essential that the five form is the only $SO(1,9)$ irreducible in a pure bispinor: The vector $(\lambda\gamma^m\lambda)\gamma_m^{\alpha\beta}$ is absent due to the pure spinor constraint, and the three form vanishes because of the antisymmetry $\gamma^{mnp}_{\alpha\beta} = \gamma^{mnp}_{[\alpha\beta]}$.

The second remarkable virtue of (10.3.50) is its compatibility with spacetime SUSY. Since λ^α is invariant and θ^α transforms as $\delta_\eta \theta^\alpha = \eta^\alpha$ (see (10.1.2) and (10.1.3)), the SUSY variation of a pure spinor amplitude can only receive a contribution like

$$\begin{aligned} \mathcal{A}_{\mathrm{S}\not{\mathrm{U}}\mathrm{SY}} \ &= \ \prod_{j=2}^{n-2} \int \mathrm{d}z_j \ \langle\, (\lambda\, \gamma^m\, \theta)\, (\lambda\, \gamma^n\, \theta)\, (\lambda\, \gamma^p\, \theta)\, (\theta\, \gamma_{mnp}\, \theta)\, \theta^\alpha\, \Phi_\alpha(z_i)\,\rangle \\ \Rightarrow \quad \delta_\eta \mathcal{A}_{\mathrm{S}\not{\mathrm{U}}\mathrm{SY}} \ &= \ \prod_{j=2}^{n-2} \int \mathrm{d}z_j \ \eta^\alpha\, \Phi_\alpha(z_i) \end{aligned} \tag{10.3.51}$$

for some θ independent spinor Φ_α. However, BRST closure (10.3.48) for the choice $f_{\alpha\beta\gamma} = (\gamma^m\theta)_\alpha(\gamma^n\theta)_\beta(\gamma^p\theta)_\gamma(\theta\gamma_{mnp}\theta)\theta^\delta\Phi_\delta$ implies

$$\prod_{j=2}^{n-2} \int \mathrm{d}z_j \ \lambda^\alpha\, \lambda^\beta\, \lambda^\gamma\, \lambda^\delta\, \Phi_\delta \ . \tag{10.3.52}$$

This forces Φ_α to be a total worldsheet derivative which makes the SUSY variation (10.3.51) vanish. Consequently, the zero mode prescription (10.3.50) firstly belongs to the BRST cohomology and secondly preserves spacetime supersymmetry.

Generically, the correlation function $\langle V_1 \prod_{j=2}^{n-2} U_j V_{n-1} V_n\rangle$ gives rise to tensor structures $\langle \lambda^\alpha\lambda^\beta\lambda^\gamma\theta^{\delta_1}\dots\theta^{\delta_5}\, F_{\alpha\beta\gamma;\delta_1\dots\delta_5}\rangle$ which differ from $(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)$. But any contraction of $\lambda^3\,\theta^5$ is necessarily proportional to (10.3.47) because there is no more than one scalar in the tensor product of three pure spinors λ^α and five unconstrained θ^α. So the task in practical computations is to determine the proportionality constant relating $\lambda^\alpha\lambda^\beta\lambda^\gamma\theta^{\delta_1}\dots\theta^{\delta_5}\, F_{\alpha\beta\gamma;\delta_1\dots\delta_5}$ to the normalized cohomology element $(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)$.[3] If the vector indices of the six gamma matrices are free, for example, then the normalization (10.3.47) implies that

$$\langle\, (\lambda\, \gamma^m\, \theta)\, (\lambda\, \gamma^n\, \theta)\, (\lambda\, \gamma^p\, \theta)\, (\theta\, \gamma_{rst}\, \theta)\,\rangle \ = \ \frac{1}{120}\, \delta^{[m}_r\, \delta^n_s\, \delta^{p]}_t \ . \tag{10.3.54}$$

Luckily, the evaluation of a generic zero mode correlator $\langle \lambda^\alpha\lambda^\beta\lambda^\gamma\theta^{\delta_1}\dots\theta^{\delta_5}\, F_{\alpha\beta\gamma;\delta_1\dots\delta_5}\rangle$ has been fully automatized [310] based on `FORM` [311, 312]. The computer program presented in this reference can extract any superfield component from the superamplitudes which we will compute in the following chapters. Some technical complications might happen in amplitudes with four or more fermions – the involved nature of ten dimensional Fierz identities leads to ambiguities in presenting the final results in terms of the spinor wave functions. For bosonic pure spinor amplitudes, the program has been checked to reliably reproduce gluon amplitudes known from RNS computations.

[3] A symbolic notation for this algorithm is

$$\langle\, \lambda^\alpha\, \lambda^\beta\, \lambda^\gamma\, f_{\alpha\beta\gamma}(\theta)\,\rangle \ \sim \ \left(\gamma^m\, \frac{\partial}{\partial\theta}\right)^\alpha \left(\gamma^n\, \frac{\partial}{\partial\theta}\right)^\beta \left(\gamma^p\, \frac{\partial}{\partial\theta}\right)^\gamma \left(\frac{\partial}{\partial\theta}\, \gamma_{mnp}\, \frac{\partial}{\partial\theta}\right)\, f_{\alpha\beta\gamma}(\theta) \ , \tag{10.3.53}$$

but one would never evaluate these θ derivatives in practice.

10.3.3 Sample calculation: The three point amplitude in components

This subsection shows a sample calculation of zero mode integration with a two–fold purpose. Firstly, we can see the zero mode integration prescription in action, and secondly, it gives an example of how a supersymmetric amplitude in pure spinor superspace reproduces component amplitudes computed in the RNS formalism.

The general formula (10.3.40) for pure spinor tree amplitudes contains no integrated vertices at $n = 3$ points:

$$\begin{aligned}
\mathcal{A}(1,2,3) \;&=\; \langle\, \lambda^\alpha \, A^1_\alpha(z_1)\, \lambda^\beta \, A^2_\beta(z_2)\, \lambda^\gamma \, A^3_\gamma(z_3)\,\rangle \\
&=\; \Bigg\langle \lambda^\alpha \, \lambda^\beta \, \lambda^\gamma \prod_{j=1}^{3} \mathrm{e}^{k_j\cdot X(z_j)} \Bigg\{ -\, \frac{\xi^1_m}{2}\, (\gamma^m\,\theta)_\alpha \, \frac{\xi^2_n}{2}\, (\gamma^n\,\theta)_\beta \, \frac{k^3_{[q}\,\xi^3_{p]}}{16}\, (\gamma_r\,\theta)_\gamma \, (\theta\,\gamma^{pqr}\,\theta) \\
&\qquad +\, \frac{\xi^1_m}{2}\, (\gamma^m\,\theta)_\alpha \, \frac{(u_2\,\gamma_n\,\theta)}{3}\, (\gamma^n\,\theta)_\beta \, \frac{(u_3\,\gamma_p\,\theta)}{3}\, (\gamma^p\,\theta)_\gamma \;+\; \text{permutations} \Bigg\} \Bigg\rangle
\end{aligned} \tag{10.3.55}$$

$$\begin{aligned}
&=\; \frac{1}{64}\, \xi^1_m\, \xi^2_n\, k^3_{[p}\, \xi^3_{q]}\, \langle\, (\lambda\,\gamma^m\,\theta)\, (\lambda\,\gamma^n\,\theta)\, (\lambda\,\gamma_r\,\theta)\, (\theta\,\gamma^{pqr}\,\theta)\,\rangle \\
&\qquad +\; \frac{1}{18}\, \xi^1_m\, u^\alpha_2\, u^\beta_3\, \langle\, (\lambda\,\gamma^m\,\theta)\, (\lambda\,\gamma^n\,\theta)\, (\lambda\,\gamma^p\,\theta)\, (\gamma_n\,\theta)_\alpha\, (\gamma_p\,\theta)_\beta\,\rangle \\
&\qquad +\; \text{permutations in } (1,2,3)
\end{aligned} \tag{10.3.56}$$

At this point, we have to compare the zero mode correlators with (10.3.50) and find the missing proportionality constants:

$$\begin{aligned}
\langle\, (\lambda\,\gamma^m\,\theta)\, (\lambda\,\gamma^n\,\theta)\, (\lambda\,\gamma_r\,\theta)\, (\theta\,\gamma^{pqr}\,\theta)\,\rangle \;&=\; \frac{1}{90}\, (\eta^{mp}\,\eta^{nq} \;-\; \eta^{mq}\,\eta^{np}) \\
\langle\, (\lambda\,\gamma^m\,\theta)\, (\lambda\,\gamma^n\,\theta)\, (\lambda\,\gamma^p\,\theta)\, (\gamma_n\,\theta)_\alpha\, (\gamma_p\,\theta)_\beta\,\rangle \;&=\; \frac{1}{160}\, \gamma^m_{\alpha\beta}
\end{aligned} \tag{10.3.57}$$

These zero mode correlators generate the tensor contractions in the component amplitudes:

$$\begin{aligned}
\mathcal{A}(1,2,3) \;&=\; \frac{1}{2880} \Big\{ (\xi^1\cdot\xi^3)\,(\xi^2\cdot k^1) \;+\; (\xi^1\cdot\xi^2)\,(\xi^3\cdot k^2) \;+\; (\xi^2\cdot\xi^3)\,(\xi^1\cdot k^3) \Big\} \\
&\qquad +\; \frac{1}{2880} \Big\{ \xi^1_m\,(u_2\,\gamma^m\,u_3) \;+\; \xi^2_m\,(u_3\,\gamma^m\,u_1) \;+\; \xi^3_m\,(u_1\,\gamma^m\,u_2) \Big\} \\
&\sim\; \frac{1}{2880} \Big\{ \mathcal{A}(g_1,g_2,g_3) \;+\; \mathcal{A}(g_1,\lambda_2,\lambda_3) \;+\; \mathcal{A}(\lambda_1,g_2,\lambda_3) \;+\; \mathcal{A}(\lambda_1,\lambda_2,g_3) \Big\}
\end{aligned} \tag{10.3.58}$$

Apart from RNS normalization factors discussed in subsection 5.3.4, we get the sum over various SUSY multiplet components like the three gluon coupling $\mathcal{A}(g_1, g_2, g_3)$ as well as all permutations of the gaugino gluon vertex $\mathcal{A}(g_1, \lambda_2, \lambda_3)$.

Performing the above steps becomes an extremely tedious task with increasing number of legs. Also at four point level, the component expressions for the pure spinor superamplitude $\mathcal{A}(1,2,3,4)$ have been computed by hand in the heroic work of [308]. Fortunately, given its systematic nature, this procedure is suitable for an automated handling with the computer program of [310]. The following chapters will therefore no longer pursue the issue of zero mode integration. The main task for the rest of this work is the computation and simplification of supersymmetric pure spinor amplitudes.

10.3.4 Independence of $\mathcal{A}$ on the (V_i, U_j) assignment

The purpose of this subsection is to prove along the lines of [289] that the pure spinor tree amplitude computed via (10.3.40) does not depend on the choice of which states are represented by an $SL(2,\mathbb{R})$ fixed vertex $V_i(z_i)$ rather than $\int \mathrm{d}z_j\, U_j(z_j)$. This independence is necessary for cyclic invariance of a color stripped amplitude.

We will make use of the so-called "cancelled propagator" argument due to [Polchinski 1] which states that terms with colliding vertex operators $V_i(z)V_j(z)$ or $V_i(z)U_j(z)$ identically vanish. Keeping this in mind, we will now show that

$$\begin{aligned} &\left\langle V_1(0) \int_0^1 \mathrm{d}z_2\; U_2(z_2) \prod_{j=3}^{n-2} \int_{z_{j-1}}^1 \mathrm{d}z_j\; U_j(z_j)\, V_{n-1}(1)\, V_n(\infty) \right\rangle \\ &\quad = \left\langle \int_{-\infty}^0 \mathrm{d}y\; U_1(y)\, V_2(0) \prod_{j=3}^{n-2} \int_{z_{j-1}}^1 \mathrm{d}z_j\; U_j(z_j)\, V_{n-1}(1)\, V_n(\infty) \right\rangle \end{aligned} \tag{10.3.59}$$

i.e. that the representation $V_i, \int \mathrm{d}z_{i+1}\, U_{i+1}$ of neighboring states can always be swapped to $\int \mathrm{d}z_i\, U_i, V_{i+1}$. Since $QU_j(w) = \oint \frac{\mathrm{d}z}{2\pi i} \lambda^\alpha d_\alpha(z) U_j(w) = \partial V_j(w)$, we can rewrite the left hand side

$$V_1(0)\, V_n(\infty) \;=\; \int_{-\infty}^0 \mathrm{d}y\; \partial V_1(y)\, V_n(\infty) \;=\; \int_{-\infty}^0 \mathrm{d}y\; Q\big(U_1(y)\big)\, V_n(\infty) \tag{10.3.60}$$

using that $V_1(\infty)V_n(\infty) = 0$ on the compactified real axis. As a next step, we deform the integration contour of the BRST current $\lambda^\alpha d_\alpha$ such that it encircles all the vertex operators apart from U_1:

$$\begin{aligned} &\left\langle V_1(0) \int_0^1 \mathrm{d}z_2\; U_2(z_2) \prod_{j=3}^{n} \int_{z_{j-1}}^1 \mathrm{d}z_j\; U_j(z_j)\, V_{n-1}(1)\, V_n(\infty) \right\rangle \\ &\quad = -\left\langle \int_{-\infty}^0 \mathrm{d}y\; U_1(y) \int_0^1 \mathrm{d}z_2\; Q\left[U_2(z_2) \prod_{j=3}^{n} \int_{z_{j-1}}^1 \mathrm{d}z_j\; U_j(z_j)\, V_{n-1}(1)\, V_n(\infty) \right] \right\rangle \end{aligned}$$

$$= \; - \left\langle \int\limits_{-\infty}^{0} \mathrm{d}y \; U_1(y) \int\limits_{0}^{1} \mathrm{d}z_2 \; \partial V_2(z_2) \prod_{j=3}^{n} \int\limits_{z_{j-1}}^{1} \mathrm{d}z_j \; U_j(z_j) \, V_{n-1}(1) \, V_n(\infty) \right\rangle$$

$$= \; + \left\langle \int\limits_{-\infty}^{0} \mathrm{d}y \; U_1(y) \, V_2(0) \prod_{j=3}^{n} \int\limits_{z_{j-1}}^{1} \mathrm{d}z_j \; U_j(z_j) \, V_{n-1}(1) \, V_n(\infty) \right\rangle \tag{10.3.61}$$

On the way to the third line, terms where Q acts on the U_j vertices with $3 \leq j \leq n-2$ were discarded due to cancelling propagator argument: It forces both boundary terms of the $\int_{z_{j-1}}^{1} \mathrm{d}z_j \, \partial V_j(z_j)$ integrals to vanish,

$$\ldots \, U_{j-1}(z_{j-1}) \big(V_j(1) \; - \; V_j(z_{j-1}) \big) \, \ldots \, V_{n-1}(1) \, \ldots \; = \; 0 \, . \tag{10.3.62}$$

$QU_2 = \partial V_2$, on the other hand, yields a nonzero contribution in the last line of (10.3.61): The upper integration limit $z_2 = 1$ cancels due to $V_2(1) \ldots V_{n-1}(1) = 0$ whereas the lower one $z_2 = 0$ generically does not coincide with the position y of U_1, i.e. $U_1(y)V_2(0) \neq 0$.

10.3.5 Equivalence to RNS computations

Before making use of the supersymmetric setup of the pure spinor formalism, we should make sure that the superfield components of its results agrees with RNS computations. At tree level, this is addressed in [289]. Let us sketch the arguments of this reference why massless RNS amplitudes with any number of bosons and up to four fermions are reproduced by the pure spinor precription (10.3.40).

In order to make statements about components, it makes sense to extract the bosonic and fermionic part of the supersymmetric vertex operators V and U. In the gauge $\theta^\alpha A_\alpha = 0$, they are determined by the θ expansion (10.2.31) and given by

$$\begin{aligned} V_{\mathrm{B}} \; &= \; \frac{1}{2} \, \xi^m \, (\lambda \, \gamma_m \, \theta) \, \mathrm{e}^{k \cdot X} \; + \; \mathcal{O}(\theta^3) \, , \qquad & U_{\mathrm{B}} \; &= \; \xi_m \, (\partial X^m \; - \; i k_n \, M^{mn}) \, \mathrm{e}^{k \cdot X} \; + \; \mathcal{O}(\theta^2) \\ V_{\mathrm{F}} \; &= \; \frac{1}{3} \, u^\alpha \, (\lambda \, \gamma_m \, \theta) \, (\gamma^m \, \theta)_\alpha \, \mathrm{e}^{k \cdot X} \; + \; \mathcal{O}(\theta^4) \, , \qquad & U_{\mathrm{F}} \; &= \; - \, u^\alpha \, p_\alpha \, \mathrm{e}^{k \cdot X} \; + \; \mathcal{O}(\theta) \end{aligned} \tag{10.3.63}$$

in obvious notation B $\equiv$ boson and F $\equiv$ fermion. The $h = 1$ primary $M_{mn} = \Sigma^{mn}_{p,\theta} + N^{mn}$ encompasses the matter- and ghost contributions to the pure spinor Lorentz current with overall level $k_{\mathrm{tot}} = 1$. At tree level, higher powers of θ which are not displayed in (10.3.63) only contribute in presence of at least six fermions.

Now consider the n point pure spinor tree amplitude and truncate the superfields to gauginos at positions $1, 2, n-1$ and n and to gluons otherwise. This component is computed via

$$\mathcal{A}^{\mathrm{PS}} \left({}^{\lambda_1, \lambda_2, \lambda_{n-1}, \lambda_n}_{g_3, \ldots, g_{n-2}} \right) \; = \; \left\langle V^1_{\mathrm{F}}(z_1) \int \mathrm{d}z_2 \, U^2_{\mathrm{F}}(z_2) \left(\prod_{j=3}^{n-2} \int \mathrm{d}z_j \; U^j_{\mathrm{B}}(z_j) \right) V^{n-1}_{\mathrm{F}}(z_{n-1}) \, V^n_{\mathrm{F}}(z_n) \right\rangle$$

$$
\begin{aligned}
= \; -\frac{1}{27} \Big\langle & u_1^\alpha \, f_\alpha(z_1) \, u_{n-1}^\beta \, f_\beta(z_{n-1}) \, u_n^\gamma \, f_\gamma(z_n) \int \mathrm{d}z_2 \; u_2^\delta \, p_\delta(z_2) \\
& \Big(\prod_{j=3}^{n-2} \int \mathrm{d}z_j \; \xi^j_{m_j} \, (\partial X^{m_j} \; - \; i k^j_{n_j} \, M^{m_j n_j})(z_j) \Big) \prod_{l=1}^{n} \mathrm{e}^{k_l \cdot X}(z_l) \Big\rangle
\end{aligned}
\tag{10.3.64}
$$

with shorthand $f_\alpha := (\lambda \gamma_m \theta)(\gamma^m \theta)_\alpha$ for the $h = 0$ field in the unintegrated fermion vertex. The analogous RNS compuation involves four fermion vertices in the canonical $-1/2$ superghost picture (3.2.42) and any gluon vertex in its zero picture (3.2.41). In the RNS conventions from earlier parts of this work:

$$
\begin{aligned}
\mathcal{A}^{\mathrm{RNS}} \left({}^{\lambda_1, \lambda_2, \lambda_{n-1}, \lambda_n}_{g_3, \ldots, g_{n-2}} \right) \; \sim \; \Big\langle & u_1^\alpha \, c \, S_\alpha \, \mathrm{e}^{-\phi/2}(z_1) \, u_{n-1}^\beta \, c \, S_\beta \, \mathrm{e}^{-\phi/2}(z_{n-1}) \, u_n^\gamma \, c \, S_\gamma \, \mathrm{e}^{-\phi/2}(z_n) \\
\int \mathrm{d}z_2 \; u_2^\delta \, S_\delta \, \mathrm{e}^{-\phi/2}(z_2) & \Big(\prod_{j=3}^{n-2} \int \mathrm{d}z_j \; \xi^j_{m_j} \, (i\partial X^{m_j} \; - \; k^j_{n_j} \, \psi^{m_j} \psi^{n_j})(z_j) \Big) \prod_{l=1}^{n} \mathrm{e}^{i k_l \cdot X}(z_l) \Big\rangle
\end{aligned}
\tag{10.3.65}
$$

Integrating out the $h = 1$ fields from the gluons gives rise to exactly the same OPE singularity structure in both formalisms. This is firstly due to the coinciding levels of the Lorentz current algebra of M^{mn} and $\psi^m \psi^n$ and secondly to the equivalence of pure spinor transformation

$$
M^{mn}(z) \, f_\alpha(w) \;\; \sim \;\; \frac{(\gamma^{mn})_\alpha{}^\beta \, f_\beta(w)}{2\,(z-w)} \; + \; \ldots \, , \qquad M^{mn}(z) \, p_\alpha(w) \;\; \sim \;\; \frac{(\gamma^{mn})_\alpha{}^\beta \, p_\beta(w)}{2\,(z-w)} \; + \; \ldots
\tag{10.3.66}
$$

to their RNS counterpart $\psi^m \psi^n(z) S_\alpha(w) \sim (\gamma^{mn})_\alpha{}^\beta S_\beta(w)/(2(z-w))$. The dependence of both amplitudes (10.3.64) and (10.3.65) on the integrated positions $z_3, \ldots, z_{n-2}$ of bosons is completely determined by these OPEs. So what is left to check is that the correlation functions of fermion fields agree. On the RNS side, we have

$$
\langle \, c \, S_\alpha \, \mathrm{e}^{-\frac{\phi}{2}}(z_1) \, c \, S_\beta \, \mathrm{e}^{-\frac{\phi}{2}}(z_{n-1}) \, c \, S_\gamma \, \mathrm{e}^{-\frac{\phi}{2}}(z_n) \, S_\delta \, \mathrm{e}^{-\frac{\phi}{2}}(z_2) \, \rangle \;\; = \;\; \frac{\gamma^\mu_{\alpha\beta} \, \gamma_{\mu\gamma\delta}}{2\, z_{n,2}} \; + \; \frac{\gamma^\mu_{\alpha\gamma} \, \gamma_{\mu\beta\delta}}{2\, z_{n-1,2}} \; + \; \frac{\gamma^\mu_{\alpha\delta} \, \gamma_{\mu\gamma\beta}}{2\, z_{12}} \; ,
\tag{10.3.67}
$$

and the analogoues PS correlator $\langle f_\alpha(z_1) f_\beta(z_{n-1}) f_\gamma(z_n) p_\delta(z_2) \rangle$ is found to be proportional to (10.3.67) by summing the p_δ singularities and performing the zero mode integration. Since also the z_2 residues agree in the two formalisms, we have established

$$
\mathcal{A}^{\mathrm{PS}} \left({}^{\lambda_1, \lambda_2, \lambda_{n-1}, \lambda_n}_{g_3, \ldots, g_{n-2}} \right) \;\; = \;\; \mathcal{A}^{\mathrm{RNS}} \left({}^{\lambda_1, \lambda_2, \lambda_{n-1}, \lambda_n}_{g_3, \ldots, g_{n-2}} \right)
\tag{10.3.68}
$$

on the level of four fermions.

The proof of equivalence for amplitudes involving two fermions is based on similar reasoning. The pure spinor- and RNS prescriptions lead to the following expressions:

$$
\mathcal{A}^{\mathrm{PS}} \left({}^{\lambda_1, \lambda_n}_{g_2, \ldots, g_{n-1}} \right) \;\; = \;\; \frac{1}{18} \Big\langle u_1^\alpha \, f_\alpha(z_1) \, u_n^\beta \, f_\beta(z_n) \, \xi_p \, (\lambda \, \gamma^p \, \theta)(z_{n-1})
$$

$$\left(\prod_{j=2}^{n-2} \int \mathrm{d}z_j \; \xi^j_{m_j} \, (\partial X^{m_j} \;-\; i k^j_{n_j} \, M^{m_j n_j})(z_j) \right) \prod_{l=1}^{n} \mathrm{e}^{k_l \cdot X}(z_l) \Bigg\rangle \tag{10.3.69}$$

$$\mathcal{A}^{\mathrm{RNS}} \left({}^{\lambda_1, \lambda_n}_{g_2, \ldots, g_{n-1}} \right) \;\sim\; \Bigg\langle \, u_1^\alpha \, c \, S_\alpha \, \mathrm{e}^{-\phi/2}(z_1) \, u_n^\beta \, c \, S_\beta \, \mathrm{e}^{-\phi/2}(z_n) \, \xi_p \, c \, \psi^p \, \mathrm{e}^{-\phi}(z_{n-1})$$

$$\left(\prod_{j=2}^{n-2} \int \mathrm{d}z_j \; \xi^j_{m_j} \, (i\partial X^{m_j} \;-\; k^j_{n_j} \, \psi^{m_j} \psi^{n_j})(z_j) \right) \prod_{l=1}^{n} \mathrm{e}^{i k_l \cdot X}(z_l) \Bigg\rangle \tag{10.3.70}$$

Integrating out the gluons at $z_2, \ldots, z_{n-2}$ gives rise to the same OPE structure in both formalisms[4], i.e. all the worldsheet integrals are guaranteed to match. Finally, the remaining three point functions are proportional to each other:

$$\langle \, c \, S_\alpha \, \mathrm{e}^{-\frac{\phi}{2}}(z_1) \, c \, S_\beta \, \mathrm{e}^{-\frac{\phi}{2}}(z_n) \, c \, \psi^p \, \mathrm{e}^{-\phi}(z_{n-1}) \, \rangle \;=\; \frac{\gamma^p_{\alpha\beta}}{\sqrt{2}} \;\sim\; \langle \, f_\alpha(z_1) \, f_\beta(z_n) \, (\lambda \, \gamma^p \, \theta)(z_{n-1}) \, \rangle \tag{10.3.72}$$

The equivalence of purely bosonic amplitudes follows from SUSY Ward identities relating the boson amplitudes in questions to two fermion amplitudes (10.3.69) and (10.3.70) which were shown to agree. The RNS realization of those Ward identities has been worked out in subsection 5.2.4, and the pure spinor generator of spacetime SUSY is given by (10.3.73).

$$\mathcal{Q}_\alpha \;=\; \oint \frac{\mathrm{d}z}{2\pi i} \, \left\{ p_\alpha \;+\; \frac{1}{2} \, \gamma^m_{\alpha\beta} \, \theta^\beta \, \partial X_m \;+\; \frac{1}{24} \, (\gamma^m \, \theta)_\alpha \, (\theta \, \gamma_m \, \partial\theta) \right\} \, . \tag{10.3.73}$$

[4] For the gluon at z_{n-1}, we need the additional information that the PS Lorentz current M^{mn} reproduces the OPE of $\psi^m \psi^n(z)$ and $\psi^p(w)$:

$$M^{mn}(z) \, (\lambda \, \gamma^p \, \theta)(w) \;\sim\; \frac{2 \, \eta^{p[n} \, (\lambda \, \gamma^{m]} \, \theta)}{z - w} \tag{10.3.71}$$

Chapter 11

SYM amplitudes in pure spinor superspace

Equipped with the pure spinor prerequisites from the previous chapter, we will now construct superfields which capture the kinematics in SYM field theory amplitudes. This chapter gathers the results of [291] and three of my pubilcations [7, 8, 11].

As emphasized several times before, the S matrix of maximally supersymmetric SYM theory ($\mathcal{N} = 1$ in $D = 10$ dimensions which is equivalent to $\mathcal{N} = 4$ in $D = 4$ dimensions) emerges as the $\alpha' \to 0$ limit of superstring amplitudes involving the massless gauge multiplet. That is why the pure spinor approach to the superstring provides a setting to describe SYM amplitudes $\mathcal{A}^{\rm SYM}$ in a manifestly supersymmetric fashion.

As explained in section 5.5, every color ordered tree level amplitude in SYM theories can be arranged into a sum over cubic diagrams of appropriate color order,

$$\mathcal{A}^{\rm SYM}(1,2,\ldots,n) \ = \ \sum_i \frac{n_i}{\prod_{\alpha_i} s_{\alpha_i}} \ . \tag{11.0.1}$$

The sum over i encompasses $\frac{1}{n-1}\binom{2n-4}{n-2}$ cubic graphs, s_{α_i} denote their $n-3$ propagators, and n_i are the associated kinematic BCJ numerators [81]. The latter can be viewed to be dual to color factors according to subsection 5.5.2, and we will construct them explicitly by pure spinor methods in the later chapter 13.

On the basis of this representation for the n-point SYM amplitude, it was suggested in [291] that the BRST cohomology of the pure spinor formalism together with the kinematic pole structure might be enough to fix the ten-dimensional SYM amplitudes. This approach bypasses the need to compute the corresponding open superstring amplitude from the general prescription (10.3.40) and to take its $\alpha' \to 0$ limit. In order for the empirical cohomology method to work, one needs to have explicit mappings between cubic diagrams and pure spinor superspace expressions, the so-called BRST building blocks. This will be accomplished in section 11.1.

As a next hierarchy level of superfield constituents, the BRST building blocks will be combined to SYM amplitudes with one off-shell leg in section 11.2. These objects were firstly considered in [237] under the name "currents" in order to derive recursion relations for gluon scattering at tree level. The pure spinor implementation of these Berende-Giele currents led to a general recursive method to construct n-point SYM amplitudes in the cohomology of the BRST operator [8]. We will present the compact result shown in figure 11.1 for the n point SYM tree level amplitude in the third section 11.3.

$$\mathcal{A}^{\mathrm{SYM}}(1,2,\ldots,n) \quad = \quad \sum_{j=1}^{n-2} \langle\, M_{12\ldots j}\, M_{j+1\ldots n-1}\, V_n \,\rangle \quad = \quad \sum_{j=1}^{n-2}$$

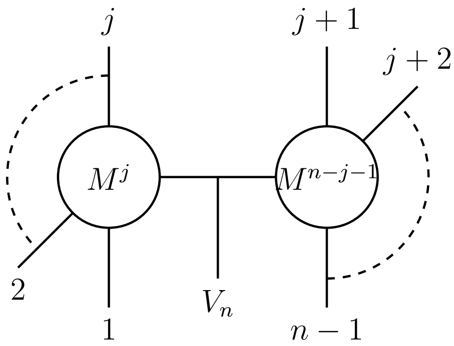


Figure 11.1: Decomposition of the color ordered n point SYM amplitude into Berends-Giele $M_{12\ldots j}$ (to be introduced in section 11.2)

So far, the pure spinor formalism as introduced in the the last chapter is applicable in $D = 10$ dimensions only. That is why the superamplitudes presented in this chapter describe ten dimensional $\mathcal{N} = 1$ SYM. But still, the superfield components of the end result (i.e. after performing the zero mode integration (10.3.50)) can be dimensionally reduced to $D = 4$ or any other dimension lower than ten by straightforward methods.

11.1 BRST building blocks

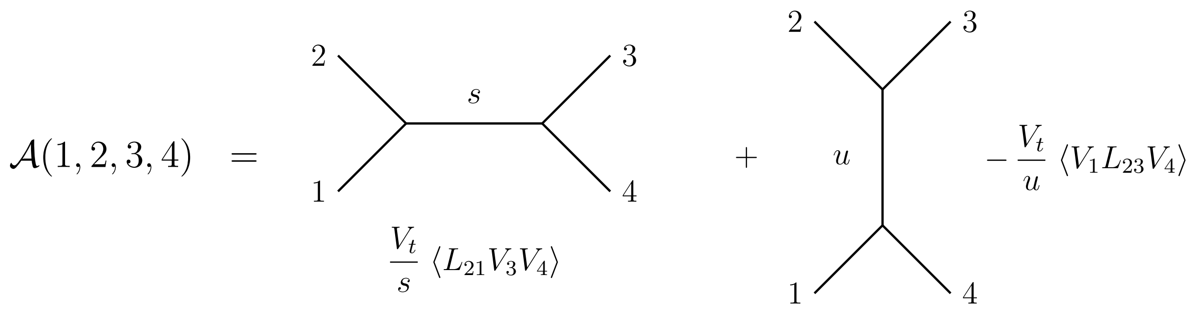
As a good motivating example, let us finish the computation of the four point superstring amplitude $\mathcal{A}(1,2,3,4)$. The CFT correlator was already obtained in (10.3.46) by summing over the OPE residues of $V_{1,3}(z_{1,3})U_2(z_2)$. That is why it makes sense to introduce the following shorthand for the residues:

$$V_1(z_1)\,U_2(z_2) \quad \sim \quad \frac{L_{21}(z_1)}{z_{21}} \,+\, \ldots\,, \qquad L_{21} \quad := \quad -\,(k_1\cdot A_2)\,V_1 \,-\, A^1_m\,(\lambda\,\gamma^m\,W_2) \,+\, Q\,(A_1\,W_2) \tag{11.1.2}$$

The z_2 integrals over $|z_2|^{-s-1}\,|1-z_2|^{-u}$ and $|z_2|^{-s}\,|1-z_2|^{-u-1}$ yield $\frac{V_t}{s}$ and $\frac{V_t}{u}$, respectively, where V_t denotes the Veneziano formfactor from four point superstring scattering, see section XXX. Hence, the color ordered four point amplitude is given by

$$\mathcal{A}(1,2,3,4) \;\;=\;\; V_t \left\{ \frac{1}{s} \, \langle \, L_{21} \, V_3 \, V_4 \, \rangle \;-\; \frac{1}{u} \, \langle \, V_1 \, L_{23} \, V_4 \, \rangle \right\} \, . \tag{11.1.3}$$

This result clearly disentangles the superfield kinematics associated with the two color ordered Feynman diagrams – the s channel and the u channel.



Moreover, it has two important messages in view of the BRST cohomology:

- BRST exact parts decouple: The usual contour deformation argument for $Q = \oint \frac{\mathrm{d}z}{2\pi i} \lambda^\alpha d_\alpha$ implies that we can integrate Q by parts within correlation functions. This can be used to show that the last piece $Q(A_1 W_2)$ of L_{21} does not contribute to the amplitude:

$$\langle \, Q \, (A_1 \, W_2) \, V_3 \, V_4 \, \rangle \;\;=\;\; - \, \langle \, (A_1 \, W_2) \, Q \, (V_3 \, V_4) \, \rangle \;\;=\;\; 0 \tag{11.1.4}$$

- We can verify by direct computation that the action of the BRST charge Q decomposes the composite superfield L_{21} into two unintegrated vertex operators

$$Q \, L_{21} \;\;=\;\; s_{12} \, V_1 \, V_2 \; . \tag{11.1.5}$$

  This is very helpful for checking BRST closedness of the amplitude (11.1.3) – the poles in $s = s_{12}$ and $u = s_{23}$ are cancelled by $\langle Q(L_{ji} V_k V_l) \rangle = s_{ij} \langle V_i V_j V_k V_l \rangle$, so the two terms in (11.1.3) give opposite contributions to the amplitude's BRST variation. Hence, knowledge of the BRST action (11.1.5) trivializes the proof of $Q\mathcal{A}(1,2,3,4) = 0$.

Both of these observations are taken seriously in the remainder of this chapter. The former guides the construction of the so-called BRST building blocks encompassing several superfields, the latter leads to severe constraints on how these building blocks can be combined to form a BRST closed SYM amplitude.

11.1.1 OPE residues

This subsection introduces composite superfields which naturally arise in the CFT computation of the superstring amplitude. These OPE residues are the first step towards the so-called BRST building blocks with diagrammatic interpretation.

As a generalization of $V_i(z_i)U_j(z_j) \sim L_{ji}(z_i)/z_{ji}$ used above, we recursively define

$$L_{21}(z_1)\, U_3(z_3) \ \sim \ \frac{L_{2131}(z_1)}{z_{31}}\,, \qquad L_{2131\ldots(p-1)1}(z_1)\, U_p(z_p) \ \sim \ \frac{L_{2131\ldots(p-1)1p1}(z_1)}{z_{p1}}\,. \tag{11.1.6}$$

The following OPE identity is very helpful fo the explicit computation of the $L_{2131\ldots p1}$:

$$(\lambda\,\gamma^m\, W_i)(z_i)\, U_j(z_j) \ \sim \ \frac{1}{z_{ji}} \Big[\mathcal{F}_i^{mn}\, (\lambda\, \gamma_n\, W_j) \ - \ (k_i \cdot A_j)\,(\lambda\,\gamma^m\, W_i) \ + \ Q\,(W_i\,\gamma^m\, W_j) \Big](z_i) \tag{11.1.7}$$

By performing BRST integration by parts in the last term, one can arrive at the following expressions at $p = 2, 3$ and $p = 4$:

$$L_{21} \ = \ -\,(k_1 \cdot A_2)\, V_1 \ - \ A^1_m\,(\lambda\,\gamma^m\, W_2) \tag{11.1.8}$$

$$\begin{aligned} L_{2131} \ = \ & \big[\,(k_1 \cdot A_2)\, V_1 \ + \ A^1_m\,(\lambda\,\gamma^m\, W_2)\big]\,\big[\,(k_1+k_2)\cdot A_3\,\big] \\ & + \ (\lambda\,\gamma^m\, W_3)\,\Big\{\, \mathcal{F}^2_{mn}\, A^n_1 \ + \ (k_1 \cdot A_2)\, A^1_m \ - \ (W_1\,\gamma_m\, W_2)\,\Big\} \end{aligned} \tag{11.1.9}$$

$$\begin{aligned} L_{213141} \ = \ & \Big\{\, \big[\,-A^1_m\,(\lambda\,\gamma^m\, W_2) \ - \ V_1 (k_1 \cdot A_2)\big]\,\big[\,(k_1+k_2)\cdot A_3\,\big] \ - \ \mathcal{F}^2_{mn}\, A^n_1\,(\lambda\,\gamma^m\, W_3) \\ & \quad - \ A^1_m\,(\lambda\,\gamma^m\, W_3)\,(k_1\cdot A_2) \ + \ (W_1\,\gamma^m\, W_2)\,(\lambda\,\gamma_m\, W_3)\,\Big\}\,\big[\,(k_1+k_2+k_3)\cdot A_4\,\big] \\ & + \ (\lambda\,\gamma^m\, W_4)\,\Big\{\,(A_1\cdot k_2)\, A^2_m\,\big[\,(k_1+k_2)\cdot A_3\,\big] \ - \ (A_1 \cdot A_2)\, k^2_m\,\big[\,(k_1+k_2)\cdot A_3\,\big] \\ & \quad - \ (A_1\cdot k_2)\,(W_2\,\gamma_m\, W_3) \ + \ (A_1\cdot k_2)\,(A_2\cdot A_3)\, k^3_m \ - \ (A_1\cdot A_2)\,(k_2\cdot A_3)\, k^3_m \\ & \quad + \ (A_1\cdot A_2)\,(k_2\cdot k_3)\, A^3_m \ - \ (A_1\cdot k_2)(A_2\cdot k_3)\, A^3_m \ + \ \mathcal{F}^3_{mn}\,(W_1\,\gamma^n\, W_2) \\ & \quad + \ (A_1\cdot k_3)\,(k_1\cdot A_2)\, A^3_m \ - \ (A_1\cdot A_3)\,(k_1\cdot A_2)\, k^3_m \ + \ (W_1\,\gamma_m\, W_3)\,(k_1\cdot A_2) \\ & \quad + \ (W_1\,\gamma_m\, W_2)\,\big[\,(k_1+k_2)\cdot A_3\,\big] \ - A^1_m\,(A_2\cdot k_1)\,\big[\,(k_1+k_2)\cdot A_3\,\big] \\ & \quad + \ \frac{1}{4}\,(W_2\,\gamma_{pq}\,\gamma_m\, W_3)\,\mathcal{F}^{pq}_1 \ - \ \frac{1}{4}\,(W_1\,\gamma_{pq}\,\gamma_m\, W_3)\,\mathcal{F}^{pq}_2\,\Big\} \end{aligned} \tag{11.1.10}$$

The next residue $L_{21314151}$ is displayed in appendix E.1.3.

We have discarded the BRST exact term $Q(A_1 W_2)$ which appeared in the definition (11.1.2) of L_{21}. Similarly, the naive computation of L_{2131} and L_{213141} involves several BRST exact pieces which are no longer displayed in (11.1.8) to (11.1.10) – they were explicitly checked to decouple from any amplitude up to six points, and the subsequent arguments strongly suggest that this pattern has to persist at higher points.

More generally, any $L_{2131\ldots p1}$ can be brought into the form

$$L_{2131\ldots p1} \ = \ -\Big[\,(k_1+k_2+\ldots+k_{p-1})\cdot A_p\,\Big]\,L_{2131\ldots(p-1)1} \ + \ (\lambda\,\gamma^m\,W_p)\,R_m^{12\ldots(p-1)} \quad (11.1.11)$$

where the first examples of remainder vector superfield $R_m^{12\ldots(p-1)}$ multiplying $(\lambda\gamma^m W_p)$ are given by $R^1_m = -A^1_m$ for $p=2$ and $R^{12}_m = \mathcal{F}^2_{mn}A^n_1 + (k_1\cdot A_2)A^1_m - (W_1\gamma_m W_2)$ for $p=3$.

An important property of the OPE residues $L_{2131\ldots p1}$ is their BRST covariance, e.g.:

$$Q\,L_{21} \ = \ s_{12}\,V_1\,V_2 \quad (11.1.12)$$

$$Q\,L_{2131} \ = \ (s_{13}+s_{23})\,L_{21}\,V_3 \ + \ s_{12}\,(L_{31}\,V_2 \ + \ V_1\,L_{32}) \quad (11.1.13)$$

$$\begin{aligned} Q\,L_{213141} \ &= \ (s_{14}+s_{24}+s_{34})\,L_{2131}\,V_4 \ + \ (s_{13}+s_{23})\,(L_{21}\,L_{43} \ + \ L_{2141}\,V_3) \\ &\quad + \ s_{12}\,(L_{3141}\,V_2 \ + \ L_{31}\,L_{42} \ + \ L_{41}\,L_{32} \ + \ V_1\,L_{3242}) \end{aligned} \quad (11.1.14)$$

There is no need to explicitly compute the BRST variation of the SYM superfields on the right hand side of (11.1.8), (11.1.9) and (11.1.10). Instead, one can obtain $QL_{2131\ldots p1}$ by directly applying Q to $L_{2131\ldots(p-1)1}U_p$ before performing OPE contractions (11.1.6). Among the three terms in $QU_p = \partial V_p = (\partial\lambda^\alpha)A^p_\alpha + \Pi^m k^p_m V_p + \partial\theta^\alpha D_\alpha V_p$, only the $(\Pi\cdot k_p)$ part has a singular OPE with $L_{2131\ldots(p-1)1}$, the residue of the single pole being proportional to $(s_{1p}+s_{2p}+\ldots+s_{p+1,p})$. This reasoning leads to a recursive method of determining $QL_{2131\ldots p1}$ from $QL_{2131\ldots(p-1)1}$:

$$\begin{aligned} Q\,L_{2131\ldots p1} \ &= \ \lim_{z_p\to z_1} z_{p1}\,\Big\{\,(Q\,L_{2131\ldots(p-1)1})(z_1)\,U_p(z_p) \ - \ L_{2131\ldots(p-1)1}(z_1)\,(\Pi\cdot k_p)\,V_p(z_p)\,\Big\} \\ &= \ \lim_{z_p\to z_1} z_{p1}\,(Q\,L_{2131\ldots(p-1)1})(z_1)\,U_p(z_p) \ + \ \sum_{j=1}^{p-1} s_{jp}\,L_{2131\ldots(p-1)1}\,V_p(z_1) \end{aligned} \quad (11.1.15)$$

The first terms evaluates as follows for $p=3$:

$$\begin{aligned} \lim_{z_3\to z_1} z_{31}\,(Q\,L_{21})(z_1)\,U_3(z_3) \ &= \ s_{12}\,\lim_{z_3\to z_1} z_{31}\,(V_1\,V_2)(z_1)\,U_3(z_3) \\ &= \ s_{12}\,(L_{31}\,V_2 \ + \ V_1\,L_{32})(z_1) \end{aligned} \quad (11.1.16)$$

When passing to the second line, U_3 can contract either V_1 or V_2 which gives rise to L_{31} and L_{32}, respectively. By iterating this procedure, one can quickly obtain the BRST variation of large building blocks $L_{2131\ldots p1}$ beyond (11.1.14), the number of terms in $QL_{2131\ldots p1}$ being $2^{p-1}-1$. The $L_{2131\ldots p1}$ themselves, however, must still be computed in the "pedestrian" way by successive application of (11.1.6) if we insist on performing zero mode integration.

11.1.2 The BRST building blocks T_{12} and T_{123}

Succesive OPE contraction of one unintegrated vertex V_1 with several integrated ones U_j defines a tower of composite superfields $L_{2131\ldots p1}$ with arbitrary number p of external legs. However, it is not clear at this point whether they have any definite symmetry properties under exchange of labels as required for making contact with cubic diagrams. This subsection focuses on the first two composites L_{21} and L_{2131} and shows a unique way to redefine them to objects $T_{12} = L_{21} + \ldots$ and $T_{123} = L_{2131} + \ldots$ with well-defined symmetries in permuting the labels.

Although the simplest OPE residue $L_{21} = -(k_1 \cdot A_2)V_1 - A^1_m(\lambda\gamma^m W_2)$ has no obvious properties under $1 \leftrightarrow 2$ exchange, we can still identify its symmetric part as a Q variation:

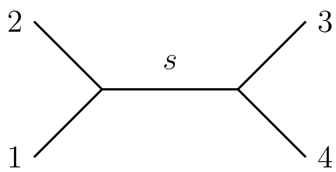
$$Q\,(A_1 \cdot A_2) \;\; = \;\; -\,L_{21} \; - \; L_{12} \tag{11.1.17}$$

We have mentioned below (11.1.10) that building blocks can be reduced to their BRST cohomological parts. BRST exact contributions were checked to decouple from superstring amplitudes up to six points. Following this philosophy, we subtract the BRST trivial symmetric part from L_{21} and define the first BRST building block T_{12} by

$$T_{12} \;\; := \;\; L_{[21]} \;, \qquad Q\,T_{12} \;\; = \;\; s_{12}\,V_1\,V_2 \;. \tag{11.1.18}$$

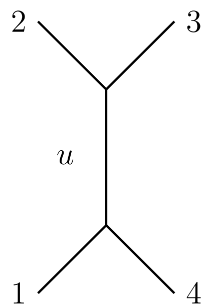
Shifting L_{21} by Q exact terms does not change its BRST variation. The four point amplitude (11.1.3) can be written in terms of T_{ij} rather than L_{kl} because BRST exact terms were explicitly shown to decouple from $\mathcal{A}(1,2,3,4)$, see (11.1.4):

$\mathcal{A}(1,2,3,4) \;\; =$



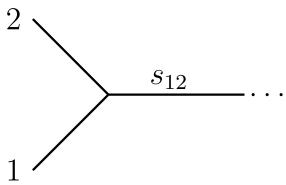
$\frac{V_t}{s}\,\langle T_{12}V_3V_4\rangle$

$+$

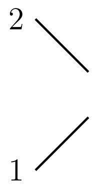


$\frac{V_t}{u}\,\langle V_1T_{23}V_4\rangle$

Their clear association with individual diagrams and their antisymmetry in $1 \leftrightarrow 2$, suggest to identify the T_{12}/s_{12} with the tailend of a cubic graph. Action of Q removes the propagator and leaves two isolated external lines



$= \frac{T_{12}}{s_{12}}\;,$



$= Q\,\frac{T_{12}}{s_{12}} \;\; = \;\; V_1V_2$

Before performing similar redefinitions of L_{2131}, we have to make sure that its Q variation reproduces the lower order BRST building blocks T_{ij} rather than the L_{kl} appearing in (11.1.13).

For this purpose, we rewrite $L_{kl} = T_{lk} - \frac{Q}{2}(A_k \cdot A_l)$ on the right hand side and define a new building block

$$\tilde{T}_{123} \;=\; L_{2131} \;+\; \frac{1}{2}\Big\{ s_{12}\big[\,(A_1\cdot A_3)\,V_2 \;-\; (A_2\cdot A_3)\,V_1\,\big] \;+\; (s_{13}+s_{23})\,(A_1\cdot A_2)\,V_3 \Big\} \tag{11.1.19}$$

which satisfies (11.1.13) with L_{kl} replaced by T_{ij} on the right hand side:

$$Q\,\tilde{T}_{123} \;=\; (s_{13}+s_{23})\,T_{12}\,V_3 \;+\; s_{12}\,(T_{13}\,V_2 \;+\; V_1\,T_{23}) \tag{11.1.20}$$

The redefinition of any higher order residue $L_{2131\ldots p1}$ also involves some combinations of $(A_i \cdot A_j)$ products, that is why we better introduce a shorthand here:

$$D_{ij} \;:=\; A_i \cdot A_j \tag{11.1.21}$$

In contrast to the superfield L_{jiki} from the OPE contraction of L_{ji} with U_k, the new object $\tilde{T}_{ijk}$ can be combined to the follwing two BRST exact quantities:

$$\tilde{T}_{123} \;+\; \tilde{T}_{213} \;=\; Q\Big(\,D_{12}\,\big[\,(k_1+k_2)\cdot A_3\,\big]\Big) \tag{11.1.22}$$

$$\tilde{T}_{123} \;+\; \tilde{T}_{312} \;+\; \tilde{T}_{231} \;=\; Q\Big(\,D_{12}\,(k_2\cdot A_3) \;+\; D_{13}\,(k_1\cdot A_2) \;+\; D_{23}\,(k_3\cdot A_1)\Big) \tag{11.1.23}$$

Hence, subtracting these BRST trivial components yields a new BRST building block

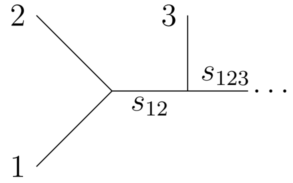
$$T_{123} \;=\; \frac{1}{3}\,(\tilde{T}_{123} \;-\; \tilde{T}_{213}) \;+\; \frac{1}{6}\,(\tilde{T}_{321} \;-\; \tilde{T}_{312} \;+\; \tilde{T}_{132} \;-\; \tilde{T}_{231}) \tag{11.1.24}$$

with symmetry properties that reduce the number of independent T_{ijk} from six down to two:

$$T_{123} \;=\; T_{[12]3}\;, \qquad T_{123} \;+\; T_{231} \;+\; T_{312} \;=\; 0 \tag{11.1.25}$$

The diagrammatic interpretation of T_{123} is guided by the Mandelstam variables s_{12} and s_{123} which appear in its BRST variation. This is necessary for BRST closedness of the overall amplitude in the end – each term in $Q\mathcal{A}(1,\ldots,n)$ is supposed to exhibit one pole less than $\mathcal{A}(1,\ldots,n)$ itself such that different cubic graphs can conspire to yield $Q\mathcal{A}(1,\ldots,n)=0$ in the end.

$$\begin{aligned} Q\,T_{123} \;&=\; s_{123}\,T_{12}\,V_3 \\ &\quad -\; s_{12}\,(T_{12}\,V_3 \;+\; T_{23}\,V_1 \;+\; T_{31}\,V_2) \end{aligned}$$



$$= \; \frac{T_{123}}{s_{12}\,s_{123}}$$

The two step redefinition $L_{2131} \to \tilde{T}_{123} \to T_{123}$ is generic for building blocks of higher rank. The simplicity of the $p=2$ case where $L_{21} = \tilde{T}_{12}$ is exceptional. The next subsection explains the generic pattern for $L_{21\ldots p1} \to \tilde{T}_{12\ldots p} \to T_{12\ldots p}$.

11.1.3 Higher rank BRST building blocks

The tower of $L_{2131\ldots p1}$ was shown in (11.1.15) to be covariant under the BRST charge – the action of Q splits the OPE residues into a sum of two smaller pieces $L_{i_2 i_1 \ldots i_q i_1}$ and $L_{i_{q+2} i_{q+1} \ldots i_p i_{q+1}}$, multiplied with some Mandelstam invariant $s_{l_1 \ldots l_r}$. The construction of BRST closed SYM amplitudes made of cubic diagrams, on the other hand, requires a redefined tower of composite superfields $T_{12\ldots p} = L_{2131\ldots p1} + \ldots$ with definite symmetry properties which is also covariant under Q action in the sense that $QT_{i_1 \ldots i_p} \sim s_{l_1 \ldots l_r} T_{i_1 \ldots i_q} T_{i_{q+1} \ldots i_p}$, e.g.

$$\begin{aligned} Q\,T_{1234} \;&=\; (s_{1234} - s_{123})\,T_{123}\,V_4 \;+\; (s_{123} - s_{12})\,(T_{12}\,T_{34} \;+\; T_{124}\,V_3) \\ &\quad\;+\; s_{12}\,(T_{134}\,V_2 \;+\; T_{13}\,T_{24} \;+\; T_{14}\,T_{23} \;+\; V_1\,T_{234}) \;=\; Q\,\tilde{T}_{1234}\,. \end{aligned} \tag{11.1.26}$$

$$\begin{aligned} Q\,T_{12345} \;&=\; (s_{12345} - s_{1234})\,T_{1234}\,V_5 \;+\; (s_{1234} - s_{123})\,(T_{1235}\,V_4 \;+\; T_{123}\,T_{45}) \\ &\quad+\; (s_{123} - s_{12})\,(T_{1245}\,V_3 \;+\; T_{124}\,T_{35} \;+\; T_{125}\,T_{34} \;+\; T_{12}\,T_{345}) \\ &\quad+\; s_{12}\,(T_{1345}\,V_2 \;+\; V_1\,T_{2345} \;+\; T_{134}\,T_{25} \;+\; T_{135}\,T_{24} \;+\; T_{145}\,T_{23} \\ &\qquad+\; T_{13}\,T_{245} \;+\; T_{14}\,T_{235} \;+\; T_{15}\,T_{234}) \end{aligned} \tag{11.1.27}$$

Hence, the first step in redefining the OPE residue $L_{2131\ldots p1}$ is to reexpress all the $L_{i_1 \ldots i_q}$ within $QL_{2131\ldots p1}$ in terms of lower order BRST building blocks $T_{i_1 \ldots i_q}$.

Suppose the redefinition of building blocks $T_{123\ldots p} = L_{2131\ldots p1} + \ldots$ is know up to order $q = p-1$. Then we can replace $L_{i_2 i_1 i_3 i_1 \ldots i_q i_1} = T_{i_1 i_2 \ldots i_q} + \ldots$ on the right hand side of $QT_{2131\ldots p1}$, and the corrections turn out to add up to a BRST variation:

$$\begin{aligned} Q\,L_{2131\ldots p1} \;&\sim\; \sum_J s^{(J)}_{l_1 \ldots l_r}\,L^{(J)}_{i_2 i_1 \ldots i_q i_1}\,L^{(J)}_{i_{q+2} i_{q+1} \ldots i_p i_{q+1}} \\ &\sim\; \sum_j s^{(J)}_{l_1 \ldots l_r}\,T^{(J)}_{i_1 \ldots i_q}\,T^{(J)}_{i_{q+1} \ldots i_p} \;+\; Q\,P_{123\ldots p} \end{aligned} \tag{11.1.28}$$

In this schematic notation, the J sum runs over the $2^{p-1} - 1$ terms of $QL_{2131\ldots p1}$. The ghost number zero superfield $P_{123\ldots p}$ appearing on the right hand side must be subtracted from $L_{21\ldots p1}$ to bring its BRST variation into the desired form:

$$\tilde{T}_{123\ldots p} \;:=\; L_{2131\ldots p1} \;-\; P_{123\ldots p} \quad\Rightarrow\quad Q\,\tilde{T}_{123\ldots p} \;=\; \sum_j s^{(J)}_{l_1 \ldots l_r}\,T^{(J)}_{i_1 \ldots i_q}\,T^{(J)}_{i_{q+1} \ldots i_p} \tag{11.1.29}$$

Hence, the difference $\tilde{T}_{123\ldots p} - L_{2131\ldots p1}$ depends on all the lower order redefinitions $T_{123\ldots q} = L_{2131\ldots q1} + \ldots$ with $q = 1, 2, \ldots, p-1$.

The symmetry properties of the lower order $T^{(J)}_{i_1 \ldots i_q}$ on the right hand side of (11.1.29) allow to identify $p-1$ BRST closed $\tilde{T}_{123\ldots p}$ combinations. The fact that composite superfields are off-shell in the sense that $(k_1 + k_2 + \ldots + k_p)^2 \neq 0$ implies that the Q cohomology at rank $p \geq 2$

is empty and each Q closed expression is automatically exact. These BRST trivial components must be subtracted from $\tilde{T}_{123\ldots p}$, i.e. the final BRST building blocks which constitute SYM- and superstring amplitudes are obtained by

$$T_{123\ldots p} \;=\; \tilde{T}_{123\ldots p} \;-\; Q\, S^{(p-2)}_{123\ldots p} \tag{11.1.30}$$

where $QS^{(p-2)}_{123\ldots p}$ collectively refers to all the BRST exact pieces. The notation with the $^{(p-2)}$ superscript will become clearer in appendix E.1.2. The superfields $S^{(0)}_{12}$ and $S^{(1)}_{123}$ of the $p = 2, 3$ examples

$$T_{12} \;=\; \tilde{T}_{12} \;-\; Q\, S^{(0)}_{12} \;, \qquad T_{123} \;=\; \tilde{T}_{123} \;-\; Q\, S^{(1)}_{123} \tag{11.1.31}$$

have already been given in the previous subsection:

$$S^{(0)}_{12} \;=\; -\frac{1}{2}\,(A_1 \cdot A_2) \;=\; -\frac{D_{12}}{2} \tag{11.1.32}$$

$$S^{(1)}_{123} \;=\; \frac{D_{12}}{3}\,\big[\,(2k_2 + k_1)\cdot A_3\,\big] \;+\; \frac{D_{13}}{6}\,\big[\,(k_1 - k_3)\cdot A_2\,\big] \;+\; \frac{D_{23}}{6}\,\big[\,(k_3 - k_2)\cdot A_1\,\big] \tag{11.1.33}$$

Of course, the subtraction (11.1.30) of BRST trivial terms does not affect the BRST variation of $\tilde{T}_{12\ldots p}$. At general rank p, the Q action on building blocks is given by

$$Q\, T_{12\ldots p} \;=\; \sum_{j=2}^{n} \sum_{\alpha \in P(\beta_j)} \big(s_{12\ldots j} - s_{12\ldots j-1}\big)\, T_{12\ldots j-1,\{\alpha\}}\, T_{j,\{\beta_j \setminus \alpha\}} \tag{11.1.34}$$

where $V_i \equiv T_i$. The set $\beta_j = \{j+1, j+2, \ldots, n\}$ encompasses the $n-j$ labels to the right of j, and $P(\beta_j)$ denotes its power set. This explicit formula follows from the recursion (11.1.15) for $QL_{2131\ldots p1}$ upon replacing $L_{j_1 i j_2 i \ldots j_p i} \mapsto T_{i j_1 j_2 \ldots j_p}$.

To summarize the chain of redefinitions above: Two steps are required to obtain proper BRST building blocks $T_{12\ldots p}$ with the necessary symmetry properties to represent cubic diagrams,

$$L_{21\ldots p1} \;\longrightarrow\; \tilde{T}_{12\ldots p} \;=\; L_{21\ldots p1} \;-\; P_{12\ldots p} \;\longrightarrow\; T_{12\ldots p} \;=\; \tilde{T}_{12\ldots p} \;-\; Q\, S^{(p-2)}_{12\ldots p} \;. \tag{11.1.35}$$

The first part $L_{21\ldots p1} \to \tilde{T}_{12\ldots p}$ makes sure that the BRST variation exclusively involves BRST building blocks of lower order. The second redefinition $\tilde{T}_{12\ldots p} \to T_{12\ldots p}$ removes BRST exact components from the intermediate superfields $\tilde{T}_{12\ldots p}$.

The explicit form of the T_{1234} and T_{12345}, in particular the associated $P_{12\ldots p}$ and $S^{(p-2)}_{12\ldots p}$ corrections, can be found in appendix E.1.

11.1.4 Diagrammatic interpretation

We had already given a diagrammatic interpretation for the simplest BRST building blocks T_{12} and T_{123} in subsection 11.1.2. This will now be generalized to higher rank $T_{12\ldots p}$: This subsection provides a dictionary between cubic tree level diagrams with arbitrary branchings and combinations of BRST building blocks $T_{12\ldots p}$.

The idea of the general dictionary between cubic diagrams and BRST building blocks $T_{12\ldots p}$ is to form combinations of building blocks whose Q variation contains the desired propagators of the graph. We have argued before that the association of a diagram to T_{123} is guided by the Mandelstam variables s_{123} and s_{12} appearing in its BRST variation. If the propagators associated with $T_{12\ldots p}$ are given by the $s_{l_1\ldots l_r}$ appearing in $QT_{12\ldots p}$, then the amplitude has a chance to be BRST closed since each term of $Q\mathcal{A}^{\rm SYM}(1,\ldots,n)$ has to vanish at the residue of the $(n-3)$ fold poles.

According to (11.1.34), the BRST variation of $T_{123\ldots p}$ involves $p-1$ Mandelstam variables encompassing an increasing number of momenta,

$$Q\,T_{123\ldots p} \quad \leftrightarrow \quad \big\{\, s_{12},\ s_{123},\ s_{1234},\ldots,s_{12\ldots p}\,\big\}\ . \tag{11.1.36}$$

This suggests the following dictionary given in figure 11.2:

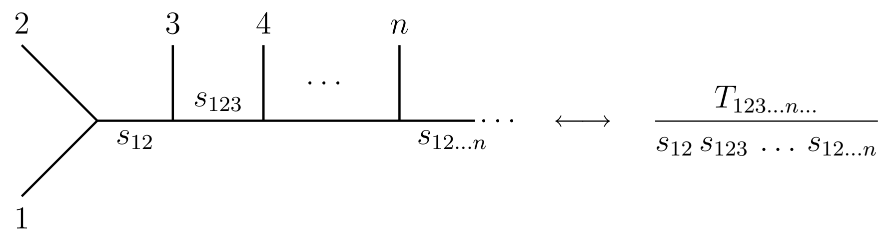


Figure 11.2: Endpoint of a cubic diagram associated with the BRST building block $T_{123\ldots n\ldots}$

But this does not cover diagrams with branches like those shown in figure 11.3:
The search for appropriate BRST building blocks to describe diagrams with branches is again guided by the $s_{i_1 i_2\ldots i_n}$ in their Q variation. BRST action on antisymmetric $T_{i_1 i_2 i_3\ldots i_p}$ combinations such as $T_{12[34]}$ involves a different set of Mandelstam invariants. Some of the variables $\{s_{i_1 i_2}, s_{i_1 i_2 i_3},\ldots, s_{i_1\ldots i_n}\}$ in $QT_{i_1 i_2\ldots i_n}$ are then replaced by other $s_{l_1\ldots l_k}$. As shown in figure 11.4, $Q(T_{1234}-T_{1243})$ contains variables s_{1234}, s_{12} and s_{34} rather than s_{123} or s_{124}:

We shall list a few more examples of how antisymmetrizing[1] the $T_{i_1 i_2\ldots i_p}$ modifies the Man-

[1]The notation $[i[jk]]$ means consecutive antisymmetrization of pairs of labels starting from the outmost label, e.g. $T_{[i[jk]]} = \frac{1}{2}(T_{i[jk]} - T_{[jk]i}) = \frac{1}{4}(T_{ijk} - T_{ikj} - T_{jki} + T_{kji})$.

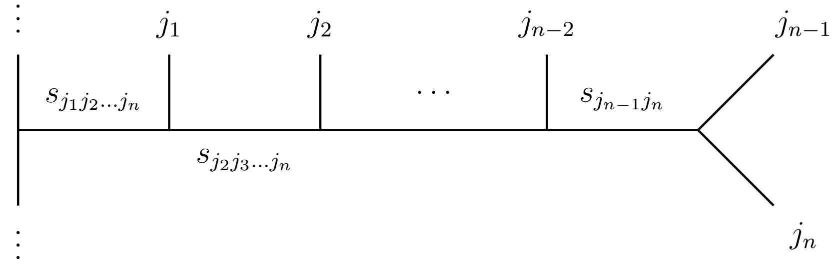


Figure 11.3: Example of a diagram with branches: Both ... lines should be followed by subdiagrams with at least one further cubic vertex.

$$
\begin{aligned}
Q\,(T_{1234}\,-\,T_{1243}) \quad &= \quad (s_{1234} - s_{12} - s_{34})\,T_{12}\,T_{34} \\
+\; s_{12}\,(T_{342}\,V_1\,-\,T_{341}\,V_2)\;+\;& s_{34}\,(T_{123}\,V_4\,-\,T_{124}\,V_3)
\end{aligned}
$$

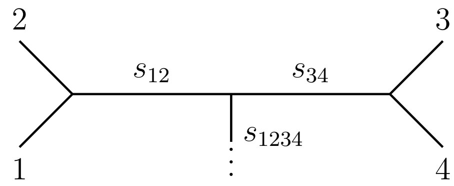
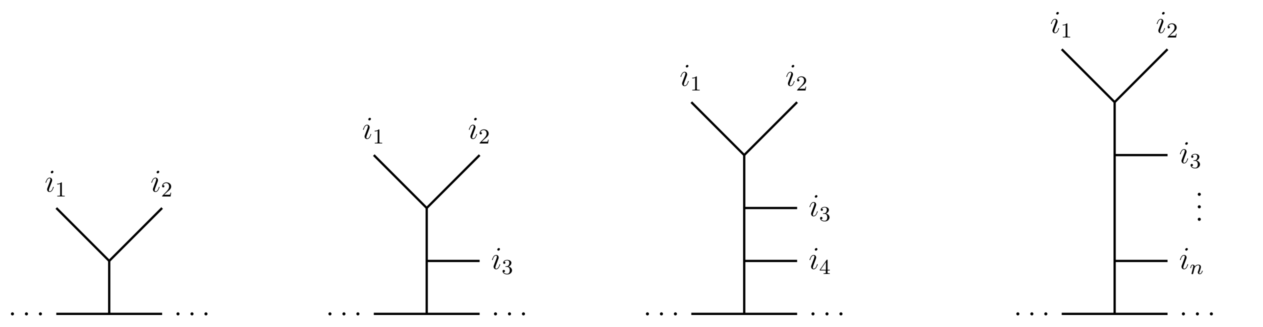


Figure 11.4: The simplest diagram with a branch

delstam variables in its BRST variation:

$$
\begin{aligned}
Q\,T_{i_1 \dots i_p [jk] r_1 \dots r_q} \quad &\longleftrightarrow \quad s_{jk} \text{ instead of } s_{i_1 i_2 \dots i_p j} \\
Q\,T_{i_1 \dots i_p [j[kl]] r_1 \dots r_q} \quad &\longleftrightarrow \quad s_{kl},\; s_{jkl} \text{ instead of } s_{i_1 \dots i_p j},\; s_{i_1 \dots i_p jk} \\
Q\,T_{i_1 \dots i_p [j[k[lm]]] r_1 \dots r_q} \quad &\longleftrightarrow \quad s_{lm},\; s_{klm},\; s_{jklm} \text{ instead of } s_{i_1 \dots i_p j},\; s_{i_1 \dots i_p jk},\; s_{i_1 \dots i_p jkl}
\end{aligned}
\tag{11.1.37}
$$

The diagrammatic interpretation of these findings and generalizations thereof are presented in figure 11.5.



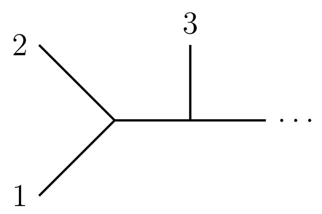
$2\,T_{\dots[i_1 i_2]\dots}$ $\quad 4\,T_{\dots[[i_1 i_2]i_3]\dots}$ $\quad 8\,T_{\dots[[[i_1 i_2]i_3]i_4]\dots}$ $\quad 2^{n-1}\,T_{\dots[[\dots[[i_1 i_2]i_3]\dots]i_n]\dots}$

Figure 11.5: The branches of cubic diagrams and their associated building blocks

11.1.5 Diagrams and symmetry properties

After the first step of the redefinition chain $L_{2131\ldots p1} \to \tilde{T}_{123\ldots p} \to T_{123\ldots p}$ from subsection 11.1.3, it is not difficult to use the BRST variation of $\tilde{T}_{123\ldots p}$ to find BRST-closed combinations for small p by trial and error. Their off-shellness forces them to be in fact BRST exact. As explained in subsection 11.1.3, the removal of the BRST exact parts of $\tilde{T}_{123\ldots p}$ gives rise to the definition of the BRST building block $T_{123\ldots p}$. Each BRST closed sum of $\tilde{T}$'s translates into a symmetry of the associated $T_{12\ldots p}$, see equations (11.1.22), (11.1.23) and (11.1.25) for $p = 3$. Therefore it is imperative to find the general BRST-closed combinations of $\tilde{T}$'s, or equivalently, the general symmetries of T's.

In this subsection we use the diagrammatic interpretation of building blocks to predict the symmetry properties of $T_{12\ldots p}$. As a first example, let us consider the diagram in figure 11.6:



$$= \begin{cases} T_{123} \\ T_{321} \; - \; T_{312} \end{cases}$$

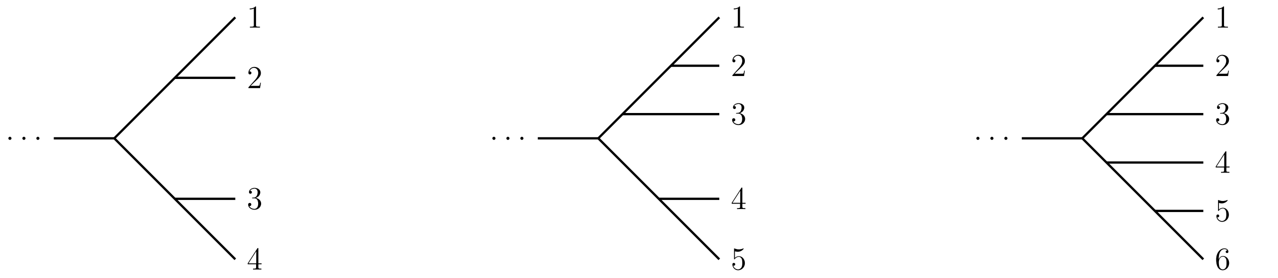
Figure 11.6: Two different ways to interpret the same diagram give rise to an identity for T_{ijk}.

In its first building block representation T_{123}, the diagram is interpreted as a tailend like the one depicted in figure 11.2. However, in the second expression the diagram is treated as a branch like the first of figure 11.5 where one of the ... legs now contains the label 3 and is therefore associated with $2T_{3[21]} = T_{321} - T_{312}$. The fact that both viewpoints have to agree implies the symmetry identity $T_{123} + T_{231} + T_{312}$.

The relative sign between the two viewpoints is fixed by the fact that diagram associated with $T_{12\ldots p}$ catch a $(-1)^{p-1}$ sign under inversion $(1, 2, 3, \ldots, (p-1), p) \leftrightarrow (p, (p-1), \ldots, 1)$. Hence, we have to make sure that the the sign of $T_{123\ldots p}$ relative to $T_{p,p-1,\ldots,21}$ is $(-1)^p$. In the $p = 3$ case, $T_{123} + (-1)^3 T_{321} + \ldots = 0$ using $T_{321} = -T_{231}$.

This same idea can be used to obtain the BRST symmetries for higher-order building blocks. In particular, the diagrams which give rise to the symmetries of $T_{123\ldots p}$ for $p = 4, 5, 6, 7$ and 8 are depicted in the figures 11.7 and 11.8 below, resulting in the following identities:

$$\begin{aligned} 0 &= 2\,T_{12[34]} \; + \; 2\,T_{43[21]} \\ 0 &= 2\,T_{123[45]} \; - \; 4\,T_{54[3[21]]} \end{aligned}$$



$$= \begin{cases} 2\,T_{12[34]} \\ -2\,T_{43[21]} \end{cases} \qquad = \begin{cases} 2\,T_{123[45]} \\ 4\,T_{54[3[21]]} \end{cases} \qquad = \begin{cases} 4\,T_{123[4[56]]} \\ -4\,T_{654[3[21]]} \end{cases}$$

Figure 11.7: Diagrammatic derivation of the BRST symmetries of higher order building blocks. The top (botton) line corresponds to the building block association which follow from reading the diagram in a counter-clockwise (clockwise) direction.

$$\begin{aligned} 0 &= 4\,T_{123[4[56]]} + 4\,T_{654[3[21]]} \\ 0 &= 4\,T_{1234[5[67]]} - 8\,T_{765[4[3[21]]]} \\ 0 &= 8\,T_{1234[5[6[78]]]} + 8\,T_{8765[4[3[21]]]} \end{aligned} \tag{11.1.38}$$

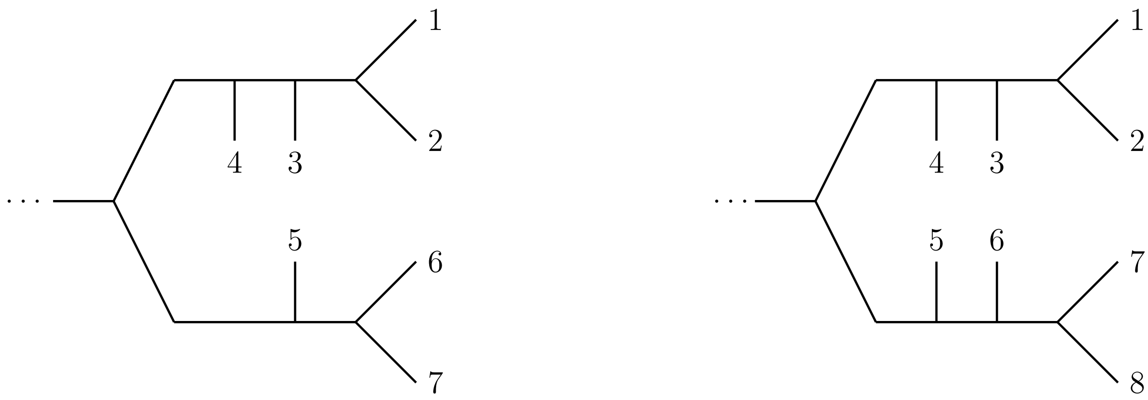
Using the BRST variations (11.1.34), we checked up to $T_{12345678}$ that these relations are indeed BRST-closed. Appendix E.1 identifies the corresponding combinations of $\tilde{T}_{1234}$ and $\tilde{T}_{12345}$ as BRST-exact, in accord with the discussion of subsection 11.1.3.

To write down the generalization of (11.1.38) to higher $p > 8$, let us distinguish between odd and even ranks for ease of notation:

$$\begin{aligned} p = 2n+1: &\quad T_{12\ldots n+1[n+2[\ldots[2n-1[2n,2n+1]]\ldots]]} - 2\,T_{2n+1\ldots n+2[n+1[\ldots[3[21]]\ldots]]} = 0 \\ p = 2n: &\quad T_{12\ldots n[n+1[\ldots[2n-2[2n-1,2n]]\ldots]]} + T_{2n\ldots n+1[n[\ldots[3[21]]\ldots]]} = 0 \end{aligned} \tag{11.1.39}$$

The relation for odd $p = 2n+1$ obviously involves $3 \cdot 2^{n-1}$ terms whereas the even one for $p = 2n$ has 2^n terms.

We should emphasize again that the lower rank identities for $T_{12\ldots q}$ carry over to larger building blocks $T_{12\ldots p}$ with $p > q$. The last labels $q+1, \ldots, p$ are then simply left untouched, e.g. $0 = T_{(12)345} = T_{[123]45} = T_{12[34]5} + T_{43[21]5}$ at rank $p = 5$. By applying the $p-1$ symmetries available at rank p, one can successively move a particular label to the first position, i.e. express $T_{i_1 i_2 \ldots i_p}$ as a combination of $T_{1 j_1 j_2 \ldots j_{p-1}}$. According to the number of $(j_1, j_2, \ldots, j_{p-1})$ permutations, we have $(p-1)!$ independent rank p building blocks $T_{i_1 i_2 \ldots i_p}$.



$$= \begin{cases} 4\, T_{1234[5[67]]} \\ 8\, T_{765[4[3[21]]]} \end{cases} \qquad\qquad = \begin{cases} 8\, T_{1234[5[6[78]]]} \\ -\,8\, T_{8765[4[3[21]]]} \end{cases}$$

Figure 11.8: The symmetries of rank 7 and 8 building blocks obtained from diagrams.

11.2 From building blocks to Berends-Giele currents

In subsection 11.1.4, we have given a superfield representation in terms of BRST building blocks $T_{i_1 \ldots i_p}$ for each color ordered cubic diagram with p on-shell external leg and one off-shell leg. In this section, we combine these diagrams to $p+1$ point field theory amplitudes with one off-shell leg. Such objects were first considered in [237] in order to derive recursion relations for QCD tree amplitudes with a large number of gluons and were referred to as "currents".

Supersymmetric analogues $M_{12\ldots p}$ of p point Berends–Giele currents allow for a compact representation of the ten-dimensional n point SYM amplitude $\mathcal{A}^{\text{SYM}}(1,\ldots,n)$ which nicely exhibits its factorization channels. The recursive nature of the Berends–Giele currents is inherited by the amplitudes and leads to the recursive method to compute higher point SYM amplitudes presented in the following section 11.3.

11.2.1 Berends Giele currents in gauge theories

In 1987, Berends and Giele have found a recursive method to compute tree level QCD amplitudes with a large number of gluons. They recursively constructed color ordered $p+1$ gluon amplitudes $J_\mu(1,2,\ldots,p)$ with one off shell leg $p+1$

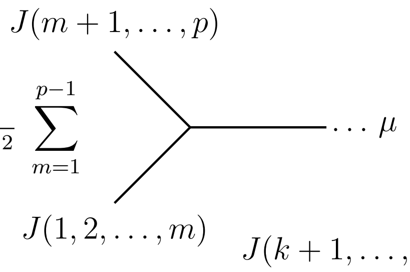
$$\begin{aligned} J_\mu(1,2,\ldots,p) \;&=\; \frac{1}{(k_1+k_2+\ldots k_p)^2} \Bigg\{ \sum_{m=1}^{p-1} \big[\, J(1,2,\ldots,m) \;,\; J(m+1,\ldots,p) \,\big]_\mu \\ &\quad + \sum_{m=1}^{p-2} \sum_{k=m+1}^{p-1} \big\{\, J(1,2,\ldots,m) \;,\; J(m+1,\ldots,k) \;,\; J(k+1,\ldots,p) \,\big\}_\mu \Bigg\} \end{aligned} \tag{11.2.40}$$

starting from the polarization vector as the one point current $J_\mu(1) = \xi^1_\mu$. The notation $[\cdot,\cdot]$ and $\{\cdot,\cdot,\cdot\}$ reflects the contributions of the three- and four gluon vertices in the Yang Mills action:

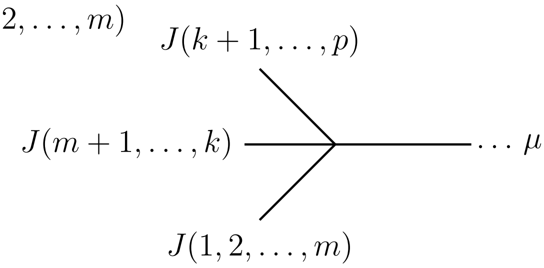
$$\begin{aligned}
\big[\, J(1)\ ,\ J(2)\,\big]_\mu \ &= \ 2\,\big(k_2\cdot J(1)\big)\,J_\mu(2) \ - \ 2\,\big(k_1\cdot J(2)\big)\,J_\mu(1) \ + \ (k_1-k_2)_\mu\,\big(J(1)\cdot J(2)\big) \\
\big\{\, J(1)\ ,\ J(2)\ ,\ J(3)\,\big\}_\mu \ &= \ J(1)\cdot\big(J(3)\,J_\mu(2) \ - \ J(2)\,J_\mu(3)\big) \\
&\quad - \ J(3)\cdot\big(J(2)\,J_\mu(1) \ - \ J(1)\,J_\mu(2)\big)
\end{aligned} \tag{11.2.41}$$

The two contribution to (11.2.40) can be depicted as

$$J_\mu(1,2,\ldots,p) \ = \ \frac{1}{(k_1+k_2+\ldots k_p)^2}\sum_{m=1}^{p-1}$$



$$+ \ \frac{1}{(k_1+k_2+\ldots k_p)^2}\sum_{m=1}^{p-2}\sum_{k=m+1}^{p-1}$$



The recursive definiton directly implies three properties: Reflection symmetry, current conservation with respect to the total momentum and finally the vanishing of its cyclic sum, similar to the identity (5.5.67) for on-shell amplitudes

$$J_\mu(1,2,\ldots,p) \ + \ (-1)^p\,J_\mu(p,p-1,\ldots,1) \ = \ 0 \tag{11.2.42}$$

$$J_\mu(1,2,\ldots,p) \ + \ \text{cyclic}(12\ldots p) \ = \ 0 \tag{11.2.43}$$

$$\sum_{i=1}^{p} k_i^\mu\,J_\mu(1,2,\ldots,p) \ = \ 0 \tag{11.2.44}$$

The color ordered n gluon amplitude can be written very compactly in terms of an $n-1$ point current:

$$\mathcal{A}^{\text{gluon}}(1,2,\ldots,n) \ \sim \ J_\mu(1,2,\ldots,n-1)\,J^\mu(n)\,(k_1+\ldots+k_{n-1})^2\,\Big|_{\sum_{i=1}^n k_i=0} \tag{11.2.45}$$

The factor of $(k_1+\ldots+k_{n-1})^2$ removes the overall propagator of the rank $n-1$ current which would diverge under momentum conservation $\sum_{i=1}^n k_i = 0$. This approach to multigluon scattering has two kinds of advantages:

- It is efficient in the sense that it makes use of lower order results and automatically captures all Feynman diagrams.

- It can be used to proof certain properties of the amplitude, such as gauge invariance, cyclicity, photon decoupling and factorization for soft and collinear gluons.

The supersymmetric generalization $M_{12\ldots p}$ which will by introduced in the following subsection share these properties and enjoy the additional benefit that the trilinear terms in the recursion (11.2.40) do not appear. This suggests that they ultimately come from a superspace action with only cubic vertices.

11.2.2 The pure spinor realization of Berends Giele currents

Each of the $\frac{1}{p}\binom{2p-2}{p-1}$ diagrams entering a rank p current has $p-2$ internal poles in Mandelstam invariants $s_{i_1\ldots i_q}$ with $q<p$ and one external propagator $1/s_{12\ldots p}$ which would diverge if the additional leg is put on-shell. Let us give the simplest examples M_{12}, M_{123} and M_{1234} of supersymmetric Berends-Giele currents in figure 11.9 and 11.10, they are associated with color ordered three-, four and five point amplitudes, respectively. Higher rank currents M_{12345} and M_{123456} are explicitly expanded in appendix E.2.

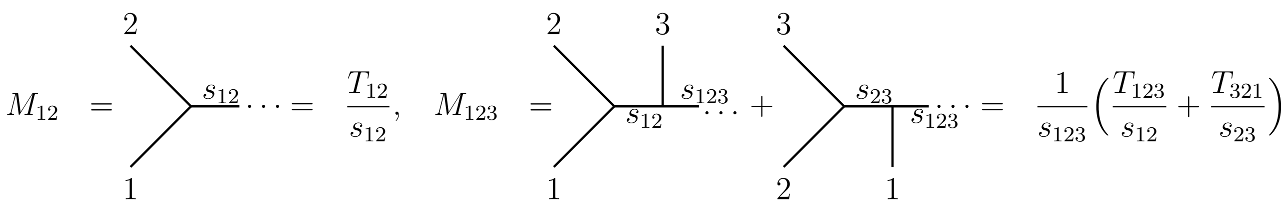
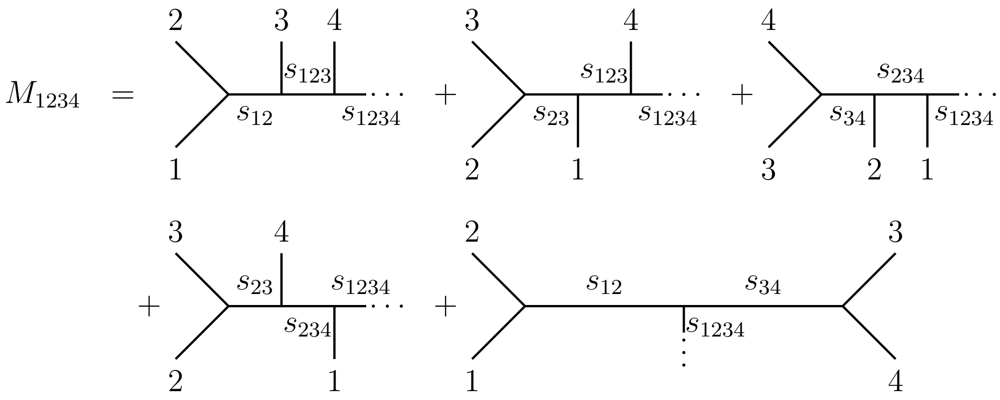


Figure 11.9: Diagrammatic construction of Berends–Giele currents M_{12}, and M_{123}

The BRST variation of the $M_{12\ldots p}$ follows from the expression (11.1.34) for $QT_{12\ldots p}$. The remarkable effect of combining the $(2p-2)!/(p!(p-1)!)$ cubic graphs is firstly the cancellation of the overall propagator $1/s_{12\ldots p}$ and secondly the conspiration of all the lower order $T_{i_1\ldots i_q}$ to full-fledged currents $M_{i_1\ldots i_q}$. In order to simplify the notation in the formulae below, let us identify M_1 with the unintegrated vertex; $V_i \equiv M_i$. Then, Q acts as follows on our examples,

$$
\begin{aligned}
Q\,M_{12} &= V_1\,V_2 = M_1\,M_2 \\
Q\,M_{123} &= \frac{V_1\,T_{23}}{s_{23}} + \frac{T_{12}\,V_3}{s_{12}} = M_1\,M_{23} + M_{12}\,M_3 \qquad (11.2.46)
\end{aligned}
$$



$$= \frac{1}{s_{1234}}\Big(\frac{T_{1234}}{s_{12}s_{123}} + \frac{T_{3214}}{s_{23}s_{123}} + \frac{T_{3421}}{s_{34}s_{234}} + \frac{T_{3241}}{s_{23}s_{234}} + \frac{2T_{12[34]}}{s_{12}s_{34}}\Big)$$

Figure 11.10: Diagrammatic construction of the rank four Berends–Giele current M_{1234}

$$\begin{aligned} QM_{1234} &= \frac{V_1}{s_{234}}\Big(\frac{T_{234}}{s_{23}} + \frac{T_{432}}{s_{34}}\Big) + \frac{T_{12}\,T_{34}}{s_{12}\,s_{34}} + \Big(\frac{T_{123}}{s_{12}} + \frac{T_{321}}{s_{23}}\Big)\frac{V_4}{s_{123}} \\ &= M_1\,M_{234} + M_{12}\,M_{34} + M_{123}\,M_4 \end{aligned}$$

Generally speaking, the BRST charge decomposes p point currents into ghost number two objects $E_{12\ldots p}$ which consist of products of two smaller currents

$$Q\,M_{12\ldots p} = \sum_{j=1}^{p-1} M_{12\ldots j}\,M_{j+1\ldots p} =: E_{12\ldots p}\ . \tag{11.2.47}$$

The action of Q cuts $M_{12\ldots p}$ in each way compatible with the color ordering, see figure 11.11.

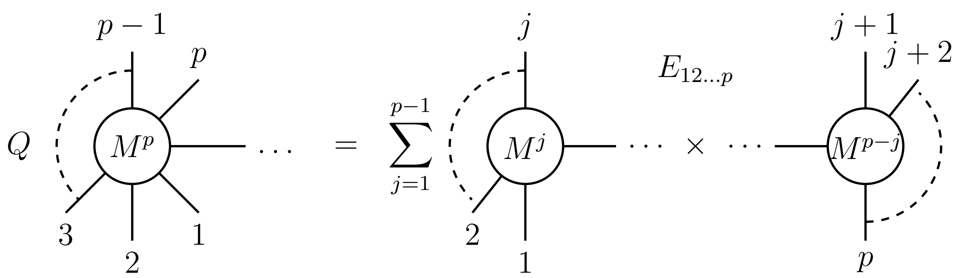


Figure 11.11: Decomposition of $M_{12\ldots p}$ into its factorization channels under Q

Equation (11.2.47) is the supersymmetric pure spinor analogue of the recursive construction (11.2.40) of gluon currents which schematically reads $J_p \sim (J^2_{q<p} + J^3_{q<p})/s_{12\ldots p}$. Its cubic term representing the four gluon vertex in the QCD action does not enter into our supersymmetric version (11.2.47). The gluon recursion in its form $s_{12\ldots p}J_p \sim J^2_{q<p} + J^3_{q<p}$ parallels (11.2.47) if we symbolically identify $s_{12\ldots p} \equiv Q$. This is plausible since the action of Q cancels the overall propagator $M_{12\ldots p} \sim s^{-1}_{12\ldots p}$.

11.2.3 Symmetry properties of Berends Giele currents

As a further motivation for identifying $M_{12\ldots p}$ with supersymmetric Berends–Giele currents, this subsection is devoted to the symmetry properties shared by $M_{12\ldots p}$ and the gluonic specialization $J_\mu(1,2,\ldots,p)$. First of all, M_{12} trivially satisfies $M_{12}+M_{21}=0$ because the building block T_{ij} is antisymmetric. Similar identities hold for M_{123}, cf. (11.1.25),

$$M_{123} \ + \ M_{231} \ + \ M_{312} \ = \ 0 \ , \qquad M_{123} \ - \ M_{321} \ = \ 0 \ , \tag{11.2.48}$$

as one can easily check by plugging the expression for M_{ijk} given in figure 11.9 and using $T_{[123]}=0$. At higher $n\geq 4$, this generalizes as follows

$$M_{12\ldots p} \ = \ (-1)^{p-1}\, M_{p\ldots 21} \ , \qquad \sum_{\sigma\in\text{cyclic}} M_{\sigma(1,2,\ldots,p)} \ = \ 0 \tag{11.2.49}$$

which reproduces properties (11.2.42) and (11.2.43) of gluon currents. The proof of these identities is most conveniently carried out on the level of the corresponding $E_{12\ldots p}=QM_{12\ldots p}=\sum_{j=1}^{p-1}M_{12\ldots j}M_{j+1\ldots n}$. Since all the BRST closed components of the $M_{12\ldots n}$ have been removed by construction of its $T_{12\ldots n}$ constituents, the BRST variation $E_{12\ldots n}$ contains all information on the symmetry properties of its $M_{12\ldots n}$ ancestor. The reflection identity can be easily checked by induction, and the vanishing cyclic sum follows from

$$\begin{aligned}\sum_{\sigma\in\text{cyclic}} E_{\sigma(1,2,\ldots,n)} \ &= \ \sum_{\sigma\in\text{cyclic}}\sum_{p=1}^{n-1} M_{\sigma(1,2,\ldots,p)}\, M_{\sigma(p+1,\ldots,n)} \\ &= \ \sum_{\sigma\in\text{cyclic}}\sum_{p=1}^{n-1}\frac{1}{2}\Big(M_{\sigma(1,2,\ldots,p)}\, M_{\sigma(p+1,\ldots,n)} \ + \ M_{\sigma(p+1,\ldots,n)}\, M_{\sigma(1,2,\ldots,p)}\Big) \ = \ 0\end{aligned} \tag{11.2.50}$$

where the last step exploits the overall cyclic sum to shift all labels of the second term by p and that the $M_{12\ldots p}$ anticommute.

The properties (11.2.49) can be naturally explained by the construction of currents $M_{123\ldots n}$ as $n+1$ point amplitudes with one off-shell leg. Inspired by this analogy, we explicitly checked up to $n=7$ that $M_{12\ldots n}$ also satisfy a relation which is obtained by removing the $(n+1)$-th leg from the $n+1$ point Kleiss-Kuijf identity (5.5.65):

$$M_{\{\beta\},1,\{\alpha\}} \ = \ (-1)^{n_\beta}\sum_{\sigma\in\text{OP}(\{\alpha\},\{\beta^T\})} M_{1,\{\sigma\}} \tag{11.2.51}$$

The summation range $\text{OP}(\{\alpha\},\{\beta^T\})$ denotes the set of all the permutations of $\{\alpha\}\bigcup\{\beta^T\}$ that maintain the order of the individual elements of both ordered subsets $\{\alpha\}$ and $\{\beta^T\}$ of $\{2,3,\ldots,n\}$. The notation $\{\beta^T\}$ represents the set $\{\beta\}$ with reversed ordering of its n_β

elements. The Kleiss-Kuijf identity is well known to reduce the number of independent color ordered $n+1$ point amplitudes[2] down to $(n-1)!$.

The specialization of (11.2.51) to sets $\{\beta\}$ with one element only, say $\{\beta\}=\{n\}$ implies the vanishing of $\sum_{\sigma \text{cyclic}} M_{\sigma(1,\ldots,n)}$. However, this so-called dual Ward identity or photon decoupling identity by itself is not sufficient for a reduction to $(n-1)!$ independent $M_{i_1 i_2 \ldots i_n}$ at $n \geq 6$. Since there are only $(n-1)!$ independent building blocks $T_{i_1 i_2 \ldots i_n}$ which constitute the $M_{i_1 i_2 \ldots i_n}$, also the latter must have a basis of no more than $(n-1)!$ elements. This counting argument suggests the Kleiss-Kuijf identity (11.2.51) to hold beyond our checks to $n \leq 7$ although we could not find an explicit proof for general n.

The reflection- and Kleiss-Kuijf identity for the $M_{12\ldots n}$ are inherited from their associated $n+1$ point amplitudes with one leg off-shell. The off-shellness of one leg is no obstruction for the aforementioned identities to hold because they do not involve any kinematic factors. However, the field theory version (5.5.69) of the monodromy relations (5.4.46) rely on having on-shell momenta, so the $M_{12\ldots n}$ do not satisfy any analogous identity and cannot be reduced to $(n-2)!$ independent permutations.

11.3 From Berends-Giele currents to SYM amplitudes

The expressions found for $QM_{12\ldots p} = E_{12\ldots p}$ might look familiar from lower order field theory amplitudes such as

$$\mathcal{A}^{\rm SYM}(1,2,3) \;=\; \langle\, V_1\, V_2\, V_3 \,\rangle \;=\; \langle\, E_{12}\, V_3 \,\rangle \tag{11.3.52}$$

$$\mathcal{A}^{\rm SYM}(1,2,3,4) \;=\; \left\langle \left(\frac{V_1\, T_{23}}{s_{23}} + \frac{T_{12}\, V_3}{s_{12}} \right) V_4 \right\rangle \;=\; \langle\, E_{123}\, V_4 \,\rangle \,, \tag{11.3.53}$$

the latter follows from the $\alpha' \to 0$ limit of (11.1.3) where $V_t \to 1$.

From $QV=0$, one might naively expect that the three-point amplitude would be BRST-exact, $\mathcal{A}^{\rm SYM}(1,2,3) \stackrel{?}{=} \langle Q(T_{12}V_3/s_{12})\rangle$, and thus doomed to vanish. However, all the Mandelstam invariants s_{ij} are zero in the momentum phase space of three massless particles – therefore writing $V_1 V_2 = Q(T_{12}/s_{12})$ is not allowed, and BRST triviality of the amplitude is avoided.

11.3.1 The field theory amplitude

More generally, the prefactor $M_{12\ldots p} \sim 1/s_{12\ldots p}$ in the p point current is incompatible with putting the external state with $k_{p+1} = -\sum_{i=1}^{p} k_i$ on-shell $k_{p+1}^2 = 0$. Therefore, n particle

[2]After using cyclicity and reflection symmetry, there are $n!/2$ so far unrelated $n+1$ point subamplitudes in the first place, see section 5.5.

kinematics $\sum_{i=1}^{n} k_i = 0$ forbids the existence of $M_{12\ldots n-1}$ and the corresponding $E_{12\ldots n-1}$ is not BRST exact. This leads to the following expression for n point field theory amplitude which is truly in the BRST cohomology [8]:

$$\mathcal{A}^{\rm SYM}(1,2,\ldots,n) \ = \ \langle\, E_{12\ldots n-1}\, V_n \,\rangle \ = \ \sum_{j=1}^{n-2} \langle\, M_{12\ldots j}\, M_{j+1\ldots n-1}\, V_n \,\rangle \tag{11.3.54}$$

Let us denote the number of kinematic pole configurations in $M_{i_1\ldots i_p}$ or $E_{i_1\ldots i_p}$ by $\wp_{p+1}$, then (11.2.47) implies the recursion relation

$$\wp_n \ = \ \sum_{i=2}^{n-1} \wp_i\, \wp_{n-i+1} \ , \qquad \wp_2 = \wp_3 \equiv 1 \ . \tag{11.3.55}$$

Its explicit solution $\wp_n = 2^{n-2} \frac{(2n-5)!!}{(n-1)!}$ shows that the right hand side of (11.3.54) has the correct number of cubic diagrams to describe a color ordered n–point SYM amplitude.

The diagrammatic representation of $\sum_{j=1}^{p-1} M_{12\ldots j} M_{j+1\ldots p}$ in figure 11.11 can be uplifted to the on-shell $n = p+1$ point amplitude $\mathcal{A}^{\rm SYM}(1,2,\ldots,n)$ where an additional cubic vertex connects the n'th leg with the two currents of rank j and $n-1-j$, respectively, see the following figure 11.12.

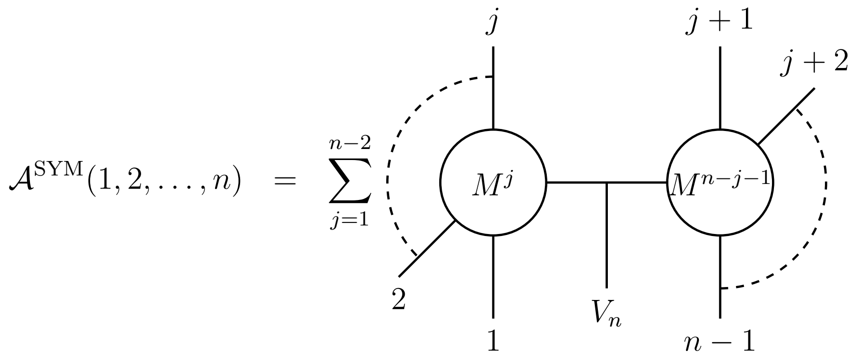


Figure 11.12: Berends-Giele decomposition of the color ordered SYM amplitude

The n point formula (11.3.54) is analogous to the Berends–Giele formula (11.2.45) for the color ordered n gluon amplitude of [237], which is written as a product of a rank $n-1$ current J_{n-1} and another J_1 for the n'th leg, multiplied by the Mandelstam factor $s_{12\ldots n-1}$ to cancel the divergent propagator. In our case, the somewhat artificial object $s_{12\ldots n-1} J_{n-1}$ is replaced by $E_{12\ldots n-1}$, which could be written as $QM_{12\ldots n-1}$ in a larger momentum phase space. This parallel also suggests the schematic identification $s_{12\ldots n-1} \equiv Q$ mentioned after (11.2.47).

11.3.2 The recursion

The families $\{M_{12\ldots p},\, E_{12\ldots p}\}$ of superfields have a manifestly recursive structure – the knowledge of all M's up to $M_{123\ldots p}$ suffices to obtain the $E_{123\ldots p+1}$ at the next level $p \mapsto p+1$ using the definition (11.2.47). Formally, this seems to give access to $M_{123\ldots p+1}$ by inverse Q action on $E_{123\ldots p+1}$, but unfortunately there is no constructive prescription for Q^{-1} at hand.

The formula (11.3.54) for the n point field theory amplitude in terms of $E_{12\ldots n-1}$ closes precisely this gap of the recursion. Having $E_{123\ldots p+1}$ due to $\{M_{123\ldots q},\, q \leq p\}$ yields the $p+2$ point amplitude $\mathcal{A}^{\rm SYM}(1,\ldots,p+2)$ with all its $(2p)!/(p!(p+1)!)$ color ordered diagrams made of cubic vertices. These diagrams, on the other hand, constitute the main input necessary for the construction of the higher point current $M_{12\ldots p+1}$ from BRST building blocks $T_{12\ldots p+1}$, see subsection 11.1.4 for the dictionary between $T_{12\ldots p+1}$ and cubic diagrams.

The following figure summarizes the aforementioned steps necessary for setting up a recursion for tree level amplitudes in pure spinor superspace.

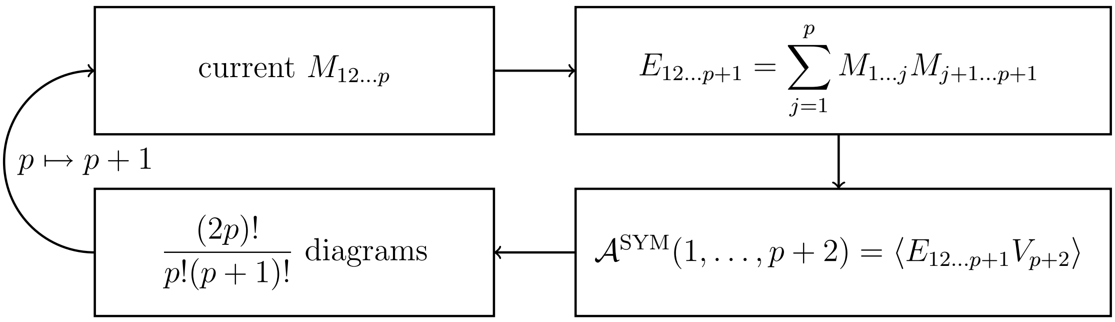


Figure 11.13: The recursive prescription for $\mathcal{A}^{\rm SYM}$

One has to admit that computing components of the supersymmetric $\mathcal{A}^{\rm SYM}$ amplitudes requires explicit expression for their BRST building blocks $T_{12\ldots p}$. Constructing the tower of $T_{12\ldots p}$ has been identified as a separate recursive problem in the earlier sections. We have explicitly given them up to T_{12345} (see appendix E.1), and there are no obstructions to getting the higher order $T_{12\ldots p}$ by straightforward application of the algorithm in section 11.1.

In the later subsection 12.1.4 we will give a string-inspired formula which allows for direct computation of $M_{12\ldots p}$ without the need for drawing the cubic diagrams of the $p+1$ point SYM amplitude.

11.3.3 BRST integration by parts and cyclic symmetry

The strength of our presentation (11.3.54) of the n point field theory amplitude is the manifestation of its factorization properties. But singling out a particular leg V_n obscures the cyclic symmetry required for color stripped amplitudes. The essential tool to restore manifest cyclicity is BRST integration by parts.

The maximum rank of $M_{i_1\ldots i_p}$ appearing in the n point amplitude (11.3.54) is $p = n-2$. By construction of $E_{12\ldots n-1}$, these terms are of the form

$$\langle\, M_{i_1\ldots i_{n-2}}\, V_{i_{n-1}}\, V_n \,\rangle \;\;=\;\; \langle\, M_{i_1\ldots i_{n-2}}\, Q\, M_{i_{n-1}n} \,\rangle \;\;=\;\; \langle\, E_{i_1\ldots i_{n-2}}\, M_{i_{n-1}n} \,\rangle \;. \tag{11.3.56}$$

After BRST integration by parts in the last step, the maximum rank of Berends–Giele currents becomes $n-3$ due to $M_{j_1\ldots j_{n-3}}$ within $E_{i_1\ldots i_{n-2}}$. More generally, the cohomology formula (11.3.54) allows enough BRST integration by parts

$$\langle\, M_{i_1\ldots i_p}\, E_{j_1\ldots j_q} \,\rangle \;\;=\;\; \langle\, E_{i_1\ldots i_p}\, M_{j_1\ldots j_q} \,\rangle \tag{11.3.57}$$

as to reduce the maximum rank of the currents to $p = [n/2]$ (where $[\cdot]$ denotes the Gauss bracket $[x] = \max_{n\in\mathbb{Z}}(n \leq x)$, which picks out the nearest integer smaller than or equal to its argument).

The benefit of performing the $[(n-3)/2]$ integrations by part in $\mathcal{A}^{\rm SYM}$ is twofold: Firstly, the complexity of superfield ingredients decreases with the rank of the $M_{i_1\ldots i_p}$ and $T_{i_1\ldots i_p}$ involved, cf. the length of the explicit formulae (11.1.8), (11.1.9) and (11.1.10) for L_{ji}, L_{jiki} and L_{jikili}. Secondly, this procedure allows the n'th leg to get involved in larger building blocks $M_{\ldots n-1,n,1\ldots}$ and to enter the amplitude on the same footing as the remaining legs, leading to manifestly cyclic-symmetric amplitudes such as

$$\begin{aligned}
\mathcal{A}^{\rm SYM}(1,2,\ldots,5) \;&=\; \langle\, M_{12}\, V_3\, M_{45} \,\rangle \;+\; \text{cyclic}(12345) \\
\mathcal{A}^{\rm SYM}(1,2,\ldots,6) \;&=\; \frac{1}{3}\,\langle\, M_{12}\, M_{34}\, M_{56} \,\rangle \;+\; \frac{1}{2}\,\langle\, M_{123}\, E_{456} \,\rangle \;+\; \text{cyclic}(123456) \\
\mathcal{A}^{\rm SYM}(1,2,\ldots,7) \;&=\; \langle\, M_{123}\, M_{45}\, M_{67} \,\rangle \;+\; \langle\, V_1\, M_{234}\, M_{567} \,\rangle \;+\; \text{cyclic}(1234567) \\
\mathcal{A}^{\rm SYM}(1,2,\ldots,8) \;&=\; \langle\, M_{123}\, M_{456}\, M_{78} \,\rangle \;+\; \frac{1}{2}\,\langle\, M_{1234}\, E_{5678} \,\rangle \;+\; \text{cyclic}(12345678)
\end{aligned} \tag{11.3.58}$$

The fractional prefactors $\frac{1}{2}$ or $\frac{1}{3}$ compensate for the fact that cyclic orbits for particularly symmetric superfield kinematics are shorter than the number n of legs. At $n=6$, for instance, $M_{12}M_{34}M_{56}$ has just one distinct cyclic image $M_{23}M_{45}M_{61}$, hence the full cyclic(123456) overcounts the occurring diagrams by a factor of three.

11.3.4 Factorization in cyclically symmetric form

In this subsection, we introduce a cyclically symmetric presentation of SYM amplitudes where their factorization into two Berends-Giele currents remains obvious. This means that all the kinematics are arranged into the form $\langle M_{i_1\ldots i_p} Q M_{j_1\ldots j_q}\rangle$ which can be interpreted as two Berends-Giele currents of rank p and q, connected by a propagator line:

$$\langle\, M_{i_1\ldots i_p}\, Q\, M_{j_1\ldots j_q}\,\rangle \quad = $$

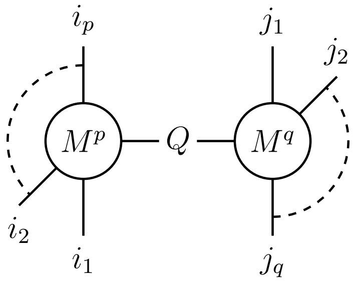


Figure 11.14: Factorization of a $p+q$ point amplitude into two Berends-Giele currents.

The role of the Q operator can be interpreted as compensating one of the two coinciding internal propagators $s^{-1}_{i_1\ldots i_p} = s^{-1}_{j_1\ldots j_q}$.

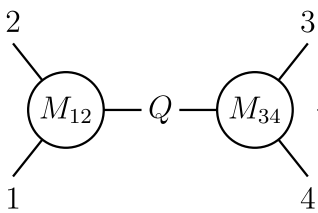
One can check by evaluating the BRST variations that the amplitudes in (11.3.58) can be equivalently written as

$$\begin{aligned}
\mathcal{A}^{\rm SYM}(1,2,3,4) &= \frac{1}{2}\,\langle\, M_{12}\, Q\, M_{34}\,\rangle \ + \ \ \text{cyclic}(1234) \\
\mathcal{A}^{\rm SYM}(1,\ldots,5) &= \frac{1}{4}\,\Big(\langle\, M_{12}\, Q\, M_{345}\,\rangle \ + \ \langle\, M_{123}\, Q\, M_{45}\,\rangle\Big) \ + \ \text{cyclic}(12345) \\
\mathcal{A}^{\rm SYM}(1,\ldots,6) &= \frac{1}{6}\,\Big(\langle\, M_{12}\, Q\, M_{3456}\,\rangle \ + \ \langle\, M_{123}\, Q\, M_{456}\,\rangle \ + \ \langle\, M_{1234}\, Q\, M_{56}\,\rangle\Big) \\
&\quad + \ \text{cyclic}(123456) \qquad\qquad (11.3.59) \\
\mathcal{A}^{\rm SYM}(1,\ldots,7) &= \frac{1}{8}\,\Big(\langle\, M_{12}\, Q\, M_{34567}\,\rangle \ + \ \langle\, M_{123}\, Q\, M_{4567}\,\rangle \\
&\quad + \ \langle\, M_{1234}\, Q\, M_{567}\,\rangle \ + \ \langle\, M_{12345}\, Q\, M_{67}\,\rangle\Big) \ + \ \text{cyclic}(1234567) \\
\mathcal{A}^{\rm SYM}(1,\ldots,8) &= \frac{1}{10}\,\Big(\langle\, M_{12}\, Q\, M_{345678}\,\rangle \ + \ \langle\, M_{123}\, Q\, M_{45678}\,\rangle \ + \ \langle\, M_{1234}\, Q\, M_{5678}\,\rangle \\
&\quad + \ \langle\, M_{12345}\, Q\, M_{678}\,\rangle \ + \ \langle\, M_{123456}\, Q\, M_{78}\,\rangle\Big) \ + \ \text{cyclic}(12345678)
\end{aligned}$$

Note that some terms are overcounted by a factor of 2 because the cyclic orbits of $\langle M_{12\ldots j} Q M_{j+1\ldots n}\rangle$ and $\langle M_{12\ldots n-j} Q M_{n-j+1\ldots n}\rangle$ are the same. The purpose of including both of them is to obtain a uniform overall coefficient in (11.3.59) and to simplify the transition to the general n point formula (11.3.60).

The factorization channels of $\mathcal{A}^{\rm SYM}(1,2,3,4)$, $\mathcal{A}^{\rm SYM}(1,\ldots,5)$ and $\mathcal{A}^{\rm SYM}(1,\ldots,6)$ can be pictorially represented as shown in the following figure:

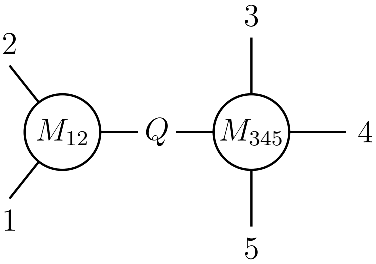
$$\mathcal{A}^{\rm SYM}(1,2,3,4) \quad = \quad \frac{\langle\, M_{12}\, Q M_{34}\,\rangle}{2} \ + \ \text{cyclic}(1234) \ = \ \frac{1}{2}$$



$+\ \text{cyclic}(1234)$

$$\mathcal{A}^{\rm SYM}(1,2,3,4,5) \;=\; \frac{1}{2}\,\langle\, M_{12}\, Q M_{345}\,\rangle \;+\; {\rm cyclic}(12345)$$

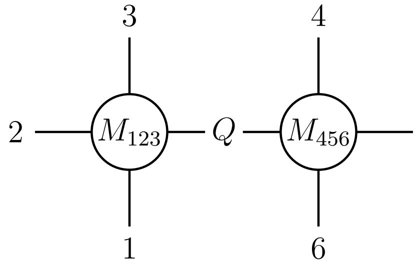
$$= \;\frac{1}{2}$$



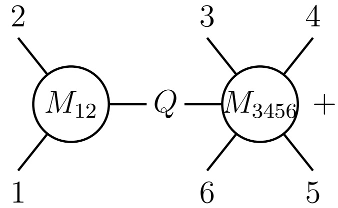
$$+ \;{\rm cyclic}(12345)$$

$$\mathcal{A}^{\rm SYM}(1,2,\ldots,6) \;=\; \frac{1}{6}\left(\frac{1}{2}\,\langle\, M_{123}\, Q M_{456}\,\rangle \;+\; \langle\, M_{12}\, Q M_{3456}\,\rangle\right) \;+\; {\rm cyclic}(123456)$$

$$= \;\frac{1}{6}$$



$$+ \;\frac{1}{3}$$



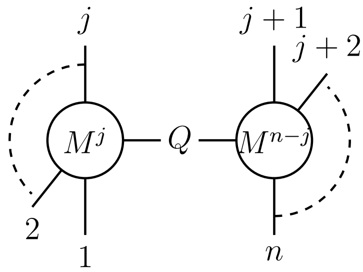
$$+ \;{\rm cyclic}(123456)$$

For n points, the expressions (11.3.59) generalize to

$$\mathcal{A}^{\rm SYM}(1,2,\ldots,n) \;=\; \frac{1}{2\,(n-3)}\sum_{j=2}^{n-2}\langle\, M_{12\ldots j}\, Q\, M_{j+1\ldots n}\,\rangle \;+\; {\rm cyclic}(12\ldots n) \tag{11.3.60}$$

which can be graphically represented as

$$\mathcal{A}^{\rm SYM}(1,2,\ldots,n) \;=\; \frac{1}{2\,(n-3)}\sum_{j=2}^{n-2}$$



$$+ \;{\rm cyclic}(12\ldots n)$$

We have explicitly checked up to $n = 10$ points that the formula (11.3.60) exactly reproduces the expression $\mathcal{A}^{\rm SYM}(1,\ldots,n) = \langle E_{12\ldots n-1}V_n\rangle$ with the right prefactors. It can also be interpreted as coming from the factorization channels of $\mathcal{A}^{\rm SYM}(1,\ldots,n)$ into two amplitudes with one leg off-shell each with the form $\langle E_{12\ldots j}V_x\rangle$ and $\langle V_x E_{j+1\ldots n}\rangle$ that are connected by a pure spinor propagator which effectively replaces $V_x\, V_x \to \frac{1}{Q}$, resulting in the symbolic expression

$$\mathcal{A}^{\rm SYM}(1,2,\ldots,n) \;\equiv\; \frac{1}{2\,(n-3)}\sum_{j=2}^{n-2}\langle\, E_{12\ldots j}\,\frac{1}{Q}\,E_{j+1\ldots n}\,\rangle \;+\; {\rm cyclic}(1\ldots n)\,. \tag{11.3.61}$$

Chapter 12

Superstring amplitudes in the pure spinor formalism

This chapter is devoted to the constructive computation of superstring tree level amplitudes of the massless SYM multiplet in the pure spinor approach. Its main purpose is to prove the main result (1.4.1) and to analyze its structure. In other words, we show that tree level amplitudes of massless states in open superstring theory are linear combinations of SYM subamplitudes associated with various color orderings. This has interesting implications in view of the duality between color and kinematics in field theory amplitudes. As a byproduct, it provides a rigorous a posteriori justification for the field theory amplitudes (11.3.54) of the last chapter which were derived by rather indirect cohomology arguments.

The five- and six point amplitudes have already been cast into BRST building block language. The former is discussed in [290, 291], whereas the six point amplitude is computed in [7]. The main advances in these works is a better understanding of the CFT correlators, in particular of the emergence of BRST building blocks and the role of double poles. This is the topic of the first section.

We then generalize to n external legs in section 12.2 and identify the color ordered field theory amplitudes among the superfield kinematics. The discovery of the BRST structure in section 11.1 and their organization in terms of Berends-Giele currents in section 11.2 are essential for seeing the emergence of various $\mathcal{A}^{\mathrm{SYM}}(1, 2_\sigma, \ldots, (n-2)_\sigma, n-1, n)$ subamplitudes (parametrized by $\sigma \in S_{n-3}$) within the string computation at fixed color ordering $(1, 2, \ldots, n-2, n-1, n)$.

12.1 The n point CFT correlator

This section explains the CFT mechanisms responsible for the appearence of BRST building block and a basis of no more than $(n-2)!$ kinematic factors in an n point tree amplitude. According to the tree level prescription (10.3.40), the first task in computing superstring amplitudes is to evaluate the CFT correlator

$$\langle\, V^1(0)\, V^{(n-1)}(1)\, V^n(\infty)\, U^2(z_2)\, U^3(z_3) \ldots U^{(n-2)}(z_{(n-2)})\,\rangle \tag{12.1.1}$$

using the techniques of subsection 10.3.1 to integrate out the $h=1$ primaries $[\partial\theta^\alpha, \Pi^m, d_\alpha, N^{mn}]$. Performing the worldsheet integral over $z_1 = 0 \leq z_2 \leq \ldots \leq z_{n-2} \leq z_{n-1} = 1$ then provides the momentum dependence of superstring amplitudes.

12.1.1 Five- and six point correlators

The five- and six point results are important data points which guided the way to higher point generalizations. They are the first instances where multiple integrated vertices U_j that have to be sequentially integrated out enter the correlator. The order in which the conformal $h=1$ fields are removed using (10.3.41) to (10.3.42) is an artifact of the algorithm at work and should not be visible in the final answer, this shall be our guiding principle in the following.

The five point amplitude requires a correlator involving U_2, U_3. If we decide to first integrate out the conformal primaries at z_2, then the correlator takes the form

$$\begin{aligned}\langle\, V^1(0)\, V^4(1)\, V^5(\infty)\, U^2(z_2)\, U^3(z_3)\,\rangle \;&=\; \prod_{i<j} |z_{ij}|^{-s_{ij}} \,\Bigg\langle\, \frac{L_{2131}\, V_4\, V_5}{z_{21}\, z_{31}} \;+\; \frac{L_{2331}\, V_4\, V_5}{z_{23}\, z_{31}} \\ +\; \frac{L_{2431}\, V_5}{z_{24}\, z_{31}} \;+\; \frac{L_{2134}\, V_5}{z_{21}\, z_{34}} \;&+\; \frac{V_1\, L_{2334}\, V_5}{z_{23}\, z_{34}} \;+\; \frac{V_1\, L_{2434}\, V_5}{z_{24}\, z_{34}} \;+\; \frac{V_1\, L_{2323}\, V_4\, V_5}{z_{23}^2} \Bigg\rangle \;. \end{aligned} \tag{12.1.2}$$

The OPE residues L_{2131} and L_{2434} where two integrated vertex operators U_i approach an unintegrated one V_j were discussed in subsection 11.1.1. The crossterms where $U_2 \to V_1$ and $U_3 \to V_4$ or vice versa are easily factorized as $L_{2134} = L_{21}L_{34}$ and $L_{2431} = L_{24}L_{31}$ because the OPEs are taken between separate pairs of fields. In addition, we have the single- and double poles in the OPE of $U_2(z_2)U_3(z_3)$ with residues L_{2331}, L_{2334} and L_{2323} in the notation of (12.1.2). Of course, the contribution from $V_n(z_n = \infty)$ is suppressed by $1/z_{2n}, 1/z_{3n} \to 0$.

The order of integrating out the $h=1$ fields at z_2 and z_3 is arbitrary and should not affect the final result. Matching (12.1.2) with its relabelling in $2 \leftrightarrow 3$ yields

$$L_{2331} \;=\; L_{3121} \;-\; L_{2131}\;, \qquad L_{2334} \;=\; L_{3424} \;-\; L_{2434} \tag{12.1.3}$$

where we made use of the antisymmetry of L_{2331}, L_{2334} in $z_2 \leftrightarrow z_3$. This allows to rewrite (12.1.2) in manifestly symmetric fashion and in terms of L_{jiki} only:

$$\begin{aligned} \langle V^1(0)\, V^4(1)\, V^5(\infty)\, U^2(z_2)\, U^3(z_3) \rangle \;&=\; \prod_{i<j} |z_{ij}|^{-s_{ij}} \left\langle \frac{L_{2131}\, V_4\, V_5}{z_{12}\, z_{23}} + \frac{L_{3121}\, V_4\, V_5}{z_{13}\, z_{32}} \right. \\ &+ \left. \frac{V_1\, L_{2434}\, V_5}{z_{42}\, z_{23}} + \frac{V_1\, L_{3424}\, V_5}{z_{43}\, z_{32}} + \frac{L_{21}\, L_{34}\, V_5}{z_{12}\, z_{43}} + \frac{L_{31}\, L_{24}\, V_5}{z_{42}\, z_{13}} + \frac{V_1\, L_{2323}\, V_4\, V_5}{z_{23}^2} \right\rangle \end{aligned} \tag{12.1.4}$$

This is the final result for the five point correlator before worldsheet integrations are taken into account.

Similar manipulations are necessary to simplify the six point amplitude. Let us display six out of the 34 terms which arise from the standard procedure to integrate out U_2, U_3 and then U_4:

$$\begin{aligned} \langle V^1(0)\, V^5(1)\, V^6(\infty)\, U^2(z_2)\, U^3(z_3)\, U^4(z_4) \rangle \;&=\; \prod_{i<j} |z_{ij}|^{-s_{ij}} \left\langle \frac{L_{213141}\, V_5\, V_6}{z_{21}\, z_{31}\, z_{41}} + \frac{L_{233141}\, V_5\, V_6}{z_{23}\, z_{31}\, z_{41}} \right. \\ &+ \left. \frac{L_{213441}\, V_5\, V_6}{z_{21}\, z_{34}\, z_{41}} + \frac{L_{233441}\, V_5\, V_6}{z_{23}\, z_{34}\, z_{41}} + \frac{L_{233145}\, V_6}{z_{23}\, z_{31}\, z_{45}} + \frac{L_{213445}\, V_6}{z_{21}\, z_{34}\, z_{45}} + \ldots \right\rangle \end{aligned} \tag{12.1.5}$$

The singe pole OPE residues of two integrated vertex operators $U_j U_k$ must be reduced to the standard superfields L_{ji}, L_{jiki} and L_{jikili}. Independence of the CFT correlator on the order of integrating out U_2, U_3 and U_4 (together with antisymmetry of the $U_i U_j$ single pole residue in i, j) implies

$$\begin{aligned} L_{233141} &= L_{312141} - L_{213141} \,, \qquad L_{213441} = L_{214131} - L_{213141} \\ L_{233441} &= L_{413121} - L_{412131} + L_{213141} - L_{312141} \\ L_{233145} &= (L_{3121} - L_{2131})\, L_{45} \,, \qquad L_{213445} = L_{21}\, (L_{4535} - L_{3545}) \end{aligned} \tag{12.1.6}$$

for the residues L_{ijklmn} of $(z_{ij} z_{kl} z_{mn})^{-1}$. Identities for L_{233545}, L_{233445} or L_{244331} follow by relabellings of (12.1.6).

The world sheet integrand of the six point amplitude then assumes the form

$$\begin{aligned} \langle V^1(0)\, V^5(1)\, V^6(\infty)\, U^2(z_2)\, U^3(z_3)\, U^4(z_4) \rangle \;&=\; \prod_{i<j} |z_{ij}|^{-s_{ij}} \left\langle \frac{L_{213141}\, V_5\, V_6}{z_{12}\, z_{23}\, z_{34}} + \frac{L_{2131}\, L_{45}\, V_6}{z_{12}\, z_{23}\, z_{54}} \right. \\ &+ \left. \frac{L_{21}\, L_{3545}\, V_6}{z_{12}\, z_{53}\, z_{34}} + \frac{V_1\, L_{253545}\, V_6}{z_{52}\, z_{23}\, z_{34}} + \text{double poles} + \mathcal{P}(2,3,4) \right\rangle \end{aligned} \tag{12.1.7}$$

where $\ldots + \mathcal{P}(2,3,4)$ denotes a symmetric sum over all S_3 permutations of $2, 3, 4$.

12.1.2 The single pole structure at higher points

Any single pole OPE residue in the five- and six point correlation functions can be reduced to the superfields $L_{2131\ldots p1}$ from subsection 11.1.1. This finding is based on two mechanisms which we now explain in the more general context of the n point correlator (12.1.1):

- The single pole contribution of the $U_j(z_j)U_k(z_k)\ldots U_l(z_l)$ OPEs between integrated vertex operators can be ultimately written in terms of $L_{jiki\ldots li}$ where $i \in \{1, n-1\}$ labels an unintegrated vertex operator. The required manipulations are based on the independence of correlation functions on the order of integrating out the $h=1$ fields. If the arguments of $V_1(z_1)U_2(z_2)U_3(z_3)\ldots U_p(z_p)$ approach each other in the order $z_2 \to z_3 \to \ldots \to z_{p-1} \to z_p \to z_1$, then the correlator (12.1.1) receives a contribution[1]

$$\frac{L_{2334\ldots p-1,p,p1}}{z_{23}\, z_{34}\ \ldots\ z_{p-1,p}\, z_{p1}} \;=\; \frac{2^{p-2} L_{[p1,[(p-1)1,[\ldots,[41,[31,21]]\ldots]]]}}{z_{23}\, z_{34}\ \ldots\ z_{p-1,p}\, z_{p1}}\ . \tag{12.1.8}$$

which for instance reads $L_{23344551} = L_{51413121} + L_{51213141} + L_{31214151} + L_{41213151} - (2 \to 3)$ at seven points.

- The two OPE cascades which terminate on the integrated vertices V_1 and V_{n-1} are taken independently such that the residue factorizes into (sums over) $L_{j_1 1 j_2 1\ldots j_p 1}$ and $L_{k_1,n-1,k_2,n-1\ldots k_q,n-1}$ with $p+q = n-3$.

On these grounds it was explicitly checked up to eight points that the correlation function (12.1.1) contains $(n-2)!$ single pole integrands (with $n-3$ distinct z_{ij}^{-1} factors each) whose residues fall into the symmetric pattern

$$\begin{aligned} \langle\, V^1(0)\, V^{(n-1)}(1)\, V^n(\infty)\, U^2(z_2)\, U^3(z_3)\ldots U^{(n-2)}(z_{(n-2)})\,\rangle \;&=\; \prod_{i<j} |z_{ij}|^{-s_{ij}} \\ \times\ \Big\langle \sum_{p=1}^{n-2} \frac{L_{2131\ldots p1}\, L_{n-2,n-1,n-3,n-1,\ldots,p+1,n-1}\, V_n}{(z_{12}\, z_{23}\ \ldots\ z_{p-1,p})\,(z_{n-1,n-2}\, z_{n-2,n-3}\ \ldots\ z_{p+2,p+1})} \Big\rangle& \\ +\ \text{double poles}\ +\ \mathcal{P}(2,3,\ldots,n-2)\ .& \end{aligned} \tag{12.1.9}$$

The next subsection explains the interplay between the double poles (which were left unspecified so far) and the $L_{jiki\ldots li}$ superfields.

12.1.3 Taming the double poles

The double pole integral $\int \mathrm{d}z_2\, \mathrm{d}z_3 \prod_{i<j} |z_{ij}|^{-s_{ij}}/z_{23}^2$ due to the five point correlator (12.1.4) is proportional to a tachyon pole $(1+s_{23})^{-1}$. It is cancelled by the superfield L_{2323} in the numerator:

$$L_{2323} \;=\; (1+s_{23})\,\big[\, A_2^\alpha\, W_\alpha^3 \;+\; A_3^\alpha\, W_\alpha^2 \;-\; (A_2 \cdot A_3)\,\big] \tag{12.1.10}$$

[1]We again make use of nested antisymmetrization brackets to streamline the notation. The results (12.1.3) and (12.1.6) for instance can be written more compactly as $L_{2331} = 2L_{[31,21]} = L_{3121} - L_{2131}$ and $L_{233441} = 4L_{[41,[31,21]]} = L_{413121} - L_{412131} - L_{312141} + L_{213141}$.

Its Q variation can be recognized as the BRST exact part of $L_{23}+L_{32}$, see (11.1.2). This can be absorbed into the other six integrals – the L_{ji} and L_{jiki} then receive the corrections (11.1.18) and (11.1.19) necessary for an upgrade to the corresponing BRST building blocks T_{ij} and $\tilde{T}_{ijk}$. The five point amplitude then simplifies to

$$\begin{aligned}
\mathcal{A}(1,2,\ldots,5) &= \int_0^1 \mathrm{d}z_2 \int_{z_2}^1 \mathrm{d}z_3 \prod_{i<j} |z_{ij}|^{-s_{ij}} \langle V^1(0)\, V^4(1)\, V^5(\infty)\, U^2(z_2)\, U^3(z_3) \rangle \\
&= \int_0^1 \mathrm{d}z_2 \int_{z_2}^1 \mathrm{d}z_3 \prod_{i<j} |z_{ij}|^{-s_{ij}} \Big\langle \frac{T_{123}\, V_4\, V_5}{z_{12}\, z_{23}} + \frac{T_{132}\, V_4\, V_5}{z_{13}\, z_{32}} \\
&\quad + \frac{V_1\, T_{423}\, V_5}{z_{42}\, z_{23}} + \frac{V_1\, T_{432}\, V_5}{z_{43}\, z_{32}} + \frac{T_{12}\, T_{34}\, V_5}{z_{12}\, z_{34}} + \frac{T_{13}\, T_{24}\, V_5}{z_{13}\, z_{24}} \Big\rangle
\end{aligned} \tag{12.1.11}$$

without any double poles. The tilde of $\tilde{T}_{ijk} = T_{ijk} + QS^{(1)}_{123}$ can be dropped because it only appears in combination with the BRST exact $V_4V_5 = QT_{45}/s_{45}$ superfields.

The six point amplitude involves ten double pole integrals which need to be absorbed into the remaining 24 integrals displayed in (12.1.7). They also carry spurious tachyon poles which cancel after some tedious manipulations based on worldsheet integration by parts, see appendix B of [7]. The corrections to the L_{ji}, L_{jiki} and L_{jikili} from double pole integrals are precisely of the form (11.1.17), (11.1.22) and (11.1.23) which converts them into the BRST building blocks T_{ij}, T_{ijk} and $\tilde{T}_{ijkl}$. Moreover, BRST exactness of V_mV_n implies $\langle \tilde{T}_{ijkl} V_m V_n \rangle = \langle T_{ijkl} V_m V_n \rangle$.

When the dust settles, the six point amplitude is found to be

$$\begin{aligned}
\mathcal{A}(1,\ldots,6) &= \int_0^1 \mathrm{d}z_2 \int_{z_2}^1 \mathrm{d}z_3 \int_{z_3}^1 \mathrm{d}z_4 \prod_{i<j} |z_{ij}|^{-s_{ij}} \langle V^1(0)\, V^5(1)\, V^6(\infty)\, U^2(z_2)\, U^3(z_3)\, U^4(z_4) \rangle \\
&= \int_0^1 \mathrm{d}z_2 \int_{z_2}^1 \mathrm{d}z_3 \int_{z_3}^1 \mathrm{d}z_4 \prod_{i<j} |z_{ij}|^{-s_{ij}} \Big\langle \frac{T_{1234}\, V_5\, V_6}{z_{12}\, z_{23}\, z_{34}} + \frac{T_{123}\, T_{54}\, V_6}{z_{12}\, z_{23}\, z_{54}} \\
&\quad + \frac{T_{12}\, T_{543}\, V_6}{z_{12}\, z_{54}\, z_{43}} + \frac{V_1\, T_{5432}\, V_6}{z_{54}\, z_{43}\, z_{32}} + \mathcal{P}(2,3,4) \Big\rangle \,.
\end{aligned} \tag{12.1.12}$$

We can draw the following general lesson from the double poles at five- and six points: The BRST trivial pieces from the $L_{jiki\ldots li}$ superfields in (12.1.9) are cancelled by the double pole residues from $U_j(z_j)U_k(z_k)$ contractions. Their conspiration requires some worldsheet integrations by parts which reshuffles Mandelstam invariants in such a way that all spurious tachyon poles cancel.

This ties two loose ends together – on the one hand, we have to understand how the integral over the correlator (12.1.9) yields a BRST closed amplitude, on the other hand, double pole integrals introduce unphysical tachyon poles. Tachyon poles always cancel due to $1+s_{i_1\ldots i_k}$ factors in the $U(z_i)U(z_j)$ double pole residues. Their kinematic factors attribute corrections of type $L_{jiki\ldots li} \to T_{ijk\ldots l}$ to the single pole part of the amplitude whose BRST variation then vanishes due to the symmetry properties of the BRST building blocks in $Q\mathcal{A}(1,2,\ldots,n)$. This pattern was rigorously checked up to the six point superstring amplitude, and consistency requires that it holds to any higher multiplicity.

The n point superstring amplitude with all double pole corrections taken into account reads:

$$\begin{aligned}\mathcal{A}(1,2,\ldots,n) \;&=\; \prod_{j=2}^{n-2}\int \mathrm{d}z_j \, \prod_{i<j}|z_{ij}|^{-s_{ij}} \\ \sum_{p=1}^{n-2}\Bigg\langle & \frac{T_{12\ldots p}\, T_{n-1,n-2,\ldots,p+1}\, V_n}{(z_{12}\, z_{23}\, \ldots\, z_{p-1,p})\,(z_{n-1,n-2}\, z_{n-2,n-3}\, \ldots\, z_{p+2,p+1})} \;+\; \mathcal{P}(2,3,\ldots,n-2) \;+\; \ldots \Bigg\rangle\end{aligned} \tag{12.1.13}$$

The z_{ij} polynomials associated with a specific BRST building block $T_{ij_1j_2\ldots j_p}$ follow an intriguing pattern:

$$T_{ij_1j_2\ldots j_p} \quad \leftrightarrow \quad \frac{1}{z_{ij_1}\, z_{j_1j_2}\, z_{j_2j_3}\, \ldots\, z_{j_{p-1},j_p}} \tag{12.1.14}$$

The first label belongs to an unintegrated vertex $i=1$ or $i=n-1$ and the remaining ones to the integrated vertices $j_k = 2,3,\ldots n-2$.

The representation (12.1.13) for the superstring amplitude can in principle be recast in terms of smaller building blocks via BRST integration by parts. But it turns out that the present form is most suitable for all our further purposes: Firstly, it provides an explicit formula for the supersymmetric Berende-Giele currents introduced in section 11.2. Secondly, it is the most appropriate starting point for decomposing the superstring amplitude into its field theory constituents. And as a third benefit, (12.1.13) is an excellent tool to generate explicit BCJ numerators n_i, see section 5.5, this construction is explained in detail in the last chapter 13.

12.1.4 A string inspired formula for Berends-Giele currents

In this subsection, we will show that the result (12.1.13) for the CFT correlator in superstring theory allows to extract a direct formula for the Berends–Giele current $M_{12\ldots p}$. The p sum in (12.1.13) runs over partitions of the legs $2,3,\ldots n-2$ into two groups – one of them gets attached to the building block $T_{1\ldots}$ of integrated vertex 1, the other one enters $T_{n-1\ldots}$. The same structure is present the field theory amplitude $\mathcal{A}^{\text{SYM}} = \sum_{p=1}^{n-2}\langle M_{12\ldots p} M_{p+1\ldots n-1} V_n\rangle$.

Let us now make use of the requirement on superstring amplitudes to reproduce their field theory counterparts in the limit $\alpha' \to 0$ of vanishing string length. This has to hold term by term in the p sums of (12.1.13) because the superfield kinematics at different p values are linearly independent. Let us pick out the $p = n-2$ term and impose $\mathcal{A}(1,\ldots,n) \to \mathcal{A}^{\rm SYM}(1,\ldots,n)$ in the field theory limit. This is one of the few instances within the pure spinor part of this work where we deviate from the $2\alpha' = 1$ convention:

$$\begin{aligned} \mathcal{A}(1,2,\ldots,n) \ &= \ (2\alpha')^{n-3} \prod_{j=2}^{n-2} \int \mathrm{d}z_j \ \prod_{i<j} |z_{ij}|^{-2\alpha' s_{ij}} \\ &\quad \times \ \left\langle \ \frac{T_{12\ldots n-2}\, V_{n-1}\, V_n}{z_{12}\, z_{23} \ \ldots \ z_{n-3,n-2}} \ + \ \mathcal{P}(2,3,\ldots,n-2) \ + \ \ldots \ \right\rangle \\ &\stackrel{!}{\to} \ \langle\, M_{12\ldots n-2}\, V_{n-1}\, V_n \,\rangle \ + \ \ldots \end{aligned} \tag{12.1.15}$$

The ellipsis $\ldots$ represents the $1 \leq p \leq n-3$ terms of the p sums with linearly independent kinematic factors. Those parts of n point amplitudes displayed in (12.1.15) can be viewed as a closed-formula solution for the rank $p = n-2$ current $M_{12\ldots p}$,

$$M_{12\ldots p} \ = \ \lim_{\alpha' \to 0} (2\alpha')^{p-1} \prod_{j=2}^{p} \int_{z_{j-1}}^{1} \mathrm{d}z_j \ \prod_{i<j}^{p+1} |z_{ij}|^{-2\alpha' s_{ij}} \ \left(\frac{T_{12\ldots p}}{z_{12}\, z_{23} \ \ldots \ z_{p-1,p}} \ + \ \mathcal{P}(2,3,\ldots,p) \right), \tag{12.1.16}$$

where $z_1 = 0$ and $z_{p+1} = 1$ as customary for a $(p+2)-$point amplitude. For example, using the momentum expansion of the five point superstring integrals 7.2 and the BRST symmetry $T_{123} + T_{213} = T_{123} + T_{231} + T_{312} = 0$, the following M_{123} is generated

$$\begin{aligned} M_{123} \ &= \ \lim_{\alpha' \to 0} (2\alpha')^2 \int_0^1 \mathrm{d}z_2 \int_{z_2}^1 \mathrm{d}z_3 \ \prod_{i<j}^{4} |z_{ij}|^{-2\alpha' s_{ij}} \ \left(\frac{T_{123}}{z_{12}\, z_{23}} \ + \ \frac{T_{132}}{z_{13}\, z_{32}} \right) \\ &= \ \frac{T_{123}}{s_{12}\, s_{123}} \ + \ \frac{T_{123}}{s_{23}\, s_{123}} \ - \ \frac{T_{132}}{s_{23}\, s_{123}} \ = \ \frac{T_{123}}{s_{12}\, s_{123}} \ + \ \frac{T_{321}}{s_{23}\, s_{123}} \end{aligned} \tag{12.1.17}$$

in agreement with the diagrams above. Similarly, we checked that this formula remains consistent with the diagramatic construction of the currents and with their BRST variation $QM_{12\ldots p} = \sum_{j=1}^{p-1} M_{12\ldots j} M_{j+1\ldots p}$ up to $p = 7$.

12.2 The structure of the n point superstring amplitude

The achievement of the representation (12.1.13) for the n point superstring amplitude is its reduction to $(n-2)!$ hypergeometric integrals and the same number of superfield kinematics $\langle T_{12_\sigma \ldots p_\sigma} T_{n-1,(n-2)_\sigma,\ldots,(p+1)_\sigma} V_n \rangle$ where $p = 1,2,\ldots,n-2$ and $\sigma \in S_{n-3}$. In this section, this expression will be reduced to a minimal basis of $(n-3)!$ elements, both on the side of worldsheet integrals and on the side of kinematics. The rather technical steps in this reduction process are organized into three subsections.

12.2.1 Trading $T_{12\ldots p}$ for $M_{12\ldots p}$

The first step in simplifying the superstring amplitude in its non-minimal representation (12.1.13) relies on eliminating the BRST building blocks $T_{12\ldots p}$ in favor of the Berends–Giele currents $M_{12\ldots p}$. This is possible because of the particular pattern (12.1.14) of z_{ij} dependences.

The lowest order example of $T \leftrightarrow M$ conversion is a triviality $\frac{T_{12}}{z_{12}} = \frac{s_{12}}{z_{12}} M_{12}$, but already the simplest generalization is the result of both partial fraction relations and the symmetry properties of T_{ijk}:

$$\frac{T_{123}}{z_{12}\, z_{23}} \ + \ \mathcal{P}(2,3) \ \ = \ \ \frac{s_{12}}{z_{12}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} \right) M_{123} \ + \ \mathcal{P}(2,3) \tag{12.2.18}$$

Similar identities have been checked at $p=4$ and $p=5$ level:

$$\begin{aligned} \frac{T_{1234}}{z_{12}\, z_{23}\, z_{34}} \ + \ \mathcal{P}(2,3,4) \ \ &= \ \ \frac{s_{12}}{z_{12}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} \right) \left(\frac{s_{14}}{z_{14}} + \frac{s_{24}}{z_{24}} + \frac{s_{34}}{z_{34}} \right) M_{1234} \ + \ \mathcal{P}(2,3,4) \\ \frac{T_{12345}}{z_{12}\, z_{23}\, z_{34}\, z_{45}} \ + \ \mathcal{P}(2,3,4,5) \ \ &= \ \ \frac{s_{12}}{z_{12}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} \right) \left(\frac{s_{14}}{z_{14}} + \frac{s_{24}}{z_{24}} + \frac{s_{34}}{z_{34}} \right) \\ &\quad \times \left(\frac{s_{15}}{z_{15}} + \frac{s_{25}}{z_{25}} + \frac{s_{35}}{z_{35}} + \frac{s_{45}}{z_{45}} \right) M_{12345} \ + \ \mathcal{P}(2,3,4,5) \end{aligned} \tag{12.2.19}$$

These identities heavily rely on the interplay of different terms in the permutation sum and on the symmetry properties (11.1.25) and (11.1.38) of the BRST building blocks which leave no more than $(p-1)!$ independent permutations $T_{i_1\ldots i_p}$ at rank p.

The natural n point generalization of (12.2.18) and (12.2.19) reads as follows:

$$\frac{T_{12\ldots p}}{z_{12}\, z_{23}\, \cdots\, z_{p-1,p}} \ + \ \mathcal{P}(2,\ldots,p) \ \ = \ \ \prod_{k=2}^{p} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}}\, M_{12\ldots p} \ + \ \mathcal{P}(2,\ldots,p) \tag{12.2.20}$$

$$\begin{aligned} \frac{T_{n-1,n-2,\ldots p+1}}{z_{n-1,n-2}\, \cdots\, z_{p+2,p+1}} \ + \ \mathcal{P}(2,\ldots,p) \ \ &= \ \ \prod_{k=p+1}^{n-2} \sum_{n=k+1}^{n-1} \frac{s_{nk}}{z_{nk}}\, M_{n-1,n-2,\ldots,p+1} \ + \ \mathcal{P}(2,\ldots,p) \\ &= \ \prod_{k=p+1}^{n-2} \sum_{n=k+1}^{n-1} \frac{s_{kn}}{z_{kn}}\, M_{p+1,p+2,\ldots,n-1} \ + \ \mathcal{P}(2,\ldots,p) \end{aligned} \tag{12.2.21}$$

In the last step, the rank $n-1-p$ current is reflected via $M_{n-1,\ldots,p+1} = (-1)^{n-p-2} M_{p+1,\ldots,n-1}$.

12.2.2 Worldsheet integration by parts

This subsection focuses on the integrals rather than the kinematic factors in the superstring amplitude. The chain of s_{mk}/z_{mk} sums which appears as a result of the general $T_{12\ldots p} \to M_{12\ldots p}$ conversion (12.2.20) and (12.2.21) is particularly suitable to perform integration by parts with respect to z_j variables.

The key idea is the vanishing of boundary terms in the worldsheet integrals:

$$\int \mathrm{d}z_2 \ldots \int \mathrm{d}z_{n-2} \, \frac{\partial}{\partial z_k} \, \frac{\prod_{i<j} |z_{ij}|^{-s_{ij}}}{z_{i_1 j_1} \, \ldots \, z_{i_{n-4} j_{n-4}}} \;\; = \;\; 0 \tag{12.2.22}$$

This identity provides relations between the integrals in an n point superstring amplitudes with $n-3$ powers of some z_{ij} in the denominator. They become particularly easy if the differentiation variable z_k does not appear in the denominator (i.e. if $k \notin \{i_l, j_l\}$) because $\frac{\partial}{\partial z_k}$ only hits the $\prod_{m \neq k} |z_{mk}|^{-s_{mk}}$ in that case:

$$\int \mathrm{d}z_2 \ldots \int \mathrm{d}z_{n-2} \, \frac{\prod_{i<j} |z_{ij}|^{-s_{ij}}}{z_{i_1 j_1} \, \ldots \, z_{i_{n-4} j_{n-4}}} \sum_{\substack{m=1 \\ m \neq k}}^{n-1} \frac{s_{mk}}{z_{mk}} \;\; = \;\; 0 \tag{12.2.23}$$

This can be directly applied to the integrands on the right hand side of (12.2.18) to (12.2.21):

$$\prod_{j=2}^{3} \int \mathrm{d}z_j \, \prod_{i<j} |z_{ij}|^{-s_{ij}} \, \frac{s_{12}}{z_{12}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} \right) \;\; = \;\; \prod_{j=2}^{3} \int \mathrm{d}z_j \, \prod_{i<j} |z_{ij}|^{-s_{ij}} \, \frac{s_{12}}{z_{12}} \, \frac{s_{34}}{z_{34}} \tag{12.2.24}$$

$$\begin{aligned} &\prod_{j=2}^{4} \int \mathrm{d}z_j \, \prod_{i<j} |z_{ij}|^{-s_{ij}} \, \frac{s_{12}}{z_{12}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} \right) \left(\frac{s_{14}}{z_{14}} + \frac{s_{24}}{z_{24}} + \frac{s_{34}}{z_{34}} \right) \\ &\quad = \prod_{j=2}^{4} \int \mathrm{d}z_j \, \prod_{i<j} |z_{ij}|^{-s_{ij}} \, \frac{s_{12}}{z_{12}} \, \frac{s_{45}}{z_{45}} \left\{ \begin{matrix} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} \right) \\ \left(\frac{s_{34}}{z_{34}} + \frac{s_{35}}{z_{35}} \right) \end{matrix} \right. \end{aligned} \tag{12.2.25}$$

$$\begin{aligned} &\prod_{j=2}^{5} \int \mathrm{d}z_j \, \prod_{i<j} |z_{ij}|^{-s_{ij}} \, \frac{s_{12}}{z_{12}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} \right) \left(\frac{s_{14}}{z_{14}} + \frac{s_{24}}{z_{24}} + \frac{s_{34}}{z_{34}} \right) \left(\frac{s_{15}}{z_{15}} + \frac{s_{25}}{z_{25}} + \frac{s_{35}}{z_{35}} + \frac{s_{45}}{z_{45}} \right) \\ &\quad = \prod_{j=2}^{5} \int \mathrm{d}z_j \, \prod_{i<j} |z_{ij}|^{-s_{ij}} \, \frac{s_{12}}{z_{12}} \, \frac{s_{56}}{z_{56}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} \right) \left(\frac{s_{45}}{z_{45}} + \frac{s_{46}}{z_{46}} \right) \end{aligned} \tag{12.2.26}$$

In the general n point case with $n-3$ integrals, it is most economic to leave the first $[n/2]-1$ factors of $\sum_{m=1}^{k-1} s_{mk}/z_{mk}$ as they are and to integrate the remaining $[(n-3)/2]$ such factors by parts:

$$\begin{aligned} &\prod_{j=2}^{n-2} \int \mathrm{d}z_j \, \prod_{i<j} |z_{ij}|^{-s_{ij}} \, \frac{s_{12}}{z_{12}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} \right) \ldots \left(\frac{s_{1,n-2}}{z_{1,n-2}} + \ldots + \frac{s_{n-1,n-2}}{z_{n-1,n-2}} \right) \\ &\quad = \prod_{j=2}^{n-2} \int \mathrm{d}z_j \, \prod_{i<j} |z_{ij}|^{-s_{ij}} \, \frac{s_{12}}{z_{12}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} \right) \ldots \left(\frac{s_{1,[n/2]}}{z_{1,[n/2]}} + \ldots + \frac{s_{[n/2]-1,[n/2]}}{z_{[n/2]-1,[n/2]}} \right) \\ &\qquad \left(\frac{s_{[n/2]+1,[n/2]+2}}{z_{[n/2]+1,[n/2]+2}} + \ldots + \frac{s_{[n/2]+1,n-1}}{z_{[n/2]+1,n-1}} \right) \ldots \left(\frac{s_{n-3,n-2}}{z_{n-3,n-2}} + \frac{s_{n-3,n-1}}{z_{n-3,n-1}} \right) \frac{s_{n-2,n-1}}{z_{n-2,n-1}} \\ &\quad = \prod_{j=2}^{n-2} \int \mathrm{d}z_j \, \prod_{i<j} |z_{ij}|^{-s_{ij}} \left(\prod_{k=2}^{[n/2]} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \right) \left(\prod_{k=[n/2]+1}^{N-2} \sum_{n=k+1}^{n-1} \frac{s_{kn}}{z_{kn}} \right) \end{aligned} \tag{12.2.27}$$

In contrast to the $T_{12\ldots p} \rightarrow M_{12\ldots p}$ reshuffling identities from the previous subsection, (12.2.24) to (12.2.26) and (12.2.27) are valid before summing over permutations of $(2, 3, \ldots, n-2)$.

12.2.3 Superstring amplitudes up to seven points

This subsection completes the derivation of the striking result (1.4.1) for the color ordered superstring n point amplitude $\mathcal{A}_n := \mathcal{A}(1, 2, \ldots, n)$ by combining the results of the previous subsections. Let us first look at the five- and six point examples to get a better feeling of the mechanisms at work.

The opening line (12.1.11) in computing the five point amplitude contains six different integrands and kinematic terms. After applying (12.2.18), the T_{ij} and T_{ijk} conspire to give M_{ij} and M_{ijk} with modified integrals, then we use integration by parts according to (12.2.24) on the way to the fourth line. Remarkably, many of the initially $(n-2)! = 6$ distinct integrals now coincide: The three kinematic terms $M_{123}V_4V_5$, $M_{12}M_{34}V_5$ and $V_1M_{234}V_5$ are multiplied by the same worldsheet functions after partial integration, the same is true for the $(2 \leftrightarrow 3)$ permutation. That is why we can identify complete field theory amplitudes in the last line:

$$\begin{aligned}
\mathcal{A}_5 \;&=\; \int \mathrm{d}z_2\, \mathrm{d}z_3 \, \prod_{i<j} |z_{ij}|^{-s_{ij}} \left\langle \frac{T_{123}\, V_4\, V_5}{z_{12}\, z_{23}} \;+\; \frac{T_{12}\, T_{43}\, V_5}{z_{12}\, z_{43}} \;+\; \frac{V_1\, T_{432}\, V_5}{z_{43}\, z_{32}} \;+\; (2 \leftrightarrow 3) \right\rangle \\
&=\; \int \mathrm{d}z_2\, \mathrm{d}z_3 \, \prod_{i<j} |z_{ij}|^{-s_{ij}} \left\langle \frac{s_{12}}{z_{12}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} \right) M_{123}\, V_4\, V_5 \;+\; \frac{s_{12}\, s_{34}}{z_{12}\, z_{34}}\, M_{12}\, M_{34}\, V_5 \right. \\
&\qquad\qquad \left. +\; \frac{s_{43}}{z_{43}} \left(\frac{s_{42}}{z_{42}} + \frac{s_{32}}{z_{32}} \right) V_1\, M_{432}\, V_5 \;+\; (2 \leftrightarrow 3) \right\rangle \\
&=\; \int \mathrm{d}z_2\, \mathrm{d}z_3 \, \prod_{i<j} |z_{ij}|^{-s_{ij}} \left\{ \frac{s_{12}\, s_{34}}{z_{12}\, z_{34}} \, \langle\, (M_{123}\, V_4 \;+\; M_{12}\, M_{34} \;+\; V_1\, M_{234})\, V_5 \,\rangle \;+\; (2 \leftrightarrow 3) \right\} \\
&=\; \int \mathrm{d}z_2\, \mathrm{d}z_3 \, \prod_{i<j} |z_{ij}|^{-s_{ij}} \left\{ \frac{s_{12}\, s_{34}}{z_{12}\, z_{34}} \, \mathcal{A}^{\mathrm{SYM}}(1,2,3,4,5) \;+\; \frac{s_{13}\, s_{24}}{z_{13}\, z_{24}} \, \mathcal{A}^{\mathrm{SYM}}(1,3,2,4,5) \right\}
\end{aligned} \tag{12.2.28}$$

Simplifying the six point amplitudes $\mathcal{A}_6$ follows similar steps. In this case, (12.2.19) takes care of the conversion of T_{ijkl} into M_{ijkl}, then integration by parts makes the four integrals within a given $(2, 3, 4)$ permutation coincide:

$$\begin{aligned}
\mathcal{A}_6 \;&=\; \prod_{j=2}^{4} \int \mathrm{d}z_j \, \prod_{i<j} |z_{ij}|^{-s_{ij}} \left\langle \frac{T_{1234}\, V_5\, V_6}{z_{12}\, z_{23}\, z_{34}} \;+\; \frac{T_{123}\, T_{54}\, V_6}{z_{12}\, z_{23}\, z_{54}} \;+\; \frac{T_{12}\, T_{543}\, V_6}{z_{12}\, z_{54}\, z_{43}} \right. \\
&\qquad\qquad \left. +\; \frac{V_1\, T_{5432}\, V_6}{z_{54}\, z_{43}\, z_{32}} \;+\; \mathcal{P}(2,3,4) \right\rangle \\
&=\; \prod_{j=2}^{4} \int \mathrm{d}z_j \, \prod_{i<j} |z_{ij}|^{-s_{ij}} \left\langle \frac{s_{12}}{z_{12}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} \right) \left(\frac{s_{14}}{z_{14}} + \frac{s_{24}}{z_{24}} + \frac{s_{34}}{z_{34}} \right) M_{1234}\, V_5\, V_6 \right. \\
&\qquad +\; \frac{s_{12}}{z_{12}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} \right) \frac{s_{45}}{z_{45}}\, M_{123}\, M_{45}\, V_6 \;+\; \frac{s_{12}}{z_{12}} \frac{s_{45}}{z_{45}} \left(\frac{s_{34}}{z_{34}} + \frac{s_{35}}{z_{35}} \right) M_{12}\, M_{543}\, V_6 \\
&\qquad \left. +\; \frac{s_{45}}{z_{45}} \left(\frac{s_{34}}{z_{34}} + \frac{s_{35}}{z_{35}} \right) \left(\frac{s_{52}}{z_{52}} + \frac{s_{42}}{z_{42}} + \frac{s_{32}}{z_{32}} \right) V_1\, M_{5432}\, V_6 \;+\; \mathcal{P}(2,3,4) \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \prod_{j=2}^{4} \int \mathrm{d}z_j \, \prod_{i<j} |z_{ij}|^{-s_{ij}} \left\{ \frac{s_{12}\, s_{45}}{z_{12}\, z_{45}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} \right) \langle\, M_{1234}\, V_5\, V_6 \; + \; M_{123}\, M_{45}\, V_6 \right. \\
&\qquad\qquad \left. + \; M_{12}\, M_{345}\, V_6 \; + \; V_1\, M_{2345}\, V_6 \,\rangle \; + \; \mathcal{P}(2,3,4) \right\} \\
&= \prod_{j=2}^{4} \int \mathrm{d}z_j \, \prod_{i<j} |z_{ij}|^{-s_{ij}} \\
&\qquad \left\{ \frac{s_{12}\, s_{45}}{z_{12}\, z_{45}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} \right) \mathcal{A}^{\mathrm{SYM}}(1,2,3,4,5,6) \; + \; \mathcal{P}(2,3,4) \right\}
\end{aligned} \tag{12.2.29}
$$

The identities (12.2.19) and (12.2.26) are sufficient to also reduce $\mathcal{A}_7$ to its field theory constituents:

$$
\begin{aligned}
\mathcal{A}_7 \;&=\; \prod_{j=2}^{5} \int \mathrm{d}z_j \, \prod_{i<j} |z_{ij}|^{-s_{ij}} \left\langle \frac{T_{12345}\, V_6\, V_7}{z_{12}\, z_{23}\, z_{34}\, z_{45}} + \frac{T_{1234}\, T_{65}\, V_7}{z_{12}\, z_{23}\, z_{34}\, z_{65}} + \frac{T_{123}\, T_{654}\, V_7}{z_{12}\, z_{23}\, z_{65}\, z_{54}} \right. \\
&\qquad \left. + \frac{T_{12}\, T_{6543}\, V_7}{z_{12}\, z_{65}\, z_{54}\, z_{43}} + \frac{V_1\, T_{65432}\, V_7}{z_{65}\, z_{54}\, z_{43}\, z_{32}} + \mathcal{P}(2,3,4,5) \right\rangle \\
&= \prod_{j=2}^{5} \int \mathrm{d}z_j \, \prod_{i<j} |z_{ij}|^{-s_{ij}} \\
&\left\{ \frac{s_{12}\, s_{56}}{z_{12}\, z_{56}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} \right) \left(\frac{s_{45}}{z_{45}} + \frac{s_{46}}{z_{46}} \right) \mathcal{A}^{\mathrm{SYM}}(1,2,3,4,5,6,7) \; + \; \mathcal{P}(2,3,4,5) \right\}
\end{aligned} \tag{12.2.30}
$$

12.2.4 The n point result

The n point generalization is based on introducing currents $M_{i_1 i_2 \ldots i_p}$ via (12.2.20) and (12.2.21) followed by integration by parts using (12.2.27). The latter makes the worldsheet integrand independent on p such that the z_{ij} can be placed outside the p sum and SYM amplitudes emerge from the kinematics.

$$
\begin{aligned}
\mathcal{A}_n \;&=\; \prod_{j=2}^{n-2} \int \mathrm{d}z_j \, \prod_{i<j} |z_{ij}|^{-s_{ij}} \left\langle \sum_{p=1}^{n-2} \frac{T_{12\ldots p}\, T_{n-1,n-2,\ldots,p+1}\, V_n}{(z_{12}\, z_{23}\, \ldots\, z_{p-1,p})\,(z_{n-1,n-2}\, \ldots\, z_{p+2,p+1})} \right. \\
&\qquad \left. + \; \mathcal{P}(2,3,\ldots,n-2) \right\rangle \\
&= \prod_{j=2}^{n-2} \int \mathrm{d}z_j \, \prod_{i<j} |z_{ij}|^{-s_{ij}} \left\langle \sum_{p=1}^{n-2} \left(\prod_{k=2}^{p} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \, M_{12\ldots p} \right) \right. \\
&\qquad \left. \left(\prod_{k=p+1}^{n-2} \sum_{m=k+1}^{n-1} \frac{s_{km}}{z_{km}} \, M_{p+1,\ldots,n-2,n-1} \right) V_n \; + \; \mathcal{P}(2,3,\ldots,n-2) \right\rangle \\
&= \prod_{j=2}^{n-2} \int \mathrm{d}z_j \, \prod_{i<j} |z_{ij}|^{-s_{ij}} \left\{ \left(\prod_{k=2}^{[n/2]} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \right) \left(\prod_{k=[n/2]+1}^{n-2} \sum_{m=k+1}^{n-1} \frac{s_{km}}{z_{km}} \right) \right.
\end{aligned}
$$

$$\sum_{p=1}^{n-2} \langle\, M_{12\ldots p}\, M_{p+1\ldots n-2,n-1}\, V_n \,\rangle \;+\; \mathcal{P}(2,3,...,n-2) \Bigg\}$$

$$= \prod_{j=2}^{n-2} \int \mathrm{d}z_j \, \prod_{i<j} |z_{ij}|^{-s_{ij}} \Bigg\{ \left(\prod_{k=2}^{[n/2]} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \right) \left(\prod_{k=[n/2]+1}^{n-2} \sum_{m=k+1}^{n-1} \frac{s_{km}}{z_{km}} \right)$$

$$\mathcal{A}^{\mathrm{SYM}}(1,2,3,\ldots,n-1,n) \;+\; \mathcal{P}(2,3,...,n-2) \Bigg\} \tag{12.2.31}$$

Hence, the color ordered n-point superstring amplitude becomes

$$\mathcal{A}(1_\sigma, 2_\sigma, \ldots, n_\sigma) \;=\; \prod_{j=2}^{n-2} \int_{\mathcal{I}_\sigma} \mathrm{d}z_j \, \prod_{i<j} |z_{ij}|^{-s_{ij}} \Bigg\{ \left(\prod_{k=2}^{[n/2]} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \right) \left(\prod_{k=[n/2]+1}^{n-2} \sum_{m=k+1}^{n-1} \frac{s_{km}}{z_{km}} \right)$$

$$\mathcal{A}^{\mathrm{SYM}}(1,2,3,\ldots,n-1,n) \;+\; \mathcal{P}(2,3,...,n-2) \Bigg\} \tag{12.2.32}$$

where only the integration region $\mathcal{I}_\sigma$ reflects the particular color ordering in question. Therefore, the end result of all these pure spinor superspace manipulations is that the n-point superstring disk amplitude is written in terms of an explicit sum over the $(n-3)!$ basis of field-theory amplitudes $\mathcal{A}^{\mathrm{SYM}}$ multiplied by an equal number of hypergeometric integrals.

12.3 The color ordered n point amplitude

This section investigates the structure and properties of the main result (12.2.32) of my research activities in Munich. The pure spinor computation of the previous section casts the complete superstring n point disk color ordered amplitude into the following form:

$$\mathcal{A}(1,2,\ldots,n) \;=\; \sum_{\sigma \in S_{n-3}} \mathcal{A}^{\mathrm{SYM}}(1, 2_\sigma, \ldots, (n-2)_\sigma, n-1, n)\, F^{\sigma}_{(1,\ldots,n)}(\alpha') \tag{12.3.33}$$

In (1.4.1), we have made the identification $F^\sigma \equiv F^\sigma_{(1,\ldots,n)}$ for the $(n-3)!$ functions. The latter will be explicitly given in subsection 12.3.4. The result (12.3.33) is valid for members of the massless vector multiplet in any spacetime dimension D, for any compactification and any amount of supersymmetry. Furthermore, it does not make any reference to kinematical or helicity choices. In the following we explore the result (12.3.33) to illuminate the role of color and kinematics.

12.3.1 Basis representations: kinematics versus color

We have explained in subsection 5.5.5 that there are in total $(n-3)!$ independent subamplitudes $\mathcal{A}^{\mathrm{SYM}}$ in field theory and given a string theory derivation of this result in section 5.4. Hence, in field theory any partial subamplitude $\mathcal{A}^{\rm SYM}(1_\Pi,\ldots,n_\Pi)$, with $\Pi\in S_n$, can be expressed as

$$\mathcal{A}^{\rm SYM}(1_\Pi,\ldots,n_\Pi) \;=\; \sum_{\sigma\in S_{n-3}} K_\Pi{}^\sigma\, \mathcal{A}^{\rm SYM}_\sigma\ , \tag{12.3.34}$$

with $i_\Pi=\Pi(i)$, some universal and state independent kinematic coefficients $K_\Pi{}^\sigma$ generically depending on the kinematic invariants, cf. (12.3.39) for a straightforward derivation. Besides, we introduced the abbreviation:

$$\mathcal{A}^{\rm SYM}_\sigma \;:=\; \mathcal{A}^{\rm SYM}(1,2_\sigma,\ldots,(n-2)_\sigma,n-1,n) \tag{12.3.35}$$

One crucial property of (12.3.33) is the fact, that the full n point amplitude may be decomposed in terms of SYM color ordered amplitudes $\mathcal{A}^{\rm SYM}_\sigma$, i.e. the whole superstring amplitude can be decomposed with respect to the kinematics described by the set of $\mathcal{A}^{\rm SYM}_\sigma,\ \sigma\in S_{n-3}$. Hence, it is evident that in (12.3.33) only $(n-3)!$ basis functions are necessary, because there are only as much independent kinematical packages. If in (12.3.33) there were more than $(n-3)!$ terms we simply would express all kinematical factors in terms of the minimal basis (12.3.35) at the cost of redefining the functions F^σ. Hence, by these results it is obvious, that in the sum of (12.3.33) only $(n-3)!$ terms and as many different multiple hypergeometric functions can appear since any additional kinematical term could be eliminated by redefining the functions F^σ thanks to the amplitude relations (12.3.34).

Moreover, the string subamplitudes (12.3.33) and its color ordered permutations solve the system of string monodromy relations

$$\begin{aligned}\mathcal{A}(1,2,\ldots,n)\ +\ e^{i\pi s_{12}}\,\mathcal{A}(2,1,3,\ldots,n)\ +\ e^{i\pi(s_{12}+s_{13})}\,\mathcal{A}(2,3,1,4,\ldots,n)&\\ +\ \ldots\ +\ e^{i\pi(s_{12}+s_{13}+\ldots+s_{1,n-1})}\,\mathcal{A}(2,3,\ldots,n-1,1,n)\ &=\ 0\ ,\end{aligned} \tag{12.3.36}$$

and permutations thereof. Furthermore, since there exists a basis of $(n-3)!$ SYM building blocks allowing for the decomposition (12.3.34), we may express any string subamplitude by *one* specific set of SYM amplitudes $\mathcal{A}^{\rm SYM}_\sigma$ referring e.g. to the string amplitude (12.3.33):

$$\mathcal{A}(1_\Pi,\ldots,n_\Pi) \;=\; \sum_{\sigma\in S_{n-3}} \mathcal{A}^{\rm SYM}_\sigma\ F^\sigma_\Pi(\alpha')\ ,\qquad \Pi\in S_n \tag{12.3.37}$$

Inserting the set (12.3.37) into the monodromy relations yields a set of relations for the functions F^σ_Π for each given $\sigma\in S_{n-3}$:

$$\begin{aligned}F^\sigma_{(1,2,\ldots,n)}\ +\ e^{i\pi s_{12}}\,F^\sigma_{(2,1,3,\ldots,n)}\ +\ e^{i\pi(s_{12}+s_{13})}\,F^\sigma_{(2,3,1,4,\ldots,n)}&\\ +\ \ldots\ +\ e^{i\pi(s_{12}+s_{13}+\ldots+s_{1,n-1})}\,F^\sigma_{(2,3,\ldots,n-1,1,n)}\ &=\ 0\end{aligned} \tag{12.3.38}$$

Hence, for a given $\sigma \in S_{n-3}$ corresponding to the given SYM amplitude $\mathcal{A}_\sigma^{\rm SYM}$, the set of functions F_Π^σ at fixed σ and $\Pi \in S_n$ enjoys the monodromy relations. As a consequence for each permutation $\sigma \in S_{n-3}$ or YM basis amplitude $\mathcal{A}_\sigma^{\rm SYM}$ there are $(n-3)!$ different functions F_Π^σ all related through the equations (12.3.38) and permutations thereof.

As a remarkable byproduct, (12.3.37) allows for an explicit expression of the kinematic coefficients $K_\Pi{}^\sigma$ introduced in (12.3.34) in the low energy limit $\alpha' \to 0$,

$$K_\Pi{}^\sigma \quad = \quad F_\Pi^\sigma(\alpha') \Big|_{\alpha'=0} \ . \tag{12.3.39}$$

This relation enables to compute the matrix elements $K_\Pi{}^\sigma$ *directly* by means of extracting the field theory limit of the string worldsheet integrals $F_\Pi^\sigma(\alpha')$ (by the method described in section 7.2) rather than by solving the monodromy relations (12.3.36).

12.3.2 Duality between color and kinematics

Further insights can be gained when looking at different representations for the same amplitude (12.3.33):

$$\mathcal{A}(1,2,\ldots,n) \quad = \sum_{\pi\in S_{n-3}} \mathcal{A}_\pi^{\rm SYM} \ F^\pi_{(1,\ldots,n)}(\alpha') \ , \tag{12.3.40}$$

In contrast to the $\mathcal{A}_\sigma^{\rm SYM}$ in (12.3.35), the set $\mathcal{A}_\pi^{\rm SYM}$ represents a more general basis of $(n-3)!$ independent subamplitudes where three legs i,j,k (possibly other than $1, n-1$ and n) are fixed and the remaining ones are permuted by $\pi \in S_{n-3}$. By applying the decomposition (12.3.34) and comparing the two expressions (12.3.40) and (12.3.33) we find the relation between the set of $(n-3)!$ new and old independent basis functions $F^\pi_{(1,\ldots,n)}$ and $F^\sigma_{(1,\ldots,n)}$:

$$F^\sigma_{(1,\ldots,N)} \quad = \sum_{\pi\in S_{N-3}} (K^{-1})_\pi{}^\sigma \ F^\pi_{(1,\ldots,N)} \ , \qquad \sigma \in S_{N-3} \tag{12.3.41}$$

In this case, the matrix $(K^{-1})_\pi{}^\sigma$ becomes a quadratic $(n-3)! \times (n-3)!$ matrix, see subsection 12.3.5 for explicit examples. Hence, for a given *fixed* color ordering $(1,\ldots,n)$, any function F^σ may be expressed in terms of a basis of $(n-3)!$ functions F^π referring to the *same* color ordering. Equation (12.3.41) is the starting point for generating sets of equation systems involving the kinematics functions F^π (of the *same* color ordering). According to (12.3.39) the field theory limits of the functions F_π^σ are enough to determine the coefficients of these equations.

The relation (12.3.41) should be compared with (12.3.34): While in the first identity one specific color ordered amplitude is decomposed with respect to a set of $(n-3)!$ independent color ordered amplitudes all referring to the *same kinematics*, in the second identity one functions referring to one specific kinematics is decomposed to with respect to a set of $(n-3)!$ independent kinematics functions all referring to the *same color ordering*. Moreover, as we shall show in subsection 12.3.5, for a fixed color ordering $(1,\ldots,n)$ an explicit set of $(n-2)!$ functions $F^{\Pi}_{(1\ldots n)}$, $\Pi\in S_{n-2}$ can be given, which fulfills (12.3.41) – just as a set of $(n-2)!$ SYM amplitudes $\mathcal{A}^{\rm SYM}_{\Pi}$ fulfills (12.3.34) for a fixed kinematics. Since the latter fact is a result of the (imaginary part) field theory monodromy relations, also the relations (12.3.41) should follow from a system of equations for the $(n-2)!$ functions.

Relations between functions $F^{\Pi}_{(1\ldots n)}$ of same color ordering are obtained by either partial fraction decomposition of their integrands or applying partial integration techniques within their $n-3$ integrals. The partial fraction expansion yields linear equations with integer coefficients for the functions F^{Π} – just like the real part of field theory monodromy relations yields linear identities (e.g. subcyclic identities) for the color ordered subamplitudes $\mathcal{A}^{\rm SYM}$. On the other hand, the partial integration techniques applied to the $(n-2)!$ functions F^{Π} provides a system of equations of rank $(n-3)!$, whose solution is given by (12.3.41). Hence, we have found a complete analogy between the monodromy relations equating subamplitudes $A^{\rm SYM}_{\Pi}$ of *different color orderings* $\Pi\in S_{n-2}$ at the *same kinematics* and a system of equations relating functions F^{Π} referring to *different kinematics* $\Pi\in S_{n-2}$ at the *same color ordering.*

To conclude, behind the expression (12.3.33) there are two sets of equations: one set, derived from the monodromy relations (12.3.36) and equating all subamplitudes of different color orderings and another set, derived from the partial fraction decomposition and partial integration relations equating all kinematics functions F^{π}. Both systems are of rank $(n-3)!$ and allow to express all colored ordered subamplitudes in terms of a minimal basis or to express all kinematic functions in terms of a minimal basis.

The color decomposition of the full n point open superstring amplitude $\mathcal{M}[1^{a_1},2^{a_2},\ldots,n^{a_n}]$ can be expressed by $(n-3)!\times(n-3)!$ different functions F^{σ}_{Π} with $(n-3)!$ SYM building blocks $\mathcal{A}^{\rm SYM}_{\sigma}$. After introducing the string generalization of (12.3.34)

$$\mathcal{A}(1_\Pi,\ldots,n_\Pi) \;=\; \sum_{\pi\in S_{n-3}} \mathcal{K}_{\Pi}{}^{\pi}(\alpha')\,\mathcal{A}(1,2_\pi,\ldots,(n-2)_\pi,n-1,n)\;, \tag{12.3.42}$$

with $\Pi\in S_N$ and $\lim_{\alpha'\to 0}\mathcal{K}_{\Pi}{}^{\pi}(\alpha')=K_{\Pi}{}^{\pi}$, we get

$$\mathcal{M}\big[1^{a_1},2^{a_2},\ldots,n^{a_n}\big] \;=\; \sum_{\Pi\in S_{n-1}} \mathrm{Tr}\big\{T^{a_1}\,T^{a_{2_\Pi}}\,\ldots\,T^{a_{n_\Pi}}\big)\sum_{\sigma,\pi\in S_{n-3}}\mathcal{A}^{\rm SYM}_{\sigma}\;\mathcal{K}_{\Pi}{}^{\pi}\;F^{\sigma}_{\pi} \tag{12.3.43}$$

with $F^{\sigma}_{\pi}:=F^{\sigma}_{(1,\pi(2),\ldots,\pi(n-2),n-1,n)}(\alpha')$. In the sum (12.3.43) the same set of basis elements $\mathcal{A}^{\rm SYM}_{\sigma}$ is used for all color orderings Π. This enables to reorganize the color decomposition sum and to interchange the two sums over color and kinematics:

$$\mathcal{M}\big[1^{a_1}, 2^{a_2}, \ldots, n^{a_n}\big] = \sum_{\sigma \in S_{n-3}} \mathcal{A}^{\rm SYM}_\sigma \sum_{\Pi \in S_{n-1}} {\rm Tr}\big\{T^{a_1}\, T^{a_{2_\Pi}} \ldots T^{a_{n_\Pi}}\big\} \sum_{\pi \in S_{n-3}} \mathcal{K}_\Pi{}^\pi \ F^\sigma_\pi \quad (12.3.44)$$

12.3.3 Yang–Mills building blocks $\mathcal{A}^{\rm SYM}$

In $D = 10$, compact expressions for $\mathcal{A}^{\rm SYM}(1, 2_\sigma, \ldots, (n-2)_\sigma, n-1, n)$ are derived in section 11.3 [8]. They can be used to describe the SYM building blocks in (12.3.33). On the other hand, for $D = 4$ compact forms for the SYM building blocks $\mathcal{A}^{\rm SYM}(1, 2_\sigma, \ldots, (n-2)_\sigma, n-1, n)$ in the spinor helicity basis can be looked up in the literature: In the maximal helicity violating (MHV) case the building blocks reduce to the famous Parke Taylor or Berends Giele formula [236,237]. For the general NMHV case the complete expressions for $\mathcal{A}^{\rm SYM}(1, 2_\sigma, \ldots, (n-2)_\sigma, n-1, n)$ can be found in [100].

Since in the sum (12.3.33) the kinematical factors $\mathcal{A}^{\rm SYM}$ and the functions F^σ encoding the string effects are multiplied together, supersymmetric Ward identities established in field theory [195, 196, 197, 198, 199] hold also for the full superstring amplitude, cf. also subsection 5.2.4 based on [150]. At any rate, by extracting SUSY components from the pure spinor result, we can obtain the n point amplitude involving any member of the SYM vector multiplet.

12.3.4 Minimal basis of multiple hypergeometric functions F^σ

The system of $(n-3)!$ multiple hypergeometric functions F^σ appearing in (12.3.33) are given as generalized Euler integrals

$$F^{(23\ldots n-2)}(s_{ij}) = \prod_{j=2}^{n-2} \int_{z_{j-1}}^{1} {\rm d}z_j \left(\prod_{i<j} |z_{ij}|^{-s_{ij}} \right) \left\{ \prod_{k=2}^{n-2} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \right\}, \quad (12.3.45)$$

with permutations $\sigma \in S_{n-3}$ acting on all indices within the curly brace (but not on the integration region). Integration by parts admits to simplify the integrand in (12.3.45). As a result the length of the sum over m becomes shorter for $k > [n/2]$:

$$\begin{aligned} F^{(23\ldots n-2)}(s_{ij}) &= \prod_{j=2}^{n-2} \int_{z_{j-1}}^{1} {\rm d}z_j \left(\prod_{i<j} |z_{ij}|^{-s_{ij}} \right) \\ &\quad \times \left(\prod_{k=2}^{[n/2]} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \right) \left(\prod_{k=[n/2]+1}^{n-2} \ \sum_{p=k+1}^{n-1} \frac{s_{kp}}{z_{kp}} \right). \end{aligned} \quad (12.3.46)$$

Above, $[\ldots]$ denotes the Gauss bracket $[x] = \max_{n \in \mathbb{Z}, n \leq x} n$, which picks out the nearest integer smaller than or equal to its argument.

The result (12.3.33) inherits its gauge invariance from its kinematic building blocks $\mathcal{A}^{\rm SYM}$. Hence, gauge invariance does not impose further restrictions on the $(n-3)!$ functions $F^\sigma_{(1,\ldots,n)}$, which would impose additional relations for them and further reduce the basis. The set (12.3.45) of $(n-3)!$ functions represents a minimal basis for the set of multiple Gaussian hypergeometric functions or Euler integrals appearing at n point and referring to the same color ordering $(1,\ldots n)$ or integration region $z_1 < \ldots < z_n$. Any function of this ordering can be expressed in terms of this basis.

The lowest terms of the α' expansion of the functions F^σ assume the form (see subsection 7.1.5):

$$F^\sigma \quad = \quad \begin{cases} 1 \;+\; \alpha'^2\, p_2^\sigma\, \zeta(2) \;+\; \alpha'^3\, p_3^\sigma\, \zeta(3) \;+\; \ldots & : \; \sigma = (23\ldots n-2) \\ \alpha'^2\, p_2^\sigma\, \zeta(2) \;+\; \alpha'^3\, p_3^\sigma\, \zeta(3) \;+\; \ldots & : \; \sigma \neq (23\ldots n-2) \end{cases} \tag{12.3.47}$$

with some polynomials p_n^σ of degree n in the kinematic invariants. Only the first term of (12.3.33) contributes to the field theory limit of the full n point superstring amplitude, in lines with the derivation (12.3.39) of the basis expansion coefficients in $\mathcal{A}_\sigma^{\rm SYM}$. Note that starting at $n \geq 7$ subsets of F^σ start at even higher order in α', i.e. $p_2^\sigma, \ldots, p_\nu^\sigma = 0$ for some $\nu \geq 2$, see appendix D.2 for further details.

The power series expansions (12.3.47) in α' is such that to each power α'^n, a transcendental function of degree n shows up. More precisely, a set of multizeta values (MZVs) of fixed weight n appears, see section 7.3 for an account on transcendentality properties. The latter are multiplied by a polynomial p_n^σ of degree n in the kinematic invariants with rational coefficients. From (12.3.47) we conclude that the whole pole structure of the amplitude (12.3.33) is encoded in the SYM subamplitudes $\mathcal{A}_\sigma^{\rm SYM}$, while the functions F^σ are finite, i.e. do not have poles in the kinematic invariants. A detailed account on multiple Gaussian hypergeometric functions can be found in [231].

12.3.5 Extended set of multiple hypergeometric functions F^Π

A system of $(n-2)!$ functions F^Π subject to (12.3.41) with $\Pi \in S_{n-2}$ can be given as follows

$$F^{(23\ldots n-1)}(s_{ij}) \quad = \quad \prod_{j=2}^{n-2} \int\limits_{z_{j-1}}^{1} dz_j \, \left(\prod_{i<j} |z_{ij}|^{-s_{ij}} \right) \left\{ \frac{1}{z_{n-1}-z_1} \prod_{k=2}^{n-2} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \right\} , \tag{12.3.48}$$

with permutations $\Pi \in S_{n-2}$ acting on all indices within the curly brace. The set of $(n-2)!$ functions (12.3.48) can be expressed in terms of the basis (12.3.45) as a consequence of the relations (12.3.41). This allows to express $(n-2)! - (n-3)! = (n-3) \times (n-3)!$ functions of (12.3.48) in terms of (12.3.45). This will be demonstrated at some examples.

- $n = 4$ functions

$$
\begin{aligned}
F^{(2)} &= \int_0^1 \mathrm{d}z_2 \left(\prod_{i<j} |z_{ij}|^{-s_{ij}} \right) \frac{s_{12}}{z_{12}} = \frac{\Gamma(1-s_{12})\,\Gamma(1-s_{23})}{\Gamma(1-s_{12}-s_{23})} \\
&= 1 - \zeta(2)\, s_{12}\, s_{23} + \zeta(3)\, s_{12}\, s_{23}\, s_{13} + \dots \,. \qquad (12.3.49)
\end{aligned}
$$

The extended set of two functions consists of (12.3.49) (with $F^{(2)} \equiv F^{(23)}$) and the additional function (12.3.48):

$$
\begin{aligned}
F^{(32)} &= \int_0^1 \mathrm{d}z_2 \left(\prod_{i<j} |z_{ij}|^{-s_{ij}} \right) \frac{1}{z_{21}} \frac{s_{13}}{z_{13}} = \frac{s_{13}}{s_{12}} \frac{\Gamma(1-s_{12})\,\Gamma(1-s_{23})}{\Gamma(1-s_{12}-s_{23})} \\
&= \frac{s_{13}}{s_{12}} - \zeta(2)\, s_{13}\, s_{23} + \zeta(3)\, s_{13}^2\, s_{23} + \dots \,. \qquad (12.3.50)
\end{aligned}
$$

With this extended set of two functions we may explicitly verify the relation (12.3.41). For the new basis $\pi = \{(1,3,2,4)\}$ in (12.3.34) we have

$$
K_{(1,3,2,4)}{}^{(2)} = \frac{s_{12}}{s_{13}} \qquad (12.3.51)
$$

with respect to the reference ordering $(1,2,3,4)$ as a consequence of the field theory relation $\mathcal{A}^{\mathrm{SYM}}(1,3,2,4) = \frac{s_{12}}{s_{13}} \mathcal{A}^{\mathrm{SYM}}(1,2,3,4)$. According to (12.3.41), we can conclude

$$
F^{(32)} = (K^{-1})_{(1,3,2,4)}{}^{(2)}\, F^{(23)} = \frac{s_{13}}{s_{12}}\, F^{(23)} \qquad (12.3.52)
$$

on the level of the functions.

- $n = 5$ functions

The set of two basis functions appearing in (12.3.33) and following from (12.3.45) is:

$$
\begin{aligned}
F^{(23)} &= \int_0^1 \mathrm{d}z_2 \int_{z_2}^1 \mathrm{d}z_3 \left(\prod_{i<j} |z_{ij}|^{-s_{ij}} \right) \frac{s_{12}}{z_{12}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} \right) \\
&= \int_0^1 \mathrm{d}z_2 \int_{z_2}^1 \mathrm{d}z_3 \left(\prod_{i<j} |z_{ij}|^{-s_{ij}} \right) \frac{s_{12}\, s_{34}}{z_{12}\, z_{34}} \\
&= 1 + \zeta(2)\,(s_{12}\, s_{34} - s_{34}\, s_{45} - s_{12}\, s_{51}) + \zeta(3)\,(s_{12}^2\, s_{34} + 2\, s_{12}\, s_{23}\, s_{34} \\
&\qquad + s_{12}\, s_{34}^2 - s_{34}^2\, s_{45} - s_{34}\, s_{45}^2 - s_{12}^2\, s_{51} - s_{12}\, s_{51}^2) + \dots \qquad (12.3.53) \\
F^{(32)} &= \int_0^1 \mathrm{d}z_2 \int_{z_2}^1 \mathrm{d}z_3 \left(\prod_{i<j} |z_{ij}|^{-s_{ij}} \right) \frac{s_{13}}{z_{13}} \left(\frac{s_{12}}{z_{12}} + \frac{s_{32}}{z_{32}} \right) \\
&= \int_0^1 \mathrm{d}z_2 \int_{z_2}^1 \mathrm{d}z_3 \left(\prod_{i<j} |z_{ij}|^{-s_{ij}} \right) \frac{s_{13}\, s_{24}}{z_{13}\, z_{24}} \\
&= \zeta(2)\, s_{13}\, s_{24} + \zeta(3)\, s_{13}\, s_{24}\,(s_{12} + s_{23} + s_{34} + s_{45} + s_{51}) + \dots
\end{aligned}
$$

The extended set of six functions consists of (12.3.53), with

$$F^{(234)} := F^{(23)} , \qquad F^{(324)} := F^{(32)} , \tag{12.3.54}$$

and the additional four functions (12.3.48):

$$\begin{aligned}
F^{(423)} &= \int_0^1 \mathrm{d}z_2 \int_{z_2}^1 \mathrm{d}z_3 \left(\prod_{i<j} |z_{ij}|^{-s_{ij}}\right) \frac{1}{z_{31}} \frac{s_{14}}{z_{14}} \frac{s_{23}}{z_{23}} \\
F^{(243)} &= \int_0^1 \mathrm{d}z_2 \int_{z_2}^1 \mathrm{d}z_3 \left(\prod_{i<j} |z_{ij}|^{-s_{ij}}\right) \frac{1}{z_{31}} \frac{s_{12}}{z_{12}} \frac{s_{34}}{z_{43}} \\
F^{(432)} &= \int_0^1 \mathrm{d}z_2 \int_{z_2}^1 \mathrm{d}z_3 \left(\prod_{i<j} |z_{ij}|^{-s_{ij}}\right) \frac{1}{z_{21}} \frac{s_{14}}{z_{14}} \frac{s_{23}}{z_{32}} \\
F^{(342)} &= \int_0^1 \mathrm{d}z_2 \int_{z_2}^1 \mathrm{d}z_3 \left(\prod_{i<j} |z_{ij}|^{-s_{ij}}\right) \frac{1}{z_{21}} \frac{s_{13}}{z_{13}} \frac{s_{24}}{z_{42}}
\end{aligned} \tag{12.3.55}$$

With this extended set of six functions we may explicitly verify the relation (12.3.41). For the new basis $\pi = \{(1,4,2,3,5),\ (1,2,4,3,5)\}$ in (12.3.34) we have

$$K_\pi{}^\sigma = \frac{1}{s_{14}\ s_{35}} \begin{pmatrix} s_{12}\, s_{34} & -\, s_{13}\,(s_{34}+s_{45}) \\ s_{14}\,(s_{12}-s_{45}) & -\, s_{14}\, s_{13} \end{pmatrix} \tag{12.3.56}$$

w.r.t. the reference basis $\sigma = \{(1,2,3,4,5),\ (1,3,2,4,5)\}$. According to (12.3.41) the following identity indeed holds:

$$\begin{pmatrix} F^{(423)} \\ F^{(243)} \end{pmatrix}^T = \begin{pmatrix} F^{(234)} \\ F^{(324)} \end{pmatrix}^T K^{-1} \tag{12.3.57}$$

On the other hand, for the new basis $\pi = \{(1,4,3,2,5),\ (1,3,4,2,5)\}$ we have

$$K_\pi{}^\sigma = \frac{1}{s_{14}\ s_{35}} \begin{pmatrix} s_{12}\,(s_{14}+s_{34}) & s_{13}\, s_{24} \\ -\, s_{12}\, s_{14} & -\, s_{14}\,(s_{12}+s_{23})\ , \end{pmatrix} \tag{12.3.58}$$

and the following relation can be checked:

$$\begin{pmatrix} F^{(432)} \\ F^{(342)} \end{pmatrix}^T = \begin{pmatrix} F^{(234)} \\ F^{(324)} \end{pmatrix}^T K^{-1} \tag{12.3.59}$$

Hence, the relations (12.3.64) and (D.1.5) allow to express the additional set of functions (12.3.55) in terms of the minimal basis (12.3.53).

- $n = 6$ functions

The set of six basis functions appearing in (12.3.33) and following from (12.3.45) is

$$F^{(234)} = \prod_{j=2}^{4} \int_{z_{j-1}}^1 \mathrm{d}z_j \left(\prod_{i<j} |z_{ij}|^{-s_{ij}}\right) \frac{s_{12}}{z_{12}} \frac{s_{45}}{z_{45}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}}\right)$$

$$
\begin{aligned}
&= 1 - \zeta(2)\,(s_4\,s_5 + s_1\,s_6 - s_4\,t_1 - s_1\,t_3 + t_1\,t_3) - \zeta(3)\,(2s_1s_2s_4 \\
&\quad + 2s_1s_3s_4 + s_4^2s_5 + s_4s_5^2 + s_1^2s_6 + s_1s_6^2 - 2s_3s_4t_1 - s_4^2t_1 \\
&\quad - s_4t_1^2 - 2s_1s_4t_2 - s_1^2t_3 - 2s_1s_2t_3 + t_1^2t_3 - s_1t_3^2 + t_1t_3^2) + \ldots \\
F^{(324)} &= \prod_{j=2}^{4}\int_{z_{j-1}}^{1}\mathrm{d}z_j\,\left(\prod_{i<j}|z_{ij}|^{-s_{ij}}\right)\frac{s_{13}}{z_{13}}\frac{s_{45}}{z_{45}}\left(\frac{s_{12}}{z_{12}}+\frac{s_{32}}{z_{32}}\right) \\
&= \zeta(2)\,s_{13}\,(s_6 - s_2 - t_3) - \zeta(3)\,s_{13}\,(s_1s_2 + s_2^2 - 2s_2s_4 - 2s_3s_4 \\
&\quad - s_1s_6 - s_6^2 + s_2t_1 - s_6t_1 + 2s_4t_2 + s_1t_3 + 2s_2t_3 + t_1t_3 + t_3^2) + \ldots \\
F^{(432)} &= \prod_{j=2}^{4}\int_{z_{j-1}}^{1}\mathrm{d}z_j\,\left(\prod_{i<j}|z_{ij}|^{-s_{ij}}\right)\frac{s_{14}}{z_{14}}\frac{s_{25}}{z_{25}}\left(\frac{s_{13}}{z_{13}}+\frac{s_{43}}{z_{43}}\right) \qquad (12.3.60) \\
&= -\zeta(2)\,s_{14}\,s_{25} + \zeta(3)\,s_{14}\,s_{25}\,(s_2 + s_3 - s_5 - s_6 - t_1 - t_2 - t_3) + \ldots \\
F^{(342)} &= \prod_{j=2}^{4}\int_{z_{j-1}}^{1}\mathrm{d}z_j\,\left(\prod_{i<j}|z_{ij}|^{-s_{ij}}\right)\frac{s_{13}}{z_{13}}\frac{s_{25}}{z_{25}}\left(\frac{s_{14}}{z_{14}}+\frac{s_{34}}{z_{34}}\right) \\
&= \zeta(2)\,s_{13}\,s_{25} + \zeta(3)\,s_{13}\,s_{25}\,(s_1 - s_2 - 2s_3 + s_6 + t_1 + 2t_2 + t_3) + \ldots \\
F^{(423)} &= \prod_{j=2}^{4}\int_{z_{j-1}}^{1}\mathrm{d}z_j\,\left(\prod_{i<j}|z_{ij}|^{-s_{ij}}\right)\frac{s_{14}}{z_{14}}\frac{s_{35}}{z_{35}}\left(\frac{s_{12}}{z_{12}}+\frac{s_{42}}{z_{42}}\right) \\
&= \zeta(2)\,s_{14}\,s_{35} - \zeta(3)\,s_{14}\,s_{35}\,(2s_2 + s_3 - s_4 - s_5 - t_1 - 2t_2 - t_3) + \ldots \\
F^{(243)} &= \prod_{j=2}^{4}\int_{z_{j-1}}^{1}\mathrm{d}z_j\,\left(\prod_{i<j}|z_{ij}|^{-s_{ij}}\right)\frac{s_{12}}{z_{12}}\frac{s_{35}}{z_{35}}\left(\frac{s_{14}}{z_{14}}+\frac{s_{24}}{z_{24}}\right) \\
&= \zeta(2)\,(s_5 - s_3 - t_1)\;s_{35} + \zeta(3)\,s_{35}\,(2s_1s_2 + 2s_1s_3 - s_3^2 - s_3s_4 \\
&\quad + s_4s_5 + s_5^2 - 2s_3t_1 - s_4t_1 - t_1^2 - 2s_1t_2 - s_3t_3 + s_5t_3 - t_1t_3) + \ldots
\end{aligned}
$$

where we use Mandelstam variables $s_i \equiv s_{i,i+1}$ and $t_i \equiv s_{i,i+1,i+2}$ subject to cyclic identification $k_{i+n} \equiv k_i$ of the external momenta.

The extended set of 24 functions consists of (12.3.60) with

$$
\begin{aligned}
F^{(2345)} &:= F^{(234)}\,, \qquad F^{(3245)} := F^{(324)}\,, \qquad F^{(4325)} := F^{(432)} \qquad (12.3.61) \\
F^{(3425)} &:= F^{(342)}\,, \qquad F^{(4235)} := F^{(423)}\,, \qquad F^{(2435)} := F^{(243)}\,,
\end{aligned}
$$

and additional 18 functions of type (12.3.48), which are listed in appendix D.1.

Switching from the reference basis $\{(1,\sigma(2),\sigma(3),\sigma(4),5,6),\ \sigma\in S_3\}$ to the new one $\pi \in \{(1,2,3,5,4,6),\ (1,3,2,5,4,6),\ (1,5,3,2,4,6),\ (1,3,5,2,4,6), (1,5,2,3,4,6),\ (1,2,5,3,4,6)\}$ involves the transformation matrix

$$
K_\pi{}^\sigma = s_{46}^{-1}
$$

$$\begin{pmatrix} s_5 - t_1 & 0 & 0 & 0 & s_{14} & -d_1 \\ 0 & s_5 - t_1 & s_{14} & s_3 + s_{14} & 0 & 0 \\ \frac{s_1 s_4 d_0}{s_{15} s_{246}} & \frac{s_4 s_{13}(s_{25}-s_{46})}{s_{15} s_{246}} & \frac{-s_{13} s_{14} s_{25}}{s_{15} s_{246}} & \frac{-s_{13} s_{25}(s_3+s_{14})}{s_{15} s_{246}} & \frac{s_{14} d_0 (s_{46}-s_1)}{s_{15} s_{246}} & \frac{s_1 (s_3+s_4) d_0}{s_{15} s_{246}} \\ \frac{-s_1 s_4}{s_{246}} & \frac{-s_4 (s_1+s_2)}{s_{246}} & \frac{s_{14} d_4}{s_{246}} & \frac{(s_{14}+s_3) d_4}{s_{246}} & \frac{s_{14}(s_1-s_{46})}{s_{246}} & \frac{-s_1 (s_3+s_4)}{s_{246}} \\ \frac{s_1 s_4 (s_{35}-s_{46})}{s_{15} s_{125}} & \frac{s_4 s_{13} d_3}{s_{15} s_{125}} & \frac{(s_{46}-s_{13}) d_3 s_{14}}{s_{15} s_{125}} & \frac{(s_4+s_{24}) s_{13} d_3}{s_{15} s_{125}} & \frac{-s_1 s_{14} s_{35}}{s_{15} s_{125}} & \frac{s_1 s_{35} d_1}{s_{15} s_{125}} \\ \frac{s_4 (s_1-t_1)}{s_{125}} & \frac{-s_4 s_{13}}{s_{125}} & \frac{s_{14}(s_{13}-s_{46})}{s_{125}} & \frac{-s_{13}(s_4+s_{24})}{s_{125}} & \frac{-s_{14} d_2}{s_{125}} & \frac{d_1 d_2}{s_{125}} \end{pmatrix} \tag{12.3.62}$$

with shorthands

$$\begin{aligned} d_1 &= s_3 - s_5 + t_1 \ , \qquad & d_2 &= s_1 - s_4 - s_5 \\ d_3 &= s_3 - s_5 - t_3 \ , \qquad & d_4 &= s_4 + s_5 - s_{13} \\ d_0 &= s_{15} + s_{35} \ . \end{aligned} \tag{12.3.63}$$

According to (12.3.41), the functions F^{σ}_{Π} at fixed Π transform with the inverse matrix

$$\begin{pmatrix} F^{(2354)} \\ F^{(3254)} \\ F^{(5324)} \\ F^{(3524)} \\ F^{(5234)} \\ F^{(2534)} \end{pmatrix}^T = \begin{pmatrix} F^{(2345)} \\ F^{(3245)} \\ F^{(4325)} \\ F^{(3425)} \\ F^{(4235)} \\ F^{(2435)} \end{pmatrix}^T K^{-1} \ . \tag{12.3.64}$$

The other two sets of basis $\pi \in \{(1, \sigma(2), \sigma(4), \sigma(5), 3, 6)\}$ and $\pi \in \{(1, \sigma(3), \sigma(4), \sigma(5), 2, 6)\}$ (with $\sigma \in S_3$) as well as their relations (12.3.41) to the reference basis $\{(1, \sigma(2), \sigma(3), \sigma(4), 5, 6)\}$ are displayed in appendix D.1.

12.4 Properties of the full amplitude

The factorization properties of tree level amplitudes are well studied in field theory [264]. These properties represent an important test of our result. An explicit check of cyclic invariance is still under research and will be written in [11]. The extended set of functions introduced in subsection 12.3.5 is believed to play a key role.

12.4.1 Soft limit

According to the SUSY argument in subsection 12.3.3, it is sufficient to focus on the n gluon amplitude. We consider the limit $k_{n-2} \to 0$. In this limit, the amplitude (12.3.33) behaves as[2]:

$$\mathcal{A}(1,2,\ldots,n) \quad \to \quad \left(\frac{\xi\, k_{n-2}}{k_{n-2}\, k} \; - \; \frac{\xi\, k_{n-3}}{k_{n-3}\, k} \right) \mathcal{A}(1,2,\ldots,n-1) \tag{12.4.65}$$

This can be proven by considering the limits of the individual summands of (12.3.33):

$$\begin{aligned}
&(i) \quad \sigma \in S_{n-4} \text{ with } (n-3)_\sigma = n-3 : \\
&\mathcal{A}^{\rm SYM}(1,2_\sigma,\ldots,(n-3)_\sigma,n-2,n-1,n)\ F^\sigma(\alpha') \\
&\to \left(\frac{\xi k_{n-2}}{k_{n-2}k} - \frac{\xi k_{n-3}}{k_{n-3}k} \right) \mathcal{A}^{\rm SYM}(1,2_\sigma,\ldots,(n-3)_\sigma,n-2,n-1)\ \tilde{F}^\sigma(\alpha')\ , \\
&(ii) \quad \sigma \in S_{n-4} \text{ with } (n-3)_\sigma \neq n-3 : \\
&\mathcal{A}^{\rm SYM}(1,2_\sigma,\ldots,(n-3)_\sigma,n-2,n-1,n)\ F^\sigma(\alpha') \\
&\to \left(\frac{\xi k_{n-2}}{k_{n-2}k} - \frac{\xi k_{(n-3)_\sigma}}{k_{(n-3)_\sigma}k} \right) \mathcal{A}^{\rm SYM}(1,2_\sigma,\ldots,(n-3)_\sigma,n-2,n-1)\ \tilde{F}^\sigma(\alpha') \\
&(iii) \quad \sigma \in S_{n-4} \text{ with } n-3 \in \{2_\sigma,\ldots,i_\sigma\} \text{ and } i=2,\ldots,n-4, \text{ i.e.} (n-3)_\sigma \neq n-3 : \\
&\mathcal{A}^{\rm SYM}(1,2_\sigma,\ldots,i_\sigma,n-2,(i+1)_\sigma,\ldots,(n-3)_\sigma,n-1,n)\ F^\sigma(\alpha') \\
&\to \left(\frac{\xi k_{(i+1)_\sigma}}{k_{(i+1)_\sigma}k} - \frac{\xi k_{i_\sigma}}{k_{i_\sigma}k} \right) \mathcal{A}^{\rm SYM}(1,2_\sigma,\ldots,(n-3)_\sigma,n-2,n-1)\ \tilde{F}^\sigma(\alpha') \\
&(iv) \quad \sigma \in S_{n-4} : \ \ \mathcal{A}^{\rm SYM}(1,n-2,2_\sigma,\ldots,(n-3)_\sigma,n-1,n)\ F^\sigma(\alpha') \quad \to \quad 0 \\
&(v) \quad \sigma \in S_{n-4} \text{ with } n-3 \in \{(i+1)_\sigma,\ldots,(n-3)_\sigma\} \text{ and } i=2,\ldots,n-4 : \\
&\mathcal{A}^{\rm SYM}(1,2_\sigma,\ldots,i_\sigma,n-2,(i+1)_\sigma,\ldots,(n-3)_\sigma,n-1,n)\ F^\sigma(\alpha') \quad \to \quad 0\ .
\end{aligned} \tag{12.4.66}$$

Above the functions $\tilde{F}^\sigma$ refer to the $n-1$ point amplitude. While the $(n-5)!$ summands of case (i) already have the right form (12.4.65) and give rise to $(n-5)!$ terms of the $n-1$ point amplitude (12.3.33), the remaining nonvanishing limits (ii) and (iii) for a given $\sigma \in S_{n-4}$ with $(n-3)_\sigma \neq n-3$ conspire to comprise the remaining $(n-5)(n-5)!$ terms of (12.3.33) thanks to the relation:

$$\left(\frac{\xi k_{n-2}}{k_{n-2}k} - \frac{\xi k_{(n-3)_\sigma}}{k_{(n-3)_\sigma}k} \right) + \sum_{\substack{i=2 \\ n-3\in\{2_\sigma,\ldots,i_\sigma\}}}^{n-4} \left(\frac{\xi k_{(i+1)_\sigma}}{k_{(i+1)_\sigma}k} - \frac{\xi k_{i_\sigma}}{k_{i_\sigma}k} \right) = \left(\frac{\xi k_{n-2}}{k_{n-2}k} - \frac{\xi k_{n-3}}{k_{n-3}k} \right) . \tag{12.4.67}$$

The remaining $\frac{1}{2}(n-3)!$ terms of the cases (iv) and (v) do not contribute in the soft limit.

[2] The vectors ξ and k refer to the transverse polarization and momentum of the soft–gluon, resepctively. One could also express the kinematic dependent soft- or eikonal factor in the $D=4$ spinor helicity basis [264, 261]

12.4.2 Collinear limit

Again, according to subsection 12.3.3 it is sufficient to focus on the n–gluon amplitude. The collinear limit is defined as two adjacent external momenta k_i and k_{i+1}, with $i+1$ mod n, becoming parallel. Due to cyclic symmetry, these can be chosen as k_{n-3} and k_{n-2}, with k_{n-3} carrying the fraction x of the combined momentum $k_{n-3}+k_{n-2} \to k_{n-3}$. Formally,

$$k_{n-3} \quad \to \quad x\ k_{n-3}\ , \qquad k_{n-2} \quad \to \quad (1-x)\ k_{n-3}\ , \tag{12.4.68}$$

where the momenta appearing in the limits describe the scattering amplitude of $n-1$ gluons. In this limit the amplitude (12.3.33) behaves as[3]

$$\mathcal{A}(1,2,\ldots,n) \quad \to \quad \frac{V^i}{k_{n-3}k_{n-2}}\,\frac{\partial}{\partial \xi^i_{n-3}}\ \mathcal{A}(1,2,\ldots,n-1)\ , \tag{12.4.69}$$

with the three gluon vertex $V^i = (\xi_{n-3}\xi_{n-2})(k^i_{n-2}-k^i_{n-3}) + 2(\xi_{n-2}k_{n-3})\xi^i_{n-3} - 2(\xi_{n-3}k_{n-2})\xi^i_{n-2}$. This can be proven by considering the limits of the individual summands of (12.3.33). First, if the two states $n-3$ and $n-2$ are not neighbors, we have:

$$\begin{aligned}
&(i) \quad \sigma \in S_{n-4} \text{ with } 2_\sigma \neq n-3: \quad \mathcal{A}^{\rm SYM}(1,n-2,2_\sigma,\ldots,(n-3)_\sigma,n-1,n) \quad \to \quad 0\\
&(ii) \quad \sigma \in S_{n-4} \text{ with } i_\sigma,(i+1)_\sigma \neq n-3 \text{ and } i=2,\ldots,n-4:\\
&\qquad \mathcal{A}^{\rm SYM}(1,2_\sigma,\ldots,i_\sigma,n-2,(i+1)_\sigma,\ldots,(n-3)_\sigma,n-1,n) \quad \to \quad 0
\end{aligned} \tag{12.4.70}$$

On the other hand, the remaining $2(n-4)!$ terms of (12.3.33) pair up into $(n-4)!$ tuples $(\sigma,\tilde\sigma)$ each giving rise to one of the $(n-4)!$ terms of the $n-1$ point amplitude (12.3.33):

$$\begin{aligned}
&\sigma,\tilde\sigma \in S_{n-4} \text{ with } i_\sigma = (i+1)_{\tilde\sigma} = n-3 \text{ and } i=2,\ldots,n-4:\\
&\mathcal{A}^{\rm SYM}(1,2_\sigma,\ldots,i_\sigma,n-2,(i+1)_\sigma,\ldots,(n-3)_\sigma,n-1,n)\ F^\sigma(\alpha')\\
&+\mathcal{A}^{\rm SYM}(1,2_{\tilde\sigma},\ldots,i_{\tilde\sigma},n-2,(i+1)_{\tilde\sigma},\ldots,(n-3)_{\tilde\sigma},n-1,n)\ F^{\tilde\sigma}(\alpha')\\
&\to \quad \frac{V^i}{k_{n-3}k_{n-2}}\,\frac{\partial}{\partial \xi^i_{n-3}}\ \mathcal{A}^{\rm SYM}(1,2_\sigma,\ldots,(n-3)_\sigma,n-2,n-1)\ F^\sigma(\alpha')
\end{aligned} \tag{12.4.71}$$

Note, that in the above combination the x–dependent parts of the two functions F^σ and $F^{\tilde\sigma}$, which stems from the limit (12.4.68), add up to zero.

12.4.3 Cyclic invariance

While the canonically ordered SYM constituent $\mathcal{A}^{\rm SYM}(1,2,\ldots,n)$ within (12.3.33) is invariant under cyclic transformations of its labels $i \to i+1$ mod n, all others transform nontrivially.

[3] One could also express the kinematic dependent factor as splitting amplitude written e.g. in the $D=4$ spinor helicity basis [264, 261].

More precisely, the set $\left\{\mathcal{A}^{\rm SYM}(1, 2_\sigma, \ldots, (n-2)_\sigma, n-1, n),\ \sigma \in S_{n-3}\right\}$ is mapped to the set $\left\{\mathcal{A}^{\rm SYM}(1, 2, 3_\sigma, \ldots, (n-2)_\sigma, (n-1)_\sigma, n),\ \sigma \in S_{n-3}\right\}$ by virtue of the cyclic properties of the $\mathcal{A}^{\rm SYM}$. The latter set belongs to the extended S_{n-2} family from subsection 12.3.5, $\left\{\mathcal{A}^{\rm SYM}(1, 2_\Pi, 3_\Pi, \ldots, (n-2)_\Pi, (n-1)_\Pi, n),\ \Pi \in S_{n-2}\right\}$ which can be expanded in terms of the original basis of $\mathcal{A}^{\rm SYM}(1, 2_\sigma, \ldots, (n-2)_\sigma, n-1, n)$ according to (12.3.34).

The cyclic transformation properties of the minimal basis functions F^σ are such that the change of $\mathcal{A}^{\rm SYM}_\sigma$ into $\mathcal{A}^{\rm SYM}_{\Pi(\sigma)} = \sum_{\pi \in S_{n-3}} K_{\Pi(\sigma)}{}^\pi \mathcal{A}^{\rm SYM}_\pi$ is compensated:

$$F^\sigma \Big|_{k_i \to k_{i+1}} \ = \ F^{\Pi(\sigma)} \ = \ \sum_{\rho \in S_{n-3}} (K^{-1})_\rho{}^{\Pi(\sigma)}\, F^\rho \tag{12.4.72}$$

The map $\Pi(\sigma)$ is defined by $(2_{\Pi(\sigma)}, \ldots, (n-1)_{\Pi(\sigma)}) = (2, 2_\sigma + 1, \ldots, (N-2)_\sigma + 1)$.

Chapter 13

Explicit BCJ numerators from pure spinors

In this chapter, we will extract lessons on scattering amplitudes in gauge theories from the pure spinor approach to the superstring. This is another example for the tight connections between superstring theory and field theories emerging in the low energy limit, in particular for the fruitful interplay on the level of the S matrix.

In section 5.5, we have introduced a parametrization of gauge theory tree amplitudes

$$\mathcal{M}^{\text{SYM}}\big[1^{a_1}, 2^{a_2} \ldots, n^{a_n}\big] \quad = \quad \sum_i \frac{c_i \, n_i}{\prod_{\alpha_i} s_{\alpha_i}} \tag{13.0.1}$$

where color- and kinematic degrees of freedom enter on the same footing. In particular, the kinematic numerators n_i can be brought into a form such that they satisfy a dual Jacobi identity $n_i + n_j - n_k = 0$ whenever $c_i + c_j - c_k = 0$ on the color side. However, the kinematic identity depends on the management of contact terms – the four point interactions of gauge fields introduce an ambiguity in defining the n_i, and generic choices cause the Jacobi identity $n_i + n_j - n_k = 0$ to fail.

The pure spinor formalism resolves these contact term ambiguities: Given the presentation (12.1.13) of the superstring tree amplitude in terms of $(n-2)!$ superfield expressions like $\langle T_{12\ldots p} T_{n-1,n-2\ldots p+1} V_n \rangle$, the low energy limit naturally expresses each n_i as a linear combination of $(n-2)!$ basis kinematics $\langle T_{12\ldots p} T_{n-1,n-2\ldots p+1} V_n \rangle$. We will show that they are guaranteed to satisfy all the dual Jacobi relations (5.5.62). It is straightforward to dimensionally reduce the superfield components to $D = 4$, and the bosonic parts describe gluon scattering independently on the existence of supersymmetries.

There is strong motivation in field theory for having explicit expressions for kinematic numerators n_i subject to $n_i + n_j - n_k = 0$. Firstly, they can be recycled from tree level to loop amplitudes by means of the unitarity method [313, 314, 78]. Jacobi identities then relate non-planar loop diagrams to the much better-understood planar sector. Secondly, the duality between color and kinematics suggests a natural reorganization of gravity tree amplitudes. As explained in section 5.6, they are built as double copies of gauge theory subamplitues [213] using KLT relations. If the Jacobi identities hold within one of the gauge theory sectors, then the numerator of each kinematic pole in the gravity amplitude is given by a square $|n_i|^2$. Ultimately, the $c_i \leftrightarrow n_i$ duality is helpful for studying the ultraviolet properties of gravity theories at higher loops.

13.1 Constructing supersymmetric BCJ numerators

The low energy limit of the pure spinor string amplitude (12.1.13) bypasses the contact term ambiguity because its BRST cohomology organization in terms of $\langle T_{12\ldots p} T_{n-1,n-2\ldots p+1} V_n \rangle$ naturally absorbs these contact terms. According to our discussion in subsection 12.1.3, they arise from the double poles in the OPE of two integrated vertices and complete the single pole OPE residues $L_{2131\ldots p1}$ to BRST building blocks $T_{123\ldots p}$.

The purpose of this section is to explain the construction of the individual BCJ numerators n_i out of the BRST ingredients $\langle T_{12\ldots p} T_{n-1,n-2\ldots p+1} V_n \rangle$ and to rigorously justify why they satisfy the dual Jacobi relations.

13.1.1 The minimal basis of BCJ numerators

As a starting point of the construction of kinematic numerators, let us give a refined version of the pure spinor superstring amplitude (12.1.13) where the α' dependence is explicitly displayed:

$$\begin{aligned}\mathcal{A}(1, 2_\sigma, \ldots, (n-1)_\sigma, n; \alpha') \quad &= \quad (2\alpha')^{n-3} \prod_{i=2}^{n-2} \int_{\mathcal{I}_\sigma} \mathrm{d}z_i \prod_{j<k} |z_{jk}|^{-2\alpha' s_{jk}} \\ \Bigg\langle \sum_{j=1}^{n-2} &\frac{T_{12\ldots j}\, T_{n-1,n-2\ldots j+1}\, V_n}{(z_{12}\, z_{23}\, \ldots\, z_{p-1,p})\,(z_{n-1,n-2}\, z_{n-2,n-3}\, \ldots\, z_{j+2,j+1})} \;+\; \mathcal{P}(2, 3, \ldots, n-2) \Bigg\rangle \end{aligned} \tag{13.1.2}$$

The ordering $\sigma \in S_{n-2}$ of the external legs is reflected in the integration region $\mathcal{I}_\sigma$ for the worldsheet positions $z_2, z_3, \ldots, z_{n-2}$, the remaining ones are fixed as $(z_1, z_{n-1}, z_n) = (0, 1, \infty)$ by $SL(2, \mathbb{R})$ invariance of the disk worldsheet. More precisely, $\mathcal{I}_\sigma$ is defined such that only those z_i which respect the ordering

$$0 \;=\; z_1 \;\leq\; z_{2_\sigma} \;\leq\; z_{3_\sigma} \;\leq\; \ldots \;\leq\; z_{(n-2)_\sigma} \;\leq\; z_{(n-1)_\sigma} \;\leq\; \infty \tag{13.1.3}$$

are integrated over. The $\alpha' \to 0$ limit of (13.1.2) extracts propagators of cubic field theory diagrams from the $n-3$ fold worldsheet integrals. Adjusting the integration region $\mathcal{I}_\sigma$ to the color ordering makes sure that the integrals in $\mathcal{A}(1, 2_\sigma, \dots, (n-1)_\sigma, n; \alpha')$ only generate those pole channels which appear in the corresponding field theory amplitude $\mathcal{A}^{\text{SYM}}(1, 2_\sigma, \dots, (n-1)_\sigma, n)$. A method to efficiently extract the field theory limit of a general n point integral was explained in section 7.2.

The set of $(n-2)!$ color ordered amplitudes $\mathcal{A}^{\text{SYM}}(1, 2_\sigma, \dots, (n-1)_\sigma, n)$ with $\sigma \in S_{n-2}$ and legs n and 1 fixed is sufficient to involve all of the $(2n-5)!!$ cubic field theory diagrams at least once. In subsection 5.5.4, we have introduced this S_{n-2} family as the KK basis because the Kleiss-Kuijf relations (5.5.65) express all the other SYM subamplitudes as sums over several $\mathcal{A}^{\text{SYM}}(1, 2_\sigma, \dots, (n-1)_\sigma, n)$ with coefficients ± 1.

The remarkable property of (13.1.2) in view of the BCJ organization is the number of independent superfield kinematics $\langle T_{12\dots j} T_{n-1,n-2,\dots,j+1} V_n \rangle$. Each of the $n-2$ terms in the j sum of (13.1.2) involves $(n-3)!$ permutations of $\langle T_{12\dots j} T_{n-1,n-2,\dots,j+1} V_n \rangle$ in the legs $(2, 3, \dots, n-2)$ such that we have $(n-2)!$ kinematic packages in total. The worldsheet integrand remains the same for any color ordering, only the integration region $\mathcal{I}_\sigma$ changes between the subamplitudes.

Hence, the $(n-2)!$ basis kinematics $\langle T_{12\dots j} T_{n-1,n-2,\dots,j+1} V_n \rangle$ combine to the kinematic factors n_i for any color stripped superstring amplitude, and in particular, they generate BCJ numerators for all the $(2n-5)!!$ pole channels of the color dressed field theory amplitude. Their coefficients are determined by the pole structure of the integrals in the corresponding integration region $\mathcal{I}_\sigma$ which is specified by the color ordering of $\mathcal{A}(1, 2_\sigma, 3_\sigma, \dots, (n-1)_\sigma, n; \alpha')$.

As we have argued in section 5.5, having a set of no more than $(n-2)!$ independent numerators is necessary for imposing the Jacobi-like identities (dual to color factors) on the $(2n-5)!!$ numerators of the pole channels in various subamplitudes. In the next paragraph we explain why the number $(n-2)!$ of kinematics in (13.1.2) is also sufficient to satisfy the Jacobi relations, i.e. to make the appropriate triplets (n_i, n_j, n_k) vanish.

13.1.2 The vanishing of numerator triplets

The fact that only $(n-2)!$ BCJ numerators can be linearly independent implies the existence of as many linear homogeneous relations between the n_i as there are Jacobi identities. Since the field theory limits of the integrals in (13.1.2) involve no other coefficients than 0 and ± 1 for the propagators, these relations must be of the form

$$n_{i_1} \ \pm \ n_{i_2} \ \mp \ n_{i_3} \ \pm \ \dots \ \mp \ n_{i_{p-1}} \ \pm \ n_{i_p} \ = \ 0, \tag{13.1.4}$$

with a so far unspecified number p of terms. In order to show that they can always be arranged into vanishing statements for triplets $n_{i_1} \pm n_{i_2} \mp n_{i_3} = 0$ one has to make use of the BCJ relations (5.5.69) between field theory subamplitudes which allow to reduce the KK subamplitudes to a basis of $(n-3)!$ independent ones. We have explained in subsection 5.5.6 that they can be recast in the form $\sum_i \frac{n_{i_k}+n_{i_l}+n_{i_m}}{\prod_{\alpha_i}^{n-4} s_{\alpha_i}} = 0$ involving $2^{n-3}(2n-7)!!(n-3)/(n-2)!$ numerator triplets $(n_{i_k}, n_{i_l}, n_{i_m})$ that vanish individually if the dual Jacobi relations hold. The important implication is the vanishing of $n_{i_k}+n_{i_l}+n_{i_m}$ at the residue of the $n-4$ poles s_{α_i} common to the n_{i_k}, n_{i_l} and n_{i_m} channels, independent on the assignment of contact terms to the numerators.

Suppose the linear dependences (13.1.4) failed to make Jacobi triplets of BCJ numerators vanish, i.e. $p > 3$, then the BCJ relations would involve terms

$$\frac{n_{i_1}+n_{i_2}+n_{i_3}}{\prod_{\alpha_i}^{n-4} s_{\alpha_i}} \quad = \quad - \frac{\sum_{j=4}^{p} n_{i_j}}{\prod_{\alpha_i}^{n-4} s_{\alpha_i}} \tag{13.1.5}$$

where each noncancelling $n_{i_{j>3}}$ is multiplied with at least one propagator outside its channels. This is clearly incompatible with the BCJ relations because there will remain contributions with a specific set of $n-4$ poles from each term like that[1].

The conclusion is that the $(2n-5)!! - (n-2)!$ independent relations (13.1.4) can always be brought into a form that reproduces all the dual Jacobi identities for the kinematic numerators n_i. If this was not the case, inconsistencies would arise in the BCJ relations (5.5.69) or (5.5.70) between color ordered field theory amplitudes. Hence, the number $(n-2)!$ of kinematics in (13.1.2) guarantees that all the n_i constructed from the $\alpha' \to 0$ limit of the integrals over $\mathcal{I}_\sigma$ satisfy the Jacobi-like relations $n_{i_k} + n_{i_l} + n_{i_m} = 0$.

13.1.3 The propagator matrix

The worldsheet integrand of (13.1.2) suggests to label the $(n-2)!$ basis kinematics in an n-point amplitude by an S_{n-3} permutation σ and an integer $l = 1, \ldots, n-2$:

$$\mathcal{K}_\sigma^l \quad := \quad \langle \, T_{12_\sigma 3_\sigma \ldots l_\sigma} \, T_{n-1,(n-2)_\sigma \ldots (l+1)_\sigma} \, V_n \, \rangle \ , \qquad l = 1, \ldots, n-2 \ , \qquad \sigma \in S_{n-3} \tag{13.1.6}$$

This makes sure that the residual S_{n-3} relabelling symmetry stays visible in the KK basis of the field theory limit. As we have mentioned before, knowing all the KK subamplitudes $\mathcal{A}^{\rm SYM}(1, 2_\rho, \ldots, (n-1)_\rho, n)$ with $\rho \in S_{n-2}$ is sufficient to address each channel and to thereby identify all the $(2n-5)!!$ numerators n_i. The superstring amplitude (13.1.2) provides a general

[1]The claimed incompatibility rests on the linear independence of the $(n-2)!$ factorial basis numerators. We wish to thank Henrik Johansson for pointing out that a loophole would arise otherwise.

prescription to construct these KK subamplitudes in terms of the basis kinematics $\mathcal{K}^l_\sigma$ defined by (13.1.6):

$$\mathcal{A}^{\rm SYM}(1,2_\rho,\ldots,(n-1)_\rho,n) \;=\; \sum_{l=1}^{n-2}\sum_{\sigma\in S_{n-3}} P_\rho{}^{(l,\sigma)}\,\mathcal{K}^l_\sigma \tag{13.1.7}$$

This introduces a $(n-2)!\times(n-2)!$ matrix $P_\rho{}^{(l,\sigma)}$ of kinematic poles whose entries are determined by the integral of the worldsheet polynomial associated with (l,σ) over the region $\mathcal{I}_\rho$:

$$P_\rho{}^{(l,\sigma)} \;=\; \lim_{\alpha'\to 0}\,(2\alpha')^{n-3}\prod_{i=2}^{n-2}\int_{\mathcal{I}_\rho}\mathrm{d}z_i\;\frac{\prod_{j<k}|z_{jk}|^{-2\alpha' s_{jk}}}{\big(z_{12_\sigma}\,z_{2_\sigma 3_\sigma}\,\ldots\,z_{(l-1)_\sigma l_\sigma}\big)\big(z_{n-1,(n-2)_\sigma}\,\ldots\,z_{(l+2)_\sigma,(l+1)_\sigma}\big)} \tag{13.1.8}$$

These $\alpha'\to 0$ limits can be straightforwardly evaluated using the methods of section 7.2.

The idea of introducing an $(n-2)!$ vector of KK amplitudes and relating it to $(n-2)!$ independent numerators by a square matrix already appeared in [315]. In our situation, the basis (13.1.6) of numerators is set by the superstring computation, and our $P_\rho{}^{(l,\sigma)}$ matrix is a specialization of the propagator matrix M in this reference to the pure spinor basis $\{\mathcal{K}^l_\sigma,\ l=1,2,\ldots,n-2,\ \sigma\in S_{n-3}\}$ of kinematics. The BCJ relations (5.5.69) between KK subamplitudes imply that the $(n-2)!\times(n-2)!$ propagator matrices M or $P_\rho{}^{(l,\sigma)}$ have nonmaximal rank $(n-3)!$.

Not all the entries of the pole matrix $P_\rho{}^{(l,\sigma)}$ have to be computed separately. The following trick relates many of them by relabelling and thereby reduces the computational effort on the way towards explicit BCJ numerators: Exclude the leg $n-1$ from the $\rho\in S_{n-2}$ permutations specifying the KK subamplitudes. They then fall into $n-2$ classes $\mathcal{A}^{\rm SYM}(1,2_\sigma,\ldots,j_\sigma,n-1,(j+1)_\sigma,\ldots,(n-2)_\sigma,n)$ with $j-1$ legs between 1 and $n-1$ and another $n-2-j$ legs between $n-1$ and n (where $j=1,2,\ldots,n-2$). It is sufficient to compute one representative of the $n-2$ classes of KK amplitudes, the others then follow as S_{n-3} permutations in $2,3,\ldots,n-2$. More precisely, once the first $n-2$ columns of (13.1.8) with $\rho=(2,3,\ldots,j,n-1,j+1,\ldots,n-2)$ and $j=1,2,\ldots,n-2$ are known, then the others follow from

$$P_{\rho=(2_\sigma,3_\sigma,\ldots,j_\sigma,n-1,(j+1)_\sigma,\ldots,(n-2)_\sigma)}{}^{(l,\tau)} = P_{\rho=(2,3,\ldots,j,n-1,j+1,\ldots,n-2)}{}^{(l,\sigma^{-1}\circ\tau)}\Big|_{k_i\mapsto k_{\sigma(i)}} \tag{13.1.9}$$

where the composition of $\sigma^{-1},\tau\in S_{n-3}$ is to be understood as $(\sigma^{-1}\circ\tau)(i)=\sigma^{-1}(\tau(i))$. The proof of (13.1.9) is a simple matter of renaming worldsheet integration variables in (13.1.8).

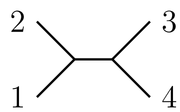
This relabelling strategy reduces the number of independent evaluations of (13.1.8) from $(n-2)!\times(n-2)!$ down to $(n-2)\times(n-2)!$, i.e. the work at this step is reduced by a factor of $(n-3)!$. But the success of these S_{n-3} relabellings does not extend to the leg $n-1$. The $P_\rho{}^{(l,\sigma)}$ entries for the $n-2$ classes of KK subamplitudes $\mathcal{A}^{\rm SYM}(1,2_\sigma,\ldots,j_\sigma,n-1,(j+1)_\sigma,\ldots,(n-2)_\sigma,n)$ with $j=1,2,\ldots,n-2$ have a different number and structure of terms as j varies. This will become more obvious from the examples in the next section 13.2. As a consequence, the n_i appearing in these subamplitudes involve more basic kinematics $\mathcal{K}^l_\sigma$ and do not follow from other BCJ numerators by relabelling.

13.1.4 Jacobi friendly notation

At higher points it is not convenient to denote the BCJ numerators sequentially by n_i for $i = 1, 2, \ldots (2n-5)!!$. Already the presentation of the fifteen numerators in the five point KK basis (5.5.63) suffers from the arbitrary assignment of numbers 1 to 15 to the cubic diagrams. It is not at all obvious from their labels which of them combine to form the Jacobi triplets (5.5.64).

Instead, we will use a notation introduced by [315] which reflects the structure of the diagram and allows the associated propagators to be reconstructed. More importantly, it makes the dual structure constant contraction available from which one can infer the symmetry properties $f^{abc} = -f^{bac}$ and the Jacobi identities $f^{b[a_1 a_2} f^{a_3]bc} = 0$. Let us clarify these properties by explicit examples:

The four point amplitude $\mathcal{M}^{\rm SYM}[1^{a_1}, 2^{a_2}, 3^{a_3}, 4^{a_4}]$ encompasses three diagrams of the form



$$\sim \ \frac{1}{s_{12}} \, f^{a_1 a_2 b} \, f^{b a_3 a_4} \, n[12, 34] \ = \ \frac{c_s \, n_s}{s}$$

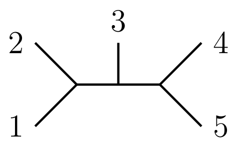
where $n[12, 34]$ has the same symmetries as the structure constants involved:

$$n[ij, kl] \ = \ -n[ji, kl] \ = \ -n[ij, lk] \ = \ n[ji, lk] \ , \qquad n[ij, kl] \ = \ n[kl, ij] \tag{13.1.10}$$

If we assign $n_t = n[13, 42]$ and $n_u = n[23, 41]$, then the Jacobi identity $n_s + n_t - n_u = 0$ can be written more compactly as

$$n[1\{2, 34\}] \ := \ n[12, 34] \ + \ n[13, 42] \ + \ n[14, 23] \ = \ 0 \ . \tag{13.1.11}$$

At five points, the first out of fifteen pole channels contributes



$$\sim \ \frac{1}{s_{12} \, s_{45}} \, f^{a_1 a_2 b} \, f^{b a_3 c} \, f^{c a_4 a_5} \, n[12, 3, 45] \ = \ \frac{c_1 \, n_1}{s_{12} \, s_{45}}$$

to $\mathcal{M}^{\rm SYM}[1^{a_1}, 2^{a_2}, 3^{a_3}, 4^{a_4}, 5^{a_5}]$ where the kinematic numerators inherit their antisymmetry under flipping a cubic vertex from the structure constants:

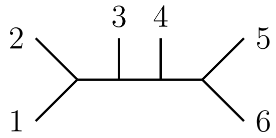
$$n[ij, k, lm] \ = \ -n[ji, k, lm] \ = \ -n[ij, k, ml] \ = \ n[ji, k, ml] \tag{13.1.12}$$

$$n[ij, k, lm] \ = \ -n[lm, k, ij] \tag{13.1.13}$$

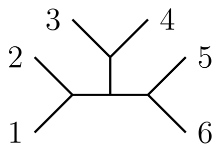
All the Jacobi identities (5.5.64) can be diagrammatically found by attaching a cubic vertex with two external legs to one of the dotted lines of figure 5.12. They can be cast into unified form

$$n[ij,\{k,lm\}] \;\; = \;\; 0 \; . \tag{13.1.14}$$

Six point amplitudes introduce two topologies of cubic diagrams



$$\sim \; \frac{f^{a_1a_2b}\, f^{ba_3c}\, f^{ca_4d}\, f^{da_5a_6}\, n[12,3,4,56]}{s_{12}\, s_{123}\, s_{56}}$$



$$\sim \; \frac{f^{a_1a_2b}\, f^{a_3a_4c}\, f^{a_5a_6d}\, f^{bcd}\, n[12,34,56]}{s_{12}\, s_{34}\, s_{56}}$$

which imprint the following symmetries on the BCJ numerators:

$$n[ij,k,l,mp] \;\; = \;\; -\,n[ji,k,l,mp] \; , \qquad n[ij,k,l,mp] \;\; = \;\; n[mp,l,k,ij] \tag{13.1.15}$$

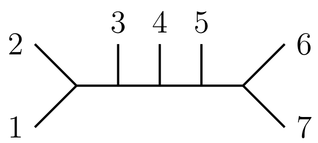
$$n[ij,kl,mp] \;\; = \;\; -\,n[ji,kl,mp] \; , \qquad n[ij,kl,mp] \;\; = \;\; -\,n[kl,ij,mp] \tag{13.1.16}$$

Also the Jacobi identities exhibit different topologies here, one can either attach three point vertices to two different dotted external lines of figure 5.12 or one color ordered four point diagram to one external line:

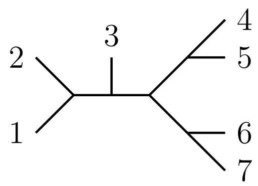
$$n[ij,k,\{l,mp\}] \;\; = \;\; 0 \; , \qquad n[ij,kl,mp] \;\; = \;\; n[ij,k,l,mp] \; - \; n[ij,l,k,mp] \tag{13.1.17}$$

The latter expresses any diagram of snowflake shape in terms of the other topology.

Cubic diagrams in seven point amplitudes again fall into two different topologies



$$\sim \; \frac{f^{a_1a_2b}\, f^{ba_3c}\, f^{ca_4d}\, f^{da_5e}\, f^{ea_6a_7}\, n[12,3,4,5,67]}{s_{12}\, s_{123}\, s_{567}\, s_{67}}$$



$$\sim \; \frac{f^{a_1a_2b}\, f^{ba_3c}\, f^{cde}\, f^{da_4a_5}\, f^{ea_6a_7}\, n\,[12,3,45,67]}{s_{12}\, s_{123}\, s_{45}\, s_{67}}$$

The associated numerators enjoy symmetry properties

$$n[ij,k,l,m,pq] \;\; = \;\; -\,n[ji,k,l,m,pq] \; , \qquad n[ij,k,l,m,pq] \;\; = \;\; -\,n[pq,m,l,k,ij] \tag{13.1.18}$$

$$n\,[ij,k,lm,pq] \;\; = \;\; -\,n\,[ji,k,lm,pq] \;\; = \;\; -\,n\,[ij,k,ml,pq] \;\; = \;\; -\,n\,[ij,k,pq,lm] \tag{13.1.19}$$

and Jacobi identities which eliminate the second topology:

$$n[ij,k,l,\{m,pq\}] \;\; = \;\; 0 \; , \qquad n\,[ij,k,lm,pq] \;\; = \;\; n[ij,k,l,m,pq] \; - \; n[ij,k,m,l,pq] \tag{13.1.20}$$

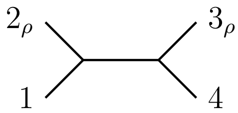
The four different topologies at eight points can still be captured by the suggestive notations $n[ij,k,l,m,p,qr], n[ij,k,lm,p,qr], n\,[ij,k,l,mp,qr]$ and $n\,[kl,ij,mp,qr]$. Jacobi identities relate diagrams of different topology such that all of them can be represented in terms of the simplest numerators class $n[ij,k,l,m,p,qr]$.

13.1.5 Reading off the BCJ numerators

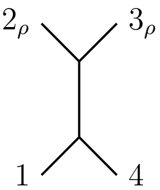
In order to finish the string inspired construction of BCJ numerators, we have to compare the field theory limits (13.1.7) with the defining equations of the numerators. The notation from the previous subsection admits to write the latter in a symmetric way once the diagramatic expansion of the color ordered n point is known.

Let us give the KK amplitudes $\mathcal{A}^{\rm SYM}(1,2_\rho,\ldots,(n-1)_\rho,n)$ for $n=4,5,6$ and $n=7$ here:

$\mathcal{A}^{\rm SYM}(1,2_\rho,3_\rho,4) =$

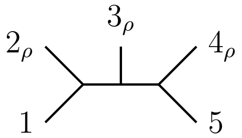


$+$



$$= \frac{n[12_\rho,3_\rho 4]}{s_{12_\rho}} + \frac{n[2_\rho 3_\rho,41]}{s_{2_\rho 3_\rho}}$$

$\mathcal{A}^{\rm SYM}(1,2_\rho,3_\rho,4_\rho,5) =$



$+ \text{cyclic}(12_\rho 3_\rho 4_\rho 5)$

$$= \frac{n[12_\rho,3_\rho,4_\rho 5]}{s_{12_\rho}\,s_{4_\rho 5}} + \text{cyclic}(12_\rho 3_\rho 4_\rho 5)$$

Figure 13.1: The four- and five point KK subamplitudes in the notation of subsection 13.1.4 for their kinematic numerators.

$$\begin{aligned}\mathcal{A}^{\rm SYM}(1,2_\rho,3_\rho,4_\rho,5_\rho,6) &= \frac{n[12_\rho,3_\rho 4_\rho,5_\rho 6]}{s_{12_\rho}\,s_{3_\rho 4_\rho}\,s_{5_\rho 6}} + \frac{n[2_\rho 3_\rho,4_\rho 5_\rho,61]}{s_{2_\rho 3_\rho}\,s_{4_\rho 5_\rho}\,s_{61}}\\ &- \frac{n[12_\rho,3_\rho,6,4_\rho 5_\rho]}{s_{12_\rho}\,s_{12_\rho 3_\rho}\,s_{4_\rho 5_\rho}} - \frac{n[2_\rho 3_\rho,4_\rho,1,5_\rho 6]}{s_{2_\rho 3_\rho}\,s_{2_\rho 3_\rho 4_\rho}\,s_{5_\rho 6}} - \frac{n[3_\rho 4_\rho,5_\rho,2_\rho,61]}{s_{3_\rho 4_\rho}\,s_{3_\rho 4_\rho 5_\rho}\,s_{61}}\\ &- \frac{n[12_\rho,6,3_\rho,4_\rho 5_\rho]}{s_{12_\rho}\,s_{3_\rho 4_\rho 5_\rho}\,s_{4_\rho 5_\rho}} - \frac{n[2_\rho 3_\rho,1,4_\rho,5_\rho 6]}{s_{2_\rho 3_\rho}\,s_{12_\rho 3_\rho}\,s_{5_\rho 6}} - \frac{n[3_\rho 4_\rho,2_\rho,5_\rho,61]}{s_{3_\rho 4_\rho}\,s_{2_\rho 3_\rho 4_\rho}\,s_{61}}\\ &+ \left(\frac{n[12_\rho,3_\rho,4_\rho,5_\rho 6]}{s_{12_\rho}\,s_{12_\rho 3_\rho}\,s_{5_\rho 6}} + \text{cyclic}(12_\rho 3_\rho 4_\rho 5_\rho 6)\right)\end{aligned} \tag{13.1.21}$$

$$\begin{aligned}\mathcal{A}^{\rm SYM}(1,2_\rho,3_\rho,4_\rho,5_\rho,6_\rho,7) &= \frac{n\,[12_\rho,3_\rho,4_\rho 5_\rho,6_\rho 7]}{s_{12_\rho}\,s_{12_\rho 3_\rho}\,s_{4_\rho 5_\rho}\,s_{6_\rho 7}} - \frac{n\,[2_\rho 3_\rho,1,4_\rho 5_\rho,6_\rho 7]}{s_{2_\rho 3_\rho}\,s_{12_\rho 3_\rho}\,s_{4_\rho 5_\rho}\,s_{6_\rho 7}}\\ &+ \frac{n[12_\rho,3_\rho,4_\rho,5_\rho,6_\rho 7]}{s_{12_\rho}\,s_{12_\rho 3_\rho}\,s_{5_\rho 6_\rho 7}\,s_{6_\rho 7}} - \frac{n[2_\rho 3_\rho,1,4_\rho,5_\rho,6_\rho 7]}{s_{2_\rho 3_\rho}\,s_{12_\rho 3_\rho}\,s_{5_\rho 6_\rho 7}\,s_{6_\rho 7}} - \frac{n[12_\rho,3_\rho,4_\rho,7,5_\rho 6_\rho]}{s_{12_\rho}\,s_{12_\rho 3_\rho}\,s_{5_\rho 6_\rho 7}\,s_{5_\rho 6_\rho}}\\ &+ \frac{n[2_\rho 3_\rho,1,4_\rho,7,5_\rho 6_\rho]}{s_{2_\rho 3_\rho}\,s_{12_\rho 3_\rho}\,s_{5_\rho 6_\rho 7}\,s_{5_\rho 6_\rho}} + \text{cyclic}(12_\rho 3_\rho 4_\rho 5_\rho 6_\rho 7)\end{aligned} \tag{13.1.22}$$

13.2 Explicit examples

In order to make the very general prescription for obtaining n_i more tractable, we shall now analyze explicit examples up to seven-point in detail. As a warm up example, let us construct the field theory limits of the four point pure spinor amplitude from its propagator matrix

$$\begin{pmatrix} P_{(2,3)}{}^1 & P_{(2,3)}{}^2 \\ P_{(3,2)}{}^1 & P_{(3,2)}{}^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{u} & \frac{1}{s} \\ -\frac{1}{u} - \frac{1}{t} & \frac{1}{t} \end{pmatrix} . \tag{13.2.23}$$

According to $\mathcal{A}^{\rm SYM}(1, 2_\rho, 3_\rho, 4) = \sum_{l=1}^{2} P_\rho{}^l \, \mathcal{K}^l$ (with the trivial $S_{n-3} = S_1$ permutation σ suppressed), the two KK subamplitudes $\mathcal{A}_4(1, 2, 3, 4) = \frac{n_s}{s} + \frac{n_u}{u}$ and $\mathcal{A}_4(1, 3, 2, 4) = -\frac{n_t}{t} - \frac{n_u}{u}$ have numerators

$$\begin{aligned} n_s &= \mathcal{K}^2 = \langle \, T_{12} \, V_3 \, V_4 \, \rangle \, , \qquad n_u = \mathcal{K}^1 = \langle \, V_1 \, T_{23} \, V_4 \, \rangle \\ n_t &= \mathcal{K}^1 - \mathcal{K}^2 = \langle \, V_1 \, T_{23} \, V_4 \, \rangle - \langle \, T_{12} \, V_3 \, V_4 \, \rangle \end{aligned} \tag{13.2.24}$$

that manifestly satisfy the Jacobi relation $n_s + n_t - n_u = 0$.

They are evaluated in superfield components in appendix C.

13.2.1 Five point numerators

The color ordered five-point superstring amplitude (12.1.11) encompasses six basis kinematics

$$\mathcal{K}^3_{\sigma(23)} = \langle \, T_{12_\sigma 3_\sigma} \, V_4 \, V_5 \, \rangle \, , \qquad \mathcal{K}^2_{\sigma(23)} = \langle \, T_{12_\sigma} \, T_{43_\sigma} \, V_5 \, \rangle \, , \qquad \mathcal{K}^1_{\sigma(23)} = \langle \, V_1 \, T_{43_\sigma 2_\sigma} \, V_5 \, \rangle \tag{13.2.25}$$

We have explained in subsection 12.1.3 that the double pole z_{23}^{-2} appearing in the five point integrand of other references [290, 210] was absorbed into the single-pole terms such that BRST building blocks T_{ij} and T_{ijk} could be formed. Absence of double poles is crucial for satisfying all the dual Jacobi relations because the counting argument of $(n-2)!$ independent numerators fails otherwise.

Performing the field theory limit of the integrals in (12.1.11) gives rise to the following six KK subamplitudes (where the permutation σ of 2 and 3 can be kept general because of (13.1.9))

$$\begin{aligned} \mathcal{A}^{\rm SYM}(1, 2_\sigma, 3_\sigma, 4, 5) &= \frac{\mathcal{K}^3_{\sigma(23)}}{s_{45} \, s_{12_\sigma}} + \frac{\mathcal{K}^1_{\sigma(23)} - \mathcal{K}^1_{\sigma(32)}}{s_{51} \, s_{2_\sigma 3_\sigma}} - \frac{\mathcal{K}^2_{\sigma(23)}}{s_{12_\sigma} \, s_{3_\sigma 4}} \\ &\quad + \frac{\mathcal{K}^3_{\sigma(23)} - \mathcal{K}^3_{\sigma(32)}}{s_{2_\sigma 3_\sigma} \, s_{45}} + \frac{\mathcal{K}^1_{\sigma(23)}}{s_{3_\sigma 4} \, s_{51}} \\ \mathcal{A}^{\rm SYM}(1, 2_\sigma, 4, 3_\sigma, 5) &= \frac{\mathcal{K}^3_{\sigma(23)} + \mathcal{K}^2_{\sigma(23)}}{s_{12_\sigma} \, s_{3_\sigma 5}} - \frac{\mathcal{K}^1_{\sigma(32)}}{s_{2_\sigma 4} \, s_{51}} + \frac{\mathcal{K}^2_{\sigma(23)}}{s_{3_\sigma 4} \, s_{12}} \end{aligned} \tag{13.2.26}$$

$$- \frac{\mathcal{K}^2_{\sigma(32)} + \mathcal{K}^1_{\sigma(32)}}{s_{3_\sigma 5}\, s_{2_\sigma 4}} - \frac{\mathcal{K}^1_{\sigma(23)}}{s_{51}\, s_{3_\sigma 4}}$$

$$\mathcal{A}^{\rm SYM}(1,4,2_\sigma,3_\sigma,5) \ = \ \frac{\mathcal{K}^3_{\sigma(23)} + \mathcal{K}^2_{\sigma(23)} + \mathcal{K}^2_{\sigma(32)} + \mathcal{K}^1_{\sigma(32)}}{s_{14}\, s_{3_\sigma 5}} + \frac{\mathcal{K}^1_{\sigma(32)}}{s_{2_\sigma 4}\, s_{51}} + \frac{\mathcal{K}^1_{\sigma(32)} - \mathcal{K}^1_{\sigma(23)}}{s_{51}\, s_{2_\sigma 3_\sigma}}$$
$$+ \frac{\mathcal{K}^3_{\sigma(23)} - \mathcal{K}^3_{\sigma(32)} - \mathcal{K}^1_{\sigma(23)} + \mathcal{K}^1_{\sigma(32)}}{s_{2_\sigma 3_\sigma}\, s_{14}} + \frac{\mathcal{K}^1_{\sigma(32)} - \mathcal{K}^2_{\sigma(32)}}{s_{3_\sigma 5}\, s_{2_\sigma 4}}$$

The last pair of color orderings $\mathcal{A}^{\rm SYM}(1,4,2_\sigma,3_\sigma,5)$ has more complicated numerators because of the coefficient $P_{(4,2_\sigma,3_\sigma)}{}^{(1,3_\sigma,2_\sigma)} = \frac{1}{s_{14}s_{3_\sigma 5}} + \text{cyclic}(142_\sigma 3_\sigma 5)$ in the propagator matrix that addresses all the five different pole channels.

By comparing (13.2.26) with $\mathcal{A}^{\rm SYM}(1,2_\rho,3_\rho,4_\rho,5) = \frac{n[12_\rho,3_\rho,4_\rho 5]}{s_{12_\rho}s_{4_\rho 5}} + \text{cyclic}(12_\rho 3_\rho 4_\rho 5)$, we can quickly read off the kinematic numerators:

$$\begin{aligned}
n[12_\sigma,3_\sigma,45] &= \mathcal{K}^3_{\sigma(23)}\,, \qquad n[3_\sigma 4,5,2_\sigma 1] = \mathcal{K}^2_{\sigma(23)}\,, \qquad n[51,2_\sigma,3_\sigma 4] = \mathcal{K}^1_{\sigma(23)}\\
n[2_\sigma 3_\sigma,4,51] &= \mathcal{K}^1_{\sigma(23)} - \mathcal{K}^1_{\sigma(32)}\,, \qquad n[2_\sigma 3_\sigma,1,45] = \mathcal{K}^3_{\sigma(32)} - \mathcal{K}^3_{\sigma(23)}\\
n[3_\sigma 4,1,2_\sigma 5] &= \mathcal{K}^1_{\sigma(23)} + \mathcal{K}^2_{\sigma(23)}\,, \qquad n[12_\sigma,4,3_\sigma 5] = \mathcal{K}^3_{\sigma(23)} + \mathcal{K}^2_{\sigma(23)}\\
n[14,3_\sigma,2_\sigma 5] &= \mathcal{K}^3_{\sigma(32)} + \mathcal{K}^2_{\sigma(23)} + \mathcal{K}^2_{\sigma(32)} + \mathcal{K}^1_{\sigma(23)}\\
n[2_\sigma 3_\sigma,5,14] &= \mathcal{K}^3_{\sigma(23)} - \mathcal{K}^3_{\sigma(32)} + \mathcal{K}^1_{\sigma(32)} - \mathcal{K}^1_{\sigma(23)}
\end{aligned} \tag{13.2.27}$$

It is sufficient to display nine of them, the rest follows from S_2 relabelling $2 \leftrightarrow 3$. The $n_1, n_2, \ldots, n_{15}$ from the parametrization (5.5.63) translate into

$$\begin{aligned}
n_1 &= n[12,3,45] & n_6 &= n[14,3,25] & n_{11} &= n[24,3,51]\\
n_2 &= n[23,4,51] & n_7 &= n[32,5,14] & n_{12} &= n[12,4,35]\\
n_3 &= n[34,5,12] & n_8 &= n[25,1,43] & n_{13} &= n[35,1,24]\\
n_4 &= n[45,1,23] & n_9 &= n[13,4,25] & n_{14} &= n[14,2,35]\\
n_5 &= n[51,2,34] & n_{10} &= n[42,5,13] & n_{15} &= n[13,2,45]
\end{aligned} \tag{13.2.28}$$

The way in which the $n[ij,k,lm]$ are built out of $\mathcal{K}^j_{\sigma(2,3)}$ trivializes the Jacobi identities (5.5.64) or $n[ij,\{k,lm\}] = 0$. However, the expressions (13.2.27) for $n[ij,k,lm]$ do not exhibit crossing symmetry including labels $1,4$ and 5.

In many instances, the symmetry properties $T_{(ij)} = T_{(ij)k} = T_{[ijk]} = 0$ of the BRST building blocks within $\mathcal{K}^l_\sigma$ allow to rewrite sums over several basic kinematics occurring in some n_i as a single superfield, e.g.

$$n_2 \ = \ \mathcal{K}^1_{(23)} - \mathcal{K}^1_{(32)} \ = \ \langle\, (T_{123} - T_{132})\, V_4\, V_5 \,\rangle \ = \ \langle\, T_{321}\, V_4\, V_5 \,\rangle \tag{13.2.29}$$

However, the right hand side is outside the five point basis of kinematics, so the Jacobi relations between numerators are rather obscured by this building block manipulations. At any number of legs, the basis of $\mathcal{K}^l_\sigma$ is designed such that all the symmetries of the building blocks are already exploited, so we refrain from performing manipulations like $T_{123}-T_{132}=T_{321}$ in higher order examples.

13.2.2 Six point numerators

In six-point amplitudes, the propagator matrix (13.1.8) can be completely constructed from the field theory limit of the four superstring subamplitudes associated with color orderings $\{1,2,3,4,5,6\}$, $\{1,2,3,5,4,6\}$, $\{1,2,5,3,4,6\}$, $\{1,5,2,3,4,6\}$. The S_3 relabelling covariance in $2,3,4$ connects them to the remaining 20 elements of the KK basis. Let us give some representative sample entries of ${P_\rho}^{(l,\sigma)}$ here,

$$\begin{aligned}
{P_{(2345)}}^{1,(423)} &= \frac{1}{s_{61}\,s_{23}\,s_{234}}\ , \qquad {P_{(2345)}}^{4,(432)} \ = \ \frac{1}{s_{56}\,s_{234}}\left(\frac{1}{s_{23}}+\frac{1}{s_{34}}\right)\\
{P_{(2345)}}^{4,(234)} &= \frac{1}{s_{56}}\left(\frac{1}{s_{12}\,s_{34}}+\frac{1}{s_{12}\,s_{123}}+\frac{1}{s_{23}\,s_{123}}+\frac{1}{s_{23}\,s_{234}}+\frac{1}{s_{34}\,s_{234}}\right)\\
{P_{(2354)}}^{3,(234)} &= \frac{1}{s_{123}}\left(\frac{1}{s_{12}}+\frac{1}{s_{23}}\right)\left(\frac{1}{s_{45}}+\frac{1}{s_{46}}\right)\\
{P_{(5234)}}^{1,(432)} &= \frac{1}{s_{61}\,s_{25}\,s_{34}}+\frac{1}{s_{15}\,s_{23}\,s_{46}}+\frac{1}{s_{15}\,s_{125}\,s_{34}}+\frac{1}{s_{125}\,s_{25}\,s_{34}}+\frac{1}{s_{15}\,s_{125}\,s_{46}}\\
&\quad+\frac{1}{s_{46}\,s_{25}\,s_{125}}+\frac{1}{s_{15}\,s_{23}\,s_{234}}+\frac{1}{s_{61}\,s_{23}\,s_{234}}+\frac{1}{s_{15}\,s_{234}\,s_{34}}+\frac{1}{s_{61}s_{234}s_{34}}\\
&\quad+\frac{1}{s_{61}\,s_{23}\,s_{235}}+\frac{1}{s_{61}\,s_{25}\,s_{235}}+\frac{1}{s_{46}\,s_{23}\,s_{235}}+\frac{1}{s_{46}\,s_{25}\,s_{235}}
\end{aligned} \tag{13.2.30}$$

and refer the reader to appendix F.1 for the complete result.

Comparing the KK subamplitudes with (13.1.21) allows to read off the 105 BCJ numerators. It is sufficient to display 25 of them in S_3-covariant form:

$$\begin{aligned}
n[12_\sigma,3_\sigma,4_\sigma,56] &= \mathcal{K}^4_{\sigma(234)}\ , & n[61,2_\sigma,3_\sigma,54_\sigma] &= \mathcal{K}^1_{\sigma(234)}\\
n[12_\sigma,3_\sigma,6,4_\sigma 5] &= \mathcal{K}^3_{\sigma(234)}\ , & n[12_\sigma,6,3_\sigma,54_\sigma] &= \mathcal{K}^2_{\sigma(234)}\\
n[12_\sigma,3_\sigma 4_\sigma,56] &= \mathcal{K}^4_{\sigma(234)}-\mathcal{K}^4_{\sigma(243)}\ , & n[2_\sigma 3_\sigma,4_\sigma 5,61] &= \mathcal{K}^1_{\sigma(324)}-\mathcal{K}^1_{\sigma(234)}\\
n[12_\sigma,3_\sigma 5,64_\sigma] &= \mathcal{K}^3_{\sigma(243)}+\mathcal{K}^2_{\sigma(243)}\ , & n[3_\sigma 4_\sigma,5,6,12_\sigma] &= \mathcal{K}^2_{\sigma(234)}-\mathcal{K}^2_{\sigma(243)}\\
n[4_\sigma 5,6,1,2_\sigma 3_\sigma] &= \mathcal{K}^3_{\sigma(324)}-\mathcal{K}^3_{\sigma(234)}\ , & n[3_\sigma 4_\sigma,5,2_\sigma,61] &= \mathcal{K}^1_{\sigma(234)}-\mathcal{K}^1_{\sigma(243)}\\
n[2_\sigma 3_\sigma,1,4_\sigma,56] &= \mathcal{K}^4_{\sigma(324)}-\mathcal{K}^4_{\sigma(234)}\ , & n[12_\sigma,3_\sigma,5,4_\sigma 6] &= \mathcal{K}^4_{\sigma(234)}+\mathcal{K}^3_{\sigma(234)}\\
n[4_\sigma 6,1,2_\sigma,3_\sigma 5] &= \mathcal{K}^2_{\sigma(423)}+\mathcal{K}^1_{\sigma(423)}
\end{aligned}$$

$$\begin{aligned}
n[2_\sigma 3_\sigma,4_\sigma,5,61] &= \mathcal{K}^1_{\sigma(324)}-\mathcal{K}^1_{\sigma(234)}+\mathcal{K}^1_{\sigma(423)}-\mathcal{K}^1_{\sigma(432)}\\
n[56,1,2_\sigma,3_\sigma 4_\sigma] &= \mathcal{K}^4_{\sigma(234)}-\mathcal{K}^4_{\sigma(243)}-\mathcal{K}^4_{\sigma(342)}+\mathcal{K}^4_{\sigma(432)}\\
n[2_\sigma 3_\sigma,5,1,4_\sigma 6] &= \mathcal{K}^2_{\sigma(432)}-\mathcal{K}^2_{\sigma(423)}-\mathcal{K}^1_{\sigma(423)}+\mathcal{K}^1_{\sigma(432)}
\end{aligned}$$

$$
\begin{aligned}
n[2_\sigma 3_\sigma, 1, 5, 4_\sigma 6] &= \mathcal{K}^4_{\sigma(234)} - \mathcal{K}^4_{\sigma(324)} + \mathcal{K}^3_{\sigma(234)} - \mathcal{K}^3_{\sigma(324)} \\
n[12_\sigma, 5, 3_\sigma, 4_\sigma 6] &= \mathcal{K}^4_{\sigma(234)} + \mathcal{K}^3_{\sigma(234)} + \mathcal{K}^3_{\sigma(243)} + \mathcal{K}^2_{\sigma(243)} \\
n[3_\sigma 4_\sigma, 6, 1, 2_\sigma 5] &= \mathcal{K}^3_{\sigma(432)} - \mathcal{K}^3_{\sigma(342)} + \mathcal{K}^1_{\sigma(342)} - \mathcal{K}^1_{\sigma(432)} \\
n[12_\sigma, 5, 6, 3_\sigma 4_\sigma] &= \mathcal{K}^4_{\sigma(243)} - \mathcal{K}^4_{\sigma(234)} + \mathcal{K}^2_{\sigma(234)} - \mathcal{K}^2_{\sigma(243)} \\
n[2_\sigma 5, 1, 3_\sigma, 4_\sigma 6] &= \mathcal{K}^3_{\sigma(342)} + \mathcal{K}^2_{\sigma(342)} + \mathcal{K}^2_{\sigma(432)} + \mathcal{K}^1_{\sigma(432)} \\
n[15, 2_\sigma, 3_\sigma, 4_\sigma 6] &= \mathcal{K}^4_{\sigma(234)} + \mathcal{K}^3_{\sigma(234)} + \mathcal{K}^3_{\sigma(243)} + \mathcal{K}^3_{\sigma(342)} \\
&\quad + \mathcal{K}^2_{\sigma(243)} + \mathcal{K}^2_{\sigma(342)} + \mathcal{K}^2_{\sigma(432)} + \mathcal{K}^1_{\sigma(432)} \\
n[2_\sigma 3_\sigma, 4_\sigma, 6, 15] &= \mathcal{K}^4_{\sigma(234)} - \mathcal{K}^4_{\sigma(324)} - \mathcal{K}^4_{\sigma(423)} + \mathcal{K}^4_{\sigma(432)} \\
&\quad + \mathcal{K}^1_{\sigma(234)} - \mathcal{K}^1_{\sigma(324)} - \mathcal{K}^1_{\sigma(423)} + \mathcal{K}^1_{\sigma(432)} \\
n[15, 2_\sigma, 6, 3_\sigma 4_\sigma] &= \mathcal{K}^4_{\sigma(243)} - \mathcal{K}^4_{\sigma(234)} - \mathcal{K}^3_{\sigma(342)} + \mathcal{K}^3_{\sigma(432)} \\
&\quad + \mathcal{K}^2_{\sigma(234)} - \mathcal{K}^2_{\sigma(243)} + \mathcal{K}^1_{\sigma(342)} - \mathcal{K}^1_{\sigma(432)} \\
n[15, 2_\sigma 3_\sigma, 4_\sigma 6] &= \mathcal{K}^4_{\sigma(234)} - \mathcal{K}^4_{\sigma(324)} + \mathcal{K}^3_{\sigma(234)} - \mathcal{K}^3_{\sigma(324)} \\
&\quad - \mathcal{K}^2_{\sigma(423)} + \mathcal{K}^2_{\sigma(432)} - \mathcal{K}^1_{\sigma(423)} + \mathcal{K}^1_{\sigma(432)}
\end{aligned}
\tag{13.2.31}
$$

They have been explicitly checked to satisfy all the 105 Jacobi relations $n[ij, k, \{l, mp\}] = 0$ and $n[ij, kl, mp] = n[ij, k, l, mp] - n[ij, l, k, mp]$ (81 of which are linearly independent). It is interesting to note that the number of K^l_σ forming the individual BCJ numerators is always a power of two, i.e. $1, 2, 4$ or 8 in this case.

13.2.3 Seven point numerators

Since the number of channels grows like $(2n-5)!!$ in an n-point amplitude, a complete list of all BCJ numerators becomes very lengthy beyond six points. Appendix F.2 gives all the 69 seven-point numerators which are not related by $2, 3, 4, 5$ relabelling, they allow to obtain all the 945 numerators by going through the $\sigma \in S_4$ permutations of $(2, 3, 4, 5)$. We also checked that all the 825 independent numerators equations (out of 1260 in total) are satisfied.

13.2.4 General observations on higher point numerators

Instead of continuing the numerator list to higher points, we conclude this section with some general remarks and observation on the structure of the string inspired expressions for the BCJ numerators.

Firstly, entries of the n point propagator matrix always factorize into sums of m propagators with $C(m) = \frac{(2m)!}{m!(m+1)!}$ terms,

$$P_\rho{}^{(l,\sigma)} \quad \sim \quad \prod \left(\sum_{j=1}^{C(m)} \frac{1}{s_{(\alpha^1)_j}\, s_{(\alpha^2)_j}\, \ldots\, s_{(\alpha^m)_j}} \right) \tag{13.2.32}$$

where $C(m)$ is the m'th Catalan number and counts the number of channels appearing in a $m+2$ point color ordered amplitude. At $n=5$, we have seen three different pole structures in (13.2.26),

$$P_\rho{}^{(l,\sigma)}\Big|_{n=5} \quad \sim \quad \left\{ \frac{1}{s_\alpha s_\beta},\, \frac{1}{s_\alpha}\left(\frac{1}{s_{\beta_1}} + \frac{1}{s_{\beta_2}}\right),\, \sum_{i=1}^{5} \frac{1}{s_{\alpha_i} s_{\beta_i}} \right\} \tag{13.2.33}$$

and the six point analogue contains the five types of products displayed in (13.2.30):

$$\begin{aligned} P_\rho{}^{(l,\sigma)}\Big|_{n=6} \quad \sim \quad &\left\{ \frac{1}{s_\alpha s_\beta s_\gamma},\, \frac{1}{s_\alpha s_\beta}\left(\frac{1}{s_{\gamma_1}} + \frac{1}{s_{\gamma_2}}\right),\, \frac{1}{s_\alpha}\left(\frac{1}{s_{\beta_1}} + \frac{1}{s_{\beta_2}}\right)\left(\frac{1}{s_{\gamma_1}} + \frac{1}{s_{\gamma_2}}\right), \right. \\ &\left. \frac{1}{s_\alpha} \sum_{i=1}^{5} \frac{1}{s_{\beta_i} s_{\gamma_i}},\, \sum_{j=1}^{14} \frac{1}{s_{\alpha_j} s_{\beta_j} s_{\gamma_j}} \right\} \end{aligned} \tag{13.2.34}$$

The pattern was observed to persist up to eight-point. However, not all possible partitions of the overall $n-3$ propagators into products of type (13.2.32) are realized. For instance, there are no terms like $\left(\frac{1}{s_{\alpha_1}} + \frac{1}{s_{\alpha_2}}\right)\left(\frac{1}{s_{\beta_1}} + \frac{1}{s_{\beta_2}}\right)$ at five points, $\left(\frac{1}{s_{\alpha_1}} + \frac{1}{s_{\alpha_2}}\right)\left(\sum_{i=1}^{5} \frac{1}{s_{\beta_i} s_{\gamma_i}}\right)$ at six points and $\left(\frac{1}{s_{\alpha_1}} + \frac{1}{s_{\alpha_2}}\right)\left(\frac{1}{s_{\beta_1}} + \frac{1}{s_{\beta_2}}\right)\left(\frac{1}{s_{\gamma_1}} + \frac{1}{s_{\gamma_2}}\right)\frac{1}{s_\delta}$ at seven points because they would involve incompatible pole channels.

Secondly, the number of $\mathcal{K}_\sigma^l$ kinematics entering the individual n point BCJ numerators up to $n=8$ is always a power of two, i.e. $1, 2, 4, \ldots, 2^{n-3}$. This can be largely explained from the flipping antisymmetry of $n[ij, k, \ldots]$ in pairs of labels i, j sharing a terminal three point vertex. If they are both from the range $i_\sigma, j_\sigma \in \{2, 3, \ldots, n-2\}$, then the $\mathcal{K}_\sigma^l$ are required to pair up with their $i_\sigma \leftrightarrow j_\sigma$ images. Moreover, if several other $2, 3, \ldots, n-2$ labels k_σ, l_σ follow, then a nested antisymmetrization emerges, e.g. $n[i_\sigma j_\sigma, k_\sigma, l_\sigma, \ldots] \leftrightarrow \mathcal{K}^l_{\sigma([[[i_\sigma j_\sigma]k_\sigma]l_\sigma]\ldots)}$.

Another source of doubling the terms is a terminal vertex with legs 1 and $n-1$: Swapping $1 \leftrightarrow n-1$ maps $\mathcal{K}_\sigma^l$ to $\mathcal{K}_{\bar\sigma}^{n-1-l}$ where $\bar\sigma$ denotes the permutation of reverse order, $\bar\sigma(23\ldots p-1, p) = \sigma(p, p-1, \ldots 32)$. That is why $n[1(n-1), \ldots]$ can only contain pairs like $\mathcal{K}_\sigma^l + \mathcal{K}_{\bar\sigma}^{n-1-l}$ which might be further antisymmetrized in some legs from $\{2, 3, \ldots, n-2\}$ due to another terminal vertex.

The following table 13.1 shows the distribution of the $(2n-5)!!$ numerators into packages of 2^j basis elements. We suspect that the grading of kinematic numerators according to their $\mathcal{K}_\sigma^l$ content is connected with the factorization pattern (13.2.32) of $P_\rho{}^{(l,\sigma)}$ entries.

# terms	4pts	5pts	6pts	7pts	8pts
1	2	6	24	120	720
2	1	6	36	240	1800
4		3	30	270	2520
8			15	210	2520
16				105	1890
32					945

Table 13.1: The number of BCJ numerators in n point amplitudes containing 2^j basis kinematics $\mathcal{K}_\sigma^l$ for $j = 0, 1, \ldots, n-3$.

Chapter 14

Epilogue

We have presented scattering amplitudes in superstring theory from various perspectives. Thanks to the exact solvability of the worldsheet SCFT in both the RNS- and the PS formalism, we could derive manifold results on tree level scattering which are relevant for string phenomenology, gauge theories and formal properties of the S matrix in superstring theory. Let us review the main results presented in this work and point out promising directions for future research.

14.1 Main results

As already sketched in section 1.4, the research achievements gathered in this thesis originate from diverse branches of the rich topic of superstring amplitudes.

14.1.1 Phenomenological results

In [1,2,6], we followed the lines of [101] and worked out some universal statements of superstring theories on hadron scattering at LHC. These are presented in chapters 8 and 9. Remarkably, the tree level physics of quark gluon scattering was recognized to be largely independent on the superstring compactification into which a SM like scenario with intersecting D branes is embedded. Apart from four chiral fermion channels, the heavy states exchanged in multiparton disk scattering are insensitive to the geometry of the internal space. The virtual particles are Regge resonances at discrete mass squares $m^2 = n/\alpha'$, set by the string scale. The latter can potentially be as low as a few TeV in scenarios with large extra dimensions [58, 103]. Kaluza Klein- or winding modes can only propagate in four quark scattering, these model dependences are fortunately suppressed by color- and luminosity effects.

More specifically, chapter 8 introduces the universality classes of model independent parton disk amplitudes – namely $\mathcal{A}(g_1, g_2, g_3, \ldots, g_n)$ and $\mathcal{A}(\lambda_1, \bar{\lambda}_2, g_3, \ldots, g_n)$ with any number of gluons g and up to two (anti-)gauginos λ ($\bar{\lambda}$) from the vector multiplets and $\mathcal{A}(q_1, \bar{q}_2, g_3, \ldots, g_n)$ as well as $\mathcal{A}(C_1, \bar{C}_2, g_3, \ldots, g_n)$ with quark- $(q, \bar{q})$ or squark- $(C, \bar{C})$ admixtures from the chiral multiplet. The striking agreement of color stripped amplitudes $\mathcal{A}(\lambda_1, \bar{\lambda}_2, g_3, \ldots, g_n)$ and $\mathcal{A}(q_1, \bar{q}_2, g_3, \ldots, g_n)$ from different multiplets clearly goes beyond SUSY. Based on spinor helicity methods [316, 317, 318] and color factor technology [265, 266], we compute full fledged cross sections for parton collisions in superstring theory up to the five point level and cast the result into a form suitable for LHC data analysis.

Production and decay of massive states is addressed in chapter 9, based on the four dimensional spectrum at mass level one which we derived in 4. Many three- and four point processes with one $m^2 = \alpha'^{-1}$ state are universal to any $\mathcal{N} = 1$ supersymmetric string compactification. These are evaluated in the spinor helicity basis (see [319, 320, 321] for helicity methods adjusted to massive states). Four point amplitudes with one higher spin state from the leading Regge trajectory are computed for completely general mass level $n \in \mathbb{N}$, with symmetries in the labels made manifest. All these processes occur at accelerators once the center of mass energies of the colliding partons exceed the mass threshold $\sim \alpha'^{-1/2}$, so their observability at LHC is again tied to a low string scale. Four point cross sections for mass level one states can be found in [2].

14.1.2 Results on spin field correlation functions

Selected results of [3, 4, 5] on correlation functions of the interacting RNS CFT are gathered in chapter 6. They address the old problem of the covariant superstring [118] that the interacting nature of spin fields makes the computation of worldsheet correlation functions inaccessible to free field methods (unless covariance is given up in favor of bosonization techniques [119] or radical field redefinitions such as [295] are applied).

The first publication [3] is specialized on tree level correlators of $SO(1,3)$ covariant NS fermions and spin fields, motivated by applications to four dimensional compactifications. The wide classes of results which could be obtained in systematic manner motivated to extend the analysis to higher genus [4] and to higher numbers of spacetime dimensions [5]. A variety of correlation functions with large numbers of fields can be found in 6, the highlight being the closed formulae (6.2.39) and (6.2.40) for the correlator $\langle \psi \ldots \psi S S \rangle$ with two spin field and any number of fermions in arbitrary even spacetime dimensions on Riemann surfaces of any genus. This is an essential CFT input for scattering amplitudes involving two massless fermions and any number of (not necessarily massless) bosons. Cases with four and more spin fields must be considered separately in $D = 4, 6, 8$ and $D = 10$ for technical reasons.

14.1.3 Explicit results in pure spinor superspace

Within the manifestly supersymmetric pure spinor formalism, we have pushed the frontier of knowledge on disk scattering of massless states from five point level [290, 291] to any number n of legs [7, 8, 9, 10, 11]. The essence of these publications is presented in part III of this work. The key advance for this achievement is a systematic construction of BRST building blocks in chapter 11 – composite superfields encompassing the kinematic degrees of freedom of several external legs. They transform covariantly under the BRST charge and allow to derive powerful constraints on superstring- and in particular field theory amplitudes based on cohomology arguments. Furthermore, a natural dictionary between tree diagrams made of cubic vertices and trilinear expressions in building blocks is proposed.

BRST building blocks can be combined into so-called Berends Giele currents, tree amplitudes with n on-shell legs and one additional off-shell leg. This objects were firstly used in [237] in the context of gluon scattering in four dimensional gauge theories and led to a recursion which proved the Parke Taylor formula for MHV amplitudes [236]. Our research in pure spinor superspace results in supersymmetric generalization of Berends Giele currents to $\mathcal{N} = 1$ SYM in $D = 10$ dimensions. We arrive at the recursive formula (11.3.54) for SYM n point amplitudes comparable with [237]. The extra virtue of our result is its manifest supersymmetry and its compactness being independent of the helicity configurations involved. Since $\mathcal{N} = 1$ SYM in $D = 10$ is straightforward to dimensionally reduce, one can carry the SUSY components of our superamplitudes over to maximally supersymmetric gauge theories in lower dimensions.

The superstring computation in the pure spinor framework makes another result for gauge theories accessible – an explicit construction of kinematic BCJ numerators, carried out in chapter 13. A striking duality between color factors c_i and kinematic factors n_i in gauge theory amplitudes has been observed [81] which led to tremendous simplification within the gauge sector but also in perturbative gravity [39, 40]. One of the main bottlenecks in exploiting this duality is the lack of constructive prescriptions towards the dual numerators n_i. This is where the pure spinor approach to the superstring turns out to be a convenient tool: By extracting the field theory limits of the hypergeometric integrals in color ordered superstring amplitudes, one can explicitly compute the kinematic numerator for each pole channel. The key identities for this purpose are (13.1.7) and (13.1.8), and the methods for taking the integrals' low energy limit can be found in chapter 7.

14.1.4 The complete n point superstring disk amplitude

The pure spinor computations within this work culminate in a strikingly short and compact expression for the n point superstring amplitude involving any external massless open string state from the SYM vector multiplet. The final expression is given in (12.3.33) and reflects a beautiful harmony of the string amplitudes. Both the kinematic building blocks and the hypergeometric integrals carrying the α' dependence of a subamplitude are reduced to a minimal basis of $(n-3)!$ elements each. Relations among the integrals and among string- or field theory subamplitudes are found to be in one-to-one correspondence, hinting a duality between color and kinematics at the level of the full fledged superstring amplitude.

We have elucidated the implications of the central result (12.3.33) both from and to field theory in section 12.3. Our result demonstrates how to efficiently compute tree level superstring amplitudes with an arbitrary number of external states. The pure spinor techniques proved to be crucial to derive (12.3.33). The methods presented in this work should be applicable to tackle any tree level disk amplitude computation involving massless states in any dimensions. Eventually, these findings may generalize to disk amplitudes with arbitrary, massive states in perturbative string theory.

The color dressed amplitude assumes a form where the role of color and kinematics can be swapped, see (12.3.43) and (12.3.44). This is a considerable step towards a stringy generalization of dual amplitudes $\mathcal{A}_n^{\mathrm{dual}}$ in which all kinematical factors are replaced by color traces with the same symmetry properties under exchange of labels [82]. However, further research is necessary to understand the kinematic dual of the original color traces $\mathrm{Tr}\{T^{a_1}T^{a_2}\dots T^{a_n}\}$ (which are referred to as $\tau(12\dots n)$ in the reference).

14.2 Future outlook

The research results open many doors for further studies. Numerous interesting and challenging subsequent projects are motivated and wait to be mastered, we shall just mention a few which are directly connected with the topics in this work:

14.2.1 Higher spin CFT operators

The huge collection of RNS correlation functions from chapter 6 almost exclusively involves the primary fields ψ^m, S_α and $S^{\dot\beta}$ with respect to the $SO(1, D-1)$ current algebra. In addition, an infinite set of Virasoro primaries in higher representations of the Lorentz group emerges from the operator algebra. They have been hardly considered in the literature, in particular the excited spin fields appearing in vertex operators of massive fermions have evaded a systematic study so far. Already leading Regge trajectory states require a handle on spin 3/2 operators in the CFT [6, 113].

Interactions of massive states are probably quicker to investigate using RNS methods, in spite of the interacting worldsheet CFT. Since the present knowledge of massive superfield formulations for higher mass levels is quite limited [292, 293], the technical requirements to apply the PS formalism are not met yet.

14.2.2 Loop amplitudes in pure spinor superspace

The remarkable success of pure spinor methods with disk amplitudes suggests to try a similar setup at loop level. So far, one loop amplitudes of the massless gauge multiplet have been computed up to five points [288]. It is desirable to identify BRST building blocks or comparable structures on higher genus such that higher leg generalizations become tractable. In particular, the superspace representation of the one loop SYM amplitude appears to be essential for understanding the structure of its string theory upgrade. Unfortunately, the tree level technology is hard to take over to loops because of the non-minimal pure spinor variables [286].

14.2.3 The structure of open- and closed string n point amplitudes

The availability of the full fledged expression (12.3.33) for the superstring n point amplitude allows a detailed study of possible recursion relations allowing to construct the amplitude with n external states from lower order amplitudes and some guiding principle. Due to the factorized form of (12.3.33), which separates the SYM part from the string part, the basic question is how to combine the field theory recursions established in the SYM sector [263, 207, 208] (see also [8]) to recursions working in the module of hypergeometric functions B_n. For the latter the following recurrence relations may be useful [322]

$$B_n \;=\; \sum B_{n_1}\, B_{n_2} \cdot \ldots \cdot B_{n_k} \ , \qquad \sum_{l=1}^{k} n_l \;=\; n \,+\, 3\,(k-1) \ , \tag{14.2.1}$$

with some partition $\{n_1, \ldots, n_k\}$ into k smaller amplitudes B_{n_l}. This equation allows to write B_n in terms of products of $(n-3)$ functions B_4, cf. figure 14.1.

Higher point gluon amplitudes (12.3.33) give rise to higher order corrections in α' to the SYM action. According to the specific form of (12.3.33) these higher order α' corrections are organized according to the field theory amplitudes $\mathcal{A}^{\mathrm{SYM}}$. Hence, the latter serve as building

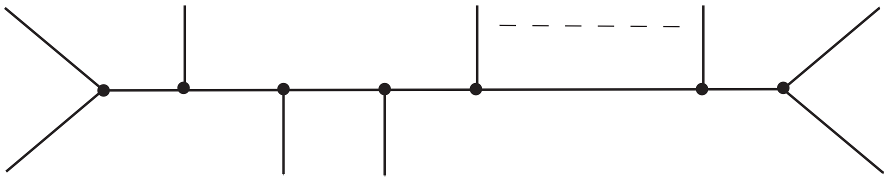


Figure 14.1: Partition of B_n into products of four point amplitudes B_4.

blocks to construct the higher order terms in the effective action with the expansion coefficients encoded in the functions F^σ. Moreover, the YM amplitudes $\mathcal{A}^{\rm SYM}$ may be arranged such that only cubic vertices contribute [81], i.e. the full superstring amplitude ultimately boils down to the three vertex in field theory. As a consequence, it should be possible to describe the higher order α' corrections in the effective action entirely in terms of the fundamental SYM three vertices dressed by the contributions from F^σ.

Together with the KLT relations [83] the open string n point amplitudes (12.3.33) can be used to obtain compact expressions for the n point closed string amplitudes [217]. The latter give rise to n graviton scattering amplitudes. Their α' expansions have been analyzed up to $n \leq 6$ through the order α'^8 in [212]. These findings proved to be crucial in constraining possible counterterms in $\mathcal{N}=8$ supergravity in $D=4$ up to seven loops [112]. Counterterms invariant under $\mathcal{N}=8$ supergravity have a unique kinematic structure and the tree level closed string amplitudes provide candidates for them, which are compatible with SUSY Ward identities and locality. The absence or restriction on higher order gravitational terms at the order α'^l together with their symmetries constrain the appearance of possible counter terms available at l–loop. With the present results it may now be possible to bolster up the results [212] and to extend the research performed in [323, 324, 325, 112].

Appendix A

Conventions

A.1 Conventions in the RNS chapters

There will be various types of indices in this work, so it is essential to keep the notation as clear and unambiguous as possible. Here is a list of occurring index types together with the preferably used alphabets and letters:

- Vector indices on the worldsheet are taken from the beginning of the latin alphabet $a, b, c, ...$ e.g. h_{ab} denotes the worldsheet metric.

- Spacetime vector indices of flat ten dimensional spacetime are taken from the middle of the latin alphabet $m, n, p, ...$

- Dirac spinor indices of $SO(1,9)$ are capital letters from the beginning of the latin alphabet $A, B, C, ...$. Indices of left handed Weyl spinors are taken from the beginning of the Greek alphabet $\alpha, \beta, \gamma, ...$, and the right handed counterparts have an additional dot $\dot{\alpha}, \dot{\beta}, \dot{\gamma}, ...$.

- $SO(6)$ vector indices for particularly symmetric internal manifolds are $i, j, k, \ldots$

- Left handed (right handed) spinors of $SO(6)$ have upper case indices I, J, K (lower case indices $\bar{I}, \bar{J}, \bar{K}$).

- Vectors of four dimensional Minkowski spacetime have indices from the middle of the Greek alphabet $\mu, \nu, \lambda, \rho, ...$.

- $SO(1,3)$ spinor indices are lower case Latin letters $a, b, c, ...$ for left handed Weyl spinors and upper case $\dot{a}, \dot{b}, \dot{c}$ for right handed Weyl spinors. They will never appear in the same context as worldsheet vector indices, so there is little chance for confusion.

- Adjoint color indices of Chan Paton generators associated with D brane stack a (b) are denoted by $a_1, a_2, \ldots$ $(b_1, b_2, \ldots)$.
- Fundamental (antifundamental) color indices associated with D brane stacks a (b) are denoted by upper case $\alpha_1, \alpha_2, \ldots$ (lower case $\beta_1, \beta_2, \ldots$)

The signature of the Dirac algebra is negative in lines with the conventions of [30], i.e.

$$(\Gamma^m)_A{}^B\,(\Gamma^n)_B{}^C \;+\; (\Gamma^n)_A{}^B\,(\Gamma^m)_B{}^C \;=\; -\,2\,\eta^{mn}\,\delta^C_A\;, \tag{A.1.1}$$

where Γ^m are the Dirac matrices of $SO(1,9)$ with 16×16 chiral blocks γ^m and $\bar\gamma^m$:

$$SO(1,9)\;:\quad (\Gamma^m)_A{}^B \;=\; \begin{pmatrix} 0 & \gamma^m_{\alpha\dot\beta} \\ \bar\gamma^{m\dot\alpha\beta} & 0 \end{pmatrix} \tag{A.1.2}$$

The lower dimensional analogues are denoted by

$$SO(6)\;:\; \begin{pmatrix} 0 & \gamma_i^{I\bar J} \\ \bar\gamma^{i\bar I J} & 0 \end{pmatrix}\;,\qquad SO(1,3)\;:\; \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot\beta} \\ \bar\sigma^{\mu\dot\alpha\beta} & 0 \end{pmatrix}\;, \tag{A.1.3}$$

they also satisfy Dirac algebras of negative signature, e.g.

$$\sigma^\mu_{\alpha\dot\beta}\,\bar\sigma^{\nu\dot\beta\gamma} \;+\; \sigma^\nu_{\alpha\dot\beta}\,\bar\sigma^{\mu\dot\beta\gamma} \;=\; -\,2\,\eta^{\mu\nu}\,\delta^\gamma_\alpha\;,\qquad \gamma_i^{I\bar J}\,\bar\gamma_{k\bar J K} \;+\; \gamma_k^{I\bar J}\,\bar\gamma_{i\bar J K} \;=\; -\,2\,\delta_{ik}\,\delta^I_K\;. \tag{A.1.4}$$

The charge conjugation matrices are $\mathcal{C}_{AB} = \left(\begin{smallmatrix} 0 & C_\alpha{}^{\dot\beta} \\ C^{\dot\alpha}{}_\beta & 0 \end{smallmatrix}\right)$ for $SO(1,9)$, $\left(\begin{smallmatrix} 0 & C^I{}_{\bar J} \\ C_{\bar I}{}^J & 0 \end{smallmatrix}\right)$ for $SO(6)$ and $\left(\begin{smallmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon^{\dot\alpha\dot\beta} \end{smallmatrix}\right)$ for $SO(1,3)$.

$$SO(1,9)\;:\quad \mathcal{C}_{AB} \;=\; \begin{pmatrix} 0 & C_\alpha{}^{\dot\beta} \\ C^{\dot\alpha}{}_\beta & 0 \end{pmatrix} \tag{A.1.5}$$

$$SO(6)\;:\; \begin{pmatrix} 0 & C^I{}_{\bar J} \\ C_{\bar I}{}^J & 0 \end{pmatrix}\;,\qquad SO(1,3)\;:\; \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon^{\dot\alpha\dot\beta} \end{pmatrix} \tag{A.1.6}$$

Useful material on spinors in various spacetime dimensions can be found in [42, 326, 327, 328].

A.2 Conventions in the pure spinor chapters

In order to follow closely the literature on pure spinor amplitudes, we have decided to change conventions in part III of this work with respect to the RNS sections. This mostly concerns $SO(1,9)$ gamma matrices and spinor indices. This somewhat schizophrenic approach provides the best guarantee to avoid errors, since most of the research papers on the pure spinor formalism deviate from the RNS conventions A.1 used in earlier chapters. It is essential for the analysis of multileg scattering amplitudes to get all the relative signs are correct.

- Left handed spinors of $SO(1,9)$ have a downstairs index from the beginning of the Greek alphabet, e.g. ψ_α, whereas right handed spinors have an undotted Greek index as a superscript, e.g. χ^β.

- The charge conjugation matrix is taken to be the Kronecker delta δ^α_β. This amounts to redefining $\mathcal{C} \mapsto \Gamma_{11}\mathcal{C}$ in the Dirac notation of the previous chapters with Γ_{11} denoting the $D = 10$ chirality matrix. This changes the symmetry properties of the even gamma matrix products $\mathcal{C}$, $(\Gamma^{mn}\mathcal{C})$ and $(\Gamma^{mnpq}\mathcal{C})$.

- Chiral gamma matrices $\gamma^m_{\alpha\beta}$, ${\gamma^{mn}}_\alpha{}^\beta$, $\gamma^{mnp}_{\alpha\beta}, \ldots$ tacitly include the (redefined) charge conjugation matrix, e.g. $(\gamma^m\, C)_{\alpha\beta} \mapsto \gamma^m_{\alpha\beta}$ can contract two right handed spinors whereas $(\gamma^{mn}\, C)_\alpha{}^\beta \mapsto {\gamma^{mn}}_\alpha{}^\beta$ requires spinors of opposite chirality:

$$(\psi_1\, \gamma^m\, \psi_2) \ = \ \psi_1^\alpha\, \gamma^m_{\alpha\beta}\, \psi_2^\beta \ , \qquad (\psi\, \gamma^{mn}\, \chi) \ = \ \psi^\alpha\, {\gamma^{mn}}_\alpha{}^\beta\, \chi_\beta \tag{A.2.7}$$

  The same notation applies to the right handed counterparts, e.g. $(\bar{\gamma}_m\, C)^{\dot\alpha\dot\beta} \mapsto \gamma_m^{\alpha\beta}$ without carrying the bar over the γ symbol along. Keeping the redefined charge conjugation matrix in mind, we can regard γ^{mn} as antisymmetric in the sense that $(\gamma^{mn})_\alpha{}^\beta = -(\gamma^{mn})^\beta{}_\alpha$.

- The Dirac algebra has a plus sign rather than the minus sign from the Wess & Bagger notation:

$$\gamma^m_{\alpha\beta}\, \gamma^{n\beta\gamma} \ + \ \gamma^n_{\alpha\beta}\, \gamma^{m\gamma\delta} \ = \ +\,2\,\eta^{mn}\, \delta^\gamma_\alpha \tag{A.2.8}$$

- In order to avoid proliferation of i factors, spacetime momenta are redefined according to $k_{\mathrm{PS}} = ik_{\mathrm{RNS}}$ such that $\mathrm{e}^{ik\cdot X}$ are replaced by $\mathrm{e}^{k\cdot X}$.

- The Regge slope is suppressed using open string conventions $2\alpha' = 1$ and can be restored by dimensional analysis .

Appendix B

Elements of superconformal field theory

The purpose of this appendix is to provide some general background on conformal field theory (CFT) necessary at various places of this work. Many of the definitions and techniques required for worldsheet computations in superstring theory are introduced in this appendix. This shall lighten the flow of reading in the main body of this work. Particular emphasis is put on the supersymmetric generalizations of CFTs, so-called superconformal field theories (SCFTs).

The basic reference on two dimensional conformal field theory is the work [329]. The $\mathcal{N} = 1$ superconformal algebra was firstly considered in [330, 331]. The most complete and detailed textbook on the subject of CFT is certainly [332], lovingly referred to as the "yellow pages". More streamlined introductions can be found in [333, 334, 335], and the first detailed accounts on the SCFT were given in [336, 337]. For readers with applications to string theory in mind, the preface of [41, 42] enumerates selected chapters which give a self contained course on CFT.

B.1 SCFT basics

This section explains the general setup and gives the basic definitions within conformal field theories.

B.1.1 What is a SCFT?

A field theory is said to be conformal if it is left invariant by angle preserving transformations. As a necessary requirement for conformal symmetry, there must not exist a preferred length scale or any other dimensionful quantity such as a mass parameter.

In d dimensional flat Minkowski space $\mathbb{R}^{p,q}$ of signature (p,q) with $p+q=d$, conformal transformations $x^a \mapsto y^a(x)$ satisfy[1]

$$\eta_{cd} \, \frac{\partial y^c}{\partial x^a} \, \frac{\partial y^d}{\partial x^b} \;\; = \;\; \Lambda \, \eta_{ab} \tag{B.1.1}$$

i.e. they reproduce the Minkowski metric η_{ab} up to a (possibly x dependent) scale function Λ. By working out the algebra of infinitesimal transformations $y^a = x^a - \eta^a(x)$ subject to (B.1.1), we find that the conformal algebra in $d \geq 3$ dimensions is isomorphic to $so(p+1, q+1)$, i.e. of dimension $\frac{1}{2}(d+1)(d+2)$.

We will be interested in the $d=2$ dimensional case for the purpose of the worldsheet SCFTs in superstring theory. In complex worldsheet coordinates $z = \sigma^1 + i\sigma^2$ and $\bar{z} = \sigma^1 - i\sigma^2$, the line element factorizes $\mathrm{d}s^2 = \mathrm{d}z \, \mathrm{d}\bar{z}$ and, by virtue of (B.1.1), the conformal group encompasses the holomorphic and antiholomorphic functions $z \mapsto f(z)$ and $\bar{z} \mapsto \bar{g}(\bar{z})$. Their infinitesimal version $z \mapsto z - \eta(z)$ can be expanded in a Laurent series such that the conformal algebra turns out to be infinite dimensional:

$$d = 2 \text{ conformal algebra:} \quad \left(z \; \mapsto \; z \, - \sum_{n \in \mathbb{Z}} \eta_n \, z^{n+1} \right) \; \oplus \; \left(\bar{z} \; \mapsto \; \bar{z} \, - \sum_{n \in \mathbb{Z}} \bar{\eta}_n \, \bar{z}^{n+1} \right) \tag{B.1.2}$$

The existence of an infinite set of Noether charges imposes rich constraints on the correlation functions of a $d=2$ CFT such that it can even be defined without making reference to an action. This rather abstract approach in terms of operator algebras and representation theory is referred to as the "boot-strap approach".

There exist supersymmetric generalizations of CFTs which can for instance be defined in the associated superspace to $d=2$ spacetime dimensions. They are naturally known as superconformal field theories (SCFTs). The superconformal algebra contains an infinite set of fermionic generators ϵ_n in addition to the bosonic set (B.1.2). Similar to the supersymmetric extension of the Poincaré group, the ϵ_n anticommute to bosonic generators η_m.

In this work, we will encounter several SCFTs on the worldsheet, most of which have a Lagrangian description. Although the action can be helpful in certain computations, see e.g. subsection B.4.2, it is not at all necessary to properly define or solve the SCFT. The Ramond spin fields firstly discussed in section 2.2 provide an excellent example of such a non-Lagrangian but still exactly solvable theory.

[1] Having the application to two dimensional worldsheets in mind, we will use letters $a, b, \ldots$ for the spacetime vector indices in this appendix.

B.1.2 The Noether charges in SCFTs

Let us go into more detail here about the infinite set of bosonic and fermionic Noether charges in SCFTs. There are two essential objects to describe a field theory's response to conformal and supersymmetry transformations – the energy momentum tensor T_{ab} and the supercurrent G_a. Superconformal symmetry forces many of their components to vanish. It is true in any number of dimensions d that the energy momentum tensor of a SCFT is traceless and the supercurrent has a vanishing γ-matrix trace:

$$\text{superconformal symmetry in } d \text{ dimensions} \quad \Rightarrow \quad T_a{}^a \;=\; 0 \quad \oplus \quad \gamma^a \, G_a \;=\; 0 \tag{B.1.3}$$

In the case of interest with $d = 2$ dimensions, both T_{ab} and G_a have two independent components which can most conveniently be distilled in complex coordinates $z = \sigma^1 + i\sigma^2$ and $\bar{z} = \sigma^1 - i\sigma^2$:

$$\begin{aligned} T_{zz}(z) \;&=:\; T(z) \quad \text{holomorphic} \, , \qquad & T_{\bar{z}\bar{z}}(\bar{z}) \;&=:\; \bar{T}(\bar{z}) \quad \text{antiholomorphic} \\ G_z(z) \;&=:\; \big(G(z), 0\big) \quad \text{holomorphic} \, , \qquad & G_{\bar{z}}(\bar{z}) \;&=:\; \big(0, \bar{G}(\bar{z})\big) \quad \text{antiholomorphic} \end{aligned} \tag{B.1.4}$$

The off diagonal part of T vanishes because of $T_{z\bar{z}} = T_{\bar{z}z} \sim T_a{}^a = 0$, and the supercurrent being a $d = 2$ Dirac spinor has two components $(\cdot, \cdot)$ for each value of $a = z, \bar{z}$.

The Noether current associated with holomorphic diffeomorphisms $\eta(z)$ and supersymmetry transformations $\epsilon(z)$ are given by products with $T(z)$ and $G(z)$ respectively. Like in any other quantum field theory (QFT), the associated Noether charges are obtained by integrating these currents over a constant time slice:

$$Q_\eta \;=\; \oint \frac{\mathrm{d}z}{2\pi i} \, \eta(z) \, T(z) \tag{B.1.5}$$

$$Q_\epsilon \;=\; \oint \frac{\mathrm{d}z}{2\pi i} \, \epsilon(z) \, G(z) \tag{B.1.6}$$

Here we are taking a result of subsection B.2.1 ahead, surfaces of constant time are mapped to circles around the origin on the complex plane, a more detailed explanation will follow later.

B.1.3 Conformal primary fields

The centrals actors in SCFTs are conformal primary fields. We define $\phi_{h,\bar{h}}(z, \bar{z})$ to be a conformal primary field of conformal weights $(h, \bar{h})$ if a change of coordinate $z \mapsto f(z)$ and $\bar{z} \mapsto \bar{g}(\bar{z})$ transforms ϕ according to

$$\phi_{h,\bar{h}}(z, \bar{z}) \;\; \mapsto \;\; \phi'_{h,\bar{h}}(z, \bar{z}) \;=\; \left(\frac{\partial f}{\partial z} \right)^h \left(\frac{\bar{\partial} \bar{g}}{\bar{\partial} \bar{z}} \right)^{\bar{h}} \phi_{h,\bar{h}}\big(f(z), \bar{g}(\bar{z})\big) \, . \tag{B.1.7}$$

The description of open superstrings mostly involves conformal fields $\phi_h(z)$ with holomorphic dependence on z only, they are called chiral. The antiholomorphic weight $\bar{h} = 0$ will be suppressed in these cases. In a unitary CFT, there are no operators with $h, \bar{h} < 0$.

The objects of main interest in QFTs are correlation functions $\langle \phi_{h_1}(z_1)\phi_{h_2}(z_2)\ldots\phi_{h_n}(z_n)\rangle$. They can be defined through a path integral weighted by some action or, in CFTs, by more axiomatic approaches. Correlation functions of conformal primary fields ϕ_{h_i} have to be covariant under the transformation property (B.1.7) which imposes severe constraints on two- and three point functions: Up to coefficients d_{ij} and C_{ijk}, they are determined to be

$$\begin{aligned} \langle\, \phi_{h_i}(z)\, \phi_{h_j}(w)\, \rangle \ &= \ \frac{d_{ij}\, \delta_{h_i,h_j}}{(z-w)^{2h_i}} \\ \langle\, \phi_{h_1}(z_1)\, \phi_{h_2}(z_2)\, \phi_{h_3}(z_3)\, \rangle \ &= \ \frac{C_{123}}{z_{12}^{h_1+h_2-h_3}\, z_{13}^{h_1-h_2+h_3}\, z_{23}^{h_2+h_3-h_1}} \ , \end{aligned} \tag{B.1.8}$$

hence, one can regard the normalizations d_{ij} and three point couplings C_{ijk} as the defining data of a CFT. For higher point correlation functions, however, conformal invariance does not provide direct formulae like (B.1.8) because they can depend on so-called cross ratios such as $\frac{z_{ij}z_{kl}}{z_{ik}z_{jl}}$. Finding the functional dependence is quite cumbersome in general. Still, the asymptotic falloff of correlation functions is independent on cross ratios and determined by the conformal dimension of the field in question:

$$\lim_{|z_i|\to\infty} \langle\, \phi_1(z_1)\, \phi_2(z_2)\, \ldots\, \phi_n(z_n)\, \rangle \ \sim \ z_i^{-2h_i} \tag{B.1.9}$$

The statements (B.1.8) also hold in higher dimensional CFTs with $d > 2$. Also, they remain true in $d = 2$ if (B.1.7) is relaxed to hold for $SL(2,\mathbb{C})/\mathbb{Z}_2$ transformations only, the globally defined holomorphic maps

$$f(z) \ = \ \frac{az\,+\,b}{cz\,+\,d}\ , \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})\ , \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ \equiv \ - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{B.1.10}$$

on the Riemann sphere $S^2 \cong \mathbb{C} \cup \{\infty\}$. In this case, the field $\phi_{h,\bar{h}}$ is called a quasi-primary of weight h.

The generalization of finite coordinate transformations to SCFT requires superspace which we won't introduce in this work. Instead, we will define superconformal primaries on the level of infinitesimal transformations such as $z \mapsto z - \eta(z)$. Let η and ϵ denote the parameters of a bosonic conformal transformations and fermionic supersymmetry transformation respectively. Two fields ϕ_0, ϕ_1 of opposite statistics are said to define a superconformal primary superfield of weight h if they exhibit the following transformation properties:

$$\delta_\eta \phi_0 \ = \ \big(\, h\, \partial\eta \ + \ \eta\partial\, \big)\, \phi_0$$

$$\begin{aligned}
\delta_\eta \phi_1 &= \Big((h + \tfrac{1}{2})\, \partial\eta \;+\; \eta\partial \Big)\, \phi_1 \\
\delta_\epsilon \phi_0 &= \frac{1}{2}\, \epsilon\, \phi_1 \\
\delta_\epsilon \phi_1 &= \frac{1}{2}\, \epsilon\, \partial\phi_0 \;+\; h\, \partial\epsilon\, \phi_0
\end{aligned} \tag{B.1.11}$$

From the non-supersymmetric point of view, ϕ_1 is simply a conformal primary of weight $h + \frac{1}{2}$. That is why we will often use the notation $(\phi_0, \phi_1) \equiv (\phi_h, \phi_{h+1/2})$.

B.2 Techniques in SCFTs

In this section, we want to exploit the rich mathematical tools offered by complex analysis. SCFTs are governed by holomorphic transformations, we can use powerful theorems such as Cauchy's formula

$$\oint_0 \frac{\mathrm{d}z}{2\pi i}\, z^n \;=\; \delta_{n,-1} \tag{B.2.12}$$

to better exploit their symmetries. In the following, the subscript $\oint_w$ along with an integral indicates that the contour encircles the point w once in counterclockwise direction.

B.2.1 Radial quantization

The most natural coordinates to parametrize a closed string's worldsheet are proper time σ^0 and a periodic position coordinate $\sigma^1 \in (0, 2\pi)$. If we collect Euclidean time σ^2 and σ^1 in the complex coordinate $z := \sigma^1 - i\sigma^2$, then z lives on a cylinder of infinite length with periodicity $z \equiv z + 2\pi$ in the direction of the real axis. Infinite past and future (with respect to Euclidean time) are found at infinite imaginary parts $\mathrm{Im}(z) \to \pm\infty$.

In order to employ the power of complex analysis and to define asymptotic states (see section B.3), it is convenient to map the cylinder to the plane via change of coordinates

$$w \;:=\; \mathrm{e}^{iz}\,, \qquad \bar{w} \;:=\; \mathrm{e}^{-i\bar{z}}\,. \tag{B.2.13}$$

Time translations on the cylinder $\mathrm{Im}(z) \mapsto \mathrm{Im}(z) + \delta\sigma^2$ correspond to dilatations on the plane $|w| \mapsto \mathrm{e}^{\delta\sigma^2}|w|$. Therefore, surfaces of constant time are circles around the origin $|w| = \text{const}$, and these are the integration contours necessary to build the superconformal Noether charges (B.1.5) and (B.1.6). Note that infinite past is the origin $|w| \to 0$ of the complex plane, and infinite future $|w| \to \infty$ can also be represented on the w coordinate patch provided we extend the plane to its on the Riemann sphere $S^2 \cong \mathbb{C} \cup \{\infty\}$ – the conformal compactification of $\mathbb{R}^2$.

Open strings have a nonperiodic worldsheet coordinate $\sigma^1 \in (0, \pi)$. Their worldsheet is an infinite strip in the z coordinate patch and the upper half plane $\{w \in \mathbb{C} : \ \mathrm{Im}\,\{w\} \geq 0\}$ in the

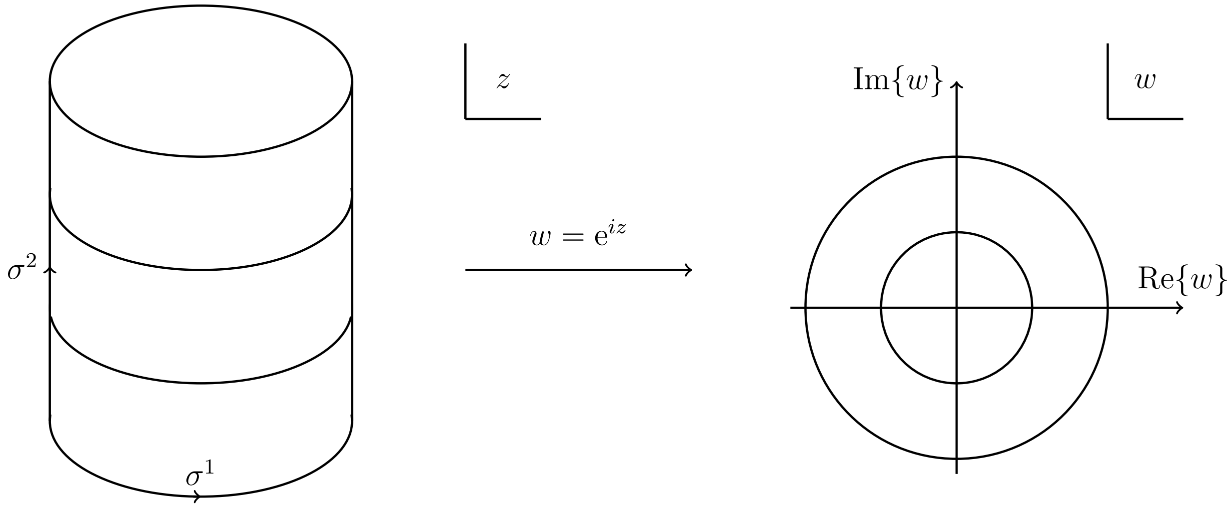


Figure B.1: Conformal mapping of the cylinder to the Riemann sphere $\mathbb{C} \cup \{\infty\}$: Surfaces of constant (Euclidean) time $\sigma^2 = const$ in the z coordinate patch become circles $|w| = const$ in the w coordinate.

w coordinate. We will say more in subsection 2.4.2 how we can still describe open strings by conformal fields defined on the full complex plane.

B.2.2 Contour integrals and operator product expansions

We have identified the Noether charges (B.1.5) and (B.1.6) associated with superconformal transformations as contour integrals involving the energy momentum tensor $T(z)$ and the supercurrent $G(z)$. As in any QFT, the transformation laws (B.1.11) obeyed by superconformal primaries should emerge from the commutator $[Q_\eta, \phi]$ or $[Q_\epsilon, \phi]$.

To make sense of a difference between $\oint \frac{\mathrm{d}z}{2\pi i}\eta(z)T(z)\phi(w)$ and $\phi(w) \oint \frac{\mathrm{d}z}{2\pi i}\eta(z)T(z)$, we need the notion of radial ordering. Correlation functions in QFTs are necessarily time-ordered, and after mapping the cylindrical worldsheet to the complex plane with time coordinate $\ln |w|$, this amounts to radial ordering of the operators. In the first contribution $\oint \frac{\mathrm{d}z}{2\pi i}\eta(z)T(z)\phi(w)$, the integration variable is understood to satisfy $|z| > |w|$ whereas the second one $\phi(w) \oint \frac{\mathrm{d}z}{2\pi i}\eta(z)T(z)$ is subject to $|z| < |w|$. This is depicted in the following figure B.2.2:

The difference gives rise to an effective contour $\oint_w \mathrm{d}z$ centered about w:

$$\begin{aligned} \big[\, Q_\eta \,,\, \phi(w) \,\big] &= \oint_{|z|>|w|} \frac{\mathrm{d}z}{2\pi i}\, \eta(z)\, T(z)\, \phi(w) \;-\; \oint_{|z|<|w|} \frac{\mathrm{d}z}{2\pi i}\, \eta(z)\, T(z)\, \phi(w) \\ &= \oint_w \frac{\mathrm{d}z}{2\pi i}\, \eta(z)\, T(z)\, \phi(w) \end{aligned} \tag{B.2.14}$$

By virtue of Cauchy's formula (B.2.12), this is compatible with the bosonic transformation laws

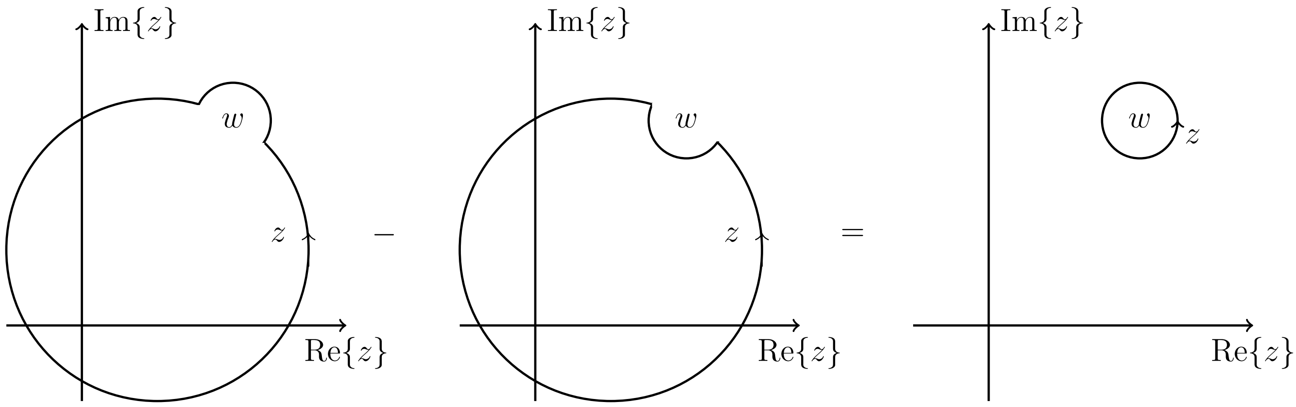


Figure B.2: The commutator of $Q_\eta = \oint \frac{\mathrm{d}z}{2\pi i}\eta(z)T(z)$ with some operator $\phi(w)$ gets contributions of opposite sign from closed integration contours with $|z| > |w|$ and $|z| < |w|$, respectively. The effective contour in z after the subtraction becomes a small circle centered about the point w.

in (B.1.11) if $T(z)$ and $\phi(w)$ give rise to the following singularity structure as $z \to w$:

$$T(z)\,\phi(w) \quad \sim \quad \frac{h\,\phi(w)}{(z-w)^2} \;+\; \frac{\partial\phi(w)}{z-w} \;+\; \ldots \tag{B.2.15}$$

The … denote regular terms in $z - w$ which do not contribute to the contour integral on the right hand side of (B.2.14). Using a similar argument, we can express a commutator with the SUSY charge as

$$\big[\, Q_\epsilon \,,\; \phi(w) \,\big] \quad = \quad \oint_w \frac{\mathrm{d}z}{2\pi i}\, \epsilon(z)\, G(z)\, \phi(w)\ , \tag{B.2.16}$$

and the list (B.1.11) of transformation laws for primary superfield components (ϕ_0, ϕ_1) translates into the following operator product expansions (with … denoting regular powers $(z - w)^{k\geq 0}$):

$$\begin{aligned}
T(z)\,\phi_0(w) \quad &\sim \quad \frac{h\,\phi_0(w)}{(z-w)^2} \;+\; \frac{\partial\phi_0(w)}{z-w} \;+\; \ldots \\
T(z)\,\phi_1(w) \quad &\sim \quad \frac{(h+\frac{1}{2})\,\phi_1(w)}{(z-w)^2} \;+\; \frac{\partial\phi_1(w)}{z-w} \;+\; \ldots \\
G(z)\,\phi_0(w) \quad &\sim \quad \frac{\frac{1}{2}\,\phi_1(w)}{z-w} \;+\; \ldots \\
G(z)\,\phi_1(w) \quad &\sim \quad \frac{h\,\phi_0(w)}{(z-w)^2} \;+\; \frac{\frac{1}{2}\,\partial\phi_0(w)}{z-w} \;+\; \ldots
\end{aligned} \tag{B.2.17}$$

The idea behind these operator product expansions (OPE) is that two operators at nearby positions can be approximated by one operator insertion at one of these points. They are only valid inside correlation functions because the argument in (B.2.14) made use of radial ordering. But then, they are exact statements whose radius of convergence extends to the next-nearest insertion of the correlator.

More generally, any two conformal primary fields ϕ_i, ϕ_j obey OPE relations

$$\phi_i(z)\,\phi_j(w) \quad \sum_k (z-w)^{h_k-h_i-h_j}\, C_{ij}{}^k\, \phi_k(w) \;+\; \mathcal{O}\big((z-w)^{h_k-h_i-h_j+1}\big) \tag{B.2.18}$$

where $C_{ij}{}^l d_{lk}$ coincides with the coefficient C_{ijk} of the three point function in (B.1.8). The higher order terms $\mathcal{O}\big((z-w)^{h_k-h_i-h_j+1}\big)$ contain descendant fields of ϕ_k (to be defined in the later subsection B.3.3).

B.2.3 The superconformal algebra

A conformal field ϕ_h is called a quasi-primary of weight h if it transform as $\phi_h(z) \mapsto \phi'_h(z,\bar{z}) = (\partial f)^h\, \phi\big(f(z)\big)$ only under the globally defined holomorphic maps $z \mapsto f(z) = \frac{az+b}{cz+d}$ on the Riemann sphere $S^2 \cong \mathbb{C} \cup \{\infty\}$ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{C})/\mathbb{Z}_2$. Since the infinitesimal version of these $SL(2,\mathbb{C})/\mathbb{Z}_2$ maps is $z \mapsto z - \eta(z)$ with $\eta(z) = \alpha + \beta z + \gamma z^2$, we can translate the defining property of quasi-primaries into the OPE

$$\begin{aligned} T(z)\,\phi_h(w) \quad &\sim \quad \sum_{n\geq 4} \frac{\mathcal{O}_{2+h-n}}{(z-w)^n} \;+\; \frac{h\,\phi_h(w)}{(z-w)^2} \;+\; \frac{\partial\phi_h(w)}{z-w} \;+\; \dots \\ &\sim \quad \dots \;+\; \frac{h\,\phi_h(w)}{(z-w)^2} \;+\; \frac{\partial\phi_h(w)}{z-w} \;+\; \dots \end{aligned} \tag{B.2.19}$$

The can deviate from the OPEs of primary fields by higher order singularities in $z-w$, with a gap of at least one power compared to the $\frac{h\phi_h(w)}{(z-w)^2}$ singularity. In any SCFT, the most prominent example of a quasi-primary (but not primary) field is the energy momentum tensor $T(z)$ (with weight $h=2$ according to the $(z-w)^{-2}$ power in (B.2.17)): Its self OPE $T(z)T(w)$ contains the terms $\frac{2T(w)}{(z-w)^2}$ and $\frac{\partial T(w)}{z-w}$. But in addition, any operator $\frac{\mathcal{O}_{h=4-n}(w)}{(z-w)^n}$ could appear on dimensional grounds.

There are two important constraints on these candidates $\mathcal{O}_h$: Firstly, unitarity restricts $h \geq 0$ such that only $n \leq 4$ are allowed. In particular, the only $h=0$ operator is the identity such that the maximal singularity $(z-w)^{-4}$ is multiplied by a constant (which is defined as $c/2$ for convenience): Since the next singularity $(z-w)^{-3}$ is incompatible with $z \leftrightarrow w$ invariance of the OPE, the only possible extra singularity is $(z-w)^{-4}$. We will call its coefficient $c/2$ for the moment.

Using similar arguments, the supercurrent self OPE $G(z)G(w)$ should not have singularities higher than $(z-w)^{-3}$, but antisymmetry in $z \leftrightarrow w$ forbids a $(z-w)^{-2}$ pole. Jacobi identities between the Laurent modes of $T(z)$ and $G(z)$ (to be explained in the following subsection B.2.4) furthermore fix the constant $(z-w)^{-3}$ coefficient to be $c/6$.

Putting all these arguments together, the most general form of the OPE algebra generated by T and G is the following:

$$\begin{aligned} T(z)\,T(w) &= \frac{c/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots \\ T(z)\,G(w) &= \frac{\frac{3}{2}\,G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w} + \dots \\ G(z)\,G(w) &= \frac{c/6}{(z-w)^3} + \frac{\frac{1}{2}\,T(w)}{z-w} + \dots \end{aligned} \tag{B.2.20}$$

The $T(z)G(w)$ OPE identifies the supercurrent as a $h = \frac{3}{2}$ primary of the conformal algebra. The constant showing up in the $T(z)T(w)$ and $G(z)G(w)$ OPEs is know as the central charge and constitutes one of the defining quantities of a SCFT, one of several physical interpretations will be given in the following subsection.

B.2.4 Mode expansions

Conformal fields $\phi_{h,\bar{h}}$ living on the cylinder have a discrete Fourier expansion with respect to the periodic coordinate $\sigma^1 \equiv \sigma^1 + 2\pi$,

$$\phi_{h,\bar{h}}(z,\bar{z}) = \sum_{n\in\mathbb{Z}} \hat{\phi}_n(\sigma^2)\, \mathrm{e}^{in\sigma^1} \; . \tag{B.2.21}$$

As we shall demonstrate, this translates into a Laurent expansion in $w = \mathrm{e}^{iz}$ on the Riemann sphere for conformal primaries $\phi_{h,\bar{h}}$. For ease of notation, let us consider chiral fields with $\bar{h} = 0$ and holomorphic dependence on the $z = \sigma^1 - i\sigma^2$ coordinate. The Fourier coefficients can then be written as $\hat{\phi}_n(\sigma^2) = i^h \mathrm{e}^{-n\sigma^2}\phi_n$ with constant ϕ_n (and a factor of i^h for convenience).

According to the transformation law $\phi'_h(w) = \left(\frac{\mathrm{d}z}{\mathrm{d}w}\right)^h \phi_h(z)$ for conformal primaries, the Fourier series (B.2.21) with $\hat{\phi}_n(\sigma^2) = i^h \mathrm{e}^{-n\sigma^2}\phi_n$ is mapped as follows to the Riemann sphere under $w = \mathrm{e}^{-iz}$:

$$\begin{aligned} \phi'_h(w) &= \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-h} \phi_h\big(z(w)\big) = (iw)^{-h} \sum_{n\in\mathbb{Z}} i^h\, \phi_n\, \mathrm{e}^{in(\sigma^1 + i\sigma^2)} \\ &= \sum_{n\in\mathbb{Z}} \phi_n\, w^{-h-n} \end{aligned} \tag{B.2.22}$$

Note that the offset of $-h$ in the w powers is a remnant of the mapping from the cylinder to the complex plane. In the following, we will always work on Riemann sphere and drop the prime of the transformed conformal fields ϕ'_h.

The energy momentum tensor $T(z)$ is a quasi-primary field and does not transform homogeneously under $z \mapsto f(z)$. Instead, the $T(z)T(w)$ OPE from (B.2.20) implies

$$\delta_\eta T = \big(2\,\partial\eta + \eta\partial\big)\,T + \frac{c}{12}\,\partial^3\eta \tag{B.2.23}$$

when inserted into the general action (B.2.14) of infinitesimal reparametrizations $z \mapsto z - \eta(z)$. Its exponentiation to finite coordinate transformations implies that the mapping $w = \mathrm{e}^{iz}$ from the cylinder to the plane causes a shift in the zero mode by $\frac{c}{24}$. Let us denote the energy momentum Laurent modes on the plane by L_n, then we have

$$T(w) \ = \ \sum_{n \in \mathbb{Z}} L_n \, w^{-n-2} \ , \qquad L_0 \ = \ L_0 \Big|_{\text{cylinder}} \ + \ \frac{c}{24} \tag{B.2.24}$$

Since the L_0 operator on the Riemann sphere takes the role of the Hamiltonian, the central charge can be interpreted as a vacuum- or Casimir energy.

B.2.5 OPEs versus commutation relations

In this subsection, we will extract commutation relations for Laurent modes from OPE methods. This demonstrates the strength of OPEs to encompass an infinite set of equations among the modes. In applications to string theory, the operator methods can be useful to obtain the physical spectrum (at least in light cone gauge) whereas interactions are most efficiently described in the path integral approach.

Let us first of all consider charges Q_k with a contour integral representation in terms of the associated currents j_k,

$$Q_k \ = \ \oint_0 \frac{\mathrm{d}z}{2\pi i} \, j_k(z) \ , \tag{B.2.25}$$

the 0 superscript indicating that the z integration countour encircles the origin once in counterclockwise direction. Their commutator can be reduced to OPEs of the currents j_k, j_l by repeating the argument of subsection B.2.2: The formal difference of two contours $\oint_{|z|>|w|} \mathrm{d}z - \oint_{|z|<|w|} \mathrm{d}z$ due to radial ordering is a small circle $\oint_w \mathrm{d}z$ around the point w, see figure B.2.2:

$$\begin{aligned} \big[\, Q_k \ , \ Q_l \, \big] \ &= \ \oint_0 \frac{\mathrm{d}z}{2\pi i} \oint_0 \frac{\mathrm{d}w}{2\pi i} \, \big[\, j_k(z) \ , \ j_l(w) \, \big] \\ &= \ \oint_0 \frac{\mathrm{d}w}{2\pi i} \left(\oint_{|z|>|w|} \frac{\mathrm{d}z}{2\pi i} \, j_k(z) \, j_l(w) \ - \ \oint_{|z|<|w|} \frac{\mathrm{d}z}{2\pi i} \, j_k(z) \, j_l(w) \right) \\ &= \ \oint_0 \frac{\mathrm{d}w}{2\pi i} \oint_w \frac{\mathrm{d}z}{2\pi i} \, j_k(z) \, j_l(w) \end{aligned} \tag{B.2.26}$$

Any Laurent mode ϕ_n of a conformal field ϕ_h can be brought into such an integral representation like (B.2.25) via

$$\phi_h(z) \ = \ \sum_{n \in \mathbb{Z}} \phi_n \, z^{-n-h} \quad \Rightarrow \quad \phi_n \ = \ \oint_0 \frac{\mathrm{d}z}{2\pi i} \, z^{n+h-1} \, \phi_h(z) \ . \tag{B.2.27}$$

In particular, the energy momentum- and supercurrent modes L_m, G_r on the Riemann sphere are given by

$$L_m \ = \ \oint_0 \frac{\mathrm{d}z}{2\pi i} \, z^{m+1} \, T(z) \ , \qquad G_r \ = \ \oint_0 \frac{\mathrm{d}z}{2\pi i} \, z^{r+1/2} \, G(z) \tag{B.2.28}$$

The OPE formulation (B.2.20) of the superconformal algebra therefore encodes any (anti-) commutation relation[2] for the modes L_m and G_r:

$$\begin{aligned} \big[\, L_m \,,\, L_n \,\big] &= (m-n)\, L_{m+n} \;+\; \frac{c}{12}\,(m^3-m)\,\delta_{m+n,0} \\ \big[\, L_m \,,\, G_r \,\big] &= \frac{m-2r}{2}\, G_{m+r} \\ \big\{\, G_r \,,\, G_s \,\big\} &= 2\, L_{r+s} \;+\; \frac{c}{12}\,(4r^2-1)\,\delta_{r+s,0} \end{aligned} \qquad \text{(B.2.29)}$$

This is valid for moding $r \in \mathbb{Z}$ or $r \in \mathbb{Z} + \frac{1}{2}$ of the supercurrent, see subsection B.3.2. The relation $G^2_{-1/2} = L_{-1}$ can be viewed as the global worldsheet supersymmetry algebra on the plane.

B.3 Fields and states

The space of states in a SCFT decomposes into irreducible representations of the super Virasoro algebra (B.2.29). This section will explain the one-to-one correspondence between conformal fields and states.

B.3.1 Primary fields and highest weight states

Each super Virasoro irreducible can be generated from a highest weight states $|h\rangle$ by action of $L_{n<0}$ and $G_{r<0}$ modes. The commutation relations $[L_0, L_n] = -nL_n$ and $[L_0, G_r] = -rG_r$ identify the $L_{n<0}$ and $G_{r<0}$ as raising operators for L_0 eigenvalues. The defining properties of highest weight states $|h\rangle$ are

$$L_{n>0}\,|h\rangle \;=\; G_{r\geq 0}\,|h\rangle \;=\; 0\;, \qquad L_0\,|h\rangle \;=\; h\,|h\rangle\;. \qquad \text{(B.3.30)}$$

Recall that infinite past in Euclidean time $\sigma^2 \to -\infty$ is located at the origin $z = 0$ of the Riemann sphere. As we will show, the highest weight properties (B.3.30) admit to identify $|h\rangle$ with an asymptotic state created by the lower component of a superconformal primary $(\phi_h, \phi_{h+1/2})$ at $z \to 0$:

$$|h\rangle \;=\; \lim_{z\to 0} \phi_h(0)\,|0\rangle \qquad \text{(B.3.31)}$$

In view of the mode expansion $\phi_h(z) = \sum_n \phi_n z^{-n-h}$, this state is only well defined if $\phi_n|0\rangle = 0$ for any $n > -h$. Regularity of $T(z)$ and $G(z)$ at the origin requires that the following super

[2] Half odd integer conformal weights like $G(z)$ give rise to anticommutation relations among themselves, this can for instance be seen from their antisymmetric two point function $\langle\phi_h(z)\phi_h(w)\rangle \sim (z-w)^{-2h} = -(w-z)^{-2h}$ for $h \in \mathbb{Z} + \frac{1}{2}$.

Virasoro modes annihilate the vacuum state $|0\rangle$:

$$L_n\,|0\rangle \;=\; 0\,, \qquad n\geq -1\,, \qquad G_r\,|0\rangle \;=\; 0\,, \qquad r\geq -\tfrac{1}{2} \tag{B.3.32}$$

The proof that the state $|h\rangle$ defined by (B.3.31) enjoys the highest weight properties (B.3.30) is based on the OPEs (B.2.17) valid for any primary field:

$$\begin{aligned} L_n\,|h\rangle \;&=\; \oint_0 \frac{\mathrm{d}z}{2\pi i}\, z^{n+1}\, T(z)\,\phi_h(0)\,|0\rangle \;=\; \oint_0 \frac{\mathrm{d}z}{2\pi i}\, z^{n+1} \left(\frac{h\,\phi_h(0)}{z^2} \;+\; \frac{\partial\phi_h(0)}{z} \;+\; \dots \right) |0\rangle \\ &=\; \begin{cases} h\,\phi_h(0)\,|0\rangle & : \; n=0 \\ \qquad 0 & : \; n>0 \end{cases} \end{aligned} \tag{B.3.33}$$

$$\begin{aligned} G_r\,|h\rangle \;&=\; \oint_0 \frac{\mathrm{d}z}{2\pi i}\, z^{r+1/2}\, G(z)\,\phi_h(0)\,|0\rangle \;=\; \oint_0 \frac{\mathrm{d}z}{2\pi i}\, z^{r+1/2} \left(\frac{\phi_{h+1/2}(0)}{2\,z} \;+\; \dots \right) |0\rangle \\ &=\; \begin{cases} \frac{1}{2}\,\phi_{h+1/2}(0)\,|0\rangle & : \; r=-1/2 \\ \qquad 0 & : \; r\geq 0 \end{cases} \end{aligned} \tag{B.3.34}$$

Hence, the upper component $\phi_{h+1/2}$ of the superconformal primary $(\phi_h,\phi_{h+1/2})$ gives rise to the asymptotic state $\phi_{h+1/2}(0)|0\rangle = 2G_{-1/2}|0\rangle$. Apart from $L_{-1}|h\rangle = \partial\phi_h(0)|0\rangle$, the action of raising operators $L_{n<0}$ and $G_{r<0}$ on $|h\rangle$ is unspecified by the singular part of the OPEs.

Let us take a look at $T(0)|0\rangle = L_{-2}|0\rangle$ as a counterexample of a highest weight state. According to its infinitesimal transformation (B.2.23), the energy momentum is not a primary field. The failure of its state correspondent $T(0)|0\rangle$ to be of highest weight type can be seen from the non-vanishing of $L_2 L_{-2}|0\rangle = [L_2, L_{-2}]|0\rangle = \frac{c}{2}|0\rangle$.

In analogy to the asymptotic in-state $|h\rangle = \phi_h(0)|0\rangle$ inserted at infinite past $z\to 0$, an asymptotic out-state $\langle h|$ created by $\phi_h(|z|\to\infty)$ at infinite future is defined by orthonormality $\langle h_i|h_j\rangle = \delta_{i,j}$[3]. The two point function $\langle\phi_h(z_1)\phi_h(z_2)\rangle = z_{12}^{-2h}$ implies that we need a compensating power of z^{2h}:

$$\langle h| \;=\; \lim_{|z|\to\infty} z^{2h}\,\langle 0|\,\phi_h(z) \quad\Rightarrow\quad \phi_n^\dagger \;=\; \phi_{-n} \tag{B.3.35}$$

This definition is quite useful in section (2.2) to derive the covariant OPE of NS fermions approaching a spin field.

B.3.2 Neveu Schwarz and Ramond sector

Conformal fields of half-odd integer conformal dimension are anticommuting (or Grassmann-odd or fermionic) variables, this follows from the two point function $\langle\phi_h(z_1)\phi_h(z_2)\rangle = z_{12}^{-2h}$ being

[3]There might exist several primary fields ϕ_i,ϕ_j of the same conformal weight $h_i = h_j$. In that case, the highest weight states $\phi_i(0)|0\rangle$ and $\phi_j(0)|0\rangle$ are understood to be orthogonal.

antisymmetric in $z_1 \leftrightarrow z_2$ if $h \in \mathbb{Z} + \frac{1}{2}$. Since observables are commuting (or bosonic) objects, fermionic variables can only enter them in bilinears. That is why the $\phi_{h\in\mathbb{Z}+1/2}$ of non-integer weight are defined up to a sign. Consequently they can be either single- or double valued in periodic directions $\phi_h(\mathrm{e}^{2\pi i}z) = \pm\phi_h(z)$.

On the complex plane, periodic fields of conformal dimension $h \in \mathbb{Z} + \frac{1}{2}$ are said to live in the Neveu Schwarz (NS) sector, $\phi_h^{\mathrm{NS}}(\mathrm{e}^{2\pi i}z) = +\phi_h^{\mathrm{NS}}(z)$ whereas the antiperiodic counterpart is referred to as the Ramond (R) sector $\phi_h^{\mathrm{R}}(\mathrm{e}^{2\pi i}z) = -\phi_h^{\mathrm{R}}(z)$. The situation on the cylinder is reversed because of the relative Jacobian factor $(iz)^{-h}$. The NS Laurent expansion on the plane must have half odd integer moding $r \in \mathbb{Z} + \frac{1}{2}$ to ensure integer z powers whereas the R Laurent expansion encompasses $r \in \mathbb{Z}$ for fractional z exponents:

$$\phi_{h\in\mathbb{Z}+1/2}(z) \ = \ \sum_r \phi_r \, z^{-r-h} \ , \qquad r \in \begin{cases} \mathbb{Z} + \frac{1}{2} & : \ \text{NS sector} \\ \mathbb{Z} & : \ \text{R sector} \end{cases} \tag{B.3.36}$$

In particular, this applies to the supercurrent modes G_r subject to the algebra (B.2.29).

Free fermions are single valued, so the state $\phi_{h\in\mathbb{Z}+\frac{1}{2}}(0)|0\rangle$ belongs to the NS sector and we have $\phi_h \equiv \phi_h^{\mathrm{NS}}$ by default. The state content of the R sector is constrained by the unitarity bound $h \geq \frac{c}{24}$ set by the $r = 0$ case of

$$0 \ \leq \ \langle h| \, G_r \, G_{-r} \, |h\rangle \ = \ \left(2h \ + \ \frac{c}{3}\left[r^2 \ - \ \tfrac{1}{4}\right]\right) \langle h|h\rangle \tag{B.3.37}$$

Since the bosonic Virasoro algebra $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$ is a subalgebra of the Ramond super Virasoro algebra, the highest weight states of the Ramond sector must be created from the vacuum by ordinary conformal fields of $h \geq \frac{c}{24}$. They are referred to as spin fields S, their role is to create branch cuts in the OPE algebra of fermionic fields such as $G(z)S(0) \sim z^{-3/2}\tilde{S}(0)$ (with $\tilde{S}$ denoting another spin field of the same conformal dimension). The main body of this work contains detailed discussions of spin fields, starting in section 2.2, that is why we will be very brief in this appendix.

By definition, spin fields $S(0)$ inserted at the origin of the plane transform the NS ground state into R ground states and therefore interpolate between the two sectors. More generally, their action on the space of states is off-diagonal

$$\begin{pmatrix} |\mathrm{NS}\rangle' \\ |\mathrm{R}\rangle' \end{pmatrix} \ = \ \begin{pmatrix} 0 & S \\ S & 0 \end{pmatrix} \begin{pmatrix} |\mathrm{NS}\rangle \\ |\mathrm{R}\rangle \end{pmatrix} \tag{B.3.38}$$

in contrast to the action of the superfields $(\phi_h, \phi_{h+1/2})$:

$$\begin{pmatrix} |\mathrm{NS}\rangle' \\ |\mathrm{R}\rangle' \end{pmatrix} \ = \ \begin{pmatrix} \phi & 0 \\ 0 & \phi \end{pmatrix} \begin{pmatrix} |\mathrm{NS}\rangle \\ |\mathrm{R}\rangle \end{pmatrix} \tag{B.3.39}$$

Presence of a spin field changes periodicity properties to $\phi^{\rm NS}({\rm e}^{2\pi i}z)S(0) = -\phi^{\rm NS}(z)S(0)$, hence we can identify $\phi^{\rm R} \equiv \phi^{\rm NS}S$ with integer Laurent modes

$$\phi^{\rm R}(z) \;=\; \phi^{\rm NS}(z)\,S(0) \;=\; \sum_{r\in\mathbb{Z}} \phi^{\rm R}_r\, z^{-r-h}\, S(0) \tag{B.3.40}$$

The R sector modes can be extracted from $\phi^{\rm NS}_h$ via

$$\phi^{\rm R}_r\, S(0) \;=\; \oint \frac{{\rm d}z}{2\pi i}\, z^{r+h-1}\, \phi^{\rm NS}(z)\, S(0)\ . \tag{B.3.41}$$

The half odd integer z powers in the $\phi^{\rm NS}(z)S(0)$ OPE compensate for $r+h \in \mathbb{Z}+\frac{1}{2}$. One should not think of $\phi^{\rm R}_{r\in\mathbb{Z}}$ and $\phi^{\rm NS}_{r\in\mathbb{Z}+1/2}$ as belonging to separate superfields. Instead, both types of Laurent modes are associated with a single superfield whose fermionic component gets modified in presence of a spin field.

B.3.3 Descendant fields

We have seen that highest weight states $|h\rangle$ of the super Virasoro algebra are in one-to-one correspondence with the lower component of primary superfields $(\phi_h, \phi_{h+1/2})$. The purpose of this subsection is to relate so-called descendant states $L_{-n}|h\rangle$ and $G_{-r}|h\rangle$ to conformal fields. The integral representation (B.2.28) of the modes allows to rewrite descendants as a field insertion at the origin:

$$L_{-n}\,|h\rangle \;=\; \oint \frac{{\rm d}z}{2\pi i}\, z^{1-n}\, T(z)\, \phi_h(0)\, |0\rangle \quad =: \quad (L_{-n}\,\phi_h)(0)\,|0\rangle \tag{B.3.42}$$

$$G_{-r}\,|h\rangle \;=\; \oint \frac{{\rm d}z}{2\pi i}\, z^{1/2-r}\, G(z)\, \phi_h(0)\, |0\rangle \quad =: \quad (G_{-r}\,\phi_h)(0)\,|0\rangle \tag{B.3.43}$$

One can read off the corresponding descendant fields which can be defined anywhere on the complex plane:

$$(L_{-n}\,\phi_h)(w) \;=\; \oint \frac{{\rm d}z}{2\pi i}\, z^{1-n}\, T(z)\, \phi_h(w) \tag{B.3.44}$$

$$(G_{-r}\,\phi_h)(w) \;=\; \oint \frac{{\rm d}z}{2\pi i}\, z^{1/2-r}\, G(z)\, \phi_h(w) \tag{B.3.45}$$

The simplest descendant fields follow from the singular parts of the OPEs (B.2.17),

$$(L_{-1}\,\phi_h) \;=\; \partial\phi_h\ , \qquad (G_{-1/2}\,\phi_h) \;=\; \tfrac{1}{2}\,\phi_{h+1/2}\ , \tag{B.3.46}$$

the general cases involve regular terms from the OPEs of $T(z)$ and $G(z)$ with $\phi_h(w)$. The procedure (B.3.44) and (B.3.45) of taking descendant fields can of course be iterated, e.g. $(L_{-1})^n\phi_h = \partial^n\phi_h$ and

$$L_{-n_1}\, L_{-n_2}\, \phi_h(w) \;=\; L_{-n_1} \oint \frac{{\rm d}z_2}{2\pi i}\, z_2^{1-n_2}\, T(z_2)\, \phi_h(w)$$

$$= \oint \frac{\mathrm{d}z_1}{2\pi i} \oint \frac{\mathrm{d}z_2}{2\pi i} \, z_1^{1-n_1} \, z_2^{1-n_2} \, T(z_1) \, T(z_2) \, \phi_h(w) \ . \tag{B.3.47}$$

Descendants with respect to the supercurrent naturally appear in the context of superghost picture changing, see section 3.1.6.

Once a correlation function among primary fields is known, then it is straightforward to get a handle on the associated descendant correlator. Superconformal Ward identities allow to derive differential operators which act on primary correlators if some fields are replaced by descendants $\phi_h \mapsto (L_{-n}\phi_h)$. That is why we restrict our attention to correlation functions of primary fields, see the following section (B.4) for free bosons and chapter 6 for the interacting CFT of the RNS superstring.

B.4 Correlation functions for free bosons

The CFT of string coordinates X^m plays a key role for the structure of superstring amplitudes. The spacetime momentum k^m of a physical state is generated by a plane wave $\mathrm{e}^{ik\cdot X}$ factor in the vertex operator which determine the kinematic poles of and relations between color ordered amplitudes, see section 5.3. Exponentials of free bosons also occur in the bosonized representation of the worldsheet fermions ψ^m and spin fields S_A of the RNS CFT in terms of chiral bosons $H^j(z)$.

B.4.1 Identifying plane waves as primary fields

It might seem a bit puzzling at first glance that the worldsheet field X^m which does not fit into a representation of the conformal group exponentiates to a conformal primary $\mathrm{e}^{ik\cdot X}$. The purpose of this subsection is to prove that $\mathrm{e}^{ik\cdot X}$ transforms as a primary of weight $h(\mathrm{e}^{ik\cdot X}) = \alpha' k^2$ like claimed in subsection 3.1.4.

The first step is to derive the OPE (3.1.18) on the basis of $i\partial X^m(z) iX^n(w) \sim \frac{2\alpha'\eta^{mn}}{z-w} + \ldots$. It rests on expanding the exponential and recognizing the number n of single contractions between $i\partial X^m$ and $(ik\cdot X)^n$:

$$\begin{aligned} i\partial X^m(z)\,\mathrm{e}^{ik\cdot X(w)} &= \sum_{n=0}^{\infty} \frac{1}{n!} \, i\partial X^m(z) \, \big(ik\cdot X(w)\big)^n \\ &\sim \sum_{n=0}^{\infty} \frac{1}{n!} \, \frac{n\,2\alpha'\,k^m}{z-w} \, \big(ik\cdot X(w)\big)^{n-1} \ + \ \ldots \\ &= \frac{2\alpha'\,k^m}{z-w} \, \mathrm{e}^{ik\cdot X(w)} \ + \ \ldots \end{aligned} \tag{B.4.48}$$

The conformal dimension follows from the OPE with the energy momentum tensor $T(z) = \frac{1}{4\alpha'} i\partial X^m i\partial X_m(z) + \ldots$:

$$\begin{aligned} T(z)\,\mathrm{e}^{ik\cdot X(w)} &= \frac{1}{4\alpha'} \sum_{n=0}^{\infty} \frac{1}{n!}\, i\partial X^m(z)\, i\partial X_m(z)\, \big(ik\cdot X(w)\big)^n \\ &= \frac{1}{4\alpha'} \sum_{n=0}^{\infty} \frac{1}{n!} \Bigg(\frac{2n\, 2\alpha'\, k_m\, i\partial X^m(z)}{z-w}\, \big(ik\cdot X(w)\big)^{n-1} \\ &\qquad + \frac{n\,(n-1)\,(2\alpha')^2\, k_m\, k^m}{(z-w)^2}\, \big(ik\cdot X(w)\big)^{n-2} \Bigg) \;+\; \ldots \\ &= \frac{\partial\big(\mathrm{e}^{ik\cdot X(w)}\big)}{z-w} \;+\; \frac{\alpha'\, k^2\, \mathrm{e}^{ik\cdot X(w)}}{(z-w)^2} \;+\; \ldots \end{aligned} \tag{B.4.49}$$

The term in the second line follows from the $2n$ single contractions $i\partial X^m \leftrightarrow (ik\cdot X)^n$ whereas the third line is a result of the $n(n-1)$ double contractions $i\partial X^m i\partial X_m \leftrightarrow (ik\cdot X)^n$. The $\alpha'\, k^2$ prefactor of the double pole $\frac{\mathrm{e}^{ik\cdot X(w)}}{(z-w)^2}$ is identified as the conformal weight of $\mathrm{e}^{ik\cdot X(w)}$.

B.4.2 Plane wave correlators

Having identified the plane waves $\mathrm{e}^{ik\cdot X}$ as conformal primary fields, we shall next derive their correlation functions starting from the worldsheet action $\mathcal{S}[X]$ given by (2.1.11). The worldsheet in the closed string sector can be represented by the Riemann sphere[4] S^2. The starting point for computing the plane wave correlator on S^2 is the path integral

$$\begin{aligned} \left\langle \prod_{j=1}^{n} \mathrm{e}^{ik_j\cdot X(z_j,\bar z_j)} \right\rangle_{S^2} &= \int \mathcal{D}X\, \prod_{j=1}^{n} \mathrm{e}^{ik_j\cdot X(z_j,\bar z_j)}\, \mathrm{e}^{-\mathcal{S}[X]} \\ &= \int \mathcal{D}X\, \exp\left(-\frac{1}{4\pi\alpha'} \int \mathrm{d}^2 z\, \partial X_m\, \bar\partial X^m \;+\; i \sum_{j=1}^{n} k_j^m\, X_m(z_j) \right) \\ &=: \int \mathcal{D}X\, \exp\left(\int \mathrm{d}^2 z\, \left\{ \frac{1}{4\pi\alpha'}\, X_m\, \partial\bar\partial X^m \;+\; iJ^m\, X_m \right\} \right) \end{aligned} \tag{B.4.50}$$

where the source term in the second line is defined by $J^m(z,\bar z) := \sum_{j=1} k_j^m \delta^2(z-z_j, \bar z - \bar z_j)$. The integral over X^m configurations is of Gaussian type, so one can solve it by an infinite dimensional generalization of the standard formula $\int \mathrm{d}^n x\, \exp\big(\frac{1}{2} xAx + iJx\big) \sim \exp\big(\frac{1}{2} JA^{-1}J\big)$ (using the Green's function $\frac{1}{4\pi} \ln|z-w|^2$ to "invert" the $\partial\bar\partial$ operator):

$$\left\langle \prod_{j=1}^{n} \mathrm{e}^{ik_j\cdot X(z_j,\bar z_j)} \right\rangle_{S^2} \;\sim\; \exp\left(\pi\alpha' \int \mathrm{d}^2 z \int \mathrm{d}^2 w\; J_m(z,\bar z)\, \frac{1}{4\pi}\, \ln|z-w|^2\, J^m(w,\bar w) \right)$$

[4]This is the one of the few instances of this work where we give a correlation function on a closed string worldsheet, that is why we attach a subscript $\langle \ldots \rangle_{S^2}$ here. Correlators without such a specification are computed on the disk by default.

$$= \prod_{i\neq j}^{n} \exp\left(\frac{\alpha'}{4}\,(k_i\cdot k_j)\,\ln|z_i-z_j|^2\right) \;=\; \prod_{i<j}^{n} |z_{ij}|^{\alpha k_i\cdot k_j} \tag{B.4.51}$$

We have omitted the zero modes x_0^m of the $X^m(z,\bar z)$ in the $\int \mathcal{D}X$ path integral. They contribute a momentum conserving delta function

$$\int \mathrm{d}^D x_0\;\exp\left(i\sum_{j=1}^{n} k_j\cdot x_0\right) \;=\; \frac{1}{(2\pi)^D}\,\delta^D\left(\sum_{j=1}^{n} k_j\right)\,, \tag{B.4.52}$$

but we will drop it for ease of notation in the following work – momentum conservation will always be implicit. So far, no care has been taken about the overall prefactor of scattering amplitudes. Normalization issues are addressed in subsection 5.3.4.

Computing the S matrix for open string states requires correlation functions of plane waves on the disk $\langle\ldots\rangle$ rather than on the sphere $\langle\ldots\rangle_{S^2}$. We can rewrite (B.4.51) more generally as

$$\left\langle \prod_{j=1}^{n} \mathrm{e}^{ik_j\cdot X(z_j,\bar z_j)}\right\rangle_{S^2} \;=\; \exp\left(\sum_{i<j}^{n} k_i^{p_i}\,k_j^{p_j}\,\langle\, iX_{p_i}(z_i,\bar z_i)\,iX_{q_j}(z_j,\bar z_j)\,\rangle_{S^2}\right) \tag{B.4.53}$$

According to subsection 2.4.3, the boundary propagator of X^m has an extra factor of two due to the image charge outside the open string worldsheet,

$$\langle\, iX_{p_i}(z_i)\,iX_{q_j}(z_j)\,\rangle \;=\; 2\alpha'\,\ln|z-w| \;=\; 2\,\langle\, iX_{p_i}(z_i,\bar z_i)\,iX_{q_j}(z_j,\bar z_j)\,\rangle_{S^2}\Big|_{z_i,z_j\in\mathbb{R}}\,, \tag{B.4.54}$$

this effectively doubles the exponents of the $|z_{ij}|$ in (B.4.51):

$$\left\langle \prod_{j=1}^{n} \mathrm{e}^{ik_j\cdot X(z_j\in\mathbb{R})}\right\rangle \;=\; \exp\left(\sum_{i<j}^{n} k_i^{p_i}\,k_j^{p_j}\,\langle\, iX_{p_i}(z_i)\,iX_{q_j}(z_j)\,\rangle\right) \;=\; \prod_{i<j}^{n}|z_{ij}|^{2\alpha' k_i\cdot k_j} \tag{B.4.55}$$

The OPE of two exponentials can be easily inferred from the limiting behavious $z_i\to z_j$:

$$\mathrm{e}^{ik_i\cdot X(z_i)}\,\mathrm{e}^{ik_j\cdot X(z_j)} \;\sim\; (z-w)^{2\alpha' k_i\cdot k_l}\,\mathrm{e}^{i(k_i+k_j)\cdot X(z_j)} \;+\;\ldots \tag{B.4.56}$$

Setting $2\alpha'=1$ reproduces the results (2.2.36) and (2.2.44) on the chiral bosons H^j subject to $H(z)H(w)\sim -\ln|z-w|$, see subsection 2.2.3 and 2.2.4.

B.4.3 Correlators with extra ∂X^m insertions

Since physical vertex operators generically contain ∂X^m fields and higher derivatives thereof, we should next discuss more general correlators involving both plane waves $\mathrm{e}^{ik_j\cdot X}$ and ∂X^m. Their rigorous derivation is based on the rewriting $i\partial X^m \mathrm{e}^{ik\cdot X} = \frac{\partial}{\partial\zeta_m}\mathrm{e}^{ik\cdot X+i\zeta\cdot\partial X}\big|_{\zeta=0}$:

$$\left\langle \prod_{l=1}^{p} i\partial X^{m_l}(z_l)\,\prod_{j=1}^{n}\mathrm{e}^{ik_j\cdot X(z_j)}\right\rangle \;=\; \prod_{l=1}^{p}\left\langle \frac{\partial}{\partial\zeta^l_{m_l}}\,\mathrm{e}^{i\zeta^l\cdot\partial X(z_l)}\,\prod_{j=1}^{n}\mathrm{e}^{ik_j\cdot X(z_j)}\right\rangle\Bigg|_{\zeta^l=0}$$

$$
\begin{aligned}
&= \prod_{l=1}^{p} \frac{\partial}{\partial \zeta^l_{m_l}} \exp\Bigg(\sum_{i<j}^{n} k_i^{r_i}\, k_j^{s_j}\, \langle\, iX_{r_i}(z_i)\, iX_{s_j}(z_j)\,\rangle \; + \; \sum_{i=1}^{p} \sum_{j\neq i}^{n} \zeta^i_{r_i}\, k_j^{s_j}\, \langle\, i\partial X^{r_i}(z_i)\, iX_{s_j}(z_j)\,\rangle \\
&\qquad\qquad + \sum_{i<j}^{p} \zeta^i_{r_i}\, \zeta^j_{s_j}\, \langle\, i\partial X^{r_i}(z_i)\, i\partial X^{s_j}(z_j)\,\rangle \Bigg) \Bigg|_{\zeta^l=0} \\
&= \prod_{l=1}^{p} \frac{\partial}{\partial \zeta^l_{m_l}} \exp\Bigg(2\alpha' \left[\sum_{i<j}^{n} (k_i\cdot k_j)\, \ln|z_{ij}| \; + \; \sum_{i=1}^{p} \sum_{j\neq i}^{n} \frac{\zeta^i\cdot k_j}{z_{ij}} \; + \; \sum_{i<j}^{p} \frac{\zeta^i\cdot\zeta^j}{z_{ij}^2} \right] \Bigg) \Bigg|_{\zeta^l=0}
\end{aligned}
\tag{B.4.57}
$$

Extracting the multilinear piece in the auxiliary variables ζ^l from the exponential is a combinatoric exercise. The result is in lines with the Wick theorem [221] for free field theories: The $h=1$ primary ∂X^m can be removed from the correlation function if we replace it by all the singularities it can produce via open string OPEs. Therefore, the $i\partial X^m(z)\mathrm{e}^{ik\cdot X(w)}$ OPE (B.4.48) suffices to determine the correlation function with one ∂X^m,

$$
\left\langle i\partial X^m(w) \prod_{j=1}^{n} \mathrm{e}^{ik_j\cdot X(z_j\in\mathbb{R})} \right\rangle \; = \; \sum_{l=1}^{n} \frac{2\alpha'\, k_l^m}{w-z_l} \left\langle \prod_{j=1}^{n} \mathrm{e}^{ik_j\cdot X(z_j\in\mathbb{R})} \right\rangle . \tag{B.4.58}
$$

The general correlator with several ∂X^m insertions additionally involves $i\partial X^m(z) i\partial X^n(w) \sim 2\alpha'\eta^{mn}/(z-w)^2 + \ldots$ in the reduction process due to the Wick rule. More precisely, summing over all OPE singularities of the $\partial X^{m_1}(w_1)$ field gives rise to the following recursive prescription:

$$
\begin{aligned}
\left\langle \prod_{l=1}^{p} i\partial X^{m_l}(w_l) \prod_{j=1}^{n} \mathrm{e}^{ik_j\cdot X(z_j\in\mathbb{R})} \right\rangle \; &= \; \sum_{i=2}^{p} \frac{2\alpha'\, \eta^{m_1 m_i}}{(w_1-w_i)^2} \left\langle \prod_{\substack{l=2\\ l\neq i}}^{p} i\partial X^{m_l}(w_l) \prod_{j=1}^{n} \mathrm{e}^{ik_j\cdot X(z_j\in\mathbb{R})} \right\rangle \\
&\quad + \sum_{i=1}^{n} \frac{2\alpha'\, k_i^{m_1}}{w_1-z_i} \left\langle \prod_{l=2}^{p} i\partial X^{m_l}(w_l) \prod_{j=1}^{n} \mathrm{e}^{ik_j\cdot X(z_j\in\mathbb{R})} \right\rangle
\end{aligned}
\tag{B.4.59}
$$

Iterating this prescription boils the correlation function on the left hand side down to a combination of momenta and Minkowski metrics (equivalent to the result of (B.4.57)) and an overall plane wave correlator $\langle \prod_{j=1}^{n} \mathrm{e}^{ik_j\cdot X}\rangle = \prod_{i<j}^{n} |z_{ij}|^{2\alpha' k_i\cdot k_j}$.

The Wick contraction rule for the fermions $\psi^m(z)\psi^n(w) \sim \frac{\eta^{mn}}{z-w} + \ldots$ is even more compact,

$$
\left\langle \prod_{l=1}^{n} \psi^{m_l}(z_l) \right\rangle \; = \; \sum_{i=2}^{n} (-1)^i\, \frac{\eta^{m_1 m_i}}{z_{1i}} \left\langle \prod_{\substack{l=2\\ l\neq i}}^{n} \psi^{m_l}(z_l) \right\rangle \tag{B.4.60}
$$

such that e.g.

$$
\langle\, \psi^m(z_1)\, \psi^n(z_2)\, \psi^p(z_3)\, \psi^q(z_4)\,\rangle \; = \; \frac{\eta^{mn}\,\eta^{pq}}{z_{12}\, z_{34}} \; - \; \frac{\eta^{mp}\,\eta^{nq}}{z_{13}\, z_{24}} \; + \; \frac{\eta^{mq}\,\eta^{np}}{z_{14}\, z_{23}} \, . \tag{B.4.61}
$$

The nontrivial correlation functions of the RNS CFT involve spin fields S_A, they are presented in chapter 6.

B.4.4 Higher spin correlators

Scattering higher spin states requires correlators with a large number of $i\partial X$ insertions. In particular, since we discuss all mass levels n on equal footing in section 9.3 we shall keep the $i\partial X$ number general. Luckily, lots of simplifications occur on the leading Regge trajectory because the $i\partial X^\mu$ are always contracted with the totally symmetric, transverse and traceless wave functions ϕ, this is why we only give a closed formula for such a contraction. Let n_i denote the mass levels in the sense that $k_i^2 = -\frac{n_i}{\alpha'}$ and s_i the number of $i\partial X(z_i)$'s (which does not necessarily coincide with the spin n_i+1 of the state for the purpose of this appendix). Of course, the tensors $\phi_{\mu_1\ldots\mu_s}$ of interest might have further free indices which will be omitted in the following result:

$$\begin{aligned}
&\phi^1_{\mu_1\ldots\mu_{s_1}}\,\phi^2_{\nu_1\ldots\nu_{s_2}}\,\phi^3_{\lambda_1\ldots\lambda_{s_3}}\,\langle \prod_{p_1=1}^{s_1} i\partial X^{\mu_{p_1}}(z_1)\prod_{p_2=1}^{s_2} i\partial X^{\nu_{p_2}}(z_2)\prod_{p_3=1}^{s_3} i\partial X^{\lambda_{p_3}}(z_3)\prod_{j=1}^{3} e^{ik_jX(z_j)}\rangle\\
&\quad = \left(\frac{z_{12}\,z_{13}}{z_{23}}\right)^{n_1-s_1}\left(\frac{z_{12}\,z_{23}}{z_{13}}\right)^{n_2-s_2}\left(\frac{z_{13}\,z_{23}}{z_{12}}\right)^{n_3-s_3} s_1!\,s_2!\,s_3!\sum_{i,j,k\in\mathcal{I}}(2\alpha')^{s_1+s_2+s_3-i-j-k}\\
&\qquad\times\ \frac{(\phi^1\cdot k_2^{s_1-i-j})\,(\phi^2\cdot k_3^{s_2-i-k})\,(\phi^3\cdot k_1^{s_3-j-k})\,\delta^i_{12}\,\delta^j_{13}\,\delta^k_{23}}{i!\,j!\,k!\,(s_1-i-j)!\,(s_2-i-k)!\,(s_3-j-k)!}
\end{aligned}\tag{B.4.62}$$

Shorthands such as $(\phi^1\cdot k_2^{s_1-i-j})\,(\phi^2\cdot k_3^{s_2-i-k})\,\delta^i_{12}$ are introduced in subsection 9.3.2. The summation range $\mathcal{I}$ for the number i,j,k of contractions among the ϕ's is defined as

$$\mathcal{I}\ :=\ \Big\{i,j,k\in\mathbb{N}_0\ :\ s_1-i-j\geq 0\ ,\quad s_2-i-k\geq 0\ ,\quad s_3-j-k\geq 0\Big\}\ . \tag{B.4.63}$$

We will also need the four particle generalization:

$$\begin{aligned}
&\phi^1_{\mu_i}\,\phi^2_{\nu_i}\,\phi^3_{\lambda_i}\,\phi^4_{\rho_i}\,\langle \prod_{p_1=1}^{s_1} i\partial X^{\mu_{p_1}}(z_1)\prod_{p_2=1}^{s_2} i\partial X^{\nu_{p_2}}(z_2)\prod_{p_3=1}^{s_3} i\partial X^{\lambda_{p_3}}(z_3)\prod_{p_4=1}^{s_4} i\partial X^{\rho_{p_4}}(z_4)\prod_{j=1}^{4} e^{ik_jX(z_j)}\rangle\\
&= |z_{12}|^{s+n_1+n_2}\,|z_{13}|^{t+n_1+n_3}\,|z_{14}|^{u+n_1+n_4}\,|z_{23}|^{u+n_2+n_3}\,|z_{24}|^{t+n_2+n_4}\,|z_{34}|^{s+n_3+n_4}\,s_1!\,s_2!\,s_3!\,s_4!\\
&\times\sum_{i,j,k,l,m,n\in\mathcal{J}}\frac{(2\alpha')^{\sum_{p=1}^4 s_p-i-j-k-l-m-n}\,(\delta_{12}/z_{12}^2)^i\,(\delta_{13}/z_{13}^2)^j\,(\delta_{23}/z_{23}^2)^k\,(\delta_{14}/z_{14}^2)^l\,(\delta_{24}/z_{24}^2)^m\,(\delta_{34}/z_{34}^2)^n}{i!\,j!\,k!\,l!\,m!\,n!\,(s_1-i-j-l)!\,(s_2-i-k-m)!\,(s_3-j-k-n)!\,(s_4-l-m-n)!}\\
&\times\left[\phi^1\cdot\left(\frac{k_2}{z_{12}}+\frac{k_3}{z_{13}}+\frac{k_4}{z_{14}}\right)^{s_1-i-j-l}\right]\left[\phi^2\cdot\left(\frac{k_1}{z_{21}}+\frac{k_3}{z_{23}}+\frac{k_4}{z_{24}}\right)^{s_2-i-k-m}\right]\\
&\times\left[\phi^3\cdot\left(\frac{k_1}{z_{31}}+\frac{k_2}{z_{32}}+\frac{k_4}{z_{34}}\right)^{s_3-j-k-n}\right]\left[\phi^4\cdot\left(\frac{k_1}{z_{41}}+\frac{k_2}{z_{42}}+\frac{k_3}{z_{43}}\right)^{s_4-l-m-n}\right]
\end{aligned}\tag{B.4.64}$$

In this case, summation variables i,j,k,l,m,n are delimited as follows:

$$\begin{aligned}
\mathcal{J}\ :=\ \Big\{&i,j,k,l,m,n\in\mathbb{N}_0\ :\ s_1-i-j-l\geq 0\ ,\quad s_2-i-k-m\geq 0\ ,\\
&s_3-j-k-n\geq 0\ ,\quad s_4-l-m-n\geq 0\Big\}
\end{aligned}\tag{B.4.65}$$

The four point amplitudes given in subsections 9.3.4 to 9.3.7 with one higher spin state of mass level n at z_4 and massless states at $z_{1,2,3}$ require

$$\begin{aligned}\phi_{\mu_1 \dots \mu_s} \langle i\partial X^{\mu_1} \dots i\partial X^{\mu_s}(z_4) \prod_{j=1}^{4} e^{ik_j X(z_j)} \rangle \quad &= \quad |z_{12}|^s \, |z_{13}|^t \, |z_{23}|^u \, |z_{14}|^{u+n} \, |z_{24}|^{t+n} \, |z_{34}|^{s+n} \\ &\quad \times \left[\phi^4 \cdot \left(\frac{k_1}{z_{41}} + \frac{k_2}{z_{42}} + \frac{k_3}{z_{43}} \right)^s \right] \qquad \text{(B.4.66)}\end{aligned}$$

and generalizations with a finite number of extra $i\partial X$ insertions for the gluons.

Appendix C

The spinor helicity formalism

This appendix introduces the spinor helicity formalism, a unifying organization scheme for polarization wave functions of bosons and fermions in $D = 4$ spacetime dimensions. Remarkably, three different groups developed these methods independently [316, 317, 318]. They give rise to severe vanishing theorems on gluon amplitudes of certain helicity configurations and extremely compact formulae for the first non-vanishing class of n gluon tree amplitudes, the so-called MHV amplitudes. Also, the formalism increases the efficiency of SUSY Ward identities: They can be solved much more explicitly for tree amplitudes involving n boson and m fermions in terms of the corresponding $n \pm 2$ boson, $m \mp 2$ fermion amplitude of the same multiplet.

The key ingredient of the spinor helicity formalism is the one-to-one correspondence between vectors and bispinors of opposite helicities in $D = 4$ spacetime dimensions. This is reflected in the relations

$$\sigma^{\mu}_{a\dot{b}}\, \sigma_{\mu c\dot{d}} \;\; = \;\; -\,2\, \varepsilon_{ac}\, \varepsilon_{\dot{b}\dot{d}} \;, \qquad \sigma^{\mu}_{a\dot{b}}\, \bar{\sigma}^{\dot{b}a}_{\nu} \;\; = \;\; -\,2\, \delta^{\mu}_{\nu} \tag{C.0.1}$$

for the four dimensional sigma matrices. Higher dimensional Lorentz groups involve higher rank p forms in the tensor product decomposition of bispionors. That is why the development of a higher dimensional spinor helicity formalism poses a difficult challenge. Recent work generalizes spinor helicity methods to $D = 6$ [338] and to $D = 10$ dimensions [339].

C.1 Massless polarization tensors

The basic building block for massless wave functions of any spin is the Weyl spinor polarization u^a of a spin $1/2$ fermion. The massless Dirac equation $u^a \, \not{k}_{a\dot{b}} = 0$ implies that it depends on the momentum, so we will use the suggestive notation

$$u^a(k) \;\; = \;\; k^a \;, \qquad \bar{u}_{\dot{a}}(k) \;\; = \;\; \bar{k}_{\dot{a}} \tag{C.1.2}$$

in the following. The massless Dirac equation for fermions implies that only one chiral half is nonzero. Therefore, the two polarization states of spin 1/2 fermions read as follows in Dirac spinor notation:

$$U^+(k) \;=\; \begin{pmatrix} k_a \\ 0 \end{pmatrix}, \qquad U^-(k) \;=\; \begin{pmatrix} 0 \\ \bar{k}^{\dot{a}} \end{pmatrix} \tag{C.1.3}$$

The 2×2 matrix $(k^\mu \sigma_\mu)_{a\dot{b}}$ has a determinant proportional to the norm $k^\mu k_\mu$ which vanishes for the lighlike momentum of a massless state. According to basic linear algebra, any 2×2 matrix with vanishing determinant can be factorized as follows:

$$\left.\begin{array}{rcl} \det(k^\mu \sigma_\mu) & \sim & k^2 \;=\; 0 \\ \det(k^\mu \bar{\sigma}_\mu) & \sim & k^2 \;=\; 0 \end{array}\right\} \;\Rightarrow\; \left\{\begin{array}{rcl} (k^\mu \sigma_\mu)_{a\dot{b}} & = & -\,k_a \bar{k}_{\dot{b}} \\ (k^\mu \bar{\sigma}_\mu)^{\dot{a}b} & = & -\,\bar{k}^{\dot{a}} k^b \end{array}\right. \tag{C.1.4}$$

These relations can be easily inverted by means of (C.0.1):

$$k_\mu \;=\; \frac{1}{2}\, k_a \bar{k}_{\dot{a}} \bar{\sigma}_\mu^{\dot{a}a} \;=\; \frac{1}{2}\, \bar{k}^{\dot{a}} k^a \sigma_{\mu a \dot{a}} \tag{C.1.5}$$

Let us introduce the following bracket notation as a shorthand for spinor products:

$$\left.\begin{array}{rcl} \langle p\, q\rangle & = & \bar{p}_{\dot{a}} \bar{q}^{\dot{a}} \;=\; -\,\langle q\, p\rangle \\ [p\, q] & = & p^a q_a \;=\; -\,[q\, p] \end{array}\right\} \;\Rightarrow\; \langle p\, q\rangle\, [q\, p] \;=\; -\,2\, p^\mu q_\mu \tag{C.1.6}$$

The antisymmetric nature of the four dimensional charge conjugation matrix $\varepsilon_{ab} = \varepsilon_{[ab]}$ and $\varepsilon^{\dot{a}\dot{b}} = \varepsilon^{[\dot{a}\dot{b}]}$ implies that, for commuting spinors p, q, also the brackets are antisymmetric in their two arguments. In particular, we have $\langle p\, p\rangle = [p\, p] = 0$ which allows for a quick check that the massless Dirac equation for spin 1/2 fermions (C.1.3) is consistent with the bispinor representation (C.1.4) of $k_\mu \sigma^\mu$.

C.1.1 Massless spin one

The polarization of massless spin one particles can be described by a transverse vector ξ_μ subject to a gauge freedom $\xi_\mu \equiv \xi_\mu + k_\mu \Lambda$ with scalar gauge parameter Λ. It has only two physically meaningful components corresponding to the $+1$ and -1 helicity states. A more natural basis for the two physical states ξ_μ^+ and ξ_μ^- involves the spinors $k_a, \bar{k}_{\dot{a}}$ which form the momentum vector k_μ via (C.1.5):

$$\xi_\mu^+(k,r) \;=\; \frac{\bar{r}_{\dot{a}}\, \bar{\sigma}_\mu^{\dot{a}b}\, k_b}{\sqrt{2}\, \langle k\, r\rangle}, \qquad \xi_\mu^-(k,r) \;=\; \frac{\bar{k}_{\dot{a}}\, \bar{\sigma}_\mu^{\dot{a}b}\, r_b}{\sqrt{2}\, [r\, k]} \tag{C.1.7}$$

The gauge freedom is represented by a reference spinor r_a which can be arbitrarily chosen as long as $r^a k_a \neq 0$. In order to see that r carries no physical information, one can compute the difference

$$\xi_\mu^\pm(k, r_1) \;-\; \xi_\mu^\pm(k, r_2) \;=\; k_\mu\, \Lambda(r_1, r_2) \tag{C.1.8}$$

which corresponds to a gauge transformation with gauge parameter Λ depending on $r_{1,2}$. Therefore, physical amplitudes cannot depend on the choice of the reference spinor r_a and we can pick the choice which leads to the maximal simplification.

C.1.2 Massless spin 3/2

Massless spin 3/2 fermions are described by a vector spinorial wavefunction χ^a_μ or $\bar{\chi}^\mu_{\dot{a}}$ subject to transversality, σ tracelessness $\chi^a_\mu \sigma^\mu_{a\dot{b}}$, massless Dirac equation and gauge freedom $\chi^a_\mu \equiv \chi^a_\mu + k_\mu \Lambda^a$. These constraints leave two physical degrees of freedom with helicities $\pm\frac{3}{2}$. In Dirac spinor notation:

$$U^+_\mu(k,r) = \frac{\bar{r}_{\dot{b}}\, \bar{\sigma}^{\dot{b}c}_\mu\, k_c}{\sqrt{2}\,\langle k\, r\rangle} \begin{pmatrix} k_a \\ 0 \end{pmatrix}, \qquad U^-_\mu(k,r) = \frac{\bar{k}_{\dot{b}}\, \bar{\sigma}^{\dot{b}c}_\mu\, r_c}{\sqrt{2}\,[r\, k]} \begin{pmatrix} 0 \\ \bar{k}^{\dot{a}} \end{pmatrix} \tag{C.1.9}$$

Again, a change of reference momentum $r_1 \mapsto r_2$ amonts to performing a gauge transformation

$$U^\pm_\mu(k,r_1) - U^\pm_\mu(k,r_2) = k_\mu\, \Lambda(r_1,r_2) \tag{C.1.10}$$

with some $r_{1,2}$ dependent Dirac spinor Λ.

C.1.3 Massless spin two

For completeless, we shall also give the helicity wave functions for massless spin two particles here. They are described by symmetric, transverse and traceless rank two tensors $\alpha_{\mu\nu}$ with gauge freedom $\alpha_{\mu\nu} \equiv \alpha_{\mu\nu} + k_\mu \Lambda_\nu + k_\nu \Lambda_\mu$ as rich as the infinitesimal diffeomorphisms. Therefore, two physical states remain

$$\alpha^+_{\mu\nu}(k,r) = \frac{\bar{\sigma}^{\dot{a}b}_\mu\, \bar{\sigma}^{\dot{c}d}_\nu\, \bar{r}_{\dot{a}}\, k_b\, \bar{r}_{\dot{c}}\, k_d}{2\,\langle k\, r\rangle\, \langle k\, r\rangle}, \qquad \alpha^-_{\mu\nu}(k,r) = \frac{\bar{\sigma}^{\dot{a}b}_\mu\, \bar{\sigma}^{\dot{c}d}_\nu\, \bar{k}_{\dot{a}}\, r_b\, \bar{k}_{\dot{c}}\, r_d}{2\,[k\, r]\, [k\, r]} \tag{C.1.11}$$

which again involve a gauge spinor r:

$$\alpha^\pm_{\mu\nu}(k,r_1) - \alpha^\pm_{\mu\nu}(k,r_2) = k_\mu\, \Lambda_\nu(r_1,r_2) + k_\nu\, \Lambda_\mu(r_1,r_2) \tag{C.1.12}$$

They can be identified as the "square" of two spin one eigenstates (C.1.7):

$$\alpha^\pm_{\mu\nu}(k,r) = \xi^\pm_{(\mu}(k,r) \otimes \xi^\pm_{\nu)}(k,r) \tag{C.1.13}$$

Note that tracelessness automatically follows from $0 = \langle p\, p\rangle = [p\, p]$.

C.2 Massive polarization tensors

Although the spinor helicity formalism is most efficient for the description of massless particles, we can still apply it to massive states and achieve a higher compactness for their scattering amplitudes [319, 320, 321]. As explained in chapter 9, helicity selection rules also exist for amplitudes which involve massive states.

Massive spin j particle in four dimensions gives rise to $2j+1$ physical degrees of freedom according to the stabilizer subgroup $SO(3)$ of the rest frame momentum $k^\mu = (M,0,0,0)$. In contrast to the massless situation where the lightlike momentum necessarily singles out a spatial direction, the rest frame momentum of massive particles does not suggest a preferred quantization axis. To handle this freedom, we decompose the timelike momentum k^μ into two lighlike ones p^μ, q^μ:

$$k^\mu \;=\; p^\mu \;+\; q^\mu \;, \qquad p^2 \;=\; q^2 \;=\; 0 \;, \qquad k^2 \;=\; -\,m^2 \;=\; 2\,p^\mu\,q_\mu \tag{C.2.14}$$

The spatial components of p^μ carry the information about which spatial angular momentum component is diagonalized. Now, $k_\mu \sigma^\mu$ can still be written as a sum over bispinors although $\det(k_\mu \sigma^\mu) \neq 0$ for massive states:

$$(k_\mu\,\sigma^\mu)_{a\dot b} \;=\; -\,p_a\,\bar p_{\dot b} \;-\; q_a\,\bar q_{\dot b} \tag{C.2.15}$$

Of course, the choice of (p^μ, q^μ) decomposition should not affect physical observables such as unpolarized cross sections: Once the modulus squared amplitudes with all the $2j+1$ polarization are summed over, all the $p_a, q_a, \bar p_{\dot c}, \bar q_{\dot d}$ dependence should drop out in favor of the full massive momentum k^μ. This is an important consistency check on the scattering amplitudes of section 9.2.

C.2.1 Massive spin 1/2

Dirac spinors for massive fermions must have two nonvanishing chiral halves because of the massive Dirac equation $(\not k + m)U = 0$. Using $\langle q\,p\rangle [p\,q] = m^2$, one can verify that

$$U\left(k, +\tfrac{1}{2}\right) \;=\; \begin{pmatrix} \frac{1}{m}\,\langle q\,p\rangle\, q_a \\ \bar p^{\dot a} \end{pmatrix} , \qquad U\left(k, -\tfrac{1}{2}\right) \;=\; \begin{pmatrix} p_a \\ \frac{1}{m}\,[q\,p]\,\bar q^{\dot a} \end{pmatrix} \tag{C.2.16}$$

indeed provides solutions to $(\not k + m)U = 0$. The second argument $\pm\frac{1}{2}$ of the wave function U indicates whether the spin polarization is parallel or antiparallel to the p^μ direction.

C.2.2 Massive spin one

Spin one particles obtain a new longitudinal degree of freedom once they have a nonzero mass. The three physical polarization can be obtained as follows

$$\begin{aligned}
\xi_\mu(k,+1) &= \frac{1}{\sqrt{2}\,m}\,\bar{p}_{\dot{a}}\,\bar{\sigma}_\mu^{\dot{a}b}\,q_b \\
\xi_\mu(k,0) &= \frac{1}{2\,m}\,\bar{\sigma}_\mu^{\dot{a}b}\,(\bar{p}_{\dot{a}}\,p_a \;-\; \bar{q}_{\dot{a}}\,q_a) \\
\xi_\mu(k,-1) &= \frac{1}{\sqrt{2}\,m}\,\bar{q}_{\dot{a}}\,\bar{\sigma}_\mu^{\dot{a}b}\,p_b
\end{aligned} \tag{C.2.17}$$

They are transverse and orthonormal in the sense that $\xi_\mu(k,h_1)\xi^\mu(k,h_2) = -\delta_{h_1+h_2,0}$.

C.2.3 Massive spin 3/2

Massive spin 3/2 particles have four polarization states subject to a massive Dirac equation and transversality:

$$\begin{aligned}
U_\mu\left(k,+\tfrac{3}{2}\right) &= \frac{1}{\sqrt{2}\,m}\begin{pmatrix} \frac{1}{m}\,\langle q\,p\rangle\,q_a \\ \bar{p}^{\dot{a}} \end{pmatrix}\bar{p}_{\dot{b}}\,\bar{\sigma}_\mu^{\dot{b}c}\,q_c \\
U_\mu\left(k,+\tfrac{1}{2}\right) &= \frac{\bar{\sigma}_\mu^{\dot{b}c}}{\sqrt{6}\,m}\left[\begin{pmatrix} \frac{1}{m}\,\langle q\,p\rangle\,q_a \\ \bar{p}^{\dot{a}} \end{pmatrix}(\bar{p}_{\dot{b}}\,p_c \;-\; \bar{q}_{\dot{b}}\,q_c) \;+\; \begin{pmatrix} \frac{1}{m}\,\langle q\,p\rangle\,p_a \\ -\,\bar{q}^{\dot{a}} \end{pmatrix}\bar{p}_{\dot{b}}\,q_c\right] \\
U_\mu\left(k,-\tfrac{1}{2}\right) &= \frac{\bar{\sigma}_\mu^{\dot{b}c}}{\sqrt{6}\,m}\left[\begin{pmatrix} p_a \\ \frac{1}{m}\,[q\,p]\,\bar{q}^{\dot{a}} \end{pmatrix}(\bar{p}_{\dot{b}}\,p_c \;-\; \bar{q}_{\dot{b}}\,q_c) \;+\; \begin{pmatrix} -\,q_a \\ \frac{1}{m}\,[q\,p]\,\bar{p}^{\dot{a}} \end{pmatrix}\bar{q}_{\dot{b}}\,p_c\right] \\
U_\mu\left(k,-\tfrac{3}{2}\right) &= \frac{1}{\sqrt{2}\,m}\begin{pmatrix} p_a \\ \frac{1}{m}\,[q\,p]\,\bar{q}^{\dot{a}} \end{pmatrix}\bar{q}_{\dot{b}}\,\bar{\sigma}_\mu^{\dot{b}c}\,p_c
\end{aligned} \tag{C.2.18}$$

C.2.4 Massive spin two

Let us finally give the five physical components of a massive spin two tensor, described by a transverse and traceless wavefunction:

$$\begin{aligned}
\alpha_{\mu\nu}(k,+2) &= \frac{1}{2\,m^2}\,\bar{\sigma}_\mu^{\dot{a}b}\,\bar{\sigma}_\nu^{\dot{c}d}\,\bar{p}_{\dot{a}}\,q_b\,\bar{p}_{\dot{c}}\,q_d \\
\alpha_{\mu\nu}(k,+1) &= \frac{1}{4\,m^2}\,\bar{\sigma}_\mu^{\dot{a}b}\,\bar{\sigma}_\nu^{\dot{c}d}\Big((\bar{p}_{\dot{a}}\,p_b \;-\; \bar{q}_{\dot{a}}\,q_b)\,\bar{p}_{\dot{c}}\,q_d \;+\; \bar{p}_{\dot{a}}\,q_b\,(\bar{p}_{\dot{c}}\,p_d \;-\; \bar{q}_{\dot{c}}\,q_d)\Big) \\
\alpha_{\mu\nu}(k,0) &= \frac{\bar{\sigma}_\mu^{\dot{a}b}\,\bar{\sigma}_\nu^{\dot{c}d}}{2\sqrt{6}\,m^2}\Big((\bar{p}_{\dot{a}}\,p_b \;-\; \bar{q}_{\dot{a}}\,q_b)\,(\bar{p}_{\dot{c}}\,p_d \;-\; \bar{q}_{\dot{c}}\,q_d) \;-\; \bar{p}_{\dot{a}}\,q_b\,\bar{q}_{\dot{c}}\,p_d \;-\; \bar{q}_{\dot{a}}\,p_b\,\bar{p}_{\dot{c}}\,q_d\Big) \\
\alpha_{\mu\nu}(k,-1) &= \frac{1}{4\,m^2}\,\bar{\sigma}_\mu^{\dot{a}b}\,\bar{\sigma}_\nu^{\dot{c}d}\Big((\bar{q}_{\dot{a}}\,q_b \;-\; \bar{p}_{\dot{a}}\,p_b)\,\bar{q}_{\dot{c}}\,p_d \;+\; \bar{q}_{\dot{a}}\,p_b\,(\bar{q}_{\dot{c}}\,q_d \;-\; \bar{p}_{\dot{c}}\,p_d)\Big) \\
\alpha_{\mu\nu}(k,-2) &= \frac{1}{2\,m^2}\,\bar{\sigma}_\mu^{\dot{a}b}\,\bar{\sigma}_\nu^{\dot{c}d}\,\bar{q}_{\dot{a}}\,p_b\,\bar{q}_{\dot{c}}\,p_d
\end{aligned} \tag{C.2.19}$$

Appendix D

Supplementing material on hypergeometric functions

D.1 Extended set of multiple hypergeometric functions for $n = 6$

In subsection 12.3.5, we have introduced an extended set of $(n-2)!$ hypergeometric functions which appear in n point amplitudes of massless open string states. They are useful for finding new representations of the superstring amplitude with alternative $\mathcal{A}_\sigma^{\text{SYM}}$ bases. Moreover, they will play an essential role in demonstrating its cyclic invariance [11].

The $n = 6$ point superstring disk amplitude can be expressed in terms of six hypergeometric functions, but changing the basis of kinematic building blocks $\mathcal{A}_\sigma^{\text{SYM}}$ involves an extended set with additional 18 functions (12.3.48). The purpose of this appendix is to list these extra functions and to give their relations (12.3.41) to the basis (12.3.45).

The six basis functions $F^{(2_\sigma 3_\sigma 4_\sigma)} := F^{(2_\sigma 3_\sigma 4_\sigma 5)}$ with $\sigma \in S_3$ can be extended by the following integrals:

$$\begin{aligned}
F^{(2354)} &= \prod_{j=2}^{4} \int_{z_{j-1}}^{1} \mathrm{d}z_j \left(\prod_{i<l} |z_{il}|^{-s_{il}} \right) \frac{1}{z_{41}} \frac{s_{12}}{z_{12}} \frac{s_{45}}{z_{54}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} \right) \\
F^{(3254)} &= \prod_{j=2}^{4} \int_{z_{j-1}}^{1} \mathrm{d}z_j \left(\prod_{i<l} |z_{il}|^{-s_{il}} \right) \frac{1}{z_{41}} \frac{s_{13}}{z_{13}} \frac{s_{45}}{z_{54}} \left(\frac{s_{12}}{z_{12}} + \frac{s_{23}}{z_{32}} \right) \\
F^{(5324)} &= \prod_{j=2}^{4} \int_{z_{j-1}}^{1} \mathrm{d}z_j \left(\prod_{i<l} |z_{il}|^{-s_{il}} \right) \frac{1}{z_{41}} \frac{s_{15}}{z_{15}} \frac{s_{24}}{z_{24}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{35}}{z_{53}} \right) \\
F^{(3524)} &= \prod_{j=2}^{4} \int_{z_{j-1}}^{1} \mathrm{d}z_j \left(\prod_{i<l} |z_{il}|^{-s_{il}} \right) \frac{1}{z_{41}} \frac{s_{13}}{z_{13}} \frac{s_{24}}{z_{24}} \left(\frac{s_{15}}{z_{15}} + \frac{s_{35}}{z_{35}} \right)
\end{aligned} \qquad \text{(D.1.1)}$$

$$
\begin{aligned}
F^{(5234)} &= \prod_{j=2}^{4}\int_{z_{j-1}}^{1} \mathrm{d}z_j \left(\prod_{i<l}|z_{il}|^{-s_{il}}\right) \frac{1}{z_{41}}\frac{s_{15}}{z_{15}}\frac{s_{34}}{z_{34}}\left(\frac{s_{12}}{z_{12}}+\frac{s_{25}}{z_{52}}\right)\\
F^{(2534)} &= \prod_{j=2}^{4}\int_{z_{j-1}}^{1} \mathrm{d}z_j \left(\prod_{i<l}|z_{il}|^{-s_{il}}\right) \frac{1}{z_{41}}\frac{s_{12}}{z_{12}}\frac{s_{34}}{z_{34}}\left(\frac{s_{15}}{z_{15}}+\frac{s_{25}}{z_{25}}\right)
\end{aligned}
$$

$$
\begin{aligned}
F^{(2453)} &= \prod_{j=2}^{4}\int_{z_{j-1}}^{1} \mathrm{d}z_j \left(\prod_{i<l}|z_{il}|^{-s_{il}}\right) \frac{1}{z_{31}}\frac{s_{12}}{z_{12}}\frac{s_{35}}{z_{53}}\left(\frac{s_{14}}{z_{14}}+\frac{s_{24}}{z_{24}}\right)\\
F^{(4253)} &= \prod_{j=2}^{4}\int_{z_{j-1}}^{1} \mathrm{d}z_j \left(\prod_{i<l}|z_{il}|^{-s_{il}}\right) \frac{1}{z_{31}}\frac{s_{14}}{z_{14}}\frac{s_{35}}{z_{53}}\left(\frac{s_{12}}{z_{12}}+\frac{s_{24}}{z_{42}}\right)\\
F^{(5423)} &= \prod_{j=2}^{4}\int_{z_{j-1}}^{1} \mathrm{d}z_j \left(\prod_{i<l}|z_{il}|^{-s_{il}}\right) \frac{1}{z_{31}}\frac{s_{15}}{z_{15}}\frac{s_{23}}{z_{23}}\left(\frac{s_{14}}{z_{14}}+\frac{s_{45}}{z_{54}}\right)\\
F^{(4523)} &= \prod_{j=2}^{4}\int_{z_{j-1}}^{1} \mathrm{d}z_j \left(\prod_{i<l}|z_{il}|^{-s_{il}}\right) \frac{1}{z_{31}}\frac{s_{14}}{z_{14}}\frac{s_{23}}{z_{23}}\left(\frac{s_{15}}{z_{15}}+\frac{s_{45}}{z_{45}}\right)\\
F^{(5243)} &= \prod_{j=2}^{4}\int_{z_{j-1}}^{1} \mathrm{d}z_j \left(\prod_{i<l}|z_{il}|^{-s_{il}}\right) \frac{1}{z_{31}}\frac{s_{15}}{z_{15}}\frac{s_{34}}{z_{43}}\left(\frac{s_{12}}{z_{12}}+\frac{s_{25}}{z_{52}}\right)\\
F^{(2543)} &= \prod_{j=2}^{4}\int_{z_{j-1}}^{1} \mathrm{d}z_j \left(\prod_{i<l}|z_{il}|^{-s_{il}}\right) \frac{1}{z_{31}}\frac{s_{12}}{z_{12}}\frac{s_{34}}{z_{43}}\left(\frac{s_{15}}{z_{15}}+\frac{s_{25}}{z_{25}}\right)
\end{aligned} \tag{D.1.2}
$$

$$
\begin{aligned}
F^{(3452)} &= \prod_{j=2}^{4}\int_{z_{j-1}}^{1} \mathrm{d}z_j \left(\prod_{i<l}|z_{il}|^{-s_{il}}\right) \frac{1}{z_{21}}\frac{s_{13}}{z_{13}}\frac{s_{25}}{z_{52}}\left(\frac{s_{14}}{z_{14}}+\frac{s_{34}}{z_{34}}\right)\\
F^{(4352)} &= \prod_{j=2}^{4}\int_{z_{j-1}}^{1} \mathrm{d}z_j \left(\prod_{i<l}|z_{il}|^{-s_{il}}\right) \frac{1}{z_{21}}\frac{s_{14}}{z_{14}}\frac{s_{25}}{z_{52}}\left(\frac{s_{13}}{z_{13}}+\frac{s_{34}}{z_{43}}\right)\\
F^{(5432)} &= \prod_{j=2}^{4}\int_{z_{j-1}}^{1} \mathrm{d}z_j \left(\prod_{i<l}|z_{il}|^{-s_{il}}\right) \frac{1}{z_{21}}\frac{s_{15}}{z_{15}}\frac{s_{23}}{z_{32}}\left(\frac{s_{14}}{z_{14}}+\frac{s_{45}}{z_{54}}\right)\\
F^{(4532)} &= \prod_{j=2}^{4}\int_{z_{j-1}}^{1} \mathrm{d}z_j \left(\prod_{i<l}|z_{il}|^{-s_{il}}\right) \frac{1}{z_{21}}\frac{s_{14}}{z_{14}}\frac{s_{23}}{z_{32}}\left(\frac{s_{15}}{z_{15}}+\frac{s_{45}}{z_{45}}\right)\\
F^{(5342)} &= \prod_{j=2}^{4}\int_{z_{j-1}}^{1} \mathrm{d}z_j \left(\prod_{i<l}|z_{il}|^{-s_{il}}\right) \frac{1}{z_{21}}\frac{s_{15}}{z_{15}}\frac{s_{24}}{z_{42}}\left(\frac{s_{13}}{z_{13}}+\frac{s_{35}}{z_{53}}\right)\\
F^{(3542)} &= \prod_{j=2}^{4}\int_{z_{j-1}}^{1} \mathrm{d}z_j \left(\prod_{i<l}|z_{il}|^{-s_{il}}\right) \frac{1}{z_{21}}\frac{s_{13}}{z_{13}}\frac{s_{24}}{z_{42}}\left(\frac{s_{15}}{z_{15}}+\frac{s_{35}}{z_{35}}\right)
\end{aligned} \tag{D.1.3}
$$

In (12.3.64) we have displayed the relation (12.3.41) for one particular basis π. Here, we want to present the relations (12.3.41) for two other choices of basis. For the new basis $\pi \in \{(1,2,4,5,3,6),\ (1,4,2,5,3,6),\ (1,5,4,2,3,6),\ (1,4,5,2,3,6),\ (1,5,2,4,3,6),\ (1,2,5,4,3,6)\}$

we have

$$K_\pi{}^\sigma \;=\; s_{36}^{-1}$$

$$\begin{pmatrix} t_1 - s_1 & s_{13} & 0 & 0 & 0 & t_1 - s_1 + s_3 \\ 0 & 0 & s_3 + s_{13} & s_{13} & t_1 - s_1 + s_3 & 0 \\ \frac{s_1(t_3-s_4)(s_{15}+s_{45})}{t_{145}s_{15}} & \frac{(s_{36}-s_1)s_{13}(s_{15}+s_{45})}{t_{145}s_{15}} & \frac{-(s_3+s_{13})s_{14}s_{25}}{t_{145}s_{15}} & \frac{-s_{13}s_{14}s_{25}}{s_{145}s_{15}} & \frac{d_8 s_{14}s_{35}}{t_{145}s_{15}} & \frac{s_1 s_{35}(s_{15}+s_{45})}{t_{145}s_{15}} \\ \frac{s_1(s_4-t_3)}{t_{145}} & \frac{(s_1-s_{36})s_{13}}{t_{145}} & \frac{(s_3+s_{13})d_5}{s_{145}} & \frac{s_{13}d_5}{t_{145}} & \frac{-(s_1+s_{24})s_{35}}{t_{145}} & \frac{-s_1 s_{35}}{t_{145}} \\ \frac{s_1 s_4(s_1-t_1)}{t_{125}s_{15}} & \frac{-s_1 s_4 s_{13}}{s_{125}s_{15}} & \frac{s_{14}(s_2+s_{35})d_3}{t_{125}s_{15}} & \frac{s_{13}d_3 d_7}{t_{125}s_{15}} & \frac{s_{14}s_{35}d_3}{t_{125}s_{15}} & \frac{s_1(s_4-s_{36})s_{35}}{s_{125}s_{15}} \\ \frac{(t_1-s_1)d_6}{t_{125}} & \frac{s_{13}d_6}{t_{125}} & \frac{-s_{14}(s_2+s_{35})}{t_{125}} & \frac{-d_7 s_{13}}{t_{125}} & \frac{-s_{14}s_{35}}{t_{125}} & \frac{d_1 s_{35}}{t_{125}} \end{pmatrix} \tag{D.1.4}$$

and the following relation can be checked:

$$\begin{pmatrix} F^{(2453)} \\ F^{(4253)} \\ F^{(5423)} \\ F^{(4523)} \\ F^{(5243)} \\ F^{(2543)} \end{pmatrix}^T = \begin{pmatrix} F^{(2345)} \\ F^{(3245)} \\ F^{(4325)} \\ F^{(3425)} \\ F^{(4235)} \\ F^{(2435)} \end{pmatrix}^T K^{-1} \tag{D.1.5}$$

On the other hand, for the third basis $\pi \in \{(1,3,4,5,2,6),\ (1,4,3,5,2,6),\ (1,5,4,3,2,6),\ (1,4,5,3,2,6),\ (1,5,3,4,2,6),\ (1,3,5,4,2,6)\}$ we have

$$K_\pi{}^\sigma \;=\; s_{26}^{-1}$$

$$\begin{pmatrix} s_1 & s_1 + s_2 & 0 & s_1 - s_3 + t_2 & 0 & 0 \\ 0 & 0 & s_1 - s_3 + t_2 & 0 & s_1 + s_{24} & s_1 \\ \frac{s_1(s_{26}-s_{13})(s_{15}+s_4)}{s_{145}s_{15}} & \frac{-d_9 s_{13}(s_{15}+s_4)}{s_{145}s_{15}} & \frac{d_{10}s_{14}s_{25}}{s_{145}s_{15}} & \frac{s_{13}s_{25}(s_{15}+s_4)}{s_{145}s_{15}} & \frac{-s_{14}(s_1+s_{24})s_{35}}{s_{145}s_{15}} & \frac{-s_1 s_{14}s_{35}}{s_{145}s_{15}} \\ \frac{s_1(s_{13}-s_{26})}{s_{145}} & \frac{d_9 s_{13}}{s_{145}} & \frac{-(s_3+s_{13})s_{25}}{s_{145}} & \frac{-s_{13}s_{25}}{s_{145}} & \frac{-d_{12}(s_1+s_{24})}{s_{145}} & \frac{-s_1 d_{12}}{s_{145}} \\ \frac{-s_1 s_4 s_{13}}{s_{246}s_{15}} & \frac{-(s_1+s_2)s_4 s_{13}}{s_{246}s_{15}} & \frac{s_{14}s_{25}d_0}{s_{246}s_{15}} & \frac{s_{13}s_{25}(s_4-s_{26})}{s_{246}s_{15}} & \frac{s_{14}(s_2+s_{25})d_0}{s_{246}s_{15}} & \frac{s_1(s_{26}-s_{14})d_0}{s_{246}s_{15}} \\ \frac{s_1 d_{11}}{s_{246}} & \frac{d_{11}(s_1+s_2)}{s_{246}} & \frac{-s_{14}s_{25}}{s_{246}} & \frac{-(s_3+s_{14})s_{25}}{s_{246}} & \frac{-s_{14}(s_2+s_{25})}{s_{246}} & \frac{s_1(s_{14}-s_{26})}{s_{246}} \end{pmatrix} \tag{D.1.6}$$

and the following relation can be checked:

$$\begin{pmatrix} F^{(3452)} \\ F^{(4352)} \\ F^{(5432)} \\ F^{(4532)} \\ F^{(5342)} \\ F^{(3542)} \end{pmatrix}^T = \begin{pmatrix} F^{(2345)} \\ F^{(3245)} \\ F^{(4325)} \\ F^{(3425)} \\ F^{(4235)} \\ F^{(2435)} \end{pmatrix}^T K^{-1} \tag{D.1.7}$$

Hence, the relations (12.3.64), (D.1.5) and (D.1.7) allow to express the additional set of 18 functions (D.1.1), (D.1.2) and (D.1.3) in terms of the minimal basis (12.3.60).

In the above matrices (D.1.4) and (D.1.6) we have introduced shorthands

$$\begin{aligned} d_1 &= s_3 - s_5 + t_1 , & d_2 &= s_1 - s_4 - s_5 \\ d_3 &= s_3 - s_5 - t_3 , & d_4 &= s_4 + s_5 - s_{13} \\ d_0 &= s_{15} + s_{35} \end{aligned} \tag{D.1.8}$$

like in subsection 12.3.5 as well as

$$\begin{aligned} d_5 &= s_1 + s_{24} - s_{36} , & d_6 &= -s_1 + s_5 + s_{35} \\ d_7 &= s_1 - s_5 + s_{24} - s_{35} , & d_8 &= s_6 - s_4 + s_{13} - s_{24} \\ d_9 &= s_2 - s_6 + t_3 , & d_{10} &= s_1 - s_3 - s_4 + s_6 \\ d_{11} &= s_3 + s_{14} - s_{26} , & d_{12} &= s_{26} - s_3 - s_{13} \end{aligned} \tag{D.1.9}$$

D.2 Power series expansion in α' for $n = 7$ point integrals

In this appendix we give the α' expansions of the 24 functions $F^{\sigma\in S_4}$ appearing in the seven point amplitude (12.3.33). While for $n = 4, 5, 6$ the latter can be found in subsection 12.3.5, here the case $n = 7$ is dealt. The strategy how to compute the power series expansion in α' for any generalized Euler integral is described in [202, 201]. Generically, this task amounts to evaluate generalized Euler–Zagier sums involving many integer sums, which becomes quite tedious for $n \geq 6$. A complementary approach to determine the α' expansion for the basis (12.3.45) can be set up by imposing the factorization properties discussed in section 12.4.

$$\begin{aligned} F^{(2345)} &= 1 - \zeta(2)\ (s_5s_6 + s_1s_7 - t_1t_4 - s_5t_5 + t_4t_5 - s_1t_7 + t_1t_7) \\ &+ \zeta(3)\ (-2s_1s_3s_5 + s_5^2s_6 + s_5s_6^2 + s_1^2s_7 + s_1s_7^2 + 2s_3s_5t_1 + 2s_4s_5t_1 + 2s_1s_5t_2 \\ &+ 2s_1s_5t_3 - 2s_5t_1t_3 + 2s_1s_2t_4 + 2s_1s_3t_4 - 2s_3t_1t_4 - t_1^2t_4 - 2s_1t_2t_4 - t_1t_4^2 - 2s_4s_5t_5 \\ &- s_5^2t_5 + t_4^2t_5 - s_5t_5^2 + t_4t_5^2 - 2s_1s_5t_6 - s_1^2t_7 - 2s_1s_2t_7 + t_1^2t_7 - s_1t_7^2 + t_1t_7^2) + \ldots \\ F^{(2354)} &= -\zeta(2)\ s_{46}\ (s_4 - s_6 + t_5) + \zeta(3)\ s_{46}\ (2s_1s_3 + s_4^2 + s_4s_5 - s_5s_6 - s_6^2 - 2s_3t_1 - 2s_4t_1 \\ &- 2s_1t_2 - 2s_1t_3 + 2t_1t_3 + s_4t_4 - s_6t_4 + 2s_4t_5 + s_5t_5 + t_4t_5 + t_5^2 + 2s_1t_6) + \ldots \\ F^{(2435)} &= \zeta(2)\ (s_3 + t_1 - t_5)\ (s_3 + t_4 - t_7) \\ &+ \zeta(3)\ (2s_1s_2s_3 + 2s_1s_3^2 - s_3^3 + 2s_3^2s_5 + 2s_3s_4s_5 - 2s_3^2t_1 + 2s_3s_5t_1 + 2s_4s_5t_1 - s_3t_1^2 \\ &- 2s_1s_3t_2 - 2s_3s_5t_3 - 2s_5t_1t_3 + 2s_1s_2t_4 + 2s_1s_3t_4 - 2s_3^2t_4 - 3s_3t_1t_4 - t_1^2t_4 - 2s_1t_2t_4 \end{aligned}$$

$$- s_3 t_4^2 - t_1 t_4^2 - 2 s_3 s_5 t_5 - 2 s_4 s_5 t_5 + 2 s_5 t_3 t_5 + s_3 t_4 t_5 + t_4^2 t_5 + s_3 t_5^2 + t_4 t_5^2 - 2 s_1 s_2 t_7$$
$$- 2 s_1 s_3 t_7 + s_3 t_1 t_7 + t_1^2 t_7 + 2 s_1 t_2 t_7 + s_3 t_5 t_7 - t_5^2 t_7 + s_3 t_7^2 + t_1 t_7^2 - t_5 t_7^2) \ + \ \ldots$$

$$F^{(2453)} \ = \ -\zeta(2) \ s_{36} \ (s_3 + t_1 - t_5)$$
$$+ \ \zeta(3) \ s_{36} \ (-2 s_1 s_2 - 2 s_1 s_3 - s_3^2 - 2 s_3 s_4 - 2 s_4 t_1 + t_1^2 + 2 s_1 t_2 + 2 s_3 t_3 + 2 t_1 t_3 + s_3 t_4$$
$$+ t_1 t_4 + 2 s_3 t_5 + 2 s_4 t_5 - 2 t_3 t_5 - t_4 t_5 - t_5^2 + s_3 t_7 + t_1 t_7 - t_5 t_7) \ + \ \ldots$$

$$F^{(2534)} \ = \ \zeta(2) \ s_{46} \ (s_3 + s_6 - t_3 - t_5) \ + \ \zeta(3) \ s_{46} \ (2 s_1 s_3 + 2 s_3^2 + s_3 s_4 - s_3 s_5 + s_3 s_6 + s_4 s_6$$
$$- s_5 s_6 - s_6^2 - 2 s_1 t_2 - 2 s_1 t_3 - 4 s_3 t_3 - s_4 t_3 + s_5 t_3 - s_6 t_3 + 2 t_3^2 - s_3 t_4 - s_6 t_4 + t_3 t_4$$
$$- 3 s_3 t_5 - s_4 t_5 + s_5 t_5 + 3 t_3 t_5 + t_4 t_5 + t_5^2 + 2 s_1 t_6) \ + \ \ldots$$

$$F^{(2543)} \ = \ -\zeta(2) \ s_{36} \ (s_3 + s_6 - t_3 - t_5) \ + \ \zeta(3) \ s_{36} \ (-2 s_1 s_3 - s_3^2 - s_3 s_4 - s_4 s_6 + s_6^2$$
$$+ 2 s_1 t_2 + 2 s_1 t_3 + 2 s_3 t_3 + s_4 t_3 - t_3^2 + s_3 t_4 + s_6 t_4 - t_3 t_4 + 2 s_3 t_5 + s_4 t_5 - 2 t_3 t_5$$
$$- t_4 t_5 - t_5^2 - 2 s_1 t_6 + s_3 t_7 + s_6 t_7 - t_3 t_7 - t_5 t_7) \ + \ \ldots$$

$$F^{(3245)} \ = \ -\zeta(2) \ s_{13} \ (s_2 - s_7 + t_7) \ + \ \zeta(3) \ s_{13} \ (s_1 s_2 + s_2^2 + 2 s_3 s_5 - s_1 s_7 - s_7^2 + s_2 t_1 - s_7 t_1$$
$$- 2 s_5 t_2 - 2 s_5 t_3 - 2 s_2 t_4 - 2 s_3 t_4 + 2 t_2 t_4 + 2 s_5 t_6 + s_1 t_7 + 2 s_2 t_7 + t_1 t_7 + t_7^2) \ + \ \ldots$$

$$F^{(3254)} \ = \ -2 \ \zeta(3) \ s_{13} \ s_{25} \ s_{46} \ + \ \ldots$$

$$F^{(3425)} \ = \ \zeta(2) \ s_{13} \ (s_3 + s_7 - t_2 - t_7) \ + \ \zeta(3) \ s_{13} \ (-s_1 s_3 + s_2 s_3 + 2 s_3^2 + 2 s_3 s_5 - s_1 s_7$$
$$+ s_2 s_7 + s_3 s_7 - s_7^2 - s_3 t_1 - s_7 t_1 + s_1 t_2 - s_2 t_2 - 4 s_3 t_2 - 2 s_5 t_2 - s_7 t_2 + t_1 t_2 + 2 t_2^2$$
$$- 2 s_5 t_3 + 2 s_5 t_6 + s_1 t_7 - s_2 t_7 - 3 s_3 t_7 + t_1 t_7 + 3 t_2 t_7 + t_7^2) \ + \ \ldots$$

$$F^{(3452)} \ = \ \zeta(2) \ s_{13} \ s_{26} \ + \ \zeta(3) \ s_{13} \ s_{26} \ (-s_1 + s_2 - s_7 - t_1 + 2 t_3 - 2 t_6 - t_7) \ + \ \ldots$$

$$F^{(3524)} \ = \ \frac{1}{4} \, \zeta(4) \ s_{13} \ s_{46} \ (10 \ s_{15} s_{24} + 3 \ s_{15} s_{26} + 27 \ s_{24} s_{35} + 10 \ s_{26} s_{35}) \ + \ \ldots$$

$$F^{(3542)} \ = \ \frac{1}{4} \, \zeta(4) \ s_{13} \ s_{26} \ (-7 \ s_{15} s_{24} - 17 \ s_{24} s_{35} + 3 \ s_{15} s_{46} + 10 \ s_{35} s_{46}) \ + \ \ldots$$

$$F^{(4235)} \ = \ -\zeta(2) \ s_{14} \ (s_3 + t_4 - t_7)$$
$$+ \ \zeta(3) \ s_{14} \ (-2 s_2 s_3 - s_3^2 - 2 s_3 s_5 - 2 s_4 s_5 + s_3 t_1 + 2 s_3 t_2 + 2 s_5 t_3 - 2 s_2 t_4 + t_1 t_4$$
$$+ 2 t_2 t_4 + t_4^2 + s_3 t_5 + t_4 t_5 + 2 s_2 t_7 + 2 s_3 t_7 - t_1 t_7 - 2 t_2 t_7 - t_5 t_7 - t_7^2) \ + \ \ldots$$

$$F^{(4253)} \ = \ \zeta(2) \ s_{14} \ s_{36}$$
$$+ \zeta(3) \ s_{14} \ s_{36} \ (2 s_2 + 3 s_3 + 2 s_4 - t_1 - 2 t_2 - 2 t_3 - t_4 - t_5 - t_7) \ + \ \ldots$$

$$F^{(4325)} \ = \ -\zeta(2) \ s_{14} \ (s_3 + s_7 - t_2 - t_7) \ + \ \zeta(3) \ s_{14} \ (-s_2 s_3 - s_3^2 - 2 s_3 s_5 - s_2 s_7 + s_7^2 + s_3 t_1$$
$$+ s_7 t_1 + s_2 t_2 + 2 s_3 t_2 + 2 s_5 t_2 - t_1 t_2 - t_2^2 + 2 s_5 t_3 + s_3 t_5 + s_7 t_5 - t_2 t_5 - 2 s_5 t_6$$
$$+ s_2 t_7 + 2 s_3 t_7 - t_1 t_7 - 2 t_2 t_7 - t_5 t_7 - t_7^2) \ + \ \ldots$$

$$F^{(4352)} \ = \ -\zeta(2) \ s_{14} \ s_{26}$$
$$+ \ \zeta(3) \ s_{14} \ s_{26} \ (-s_2 + s_3 + s_7 + t_1 - t_2 - 2 t_3 + t_5 + 2 t_6 + t_7) \ + \ \ldots$$

$$
\begin{aligned}
F^{(4523)} &= \zeta(2)\ s_{14}\ s_{36} \\
&\quad + \zeta(3)\ s_{14}\ s_{36}\ (2s_2 + 2s_4 - t_1 + t_2 + t_3 - t_4 - t_5 - 3t_6 - t_7)\ +\ \ldots \\
F^{(4532)} &= -\zeta(2)\ s_{14}\ s_{26} \\
&\quad + \zeta(3)\ s_{14}\ s_{26}\ (-s_2 - s_4 + s_7 + t_1 - t_2 - t_3 + t_5 + 2t_6 + t_7)\ +\ \ldots \\
F^{(5234)} &= \zeta(2)\ s_{15}\ s_{46} \\
&\quad + \zeta(3)\ s_{15}\ s_{46}\ (s_4 - s_5 - s_6 + 2t_2 - t_4 - t_5 - 2t_6)\ +\ \ldots \\
F^{(5243)} &= -\zeta(2)\ s_{15}\ s_{36} \\
&\quad + \zeta(3)\ s_{15}\ s_{36}\ (s_3 - s_4 + s_6 - 2t_2 - t_3 + t_4 + t_5 + 2t_6 + t_7)\ +\ \ldots \\
F^{(5324)} &= \frac{1}{4}\,\zeta(4)\ s_{15}s_{46}\ \ (10\ s_{13}s_{24} + 3\ s_{13}s_{26} - 17\ s_{24}s_{35} - 7\ s_{26}s_{35})\ +\ \ldots \\
F^{(5342)} &= \frac{1}{4}\,\zeta(4)\ s_{15}s_{26}\ \ (-7\ s_{13}s_{24} + 3\ s_{13}s_{46} + 10\ s_{24}s_{35} - 7\ s_{35}s_{46})\ +\ \ldots \\
F^{(5423)} &= -\zeta(2)\ s_{15}\ s_{36} \\
&\quad + \zeta(3)\ s_{15}\ s_{36}\ (-s_2 - s_4 + s_6 - t_2 - t_3 + t_4 + t_5 + 2t_6 + t_7)\ +\ \ldots \\
F^{(5432)} &= \zeta(2)\ s_{15}\ s_{26}\ +\ \zeta(3)\ s_{15}\ s_{26}\ (-s_6 - s_7 + t_2 + t_3 - t_5 - t_6 - t_7)\ +\ \ldots
\end{aligned}
$$

There is one function $F^{(3254)}$ starting only at $\zeta(3)\alpha'^3$ and a set of four functions starting not until at $\zeta(4)\alpha'^4$.

Appendix E

Further material for pure spinor computations

This appendix gathers additional material which is relevant for higher point scattering amplitudes computed in the pure spinor formalism. It contains some lengthy expressions which we decided to outsource from chapters 10, 11 and 12 in order to improve the flow of reading.

E.1 The BRST building blocks T_{1234} and T_{12345}

The evaluation of pure spinor superspace expressions in SUSY components requires knowledge of the theta expansion of the superfields involved. Hence, the BRST building blocks must be explicitly expressed in terms of the SYM superfield $[A_\alpha, A_m, W^\alpha, \mathcal{F}_{mn}]$ with θ expansions (10.2.31) in order to make checks against RNS results possible. The rank three building block's superfield content can be gathered by combining equations (11.1.9), (11.1.19) and (11.1.24). Since higher rank analogues become quite lengthy, we decided to defer their presentation to this appendix.

We will give the superfield expressions for T_{ijkl} and T_{ijklm} in the following which allows to obtain the SUSY components of any superstring (and field-theory) amplitude up to $n = 11$ legs in terms of momenta and polarization vectors or -spinors[1].

E.1.1 Redefinition $L_{213141} \mapsto \tilde{T}_{1234}$

The definition of T_{1234} is explained in subsection 11.1.3. The first step from L_{213141} to $\tilde{T}_{1234}$ uses the information from the lower-order redefinitions of L_{21} and L_{2131}. First one rewrites L_{ji}

[1]The use of a `FORM` [311, 312] program like [310] is useful for these kind of calculations.

and L_{jiki} in terms of T_{ij} and T_{ijk} in the RHS of the identity for QL_{213141} given in (11.1.14):

$$\begin{aligned} L_{21} &= T_{12} - \frac{1}{2}\, Q\, D_{12} \\ L_{2131} &= \tilde{T}_{123} - \frac{s_{13}+s_{23}}{2}\, D_{12}\, V_3 + \frac{s_{12}}{2}\, (D_{23}\, V_1 - D_{13}\, V_2) + Q\, S^{(1)}_{123} \end{aligned} \tag{E.1.1}$$

Recall the shorthand $D_{ij} = A_i{\cdot}A_j$. After some algebra one finds the appropriate first redefinition

$$\begin{aligned} \tilde{T}_{1234} &= L_{213141} - \frac{1}{4}\Big((s_{13}+s_{23})\, D_{12}\, QD_{34} + s_{12}\, (D_{13}\, QD_{24} + D_{14}\, QD_{23}) \Big) \\ &+ \frac{1}{2}\Big((s_{13}+s_{23})\, (D_{12}\, T_{34} - D_{34}\, T_{12}) + s_{12}\, (D_{13}\, T_{24} + D_{14}\, T_{23} - D_{23}\, T_{14} - D_{24}\, T_{13}) \Big) \\ &- (s_{14}+s_{24}+s_{34})\, S^{(1)}_{123}\, V_4 - (s_{13}+s_{23})\, S^{(1)}_{124}\, V_3 + s_{12}\, (S^{(1)}_{234}\, V_1 - S^{(1)}_{134} V_2) \end{aligned} \tag{E.1.2}$$

with the required property of

$$\begin{aligned} Q\, \tilde{T}_{1234} &= s_{12}\, (T_{134}\, V_2 + T_{13}\, T_{24} + T_{14}\, T_{23} + V_1\, T_{234}) \\ &\quad + (s_{13}+s_{23})\, (T_{12}\, T_{34} + T_{124}\, V_3) + (s_{14}+s_{24}+s_{34})\, T_{123}\, V_4 \end{aligned} \tag{E.1.3}$$

E.1.2 Redefinition $\tilde{T}_{1234} \mapsto T_{1234}$

It is easy to check that the BRST closed combinations $\tilde{T}_{(ij)k}$ and $\tilde{T}_{[ijk]}$ are inherited by the first three labels of $\tilde{T}_{1234}$, i.e.

$$Q\, (\tilde{T}_{1234} + \tilde{T}_{2134}) = Q\, (\tilde{T}_{1234} + \tilde{T}_{3124} + \tilde{T}_{2314}) = 0\ , \tag{E.1.4}$$

and that there is one additional BRST identity involving the fourth label,

$$Q\, (\tilde{T}_{1234} - \tilde{T}_{1243} + \tilde{T}_{3412} - \tilde{T}_{3421}) = 0\ . \tag{E.1.5}$$

Using the SYM equations of motion in a long sequence of calculations identifies these combinations as BRST-exact

$$\begin{aligned} \tilde{T}_{1234} + \tilde{T}_{2134} &= Q\, R^{(1)}_{1234} \\ \tilde{T}_{1234} + \tilde{T}_{3124} + \tilde{T}_{2314} &= Q\, R^{(2)}_{1234} \\ \tilde{T}_{1234} - \tilde{T}_{1243} + \tilde{T}_{3412} - \tilde{T}_{3421} &= Q\, R^{(3)}_{1234} \end{aligned} \tag{E.1.6}$$

where

$$R^{(1)}_{1234} = -R^{(1)}_{123}\, (k^{123}\cdot A^4) - \frac{s_{12}}{4}\, (D_{13}\, D_{24} + D_{14}\, D_{23}) \tag{E.1.7}$$

$$R^{(2)}_{1234} = -R^{(2)}_{123}\, (k^{123}\cdot A^4) - \frac{1}{4}\big(s_{12}\, D_{23}\, D_{14} + s_{23}\, D_{24}\, D_{13} + s_{13}\, D_{34}\, D_{12}\big) \tag{E.1.8}$$

$$R^{(3)}_{1234} = (k^1\cdot A^2)\big(D_{14}\, (k^4\cdot A^3) - D_{13}\, (k^3\cdot A^4)\big) - (k^2\cdot A^1)\big(D_{24}\, (k^4\cdot A^3) - D_{23}\, (k^3\cdot A^4)\big)$$

$$
\begin{aligned}
&+\ D_{12}\big((k^4\cdot A^3)\,(k^2\cdot A^4)\ -\ (k^3\cdot A^4)\,(k^2\cdot A^3)\big)\\
&+D_{34}\ \big((k^2\cdot A^1)\,(k^4\cdot A^2)\ -\ (k^1\cdot A^2)\,(k^4\cdot A^1)\big)\\
&+\frac{1}{4}\,D_{12}\,D_{34}\,(s_{14}+s_{23}-s_{13}-s_{24})\ +\ (W^1\,\gamma^m\,W^2)\,(W^3\,\gamma_m\,W^4)\ .
\end{aligned}
\tag{E.1.9}
$$

Removing these BRST-exact parts is accomplished by the second redefinition $\tilde{T}_{1234}\longrightarrow T_{1234}$, leading to the rank-four BRST building block

$$
T_{1234}\ =\ \tilde{T}_{1234}\ -\ Q\,S^{(2)}_{1234}\ ,
\tag{E.1.10}
$$

where $S^{(2)}_{1234}$ is defined recursively by

$$
S^{(2)}_{1234}\ =\ \frac{3}{4}\,S^{(1)}_{1234}\ +\ \frac{1}{4}\,(S^{(1)}_{1243}\ -\ S^{(1)}_{3412}\ +\ S^{(1)}_{3421})\ +\ \frac{1}{4}\,R^{(3)}_{1234}
\tag{E.1.11}
$$

$$
S^{(1)}_{1234}\ =\ \frac{1}{2}\,R^{(1)}_{1234}\ +\ \frac{1}{3}\,R^{(2)}_{[12]34}\ .
\tag{E.1.12}
$$

This a posteriori justifies the notation $S^{(p-2)}_{12\ldots p}$ in (11.1.30). To see that (E.1.10), (E.1.11) and (E.1.12) imply $T_{(12)34}=T_{[123]4}=0$ and $T_{1234}-T_{1243}+T_{3412}-T_{3421}=0$, it suffices to check that the following identities hold:

$$
\begin{aligned}
S^{(2)}_{1234}\ +\ S^{(2)}_{2134}\ &=\ R^{(1)}_{1234}\\
S^{(2)}_{1234}\ +\ S^{(2)}_{3124}\ +\ S^{(2)}_{2314}\ &=\ R^{(2)}_{1234}\\
S^{(2)}_{1234}\ -\ S^{(2)}_{1243}\ +\ S^{(2)}_{3412}\ -\ S^{(2)}_{3421}\ &=\ R^{(3)}_{1234}
\end{aligned}
\tag{E.1.13}
$$

E.1.3 The OPE residue $L_{21314151}$

Obtaining $L_{21314151}$ from the expression (11.1.10) for L_{213141} is based on the OPE identity (11.1.7). The most economic way of writing the result packages many of its terms into smaller OPE residues L_{ji}, L_{jiki} and L_{jikili}. Moreover, we will use the shorthand $k^{ij\ldots l}=k^i+k^j+\ldots+k^l$.

$$
\begin{aligned}
L_{21314151}\ &=\ -L_{213141}\,(k^{1234}\cdot A^5)\ -\ \big(L_{213151}\ +\ L_{2131}\,(k^{123}\cdot A^5)\big)\,(k^{123}\cdot A^4)\\
&-\ \Big(L_{214151}\ +\ L_{2141}\,(k^{124}\cdot A^5)\ +\ \big(L_{2151}\ +\ L_{21}\,(k^{12}\cdot A^5)\big)\,(k^{12}\cdot A^4)\Big)\,(k^{12}\cdot A^3)\\
&-\ \Big\{(\lambda\,\gamma^m\,W^5)\,\Big(\frac{1}{4}\,(W^1\,\gamma^{pq}\,\gamma^n\,W^3)\,\mathcal{F}^2_{pq}\,\mathcal{F}^4_{mn}\ +\ \frac{1}{16}\,(W^4\,\gamma_m\,\gamma^{pq}\,\gamma^{rs}\,W^1)\,\mathcal{F}^2_{rs}\,\mathcal{F}^3_{pq}\Big)\\
&\quad+\ (k^1\cdot A^2)\Big[\Big(L_{4151}\ +\ L_{41}\,(k^{14}\cdot A^5)\ +\ \big(L_{51}\ +\ V^1\,(k^1\cdot A^5)\big)\,(k^1\cdot A^4)\Big)\,(k^1\cdot A^3)\\
&\quad+\ L_{314151}\ +\ L_{3141}\,(k^{134}\cdot A^5)\ +\ \big(L_{3151}\ +\ L_{31}\,(k^{13}\cdot A^5)\big)\,(k^{13}\cdot A^4)\Big]\ -\ (1\leftrightarrow 2)\Big\}\\
&+\ (\lambda\,\gamma^m\,W^5)\,\Big((W^1\gamma^nW^2)\,\big(\mathcal{F}^4_{mp}\,\mathcal{F}^{3p}_{n}\ -\ (W^3\,\gamma_m\,W^4)\,k^3_n\ -\ \frac{1}{2}\,(W^4\,\gamma_m\,\gamma_n\,\gamma^p\,W^3)\,(k^1_p+k^2_p)\big)\\
&-\ \frac{1}{2}\,(W^3\,\gamma^{pq}\,\gamma_m\,W^4)\,\mathcal{F}^1_{pr}\,\mathcal{F}^{2r}_{q}\ +(A^1\cdot A^2)\,\big(\mathcal{F}^{3q}_{p}\,\mathcal{F}^4_{mp}\,k^2_q\ +\ (W^3\,\gamma_m\,W^4)\,(k^2\cdot k^3)\big)\Big)
\end{aligned}
\tag{E.1.14}
$$

E.1.4 Redefinition $L_{21314151} \mapsto \tilde{T}_{12345}$

In order to find the appropriate redefinition of $L_{21314151}$ leading to $\tilde{T}_{12345}$ one simply uses the known redefinitions of $[L_{21}, L_{2131}, L_{213141}] \mapsto [T_{12}, T_{123}, T_{1234}]$ in the right-hand side of

$$
\begin{aligned}
Q\, L_{21314151} \;\; = \;\; & (s_{12345} - s_{1234})\, L_{213141}\, V_5 \; + \; (s_{1234} - s_{123})\, (L_{213151}\, V_4 \; + \; L_{2131}\, L_{54}) \\
& + \; (s_{123} - s_{12})\, (L_{214151}\, V_3 \; + \; L_{2141}\, L_{53} \; + \; L_{2151}\, L_{43} \; + \; L_{21}\, L_{4353}) \\
& + \; s_{12}\, (L_{314151}\, V_2 \; + \; V_1\, L_{324252} \; + \; L_{3141}\, L_{52} \; + \; L_{3151}\, L_{42} \; + \; L_{4151}\, L_{32} \\
& \quad + \; L_{31}\, L_{4252} \; + \; L_{41}\, L_{3252} \; + \; L_{51}\, L_{3242}) \; .
\end{aligned} \tag{E.1.15}
$$

Even though it is not obvious, all correction terms from these lower-order redefinitions group together into a BRST-exact combination which can be moved to the left-hand side of (E.1.15), see (11.1.28) and (11.1.29). This procedure leads to the following definition of $\tilde{T}_{12345}$,

$$
\begin{aligned}
\tilde{T}_{12345} \;\; = \;\; & L_{21314151} \\
& - \frac{1}{4}\, (s_{13} + s_{23})\, \big[D_{12}\, D_{34}\, V_5\, (s_{35} + s_{45}) \; + \; D_{12}\, D_{35}\, V_4\, s_{34} \; - \; D_{12}\, D_{45}\, V_3\, s_{34} \big] \\
& - \frac{1}{4}\, s_{12} \Big[\, D_{13}\, D_{24}\, V_5\, (s_{25} + s_{45}) \; + \; D_{14}\, D_{23}\, V_5\, (s_{25} + s_{35}) \; + \; D_{15}\, D_{23}\, V_4\, (s_{24} + s_{34}) \\
& \qquad + \; s_{24}\, (D_{13}\, D_{25}\, V_4 \; - \; D_{13}\, D_{45}\, V_2) \; + \; s_{13}\, (D_{34}\, D_{25}\, V_1 \; + \; D_{35}\, D_{24}\, V_1) \\
& \qquad + \; s_{23}\, (D_{14}\, D_{25}\, V_3 \; - \; D_{14}\, D_{35}\, V_2 \; + \; D_{15}\, D_{24}\, V_3 \; - \; D_{15}\, D_{34}\, V_2) \; + \; s_{14}\, D_{45}\, D_{23}\, V_1 \Big] \\
& - (s_{15} + s_{25} + s_{35} + s_{45})\, S^{(2)}_{1234}\, V_5 \; - \; (s_{14} + s_{24} + s_{34})\, (S^{(1)}_{123}\, L_{54} \; + \; S^{(2)}_{1235}\, V_4) \\
& - (s_{13} + s_{23})\, (S^{(1)}_{124}\, L_{53} \; + \; S^{(1)}_{125}\, L_{43} \; - \; S^{(1)}_{345}\, L_{21} \; + \; S^{(2)}_{1245}\, V_3) \\
& - s_{12} \Big[S^{(1)}_{134} L_{52} \; + S^{(1)}_{135} L_{42} \; + S^{(1)}_{145} L_{32} \; + S^{(2)}_{1345} V_2 \; - \; (1 \leftrightarrow 2) \Big] \\
& - \frac{1}{2} \Big[T_{123}\, D_{45}\, (s_{14} + s_{24} + s_{34}) \; + \; (T_{125}\, D_{34} \; - \; T_{345}\, D_{12} \; + \; T_{124}\, D_{35})\, (s_{13} + s_{23}) \\
& + s_{12}\, \big(T_{134} D_{25} \; + \; T_{135} D_{24} \; + \; T_{145} D_{23} \; - \; (1 \leftrightarrow 2) \big) \Big]
\end{aligned} \tag{E.1.16}
$$

which, by construction, has the same BRST variation (11.1.27) as T_{12345}.

E.1.5 Redefinition $\tilde{T}_{12345} \mapsto T_{12345}$

Four independent combinations of $\tilde{T}_{12345}$ are BRST exact,

$$
\begin{aligned}
\tilde{T}_{12345} \; + \; \tilde{T}_{21345} \;\; &= \;\; Q\, R^{(1)}_{12345} \\
\tilde{T}_{12345} \; + \; \tilde{T}_{23145} \; + \; \tilde{T}_{31245} \;\; &= \;\; Q\, R^{(2)}_{12345} \\
\tilde{T}_{12345} \; - \; \tilde{T}_{12435} \; + \; \tilde{T}_{34125} \; - \; \tilde{T}_{34215} \;\; &= \;\; Q\, R^{(3)}_{12345} \\
\tilde{T}_{12345} \; - \; \tilde{T}_{12354} \; + \; \tilde{T}_{45123} \; - \; \tilde{T}_{45213} \; - \; \tilde{T}_{45312} \; + \; \tilde{T}_{45321} \;\; &= \;\; Q\, R^{(4)}_{12345}
\end{aligned} \tag{E.1.17}
$$

where the ghost number zero superfields $R^{(i)}_{12345}$ on the right hand side are given by

$$
\begin{aligned}
R^{(1)}_{12345} \;&=\; D_{12}\,(k^{12}\cdot A^3)\,(k^{123}\cdot A^4)\,(k^{1234}\cdot A^5) \;+\; \frac{1}{6}\,(s_{13}+s_{23})\,D_{12}\Big[\,D_{45}\,\big((k^4\cdot A^3)\;-\;(k^5\cdot A^3)\big) \\
&\quad + D_{35}\,\big((k^5\cdot A^4)\;-\;(k^3\cdot A^4)\big)\;-\;2D_{34}\,\big((k^3\cdot A^5)\;+\;2(k^4\cdot A^5)\big)\Big] \qquad \text{(E.1.18)} \\
R^{(2)}_{12345} \;&=\; D_{12}\,(k^{2}\cdot A^3)\,(k^{123}\cdot A^4)\,(k^{1234}\cdot A^5) \;+\; \frac{1}{6}\Big[\,s_{12}\,D_{13}\Big(\,D_{45}\,\big((k^4\cdot A^2)\;-\;(k^5\cdot A^2)\big) \\
&\quad + D_{25}\,\big((k^5\cdot A^4)\;-\;(k^2\cdot A^4)\big)\;-\;2D_{24}\,\big((k^2\cdot A^5)\;+\;2(k^4\cdot A^5)\big)\Big)\;+\;\text{cyclic}(123)\Big] \qquad \text{(E.1.19)} \\
R^{(3)}_{12345} \;&=\; -\,(W^1\gamma^m W^2)\,(W^3\gamma^m W^4)\,(k^{1234}\cdot A^5) \\
&\quad + \big[D_{12}\,(k^3\cdot A^4)\,(k^2\cdot A^3)\,(k^{1234}\cdot A^5)\;+\;\frac{1}{3}\,(s_{24}-2s_{23})\,D_{34}\,D_{12}\,(k^4\cdot A^5)\;-\;(3\leftrightarrow 4)\big] \\
&\quad + \frac{1}{6}\,(s_{14}+s_{24})\Big[\,D_{25}\,D_{34}\,\big((k^2\cdot A^1)\;-\;(k^5\cdot A^1)\big)\;+\;D_{15}\,D_{34}\,\big((k^5\cdot A^2)\;-\;(k^1\cdot A^2)\big)\Big] \\
&\quad + \frac{1}{6}\,(s_{23}+s_{24})\Big[\,D_{45}\,D_{12}\,\big((k^4\cdot A^3)\;-\;(k^5\cdot A^3)\big)\;+\;D_{35}\,D_{12}\,\big((k^5\cdot A^4)\;-\;(k^3\cdot A^4)\big)\Big] \\
&\quad + \Big[\,\big(D_{13}\,(k^1\cdot A^2)\,(k^3\cdot A^4)\;+\;D_{24}\,(k^2\cdot A^1)\,(k^4\cdot A^3)\;+\;D_{34}\,(k^1\cdot A^2)\,(k^4\cdot A^1)\big)\,(k^{1234}\cdot A^5) \\
&\qquad + \;\frac{1}{3}\,(s_{24}-2s_{14})\,D_{34}\,D_{12}\,(k^2\cdot A^5)\;-\;(1\leftrightarrow 2)\Big] \qquad \text{(E.1.20)} \\
R^{(4)}_{12345} \;&=\; (W^1\gamma^m W^2)\,\big[(W^4\gamma^n W^5)\,\mathcal{F}^3_{mn}\;-\;(W^4\gamma^m W^5)\,(k^{12}\cdot A^3)\big] \\
&\quad + \Big[\,(W^1\gamma^m W^2)\,(W^3\gamma^m W^5)\,(k^5\cdot A^4)\;+\;\frac{1}{4}\,(W^1\gamma^m W^2)\,(W^5\gamma^{np}\gamma^m W^3)\mathcal{F}^4_{np} \\
&\qquad + \;D_{12}\,(k^2\cdot A^3)\,(k^{23}\cdot A^4)\,(k^4\cdot A^5)\;+\;D_{12}\,(k^1\cdot A^3)\,(k^2\cdot A^4)\,(k^4\cdot A^5) \\
&\qquad + \;\frac{1}{6}\,D_{12}\,D_{35}\,(k^3\cdot A^4)\,s_{23}\;+\;\frac{5}{6}\,D_{12}\,D_{35}\,(k^5\cdot A^4)\,s_{23}\;+\;\frac{1}{3}\,D_{12}\,D_{45}\,(k^4\cdot A^3)\,s_{23} \\
&\qquad + \;D_{14}\,(k^1\cdot A^2)\,(k^{12}\cdot A^3)\,(k^4\cdot A^5)\;+\;D_{25}\,(k^2\cdot A^1)\,(k^{12}\cdot A^3)\,(k^5\cdot A^4) \\
&\qquad + \;D_{34}\,(k^2\cdot A^1)\,(k^3\cdot A^2)\,(k^4\cdot A^5)\;+\;D_{35}\,(k^3\cdot A^1)\,(k^1\cdot A^2)\,(k^5\cdot A^4)\;-\;(4\leftrightarrow 5)\Big] \\
&\quad + \Big[\,(W^2\gamma^m W^3)\,(W^4\gamma^m W^5)\,(k^2\cdot A^1)\;+\;\frac{1}{4}\,(W^4\gamma^m W^5)\,(W^1\gamma^{np}\gamma^m W^3)\,\mathcal{F}^2_{np} \\
&\qquad + \;D_{13}\,(k^1\cdot A^2)\,(k^3\cdot A^4)\,(k^4\cdot A^5)\;-\;D_{13}\,(k^5\cdot A^4)\,(k^1\cdot A^2)\,(k^3\cdot A^5) \\
&\qquad + \;D_{45}\,(k^2\cdot A^1)\,(k^3\cdot A^2)\,(k^5\cdot A^3)\;+\;D_{45}\,(k^5\cdot A^1)\,(k^1\cdot A^2)\,(k^{12}\cdot A^3) \\
&\qquad + \;\frac{1}{3}\,D_{12}\,D_{45}\,(k^2\cdot A^3)\,(-2s_{15}+s_{25}+s_{35})\;+\;\frac{1}{6}\,D_{13}\,D_{45}\,(k^3\cdot A^2)\,(s_{15}+s_{25}+s_{35}) \\
&\qquad - \;\frac{1}{6}\,D_{13}\,D_{45}\,(k^1\cdot A^2)\,(s_{15}+s_{25}-5s_{35})\;-\;(1\leftrightarrow 2)\Big] \qquad \text{(E.1.21)}
\end{aligned}
$$

Removing these BRST-exact parts is accomplished by the second redefinition $\tilde{T}_{12345} \longrightarrow T_{12345}$, leading to the rank-five BRST building block

$$T_{12345} \;=\; \tilde{T}_{12345} \;-\; Q\,S^{(3)}_{12345}\,. \qquad \text{(E.1.22)}$$

The expression for $S^{(3)}_{12345}$ can be written recursively as

$$S^{(3)}_{12345} \;=\; \frac{4}{5}\,S^{(2)}_{12345} \;+\; \frac{1}{5}\,\big(S^{(2)}_{12354}\;-\;S^{(2)}_{45123}\;+\;S^{(2)}_{45213}\;+\;S^{(2)}_{45312}\;-\;S^{(2)}_{45321}\big)\;+\;\frac{1}{5}R^{(4)}_{12345}$$

$$\begin{aligned} S^{(2)}_{12345} &= \frac{3}{4}\, S^{(1)}_{12345} + \frac{1}{4}\,(S^{(1)}_{12435} - S^{(1)}_{34125} + S^{(1)}_{34215}) + \frac{1}{4}\, R^{(3)}_{12345} \\ S^{(1)}_{12345} &= \frac{1}{2}\, R^{(1)}_{12345} + \frac{1}{3}\, R^{(2)}_{[12]345} \end{aligned} \tag{E.1.23}$$

To see that (E.1.22) and (E.1.23) imply all the BRST-symmetries of T_{12345}

$$\begin{aligned} 0 &= T_{12345} + T_{21345} = T_{12345} + T_{31245} + T_{23145} = T_{12345} - T_{12435} + T_{34125} - T_{34215} \\ &= T_{12345} - T_{12354} + T_{45123} - T_{45213} - T_{45312} + T_{45321} \end{aligned} \tag{E.1.24}$$

it suffices to check that the following identities hold,

$$\begin{aligned} S^{(3)}_{12345} + S^{(3)}_{21345} &= R^{(1)}_{12345} \\ S^{(3)}_{12345} + S^{(3)}_{31245} + S^{(3)}_{23145} &= R^{(2)}_{12345} \\ S^{(3)}_{12345} - S^{(3)}_{12435} + S^{(3)}_{34125} - S^{(3)}_{34215} &= R^{(3)}_{12345} \\ S^{(3)}_{12345} - S^{(3)}_{12354} + S^{(3)}_{45123} - S^{(3)}_{45213} - S^{(3)}_{45312} + S^{(3)}_{45321} &= R^{(4)}_{12345}\ . \end{aligned} \tag{E.1.25}$$

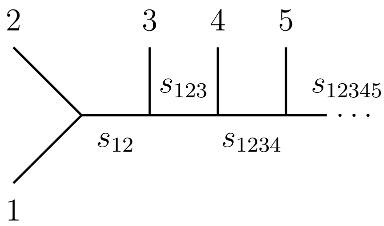
Having the explicit superfield expressions for the building blocks up to T_{12345} allows all component amplitudes up to $n = 11$ to be evaluated.

E.2 Higher rank Berends Giele currents

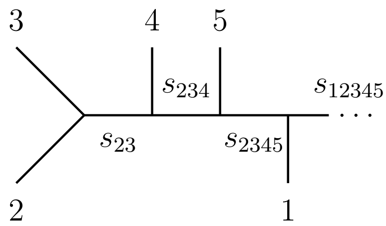
In subsection 11.2.2, we have given explicit expressions for the supersymmetric Berends-Giele currents $M_{12\ldots p}$ up to $p = 4$. Higher rank currents encompass $\frac{(2p-2)!}{(p-1)!p!}$ diagrams, that is why defer these lengthy expressions to this appendix. We display the fourteen diagrams entering M_{12345} and the expansion of M_{123456} in terms of $T_{i_1 i_2 i_3 i_4 i_5 i_6}$. The 42 diagrams within M_{123456} as well as the formula for $M_{1234567}$ can be found in an appendix of [11].

E.2.1 Diagrammatic representation of M_{12345}

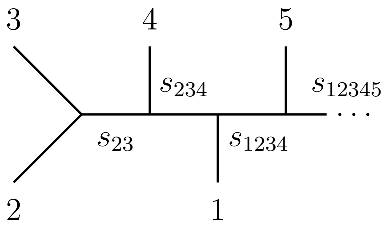
The rank five Berends-Giele current contains the fourteen color ordered diagrams of a six point amplitude with one off-shell leg:



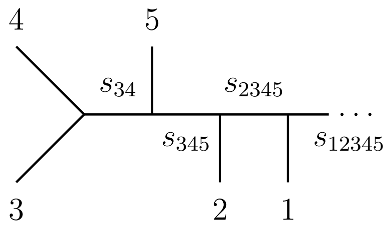
$$= \frac{T_{12345}/s_{12345}}{s_{12}\, s_{123}\, s_{1234}}$$



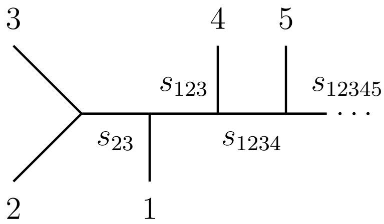
$$= \frac{T_{32451}/s_{12345}}{s_{23}\, s_{234}\, s_{2345}}$$



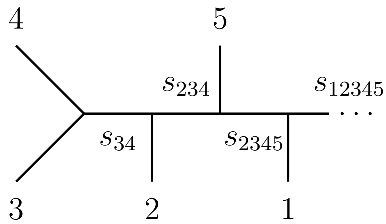
$$= \frac{T_{32415}/s_{12345}}{s_{23}\, s_{234}\, s_{1234}}$$



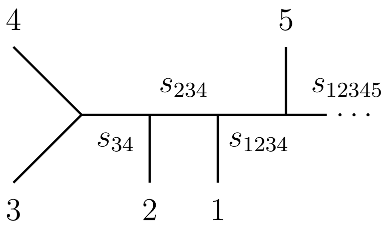
$$= \frac{T_{34521}/s_{12345}}{s_{34}\, s_{345}\, s_{2345}}$$



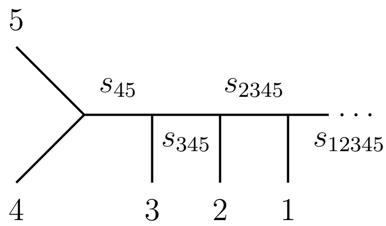
$$= \frac{T_{32145}/s_{12345}}{s_{23}\, s_{123}\, s_{1234}}$$



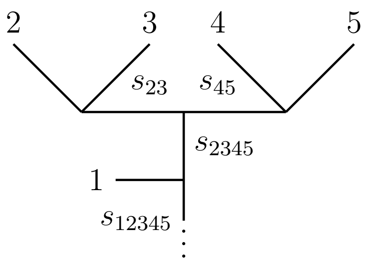
$$= \frac{T_{34251}/s_{12345}}{s_{34}\, s_{234}\, s_{2345}}$$



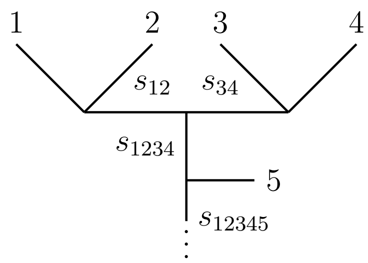
$$= \frac{T_{34215}/s_{12345}}{s_{34}\, s_{234}\, s_{1234}}$$



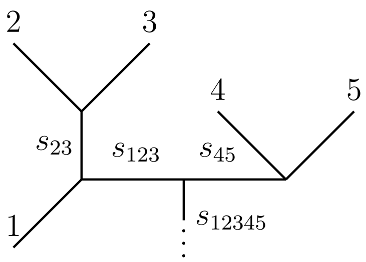
$$= \frac{T_{54321}/s_{12345}}{s_{45}\, s_{345}\, s_{2345}}$$



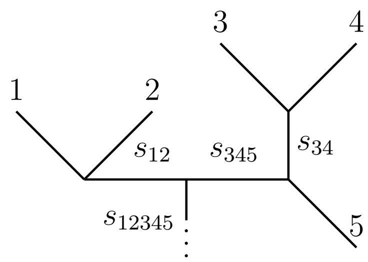
$$= \frac{2T_{54[32]1}/s_{12345}}{s_{23}\, s_{45}\, s_{2345}}$$



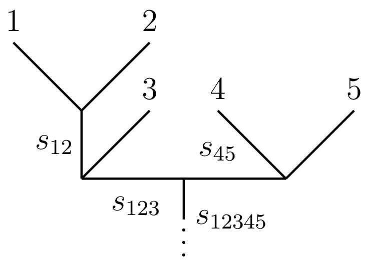
$$= \frac{2T_{12[34]5}/s_{12345}}{s_{12}\, s_{34}\, s_{1234}}$$



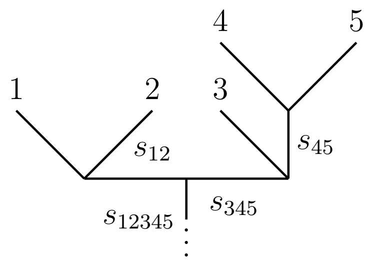
$$= \frac{2T_{321[45]}/s_{12345}}{s_{23}\, s_{123}\, s_{45}}$$



$$= \frac{2T_{435[12]}/s_{12345}}{s_{12}\, s_{34}\, s_{345}}$$



$$= \frac{2T_{123[45]}/s_{12345}}{s_{12}\, s_{123}\, s_{45}}$$



$$= \frac{2T_{453[12]}/s_{12345}}{s_{12}\, s_{45}\, s_{345}}$$

E.2.2 The formula for M_{123456}

The rank six current receives 42 contributions from BRST building blocks $T_{i_1 i_2 i_3 i_4 i_5 i_6}$, according to the cubic diagrams in a color ordered seven point SYM amplitude:

$$
\begin{aligned}
M_{123456} \;=\; & \frac{1}{s_{123456}} \Big[\frac{4T_{12[34][56]}}{s_{12}s_{34}s_{56}s_{1234}} + \frac{4T_{34[56][21]}}{s_{12}s_{34}s_{56}s_{3456}} + \frac{4T_{123[[45]6]}}{s_{12}s_{45}s_{123}s_{456}} + \frac{4T_{123[4[56]]}}{s_{12}s_{56}s_{123}s_{456}} \\
& + \frac{4T_{231[[54]6]}}{s_{23}s_{45}s_{123}s_{456}} + \frac{4T_{231[4[65]]}}{s_{23}s_{56}s_{123}s_{456}} + \frac{2T_{345[21]6}}{s_{12}s_{34}s_{345}s_{12345}} + \frac{2T_{3456[21]}}{s_{12}s_{34}s_{345}s_{3456}} \\
& + \frac{2T_{12[34]56}}{s_{12}s_{34}s_{1234}s_{12345}} + \frac{2T_{123[45]6}}{s_{12}s_{45}s_{123}s_{12345}} + \frac{2T_{543[21]6}}{s_{12}s_{45}s_{345}s_{12345}} + \frac{2T_{5436[21]}}{s_{12}s_{45}s_{345}s_{3456}} \\
& + \frac{2T_{4563[12]}}{s_{12}s_{45}s_{456}s_{3456}} + \frac{2T_{1234[56]}}{s_{12}s_{56}s_{123}s_{1234}} + \frac{2T_{5643[21]}}{s_{12}s_{56}s_{456}s_{3456}} + \frac{2T_{231[54]6}}{s_{23}s_{45}s_{123}s_{12345}} \\
& + \frac{2T_{456[23]1}}{s_{23}s_{45}s_{456}s_{23456}} + \frac{2T_{34[56]21}}{s_{34}s_{56}s_{23456}s_{3456}} + \frac{2T_{23[54]16}}{s_{23}s_{45}s_{12345}s_{2345}} + \frac{2T_{23[54]61}}{s_{23}s_{45}s_{2345}s_{23456}} \\
& + \frac{2T_{2314[65]}}{s_{23}s_{56}s_{123}s_{1234}} + \frac{2T_{2341[65]}}{s_{23}s_{56}s_{234}s_{1234}} + \frac{2T_{234[65]1}}{s_{23}s_{56}s_{234}s_{23456}} + \frac{2T_{564[32]1}}{s_{23}s_{56}s_{456}s_{23456}} \\
& + \frac{2T_{3421[56]}}{s_{34}s_{56}s_{234}s_{1234}} + \frac{2T_{342[56]1}}{s_{34}s_{56}s_{234}s_{23456}} + \frac{T_{321456}}{s_{23}s_{123}s_{1234}s_{12345}} + \frac{T_{324156}}{s_{23}s_{234}s_{1234}s_{12345}} \\
& + \frac{T_{324516}}{s_{23}s_{234}s_{12345}s_{2345}} + \frac{T_{324561}}{s_{23}s_{234}s_{2345}s_{23456}} + \frac{T_{342156}}{s_{34}s_{234}s_{1234}s_{12345}} + \frac{T_{342516}}{s_{34}s_{234}s_{12345}s_{2345}} \\
& + \frac{T_{342561}}{s_{34}s_{234}s_{2345}s_{23456}} + \frac{T_{345216}}{s_{34}s_{345}s_{12345}s_{2345}} + \frac{T_{345261}}{s_{34}s_{345}s_{2345}s_{23456}} + \frac{T_{345621}}{s_{34}s_{345}s_{23456}s_{3456}} \\
& + \frac{T_{543216}}{s_{45}s_{345}s_{12345}s_{2345}} + \frac{T_{543261}}{s_{45}s_{345}s_{2345}s_{23456}} + \frac{T_{123456}}{s_{12}s_{123}s_{1234}s_{12345}} + \frac{T_{543621}}{s_{45}s_{345}s_{23456}s_{3456}} \\
& + \frac{T_{546321}}{s_{45}s_{456}s_{23456}s_{3456}} + \frac{T_{564321}}{s_{56}s_{456}s_{23456}s_{3456}} \Big]
\end{aligned}
\tag{E.2.26}
$$

Appendix F

Supplementing material on BCJ numerators

This appendix contains some lengthy material for the explicit construction of BCJ numerators in chapter 13: the field theory limit of the six point integrals and the seven point BCJ numerators.

F.1 Field theory limits of six point integrals

The list (13.2.31) of 25 six point numerators are constructed by comparing the expansion (13.1.21) of $\mathcal{A}^{\text{SYM}}(1, 2_\rho, 3_\rho, 4_\rho, 5_\rho, 6)$ in terms of fourteen cubic diagrams with the field theory limit of the superstring amplitude (12.1.12) integrated over the appropriate worldsheet regions $\mathcal{I}_\rho$. The latter assumes the following form once the $\alpha' \to 0$ limit of the worldsheet integrals is taken:

$$
\begin{aligned}
\mathcal{A}(1,2_\sigma,3_\sigma,4_\sigma,5,6) \;=\;& \left(\frac{1}{s_{2_\sigma 3_\sigma}} + \frac{1}{s_{3_\sigma 4_\sigma}}\right) \frac{\mathcal{K}^4_{\sigma(432)}}{s_{56}s_{2_\sigma 3_\sigma 4_\sigma}} - \left(\frac{1}{s_{12_\sigma 3_\sigma}} + \frac{1}{s_{2_\sigma 3_\sigma 4_\sigma}}\right) \frac{\mathcal{K}^4_{\sigma(324)}}{s_{2_\sigma 3_\sigma}s_{56}} \\
&- \left(\frac{1}{s_{12_\sigma}} + \frac{1}{s_{2_\sigma 3_\sigma 4_\sigma}}\right) \frac{\mathcal{K}^4_{\sigma(243)}}{s_{3_\sigma 4_\sigma}s_{56}} - \frac{\mathcal{K}^4_{\sigma(423)}}{s_{2_\sigma 3_\sigma}s_{56}s_{2_\sigma 3_\sigma 4_\sigma}} - \frac{\mathcal{K}^4_{\sigma(342)}}{s_{3_\sigma 4_\sigma}s_{56}s_{2_\sigma 3_\sigma 4_\sigma}} \\
&+ \left(\frac{1}{s_{12_\sigma}s_{3_\sigma 4_\sigma}} + \frac{1}{s_{12_\sigma}s_{12_\sigma 3_\sigma}} + \frac{1}{s_{2_\sigma 3_\sigma}s_{12_\sigma 3_\sigma}} + \frac{1}{s_{2_\sigma 3_\sigma}s_{2_\sigma 3_\sigma 4_\sigma}} + \frac{1}{s_{3_\sigma 4_\sigma}s_{2_\sigma 3_\sigma 4_\sigma}}\right) \frac{\mathcal{K}^4_{\sigma(234)}}{s_{56}} \\
&+ \frac{\mathcal{K}^3_{\sigma(324)}}{s_{2_\sigma 3_\sigma}s_{4_\sigma 5}s_{12_\sigma 3_\sigma}} - \left(\frac{1}{s_{12_\sigma}} + \frac{1}{s_{2_\sigma 3_\sigma}}\right) \frac{\mathcal{K}^3_{\sigma(234)}}{s_{4_\sigma 5}s_{12_\sigma 3_\sigma}} + \left(\frac{1}{s_{3_\sigma 4_\sigma}} + \frac{1}{s_{4_\sigma 5}}\right) \frac{\mathcal{K}^2_{\sigma(234)}}{s_{12_\sigma}s_{3_\sigma 4_\sigma 5}} \\
&- \frac{\mathcal{K}^2_{\sigma(243)}}{s_{12_\sigma}s_{3_\sigma 4_\sigma}s_{3_\sigma 4_\sigma 5}} - \left(\frac{1}{s_{2_\sigma 3_\sigma}} + \frac{1}{s_{3_\sigma 4_\sigma}}\right) \frac{\mathcal{K}^1_{\sigma(432)}}{s_{61}s_{2_\sigma 3_\sigma 4_\sigma}} + \left(\frac{1}{s_{4_\sigma 5}} + \frac{1}{s_{2_\sigma 3_\sigma 4_\sigma}}\right) \frac{\mathcal{K}^1_{\sigma(324)}}{s_{61}s_{2_\sigma 3_\sigma}} \\
&+ \left(\frac{1}{s_{2_\sigma 3_\sigma 4_\sigma}} + \frac{1}{s_{3_\sigma 4_\sigma 5}}\right) \frac{\mathcal{K}^1_{\sigma(243)}}{s_{61}s_{3_\sigma 4_\sigma}} + \frac{\mathcal{K}^1_{\sigma(423)}}{s_{61}s_{2_\sigma 3_\sigma}s_{2_\sigma 3_\sigma 4_\sigma}} + \frac{\mathcal{K}^1_{\sigma(342)}}{s_{61}s_{3_\sigma 4_\sigma}s_{2_\sigma 3_\sigma 4_\sigma}} \\
&- \left(\frac{1}{s_{2_\sigma 3_\sigma}s_{4_\sigma 5}} + \frac{1}{s_{2_\sigma 3_\sigma}s_{2_\sigma 3_\sigma 4_\sigma}} + \frac{1}{s_{3_\sigma 4_\sigma}s_{2_\sigma 3_\sigma 4_\sigma}} + \frac{1}{s_{3_\sigma 4_\sigma}s_{3_\sigma 4_\sigma 5}} + \frac{1}{s_{4_\sigma 5}s_{3_\sigma 4_\sigma 5}}\right) \frac{\mathcal{K}^1_{\sigma(234)}}{s_{61}}
\end{aligned} \tag{F.1.1}
$$

$$
\begin{aligned}
\mathcal{A}(1,2_\sigma,3_\sigma,5,4_\sigma,6) \;=\;& -\frac{\mathcal{K}^4_{\sigma(324)}}{s_{12_\sigma 3_\sigma} s_{2_\sigma 3_\sigma} s_{4_\sigma 6}} + \left(\frac{1}{s_{12_\sigma}}+\frac{1}{s_{2_\sigma 3_\sigma}}\right)\frac{\mathcal{K}^4_{\sigma(234)}}{s_{4_\sigma 6} s_{12_\sigma 3_\sigma}} - \frac{\mathcal{K}^3_{\sigma(243)}}{s_{12_\sigma} s_{3_\sigma 5} s_{4_\sigma 6}} \\
& - \left(\frac{1}{s_{4_\sigma 5}}+\frac{1}{s_{4_\sigma 6}}\right)\frac{\mathcal{K}^3_{\sigma(324)}}{s_{2_\sigma 3_\sigma} s_{12_\sigma 3_\sigma}} + \left(\frac{1}{s_{12_\sigma}}+\frac{1}{s_{2_\sigma 3_\sigma}}\right)\left(\frac{1}{s_{4_\sigma 5}}+\frac{1}{s_{4_\sigma 6}}\right)\frac{\mathcal{K}^3_{\sigma(234)}}{s_{12_\sigma 3_\sigma}} \\
& - \frac{\mathcal{K}^2_{\sigma(432)}}{s_{2_\sigma 3_\sigma} s_{4_\sigma 6}\, s_{2_\sigma 3_\sigma 5}} - \left(\frac{1}{s_{4_\sigma 6}}+\frac{1}{s_{3_\sigma 4_\sigma 5}}\right)\frac{\mathcal{K}^2_{\sigma(243)}}{s_{12_\sigma} s_{3_\sigma 5}} + \left(\frac{1}{s_{2_\sigma 3_\sigma}}+\frac{1}{s_{3_\sigma 5}}\right)\frac{\mathcal{K}^2_{\sigma(423)}}{s_{4_\sigma 6} s_{2_\sigma 3_\sigma 5}} \\
& - \frac{\mathcal{K}^2_{\sigma(234)}}{s_{12_\sigma} s_{4_\sigma 5} s_{3_\sigma 4_\sigma 5}} - \left(\frac{1}{s_{61}}+\frac{1}{s_{4_\sigma 6}}\right)\frac{\mathcal{K}^1_{\sigma(432)}}{s_{2_\sigma 3_\sigma} s_{2_\sigma 3_\sigma 5}} - \frac{\mathcal{K}^1_{\sigma(324)}}{s_{61} s_{2_\sigma 3_\sigma} s_{4_\sigma 5}} + \frac{\mathcal{K}^1_{\sigma(243)}}{s_{61} s_{3_\sigma 5} s_{3_\sigma 4_\sigma 5}} \\
& + \left(\frac{1}{s_{61}}+\frac{1}{s_{4_\sigma 6}}\right)\left(\frac{1}{s_{2_\sigma 3_\sigma}}+\frac{1}{s_{3_\sigma 5}}\right)\frac{\mathcal{K}^1_{\sigma(423)}}{s_{2_\sigma 3_\sigma 5}} + \left(\frac{1}{s_{2_\sigma 3_\sigma}}+\frac{1}{s_{3_\sigma 4_\sigma 5}}\right)\frac{\mathcal{K}^1_{\sigma(234)}}{s_{61} s_{4_\sigma 5}}
\end{aligned}
\tag{F.1.2}
$$

$$
\begin{aligned}
\mathcal{A}(1,2_\sigma,5,3_\sigma,4_\sigma,6) \;=\;& -\frac{\mathcal{K}^4_{\sigma(243)}}{s_{12_\sigma} s_{3_\sigma 4_\sigma} s_{12_\sigma 5}} + \left(\frac{1}{s_{3_\sigma 4_\sigma}}+\frac{1}{s_{4_\sigma 6}}\right)\frac{\mathcal{K}^4_{\sigma(234)}}{s_{12_\sigma} s_{12_\sigma 5}} + \frac{\mathcal{K}^3_{\sigma(432)}}{s_{2_\sigma 5} s_{3_\sigma 4_\sigma} s_{12_\sigma 5}} \\
& + \left(\frac{1}{s_{3_\sigma 5}}+\frac{1}{s_{12_\sigma 5}}\right)\frac{\mathcal{K}^3_{\sigma(243)}}{s_{12_\sigma} s_{4_\sigma 6}} - \left(\frac{1}{s_{3_\sigma 4_\sigma}}+\frac{1}{s_{4_\sigma 6}}\right)\frac{\mathcal{K}^3_{\sigma(342)}}{s_{2_\sigma 5} s_{12_\sigma 5}} + \frac{\mathcal{K}^3_{\sigma(234)}}{s_{12_\sigma} s_{4_\sigma 6} s_{12_\sigma 5}} \\
& + \left(\frac{1}{s_{3_\sigma 5} s_{4_\sigma 6}}+\frac{1}{s_{3_\sigma 4_\sigma} s_{12_\sigma 5}}+\frac{1}{s_{4_\sigma 6} s_{12_\sigma 5}}+\frac{1}{s_{3_\sigma 4_\sigma} s_{3_\sigma 4_\sigma 5}}+\frac{1}{s_{3_\sigma 5} s_{3_\sigma 4_\sigma 5}}\right)\frac{\mathcal{K}^2_{\sigma(243)}}{s_{12_\sigma}} \\
& - \left(\frac{1}{s_{12_\sigma 5}}+\frac{1}{s_{2_\sigma 3_\sigma 5}}\right)\frac{\mathcal{K}^2_{\sigma(432)}}{s_{2_\sigma 5} s_{4_\sigma 6}} - \frac{\mathcal{K}^2_{\sigma(423)}}{s_{3_\sigma 5} s_{4_\sigma 6} s_{2_\sigma 3_\sigma 5}} - \left(\frac{1}{s_{12_\sigma 5}}+\frac{1}{s_{3_\sigma 4_\sigma 5}}\right)\frac{\mathcal{K}^2_{\sigma(234)}}{s_{12_\sigma} s_{3_\sigma 4_\sigma}} \\
& - \frac{\mathcal{K}^2_{\sigma(342)}}{s_{2_\sigma 5} s_{4_\sigma 6} s_{12_\sigma 5}} - \left(\frac{1}{s_{3_\sigma 4_\sigma}}+\frac{1}{s_{3_\sigma 5}}\right)\frac{\mathcal{K}^1_{\sigma(243)}}{s_{61} s_{3_\sigma 4_\sigma 5}} - \left(\frac{1}{s_{61}}+\frac{1}{s_{4_\sigma 6}}\right)\frac{\mathcal{K}^1_{\sigma(423)}}{s_{3_\sigma 5} s_{2_\sigma 3_\sigma 5}} \\
& - \left(\frac{1}{s_{61} s_{3_\sigma 4_\sigma}}+\frac{1}{s_{3_\sigma 4_\sigma} s_{12_\sigma 5}}+\frac{1}{s_{4_\sigma 6} s_{12_\sigma 5}}+\frac{1}{s_{61} s_{2_\sigma 3_\sigma 5}}+\frac{1}{s_{4_\sigma 6} s_{2_\sigma 3_\sigma 5}}\right)\frac{\mathcal{K}^1_{\sigma(432)}}{s_{2_\sigma 5}} \\
& + \left(\frac{1}{s_{61}}+\frac{1}{s_{12_\sigma 5}}\right)\frac{\mathcal{K}^1_{\sigma(342)}}{s_{2_\sigma 5} s_{3_\sigma 4_\sigma}} + \frac{\mathcal{K}^1_{\sigma(234)}}{s_{61} s_{3_\sigma 4_\sigma} s_{3_\sigma 4_\sigma 5}}
\end{aligned}
\tag{F.1.3}
$$

$$
\begin{aligned}
\mathcal{A}(1,5,2_\sigma,3_\sigma,4_\sigma,6) \;=\;& \left(\frac{1}{s_{2_\sigma 3_\sigma}}+\frac{1}{s_{3_\sigma 4_\sigma}}\right)\frac{\mathcal{K}^4_{\sigma(432)}}{s_{15} s_{2_\sigma 3_\sigma 4_\sigma}} - \left(\frac{1}{s_{12_\sigma 5}}+\frac{1}{s_{2_\sigma 3_\sigma 4_\sigma}}\right)\frac{\mathcal{K}^4_{\sigma(243)}}{s_{15} s_{3_\sigma 4_\sigma}} \\
& + \left(\frac{1}{s_{2_\sigma 3_\sigma} s_{4_\sigma 6}}+\frac{1}{s_{3_\sigma 4_\sigma} s_{12_\sigma 5}}+\frac{1}{s_{4_\sigma 6} s_{12_\sigma 5}}+\frac{1}{s_{2_\sigma 3_\sigma} s_{2_\sigma 3_\sigma 4_\sigma}}+\frac{1}{s_{3_\sigma 4_\sigma} s_{2_\sigma 3_\sigma 4_\sigma}}\right)\frac{\mathcal{K}^4_{\sigma(234)}}{s_{15}} \\
& - \frac{\mathcal{K}^4_{\sigma(423)}}{s_{15} s_{2_\sigma 3_\sigma} s_{2_\sigma 3_\sigma 4_\sigma}} - \frac{\mathcal{K}^4_{\sigma(342)}}{s_{15} s_{3_\sigma 4_\sigma} s_{2_\sigma 3_\sigma 4_\sigma}} - \left(\frac{1}{s_{4_\sigma 6}}+\frac{1}{s_{2_\sigma 3_\sigma 4_\sigma}}\right)\frac{\mathcal{K}^4_{\sigma(324)}}{s_{15} s_{2_\sigma 3_\sigma}} - \frac{\mathcal{K}^3_{\sigma(324)}}{s_{15} s_{2_\sigma 3_\sigma} s_{4_\sigma 6}} \\
& - \left(\frac{1}{s_{15}}+\frac{1}{s_{2_\sigma 5}}\right)\frac{\mathcal{K}^3_{\sigma(432)}}{s_{3_\sigma 4_\sigma} s_{12_\sigma 5}} + \frac{\mathcal{K}^3_{\sigma(243)}}{s_{15} s_{4_\sigma 6} s_{12_\sigma 5}} + \left(\frac{1}{s_{2_\sigma 3_\sigma}}+\frac{1}{s_{12_\sigma 5}}\right)\frac{\mathcal{K}^3_{\sigma(234)}}{s_{15} s_{4_\sigma 6}} \\
& + \left(\frac{1}{s_{15}}+\frac{1}{s_{2_\sigma 5}}\right)\left(\frac{1}{s_{3_\sigma 4_\sigma}}+\frac{1}{s_{4_\sigma 6}}\right)\frac{\mathcal{K}^3_{\sigma(342)}}{s_{12_\sigma 5}} + \left(\frac{1}{s_{3_\sigma 4_\sigma}}+\frac{1}{s_{4_\sigma 6}}\right)\frac{\mathcal{K}^2_{\sigma(243)}}{s_{15} s_{12_\sigma 5}} \\
& + \left(\frac{1}{s_{15} s_{2_\sigma 3_\sigma}}+\frac{1}{s_{15} s_{12_\sigma 5}}+\frac{1}{s_{2_\sigma 5} s_{12_\sigma 5}}+\frac{1}{s_{2_\sigma 3_\sigma} s_{2_\sigma 3_\sigma 5}}+\frac{1}{s_{2_\sigma 5} s_{2_\sigma 3_\sigma 5}}\right)\frac{\mathcal{K}^2_{\sigma(432)}}{s_{4_\sigma 6}} \\
& - \left(\frac{1}{s_{15}}+\frac{1}{s_{2_\sigma 3_\sigma 5}}\right)\frac{\mathcal{K}^2_{\sigma(423)}}{s_{2_\sigma 3_\sigma} s_{4_\sigma 6}} + \left(\frac{1}{s_{15}}+\frac{1}{s_{2_\sigma 5}}\right)\frac{\mathcal{K}^2_{\sigma(342)}}{s_{4_\sigma 6} s_{12_\sigma 5}} - \frac{\mathcal{K}^2_{\sigma(234)}}{s_{15} s_{3_\sigma 4_\sigma} s_{12_\sigma 5}} \\
& + \left(\frac{1}{s_{61} s_{2_\sigma 5} s_{3_\sigma 4_\sigma}}+\frac{1}{s_{15} s_{2_\sigma 3_\sigma} s_{4_\sigma 6}}+\frac{1}{s_{15} s_{12_\sigma 5} s_{3_\sigma 4_\sigma}}+\frac{1}{s_{12_\sigma 5} s_{2_\sigma 5} s_{3_\sigma 4_\sigma}}+\frac{1}{s_{15} s_{12_\sigma 5} s_{4_\sigma 6}}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\ \frac{1}{s_{4_\sigma 6} s_{2_\sigma 5} s_{12_\sigma 5}} + \frac{1}{s_{15} s_{2_\sigma 3_\sigma} s_{2_\sigma 3_\sigma 4_\sigma}} + \frac{1}{s_{61} s_{2_\sigma 3_\sigma} s_{2_\sigma 3_\sigma 4_\sigma}} + \frac{1}{s_{15} s_{2_\sigma 3_\sigma 4_\sigma} s_{3_\sigma 4_\sigma}} + \frac{1}{s_{61} s_{2_\sigma 3_\sigma 4_\sigma} s_{3_\sigma 4_\sigma}} \\
&+\ \frac{1}{s_{61} s_{2_\sigma 3_\sigma} s_{2_\sigma 3_\sigma 5}} + \frac{1}{s_{61} s_{2_\sigma 5} s_{2_\sigma 3_\sigma 5}} + \frac{1}{s_{4_\sigma 6} s_{2_\sigma 3_\sigma} s_{2_\sigma 3_\sigma 5}} + \frac{1}{s_{4_\sigma 6} s_{2_\sigma 5} s_{2_\sigma 3_\sigma 5}} \Bigg) \mathcal{K}^1_{\sigma(432)} \\
&-\ \left(\frac{1}{s_{15}} + \frac{1}{s_{16}} \right) \frac{\mathcal{K}^1_{\sigma(324)}}{s_{2_\sigma 3_\sigma} s_{2_\sigma 3_\sigma 4_\sigma}} - \left(\frac{1}{s_{15}} + \frac{1}{s_{16}} \right) \frac{\mathcal{K}^1_{\sigma(243)}}{s_{3_\sigma 4_\sigma} s_{2_\sigma 3_\sigma 4_\sigma}} \\
&-\ \left(\frac{1}{s_{15} s_{4_\sigma 6}} + \frac{1}{s_{15} s_{2_\sigma 3_\sigma 4_\sigma}} + \frac{1}{s_{16} s_{2_\sigma 3_\sigma 4_\sigma}} + \frac{1}{s_{16} s_{2_\sigma 3_\sigma 5}} + \frac{1}{s_{4_\sigma 6} s_{2_\sigma 3_\sigma 5}} \right) \frac{\mathcal{K}^1_{\sigma(423)}}{s_{2_\sigma 3_\sigma}} \\
&-\ \left(\frac{1}{s_{16} s_{2_\sigma 5}} + \frac{1}{s_{15} s_{12_\sigma 5}} + \frac{1}{s_{2_\sigma 5} s_{12_\sigma 5}} + \frac{1}{s_{15} s_{2_\sigma 3_\sigma 4_\sigma}} + \frac{1}{s_{16} s_{2_\sigma 3_\sigma 4_\sigma}} \right) \frac{\mathcal{K}^1_{\sigma(342)}}{s_{3_\sigma 4_\sigma}} \\
&+\ \left(\frac{1}{s_{15}} + \frac{1}{s_{16}} \right) \left(\frac{1}{s_{2_\sigma 3_\sigma}} + \frac{1}{s_{3_\sigma 4_\sigma}} \right) \frac{\mathcal{K}^1_{\sigma(234)}}{s_{2_\sigma 3_\sigma 4_\sigma}} \qquad \text{(F.1.4)}
\end{aligned}
$$

F.2 Seven point BCJ numerators

Seven point tree amplitudes in gauge theories involve 945 pole channels in total. It is sufficient to display the following 69 associated numerator, the rest can be obtained from S_4 permutations σ of the labels $2, 3, 4, 5$:

$$
\begin{aligned}
n[12_\sigma, 3_\sigma, 4_\sigma, 5_\sigma, 67] &= \mathcal{K}^5_{\sigma(2345)}\ , & n[5_\sigma 6, 4_\sigma, 7, 3_\sigma, 12_\sigma] &= \mathcal{K}^3_{\sigma(2345)} \\
n[71, 2_\sigma, 3_\sigma, 4_\sigma, 5_\sigma 6] &= \mathcal{K}^1_{\sigma(2345)}\ , & n[12_\sigma, 3_\sigma, 4_\sigma, 7, 5_\sigma 6] &= \mathcal{K}^4_{\sigma(2345)} \\
n[12_\sigma, 7, 3_\sigma, 4_\sigma, 5_\sigma 6] &= \mathcal{K}^2_{\sigma(2345)}\ , & &
\end{aligned}
$$

$$
\begin{aligned}
n[12_\sigma, 3_\sigma, 4_\sigma 5_\sigma, 67] &= \mathcal{K}^5_{\sigma(2345)} - \mathcal{K}^5_{\sigma(2354)}\ , & n[4_\sigma 6, 3_\sigma, 5_\sigma 7, 2_\sigma 1] &= \mathcal{K}^2_{\sigma(2534)} + \mathcal{K}^3_{\sigma(2534)} \\
n[12_\sigma, 7, 3_\sigma, 6, 4_\sigma 5_\sigma] &= \mathcal{K}^2_{\sigma(2354)} - \mathcal{K}^2_{\sigma(2345)}\ , & n[2_\sigma 3_\sigma, 1, 4_\sigma, 7, 5_\sigma 6] &= \mathcal{K}^4_{\sigma(3245)} - \mathcal{K}^4_{\sigma(2345)} \\
n[5_\sigma 6, 7, 12_\sigma, 3_\sigma 4_\sigma] &= \mathcal{K}^4_{\sigma(2435)} - \mathcal{K}^4_{\sigma(2345)}\ , & n[71, 2_\sigma, 3_\sigma 4_\sigma, 5_\sigma 6] &= \mathcal{K}^1_{\sigma(2345)} - \mathcal{K}^1_{\sigma(2435)} \\
n[12_\sigma, 3_\sigma, 4_\sigma 6, 75_\sigma] &= \mathcal{K}^3_{\sigma(2354)} + \mathcal{K}^4_{\sigma(2354)}\ , & n[12_\sigma, 3_\sigma, 4_\sigma, 6, 5_\sigma 7] &= \mathcal{K}^4_{\sigma(2345)} + \mathcal{K}^5_{\sigma(2345)} \\
n[12_\sigma, 7, 3_\sigma 4_\sigma, 5_\sigma 6] &= \mathcal{K}^2_{\sigma(2345)} - \mathcal{K}^2_{\sigma(2435)}\ , & n[2_\sigma 3_\sigma, 1, 4_\sigma, 5_\sigma, 67] &= \mathcal{K}^5_{\sigma(3245)} - \mathcal{K}^5_{\sigma(2345)} \\
n[4_\sigma 5_\sigma, 6, 7, 3_\sigma, 12_\sigma] &= \mathcal{K}^3_{\sigma(2354)} - \mathcal{K}^3_{\sigma(2345)}\ , & n[5_\sigma 6, 4_\sigma, 71, 2_\sigma 3_\sigma] &= \mathcal{K}^1_{\sigma(3245)} - \mathcal{K}^1_{\sigma(2345)} \\
n[5_\sigma 6, 4_\sigma, 7, 1, 2_\sigma 3_\sigma] &= \mathcal{K}^3_{\sigma(3245)} - \mathcal{K}^3_{\sigma(2345)}\ , & n[5_\sigma 7, 1, 2_\sigma, 3_\sigma, 64_\sigma] &= \mathcal{K}^1_{\sigma(5234)} + \mathcal{K}^2_{\sigma(5234)} \\
n[67, 5_\sigma, 12_\sigma, 3_\sigma 4_\sigma] &= \mathcal{K}^5_{\sigma(2435)} - \mathcal{K}^5_{\sigma(2345)}\ , & n[71, 2_\sigma, 3_\sigma, 6, 4_\sigma 5_\sigma] &= \mathcal{K}^1_{\sigma(2354)} - \mathcal{K}^1_{\sigma(2345)}
\end{aligned}
$$

$$
\begin{aligned}
n[4_\sigma 5_\sigma, 6, 71, 2_\sigma 3_\sigma] &= \mathcal{K}^1_{\sigma(2345)} - \mathcal{K}^1_{\sigma(2354)} - \mathcal{K}^1_{\sigma(3245)} + \mathcal{K}^1_{\sigma(3254)} \\
n[4_\sigma 5_\sigma, 6, 7, 1, 2_\sigma 3_\sigma] &= \mathcal{K}^3_{\sigma(2345)} - \mathcal{K}^3_{\sigma(2354)} - \mathcal{K}^3_{\sigma(3245)} + \mathcal{K}^3_{\sigma(3254)} \\
n[2_\sigma 3_\sigma, 1, 4_\sigma 5_\sigma, 67] &= \mathcal{K}^5_{\sigma(2354)} - \mathcal{K}^5_{\sigma(2345)} + \mathcal{K}^5_{\sigma(3245)} - \mathcal{K}^5_{\sigma(3254)} \\
n[2_\sigma 3_\sigma, 4_\sigma, 5_\sigma 6, 71] &= \mathcal{K}^1_{\sigma(2345)} - \mathcal{K}^1_{\sigma(3245)} - \mathcal{K}^1_{\sigma(4235)} + \mathcal{K}^1_{\sigma(4325)} \\
n[3_\sigma 4_\sigma, 5_\sigma, 6, 7, 12_\sigma] &= \mathcal{K}^2_{\sigma(2435)} - \mathcal{K}^2_{\sigma(2345)} + \mathcal{K}^2_{\sigma(2534)} - \mathcal{K}^2_{\sigma(2543)} \\
n[3_\sigma 4_\sigma, 5_\sigma, 67, 12_\sigma] &= \mathcal{K}^5_{\sigma(2345)} - \mathcal{K}^5_{\sigma(2435)} - \mathcal{K}^5_{\sigma(2534)} + \mathcal{K}^5_{\sigma(2543)}
\end{aligned}
$$

$$
\begin{aligned}
n[3_\sigma 4_\sigma, 6, 5_\sigma 7, 12_\sigma] &= \mathcal{K}^2_{\sigma(2534)} - \mathcal{K}^2_{\sigma(2543)} + \mathcal{K}^3_{\sigma(2534)} - \mathcal{K}^3_{\sigma(2543)} \\
n[4_\sigma 5_\sigma, 7, 1, 2_\sigma, 3_\sigma 6] &= \mathcal{K}^1_{\sigma(5423)} - \mathcal{K}^1_{\sigma(4523)} + \mathcal{K}^3_{\sigma(4523)} - \mathcal{K}^3_{\sigma(5423)} \\
n[4_\sigma 5_\sigma, 7, 12_\sigma, 3_\sigma 6] &= \mathcal{K}^2_{\sigma(2453)} - \mathcal{K}^2_{\sigma(2543)} - \mathcal{K}^4_{\sigma(2453)} + \mathcal{K}^4_{\sigma(2543)} \\
n[5_\sigma 6, 7, 1, 2_\sigma, 3_\sigma 4_\sigma] &= \mathcal{K}^4_{\sigma(2435)} - \mathcal{K}^4_{\sigma(2345)} + \mathcal{K}^4_{\sigma(3425)} - \mathcal{K}^4_{\sigma(4325)} \\
n[5_\sigma 7, 1, 2_\sigma 3_\sigma, 4_\sigma 6] &= \mathcal{K}^1_{\sigma(5324)} - \mathcal{K}^1_{\sigma(5234)} - \mathcal{K}^2_{\sigma(5234)} + \mathcal{K}^2_{\sigma(5324)} \\
n[12_\sigma, 3_\sigma, 6, 4_\sigma, 5_\sigma 7] &= \mathcal{K}^3_{\sigma(2354)} + \mathcal{K}^4_{\sigma(2345)} + \mathcal{K}^4_{\sigma(2354)} + \mathcal{K}^5_{\sigma(2345)} \\
n[12_\sigma, 3_\sigma, 6, 7, 4_\sigma 5_\sigma] &= \mathcal{K}^3_{\sigma(2345)} - \mathcal{K}^3_{\sigma(2354)} - \mathcal{K}^5_{\sigma(2345)} + \mathcal{K}^5_{\sigma(2354)} \\
n[2_\sigma 3_\sigma, 1, 4_\sigma 6, 5_\sigma 7] &= \mathcal{K}^3_{\sigma(2354)} - \mathcal{K}^3_{\sigma(3254)} + \mathcal{K}^4_{\sigma(2354)} - \mathcal{K}^4_{\sigma(3254)} \\
n[2_\sigma 3_\sigma, 1, 4_\sigma, 6, 5_\sigma 7] &= \mathcal{K}^4_{\sigma(3245)} - \mathcal{K}^4_{\sigma(2345)} - \mathcal{K}^5_{\sigma(2345)} + \mathcal{K}^5_{\sigma(3245)} \\
n[3_\sigma 4_\sigma, 5_\sigma, 6, 2_\sigma, 71] &= \mathcal{K}^1_{\sigma(2435)} - \mathcal{K}^1_{\sigma(2345)} + \mathcal{K}^1_{\sigma(2534)} - \mathcal{K}^1_{\sigma(2543)} \\
n[5_\sigma 7, 1, 2_\sigma, 6, 3_\sigma 4_\sigma] &= \mathcal{K}^1_{\sigma(5234)} - \mathcal{K}^1_{\sigma(5243)} + \mathcal{K}^2_{\sigma(5234)} - \mathcal{K}^2_{\sigma(5243)} \\
n[5_\sigma 7, 6, 12_\sigma, 3_\sigma 4_\sigma] &= \mathcal{K}^4_{\sigma(2435)} - \mathcal{K}^4_{\sigma(2345)} - \mathcal{K}^5_{\sigma(2345)} + \mathcal{K}^5_{\sigma(2435)} \\
n[67, 5_\sigma, 1, 2_\sigma, 3_\sigma 4_\sigma] &= \mathcal{K}^5_{\sigma(2435)} - \mathcal{K}^5_{\sigma(2345)} + \mathcal{K}^5_{\sigma(3425)} - \mathcal{K}^5_{\sigma(4325)} \\
n[5_\sigma 7, 4_\sigma, 1, 2_\sigma, 63_\sigma] &= \mathcal{K}^1_{\sigma(5423)} + \mathcal{K}^2_{\sigma(4523)} + \mathcal{K}^2_{\sigma(5423)} + \mathcal{K}^3_{\sigma(4523)} \\
n[5_\sigma 7, 4_\sigma, 12_\sigma, 3_\sigma 6] &= \mathcal{K}^2_{\sigma(2543)} + \mathcal{K}^3_{\sigma(2453)} + \mathcal{K}^3_{\sigma(2543)} + \mathcal{K}^4_{\sigma(2453)}
\end{aligned}
$$

$$
\begin{aligned}
n[2_\sigma 3_\sigma, 1, 6, 7, 4_\sigma 5_\sigma] &= -\mathcal{K}^3_{\sigma(2345)} + \mathcal{K}^3_{\sigma(2354)} + \mathcal{K}^3_{\sigma(3245)} - \mathcal{K}^3_{\sigma(3254)} \\
&\quad + \mathcal{K}^5_{\sigma(2345)} - \mathcal{K}^5_{\sigma(2354)} - \mathcal{K}^5_{\sigma(3245)} + \mathcal{K}^5_{\sigma(3254)} \\
n[67, 1, 2_\sigma 3_\sigma, 4_\sigma 5_\sigma] &= +\mathcal{K}^5_{\sigma(2345)} - \mathcal{K}^5_{\sigma(2354)} - \mathcal{K}^5_{\sigma(3245)} + \mathcal{K}^5_{\sigma(3254)} \\
&\quad - \mathcal{K}^5_{\sigma(4523)} + \mathcal{K}^5_{\sigma(4532)} + \mathcal{K}^5_{\sigma(5423)} - \mathcal{K}^5_{\sigma(5432)} \\
n[12_\sigma, 6, 3_\sigma 4_\sigma, 5_\sigma 7] &= -\mathcal{K}^2_{\sigma(2534)} + \mathcal{K}^2_{\sigma(2543)} - \mathcal{K}^3_{\sigma(2534)} + \mathcal{K}^3_{\sigma(2543)} \\
&\quad + \mathcal{K}^4_{\sigma(2345)} - \mathcal{K}^4_{\sigma(2435)} + \mathcal{K}^5_{\sigma(2345)} - \mathcal{K}^5_{\sigma(2435)} \\
n[2_\sigma 3_\sigma, 4_\sigma, 5_\sigma, 6, 71] &= +\mathcal{K}^1_{\sigma(2345)} - \mathcal{K}^1_{\sigma(3245)} - \mathcal{K}^1_{\sigma(4235)} + \mathcal{K}^1_{\sigma(4325)} \\
&\quad - \mathcal{K}^1_{\sigma(5234)} + \mathcal{K}^1_{\sigma(5324)} + \mathcal{K}^1_{\sigma(5423)} - \mathcal{K}^1_{\sigma(5432)} \\
n[2_\sigma 6, 1, 3_\sigma, 7, 4_\sigma 5_\sigma] &= +\mathcal{K}^1_{\sigma(4532)} - \mathcal{K}^1_{\sigma(5432)} + \mathcal{K}^2_{\sigma(3452)} - \mathcal{K}^2_{\sigma(3542)} \\
&\quad - \mathcal{K}^3_{\sigma(4532)} + \mathcal{K}^3_{\sigma(5432)} - \mathcal{K}^4_{\sigma(3452)} + \mathcal{K}^4_{\sigma(3542)} \\
n[3_\sigma 4_\sigma, 5_\sigma, 7, 1, 2_\sigma 6] &= -\mathcal{K}^1_{\sigma(3452)} + \mathcal{K}^1_{\sigma(4352)} + \mathcal{K}^1_{\sigma(5342)} - \mathcal{K}^1_{\sigma(5432)} \\
&\quad - \mathcal{K}^4_{\sigma(3452)} + \mathcal{K}^4_{\sigma(4352)} + \mathcal{K}^4_{\sigma(5342)} - \mathcal{K}^4_{\sigma(5432)} \\
n[4_\sigma 5_\sigma, 7, 1, 6, 2_\sigma 3_\sigma] &= +\mathcal{K}^1_{\sigma(4523)} - \mathcal{K}^1_{\sigma(4532)} - \mathcal{K}^1_{\sigma(5423)} + \mathcal{K}^1_{\sigma(5432)} \\
&\quad - \mathcal{K}^3_{\sigma(4523)} + \mathcal{K}^3_{\sigma(4532)} + \mathcal{K}^3_{\sigma(5423)} - \mathcal{K}^3_{\sigma(5432)}
\end{aligned}
$$

$$
\begin{aligned}
n[5_\sigma 7, 4_\sigma, 1, 6, 2_\sigma 3_\sigma] &= +\mathcal{K}^1_{\sigma(5423)} - \mathcal{K}^1_{\sigma(5432)} + \mathcal{K}^2_{\sigma(4523)} - \mathcal{K}^2_{\sigma(4532)} \\
&\quad + \mathcal{K}^2_{\sigma(5423)} - \mathcal{K}^2_{\sigma(5432)} + \mathcal{K}^3_{\sigma(4523)} - \mathcal{K}^3_{\sigma(4532)} \\
n[67, 1, 2_\sigma, 3_\sigma, 4_\sigma 5_\sigma] &= +\mathcal{K}^5_{\sigma(2345)} - \mathcal{K}^5_{\sigma(2354)} - \mathcal{K}^5_{\sigma(2453)} + \mathcal{K}^5_{\sigma(2543)} \\
&\quad - \mathcal{K}^5_{\sigma(3452)} + \mathcal{K}^5_{\sigma(3542)} + \mathcal{K}^5_{\sigma(4532)} - \mathcal{K}^5_{\sigma(5432)} \\
n[71, 6, 2_\sigma 3_\sigma, 4_\sigma 5_\sigma] &= -\mathcal{K}^1_{\sigma(2345)} + \mathcal{K}^1_{\sigma(2354)} + \mathcal{K}^1_{\sigma(3245)} - \mathcal{K}^1_{\sigma(3254)} \\
&\quad + \mathcal{K}^1_{\sigma(4523)} - \mathcal{K}^1_{\sigma(4532)} - \mathcal{K}^1_{\sigma(5423)} + \mathcal{K}^1_{\sigma(5432)} \\
n[12_\sigma, 6, 3_\sigma, 4_\sigma, 5_\sigma 7] &= +\mathcal{K}^2_{\sigma(2543)} + \mathcal{K}^3_{\sigma(2354)} + \mathcal{K}^3_{\sigma(2453)} + \mathcal{K}^3_{\sigma(2543)} \\
&\quad + \mathcal{K}^4_{\sigma(2345)} + \mathcal{K}^4_{\sigma(2354)} + \mathcal{K}^4_{\sigma(2453)} + \mathcal{K}^5_{\sigma(2345)} \\
n[12_\sigma, 6, 3_\sigma, 7, 4_\sigma 5_\sigma] &= +\mathcal{K}^2_{\sigma(2453)} - \mathcal{K}^2_{\sigma(2543)} + \mathcal{K}^3_{\sigma(2345)} - \mathcal{K}^3_{\sigma(2354)} \\
&\quad - \mathcal{K}^4_{\sigma(2453)} + \mathcal{K}^4_{\sigma(2543)} - \mathcal{K}^5_{\sigma(2345)} + \mathcal{K}^5_{\sigma(2354)} \\
n[2_\sigma 3_\sigma, 1, 6, 4_\sigma, 5_\sigma 7] &= -\mathcal{K}^3_{\sigma(2354)} + \mathcal{K}^3_{\sigma(3254)} - \mathcal{K}^4_{\sigma(2345)} - \mathcal{K}^4_{\sigma(2354)} \\
&\quad + \mathcal{K}^4_{\sigma(3245)} + \mathcal{K}^4_{\sigma(3254)} - \mathcal{K}^5_{\sigma(2345)} + \mathcal{K}^5_{\sigma(3245)} \\
n[2_\sigma 3_\sigma, 4_\sigma, 6, 1, 5_\sigma 7] &= +\mathcal{K}^1_{\sigma(5234)} - \mathcal{K}^1_{\sigma(5324)} - \mathcal{K}^1_{\sigma(5423)} + \mathcal{K}^1_{\sigma(5432)} \\
&\quad + \mathcal{K}^2_{\sigma(5234)} - \mathcal{K}^2_{\sigma(5324)} - \mathcal{K}^2_{\sigma(5423)} + \mathcal{K}^2_{\sigma(5432)} \\
n[2_\sigma 6, 1, 3_\sigma 4_\sigma, 5_\sigma 7] &= -\mathcal{K}^1_{\sigma(5342)} + \mathcal{K}^1_{\sigma(5432)} - \mathcal{K}^2_{\sigma(5342)} + \mathcal{K}^2_{\sigma(5432)} \\
&\quad + \mathcal{K}^3_{\sigma(3452)} - \mathcal{K}^3_{\sigma(4352)} + \mathcal{K}^4_{\sigma(3452)} - \mathcal{K}^4_{\sigma(4352)} \\
n[3_\sigma 4_\sigma, 5_\sigma, 7, 6, 12_\sigma] &= -\mathcal{K}^2_{\sigma(2345)} + \mathcal{K}^2_{\sigma(2435)} + \mathcal{K}^2_{\sigma(2534)} - \mathcal{K}^2_{\sigma(2543)} \\
&\quad - \mathcal{K}^5_{\sigma(2345)} + \mathcal{K}^5_{\sigma(2435)} + \mathcal{K}^5_{\sigma(2534)} - \mathcal{K}^5_{\sigma(2543)} \\
n[5_\sigma 7, 6, 1, 2_\sigma, 3_\sigma 4_\sigma] &= -\mathcal{K}^4_{\sigma(2345)} + \mathcal{K}^4_{\sigma(2435)} + \mathcal{K}^4_{\sigma(3425)} - \mathcal{K}^4_{\sigma(4325)} \\
&\quad -\mathcal{K}^5_{\sigma(2345)} + \mathcal{K}^5_{\sigma(2435)} + \mathcal{K}^5_{\sigma(3425)} - \mathcal{K}^5_{\sigma(4325)} \\
n[2_\sigma 6, 1, 3_\sigma, 4_\sigma, 5_\sigma 7] &= +\mathcal{K}^1_{\sigma(5432)} + \mathcal{K}^2_{\sigma(3542)} + \mathcal{K}^2_{\sigma(4532)} + \mathcal{K}^2_{\sigma(5432)} \\
&\quad + \mathcal{K}^3_{\sigma(3452)} + \mathcal{K}^3_{\sigma(3542)} + \mathcal{K}^3_{\sigma(4532)} + \mathcal{K}^4_{\sigma(3452)}
\end{aligned}
$$

$$
\begin{aligned}
n[4_\sigma 5_\sigma, 7, 16, 2_\sigma 3_\sigma] &= +\mathcal{K}^1_{\sigma(4523)} - \mathcal{K}^1_{\sigma(4532)} - \mathcal{K}^1_{\sigma(5423)} + \mathcal{K}^1_{\sigma(5432)} - \mathcal{K}^5_{\sigma(3245)} + \mathcal{K}^5_{\sigma(3254)} \\
&\quad - \mathcal{K}^3_{\sigma(2345)} + \mathcal{K}^3_{\sigma(2354)} + \mathcal{K}^3_{\sigma(3245)} - \mathcal{K}^3_{\sigma(3254)} - \mathcal{K}^3_{\sigma(4523)} \\
&\quad + \mathcal{K}^3_{\sigma(4532)} + \mathcal{K}^3_{\sigma(5423)} - \mathcal{K}^3_{\sigma(5432)} + \mathcal{K}^5_{\sigma(2345)} - \mathcal{K}^5_{\sigma(2354)} \\
n[16, 2_\sigma, 3_\sigma 4_\sigma, 5_\sigma 7] &= -\mathcal{K}^1_{\sigma(5342)} + \mathcal{K}^1_{\sigma(5432)} - \mathcal{K}^2_{\sigma(2534)} + \mathcal{K}^2_{\sigma(2543)} + \mathcal{K}^5_{\sigma(2345)} - \mathcal{K}^5_{\sigma(2435)} \\
&\quad - \mathcal{K}^2_{\sigma(5342)} + \mathcal{K}^2_{\sigma(5432)} - \mathcal{K}^3_{\sigma(2534)} + \mathcal{K}^3_{\sigma(2543)} + \mathcal{K}^3_{\sigma(3452)} \\
&\quad - \mathcal{K}^3_{\sigma(4352)} + \mathcal{K}^4_{\sigma(2345)} - \mathcal{K}^4_{\sigma(2435)} + \mathcal{K}^4_{\sigma(3452)} - \mathcal{K}^4_{\sigma(4352)} \\
n[16, 7, 2_\sigma 3_\sigma, 4_\sigma 5_\sigma] &= +\mathcal{K}^1_{\sigma(2345)} - \mathcal{K}^1_{\sigma(2354)} - \mathcal{K}^1_{\sigma(3245)} + \mathcal{K}^1_{\sigma(3254)} - \mathcal{K}^5_{\sigma(5423)} + \mathcal{K}^5_{\sigma(5432)} \\
&\quad - \mathcal{K}^1_{\sigma(4523)} + \mathcal{K}^1_{\sigma(4532)} + \mathcal{K}^1_{\sigma(5423)} - \mathcal{K}^1_{\sigma(5432)} - \mathcal{K}^5_{\sigma(2345)}
\end{aligned}
$$

$$
\begin{aligned}
& + \mathcal{K}^5_{\sigma(2354)} + \mathcal{K}^5_{\sigma(3245)} - \mathcal{K}^5_{\sigma(3254)} + \mathcal{K}^5_{\sigma(4523)} - \mathcal{K}^5_{\sigma(4532)} \\
n[2_\sigma 3_\sigma, 4_\sigma, 5_\sigma, 7, 16] = & - \mathcal{K}^1_{\sigma(2345)} + \mathcal{K}^1_{\sigma(3245)} + \mathcal{K}^1_{\sigma(4235)} - \mathcal{K}^1_{\sigma(4325)} + \mathcal{K}^5_{\sigma(5423)} - \mathcal{K}^5_{\sigma(5432)} \\
& + \mathcal{K}^1_{\sigma(5234)} - \mathcal{K}^1_{\sigma(5324)} - \mathcal{K}^1_{\sigma(5423)} + \mathcal{K}^1_{\sigma(5432)} + \mathcal{K}^5_{\sigma(2345)} \\
& - \mathcal{K}^5_{\sigma(3245)} - \mathcal{K}^5_{\sigma(4235)} + \mathcal{K}^5_{\sigma(4325)} - \mathcal{K}^5_{\sigma(5234)} + \mathcal{K}^5_{\sigma(5324)} \\
n[2_\sigma 3_\sigma, 4_\sigma, 5_\sigma 7, 16] = & + \mathcal{K}^1_{\sigma(5234)} - \mathcal{K}^1_{\sigma(5324)} - \mathcal{K}^1_{\sigma(5423)} + \mathcal{K}^1_{\sigma(5432)} - \mathcal{K}^5_{\sigma(4235)} + \mathcal{K}^5_{\sigma(4325)} \\
& + \mathcal{K}^2_{\sigma(5234)} - \mathcal{K}^2_{\sigma(5324)} - \mathcal{K}^2_{\sigma(5423)} + \mathcal{K}^2_{\sigma(5432)} + \mathcal{K}^4_{\sigma(2345)} \\
& - \mathcal{K}^4_{\sigma(3245)} - \mathcal{K}^4_{\sigma(4235)} + \mathcal{K}^4_{\sigma(4325)} + \mathcal{K}^5_{\sigma(2345)} - \mathcal{K}^5_{\sigma(3245)} \\
n[16, 2_\sigma, 3_\sigma, 4_\sigma, 5_\sigma 7] = & + \mathcal{K}^1_{\sigma(5432)} + \mathcal{K}^2_{\sigma(2543)} + \mathcal{K}^2_{\sigma(3542)} + \mathcal{K}^2_{\sigma(4532)} + \mathcal{K}^4_{\sigma(3452)} + \mathcal{K}^5_{\sigma(2345)} \\
& + \mathcal{K}^2_{\sigma(5432)} + \mathcal{K}^3_{\sigma(2354)} + \mathcal{K}^3_{\sigma(2453)} + \mathcal{K}^3_{\sigma(2543)} + \mathcal{K}^3_{\sigma(3452)} \\
& + \mathcal{K}^3_{\sigma(3542)} + \mathcal{K}^3_{\sigma(4532)} + \mathcal{K}^4_{\sigma(2345)} + \mathcal{K}^4_{\sigma(2354)} + \mathcal{K}^4_{\sigma(2453)} \\
n[16, 2_\sigma, 3_\sigma, 7, 4_\sigma 5_\sigma] = & + \mathcal{K}^1_{\sigma(4532)} - \mathcal{K}^1_{\sigma(5432)} + \mathcal{K}^2_{\sigma(2453)} - \mathcal{K}^2_{\sigma(2543)} - \mathcal{K}^5_{\sigma(2345)} + \mathcal{K}^5_{\sigma(2354)} \\
& + \mathcal{K}^2_{\sigma(3452)} - \mathcal{K}^2_{\sigma(3542)} + \mathcal{K}^3_{\sigma(2345)} - \mathcal{K}^3_{\sigma(2354)} - \mathcal{K}^3_{\sigma(4532)} \\
& + \mathcal{K}^3_{\sigma(5432)} - \mathcal{K}^4_{\sigma(2453)} + \mathcal{K}^4_{\sigma(2543)} - \mathcal{K}^4_{\sigma(3452)} + \mathcal{K}^4_{\sigma(3542)} \\
n[3_\sigma 4_\sigma, 5_\sigma, 7, 2_\sigma, 16] = & - \mathcal{K}^1_{\sigma(3452)} + \mathcal{K}^1_{\sigma(4352)} + \mathcal{K}^1_{\sigma(5342)} - \mathcal{K}^1_{\sigma(5432)} + \mathcal{K}^5_{\sigma(2534)} - \mathcal{K}^5_{\sigma(2543)} \\
& - \mathcal{K}^2_{\sigma(2345)} + \mathcal{K}^2_{\sigma(2435)} + \mathcal{K}^2_{\sigma(2534)} - \mathcal{K}^2_{\sigma(2543)} - \mathcal{K}^4_{\sigma(3452)} \\
& + \mathcal{K}^4_{\sigma(4352)} + \mathcal{K}^4_{\sigma(5342)} - \mathcal{K}^4_{\sigma(5432)} - \mathcal{K}^5_{\sigma(2345)} + \mathcal{K}^5_{\sigma(2435)} \\
n[5_\sigma 7, 4_\sigma, 16, 2_\sigma 3_\sigma] = & + \mathcal{K}^1_{\sigma(5423)} - \mathcal{K}^1_{\sigma(5432)} + \mathcal{K}^2_{\sigma(4523)} - \mathcal{K}^2_{\sigma(4532)} - \mathcal{K}^5_{\sigma(2345)} + \mathcal{K}^5_{\sigma(3245)} \\
& + \mathcal{K}^2_{\sigma(5423)} - \mathcal{K}^2_{\sigma(5432)} - \mathcal{K}^3_{\sigma(2354)} + \mathcal{K}^3_{\sigma(3254)} + \mathcal{K}^3_{\sigma(4523)} \\
& - \mathcal{K}^3_{\sigma(4532)} - \mathcal{K}^4_{\sigma(2345)} - \mathcal{K}^4_{\sigma(2354)} + \mathcal{K}^4_{\sigma(3245)} + \mathcal{K}^4_{\sigma(3254)}
\end{aligned}
$$

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