Microscopic Calabi-Yau Black holes in String Theory

Saeid Ansari

München 2011

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Saeid Ansari

Dissertation
an Fakultät Physik
der Ludwig-Maximilians-Universität
München

vorgelegt von
Saeid Ansari
aus Teheran

München, den 1. Dezember 2010

Erstgutachter: Prof. Dr. Ivo Sachs

Zweitgutachter: Prof. Dr. Christian Römelsberger

Tag der mündlichen Prüfung: 22 Juli 2011

To my parents

Zusammenfassung

In dieser Arbeit untersuchen wir mikroskospische Aspekte von Schwarzen Löchern mit Calabi-Yau-Geometrie in Typ-IIA-Stringtheorie. Wir berechnen den Absorptionsquerschnitt der masselosen Raumzeitskalare durch das D2-Branen-Weltvolumen, welches um eine S^2 einer $AdS_2 \times S^2 \times CY_3$-Geometrie eines vier-dimensionalen Schwarzen Lochs mit Calabi-Yau-Geometry gewickelt ist. Die D2-Brane kann auch eine gewöhnliche D0-Probebranen-Ladung besitzen. Wir beschränken uns jedoch auf D2-Branen mit kleiner D0-Ladung, sodass Störungstheorie anwendbar ist. Der Kandidat für die duale Theorie gemäß der vorgeschlagenen AdS_2/QM-Korrespondenz ist die Quantenmechanik einer Menge von D0-Probebranen in der AdS_2-Geometrie. Für kleine aber von Null verschiedene D0-Probeladungen finden wir den quantenmechanischen Absorptionsquerschnitt, der von einem asymptotischen anti-de Sitter-Beobachter gesehen wird. Wir wiederholen die Rechnungen für verschwindende D0-Probeladungen und diskutieren unser Ergebnis im Vergleich mit dem klassischen Absorptionsquerschnitt. In einem weiteren Projekt ermitteln wir für ein gegebenes vier-dimensionales Schwarzes Loch mit Calabi-Yau-Geometrie und gewöhnlichen D6-D4-D2-D0 Ladungen die Menge der supersymmetrischen Branen der korrespondierenden elf-dimensionalen Geometrie in der Nähe des Horizonts, die in globalen Koordinaten statisch oder stationär sind. Die Menge dieser BPS-Zustände, die Branen miteinschließt, die teilweise oder ganz den Horizont einhüllen, sollten für das Verständnis der Zustandssumme von Schwarzen Löchern mit D6-Ladungen von Bedeutung sein.

Abstract

In this thesis we study microscopic aspects of Calabi-Yau black holes in string theory. We compute the absorption cross-section of the space-time massless scalars by the world-volume of D2-branes, wrapped on the S^2 of an $AdS_2 \times S^2 \times CY_3$ geometry of a four-dimensional D4-D0 Calabi-Yau black hole. The D2-brane can also have a generic D0 probe-brane charge. However, we restrict ourselves to D2-branes with small D0-charge so that the perturbation theory is applicable. According to the proposed AdS_2/QM correspondence the candidate for the dual theory is the quantum mechanics of a set of probe D0-branes in the AdS_2 geometry. For small but non-zero probe D0-charge we find the quantum mechanical absorption cross-section seen by an asymptotic anti-de Sitter observer. We repeat the calculations for vanishing probe D0-charge as well and discuss our result by comparing with the classical absorption cross-section. In other project, for a given four-dimensional Calabi-Yau black hole with generic D6-D4-D2-D0 charges we identify a set of supersymmetric branes, which are static or stationary in the global coordinates, of the corresponding eleven-dimensional near horizon geometry. The set of these BPS states, which include the branes partially or fully wrap the horizon, should play a role in understanding the partition function of black holes with D6-charge.

Contents

CHAPTER 1

Introduction

One of the most fascinating objects that general relativity predicts are the black holes. In nature, black holes are formed from the collapse of gravitating matter. The simplest model for black hole formation involves a collapsing thin spherical shell of a massless matter. A shell of photons with very small extension and total mass M provides an example.

Black holes are, roughly speaking, solutions of the equation of motion of Einstein's theory of General Relativity that exhibit a region of the space-time which one can not escape. More precisely, a black hole is an asymptotically flat space-time containing a region which is not in the backward lightcone of the future time-like infinity. Classically, the black holes are completely black. Objects inside their event horizon are eternally trapped and nothing can emerge from inside the event horizon to the outside. Even light rays are confined by the gravitational force. In the early seventies, number of laws that govern the physics of black holes were established.In particular, it was found that there is a very close analogy between these laws and the four laws of thermodynamics [1]. The black hole laws become that of the thermodynamics if one replaces the surface gravity κ of the black hole by the temperature T of a body in thermal equilibrium, the area of the black hole A by the entropy S, the mass of the black hole M by the energy of the system E etc.

Considering thermodynamical behavior for the black holes, it is natural to wonder whether this formal similarity is more than just an analogy. Classically the identification between the black hole mechanics and the laws of thermodynamics does not seem to have physical content, because a classical black hole is just black and, therefore, the mass can only increase as matter falls through the horizon, it cannot radiate and therefore one should assign temperature zero to it so that the interpretation of the surface gravity as temperature fails to be correct. However, when the quantum effects are taken into account one can analyze the black holes in the context of the quantum field theory in the curved space-time, where the matter is described by the quantum field theory while the gravity treated as a classical background field, so-called the semi-classical approximation, of a full theory of the quantum gravity. In this framework it was discovered [1] that the black holes can emit (Hawking) radiation and consequently, can loses mass via Hawking radiation which allows to assign the so-called Hawking temperature. This gives the Bekenstein-Hawking entropy as

$$S_{macro} = \frac{A}{4} \tag{1.1}$$

On the other hand, the entropy is also a measure for the number of internal microstates of the system. To compute the entropy on the microscopic side one needs to identify the internal microstates and count the degeneracy N of microstates which give rise to a same macrostate. The microscopic entropy is, accordingly, given by

$$S_{micro} = \log N \tag{1.2}$$

This raise the question about the nature of the microstates of a black hole. One would like to understand whether there exists a fundamental, microscopic level describing the black holes.

String theory, as the leading candidate for the quantum theory of gravity, should be able to tell us about the microscopic configuration or, in other word, about the quantum statistical mechanics of the black holes. The detailed study of matching the thermodynamic entropy with state counting in string theory is, however, only possible for the supersymmetric black holes [2, 3]. These are the black holes that are asymptotically flat, charged and extremal, also called the BPS black holes. The key point is that as long as the supersymmetry is preserved certain quantities can be calculated at zero coupling and the result remains valid for all values of the string coupling, g_s. The BPS property, especially, ensures that the number of micro-states will be conserved under varying the coupling. Therefore, in case of the supersymmetric black holes, it is meaningful to count the microstates in non-interacting regime, where the coupling is zero, and compare it to the macroscopic entropy of black hole, where the interactions are turned on.

Four Dimensional Physics

Since, string theory lives in a ten-dimensional space-time, or 11 dimensions from M-theory point of view, if we want to describe the four-dimensional black holes we should consider a space-time where the extra dimensions have been compactified. With this assumption, the original ten-dimensional space-time would be split in two sub-spaces as

$$M^4 \times X$$

where M^4 is a four-dimensional space-time corresponding to the world we know, and X is some compact six-dimensional manifold which is too small for us to observe directly. We can visualize this geometry by thinking that at every point in M^4 there is corresponding space X. Although, the compactified space can not be seen by the observers living in M, but properties of the internal space X, lead to physical consequences in the four-dimensional space-time.

In principle, there are large number of possible six-dimensional compact manifolds one could choose from, but, the requirement that the four-dimensional theory resemble the observed world would limit our choice of X some what. Indeed, demanding that the supersymmetry be preserved when we compactify our ten-dimensional theory to four dimensions, lead us to the requirement that our compact space be a Calabi-Yau manifold. The Calabi-Yau manifolds are complex and are even-dimensional spaces which the six-dimensional one is called Calabi-Yau three-fold, briefly CY_3. This leads us to the so-called Calabi-Yau black holes in string theory.

The four-dimensional black holes in string theory are typically engineered in terms of the branes wrapped around the appropriate cycles of the internal space [4].
Suppose we have compactified the space-time, on a manifold, down to four dimensions, then the branes wrapped around the directions in the compact dimensions will look like point-like objects, or so to say (charged) particles, in the four-dimensional space-time, which are considered at a point in the space which is the center of black hole. These configurations have huge number of the internal excitations which all lead to a same four-dimensional black holes are count for the microscopic description.
One of the special features of the supersymmetric black holes in four dimensions is that they have a residual supersymmetry and interpolate between two maximally supersymmetric vacua, namely the Minkowski space-time at infinity and the $AdS_2 \times S_2$ at the horizon. This lead us to the $AdS_2 \times S^2 \times CY_3$ geometry at the near horizon of the Calabi-Yau black hole. In more modern language, string theory in AdS background is conjectured to be dual to a conformal quantum mechanics [5], known as AdS/CFT correspondence. The AdS/CFT relates a theory with gravity to theory without gravity [6]. In other word, the macroscopic gravitational dynamics are holographically encoded in the microscopic gauge theoretical degrees of freedom living at the conformal boundary of the near-horizon region.

This duality provide us a promising approach for reproducing the macroscopic entropy by state counting, where for AdS_2 is then AdS_2/CFT_1. In particular, it was shown [7] that using AdS_2/CFT_1 proposal for a class of black holes in IIA on the CY_3 carrying D0 and D4-branes yield the result agrees with the Bekenstein-Hawking entropy formula, where the CFT_1 takes the form of a quantum mechanics of a set of probe D0-branes moving in the AdS_2 near-horizon geometry.

Apart from reproducing the entropy of the black hole one expects that the dual quantum mechanics should also provide a microscopic description of the absorption- and Hawking emission of the space-time fields by a black hole. This motivated us to compute the low energy absorption cross-section of the space-time scalars on an static D2-brane, wrapped on the S^2 of an $AdS_2 \times S^2 \times CY_3$ geometry [8]. In principle these D2-branes can also have a generic D0 probe-brane charge. However, for large probe D0-brane charge the coupling of the space-time fields to the world-volume quantum mechanics becomes large so that the linearized perturbation theory is no longer applicable and back-reaction on the geometry has to be taken into account.

Furthermore, though, the D4-D0 Calabi-Yau black hole and its dual quantum mechanics have produced the results in agreement with the Bekenstein-Hawking entropy but since the D6-charge is taken to be zero, it is not the most general black holes we can construct in IIA. In general, we would like to get a precise microscopic description of the four dimensional black hole with all possible charges of IIA theory, namely non-vanishing D6-D4-D2-D0 charges. Computing the entropy of such a CY_3 black hole with generic charges and their relation to the conformal field theories are yet to be understood.

One can prepare the ground for extending the D4-D0 black hole to the black holes with D6-charge by describing the supersymmetric probe branes in the background of D6-charge. The classification of possible supersymmetric branes can play a role for understanding the degeneracies of states, where will give some insight about the underlaying microscopic theory. This is the work done in our other project [9], where for a given black hole with generic D6-D4-D2-D0-charges in four dimensions, the set of supersymmetric branes, static or stationary in global coordinates, of the corresponding eleven-dimensional near horizon geometry are identified.

This thesis is structured as follows: In chapter 2, first we review the black holes in general relativity, namely the Schwarzschild and the Reissner-Nordström black holes. We look deeper in case of the RN black hole and discuss its near horizon geometry. Then we address the thermodynamic laws of black holes, and specially the entropy. Since, we are interested in the absorption of black holes, we provide the recipe of how to find the absorption cross-section for black holes and then we employ it to find the absorption cross-section for the Schwarzschild and Reissner-Nordström black holes. In case of RN we treat the absorption for the extremal and the near-extremal case separately. In last section we review the universality of low energy absorption cross-section, namely the absorption cross-section independency of the falling wave's frequency. In chapter 3, we provide some background of how to make a back hole in string theory. Specially we are interested in the Calabi-Yau black holes in IIA string theory with various wrapped-branes around the cycles of the Calabi-Yau. To do this we give the bosonic field content of IIA and then we talk about the D4-D0 black hole. As a simple example, we first illustrate the structure of the configuration of limited number of D4-branes and D0-branes in the context of toroidal compactifications, where the six-dimensional compact space is six-torus, T^6.We see that the near horizon of such a class of black holes has an $AdS_2 \times S^2$ geometry. To have a better understanding of the possible brane interactions first we review the dynamics of the D-branes and their coupling to the various background fields. Specially we consider a D2-brane which is wrapped around the horizon and couples to the background gauge field arising from the D4 and D0 branes in the internal space. These branes can also have a D0-brane charge on them, we show that the value of D0-charge determines the radial position of the D2-brane. Also we observe that the horizon-wrapped branes are static in global coordinates while they pup out and in of the horizon in the Poincaré coordinates. After making a Calabi-Yau black hole, we then address the entropy of black hole in string theory. The goal is to use string theory to reproduce the macroscopic entropy law by counting the states of the underlaying microscopic theory. We address why we are interested in the supersymmetric black holes in string theory and then we give an example of how we count the degeneracy of a black hole in the D-brane description. We consider three D4 branes where each of them are wrapped the four-cycles of six-torus plus some additional D0-branes and by counting the degeneracies we lead to precise agreement between the macroscopic and microscopic black hole entropy.

As the main part of the chapter we compute absorption of the space-time scalars by the world-volume of D2-branes, wrapped on the S^2 of a global $AdS_2 \times S^2$ geometry. The D2-branes can also have a generic D0 probe-brane charge. However, we will restrict ourselves to the D2-branes with small D0-charge so that the perturbation theory is applicable. First we identify the vibration modes of the brane then we compute the cross-section for the absorption of dilatons on the two-brane. Starting with the s-wave absorption of the D2-

brane without any D0-charge, then we follow to find the absorption cross-section of higher partial waves and also we compute the absorption on D2-brane in case of non-vanishing D0-charges. For small but non-zero probe D0-charge we find that the quantum mechanical absorption cross-section seen by an asymptotic anti-de Sitter observer, static in the Poincaré time vanishes linearly in ω for the small frequencies. We find the result which is in disagreement with the classical s-wave absorption cross-section by the black hole, which vanishes quadratically in ω forthe small frequencies [10]. We will try to clarify this point in the text.

Finally, in the last chapter we dealing with the problem of supersymmetric branes in the Calabi-Yau background. In this chapter we are interested in the possible supersymmetric M2 and M5-branes in M-theory which by the Calabi-Yau compactification give raise to five-dimensions. To provide a base for our discussion, we start with describing the conditions which yield the BPS branes. In order to be self-contained and to fix the conventions we first review the relevant static half BPS solutions of four-dimensional IIA supergravity. Then, we will describe the eleven-dimensional near horizon geometry of a 4D black hole with generic D6-D4-D2-D0 charge. We obtain the near horizon killing spinor in the global coordinates and analyze the κ-symmetry for stationary probe branes in the global time. In particular we find BPS two-branes wrapped on a two-cycle in the Calabi-Yau. Furthermore, we consider five-branes which can potentially wrap the horizon partially or completely. We determine the trajectories of the five-brane which preserve supersymmetry for the case of either wrap a holomorphic four-cycle in the Calabi-Yau and an S^1 in space-time (hence partially wrapping the horizon), or fully wrap the horizon $S^3/\mathbb{Z}_{p^0}$ and a holomorphic two-cycle in the Calabi-Yau. At the end we provide some useful material at the appendices.

CHAPTER 2

Black Holes in General Relativity

Gravity, as a classical theory, is described by the Einstein classical theory of gravity. The basic idea of the Einstein gravity is that the geometry of space-time is dynamical and is determined by the matter distribution. Conversely the motion of matter is determined by the space-time geometry. Einstein gravity without any sources, is based on the Einstein-Hilbert action

$$S_{EH} = \frac{1}{16\pi G} \int d^4 \sqrt{-g} R \tag{2.1}$$

where G is the Newton gravitational constant. The classical equations of motion following from this action are the source-free ($T_{\mu\nu}$) Einstein equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{2.2}$$

which comes from varying the action with respect to the metric. Here $R_{\mu\nu}$ and R are the Ricci tensor and the Ricci scalar, respectively. In case of presence of the matter coupled to the gravity, the energy-momentum tensor is non-zero and so the Einstein equations would be

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{2.3}$$

Some of the most interesting solutions to the equation of motion describe black holes. These solutions contains singularities at which certain curvature diverge.

The generalizations of the Einstein-Hilbert action are provided by considering the electromagnetic fields, spinors or tensor fields, such as those that appear in the supergravity theories. In case of the matter coupling to the electromagnetic field, which is relevant for us, one gets the, so called, Einstein-Maxwell action

$$S = \int d^4 x \sqrt{-|g|} \left(\frac{1}{16\pi G_4} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \tag{2.4}$$

2.0.1 Schwarzschild black hole

The first exact solution to the Einstein's equations of motion is the Schwarzschild solution, which describes a black hole. The Schwarzschild solution is the unique, spherically symmetric and, by the Birkhoff's theorem, static solution of the vacuum Einstein equation in the four space-time dimensions.

Solution to the vacuum, with a vanishing energy momentum tensor ($T_{\mu\nu} = 0$), Einstein's equation is

$$R_{\mu\nu} = 0 \tag{2.5}$$

that describes the geometry outside of the mass distribution which is precisely the Ricci-flat space-time. In the standard Schwarzschild coordinates (t, r, θ, ϕ) the Schwarzschild solution is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\left(1 - \frac{r_H}{r}\right) dt^2 + \left(1 - \frac{r_H}{r}\right)^{-1} dr^2 + r^2 d\omega_2^2 \tag{2.6}$$

where

$$r_H = 2G_4 M \tag{2.7}$$

is known as the *Schwarzschild radius* and G_4 is the Newton's constant.

$$d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{2.8}$$

is the metric of unit two-sphere. The surface $r = r_H$ is called the event horizon. Let us note that r_H is very small, for example one finds $r_H = 2.9$km for the sun and $r_H = 9$mm for the earth. Thus for atomic matter the Schwarzschild radius is inside the matter distribution and therefore does not threat as a black hole. The Schwarzschild metric is only a function of the mass and it reduces to the Minkowski metric as the mass goes to zero. Furthermore the metric is asymptotically flat

$$g_{\mu\nu}(r) \xrightarrow{r\to\infty} \eta_{\mu\nu} \tag{2.9}$$

To see if M really has interpretation of a mass, we should consider the weak coupling limit that is the asymptotic behavior of the metric. In general, in this limit the Newtonian potential Φ in the stationary coordinates can be read off from the tt component of the metric

$$g_{tt} \sim -(1 - 2\Phi) \tag{2.10}$$

So by taking $r \to \infty$ limit of the g_{tt} component of the Schwarzschild black hole we have

$$g_{tt}(r \to \infty) \sim 1 - \frac{2MG_4}{r} \tag{2.11}$$

which correctly produce the Newtonian potential

$$\Phi = -\frac{2MG_4}{r} \tag{2.12}$$

2.0.2 Reissner-Nordström black hole

Generalization of the Schwarzschild black hole is the electrically charged one, which is known as the Reissner-Nordström black hole.

An spherically symmetric solution of the coupled equations of the Einstein and of the Maxwell is the that of Reissner-Nordström. The Reissner-Nordström solution is the most general static black hole of the Einstein-Maxwell theory

$$S = \int d^4x\sqrt{-|g|}\left(\frac{1}{16\pi G_4}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \tag{2.13}$$

combined with the equation of motion and the Bianchi identity for the gauge field

$$\begin{aligned} \nabla_\mu F^{\mu\nu} &= 0 & (2.14)\\ \varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} &= 0 & (2.15) \end{aligned}$$

The Einstein's equations in the presence of an electric/magnetic field are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_4\left(F_{\mu\rho}F_{\rho_\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right) \tag{2.16}$$

where $F_{\mu\nu}$ denotes the components of Maxwell tensor. Taking the trace we get $R = 0$. This is always the case if the energy momentum tensor is traceless.
The unique spherically symmetric solution of (2.16) is the Reissner-Nordström solution

$$\begin{aligned} &ds^2 = -\Delta dt^2 + \Delta^{-2}dr^2 + r^2 d\Omega_2^2\\ &F_{tr} = -\frac{q}{r^2}, \qquad F_{\theta\phi} = p\sin\theta\\ &\Delta = 1 - \frac{2MG_4}{r} + \frac{(q^2+p^2)G_4}{r^2} \end{aligned} \tag{2.17}$$

q and p are the electric and magnetic charges of the black hole. This can be checked by recalling the definition of electric and magnetic charge in terms of the Maxwell tensor

$$q = \frac{1}{4\pi}\oint *F, \qquad p = \frac{1}{4\pi}\oint F \tag{2.18}$$

where

$$F = \frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu \tag{2.19}$$

is the two-form field strength , $*F$ is the dual field strength and integration surface surrounds the sources. It is worthwhile to note that the solution is static and asymptotically flat.

The Reissner-Nordström black hole is characterized by three parameters, namely, the mass M, the electric q and the magnetic charge p, where the charges can be conveniently combined into $Q = q + ip$.
The Reissner-Nordström metric has two horizons at

$$r = r_\pm = MG_4 \pm \sqrt{(MG_4)^2 - Q^2G_4} \tag{2.20}$$

This encourage us to rewrite the metric in more convenient form as

$$\begin{aligned} ds^2 &= -\Delta_+\Delta_- dt^2 + \Delta_+^{-1}\Delta_-^{-1}dr^2 + r^2 d\Omega_2^2, \\ &\Delta_\pm = 1 - \frac{r_\pm}{r} \end{aligned} \tag{2.21}$$

It should be clear that there is a singularity at $r=0$. We have assumed that $M \geq Q$, since otherwise there is no horizon and the solution has a naked singularity and thus is not physically acceptable. This is known as the cosmic censorship hypothesis, which says that the gravitational collapse does not lead to a naked singularity. There is a very important special case arising when we saturate the lower bound on the mass, M, by making it equal to the charge, Q, which is call the extremal case. Then we see that the both horizons coincide at $r=Q$. The extremal metric takes the form of

$$ds^2 = -\left(1-\frac{r_H}{r}\right)^2 dt^2 + \left(1-\frac{r_H}{r}\right)^{-2} dr^2 + r^2 d\Omega_2^2 \tag{2.22}$$

where $r_H = MG_4$. Let us change the radial coordinate to

$$y \equiv r - r_H \tag{2.23}$$

then the metric can be written as

$$ds_{ext}^2 = -H(y)^{-2}dt^2 + H(y)^2(dy^2 + y^2 d\Omega_2^2), \qquad H(y) \equiv 1 + \frac{r_H}{y} \tag{2.24}$$

which, now, the horizon is located at $y=0$. We see that in these coordinates there is a manifest $SO(3)$ symmetry. These are known as the isotropic coordinates.

2.0.2.1 Near horizon behavior

Another property of the extremal case, is when we are close to the horizon. Near the horizon, where $y \approx 0$, the extremal metric (2.24) can be approximated by

$$ds_{ext}^2 \xrightarrow{y\to 0} -\left(\frac{y}{r_H}\right)^2 dt^2 + \left(\frac{y}{r_H}\right)^{-2} dy^2 + r_H^2 d\Omega_2^2 \tag{2.25}$$

We see that the spatial part of the solution has degenerated into the product of an infinitely long tube or 'throat'of topology $\mathbb{R} \times S^2$ with fixed radius set by the value of horizon radius, or equivalently by the charge. The whole geometry, called the 'Bertotti-Robinson'universe. Defining yet another coordinate as

$$z \equiv \frac{r_H^2}{r} \tag{2.26}$$

we find that the geometry approaches a direct product of a two sphere, parametrized by (θ, ϕ) and a two-dimensional anti-de Sitter space parametrized by (r,t):

$$ds_{ext}^2 = \underbrace{\frac{r_H^2}{z^2}(-dt^2 + dz^2)}_{AdS_2} + \underbrace{r_H^2 d\Omega_2^2}_{S^2} \qquad AdS_2 \times S^2 \tag{2.27}$$

The anti-de Sitter space-time is the most symmetric vacuum solution to the two-dimensional Einstein equations with a negative cosmological constant.
Both the sphere and the anti-de Sitter spaces have a same curvature radius, and because the AdS space has negative curvature, the curvature of full the metric is zero and so it is flat. Since the Reissner-Nordström space-time is also asymptotically flat, we see that it interpolates between two maximally symmetric space-time.

2.1 Black hole thermodynamics and quantum aspects

One of the remarkable results of the black hole physics is that one can derive a set of laws, called the laws of black hole mechanics, which have a same structure as the laws of thermodynamics [11]. The black hole laws are derived using the geometrical properties of event horizon and has not linked to the thermodynamics from the beginning. The laws are statements about the solution of the field equations, and in the original proofs the Einstein equations are used.
The zeroth black hole mechanics law states that the so-called surface gravity κ is constant over the event horizon of an stationary black hole.

$$\kappa = \text{const.} \tag{2.28}$$

The surface gravity of an stationary black hole is the acceleration of an static observer at the horizon, as measured by an observer at infinity).
The first law is an energy conservation. It is simply an identity , relating the change in mass, M of an stationary black hole to the changes of the angular momentum, J, the horizon area, A, and the electric charge, Q,

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J + \Phi_H \delta Q \tag{2.29}$$

where Ω_H is the angular velocity at the horizon and Φ_H is the electrostatic potential at the horizon,$\Phi_H = Q/r_H$.
The second law says that the area of the horizon, A is non-decreasing function of time

$$\delta A \geq 0 \tag{2.30}$$

The zeroth law of the black hole mechanics resembles the zeroth law of thermodynamics

$$T = 0 \tag{2.31}$$

which says that the temperature is constant in a thermodynamic equilibrium, and also the first law of black hole has the same form as the first law of the thermodynamics

$$\delta E = T\delta S + pdV + \mu dN \tag{2.32}$$

The comparison of the first two laws suggests us to identify the surface gravity with the temperature and consequently the area of horizon with the entropy

$$\kappa \sim T\,, \qquad A \sim S \tag{2.33}$$

such that the temperature of black hole is its surface gravity and the entropy of black hole is equal to the size of the horizon. The analogy of the horizon area and the entropy is confirmed by comparing the second law of black hole mechanics with the one from the thermodynamics.

Classically the identification between the black hole mechanics and the laws of thermodynamics does not seem to have a physical content, because a classical black hole is just black and therefore, the mass can only increase as the matter falls through the horizon, it cannot radiate and therefore one should assign temperature zero to it, so that the interpretation of the surface gravity as a temperature has no physical content, since the surface gravity is non-zero.
However, when the quantum effects are taken into account one can analyze the black holes in the context of quantum field theory in a curved space-time, where matter is described by the quantum field theory while gravity treated as a classical background field [1]. In this framework it was discovered [1] that the black holes can emit (Hawking) radiation. Hawking argued that because the gravitational fields at the horizon are strong for a quantum mechanical pair production in the vicinity of the horizon, one particle of the virtual pair falls into the black hole and the other one is emitted as a radiation. Consequently, a black hole can loses mass via the Hawking radiation [2]. This allows us to assign the so-called Hawking temperature

$$T_H = \frac{\hbar\kappa}{2\pi} \tag{2.34}$$

to a black hole, which is indeed proportional to the surface gravity. For example one can compute the Hawking temperature for the sun, which is $T \sim 6 \times 10^{-8} K$. Now we can look at the first law. Since the Hawking temperature fixes the factor between temperature and surface gravity, the entropy-area identification becomes precise

$$S_{BH} = \frac{A}{4\hbar G_4} \tag{2.35}$$

This relation is known as the area law and S_{BH} is called the Bekenstein-Hawking entropy. We reinserted the Newton's constant to show that the black hole entropy is dimensionless.

The area law is *universally* valid result for any black hole in any dimension

$$S_{BH} = \frac{A_d}{4G_d} \tag{2.36}$$

where A_d is the, $(d-2)$-dimensional, area of the horizon and G_d is the Newton's constant in a d-dimensional space-time. Let us comment on the second law in the quantum realm. From one side, the Hawking radiation decreases the mass of the black hole and the horizon will shrink. This violates the second law of the the black hole mechanics, which states that the area of the horizon is non-deceasing. On the other side, the classical second law of thermodynamics violates because the entropy of black hole could be reduced by a matter moving adiabatically, into the black hole. This contradiction, leads us to consider the, so-called, generalized entropy, which includes the entropy of black hole plus the other stuff such as the Hawking radiation and then, the second law of thermodynamics states that the total entropy is non-decreasing. One example of the unusual thermodynamic behavior of

[1] Of course, the quantum field theory in a curved space-time is only an approximation, so-called the semi-classical approximation, of a full theory of quantum gravity.

[2] Since, the radiation is thermal, the back-reaction can be neglected.

the uncharged black hole is the dependence of temperature of the black hole to its mass. In case of the Schwarzschild black hole one finds $T = 1/(8\pi M)$, which means that the specific heat is negative and so the black hole heats up by loosing mass, till it fully decays into the radiation. This unusual behavior of the uncharged black holes leads to the information problem of quantum gravity. For the charged black holes, radiation can not destroys the black hole, since it radiates till the extremality is reached. The Hawking temperature vanishes in the extremal limit and therefore the extremal black holes are stable.
Having established the thermodynamical laws for the black holes and knowing about the macroscopic parameters, one would like to understand the underlying microscopic theory, such as, where does the entropy of the black hole come from and what are the microscopic degrees of freedom make up the black hole?
In the other word, whether there exists a fundamental, microscopic level of black hole's description, where one can identify the microstates and counts how many of them lead to a same macrostate. At the microscopic level one can define the microscopic or statistical entropy by counting the degeneracy of microstates which give rise to a same macrostate. The macrostates of a black hole are characterized by its mass M, charge Q and angular momentum J. Letting $d(M,Q,J)$ be the number of the microstates of a black hole with macroscopic parameters, M, Q, J, the microscopic black hole entropy is defined by

$$S_{micro} = \log d(M,N,J) \tag{2.37}$$

If the interpretation of the Bekenstein-Hawking entropy, S_{BH}, as macroscopic entropy is correct, then it must be equal to the microscopic entropy, namely

$$S_{BH} = S_{micro} \tag{2.38}$$

We deal with the microscopical entropy in next chapter when we study the black holes in string theory.

2.2 Classical absorption on black holes

Let us look at the computation on the gravity side. Consider an spherical black hole with the horizon area A. Suppose that we have a minimally coupled massless scalar in the theory:

$$\Box\Phi = 0 \tag{2.39}$$

$$\Box = \frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}g^{\alpha\beta}\partial_\beta\right)$$

We wish to find the cross-section for absorption of such scalars into the black hole. To do this, we must solve the wave equation (2.39), with some boundary conditions. We have a plane wave incident from infinity. We put the boundary condition at the horizon which says that the quanta are falling in but not coming out. Some part of the incident plane wave will be reflected from the metric around the black hole and give rise to an outgoing waveform at infinity. The rest goes to the horizon and represents the part that is absorbed. From this

absorbed part we deduce the absorption cross-section σ_{abs}.
In general, this is a hard calculation to do, but it becomes simple in the limit where the wavelength of the incident particle becomes much larger than the size of the black hole. We call this limit the *low-energy absorption cross-section*

$$\omega \ll \frac{1}{M} \tag{2.40}$$

To calculate the absorption cross-section we need to find the so-called *greybody factor* $\mathscr{F}$ defined as

$$\mathscr{F} = \left| \frac{f_{tr}}{f_{in}} \right| \tag{2.41}$$

where f_{tr} and f_{in} are, respectively, the flux passing the horizon and the incoming flux from infinity. The greybody factor is an important quantity to understand the absorption and the emission phenomena of a black hole. It is this factor which makes a black hole to be different from a black body. The physical origin of this factor is the effective potential barrier generated by a black hole space-time. For example, the potential for the s-wave massless scalar generated by a Schwarzschild space-time is

$$V_{eff}(r_*) = \frac{r_H}{r^3}\left(1 - \frac{r_H}{r}\right) \tag{2.42}$$

when the wave equation is expressed in terms of the “tortoise“ coordinate $r_* = r + r_H \ln(r/r_H - 1)$. This potential generally backscatters a part of the outgoing radiation quantum mechanically, which results a frequency-dependent greybody factor. To find the absorption cross-section we need to project a plane wave into an s-wave. In four dimension, this gives

$$\sigma_{abs} = \frac{\pi}{\omega^2}\mathscr{F} \tag{2.43}$$

2.3 Computing the low energy absorption cross-section

The computational procedure most people adopted is the *matching procedure*. In matching procedure we divide the space-time outside the horizon,$r \geqslant r_H$, into two overlapping regions defined by

$$\text{Near Horizon Region}: r - r_H \ll \frac{1}{\omega} \tag{2.44}$$

$$\text{Far Region}: M \ll r - r_H \tag{2.45}$$

where in case of the RN black hole r_H is the outer horizon. In each region the wave equation (2.39) can be approximated by using the above near horizon and far region properties and then we can solve it exactly.
Each of the, near horizon and far region, solutions would have undetermined coefficients. The complete solution obtained by fixing the constants and we find them by imposing the boundary conditions and by matching the solutions.

To match the near and far solutions we take the large r limit of the near horizon solution and match it to the small r limit of the far solution, namely

$$\Phi_{nh}(r \to \infty) \Leftrightarrow \Phi_{far}(r \to 0) \tag{2.46}$$

where Φ_{nh} and Φ_{far} are, respectively, the near horizon and the far solutions of the wave equation. By matching the solutions we obtain a complete solutions of the wave equation. Now we can calculate the greybody factor, $\mathscr{F}$. As already mentioned, the greybody factor is the ratio of the flux passing across the horizon, f_{tr} to the incoming flux from infinity, f_{in}

$$\mathscr{F} = \left| \frac{f_{tr}}{f_{in}} \right| \tag{2.47}$$

and the conserved flux is defined as

$$f \equiv \frac{4\pi}{2i} \sqrt{|g|}\, g^{rr} \left(R^*(r)\partial_r R(r) - R\partial_r R^* \right) \tag{2.48}$$

where r is for the radial coordinate and $R(r)$ is the radial part of the wave equation Φ. We should notice that for computing the incoming flux from infinity, we must insert the incoming part of the Φ_{far} in the flux equation. In case of the flux ingoing to the horizon we have the boundary condition that the is no reflected wave and so we use the complete form of Φ_{nr} to determine the f_{in}. Finally, we insert the computed greybody factor in (2.47) to find the absorption cross-section.
In what follows we use the technique described here to calculate the absorption cross-section for massless scalars in case of the Schwarzschild and the Reissner-Nordström black holes.

2.4 Absorption on 4D Schwarzschild black holes

As a first example we consider a four-dimensional Schwarzschild black hole and compute the absorption cross-section of a massless scalar.
Absorption spectra of a Schwarzschild black hole firstly calculated in [12]. We look at the low energy absorption, where only the mode with lowest angular momentum will contribute to the cross-section which for the scalars this is the s-wave.
The Schwarzschild black hole metric is

$$ds^2 = -h(r)dt^2 + h(r)^{-1}dr^2 + r^2 d\Omega_2^2 \qquad\qquad h(r) := 1 - \frac{r_H}{r} \tag{2.49}$$

where r_H, the Schwarzschild radius, is related to the black hole mass M as

$$r_H = 2M \tag{2.50}$$

Now we follow the steps described in the previous section to calculate the absorption cross-section for the Schwarzschild black hole. First we should solve the wave equation (2.39) for the near horizon and asymptotic regimes.

2.4.1 Wave equation and the solution

The general form of the wave equation is

$$\Box\Phi(r,t) = 0 = \partial_t\left[\sqrt{|g|}g^{tt}\partial_t\right]\Phi(r,t) + \partial_r\left[\sqrt{|g|}g^{rr}\partial_r\right]\Phi(r,t) \tag{2.51}$$

We adopt the separation of variation as

$$\Phi(r,t) = e^{-i\omega t}R_\omega(r) \tag{2.52}$$

With this, the wave equation would be

$$r^2h(r)^{-1}\omega^2R(r) + \partial_r\left[r^2h(r)\partial_r\right]R(r) = 0 \tag{2.53}$$

Next, we should solve the above wave equation in near horizon and far regions.

Far region

In the region far from the black hole, $r \to \infty$, the black hole and its effects disappear. We have

$$h(r) \simeq 1 \qquad\qquad r \to \infty \tag{2.54}$$

and so the metric becomes a flat metric

$$ds^2 = -dt^2 + dr^2 + r^2d\Omega_2^2 \tag{2.55}$$

thus the wave equation is simply

$$r^2\partial_r^2R_{far}(r) + 2r\partial_rR_{far}(r) + (r\omega)^2R(r) = 0 \tag{2.56}$$

where we have added the index "*far*“ to $R(r)$ to remember that we are looking at the solution of wave equation in the far region. Above differential equation has a complete solution in the form of Bessel functions

$$R_{far}(r) = \sqrt{\frac{\pi}{2\omega r}}\left[\alpha J_{\frac{1}{2}}(\omega r) - \beta J_{-\frac{1}{2}}(\omega r)\right] \tag{2.57}$$

α and β are constants and J is Bessel function.

Near horizon

To solve the wave equation in the region close to the horizon, $r \to r_H$, we define a new radial coordinate as

$$y := \frac{1}{r_H}\ln(h(r)) \tag{2.58}$$

which simplifies the wave equation to

$$\partial_y^2R_{nh}(y) + {r_H}^4\omega^2R_{nh}(y) = 0 \tag{2.59}$$

Solution to the above equation is

$$R_{nh}(y) = Ae^{-i\omega {r_H}^2 y} \tag{2.60}$$

A is a constant which will be related to the constants in (2.57) when we match the solutions.

2.4.2 Matching the far and near solutions

Now we must match the far and near solutions or more precisely we should match the near horizon solution when $r \to \infty$ to the far region solution when $r \to 0$.
First, we look at the far region solution eq. (2.57) and take the $r \to 0$ limit, which is easily carried out by using the series expansion of the Bessel functions [3]

$$R_{far}(r \to 0) \simeq \frac{1}{\sqrt{r}} \left[\alpha \left(\frac{\omega r}{2}\right)^{\frac{1}{2}} \frac{1}{\Gamma\left(\frac{3}{2}\right)} + \beta \left(\frac{\omega r}{2}\right)^{-\frac{1}{2}} \frac{1}{\Gamma\left(\frac{1}{2}\right)} \right] \tag{2.61}$$

For $r \to \infty$ limit of the near horizon solution we notice that

$$r \to \infty \Rightarrow h(r) \to 1 \Rightarrow y \to 0 \tag{2.62}$$

so (2.60) can be approximated by

$$R_{nh}(y \to 0) \simeq A[1 - i\omega r_H^2 y] \tag{2.63}$$

or equally

$$R_{nh}(r \to \infty) \simeq A \left[1 - i\omega r_H \ln\left(1 - \frac{r_H}{r}\right)\right] \simeq A(1 + i\omega \frac{r_H^2}{r}) \tag{2.64}$$

Now by comparing (2.61) and (2.64) we straightforwardly lead to

$$\alpha = \left(\frac{2\omega}{\pi}\right)^{\frac{1}{2}} A \tag{2.65}$$

$$\beta = i r_H^2 \left(\frac{\pi \omega^3}{2}\right)^{\frac{1}{2}} A \tag{2.66}$$

Also, since we are looking at the low energy limit where $\omega \ll 1$, we find that $\beta \ll \alpha$, which will use it later.

2.4.3 Computing absorption cross-section

To compute the absorption cross-section we need to know the incoming flux from infinity and the flux transmitted to the horizon. First, we compute the incoming flux from infinity, f_{in}.
In the far region, for $r \to \infty$ we can approximate the solution (2.57) by using the asymptotic form of the Bessel functions

$$R_{far}(r \to \infty) \simeq \frac{\pi}{2\omega r} [\sin \omega r + \cos \omega r] \tag{2.67}$$

we need to compute the *incoming* flux, thus we should extract the incoming part of the wave equation. The above Bessel function can be rewritten as

$$R_{far}(r \to \infty) \simeq \frac{\pi}{2\omega r} \frac{1}{2} \left[(i\alpha + \beta) e^{-i\omega r} (-i\alpha + \beta) e^{i\omega r}\right] \tag{2.68}$$

[3] see appendix D there you can find more about Bessel functions and some useful formula.

this is of the form of

$$R_{far}(r \to \infty) = \phi_{in} + \phi_{ref} \tag{2.69}$$

where ϕ_{in} and ϕ_{ref} are, respectively, the incident and the reflected waves at infinity

$$\phi_{in} = \frac{\pi}{2\omega r}\frac{1}{2}(i\alpha + \beta)e^{-i\omega r} \tag{2.70}$$

$$\phi_{ref} = \frac{\pi}{2\omega r}\frac{1}{2}(-i\alpha + \beta)e^{i\omega r} \tag{2.71}$$

Now we have the incident part of the wave equation at infinity and we can compute the incoming flux from infinity by inserting ϕ_{in} in the definition of conserved flux (2.48)

$$f_{in}(r \to \infty) = \frac{2\pi}{i} r^2 \left[\phi_{in}^* \partial_r \phi_{in} - \phi_{in} \partial_r \phi_{in}^*\right] \tag{2.72}$$

$$= 2(|\alpha|^2 + |\beta|^2) \tag{2.73}$$

where we have used the asymptotic form of the metric in far region (2.4.1). Also, since we are looking at the low energy limit where $\omega \ll 1$, according to (2.57) we find that $\beta \ll \alpha$ and so we can neglect the second term. Finally, the incoming flux from infinity is

$$f_{in}(r \to \infty) \simeq 2|\alpha|^2 \tag{2.74}$$

The flux passing the horizon, f_{abs} can be computed in a similar way.

$$f_{abs} = 4\pi\omega r_H A^2 \tag{2.75}$$

So by taking (2.70) the absorption cross-section found to be

$$\sigma_{abs} = \frac{\pi}{\omega^2}\left|\frac{f_{tr}}{f_{in}}\right| = 4\pi r_H^2 = A_H \tag{2.76}$$

The absorption cross-section of the low energy massless scalars equals the area of the horizon.

2.5 Absorption on 4D Reissner-Nordström black holes

Our next and more realistic example is the Reissner-Nordström black hole in a four-dimensional space-time. Again, we look at the absorption of a minimally coupled massless scalar. We consider the extremal and the near-extremal Reissner-Nordström separately and calculate the absorption cross-section for each of them.

2.5.1 Extremal

The metric of an extremal Reissner-Nordström black hole is

$$ds^2 = -\left(1 - \frac{r_H}{r}\right)^2 dt^2 + \left(1 - \frac{r_H}{r}\right)^{-2} dr^2 + r^2 d\Omega^2 \tag{2.77}$$

which for later convenience we define a new radial coordinate as

$$\tilde{r} = r - r_H \tag{2.78}$$

and dropping the tilde, then the metric transforms to

$$ds^2 = -\left(1 + \frac{r_H}{r}\right)^{-2} dt^2 + \left(1 + \frac{r_H}{r}\right)^2 \left(dr^2 + r^2 d\Omega_2{}^2\right) \tag{2.79}$$

Now the horizon is located at zero $r_H = 0$.
which $d\Omega_2^2$ is the metric of an unit two-dimensional sphere.

Near horizon geometry

In the near horizon regime we can approximate the metric (2.79) as

$$ds_{nh}^2 = -\left(\frac{r_H}{r}\right)^{-2} dt^2 + \left(\frac{r_H}{r}\right)^2 dr^2 + r_H{}^2 d\Omega_2{}^2 \tag{2.80}$$

which ds_{nh}^2 stands for the near horizon metric. Again we change the radial coordinate as

$$\frac{r_H}{r} = y \tag{2.81}$$

then the metric simplifies as

$$ds_{nh}{}^2(y) = -\frac{1}{y^2} dt^2 + \frac{r_H{}^2}{y^2} dy^2 + r_H{}^2 d\Omega_2{}^2 \tag{2.82}$$

Wave equation and the solution

By inserting (2.79) in the wave equation, the radial part would be

$$\partial_r \left[r^2 \partial_r R(r)\right] - l(l+1)R(r) - r^2 \omega^2 R(r) = 0 \tag{2.83}$$

where we have assumed separability condition

$$\Phi = e^{-i\omega t} R(r) Y_l(\Omega) \tag{2.84}$$

$Y_l(\Omega)$ is spherical harmonics following

$$r^2 \nabla^2 Y_l(\Omega) = -l(l+1) Y_l(\Omega) \tag{2.85}$$

Now we look at the asymptotic and the near horizon form of the wave equation and also we find the solution for these regions.

Near horizon

To compute the wave equation for the near horizon we use the metric (2.82), the result of radial part would be

$$\Box\Phi = \partial_y{}^2 + (r_H\omega)^2 R(r) + \frac{1}{y^2} l(l+1) R_{nh}(y) = 0 \tag{2.86}$$

$R_{nh}(r)$ denotes the near horizon version of $R(r)$. Above equation is a Bessel differential equation and it has a complete solution as

$$R_{nh}(y) = \sqrt{y}\left[AJ_{l+\frac{1}{2}}(r_H\omega y) + BY_{l+\frac{1}{2}}(r_H\omega y)\right] \tag{2.87}$$

or equally

$$R_{nh}(y) = \sqrt{y}\left[AJ_{l+\frac{1}{2}}(r_H\omega y) + BJ_{-l-\frac{1}{2}}(r_H\omega y)\right] \tag{2.88}$$

Far region

In the region far from the black hole, $r \gg 1$, the black hole, and its effects, disappear and one leave with a flat metric

$$ds^2_{far} = -dt^2 + dr^2 + r^2 d\Omega_2^2 \tag{2.89}$$

using the flat metric, the wave equation would be

$$\Box\Phi(r,t) = \partial_r\left[r^2\partial_r\Phi(r,t)\right] - \partial_t\left[r^2\partial_t\Phi(r,t)\right] = 0 \tag{2.90}$$

or

$$\partial_r^2 R(r) + 2r\partial_r R(r) + (r\omega)^2 R(r) = 0 \tag{2.91}$$

where we have assumed the separation of variables similar to the near-horizon case. Solution to the above differential is, again, a linear combination of Bessel functions

$$R_{far}(r) = \frac{1}{\sqrt{r}}\left[\alpha J_{l+\frac{1}{2}}(\omega r) + \beta J_{-l-\frac{1}{2}}(\omega r)\right] \tag{2.92}$$

we can simplify the solution by applying $r \to \infty$ to the solution. Then the solution behaves like

$$R_{far}(r) \simeq \frac{1}{r}\sqrt{\frac{2}{\pi\omega}}\left[\alpha\sin\left(\omega r - \frac{l\pi}{2}\right) + \beta\cos\left(\omega r + \frac{l\pi}{2}\right)\right] \tag{2.93}$$

Matching the far and near solutions

As explained before, we need to match the small r far region solution (2.92) to the large r ($y \to 0$) near horizon solution (2.88). Using the series expansion of Bessel functions

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m+\alpha} \tag{2.94}$$

the asymptotic solution at small r (2.92) can be replaced by

$$R_{far}(r\to 0)\simeq \frac{1}{\sqrt{r}}\left[\alpha\left(\frac{\omega r}{2}\right)^{l+\frac{1}{2}}\frac{1}{\Gamma\left(l+\frac{3}{2}\right)}+\beta\left(\frac{\omega r}{2}\right)^{-l-\frac{1}{2}}\frac{1}{\Gamma\left(-l+\frac{1}{2}\right)}\right] \tag{2.95}$$

In the same way, the large r near horizon solution can be approximated by

$$R_{nh}(r\to\infty)\simeq \frac{1}{\sqrt{r}}\left[A\left(\frac{\omega {r_H}^2}{r}\right)^{l+\frac{1}{2}}\frac{1}{\Gamma\left(l+\frac{3}{2}\right)}+B\left(\frac{\omega {r_H}^2}{r}\right)^{-l-\frac{1}{2}}\frac{1}{\Gamma\left(-l+\frac{1}{2}\right)}\right] \tag{2.96}$$

where we have resubstituted r from (2.81). Matching (2.95) at small r to (2.96) at large r, we get

$$\frac{\alpha}{\Gamma\left(l+\frac{3}{2}\right)}\left(\frac{\omega r}{2}\right)^{l+\frac{1}{2}} = \frac{B}{\Gamma\left(-l+\frac{1}{2}\right)}\left(\frac{\omega {r_H}^2}{r}\right)^{-l-\frac{1}{2}} \tag{2.97}$$

$$\frac{\beta}{\Gamma\left(-l+\frac{1}{2}\right)}\left(\frac{\omega r}{2}\right)^{-l-\frac{1}{2}} = \frac{A}{\Gamma\left(l+\frac{3}{2}\right)}\left(\frac{\omega {r_H}^2}{r}\right)^{l+\frac{1}{2}} \tag{2.98}$$

In the low energy approximation ($\omega \ll 1$) one finds $\beta \ll \alpha$.

Computing absorption cross-section

To compute the absorption cross-section we need to know the incoming flux from infinity and the flux transmitted to the horizon. First, we compute the incoming flux from infinity, f_{in}, by using the general form of the conserved flux associated to the wave equation (2.48). The Bessel function corresponding to the far region (2.93) can be writtien as

$$R_{far}(r\to\infty)\simeq \frac{1}{2r}\sqrt{\frac{2}{\pi\omega}}\left[(-i\alpha+\beta)e^{-i\left(\omega r-\frac{l\pi}{2}\right)}+(i\alpha+\beta)e^{+i\left(\omega r-\frac{l\pi}{2}\right)}\right] \tag{2.99}$$

which can be decomposed into two terms as

$$R_{far}(r\to\infty)=\phi_{in}+\phi_{ref} \tag{2.100}$$

where ϕ_{in} and ϕ_{ref} are the incident and reflected waves, respectively. So, the incoming flux from infinity can be derived as

$$\begin{aligned} f_{in} &= \frac{2\pi}{i}r^2\left(\phi_{in}^*\partial_r\phi_{in}-\phi_{in}\partial_r\phi_{in}^*\right) \\ &= 2|\alpha|^2+2|\beta|^2 \\ &\simeq 2|\alpha|^2 \end{aligned} \tag{2.101}$$

which in the last step we have used that $\alpha \gg \beta$. The flux passing across the horizon, f_{abs}, can be computed by using the near horizon metric (2.80). For $r\to 0$, the Bessel function

(2.88) can be approximated by

$$\begin{aligned} R_{nh}(r\to 0) &= \sqrt{\frac{1}{\pi\omega r_H}}\left[A\sin\left(\frac{1}{2}(r_H^2\omega - l\pi)\right) + B\cos\left(\frac{1}{2}(r_H^2\omega + l\pi)\right)\right] \\ &= \sqrt{\frac{1}{\pi\omega r_H}}\left[(-iA+B)e^{-i\left(\frac{\omega r_H^2}{r}-\frac{l\pi}{2}\right)} + (iA+B)e^{i\left(\frac{\omega r_H^2}{r}+\frac{l\pi}{2}\right)}\right] \end{aligned} \tag{2.102}$$

First and second term in last line, are the transmitted and the reflected waves respectively. Demanding no reflection at the horizon, cause the restriction $A = -iB$. Now we can calculate the transmitted flux by using

$$\phi_{tr} = (2B)e^{-i\left(\frac{\omega r_H^2}{r}-\frac{l\pi}{2}\right)} \tag{2.103}$$

The result is then

$$f_{abs} = 4|A|^2 \tag{2.104}$$

So the greybody factor, $\mathscr{F}$, is

$$\mathscr{F} = \left|\frac{f_{abs}}{f_{in}}\right| = \frac{2|B|^2}{|\alpha|^2} = 2\left|\frac{\Gamma\left(-l+\frac{1}{2}\right)}{\Gamma\left(l+\frac{3}{2}\right)}\right|^2 \left(\frac{\omega r_H}{2}\right)^{4l+2} \tag{2.105}$$

We can convert the partial wave cross-section to the plane wave cross-section by multiplying the above result by π/ω^2. For s-wave, $l = 0$ we find

$$\sigma_{abs}^0 = A_H \tag{2.106}$$

Absorption cross-section in the low energy limit is proportional to the area of the horizon.

2.5.2 Near-extremal

After dealing with the simple case of an extremal black hole, now we consider a general RN black hole with no restriction on the charge. We follow a same procedure as the previous section. As already mentioned, the complete RN metric is

$$ds^2 = -\Delta dt^2 + \Delta^{-1}dr^2 + r^2 d\Omega^2 \tag{2.107}$$

with

$$\Delta = \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right) \tag{2.108}$$

and

$$r_\pm = M \pm \sqrt{M^2 - Q^2} \tag{2.109}$$

Wave equation and the solution

By using the general form of wave equation we get

$$\left[\partial_r \Delta \partial_r + r\Delta^{-1}\partial_t^2\right]\Phi = 0 \tag{2.110}$$

We need to find the solution for above wave equation at the near horizon and far region, which will be done in the next two sections.

Near horizon

For the near horizon region we have

$$r - r+ \ll \frac{1}{\omega} \tag{2.111}$$

the wave equation can be written as

$$\frac{\Delta}{r^2}\partial_r \frac{\Delta}{r^2}\partial_r R + \omega^2 r_+^4 - \frac{l(l+1)}{r^2}\Delta R = 0 \tag{2.112}$$

To solve the near horizon wave equation we define a new variable as

$$z = \frac{r - r_+}{r - r_-} \qquad , \qquad 0 \leqslant z \leqslant 1 \tag{2.113}$$

where $z = 0$ would be the horizon. The wave equation (2.110) becomes

$$\begin{aligned} 0 &= (r_+ - r_-)z\partial_z(r_+ - r_-)z\partial_z R + \omega^2 r_+^4 R - l(l+1)(r - r+)(r - r_-)R \\ &= (r_+ - r_-)^2 z^2 \partial_z^2 R + (r_+ - r_-)^2 z\partial_z R + \omega^2 r_+^4 R - l(l+1)(r - r+)(r - r_-)R \end{aligned} \tag{2.114}$$

and by dividing by $z(r_+ - r_-)(r - r_-)$ we have

$$\begin{aligned} 0 &= \frac{r_+ - r_-}{r - r_-} z\partial_z^2 R - \frac{r_+ - r_-}{r - r_-}\partial_z R + \frac{\omega^2 r_+^4}{z(r + - r_-)(r - r_-)} R - \frac{l(l+1)}{z}\frac{r - r_+}{r_+ - r_-} R \\ &= z(1-z)\partial_z^2 R + (1-z)\partial_z R + \frac{\omega^2 r_+^4}{(r_+ - r_-)^2}\left(\frac{1-z}{z}\right) R - \frac{l(l+1)}{1-z} R \end{aligned} \tag{2.115}$$

Above near horizon wave equation transforms into an standard hypergeometric form by defining

$$R = A z^{i\omega\tilde{r}}(1-z)^{l+1} F\,, \qquad \tilde{r} := \frac{r_+^2}{r_+ - r_-} \tag{2.116}$$

where A is a normalization constant which will be determined later.
By applying above definition, F obeys

$$z(1-z)\partial_z^2 F + [1 + i\omega\tilde{r} - (1 + 2(l+1) + i\omega\tilde{r})z]\,\partial_z F - \left[(l+1)^2 - i\omega\tilde{r}(l+1)\right] F = 0 \tag{2.117}$$

This is of the form of hypergeometric differential equation

$$z(1-z)\partial_z^2 F + [c - (a+b+1)z]\,\partial_z F - abzF = 0$$

with

$$\begin{aligned} a &= l + 1 + i\omega r\tilde{r} \\ b &= l + 1 \\ c &= 1 + i\omega\tilde{r} \end{aligned} \tag{2.118}$$

(2.118) has following complete solution

$$F = A_1 2F_1(a,b;c;z) + A_2 z_2^{1-c} F_1(a+1-c, b+1-c; 2-c; z) \tag{2.119}$$

By imposing only the ingoing flux at the horizon, $z = 0$, we find $A_2 = 0$ and therefore R is

$$R_n = A z^{i\omega\tilde{r}} (1-z)^{l+1} F(l+1+i\omega\tilde{r}, l+1; 1+i\omega\tilde{r}; z) \tag{2.120}$$

Finally by substitution $\tilde{r}$ from (2.116) we find the solution of wave equation for near horizon regime as

$$R_n = A z^{i\omega\left(\frac{r_+^2}{r_+-r_-}\right)} (1-z)^{l+1} F(l+1+i\omega\left(\frac{r_+^2}{r_+-r_-}\right) r, l+1; 1+i\omega\left(\frac{r_+^2}{r_+-r_-}\right); z) \tag{2.121}$$

where A subjects to be determined.

Far region

For far region, $r \to \infty$, wave equation (2.110) takes simple form

$$\left[\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{l(l+1)}{r^2} + \omega^2\right] R_f = 0 \tag{2.122}$$

or

$$\left[r^2\partial_r^2 + 2r\partial_r + \omega^2 r^2 - l(l+1)\right] R_f = 0 \tag{2.123}$$

which is the equation for a massless scalar field with frequency ω and angular momentum l in a flat space. (2.122) is of the form of Helmholtz equation in spherical coordinates

$$x^2\partial_x^2 f + 2x\partial_x f + [x^2 - n(n+1]f = 0 \tag{2.124}$$

which its solution is a linear combination of Bessel functions

$$f = \sqrt{\frac{\pi}{2x}}\left[J_{n+\frac{1}{2}}(x) - J_{-n-\frac{1}{2}}(x)\right] \tag{2.125}$$

so the solution of far region wave equation (2.122) is

$$R_f = \sqrt{\frac{\pi}{2r\omega}}\left[\alpha J_{l+\frac{1}{2}}(\omega r) + \beta J_{-l-\frac{1}{2}}(\omega r)\right] \tag{2.126}$$

which for $r \to \infty$ simplifies to

$$R_f(r \to \infty) = \frac{1}{r\omega}\left[-\alpha \sin\left(\omega r - \frac{l}{2}\right) + \beta \cos\left(\omega r + \frac{l}{2}\right)\right] \tag{2.127}$$

Matching the far and near solutions

For the small r far region, we use the series expansion of Bessel function for the small arguments

$$R_f(r \to 0) \simeq \sqrt{\frac{1}{r}} \left[\frac{\alpha}{\Gamma\left(l+\frac{3}{2}\right)} \left(\frac{\omega r}{2}\right)^{l+\frac{1}{2}} + \frac{\beta}{\Gamma\left(-l+\frac{1}{2}\right)} \left(\frac{\omega r}{2}\right)^{-l-\frac{1}{2}} \right] \tag{2.128}$$

According to (2.113), the near horizon large r means $z \to 1$ or $z-1 \to 0$. We use the transformation law for $z \to z-1$ which is

$$\begin{aligned} F(a,b;c;z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,b,a+b-c+1,1-z) \\ & (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F(c-a,c-b;c-a-b+1;1-z) \end{aligned} \tag{2.129}$$

By using the series expansion of hypergeometric function, F, and keep only the first term, we have

$$\begin{aligned} R_n(r \to \infty) &= A \left(\frac{r}{r_+ - r_-}\right)^{-l-1} \Gamma(1+\omega\tilde{r}) \times \\ & \left[\frac{\Gamma(-2l-1)}{\Gamma(-l)\Gamma(i\omega\tilde{r}-l)} + \left(\frac{r}{r_+ - r_-}\right)^{2l+1} \frac{\Gamma(2l+1)}{\Gamma(l+1)\Gamma(l+1+i\omega\tilde{r})} \right] \end{aligned} \tag{2.130}$$

where, for $r \to \infty$, we have used that

$$\begin{aligned} 1-z &\approx \frac{r_+ - r_-}{r} \\ z &\approx 1\,. \end{aligned} \tag{2.131}$$

Matching (2.128) at small r to (2.130) at large r we find $\beta \ll \alpha$ and

$$A = \alpha (r_+ - r_-)^l \frac{\Gamma(l+1)\Gamma(l+1+i\omega\tilde{r})}{\Gamma(l+\frac{3}{2})\Gamma(2l+1)\Gamma(1+i\omega\tilde{r})} \left(\frac{\omega}{2}\right)^{l+\frac{1}{2}} \equiv N\alpha \tag{2.132}$$

where $\tilde{r}$ is defined in (2.116).

Computing absorption cross-section

Same as the extremal case, to find the cross-section, first we should compute the incoming flux from infinity and the flux passing the horizon.

$$R = \sqrt{\frac{2}{\pi\omega}} \frac{1}{r} \left[-\alpha \sin\left(\omega r - \frac{l\pi}{2}\right) + \beta \cos\left(\omega r + \frac{l\pi}{2}\right) \right] \tag{2.133}$$

$$= \sqrt{\frac{2}{\pi\omega}} \frac{1}{r} \left[(-i\alpha + \beta) e^{-i\left(\omega r - \frac{l\pi}{2}\right)} + (i\alpha + \beta) e^{i\left(\omega r + \frac{l\pi}{2}\right)} \right] \tag{2.134}$$

In last line, terms in the bracket are the incident wave, $R_{f,in}$, and the reflective wave, $R_{f,ref}$, namely:

$$R_{f,in} = \sqrt{\frac{2}{\pi\omega}}\frac{1}{r}(-i\alpha+\beta)e^{-i\left(\omega r-\frac{l\pi}{2}\right)} \tag{2.135}$$

$$R_{f,ref} = \sqrt{\frac{2}{\pi\omega}}\frac{1}{r}(i\alpha+\beta)e^{i\left(\omega r+\frac{l\pi}{2}\right)} \tag{2.136}$$

Since we are interested in computing the incoming flux from infinity, we take only the R_{in} part:

$$\begin{aligned} f_{in} &= \frac{2\pi}{i}(R^*_{f,in}\partial R - R_{f,in}\partial R^*_{f,in}) && (2.137)\\ &= \frac{2\pi}{i}\frac{1}{2\pi\omega}r^2\Bigg[(\beta^*-i\alpha^*)\frac{e^{i\omega r}}{r}(\beta+i\alpha)\left(-\frac{1}{r^2}-\frac{i\omega}{r}\right)e^{-i\omega r} - \\ &\qquad (\beta+i\alpha)\frac{e^{-i\omega r}}{r}(\beta^*-i\alpha^*)\left(-\frac{1}{r^2}+\frac{i\omega}{r}\right)e^{i\omega r}\Bigg] \\ &= 2(|\alpha|^2+|\beta|^2) && (2.138) \end{aligned}$$

and finally by considering $\beta \ll \alpha$, the incoming flux from infinity is approximately

$$f_{in} \simeq 2|\alpha|^2 \tag{2.139}$$

To calculate the flux passing the horizon, we take the near horizon form of metric (2.107)

$$\begin{aligned} f_{abs} = \; & \frac{2\pi}{i}\Delta|A|^2\Bigg[\left(z^{-i\omega\tilde{r}}(1-z)^{l+1}F\right) \\ & \times\left(i\omega\tilde{r}\frac{1}{z}(1-z)^{l+1}F+(l+1)(1-z)^{l}F+(1-z)^{l+1}\partial_z F\right)z^{i\omega\tilde{r}} \\ & -\left(z^{i\omega\tilde{r}}(1-z)^{l+1}F\right) \\ & \times\left(-i\omega\tilde{r}\frac{1}{z}(1-z)^{l+1}F+(l+1)(1-z)^{l}F+(1-z)^{l+1}\partial_z F\right)z^{-i\omega\tilde{r}}\Bigg] \end{aligned} \tag{2.140}$$

for the near horizon $r \to r_+$ we have

$$\begin{aligned} z &\to 0 \\ F &\to 1 \end{aligned} \tag{2.141}$$

and we substitute back r, we obtain

$$f_{abs} \simeq 4\pi\omega r_+^2|A|^2 = \omega|A|^2 A_H \tag{2.142}$$

A_H is the area of horizon. So the greybody factor is

$$\mathscr{F} = \frac{|f_{abs}|}{|f_{in}|} = \frac{1}{2}A_H\omega|N|^2 \tag{2.143}$$

where, we have defined N in (2.132). Consequently

$$\sigma^l_{abs} = \frac{\pi}{\omega}^{-1} |N|^2 A_H \tag{2.144}$$

and as we expect for the s-wave, the absorption cross-section is independent of the frequency

$$\sigma^0_{abs} = A_H \tag{2.145}$$

2.6 Universality of low energy absorption cross-section

In last two sections, we studied the absorption cross-section of a Schwarzschild and a Reissner-Nordström black holes. we saw that in the low energy limit, where the wave length of the incident wave is much larger than the size of the horizon, absorption cross-section of a massless space-time scalars on the four-dimensional Schwarzschild and Reissner-Nordström black holes, both, are equal to the area of the black hole horizon

$$\sigma_{abs} = A_H \tag{2.146}$$

and thus is independent of the frequency of falling wave. Indeed in [10] it was shown that for the all spherically symmetric black holes the low energy cross-section for massless minimally coupled scalar fields is equal to the area of the horizon in any dimensions. This is follow from

$$\sigma^d_{abs} = \frac{2\pi^{(d-1)/2} r_H^{d-2}}{\Gamma[(d-1)/2]} = A_H \tag{2.147}$$

in a d-dimensional space-time, which now A_H is the area of the horizon hypersurface.
The fact that the low energy absorption cross-section for an s-wave massless scalar equals to the horizon area is generally proved for the higher dimensional asymptotically flat and spherically symmetric black holes. For completeness, we review the calculation of the absorption cross-section for an spherically symmetric black hole [10].
In d-dimensional space-time, the spherically symmetric black hole metric takes the form of

$$ds^2 = -f(r)dt^2 + g(r)[dr^2 + r^2 d^2_{d-2}] \tag{2.148}$$

in Einstein frame. The functions $f(r)$ and $g(r)$ are chosen to ensure that the metric is asymptotically flat

$$f(r), g(r) \xrightarrow{r\to\infty} 1 \tag{2.149}$$

If the metric is not already in this form, always with a coordinate transformation we can do it. We let the horizon to be at $r = r_H$, then the area of the horizon is

$$A_H = \left(r_H\sqrt{g(r_H)}\right)^{d-2} \Omega_{d-2} \equiv R_H^{d-2}\Omega_{d-2} \tag{2.150}$$

where Ω_{d-2} is the volume of the unit $(d-2)$-sphere

$$\Omega_{d-2} = \frac{2\sqrt{\pi}^{(d-1)}}{\Gamma\left[\frac{1}{2}(d-1)\right]} \tag{2.151}$$

and R_H defined via

$$R_H^{2(d-2)} \equiv r_H^2 g(r_H)^{d-2} \tag{2.152}$$

For a minimally coupled massless scalars the wave equation is

$$\Box\Psi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right)\Psi = 0 \tag{2.153}$$

At low energies only the mode with lowest angular momentum will contribute to the cross-section, which for the scalars is the s-wave. Thus $\Psi = \Psi_\omega(r)e^{-i\omega t}$, and so

$$\partial_t\left(g^{tt}\partial_t\right)\Psi_\omega - \frac{1}{\sqrt{f(r)g(r)^{d-1}}}r^{d-2}\partial\left(\sqrt{f(r)g^{d-1}}r^{d-2}g(r)^{-1}\right)\Psi_\omega = 0 \tag{2.154}$$

Since we are working in the low energy approximation, frequency of the wave ω is much smaller than any energy scale set by the black hole. Now, by defining

$$\partial_\rho \equiv \sqrt{f(r)g(r)^{d-3}}r^{d-2}\partial_r \tag{2.155}$$

the wave equation takes the form of

$$\left[\partial_\rho^2 + \left(r^2(\rho)g(r(\rho))^{d-2}\omega^2\right)\right]\Psi_\omega(\rho) = 0 \tag{2.156}$$

Near the horizon, where $r \approx r_H$, we can write the wave equation as

$$\left[\partial_\rho^2 + \omega^2 R_H^{2(d-2)}\right]\Psi_\omega^{\text{near}}(\rho) = 0 \tag{2.157}$$

Solution to the wave equation at the near horizon, which must be purely ingoing, is

$$\Psi_\omega^{\text{near}}(\rho) = \alpha e^{-i\omega R_H^{(d-2)}\rho} \tag{2.158}$$

For later use, when we match the far and the near solutions, we need to know the large r behavior of the near horizon solution. By studying (2.155), we can see that when $r \gg 1$, $f(r)$ and $g(r)$ can be approximated by 1, and so $\rho \ll 1$. So the near horizon wave function for large r, can be be written as

$$\Psi_\omega^{\text{near}}(r)_{r\gg 1} \sim \alpha\left[1 - i\omega R_H^{d-2}\frac{r^{3-d}}{3-d}\right] \tag{2.159}$$

Next, we should determine the far region solution of the wave equation. For large r, the wave equation (2.154) simplifies as

$$\left[r^{r-2}\partial_r\left(r^{d-2}\partial_r\right) + \omega^2 r^{2(d-2)}\right]\Psi_\omega^{\text{far}}(r) = 0 \tag{2.160}$$

To eliminate the linear derivative we change variables as

$$\Psi_\omega^{\text{far}}(r) \equiv r^{-\frac{1}{2}(d-2)}\chi_\omega(r) \tag{2.161}$$

and define

$$z \equiv \omega r \tag{2.162}$$

which gives the wave equation

$$\left[\partial_z^2 + 1 - \frac{(d-2)(d-4)}{4z^2}\right]\chi_\omega(r) = 0 \tag{2.163}$$

Solution to this differential equation is Bessel function, so Ψ_ω^{far} determines to be

$$\Psi_\omega^{\text{far}}(z) = z^{\frac{1}{2}(3-d)}\left[AJ_{\frac{1}{2}(d-3)}(z) + BJ_{-\frac{1}{2}(d-3)}(z)\right] \tag{2.164}$$

In order to find the small r behavior of far solution, we use the series expansion of the Bessel function to get

$$\Psi_\omega^{\text{far}}(\omega r)_{r\ll 1} \sim A\frac{2^{\frac{1}{2}(3-d)}}{\Gamma\left[\frac{1}{2}(d-1)\right]} + B\frac{2^{\frac{1}{2}(d-3)}}{\Gamma\left[\frac{1}{2}(5-d)\right](\omega r)^{d-3}} \tag{2.165}$$

Matching the near horizon and far region wave function yields

$$A = 2^{\frac{1}{2}(d-3)}\Gamma\left[\frac{1}{2}(d-1)\right]\alpha \tag{2.166}$$

$$B = i\frac{2^{\frac{1}{2}(3-d)}\Gamma\left[\frac{1}{2}(5-d)\right](\omega R_H)^{d-2}}{3-d}\alpha \tag{2.167}$$

Now, we can compute the absorption probability, $\mathscr{F}$,

$$\mathscr{F} = 1 - \left|\frac{A + e^{i\pi\frac{1}{2}(d-3)}B}{A + e^{-i\pi\frac{1}{2}(d-3)}B}\right| \tag{2.168}$$

$$= 4\frac{2^{-(d-3)}}{d-3}\frac{\Gamma\left[\frac{1}{2}(5-d)\right]}{\Gamma\left[\frac{1}{2}(d-1)\right]}\sin\left[\pi(d-3)/2\right](\omega R_H)^{d-2} \tag{2.169}$$

In last step, to convert the spherical wave absorption probability into absorption cross-section we must extract the ingoing spherical wave from the plane wave. This can be done by normalizing the absorption probability as

$$\sigma_{abs} = |N|^2\mathscr{F}, \qquad |N|^2 = \frac{(2\pi)^{d-2}}{\omega^{d-2}\Omega_{d-2}} \tag{2.170}$$

in d-dimensional space-time. So, the absorption cross-section is

$$\sigma_{abs} = \frac{2\sqrt{\pi}^{d-1}R_H^{d-2}}{\Gamma\left[\frac{1}{2}(d-1)\right]} = A_H \tag{2.171}$$

which, last equality is deduced by considering the area of horizon (2.150). This result for the low energy cross-section for massless minimally coupled scalars is completely universal for the spherically symmetric black holes.

It is worthwhile to note that for the *massive* scalars there is also a universality in low energy absorption cross-section. In case of the massive scalars the equality in (2.147) is replaced by a proportionality, where the proportionality constant is a velocity parameter. This universal property was also re-examined for the extended objects [13, 14].

CHAPTER 3

Black Holes in String Theory

3.1 Introduction

We have seen that the semi-classical considerations of Einstein gravity yield that the black holes have thermodynamical behavior and obey the laws of thermodynamics. The Bekenstein entropy only deals with the macroscopical parameters of black holes but does not tell us about the underlying microscopic description of black holes, that, what are the degrees of freedom and how can we count them. A theory of quantum gravity should be able to tell us about the microscopic configuration or,in other word, about the quantum statistical mechanics of black holes. String theory, as the leading candidate for quantum theory of gravity, ought to address these issues. This motivates us to review the progress in understanding black holes in the context of string theory and study some quantum mechanical aspects of black holes.
Black holes arise in string theory are the solutions of the corresponding low-energy supergravity theory. They can be neutral or charged with various charges that string theory permits. String theory lives in 10 dimensional space-time (or 11 dimensions from M-theory view), so, if we want to describe the four-dimensional black holes we should consider the space-times where the extra dimensions have been compactified. The compact space is taken to be small such that it can not be observed and thus we would only see the four-dimensional space-time. With this assumption, the original ten-dimensional space-time would be split in two spaces

$$M^4 \times X$$

where M^4 is a four-dimensional space-time corresponding to the world we know, and X is some compact six-dimensional manifold. Although, the compactified space can not be seen by the observers living in M, but the properties of internal space X, lead to physical consequences in the four-dimensional space-time, or in other word, the particle content of the resulting four-dimensional theory is intimately related to the topological properties of the compactified manifold. Also, conversely, the computations in four-dimensions should give us information about the geometry of the internal space. As an example we will see that the massless four-dimensional fields are associated with the harmonic forms on X.

In string theory, black holes are typically engineered in terms of the branes wrapped around the appropriate cycles of the internal space[4]. Suppose we have compactified the space-time, on a manifold, down to four dimensions. The branes wrapped around the directions in the compact dimensions will look like point-like objects, or so to say (charged) particles, in the four-dimensional space-time, where coincide at a point in space which is the center of black hole. Indeed, this leads us how to have charged black holes in string theory: the charges arise in black holes of string theory are sourced by the different charges carried by the extended objects of the theory.

So the idea of making black holes in string theory is to construct an intersecting of branes wrapped the various cycles of compactified Calabi-Yau three-fold, which upon the dimensional reduction yields a black hole in four dimensions. If the brane intersection be supersymmetric then the black hole will be extremal supersymmetric.
We should remark that, though, the string theory, in ten-dimensional space-time, has supersymmetry but since we compactify the extra dimensions there is no guarantee that, in general, the supersymmetry be preserved in four-dimension. The geometry of compact space determines that how many of supersymmetries survive in the four-dimension.

3.2 Calabi-Yau black holes in IIA

3.2.1 IIA string theory field content

We start with the ten-dimensional bosonic field content of type IIA string theory. Since we are interested in low energy, only the massless fields are relevant. The bosonic massless sector of IIA consist of following fields

$$G_{MN}, \quad B_{MN}, \quad \Phi \qquad\qquad C^{(1)}_M, \quad C^{(3)}_{MN\lambda} \tag{3.1}$$

where the indices M and N run over the whole ten-dimensional space-time coordinates, $M,N = 0,1,..,9$. G_{MN} is the graviton, B_{MN} is an antisymmetric two-form tensor and Φ is the dilaton which make up the NS-NS sector. The string coupling constant, g_s, of IIA is identified with the value at infinity of the dilaton field as

$$g_s = \exp\big(\Phi(r \to \infty)\big) = \exp(\Phi_0) \tag{3.2}$$

$C^{(p)}$ are the R-R sector p-form antisymmetric gauge fields, yielding two and four-form field strength

$$F^{(2)} = dC^{(1)} \qquad\qquad F^{(4)} = dC^{(3)} \tag{3.3}$$

The NS-NS sector fields couple to strings. The R-R sector fields, however, do not couple to strings but rather to Dp-branes which are the extended objects in the theory. More precisely $C^{(p+1)}$ gauge fields can electrically couple to the world-volume of the Dp-branes

$$\mu_p \int\limits_W C^{(p+1)}_{\mu_1\ldots\mu_{p+1}} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{p+1}} \tag{3.4}$$

where W is the world-volume of the brane and μ_p is the charge of p-brane. Such Dp-branes are called the electrically charged branes as can be seen by evaluating the electric charge by using the Gauss's law. In D dimensions, a p-brane has a R^{D-p+1} transverse space. So to measure the electric charge of the p-brane we should integrate over an sphere S^{D-p-2}

$$\mu_p = \int_{S^{D-p-2}} *F_{p+2} \tag{3.5}$$

Furthermore, in ten dimensions we have duality as

$$*dC^{(p+1)} = d\tilde{C}^{(7-p)} \tag{3.6}$$

which is followed from the duality of field strength in 10 dimension

$$F^{(p+2)} = *F^{(8-p)} \tag{3.7}$$

means that to each electrically charged Dp-brane there is also a dual magnetically charged D$(6-p)$-brane. It turns out that $C^{(1)}$ couples to a D0-brane electrically and to a D6-brane magnetically. Respectively, D2-brane and D4-brane become electrically and magnetically charged under $C^{(3)}$.
Consequently, in IIA string theory the stable Dp-branes, which carry conserved charges, are: [1]

$$\begin{array}{ll} \text{D0, D2} & \text{electrically charged} \\ \text{D4, D6} & \text{magnetically charged} \end{array} \tag{3.8}$$

As already mentioned, to have a black hole in four dimensions, we should wrap branes around the non-trivial cycles of the CY_3 at a particular position in M^4, where reflect as point-like objects in the context of four-dimensional effective field theory.
Compactifying on a Calabi-Yau manifold breaks $3/4$ of the original 32 component supersymmetries of type IIA, and the remaining 8 supersymmetries give $N = 2$ supergravity in four dimensions.
To have preserved supersymmetry in $4d$ we should wrap the branes around supersymmetric cycles. Now the question is that how the wrapped branes represent themselves in the effective action?
First step is to know that the massless four-dimensional fields are associated with the harmonic forms on Calabi-Yau space. The number of linearly independent harmonic p-forms for a given p is given by the so-called Betti numbers, b_p, which are fixed by the topology of CY_3. This is precisely relevant for the wrapped branes which curl up the submanifolds (various dimensional non-trivial cycles) of CY_3. According to the Poincaré duality there exists a dual relationship between harmonic p-forms and $(d_X - p)$-cycles, where d_X is the dimension of X. This is the way we obtain charged particles in string theory compactified on a Calabi-Yau manifold, namely, by wrapping the D-branes on the various cycles of Calabi-Yau. In string theory D-branes source the $10d$ R-R fields, which from the $4d$ point of view look like point particle charges that source the different gauge fields which

[1] Since all the stable branes are even dimensional, it is natural to consider $p = 8$, as well. Stability of D8-brane is due to existence of $C^{(9)}$ gauge fields which occur in special circumstances.

come from the dimensional reduction of the $10d$ fields. The number of units of $4d$ charge is determined by times we warp the D-branes around the particular cycle, called winding number.

In type IIA one can produce any electric and magnetic charges by wrapping D6, D4, D2 and D0 branes on the various cycles in Calabi-Yau. As mentioned the, number of independent cycles is determined by Betti number. If $A \in 1,...,b_2$ [2] labels the 4 (and also the dual 2)-cycles, then the most general set of charges we can get is

$$p^0, \quad p^A, \quad q_A, \quad q_0 \tag{3.9}$$

which stands for D6, D4 (magnetic) and D2, D0 (electric) charges respectively. This means that by compactifying IIA on the Calabi-Yau manifold with specific Betti number, b_2, we have at most total of $2b_2+2$ different electric and magnetic charges in the non-compact space. If we have a large number of the wrapped-branes in Calabi-Yau sitting at the same point of $4d$ space, we should consider the back-reaction of the metric (and other supergravity fields). It turns out that for large charges one can obtain black holes in the four-dimensional space-time [15, 16] (or five-dimensional black hole in M-theory).

The black holes made by compactifying the branes on Calabi-Yau are not necessarily large black hole, but can be appear as an small black holes too. It turns out that the CY compactification of a single brane, yields a four-dimensional black hole with vanishing event horizon. Indeed to describe a four-dimensional black holes with a finite event horizon, and hence a finite entropy we have to consider more complicated D-branes configuration. The four-dimensional large black holes are obtained by combining four different charges, which, happens when they have different dimensions, or if they wrap on the different independent cycles. In following we will give explicit examples of such black holes in string theory.

3.2.2 D4-D0 Black hole

One of the interesting case of Calabi-Yau black holes constructs by considering charges as

$$q^0, p^A \neq 0, \qquad q^0, q_A = 0 \tag{3.10}$$

this configuration corresponds to wrapping the q^A D4-branes on non-contractible four-cycles $\mathscr{P}$ in the internal manifold and some additional q^0 D0-charges. [3] This pattern then describes the extremal black holes of $N=2$ supergravity for compactification on Calabi-Yau [2, 3]. The four-cycle has to be holomorphic in order that the configuration be BPS state [18]. The validity of the macroscopic (and the state counting) black hole solutions require to impose two conditions: First, we work in the large volume limit which, geometrically, means to take the size of the manifold and of all its four-cycles (and two-cycles if

[2] for CY_3 we have $b_2 = h_{1,1}$

[3] We can generalize it to the black holes which include q_A charges. This generalization can be achieved by adding D2-branes to the configuration which are wrapped on two-cycles in CY [17].

$q^A \neq 0$) are large and therefore the curvature is small. This can be achieved by taking the q_0 to be much larger than the charges p^A:

$$|q_0| \gg p^A \tag{3.11}$$

Second, to make sure that the space-time loops are suppressed we need to have small curvature at the horizon so that the higher curvature terms can be neglected. This can be done by taking all the black hole's charges to be large

$$|q_0|, |p^A| \gg 0 \tag{3.12}$$

which indeed is the limit where the supergravity analysis is valid. So we have to impose

$$|q_0| \gg |p^A| \gg 0 \tag{3.13}$$

As a simple example, we first illustrate the structure of the configuration of three D4-branes with charges p^1,p^2,p^3 and q_0 D0-branes in IIA in the context of toroidal compactification, where the six-dimensional compact space is a six-torus, T^6. This has a generalization to the general Calabi-Yau manifold. The geometry is such that the D4-branes are wrapped around the three different four-cycles of the internal six-torus, such that they intersect transversely on two-cycles and triple-intersect over a zero-cycle yielding to non-zero triple intersection number. The corresponding ten-dimensional string frame metric is [19]

$$\begin{aligned} ds_{10}^2 &= -\frac{1}{\sqrt{H_0 h}}dt^2 + \sqrt{H_0 h}\left(dr^2 + r^2 d\Omega_2^2\right) \qquad (3.14)\\ &+\sqrt{\frac{H_0 H^1}{H^2 H^3}}(dy_1^2 + dy_2^2) + \sqrt{\frac{H_0 H^2}{H^1 H^3}}(dy_3^2 + dy_4^2) + \sqrt{\frac{H_0 H^3}{H^1 H^2}}(dy_5^2 + dy_6^2) \end{aligned}$$

where y_i denote the coordinates along the torus and by definition

$$\begin{aligned} H_0 &= 1 + \frac{q_0}{r}, \qquad h = H^1 H^2 H^3, \\ H^i &= 1 + \frac{p^A}{r}, \qquad A = 1,2,3 \end{aligned} \tag{3.15}$$

H_0 and H^A are harmonic functions. With convention, charges have length, L, dimension. By toroidal compactification, we get a four-charge extremal black hole in four-dimensions with charges q_0,p^A and metric

$$ds_4^2 = -\frac{1}{\sqrt{H_0 h}}dt^2 + \sqrt{H_0 h}\left[dr^2 + r^2 d\Omega_2^2\right] \tag{3.16}$$

and the dilaton is

$$e^\Phi = \sqrt[4]{\frac{H_0}{h}} \tag{3.17}$$

This background also contains the R-R fields produced by the branes as

$$\begin{aligned} A^{(1)} &= \left(1 - \frac{1}{H_0}dt\right) \qquad (3.18)\\ A^{(3)} &= \frac{1}{2}\sin\theta\, d\theta\, d\phi \left(p^1[y^1 dy^2 - y^2 dy^1] - p^2[y^3 dy^4 - y^4 dy^3] - p^3[y^5 dy^6 - y^6 dy^5]\right) \end{aligned}$$
(3.19)

where we have assumed that the D4-branes are wrapped on the directions $y^1y^2y^3y^4$, $y^1y^2y^5y^6$ and $y^3y^4y^5y^6$ of the six-torus.

To determine the mass of black hole we should study the large distance behavior of g_{tt} component of the metric same as what we did in case of Schwarzschild black hole following from (2.11) and (2.12). For large r we can approximate g_{tt} as

$$g_{tt}(r)_{r\to\infty} \sim 1 + \frac{1}{2}\frac{q_0 + p^1 + p^2 + p^3}{r} \tag{3.20}$$

now the mass can be read off

$$M = M_0 + M_1 + M_2 + M_3, \qquad M_0 = \frac{q_0}{4G_4}, \quad M_i = \frac{p^i}{4G_4} \tag{3.21}$$

where, since we are dealing with an extremal black hole, gives us, also, the total charge of the black hole Q.

3.2.2.1 Near horizon geometry of D4-D0 black hole

As we see, the near horizon limit of this metric depends on whether all the charges are non-vanishing or not. If one of the charges vanishes, this has an null singularity at $r = 0$ which means the black hole has zero horizon area. For non-vanishing case, $q_0, p^A \neq 0$, the near horizon geometry in non-compact space reduces to $AdS_2 \times S^2$. This can be seen by looking at the metric in the limit where

$$r \ll q_0, p^A \tag{3.22}$$

then metric appears to be

$$ds_4^2 = -\frac{r^2}{\sqrt{q_0p^1p^2p^3}}dt^2 + \frac{\sqrt{q_0p^1p^2p^3}}{r^2}dr^2 + \sqrt{q_0p^1p^2p^3}\left[dr^2 + r^2d\Omega_2^2\right] \tag{3.23}$$

or equally

$$ds_4^2 = -\frac{r^2}{R^2}dt^2 + \frac{R^2}{r^2}dr^2 + R^2\left[dr^2 + r^2d\Omega_2^2\right] \tag{3.24}$$

where we have defined

$$R = \sqrt[4]{q_0p^1p^2p^3} \tag{3.25}$$

The radius of the of AdS_2 and S^2 depends only on the value of charges. Value of dilaton in the near horizon also is fixed by the background charges

$$e^{\Phi} = \frac{q_0}{R} \tag{3.26}$$

In four-dimensional point of view $A^{(1)}$ give us an space-time gauge field on AdS_2 and $A^{(3)}$ with two component tangent to the S^2 responsible for two-form field strength on the horizon.

The above pattern of compactification with four charges can be generalized to a compactification on Calabi-Yau three-fold and also cover all possible D0-D4 charges of the theory, so to say, q_0, p^A for $A = 1,..,b_2$. Indeed, in [2, 3] it has shown that the compactification of type II string theory on a Calabi-Yau manifold contains the extremal black hole arising from wrapping branes around the cycles of CY_3, whose near-horizon region is, similarly, an $AdS_2 \times S^2 \times CY_3$.
Metric in (3.24) is the metric in Poincaré coordinate at the near horizon. In string theory we are interested in the global geometry. Using the coordinate transformations

$$\begin{aligned} \frac{R}{r} &= \frac{1}{\cosh\chi\cos\tau + \sinh\chi} \\ t &= \frac{R\cosh\chi\sin\tau}{\cosh\chi\cos\tau} \end{aligned} \tag{3.27}$$

we move from Poincaré, $(r,t,)$ to global, (χ,τ). With this, indeed we embedding the near horizon geometry into the global AdS_2. This can be seen in Figure (3.1), where in the left, the white band is the near horizon geometry in poincaré which can be embedded in the global AdS_2 geometry, as in the right figure.

the metric in global coordinates is then

$$ds^2 = R^2\left(-\cosh^2\chi d\tau^2 + d\chi^2\right) + R^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{3.28}$$

The radius R of the near horizon $AdS_2 \times S^2$ is given by

$$R = \sqrt{2}(D)^{\frac{1}{4}}, \qquad D \equiv D_{ABC}p^A p^B p^C \tag{3.29}$$

which depends on the charges of background. C_{ABC} are the triple intersection numbers on the CY_3. Requiring that the black hole has a non-vanishing horizon area in the leading supergravity approximation restrict us to have non-zero triple intersection number, $D \neq 0$. There is also four-form field strength

$$F^{(4)} = \omega_{S^2} \wedge F_{CY} = R^2 \sin\theta d\theta \wedge d\phi \wedge F^{(2)}_{CY_3} \tag{3.30}$$

and two form field strength

$$dC^{(1)} = \frac{1}{R}\omega_{AdS_2} = R\cosh\chi\, d\tau \wedge d\chi \tag{3.31}$$

where $F^{(2)}_{CY_3}$ is the two-form field strength in CY_3, ω_{S^2} and ω_{AdS_2} are, respectively, volume form on S^2 and AdS_2 defined as

$$\omega_{S^2} = R^2 \sin\theta d\theta d\phi, \qquad \omega_{AdS_2} = R^2 \cosh\chi d\tau d\chi \tag{3.32}$$

3.2.3 Branes in $AdS_2 \times S^2 \times CY_3$ background geometry

In previous section we constructed an extremal four-dimensional black hole arising from compactifying IIA string theory on a CY manifold and wrapping branes around the cycles of CY_3 whose near-horizon region is an $AdS_2 \times S^2 \times CY_3$ geometry. In addition, the background, in general contains dC^{p+1} form fields, where in case of a D4-D0 black hole give raise two and four-form field strength. The four-form has two component in non-compact space, tangent to the horizon and other two in the Calabi-Yau.

This geometry admits supersymmetric probe D2-branes wrapped on the black hole horizon, S^2, with arbitrary D0-charges bound to them [20]. These are the branes which preserve some space-time supersymmetries[4]. Such a horizon-wrapped branes which wrap the trivial cycles do not give any asymptotic charges due to their dimensionality in the full black hole geometry, rather they carry charges of the lower dimension. Consider a two-brane which is wrapped on the S^2. The wrapped bane behaves like a point particle in Calabi-Yau and anti-de Sitter space and in general can move along them. Furthermore the brane can couple to the gauge fields on its world-volume as well as to the space-time fields. To have a better understanding of possible brane interactions first we review the dynamics of D-branes and their coupling to the various background fields and then, explicitly, we deal with our case of interest.

3.2.3.1 D-brane dynamics

Dynamics of the gauge field living on the brane and fluctuation of the brane itself is governed by Dirac-Born-Infeld action which is usually referred to as the DBI action.
If we introduce coordinates ξ^a, $a = 0,...,p$ to be the world-volume coordinates on the brane, bosonic part of the DBI action is

$$S_{DBI} = -\mu_p \int d^{p+1}\xi \sqrt{\det(G_{ab} + 2\pi\alpha' F_{ab})} \tag{3.33}$$

G_{ab} is the induced metric on the brane, known as the pull-back of the space-time metric onto the world-volume

$$G_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\xi^b}\eta_{\mu\nu} \tag{3.34}$$

where X^μ are space-time coordinates, with $\mu, \nu = 0,...,D-1$. μ_p is the Dp-brane tension

$$\mu_p = \frac{1}{(2\pi)^p l_s^{p+1}} \tag{3.35}$$

which is analogue of the mass, M_p

$$M_p = \mu_p R^p \Omega_p \tag{3.36}$$

and F_{ab} is the field strength corresponding to the gauge field, $A_a(\xi)$, on the brane. $A_a(\xi)$ is sourced by the massless open strings attached to the brane.
Above action describes the low energy dynamics of a Dp-brane in a flat space. The motion

[4] We study the supersymmetric branes in next chapter.

of the D-brane can be affected if it moves in a non-flat background created by the close string modes $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ. Presence of a background can be imposed to the action as

$$S_{DBI} = -\mu_p \int d^{p+1}\xi \, e^{-\Phi} \sqrt{\det(G_{ab} + 2\pi\alpha' F_{ab} + B_{ab})} \tag{3.37}$$

where now, in (3.34) to compute the induced metric we should use background metric, $G_{\mu\nu}$, instead of the flat metric $\eta_{\mu\nu}$. Furthermore, apart from the gauge fields living on the brane, coupling of the brane to the space-time gauge fields should also be considered. These kind of gauge fields, which are again sourced by the massless open strings, describe the motion of the D-brane in the transverse dimensions, we call them $C_I, I = p+1, ..., 10$. [5]

The space-time gauge fields do not contribute to the DBI action, but rather they appear as an extra term, the Chern-Simons term (CS term in brief), in the action as

$$\tilde{\mu}_p \int P[C^{(p+1)}] \tag{3.38}$$

where, $\tilde{\mu}_p$ denotes the Dp-brane charge and $P[C^{(p+1)}]$ stands for the pullback of the ten-dimensional gauge field to the world-volume. However, in the presence of a world-volume gauge fields (or background B field), the brane also couples to lower rank R-R gauge fields in the form of [6]

$$\mu_p \int C^{(p-1)} \wedge F^{(2)} \tag{3.39}$$

which performs a $(p+1)$-form as it should. The complete Chern-Simons term is then

$$S_{CS} = \tilde{\mu}_p \int P[C^{(p+1)} + 2\pi\alpha' F^{(2)} \wedge C^{(p-1)}] \tag{3.40}$$

The Chern-Simons term, in particular, encodes the fact that the Dp-brane can carry charges of lower dimensional D-branes by having the world-volume field strength turned on. So the bosonic part of full D-brane dynamics governs the sum of DBI action and CS term as

$$\begin{aligned} S_{D_p} &= S_{DBI} + S_{CS} \qquad (3.41) \\ S_{D_p} &= -\mu_p \int d^{p+1}\xi \sqrt{\det(G_{ab} + 2\pi\alpha' F_{ab})} + \tilde{\mu}_p \int P[C^{(p+1)} + 2\pi\alpha' F^{(2)} \wedge C^{(p-1)}] \end{aligned}$$

We stress that $F^{(2)}$ is the field strength of gauge field, denoted by $A_a(\xi)$, living on the brane and $C^{(p-1)}$ is the space-time gauge field. There is also an extremal limit where the tension of brane, μ_p, equals its charge, $\tilde{\mu}_p$. So for an extremal Dp-brane the action is

$$S_{D_p} = -\mu_p \int d^{p+1}\xi \sqrt{\det(G_{ab} + 2\pi\alpha' F_{ab})} + \mu_p \int P[C^{(p+1)} + 2\pi\alpha' F^{(2)} \wedge C^{(p-1)}] \tag{3.42}$$

[5] The massless open strings have a vector index (A_p,C_I). If the indices lie in the directions parallel to the brane they describe gauge fields living on the brane and if the indices are perpendicular to the brane, they are related to oscillations of the brane in the perpendicular directions. For $p > 1$, the massless open strings also describe fluctuations in the world-brane gauge field on Dp-brane.

[6] Since F is two-form, In IIA, only odd-rank RR fields contribute for even p.

Now, we can write the action describes the dynamics of D2-brane we have considered in this section: a probe D2-brane in the $AdS_2 \times S^2 \times CY_3$ background (3.28)together with the R-R one-form gauge field

$$F^{(2)} = dC^{(1)} = \frac{R^2}{q_0} \cosh\chi d\tau \wedge d\chi \tag{3.43}$$

in the global AdS_2 coordinates. Fixing static gauge as

$$\xi^a \equiv X^a, \qquad a = 0,1,2 \tag{3.44}$$

where the world-volume time is chosen to be the target space time and the world-volume angles are chosen to be the angles on S^2, then we get the following action for the D2-brane

$$S_{D_2} \;\; = \;\; -\mu_2 \int d\xi^3 R \sin\theta \sqrt{(\cosh^2\chi - \dot{\chi}^2)R^4} \tag{3.45}$$

The Chern-Simons term vanishes because there is no R-R three-form, $C^{(3)}$, in the background and also we do not have any gauge field on the two-brane. In principle the D2-brane can also have D0-brane charge on it. This can be done by, adiabatically, moving D0-branes from infinity to the surface of the two-brane. The resulting D2-D0 bound state can be easily described: the D0-brane dissolves in the D2-brane, leaving a flux, where the D0-charge appears as U(1) gauge field potential, A,

$$A = -\frac{f}{2\pi\alpha'} \cos\theta d\phi \quad \Rightarrow \quad F = dA = \frac{f}{2\pi\alpha'} d\theta \wedge d\phi \tag{3.46}$$

on the D2-brane. The resulting action, in presence of the above gauge field on the brane, is then

$$\begin{aligned} S_{D_2} &= -\mu_2 \int d\xi^3 R \sin\theta \sqrt{(\cosh^2\chi - \dot{\chi}^2)(R^4 + f^2)} + \mu_2 \frac{R^2}{q_0} \int d\xi^3 \sinh\chi \\ &= -4\pi\mu_2 R \int d\tau \sqrt{(\cosh^2\chi - \dot{\chi}^2)(R^4 + f^2)} + 4\pi\mu_2 f \frac{R^2}{q_0} \int d\tau \sinh\chi \end{aligned} \tag{3.47}$$

where now we have contribution from the Chern-Simons term. Since we are working in the approximation (3.13) the internal excitation levels are suppressed and can be neglected. In this case the global Hamiltonian of the D2-brane is

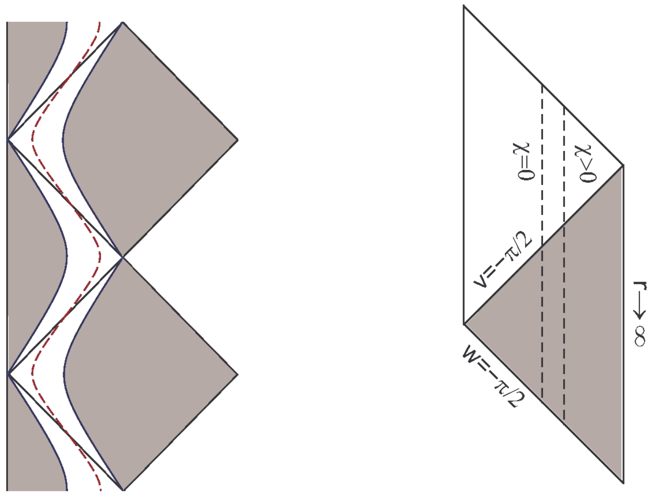
$$H = \cosh\chi \left[(M_2^2 + M_0^2)e^{-2\Phi_0} + P_\chi^2\right]^{1/2} + M_0 \frac{R}{q_0}[1 - \sinh\chi] \tag{3.48}$$

where M_0 and M_2 are defined as

$$M_0 = 4\pi\mu_2 f, \qquad M_2 = 4\pi\mu_2 R^2 \tag{3.49}$$

The D2-brane Hamiltonian has an static solution at $\chi = \chi_0$ given by

$$\sinh\chi_0 = \frac{M_0}{M_2} \tag{3.50}$$



(a) (b)

Figure 3.1: (a) Penrose diagram for an extremal black hole. The white band is the near horizon AdS_2 geometry in Poincaré coordinates and dashed line shows the brane position which oscillates around the horizon. (b) Brane in global AdS, dashed line represent the brane position.

which is the minimum of brane potential. In the Poincaré coordinates this brane is not static, rather it oscillates in and out of the horizon as can be seen in Carter-Penrose diagram in figure (3.1). As we see, in global coordinates the radial position of the brane is determined by the value of D0-charge, f. This means that for large f, the D-brane can potentially go away from the near horizon region. Indeed the brane remains close to the horizon as long as

$$M_0 \ll M_2 \tag{3.51}$$

but otherwise goes out of this region.

3.3 Black holes entropy from string theory

After making black hole in string theory, now we should address entropy of the black hole in string theory. The goal is to use string theory to reproduce the macroscopic entropy law by counting the states of underlaying microscopic theory. While the original string-black hole correspondence, to understand the entropy of black holes, invoked only the fundamental string states [21], it generalized to the D-brane description of black holes which yield a precise derivation of a black hole entropy.

In string theory, one can test the expected relation between the macroscopic and microscopic entropy by counting the ten-dimensional states which give raise to a same four-dimensional black hole, This comparison generally involves the variation of parameters, as for the string coupling we have

$$\text{Macroscopic regime:} \quad R_H \gg l_s \Rightarrow |Q| g_s \gg 1 \tag{3.52}$$

$$\text{Microscopic regime:} \quad |Q| g_s \ll 1 \tag{3.53}$$

We see that for comparing the two regimes we have to go from the weak to strong coupling and it is not priori clear that whether the number of states are preserved under this interpolation. In the other word, in string theory computation of the entropy in microscopic level can be preformed in the limit when we neglect the effects of interactions, $g_s \ll 1$. Since a black hole can only exist once the interactions are turned on. This means that exact computation of the entropy of a black hole in string theory requires that counting of the states in weak coupling remains valid for all values of the coupling.
As an example we compare the entropy of a Schwarzschild black hole with an extremal black hole ($c = \hbar = 1$)

$$S = \frac{A}{4G} = \begin{cases} 4\pi M^2 G & \text{Schwarzchild} \\ \pi(q^2 + p^2) & \text{Extremal} \end{cases} \tag{3.54}$$

The entropy of black hole is not independent of Newton constant, G, but interestingly, in case of the extremal black hole it is. The same applies to a supersymmetric black hole in general $N = 2$ supergravity in four dimensions.
To see what is the consequence of this property, we remind the relation between Newton constant and the string coupling constant which is given by $G \sim g_s^2 l_s^2$, where g_s and l_s denote the string coupling and the string length, respectively. So, since the entropy of a supersymmetric black holes is independent of g_s, it is meaningful to compare the macroscopic entropy with the microscopic description.
This remarkable property is present in supersymmetric black holes. In other word, as long as supersymmetry is preserved, the results obtained in non-interacting regime are valid for any couplings. However, we note that when we compactify space-time, the supersymmetry preserved if we curl up dimensions into cycles. To make black hole we wrap extended objects around cycles where can cause breaking of supersymmetry too, so we consider special configurations of wrapped objects which some supersymmetry survives.
Moreover, there is a mechanism that ensures that the macroscopic entropy of a supersymmetric black hole is entirely specified in terms of the charges q_I, p^I

$$S_{macro} = S_{macro}(q_I, p^I) \tag{3.55}$$

This is the attractor mechanism which first discovered for supersymmetric black holes [2, 22, 23] and later shown that it holds for extremal black hole even if not supersymmetric [24].

In the microscopic level, in D-brane picture, the large degeneracy which gives the statistical interpretation to the thermodynamic entropy comes from all the possible internal excitations of the wrapped branes which lead to a same four dimensional black hole in the macroscopic level. It has been shown [4, 25] that the entropy calculated using the D-brane description agrees precisely with the classical Bekenstein-Hawking entropy. For a D4-D0 Calabi-Yau black hole the problem of microscopically computing the entropy has also been solved [26]. Specially, recently it was shown [7] that one can derive the entropy by doing calculations in the proposed dual conformal field theory, CFT_1. The CFT_1 takes the form of quantum mechanics of a set of probe D0-branes moving in the AdS_2 near-horizon geometry. The non-abelian N-D0 configurations corresponding to D2 branes wrap the black hole horizon and carry N units of world-volume magnetic flux. The D2-D0 brane is in the background magnetic flux on the CY_3 and therefore acquires large degeneracy corresponding to the lowest Landau levels. Number of degeneracies found to be exactly reproduce the leading order area-entropy formula for a D4-D0 black hole.

In following we see how one can count the microscopic degeneracy of a black hole in string theory and we will see it precisely matches with the Bekenstein-Hawking entropy.

3.3.1 Example: D4-D0 black hole on T^6

To have a better view of the underlaying microscopic description and to see that if it agrees with the macroscopic entropy we study the D4-D0 black hole which we considered earlier, namely, the configuration of three D4-branes with charges p^1,p^2,p^3 and q_0 D0-branes in IIA compactified on T^6. Macroscopic entropy can be calculated easily by looking at the area of horizon. The horizon of the black hole is located in $r=0$ and the area of the horizon can be computed as

$$A = 4\pi \lim_{r\to 0}\left(r^2\sqrt{\frac{q^0p^1p^2p^3}{r^4}}\right) = 4\pi\sqrt{q^0p^1p^2p^3} \tag{3.56}$$

In order to write down the Bekenstein-Hawking entropy it is instructive to restore dimensions. Specially, since the spectrum of charges are discrete one can introduce fundamental charge units c_0, c^i and express q_0, p^1, p^2, p^3 (which have length dimension) to be integers as

$$q_0,\ p^i \quad \Rightarrow \quad c_0q_0,\ c^ip^i \tag{3.57}$$

With this description now we have p^1 D4-branes wrapped on the y_1, y_2, y_3, y_4, p^2 D4-branes wrapped on the y_1, y_2, y_5, y_6, p^3 D4-branes wrapped on the y_3, y_4, y_5, y_6 and q_0 D0-branes. One straightforward way is first to write down the entropy in terms of the masses via (3.21)

$$S_{macro} = \frac{A}{4G_4} = 16\pi G_4\sqrt{M_0M_1M_2M_3} \tag{3.58}$$

Now we can use (3.35) and (3.36) as

$$\begin{aligned} M^0 &= \frac{1}{g_s l_s} q_0 \\ M^1 &= \frac{1}{g_s (2\pi)^4 l_s^5} (2\pi R_1)(2\pi R_2)(2\pi R_3)(2\pi R_4) p^1 \\ M^2 &= \frac{1}{g_s (2\pi)^4 l_s^5} (2\pi R_1)(2\pi R_2)(2\pi R_5)(2\pi R_6) p^2 \\ M^3 &= \frac{1}{g_s (2\pi)^4 l_s^5} (2\pi R_3)(2\pi R_4)(2\pi R_5)(2\pi R_6) p^3 \end{aligned} \tag{3.59}$$

where $R_1, ..., R_6$ are the radii of the T^6 cycles. Furthermore we need to know the expresion for four-dimensional Newton constants, G_4

$$G_4 = \frac{G_{10}}{V_{T^6}}, \qquad G_{10} = 8\pi^6 g_s^2 l_s^8 \tag{3.60}$$

the resulting macroscopic entropy is

$$S_{BH} = 2\pi\sqrt{q_0 p^1 p^2 p^3} \tag{3.61}$$

All dimensionful constants and all continuous parameters cancel precisely and the entropy is a pure number which is given by the number of charges. Indeed, this indicates that an interpretation in terms of the microscopic D-brane states is possible.
The goal is to use the D-brane configuration to produce the entropy (3.61) by counting the states, which originally performed in [27]. We should find, so to say, how this black hole can be constructed in many possible ways. We count the states when there are no interactions and the supersymmetry guarantees that the zero coupling counting holds for nonzero coupling, where we have a black hole.
As we already mentioned, the microscopic entropy is the logarithm of states degeneracy, which leads to the same macroscopic quantities. In our case

$$S_{micro} = \log N(q_0, p^1, p^2, p^3) \tag{3.62}$$

where $N(q_0, p^1, p^2, p^3)$ is the number of microstates of a black hole with macroscopic parameters q_0, p^1, p^2, p^3. Lets start with an special case where we have only three D4-branes, $p^1 = p^2 = p^3 = 1$. The three D4-branes intersect only at one point in T^6. Now we bind q_0 D0-brane to the mutual intersection point by letting, massless, strings run between the zero-branes and each of the four-branes. To count the zero modes of the D0-brane in the intersection point, we note that two four-branes brake 3/4 of the original 32 supersymmetries, The third four-brane also breaks half of the remaining supersymmetry and hence we are left with four bosonic modes. Furthermore, the unbroken supersymmetry implies an equal number of fermionic degrees of freedom. The degeneracy of states, $d(n)$, with charge n is given by the coefficient of q^n of generating function

$$\sum_n d(n) q^n = \prod_{k=1}^{\infty} \left(\frac{1+q^k}{1-q^k}\right)^4 \tag{3.63}$$

So, the degeneracy of states with three four-branes and q_0 D0-charge, where $q_0 \gg 1$, is

$$d(q_0) = \exp\left(2\pi\sqrt{q_0}\right) \tag{3.64}$$

Returning to our example where p^1, p^2, p^2 are greater than unity, we have three set of four-branes which each of them consists of many parallel four-branes. Separating four-branes gives $p^1p^2p^3$ distinct points for the intersecting three four-branes. Binding q_0 D0-brane to each intersection points gives four bosonic and four fermionic modes, and hence total of $4p^1p^2p^3$ bosonic/fermionic modes. The generating function of degeneracy is then

$$\sum_n d(n)q^n = \prod_{k=1}^{\infty}\left(\frac{1+q^k}{1-q^k}\right)^{4p^1p^2p^3} \tag{3.65}$$

for large q_0 is leads to degeneracy $d(q_0) = \exp 2\pi\sqrt{q_0p^1p^2p^3}$ and the entropy is then

$$S_{micro} = 2\pi\sqrt{q_0p^1p^2p^3} \tag{3.66}$$

This coincides exactly with the Bekenstein-Hawking entropy (3.61) extracted from the metric. Our example can be generalized to Calabi-Yau black holes with arbitrary charges q_0, p^A. This has been done in [26]. Without going to details we give the result

$$S = 2\pi\sqrt{|q_0|\left(C_{ABC}p^Ap^Bp^C + c_{2A}p^A\right)} \tag{3.67}$$

c_2 is the second Chern class and C_{ABC} is the intersection number, which is fixed by the topology of internal manifold. This holds both for the macroscopic and the microscopic description. Finally, returning to our example, since the torus T^6 is flat, the second Chern class vanishes and the intersection form is $C_{ABC} = \varepsilon_{ABC}$ and $A,B,C = 1,2,3$ and hence we get same formula as before.

Apart from computing the statistical entropy of black holes, the D-brane method of black holes can be also applied to derive the Hawking radiation and to compute the greybody factor. In what follows we mainly focus on the black hole absorption cross section in D-brane picture.

3.4 Absorption of black holes in string theory

We have made black holes in string theory and found that the microscopic physics of branes reproduces the Bekenstein entropy for near-extremal holes. A natural step beyond the comparison of entropies is to interpret absorption cross sections for massless particles in terms of the D-brane world-volume theories. In oder word, we can ask about the dynamics of black holes: Can we compute the probability of the string state to absorb or emit quanta, and then compare this to the probability for the black hole to absorb infalling quanta or emit Hawking radiation?

In D-brane picture, absorption and Hawking radiation can be understood in terms of closed strings. The absorption of massless scalars into black holes can be described in the effective string language as a massless closed string state hitting a set of intersecting D-branes and turning into a pair of open strings that run in opposite directions along the 1 + 1-dimensional intersection manifold. Emission happens for near-extremal black holes, where we have, for example, sea of left movers and a right mover. Joining right mover with a left mover form a massless closed string which leave the brane as Hawking radiation.

In order to calculate the absorption cross section in D-brane formalism one needs the low energy world-volume action for D-branes coupled to the massless bulk fields. These couplings may be deduces from the D-brane Born-Infeld action described earlier in this chapter.

3.5 Absorption on horizon-wrapped branes

Here we will be interested in the absorption cross section on such D2-branes as seen by an asymptotic observer. For this we need the coupling of the D2-branes to the space-time fields. This can be inferred from the Born-Infeld action

$$S = -\mu_2 \int d^3\xi e^{-\Phi} \sqrt{-|G+2\pi\alpha' F|} + \mu_2 \int_{\Sigma_3} [P[C^{[3]}] + 2\pi\alpha' F \wedge P[C^{[1]}]] \,, \tag{3.68}$$

where G_{ab} is the induced string frame world-volume metric for a given 10-dimensional string metric and $C^{[1]}$ is the RR 1-form in IIA theory with

$$dC^{[1]} = \frac{R^2}{q_0} \cosh\chi d\tau \wedge d\chi \ . \tag{3.69}$$

in global AdS_2 coordinates. F_{ab} is the field strength of the world-volume $U(1)$ gauge potential, A with background value

$$A = -\frac{f}{2\pi\alpha'} \cos(\theta) d\phi \ . \tag{3.70}$$

Finally the background value of the dilaton is given by

$$e^{\Phi_0} = \frac{q_0}{R} . \tag{3.71}$$

3.5.1 Vibration modes

In what follows we will work in the static gauge and we will neglect internal excitations levels in X which are suppressed in the approximation (3.13). In this case there is exactly one transverse scalar field parameterizing the radial position of the brane in AdS_2 as well a gauge field A_a.There are two kind of fluctuations to be considered, one from radial position, $\delta\chi$, and other one is the fluctuations of gauge field living on the brane

$$\chi \rightarrow \chi_0 + \varepsilon\delta\chi \tag{3.72}$$
$$F_{ab} \rightarrow F_{ab} + \varepsilon f_{ab} \tag{3.73}$$

We take the D0-charge on the two-brane, $\tilde{q}_0 \propto \int_{S^2} F$ to be fixed. In this case the quantum mechanical (s-wave) excitation $\delta\chi$ decouples from all other excitations and can thus be treated separately. Upon substitution of the $AdS_2 \times S^2$ metric

$$ds^2 = R^2[-\cosh^2\chi d\tau^2 + d\chi^2] + R^2 d\Omega_2^2 \tag{3.74}$$

in the Born-Infeld action and expanding up to second order in derivatives we find the action for the transverse scalar χ

$$S_{D2} = \frac{1}{g^2}\int\left[\frac{1}{2}\dot{\chi}^2 - V_1(\chi) + V_2(\chi)\right] d\tau\,, \tag{3.75}$$

with

$$V_1(\chi) = \frac{\sqrt{M_2^2 + M_0^2}}{M_2}\cosh(\chi)\,, \qquad V_2(\chi) = \frac{M_0}{M_2}\sinh(\chi)\,, \tag{3.76}$$

and

$$g^{-2} = M_2 R e^{-\Phi_0} = 16\pi\mu_2 D \tag{3.77}$$

$V_1\chi$ is sourced by DBI part of the action and $V_2(\chi)$ originates from the Chern-Simons term in (3.68). Upon expanding the potential V to second order in $\delta\chi$ we obtain a harmonic oscillator with frequency 1 (in units of $1/R$) which can be quantized in standard way as

$$\delta\chi(\tau) = \frac{g}{\sqrt{2}}\left(e^{i\tau}a + e^{-i\tau}a^\dagger\right)\,, \tag{3.78}$$

where a and $a^\dagger$ are, respectively, the annihilation- and creation operators for the harmonic oscillator.

Next we consider quadratic fluctuations with non-vanishing angular momentum. By expanding the action (3.68) to second order in the fluctuation in the position $\delta\chi$ and the gauge field, $f_{ab} = \partial_a A_b - \partial_b A_a$, we obtain

$$\begin{aligned} S^{(2)} &= \frac{1}{g^2}\int d^3\xi R^3 \sin\theta \frac{1}{2}(\delta\chi)^2 \qquad (3.79)\\ &+\frac{1}{g^2}\int d^3\xi R^3\sin\theta\left(g^{ij}\partial_i\delta\chi\partial_j\delta\chi + (\partial_\tau\delta\chi)^2\right) \qquad i,j=\theta,\phi\\ &-\frac{1}{g^2}\int d^3\xi R^3\sin\theta\frac{(2\pi\alpha')^2}{f\sqrt{R^4+f^2}}F^{ab}f_{ab}\delta\chi\\ &+\frac{1}{g^2}\int d^3\xi R^3\sin\theta\frac{(2\pi\alpha')^2}{R^4+f^2}\left(\frac{1}{\sin^2\theta}f_{12}f_{12} - f_{01}f_{01} - \frac{1}{\sin^2\theta}f_{02}f_{02}\right)\end{aligned}$$

Here $S^{(2)}$ refers to the action corresponding to second order fluctuations. We can simplify the above action by introducing a modified metric as

$$\tilde{g}_{ab} = R^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sin^2(\theta) \end{pmatrix} . \tag{3.80}$$

action is then

$$\begin{aligned} S^{(2)} &= \frac{1}{g^2 4\pi} \int d^3\xi \sqrt{-\tilde{g}} \left[-\frac{1}{2}(\delta\chi)^2 - \frac{1}{2}\tilde{g}^{ab}\partial_a \delta\chi \partial_b \delta\chi - \right. \\ & \left. -\frac{1}{4}\frac{(2\pi\alpha')^2}{R^4+f^2}\tilde{g}^{ac}\tilde{g}^{bd} f_{ab} f_{cd} + \frac{1}{2}\frac{(2\pi\alpha')^2}{f\sqrt{R^4+f^2}} F^{ab} f_{ab}\delta\chi \right] , \end{aligned} \tag{3.81}$$

Thus the dynamics of the quadratic fluctuations on a two-brane in $AdS_2 \times S^2$ wrapped on S^2 is identical to that of a brane fluctuating in $\mathbb{R}^{1,1} \times S^2$ with a non-trivial potential $V(\delta\chi, f_{ab})$. The coupling between f_{ab} and χ persists in the absence of a D0 probe brane charge as a consequence of the Chern-Simons term in (3.68). The equations of motion for the fluctuations obtained from (3.81) are then found to be

$$\begin{aligned} \frac{\delta S^{(2)}}{\delta A_e} &= (2\pi\alpha')^2 \tilde{\nabla}_a \left(f^{ae} - \frac{\sqrt{R^4+f^2}}{f} F^{ae}\delta\chi \right) = 0 , \\ \frac{\delta S^{(2)}}{\delta(\delta\chi)} &= \left(\tilde{\nabla}^2 - 1\right)\delta\chi + \frac{1}{2}\frac{(2\pi\alpha')^2}{f\sqrt{R^4+f^2}} F^{ab} f_{ab} = 0 , \end{aligned} \tag{3.82}$$

where the indices are now lifted with $\tilde{g}$ and $\tilde{\nabla}^2$ is with respect to $\tilde{g}$. To continue, using (3.82), we express f_{ab} in terms of a new scalar field, ψ, through

$$f^{ae} = \frac{R^2}{(2\pi\alpha')} \frac{1}{\sqrt{\tilde{g}}} \varepsilon^{aeg}\partial_g \psi + \frac{\sqrt{R^4+f^2}}{f} F^{ae}\delta\chi . \tag{3.83}$$

Note that since we have defined ψ, via equation of motion this change of variables is valid on-shell only. In terms of these new fields the equations of motion then take the form

$$\begin{aligned} \tilde{\nabla}^2 \psi - \frac{\sqrt{R^4+f^2}}{R^2}\partial_\tau \delta\chi &= 0 , \\ \tilde{\nabla}^2 \delta\chi + \frac{R^2}{\sqrt{R^4+f^2}}\partial_\tau \psi &= 0 . \end{aligned} \tag{3.84}$$

In order to diagonalize this system we expand ψ and $\delta\chi$ in spherical harmonics as

$$\begin{aligned} \delta\chi(\tau,\theta,\phi) &= \delta\chi_{lm} e^{-i\Omega\tau} Y_{lm}(\theta,\phi) \\ \delta\psi(\tau,\theta,\phi) &= \psi_{lm} e^{-i\Omega\tau} Y_{lm}(\theta,\phi) \end{aligned} \tag{3.85}$$

Then the equation of motion leads to

$$\begin{pmatrix} \frac{\sqrt{R^4+f^2}}{R^2}\left(-l(l+1)+\Omega^2\right) & -i\Omega \\ i\Omega & \frac{R^2}{\sqrt{R^4+f^2}}\left(-l(l+1)+\Omega^2\right) \end{pmatrix} \begin{pmatrix} \delta\chi_{lm} \\ \psi_{lm} \end{pmatrix} = 0 . \tag{3.86}$$

Now we can derive the eigenfrequencies by finding the zero's of the determinant. They are given by

$$\Omega = l \quad \text{and} \quad \Omega = l+1 \tag{3.87}$$

with their corresponding eigenmodes

$$\Phi^{0,1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \frac{\sqrt{R^4+f^2}}{R^2} \end{pmatrix} \varphi^{0,1}_{lm}\,, \tag{3.88}$$

respectively, where $\varphi^{0,1}_{lm}$ are canonically normalized free fields with dispersion relation $\Omega = l$ for φ^0_{lm} and $\Omega = l+1$ for φ^1_{lm}. We can thus quantize $\delta\chi$ and ψ in terms of free fields a_{lm} and b_{lm} as

$$\begin{aligned}\begin{pmatrix} \delta\chi_{lm}(\tau) \\ \psi_{lm}(\tau) \end{pmatrix} = \frac{g}{2}\Bigg[&\frac{a_{lm}}{\sqrt{l}} e^{il\tau} Y_{lm} \begin{pmatrix} 1 \\ -i\frac{\sqrt{R^4+f^2}}{R^2} \end{pmatrix} + \frac{a^\dagger_{lm}}{\sqrt{l}} e^{-il\tau} Y^*_{lm} \begin{pmatrix} 1 \\ i\frac{\sqrt{R^4+f^2}}{R^2} \end{pmatrix} \\ &+ \frac{b_{lm}}{\sqrt{l+1}} e^{i(l+1)\tau} Y_{lm} \begin{pmatrix} 1 \\ i\frac{\sqrt{R^4+f^2}}{R^2} \end{pmatrix} + \frac{b^\dagger_{lm}}{\sqrt{l+1}} e^{-i(l+1)\tau} Y^*_{lm} \begin{pmatrix} 1 \\ -i\frac{\sqrt{R^4+f^2}}{R^2} \end{pmatrix} \Bigg].\end{aligned} \tag{3.89}$$

where we adopt the convention $\int_{S^2} Y_{lm} Y^*_{l'm'} = 4\pi \delta_{ll'} \delta_{mm'}$. The integer valuedness of the spectrum is to be expected since in global coordinates time $\tau \simeq \tau + 2\pi$ is compactified.

3.5.2 Absorption

The dilaton and the volume moduli of the CY (the latter is a fixed scalar) couple to the radial position χ of the two-branes in AdS_2 as well as to the world-volume gauge potential through the DBI-term in (3.68). The RR 1-form field couples to the world-volume gauge potential and the radial position through the CS-term in (3.68). We will focus on the dilaton absorption at present. To begin with we consider the quantum mechanical (s-wave) absorption of a dilaton $\delta\phi$, which then couples to the transverse position as

$$S_{D2} = \frac{1}{g^2} \int (-\delta\phi) \left(\frac{1}{2}\dot{\chi}^2 - V_1(\chi) \right) d\tau\,. \tag{3.90}$$

The potential V_2 is induced by the CS-term in (3.68) and thus does not couple to the dilaton. To continue we need to distinguish between small- and large D0 probe-brane charge since $\delta\phi$ does not couple to V_2. For $M_0 << M_2$ we have $V_1(\chi_0) \simeq 1$ so that the back-reaction of the probe brane on the dilaton Φ can be neglected. For $M_2 << M_0$ on the other hand $V_1(\chi_0) \propto \frac{f^2}{R^4} >> 1$ so that in the linearized approximation back reaction of the probe brane destabilizes the supergravity background[7]. We will thus not consider this possibility. On the other hand, for small D0 probe brane charge $\tilde{q}_0$, the probe brane trajectory is within the near horizon region of the Poincaré patch of AdS_2. In this case it is interesting to compute the absorption cross section as observed by an asymptotic observer and compare it to the classical black hole absorption cross section.

[7]Of course, in the full non-linear theory a (constant) deformation of the dilaton is a marginal deformation since the position of the probe brane is a smooth function of the string coupling.

3.5.3 Spherical excitations without D0-charge

We will be interested in the absorption cross section seen by an observer static in AdS_2 with respect to Poincaré time[8]. We first consider the case of a probe brane without D0-charge, $M_0 = 0$. In the near horizon AdS_2 the classical solutions for an s-wave dilaton perturbation with frequency ω with respect to the Poincaré time are given by

$$\delta\phi_\omega(t) = e^{i\omega(t\mp\frac{R^2}{r})} . \tag{3.91}$$

3.5.3.1 Quantum mechanical mode

As a warm-up we first treat the absorption cross section in the harmonic oscillator approximation (3.90), ie. assuming vanishing angular momentum of the excitations on the probe-brane. In this case, by Taylor expansion the Lagrangian, the coupling of the dilaton perturbation to the transverse excitation of the two-brane is given by

$$S_{int} = \frac{1}{2g^2}\int \delta\phi : \left(-\delta\dot{\chi}^2 + \delta\chi^2\right) : , \tag{3.92}$$

where " : : " denotes normal ordering and " $\dot{}$ " indicates derivative w.r.t. global time τ. To leading order in g we get for an ingoing dilaton upon inserting (3.78)

$$\langle 2|S_{int}|0\rangle = \frac{\sqrt{2}}{2}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{i\omega(t-\frac{R^2}{r})} e^{-2i\tau} d\tau . \tag{3.93}$$

Note that for $M_0 = 0$, first order perturbation in $\delta\chi$ vanishes and hence there is no transition from the ground state to the first excited state $|1\rangle$. In order to evaluate the integral we change to Kruskal coordinates which for $\chi_0 = 0$ are given by

$$\begin{aligned} vR &= t - \frac{R^2}{r} = R\frac{\sin\tau - 1}{\cos\tau} = R\tan(\frac{\tau - \frac{\pi}{2}}{2}) , \\ wR &= t + \frac{R^2}{r} = R\frac{\sin\tau + 1}{\cos\tau} = R\tan(\frac{-\tau + \frac{\pi}{2}}{2}) . \end{aligned} \tag{3.94}$$

Then

$$\begin{aligned} \langle 2|S_{int}|0\rangle &= -\sqrt{2}\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \frac{1}{1+v^2} e^{i\omega R v}\left(\frac{1-iv}{1+iv}\right)^2 dv \\ &= 2\sqrt{2}\pi R\omega(R\omega - 1)e^{-R\omega} , \end{aligned} \tag{3.95}$$

where the contour C closes in the upper half plane for $\omega > 0$ and passes above the pole at the origin in accordance with the $i\varepsilon$ prescription for absorption. The boundary term in (3.95) ensures the correct fall-off at $v = \pm\infty$ required in order to recast this amplitude as a

[8] This time coincides with the time of an asymptotic observer in Minkowski space-time.

contour integral. Excitation to higher level is also possible and the corresponding amplitude can be calculated by expanding V_1 to higher order in χ. The corresponding amplitude is sub-leading in g.

To determine the cross section for an s-wave dilaton into an s-wave excitation on the brane for an asymptotic observer in $AdS_2 \times S^2$ geometry we should compute the ingoing flux of the field ϕ, given in (3.91). By recalling the definition of conserved flux

$$\mathscr{F} = 4\pi \frac{1}{2i}\sqrt{g}g^{rr}\left(\bar{\phi}\partial_r\phi - \phi\partial_r\bar{\phi}\right) \tag{3.96}$$

and $AdS_2 \times S^2$ metric

$$ds^2 = -\frac{r^2}{R^2}dt^2 + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_2^2 \tag{3.97}$$

the ingoing flux is then

$$\mathscr{F}_{AdS_2} = 4\pi\omega \tag{3.98}$$

The AdS_2 cross section for this process is thus given by

$$\begin{aligned} \sigma_{AdS_2} &= \frac{|\mathscr{A}_{0\to 2}|^2}{T\mathscr{F}}. \\ &= R\omega(R\omega - 1)^2 e^{-2\omega R} \end{aligned} \tag{3.99}$$

Here $T = 2\pi R$ is the time interval in global coordinates. Of course, this is only part of the complete absorption cross section for an s-wave dilaton, since we ignored higher angular momentum excitation so far. Never the less this partial cross section allows us to discuss some qualitative features. First we note that in spite of the discreteness of the spectrum of the D2-brane Hamiltonian, the D2-brane can absorb arbitrarily small frequencies with respect to Poincaré time. This is in agreement with the classical picture of black hole absorption. On the other hand, the low frequency behavior of (3.99) differs from the classical s-wave absorption cross section, which is proportional to the square of transmission coefficient T that has a universal form for small frequencies [9] [10], $|T|^2 \simeq (R\omega)^2$. We will come back to this point in the conclusions. We should emphasize that the details of the absorption process described here are qualitatively different from the world-volume absorption in flat space. In flat space, the low energy behavior is dominated by goldstone bosons and possible other massless fields whereas here no massless degrees of freedom are present. The fact that the cross section vanishes linearly for $\omega \to 0$ is due precisely to the absence of massless degrees of freedom.

3.5.3.2 Higher partial waves

After computing the absorption cross section for simple case of vanishing angular excitations, now we include such possibility to have total cross section. Recalling that the

[9] We have shown this universality in previous chapter.

Chern-Simons term does not couple to $\delta\phi$ we get the following interaction including all angular momenta on S^2:

$$S_{int} = \frac{1}{2g^2 4\pi} \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{S^2} \delta\phi(t) \Bigg[\sqrt{-\tilde{g}} \left(\tilde{g}^{ab}\partial_a\delta\chi\partial_b\delta\chi - \frac{R^4}{R^4+f^2}\tilde{g}^{ab}\partial_a\delta\psi\partial_b\delta\psi \right) + +2\frac{R^2}{\sqrt{R^4+f^2}}\delta\chi\partial_\tau\psi \Bigg] d\tau \,. \tag{3.100}$$

which can be obtained easily by using (3.83) and inserting, ψ in action of second order perturbations (3.81). Now we use (3.89) to have in interaction in terms of free fields, we then end up with

$$S_{int} = \frac{1}{4}\sum_{l\neq 0,m} \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{i\omega(t-\frac{R^2}{r})}(-1)^m \Bigg[\frac{1}{l} a^\dagger_{lm} a^\dagger_{l-m} e^{-i2l\tau} + + \frac{1}{\sqrt{l(l+1)}} a^\dagger_{lm} b^\dagger_{l-m} e^{-i(2l+1)\tau} + \frac{1}{(l+1)} b^\dagger_{lm} b^\dagger_{l-m} e^{-i2(l+1)\tau} \Bigg] d\tau \,. \tag{3.101}$$

Here we have ignored terms that contain the annihilation operators a_{lm} and b_{lm} since we take the initial state to be the ground state so that they do not contribute to the absorption amplitude. Note that if we include the $l=0$ mode for the last term then we indeed recover the quantum mechanical mode discussed before. The transition amplitudes for fixed $l>0$ and m in leading order, in g, are then found to be [10]

$$\begin{aligned} \langle a, l\ m; a, l\ -m|S_{int}|0\rangle &= c_m \frac{(-1)^m}{l} \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{i\omega(t-\frac{R^2}{r})} e^{-i2l\tau} d\tau \\ &= 2c_m \frac{(-1)^m}{l} \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \frac{1}{1+v^2} e^{i\omega R v} \left(\frac{1-iv}{1+iv}\right)^{2l} dv \\ &= 2\pi c_m \frac{(-1)^{l+m+1}}{l} M_{2l,\frac{1}{2}}(2R\omega)\,, \end{aligned} \tag{3.102}$$

with $c_m = 1/2$ for $m\neq 0$, $c_m = \sqrt{2}/4$ for $m=0$ and $\langle a, l\ m; a, l\ -m|$ denotes the final state consisting of two excitations of type a with angular momentum l and $L_3 = \pm m$ respectively. Furthermore

$$M_{\lambda,\mu}(z) = \frac{z^{\mu+\frac{1}{2}}}{2^{2\mu}B(\frac{1}{2}+\mu+\lambda, \frac{1}{2}+\mu-\lambda)} \int_{-1}^{1} e^{\frac{1}{2}zt}(1+t)^{\mu-\lambda-\frac{1}{2}}(1-t)^{\mu+\lambda-\frac{1}{2}} dt \tag{3.103}$$

[10] At the end of the section we give the details of how to compute the integral

is the Whittaker function [28] and $B(x,y)$ is the beta function. Similarly

$$\begin{aligned}\langle b,lm;b,l\ -m|S_{int}|0\rangle &= c_m\frac{(-1)^m}{l+1}\int\limits_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{i\omega(t-\frac{R^2}{r})}e^{-i2(l+1)\tau}d\tau \\ &= 2\pi c_m\frac{(-1)^{l+m}}{l+1}M_{2l+2,\frac{1}{2}}(2R\omega)\,, \qquad (3.104)\end{aligned}$$

for two b-type excitations in the final state and

$$\begin{aligned}\langle b,lm;a,l\ -m|S_{int}|0\rangle &= c_m\frac{(-1)^m}{\sqrt{l(l+1)}}\int\limits_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{i\omega(t-\frac{R^2}{r})}e^{-i(2l+1)\tau}d\tau \\ &= \frac{1}{2}\pi i\frac{(-1)^{l+m+1}}{\sqrt{l(l+1)}}M_{2l+1,\frac{1}{2}}(2R\omega)\,, \qquad (3.105)\end{aligned}$$

for one a-type and one b-type excitation in the final state. In order to obtain the total cross section for absorption of an s-wave dilaton on a two-brane without D0-charge we have to sum the partial cross section over l and m. Note that taking into account the quantum mechanical mode the sum over l starts from 1 for (3.102) and (3.105) and from 0 for (3.104). Furthermore $m=0,\cdots,l$ for (3.102) and (3.104) due to symmetry under $m\to -m$, and $m=-l,\cdots,l$ for (3.105). Putting all this together and dividing by the incoming flux, $\mathscr{F}_{\mathscr{A}\lceil\mathscr{S}_{\in}}$, we end up with

$$\sigma_{AdS_2} = \frac{1}{4R\omega}\sum_{l=1}^{\infty}\frac{1}{l}|M_{2l,\frac{1}{2}}(2R\omega)|^2+\frac{1}{32R\omega}\sum_{l=1}^{\infty}\frac{2l+1}{l(l+1)}|M_{2l+1,\frac{1}{2}}(2R\omega)|^2\,. \qquad (3.106)$$

While we are not aware of any closed expression for the above sum we can never the less extract the low frequency behavior with the help of an integral approximation of the sums in (3.106). First we give the result and then describe how we get to it.

$$\sigma_{AdS_2} \simeq -5\,R\omega(\log(R\omega)+const)\,,\qquad \omega\to 0\,. \qquad (3.107)$$

The total absorption cross section is thus non-analytic at $\omega=0$. The absorption cross section for space-time scalars with non-vanishing angular momentum can be obtained along the same lines. We indicate the modifications later in this chapter.

In order to isolate the low frequency behavior of the cross section (3.106) we use a convergent expansion of the Whittaker function $M_{\lambda,\mu}(z)$ in a series of Bessel functions given by Buchholz [29]. It reads

$$M_{\lambda,\mu}(z)=\Gamma(2\mu+1)2^{2\mu}z^{\mu+\frac{1}{2}}\sum_{n=0}^{\infty}p_n^{(2\mu)}(z)\frac{J_{2\mu+n}(2\sqrt{\lambda z})}{(2\sqrt{\lambda z})^{2\mu+n}}\,, \qquad (3.108)$$

where $p_n^{(2\mu)}(z)$ are the Buchholz polynomials.

Assume that $f(l)$ is a function such that $f(l)\simeq 1/l$ for $l>>1$ and let now $L>>1$ be an integer. Then we have for $z \to 0$

$$\frac{1}{z}\sum_{l=1}^{\infty} f(l)|M_{2l,\frac{1}{2}}(z)|^2 \simeq \frac{1}{z}\sum_{l=1}^{L} f(l)|M_{2l,\frac{1}{2}}(z)|^2 + \frac{1}{z}\sum_{l=L}^{\infty}\frac{1}{l}|M_{2l,\frac{1}{2}}(z)|^2 = \tag{3.109}$$
$$= C(L)z + 4z^2\sum_{l=L}^{\infty}\frac{1}{zl}\sum_{m,n} p_n^{(1)}(z)p_m^{(1)}(z)\frac{J_{1+n}(2\sqrt{2lz})J_{1+m}(2\sqrt{2lz})}{(2\sqrt{2lz})^{2+n+m}},$$

where $C(L)$ is a z-independent constant that depends logarithmically on L. To continue we note that $p_n^{(1)}(z)$ is bounded by z^n with $p_0^{(1)}(z) = 1$. Furthermore for $z \to 0$ we can replace the sum over l by an integral so that

$$\frac{1}{z}\sum_{l=1}^{\infty} f(l)|M_{2l,\frac{1}{2}}(z)|^2 \simeq Cz + 4z\sum_{m,n} z^{n+m}\int_{8zL}^{\infty}\frac{dx}{x}\frac{J_{2\mu+n}(\sqrt{x})J_{2\mu+m}(\sqrt{x})}{(\sqrt{x})^{2+n+m}}. \tag{3.110}$$

This integral is well defined and finite apart from a logarithmic divergence for $x \to 0$. The L-dependence between the first and second line cancels and we are left with

$$\frac{1}{z}\sum_{l=1}^{\infty} f(l)|M_{2l,\frac{1}{2}}(z)|^2 \quad \simeq \quad 4z(\log(z) + const + ..). \tag{3.111}$$

3.5.4 Spherical excitations with D0-charge

Let us now consider a D2-brane which is charged under the D0-branes are bounded to it, with $0 < M_0 << M_2$. In this case the interaction term at leading order in g is linear and we have absorption to the first excited state,$|1\rangle$. In particular, for the absorption of an s-wave dilaton perturbation

$$S_{int} \quad = \quad \frac{M_0}{M_2 g^2}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \delta\phi(t)\cosh(\chi_0)\delta\chi d\tau, \tag{3.112}$$

so that leading order absorption amplitude becomes

$$\langle 1|S_{int}|0\rangle \quad = \quad \frac{M_0\cosh(\chi_0)}{\sqrt{2}gM_2}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{i\omega(t-\frac{1}{r})}e^{-i\tau}d\tau. \tag{3.113}$$

This integral can again be brought in closed form with a suitable transformation of variables,

$$\langle 1|S_{int}|0\rangle = \sqrt{2}\frac{M_0\cosh(\chi_0)}{\sqrt{2}\,gM_2}\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} e^{i\omega Rv}\left(\frac{1}{1+iv}\right)^2 dv, \tag{3.114}$$

leading to

$$\begin{aligned}\langle 1|S_{int}|0\rangle \quad &= \quad -i\frac{M_0\cosh(\chi_0)}{gM_2}2\sqrt{2}\pi R\omega e^{-R\omega} \\ &\simeq \quad -i\frac{f}{gR^2}2\sqrt{2}\pi R\omega e^{-R\omega},\end{aligned} \tag{3.115}$$

where we have assumed $f/R^2 << 1$ (which is concluded from $0 < M_0 \ll M_2$). Proceeding as above we then obtain the AdS cross section

$$\sigma_{AdS_2} = \frac{f^2}{g^2R^4} R\omega e^{-2\omega R} . \qquad (3.116)$$

3.5.5 Dilaton with arbitrary angular momentum

So far we have only considered the absorption of *s-wave* dilaton, but this can be generalized to find the absorption cross-section of a dilaton with an arbitrary angular momentum on a probe brane with or with-out D0-charge, f. We compute the absorption of such a dilation for non-vanishing D0-charge case and also give a overall view of how to compute the absorption for uncharged D2-brane.

3.5.5.1 With D0-charge

Concretely we take the dilaton perturbation of the four space-time dilaton with arbitrary angular momentum

$$\delta\phi(t,\theta,\varphi) = Y_{lm}(\theta,\varphi)e^{i\omega(t\mp\frac{R^2}{r})} . \qquad (3.117)$$

For $f \neq 0$ the dilaton couples to the probe brane oscillation through the first order interaction

$$\begin{aligned} S_{int} &= \frac{f}{R^2g^24\pi} \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{S^2} \sqrt{-\tilde{g}}\, \delta\phi \left[\cosh(\chi_0)\delta\chi + \frac{(2\pi\alpha')^2}{2f^2}\sinh(\chi_0)F^{ab}f_{ab}\right] d\tau \\ &= \frac{f}{g^2R^2\pi} \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{S^2} \sqrt{-\tilde{g}}\, \delta\phi(t,\theta,\varphi) \left[2\frac{\sqrt{R^4+f^2}}{R^2}\delta\chi + \partial_0\psi\right] d\tau . \qquad (3.118) \end{aligned}$$

The leading order, non-vanishing components of the transition amplitude for absorption of a dilaton with angular momentum l,m into an a-type excitation is thus given by

$$\begin{aligned} \langle a; lm|S^{lm}_{int}|0\rangle &= \sqrt{R^4+f^2}\frac{f}{2gR^4\sqrt{l}} \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{i\omega(t-\frac{R^2}{r})}(2+l)e^{-il\tau}d\tau \\ &\simeq -\frac{f(2+l)i^l}{gR^2\sqrt{l}}\pi M_{l,\frac{1}{2}}(2R\omega) , \qquad (3.119) \end{aligned}$$

where we have ignored terms of order f^2 and higher when going from the first to the second line. Indeed we have argued above that for charged probe branes the absorption cannot be treated perturbatively unless $f << R^2$. Similarly we get for absorption into an b-type excitation

$$\langle b; lm|S^{lm}_{int}|0\rangle = -\frac{f(1-l)i^{l+1}}{gR^2\sqrt{l+1}}\pi M_{l+1,\frac{1}{2}}(2R\omega) + O(f^2) . \qquad (3.120)$$

Note that a naive application of (3.120) for $l=0$ leads to a $\sqrt{2}$ discrepancy with the earlier result (3.113). This apparent contradiction is resolved by recalling that for $l=0$ there is no a-mode so that there is an extra $\sqrt{2}$ in the normalization of (3.89). Note also that for $l=1$ the amplitude for absorption in to a b-type excitation vanishes.

The total cross section for the absorption of ϕ_{am} is the again obtained by adding the squares of the a- and b-type amplitudes (without summing over m). This gives

$$\sigma_{AdS_2} = \frac{f^2}{8gR\omega R^4}\left[\frac{(2+l)^2}{l}|M_{l,\frac{1}{2}}(2R\omega)|^2 + \frac{(1-l)^2}{(l+1)}|M_{l+1,\frac{1}{2}}(2R\omega)|^2\right], \tag{3.121}$$

which vanishes linearly for small ω.

3.5.5.2 Without D0-charge

To find the absorption cross-section for a dilaton with arbitrary angular momentum for vanishing D0-charge, $f=0$, we consider the dilaton perturbation of the form

$$\delta\phi(t,\theta,\varphi) = Y_{LM}(\theta,\varphi)e^{i\omega(t\mp\frac{R^2}{r})}\,. \tag{3.122}$$

taking (3.100), since now we have set of three spherical harmonic functions, the orthogonality condition we have used to write down (3.101) does not apply here, hence we have to keep summation over l_1, m_1, l_2, m_2 and for example $a^\dagger_{lm}a^\dagger_{l-m}$ replaces by $a^\dagger_{l_1m_1}a^\dagger_{l_2m_2}$ and so on.
The integral over sphere can be expressed by $3j$ *symbol* as

$$\int_{S^2} y^*_{m_1l_1}y^*_{m_2l_2}y_{LM} = (-1)^{l_1+l_2-L+M}\sqrt{\frac{(2l_1+1)(2l_2+1)(2L+1)}{4\pi}} \times\begin{pmatrix} l_1 & l_2 & L\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l_1 & l_2 & L\\ m_1 & m_2 & -M\end{pmatrix}$$

which, in turn can be calculated as a finite sum by using the *Racahformula* [30]

$$\begin{pmatrix} l_1 & l_2 & L\\ m_1 & m_2 & -M\end{pmatrix} = (-1)^{l_1-l_2+M}\sqrt{\Delta(l_1l_2L)} \tag{3.123}$$
$$\times\sqrt{(l_1-m_1)!(l_1+m_1)!(l_2-m_2)!(l_2+m_2)!(L-M)!(L+M)!}\times\sum_t\frac{1}{f(t)}$$

where $\Delta(l_1l_2L)$ is a triangle coefficient

$$\Delta(l_1l_2L) := \frac{(l_1+l_2-L)!(l_2+L-l_1)!(L+l_1-l_2)!}{(1+l_1+l_2+L)!} \tag{3.124}$$

and

$$f(t) := t!(L-l_2+m_1+t)!(L-l_1-m_2+t)!(l_1+l_2-L-t)!(l_1-m_1-t)!(l_2+m_2-t)! \tag{3.125}$$

note that sum goes over all integer values of t for which the arguments of factorials are non-negative.

3.5.6 Detailed calculation of absorption amplitude's integral

For completeness we include the detailed calculation of the integrals appeared in (3.102) and (3.113). The integrals are in the form of

$$\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} d\tau e^{i\omega(t-\frac{R^2}{r})} e^{-iN\tau} \tag{3.126}$$

where N is an arbitrary integer. We define variable x through

$$x := t - \frac{R^2}{r} = R\frac{\cosh\chi_0 \sin\tau - 1}{\cosh\chi_0\cos\tau + \sinh\chi_0}, , \tag{3.127}$$

Then we have

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} d\tau e^{i\omega(t-\frac{R^2}{r})} e^{-iN\tau} &= \oint dx e^{i\omega Rx} e^{-iN\tau(x)} \frac{d\tau}{dx} \\
&= 2A \oint dx e^{i\omega Rx} \frac{1}{1+x^2}\left(\frac{1-ix}{1+ix}\right)^N \\
&= 2A\frac{2\pi i}{N!}(-i)^{N+1}\left[e^{i\omega Rx}(1-ix)^{N-1}\right]^{(N)}_{x=i}, \\
&= (2\pi)\left(i\frac{1}{\cosh\chi_0} - \tanh\chi_0\right)^N \sum_{k=0}^{N-1}\binom{N-1}{k}\frac{(-2\omega)^{N-k}}{(N-k)!}e^{-\omega}.
\end{aligned} \tag{3.128}$$

where "(N)" denotes the number of derivatives and A is a constant

$$A = \left(\tanh\chi_0 - \frac{i}{\cosh\chi_0}\right)^N. \tag{3.129}$$

This sum which appears in the result of integral is of the form

$$\sum_{k=0}^{N}\binom{N}{k}\frac{1}{(N+1-k)!}(-2\omega)^{N+1-k}, \tag{3.130}$$

we can write the sum in terms of Hypergeometric function, ie.

$$\begin{aligned}
\cdots &= (-2\omega)\sum_{k=0}^{N}\frac{N!}{(N-k)!(N+1-k)!k!}(-2\omega)^{N-k} \\
&= (-2\omega)\sum_{p=0}^{N}\frac{N!}{(p)!(p+1)!(N-p)!}(-2\omega)^p \qquad p := N-k \\
&= (-2\omega)\sum_{p=0}^{N}\frac{(-N)_p}{(2)_p}\frac{(2\omega)^p}{p!} \\
&= (-2\omega)M(-N,2;2\omega) = -e^{\omega}M_{1+N,\frac{1}{2}}(2\omega),
\end{aligned} \tag{3.131}$$

where $M(a,b,;z)$ is Confluent Hypergeometric function which is related to Whittaker function, $M_{\lambda,\mu}(z)$, by [28]

$$M_{\lambda,\mu}(z) = z^{\mu+\frac{1}{2}} e^{-z/2} M\left(\mu - \lambda + \frac{1}{2}, 2\mu + 1; z\right) \tag{3.132}$$

Therefore (3.126) has a solution as

$$\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} d\tau e^{i\omega(t-\frac{1}{r})} e^{-iN\tau} = -(2\pi)\left(i\frac{1}{\cosh\chi_0} - \tanh\chi_0\right)^N M_{N,\frac{1}{2}}(2\omega)\,. \tag{3.133}$$

3.6 Discussion

In this chapter we reviewed the Calabi-Yau black holes arise in IIA string theory and particularly we concerned with the special case of D4-D0 black hole. Afterward, by looking at the entropy, we explained why we are interested in the supersymmetric black holes and how they provide us the capability of identifying and counting the microstates of black hole and examining its equality to the macroscopic entropy.

As the main point, we have obtained analytic expressions for the low energy absorption cross-section of space-time scalars on the horizon wrapped D2-branes, static in global coordinates of the near horizon AdS_2 geometry. The fact that these amplitudes can be computed exactly may come as a surprise since the probe two-brane describes a complicated trajectory in the asymptotic Poincaré coordinates.

An interesting feature is that although the Hamiltonian of the D2-brane has a discrete spectrum with spacing given by the inverse of the radius of the horizon the D2-brane can absorb arbitrarily small frequencies with respect to an asymptotic observer. We should mention that we only considered the bosonic sector of the world-volume theory. However, it is not hard to see that fermions give a vanishing contribution at the lowest (quadratic) level. Also we have not considered fixed scalars in this paper although their inclusion should be straight forward. As mentioned before, the details of the absorption process described here are qualitatively different from the world-volume absorption on D-branes in flat space. In flat space, the low energy behavior is dominated by goldstone bosons and possible other massless fields whereas here no massless degrees of freedom are present. The fact that the cross section vanishes linearly for $\omega \to 0$ is due precisely to the absence of massless degrees of freedom.

In view of a possible interpretation for a dual interpretation of 4-dimensional CY-black holes in terms of the quantum mechanics of probe D2-branes wrapped on the S^2 of their near horizon geometry an encouraging result would have been to find agreement for the low energy absorption cross section on both sides. Our concrete calculation shows however that this is not the case since the microscopic absorption cross section on the two-brane does not have the correct behavior at small frequencies compared to the classical absorption cross section of massless scalars which vanishes quadratically in ω. However, the comparison between these two calculations is more subtle. First we note that our absorption cross section was computed for wrapped D2-branes with small D0-charge whereas the dominant contribution to the entropy comes from wrapped branes with large D0-charge. It would be desirable to know the result in that case although the calculation appears to be more involved since, as we showed, linearized perturbation theory breaks down in this situation. A possible application of the present calculation is to interpret a single wrapped D2-brane with small D0-charge as a small non-extremal perturbation of the extremal black hole. This is sensible since for small D0-charge the two-brane is confined to the near horizon AdS_2 geometry of the asymptotically flat global geometry. In this case we should compare the classical absorption cross section for the near extremal black hole with the the product of the transmission coefficient from asymptotically flat space to the near horizon AdS_2 region and the AdS_2 absorption cross section computed in this paper. The microscopic cross section obtained in this way vanishes like $(R\omega)^3$. So we still have disagreement. One possible explanation for the disagreement could be that there are microscopic configurations, other than the wrapped D2-branes considered here, correspond to a non-extremal black hole which reproduce the correct low energy behavior. One such generalization is to consider multi-branes wrapping horizon, however this does not change qualitative small frequency behavior. Another possibility is to consider the scattering of massless space-time scalers on individual probe D0-branes. However in that case the brane absorption amplitude vanishes due to energy-momentum conservation.

CHAPTER 4

Supersymmetric Branes in Calabi-Yau Black hole

4.1 Introduction

In previous chapter, we saw that the extremal black holes in string theory play an important role to understand the precise microscopic description of four dimensional black holes obtained by the compactification of type IIA string theory on Calabi-Yau three-fold X. There, we restricted ourselves to the case where only the D4-D0 charges were non-zero on the background geometry. In general, we would like to get a precise microscopic description of a four dimensional black hole with all possible charges of IIA theory, namely D6-D4-D2-D0 charges, all non-vanishing.

This is actually an outstanding problem in studying the extremal black holes in string theory and their relation to the conformal field theories to get a precise microscopic description of four-dimensional black holes with D6-D4-D2-D0-charges obtained by the compactification of type IIA string theory on a Calabi-Yau three-fold X. The charges are due to the D-branes completely wrapped on the non-trivial cycles of X. For generic charges one expects that this black hole geometry to be dual to some conformal quantum mechanics on the boundary of the near horizon AdS_2 geometry.

For vanishing D6-charge the geometric entropy of such black holes can be given a microscopic understanding upon lifting this solution to M-theory. From the M-theory perspective this class of black holes are obtained by wrapping M5-branes with fluxes and momentum along the M-theory circle and on a four-cycle in X. The corresponding near-horizon geometry is dual to some $1+1$-dimensional conformal field theory which lives on the dimensionally reduced five-brane world-volume [26]. This observation allowed the authors of [26] to derive the asymptotic degeneracy of states using standard methods of conformal field theory. Upon compactification to IIA theory the near horizon geometry obtained is $\mathrm{AdS}_2 \times S^2$. For this model a candidate for a dual quantum mechanics for the D4-D0 black hole has been proposed [7] in terms of the degrees of freedom of probe D0-branes in this background (see also [31, 32] for related discussions).

When the D6-brane charge is non-zero, the 4D black hole in IIA compactified on the Calabi-Yau may be lifted to a 5D black hole in M-theory on $CY_3 \times TN_{p^0}$, where TN denotes the four-dimensional Euclidean Taub-NUT space with charges p^0. While for large distances compared to the size of the asymptotic Taub-NUT circle these black hole geometries asymptotes to $\mathbb{R}^3 \times S^1$, and therefore are effectively four-dimensional, the M-theory circle near the horizon is proportional to the size of the horizon so that the M-theory perspective is more appropriate. Essentially, the near horizon geometry is a five-dimensional spinning black hole sitting at the center of the Taub-NUT geometry, where the black hole is assumed to sit. This 4D-5D connection has been exploited in [33] to relate a certain partition function of a class of four-dimensional black holes to that of five-dimensional black holes. The near horizon-geometry in five dimensions is essentially $\mathrm{AdS}_2 \times S^3/\mathbb{Z}_{p^0}$. Consequently for non-vanishing D6-charge the problem is not directly related to a $1+1$-dimensional CFT [1].

On the other hand it appears that the supersymmetric probe branes in the near horizon geometry play a role in understanding the dual quantum mechanics and the black hole partition function. They can be thought of as the "constituents" of the black hole in question. The asymptotic degeneracy of the electric constituents in the background flux geometry supported by the magnetic charges accounts for the black hole entropy in some cases [7, 32]. For instance it has been shown in [7] that the ground state degeneracy of D0-branes in a D4-brane flux background reproduces the correct asymptotic degeneracy for D4-D0 charge black holes. Here the relevant Hamiltonian is the one of conformal quantum mechanics on the moduli space of D0-branes in the flux background. An important subtlety is though that the appropriate Hamiltonian appears to be that which generates translation in "global" time rather than Poincaré time which coincides with asymptotic time[2]. The dominant contribution to the entropy comes from D0-branes bound to two-branes wrapping the horizon of the black hole.

On another front it has been shown [37] that the elliptic genus of the $(0,4)$-CFT [26] dual to black holes with D4-D2-D0-charge has a dilute gas expansion dominated by multi-particle chiral primaries which are just the stationary M2 (and anti-M2) branes in global AdS_3-coordinates, wrapped on holomorphic curves in the Calabi-Yau and sitting at the center of AdS_3. This provides a derivation of the OSV-conjecture relating the mixed partition function of the black hole to the square of the topological string partition function [38].

In what follows we prepare the ground for extending the above-mentioned results [7, 37] to black holes with D6-charge by describing the supersymmetric probe branes in D6-charge backgrounds [9]. Of course, in this case we do not have a known "parent" $1+1$-dimensional CFT to compare the probe-brane degeneracies with. Nevertheless one can hope that understanding the degeneracies of these states will give some insight about the underlying microscopic theory. In Section 2 we will describe the eleven-dimensional near horizon geometry of a 4D black hole with generic D6-D4-D2-D0 charge. While the full space-time geometry of a generic black hole with D6-charge is a solution of five-dimensional $\mathcal{N}=2$ supergravity with n_v-1 vector-multiplets, the attractor mechanism

[1] In fact it has been argued in [34] that black holes with D6-charge are related through a chain of string dualities to BPS states without D6-charge. A similar relation was also conjectured in [35] based on an embedding of space-time in the total space of the $U(1)$-gauge bundle over near horizon geometry of the black hole. It would be interesting to see how these two approaches are related.

[2] The Poincare-Hamiltoinian has a continuos spectrum with no ground state as a consequence of the incompleteness of classical dynamics in Poincaré coordinates [36].

ensures that its near-horizon geometry is equivalently described in terms of $\mathcal{N}=2$ supergravity with just one vector multiplet - the graviphoton, i.e. minimal supergravity in five dimensions. A classification of the solutions of minimal supergravity in five dimensions can be found in [39]. This property simplifies the task of finding the relevant Killing spinors for these black holes. In Section 3 we obtain the near horizon killing spinor in global coordinates and analyze the κ-symmetry for an stationary probe branes in global time along the lines of [20, 40]. In particular we find BPS two-branes wrapped on a holomorphic two-cycle in the Calabi-Yau. These correspond to the zero-branes found in [40] and have the right properties to be the relevant degrees of freedom for deriving the OSV-relation for black holes with D6-charge. In addition we find BPS five-branes which wrap either a holomorphic four-cycle in the Calabi-Yau and an S^1 in space-time or wrap the horizon $S^3/\mathbb{Z}_{p^0}$ completely and a holomorphic two-cycle in the Calabi-Yau. These may play a role analogous to the horizon wrapped two-branes for D4-D0-black holes [7]. We plan to report on these issues in subsequent work.

4.2 Supersymmetric properties of the branes

The world-volume action of D-branes have two symmetries. First, there is κ-symmetry which is a local symmetry and, second, global space-time supersymmetry which parametrized by Killing spinors ε.

In general, a brane configuration trajectory will preserve the space-time supersymmetry if the action on the world-volume fermions can be compensated by the κ transformation. In order for the D-brane to be supersymmetric we only need to check that the κ-symmetry condition

$$\Gamma\varepsilon = \varepsilon \tag{4.1}$$

is satisfied. ε is the pull-back of Killing spinor on to the brane world-volume, corresponding to the unbroken supersymmetry and Γ is called the κ projection operator. The expression for κ projection matrix, Γ, depends on the embedding map from the world-volume of brane into the space-time. For any Dp-brane, Γ is given by

$$\Gamma = \frac{\sqrt{\det h}}{\sqrt{\det(G+\mathscr{F})}}\sum_n \frac{1}{2^n n!}\Gamma^{\hat{\mu}_1\hat{\nu}_1\ldots\hat{\mu}_n\hat{\nu}_n}\mathscr{F}_{\hat{\mu}_1\hat{\nu}_1\ldots\hat{\mu}_n\hat{\nu}_n}\Gamma_{(11)}^{n+\frac{p-2}{2}}\Gamma_{(0)} \tag{4.2}$$

$$\Gamma_{(0)} = \frac{1}{(p+1)!\sqrt{\det\hat{h}}}\varepsilon^{\hat{\mu}_0\ldots\hat{\mu}_p}\Gamma_{\hat{\mu}_0\ldots\hat{\mu}_p} \tag{4.3}$$

where the hatted indices are the world-volume coordinates and $\hat{h}$ is the determinant of the pull-back of the space-time metric to the world-volume, $h_{\hat{a}\hat{b}}$, and $\mathscr{F}=\mathscr{F}+\mathscr{B}$. Killing spinors are the solution of Killing spinor equations, which are the equations of vanishing supersymmetry transformations for the gravitino. The Killing spinor equation in eleven dimensions is given by

$$
\begin{aligned}
0 &= \left[\nabla_M + \frac{1}{288}\left(\Gamma_M{}^{N_1N_2N_3N_4} - 8\delta_M^{N_1}\Gamma^{N_2N_3N_4}\right)G_{N_1N_2N_3N_4}\right]\varepsilon_{11} \\
\nabla_M &= \partial_M + \frac{1}{4}\omega_{MAB}\Gamma^{AB} \qquad \text{and} \qquad G = \mathrm{d}C^{[3]}\,, \qquad (4.4)
\end{aligned}
$$

where the capital indices run from zero to ten and ω_{MBA} is the spin connection. Since we are interested in black holes in lower dimensions, we would like to write the ten dimensional spinor,ε_{11}, as the tensor product of internal and non-compact spinors. In case of M-theory, this is to write the spinor as

$$
\varepsilon_{11} = \varepsilon \otimes \eta \qquad (4.5)
$$

where, respectively, ε and η are the four dimensional and the internal (Calabi-Yau) spinors. The Killing equation on the Calabi-Yau is solved by the covariant constant spinors $\eta_\pm$. We have chosen the following conventions: the γ-matrices on the X w.r.t. an orthonormal frame we denote by $\rho^{\mathbf{i}}$ with

$$
\{\rho^{\mathbf{i}}, \rho^{\mathbf{j}}\} = 2\delta^{\mathbf{ij}} \qquad (4.6)
$$

The spinors $\eta_\pm$ obey the relations

$$
\begin{aligned}
\rho_{(7)}\eta_\pm &= \pm\eta_\pm\,, \\
\rho_{\bar{i}}\eta_+ = 0, \qquad & \rho_i\eta_- = 0\,, \qquad (4.7)
\end{aligned}
$$

where i and $\bar{i}$ are indices w.r.t. complex coordinates on X. The γ-matrices of five-dimensional space w.r.t. an orthonormal frame we denote by γ^a, such that the eleven-dimensional γ-matrices Γ^M, $M = 0, ..., 11$, decompose the Γ^M into a tensor product of five and six-dimensional γ-matrices,as

$$
\begin{aligned}
\Gamma^a &= \gamma^a \otimes \rho_{(7)}\,, \qquad & a = 0, ..., 4 \qquad (4.8) \\
\Gamma^{\mathbf{i}} &= \gamma_{(5)} \otimes \rho^{\mathbf{i}} = \mathbb{1} \otimes \rho^{\mathbf{i}}\,, \qquad & i = 5, ..., 11
\end{aligned}
$$

where in the second line we have used that $\gamma_{(5)} = \mathrm{i}\gamma^{01234} = \mathbb{1}$. Decomposition of spinors leads to the separation of Killing spinor equations and to treat their corresponding solutions independently. The five-dimensional space-time supersymmetry arises from the solutions of Killing spinor equation are then

$$
\left[\partial_M + \frac{1}{4}\omega_{MRP}\Gamma^{RP} - \frac{i}{4\sqrt{3}}\left(F_{AB}\Gamma_M^{AB} - 4F_{MN}\Gamma^N\right)\right]\varepsilon = 0 \qquad (4.9)
$$

ε is a five-dimensional spinor. This leads to the five-dimensional equations and determines the Killing spinor ε. So the recipe for recognizing if a D-brane preserves some supersymmetry, is to find the Killing spinor and also determine the expression for Γ and see if the κ-symmetry condition (4.1) holds.

4.3 Supersymmetric branes in M-theory attractor geometries

4.3.1 Near horizon geometry of D6-D4-D2-D0 black holes

In order to be self-contained and to fix the conventions we first review the relevant static half BPS solutions of four-dimensional $\mathcal{N}=2$ supergravity with n_v vector multiplets [2]. The general stationary BPS configurations were derived in [41, 42, 43]. We then describe the lift of these solutions to five dimensions [33] and determine the near horizon geometry for a given set of four-dimensional charges.

We consider static single-centered BPS solutions in four dimensions. These solutions are characterized by their asymptotic magnetic and electric charges (p^I, q_I), $I=0,\ldots,n_v$ and their asymptotic moduli. As such they are completely determined in terms of $2n_v+2$ real harmonic functions on $\mathbb{R}^3$

$$H^I(r) = h^I + \frac{p^I}{r}\,, \qquad H_I(r) = h_I + \frac{q_I}{r}\,, \tag{4.10}$$

subjected to the condition

$$p^I h_I - q_I h^I = 0\,. \tag{4.11}$$

The corresponding metric is given by

$$\mathrm{d}s^2_{(4)} = -\frac{\pi}{S(r)}\mathrm{d}t^2 + \frac{S(r)}{\pi}\mathrm{d}\vec{x}^2\,. \tag{4.12}$$

The function S can be expressed as

$$S = 2\pi\sqrt{H^0 Q^3 - (H^0 L)^2}\,, \tag{4.13}$$

with

$$\begin{aligned} L &= \frac{H_0}{2} + \frac{H^A H_A}{2H^0} + \frac{C_{ABC}H^A H^B H^C}{6(H^0)^2}\,, \\ Q^{3/2} &= \frac{1}{6}C_{ABC}y^A y^B y^C\,. \end{aligned} \tag{4.14}$$

Here $A,B,C \in \{1,\ldots,n_v\}$ and y^A are implicitly determined by the equation

$$C_{ABC}y^B y^C = 2H_A + \frac{C_{ABC}H^B H^C}{H^0}\,. \tag{4.15}$$

The gauge potentials are determined again by the harmonic functions (4.10) and $S(r)$ (4.13)

$$A^I_{(4)} = \frac{1}{S}\frac{\partial S}{\partial H_I}\mathrm{d}t + \mathscr{A}^I\,, \qquad \mathrm{d}\mathscr{A}^I = *_3\mathrm{d}H^I\,. \tag{4.16}$$

To complete the four-dimensional description of generic D6-D4-D2-D0 attractor black holes we give the complex scalar fields

$$t^A = \frac{H^A + \frac{\mathrm{i}}{\pi}\frac{\partial S}{\partial H_A}}{H^0 + \frac{\mathrm{i}}{\pi}\frac{\partial S}{\partial H_0}}\,. \tag{4.17}$$

As mentioned above for generic values of these charges the string coupling becomes large in the near horizon regime. To allow for a unified description for generic charges we now give the lift of these solutions to five dimensions. For a nice discussion of this lift see for example [44].

The five dimensional metric is given by

$$\begin{aligned} ds^2_{(5)} &= 2^{2/3}\mathscr{V}^2(d\psi + A^0_{(4)})^2 + 2^{-1/3}\mathscr{V}^{-1}ds^2_{(4)} \\ &= -(2^{2/3}Q)^{-2}(dt + 2L(d\psi+\mathscr{A}^0))^2 + (2^{2/3}Q)(\frac{1}{H^0}(d\psi+\mathscr{A}^0)^2 + H^0 d\vec{x}^2), \end{aligned} \tag{4.18}$$

with

$$\mathscr{V} = \left(\frac{1}{6}C_{ABC}\Im t^A \Im t^B \Im t^C\right)^{1/3} = \frac{S}{2\pi H^0 Q}. \tag{4.19}$$

The four and five dimensional gauge potentials are related by

$$\begin{aligned} A^A_{(5)} &= \Re t^A (d\psi + A^0_{(4)}) - A^A_{(4)} \\ &= -\frac{Y^A}{2Q}dt + \left(\frac{H^A}{H^0} - \frac{L}{Q}Y^A\right)(d\psi + \mathscr{A}^0) - \mathscr{A}^A \end{aligned} \tag{4.20}$$

where we introduced the five dimensional scalars

$$Y^A = \frac{y^A}{Q^{1/2}}. \tag{4.21}$$

They obey the relation

$$\frac{1}{6}C_{ABC}Y^A Y^B Y^C = 1. \tag{4.22}$$

Let us now take the near horizon limit, $r \ll \{p^I, q_I\}$, of the five-dimensional solution. For this we define

$$\begin{aligned} \sigma &= \frac{1}{r}, \\ d\Omega_3^2 &= d\theta^2 + \sin^2\theta d\phi^2 + (d\psi/p^0 + \cos\theta d\phi)^2, \\ R^2_{\mathrm{AdS}} &= \lim_{r\to 0}\left(2^{2/3}H^0 Q\right), \\ J &= \lim_{r\to 0}\left(\frac{(H^0)^{1/2}L}{Q^{3/2}}\right) \qquad \text{and} \\ Y_0^A &= \lim_{r\to 0} Y^A. \end{aligned} \tag{4.23}$$

Then, rescaling t appropriately (denoted again by t) we obtain [45]

$$ds^2_{(5)} = R^2_{\mathrm{AdS}}\left(-\left(\frac{dt}{\sigma} + J(d\psi/p^0 + \cos\theta d\phi)\right)^2 + \frac{d\sigma^2}{\sigma^2} + d\Omega_3^2\right), \tag{4.24}$$

$$A^A_{(5)} = -\frac{Y_0^A 2^{2/3}}{\sqrt{3}}A + \frac{p^A}{p^0}d\psi, \tag{4.25}$$

$$A = \frac{\sqrt{3}}{2}R_{\mathrm{AdS}}\left(\frac{dt}{\sigma} + J(d\psi/p^0 + \cos\theta d\phi)\right). \tag{4.26}$$

The near horizon geometry depends on three parameters: the D6 charge p^0, the AdS_2 radius $R_{\rm AdS}$ which is determined by the value of Q at the horizon and the five-dimensional angular momentum J. These are also the quantities that appear in the Bekenstein-Hawking entropy. In other words, as pointed out in [33] (see also [46]) the Taub-NUT fibration, which interpolates between the four-dimensional and the five-dimensional geometry gives a simple geometric representation of the entropy formula for the 4D-entropy of D6-D4-D2-D0 black holes based on special geometry [47]. For $J \to 1$ a closed light-like curve develops. This singular limit has recently been analyzed in [35].

4.3.2 Global coordinates and half BPS-branes, five-dimensional analysis

Next we introduce global coordinates. For this we start with the expressions (4.24) and (4.26) for the metric and gauge field respectively and change coordinates as ($\sin B := J$)

$$\begin{aligned} t &= \frac{\cos B \cosh\chi \sin\tau}{\cosh\chi\cos\tau + \sinh\chi}\,, \\ R_{\rm AdS}\,\sigma &= \frac{1}{\cosh\chi\cos\tau + \sinh\chi}\,, \\ \psi^{poinc} &= \psi + 2\tan B \tanh^{-1}(e^{-\chi}\tan(\frac{\tau}{2}))\,. \end{aligned} \tag{4.27}$$

The metric and field strength of the graviphoton then take the form

$$\begin{aligned} \mathrm{d}s^2_{(5)} &= R^2_{\rm AdS}\left(-\cosh^2\chi \mathrm{d}\tau^2 + \mathrm{d}\chi^2 + (\sin B \sinh\chi \mathrm{d}\tau - \cos B \sigma_3)^2 + \mathrm{d}\Omega_2^2\right)\,, \\ F &= \frac{\sqrt{3}}{2} R_{\rm AdS}\left(\cos B \cosh\chi \mathrm{d}\chi\wedge\mathrm{d}\tau - \sin B \sin\theta \mathrm{d}\theta\wedge\mathrm{d}\phi\right)\,, \end{aligned} \tag{4.28}$$

where $\mathrm{d}\Omega_2^2$ is the line element of the unit two-sphere and

$$\sigma_3 = \mathrm{d}\psi/p^0 + \cos\theta\mathrm{d}\phi\,.$$

It is straightforward to lift this near horizon geometry to eleven dimensions. The lifted geometry is a direct product of the five-dimensional space and a Calabi-Yau three-fold X with Kähler form $Y_0^A\omega_A$ where $\omega_A \in H^2(X,\mathbb{Z})$ is a basis. The three-form $C^{[3]}$ in eleven dimensions is proportional to the wedge product of the gauge field (4.26) with the Kähler form. Now to derive the Killing spinors from equations (4.9) we need first to have the fünf-beine and the spin connections. With a convenient choice of the fünf-beine

$$\begin{aligned} e^0 &= \cosh\chi\mathrm{d}\tau, \qquad e^1 = \mathrm{d}\chi\,, \\ e^2 &= \mathrm{d}\theta, \qquad e^3 = \sin\theta\mathrm{d}\phi\,, \\ e^4 &= \cos B\left(\frac{\mathrm{d}\psi}{p^0} + \cos\theta\mathrm{d}\phi\right) - \sin B\sinh\chi\mathrm{d}\tau\,, \end{aligned} \tag{4.29}$$

The spin connections are then given by

$$
\begin{aligned}
\omega_{34} &= -\frac{1}{2}\cos B e^2, \qquad \omega_{24} = \frac{1}{2}\cos B e^3 \\
\omega_{14} &= \frac{1}{2}\sin B e^0, \qquad \omega_{04} = -\frac{1}{2}\sin B e^1 \\
\omega_{23} &= -\frac{\cos\theta}{\sin\theta} e^3 + \frac{1}{2}\cos B e^4 \\
\omega_{01} &= -\frac{\sinh\chi}{\cosh\chi} e^0 - \frac{1}{2}\sin B e^4 \\
\omega_{12} &= \omega_{02} = \omega_{13} = \omega_{03} = 0
\end{aligned}
\tag{4.30}
$$

Now, to solve the Killing equation we substitute (4.30) and (4.28) in the Killing spinor equation 4.4. The five-dimensional part of the Killing equation becomes

$$
\begin{aligned}
0 &= \partial_\psi \varepsilon\,, \\
0 &= \left(\partial_\phi + \frac{1}{2}\cos B\sin\theta\gamma^{24} - \frac{1}{2}\cos\theta\gamma^{23} + \frac{i}{2}\sin B\sin\theta\gamma^2\right)\varepsilon\,, \\
0 &= \left(\partial_\theta - \frac{1}{2}\cos B\gamma^{34} - \frac{i}{2}\sin B\gamma^3\right)\varepsilon\,, \\
0 &= \left(\partial_\tau + \frac{1}{2}\sin B\cosh\chi\gamma^{14} - \frac{1}{2}\sinh\chi\gamma^{01} - \frac{i}{2}\cosh\chi\cos B\gamma^1\right)\varepsilon\,, \\
0 &= \left(\partial_\chi - \frac{1}{2}\sin B\gamma^{04} + \frac{i}{2}\cos B\gamma^0\right)\varepsilon\,.
\end{aligned}
\tag{4.31}
$$

These equations are solved by (see also [40])

$$
\begin{aligned}
\varepsilon &= S(B,\chi,\tau,\theta,\phi)\varepsilon_0\,, \\
S(B,\chi,\tau,\theta,\phi) &= e^{-\frac{i}{2}B\gamma^4} e^{-\frac{i}{2}\chi\gamma^0} e^{\frac{i}{2}\tau\gamma^1} e^{\frac{1}{2}\theta\gamma^{34}} e^{\frac{1}{2}\phi\gamma^{23}}
\end{aligned}
\tag{4.32}
$$

and ε_0 is an arbitrary, constant four-component spinor.

We find the BPS configurations of M-branes that are wrapped on compact portions of our background and, possibly, also (in case of M5-brane) have components tangent to the horizon, and are point-like in the AdS space. In the following we classify the stationary supersymmetric probe branes in this background.

This implies in particular that they are static in AdS$_2$, i.e. $\dot\chi = 0$ but allows for M2-branes orbiting around the three-dimensional horizon as well as M5-branes partially or fully wrapping the horizon.

4.3.3 Half BPS M2-branes in global coordinates

We begin with the set of stationary, supersymmetric M2-branes wrapping a holomorphic two-cycle in X. The κ-symmetry condition is [18]

$$
\Gamma\,\varepsilon\otimes\eta = \varepsilon\otimes\eta \tag{4.33}
$$

with

$$\begin{aligned}\Gamma &= \frac{1}{(p+1)!\sqrt{\det h}}\varepsilon^{\hat{a}\hat{b}\hat{c}}\Gamma_{\hat{a}\hat{b}\hat{c}} \\ &= \frac{1}{\sqrt{h_{00}}}\frac{\mathrm{d}X^{\mu}}{\mathrm{d}\tau}e^{a}_{\mu}\gamma_a\otimes \mathrm{i}\mathbb{1}\,, \end{aligned} \tag{4.34}$$

where we have assumed that the M2-brane has positive orientation. The hatted indices are the world-volume coordinates and $h_{\hat{a}\hat{b}}$ is the pull-back of the space-time metric to the world-volume. The second line is expressed in static gauge $\dot{X}^0 = 1$. Having the form of Killing spinor form (4.32) and plug it in the κ-symmetry condition (4.33) the BPS condition is then

$$S^{-1}\Gamma S\,\varepsilon_0\otimes\eta \;=\; \varepsilon_0\otimes\eta \tag{4.35}$$

We need to know the component of $S^{-1}\Gamma S$ which are of the form $S^{-1}e^{a}_{b}\gamma_a S$ for different values of b. The explicit prefactors are[3]

$$\begin{aligned}
S^{-1}\,e^{a}{}_{0}\,\gamma_a\,S &= \cos B\cosh\chi\cos\tau\gamma_0 + i\sin B\cos\theta\gamma^{04} + i\cos B\cosh\chi\sin\tau\gamma^{10}\\
&\quad - i\sin B\sin\theta(\cos\phi\gamma^{03}-\sin\phi\gamma^{02})\\[1em]
S^{-1}\,e^{a}{}_{1}\,\gamma_a\,S &= \cos B\cosh\chi\gamma_1 + \sin B\sin\tau\sin\theta(\cos\phi\gamma_3-\sin\phi\gamma_2) - \sin B\sin\tau\cos\theta\gamma_4\\
&\quad + \cos B\sinh\chi\sin\tau\gamma_0 - i\sin B\cos\tau\cos\theta\gamma^{14} - i\cos B\sinh\chi\cos\tau\gamma^{10}\\
&\quad + i\sin B\cos\tau\sin\theta(\cos\phi\gamma^{13}-\sin\phi\gamma^{12})\\[1em]
S^{-1}\,e^{a}{}_{2}\,\gamma_a\,S &= \cos B\cosh\chi\cos\tau(\cos\phi\gamma_2+\sin\phi\gamma_3) - i\sin B\sin\theta\gamma^{32}\\
&\quad + i\cos B\sinh\chi(\cos\phi\gamma^{02}+\sin\phi\gamma^{03}) + i\sin B\cos\theta(\cos\phi\gamma^{42}+\sin\phi\gamma^{43})\\
&\quad - i\cos B\cosh\chi\sin\tau(\cos\phi\gamma^{12}+\sin\phi\gamma^{13})\\[1em]
S^{-1}\,e^{a}{}_{3}\,\gamma_a\,S &= \cos B\cosh\chi\cos\tau\gamma_4 - i\cos B\cosh\chi\sin\tau\gamma^{14} + i\cos B\sinh\chi\gamma^{04}\\
&\quad + i\sin B\sin\theta(\cos\phi\gamma^{43}-\sin\phi\gamma^{42})\\[1em]
S^{-1}\,e^{a}{}_{4}\,\gamma_a\,S &= \frac{\cos B}{p_0}\{\cosh\chi\cos\tau\cos\theta\gamma_4 - \cosh\chi\cos\tau\sin\theta(\cos\phi\gamma_3-\sin\phi\gamma_2)\\
&\quad - i\cosh\chi\sin\tau\cos\theta\gamma^{14} + i\cosh\chi\sin\tau\sin\theta(\cos\phi\gamma^{13}-\sin\phi\gamma^{12})\\
&\quad + i\sinh\chi\cos\theta\gamma^{04} - i\sinh\chi\sin\theta(\cos\phi\gamma^{03}-\sin\phi\gamma^{02})\}
\end{aligned} \tag{4.36}$$

Let us first consider the case where the two-brane sits at fixed θ and B but rotates in the ϕ direction. For $\dot{\phi}\neq\pm1$ the BPS condition can never be satisfied. For $\dot{\phi}=1$ we have $\sqrt{|h_{00}|} = |\cos(B)\sinh(\chi)+\sin(B)\cos(\theta)|$ and

$$\Gamma_{(0)} := \frac{\mathrm{d}X^{\mu}}{\mathrm{d}\tau}e^{a}_{\mu}\gamma_a \;=\; -\cosh\chi\gamma^0 + \sin\theta\gamma^3 - \sin B\sinh\chi\gamma^4 + \cos B\cos\theta\gamma^4 \tag{4.37}$$

[3]In Appendix B we provide the a list of γ-matrix identities which are useful for computing the prefactors of κ-symmetry condition

and

$$\begin{aligned}S^{-1}\Gamma_{(0)}S &= \cos B\cosh\chi\cos\tau(-\gamma^0+\gamma^4)+i(\cos B\sinh\chi+\sin B\cos\theta)\gamma^{04}\\ &\quad +i\cos B\cosh\chi\sin\tau(\gamma^{10}-\gamma^{14})-i\sin B\sin\theta(\cos\phi\gamma^{03}-\sin\phi\gamma^{02})\\ &\quad +i\sin B\sin\theta(\cos\phi\gamma^{43}-\sin\phi\gamma^{42})\,. \end{aligned} \tag{4.38}$$

Requiring the κ-symmetry condition be independent of τ implies

$$\gamma^{04}\varepsilon_0=-\varepsilon_0\,. \tag{4.39}$$

Furthermore if the latter condition is fulfilled we have

$$\Gamma\varepsilon\otimes\eta = \left(\frac{i\Gamma_{(0)}}{\sqrt{h_{00}}}\varepsilon\right)\otimes\eta=\varepsilon\otimes\eta \tag{4.40}$$

which is just the BPS condition (4.33). These solutions correspond to the zero branes found in [40]. Note that this brane is BPS for $\dot\phi=1$ while sitting at the north pole $\theta=0$ on the base S^2. This does not mean that this brane is static. Indeed, as the velocity along the fiber is given by $\dot\psi/p^0+\cos\theta\dot\phi$, this configuration is geometrically equivalent to that with $\dot\phi=0$ and $\dot\psi=p^0$ which is a trajectory along the fiber of the $S^3/\mathbb{Z}_{p^0}$-bundle, i.e. a "great circle" on $S^3/\mathbb{Z}_{p^0}$. If we assume instead that $\phi=const,\theta=const$, then $\dot\psi=p^0$ necessarily and the BPS condition reads

$$[-\cos(\theta)\gamma^{04}+\sin(\theta)(\cos(\phi)\gamma^{03}-\sin(\phi)\gamma^{02})]\varepsilon_0 = \varepsilon_0\,. \tag{4.41}$$

So the M2-brane can move along the fiber with constant velocity p^0 and sits at any point of the base space S^2. The condition (4.41) reduces to (4.39) for $\theta=0$.

For $\dot\psi=0$, $\phi=\phi_0$ ($\phi=\phi_0+\pi$) constant and necessarily $\dot\theta=1$ ($\dot\theta=-1$) the BPS condition becomes

$$(\cos(\phi)\gamma^{20}+\sin(\phi)\gamma^{30})\varepsilon_0=\varepsilon_0. \tag{4.42}$$

Geometrically, this is the case where the M2-brane moves along a meridian of the base S^2 with constant velocity one and does not move along the fiber of the $S^3/\mathbb{Z}_{p^0}$-bundle over S^2.

To summarize, an M2-brane on C_2 is BPS if and only if it rotates with unit angular velocity on the covering space S^3. For $\chi=0$ they describe uncharged null-geodesics on S^3, while for $\chi>0$ the $M2$-branes are charged and follow a time-like trajectory [4]. The rotation is required to stabilize them at fixed χ. This interpretation is compatible with the four-dimensional analysis in [20] where it was observed that existence of static half-BPS branes requires that the symplectic product of the charge vector of the probe brane with the background charges does not vanish. Let us consider rotation along the fiber first. Then, since the above results for the wrapped M2-branes are independent of B we can consider vanishing B. In this case the non-vanishing of the symplectic product in four dimensions

[4] See [40] for a detailed analysis of the corresponding Born-Infeld action.

requires that the two-brane rotates along the fiber. Invoking rotational invariance on S^3 we then conclude that rotation along any geodesic circle of the S^3 will lead to a half BPS-state.

Note also that a M2-brane sitting for example at $\theta = 0$ and rotating in the fiber with $\psi = p^0$ preserves the same supersymmetry as an anti-M2-brane (i.e. negative orientation) at $\theta = \pi$ and $\psi = p^0$. Thus these branes are mutually BPS. This property makes them natural candidates to extend the observation of [37] to black holes with D6-charge.

4.3.4 Half BPS five-branes

We now consider static M5-branes which partially, or fully wrap the horizon of the five-dimensional black hole. The remaining dimensions of the five-brane are wrapped on a holomorphic cycle in X.

4.3.4.1 M5 on $C_4 \times Y$

For the M5-brane just to wrap the horizon partially, we wrap four dimensions of the brane on the four-cycle of X and the the rest dimension wrap each cycles of S^3. We will assume that the four-cycle $C_4 \in X$ is chosen such that the pull back of the RR-field $\mathrm{d}C^{[3]}$ to the world-volume of the five-brane vanishes. Then, since the coupling of the bulk to the world-volume three form $F^{\hat{a}\hat{b}\hat{c}}$ involves $f^*(\mathrm{d}C^{[3]})$, we can consistently set $F^{\hat{a}\hat{b}\hat{c}} = 0$. In addition we will assume that the five-brane is wrapped holomorphically on C_4. Then the CY-part of Γ is just -1 [20], so that

$$\begin{aligned}
\Gamma &= \frac{1}{(p+1)!\sqrt{\det h}} \varepsilon^{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}\hat{f}} \Gamma_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}\hat{f}} \\
&= -\frac{1}{2\sqrt{h}} \frac{\varepsilon^{\hat{a}\hat{b}} \partial X^\mu \partial X^\nu}{\partial \sigma^{\hat{a}} \partial \sigma^{\hat{b}}} e^a_\mu e^b_\nu \gamma_{ab} \otimes \mathbb{1}\,. \qquad (4.43)
\end{aligned}$$

The generic situation can be understood by distinguishing three different S^1-wrappings:

i) $Y = (\tau, \theta)$: here

$$\sqrt{h} = \sqrt{\cosh^2(\chi) - \sin^2(B)\sinh^2(\chi)}\,. \qquad (4.44)$$

We then conclude that the brane is BPS ($\Gamma\varepsilon = \varepsilon$) for $\chi = 0$ provided

$$e^{-\frac{1}{2}\phi\gamma_{23}} \gamma_{02} e^{\frac{1}{2}\phi\gamma_{23}} \varepsilon_0 = -\varepsilon_0\,. \qquad (4.45)$$

There is no condition on B and ψ. This brane wraps a geodesic circle in the horizon $S^3/\mathbb{Z}_{p^0}$ and is uncharged w.r.t. background fluxes.

ii) $Y = (\tau, \phi)$: For $B = 0$ and $\chi = 0$ the brane is BPS for all values of θ provided
We look at the case where $\chi = 0$.

We have

$$\begin{aligned}
\Gamma &= -\frac{1}{2\sqrt{h}}\frac{\varepsilon^{\tau\phi}\partial X^{\mu}\partial X^{\nu}}{\partial\tau\partial\phi}e^{a}_{\mu}e^{b}_{\nu}\gamma_{ab} \\
&= -\frac{1}{2\sqrt{h}}(e^{a}_{\tau}e^{b}_{\phi}-e^{a}_{\phi}e^{b}_{\tau})\gamma_{ab} = -\frac{1}{2\sqrt{h}}(e^{a}_{0}e^{b}_{3}-e^{a}_{3}e^{b}_{0})\gamma_{ab} \\
&= -\frac{1}{\sqrt{h}}(e^{0}_{0}e^{3}_{3}\gamma_{03}+e^{4}_{0}e^{3}_{3}\gamma_{43}+e^{0}_{0}e^{4}_{3}\gamma_{04}) \\
&= -\frac{1}{\sqrt{h}}(\cosh\chi\sin\theta\gamma_{03}-\sin B\sinh\chi\sin\theta\gamma_{43}+\cos B\cosh\chi\cos\theta\gamma_{04}) \\
&= -(\sin\theta\gamma_{03}+\cos B\cos\theta\gamma_{04})
\end{aligned} \tag{4.46}$$

where in last line we imposed the restriction $\chi = 0$. We need

$$\begin{aligned}
S^{-1}\gamma_{04}S &= -\cos B\cos\theta\gamma^{04}+\cos B\sin\theta(\cos\phi\gamma^{03}-\sin\phi\gamma^{02}) \\
&\quad -i\sin B(\cos\tau\gamma_{0}-i\sin\tau\gamma^{01})
\end{aligned} \tag{4.47}$$

$$S^{-1}\gamma_{03}S = -\cos\theta(\cos\phi\gamma^{03}-\sin\phi\gamma^{02})-\sin\theta\gamma^{04} \tag{4.48}$$

So

$$\begin{aligned}
S^{-1}\Gamma_{(0)}S &= (\sin^2\theta+\cos^2 B\cos^2\theta)\gamma^{04}+i\cos B\sin B\cos\theta(\cos\tau\gamma_{0}-i\sin\tau\gamma^{01}) \\
&\quad -(1-\cos^2 B)\sin\theta\cos\theta(\cos\phi\gamma^{03}-\sin\phi\gamma^{02})
\end{aligned} \tag{4.49}$$

The BPS condition (for $\chi = 0$) holds in two cases: First, for $B = 0$ the brane is BPS for all values of θ. In this case the brane wraps a geodesic in $S^3/\mathbb{Z}_{p^0}$ and is uncharged. Second, if $B \neq 0$ the brane is BPS only for $\theta = \frac{\pi}{2}$. Both cases provided

$$\gamma_{04}\varepsilon_0 = -\varepsilon_0\,. \tag{4.50}$$

iii) $Y = (\tau, \psi)$: here the brane is BPS for $\chi = 0$ and $B = 0$ provided

$$e^{-\frac{1}{2}\phi\gamma_{23}}e^{-\frac{1}{2}\theta\gamma_{34}}\gamma_{04}e^{\frac{1}{2}\theta\gamma_{34}}e^{\frac{1}{2}\phi\gamma_{23}}\varepsilon_0 = -\varepsilon_0. \tag{4.51}$$

There is no solution for $B \neq 0$.

To summarize, a M5 on $C_4 \times Y$ is BPS provided it wraps a maximal geodesic circle in the squashed horizon, $S^3/\mathbb{Z}_{p^0}$. From the ten-dimensional perspective these results may be interpreted as follows: An M5-brane wrapped along the S^2 base becomes an NS5-brane in ten-dimensions which is clearly uncharged and therefore static at $\chi = 0$. If the M5-branes is wrapped along the S^3-fiber instead, this will become a $D4$-brane with charge vector aligned with that of the background. This brane cannot be static in global time unless the background flux vanishes, i.e. $B = 0$.

Note that the absence of static branes wrapped along the fiber for $B \neq 0$ does not exclude stationary branes. Indeed for rotation in the ϕ direction, $\dot{\phi} = \pm 1$, we have

$$\Gamma = \frac{-1}{\sqrt{h}}[\cosh\chi\cos B\gamma_{04} \pm \sin\theta\cos B\gamma_{34}] \tag{4.52}$$

with

$$\sqrt{det\ h} = \sqrt{\cos^2 B(\sinh^2\chi + \cos^2\theta)}\,. \tag{4.53}$$

We need to have explicit form of $S^{-1}\gamma_{34}S$ and $S^{-1}\gamma_{04}S$, they are as follows

$$\begin{aligned} S^{-1}\gamma_{34}S &= \cos B(\cos\phi\gamma_{34} - \sin\phi\gamma_{42}) - i\sin B\cosh\chi\cos\tau\cos\theta(\cos\phi\gamma_3 - \sin\phi\gamma_2) \\ &\quad -i\sin B\cosh\chi\cos\tau\sin\theta\gamma_4 - \sin B\cosh\chi\sin\tau\cos\theta(\cos\phi\gamma_{13} - \sin\phi\gamma_{12}) \\ &\quad +\sin B\sinh\chi\cos\theta(\cos\phi\gamma_{03} - \sin\phi\gamma_{02}) - \sin B\cosh\chi\sin\tau\sin\theta\gamma_{14} \\ &\quad +\sin B\sinh\chi\sin\theta\gamma^{04} \end{aligned} \tag{4.54}$$

$$\begin{aligned} S^{-1}\gamma_{04}S &= -\cos B\cosh\chi\cos\theta\gamma^{04} + \cos B\cosh\chi\sin\theta(\cos\phi\gamma^{03} - \sin\phi\gamma^{02}) \\ &\quad -i\cos B\sinh\chi\cos\tau\sin\theta(\cos\phi\gamma_3 - \sin\phi\gamma_2) + \cos B\sinh\chi\sin\tau\cos\theta\gamma^{14} \\ &\quad -\cos B\sinh\chi\sin\tau\sin\theta(\cos\phi\gamma^{13} - \sin\phi\gamma^{12}) - i\sin B(\cos\tau\gamma_0 - i\sin\tau\gamma^{01}) \\ &\quad +i\cos B\sinh\chi\cos\tau\cos\theta\gamma^4 \end{aligned} \tag{4.55}$$

So κ-symmetry condition derives as

$$\begin{aligned} &\big[(-i\cos^2 B\cosh\chi\sinh\chi\cos\tau\sin\theta \mp i\cos B\sin B\cosh\chi\cos\tau\sin\theta\cos\theta)(\cos\phi\gamma_3 - \sin\phi_2) \\ &+(-\cos^2 B\cosh\chi\sinh\chi\sin\tau\sin\theta \mp \cos B\sin B\cosh\chi\sin\tau\cos\theta\sin\theta)(\cos\phi\gamma_{13} - \sin\phi_{12}) \\ &+(\cos^2 B\cosh\chi\sinh\chi\cos\theta \mp \cos B\sin B\cosh\chi\sin^2\theta)\sin\tau\gamma^{14} \\ &+(i\cos^2 B\cosh\chi\sinh\chi\cos\theta \mp i\cos B\sin B\cosh\chi\sin^2\theta)\cos\tau\gamma_4 \\ &+\cos B\sin B\cosh\chi(\sin\tau\gamma_1 + i\cos\tau\gamma^0)\big]\varepsilon_0 = 0\varepsilon_0 \end{aligned} \tag{4.56}$$

for τ-dependent terms, and

$$\begin{aligned} &\frac{-1}{\sqrt{\cos^2 B(\sinh^2\chi + \cos^2\theta)}}[(-\cos^2 B\cosh^2\chi\cos\theta \pm \cos B\sin B\sinh\chi\sin^2\theta)\gamma^{04} \\ &+(\cos^2 B\cosh^2\chi\sin\theta \pm \cos B\sin B\sinh\chi\cos\theta\sin\theta)(\cos\phi\gamma^{43} - \sin\phi\gamma^{42}) \\ &\mp\cos^2 B\sin\theta(\cos\phi\gamma^{03} - \sin\phi\gamma^{02})]\varepsilon_0 = \varepsilon_0 \end{aligned} \tag{4.57}$$

for τ-independent terms.
The BPS condition is then given by

$$\begin{aligned} \gamma^{04}\varepsilon_0 &= \mp\varepsilon_0\,, \\ \sinh\chi &= \mp\tan B\cos\theta\,. \end{aligned} \tag{4.58}$$

It is clear that since $\chi = 0$, for $\theta = \pi/2$ the determinant vanishes and hence the brane can not stay at the geodesic. These solutions correspond to rotating BPS configurations found in [40].

4.3.4.2 M5 on $C_2 \times S^3/\mathbb{Z}_{p^0}$

The induced metric $h_{\hat{a}\hat{b}}$ is in this case

$$\begin{pmatrix} -\cosh^2(\chi)+\sin^2(B)\sinh^2(\chi) & 0 & -\sin(B)\sinh(\chi)\cos(B)\cos(\theta) & -\frac{\sin(B)\sinh(\chi)\cos(B)}{p^0} \\ 0 & 1 & 0 & 0 \\ -\sin(B)\sinh(\chi)\cos(B)\cos(\theta) & 0 & \sin^2(\theta)+\cos^2(\theta)\cos^2(B) & \frac{\cos(\theta)\cos^2(B)}{p^0} \\ -\frac{\sin(B)\sinh(\chi)\cos(B)}{p^0} & 0 & \frac{\cos(\theta)\cos^2(B)}{p^0} & \frac{\cos^2(B)}{(p^0)^2} \end{pmatrix}$$

and

$$\sqrt{|h|} = |\cosh(\chi)\cos(B)\sin(\theta)/p^0|. \tag{4.59}$$

We will again assume that the brane is wrapped holomorphically on C_2 so that the CY-part of Γ is $\mathbb{1}\otimes \mathrm{i}\rho_{(7)}$.

For $B=0$ we then have

$$\begin{aligned} \Gamma &= \frac{1}{4!\sqrt{h}} \frac{\varepsilon^{\hat{a}\hat{b}\hat{c}\hat{d}}\partial X^\mu \partial X^\nu \partial X^\rho \partial X^\delta}{\partial\sigma^{\hat{a}}\partial\sigma^{\hat{b}}\partial\sigma^{\hat{c}}\partial\sigma^{\hat{d}}} e^a_\mu e^b_\nu e^c_\rho e^d_\delta \Gamma_{abcd}\left(\mathbb{1}\otimes \mathrm{i}\rho_{(7)}\right) \\ &= \mathrm{i}\gamma_{0234}\otimes\rho_{(7)} \end{aligned} \tag{4.60}$$

so that the brane is BPS for $\chi=0$ and $(i\gamma_{0234}\otimes\rho_{(7)})(\varepsilon_0\otimes\eta)=\varepsilon_0\otimes\eta$.

Next we consider the possibility of non-vanishing world-volume three-form flux $F^{\hat{a}\hat{b}\hat{c}}$ corresponding to M2-branes wrapping C_2 and bound to the M5-brane. For this we write

$$F = -f\, e^2\wedge e^3\wedge e^4 - f\, {}^{*_6}(e^2\wedge e^3\wedge e^4)\,. \tag{4.61}$$

Here f is proportional to the number of two-branes. Note that e^a, $a=2,3,4$ are the vielbeine on the unit three-sphere, not the three-sphere with radius R_{AdS} on which the world-volume is wrapped and which determines the induced metric relevant for the ^{*_6} operation. Thus

$${}^{*_6}(e^2\wedge e^3\wedge e^4) = \frac{1}{R^3_{\text{AdS}}} e^0\wedge e^5\wedge e^6\,, \tag{4.62}$$

where e^0 is as in (4.30) and e^5 and e^6 are the zwei-beine on C_2 with unit volume. Since $R_{\text{AdS}} >> 1$ we can neglect the last term in (4.61).

With this in mind we will now analyze the κ-symmetry condition. We find the representation of [48] most convenient. Adapting the corresponding projector Γ to our situation we get

$$\begin{aligned} \Gamma &= \frac{1}{\sqrt{1-\frac{1}{4}f^2}}\left(\frac{1}{6!\sqrt{\det h}}\varepsilon^{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}\hat{f}}\Gamma_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}\hat{f}} - \frac{1}{2\cdot 3!}F^{\hat{a}\hat{b}\hat{c}}\gamma_{\hat{a}\hat{b}\hat{c}}\right) \\ &= \frac{1}{\sqrt{1-\frac{1}{4}f^2}}\left(i\gamma_{0234}\otimes\rho_{(7)} + \frac{1}{2}f\gamma_{234}\right). \end{aligned} \tag{4.63}$$

To study the BPS condition $\Gamma\varepsilon = \varepsilon$ we need to know that

$$\begin{aligned} S^{-1}\gamma_{0234}S &= \cosh\chi\,\gamma_{0234} + i\sinh\chi(\cos\tau\gamma_{234} - i\sin\tau\gamma_{1234}) \\ S^{-1}\gamma_{234}S &= \cosh\chi(\cos\tau\gamma_{234} - i\sin\tau\gamma_{1234}) - i\sinh\chi\,\gamma_{0234} \end{aligned}$$

then we can compute

$$\begin{aligned} S^{-1}\Gamma S &= \frac{1}{\sqrt{1-\frac{1}{4}f^2}}[-(\cos\tau\gamma_{234} - i\sin\tau\gamma_{1234})(\sinh\chi\otimes\rho^{(7)} - \frac{1}{2}f\cosh\chi) \\ &\quad + i(\cosh\chi\otimes\rho^{(7)} - \frac{1}{2}f\sinh\chi)\gamma_{0234}]\varepsilon_0 = \varepsilon_0 \end{aligned} \tag{4.64}$$

The BPS condition then implies that

$$(\sinh\chi\otimes\rho^{(7)} - \frac{1}{2}f\cosh\chi)\varepsilon_0 = 0 \tag{4.65}$$

$$\frac{i}{\sqrt{1-\frac{1}{4}f^2}}(\cosh\chi\otimes\rho^{(7)} - \frac{1}{2}f\sinh\chi)\gamma_{0234}\varepsilon_0 = \varepsilon_0 \tag{4.66}$$

Satisfying first line of above condition, we get

$$|f| = 2\tanh(\chi) \tag{4.67}$$

For $\frac{1}{2}f = \pm\tanh\chi$ we conclude that $\rho^{(7)}$ must act on ε_0 as $\rho^{(7)}\varepsilon_0 = \pm\varepsilon_0$ and the BPS condition is

$$\begin{cases} i\gamma_{0234}\varepsilon_0 = \varepsilon_0 \quad \text{and} \quad \rho_{(7)}\eta = \eta \quad \text{for} \quad f > 0 \\ i\gamma_{0234}\varepsilon_0 = -\varepsilon_0 \quad \text{and} \quad \rho_{(7)}\eta = -\eta \quad \text{for} \quad f < 0\,. \end{cases} \tag{4.68}$$

Upon double dimensional reduction along the fiber of $S^3/\mathbb{Z}_{p^0}$ to ten dimensions we get a D4-brane with

$$F_{\hat{a}\hat{b}} = F_{4\hat{a}\hat{b}} \tag{4.69}$$

which in turn is SUSY according to [20].

Let us now analyze the case with non-vanishing four-brane flux $B \neq 0$. We make the Ansatz

$$F = g\,e^0\wedge e^2\wedge e^3\,. \tag{4.70}$$

The the κ-symmetry projector

$$\Gamma = \frac{1}{\sqrt{1+\frac{1}{4}g^2}}\left(i\gamma_{0234}\otimes\rho_{(7)} - \frac{1}{2}g\gamma_{023}\right)\,. \tag{4.71}$$

calculating $S^{-1}\gamma_{0234}S$ and $S^{-1}\gamma_{234}S$ for this case, we get

$$\begin{aligned} S^{-1}\gamma_{0234}S = & \cos B\cosh\chi\gamma_{0234} + i\cos B\sinh\chi(\cos\tau\gamma_{423} - i\sin\tau\gamma_{1423}) \\ & -i\sin B\cos\tau\sin\theta(\cos\phi\gamma_{024} + \sin\phi\gamma_{034}) + \sin B\sin\tau\cos\theta\gamma_{0312} \\ & +\sin B\sin\tau\sin\theta(\cos\phi\gamma_{0241} + \sin\phi\gamma_{0341}) - i\sin B\cos\tau\cos\theta\gamma_{023} \end{aligned} \quad (4.72)$$

$$\begin{aligned} S^{-1}\gamma_{023}S = & \cos B\cos\tau\cos\theta\gamma_{023} + \cos B\cos\tau\sin\theta(\cos\phi\gamma_{024} + \sin\phi\gamma_{034}) \\ & +i\cos B\sin\tau\cos\theta\gamma_{0312} + i\cos B\sin\tau\sin\theta(\cos\phi\gamma_{0241} + \sin\phi\gamma_{0341}) \\ & -i\sin B\cosh\chi\gamma_{0234} + \sin B\sinh\chi(\cos\tau\gamma_{423} - i\sin\tau\gamma_{1423}) \end{aligned} \quad (4.73)$$

By imposing the κ-symmetry condition we have

$$\begin{aligned} \big[& i\sin\tau\sin\theta(\sin B\otimes\rho^{(7)} - \frac{1}{2}g\cos B)(\cos\phi\gamma_{0241} + \sin\phi\gamma_{0341}) \\ & +\cos\tau\sin\theta(\sin B\otimes\rho^{(7)} - \frac{1}{2}g\cos B)(\cos\phi\gamma_{024} + \sin\phi\gamma_{034}) \\ & -(\cos B\sinh\chi\otimes\rho^{(7)} + \frac{1}{2}g\sin B\sinh\chi)(\cos\tau\gamma_{423} - i\sin\tau\gamma_{1423}) \\ & -i\sin\tau\cos\theta(\sin B\otimes\rho^{(7)} - \frac{1}{2}g\cos B)\gamma_{0312} \\ & +\cos\tau\cos\theta(\cos B\otimes\rho^{(7)} - \frac{1}{2}g\cos B)\gamma_{023}\big]\varepsilon_0 = 0 \end{aligned} \quad (4.74)$$

$$\frac{i}{\sqrt{1+\frac{1}{4}g^2}}(\cos B\cosh\chi\otimes\rho^{(7)} + \frac{1}{2}g\sin B\cosh\chi)\gamma_{0234}\varepsilon_0 = \varepsilon_0 \quad (4.75)$$

(4.74) and (4.75) implies the BPS condition for $\chi = 0$ and

$$\begin{cases} i\gamma_{0234}\varepsilon_0 = \varepsilon_0\,, \quad \rho_{(7)}\eta = \eta & \frac{1}{2}g = \tan(B) \quad \text{or} \\ i\gamma_{0234}\varepsilon_0 = -\varepsilon_0\,, \rho_{(7)}\eta = -\eta & \frac{1}{2}g = -\tan(B) \quad . \end{cases} \quad (4.76)$$

For $g \neq 0$ these configurations describe M5-branes with delocalized M2-branes ending on them.

If both, f and g are non-vanishing then the τ-dependent/independent terms of BPS condition reads off the be

$$\begin{aligned} \big[& i\sin\tau\sin\theta(\sin B\otimes\rho^{(7)} - \frac{1}{2}g\cos B)(\cos\phi\gamma_{0241} + \sin\phi\gamma_{0341}) \\ & +\cos\tau\sin\theta(\sin B\otimes\rho^{(7)} - \frac{1}{2}g\cos B)(\cos\phi\gamma_{024} + \sin\phi\gamma_{034}) \\ & -(\sinh\chi\cos B\otimes\rho^{(7)} + \frac{1}{2}g\sin B\sinh\chi - \frac{1}{2}f\cosh\chi)(\cos\tau\gamma_{423} - i\sin\tau\gamma_{1423}) \\ & -i\sin\tau\cos\theta(\sin B\otimes\rho^{(7)} - \frac{1}{2}g\cos B)\gamma_{0312} \\ & +\cos\tau\cos\theta(\sin B\otimes\rho^{(7)} - \frac{1}{2}g\cos B)\gamma_{023}\big]\varepsilon_0 = 0 \end{aligned} \quad (4.77)$$

$$\frac{i}{\sqrt{1+\frac{1}{4}g^2}}(\cos B\cosh\chi\otimes\rho^{(7)} + \frac{1}{2}g\sin B\cosh\chi - \frac{1}{2}f\sinh\chi)\gamma_{0234}\varepsilon_0 = \varepsilon_0 \quad (4.78)$$

The κ-symmetry projector takes the form

$$\Gamma = \frac{1}{\sqrt{1+\frac{1}{4}(g^2-f^2)}}\left(i\gamma_{0234}\otimes\rho_{(7)}+\frac{1}{2}(f\gamma_{234}-g\gamma_{023})\right). \tag{4.79}$$

and the BPS-condition reads

$$\begin{cases} i\gamma_{0234}\varepsilon_0=\varepsilon_0\,, & \rho_{(7)}\eta=\eta\,, \quad \frac{1}{2}g=\tan(B), \quad \frac{1}{2}f=\frac{\tanh(\chi)}{\cos B}\,, \qquad \text{or} \\ i\gamma_{0234}\varepsilon_0=-\varepsilon_0\,, & \rho_{(7)}\eta=-\eta\,, \quad \frac{1}{2}g=-\tan(B), \quad \frac{1}{2}f=-\frac{\tanh(\chi)}{\cos B}\,. \end{cases} \tag{4.80}$$

Thus, an static M5-brane wrapped on the horizon and a two-cycle in X with M2-branes on C_2 bound to it is BPS for certain values of χ.

4.4 Discussion

I this chapter we constructed supersymmetric probe branes, stationary in global coordinates of the eleven-dimensional near-horizon geometry, of a generic four-dimensional, single-centered attractor black hole. The motivation for this study came from the success of [7, 37] in approximating the black hole partition function by a dilute gas of non-interacting probe branes in the near horizon geometry of attractor black holes without D6-charge. Our results should provide the necessary ingredients for extending this approach to include D6-charge as well. In particular, we expect the M2-branes found here to be relevant for understanding the OSV partition function in the presence of D6-charge. Similarly the horizon wrapping M5-branes should contribute, as collective excitations, to the partition function of the conformal quantum-mechanical system dual to the AdS_2 near horizon geometry.

CHAPTER 5

Discussion and Conclusion

In this thesis we studied the microscopic behavior of Calabi-Yau black holes in IIA string theory. First in chapter 2, we reviewed the classical black holes and their thermodynamical properties. We calculated the absorption cross-section of massless scalars on Schwarzschild and Reissner-Nordström black holes and also reviewed the universality of low energy absorption cross-section. In chapter 3, by employing AdS/CFT correspondence, we obtained an analytic expressions for the low energy absorption cross-section of a massless space-time scalars on the horizon-wrapped D2-branes, static in global coordinates of the near horizon AdS_2 geometry of Calabi-Yau black hole. The fact that these amplitudes can be computed exactly may come as a surprise since the probe two-brane describes a complicated trajectory in the asymptotic Poincaré coordinates. For small but non-zero probe D0-charge we found that the quantum mechanical absorption cross section seen by an asymptotic anti-deSitter observer, static in Poincaré time vanishes linearly in ω for small frequencies. For vanishing probe D0-charge the absorption cross section has non-analytic ($\omega \log(\omega)$) behavior. This is in disagreement with the classical s-wave absorption cross section by the black hole which vanishes quadratically in ω for small frequencies. However, the comparison with the classical result is more subtle since for the classical absorption cross section on near extremal black holes the potentials outside the near horizon AdS_2 geometry is the only relevant one. This is because there is no reflection in AdS_2.

An interesting feature is that although the Hamiltonian of the D2-brane has a discrete spectrum with spacing given by the inverse of the radius of the horizon the D2-brane can absorb arbitrarily small frequencies with respect to an asymptotic observer. We should mention that we only considered the bosonic sector of the world-volume theory. However, it is not hard to see that fermions give a vanishing contribution at the lowest (quadratic) level. Also we have not considered fixed scalars in this paper although their inclusion should be straight forward. As mentioned before, the details of the absorption process described here are qualitatively different from the world-volume absorption on D-branes in flat space. In flat space, the low energy behavior is dominated by goldstone bosons and possible other massless fields whereas here no massless degrees of freedom are present. The fact that the cross section vanishes linearly for $\omega \to 0$ is due precisely to the absence of massless degrees of freedom.

In view of a possible interpretation for a dual interpretation of four-dimensional CY black holes in terms of the quantum mechanics of probe D2-branes wrapped on the S^2 of their near horizon geometry an encouraging result would have been to find agreement for the low energy absorption cross section on both sides. Our concrete calculation shows how-

ever that this is not the case since the microscopic absorption cross section on the two-brane does not have the correct behavior at small frequencies compared to the classical absorption cross section of massless scalars which vanishes quadratically in ω. However, the comparison between these two calculations is more subtle. First we note that our absorption cross section was computed for wrapped D2-branes with small D0-charge whereas the dominant contribution to the entropy comes from wrapped branes with large D0-charge. A possible application of the present calculation is to interpret a single wrapped D2-brane with small D0-charge as a small non-extremal perturbation of the extremal black hole. This is sensible since for small D0-charge the two-brane is confined to the near horizon AdS_2 geometry of the asymptotically flat global geometry. In this case we should compare the classical absorption cross section for the near extremal black hole with the the product of the transmission coefficient from asymptotically flat space to the near horizon AdS_2 region and the AdS_2 absorption cross section computed in this paper. The microscopic cross section obtained in this way vanishes like $(R\omega)^3$. So we still have disagreement. One possible explanation for the disagreement could be that there are microscopic configurations, other than the wrapped D2-branes considered here, correspond to a non-extremal black hole which reproduce the correct low energy behavior. One such generalization is to consider multi-branes wrapping horizon, however this does not change qualitative small frequency behavior. Another possibility is to consider the scattering of massless space-time scalers on individual probe D0-branes. However in that case the brane absorption amplitude vanishes due to energy-momentum conservation.
It would be desirable to know the result in that case although the calculation appears to be more involved since, as we showed, linearized perturbation theory breaks down in this situation.

In chapter 4, we constructed supersymmetric probe branes, stationary in global coordinates of the eleven-dimensional near-horizon geometry of a four-dimensional black hole with generic D6-D4-D2-D0-charges. The motivation for this study came from the success [7, 37] in approximating the black hole partition function by a dilute gas of non-interacting probe branes in the near horizon geometry of attractor black holes without D6-charge. In particular we found BPS two-branes wrapped on a holomorphic two-cycle in the Calabi-Yau. Also, we determine the trajectories of the five-brane which preserve supersymmetry for the case of wrapping a holomorphic four-cycle in the Calabi-Yau and an S^1 in space-time (hence partially wrapping the horizon) and for the case of fully wrapped the horizon $S^3/\mathbb{Z}_{p^0}$ and a holomorphic two-cycle in the Calabi-Yau. Our results should provide the necessary ingredients for extending this approach to include D6-charge as well. In particular, we expect the M2-branes found here to be relevant for understanding the OSV partition function in the presence of D6-charge. Similarly the horizon wrapping M5-branes should contribute, as collective excitations, to the partition function of the conformal quantum-mechanical system dual to the AdS_2 near horizon geometry.

Appendix A

Calabi-Yau spaces

Here we give a brief overview of essential concepts leading to the definition of the Calabi-Yau manifold, we follow mostly [49, 50, 51]. Calabi-Yau space is a manifold X with Riemannian metric g which satisfies following three conditions:

- X is complex manifold.
  This means that Calabi-Yau looks locally like $\mathbb{C}^n$ for some n, in the sense that it can be covered by patches admitting local complex coordinates

  $$z_1, \dots, z_n \tag{A.1}$$

  and the transition functions between patches are holomorphic. Consequently, the real dimension of X is $2n$, and hence Calabi-Yau spaces are even-dimensional manifolds, known as *Calabi-Yau n-folds*. Furthermore, the metric g should be Hermitian with respect to the complex structure. This means

  $$g_{ij} = g_{\bar{i}\bar{j}} = 0 \tag{A.2}$$

  so the only non-vanishing components of metric are $g_{i\bar{j}}$.

- X is Kähler.
  This means that there is a real function K locally on X such that

  $$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K \tag{A.3}$$

  where together with a Hermitian metric g one can define its associated kähler form

  $$k = g_{i\bar{j}} \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j \tag{A.4}$$

  The Kähler condition is then $\mathrm{d}k = 0$.

- X admits a global holomorphic n-forms In each local coordinate patch of X one can write many such forms

  $$\Omega = f(z_1, \dots, z_n)\, \mathrm{d}z_1 \wedge \dots \wedge \mathrm{d}z_n \tag{A.5}$$

  where f is an arbitrary holomorphic function. For compact Calabi-Yau manifold there is at most on nowhere vanishing one such form, up to an overall scalar rescaling. Topologically this is equivalent to vanishing first Chern class.

So, briefly, a Calabi-Yau n-fold is a Kähler manifold having n complex dimensions and vanishing first Chern class.

Calabi-Yau manifold, defined above, has an important property: Metric on the Calabi-Yau manifold is Ricci-flat

$$R_{i\bar{j}} = 0 \tag{A.6}$$

Compact Calabi-Yau manifold for $n = 1$ is torus T^2, for $n = 2$ just T^4 and $K3$, but for $n = 3$it is not even known whether the number of compact Calabi-Yau spaces is finite. T^6 is an example

A Calabi-Yau n-fold is characterized by the values of its Hodge numbers $h^{p,q}$, which count the number of harmonic (p,q)-forms on the manifold. These numbers do not characterize Calabi-Yau manifold completely but specify a class of manifolds which have same Hodge numbers. Hodge numbers satisfy following relations

$$\begin{aligned} h^{p,0} &= h^{n-p,0} \\ h^{p,q} &= h^{q,p} \\ h^{p,q} &= h^{n-q,n-p} \end{aligned} \tag{A.7}$$

First relation follows from the fact that the manifolds with de Rham cohomology $H^p(X)$ and $H^{n-p}(X)$ are isomorphic, second relation comes from complex conjugation and last one is given by Poincaré duality. Furthermore, for any complex manifold $h^{1,1}$=1. In addition we have $h^{1,0} = h^{0,1} = 0$.
There is also Betti number b_p determines the dimension of $H^p(X)$, which counts the number of lineally independent harmonic p-forms on the manifold

$$b_k = \sum_{p=0}^{k} h^{p,k-p}(-1)^p b_p \tag{A.8}$$

Therefore, in case of Calabi-Yau three-fold one requires only to specify $h^{1,1}$ and $h^{2,1}$. Finally, the Euler characteristic of our manifold is just the alternating sum of Betti numbers:

$$\chi(X) = \sum_{p=0}^{2n} \tag{A.9}$$

where for $n = 3$ gives $\chi = 2(h^{1,1} - h^{2,1})$.

APPENDIX B

Useful identities for γ-matrices

Here, we provide some identities for γ-matrices which are useful for calculating κ-symmetry projection operator in chapter 4. α is some integer.

$$e^{\pm i\frac{\alpha}{2}\gamma^a}\;\gamma^a\;e^{\mp i\frac{\alpha}{2}\gamma^a}=\gamma^a \tag{B.1}$$

$$e^{\pm\frac{\alpha}{2}\gamma^{ab}}\;\gamma^{ab}\;e^{\mp\frac{\alpha}{2}\gamma^{ab}}=\gamma^{ab} \tag{B.2}$$

$$e^{-\frac{\alpha}{2}\gamma^{bc}}\;\gamma^a\;e^{\frac{\alpha}{2}\gamma^{bc}}=\gamma^a \tag{B.3}$$

$$e^{\pm i\frac{\alpha}{2}\gamma^{bc}}\;\gamma^a\;e^{\mp i\frac{\alpha}{2}\gamma^{bc}}=\gamma^a \qquad b,c\neq a \tag{B.4}$$

$$e^{\pm i\frac{\alpha}{2}\gamma^a}\;\gamma^{bc}\;e^{\mp i\frac{\alpha}{2}\gamma^a}=\gamma^{bc} \qquad b,c\neq a \tag{B.5}$$

$$\begin{aligned} e^{i\frac{\alpha}{2}\gamma^a}\;\gamma^{ab}\;e^{-i\frac{\alpha}{2}\gamma^a} &= \gamma^{ab}\cosh\alpha - i\gamma^b\sinh\alpha \qquad a=0 \\ &= \gamma^{ab}\cos\alpha + i\gamma^b\sinh\alpha \qquad a\neq 0 \end{aligned} \tag{B.6}$$

$$\begin{aligned} e^{i\frac{\alpha}{2}\gamma^{ba}}\;\gamma^{a}\;e^{-i\frac{\alpha}{2}\gamma^{ba}} &= \gamma^{a}\cos\alpha - i\gamma^b\sin\alpha \qquad a \text{ or } b=0 \\ &= \gamma^{a}\cosh\alpha + i\gamma^b\sinh\alpha \qquad a,b\neq 0 \end{aligned} \tag{B.7}$$

$$\begin{aligned} e^{\pm i\frac{\alpha}{2}\gamma^a}\;\gamma^{b}\;e^{\mp i\frac{\alpha}{2}\gamma^a} &= \gamma^{b}\cosh\alpha \pm i\gamma^{ab}\sinh\alpha \qquad a=0 \\ &= \gamma^{b}\cos\alpha \pm i\gamma^{ab}\sin\alpha \qquad a\neq 0 \end{aligned} \tag{B.8}$$

$$
\begin{aligned}
e^{-\frac{\alpha}{2}\gamma^{ba}}\,\gamma^{a}\,e^{\frac{\alpha}{2}\gamma^{ba}} &= \gamma^{a}\cosh\alpha+\gamma^{b}\sinh\alpha \qquad a \text{ or } b=0 \\
&= \gamma^{a}\cos\alpha-\gamma^{b}\sin\alpha \qquad a,b\neq 0
\end{aligned}
\tag{B.9}
$$

$$
\begin{aligned}
e^{-\frac{\alpha}{2}\gamma^{ab}}\,\gamma^{ac}\,e^{\frac{\alpha}{2}\gamma^{ab}} &= \gamma^{ac}\cos\alpha+\gamma^{bc}\sin\alpha \qquad a,b\neq 0 \\
e^{-\frac{\alpha}{2}\gamma^{ba}}\,\gamma^{ca}\,e^{\frac{\alpha}{2}\gamma^{ba}} &= \gamma^{ca}\cos\alpha-\gamma^{cb}\sin\alpha \qquad a,b\neq 0
\end{aligned}
\tag{B.10}
$$

APPENDIX C

Anti-de Sitter space geometry

The D-dimensional Anti-de Sitter space, AdS_D is usually defined as a surface embedded in a (D+1)-dimensional flat space $\mathbb{R}^{2,D-1}$ with, signature $(-,-,+,+,...)$, two time coordinates X_0 and X_D plus D-1 space coordinates X_i :

$$ds^2 = -dX_0^2 - dX_D^2 + \sum_{i=1}^{D-1} dX_i^2\,, \tag{C.1}$$

together with the constraint

$$-X_0^2 - X_D^2 + \sum_{i=1}^{D-1} X_i^2 = -R^2 \tag{C.2}$$

where R is some constant.

A solution of constraint equation is

$$\begin{aligned} X_0 &= R\cosh\rho\cos(\tau/R) \\ X_D &= R\cosh\rho\sin(\tau/R) \\ X_i &= R\Omega_i\sinh\rho \end{aligned} \tag{C.3}$$

where Ω_i's are chosen such that $\sum_{i=1}^{D-2}\Omega_i = 1$. In order to determine the metric on the hyperboloid we should substitute above solution into the space-time metric (C.1), and we find the global AdS_D metric

$$ds^2 = -\cosh^2\rho dt^2 + R^2 d\rho^2 + R^2\sinh^2\rho d\Omega_{D-2}^2 \tag{C.4}$$

where, with

$$\begin{aligned} 0 \leq \quad \tau \quad &\leq 2\pi \\ \rho \quad &\geq 0 \end{aligned} \tag{C.5}$$

our solution covers the entire hyperboloid, and this is why $(\tau,\,\rho)$ coordinates are called the *global* on AdS. The time τ is usually taken not as a circle, which gives closed time-like curves, but on the real line such that it is analytic everywhere, $-\infty \leq \tau \leq \infty$, giving the universal cover of the hyperboloid.

Another solution to the hyperboloid equation (C.8) is

$$\begin{aligned} X_0 &= \frac{1}{2r}\left(1+r^2(R^2+\vec{x}^2-t^2)\right) \\ X_{D-1} &= \frac{1}{2r}\left(1-r^2(R^2-\vec{x}^2+t^2)\right) \\ X_D &= rt \\ X_i &= rx_i \end{aligned} \tag{C.6}$$

These coordinates cover only *half* of the hyperboloid. The resulting metric, on the hyperboloid, after substitution into space-time metric (C.1) is called the *Poincaré* form of the metric

$$ds^2 = \frac{r^2}{R^2}\left(-dt^2+dx_1^2+\ldots+dx_{D-1}^2\right)+\frac{R^2}{r^2}dr^2 \tag{C.7}$$

(t, r) are the 'local 'coordinates.
It should also be noted that, in global coordinates, there is no horizon. The horizon is a feature of the description in terms of the coordinates of the Poincaré patch but not of the global space-time.

AdS_D has a very natural geometric representation. We can visualize it using a two-dimensional surface on which each point represents a sphere. We write the constraint equation (C.8) as

$$X_0^2+X_D^2 = R^2+\sum_{i=1}^{D-1}X_i^2 \tag{C.8}$$

and plot the space using axes for X_0, X_D and $\rho = \sqrt{\vec{X}.\vec{X}}$. A point on the surface, generated by above constraint, is determined by X_0 and X_D, while then the value of ρ is fixed. To see the whole AdS space we must include at each point on the surface a sphere S^{D-2}, defined by the points $\vec{X}$ that satisfy $\vec{X}.\vec{X} = \rho^2 = X_0^2+X_D^2-R^2$. For AdS_2 this is just a hyperboloid with no sphere.

APPENDIX D

Bessel function

The Bessel functions are more frequently defined as solutions to the differential equation [28]

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}+(x^2-\alpha^2)y=0 \tag{D.1}$$

There are two classes of solution, called the Bessel function of the first kind $J_n(x)$ and Bessel function of the second kind $Y_\alpha(x)$. When α is an integer, the general solution is of the form

$$y(x)=C_1J_\alpha(x)+C_2Y_\alpha(x) \tag{D.2}$$

For Bessel functions of order equal to an integer plus one-half,$\alpha=n+1/2$, the two class are related as

$$Y_{n+\frac{1}{2}}(x) = (-1)^{n-1}J_{-n-\frac{1}{2}}(x) \tag{D.3}$$

$$Y_{-n-\frac{1}{2}}(x) = (-1)^{n-1}J_{n+\frac{1}{2}}(x) \tag{D.4}$$

where

$$J_{n+\frac{1}{2}}(x) = (-1)^n x^{n+\frac{1}{2}}\sqrt{\frac{2}{\pi}}\frac{\mathrm{d}^n}{(x\mathrm{d}x)^n}\left(\frac{\sin x}{x}\right) \tag{D.5}$$

$$J_{-n-\frac{1}{2}}(x) = x^{n+\frac{1}{2}}\sqrt{\frac{2}{\pi}}\frac{\mathrm{d}^n}{(x\mathrm{d}x)^n}\left(\frac{\cos x}{x}\right) \tag{D.6}$$

The special cases of order $\pm1/2$ are therefore defined as

$$J_{-1/2}(x) \equiv \sqrt{\frac{2}{\pi x}}\cos x \tag{D.7}$$

$$J_{1/2}(x) \equiv \sqrt{\frac{2}{\pi x}}\sin x \tag{D.8}$$

So the complete solution for (D.1) is then

$$y(x) = A\,J_{n+\frac{1}{2}}(x)+B\,Y_{n+\frac{1}{2}}(x) \tag{D.9}$$

$$y(x) = A\,J_{n+\frac{1}{2}}(x)+B\,(-1)^{n-1}J_{-n-\frac{1}{2}}(x) \tag{D.10}$$

Γ is gamma function.
Bessel function has following series expansion

$$J_\alpha(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m+\alpha} \tag{D.11}$$

So the complete solution $y(x)$ has following small argument approximation

$$y(x) \approx A \frac{1}{\Gamma(\alpha+1)} \left(\frac{x}{2}\right)^{\alpha} + B(-1)^{n-1} \frac{1}{\Gamma(-\alpha+1)} \left(\frac{x}{2}\right)^{-\alpha}, \quad x \ll 1 \tag{D.12}$$

where we have kept only first term of expansion. Finally, the Bessel function $J_\alpha(x)$ has large x form as

$$J_\alpha(x) \approx \sqrt{\frac{2}{\pi z}} \cos\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right), \qquad x \gg |\alpha^2 - 1/4| \tag{D.13}$$

$$Y_\alpha(x) \approx \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) \tag{D.14}$$

which leads to following approximate solution of the differential equation for $\alpha = n + 1/2$

$$y(x) \approx \sqrt{\frac{2}{\pi z}} \left[-A \sin\left(x - \frac{n\pi}{2}\right) + B \cos\left(x + \frac{n\pi}{2}\right)\right] \tag{D.15}$$

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