Three Essays in Dynamic Economic Analysis

Inaugural-Dissertation
zur Erlangung des Grades
Doctor oeconomiae publicae (Dr. oec. publ.)
an der Ludwig-Maximilians-Universität München

2009

vorgelegt von

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Promotionsabschlussberatung:	19. Mai 2010

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[1] This chapter is a joint work with Ray Rees.

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Preface

Still, intuitive assumptions about behavior is only the starting point of systematic analysis, for alone they do not yield many interesting implications.[2]

This dissertation consists of three self-contained chapters that are contributions to the fields of industrial organization, family economics, and energy economics. Each chapter has its own introduction and can be read independently of the other two chapters. All three chapters have the common theme “dynamic optimization” of either an individual’s or firm’s objective. The first two chapters apply the tools of optimal control theory to study different aspects of learning and timing, one in the field of industrial organization, the other in family economics.

The first model studies the optimal timing for a firm to adopt a new technology. Infant industries have often rendered positive externalities, which justify subsidies. Examples can be found in the renewable energy sector; these technologies not only provide electricity to their owners, but they also reduce the carbon dioxide content in the atmosphere. Thus, it is in the public’s interest to support this sector, such that it can reduce its costs to a level where it can compete with conventional, $C0_2$ emitting technologies. A policy that has been implemented by governments throughout the world to reduce the cost level, is to either subsidize the research of these technologies or their distribution. Till this day the economic literature lacks a model that can evaluate these instruments in a suitable way, allowing predictions of their effects on consumer and producer surplus, and welfare in general. Chapter 1 demonstrates how government interventions can affect the optimal timing for adoption of a new technology. The timing increases positive externalities, as for example in the renewable energy field. It is not only relevant that renewable energy technologies reach a low cost level in the future, it is also important to know how many products are distributed before this future date is reached. In addition, the timing of distribution matters. Imagine

[2] Gary Becker (Nobel laureate in Economics, 1992)

there are two production plans for distributing solar panels, where both reach the same cost level at a distant date in the future with the same number of solar panels installed. Then the production plan, according to which distribution is larger in the beginning, is preferred to one, where distribution takes place later, because the first more greatly reduces the amount of carbon dioxide being released in the atmosphere. Furthermore this first chapter makes predictions on how the effects change, when the total quantity that can be produced is fixed; the installations of wind powered energy plants in Germany exemplify this point. Onshore, there have only been a few new instalments of wind power plants lately, because of a lack of suitable space. Sales and research subsidies have a very different effect in this case, when the total production quantity is not endogenous. Depending on whether producer rents, consumer rents or early implementation are more important to the government, the chapter offers the appropriate tools to attain its objective.

The second model analyzes the optimal timing for a woman to give birth. Malthusianism has become a widely used term, one that stems from the paper 'Essay on the Principle of Population' written by the infamous Thomas Malthus. He is one of the main founders of population economics; forecasting that population growth would ultimately outstrip the world's food supply in 1798. With the immigration to the Americas and Industrialization, the arguments of his essay became quickly neutralized. As Industrialization advanced on the world, fertility began to stagnate and then to the surprise of many avid Malthusians, recede.

Starting in the richer countries, fertility first began to fall in industrialized countries. Within the last decades it has finally become possible to see a decline in growth rates in the developing world, as they slowly have become richer. What is astonishing is not only the rate at which this is happening but the scale of the decline. Developing countries are changing so rapidly that the demographic transition has become one of the largest social changes taking place. An example of this is Iran, in 1984 the fertility rate was still relatively high at 7 children, in 2006 is had dropped to 1.9. With the worldwide debate over the threats and solutions to climate change larger than ever before, the Maltusian worries are resurfacing. Fears of a growing, richer, more consumer driven population have steered economists to take a closer look at population growth. With fertility rates falling in India, Brazil and Indonesia, the fertility rate has now reached the replacement rate of 2.1 in half of the world. The trepidation of the environmental impact due to high fertility can, at least for now, be negated.

A decrease of fertility along with a longer life expectancy, leads to an increase in the proportion of people that are retired compared to those that are working. A pay-as-you-go (PAYG) social security system is characterized by contributions to their beneficiaries, financed through the regular payment flows of working individuals, such that reserves do not need to be created. If a fall in fertility occurs too quickly, then a PAYG financed pension system could collapse. Retired people generally have accumulated own savings, but these are often insufficient to cover the entire retirement phase if the share of the working population is relatively too low. Fewer people being born means, there is less income during those periods, when they would have been contributing on the job market. Lower old age pensions mean that the society's welfare decreases. In most pension systems there is no coherence between fertility and pensions. Chapter two evaluates government policies that increase the incentive to have children in order to smooth the digression of fertility.

This chapter is a joint work with Ray Rees, and contains a model to solve for the optimal timing of childbirth and the optimal number of children in a continuous time framework simultaneously. The model depicts how changes in wage at different stages of an individual's life, influence the timing decision of childbirth and the optimal number of children. Some of the numerous findings are quite surprising. When a woman would like to have more children, she decides to have them at a younger age. Medical research that extends the fecund life span induces women to have fewer children. A reduction of the parental leave due to day-care centers, and a reduction in the costs of leave due to child benefits, increases the number of children. Women value labor more, when they face the risk of an unknown divorce. This paper also shows that divorce does not change the timing of childbirth directly, however it influences the number of children negatively, and the reduced number of children delays the timing. The model can be used to predict upper bound fertility rates, when the expected divorce rate continues to increase.

While the first two models are framed in continuous time, the third is framed in discrete time. It studies the effect of default risk on a market, where its players meet twice; on a contract and on a spot market. The financial crisis has shown that there are market players, which are "too big to fail". In order to preserve the financial system's stability, banks and insurances that have incurred speculation losses, have been bailed out. Together with the nationalization of the firm, its debts are refinanced through taxpayers' money. In conjunction with this issue revealed by the banking crisis, the risk of bankruptcy alone, can affect welfare negatively, which is demonstrated in chapter 3.

The chapter illustrates this point for the electricity sector, using a method that can be easily applied to other sectors as well. Upstream producers that possess market power, sell forwards with a lengthy duration to regional electricity companies (REC). As part of the liberalization of the electricity market, RECs have been privatized and exposed to a possible bankruptcy risk, if spot prices fall below their expected value. An interesting observation is that the downstream firm's expected profit is larger, when it is less likely to be bailed out. The intuition behind this result is that producers adopt the spot price upwards to lower the retailer's default risk that is positively correlated to the REC's loss from its contracts. The effect on upstream profits is ambiguous while consumers loose. Options are less welfare increasing than forwards, but the difference is minimal. In the presence of bankruptcy, options are the preferred welfare maximizing market instrument.

Chapter 1

Learning and Technology Adoptions

1.1 Introduction

Typical infant industries are characterized by cost reductions through learning in the production process, and continuous new technology adoptions. Mature industries are often characterized by numerous technology generations, while learning takes place at the same time.[1] Market players try to find new technologies that are more sustainable, efficient and safer, however, at the same time they are improving existing technologies. Thus it is important to connect both: experience and innovations in a single model. In this dynamic framework a firm can adopt innovation breakthroughs from its research department. In addition, the firm decides upon a pricing rule for each point in time. It is assumed that experience spills over to the next technology generation after an innovation breakthrough has been adopted. The empirical literature till present, has concentrated on learning models, in which technology spillovers were absent (Irwin and Klenow, 1994). Jamasb (2007) is an exception: in his purely econometric analysis, he estimates learning by doing and research rates for a range of energy technologies in different stages of technical progress. He separates the cost reduction effect caused by learning and research, expressed by cumulative sales and patents. Unfortunately, it is difficult to obtain data on costs, which makes the study rely on very few data points.

This model shall be the theoretical foundation of applied work, in which firms can use the experience, they have accumulated thus far, for the next technology generation. In macroeconomics, there are studies, where the experience gained from learning, is passed on from one generation to the next. Examples are Young (1993) and Parente

[1] Currently produced nuclear power plants for example use the 3rd technology generation, the 4th generation will be deployed some time around 2030.

(1994). Young makes clear that innovations occur in markets that are large. In this setup, production costs do not decrease with new technology adoptions, but rather through learning. In Parente's model, learning and technology adoptions occur both after the product introduction. A firm faces a trade-off between learning at a decreasing rate or switching to a new technology, which is costly as not all expertise can be transferred. In return, the learning curve becomes steeper. From a microeconomic perspective, both models face one problem in particular: learning occurs only through time and not through cumulative production or "by doing". Therefore the strategic pricing behavior of firms can not be analyzed. This paper introduces a model, where firms simultaneously choose a research budget and the optimal production quantity, exploiting the learning effect optimally. The production and the time of technology adoptions are control variables of the firm.

This research has two main objectives: firstly, to describe the market equilibrium of a setup that accounts for innovations and learning; and secondly, to show the effects of subsidies on the market equilibrium. The second objective is based on the observation that products produced by learning industries have often rendered positive externalities in the past; renewable energy technologies can be cited as examples. The production cost per unit of electricity has been reduced significantly for technologies that are powered by wind, sunlight and biomass. The positive externality is the deduction of the carbon dioxide level in the atmosphere, because electricity from renewable energy is a perfect substitute to conventionally generated electricity.[2] It is illustrated that sales and innovation subsidies have the same effect on the innovation date and prices, if and only if the innovation date and total cumulative production quantity are endogenous. Effects differ significantly, when the total quantity that can be produced in a market is fixed. An example for such products, whose costs are affected by learning are wind power plants. In Germany the installation of onshore wind power plants reached its peak in 2002, with an installed capacity of over 3000 MW. The installed capacity in 2009 was estimated to be less than 1000 MW due to a lack of suitable space (Dena, 2005). The cost of producing wind power capacity has fallen drastically; the price of 1 KW wind energy capacity fell by 29% between 1990 and 2004 (Iset, 2005).

The layout of this model is as follows: a social planner or monopolist learn with some learning parameter λ, and it can choose any particular date in the future, when they would like to adopt a new technology. This is characterized by an increase of

[2] Another example is the aerospace technology, which was mainly developed for military purposes during the 1930s and 40s. This was a stepping stone for the development of commercially used airplanes, which has enabled societies to travel and trade at an increased pace. The learning effect in this industry was described by Wright (1936).

the learning parameter to γ, where $\gamma \geq \lambda$. After the innovation date, newly gained experience reduces the present level of cost by more, than before the innovation date. Thus the process innovation described here, is a substitute to learning. A firm can adopt a new technology when its research department has been successful. The cost of research is given by a convex and decreasing function $a(t_1)$, where t_1 is the date of innovation. When firms prefer an earlier innovation date over that of a late one, then they employ more researchers; this is reflected by a higher innovation cost in the model. The setting is deterministic to avoid unnecessary complications, which would not add any further results. Players choose the date of innovation at the beginning of the planning horizon; production starts thereafter. In the first step, a pricing rule is derived for an exogenous innovation date, which is endogenized thereafter.

The findings of this paper are: the social planner/ monopolist charges two different prices, for the time phases before the innovation and after the innovation. Both prices are constant for a constant price elasticity of demand. After the innovation has occurred, the decision maker's price rule, is such that the price (social planner) or the marginal revenue (monopolist) equal marginal cost at the last unit produced. This result is analogous to the findings of Spence (1981) who examined learning in the absence of innovations. However before the innovation occurs, the social planner's (monopolist's) price rule is such that the price (marginal cost) equals marginal cost at t_1 plus a negative constant. At the time of innovation, the costate variables of the two phases equal the ratio of the learning parameters λ/γ. Thus there is a downward jump in prices at the innovation date. In a second step, a subsidy on innovation cost and a subsidy on sales are introduced. The central results of these market interventions are: innovation subsidies and distribution subsidies reduce the prices of both phases if all variables (the timing of innovation t_1, the cumulative production quantities at the innovation date; $y(t_1)$ and at the end of the planning horizon $y(T)$) are endogenous. Both subsidy types induce innovation to proceed earlier. Consequently the total quantity produced during the entire planning horizon increases. The production plan in the presence of subsidies lies entirely above the production plan without subsidies. The result being, if early distribution yield positive externalities, then subsidies on sales and on innovation contain an additional positive effect.[3]

It is also shown that a subsidy on innovation cost (sales), which is financed through a tax on sales (innovation cost) changes the proportion of consumer and producer rents. Customers generally benefit more from sales subsidies, producers from innovation

[3]For technologies in the renewable energy sector holds that early distributions increase their positive externality on the atmosphere more. The total carbon dioxide emissions are reduced more, because renewable energy sources can substitute conventional $C0_2$ emitting ones earlier.

subsidies. Another central result emerges, when the total production $y(T)$ is restricted. In this case the two kinds of subsidies that are analyzed have different effects.

The next section introduces the model and solves for an optimal pricing rule, which is analyzed in detail. Section 1.3 endogenizes the timing of innovation. Section 1.4 continues with a welfare analysis. Section 1.5 concludes.

1.2 The Model with an exogenous innovation date

This model is solved for different market structures, at first the social planner's problem is solved, which can be easily extended to account for a market with perfect competition that yields quite similar results. The learning by doing case without innovation has been examined similarly by Brueckner et al. (1983). Later a monopolist takes the place of the social planner. This scenario is more relevant to reality, because in an environment of innovations, patents guarantee that their holders are able to execute market power. It has been rarely observed that a state runs a public firm in a learning industry, nevertheless a social planner's actions are examined as though they are almost identical to those of a monopolist.

1.2.1 The Social Planner's Problem

Assume there is a publicly owned firm, which faces the demand function: $x(p(t), t)$ for a non-storable output $x(t)$ that is sold at a price $p(t)$. Time is denoted by $t \in R^0_+$. The beginning of the first phase, when the planning horizon begins is t_0. The time when the innovation takes place is t_1. It is the end of the first phase and the beginning of the second phase. The planning horizon ends at $t = T$. The firm chooses an optimal time path for its control variables during the first phase, $p_0(t)$ and the second phase, $p_1(t)$; where $p(t) = \{p_0(t), p_1(t)\}$ The instantaneous production flows of the first and second phase are $x_0(p_0(t), t)$ and $x_1(p_1(t), t)$ respectively. They are the derivatives of the state variables $y_0(t)$ and $y_1(t)$, which are the cumulative production quantities for a period t before and after the innovation. Over the intervals $[t_0, t_1]$ and $[t_1, T]$, the social planner receives a stream of consumption benefits discounted back to $t = t_0$,

$$\int_{t_0}^{t_1} B_0(p_0(t), t)e^{-r(t-t_0)}dt \text{ and } \int_{t_1}^{T} B_1(p_1(t), t)e^{-r(t-t_0)}dt \tag{1.1}$$

where $B_0(p_0(t), t)$ and $B_1(p_1(t), t)$ denote the per-period social surplus during the first and second phase. They are each equivalent to the area below the inverse demand function at some t, before and after t_1, respectively. The social planner faces a marginal cost that consists of two parts; a fixed part denoted by the parameter m, and a variable part that is equal to c at the beginning of the first phase, when experience $y_0(t)$ equals zero. This variable part decreases with a learning parameter λ before an innovation occurs. Intuitively there is continuous discounting involved, which is expressed by the exponential term.

$$MC_0(y_0(t)) = m + ce^{-\lambda y_0(t)} \qquad for\ t_0\ \leq t \leq t_1 \tag{1.2}$$

The social planner can adopt a new technology, when the research department has been successful. The faster an innovation occurs, the more costly it is. For now, the innovation cost function depends solely on the innovation date t_1. When no innovation occurs and the firm produces with the same technology during the entire planning horizon, then the innovation cost is zero; $a(t_1) > 0, \nabla t \setminus t = T$ where $a(T) = 0$. $a' < 0$, $a'' > 0$. A new technology is adopted right after the innovation. Otherwise, if a later date of innovation is preferred, the planner could reduce its cost by devoting fewer resources to its research department. A different cost function is introduced in section 1.2.4. The innovation cost is assumed to be paid in advance at t_0. After t_1 the firm faces more intensive learning; it learns with a learning parameter $\gamma \geq \lambda$. Experience completely transfers to the new technology . Switching costs are ignored, because they do not yield results, which extend the knowledge of the existing literature (see Parente, 1993). The second phase's marginal costs are

$$MC_1(y_1(t)) = m + ce^{-\lambda y_0(t_1) - \gamma[y(t) - y_0(t_1)]} \qquad for\ t_1\ < t \leq T \tag{1.3}$$

Thus the social planner's objective is,

$$\underset{p_0(t),\, p_1(t),\, t_1}{Max} \; SP \equiv \int_{t_0}^{t_1} e^{-r(t-t_0)} \left\{ B_0(p_0(t), t) - (m + ce^{-\lambda y_0(t)}) x_0[p_0(t), t] \right\} dt \tag{1.4}$$

$$-a(t_1)e^{-rt_0} + \int_{t_1}^{T} e^{-r(t-t_0)} \left\{ B_1(p_1(t), t) - (m + ce^{-\lambda y_0(t_1) - \gamma[y(t) - y_0(t_1)]}) x_1[p_1(t), t] \right\} dt,$$

where $\frac{\partial B_i}{\partial p_i} = p_i(t) \frac{\partial x_i[p_i(t), t]}{\partial p_i(t)}$ for $i \in (0, 1)$. The constraints of the problem are given by

$$\dot{y_0}(t) = x_0[p_0(t), t] \quad t \in [t_0, t_1] \tag{1.5}$$

$$\dot{y_1}(t) = x_1[p_1(t), t] \quad t \in [t_1, T] \tag{1.6}$$

$$y_0(t_0) = 0 \tag{1.7}$$

$$y_0(t_1) = y_1(t_1) = y(t_1) \text{ is free} \tag{1.8}$$

$$y_1(T) \text{ is free} \tag{1.9}$$

$\dot{y_0}(t)$ and $\dot{y_1}(t)$ are time derivatives of cumulative production quantities or experience stocks. To keep this analysis simple, a real interest rate of zero is assumed. In the appendix it is shown, how the equilibrium changes when $r \neq 0$. Flows are functions of the price and time, where the price itself is a function of time. The cumulative quantity cannot change over night, when the innovation takes place and the new production process is adopted (1.8). Condition (1.9) is used as a transversality condition for the second phase. Necessary conditions of this problem are derived in two steps. Firstly this study examines some innovation date $t_1 \in [t_0, T]$ and solves for the price paths $p_0(t)$ and $p_1(t)$ with t_1 being fixed. In the next step the innovation date is endogenized.

Proposition 1.1 *A social planner chooses a constant price for each period of phase one and two respectively. The two prices are different across phase one* $[t_0, t_1]$ *and phase two* $[t_1, T]$.

Proof By a theorem of Hestens, take SP (1.4) with a fix t_1 and define $\eta_0(t)$ on the interval $[t_0, t_1]$ and $\eta_1(t)$ on the interval $[t_1, T]$ as the costate variables of the cumulative quantities $y_0(t)$ and $y_1(t)$ respectively.[4] The innovation cost function is $a(t_1)$. It can be ignored during the time the pricing rule is analyzed, because t_1 is fixed. Thus $a(t_1)$ is constant and drops out of the first order condition that describes the optimal pricing rule. The Hamiltonian is

$$H\ [p_0(t),\ p_1(t),\ \eta_0(t),\ \eta_1(t)] = B_0(t) - C_0(t) + B_1(p_1(t), t) - C_1(t) \tag{1.10}$$

$$-a(t_1) + \ \eta_0(t)x[p_0(t), t] + \eta_1(t)x[p_1(t), t]$$

where $C_i(t) = x_i(t)MC_i(y_i(t))$ for $i \in (0, 1)$ is the per-period cost. $p_0^*(t)$ and $p_1^*(t)$ maximize (1.10) such that

[4] see Takayama p.658

$$H\ [p_0^*(t),\ p_1^*(t),\ \eta_0(t),\ \eta_1(t)] \geq H\ [p_0(t),\ p_1(t),\ \eta_0(t),\ \eta_1(t)] \tag{1.11}$$
$$\text{for all }\ p_0(t) \geq 0,\ p_1(t) \geq 0.$$

The pricing rule for the first phase ($t \leq t_1$)

As $p_0^*(t)$ maximizes H for ($t \leq t_1$), the necessary condition is

$$\frac{\partial H}{\partial p_0(t)} = p_0(t)\frac{\partial x_0(p_0(t),t)}{\partial p_0(t)} - \frac{\partial x_0(p_0(t),t)}{\partial p_0(t)}(m + ce^{-\lambda y_0(t)}) + \eta_0(t)\frac{\partial x_0(p_0(t),t)}{\partial p_0(t)} \doteq 0$$

$$\Leftrightarrow p_0(t) = m + ce^{-\lambda y_0(t)} - \eta_0(t) \tag{1.12}$$

The social planner sets a price that equals the marginal cost minus the shadow price of cumulative quantity at some t. The second necessary condition is

$$\dot{\eta_0}(t) = -\frac{\partial H}{\partial y_0(t)}$$

$$\Leftrightarrow \eta_0(t) = ce^{-\lambda y_0(t)} + const_1 \tag{1.13}$$

The third necessary condition is (1.5).

Lemma 1.1 *The shadow price at the end point of the first phase equals* $\eta_0(t_1) = \frac{\lambda}{\gamma}ce^{-\lambda y(t_1)} - \frac{\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)}$.

Proof. See appendix. ■

The second necessary condition (1.13) can be solved for $const_1$ with the transversality condition $\eta_0(t_1) = -\frac{\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)} + \frac{\lambda}{\gamma}ce^{-\lambda y(t_1)}$. Evaluating $\eta_0(t)$ at t_1

$$const_1 = -\frac{\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)} + \frac{\lambda-\gamma}{\gamma}ce^{-\lambda y(t_1)}$$

$$\Longrightarrow \eta_0(t) = ce^{-\lambda y_0(t)} - \frac{\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)} + \frac{\lambda-\gamma}{\gamma}ce^{-\lambda y(t_1)} \tag{1.14}$$

(1.13) and (1.14) solve for the price of the first phase

$$p_0 = m + \frac{\gamma-\lambda}{\gamma}ce^{-\lambda y(t_1)} + \frac{\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)}; \qquad t_0 \leq t \leq t_1 \tag{1.15}$$

During the first phase, p_0 is independent of time, which completes the first part of the proof of 1.1

The pricing rule for the second phase ($t > t_1$)

The necessary first order condition with respect to $p_1(t)$ can be solved for a function of the the second phase's costate

$$\frac{\partial H}{\partial p_1(t)} = p_1(t)\frac{\partial x_1(p_1(t),t)}{\partial p_1(t)} - \frac{\partial x_1(p_1(t),t)}{\partial p_1(t)}(m+ce^{-\lambda y(t_1)-\gamma[y(t)-y(t_1)]}) + \eta_1(t)\frac{\partial x_1(p_1(t),t)}{\partial p_1(t)} \doteq 0$$

$$\Leftrightarrow p_1(t) = m + ce^{-\lambda y(t_1)-\gamma[y(t)-y_1(t_1)]} - \eta_1(t) \tag{1.16}$$

The social planner's price is equal to the marginal cost minus the shadow price of cumulative quantity. The second condition that needs to be fulfilled is

$$\dot{\eta_1}(t) = -\frac{\partial H}{\partial y_1(t)}$$

$$\Leftrightarrow \eta_1(t) = ce^{-\lambda y(t_1)-\gamma[y(t)-y(t_1)]} + const_2 \tag{1.17}$$

The third condition is given by (1.6). $\eta(T) = 0$, because the value of experience at the end of the second phase is zero. The cumulative quantity at the end of the second phase is not restricted, hence (1.9) can be used to set up the following transversality condition, which solves for $const_2$.

$$const_2 = -ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)}$$

$$\Longrightarrow \eta_1(t) = ce^{-\lambda y(t_1)-\gamma[y(t)-y(t_1)]} - ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)} \tag{1.18}$$

(1.17) and (1.18) are used to express the second phase's price

$$p_1 = m + ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)}; \qquad t_1 \le t \le T \tag{1.19}$$

For any t where $t_1 \le t \le T$, the price of phase 2 is constant. This completes the second part of proposition 1.1's proof.

It is assumed that the demand function does not change over time, thus the planner produces the same quantity in each period within the first phase and the same quantity within the second phase. The intuition behind this result is: although costs decrease through time, which would yield lower prices in a static model, the decrease of costs is completely offset by the decrease of the experience value in this dynamic framework. When either $\gamma = \lambda$ or $t_1 = T$, then (1.15) and (1.19) are equal: $p_0(t) = p_1(t) = m + ce^{-\lambda y(T)}$. For $\gamma > \lambda$ or $t_1 < T$, the price of the first phase exceeds the price of the second, which is as follows

$$p_0 > p_1$$
$$\Leftrightarrow \frac{\lambda}{\gamma} e^{(\gamma-\lambda)y(t_1)-\gamma y(T)} - \frac{\lambda-\gamma}{\gamma} e^{-\lambda y(t_1)} > e^{(\gamma-\lambda)y(t_1)-\gamma y(T)}$$
$$\Leftrightarrow \frac{\lambda-\gamma}{\gamma} e^{(\gamma-\lambda)y(t_1)-\gamma y(T)} > \frac{\lambda-\gamma}{\gamma} e^{-\lambda y(t_1)}$$
$$\gamma[y(t_1) - y(T)] < 0$$

As the last expression holds, it follows that the claim $p_0 > p_1$ is correct. A social planner encounters a loss, because during the first phase, the price is below the marginal cost at t_1, during the second phase the price just covers its cost at $t = T$ and is below that level for all preceding periods. Therefore one would need to introduce a tax on a different market to compensate for the loss. The monopolist's problem is solved, before the results, which are quite similar are interpreted further.

1.2.2 The Monopolist's Problem

The monopolist's instantaneous profit functions for the two phases are,

$$\pi^0(t) \equiv \left[p_0(t) - (m + ce^{-\lambda y_0(t)})\right] x_0(p_0(t), t) : \qquad t \in [t_0, t_1] \tag{1.20}$$

$$\pi^1(t) \equiv \left[p_1(t) - (m + ce^{-\lambda y_0(t_1)-\gamma[y(t)-y_0(t_1)]})\right] x_1(p_1(t), t) : \qquad t \in [t_1, T] \tag{1.21}$$

where the variables and parameters are defined and interpreted in the social planner's problem. The firm's objective is,

$$Max\ MP = \int_{t_0}^{t_1} \pi^0(t)e^{-r(t-t_0)}dt - a(t_1)e^{-rt_0} + \int_{t_1}^{T} \pi^1(t)e^{-r(t-t_0)}dt \tag{1.22}$$

subject to constraints (1.5) to (1.9). In the absence of discounting, the Hamiltonian equals

$$H\ [p_0(t),\ p_1(t),\ \eta_0(t),\ \eta_1(t)] = \pi^0(t) + \pi^1(t) - a(t_1) + \ \eta_0(t)x[p_0(t), t] + \eta_1(t)x[p_1(t), t] \tag{1.23}$$

One can use the same methods that were used to derive (1.15) and (1.19) to derive the pricing rules when a monopolist is the decision maker

$$p_0\left(1 - \frac{1}{\varepsilon(t)}\right) = m + ce^{-\lambda y(t_1)} - \eta_0(t)$$
$$\text{where } \varepsilon(t) = -\frac{\partial x_0}{\partial p_0}\frac{p_0}{x_0}$$

$$\Rightarrow MR_0(t) = m + ce^{-\lambda y(t_1)} - \left[\frac{\lambda}{\gamma}ce^{-\lambda y(t_1)} - ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)}\right]; \qquad t_0 \leq t \leq t_1 \tag{1.24}$$

(1.24) is the pricing rule before the innovation date,

$$p_1\left(1 - \frac{1}{\varepsilon(t)}\right) = m + ce^{-\lambda y(t_1)-\gamma[y(t)-y_1(t_1)]} - \eta_1(t)$$
$$\text{where } \varepsilon(t) = -\frac{\partial x_1}{\partial p_1}\frac{p_1}{x_1}$$

$$\Rightarrow MR_1(t) = m + ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)} \tag{1.25}$$

and (1.25) the innovation date after t_1. Therefore the monopolist sets the price where the marginal revenue equals marginal cost minus the shadow price of cumulative quantity. The only difference to the social planner's problem is that the optimal rule contains the multiplier $(1 - \frac{1}{\varepsilon(t)})$, and thus the marginal revenue and not the price, appears in the optimality condition. Prices in the monopoly model are constant for constant elasticities $\varepsilon(t) = \varepsilon$, hence the same holds for the per period production quantities. When either the equality $\gamma = \lambda$ or $t_1 = T$ hold, then (1.24) reduces to (1.25): $MR_0(t) = MR_1(t) = m + ce^{-\lambda y(T)}$. This is the classical optimal pricing behavior of a learning monopolist in the absence of innovations shown by Spence (1981): "At every time, output should be profit maximizing output, given that marginal cost is the unit cost that obtains at the end of the period".[5] The total cost that a firm faces is

[5] See page 52.

the area underneath the learning curve or marginal cost curve between t_0 and T. If the firm increases output by ϵ_0, in any interval within $[t_0, t_1]$ or by ϵ_1, in any interval within $[t_1, T]$, then the incremental cost is not the cost that arises during that time. It is rather the change of the total area below the learning curve.

Proposition 1.2 *In the presence of innovations, a monopolist charges a price such that marginal revenue equals incremental cost at any point in time.*

Proof. See appendix. ■

Consequently, within the interval $[t_1, T]$ the monopolist prices optimally, when the marginal revenue at each point in time equals the marginal cost of the last unit produced. $MR_1(t) = MC|_{t=T}$ by (1.25). In the interval $[t_0, t_1]$ the monopolist charges $MR_0(t) = MC|_{t=t_1} - \frac{\lambda}{\gamma}\left[ce^{-\lambda y(t_1)} - ce^{-\lambda y(t_1) - \gamma[y(T) - y(t_1)]}\right]$ by (1.24), where the sum in brackets is positive. Consequently, the monopolist charges a lower price such that marginal revenue at each point in time is below the marginal cost of the last unit produced at t_1, because production continues beyond t_1. The "price discount" equals $\frac{\lambda}{\gamma}\left[ce^{-\lambda y(t_1) - \gamma[y(T) - y(t_1)]} - ce^{-\lambda y(t_1)}\right]$.[6] It contains information about how much the experience level $y(t_1)$ is worth for the production after t_1. In the next section this term is analyzed further.

If discounting is included in the analysis, then prices increase compared to those in (1.24) and (1.25) for all t. When future profits are discounted, then learning is valued less, because the experience payoff decreases. Thus in the presence of a positive discount rate, the firm increase its price over the entire planning horizon. In a model without discounting, learning is appreciated most in the beginning of the planning horizon, because its return lasts for a long period of time. In the absence of innovations the price difference between a model with and without a discount rate, reaches its peak at t_0. In this model, where innovation increases the learning parameter, the price difference could even be larger at t_1 than at t_0, because the learning intensity jumps. At T, prices that include discounting are equal to those where discounting is absent, because the return to experience is non-existing.

Proposition 1.3 *When $r \neq 0$, then a monopolist sets its price during the first phase, such that the following condition holds,* $MR_0(t) = m + \frac{\lambda}{\gamma} ce^{(\gamma - \lambda) y(t_1) - \gamma y(T)} + \frac{\gamma - \lambda}{\gamma} ce^{-\lambda y(t_1)} +$

[6] To be precise, the price discount also contains the constant multiplier $\left(1 - \frac{1}{\varepsilon}\right)^{-1}$, which is ignored in the following partial analysis.

$r \int_t^{t_1} \left[m + ce^{-\lambda y_0(\tau)}\right] e^{-r(\tau - t)} d\tau + r \int_{t_1}^{T} [m + ce^{(\gamma-\lambda)y(t_1)-\gamma y(\tau)}] e^{-r(\tau-t)} d\tau.$ *During the second phase the optimality condition is* $MR_1(t) = m + ce^{(\gamma-\lambda)y(t_1)-\gamma y_1(T)} + r \int_t^{T} \left[m + ce^{(\gamma-\lambda)y(t_1)-\gamma y_1(\tau)}\right] e^{-r(\tau-t)} d\tau$

Proof. See appendix. ■

The additional terms on the right hand side are positive, which implies that the price increases. At $r = 0$, the conditions reduce to (1.24) and (1.25).

The market equilibrium of a social planner and a monopolist are quite similar. In the past, infant industries have been heavily subsidized by governments, but they were not run as public firms. Examples are the aerospace and defense industry during and after World War Two, computer industries in the 1980/90s and firms that have operated in the renewable energy sector during the last 10 years. In the presence of learning and innovations, where the later can be protected by property rights, there are either monopolies or oligopolies in the market. This holds true for all industries mentioned above: Airbus and Boeing (aerospace market), Microsoft and IBM (software and hardware) and the renewable energy sector, where for instance five producers have a market share of over 90% of worldwide wind turbine sales.[7] Based on these real world observations, for the rest of this article, it seems reasonable to assume that a monopolist is the decision maker. Furthermore it does not matter much, because the pricing rules differ by a multiplier that depends on the demand elasticity.

1.2.3 A partial comparative analysis

The price discount of the first phase

The first phase's price discount is $\frac{\lambda}{\gamma}\left[ce^{-\lambda y(t_1)} - ce^{-\lambda y(t_1)-\gamma[y(T)-y(t_1)]}\right]$. This section studies the discount's size based on the underlying parameters. It follows a comparative analysis; the discount is partially differentiated with respect to the parameters λ, γ and c. It is important to note that all parameters affect the three variables t_1, $y(t_1)$ and $y(T)$, which are fixed here. This analysis is meant to explain intuitively the results that are derived later, when t_1 and $y(t_1)$ are endogenous, but $y(T)$ is not.

$$\frac{\partial(\cdot)}{\partial\lambda}[ce^{-\lambda y(t_1)} - ce^{-\lambda y(t_1)-\gamma[y(T)-y(t_1)]}]\left(\frac{1}{\gamma} - \frac{\lambda}{\gamma}y(t_1)\right) \tag{1.26}$$

[7] Press release of BTM Consult ApS (27.3.2008).

The first bracket of (1.26) is positive, the second is positive for $\lambda y(t_1) < 1$, which is satisfied in the numerical simulation later. The price discount of the first phase rises with the learning parameter of the first phase, λ. The monopolist reduces its first phase's price to exploit a larger learning intensity.

$$\frac{\partial(\cdot)}{\partial\gamma} = -\frac{\lambda}{\gamma^2}\left[ce^{-\lambda y(t_1)} - ce^{-\lambda y(t_1)-\gamma[y(T)-y(t_1)]}\right] + \frac{\lambda}{\gamma}ce^{-\lambda y(t_1)-\gamma[y(T)-y(t_1)]}\left[y(T) - y(t_1)\right] \tag{1.27}$$

In (1.27), the first summand is negative, because its bracket term is positive. The second summand is positive. The first summand exceeds the second in absolute value conditional on $e^{\gamma[y(T)-y(t_1)]} - 1 > \gamma[y(T) - y(t_1)]$. This condition is met when the produced quantity after t_1 is large enough. A large learning parameter after the innovation, γ decreases the incentive to reduce the incentive to reduce cost before the innovation date.

$$\frac{\partial(\cdot)}{\partial c} = \frac{\lambda}{\gamma}\left[e^{-\lambda y(t_1)} - e^{-\lambda y(t_1)-\gamma[y(T)-y(t_1)]}\right] \tag{1.28}$$

(1.28) shows, how the variable part of the marginal cost level affects the price discount of the first phase.[8] The derivative is positive, because the return to experience increases when the original cost level is high. The price discount on p_0 increases with λ and c, it decreases with γ.

The Costates

The costate variables are positive for all t, however they decrease. $\eta_0(t)$ declines at a rate of the marginal cost's derivative for the first phase, $\eta_1(t)$ at a rate of the marginal cost's derivative for the second phase. An interesting result is that the quotient of the two costates at the optimal innovation time t_1 is the quotient of the learning parameters:

$$\eta_0(t_1) = \frac{\lambda}{\gamma}ce^{-\lambda y(t_1)} - \frac{\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)} = \frac{\lambda}{\gamma}\eta_1(t_1) \Rightarrow \frac{\eta_0(t_1)}{\eta_1(t_1)} = \frac{\lambda}{\gamma} \tag{1.29}$$

Figure 1.1 shows the course of two costates, given a cumulative production quantity at the innovation date of 50 and 100. The cumulative quantity at the end of the planning horizon is 150, thus experience becomes worthless and both costate functions converge

[8] Recall that $MC|_{t=0} = m + c$

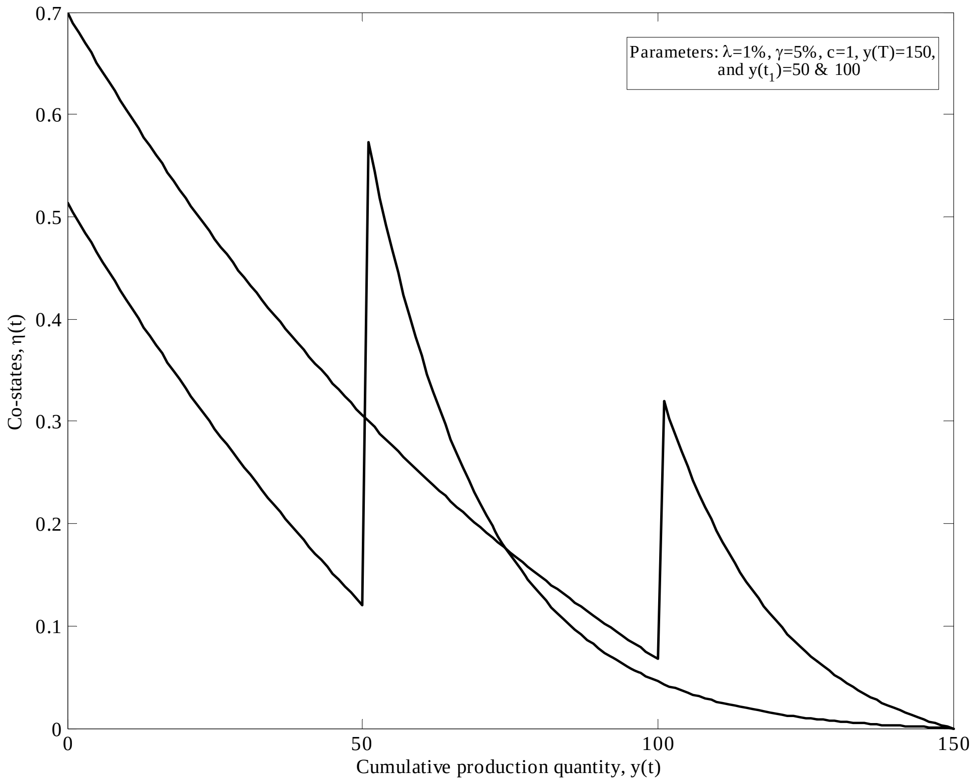

Figure 1.1: Costates jump at the date of innovation

to the horizontal axis. At $y(t_1)$ the costates jump vertically upward such that the costate at $\lim_{\epsilon \longrightarrow 0} [y(t_1) + \epsilon]$ is $\frac{\gamma}{\lambda}$ times larger compared to its value at $\lim_{\epsilon \longrightarrow 0} [y(t_1) - \epsilon]$. Prices have been derived as functions of costates during phase one, $(1 - \frac{1}{\varepsilon})^{-1} p_0(t) = m + ce^{-\lambda y(t)} - \eta_0(t)$ [see (1.24)] and phase two, $(1 - \frac{1}{\varepsilon})^{-1} p_1(t) = m + ce^{-\lambda y(t_1) - \gamma[y(t) - y_1(t_1)]} - \eta_1(t)$ [see (1.25)]. At t_1, the prices $p_0(t)$ and $p_1(t)$ reduce to $\left(m + ce^{-\lambda y(t_1)}\right)(1 - \frac{1}{\varepsilon})$, subtracted by $\eta_i(t)(1 - \frac{1}{\varepsilon})$ for $i \in (0, 1)$. Figure 1 clearly shows an upward jump of costates, which means that prices drop discontinuously by the amount that the costates jump with their constant multiplier.

In the past there have been government interventions that aimed to sell a fix number of products, which are characterized through positive externalities e.g. solar panels.[9] Figure 1.1 shows a decrease of $y(t_1)$ from 100 to 50, keeping $y(T)$ and all parameters constant. The costate of the function, where the innovation occurs earlier is lower

[9]The "100,000 roof-program" was part of the Renewable Energy Law in Germany. It intended to install 100,000 solar panels (which would be equivalent to $y(T)$) in a given time (T).

during its first phase compared to the other costate. It exceeds the other costate thereafter before $y(t_1) = 75$ is reached. Afterwards it is lower again. Therefore the price during the first phase decreases with $y(t_1)$. The next proposition shows that the same holds true for t_1.

Proposition 1.4 *An earlier innovation date t_1 increases the price p_0 and decreases the price p_1, iff $y(T)$ is fixed.*

Proof. The elasticity of demand is assumed to be constant and the monopolist chooses the optimal innovation date t_1 before production starts. Given p_0 (1.24) and p_1 (1.25), prices are differentiated with respect to t_1.

$$\frac{\partial p_0}{\partial t_1} = \underbrace{\frac{(\gamma-\lambda)\lambda}{\gamma}}_{>0} c \underbrace{\left[e^{(\gamma-\lambda)y(t_1)-\gamma y(T)} - e^{-\lambda y(t_1)}\right]}_{<0} \underbrace{\frac{\varepsilon}{\varepsilon-1}}_{>0} \underbrace{\frac{y(t_1)}{\partial t_1}}_{>0} < 0 \tag{1.30}$$

In (1.30) all terms except of one, are positive for $\gamma > \lambda$, $c > 0$, $y(t_1) < y(T)$. Later it is shown numerically that a delay of the innovation date increases the cumulative quantity up to the innovation date: $\frac{y(t_1)}{\partial t_1} > 0$.

$$\frac{\partial p_1}{\partial t_1} = \frac{\partial p_1}{\partial y(t_1)} \frac{y(t_1)}{\partial t_1} = c \underbrace{\left[(\gamma-\lambda)e^{(\gamma-\lambda)y(t_1)-\gamma y(T)}\right]}_{>0} \underbrace{\frac{\varepsilon}{\varepsilon-1}}_{>0} \underbrace{\frac{y(t_1)}{\partial t_1}}_{>0} > 0 \tag{1.31}$$

The price after t_1 increases, when the innovation occurs earlier. ■

This result seems to be puzzling at first glance. In the presence of learning without process innovations, the learning effect is smaller than the level effect of costs. Fudenberg and Tirole (1983) show that "output increases over time, ... [but] produce a lot now to lower costs, then ease off as an optimal control strategy" [was ruled out by their results].[10] This model also contains the same effects as in the Fundenberg and Tirole model: A firm chooses a production plan that maximizes today's profits taking account of all future cost reductions, where the later is determined by the learning effect. In the presence of innovations, the learning effect is stronger, when the date of innovation occurs later. Therefore a firm reduces p_0 when t_1 increases. At the same time it increases p_1, because future time $(T - t_1)$ decreases along with the benefit of future cost reduction. The "today's-profit maximizing-effect" is stronger than the learning effect and a decrease of t_1 comes with an increase of p_0. The intuitions provided by this partial analysis are helpful for section 1.3, where t_1^* is endogenous. The numerical

[10] See proposition 2 on p. 525.

part of section 1.3 contains two parts; in the first part all variables are endogenous, and it is shown that (1.30) and (1.31) do not hold. In the second part, where t_1^* and $y(t_1^*)$ are endogenous, but $y(T)$ is not, the results provided here, appear again.

1.2.4 A different innovation cost function

In this section the thus far neglected innovation cost function with a different structure is introduced. It was able to be neglected, because it depended solely on the innovation date, which was exogenous. What happens, when one assumes that the innovation cost function also depends on the experience accumulated before t_1 and $a = a[y(t_1), t_1]$, with $\frac{\partial a}{\partial y(t_1)} < 0$? This assumption is reasonable, when the research department works closely together with the production floor. The altered transversality condition of the first phase is $\eta_0(t_1) = -\frac{\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)} + \frac{\lambda}{\gamma}ce^{-\lambda y(t_1)} - \frac{\partial a[y(t_1),t_1]}{\partial y(t_1)}$. As innovation cost decreases with $y(t_1)$, so does the marginal revenue

$$MR_0(t) = m + \frac{\gamma-\lambda}{\gamma}ce^{-\lambda y(t_1)} + \frac{\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)} + \frac{\partial a[y(t_1), t_1]}{\partial y(t_1)}; \qquad t_0 \le t \le t_1 \tag{1.32}$$

Accordingly the price during the first phase decreases. The intuition behind the changes are obvious; experience does not only reduce future production cost, but also the cost of research.

1.3 The model with an endogenous innovation date

1.3.1 Analytical part

The optimal timing of innovation has been exogenous thus far, in this section it will be endogenized. The problem is solved for an innovation cost function of the form $a = a(t_1)$. With t_1 fixed, consider the following problem with the two segments,

$$S.0.\max_{p_0(t)} \int_{t_0}^{t_1} \{x_0(p_0,t)(p_0 - m - ce^{-\lambda y_0(t)})\}dt \tag{1.33}$$

$$s.t.\dot{y}_0(t) = x_0(p_0,t),\ t \in [t_0,t_1] \quad t_0, t_1 \text{ fixed}$$

and

$$S.I.\max_{p_1(t)} \int_{t_1}^{T} \{x_1(p_1,t)(p_1 - m - ce^{(\gamma-\lambda)y(t_1)-\gamma y_1(t)})\}dt \tag{1.34}$$

$$s.t.\dot{y_1}(t) = x_1[p_1,t],\ t \in [t_1,T] \quad t_1, T \text{ fixed}$$

When p_0^* and p_1^* are solutions to the problem MP (1.22), given conditions (1.5) to (1.9), then with the Hamiltonians being defined as usual,

$$H_0(p_0^*,\ \eta_0(t)) \geq H_0(p_0,\ \eta_0(t)) \qquad \nabla p_0(t) \geq 0 \tag{1.35}$$

$$H_1(p_1^*,\ \eta_1(t)) \geq H_1(p_1, \eta_1(t)) \qquad \nabla p_1(t) \geq 0 \tag{1.36}$$

where

$$H_0(p_0,\ \eta_0(t)) = \pi^0(t) + \eta_0(t)x(p_0,t)$$

$$H_1(p_1,\ \eta_1(t)) = \pi^1(t) + \eta_1(t)x(p_1,t)$$

Adding (1.35) and (1.36), one can see that p_0^* and p_1^* also satisfy Hamiltonian condition (1.11). Therefore the control variables that solve (1.22) also solve problems (1.35) and (1.36) respectively. Denote the maximized values of the objectives $S.0.$ (1.33) by $V_0^*(t)$ and $S.I.$ (1.34) by $V_1^*(t)$. A standard result of optimal control theory is

$$\frac{\partial V_0^*(t)}{\partial t_1} = H_0(t_1) \text{ and } \frac{\partial V_1^*(t)}{\partial t_1^*} = -H_1(t_1^*) \tag{1.37}$$

Consider the optimal value of t_1^*, denoted by $t_1^* \in (t_0, T)$. If t_1^* is optimal, it must solve

$$\max_{t_1} \left\{ \int_{t_0}^{t_1} \pi^0(p_0,t)dt + \int_{t_1}^{T} \pi^1(p_1,t)dt - a(t_1) \right\} = V_0^*(t) + V_1^*(t) - a(t_1) \tag{1.38}$$

$$\frac{\partial}{\partial t_1}[V_0^*(t) + V_1^*(t) - a(t_1)] = H_0(t_1^*) - H_1(t_1^*) - \frac{\partial a(t_1^*)}{\partial t_1^*} \doteq 0$$

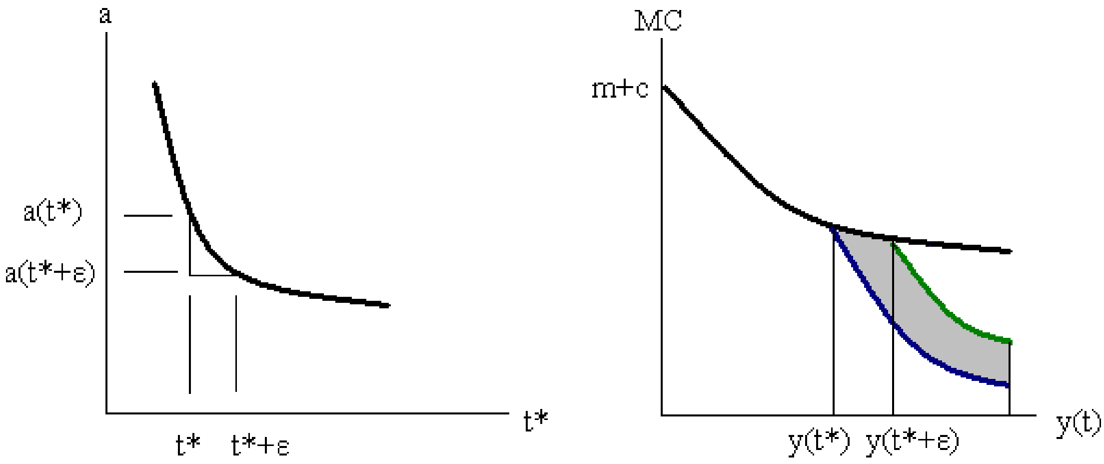

Figure 1.2: Benefits and costs of delaying innovation

$$\Leftrightarrow a'(t_1^*) = \pi^0(t_1^*) + \ \eta_0(t_1^*)x[p_0, t_1^*] - \pi^1(t_1^*) - \eta_1(t_1^*)x[p_1, t_1^*] \qquad (1.39)$$

(1.39) reveals that the optimal innovation date t_1^*, the difference of the first and second phase's returns equal the derivative of the innovation function with respect to t_1^*, where the word 'return' circumscribes the instantaneous profit flow $\pi(t_1^*)$ and the return to experience $\eta(t_1^*)x[p, t_1^*]$. A simple innovation cost function $a(\cdot)$ that shall mimic reality has the following characteristics: $a(t_1) > 0 \ \nabla t \setminus t = T$ where $a(T) = 0$, $a' < 0$, $a'' > 0$. Thus innovation is costly if it is implemented within the planning horizon. The cost is proportionally larger, the sooner innovation takes place. The right side of (1.39) is moderately negative over the entire planning horizon, which can be shown numerically. Therefore the equality of both sides is guaranteed for an appropriate set of parameters. Figure 1.2 demonstrates that postponing innovation comes along with lower innovation cost (left diagram), but an increase of the production cost (on the right). The optimality condition for t_1^* (1.39) conveys the same result; in the optimum the cost savings of delaying innovation per period (left side) equals the differences of return of phase 1 and 2 (right side).

In section 1.2.4, the innovation cost function was changed to $a[y(t_1), t_1]$. If one accounts for these changes here, (1.39) adjusts to

$$\frac{\partial a[y(t_1^*), t_1^*]}{\partial t_1^*} + \frac{\partial a[y(t_1^*), t_1^*]}{\partial y(t_1^*)} \frac{\partial y(t_1^*)}{\partial t_1^*} = \pi^0(t_1^*) + \eta_0(t_1^*)x(p_0) - \pi^1(t_1^*) - \eta_1(t_1^*)x(p_1, t_1^*) \quad (1.40)$$

If $\frac{\partial a[y(t_1),t_1]}{\partial t_1}$ has not changed and the assumptions $\frac{\partial a[y(t_1),t_1]}{\partial y(t_1)} < 0$ and $\frac{\partial y(t_1)}{\partial t_1} > 0$ hold, then the left side of (1.40) decreases. The optimal date of innovation would increase and the monopolist's research budget would decrease. When an increase in $y(t_1)$ decreases the innovation cost, then the value of waiting with an innovation adoption increases.

1.3.2 Numerical part

After introducing a specific demand function, a numerical simulation method is used to find specific values of t_1^*, $y(t_1^*)$ and $y(T)$. Tax and subsidy parameters are added simultaneously. They are also helpful for the next section, when a welfare analysis is carried out.

Demand: The per period inverse demand function is,

$$p(x_i) = \frac{{x_i}^{-\alpha}}{1-\alpha} \tag{1.41}$$

for $i \in (0,1)$. For $p(x_i)$ as defined in (1.41), the price elasticity of demand is $\varepsilon = -\frac{\partial x}{\partial p}\frac{p}{x} = \frac{1}{\alpha}$.[11] It follows from (1.24) and (1.25) that $MR_i = k_i \Leftrightarrow p_i(1-\alpha) = k_i \Leftrightarrow x_i^{-\alpha} = k_i$, where

$$k_0 = \left(m + \frac{\lambda}{\gamma} c e^{(\gamma-\lambda)y(t_1^*) - \gamma y(T)} + \frac{\gamma-\lambda}{\gamma} c e^{-\lambda y(t_1^*)}\right)$$

$$k_1 = \left(m + c e^{(\gamma-\lambda)y(t_1^*) - \gamma y(T)}\right)$$

Sales subsidy/ tax: An ad valorem subsidy ($\tau > 1$) or tax parameter ($\tau < 1$) is added, a subsidy shifts the demand function out, a tax shifts it in. A demand function of the the type in (1.41), which includes the parameter τ is

$$x_i = \left(\frac{\tau}{p_i(1-\alpha)}\right)^{\beta} \tag{1.42}$$

It is helpful to transform (1.39) such that the t_2-optimality condition becomes

$$x_0 \left[p_0 - MC(t_1^*) + \eta_0(t_1^*)\right] - x_1 \left[p_1 - MC(t_1^*) + \eta_1(t_1^*)\right] = a'(t_1^*) \tag{1.43}$$

(1.29), (1.42) and (1.2) or (1.3) are substituted in (1.43)

[11] Define $\beta = \frac{1}{\alpha}$. One can easily account for a time-varying demand function by multiplying $p(x_i)$ with $b(t)$ where $b(t)^{\beta} = be^{\delta t}$. But this does not add much to this analysis.

$$\frac{\alpha\tau^{\beta}}{1-\alpha}\left(k_0^{1-\beta}-k_1^{1-\beta}\right)=a'(t_1^*) \tag{1.44}$$

Innovation cost: An innovation cost function that fulfills the requirements; $a(t_1^*) > 0, \nabla t \setminus t = T$ where $a(T) = 0$. $a' < 0$, $a'' > 0$ is

$$a(t_1^*) = ie^{-\mu t_1^*} - ie^{-\mu T} \tag{1.45}$$

The parameter i affects the level of the innovation cost, the parameter μ the slope with respect to t_1^*.

Innovation subsidy/ tax: The innovation subsidy is constructed in a way such that the government either pays a part of the innovation cost, $s > 0$ or charges a tax, $s < 0$. Thus the gross innovation cost is

$$a(t_1^*) = \rho[ie^{-\mu t_1^*} - ie^{-\mu T}] \tag{1.46}$$

where $\rho = 1 - s$. $a'(t_1^*)$ decreases with ρ, hence it becomes less beneficial for the firm to procrastinate the innovation timing and the innovation date occurs earlier when innovation is subsidized. Equation (1.44) can be rewritten as

$$\frac{\alpha\tau^{\beta}}{1-\alpha}\left(k_0^{1-\beta}-k_1^{1-\beta}\right)=-\rho\mu ie^{-\mu t_1^*} \tag{1.47}$$

(1.47) is the optimality conditions for t_1. In addition, one needs to express $y(t_1^*)$ and $y(T)$ to solve for these three variables simultaneously. The demand function (1.42) and the optimality condition $p_i = (1-\alpha)^{-1}k_i$ solve for the per period demand of phase 1, $x_0 = \left(\frac{\tau}{k_0}\right)^{\beta}$. As the per period production is constant, the cumulative production quantity up to the innovation date t_1^* is simply

$$y^M(t_1^*) = \left(\frac{\tau}{k_0}\right)^{\beta}(t_1^* - t_0) \tag{1.48}$$

Similarly, $y(T)$ equals the integral of the second phase's per period production flows between t_1^* and T, which is added to the cumulative quantity up to t_1^*

$$y^M(T) = y(t_1^*) + \left(\frac{\tau}{k_1}\right)^{\beta}(T - t_1^*) \tag{1.49}$$

Equations (1.47), (1.48) and (1.49) are three independent equations that contain as many unknowns t_1^*, $y(t_1^*)$ and $y(T)$. The parameters c, m, i, γ, λ, α, μ, δ and the planning horizon, t_0 and T are known to the firm. Thus the model can be solved numerically.[12]

Results

When $y(T)$ is endogenous then (1.30) does not hold. Figure 1.3 illustrates, how the variables of the model t_1^*, $y(t_1^*)$, $y(T)$ and the prices p_0, p_1 are affected by τ, ρ (both upper part), λ, and γ (both lower part). The vertical axis of the cumulative quantities is on the right side of each of the four panels, the vertical axis of all other variables on the left side. In the upper left panel sales taxes ($\tau < 1$) and subsidies ($\tau > 1$), in the upper right panel innovation subsidies ($\rho < 1$) and taxes ($\rho > 1$) vary. Taxes/ subsidies are absent when $\tau = \rho = 1$.[13] Taxes and subsidies, either on sales or innovation cost have the same effect. Taxes increase the prices of both phases and decrease the total cumulative production. The cumulative quantity at the innovation date increases, whenever innovation is postponed.

The lower panel demonstrates what happens when the learning parameter λ varies in a range of $[0.5\%, 1.5\%]$, the lower right panel shows γ varying in a range of $[4.5\%, 5.5\%]$. A rise in either learning parameter reduces both prices and increases total cumulative quantity. Innovation is delayed when λ increases, it occurs earlier, when γ increases. $y(t_1^*)$ moves into the same direction as t_1^*. Subsidies and more intense learning therefore are shown to reduce prices. A large λ decreases both prices but the larger price p_0 is charged by the monopolist longer, because a postponement of innovation means that the length of phase 1 increases. In the absence of discounting the average per period price is still smaller. But if consumer discounting is high, a large λ might not be beneficial for buyers.

So far positive externalities, which justify subsidies have been ignored. An early distribution of solar panels is preferred to a later date, to reduce the total carbon dioxide concentration in the atmosphere. Both types of subsidies yield a lower p_0 and a lower t_1^*, which together guarantee that with regards to time, more products are sold

[12] So far a process innovation was analyzed, it is however fairly simple to account for a product innovation in this model. When a firm does not change its cost structure, but its product features, then it is possible to change the demand function from one with an exponent of α to one with an exponent different from α after t_1^*. This article considers the case, where consumers do not anticipate price changes, thus the demand is equal before and after t_1^*.

[13] The command 'fsolve' of the computer program MATLAB solves systems of nonlinear equations. It was used to derive t_1, $y(t_1)$ and $y(T)$ based on (1.47), (1.48) and (1.49). The parameters have been arbitrarily chosen. The results hold for other parameter sets, which yields a solution. The parameters used here are: $\alpha = 0.9$, $\gamma = 0.05$, $\lambda =$ 0.01, $t_0 = 0$, $\mu = 0.1$, $m = 1$, $c = 5$, $T = 50$ and $i = 20$.

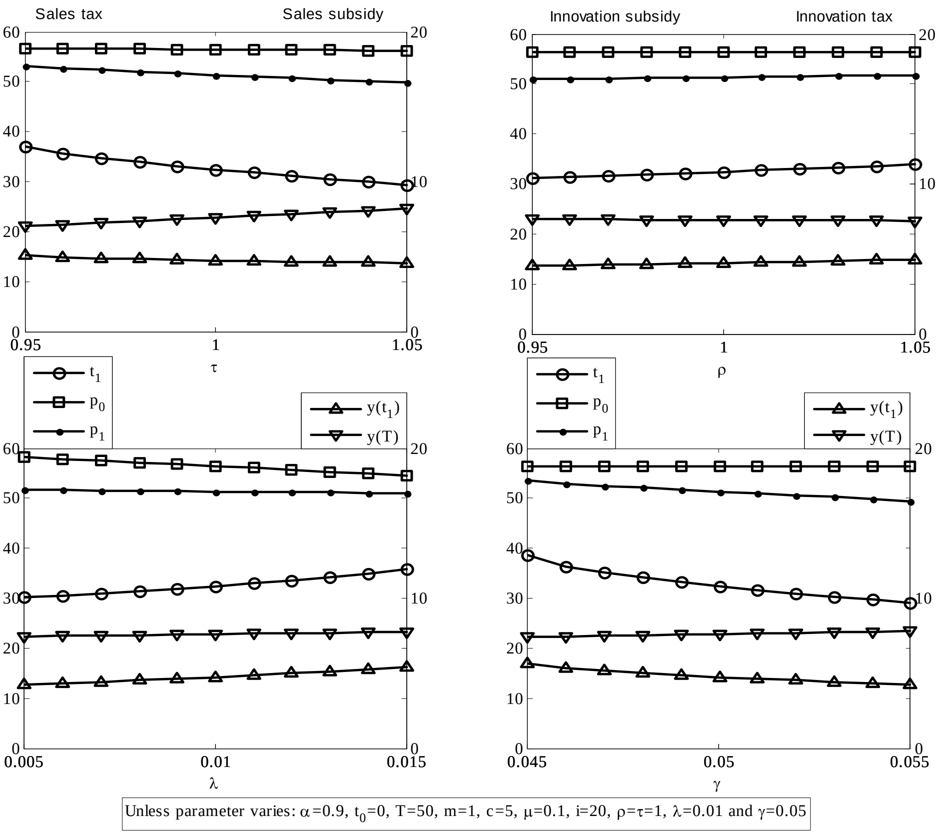

Figure 1.3: A comparative analysis, when all variables are endogenous

earlier, because $p_0 > p_1$. Thus both subsidies are beneficial, when early distributions play a role.

Results change drastically, when $\overline{y(T)}$ is given. The results of the partial analysis, (1.30) and (1.31) are supported by a numerical analysis, illustrated in figure 1.4. The structure and the parameters of this figure are the same as in figure 1.3. In figure 1.3 a total cumulative production quantity of around $y(T) = 8$ was derived. Figure 1.4 shows the results, when $y(T) = 8$ is assumed to be given exogenously. Thus the optimality condition (1.49) is excluded from the analysis. The results here have changed dramatically: innovation is postponed when sales are subsidized. It occurs earlier, when the innovation cost is subsidized. p_0 increases when innovation is subsidized, it decreases, when sales are subsidized. p_1 moves into the opposite direction. These results correspond with the analytical study above, where it was shown that $\frac{\partial p_0}{\partial t_1} < 0$ and $\frac{\partial p_0}{\partial t_1} > 0$ given $y(T)$ being exogenous. In the absence of subsidies the average production unit price equals $p^{AV} = y(t_1^*)p_0 + [y(T) - y(t_1^*)]\, p_1 = 53.47$. If a low sales subsidy ($\tau = 1.05$) is introduced, then the average price increases to $p^{AV,S} = 53.89$. An innovation subsidy ($\rho = 0.95$) decreases the average price $p^{AV,I} = 53.27$. If a regulator cares about consumer prices, but it wants to subsidize the industry as it yields positive externalities, then an innovation subsidy is preferred over a sales subsidy. Even though an innovation subsidy increases p_0, the average price decreases. In order to keep p_0 from rising, a price constraint could be implemented for the first time phase.

Proposition 1.5 *A price constraint during the first phase increases the consumer surplus and decreases t_1^*.*

Proof. See appendix. ■

The lower panels show the affect of the learning parameters. An increase of λ reduces p_0 strongly, while p_1 increases moderately. An increase of the second phase's learning parameter γ decreases p_1 and increases p_0. This observation is in line with the results from the partial analysis (1.26) and (1.27). A change of any learning parameter affects the timing of innovation, which influences the cumulative quantity at the innovation date positively if t_1^* increases and negatively if t_1^* decreases. A change of $y(t_1^*)$ affects the prices as shown by (1.30) and (1.31). The average price decreases, when either parameter increases. In the baseline case $\lambda = 1\%$, $\gamma = 5\%$ and the average price is $p^{AV} = 53.47$. When $\lambda = 1.5\%$, the average price increases to $p^{AV,\lambda} = 52.87$, when $\gamma = 5.5\%$ the average prices increases to $p^{AV,\gamma} = 52.67$.

When the timing of early distribution plays a role, one has to examine a change of t_1^* and p_0 to evaluate a policy or change of parameters in the same way as it has been done, when $y(T)$ was endogenous. An innovation subsidy yields a higher p_0, but innovation occurs earlier than in the presence of a sales subsidy, where p_0 is lower and t_1 is larger. With an innovation subsidy, fewer products are sold during the first phase. However, the second phase, during which the number of products sold per period is larger (as $p_1 < p_0$) begins earlier. Thus a regulator who is concerned about early distributions prefers a sales subsidy towards an innovation subsidy if she is very impatient.

1.4 More on welfare effects

This section contains a welfare analysis based on consumer and producer rents, which are first derived. It is illustrated that the two types of subsidies of either innovation or sales have different effects on consumer/ producer rents. This article does not contain a general welfare analysis, which needs to verify clearly the positive externalities that would induce a state to intervene. It would only be rational to do this, if one considers

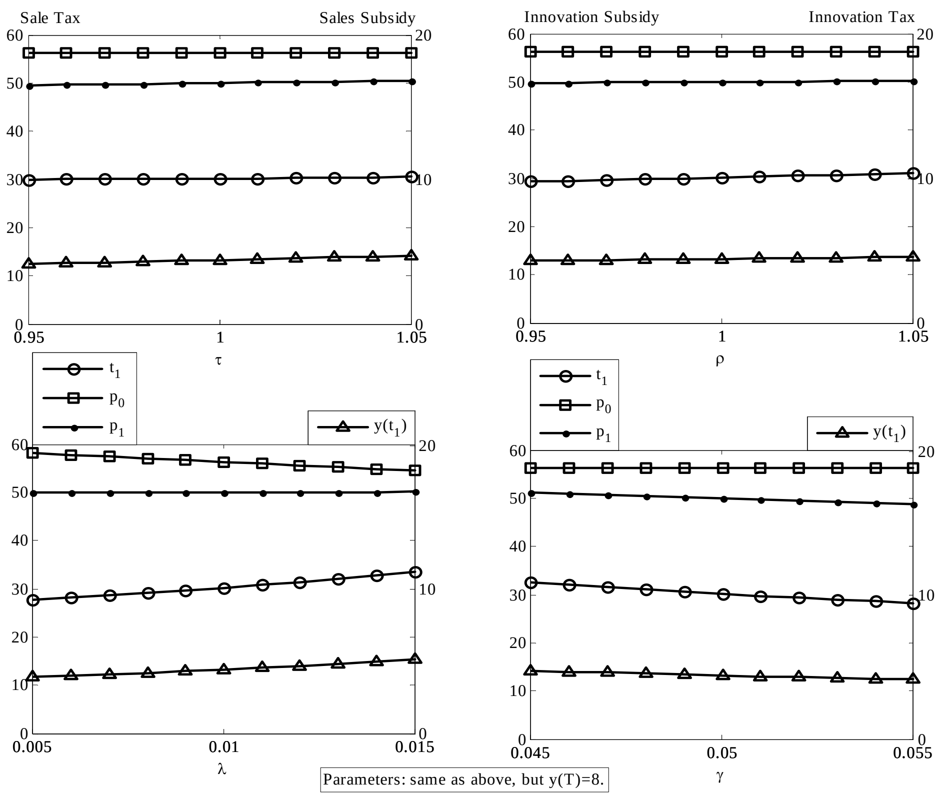

Figure 1.4: Comparative statics with $\overline{y(T)}$ being exogenously determined

a specific industry with information pertaining to the model's underlying parameters, which is beyond the scope of this paper. This section shows that an innovation (sales) subsidy, which is fully financed through a sales (innovation) tax, changes the proportion of consumer and producer rents. This way consumer rents could be increased, which has a market power mitigating effect.[14] In this part all variables are endogenous. The same inverse demand function holds; $p = \frac{\tau}{1-\alpha}x^{-\alpha}$ and in the numerical analysis, the same set of parameters is used as before.

Consumer Rents: The per-period consumer rent during the first phase, CR_t^0 equals the area underneath the inverse demand function above the monopoly price.[15]

$$CR_t^0 = \int_{p_0}^{\infty} x_0[p_0(t)]dp = \left(\frac{\tau}{1-\alpha}\right)^{\beta} \int_{p_0}^{\infty} p^{-\beta}dp = \left(\frac{\tau}{1-\alpha}\right)^{\beta} \frac{-1}{1-\beta}p_0^{1-\beta}$$

$$= \frac{\alpha}{(1-\alpha)^2}k_0^{1-\beta}\tau^{\beta} \tag{1.50}$$

The price is substituted for $k_0/(1-\alpha)$ and $k_0 = m + \frac{\gamma-\lambda}{\gamma}ce^{-\lambda y(t_1^*)} + \frac{\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1^*)-\gamma y(T)}$.[16] The consumer rent over the entire first phase CR^0 is

$$CR^0 = \int_{t_0}^{t_1^*} CR_t^0 dt = \frac{\alpha\tau^{\beta}k_0^{1-\beta}}{(1-\alpha)^2}(t_1^* - t_0) \tag{1.51}$$

Applying the same steps again, the per period consumer rent after the innovation has taken place CR_t^1 and the consumer rent over all these periods, CR^1 are

$$CR_t^1 = \frac{\alpha\tau^{\beta}k_1^{1-\beta}}{(1-\alpha)^2} \tag{1.52}$$

$$CR^1 = \frac{\alpha\tau^{\beta}k_1^{1-\beta}}{(1-\alpha)^2}(T - t_1^*) \tag{1.53}$$

where $k_1 = m + ce^{(\gamma-\lambda)y(t_1^*)-\gamma y(T)}$.

[14] A valuation of such a measure is not part of this analysis. It depends on the specific industry to judge, if such a procedure is justifiable.

[15] A necessary assumption is that the elasticity of demand exceeds one; $\beta = \frac{1}{\alpha} > 1$, to guarantee $\lim_{p\to\infty} p^{1-\beta} = 0$.

[16] An increase of α is equivalent to a decrease of the demand elasticity. Consider (1.50), an increase of α increases CR_t^0, which is quite intuitive. A monopolist picks a lower price when the elasticity is large.

Producer Rents: The producer rent of the monopoly, which equals its profit over the entire planning horizon is

$$PR = \int_{t_0}^{t_1^*} x_0 \left(p_0 - ce^{-\lambda y(t)} - m\right) dt + \int_{t_1^*}^{T} x_1 \left(p_1 - ce^{(\gamma-\lambda)y(t_1^*)-\gamma y(t)} - m\right) dt$$

$$= \left(\frac{k_0}{1-\alpha} - m\right)\left(\frac{\tau}{k_0}\right)^{\beta}(t_1^* - t_0) + \left(\frac{k_1}{1-\alpha} - m\right)\left(\frac{\tau}{k_1}\right)^{\beta}(T - t_1^*)$$

$$+\frac{1}{\lambda}\left[ce^{-\lambda y(t_1^*)} - ce^{-\lambda y(t_0)}\right] + \frac{1}{\gamma}\left[ce^{(\gamma-\lambda)y(t_1^*)-\gamma y(T)} - ce^{-\lambda y(t_1^*)}\right] \quad (1.54)$$

Results

The results are summarized by figure 1.5. In the absence of government interventions consumer and producer rents are given by the horizontal line. After a sales subsidy is introduced to the market, consumer and producer rents jump upwards to the level, where the falling curves touch the vertical axis. Then an innovation tax comes into place. It is depicted on the horizontal axis of each panel. An increase of the innovation tax reduces consumer and producer welfare, which is expressed by the falling graph. Producer rents fall much faster than consumer rents. Hence a sales subsidy, which is financed through an innovation tax has market power mitigating effects. This illustration shall not propose such a market intervention, rather it shows that either subsidy type has different effects on producer and consumer rents. A similar figure could be shown for welfare changes through an innovation subsidy that is financed by a sales tax, where the upper curve is steeper for consumers.

1.5 Conclusion

The innovation of this model is that it is able to evaluate distribution and innovation subsidies, while innovation costs depend on time, and learning depends on cumulative production. This article examines the pricing of a monopolist and a public owned firm in an environment, where the unit cost of production decreases through learning. The learning intensity decreases with cumulative production. The firm can invest in an innovation process. At the time, when research is successful, learning for a given production quantity jumps. Thus a lower cost level can be achieved with less cumulative

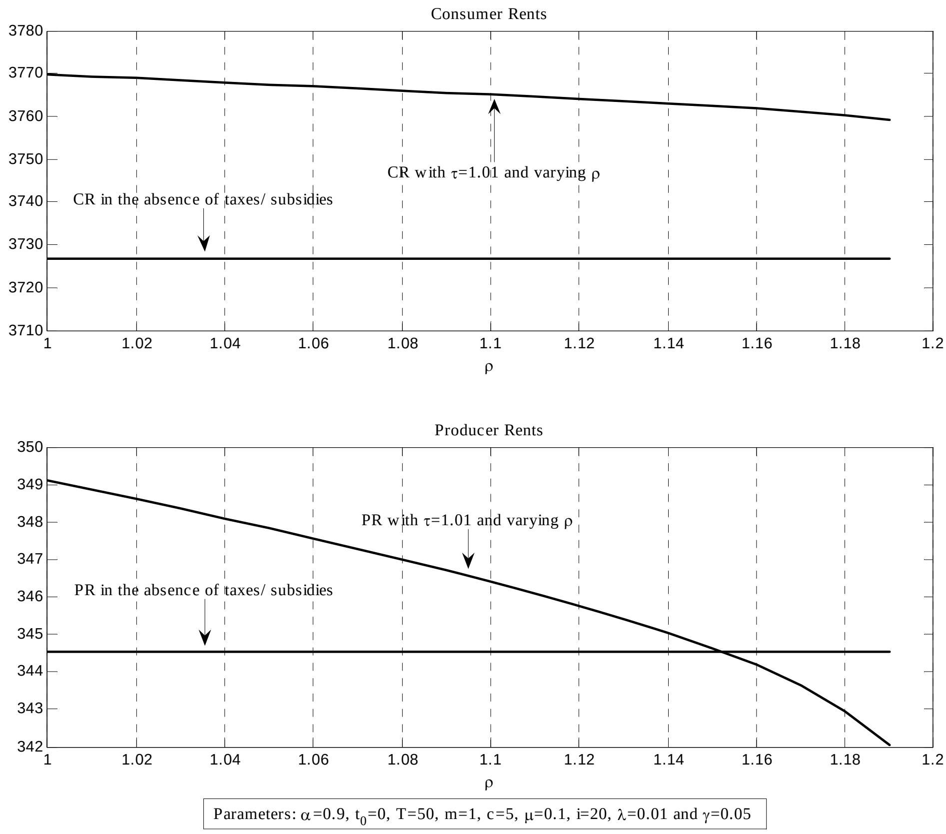

Figure 1.5: Welfare effects of government interventions

production added. So far models, which have included learning and innovation have assumed that learning occurs through time. Hence these models are not applicable to questions in the area of industrial organization, because learning has no effect on production or pricing.

In the absence of discounting it is shown that prices before and after the innovation date are constant. The price of the first phase exceeds the price of the second phase, hence there is a downward jump when the new technology is implemented. If discounting is included in the analysis, prices before and after the innovation date rise, because the return on learning falls. The problem of a monopolist and social planner are quite similar, however, this analysis concentrates on the monopolist, because learning industries, although often severely subsidized, are generally not publicly owned. Other central results of this paper are: innovation subsidies and sales subsidies reduce the prices of both phases if all variables (the timing of innovation t_1, the cumulative production quantities at the innovation date, $y(t_1)$ and at the end of the planning horizon, $y(T)$) are endogenous. Both types of subsidies induces the date of innovation to occur earlier. Therefore the total quantity produced during the entire planning horizon increases. The production plan in the presence of subsidies lies entirely above the production plan without subsidies. Thus if early distributions yield positive externalities, then subsidies on sales and on innovation contain an additional positive effect.[17] Another central result emerges, when the total production $y(T)$ is restricted.[18] In this case the two kinds of subsidies analyzed have different effects. Innovation subsidies decrease the innovation date, but the price during the first phase increases (thus sales decrease). It is shown that a price cap can reduce the first phase's price, furthermore it induces innovation to occur earlier. In order to evaluate an innovation subsidy one would need to consider the negative effect of the "early-distribution argument". Sales subsidies induce the innovation timing to occur later, which means that for a longer period of time, consumer pay the higher first phase's price p_0. The positive effect is that p_0 falls, which is why the "early-distribution argument" might be in favour of sales subsidies. The two subsidies considered have different effects on consumer and producer surplus. It is also shown that a subsidy on innovation (sales), which is financed through a tax on sales (innovation), changes the proportion of consumer and producer rents. Customers generally benefit more from sales subsidies, producers from innovation subsidies.

[17] In the renewable energy sector early installations increase the positive externality. The total amount of carbon dioxide in the atmosphere is reduced, because renewable energy sources are substitutes to conventional energy sources that emit $C0_2$.

[18] For example in medium-sized countries, there is a fixed number of places, where wind energy plants can be built.

This paper does not conduct a general welfare analysis; for that one would need to measure the cause for government subsidies (positive externalities). This would only be feasible, if one would have specific learning parameters and the parameters of the innovation cost function, which describe a specific market. This goes beyond the scope of this article and is left for future work. It would also be interesting to show, how the market equilibrium changes, when the date of innovation is anticipated by customers. Most likely it would be optimal for some to wait and purchase the product after the innovation date. This would reduce the learning before t_1 unless innovation is delayed by the firm. Another extension of this paper could add more insight by accounting for a stochastic innovation process, market entry and technology switching cost.

Appendix to Chapter 1

Proof of lemma 1.1

A transversality condition for the state variable solves for $const_1$ in (1.13). The total cost before t_1 is sunk and can be ignored, $\eta_0(t_1)$ has an influence only on future cost. The area below the learning curve between $y(t_1)$ and $y(t)$, for $t \in [t_1, T]$ is defined as

$$\Gamma\left[y(t)\right] = \int_{y(t_1)}^{y(t)} m + ce^{-\lambda y(t_1) - \gamma[v - y(t_1)]}\ dv = m[y(t) - y(t_1)] + \frac{c}{\gamma}[e^{-\lambda y(t_1)} - e^{-\lambda y(t_1) - \gamma[y(t) - y(t_1)]}] \tag{1.55}$$

The time derivative is $\dot{\Gamma} = \frac{d\Gamma}{dt} = x(t)[m + ce^{-\lambda y(t_1) - \gamma[y(t) - y(t_1)]}]$, thus the time dependent area under the learning curve or the total cost of the second phase is

$$\int_{t_1}^{T} \dot{\Gamma} dt = [y(T) - y(t_1)]m + \frac{c}{\gamma}[e^{-\lambda y(t_1)} - e^{-\lambda y(t_1) - \gamma[y(T) - y(t_1)]}]$$

The usual methods of the principle of variations are used. At first the optimal path of the production flow is displaced for the cost that occurs after t_1; $x(p_1(t), t) \rightarrow x(p_1(t), t) + \delta\phi(t)$, $-\frac{\lambda}{\gamma}ce^{-\lambda y(t_1)} + \frac{\lambda}{\gamma}ce^{(\gamma - \lambda)y(t_1) - \gamma y(T)}$

$$\Gamma\left[y(t)\right] = y(T)m - y(t_1)m - \frac{1}{\gamma}ce^{(\gamma - \lambda)y(t_1) - \gamma y(T)} + \frac{1}{\gamma}ce^{-\lambda y(t_1)}$$

$$\Gamma\left[y(t)\right]^{\delta} = [\int_{t_0}^{T}[x(t) + \delta\phi(t)]dt]m - [\int_{t_0}^{t_1}[x(t) + \delta\phi(t)]dt]m$$

$$+\frac{1}{\gamma}ce^{-\lambda\int_{t_0}^{t_1}[x(t) + \delta\phi(t)]dt} - \frac{1}{\gamma}ce^{(\gamma - \lambda)\int_{t_0}^{t_1}[x(t) + \delta\phi(t)]dt - \gamma\int_{t_0}^{T}[x(t) + \delta\phi(t)]dt}$$

In a second step the displaced total cost after t_1 (called $\Gamma\left[y(t)\right]^{\delta}$) is differentiated with respect to δ. The derivative is evaluated at $\delta = 0$, employing the standard calculus of variations approach,

$$\frac{\partial \Gamma\left[y(t)\right]^{\delta}}{\partial \delta}|_{(\delta=0)} = \left\{-\frac{\lambda}{\gamma}ce^{-\lambda y(t_1)} + \frac{\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)}\right\}\int_{t_0}^{t_1}\phi(t)dt \tag{1.56}$$

$$+\{m + ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)}\}\int_{t_1}^{T}\phi(t)dt$$

where (1.56) separates the terms multiplied by $\int_{t_0}^{t_1}\phi(t)dt$ and $\int_{t_1}^{T}\phi(t)dt$ respectively. For any $t > t_1$, the marginal cost at T are collected. This result is similar to Spence's (1981) learning model. Given a production plan, he shows that when a firm extends its production by one unit at any time, then the incremental cost is equal to marginal cost at the end of the planning horizon. For $t_0 \le t \le t_1$ we collect another term; $-\frac{\lambda}{\gamma}ce^{-\lambda y(t_1)} + \frac{\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)} < 0$. For $t = t_1$ the incremental cost is thus the marginal cost of the second phase at T plus $\left[-\frac{\lambda}{\gamma}ce^{-\lambda y(t_1)} + \frac{\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)}\right]$. The value of delaying the innovation by one production unit is obtained when this term is multiplied by -1. Thus the first phase's costate at the innovation date is $\eta_0(t_1) = \frac{\lambda}{\gamma}ce^{-\lambda y(t_1)} - \frac{\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)}$.

Proof of proposition 1.2

When a firm produces an additional quantity ϵ_0 *before* the innovation occurs ($t_0 \le t \le t_1$) then the pricing rule (1.24) holds. The incremental cost for the first time phase is computed next. The total cost is equal to

$$y(T)m + \frac{c}{\lambda}[1 - e^{-\lambda y(t_1)}] + \frac{c}{\gamma}[e^{-\lambda y(t_1)} - e^{-\lambda[y(t_1)]-\gamma[y(T)-y(t_1)]}] \tag{1.57}$$

When the firm produces an additional unit ϵ_0 before the innovation takes place, then total cost increases to

$$(y(T) + \epsilon_0)m + \frac{c}{\lambda}[1 - e^{-\lambda[y(t_1)+\epsilon_0]}] + \frac{c}{\gamma}[e^{-\lambda[y(t_1)+\epsilon_0]} - e^{-\lambda[y(t_1)+\epsilon_0]-\gamma[y(T)-y(t_1)]}] \tag{1.58}$$

The incremental cost during the first phase is denoted by IC_0

$$IC_0 = \frac{(1.58) - (1.57)}{\epsilon_0} = c\{-\frac{e^{-\lambda[y(t_1)+\epsilon_0]}}{\lambda\epsilon_0} - e^{-\lambda y(t_1)}] + \frac{e^{-\lambda[y(t_1)+\epsilon_0]} - e^{-\lambda y(t_1)}}{\gamma\epsilon_0}.$$

$$-\frac{e^{-\lambda[y(t_1)+\epsilon_0]-\gamma[y(T)-y(t_1)]} - e^{-\lambda[y(t_1)]-\gamma[y(T)-y(t_1)]}}{\gamma\epsilon_0}\} + m \tag{1.59}$$

It needs to be shown that (1.24) is equivalent to setting the marginal revenue equal to the incremental cost (1.59). $ce^{-\lambda y(t_1)}$ is equivalent to $\lim_{\epsilon_0 -> 0} -\frac{c}{\lambda\epsilon_0}[e^{-\lambda[y(t_1)+\epsilon_0]} - e^{-\lambda y(t_1)}]$ and $-\frac{\lambda}{\gamma}e^{-\lambda y(t_1)}$ to $\lim_{\epsilon_0 -> 0} \frac{e^{-\lambda[y(t_1)+dx]} - e^{-\lambda y(t_1)}}{\gamma\epsilon_0}$. In order to show the equivalence of the third term, it has to be transformed: $\lim_{\epsilon_0 -> 0} - \frac{1}{\gamma\epsilon_0}[e^{-\lambda[y(t_1)+\epsilon_0]-\gamma[y(T)-y(t_1)]} - e^{-\lambda[y(t_1)]-\gamma[y(T)-y(t_1)]}] = -\frac{1}{\gamma\epsilon_0}[e^{(\gamma-\lambda)y(t_1)-\gamma y(T)-\lambda\epsilon_0} - e^{-\lambda[y(t_1)]-\gamma[y(T)-y(t_1)]}]$. It follows that $\lim_{\epsilon_0 -> 0} -\frac{1}{\gamma\epsilon_0}[e^{(\gamma-\lambda)y(t_1)-\gamma y(T)-\lambda\epsilon_0} - e^{-\lambda[y(t_1)]-\gamma[y(T)-y(t_1)]}] = \frac{\lambda}{\gamma}e^{(\gamma-\lambda)y(t_1)-\gamma y(T)}$, where the following rule was applied: $\lim_{\epsilon_0 -> 0} \frac{e^{a-\lambda\epsilon_0} - e^a}{\epsilon_0} = e^a \lim_{\epsilon_0 -> 0} \frac{e^{-\lambda\epsilon}-1}{\epsilon} = e^a\lambda$, for any constant $a \in \mathbb{R}$. This completes the first part of the proof. During the first time phase, the monopolist behaves optimally, when it sets marginal revenue equal to incremental cost at each instant of time. Next, the second time phase $\nabla t_1 \leq t \leq T$ is analyzed. (1.25) is the optimality condition during this phase. When the firm produces an additional unit ϵ_1 *after* the innovation has occurred, total cost (1.57) increases to

$$(y(T) + \epsilon_1)m + \frac{c}{\lambda}[1 - e^{-\lambda y(t_1)}] + \frac{c}{\gamma}[e^{-\lambda y(t_1)} - e^{-\lambda y(t_1)-\gamma[y(T)+\epsilon_1-y(t_1)]}] \tag{1.60}$$

IC_1 is the incremental cost that occurs through an additional ϵ_1 after t_1

$$IC_1 = \frac{(1.60) - (1.57)}{\epsilon_1} = \frac{c_0}{\gamma\epsilon_1}[e^{-\lambda y(t_1)-\gamma[y(T)-y(t_1)]} - e^{-\lambda y(t_1)-\gamma[y(T)+\epsilon_1-y(t_1)]}] \tag{1.61}$$

(1.25) is equivalent to (1.61), because $ce^{(\gamma-\lambda)y(t_1)-\gamma[y(T)-y(t_1)]} = \lim_{\epsilon_1 -> 0} \frac{c}{\gamma\epsilon_1}[e^{-\lambda y(t_1)-\gamma[y(T)-y(t_1)]} - e^{-\lambda y(t_1)-\gamma[y(T)+\epsilon_1-y(t_1)]}]$.

Proof of proposition 1.3

Pricing during the first phase ($t \leq t_1$); when $r \neq 0$

This section reconstructs equation (1.24) in the presence of discounting with $t_0 = 0$. The costate that contains a discount rate is denoted by $\psi_0(t) = e^{-rt}\ \eta_0(t)$. The transversality condition for $\psi_0(t_1)$ has to be derived. It is shown that (1.12) holds, when $\eta_0(t)$ is substituted for $\psi_0(t)$

$$\frac{\partial H}{\partial p_0(t)} = x_0(p_0(t),t) + p_0(t)\frac{\partial x_0(p_0(t),t)}{\partial p_0(t)} - \frac{\partial x_0(p_0(t),t)}{\partial p_0(t)}(m + ce^{-\lambda y(t)}) + \psi_0(t)\frac{\partial x_0(p_0(t),t)}{\partial p_0(t)} \doteq 0$$

$$\Leftrightarrow p_0(1 - \frac{1}{\varepsilon(t)}) = m + ce^{-\lambda y(t)} - \psi_0(t) \tag{1.62}$$

The second first order condition (1.13), becomes

$$\dot{\psi}_0(t) - r\psi_0(t) = -\frac{\partial H}{\partial y_0(t)}$$

multiplying this equation by e^{-rt} and transforming yields

$$\Leftrightarrow e^{-rt}\dot{\psi}_0(t) - re^{-rt}\psi_0(t) = -e^{-rt}\frac{\partial H}{\partial y_0(t)}$$

$$\Leftrightarrow e^{-rt}\psi_0(t) + const_3 = \int \left[e^{-rt}(-\lambda)ce^{-\lambda y_0(t)}x_0(t)\right] dt$$

$$\Leftrightarrow e^{-rt}\psi_0(t) + const_3 = e^{-rt}ce^{-\lambda y_0(t)} + r\int e^{-rt}ce^{-\lambda y_0(t)}dt$$

$$\Leftrightarrow \psi_0(t) = ce^{-\lambda y_0(t)} + re^{rt}\int e^{-rt}ce^{-\lambda y_0(t)}dt - e^{rt}const_3 \tag{1.63}$$

In order to find an expression of the costate at the innovation date $\psi_0(t_1)$, the total cost that occurs after the innovation is expressed by V.

$$V = \int_{t_1}^{T} \left\{x_1(p_1(t)(m + ce^{(\gamma-\lambda)y(t_1)-\gamma y(t)})e^{-rt}\right\} dt$$

Integration by parts yields:

$$V = \int_{t_1}^{T} re^{-rt}\left\{my_1(t) - \frac{1}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(t)}\right\} dt$$

$$+e^{-rt}\left\{my(T) - \frac{1}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)} - my(t_1) + \frac{1}{\gamma}ce^{-\lambda y(t_1)}\right\}$$

Applying the usual methods of the principle of variations; the optimal path of the production flow $x(p_1(t),t) \to x(p_1(t),t) + \delta\phi(t)$ in V is displaced, before its derivative is evaluated at $\delta = 0$ according to the standard calculus of variations approach.

$$V = \int_{t_1}^{T} re^{-rt}\left\{m\int_{t_0}^{t}\left[x_1(t) + \delta\phi(t)\right]dt - \frac{1}{\gamma}ce^{(\gamma-\lambda)\int_{t_0}^{t_1}[x_1(t)+\delta\phi(t)]dt - \gamma\int_{t_0}^{t}[x_1(t)+\delta\phi(t)]dt}\right\} dt$$

$$+e^{-rt}\{m\int_{t_0}^{T}\left[x_1(t) + \delta\phi(t)\right]dt - \frac{1}{\gamma}ce^{(\gamma-\lambda)\int_{t_0}^{t_1}[x_1(t)+\delta\phi(t)]dt - \gamma\int_{t_0}^{T}[x_1(t)+\delta\phi(t)]dt}$$

$$-m\int_{t_0}^{t_1}\left[x_1(t) + \delta\phi(t)\right]dt + \frac{1}{\gamma}ce^{-\lambda\int_{t_0}^{t_1}[x_1(t)+\delta\phi(t)]dt}\}$$

$$\frac{\partial V}{\partial \delta}]_{\delta=0} = \int_{t_1}^{T} re^{-rt}\left\{m\int_{t_0}^{t}\phi(t)dt - \frac{\gamma-\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-y(t)}\int_{t_0}^{t_1}\phi(t)dt + ce^{(\gamma-\lambda)y(t_1)-y(t)}\int_{t_0}^{t}\phi(t)dt\right\} dt$$

$$+e^{-rt}\{m\int_{t_0}^{T}\phi(t)dt-\frac{\gamma-\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)}\int_{t_0}^{t_1}\phi(t)dt+ce^{(\gamma-\lambda)y(t_1)-y(T)}\int_{t_0}^{T}\phi(t)dt$$
$$-m\int_{t_0}^{t_1}\phi(t)dt-\frac{\lambda}{\gamma}ce^{-\lambda y(t_1)}\int_{t_0}^{t_1}\phi(t)dt\}$$

$$=\int_{t_1}^{T}re^{-rt}\left\{\left[m+ce^{(\gamma-\lambda)y(t_1)-y(t)}\right]\int_{t_0}^{t}\phi(\tau)d\tau-\frac{\gamma-\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-y(t)}\int_{t_0}^{t_1}\phi(t)dt\right\}dt$$
$$+e^{-rt}\{\int_{t_0}^{t_1}\left[\frac{\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)}-\frac{\lambda}{\gamma}ce^{-\lambda y(t_1)}\right]\phi(t)dt+\int_{t_1}^{T}\left[m+ce^{(\gamma-\lambda)y(t_1)-y(T)}\right]\phi(t)dt\} \tag{1.64}$$

The term $\int_{t_1}^{T}re^{-rt}\left\{\frac{\gamma-\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-y(t)}\int_{t_0}^{t_1}\phi(t)dt\right\}dt$ can be ignored. It originates from displacing the cumulative quantity before t_1 after the innovation has already taken place. As the firm cannot change $y(t_1)$ after t_1, it cannot effect the cost during the first time phase at t_1. It just has an effect on the second phase's cost. One can also examine this term itself and recognize that for any t it is zero, because when $t<t_1$ then the first integral becomes zero, when $t>t_1$, then the second integral is zero. Hence (1.64) becomes

$$\int_{t_1}^{T}re^{-rt}\left\{\left[m+ce^{(\gamma-\lambda)y(t_1)-y(t)}\right]\int_{t_0}^{t}\phi(\tau)d\tau\right\}dt$$
$$+e^{-rt}\{\int_{t_0}^{t_1}\left[\frac{\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)}-\frac{\lambda}{\gamma}ce^{-\lambda y(t_1)}\right]\phi(t)dt+\int_{t_1}^{T}\left[m+ce^{(\gamma-\lambda)y(t_1)-y(T)}\right]\phi(t)dt\} \tag{1.65}$$

One can replace the variables of the first term above. The range of τ is $[t_0,t]$ and that of t is: $[t_1,T]\ \rightarrow t_0\leq\tau\leq t\leq T$. After replacing the variables $t_0\leq t\leq\tau\leq T$, the range is $[t,T]$ for τ, when $t>t_1$ and it is $[t_1,T]$ when $t\leq t_1$. Before the swap, t was larger than t_1 hence τ is larger than t_1 after the swap. When t is smaller than t_1, then τ's lower limit is t_1. When t is larger than t_1, then τ's lower limit is t. The new range of t is $[t_0,T]$. Figure 1.6 illustrates the range before and after replacing the variables t and τ.

(1.65) can be transformed to

$$\int_{t_0}^{t_1}re^{-rt}\left\{\left[m+ce^{(\gamma-\lambda)y(t_1)-y(t)}\right]\int_{t_0}^{t}\phi(\tau)d\tau\right\}dt+$$
$$\int_{t_1}^{T}re^{-rt}\left\{\left[m+ce^{(\gamma-\lambda)y(t_1)-y(t)}\right]\int_{t_0}^{t}\phi(\tau)d\tau\right\}dt$$

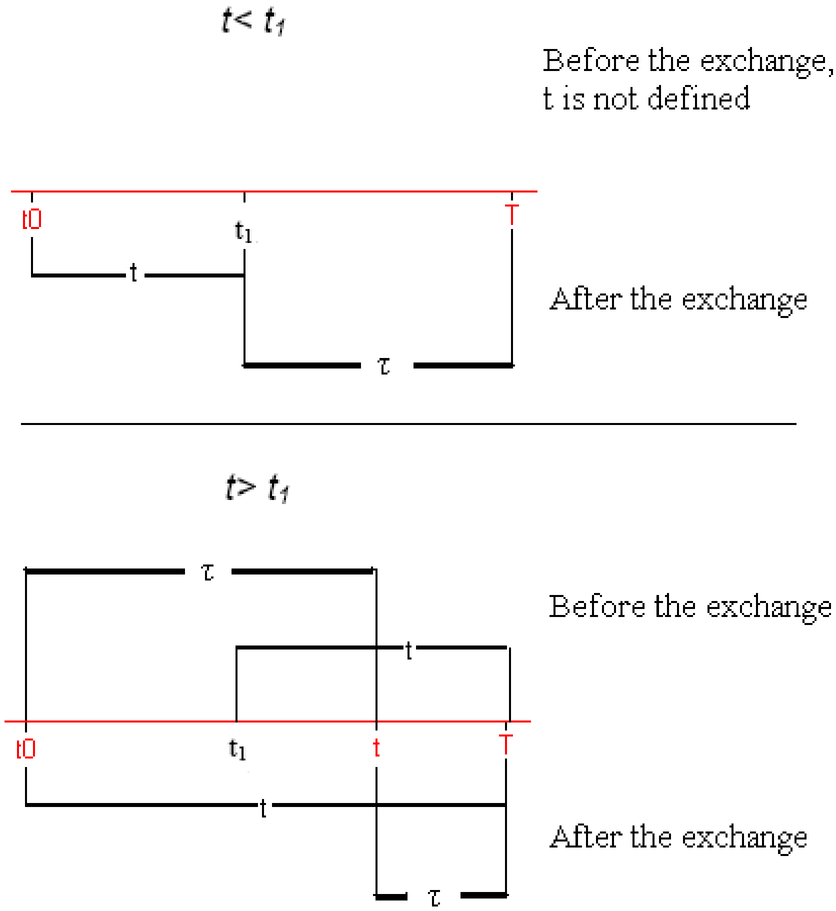

Figure 1.6: Exchanging variables

$$+e^{-rt}\{\int_{t_0}^{t_1}\left[\frac{\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)}-\frac{\lambda}{\gamma}ce^{-\lambda y(t_1)}\right]\phi(t)dt+\int_{t_1}^{T}\left[m+ce^{(\gamma-\lambda)y(t_1)-y(T)}\right]\phi(t)dt\}$$

The variables are replaced next.

$$=\int_{t_0}^{t_1}\left\{\int_{t_1}^{T}r\left[m+ce^{(\gamma-\lambda)y(t_1)-y(T)}\right]e^{-r\tau}d\tau\right\}\phi(t)dt+\int_{t_1}^{T}\left\{\int_{t}^{T}r\left[m+ce^{(\gamma-\lambda)y(t_1)-y(T)}\right]e^{-r\tau}d\tau\right\}$$

$$\phi(t)dt+e^{-rt}\{\int_{t_0}^{t_1}\left[\frac{\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)}-\frac{\lambda}{\gamma}ce^{-\lambda y(t_1)}\right]\phi(t)dt+\int_{t_1}^{T}\left[m+ce^{(\gamma-\lambda)y(t_1)-y(T)}\right]\phi(t)dt\}$$

$$=\int_{t_0}^{t_1}\left\{\int_{t_1}^{T}r\left[m+ce^{(\gamma-\lambda)y(t_1)-y(T)}\right]e^{-r\tau}d\tau\right\}\phi(t)dt$$

$$+e^{-rt}\{\int_{t_0}^{t_1}\left[\frac{\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)}-\frac{\lambda}{\gamma}ce^{-\lambda y(t_1)}\right]\phi(t)dt$$

$$+\int_{t_1}^{T}\left\{\int_{t}^{T}r\left[m+ce^{(\gamma-\lambda)y(t_1)-y(T)}\right]e^{-r\tau}d\tau\right\}\phi(t)dt+\int_{t_1}^{T}e^{-rt}\left[m+ce^{(\gamma-\lambda)y(t_1)-y(T)}\right]\phi(t)dt\}$$

$$=\int_{t_0}^{t_1}\left\{e^{-rt}\left[\frac{\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)}-\frac{\lambda}{\gamma}ce^{-\lambda y(t_1)}\right]+\int_{t_1}^{T}r\left[m+ce^{(\gamma-\lambda)y(t_1)-y(T)}\right]e^{-r\tau}d\tau\right\}\phi(t)dt$$

$$+\int_{t_1}^{T}\left\{e^{-rt}\left[m+ce^{(\gamma-\lambda)y(t_1)-y(T)}\right]+\int_{t}^{T}r\left[m+ce^{(\gamma-\lambda)y(t_1)-y(T)}\right]e^{-r\tau}d\tau\right\}\phi(t)dt\} \tag{1.66}$$

The same reasoning as in (1.56) applies here hence the costate of the first phase with a non-zero discount rate at t_1 equals,

$$\rightarrow\psi_0(t_1)=e^{-rt_1}\left[\frac{\lambda}{\gamma}ce^{-\lambda y(t_1)}-\frac{\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)}\right]-\int_{t_1}^{T}r\left[m+ce^{(\gamma-\lambda)y(t_1)-y(T)}\right]e^{-r\tau}d\tau \tag{1.67}$$

The constant $const_3$ is found in the following.

$$\Leftrightarrow\psi_0(t)=ce^{-\lambda y_0(t)}+re^{rt}\int e^{-rt}ce^{-\lambda y_0(t)}dt-e^{rt}const_3$$

$$\rightarrow\psi_0(t_1)=ce^{-\lambda y_0(t_1)}+\left[re^{rt}\int e^{-rt}ce^{-\lambda y_0(t)}dt\right]_{t=t_1}-e^{rt_1}const_3$$

$$\rightarrow ce^{-\lambda y_0(t_1)}+\left[re^{rt}\int e^{-rt}ce^{-\lambda y_0(t)}dt\right]_{t=t_1}-e^{rt_1}const_3=$$

$$e^{-rt_1}\left[\frac{\lambda}{\gamma}ce^{-\lambda y(t_1)}-\frac{\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)}\right]-\int_{t_1}^{T} r\left[m+ce^{(\gamma-\lambda)y(t_1)-y(T)}\right]e^{-r\tau}d\tau$$

$$-\int_{t_1}^{T} r\left[m+ce^{(\gamma-\lambda)y(t_1)-y(T)}\right]e^{-r\tau}d\tau$$

$$\Leftrightarrow e^{rt_1}const_3=\left[\frac{\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)}-\frac{\lambda}{\gamma}ce^{-\lambda y(t_1)}\right]e^{-rt_1}+\int_{t_1}^{T} r[m+ce^{(\gamma-\lambda)y(t_1)-\gamma y(\tau)}]e^{-r\tau}d\tau$$

$$+ce^{-\lambda y_0(t_1)}+\left[re^{rt}\int e^{-rt}ce^{-\lambda y_0(t)}dt\right]_{t=t_1}$$

It is substituted into the costate function (1.67), which is

$$\rightarrow \psi_0(t)=ce^{-\lambda y_0(t)}+re^{rt}\int e^{-rt}ce^{-\lambda y_0(t)}dt$$

$$-\left[\frac{\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)}-\frac{\lambda}{\gamma}ce^{-\lambda y(t_1)}\right]e^{-rt_1}-\int_{t_1}^{T} r[m+ce^{(\gamma-\lambda)y(t_1)-\gamma y(\tau)}]e^{-r\tau}d\tau-ce^{-\lambda y_0(t_1)} \tag{1.68}$$

Finally one can substitute the costate in equation (1.68) to find an expression, how the monopolist sets the price during the first time phase.

$$p_0(1-\frac{1}{\varepsilon(t)})=m+ce^{-\lambda y_0(t)}-ce^{-\lambda y_0(t)}-re^{rt}\int e^{-rt}ce^{-\lambda y_0(t)}dt$$

$$+\left[\frac{\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)}-\frac{\lambda}{\gamma}ce^{-\lambda y(t_1)}\right]e^{-rt_1}+\int_{t_1}^{T} r[m+ce^{(\gamma-\lambda)y(t_1)-\gamma y(\tau)}]e^{-r\tau}d\tau+ce^{-\lambda y_0(t_1)}$$

$$\left[re^{rt}\int e^{-rt}ce^{-\lambda y_0(t)}dt\right]_{t=t_1}$$

$$\Leftrightarrow MR_0(t)=m+\left[\frac{\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)}-\frac{\lambda}{\gamma}ce^{-\lambda y(t_1)}\right]e^{-rt_1}+ce^{-\lambda y_0(t_1)}$$

$$+\left[re^{rt}\int e^{-rt}ce^{-\lambda y_0(t)}dt\right]_{t=t_1}-re^{rt}\int e^{-rt}ce^{-\lambda y_0(t)}dt+r\int_{t_1}^{T}[m+ce^{(\gamma-\lambda)y(t_1)-\gamma y(\tau)}]e^{-r\tau}d\tau$$

$$\Leftrightarrow MR_0(t)=m+\frac{\lambda}{\gamma}ce^{(\gamma-\lambda)y(t_1)-\gamma y(T)}+\frac{\gamma-\lambda}{\gamma}ce^{-\lambda y(t_1)}$$

$$+r\int_t^{t_1}\left[m+ce^{-\lambda y_0(\tau)}\right]e^{-r(\tau-t)}d\tau+r\int_{t_1}^{T}[m+ce^{(\gamma-\lambda)y(t_1)-\gamma y(\tau)}]e^{-r(\tau-t)}d\tau \tag{1.69}$$

With a positive discount rate, there are two additional terms in the optimality condition. These reflect the fact that the return to experience or learning, which is expressed by a lower future cost, matters less, due to the presence of discounting. The optimal plan is to increase p_0 and thus present profits at the expense of future profits. Note that (1.69) reduces to (1.24), when $r = 0$.

Pricing during the second phase ($t > t_1$); when $r \neq 0$

(1.25) and (1.17) can be rewritten to account for discounting

$$MR_1(t) = m + ce^{(\gamma-\lambda)y(t_1)-\gamma y(t)} - \psi_1(t) \tag{1.70}$$

$$\psi_1(t) = ce^{(\gamma-\lambda)y(t_1)-\gamma y_1(t)} + re^{rt}\int e^{-rt}e^{(\gamma-\lambda)y(t_1)-\gamma y(t)}dt - e^{-rt}const_4 \tag{1.71}$$

Extending production at $t = T$ does not increase the profit, if the firm behaves optimally. Hence $\psi_1(T) = 0$.

$$\rightarrow \psi_1(T) = ce^{(\gamma-\lambda)y(t_1)-\gamma y_1(T)} + \left[re^{rt}\int e^{-rt}e^{(\gamma-\lambda)y(t_1)-\gamma y(t)}dt\right]_{t=T} - e^{-rT}const_4 = 0$$

$$\Leftrightarrow ce^{(\gamma-\lambda)y(t_1)-\gamma y_1(T)} + \left[re^{rt}\int e^{-rt}e^{(\gamma-\lambda)y(t_1)-\gamma y(t)}dt\right]_{t=T} = e^{-rT}const_4$$

$$\rightarrow \psi_1(t) = ce^{(\gamma-\lambda)y(t_1)-\gamma y_1(t)} + re^{rt}\int e^{-rt}e^{(\gamma-\lambda)y(t_1)-\gamma y(t)}dt - ce^{(\gamma-\lambda)y(t_1)-\gamma y_1(T)}$$

$$-\left[re^{rt}\int e^{-rt}e^{(\gamma-\lambda)y(t_1)-\gamma y(t)}dt\right]_{t=T}$$

$$MR_1(t) = m+ce^{(\gamma-\lambda)y(t_1)-\gamma y_1(T)} - re^{rt}\int e^{-rt}e^{(\gamma-\lambda)y(t_1)-\gamma y(t)}dt + \left[re^{rt}\int e^{-rt}e^{(\gamma-\lambda)y(t_1)-\gamma y(t)}dt\right]_{t=T}$$

$$\Leftrightarrow MR_1(t) = m+ce^{(\gamma-\lambda)y(t_1)-\gamma y_1(T)} - re^{rt}\int e^{-rt}e^{(\gamma-\lambda)y(t_1)-\gamma y(t)}dt + \left[re^{rt}\int e^{-rt}e^{(\gamma-\lambda)y(t_1)-\gamma y(t)}dt\right]_{t=T}$$

During the second phase, the monopolist sets p_1 such that

$$\Leftrightarrow MR_1(t) = m + ce^{(\gamma-\lambda)y(t_1)-\gamma y_1(T)} + r\int_t^T\left[m+ce^{(\gamma-\lambda)y(t_1)-\gamma y_1(\tau)}\right]e^{-r(\tau-t)}d\tau \tag{1.72}$$

With a positive discount rate, there is an additional term. Again, the optimality condition reflects the fact that the return to experience is in the future, which is discounted, thus p_1 rises. (1.72) reduces to (1.25), when $r = 0$.

Proof of proposition 1.5

For a fix total production quantity $\overline{y(T)}$, it is shown analytically and numerically that a subsidy on innovation cost, induces the innovation to occur earlier and thus p_0 to increase. The welfare loss could be encountered by a price ceiling during this phase. The price ceiling would have a counter effect, when the date of innovation is delayed by the introduction of a price constraint. In this case, consumers would pay the higher price p_0 for a longer period of time. The answer is a straightforward extension of the preceding section and is based on a method that is used in Rees (1986). p_0 is constant, therefore either the price constraint $\overline{p} - p_0 \geq 0$, where $\overline{p}$ is the price ceiling binds over the entire interval $[t_0, t_1]$ or it does not bind at all. When it does not bind, it has no effect. Assume it does bind, and $p_0^* > \overline{p}$, where p_0^* is the optimal price set by the monopolist; in this case the monopolist looses some of its profit due to the price cap. This loss is denoted by $R[x_0(\overline{p})]$, with $R_{x_0} < 0$, and $R > 0$ if $x < x(\overline{p_0})$. The per period quantity x_0 on which R depends upon is considered as a function of $\overline{p}$. Whenever $p_0 \leq \overline{p}$, then $x > x(\overline{p_0})$ and $R \equiv 0$. The new objective is $\underset{x_0, x_1, t_1}{Max}\ MP' = \int_{t_0}^{t_1} \left[\pi^0(t) - R(x(\overline{p}))\right] dt + \int_{t_1}^{T} \pi^1(t) dt - a(t_1)$ subject to the constraints (1.5)-(1.9). The problem can be divided into two segments. The maximand over the first phase $[t_0, t_1]$ is $V_0^* \equiv Max \int_{t_0}^{t_1} \left[\pi^0(t) - R(x_0(\overline{p_0}))\right] dt - a(t_1)$, that of the second phase is $V_1^* \equiv Max \int_{t_1}^{T} \pi^1(t) dt$. t_1^* maximizes the sum $V_0^* + V_1^*$, and satisfies $\partial \left(V_0^* + V_1^*\right)/t_1 \overset{\circ}{=} 0$, from which the optimality condition $a'(t_1^*) + R(x(\overline{p_0})) = \pi^0(t_1^*) + \eta_0(t_1^*) x[p_0(t_1^*), t_1^*] - \pi^1(t_1^*) - \eta_1(t_1^*) x[p_1(t_1^*), t_1^*]$ can be derived. Aside from R the optimality condition is the same as (1.39). The introduction of a price constraint decreases t_1^*. The low price p_1 is charged for a longer time span, because the length of phase 2 increases. In addition, an earlier innovation date causes p_1 to decrease further.

Summing up one can say that the introduction of a price constraint along with an innovation subsidy increases consumer rents during the entire planning horizon. Producers receive lower revenues, when $\overline{y(T)}$ is fixed, because prices do not increase, however, their innovation cost decline, due to the subsidy. Their production cost decrease, because producers learn with a larger learning parameter sooner.

Chapter 2

Optimal Fertility Decisions in a Life-Cycle Model *

2.1 Introduction

Three of the most significant socioeconomic developments in virtually all the developed economies in the second half of the 20'th century were the large increases in female labor force participation, the falls in fertility rates and the increases in divorce rates. A number of exogenous factors clearly have played an important role in these, for example the growth in demand for female labor, the availability of the contraceptive pill, and changes in divorce laws that have made divorce easier and less costly to obtain. It seems also clear however that there are several possible interrelationships among these three developments: child care and work in the market are alternative uses of a mother's time and increasing wage rates raise the opportunity cost of children; the attempt to build a career could lead to postponing childbirth and having fewer children as a result of this; the perception of an increased chance that the marriage might end in divorce could lead to a decision to have fewer children. At the same time, there is considerable heterogeneity across households in respect of female market labor supply, even after controlling for wage rates and number and ages of children, and it does not seem adequate simply to regard this as due to preference heterogeneity.[19]

In this paper we develop a new theoretical framework to try to explore some of these interrelationships, and to consider possible explanations for them, that are rooted in optimal intertemporal decision taking over the life cycle. A woman's human capital,

*This chapter is a joint work with Ray Rees.

[19] See Apps and Rees (2009), chapters 1 and 5, where this is discussed at some length.

and therefore her wage rate, is endogenous and depends first on the choice of how much formal education to acquire, and secondly on how much work experience to gain in the labor market. Both these decisions affect the timing and number of births, and in turn are affected by them because of the demands on time made by child care. We first set out a model which allows these interacting decisions to be formally analyzed. We then extend it by analyzing the effect on the timing and number of births of perceptions of the likelihood of divorce.

There is a large literature that asks how children affect such economic variables as demand patterns and consumption. In that context they examine intertemporal decisions and equality questions. For an overview of this literature see Browning (1992) and (Becker 1993). Most of the literature that deals with the effect of children on labor supply concentrates on female labor participation, because the effect on male labor market participation has so far been quite low.[20] Ward and Butz (1980) show empirically that couples time their births to avoid periods when the female's income is high. Heckman and Walker (1990) show that the negative (positive) relation between the optimal number of children (fertility timing) and female wages is robust across a variety of empirical specifications, while they cannot prove that the same holds for male wages. Based on this literature we focus on the female as the utility maximizing individual throughout this paper.

In order to assess the costs of raising children, one has to take account of the timing of births. Labor market earnings depend on work experience. In an early study Happel et al. (1984) set up a model in which a woman works before she gives birth and gains labor market experience, and her income increases with experience. After giving birth a woman takes some time off to raise her child or children. When she re-enters the labor market, some of her experience has decayed by some constant factor. It is assumed to be zero for unskilled workers, in which case there is no timing preference. Otherwise a woman would want to either have children in the very beginning of her marriage, when she has not accumulated any labor experience before her marriage or shortly before her period of fecundity ends. In an empirical paper using Swedish data, Walker (1995) decomposes the total costs of children into the opportunity costs of not working, the foregone return for foregone human capital investment and the net direct. The model in this paper will take account of this decomposition and solve for the optimal timing in a continuous time framework.

Gustafsson (2001) gives a nice overview of the past theoretical and empirical research on the optimal timing of childbirth. Cigno (1991) analyses a dynamic model

[20] Browning (1992) pp. 1449-1464

in discrete time, in which the female's income depends on her education level as well as on labor market experience. He derives the optimality conditions that describe an optimal fertility profile, with the value of the number of children growing at the rate of interest. Along these lines he demonstrates that postponing childbirth raises the income loss and lowers the human capital loss of a birth, because income rises with labor experience. In order to go a step further in this paper we set up a model in continuous time, which allows us to find an explicit solution for the fertility timing and number of children. Blackburn et al. (1993) show theoretical linkages between a woman's fertility timing and her investments in human capital and income profile. A late child bearer accumulates more human capital when the discount rate is larger than the economy-wide growth rate of wages for late child bearers.

In our baseline model in the next section, we examine the effects of the income level on our two variables of interest: the timing of fertility and the number of children. We then go on to analyze how the return to labor market experience within the different life cycle phases affects the timing and number of births, which is new in this literature. We also have various cost parameters included for the purpose of deriving some policy implications. Empirically it can be shown that less educated families decide to have more children (De la Croix and Doepke, 2003). This model can be extended with an education phase. Empirically it can be shown that less educated families decide to have more children (De la Croix and Doepke, 2003). This model can be extended to include an education phase, where ability plays a role. Individuals that would benefit from a higher return to education, enter the labor market later, and have later, fewer children. We waived this addition though as it does not add much to the existing literature. The major part of the fertility literature is embedded in a deterministic framework. Exceptions are Newman (1983) and Hotz and Miller (1986). Drastic simplifications have to be made to keep these models manageable. As a consequence these models have bang-bang solutions, where the probability of giving birth is piled up either at the beginning of marriage or at the end of a woman's period of fecundity. Our model introduces some stochastic elements by introducing the possibility of divorce. We then show how this possibility influences the optimal timing and number of childbirths, and this appears to be new to the literature.

2.2 The Baseline Model

We assume that the working life of a representative woman falls into 3 stages (Figure: 2.1):

1. During the first phase $t \in [t_1; t_2]$ she works full-time. A utility function that accounts for leisure and consumption that can solve for the optimal control problem is quasilinear in leisure and linear in consumption $x_i(t)$. Total time is assumed to be Ψ, and labor is denoted by $l_i(t)$, where i is the subscript for the present phase the representative is in. An individual gains utility from consuming the representative good and leisure: $u_1\left[x_1(t), l_1(t)\right] = x_1(t) + \ln\left[\Psi - l_1(t)\right]$. The price of consumption is normalized to 1. All income is consumed, hence the budget constraint is given by $w(\theta, L(t))l_1(t) = x_1(t)$, where the income $w(\theta, L(t))$ depends on ability θ and labor experience gained thus far, $L(t) = \int_{t_1}^{t} l(t)dt$. Labor experience $L(t)$ is the state variable of this problem and to simplify notations it is denoted $L(t) = L_t$. L_0 is assumed to be zero, hence the the first income $w(\theta, 0)$ depends solely on ability. Education could also be part of this ability parameter. It can be shown how a proceeding education phase influences fertility; the timing when she enters the labor market and her initial income becomes endogenous. This reflects how, flexible this model setup is, and that it can be used for a wide variety of policy evaluations that affect fertility. In order to keep the model manageable to avoid adding more phases, we make the simplifying assumption that all children are born at the same time t_2 and do not require any child-care after t_3. The length of phase 3 has length $h(k)$ and depends on the number of children k. The decisions, how when to have children and how many children one wants to have depend on each other in real life. This is also reflected by this model setup as that t_2 and k are derived simultaneously.

2. During phase two, when $t \in [t_2, t_3]$, the woman has children and works part-time. When she is married and does not get divorced, which we assume in the baseline model, then time costs for k children that have been born at t_2 are $c(k, t_2)$ and the monetary costs are $m(k, t_2)$, which are lower that full costs. The father bears the rest of the costs. For the purpose of this article, we do not need to model the proportions. After divorce a woman's time costs and monetary costs increase to $c^d(k, t_2)$ and $m^d(k, t_2)$, respectively. c^d and m^d are strictly less than full costs as the father has to bear some part that can be specified with appropriate parameters. Having k children introduces not just costs but also benefits from having children during phase two and three $v_i(k)$; $i \in (2, 3)$. The utility function is given by $u_2\left[x_2(t), l_2(t)\right] = x_2(t) + \ln\left[\Psi - l_2(t) - c(k, t_2)\right] + v_2(k)$. The labor income is consumed partly by the mother and partly by her children, the budget constraint is therefore $w(L_t)l_2(t) = x_2(t) + m(k, t_2)$, where the monetary costs for the mother are smaller than the total monetary costs of having k children, because

the husband is assumed to contribute his part as well. How much he contributes depends on different aspects such as his own income, outside options for having k children with this particular woman, and the intra-household distribution. This model could be extended to take these complex issues into consideration. They are left open for further research.

3. During the last phase $t \in [t_3, T]$, the individual works full-time again. After t_3 children are older and do not have to be looked after. A woman consumes the consumption good $x_3(t)$, leisure $[\Psi - l_3(t)]$ and retrieves utility from having k children $v_3(k)$, thus $u_3 [x_3(t), l_3(t)] = x_3(t) + \ln [\Psi - l_3] + v_3(k)$. The budget constraint in this phase is $w(L_{t_3})l_3(t) = x_3(t)$. The wage depends on the labor experience accumulated until the end of phase 2. We assume that the wage is constant during this phase for simplicity. We also solved the model for a non-constant wage, but the main results do not change. Empirically one can observe that wage often even decreases before retirement, hence labor experience gained then does not pay off. At time T the planning horizon ends. The retirement shall not play any role in this analysis.

The Hamiltonian for phase $i \in [1, 2, 3]$ is given by $H [x_i(t), l_i(t), \eta_i(t)] = u_i + \eta_i(t)l_i(t)$, where $\eta_i(t)$ is the costate function of this optimal control problem. During the last phase $\eta_3(t) = 0$, because the wage rate is constant then. The derivative of the income with respect to labor experience is denoted as $\frac{\partial w_i(L_t)}{\partial L_t} = \alpha_i (, L_t)$. $\alpha_i (L_t)$ is larger during phase 1 than during phase 2 when a mother works part-time. A possible income scheme is shown by figure 2.1, where we show income per time period. There are no discontinuous vertical movements, because we assume the individual keeps earning the same hourly wage rate, when she enters a new phase, because experience does not decay overnight.

The planning horizon begins at $t = t_1$ and ends at $t = T$; both exogenous. t_2 is determined in the baseline model, t_3 shall be equal to t_2 plus $h(k)$, which is time independent and depends on the number of k children; $t_3 = t_2 + h(k)$. $h(k)$ characterizes the length of time of parental leave. For simplicity however, and because we are not interested in the choice of interval between births, we assume that all children are born at t_2. We do not assume that skills deteriorate during phase two as Happel et al. (1984), but that could be another possible extension.

We solve the problem for each of the three phases of a woman's life backward from the last. We develop necessary conditions for this problem. First we take $t_2 \in [t_1, T]$ and $k > 0$ as given and solve for the optimal consumption and labor supply. In a next

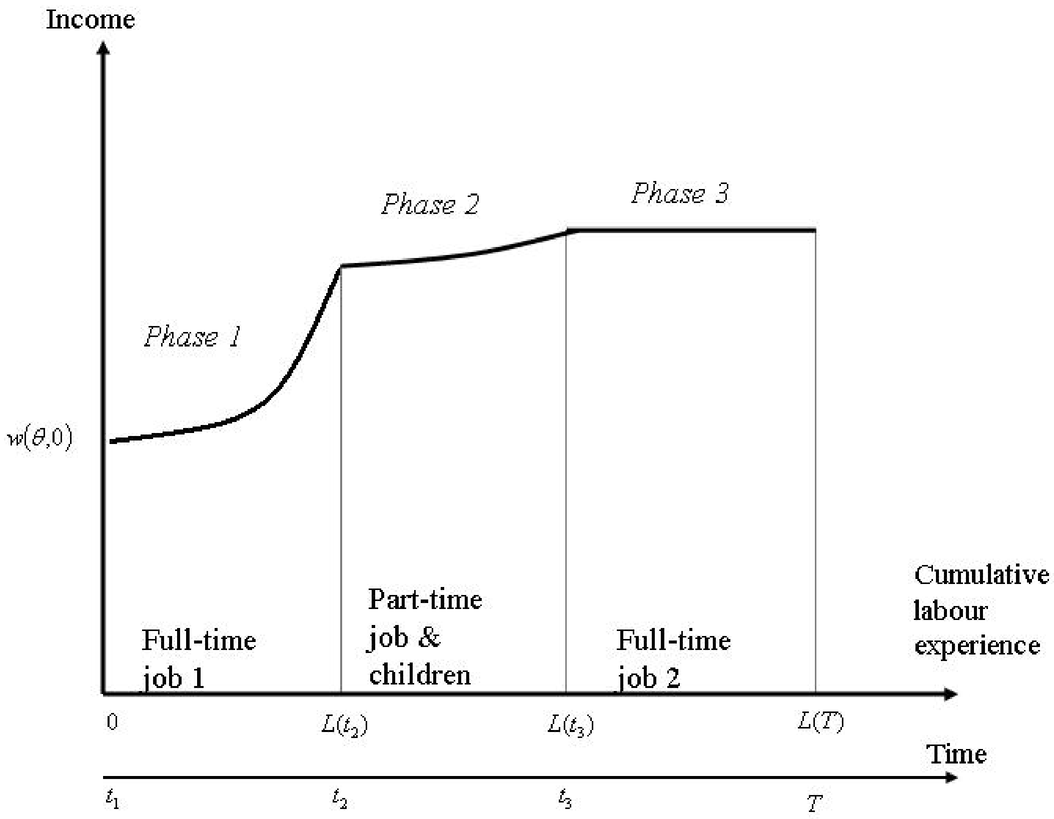

Figure 2.1: The per period income over a life-cycle.

step we characterize the optimal time of childbirth t_2. By a theorem of Hestens, given the problem with t_2 and k fixed, we can define $\eta_i(t)$ on $[t_i, t_{i+1}]$; $i = 1, 2$ as the costate variables of labor experience.[21]

2.2.1 Solving the model

Phase 3: $t \in [t_3, T]$

An individual's objective is to maximize $\int_{t_3}^{T} \{x_3(t) + \ln[\Psi - l_3] + v_3(k)\}\, dt$ subject to the budget constraint. The Lagrangian is

$$\Gamma[x_3(t), l_3(t)] = x_3(t) + \ln[\Psi - l_3] + v_3(k) + \lambda_3(t)[w(L_{t_3})l_3(t) - x_3(t)] \tag{2.1}$$

where $\lambda_3(t)$ is the Lagrangian multiplier for phase three. For simplicity we assume no discounting. A positive discount rate complicates the analysis unnecessarily and leads to a decrease in labor supply, because experience is valued less. A proof follows the same lines as proposition 1.3 of chapter 1. The constant labor supply and consumption can be expressed in terms of the wage rate achieved at t_3.

$$l_3^* = \Psi - \frac{1}{w(L_{t_3})} \tag{2.2}$$

$$x_3^* = \Psi w(L_{t_3}) - 1 \tag{2.3}$$

Phase 2: $t \in [t_2, t_3]$

The computations are more refined in this section as that labor experience obtained within this phase has a future return. The objective here is to maximize $\int_{t_2}^{t_3} \{x_2(t) + \ln[\Psi - l_2(t) - c(k, t_2)] + v_2(k)\}\, dt + V_3^*$ subject to the budget constraint $w(L_t)l_2(t) = x_2(t) + m(k, t_2)$ and $\dot{L}_t = l_2(t)$. V_3^* is the optimally chosen utility stream from t_3 to T, given some labor experience level L_{t_2}. The choice of labor in this phase determines L_{t_3} and thus effects V_3^*. The Lagrangian is

$$\begin{aligned} \Gamma_2[x_2(t), l_2(t)] &= x_2(t) + \ln[\Psi - l_2(t) - c(k, t_2)] + v_2(k) \\ &\quad + \eta_2(t)l_2(t) + \lambda_2(t)[w(L_t)l_2(t) - x_2(t) - m(k, t_2)] \end{aligned} \tag{2.4}$$

[21] see Takayama p.658

From the first order condition of labor and the general optimal control condition, where the time derivative of the costate is equal to the negative Hamiltonian's derivative with respect to the state variable (labor experience), we determine the following two expressions after substituting the optimality condition for consumption $\lambda_2(t) = 1$. Time derivatives are denoted by a dot above a time dependent function.

$$l_2(t) = \Psi - c(k, t_2) - \frac{1}{\eta_2(t) + w(L_t)} \tag{2.5}$$

$$\dot{\eta_2}(t) = -\dot{w}(L_t) \tag{2.6}$$

The transversality condition here is an expression of the costate at t_3. Working an additional hour at t_3 increases her income and has a future return of

$$\eta_2(t_3) = \frac{\partial w(L_t)}{\partial L_t}|_{t=t_3} \int_{t_3}^{T} l_3 dt \tag{2.7}$$

Given the transversality condition (2.7) and the transformation

$$l_2(t)\frac{\partial w(\theta, L_t)}{\partial L_t} = \dot{w}(L_t) \tag{2.8}$$

we can transform (2.6) in a way such that the costate function becomes

$$\eta_2(t) = w(L_{t_3}) - w(L_t) + \alpha_2(t_3)\,[L_T - L_{t_3}] \tag{2.9}$$

where $\alpha_2(t_3) = \frac{\partial w(L_t)}{\partial L_t}|_{t=t_3}$. Using (2.5) and (2.9) one can solve for the optimal labor supply, which is time independent and its consumption counterpart, which does depend on time,

$$l_2^* = \Psi - c(k, t_2) - \frac{1}{w(L_{t_3}) + \alpha_2(t_3)\,[L_T - L_{t_3}]} \tag{2.10}$$

$$x_2^*(t) = w(L_t)[\Psi - c(k, t_2) - \frac{1}{w(L_{t_3}) + \alpha_2(t_3)\,[L_T - L_{t_3}]}] - m(k, t_2) \tag{2.11}$$

The labor supply is also independent from time in phase 1, which we show next. This result is driven by a decreasing return of experience, as the length of time between any t and T, when earlier accumulated experience pays off, decreases. On the other side income increases with experience, which would increase labor supply. Both effects are equally strong and cancel out. This result can be compared to the pricing of a monopolist that produces a single good and learns through production, which is reflected by decreasing unit costs. At each period it sets an optimal price such that its marginal revenue equals the marginal costs at the end of its planning horizon. Given a constant elasticity of demand its price is constant, even though its marginal costs decrease (Spence, 1981). One can also draw the parallel to chapter 1, where in section 1.2.1 it is shown that a social planner sets different prices across phases, but the same price across periods within a phase.

This feature is useful considering the fact that we do not view changes in labor supply from period to period in reality either. Hence this model is more realistic owing to a derivable constant labor supply. Furthermore we derive an increasing consumption function mimicing reality.

Phase 1: $t \in [t_1, t_2]$

An individual's objective is to maximize $V_1 = \int_{t_1}^{t_2} \{x_1(t) + \ln[\Psi - l_1(t)]\}\, dt + V_2^* dt$ subject to the budget constraint and $\dot{L}_t = l_1(t)$. The choice of labor in this phase determines L_{t_2} and influences the utility stream after $t = t_2$, which is denoted by V_2^*. The solution to the problem is

$$l_1^* = \Psi - \frac{1}{w(L_{t_2}) + \alpha_1(t_2)\left[L_T - L_{t_2}\right]} \tag{2.12}$$

$$x_1^*(t) = w(L_t)\left[\Psi - \frac{1}{w(L_{t_2}) + \alpha_1(t_2)\left[L_T - L_{t_2}\right]}\right] \tag{2.13}$$

The costate functional for phase 2 has been derived following the same lines that have led to (2.7)

$$\eta_1(t) = w(L_{t_2}) - w(L_t) + \alpha_1(t_2)\left[L_T - L_{t_2}\right] \tag{2.14}$$

Conclusively we are able to determine the labor supplies for each phase and thus expressions for cumulative labor supplies at the end of phases 1-3. These expressions are needed, when solving for the timing of fertility. They are given by the integrals of instantaneous labor supplies (2.2), (2.10) and (2.12). Since the per period labor supplies are all constants, we can multiply them with the length of each respective phase and add the experience gained in former phases to find the labor experience at the end of each phase.

$$L_{t_2} = \left[\Psi - \frac{1}{w(L_{t_2}) + \alpha_1(t_2)\left[L_T - L_{t_2}\right]}\right](t_2 - t_1) \tag{2.15}$$

$$L_{t_3} = L_{t_2} + \left[\Psi - c(k, t_2) - \frac{1}{w(L_{t_3}) + \alpha_2(t_3)\left[L_T - L_{t_3}\right]}\right](t_3 - t_2) \tag{2.16}$$

$$L_T = L_{t_3} + \left[\Psi - \frac{1}{w(L_{t_3})}\right](T - t_3) \tag{2.17}$$

Jumps of Costates

Proposition 2.1 *There is a discontinuous downward jump (upward) jump, when the return of labor experience is larger (smaller) during the first of the two phases. Furthermore one can show that the quotient of the two consecutive phases 1 and 2 is constant at t_2, when the experience derivative of income is constant within each phase.*

Proof. For $\alpha_i(t) \neq const$

$$\frac{\eta_1(t_2)}{\eta_2(t_2)} = \frac{\alpha_1(t_2)\left[L_T - L_{t_2}\right]}{w(L_{t_3}) - w(L_{t_2}) + \alpha_2(t_3)\left[L_T - L_{t_3}\right]} = \frac{\alpha_1(t_2)\left[L_T - L_{t_2}\right]}{\int_{t_2}^{t_3} \alpha_2(t) l_t dt + \alpha_2(t_3) \int_{t_3}^{T} L_t dt} \tag{2.18}$$

When the experience return is larger at a given point in time during phase 1 (in particular at t_2) than during phase 2, then the quotient (2.18) must be greater than one. Hence there is a downward jump of labour supply at t_2.

For $\alpha_i(t) = \alpha_i = const$

$$\frac{\eta_1(t_2)}{\eta_2(t_2)} = \frac{\alpha_1\left[L_T - L_{t_2}\right]}{w(L_{t_3}) - w(L_{t_2}) + \alpha_2\left[L_T - L_{t_3}\right]} = \frac{\alpha_1}{\alpha_2} \tag{2.19}$$

If $\alpha_2(t)$ decreases with time, then the denominator of (2.18) is smaller than that of (2.19), hence (2.18) must be larger than (2.19), which means that the upward jump is larger when $\alpha_i(t) \neq const.$ ■

In order to simplify the continuative analysis, we assume that a_1 and α_2 are independent of time but $a_1 > \alpha_2$ as discussed earlier. The income payments at the end of phase one and two are then equal to the expressions,

$$w(L_{t_2}) = w(L_{t_1}) + \alpha_1 L_{t_2} \tag{2.20}$$

$$w(L_{t_3}) = w(L_{t_2}) + \alpha_2 [L_{t_3} - L_{t_2}] \tag{2.21}$$

$$w(L_T) = w(L_{t_3}) \tag{2.22}$$

where $w(L_{t_1}) = w(0)$ is the income of an individual who has recently commenced working. (2.22) reminds us that there is no return on experience gained during phase 3. How results change, when we substitute $w(L_T) = w(L_{t_3}) + \alpha_3 [L_T - L_{t_3}]$ for (2.22) where the experience return during phase 3 is $\alpha_3 \neq 0$, is briefly explored later.

2.2.2 The optimality condition for the timing of childbirth

There is the desire to have children earlier in life; and the probability that a child has a disability increases with the mother's age. This is modelled by a change in the expected cost. To keep things simple, we assume that $c(k, t_2)$ and $m(k, t_2)$ increase with certainty, when childbirth is delayed. Advanced medical research makes it feasible to give birth later in life, but such procedures are expensive. In addition to which, parents that are wealthier spend more money on raising their children. Since income increases in this model continuously, monetary costs $m(k, t_2)$ increase with t_2. Besides a positive derivative of $m(k, t_2)$ with respect to t_2, we argue for a positive relation of time costs $c(k, t_2)$ and childbirth. The same rules that apply on the labor market also apply when people raise children: younger people can generally adopt better to changing market conditions and learn faster. A mother in her early 20s might be still able to drop off her children at the kindergarten, before going to her part-time job and pick them up again in the afternoon. Furthermore we assume that the length of time required to raise children is longer, when there are more children; $h'(k) > 0$. This term can be used later to evaluate policy implications for schools, where children can stay all day long. Once children are old enough to go to these schools, both parents could begin to work full-time again. In the model the individual then enters phase 3. We included monetary costs for phase 3 in an earlier working paper. Results shall be briefly discussed below.

With t_2 fixed, one can take the utility stream from $t_1$up to T and differentiate this expression with respect to t_2. This expression must be equal to zero at the optimal time of childbirth t_2^*. Now consider the following three sub-problems:

For $t \in [t_1, t_2]$ t_1 and t_2 fixed

$$SP_1^* = \max_{x_1(t)} \int_{t_1}^{t_2} \{x_1(t) + \ln[\Psi - l_1(t)]\} \, dt \tag{2.23}$$

$$s.t. \; \dot{l_1}(t) = l_1(t) \text{ and } w(L_t)l_1(t) - x_1(t) = 0$$

For $t \in [t_2, t_2 + h(k)]$ t_2 *and* $h(k)$ *fixed*

$$SP_2^* = \max_{x_2(t)} \int_{t_2}^{t_2+h(k)} \{x_2(t) + \ln[\Psi - l_2(t) - c(k, t_2)] + v_2(k)\} \, dt \tag{2.24}$$

$$s.t. \; \dot{l_2}(t) = l_2(t) \text{ and } w(L_t)l_2(t) - x_2(t) - m(k, t_2) = 0$$

For $t \in [t_2 + h(k), T]$ $\quad t_2$, $h(k)$ *and* T *fixed*

$$SP_3^* = \max_{x_3(t)} \int_{t_2+h(k)}^{T} \{x_3(t) + \ln[\Psi - l_3(t)] + v_3(k)\} \, dt \tag{2.25}$$

$$s.t. \dot{l_3}(t) = l_3(t) \text{ and } w(L_t)l_3(t) - x_3(t) = 0$$

We need to use the Leibniz Rule to derive $\frac{\partial SP_i^*}{\partial t_2}$ for $i = 1, 2$ and 3. For each phase i we receive three terms:

1. The integral of $\frac{\partial SP_i^*}{\partial t_2}$ with the corresponding phase's bounds.
2. We subtract the t_2 derivative of the lower bound of phase i, which is multiplied by the Hamiltonian evaluated at the lower bound.
3. Finally we add the derivative of the upper bound with respect to t_2, which is multiplied by the Hamiltonian evaluated at that point.

Phase 1

$$\frac{\partial SP_1^*}{\partial t_2} = H_1^*(t_2) \tag{2.26}$$

Applying the envelope theorem, the first term is zero. The lower bound is independent of childbirth, hence term two is zero. The third term; $H_1^*(t_2)$ intuitively means that an incremental increase in t_2 comes along with additional per period utility gained during phase one at t_2.

Phase 2

$$\frac{\partial SP_2^*}{\partial t_2} = -\frac{\partial c(k,t_2)}{\partial t_2}\frac{h(k)}{\Psi - l_2^* - c(k,t_2)} - H_2^*\left(t_2\right) + H_2(t_3) \tag{2.27}$$

One can show that $H_2^*(t_3) - H_2^*\left(t_2\right) = 0$. This result is due to the fact that in the presence of learning, the per period utility within each phase is constant. The change in utility through an increase in consumption is completely offset by the change of utility through the decrease of the experience value. One can draw a parallel to the earlier discussion in section 2.2.1. Hamiltonians within any phase are of equal value independent of the period in which they are evaluated.

Applying the envelope theorem, the first term is $-\frac{\partial c(k,t_2)}{\partial t_2}\frac{h(k)}{\Psi - l_3^* - c(k,t_2)}$ and does not vanish here, because the derivative with respect to $c(k,t_2)$ is not equal to zero. However the derivatives of the per period Hamiltonian with respect to $x_2^*(t)$, l_2^* and $\eta_2^*(t)$, which have already been chosen optimally are zero. $c(k,t_2)$ depends on the number of children and the timing of childbirth, which are not optimal at this stage yet. The change of time costs has to be paid for the length of this phase, $h(k)$. The second term comes from a decrease of phase two's utility at the original t_2 before the change, the third term from an increase of phase two's utility at t_3. Phase two can be seen as shifted to the right within the time interval.

Phase 3

$$\frac{\partial SP_3^*}{\partial t_2} = -H_3^*\left(t_3\right) \tag{2.28}$$

The envelope theorem allows the first term to vanish, the third term does not occur here either, because the upper bound of phase four T is exogenously given and hence independent of t_2. $-H_3^*\left(t_3\right)$ expresses the fact that phase three becomes shorter and loses an incremental period at t_3.

Adding (2.26), (2.27), (2.28) and setting them equal to zero gives the optimality condition for the optimal timing of childbirth, where k is still assumed to be fixed.

$$H_1^*\left(t_2\right) - \frac{\partial c(k,t_2)}{\partial t_2}\frac{h(k)}{\Psi - l_3^* - c(k,t_2)} - H_3^*\left(t_3\right) \doteq 0 \tag{2.29}$$

2.2.3 The optimality condition for the number of children

Again we use the Leibniz rule and the Envelope theorem with the same method used to derive the t_2^*-optimality condition. The timing of childbirth depends on phase one's utility stream, but the number of children k does not, thus $\frac{\partial SP_1^*}{\partial k} = 0$.

The length of phase two and three changes with the number of children. The terms that affect the number of children are the costs and benefits, while children are young (phase 2), the benefits when they are older (phase 3), and the length of phase 2, $h(k)$.

Phase 2

$$\frac{\partial SP_2^*}{\partial k} = \left[\frac{\partial v_2(k)}{\partial k} - \frac{\partial c(k,t_2)}{\partial k}\frac{1}{\Psi - l_2^* - c(k,t_2)}\right] h(k) + H_2^*(t_3)h'(k) \tag{2.30}$$

The first term is the change of the per period utility of phase two from an increase of benefits from having more children, subtracted by additional costs multiplied by the length of this phase $h(k)$. The second term is the additional utility from an increase of length of phase two.

Phase 3

$$\frac{\partial SP_3^*}{\partial k} = (T - t_3)\frac{\partial v_3(k)}{\partial k} - h'(k)H_3^*(t_3) \tag{2.31}$$

When more children are born, the additional benefit from having them is accounted for by the first term. Phase 3 becomes shorter through an increase of length in phase 2 when more children are present (second term) .

The k^*-optimality condition is thus given by

$$h(k)\left[\frac{\partial v_2(k)}{\partial k} - \frac{\partial c(k,t_2)}{\partial k}\frac{1}{\Psi - l_2^* - c(k,t_2)}\right] + (T - t_3)\frac{\partial v_3(k)}{\partial k} + h'(k)\left[H_2^*(t_3) - H_3^*(t_3)\right] \doteq 0 \tag{2.32}$$

We derive the optimal number of children and the optimal timing of childbirth simultaneously. The equation that describes the optimal number of children is given by (2.32), which depends on t_2 just in the same way as (2.29), the equation that characterizes the optimal date of childbirth.

Given (2.15), (2.16), (2.17), (2.20), (2.21), (2.22), (2.29) and (2.32) we can solve for the optimal number of children and timing of childbirth numerically. Besides these two variables, we can also solve for cumulative labor experience at t_2, t_3, and T and the per period income level at these points. The characterization of an analytical solution would be extremely tedious, because one would have to apply the implicit function theorem for eight equations, where each of them depends on all other seven equations.

	Functional form	*Phase*
Time costs	$c(k,t_2) = c_1 k^{\beta_1} + c_2^{-1} t_2^{\beta_2}$	2
Utility from Children	$v_2(k) = c_3 k^{\beta_3}$	2
Utility from Children	$v_3(k) = c_4 k^{\beta_3}$	3
Length of phase 2	$h(k) = c_5 k^{\beta_4}$	2
Monetary costs	$m(k,t_2) = c_6 k^{\beta_5} + c_2^{-1} t_2^{\beta_2}$	2

Table 2.1: Functional forms

2.2.4 Results

We need to make assumptions regarding the functional forms of the cost functions, utility derived from children and the length of phase two. These are presented in table 2.1.[22]

All functions in table 2.1 are concave in the number of children k. When they depend on the timing of childbirth, then they are convex in t_2. The parameters have also been chosen such that the optimal number of children is 2 .2 to reflect the number of children a woman must have on average to keep the population at a constant level. In 2006, the average age of a woman receiving her first child in the 25 European Union member states was approximately 29 years of age.[23] The parameters of the baseline model are chosen to have an optimal number of years spent on the labor market of about 7.4 years, because an average age, when entering the labor market of 21.6 seems reasonable.[24] T, the total number of years spent on the labor market is assumed to be 40. The age at retirement is thus 61.6. The parameters α_1 and α_2 are 5% and 2%, reflecting the observation that income increases with experience more during phase 1 when no children are present and less when she works part-time and looks after her children (phase 2). Empirically one does not observe an increase of real income during phase 3, hence we set $\alpha_3 = 0$. We start at an exogenously given wage of 10. It endogenously increases to 13.7 until t_2^*, furthermore goes up to 14.1 during phase 2 and remains at this level until T. Comparative static results are summarized in table 2.2. To save space we left out how other variables such as labor experience and the wage rate are affected through a parameter change. Bold (italic) values represent increasing (decreasing) t_2^*'s or k^*'s due to a 1% increasing parameter.

[22] We use Matlab to find numerical solutions for the eight conditions; the command "fsolve" finds solutions for nonlinear systems.

[23] Eurostat (2006): Population statistics

[24] Within the EU-15 countries over 40% of the cohort aged 22 years has entered the labour force.

Values of the baseline model					
t_2^*	7.447	$L_{t_2}^*$	74.200	$w(t_2^*)$	13.710
k^*	2.189	$L_{t_3}^*$	95.806	$w(t_3^*)$	14.142
		L_T	345.583	$w(T)$	14.142
How a 1% increase of the parameters below affects t_2^* and k^*					
Variables	$w_{t_1} = 10$	$\alpha_1 = 5\%$	$\alpha_2 = 2\%$	$t_1 = 0\ (+0.1)$	$T = 40$
t_2^*	**7.5035**	**7.7503**	*7.4299*	*7.4192*	**7.6150**
k^*	*2.1604*	*2.1147*	**2.1901**	**2.2030**	*2.1829*
	$c_1 = 4$	$c_2 = 50$	$c_3 = 70$	$c_4 = 70$	$c_5 = 5$
t_2^*	**7.5908**	**7.5228**	*7.3385*	*7.1978*	**7.4841**
k^*	*2.1307*	*2.1742*	**2.2309**	**2.2565**	*2.1521*
	$c_6 = 20$	$\beta_1 = 5\%$	$\beta_2 = 2$	$\beta_3 = 3\%$	$\beta_4 = 3\%$
t_2^*	**7.474**	**7.5896**	*7.0766*	*7.1819*	**7.5548**
k^*	*2.1307*	*2.133*	**2.2636**	**2.2854**	*2.1392*
	$\beta_5 = 5\%$	$\Psi = 10$			
t_2^*	**7.4619**	**8.0552**			
k^*	*2.1831*	*2.0429*			

Table 2.2: How the optimal number of children and the timing is affected by the underlying parameters.

Changing one of the underlying parameters affects all optimality conditions. A first observation is that when k^* increases (decreases) due to a change of one parameter, then the timing of childbirth t_2^* decreases (increases). Besides the negative correlation between these variables, there is a negative correlation between k^* and all other variables; the optimal number of children increases only, when the optimal cumulative labor supplies and incomes at the end of all phases decrease. We interpret the results one for one and concentrate on the timing of childbirth and the number of children. An increase in the income level decreases the number of children wanted. The opportunity costs of having children increases, thus less children are born. An increase in α_1 delays the optimal timing of childbirth, because an individual wants to exploit income increases during phase 1, which are larger than in any other phase. A delayed timing of childbirth is automatically connected to fewer children. An increase in α_2 on the other hand increases the number of children wanted, because an early childbirth is not as expensive, when her wage can still increase sufficiently after t_2^*. In an earlier version, we accounted for $\alpha_3 > 0$; labor experience gained during phase 3 increases the future income. Increasing α_3 has the same comparative effects on the choice variables as increasing α_2 with the same intuition behind it. Kreyenfeld (2003) examines the difference of fertility rates between East and West Germany after the reunification in 1990. She shows that the East German cohort of young people has its first child at a younger age compared to the West German cohort, even though it has fewer children in total. Kreyenfeld (2003) claims that the increase in uncertainty about future income was the main cause for this observation. Another reason seems compelling; many young East Germans, who worked in areas for which labor experience mattered, moved to West Germany after the re-unification, leaving those behind, whose opportunity costs of having children early were low.

An increasing working-span of an individual (changes in t_1, T and Ψ) has a negative effect on fertility. An increase of the working life raises life-time income and income per period. Thus the opportunity costs of having children are larger. An increase of c_1 or β_1 means that the marginal time cost of an additional child increases. Not surprisingly, if these costs increase, the number of children goes down. Governments that offer placements in kindergartens, where children can stay until the afternoon, give the mother the opportunity to take a longer part-time job and hence decrease c_1. c_2 and β_2 are parameters that are connected to the time cost burden of raising children, when children come late. Up to a number of $\sqrt{c_2}$ years, the time costs reflected by the second term of $c(k, t_2)$ are less than one. Since they increase exponentially though, they do matter at some point and induce her to enter phase 2. When c_2 increases or β_2

decreases, the marginal time cost of giving birth late decreases. Therefore women have fewer children but later. Medical research enabling late childbirth has a negative effect on fertility. Soares (2005) shows why advances in medical research corresponds with lower fertility in developing countries. When child mortality is reduced, the expected costs of large families increase and the marginal benefits decrease. An increase of benefits from young and old children c_3, c_4 and β_3 increase the number of children. If the length of phase 2 is long (large c_5 and β_4), then the individual's number of children decreases. The results come from the underlying structure of the model based on costs mainly occurring due to leaving phase 1 and entering phase 2 (decrease of cumulative experience return), but benefits also occur during the last phase. A government that offers sufficient placements of full-day care centres or full-time schools increases its country's fertility, by shortening phase 2. β_5 and c_6 are connected to the monetary costs she has to encounter, when children are young. An increase of child benefits increases the number of children. It is straightforward to include monetary costs for phase 3 as well. Changing the parameters of these, when they have the same functional form as $m(k, t_2)$ also has the same effect as changing β_5 and c_6. Child benefits are reflected by a lower c_6. A financial incentive given to parents in Germany is the so-called "Elterngeld" (parental benefits). Parents receive up to 2/3 of one of the partner's last net income for one year, if one parent stays at home during that time and looks after the child. Parents can choose between a one-year-parental-leave and a day-care centre. In our model this would be reflected by the choice between a positive c_6 and a lower $c(k, t_2)$ if the parental leave is rejected and a negative c_6 and a very large $c(k, t_2)$ such that $l_2^* = 0$ if it is accepted. Apps and Rees (2004) also show how specific government policies affect fertility choices.

2.3 Extension A: Divorce

Marriage may not last until the end of a woman's planning horizon T. [25] When the probability of divorce increases through an exogenous change, then Grossbard-Shechtman (1984) argues that women have more outside options and reduce their supply of household goods which includes the number of children. In our setup divorce causes the number of children to be reduced as well, but for a different reason. Divorce is more costly for a woman when she has more children. A woman with many

[25] Sweden and the United Kingdom have the highest divorce rates in Europe with over 50%. Austria, Belgium, Czech Republic, Denmark, Finland, France, Germany, Hungary, Norway and Switzerland have divorce rates between 40%-50%. Ireland, Italy, Poland and Spain have the lowest divorce rates of less than 20% according to Eurostat (2006) "Population Statistics".

children has less labor experience and hence a lower income. At the same time the costs of having children increase after divorce, because she has to raise them by herself $c^d(k,t_2) > c(k,t_2)$ and receives less monetary support from the father, hence her monetary contribution to children increases $m^d(k,t_2) > m(k,t_2)$. Benefits from children for the mother do not change after divorce. A divorce solely effects the woman's utility, when it occurs during phase 2, therefore we also restrict it to that phase.

Phase 2 $t \in [t_2, t_3]$ is solved in two steps

1. The date of divorce d is known and $d \in [t_2, t_3]$
2. The date of divorce is uncertain.

2.3.1 Step 1: The Optimal Plan before and after Divorce known to occur at time d.

Optimal Plan after d We begin to solve the problem by finding the individual's optimal plan after divorce has occurred. Later it is shown, how the individual acts before the known date d. The objective that needs to be maximized is

$V_2^d = \int_d^{t_3} \left\{ x_2(t) + \ln\left[\Psi - l_2(t) - c^d(k,t_2)\right] + v_2(k) \right\} dt + V_3^*$ subject to the budget constraint $w(L_t)l_2(t) = x_2(t) + m^d(k,t_2)$ and as before $\dot{L}_t = l_2(t)$. The Lagrangian after divorce is

$$\begin{aligned} \Gamma_2^d\left[x_2(t), l_2(t)\right] &= x_2(t) + \ln\left[\Psi - l_2(t) - c^d(k,t_2)\right] + v_2(k) \\ &\quad + \eta_2(t) l_2(t) + \lambda_2(t)\left[w(L_t)l_2(t) - x_2(t) - m^d(k,t_2)\right] \end{aligned} \tag{2.33}$$

Substituting $\lambda_2(t) = 1$ the equilibrium conditions of this problem are

$$l_2^d(t) = \Psi - c^d(k,t_2) - \frac{1}{\eta_2(t) + w(L_t)} \tag{2.34}$$

$$\dot{\eta}_2(t) = -\dot{w}(L_t) \tag{2.35}$$

The transversality condition here is an expression of the costate at t_3. Working an additional hour at t_3 increases her income and has a future return of

$$\eta_2^d(t_3) = \frac{\partial w(L_t)}{\partial L_t}|_{t=t_3} \int_{t_3}^{T} L_T dt = \frac{\partial w(L_t)}{\partial L_t}|_{t=t_3} \left[L_T - L_{t_3}\right] \tag{2.36}$$

Given transversality condition (2.36), equation (2.35) can be re-written such that the costate becomes

$$\eta_2^d(t) = w(L_{t_3}) - w(L_t) + \alpha_2(t_3)\left[L_T - L_{t_3}\right] \tag{2.37}$$

where $\alpha_2(t_3) = \frac{\partial w_3(L_t)}{\partial L_t}|_{t=t_3}$.

Using (2.34) and (2.37) we solve for the optimal labor supply, which is independent of time and its consumption counterpart, which does depend on time just as in the absence of divorce,

$$l_2^d = \Psi - c^d(k,t_2) - \frac{1}{w(L_{t_3}) + \alpha_2(t_3)\left[L_T - L_{t_3}\right]} \tag{2.38}$$

$$x_2^d(t) = \Psi w(L_t) - c^d(k,t_2)w(L_t) - \frac{w(L_t)}{w(L_{t_3}) + \alpha_2(t_3)\left[L_T - L_{t_3}\right]} - m^d(k,t_2) \tag{2.39}$$

The direct utility after divorce $V_2^d(d)$, which is needed to find the optimal number of children later is

$$V_2^d(d) = \int_d^{t_3} \left\{x_2^d(t) + \ln\left[\Psi - l_2^d - c^d(k,t_2)\right] + v_2(k)\right\} dt + V_3^* \tag{2.40}$$

and the per-period direct utility, needed for the same reason, is

$$V_2^d(t) = \begin{array}{c} \Psi w(L_t) - c^d(k,t_2)w(L_t) - \frac{w(L_t)}{w(L_{t_3})+\alpha_2(t_3)\left[L_T - L_{t_3}\right]} - m^d(k,t_2) \\ + \ln\left[\frac{1}{w(L_{t_3})+\alpha_2(t_3)\left[L_T-L_{t_3}\right]}\right] + v_2(k) \end{array} \tag{2.41}$$

Optimal Plan before d We solve for an optimal plan for a known date of divorce d. The individual maximizes the objective $\int_{t_2}^{d} \{x_2(t) + \ln\left[\Psi - l_2(t) - c(k,t_2)\right] + v_2(k)\} dt$ subject to the constraint $w(L_t)l_2(t) = x_2(t) + m(k,t_2)$.

$$\dot{\eta_2}(t) = -\dot{w(L_t)} \tag{2.42}$$

together with the transversality condition

$$\eta_2(d) = \alpha_2(d) \int_d^T l(t)dt \tag{2.43}$$

yields the costate's functional equation

$$\eta_2(t) = w(L(d)) - w(L_t) + \alpha_2(d)\left[L_T - L(d)\right] \tag{2.44}$$

Proposition 2.2 *The costate function does not change after a known date of divorce, when the experience derivative of income is time independent $\alpha_2(t) = \alpha_2$.*

Proof. Substituting α_2 for $\alpha_2(t)$ in (2.37)

$$\begin{aligned}
\eta_2^d(t) &= w(L_{t_3}) - w(L_t) + \alpha_2\left[L_T - L_{t_3}\right] \\
&= w(L(d)) + \int_d^{t_3} \dot{w}(L_t)dt - w(L_t) + \alpha_2\left[L_T - L_{t_3}\right] \\
&= w(L(d)) + \int_d^{t_3} \alpha_2 l(t)dt - w(L_t) + \alpha_2\left[L_T - L_{t_3}\right] \\
&= w(L(d)) + \alpha_2\left[L_{t_3} - L(d)\right] - w(L_t) + \alpha_2\left[L_T - L_{t_3}\right]
\end{aligned}$$

which is equal to (2.44). ■

The result here is also due to the utility's functional form. If it were not quasi-linear in the consumption good, then $\lambda_2(t) \neq 1$ and the costate would depend on per-period labor or consumption.

$$l_2 = \Psi - c(k, t_2) - \frac{1}{w(L_{t_3}) + \alpha_2(t_3)\left[L_T - L_{t_3}\right]} \tag{2.45}$$

$$x_2(t) = \Psi w(L_t) - c(k, t_2)w(L_t) - \frac{w(L_t)}{w(L_{t_3}) + \alpha_2(t_3)\left[L_T - L_{t_3}\right]} - m(k, t_2) \tag{2.46}$$

(2.46) shows that consumption is larger before than after divorce has occurred, because $c(k, t_2) < c^d(k, t_2)$ and $m(k, t_2) < m^d(k, t_2)$. $l_2 > l_2^d$ because children demand more time for their child care. Future benefits of labor remain unchanged.

2.3.2 Step 2: The optimal plan before an unknown date of divorce

Decisions after d are given in the last section; see (2.38) and (2.39). They do not vary, when divorce is uncertain, because after d all uncertainty is cleared. Expectations about divorce are uniform among all representatives, the subjective probability that divorce occurs at time t is $\phi(t)$. The perceived probability that the marriage will persist at least until time t is consequently calculated as

$$G(t) = \int_{t}^{t_3} \phi(t)dt \tag{2.47}$$

The date of divorce is unknown; the individual is obliged to maximize her expected utility,

$$\int_{t_2}^{t_3} \phi(d)H(d)dd + \int_{t_2}^{t_3} \phi(d)V_2^d(d)dd \tag{2.48}$$

where $V_2^d(d)$ is given by (2.40) and $H(d) = \int_{t_2}^{d} u_2\left[x_2(t), l_2\right] dt = \int_{t_2}^{d} \left\{x_2(t) + \ln\left[\Psi - l_2(t) - c(k, t_2)\right] + v_2(k)\right\} dt$. (2.48), upon integration by parts may be expressed as

$$\int_{t_2}^{t_3} \left\{G(t)u_2\left[x_2(t), l_2\right] + \phi(t)V_2^d(t)\right\} dt \tag{2.49}$$

where $V_2^d(t)$ is given by (2.41).

Therefore an individual maximizes

$$\int_{t_2}^{t_3} \left\{G(t)u_2\left[x_2(t), l_2\right] + \phi(t)V_2^d(t)\right\} dt + V_3^* \tag{2.50}$$

subject to the known constraints. Consequently, the Lagrangian from which the socially optimal plan before divorce can be derived is

$$\begin{aligned} \Gamma \;=\; & G(t)\left\{x_2(t) + \ln\left[\Psi - l_2(t) - c(k, t_2)\right] + v_2(k)\right\} + \phi(t)V_2^d(L_t) \\ & + \eta_2^{bd}(t)l_2(t) + \lambda_2(t)\left[w(L_t)l_2(t) - x_2(t) - m(k, t_2)\right] \end{aligned} \tag{2.51}$$

where the equilibrium conditions are

$$\lambda_2(t) = G(t) \tag{2.52}$$

$$l_2(t) = \Psi - c(k, t_2) - \frac{G(t)}{\eta^{bd}(t) + G(t)w(L_t)} \tag{2.53}$$

$$\dot{\eta_2^{bd}}(t) = -\phi(t)\frac{\partial V_2^d}{\partial L_t} - \dot{w}(L_t)G(t) \tag{2.54}$$

where $\eta_2^{bd}(t)$ is the costate of phase 3 before divorce, when divorce is uncertain. The per period consumption is

$$x_2(t) = \Psi w(L_t) - c(k, t_2)w(L_t) - \frac{G(t)w(L_t)}{\eta_2^{bd}(t) + \lambda_2(t)w(L_t)} - m(k, t_2) \tag{2.55}$$

The expected direct utility in the presence of uncertainty (index U) at t_2 for all future periods of phase 3 is

$$V_2^U(k, L_t) = \int_{t_2}^{t_3} \left\{G(t)u_2\left[x_2^*(t), l_2^*\right] + \phi(t)V_2^d(L_t)\right\} dt + V_3^* \tag{2.56}$$

The costate's time derivatives before and after divorce in the absence of uncertainty (2.42) and (2.35) respectively denoted by $\dot{\eta_2}(t)$ are equal. Comparing these with (2.54) denoted by $\dot{\eta_2^{bd}}(t)$ indicates the timing of childbirth, when divorce is uncertain. Both equations are used to derive

$$\dot{\eta_2}(t) = \frac{\dot{\eta_2^{bd}}(t) + \phi(t)\frac{\partial V_2^d}{\partial L_t}}{G(t)} \tag{2.57}$$

The costates' time derivative and therefore also the costates themselves are equal, when the probability of divorce at some time t, $\phi(t) = 0$, and the perceived probability that marriage will persist at least until time t, $G(t) = 1$. The second term in the nominator of (2.57) is small, because the instantaneous probability of divorce $\phi(t)$ is small. $G(t)$ is the probability that a couple is still married at time t. In most EU countries except the UK and Sweden this value is at least 0.5 for all $t \in [t_2, t_3]$. Thus one can assume that $G(t) > \phi(t)\frac{\partial V_2^d}{\partial L_t}$. Both time derivatives are negative, because within this phase and any other phase, experience pays off less and less the sooner she

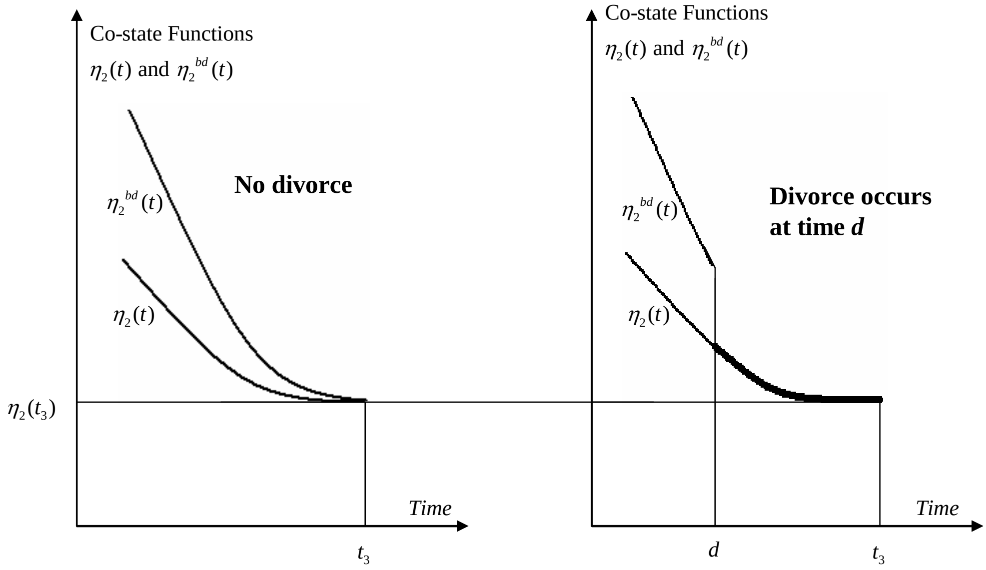

Figure 2.2: The co-states of phase two, when a divorce does not occur and when it does for a known and an unknown d.

reaches her retirement, therefore $\dot{\eta}_2^{bd}(t) < \dot{\eta}_2(t)$. Both costate functions have the same functional value at t_3, because all uncertainty is resolved at t_3. In the case of no divorce $\eta_2^{bd}(t)$ must lie entirely above $\eta_2(t)$. They coincide at t_3. In case a divorce occurs, $\eta_2^{bd}(t)$ must jump downwards such that both costates can coincide. This is shown in figure 2.2.

When the date of divorce is known, then the costate before and after divorce is unchanged. It is only affected, when d is unknown. This shows that our individual values labor more, when she faces the risk of divorce. She therefore has a higher labor supply in the presence of uncertainty. A known date of divorce would therefore lead to a lower labor supply and more children due to the negative correlation between these variables. Next we answer the question whether a woman reduces the number of children in the presence of divorce and if she consequently delays the timing of childbirth.

2.4 Extension B: Divorce, a numerical simulation

After illustrating divorce within this model setup analytically such that there is a positive probability of divorce in every period of phase 2 (extension A), we continue to show a simplified method where divorce occurs with a positive probability at varying points in time between t_2 and t_3. Derivations from extension A are needed in this section. The divorce probability is zero for all other periods as in extension A, because a woman would not be affected by it in this setup. Again it's a straightforward extension to include divorce for phase 3, when there are monetary costs connected to children in that phase. Our main results do not change, hence we leave it out, however we discuss them briefly below. Within this framework, we can solve for the timing of fertility and the number of children numerically as we have done in the baseline model. Extension A was more general therefore less precise, because it only characterizes the costate during phase 2 in the presence of divorce, but does not find a solution for t_2^* and k^* explicitly, which this section does. With a probability of $p < 1$ there is a divorce during phase 2. Re-marriages are excluded for simplicity. The possible date of divorce d during phase 2 is given by

$$d = t_2 + \frac{h(k)}{c_7} \tag{2.58}$$

where $c_7 \in (1, \infty)$. (2.58) means that divorce occurs after a certain portion of phase 2 is over, which depends on c_7. The longer phase 3 the more children are present; $h'(k) > 0$. Divorce occurs then later as it is more costly, when more children are present. Next we derive the t_2 and k- optimality conditions. Again we differentiate utility streams. The first and third phases' utilities do not change through divorce but their utility stream needs to be added to the two cases: divorce and no-divorce. The utility streams from t_1 to T are thus;

1. No divorce: (2.23)+(2.24)+(2.25)
2. Divorce during phase 2: (2.23)+$DP^*_{2.1} + DP^*_{2.2d}$+(2.25).[26]

F*or* $t \in [t_2, d]$, t_2 *and* d *fixed*

$$DP^*_{2.1} = \max_{x_2(t)} \int_{t_2}^{d} \{x_2(t) + \ln[\Psi - l_2(t) - c(k, t_2)] + v_2(k)\}\, dt \tag{2.59}$$

$$s.t.\ \dot{l_2}(t) = l_2(t) \text{ and } w(L_t)l_2(t) - x_2(t) - m(k, t_2) = 0$$

For $t \in [d, t_2 + h(k)]$ t_2, $h(k)$ *and* d *fixed*

[26]The subscript 2.1 is attached to the utility during phase 2 up to d and 2.2d to the utility during phase 2 after d.

$$DP_{2.2d}^{*} = \max_{x_2(t)} \int_{d}^{t_2+h(k)} \left\{ x_2(t) + \ln\left[\Psi - l_2(t) - c^d(k,t_2)\right] + v_2(k) \right\} dt \tag{2.60}$$

$$s.t.\ \dot{l_2}(t) = l_2(t) \text{ and } w(L_t) l_2(t) - x_2(t) - m^d(k,t_2) = 0$$

Case 1, the no divorce case is described by the baseline model. The left hand side of (2.29) multiplied by the no-divorce probability is the first part of the expected utility. Case 2: We have already solved for $DP_{2.1}^{*} + DP_{2.2d}^{*}$ in extension A. The t_2-optimality conditions can be derived when adding the terms of our two cases:

1. The expected utility from "no-divorce" case for the t_2-optimality condition is given by

$$(1-p)\left[H_2^{*}(t_2) - \frac{\partial c(k,t_2)}{\partial t_2} \frac{h(k)}{\Psi - l_2^{*} - c(k,t_2)} - H_3^{*}(t_3)\right] \tag{2.61}$$

2. The part, when divorce occurs at d during phase 3 is

$$p \left\{ \begin{array}{c} H_2^{*}(t_2) - H_2^{d*}(t_2 + h(k)) - \frac{1}{\Psi - l_2^{*} - c(t_2,k)} * \\ \left[\frac{h(k)}{c_7}\frac{\partial c(k,t_2)}{\partial t_2} + h(k)\left(1 - c_7^{-1}\right)\frac{\partial c^d(k,t_2)}{\partial t_2}\right] \end{array} \right\} \tag{2.62}$$

The quotient $\frac{1}{\Psi - l_2^{*} - c(t_2,k)}$ is equal after and before divorce, because the change of the labor supply and the change of the children's time costs $c(t_2,k)$ cancel. For the not-divorce and for the divorce case, Hamiltonians of the same phase evaluated at different periods are equal such that $H_2^{*}(t_2 + \frac{h(k)}{c_7}) - H_2^{*}(t_2) = 0$ and $H_2^{d*}(t_2 + h(k)) - H_2^{d*}(t_2 + \frac{h(k)}{c_7}) = 0$.

Adding (2.61) and (2.62), and setting these terms equal to zero is the t_2-optimality condition, when divorce is a possibility within a marriage. The k-optimality condition is derived next.

1. *Case 1: the probability of no-divorce is multiplied with the LHS of equation (2.32);*

$$(1-p)\left\{ \begin{array}{c} h(k)\left[\frac{\partial v_2(k)}{\partial k} - \frac{\partial c(k,t_2)}{\partial k}\frac{1}{\Psi - l_2^{*} - c(k,t_2)}\right] \\ +(T-t_3)\frac{\partial v_3(k)}{\partial k} + h'(k)\left[H_2^{*}(t_3) - H_3^{*}(t_3)\right] \end{array} \right\} \tag{2.63}$$

		Parameter increases by 1%			
Variables	Baseline	$c_7 = 2$	$c_8 = 2$	$c_9 = 1.5$	$p = 40\%$
$L_{t_2^*}$	75.6536	**75.6980**	**75.6994**	**75.7193**	**75.6926**
$L_{t_3^*}$	98.6950	**98.7433**	**98.7523**	**98.7553**	**98.7389**
L_T	354.0908	**354.1627**	**354.2116**	**354.1273**	**354.1630**
$w_{t_2^*}$	13.7827	**13.7849**	**13.7850**	**13.7860**	**13.7846**
$w_{t_3^*}$	14.2435	**14.2458**	**14.2460**	**14.2467**	**14.2456**
w_T	14.2435	**14.2458**	**14.2460**	**14.2467**	**14.2456**
t_2^*	7.5928	**7.5972**	**7.5974**	**7.5994**	**7.5967**
k^*	1.7887	*1.7850*	*1.7828*	*1.7865*	*1.7851*

Table 2.3: The effect of divorce related parameters on the variables of the model

2. *Case 2: divorce at d:*

$$p\left\{\begin{array}{c} h(k)\left[\frac{\partial v_2(k)}{\partial k}-\frac{\partial c(k,t_2)}{\partial k}\frac{1}{\Psi-l_2^*-c(k,t_2)}\right]+(T-t_3)\frac{\partial v_3(k)}{\partial k} \\ +\frac{h'(k)}{c_7}\left[H_2^*(d)-H_2^{d*}(d)\right]+h'(k)\left[H_2^{d*}(t_3)-H_3^{d*}(t_3)\right] \end{array}\right\} \tag{2.64}$$

The first two terms are the same as in case 1. The length of phase 2 increases with k by $h'(k)$, also remember that d is positively dependent on $h(k)$. When divorce occurs at d, then the first part of phase 2 $[t_2, d_3]$ increases, because divorce occurs later (term 3). At the same time phase 2 becomes longer and phase 3 becomes shorter (term 4).

Setting the sum of (2.63) and (2.64) equal to zero, is the k- optimality condition in the presence of divorce. We can continue with the numerical simulation to find k^* and t_2^*. We assume that a woman's time cost, which occur during phase 2, when raising children increase to $c_8c(k,t_2)$ after she had a divorce. Monetary costs during phase 2 change to $c_9m(k,t_2)$. The parameters c_8 and c_9 must all be larger than one. Values of newly introduced parameters, where p is the divorce probability and c_7 the timing when divorce occurs within this phase are given in the second line of table 2.3. The probability of divorce is assumed to be 40%. Divorce occurs half way through within each phase, time costs are doubled and monetary costs increase by one half. All other parameters are the same as in the baseline model. The results for divorce are summarized by table 2.3.

The number of children in the presence of divorce decreases to $k^* = 1.79$ from around 2.2 in the baseline model, where divorce was excluded from the analysis. $k^* = 1.79$ is closer to the average of the number of children a woman within the European Union countries gave birth to in 2007. In 2007 the fertility rate within the 27 European member states was between 1.25 (Slovakia) and 1.98 (France).[27] Not surprisingly, an increase of all divorce related parameters delays childbirth and yields a decrease of the optimal number of children. The burden of divorce is largest when children are young. We extended this present model to allow for divorce during phase 3; children would not need to be looked after, however they still receive a monetary transfer from their parents: $c_3(k, t_2) = 0$ and $m_3(k, t_2) > 0$. A change of the divorce probability of phase 3 affects fertility less than a change of the divorce probability of phase 2, because costs are larger during phase 2, when time is devoted to raising children c_8. If divorce occurs with certainty, then fertility decreases to 1.38 in this setting. One can further show that the second derivative of the fertility rate as a function of the divorce rate is positive. Our results are in line with empirical observations. Bedard and Deschenes (2003) use data from the 1980 U.S. Census Public-Use Micro Samples and show that the ever-divorced women have higher wages, which are reflected by increased labor supply intensities. In table 2.3, cumulative labor supplies or experience levels L_{t_2}, L_{t_3} and L_T and corresponding wage levels are larger when the divorce probability increases marginally. Our results still hold, when p would increase to 100%.

2.5 Conclusions

This model has been the first to solve simultaneously for the optimal timing of childbirth and number of children in a continuous time framework, where the wage is determined by work experience in a way that depends on the life phase in which it is accumulated. It shows that the date of childbirth and the number of children are negatively related. The marginal value of labor jumps when labor experience influences income differently, which is most likely to be the case when one changes from a full-time to a part-time job. A steep income profile right after leaving school has a negative effect on fertility, while a steep income profile when raising children and afterwards affects fertility positively.

We have shown the effects of the different types of cost of raising children, time costs and money costs. Individuals with high returns from education spend more time in education and have fewer children. Women value market work more when they face the risk of divorce, and so fertility is delayed and fewer children are born. The largest impact of divorce is when the probability of divorce during the phase in which the children are at home is large. Then a woman has to bear larger monetary costs, but

[27] European Commission, Eurostat: Statistics in focus 81/2008, Population and social conditions.

even more importantly she has to devote more of her time towards child care. This has two negative effects: her current and future income decrease, because she is forced to work less on the labor market. Overall, the results of our model appear to be consistent with what empirical evidence is available on these relationships.

Chapter 3

Derivatives and Default Risk in the Electricity Market

3.1 Introduction

Motivation

The European Commission and the USA want to regulate the off-market trade of derivatives that covers 592,000 billion US-\$. This reform is one of the largest tasks for governments and regulators to come. After the insurance company American International Group (AIG) had to be backed up by the US government, due to its risky bets with derivatives in September 2008, the USA and Europe have been working on stricter regulations. Fundamental elements of the reform are Central Counter Parties (CCPs) that take over the risk in case of liquidity shortages. According to EU and US regulatory suggestions, standardized derivative contracts need to go through CCPs. Derivatives of this kind are often used by energy producers. Thus, it is not surprising that Eon, one of Europe's largest electricity and gas suppliers claims that it needs an additional 7.5 billion US\$ in capital, when the CCP requirements are enforced. (Financial Times, 7/10/ 2009).

This paper is a first attempt to evaluate defaults and forwards in the presence of an upstream oligopoly and downstream firms, operating in a competitive environment. In addition to potential government bailouts, the model shows that welfare decreases for another reason: the threat of market exit through insolvency affects the market equilibrium in itself. If an upstream oligopolist has sold forwards to a downstream firm, and the spot price has unexpectedly fallen, then the downstream firm might not be able to discharge its payment obligations to the oligopolist. An oligopolist reduces this risk by increasing the spot price, which has an immediate negative effect on customers. This model is applied to the electricity sector, and uses parameters that are based on historic data from England and Wales. It is based on regional electricity companies (RECs) that purchase electricity from oligopolistic generators. Before liberalization took place, RECs had local monopolies to supply residential customers. The interaction between producers and RECs takes place on a contract and spot market.

Literature

The market environment of this model can be well framed into a branch of the industrial organization literature that was initiated by Allaz and Villa (1993) [AV], and is summarized in the following. AV's influential article shows that the presence of a contract market increases welfare, because the competition among firms is intensified. It creates a prisoner's dilemma, in which firms voluntarily sell forward some of their production on the contract market. Once they have engaged on the contract market, they find it profitable to extend production on the spot market; the marginal revenue increases with the amount that has been contracted before. Sustaining from contracting is a dominated strategy, because the other firm could increase its profits by writing contracts alone, to then become the Stackelberg leader of the game.

Mahenc and Salanié (2004) [MS] challenge the view that contract markets increase welfare. If risk neutral producers are allowed to buy their own quantity on the contract market, then it is a dominant strategy to do so in order to increase prices on the spot market. The intuition here is that producers want to increase their profits on the contract market, by increasing the spot price. In AV, producers compete in quantities on the contract and spot market, but in MS, producers compete in quantities on the future market and prices on the spot market. It is a necessary assumption that the spot market is modeled as a differentiated goods Bertrand model to ensure the strategic complementarity of prices. Another well-known method to avoid the prisoner's dilemma is to increase the time horizon, either to infinity or to a finite number of periods, where firms use trigger strategies. This has been done by Liski and Montero (2006) [LM], who extend the two-stage model of AV and MS to a multiple period game with Bertrand and Cournot competition. Contracts are traded first, the corresponding spot market takes place one period later. In their model firms can use a trigger strategy to sustain collusion: they have to charge the monopoly price on both markets, or the price is otherwise set equal to marginal costs for all subsequent periods. Contract markets help to sustain collusion, because the spot market share decreases. Furthermore firms sell more forward, when they compete in prices, and less, when they compete in quantities to stay on the collusion path. Le Coq (2004) exhibits similar results under a different setting. Firms trade on the future market once. Quantities are delivered at multiple subsequent spot markets.

Newbery (1998) introduces contracts in a supply function model, which is more suitable to picture the electricity market. He shows that contracts that drive down the expected spot price, reduce the incentive for competitors to enter the market. Entry can thus be deterred, if incumbents hold sufficient capacity; a conception first illustrated by Dixit (1980). Murphy and Smeers (2005) [MS] introduce investment decisions in the two-markets setup. They prove that the equilibrium of a model with, and without a contract market is the same, when players have to choose capacities before they produce. The intuition behind the result is the same as in the Kreps and Scheinkman (1983) model; firms choose low capacities to avoid destructive competition and restore the Cournot equilibrium. Bushnell (2007) extends AV's model to n firms that face increasing marginal costs. He demonstrates, how the equilibrium changes, when an additional firm enters the market in the presence of a contract market, as opposed to the change in the absence of a contract market. Grimm and Zoettl (2006) [GZ] establish that a contract market decreases investment capacity in a time-varying demand model. Capacity choices decrease the positive competition effect of contract markets. Firms choose lower capacities to avoid competition, but when demand is low and capacity is not a binding constraint, then contracts do increase competition. Only when demand is certain and capacity binds, then contracts do not affect the efficiency outcome. The model of MS shows that capacity investment decisions under perfect foresight yield the same market outcome with and without a contract market. Newbery (2008) studies the effect of mergers in the presence of a contract market. He demonstrates that market power increases more after a merger, when a contract market is present. Furthermore he proves that contracts reduce capacity, which is consistent with GZ. They also increase the fraction of time that capacity is constrained, but still lower the time-weighted average price. The later finding shows that future markets increase at least consumer rents, in the presence of capacity investments, and come therefore closest to a positive contract market welfare analysis, even in the presence of capacity constraints.

GZ and Newbery (1998) are the only papers that reasonably allude, future markets could possible be welfare decreasing, because producers scale down their installed capacity. All other model that claim, forwards are welfare decreasing, rest on very strict assumptions: differentiated goods, perfect information and collusion (LM) or allowing producers to buy forward (MS). However these assumptions can be counteracted by a regulator, if they prove to be realistic. In line with the majority of articles, forwards reduce the spot price in this essay, however, the efficiency gain is lower, when buyers face an insolvency risk for low spot prices. The models examined so far, assume "no-arbitrage profits" from futures. They model the upstream market and assume that buyers accept any forward price, as long as it is not below the expected spot price. In equilibrium, the forward price equals the spot price, an assumption that often does not hold empirically.[28] This essay models both market participants, and allows the forward price to be different from the spot price.

The electricity sector

Few papers have studied the impact of retail competition on contracts. Exceptions are Powell (1993) and Green (2004), on which this model is closely based. Powell shows that there are more forwards sold, when producers coordinate on the forward and spot market, as opposed to a market, where producers exclusively coordinate on the forward market. Green finds that the number of contracts sold is higher in an industry, where an incumbent does not face any competition (in the presence of yardstick regulation) as compared to an incumbent that is faced by a competitive fringe, which always charges the spot price (in the presence of switching costs).

After the electricity sector has been liberalized, incumbent retailers have faced fierce competition as opposed to producers, which have remained in an oligopoly position. A famous retail bankruptcy example for the British market is the failure of 'Independent Energy' that collapsed in 2000. Thus, RECs have become vulnerable to the risk of spot prices that have fallen below the expected level at the time, when contracts were written. If they charge a retail price that exceeds the spot price substantially, then some of their clients leave their previous electricity supplier to be supplied by a competitive fringe, which buys and re-sells electricity for the current spot price. Green takes account of the market reforms and calibrates his model with historic data from the English/ Welsh electricity sector in the 1990s that this model utilizes.

In the course of the 1990 electricity market liberalization of the UK, the RECs were privatized. They became either public limited companies (plc) or they were bought by large domestic producers (e.g. Powergen and Scottish Power) and foreign firms (e.g. Eon and EDF). According to the Utilities Act in 2000, all former RECs had to separate their supply and distribution businesses. The forwards studied here, are "over-the-counter" (OTC) contracts that exclusively concern the supply business part,

[28] One of the first empirical essays on this issue is Protopapadakis and Stoll (1983).

which supplies management services, such as billing, customer service, metering, debt collection and administration. There is not much capital bound in the newly formed retailer's business, a miscalculation of past forward purchases can easily destabilize the financial condition and force a retailer to exit the market.

Objectives

This model does not reconstruct bankruptcy probabilities for the electricity market in England and Wales, it uses the noise that was generated by its liberalization to justify the assumption that the spot market alone is affected by the threat of insolvency. Before the liberalization, incumbent retailers held monopoly positions and bankruptcies were highly unlikely. The forward market, described in this model, has a very long time horizon, such that the liberalization was not anticipated, when the contract market opened. I study the market equilibrium, where the bankruptcy threat is anticipated, in a different paper (Scholz, 2009). That model uses the same assumptions as the literature described in the beginning; retailers are modeled just implicitly and buy any number of forwards, offered by producers, but it lacks the adoptability to the electricity market. Furthermore closed-form solutions cannot be derived, when the default risk is endogenous. It shows that the anticipation of bankruptcy at the closure of contracts reduces the number of contracts. This induces the negative welfare effect of the insolvency risk to be even larger. The results presented here, can thus be interpreted as being a conservative estimation.

Furthermore this model compares welfare effects between forwards and options. It demonstrates that options yield a slightly lower welfare, but are easily the preferred instrument in the presence of bankruptcy. It is the first model that allows a welfare analysis, in which forwards are compared to options, whose strike price is endogenous. The model of this essay has two parts; the first part (section 3.2) studies the impact of bankruptcy in the presence of a forward market. The second part (section 3.3) compares the market equilibrium in the presence of forwards, with the one in the presence of options. Section 3.4 concludes.

3.2 Forwards

3.2.1 Pre-liberalization period

There is an upstream market with producers, who sell forwards to incumbent regional electricity companies. Producers set the price on the forward market and the quantities on the spot market.[29] RECs decide, how many forwards they want to buy.

Producers cooperate on the forward market, but do not coordinate on the spot market. If there was no coordination among producers on the forward market, the price would equal marginal costs, which is unrealistic for the electricity market. If there was coordination on the forward and spot market, such that the total spot quantity is the monopoly quantity, then the number of contracts increases compared to a market, where coordination is restricted to the forward market.[30] When a REC has paid a high forward price p^f, and the spot price is unexpectedly low, then the REC makes a loss. The more contracts have been traded in the past, the larger the loss and the bankruptcy probability; a positive correlation of these two variables is assumed. Thus if producers cooperate on the spot market, retailers would have bought more contracts, and the default probability would be even larger. The results presented here, can then again be interpreted, as being a conservative estimation.

Production sector

Producers maximize their expected profits $E\pi^P$, while RECs maximize a mean-variance utility function of their profit π^R. There are two symmetric producers and RECs, such that in equilibrium the production quantity of producer i equals that of producer j and the number of forwards sold to each REC is equal. Call $f_i(f_j)$ the number of forwards sold by producer i (j) and purchased by REC i (j).[31] The game is solved by backward induction. When producers set their spot market quantities, they do so given the number of forwards f sold. Producers maximize their profits. Producer $i's$ objective is

[29] Unitil 1995, the generation duopoly in the UK, even though it held less than 50% of generation capacity, set the price 90% of the time, see Wolfram (1999). Section 3.2.2 explores this issue in greater depth.

[30] The proofs are given in Powell (1993) on p. 449-450.

[31] In Germany RECs ("Stadtwerke") often still buy all electricity exclusively from one generator, even though they are not owned by it anymore.

$$\max_{q_i} \pi^P = (p - c)q_i - f_i(p - p^f) \tag{3.1}$$

where q_i is the spot quantity of producer i, c is marginal cost and p^f is the forward price. The first part of (3.1) is the spot market profit, and the second the contract market profit. The linear inverse residual demand function with an intercept A and slope $-b$ can be expressed as

$$p = A - bq_i - bq_j + \epsilon \tag{3.2}$$

where $\epsilon \sim N(0, \sigma^2)$. All customers that do not pay the retail price are described by the term "residual"; in particular large industrial customers, who can buy electricity from the production sector directly. It is straightforward to solve (3.1) for the expected spot quantity of producer i;

$$Eq_i = \frac{A - c + 2bf_i - bf_j}{3b}. \tag{3.3}$$

As producers are symmetric, the expected spot price is

$$Ep = \frac{A + 2c - bf_i - bf_j}{3} \tag{3.4}$$

As mentioned before, producers set the forward price and maximize their objective accordingly.

$$\max_{p^f} E\pi^P = (Ep - c)Eq - f(Ep - p^f) + Cov(p, q) \tag{3.5}$$

$Cov(p, q)$ is the constant covariance of the spot price and quantity. The first order condition of (3.5) can be solved for p^f

$$p^f = Ep + \left(\frac{\partial f_i}{\partial p^f}\right)^{-1} \left[-f_i - (Eq_i - f_i)\frac{\partial Ep}{\partial p^f} - (Ep - c)\frac{\partial Eq_i}{\partial p^f}\right] \tag{3.6}$$

Powell (1993) shows that the forward price is larger than the expected spot price. The capacity literature can be viewed parallel to this observation; in order to mitigate the negative effect of forwards on their market power, producers charge a higher price than Ep, whereas in the capacity literature, incumbents might have an incentive to over-invest in capacity as a strategic device; see Spence (1977), Dixit (1980) and Newbery (1998) as a more recent application to the electricity market. In order to find

$\frac{\partial f}{\partial p^f}$, RECs are modeled that choose the optimal number of contracts, given the forward price that is offered by the production sector.

Retail sector

A fix number of customers served by an incumbent REC, V purchases electricity for a regulated price r before the market was reformed.

The two RECs that have been characterized by subindexes i and j in the last section, operate in separate markets but are symmetric. In reality there were 12, and not two heterogeneous risk averse RECs in England and Wales; cooperation would thus have been very difficult to implement, and is not assumed in this model. A REC maximizes a mean-variance function applied to its profit as in Powell (1993),

$$U_i = E(\pi_i^R) - \frac{1}{2}\lambda Var(\pi_i^R) \tag{3.7}$$

where the expected profit is

$$E(\pi_i^R) = V[r - E(p)] + f_i[E(p) - p^f] \tag{3.8}$$

The variance is $Var(\pi^R) = Var[V(r-p) + f_i(p-p^f)] = Var[p(f_i - V)] = (V - f_i)^2\sigma^2$, where the only variable part is the price. RECs choose the optimal number of contracts, they purchase. REC $i's$ objective is $\max\limits_{f_i} U_i = E(\pi_i^R) - \frac{1}{2}\lambda Var(\pi_i^R)$, which is solved for f_i

$$f_i = V + \frac{E(p) - p^f}{\lambda\sigma^2 - [\partial E(p)/\partial f_i]} \tag{3.9}$$

(3.4) is used to manipulate (3.9), in order to derive REC $i's$ demand for contracts as a function of the number of contracts bought by the other REC.

$$f_i(f_j) = \frac{V(b + 3\lambda\sigma^2) + A + 2c - bf_j - 3p^f}{2b + 3\lambda\sigma^2} \tag{3.10}$$

Due to symmetry, $f_i(f_j)$ and $f_j(f_i)$ solve for REC's demand function of contracts, given the forward price:

$$f(p^f) = \frac{V(b + 3\lambda\sigma^2) + A + 2c - 3p^f}{3(b + \lambda\sigma^2)} \tag{3.11}$$

Equation (3.6) and (3.11) can be solved for the optimal number of forwards, $f_i^* = f_j^* = f^*$, based on the underlying parameters. The derivatives in (3.6) can easily be derived, using (3.3), (3.4) and (3.11).

$$f^* = \frac{V(\frac{b}{3} + \lambda\sigma^2) - \frac{1}{9}(A - c)}{\frac{10}{9}b + 2\lambda\sigma^2} \tag{3.12}$$

Furthermore the first order condition of p^f, (3.6) is solved with (3.3), (3.4), (3.11) and (3.12) to express p^f based on the expected spot price and the number of contracts signed

$$p^f = Ep + \frac{1}{9}(A - c) + f^*(\frac{7}{9}b + \lambda\sigma^2) \tag{3.13}$$

This shows that the forward price exceeds the level of the expected spot price. The difference increases with the risk aversion parameter. Even for $\lambda = 0$, the forward price exceeds the expected price, because contracts decrease future spot prices (see also Powell, 1993). The demand for contracts decreases with the number of contracts the other REC purchases, (3.10), which is a justified result, as RECs were of considerable size. The larger the demand elasticity, the more RECs hedge, because the negative impact on the spot price per forward contract, increases with b. (see (3.4)) Another reason for price divergence is the large percentage of OTC trade in the electricity sector, which implicates non-transparent pricing.[32]

3.2.2 Post liberalization period

Since the market was reformed, residential customers have been able to choose their electricity supplier. If a customer chooses to find a new supplier in this model, then she would receive her electricity from the competitive fringe. Costumers are assumed to face switching costs, such that some are willing to remain with their regional electricity company and pay a higher price.

The market share of the incumbent retailer decreases, when the retail price, which

[32] In Germany for instance the liberalization of the electricity market has not yet reached the same level as in the UK, because RECs ("Stadtwerke") still hold both: distribution and supply. Over 80% of electricity is sold through bilateral contracts, most with a single incumbent generator based on historical ties. Due to commercial confidentiality, neither price nor quality information are revealed. (WIK, 2008)

is assumed not to be regulated after liberalization, is above the current spot price. Green's (2004) simple demand expression that an incumbent retailer faces, after a competitive fringe has entered the market, is also useful for this work

$$V^{NEW} = V - h(r - p) \tag{3.14}$$

A high constant parameter h is interpreted by low switching costs. When switching costs are low, the incumbent's market share decreases more for $(r - p) > 0$.

Retail sector

The former RECs are allowed to choose the retail price r, which has been dictated by a regulator before the liberalization. Thus the new retail objective becomes

$$\max_{r} \pi^R = V^{NEW}(r - p) + f(p - p^f) \tag{3.15}$$

The optimal new retail price is

$$r^* = p + \frac{V}{2h} \tag{3.16}$$

There is no expectation operator in (3.15), because retailers know the realization of ϵ, when they choose r^*. In the past, forward contracts were written to protect RECs from volatile pool prices, because they had to sell into a regulated market with a formerly fix retail price. After the liberalization, this alleged protection has jeopardized retailers that now have to act in a volatile retail price environment. Meanwhile the market has become more competitive and retailers have to carry the burden of contracts. This model assumes that there is a positive probability of bankruptcy, when a retailer incurs a loss based on the contract of differences. The return of forwards is negative, when the spot price is below the forward price, otherwise contracts yield positive returns. If $p < p^f$ the situation worsens with low switching costs (large h), because in that case, retailers can just charge a low mark-up, see (3.16). The bankruptcy probability consists of an exogenous part s, which contains information about its ownership structure, how likely the retailer is able to raise loans from banks, and how much savings it holds. Incumbent retailers might also be bailed out by their owners, when these are able to raise sufficient funds. Owners are generally less willing to vouch for the retailers, when the loss $-\pi^R$ is very large, which is incorporated in the bankruptcy probability. But there are also different warrantors as such; public entities are generally more willing to burn (taxpayer's) money than private entities, to preserve trust. In the English/ Welsh market all RECs were bought by private companies, some of them very large and operating worldwide, thus they would be reluctant not to act as a guarantor for their retailer, registered as public limited company, to maintain their reputation. The different owner types are expressed by the exogenous multiplier s. Thus the default probability is defined as

$$\alpha = \alpha(s\pi^R) = \begin{array}{ccc} 0 & if & \pi^R > 0 \\ -s\pi^R & if & \pi^R < 0 \end{array} \tag{3.17}$$

The survival probability is denoted by $\eta(\pi^R) = 1 - \alpha(\pi^R)$. If a retailer has a low s then it is owned by an entity that is more likely to guarantee for its retailer's payments, when $\pi^R < 0$. If π^R is positive, then the bankruptcy threat is absent, $\alpha = 0$ and $\eta = 1$. If bankruptcy occurs or not, is irrelevant in this model; it is the risk that affects the spot market equilibrium.

Production sector

The retailer's ownership structure is known in the UK, hence s can be estimated. Furthermore the number of contracts can be assessed, based on the market that the former REC operated, allowing the probability of default to be derived. Producers maximize their expected profit by choosing an optimal production quantity, where the expectation is based on, how likely it is that the retailer manages to transfer $p^f - p$, for the contracts signed. There is no uncertainty about the demand intercept at this stage. If a retailer fails, contracts become worthless, but producers still sell an unconstraint quantity on the spot market. The residual demand is not affected by bankruptcy, because there are other generators that can absorb customers from bankrupt, incumbent retailers. The generation capacity of the duopoly, which covered 73% of total capacity in 1990-91 decreased to 46% in 1995-96 and an estimated 38% in 2000-01 (Monopolies and Merger Commission, 1996). But until 1995, the duopoly set the pool's electricity price 90% of the time, which justifies this model's assumption that the duopoly sets a quantity that reflects the market price.[33]

$$\max_{q_i} E\pi_i^P = pq_i - cq_i - f_i(p - p^f)\eta_i(\pi_i^R) \tag{3.18}$$

[33] For background information see Wolfram (1999) and Newbery (1995, 1998).

To find the spot quantity of (3.18), a function is maximized that depends on the optimal outcome, as $\pi^R(q)$ is a function of $q = (q_i, q_j)$. The optimal value of q is found by solving two separate maximization problems with $\eta(\pi^R) < 1$ and $\eta(\pi^R) = 1$ respectively, because $\eta(\pi^R)$ is not continuous. Define $\pi^{P,B}(q^B)$ to be the producer's objective, when $\eta(\pi^R) < 1$ and $\pi^{P,NB}(q^{NB})$ the objective, when $\eta(\pi^R) = 1$.[34] q^{B*} and q^{NB*} are the corresponding optimal values, which are compared in the four different equilibria, possible;

1. $q^{max} = q^{B*}$ if $E\pi^{P,B}(q^{B*}) > \pi^{P,NB}(q^{NB*})$ and $E\pi^R(q^{B*}) < 0$.[35]

2. $q^{max} = q^{NB*}$ if $\pi^{P,NB}(q^{NB*}) > E\pi^{P,B}(q^{B*})$ and $E\pi^R(q^{B*}) > 0$.

3. $\pi^{P,NB}(q^{NB*}) > E\pi^{P,B}(q^{B*})$ and $\pi^R(q^{NB*}) < 0$: when this outcome occurs, producers prefer that retailers have a zero probability to go bust. If $\pi^{P,NB}(q^{NB*}) + \pi^{R,NB}(q^{NB*}) > E\pi^{P,B}(q^{B*}) + E\pi^{R,B}(q^{B*})$, producers and retailers might consider to either merge or renegotiate their contracts.

4. $E\pi^{P,B}(q^{B*}) > \pi^{P,NB}(q^{NB*})$ and $\pi^R(q^{B*}) > 0$: In this case producers rather maximize the objective when their retailers could possibly default. Producers produce $q^{max} = q^{NB}$ as they cannot force retailers to go bankrupt, when $\pi^R(q^{B*}) > 0$. Furthermore computing $E\pi^{P,B}(q^{B*})$ does not make sense, because one would assume that $\eta > 1$. Thus this equilibrium is not realistic.

For a retailer's survival probability of $\eta(\pi^R) < 1$

First, the optimal spot market quantity is solved, which is set by producers. When an incumbent retailer goes bust, then the producer does not receive the forward price for the contract coverage, but sells to customers directly or through the competitive fringe. A retailer is threatened by a loss when $p \ll p^f$, where the price difference has to be sufficient, because retailers realize a profit from those customers who do not switch, and pay a retail price above the spot price, see (3.16). After the bankruptcy of a retailer, whom a producer has written contracts with, the positive transfer of $f(p - p^f)$ would not be obtained. Thus producers minimize the default risk, by keeping the spot price up. (3.15) and (3.17) are converted, to rewrite the producer's profit as a function of the retailer loss

[34] Throughout the rest of this chapter, the superscript B stands for, "there exists a bankruptcy risk", and NB stands for, "there exists no bankruptcy risk".

[35] The expectation operator for profits applies to the B- case only, because there is no uncertainty, when retailers cannot possible go bankrupt.

$$\pi_1^{P,B} = (p-c)q_1 - f_1(p-p^f)\left\{1 + s\left[V(r^* - p) + f_1(p-p^f)\right]\right\}$$

The first term is the spot market profit, the second term is the expected contract market return. The later simple contains the survival probability, $\eta(\pi^R)$ as a multiplier. Substituting r^*

$$\pi_1^{P,B} = (p-c)q_1 - f_1(p-p^f)\left\{1 + s\left[\frac{V^2}{4h} + f_1(p-p^f)\right]\right\}$$

The spot price p contains $A^* = A + \epsilon$. The demand is known at this point and producers play the Cournot game on the spot market. At the contract market though conjectural variations $\frac{\partial f_i}{\partial f_j}$ are assumed to equal zero.

$\frac{\partial \pi^{P,B}}{\partial q_i} = A^* - 2bq_i - bq_j - c + bf_i\left\{1 + s\frac{V^2}{4h} + sf_i\left[(A^* - bq_i - bq_2) - p^f\right]\right\} + sbf_i^2\left[(A^* - bq_i - bq_j) - p^f\right] \doteq 0$

The foc can be rewritten for identical retailers and producers, $f_i = f_j = f$ and $q_i^B = q_j^B = q^B$

$$q^{B*} = \frac{A^* - c + bf\left(1 + s\frac{V^2}{4h}\right) + 2sbf^2(A^* - p^f)}{3b + 4sb^2f^2} \tag{3.19}$$

If q^{B*} is realized, then the spot price is equal to

$$p^{B*} = A^* - 2bq^{B*} = \frac{A^* + 2c - bf\left(2 + s\frac{V^2}{2h}\right) + 4sbf^2p^f}{3 + 4sbf^2} \tag{3.20}$$

(3.20) is the optimal production quantity, when bankruptcy is possible. One can substitute (3.12), (3.16), (3.19) and (3.20) in (3.15) and (3.18) to derive $\pi^R(q^{B*})$ and $E\pi^P(q^{B*})$ that only depend on the underlying parameters of the model. If $\pi^R < 0$ and $\pi^{P,B}(q^{B*}) > \pi^{P,NB}(q^{NB*})$, then it is for generators optimal to take the risk, that incumbent retailers are exposed to the bankruptcy threat.

For a retailer's survival probability of $\eta(\pi^R) = 1$

If a former REC realizes a profit, then it survives by definition and the survival probability equals one. Again, producers play the Cournot game on the spot market, while conjectural variations at the forward market are zero. The equilibrium is described through (3.3) and (3.4), where A is substituted for the realized intercept A^*.

$\pi^R(q^{AV*})$ and $\pi^P(q^{AV*})$ can thus also be expressed by the model's parameters. When $\pi^R > 0$ and $\pi^{P,C}(q^{C*}) < \pi(q^{AV*})$, generators choose a spot quantity such that the incumbent retailer survives with certainty.

3.2.3 Results

This section presents a numerical solution of this model, based on data of the electricity sector in England and Wales. In the early 1990s, there was a generation duopoly, and there were 12 incumbents in the retail sector. The two privatized firms, National Power and Powergen, held respectively 50% and 30% of the total generation capacity. The Electricity Supply Industry in England and Wales was reformed in 1990. Before its restructuring took place, there had been a state-owned Central Electricity Generating Board, responsible for generation and transmission, selling to 12 state-owned Area Electricity Boards, which were responsible for distribution. Nearly 80% of the industry's generation came from coal-fired stations, and most of the remaining electricity from nuclear power. Green's (2004) parameter values are applied in this model. He assumes marginal cost c being equal to £20/ MWh. The parameters of the residual demand curve are set to $A = 50$ and $b = \frac{2}{3}$.

The welfare analysis, which is conducted later, estimates the consumer surplus based on "residual" demand.[36] Customers that remained with the incumbent are ignored, because this is not a general welfare analysis of the liberalization process as such. This model rather analyzes, how bankruptcy affects the market equilibrium. In the 1990s, there were already some small generators on the market, which were price takers. Therefore the profit and expected profit that are derived are "residuals", too.

Green sets the sales volume per REC to V=2.5 GW representing the total sales to small customers of the 12 RECs equal to 30 GW. There are two retailers in this model; each writes contracts with one generator, nevertheless the same volume per REC of V=2.5 GW is adopted, as it can be shown that even with a relatively small contract coverage, the market equilibrium is changed by the risk of default, significantly. This makes this model's findings even more meaningful. The switching cost parameter is set to $h = 0.15$, because incumbents lost approximately one third of their market share or 0.9 GW of sales due to a 10% retail price difference at that time, when the retail price was around £60/MWh. Green claims that the variance of the annual

[36] In addition to consumer rents, welfare includes expected producer and retailer profits. Consumer rents equal the area between the inverse demand function and the spot price.

pool's price, $\sigma^2 = 5.76$ (from 01/1990 to 01/2000) was distinctively low, because of high level contracting, market power and regulatory pressure. It is contrary to the volatility in Nordic countries, which depend heavily on rainfall, due to the importance of hydro power plants. There, the variance was equal to 34.9 between 1993 and 2003. A volatility of somewhere in between is used; $\sigma^2 = 30$. Green derives a risk aversion parameter of $\lambda = 0.178$.[37]

The default probability consists of the retailer's profit π^R and the exogenous parameter s.[38] In the first numerical simulation, which is summarized by figure 3.1, the intercept is 20% below the expected value, thus $\epsilon = -10$. In order to analyze bankruptcy, it must occur with positive probability. Accordingly, the necessary condition is that retailers incur a loss. Instead of choosing a small ϵ, one could have lowered switching costs though increasing h. Former RECs make a small profit, when the switching parameter is lifted from $h = 0.15$ to $h = 0.14$. Thus in the absence of the unexpected market entry, RECs would have never had to face a loss, which justifies the assumption that default was never contemplated, when the contracts were signed. Finally s , the multiplier of π^R for $\pi^R < 0$ is chosen to arrive at the bankruptcy probability α. This essay is not interested in finding a potential default rate of former UK RECs, however in the question, how the market equilibrium is affected, when different s-parameters are considered. Different values of s are chosen that determine a reasonably bankruptcy rate α.

In this setup, the number of forwards is not affected by the default probability. It is just the production quantity that producers can influence. The number of contracts has already been chosen before market entry took place, when RECs made profits even when the spot price was below the forward price. Based on the underlying parameters, the expected price is $Ep = 29.59$ by (3.4) and the optimal number of contracts $f^* = 0.93$, see (3.12). Thus 37% of total expected sales are bought on the forward market for a forward price of $p^f = 38.35$, see (3.13). One can alter the multiplier s to show, how total welfare, producer profits, α and thus the retailer's loss are affected. When $s = 0$, then $q^{B*} = q^{NB*}$, because the objective (3.18) reduces to objective (3.1). When the exogenous multiplier s of the bankruptcy probability $\alpha(\pi^{R*})$ increases by some percentage, then the default probability strictly increases by less or even decreases, as long as $b < 1$. Producers lower the difference of p^f and p by increasing p, (which lowers

[37] Green (2004) applies Grinold (1996)'s "grapes from wine" method, pp. 16-17.

[38] In order to derive profits and consumer surplus in monetary values one multiplies the values we derive by 8.780 million. Prices are in MWh and based on hourly consumption, while V is an annual capacity measured in GW. 1 GW annual capacity of electricity is equivalent to 8,780 GWh and 1 GWh=1000 MWh.

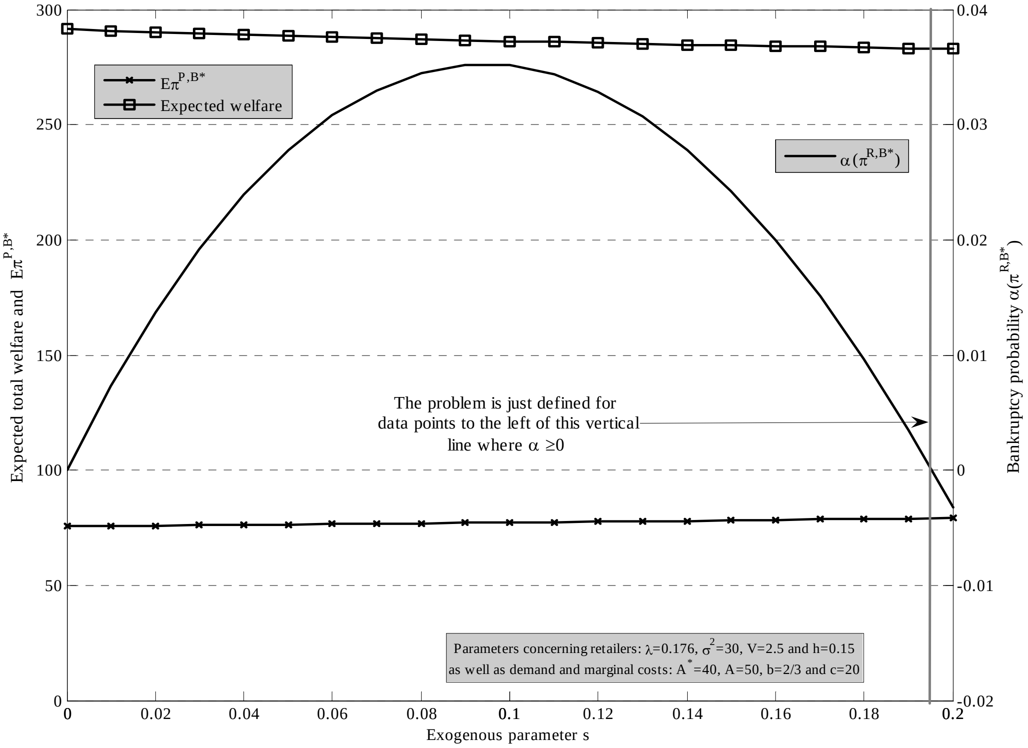

Figure 3.1: The effect of s on α, expected welfare and profits

the retailer's loss) to scale down the probability that retailers fail. The intuition behind this result is simple; when a retailer goes bust, producers do not receive the transfer from contracts.

The expected producer profit increases with s as the spot price goes up. The retailer's expected loss is largest when $s = 0$ and $E\pi^R = -0.775$. Retailers break even at $s \approx 0.195$, where producers have their largest profit of $E\pi^P = 79.1$ up by 4.5% compared to its profit at $s = 0$. The bankruptcy probability reaches its peak for intermediate values of s. Total expected welfare, consisting of consumer surplus, producer and incumbent retail profits decreases from 291.5 to 283.1, when s increases from zero to 0.195 (and $\alpha = 0$ for the second time after $s = 0$) at the expense of the residual consumer surplus that decreases when p goes up. Thus a producer prefers a less solvent retailer, as producers are then committed to set a lower production quantity to keep the possible loss from retailers low. This has been the realization of equilibrium 1 at each data point on the left hand side of the gray vertical line. Equilibrium 4, which is not reasonable in reality occurs to the right of the vertical line. Next, examples for equilibrium 3, which is realized when ϵ is smaller, are demonstrated.

The parameters of the simulation that figure 3.2 is based on are the same as before, except that $\epsilon = -15$ (upper part) and $\epsilon = -25$ (lower part). In the upper part of figure 3.2, one can notice again that α is a concave function of s. Producers reduce the spot price when they face a less solvent retailer to increase the probability that they receive a payment when $\epsilon < 0$ and $p^f \gg p$. Retailers benefit from increasing s, thus less solvent retailers have a lower loss. Producers do not benefit from low retailer reserves. $\pi^{P,B}(q^{B*})$ falls with s up to a level where $s = 0.12$, which is equivalent to a bankruptcy probability of 20%. The sum of expected producer and retailer profits is illustrated. It is a convex increasing function of s. The sum decreases with s for small s and increases once $s \approx 0.04$, which corresponds to $\alpha \approx 9\%$. Thus in a range of $s \in (0, 0.04]$, producers and retailers prefer to merge, as $E\pi^{P,B}(q^{B*}) + E\pi^R(q^{B*}) < \pi^{P,NB}(q^{NB*}) + \pi^{R*}(q^{NB*})$, which is equilibrium 3.

The lower part of figure 3.2, where ϵ is even smaller, shows that for any $\alpha > 0$ or $s > 0$, the sum of the retail profit and expected producer profit is lower as when $\alpha = 0$ and $s = 0$. Thus a merger might always be a preferable solution. It is straightforward to derive the solution for asymmetric retailers with respect to h and s. A producer, who has written contracts with a retailer that is less likely to be bailed out (high s), or one that has customers that are more likely to switch (high h), is more prepared to adjust the spot price downward. The other producer increases its production quantity, due to the strategic substitutability. Accordingly, the profit of the producer with the more

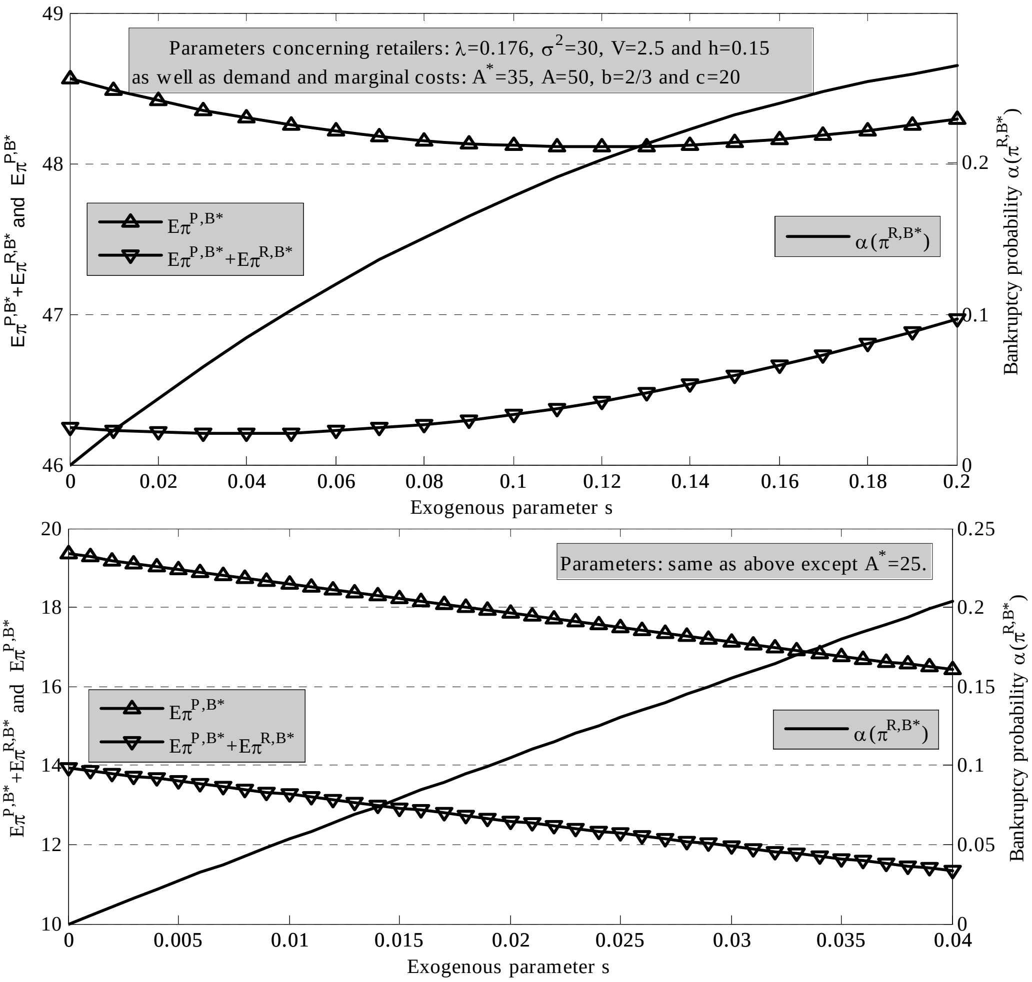

Figure 3.2: When renegotiations or mergers are preferable

solvent retailer, rises, while that of its competitor falls. The same holds for producers, who wrote contracts with retailers that used to cover a larger market or were more risk averse. These retailers purchased larger stakes in production plants and hold more forwards; thus they are more exposed to the risk of low spot prices today.

3.3 Options

'Contracts for differences' (CfDs) are pure financial contracts that resemble two-way forward contracts, which have been examined in section 3.2. One-way CfDs, which are call options are examined next. The first type has been studied by a broad literature mentioned in the introduction, the second type only by very few authors. This model demonstrates, how options can reduce market power, just in the same way, as two-way contracts can. Retailers do not transfer $(p^f - p)f$, when the spot price p is low, thus they have the advantage that an incumbent retailer cannot be underbid by a competitive fringe, which might enter the market after privatization takes place. When the spot price is low, the option holder purchases its demand at the spot market, when the price is high, the option holder pays the lower strike price p^s. The cost of an option, paid in any demand state, is p^o. Indeed if the spot price is lower than the strike price, the REC looses p^o on each option bought. The option price is generally paid before the spot market opens, thus illiquidity does not occur when the spot price is lower than expected.[39]

In this section, a model is introduced that compares the market equilibrium with forwards, to one with options. It is also shown, if welfare is higher in the forward or option model. This is the first model that allows a welfare analysis between an option and a forward market in the Allaz Vila (1993) framework, where the strike price is endogenously determined. Vázquez et al. (2002) propose options as a long-term security of supply mechanism. Few papers have compared welfare effects between markets that use options and those that use forwards. Exceptions are Chao/ Wilson and Willems (both 2004). The first of the two papers proposes

> "an annual auction of a specified quantity of multi-year option contracts at each strike price in a specified range. Each contract is an option on physical capacity since it requires the supplier to back the contract with available

[39] Schmidt (1997) shows that liquidation risk increases managerial incentives. If a firm would have had a low liquidity in the past, previous payments would have increased managerial incentives. Hence generally payments, made in the past are less harmful to a firm as they could be balanced through managerial effort in consecutive periods.

capacity, to submit a standing bid at the ISO for the contracted quantity at a price no higher than the strike price, and to be dispatchable for either energy or reserve capacity." (p.3)

Willems took the idea of using auctions and embedded this into a model that can be compared to the AV model. Willems introduced an exogenous strike price and retrieved the following results from his model: When the strike price is above the Cournot price, options are out of money and the producers settle at the Cournot equilibrium. For strike prices below the spot price achieved in AV, the AV price is achieved. For intermediate strike prices, producers flood the market until the market price reaches the strike price. The main result of Willems (2004) was that the market price is never lower in an option market than in a forward market. This model supports this view, but shows that if a model is extended by a possible bankruptcy threat, an option market is preferred to a forward market from a welfare perspective.

One needs to make some changes to the setup, to be able to compare the option with the forward market, while keeping it simple. If one continues to use an $\epsilon \sim N(0, \sigma^2)$, the spot price, strike price and the optimal number of contracts would depend on the probability $Pr(p > p^s)$ and its derivative with respect to these variables. To avoid the resulting complications that do not add further insight to our questions, it helps to assume that ϵ takes on discrete values, which act as demand shocks; $\epsilon \in \{0, H\}$ with probabilities ϕ for $\epsilon = H$ and $(1 - \phi)$ for $\epsilon = 0$. Thus $E\epsilon = \phi H$ and $Var(\epsilon) = \sigma^2 = \phi(1 - \phi)H^2$. If the risk aversion parameter in (3.12) is set equal to zero, the optimal number of forwards becomes: $f^* = \frac{3Vb-(A-c)}{10b}$. Just for $3Vb = A - c$, $f^* = 0$ otherwise hedging still takes place; there are short hedges when $3Vb > A - c$ and long hedges when $3Vb < A - c$. Powell (1993) shows that in the absence of risk aversion and when generators sell the monopoly quantity on the spot market; $f^* = \frac{1}{2}V > 0$, which implies that there is always short, never long hedging. Meaning even risk neutral RECs hedge to keep the future spot price low, assuming that they are not too small in relative size to the market. It is legitimate to assume that RECs buy contracts to lower the future spot price, as their size was significant in the UK, before they were privatized. Producers coordinate on the future market as before. This model has two parts again. First, the market equilibrium before the liberalization is solved, second, it is modified to account for bankruptcy after the liberalization. The first part of the model is solved for producers that coordinate on the spot market, as only then the number of forwards and options is guaranteed to be positive. For a study on market power of UK's generation duopoly, see Wolfram (1999).[40] The equilibrium is shown, when there

[40] Müsgens (2006) shows that there have been price agreements among German producers in par-

is no coordination likewise. After the liberalization of the market, producers do not coordinate, as they did in section 3.2. The liberalization of the electricity market has come along with strict actions by regulators against price agreements among incumbent generators. [41] RECs are risk neutral, $\lambda = 0$, which reduces the demand for options and forwards in the same way, a comparison of the two models is thus still possible. They maximize their expected profit instead of a mean-variance utility function. First, the forward market has to be solved under the changed market setup to be able to compare it to the option market, then the option market is solved.

3.3.1 Pre-liberalization period

Forward Market

Production sector

Producers maximize the expected monopoly profit $E\pi^M$ by choosing the expected monopoly spot quantity EQ, taking the total number of forwards f_Σ as given; $\max\limits_{EQ} E\pi^M = (Ep - c)EQ - f_\Sigma(Ep - p^f) + Cov(p, Q)$ where $Ep = A + E\epsilon - bEQ$ and $Cov(p, Q) = \frac{\sigma^2}{4b}$. Thus the expected forward price and monopoly quantity are

$$Ep = \frac{A + \phi H + c - bf_\Sigma}{2} \tag{3.21}$$

$$EQ = \frac{A + \phi H - c + bf_\Sigma}{2b} \tag{3.22}$$

The first order condition of the forward price (3.6), is thus $p^f = Ep + \left(\frac{\partial f_\Sigma}{\partial p^f}\right)^{-1} \left[f_\Sigma - (EQ - f_\Sigma)\frac{\partial Ep}{\partial p^f} - (Ep - c)\frac{\partial EQ}{\partial p^f}\right]$.

Retail sector

RECs maximize their utility functions of the form (3.7) (with $\lambda = 0$) where the foc can be solved for the number of contracts f (see (3.9)), which can be further transformed to

$$f^*(p^f) = \frac{Vb + A + \phi H + c - 2p^f}{3b} \tag{3.23}$$

ticular during peak periods.

[41] The former generation duopoly National Power and Powergen was forced to sell generation units to reduce their market power. Finally they were bought by foreign competitors after National Power demerged in 2001.

This is the number of forwards as a function of p^f, which each REC requests. One can now transform the p^f-foc using (3.21), (3.22) and (3.23) to solve for the future spot price premium,

$$p^f - Ep = \frac{3bf_\Sigma}{4} \tag{3.24}$$

and the optimal number of contracts, which depends on the REC's market size alone

$$f_\Sigma^* = \frac{V}{2} \tag{3.25}$$

Option Market

Production sector

The spot price is defined to exceed the strike price, $p > p^s$ when $\epsilon = H$, otherwise a REC would never want to exercise its option. Thus H must be sufficiently large, because $p^f > Ep$. Later it is proven that the same holds for options; the expected unit price, covered by an option is larger than the expected spot price: $(1-\phi)p^L + \phi p^s + p^o > Ep$. RECs are willing to pay an option price premium, just as they pay a forward price premium. In return, to receive one unit for the lower strike price, the REC pays an option price p^o to the producer in any state of the world. The total number of options sold to both RECs is o_Σ. The generator's monopoly profit is $\pi^M = (p^H - c)Q^H - \phi o_\Sigma(p^H - p^s) + o_\Sigma p^o$ with probability ϕ where $p^H = A + H - bQ^H$. It is $\pi^M = (p^L - c)Q^L + o_\Sigma p^o$ with probability $(1-\phi)$ and $p^L = A - bQ^L$, thus the expected profit function is $E\pi^M = (Ep - c)EQ - \phi o_\Sigma(p^H - p^s) + o_\Sigma p^o + Cov(p, Q)$, where Ep and $Cov(p, Q)$ are defined in the forward model. The expected price, the corresponding spot quantity and the high spot price are

$$Ep = \frac{A + \phi H + c - \phi b o_\Sigma}{2} \tag{3.26}$$

$$EQ = \frac{A + \phi H - c + \phi b o_\Sigma}{2b} \tag{3.27}$$

$$p^H = \frac{A + H + c - b o_\Sigma}{2} \tag{3.28}$$

Besides the spot quantity, producers choose p^o and p^s, while RECs choose the number of options they want to buy. The first order conditions are

$$\frac{\partial E\pi^M}{\partial p^o} = \frac{\partial Ep}{\partial p^o}EQ + (Ep-c)\frac{\partial EQ}{\partial p^o} + \frac{\partial o_\Sigma}{\partial p^o}\left[p^o - \phi(p^H - p^s)\right] + o_\Sigma\left(1-\phi\frac{\partial p^H}{\partial p^o}\right) \doteq 0 \tag{3.29}$$

$$\frac{\partial E\pi^M}{\partial p^s} = \frac{\partial Ep}{\partial p^s}EQ + (Ep-c)\frac{\partial EQ}{\partial p^s} + \frac{\partial o_\Sigma}{\partial p^s}p^o - \phi\frac{\partial o_\Sigma}{\partial p^s}(p^H - p^s) + \phi o_\Sigma\left(1-\phi\frac{\partial p^H}{\partial p^s}\right) \doteq 0 \tag{3.30}$$

Retail sector

RECs choose the number of options they buy, REC $i's$ objective (3.7) reduces to $\max_{o_i} U_i = E(\pi_i^R)$.[42] Retailer $i's$ expected profit is

$$E\pi_i^R = V(r - Ep) + \phi o_i(p^H - p^s) - o_i p^o \tag{3.31}$$

The first order condition of (3.31) is

$$o_i = \frac{V\frac{\partial Ep}{\partial o_i} + p^o - \phi(p^H - p^s)}{\phi\frac{\partial p^H}{\partial o_i}} \tag{3.32}$$

As the denominator of (3.32) is negative, the REC's demand for options increases with p^H and decreases with the strike and option price. (3.26)-(3.28) solve for $o_1(o_2) = \frac{1}{2}\left[V - o_2 + b^{-1}\left(A + H + c - 2p^s - 2\phi^{-1}p^o\right)\right]$, taking p^s and p^o as given. The demand for options decreases with the number of options that the other REC buys. The public good attributes that are observed on the forward market also apply to options. Due to symmetry across RECs, the optimal number of options based on the underlying parameter set is

$$o(p^s, p^o) = \frac{1}{3}\left[V + b^{-1}\left(A + H + c - 2p^s - 2\phi^{-1}p^o\right)\right] \tag{3.33}$$

(3.26)-(3.28) and (3.33) transform (3.29) and (3.30) in order to find expressions for p^s and p^o, keeping in mind that $o_\Sigma = 2o(p^s, p^o)$. The first order condition for the strike price p^s can be expressed as

[42] If we would allow $\lambda > 0$, the variance of the REC's profit is $(\phi - \phi^2)[VH - o(p^H - p^s)]^2$. The high demand price is a function of o. The derivative of $Var(\pi^R)$ with respect to o depends on cubed and quadratic o_i,- and o_j,- terms, which would not allow us to have closed form solutions.

$$p^s(p^o, o_\Sigma) = \frac{1}{4}\left[2(A + H + c - 2\phi^{-1}p^o) + o_\Sigma(2b\phi - b)\right] \tag{3.34}$$

It is linearly dependent to the first order condition for the option price p^o

$$p^o(p^s, o_\Sigma) = \frac{\phi}{4}\left[2(A + H + c - 2p^s) + o_\Sigma(2b\phi - b)\right] \tag{3.35}$$

This is not surprising as $(p^H - p^s)$ is a transfer from the producer to the REC with probability ϕ, and p^o is a transfer from the REC to the producer with certainty, while both pairs of players are risk neutral and maximize their expected profit, p^s must be negatively correlated to p^o, and one variable can be expressed through the other. One can write $p^o(o, p^s)$ as $p^o(p^s)$ using $o(p^s, p^o)$ and substitute $p^o(p^s)$ in $o(p^s, p^o)$ to receive the optimal number of options o^*, independent of p^o and p^s, based on the underlying parameter set. $p^o(p^s)$ can be transformed to

$$p^o + \phi p^s = \frac{\phi}{2}(A + H + c) + \frac{\phi b(2\phi - 1)}{2(\phi + 1)}V \tag{3.36}$$

which is the expected option payment made to the generator, to avoid paying p^H. Adding $(1 - \phi)p^L$ to $p^o + \phi p^s$, gives an expected unit price; when that unit is covered with an option. $(1 - \phi)p^L + p^o + \phi p^s - Ep = \frac{(4\phi+1)b\phi V}{6(1+\phi)} > 0$ does not depend on the size of the demand shock. This corresponds to the observation, first made by Powell (1993) for the forward market, who shows that $p^f > Ep$, which goes back to Allaz and Vila's (1993) article. But so far, it has not been shown for the option market. The reason behind this solution is the same; forwards and options lower the expected future spot price. Substituting p^o in (3.33) by (3.36) gives an expression for the optimal number of options

$$o^*_\Sigma = \frac{2}{3}V\frac{2 - \phi}{1 + \phi} \tag{3.37}$$

There are all the ingredients, one needs to compare the expected spot price for futures and options, stated here again for convenience: $Ep^{Futures} = \frac{A+\phi H+c-bf_\Sigma}{2}$ with $f^*_\Sigma = \frac{V}{2}$ and $Ep^{Options} = \frac{A+\phi H+c-\phi bo_\Sigma}{2}$. The expected spot price is larger in the presence of options than forwards when: $\phi o^*_\Sigma < f^*_\Sigma \Leftrightarrow \frac{2\phi(2-\phi)}{3(1+\phi)} < 1$, which holds for all $0 \leq \phi \leq 1$. The expected spot price is smaller in the forward model. For the simulation of both models, the same parameters are used as before: $h = 0.15$, $V = 2.5$, $A = 50$, $b = 0.67$ and $c = 20$. The retail price did not matter in the analysis of the first part of this article, here it equals r=£60/MWh as in Green (2004). Furthermore the probability of

	f_Σ^*, o_Σ^*	Ep	$p^f, Ep^{w.\ Options}$	EQ	$E\pi^M$	$E\pi^R$	$E(CS)$
Forwards	1.25	39.48	40.21	30.63	609.90	50.65	159.51
Options	$1.\overline{6}$	$39.7\overline{2}$	40.00	30.42	609.73	50.23	156.31

Table 3.1: Welfare in the absence of bankruptcy

a positive demand shock, $H = 20$, is $\phi = 50\%$. Before the liberalization takes place, market participants choose the values given by table 3.1.

Welfare is hardly smaller in the one-way contract model, residual consumers loose about 2% in expectation $[E(CS)]$, while expected producer and REC profits are barely different. Note that the expected price of a unit purchased with an option is less expensive than p^f, which holds when $8\phi^2 - 5\phi - 1 < 0$. One can repeat the derivations described in this section so far, when producers do not coordinate at the spot market. In that case, the analog equation for (3.36) and (3.37) are

$$p^o + \phi p^s = \frac{\phi}{18 + 12\phi}\left[A\,(9 + 4\phi) + c\,(9 + 8\phi) + H\,(6 + 7\phi) + Vb\,(-3 + 4\phi)\right] \tag{3.38}$$

and

$$\frac{o_\Sigma^*}{2} = \frac{3Vb + c - A - \phi H}{b(6 + 4\phi)} \tag{3.39}$$

One can easily see that there are long hedges possible, when producers do not cooperate at the spot market and $3Vb+c < A+\phi H$. The rest of the analysis uses the model, where producers cooperate, and shows under what circumstances a welfare maximizing regulator prefers one-way contracts over two-way contracts, when the default threat is included. In the absence of bankruptcy, one-way contracts have the disadvantage that just in the high demand state, a fix price is paid for the production that was covered. When options are “out-of-the-money” and producers play the Cournot game. Thus intuitively it is clear, why one-way contracts can not reduce market power to the same extend as two-way contracts do.

3.3.2 Post liberalization period

This section contains the same structure previously used. After a competitive fringe has entered the market, the new demand for incumbent retailers is (3.14) as before, and the optimal retail price is (3.16).

Forwards

The retailer's profit in the absence and presence of a demand shock are

$$\begin{array}{ll} \pi^R = \frac{V^2}{4h} + f^*(p^L - p^f) & if \quad \epsilon = 0 \\ \pi^R = \frac{V^2}{4h} + f^*(p^H - p^f) & if \quad \epsilon = H \end{array} \tag{3.40}$$

$\pi^R > 0$, when $\epsilon = H$ in the forward model, because $p^f > Ep > p^L$. Bankruptcy is possible, when $\epsilon = 0$ in the forward model when switching costs are low (large value of h), V is small and p^L is much smaller than p^f, such that π^R becomes negative. When $\epsilon = H$, and thus bankruptcy does not occur with certainty, (3.3) and (3.4) continue to hold, where A is substituted by the realized intercept $A+H$. Bankruptcy does not play any role. When $\epsilon = 0$, then a retailer's profit is $\pi_i^R = \frac{V^2}{4h} + f_i^*(p^L - p^f)$, where $f_i^* = \frac{f_\Sigma^*}{2}$. Each producer maximizes its profit, $\max_{q_i^L} \pi_i^P = (p^L - c)q_i^L - f_i^*(p^L - p^f)(1 + s\pi_i^R)$, where $(1 + s\pi_i^R)$ is the survival probability of retailer i, when $\pi_i^R < 0$. (3.19) and (3.20) describe the equilibrium, where the realized intercept is $A^* = A$.

Options

The possible retail profits in the option model are

$$\begin{array}{ll} \pi^R = \frac{V^2}{4h} - o^* p^o & if \quad \epsilon = 0 \\ \pi^R = \frac{V^2}{4h} + o^*[p^H - (p^s + p^o)] & if \quad \epsilon = H \end{array} \tag{3.41}$$

When $\epsilon = 0$ the producer's objective is $\max_{q^L} \pi^P = (p^L - c)q^L + op^o$ and the equilibrium values are $q^{L*} = \frac{A-c}{3b}$ and $p^{L*} = \frac{A+2c}{3}$. Thus producers offer the regular Cournot price. It is a reasonable assumption that the option price has been paid in advance, thus a possible bankruptcy does not affect the producer's objective. When $\epsilon = H$, the producer's objective is

$$\max_{q_i^H} \pi_i^P = (p^H - c)q_i^H + o_i[p^o + p^s - p^H] \tag{3.42}$$

If $\epsilon = H$, options are "in the money" and the second term of π^P is strictly positive by definition, as $p^o + p^s > p^H$. Otherwise the option would never be exercised. In the preceding analysis, the relevant value has been, $p^o + \phi p^s$. The problem was defined in such a way that options are exercised when $\epsilon = H$. It would not have been a realistic assumption from the REC's point of view, when it would make a loss through exercising

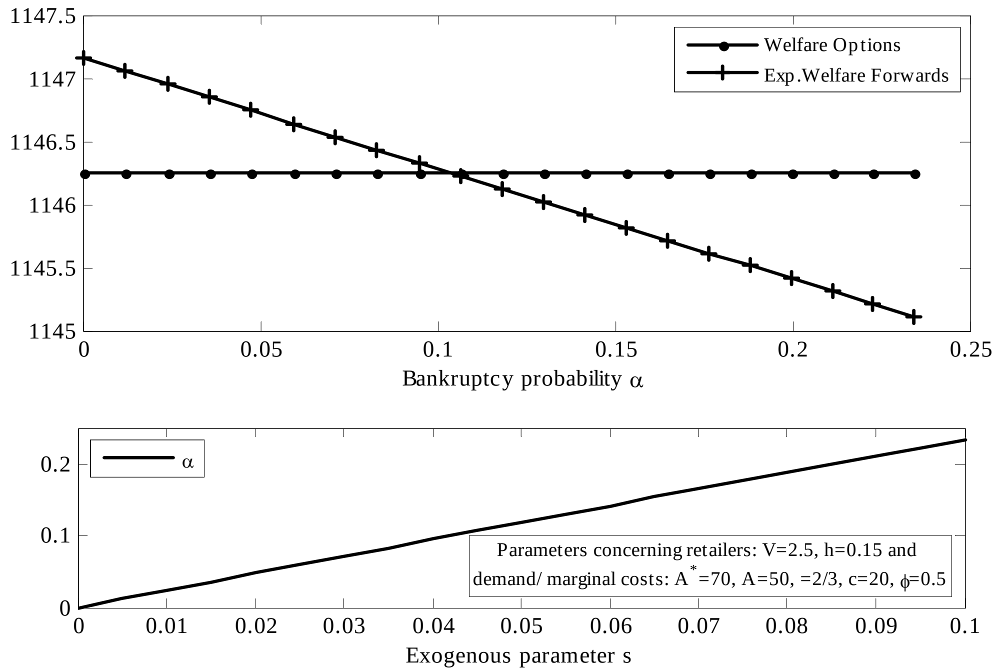

Figure 3.3: Welfare: Forwards vs. Options

an option, when the demand is high. Options just lower producer's market power on the spot market if they are exercised for $\epsilon = H$. If it would not be in the REC's interest to exercise the option, it would be known to the producers. Thus options would not reduce market power and RECs would not purchase them in the first place.

The optimal values are $q^{H*} = \frac{A+H-c+bo^*}{3b}$ and $p^{H*} = \frac{A+H+2c-2bo^*}{3}$, where $o^* = \frac{o_\Sigma}{2}$. They have the same structure as in the Allaz Vila equilibrium. The buyer of one-way contracts pays in either state of the world p^o. Thus the difference between profit and loss is smaller with options than with forwards.

3.3.3 Results

There are two different equilibria for forwards and options, respectively when $\epsilon = 0$ and $\epsilon = H$. It has already been proven that forwards are preferable to options in the absence of bankruptcy. After weighting the two possible outcomes of the forward model equilibrium, one can determine, if either options or forwards are preferable in the presence of the threat of bankruptcy.

The same parameters are used to generate figure 3.3, which summarizes the main result: in the absence of a default threat ($s = \alpha = 0$), the option model yields a lower welfare than the forward model, which has been shown algebraically before. But note that the welfare difference is quite small. The lower diagram shows the relation between s and α, the upper diagram between α and welfare.[43] As the bankruptcy threat of former RECs just concerns producers in the forward model, welfare in the option model is not affected by the exogenous parameter s. The higher the bankruptcy probability $\alpha = -s\pi^R$ becomes, the lower the welfare in the two-way contract model just as in figure 3.1 and 3.2. For a default probability of around 10%, a regulator that maximizes total welfare would prefer the option model over that of the forward model. One has to bear in mind that this estimate is very conservative, as in this model, default only has an effect on the spot, but not contract market.

3.4 Conclusion

This paper introduces a simple model, where downstream firms, operating in a competitive environment, may go bankrupt after incurring a loss on forward contracts that have been signed with upstream firms. The first part of the model shows that former RECs, which still hold long term forward contracts, benefit when they have an owner that is less likely to bail them out. The bankruptcy probability consists of the retailer's loss, and an exogenous multiplier that reflects the willingness for bail-out of the retailer's owner. Producers minimize the risk of a retailer not meeting its contract of difference payments by reducing the production quantity and hence increasing the spot price, which in return reduces the difference. For reasonable parameters, when an owner is less likely to bail its subsidiary out, the upward spot price shift is sufficiently large to *turn a loss* (in the absence of a default threat) *into a profit* (in the presence of a default threat). Depending on the extent of the upward price adoption, producers benefit or loose. Consumers always loose more than firms gain, thus welfare decreases.

The second part of the model introduces one-way contracts, and demonstrates that options lower the spot price, as forwards do. In the absence of risk aversion, the expected price of a unit that is bought with an option, exceeds the expected price. Once again, the parallel can be drawn to the forward market. Options reduce the profit of an upstream firm less than forwards do; thus the spot price is larger in an option model than a forward model. The model simulates the English/ Welsh electricity market and shows that the welfare difference is very small. Producer and retailer profits are hardly

[43] Welfare equals the residual consumer surplus, which is the area below the inverse demand function and above the spot price. In addition, the producer and retail profits are added.

differently affected by the instruments, and consumer surplus decreases by a mere 2%. Including bankruptcy and comparing the option and forward model again, one-way contracts are preferred to two-way contracts for bankruptcy probabilities of 10%. If one includes other costs that are connected to bankruptcy, and considering that this estimate is very conservative, options might quickly become preferred to forwards.

When regulators decide how to treat off-market trade in the near future, they shall have to keep in mind that depending on the industry structure, insolvency might not only cost taxpayers' money, it may also reduce the market power mitigating effect of two-way contracts. If two-way contracts become more heavily regulated to avoid bankruptcies, and require a large amount of capital as market participants claim, one-way contracts could step in, and play the same role that forwards have done in the past.

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Ich versichere hiermit eidesstattlich, dass ich die vorliegende Arbeit selbständig und ohne fremde Hilfe verfasst habe. Die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sowie mir gegebene Anregungen sind als solche kenntlich gemacht.

Die Arbeit wurde bisher keiner anderen Prüfungsbehörde vorgelegt und auch noch nicht veröffentlicht.

Datum: 18.12.2009

Unterschrift:

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