On Convex Subcomplexes of Spherical Buildings and Tits' Center Conjecture

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Zusammenfassung

In dieser Arbeit untersuchen wir konvexe Unterkomplexe sphärischer Gebäude. Insbesondere interessieren wir uns für eine Frage von J. Tits aus den 50er Jahren, die *Zentrumsvermutung*. Sie behauptet, dass ein konvexer Unterkomplex eines sphärischen Gebäudes ein Untergebäude ist oder die Gebäude-Automorphismen, die den Unterkomplex erhalten, einen gemeinsamen Fixpunkt besitzen.

Ein Beweis der Zentrumsvermutung für die Gebäude klassischen Typs (A_n, B_n und D_n) wurde von B. Mühlherr und J. Tits in [MT06] gegeben. Der F_4-Fall wurde von C. Parker und K. Tent in einem Vortrag in Oberwolfach präsentiert [PT08]. Beide Argumente verwenden kombinatorische Methoden aus der Inzidenzgeometrie. B. Leeb und der Autor gaben in [LR09] differentialgeometrische Beweise für die Fälle F_4 und E_6 aus der Sicht der Theorie metrischer Räume mit oberen Krümmungsschranken.

In dieser Arbeit wird der differentialgeometrische Zugang weiterentwickelt. Unser Hauptresultat ist der Beweis der Zentrumsvermutung für Gebäude vom Typ E_7 und E_8, deren Geometrie noch wesentlich komplexer ist. Insbesondere wird dadurch der Beweis der Zentrumsvermutung für alle dicken sphärischen Gebäude abgeschlossen. Wir geben auch einen kurzen differentialgeometrischen Beweis für die klassischen Typen. Schliesslich zeigen wir noch, wie man die Fälle F_4, E_6 und E_7 aus dem E_8-Fall folgern kann.

Abstract

In this thesis we study convex subcomplexes of spherical buildings. In particular, we are interested in a question of J. Tits which goes back to the 50's, the so-called *Center Conjecture*. It states that a convex subcomplex of a spherical building is a subbuilding or the building automorphisms preserving the subcomplex have a common fixed point in it.

A proof of the Center Conjecture for the buildings of classical types (A_n, B_n and D_n) has been given by B. Mühlherr and J. Tits in [MT06]. The F_4-case was presented by C. Parker and K. Tent in a talk in Oberwolfach [PT08]. Both approaches use combinatorial methods from incidence geometry. B. Leeb and the author gave in [LR09] differential-geometric proofs for the cases F_4 and E_6 from the point of view of the theory of metric spaces with curvature bounded from above.

In this work we develop the differential-geometric approach further. Our main result is the proof of the Center Conjecture for buildings of type E_7 and E_8, whose geometry is considerably more complicated. In particular, this completes the proof of the Center Conjecture for all thick spherical buildings. We also give a short differential-geometric proof for the classical types. Finally, we show how the cases F_4, E_6 and E_7 can be deduced from the E_8-case.

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Contents

Introduction

Buildings were first introduced by J. Tits in order to give geometric interpretations to algebraic groups and the pattern of certain kinds of subgroups. In this work, we will only consider buildings of spherical type. From the point of view of differential geometry these can be thought of as a special kind of singular metric spaces with upper curvature bound one in the sense of Aleksandrov. They are characterized by the property that they contain many top-dimensional convex subsets isometric to unit round spheres, the so-called *apartments* (see Section 1.3 for the formal definition). Spherical buildings occur in Riemanninan geometry as boundaries at infinity of symmetric spaces of noncompact type and play a prominent role in rigidity questions.

A spherical building carries a natural structure as a piecewise spherical polyhedral complex. Its top-dimensional faces, the so-called *chambers*, are all isometric. Their isometry type is called the *model Weyl chamber*. In this thesis we study closed convex subsets of spherical buildings, which are also subcomplexes. In particular, we consider a conjecture first proposed by J. Tits in the 50's which is known as the *Center Conjecture*. It is now formulated as follows (compare [MT06] and [Se05, Conjecture 2.8]).

Conjecture 1 (Center Conjecture). *Suppose that B is a spherical building and that $K \subseteq B$ is a convex subcomplex. Then K is a subbuilding or the action $Stab_{Aut(B)}(K) \curvearrowright K$ of the automorphisms of B preserving K has a fixed point.*

A building automorphism is an isometry, which preserves the polyhedral structure of the building. In particular, it induces an isometry of the model Weyl chamber, which may be nontrivial. If it is trivial, the automorphism is type preserving. The isometries of the model Weyl chamber can be identified with the symmetries of the Dynkin diagram.

A fixed point of the action $Stab_{Aut(B)}(K) \curvearrowright K$ is called a *center* of the subcomplex K.

Apparently, the first motivation of Tits for considering the Center Conjecture came from algebraic group theory. Namely, he wanted to prove a result associating a parabolic subgroup P to a unipotent subgroup U of a reductive algebraic group G [Ti62, Lemma 1.2]. This result is a direct consequence of the Center Conjecture. The desired parabolic subgroup is obtained as the center of the fixed point set of the action $U \curvearrowright B$, where B is the building associated to the group G. This result was later obtained by Borel and Tits in [BT71] using other methods.

In Geometric Invariant Theory a special case of the Center Conjecture is used to find parabolic subgroups that are *most* responsible for the instability of a point (see [Mu65]). This special case was proven by Rousseau [Rou78] and Kempf [Ke78].

From the point of view of metric geometry and CAT(1) spaces, a natural generalization of Conjecture 1 is to drop the assumption of K being a subcomplex and consider arbitrary closed convex subsets $C \subset B$. Such a subset C is a CAT(1) space itself. We can also forget the ambient building and look for fixed points for the whole group of isometries $Isom(C)$.

Conjecture 2. *If C is a closed convex subset of a spherical building B, then C is a subbuilding or the action $Isom(C) \curvearrowright C$ has a fixed point.*

Conjecture 2 was answered positively in [BL05] for the case $dim(C) \leq 2$. The strategy of their proof is basically to consider a smallest $Isom(C)$-invariant closed convex subset $Y \subset C$ and then prove that if Y is not a subbuilding, it has intrinsic radius $\leq \frac{\pi}{2}$ (by intrinsic radius of Y we denote the infimum of the radii of balls centered at Y and containing Y). If a CAT(1) space X has intrinsic radius $\leq \frac{\pi}{2}$, it was also shown in [BL05] that the set Z of circumcenters of X is not empty and has radius $< \frac{\pi}{2}$, in particular, $Z \subset X$ has a unique circumcenter and it is fixed by $Isom(X)$. It follows that $Isom(C)$ fixes a point in $Y \subset C$.

If $C \subset B$ has intrinsic radius π then it must be a building (see [BL06]) and if it has intrinsic radius $\leq \frac{\pi}{2}$, it satisfies the fixed point property asserted in Conjecture 2 as already mentioned above. It is natural to ask if there are closed convex subsets between these two possibilities or if Conjecture 2 is just a consequence of a more general "gap phenomenon" (cf. [KL06, Question 1.5]).

Conjecture 3. *If C is a closed convex subset of a spherical building B, then C is a subbuilding or $rad_C(C) \leq \frac{\pi}{2}$.*

If $dim(C) \leq 1$, then it is easy see that Conjecture 3 holds, namely, a one-dimensional convex subset is a building or a tree of radius $\leq \frac{\pi}{2}$. Another easy case is when the building B is just a spherical Coxeter complex, i.e. B is a round sphere with curvature $\equiv 1$, then C is also a round sphere with curvature $\equiv 1$ or it has intrinsic radius $\leq \frac{\pi}{2}$.

Unfortunately, we do not know more positive results for the Conjectures 2 and 3, other than those mentioned above. Notice that we have the implications $3 \Rightarrow 2 \Rightarrow 1$.

If K is a convex subcomplex of a reducible building $B = B_1 \circ \cdots \circ B_k$, then K decomposes as a spherical join $K = K_1 \circ \cdots \circ K_k$ where $K_i \subset B_i$ is a convex subcomplex for $i = 1, \ldots, k$. Thus, the Center Conjecture easily reduces to the case of irreducible buildings. For irreducible buildings of classical type (i.e. A_n, B_n and D_n) the Center Conjecture was shown in [MT06]. The F_4-case was presented in a talk in Oberwolfach in [PT08]. The proof uses the incidence-geometric realizations of the corresponding different types of buildings.

Our approach to these problems is of differential-geometric nature, using methods from the theory of metric spaces with curvature bounded above. In Section 4.1 we give another proof for the case of buildings of classical type from the point of view of comparison geometry. The cases of buildings of type F_4 and E_6 are settled in [LR09], we reproduce the proofs with some minor modifications in Sections 4.3 and 4.4 for the sake of completeness.

The main result in this work is:

Theorem 4. *The Center Conjecture 1 holds for spherical buildings of type E_7 and E_8.*

We give first a direct proof of the E_7-case in Section 4.5. The E_8-case is proven in Section 4.6, where we also give alternative proofs for the cases of buildings of type F_4, E_6 and E_7 as consequences of the E_8-case. The case of buildings of type H_3 can be easily treated with our methods (Section 4.2) or just be considered as a consequence of the main result in [BL05]. Hence we have the following result.

Corollary 5. *The Center Conjecture 1 holds for spherical buildings without factors of type H_4.*

Our proofs of these results actually show a more general version of the Center Conjecture (something between Conjecture 1 and 2 as far as group actions are concerned):

Corollary 6. *If B is a spherical building without factors of type H_4 and $K \subseteq B$ is a convex subcomplex, then K is a subbuilding or the action $Aut_B(K) \curvearrowright K$ has a fixed point.*

The automorphisms in $Aut_B(K)$ are defined to be isometries of K preserving its polyhedral structure induced by B and such that the permutation of the labelling of its vertices is induced by a symmetry of the Dynkin diagram of B. They need not be extendable to automorphisms of B (see Section 1.3).

While any spherical Coxeter complex is a spherical building, not all spherical Coxeter complexes occur as Coxeter complexes for thick spherical buildings ([Ti77]). Namely, there are no thick spherical buildings of type H_3 ($\overset{1}{\bullet}\overset{5}{-}\overset{2}{\bullet}-\overset{3}{\bullet}$) and H_4 ($\overset{1}{\bullet}\overset{5}{-}\overset{2}{\bullet}-\overset{3}{\bullet}-\overset{4}{\bullet}$), these being the only cases. On the other hand, any spherical building has a canonical thick structure (depending only on its isometry type) which results from restricting to a subgroup of the Weyl group ([Sch87], [KL98, Sec. 3.7]). The polyhedral structure thus obtained is (possibly) coarser. The Center Conjecture is most natural when posed for thick spherical buildings, because then K is a subcomplex of the natural polyhedral structure of B. In this case we have:

Corollary 7. *The Center Conjecture 1 holds for all thick spherical buildings.*

A completely different approach to the special case of the Center Conjecture for spherical buildings B associated to algebraic groups G and subcomplexes K which are fixed point sets of the action of a subgroup $H \subset G$ can be found in [BMR09]. They show that such a subcomplex is a subbuilding or the action $Stab_G(K) \curvearrowright K$ fixes a point. In [BMRT09] this result is extended to the action $Stab_{Aut(G)}(K) \curvearrowright K$.

We give now a short description of the structure of this work. In Chapter 1 we present the definitions and known facts used in this thesis about CAT(1) spaces, spherical Coxeter complexes and spherical buildings. In Chapter 2 we study some geometric properties of the different spherical Coxeter complexes. In Chapter 3 we gather some lemmata about convex subcomplexes of buildings and isometric actions on them that will be used in the proofs of the different cases of the Center Conjecture in Chapter 4. The Appendices collect all the information on the spherical Coxeter complexes that is used to deduce the properties in Chapter 2.

Chapter 1

Preliminaries

1.1 CAT(1) spaces

A complete metric space X is said to be a *$CAT(1)$ space* if it is π-geodesic and the geodesic triangles of perimeter less than 2π are not *thicker* than those in the round sphere with curvature $\equiv 1$. The formal definition can be stated in several equivalent ways, we refer to [BH99, Chapter II.1].

For two distinct points x, y in a CAT(1) space X at distance $< \pi$, we denote by xy the unique segment connecting both points. Let $m(x, y)$ denote the midpoint of the segment xy. Two points at distance $\geq \pi$ are called *antipodal*.

The *link* $\Sigma_x X$ at a point $x \in X$ is the space of directions at x with the angle metric. It is again a CAT(1) space. If $y \neq x$ and y is not antipodal to x, we denote with $\overrightarrow{xy} \in \Sigma_x X$ the direction at x of the segment xy.

A subset C of a CAT(1) space is called *convex*, if for any $x, y \in C$ at distance $< \pi$ the segment xy is contained in C. A closed convex subset of a CAT(1) space is itself a CAT(1) space. A closed ball of radius $\leq \frac{\pi}{2}$ in a CAT(1) space is always convex. The *closed convex hull* $CH(A)$ of a subset A is the smallest closed convex subset containing A.

Let A be a subset of a CAT(1) space X and let $x \in X$. The *radius of A with respect to x* is defined as $rad(x, A) := \sup\{d(x, y) | y \in A\}$ and the *circumradius* (or just *radius*) of A in X is $rad_X(A) = \inf\{rad(x, A) | x \in X\}$. For a closed convex subset C the radius $rad_C(C)$ is called the *intrinsic radius* of C. A point $x \in CH(A)$, such that $rad(x, A) = rad_{CH(A)}(A)$ is called a *circumcenter* of A.

A classical result of comparison geometry states that a closed convex subset of a CAT(1) space with intrinsic radius $< \frac{\pi}{2}$ has a unique circumcenter (see e.g. [BH99, Ch. 2, Prop. 2.7]).

For more information and properties of CAT(1) spaces we refer to [BH99].

1.2 Coxeter complexes

A *spherical Coxeter complex* (S, W) is a pair consisting of a round sphere S with curvature $\equiv 1$ together with a finite group of isometries W, called the *Weyl group*, generated by reflections on great spheres of codimension one.

There is a natural structure of spherical polyhedral complex on S induced by W. The spheres of codimension one, that are the fixed point sets of the reflections in W are called the *walls*. The *Weyl chambers* are the closures of the connected components of S minus the union of all the walls. A Weyl chamber is a convex spherical polyhedron. The Weyl chambers are fundamental domains for the action of the Weyl group on S and therefore isometric to the *model Weyl chamber* $\triangle_{mod} := S/W$. A *root* is a top-dimensional hemisphere bounded by a wall. A *singular* sphere is an intersection of walls. The intersections of a singular sphere and a Weyl chamber is called a *face* of the Weyl chamber. A *vertex* is a 0-dimensional face. A segment contained in a singular 1-sphere is called a *singular segment*. The face *spanned* by a point is the smallest face containing it. The *type* of a point $x \in S$ is its image in the model Weyl chamber under the natural map $\theta_S : S \to S/W = \triangle_{mod}$.

The geometry of a spherical Coxeter complex (S, W) can be encoded in a weighted graph Γ, the so-called *Dynkin diagram*, as follows. The vertices of Γ correspond to the codimension one faces of $\triangle_{mod}$. Two codimension one faces of $\triangle_{mod}$ intersect with a dihedral angle $\frac{\pi}{k}$ for $k \geq 2$ an integer. Two vertices of Γ are connected by a simple edge if the angle between the corresponding faces is $\frac{\pi}{k}$ for $k = 3$; they are connected by a double edge, if $k = 4$; by a triple edge, if $k = 6$; and by an edge with label k, if $k = 5, 7, 8, \ldots$. A labelling by an index set I of the vertices of the Dynkin diagram induces a labelling of the vertices of $\triangle_{mod}$, by giving a vertex $v \in \triangle_{mod}$ the label of the vertex of Γ corresponding to the face opposite to v. We say that a vertex in S is an i-vertex for $i \in I$, if its type in $\triangle_{mod}$ has label i.

The group $Isom(\triangle_{mod})$ is canonically identified with the group of symmetries of the Dynkin diagram. An *automorphism* of (S, W) is an isometry of S preserving its polyhedral structure, that is, $Aut(S, W)$ is the normalizer of W in $Isom(S)$. The group $Aut(S, W)/W$ can be canonically identified with the isometries of the model Weyl chamber $\triangle_{mod}$ and therefore also with the symmetries of the Dynkin diagram. Notice that the antipodal involution of S is always an automorphism of (S, W). The *canonical involution* of $\triangle_{mod}$ is the image of the antipodal involution under the identification mentioned above.

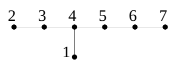
The *rank* of (S, W) is the number of vertices of its Dynkin diagram. One can show that $rank(S, W) = dim(S) + 1$ if and only if W has no fixed points in S, equivalently, if and only if $diam(\triangle_{mod}) \leq \frac{\pi}{2}$. In this case the Dynkin diagram determines $\triangle_{mod}$ up to isometry. Otherwise, $rank(S, W) < dim(S) + 1$ and the Coxeter complex (S, W) is the spherical join of the spherical Coxeter complex with the same Dynkin diagram as (S, W) and dimension $rank(S, W) - 1$, and a sphere of dimension $dim(S) - rank(S, W)$. In this case $diam(\triangle_{mod}) = \pi$. If $diam(\triangle_{mod}) = \frac{\pi}{2}$, then (S, W) decomposes as a spherical join of spherical Coxeter complexes, their Dynkin diagrams correspond to the connected components of the Dynkin diagram of (S, W). Thus we say that a spherical Coxeter complex (S, W) is *irreducible* if $rank(S, W) = dim(S) + 1$ and its Dynkin diagram is connected. A Coxeter complex is irreducible if and only if $diam(\triangle_{mod}) < \frac{\pi}{2}$.

The irreducible spherical Coxeter complexes of rank $n \geq 3$ have Dynkin diagrams of type A_n, B_n, D_n (for $n \geq 4$), H_3, H_4, F_4, E_6, E_7 and E_8 (see Appendix A, p. 83 for a figure of the Dynkin diagrams).

Let $x \in S$ and let σ be the face spanned by x. The link $\Sigma_x S$ decomposes as the spherical join $\Sigma_x S \cong \Sigma_x \sigma \circ \nu_x \sigma$ of the sphere $\Sigma_x \sigma$ of directions tangent to σ and the sphere $\nu_x \sigma$ of orthogonal directions. Suppose $x' \in S$ is another point spanning σ, we can canonically identify $\nu_x \sigma$ and $\nu_{x'} \sigma$ by identifying *parallel* directions (in the Riemannian sense), or equivalently, if $c, c' : [0, \epsilon) \to S$ are unit speed geodesics with $c(0) = x$, $c'(0) = x'$ and orthogonal to σ, we identify the directions $\dot{c}(0)$ and $\dot{c}'(0)$ if and only if the convex hulls $CH(\sigma \cup \{\dot{c}(t)\})$ and $CH(\sigma \cup \{\dot{c}'(t)\})$ coincide near x and x' for small $t > 0$. We can therefore define the link $\Sigma_\sigma S$ of σ in S as the identification space of the spheres $\nu_x \sigma$ for $x \in S$ spanning σ. It is again a spherical Coxeter complex with Weyl group $W_\sigma := Stab_W(\sigma)$ and model Weyl chamber $\triangle_{mod}^{(\Sigma_\sigma S, W_\sigma)} \cong \Sigma_\sigma \triangle_{mod}^{(S,W)}$. Its Dynkin diagram can be obtained from the Dynkin diagram of (S, W) by deleting the vertices corresponding to the vertices of σ.

Consider a singular sphere $s \subset S$. Then s has a natural structure of Coxeter complex induced by (S, W) as follows. The *induced Weyl group* $W_s \subset Isom(s)$ on s is the subgroup generated by the reflections on s induced by isometries in W. Then (s, W_s) is a Coxeter complex and we call it a *Coxeter subcomplex* of (S, W). The polyhedral structure of (s, W_s) is in general coarser than the one induced by the polyhedral structure of (S, W). A singular sphere $s' \subset s$ of codimension one in s is a wall of (s, W_s) if and only if for any top-dimensional face in s' (with respect to the polyhedral structure of (S, W)) the two top-dimensional faces in s (again with respect to (S, W)) adjacent to it have the same type, i.e. the same image in $\triangle_{mod}^{(S,W)}$.

Remark 1.2.1. The induced Weyl group W_s can be strictly smaller than the image of $Stab_W(s) \to Isom(s)$ as shown in the following example.

Example 1.2.2. Consider the Coxeter complex of type E_7 with the labelling 

of its Dynkin diagram. We find a singular 1-sphere s of type 13756137561 (see Section 2.6). The induced Weyl group W_s is trivial, but the antipodal involution on s is induced by isometries in $Stab_W(s)$.

We refer to [GB71] and [KL98, Sec. 3.1, 3.3] for further information on spherical Coxeter complexes.

1.3 Spherical buildings

We refer to [AB08], [KL98] and [Ti74] for more information on spherical buildings. We will consider spherical buildings from the point of view of CAT(1) spaces as presented in [KL98].

A *spherical building* B modelled on a spherical Coxeter complex (S, W) is a CAT(1) space together with an atlas $\mathcal{A}$ of isometric embeddings $S \hookrightarrow B$ (the images of these embeddings are called *apartments*) with the following properties: any two points in B are contained in a common apartment, the atlas $\mathcal{A}$ is closed under precomposition with isometries in W and the coordinate changes are restrictions of isometries in W. The empty set is considered to be a building.

The polyhedral structure of (S, W) induces a polyhedral structure on the building B. The objects (walls, roots,...) defined for spherical Coxeter complexes can be defined for the building B as the corresponding images in B.

A building is called *thick* if every wall is the boundary of at least three different roots.

Let $a \subset B$ be an apartment and let $\sigma \subset a$ be a Weyl chamber. There is a natural 1-Lipschitz retraction $\rho_{a,\sigma} : B \to a$, such that $\rho_{a,\sigma}|_a = id_a$, defined as follows. For $y \in B$ let $x \in \sigma$ be an interior point of σ not antipodal to y. Then $\rho_{a,\sigma}(y)$ is the point in a, such that $d(x, y) = d(x, \rho_{a,\sigma}(y))$ and $\overrightarrow{xy} = \overrightarrow{x\rho_{a,\sigma}(y)}$. For an apartment a' containing σ, $\rho_{a,\sigma}|_{a'}$ is the unique isometry from a' to a fixing σ pointwise.

There is also a natural 1-Lipschitz *anisotropy map* $\theta_B : B \to \triangle_{mod}$. It is characterized by the property that for any chart $\iota : S \to B$ we have $\theta_B \circ \iota = \theta_S$. If $\sigma \subset \iota(S)$ is any chamber, then we also have $\theta_S \circ \iota^{-1} \circ \rho_{\iota(S),\sigma} = \theta_B$. The anisotropy map restricts to an isometry on any Weyl chamber. We define the *type* of a point in B as its image under θ_B. As for Coxeter complexes, a labelling of the vertices of the Dynkin diagram of (S, W) induces a labelling of the vertices of B.

The following proposition gives a criterion for the existence of a structure as a spherical building on a CAT(1) space in terms of the anisotropy map (compare with [KL98, Prop. 3.5.1]).

Proposition 1.3.1 ([LR09, Prop. 2.2]). *Let (S^n, W) be a spherical Coxeter complex and let X be a CAT(1) space with a structure of spherical polyhedral complex of dimension n. Suppose that there is a 1-Lipschitz map $\theta_X : X \to \triangle_{mod} = S/W$, such that it restricts to an isometry on any top-dimensional face of X. Suppose furthermore that any two points of X lie in an isometrically embedded copy of S. Then X has a natural structure as a spherical building modelled in (S, W) with anisotropy map θ_X.*

Proof. Let us call a top-dimensional face of X a chamber and an isometrically embedded copy of S an apartment. By the assumptions, all chambers are isometric to $\triangle_{mod}$ and the apartments are tesselated by chambers. If σ_1, σ_2 are two adjacent chambers contained in an apartment a, then the isometry $(\theta_X|_{\sigma_2})^{-1} \circ \theta_X|_{\sigma_1} : \sigma_1 \to \sigma_2$ coincides with the reflection at the common face of codimension one. It follows that the tesselation of a by chambers *coincides* with the polyhedral structure of (S, W), that is, there is an isometry $\iota_a : S \to a$ with $\theta_X \circ \iota_a = \theta_S$, which is unique up to precomposition with isometries in W. All these isometries constitute an atlas and the compatibility of the charts is clear. □

If $x, x' \in B$ lie in a common Weyl chamber σ, then the convex hull $CH(x, x', y)$ is a spherical triangle for all $y \in B$ (just consider the apartment containing y and σ).

Let $x \in B$ and let σ be the face of B spanned by x. The link $\Sigma_x B$ decomposes as the spherical join $\Sigma_x B \cong \Sigma_x \sigma \circ \nu_x \sigma$ of the sphere $\Sigma_x \sigma$ of directions tangent to σ and the space $\nu_x \sigma$ of orthogonal directions to σ. If $x' \in B$ is another point spanning σ, then the spaces $\nu_x \sigma$ and $\nu_{x'} \sigma$ are canonically isometric as follows. If $c, c' : [0, \epsilon) \to B$ are unit speed geodesics with $c(0) = x$, $c'(0) = x'$ and orthogonal to σ, we identify the directions $\dot{c}(0)$ and $\dot{c}'(0)$ if and only if there is a chamber τ containing $c(t)$ and $c'(t)$ for small $t > 0$ and the directions $\dot{c}(0)$ and $\dot{c}'(0)$ are *parallel* in τ, equivalently, if and only if the convex hulls $CH(\sigma \cup \{\dot{c}(t)\})$ and $CH(\sigma \cup \{\dot{c}'(t)\})$ coincide near x and x' for small $t > 0$. We can therefore define the link $\Sigma_\sigma B$ of σ in B as the corresponding identification space. It has a structure as a spherical building modelled on the spherical Coxeter complex $(\Sigma_{\iota^{-1}(\sigma)} S, Stab_W(\iota^{-1}(\sigma)))$, where $\iota : S \hookrightarrow B$ is a chart with $\sigma \subset \iota(S)$.

For $x \in B$ and sufficiently small $\epsilon > 0$, the ball $B_\epsilon(x) \subset B$ is canonically isometric to the spherical cone of height ϵ over the link $\Sigma_x B$. Thus, spherical buildings have a *local conicality* property.

A *building automorphism* is an isometry preserving the polyhedral structure. We denote by $Aut(B)$ the group of automorphisms of B and by $Aut_0(B) \subseteq Aut(B)$ the subgroup of type preserving automorphisms. An automorphism of B induces an isometry of the model Weyl chamber $\triangle_{mod}$. This isometry is trivial if the automorphism is type preserving. The quotient $Aut(B)/Aut_0(B)$ embeds as a subgroup of $Isom(\triangle_{mod})$. Notice that if the building B is thick, then $Aut(B) = Isom(B)$.

A *convex subcomplex* K is a closed convex subset of B which is a subcomplex with respect to the polyhedral structure of B. Let $Aut_B(K)$ denote the group of isometries of K preserving the polyhedral structure of K induced by the polyhedral structure of B and such that the permutation in the labelling of the vertices of K is induced by a symmetry of the Dynkin diagram of B. Notice that the automorphisms in $Aut_B(K)$ are not necessarily extendable to automorphisms of B, as the following example shows. In particular, $Aut_B(K)$ is possibly a larger group than $Stab_{Aut(B)}(K)$.

Example 1.3.2. Let $\sigma \subset B$ be a panel and let K_σ be the union of the Weyl chambers in B containing σ. It is a convex subcomplex of B and $Aut_B(K_\sigma)$ is the group of permutations of the set of Weyl chambers containing σ. This group is very large if e.g. the set of Weyl chambers containing σ is uncountable.

Although the automorphisms in $Aut_B(K)$ must not be extendable to automorphisms of B, the group $Aut_B(K)$ depends on the ambient building B in the sense illustrated by the following example.

Example 1.3.3. Let B be a building of type F_4 and let $K \subset B$ be a convex subcomplex. We can embed B in a building $\widetilde{B}$ of type E_8, such that the polyhedral structure of B coincides with the structure induced by the polyhedral structure of $\widetilde{B}$. The image of K under this embedding is a convex subcomplex of $\widetilde{B}$. Then $Aut_{\widetilde{B}}(K)$ is the possible smaller subgroup of $Aut_B(K)$ of type preserving automorphisms. (See Sections 2.4 and 4.6.1 for more details.)

The *simplicial convex hull* of a subset $A \subset B$ is the smallest convex subcomplex containing A.

A *subbuilding* is a convex subcomplex K of a building B, such that any two points in K are contained in a singular sphere $s \subset K$ of the same dimension as K. The next result justifies the term *subbuilding*, namely, a subbuilding carries a natural structure as a spherical building induced by B. Its associated Coxeter complex can be described as follows. Let $s \subset K$ be a singular sphere of dimension $dim(K)$ and let $a \subset B$ be an apartment containing s. If $\iota : S \to a$ is a chart, then $(\iota^{-1}(s), W_{\iota^{-1}(s)})$ is a Coxeter complex unique determined up to isomorphism.

Proposition 1.3.4 ([LR09, Proposition 2.3]). *The subbuilding K carries a natural structure as a spherical building modelled on $(\iota^{-1}(s), W_{\iota^{-1}(s)})$.*

Proof. Let $a \subset B$ be an apartment containing s and let $\sigma \subset a$ be a chamber, such that $\tau := \sigma \cap s$ is a top-dimensional face of K. The retraction $\rho_{a,\sigma} : B \to a$ restricts to a retraction $\rho_{s,\tau} : K \to s$ of K in s. By Proposition 1.3.1 it suffices to give K a polyhedral structure such that the map

$$K \xrightarrow{\rho_{s,\tau}} s \to \iota^{-1}(s) \to \iota^{-1}(s)/W_{\iota^{-1}(s)} = \triangle_{mod}^{(\iota^{-1}(s), W_{\iota^{-1}(s)})} \qquad (*)$$

restricts to an isometry in each top-dimensional face of this polyhedral structure.

Let $s' \subset K$ be a singular sphere containing τ. We can pull back the polyhedral structure of the Coxeter complex $(\iota^{-1}(s), W_{\iota^{-1}(s)})$ to s' via the map $\iota^{-1} \circ \rho_{s,\tau}|_{s'}$. We call this structure the Coxeter polyhedral structure on s'. With this structure it is automatic that the restriction of the map $(*)$ to s' restricts to an isometry in each top-dimensional face of s'. Thus it remains to show that the Coxeter polyhedral structures on all singular spheres in K containing τ match and yield a polyhedral structure on K.

Consider the polyhedral structure of K induced by B (for short, we say w.r.t. B). Let ϕ be a codimension one face of K w.r.t. B. We say that K *branches* along ϕ, if K contains at least three distinct top-dimensional faces (w.r.t. B) τ_1, τ_2 and τ_3 adjacent to ϕ. By the convexity of K and because the τ_i are top-dimensional in K we conclude that the unions $\tau_i \cup \tau_j$ are convex and contained in apartments. Let $\iota_{ij} : S \to a_{ij}$ be charts of apartments a_{ij} containing $\tau_i \cup \tau_j$ for $i \neq j$. We may choose these charts, so that $\iota_{12}^{-1}(\tau_1) = \iota_{13}^{-1}(\tau_1)$. This implies that $\iota_{12}^{-1}(\tau_2) = \iota_{13}^{-1}(\tau_3)$ and in particular, $\theta_B(\tau_2) = \theta_B(\tau_3)$.

Analogously, $\theta_B(\tau_1) = \theta_B(\tau_2)$. It follows that the τ_i have the same type, i.e. the same image under θ_B.

Let $s' \subset K$ be a singular sphere containing τ. The discussion above implies that if K branches along a codimension one face (w.r.t. B) $\phi \subset s'$, then ϕ is contained in a wall of s' with respect to the Coxeter polyhedral structure. This implies that the intersection of two singular spheres $s_1, s_2 \subset K$ containing τ intersect in a top-dimensional convex subcomplex with respect to the Coxeter polyhedral structure, because any two top-dimensional faces $\tau_i \subset s_i$ (again w.r.t. the Coxeter structure) either have disjoint interiors or coincide since K cannot branch in the interior of τ_i. It follows that the Coxeter polyhedral structures on all singular spheres in K containing τ match and give K the desired polyhedral structure. $\square$

Chapter 2

Spherical Coxeter complexes

This section contains some geometric properties of spherical Coxeter complexes.

In our arguments later, we will need some information on singular spheres of codimension ≤ 2 in the different Coxeter complexes.

If the Coxeter complex (S, W) is irreducible and its Dynkin diagram has no weights on its edges, i.e. if it is of type A_n, D_n, E_6, E_7 or E_8, then it is easy to see, that the Weyl group acts transitively on the set of roots ([GB71, Proposition 5.4.2]). In particular all walls (singular spheres of codimension 1) are equivalent modulo the action of W. If there is more than one orbit of roots, then we define the *type* of a wall as the type of the center of the corresponding root. Note that this definition is independent of which of both roots we take.

A singular sphere of codimension 2 is the intersection of two different walls. We define the *type* of a sphere of codimension 2 as the type of the circle spanned by the centers of the corresponding roots.

We gather in the next sections some of the geometric properties of the different Coxeter complexes. This information can be deduced from the data in the Appendix A.

2.1 The Coxeter complex of type A_n

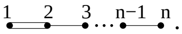
For $n \geq 2$ let (S, W_{A_n}) be the spherical Coxeter complex of type A_n with Dynkin diagram $\overset{1}{\bullet}\!-\!\overset{2}{\bullet}\cdots\overset{n-1}{\bullet}\!-\!\overset{n}{\bullet}$. It has dimension $n-1$.

The Dynkin diagram has only one symmetry, it exchanges the vertices $i \leftrightarrow (n-i)$ for $i = 1, ..., [\frac{n}{2}]$. This symmetry corresponds to the canonical involution of the Weyl chamber $\triangle_{mod}^{A_n}$. In particular, the antipodes of i-vertices are $(n-i)$-vertices for $i = 1, ..., [\frac{n}{2}]$.

The centers of the roots are the midpoints of edges of type $1n$.

Let $x \in S$ be a 1-vertex and $\widehat{x}$ the n-vertex antipodal to x. Any other vertex $y \neq x, \widehat{x}$ in S is adjacent either to x or $\widehat{x}$.

2.2 The Coxeter complex of type B_n

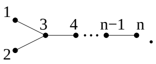
For $n \geq 2$ let (S, W_{B_n}) be the spherical Coxeter complex of type B_n with Dynkin diagram 

. It has dimension $n-1$.

The Dynkin diagram of type B_n for $n \geq 3$ has no symmetries, therefore all automorphisms of (S, W_{B_n}) are *type preserving*. If $n = 2$, the Dynkin diagram has one symmetry, it exchanges the vertices $1 \leftrightarrow 2$. The canonical involution of the Weyl chamber $\triangle_{mod}^{B_n}$ is trivial.

There are two orbits of roots under the action of the Weyl group. Their centers are vertices of type n or $n-1$ respectively.

2.3 The Coxeter complex of type D_n

For $n \geq 4$ let (S, W_{D_n}) be the spherical Coxeter complex of type D_n with Dynkin diagram 

. It has dimension $n-1$.

The $(n-1)$-vertices are the vertices of *root type*. All hemispheres bounded by walls are centered at a $(n-1)$-vertex.

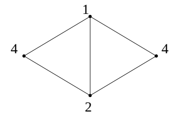
For $n \geq 5$ the Dynkin diagram has one symmetry: it exchanges the vertices $1 \leftrightarrow 2$ and fixes the others. This symmetry is induced by the canonical involution of the Weyl chamber $\triangle_{mod}^{D_n}$ if n is odd. If n is even, then the canonical involution is trivial. For $n = 4$ the Dynkin diagram has six symmetries, they permute the vertices 1, 2, 4 and fix the vertex 3.

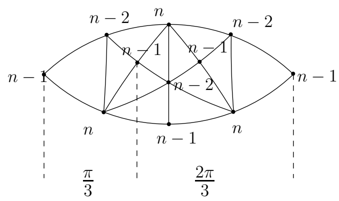
We describe now the possible lengths and types (modulo the action of the Weyl group) of segments between vertices. We list only the ones that we will need later.

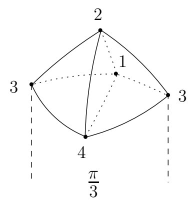
- Distances between two n-vertices x and x':

Distance	Simplicial convex hull of segments xx'
$0, \pi$	
$\frac{\pi}{2}$	singular segment of type $n(n-1)n$

- Distances between two $(n-1)$-vertices x and x':

<table>
<tr><th>Distance</th><th>Simplicial convex hull of segments xx'</th></tr>
<tr><td>$0, \pi$</td><td></td></tr>
<tr><td>$\frac{\pi}{2}$</td><td>singular segment of type $(n-1)n(n-1)$ for $n \geq 4$/
singular segment of type $(n-1)(n-3)(n-1)$, if $n \geq 6$;


, if $n = 5$.
singular segment of type 313 or 323, if $n = 4$.</td></tr>
<tr><td>$\frac{\pi}{3}$ $(\frac{2\pi}{3})$</td><td>

, if $n \geq 5$;
if n=4, the simplicial convex hull of a segment xx' of length $\frac{\pi}{3}$ is 3-dimensional:


A segment xx' of length $\frac{2\pi}{3}$ consists of two segments of length $\frac{\pi}{3}$ as above.</td></tr>
</table>

- Distances between two 1- (2)-vertices x and x':

Distance	Simplicial convex hull of segments xx'
$\arccos(\frac{n-4k}{n})$ for $k = 0, 1, \ldots, [\frac{n}{2}]$	singular segment of type $1(2k+1)1$, $(2(2k+1)2$ resp.)

- Distances between a 1- (2)-vertex x and a n-vertex y:

Distance	Simplicial convex hull of segments xy
$\arccos(\frac{1}{\sqrt{n}})$	singular segment of type $1n$, ($2n$ resp.)
$\arccos(-\frac{1}{\sqrt{n}})$	singular segment of type $12n$, ($21n$ resp.)

The following properties of singular spheres in D_n can be easily seen in the vector space realization of the Coxeter complex presented in Appendix A.

A wall in S contains a singular sphere of codimension 1 spanned by $n-2$ pairwise orthogonal n-vertices.

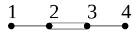
The convex hull of $n-1$ pairwise orthogonal n-vertices and their antipodes is a $(n-2)$-sphere, but it is not a wall, in particular, it is not a subcomplex. Its simplicial convex hull is S.

If $n \geq 5$ ($n = 4$) there are three (four) types of singular spheres of codimension 2. They correspond to the two (three) types of segments connecting two $(n-1)$-vertices at distance $\frac{\pi}{2}$ and the unique type of segments connecting two $(n-1)$-vertices at distance $\frac{\pi}{3}$. We say that a sphere of the last type is a $(n-3)$-sphere *of type* $\frac{\pi}{3}$.

A singular sphere of codimension 2 always contains a singular $(n-5)$-sphere spanned by $n-4$ pairwise orthogonal n-vertices.

Let h be a singular hemisphere of codimension 1 bounded by a singular $(n-3)$-sphere s. It is the intersection of a wall and a root bounded by a different wall. If $n \geq 6$ and s is of type $(n-1)n(n-1)$ (or $(n-1)(n-3)(n-1)$), then h is centered at a $(n-1)$-vertex x. The link $\Sigma_x h$ in the Coxeter complex $\Sigma_x S$ of type $D_{n-2} \circ A_1$ is a wall of type n (or $(n-3)$). If $n \geq 5$ and s is of type $\frac{\pi}{3}$, then h is centered at a point contained in a singular segment of type $n(n-2)$, it is the midpoint of two $(n-1)$-vertices at distance $\frac{\pi}{3}$.

2.4 The Coxeter complex of type F_4

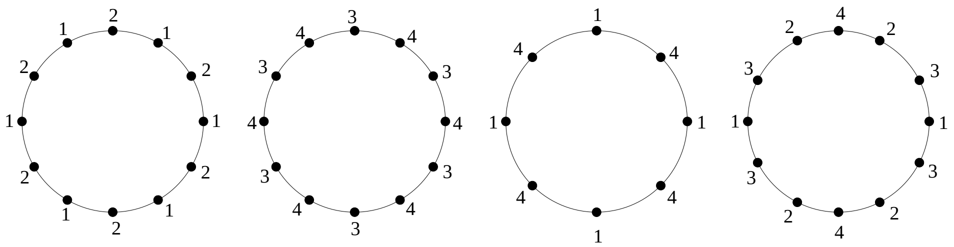
Let (S, W_{F_4}) be the spherical Coxeter complex of type F_4 with Dynkin diagram 

. It has dimension 3.

The Dynkin diagram has only one symmetry, it exchanges the vertices $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$. The canonical involution of the Weyl chamber $\triangle_{mod}^{F_4}$ is trivial, in particular, the antipodes of i-vertices are i-vertices.

There are two orbits of roots under the action of the Weyl group. Their centers are vertices of type 1 or 4.

These are the one dimensional singular spheres in (S, W_{F_4}):



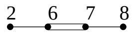
We describe now the possible lengths and types (modulo the action of the Weyl group) of segments between vertices. We list only the ones that we will need later.

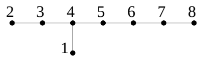
- Distances between two 1- (4-)vertices x and x':

Distance	Simplicial convex hull of segments xx'
$0, \pi$	
$\frac{\pi}{3}$	singular segment of type 121 (434)
$\frac{\pi}{2}$	singular segment of type 141 (414)
$\frac{2\pi}{3}$	singular segment of type 12121 (43434)

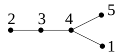
- Distances between a 1-vertex x and a 4-vertex y:

Distance	Simplicial convex hull of segments xy
$\frac{\pi}{4}$	singular segment of type 14
$\frac{\pi}{2}$	singular segment of type 1324
$\frac{3\pi}{4}$	singular segment of type 1414

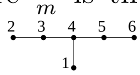
Consider now the following labelling for the Dynkin diagram of type F_4: 

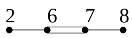
. With this labelling the Coxeter complex of type F_4 can be considered as a Coxeter subcomplex of the Coxeter complex (S, W_{E_8}) of type E_8 with Dynkin diagram 

. It is a singular sphere S' spanned by a simplex σ of type 2678.

Let us verify first that S' is indeed tesselated by simplices of type 2678. Let $i \in \{2, 6, 7, 8\}$ and let τ_i be the face of σ opposite to the vertex of type i. Let $\sigma_i \neq \sigma$ be the simplex in S' sharing the face τ_i. We just have to check that the vertex of σ_i opposite to the face τ_i has type i for $i = 2, 6, 7, 8$. This can be seen by considering the Dynkin diagram of the link in (S, W_{E_8}) of the face τ_i. For example, for $\Sigma_{\tau_2} S$, it is 

, and the antipodes of 2-vertices in $\Sigma_{\tau_2} S$ are 2-vertices.

Finally, one has to check that the geometry of S' correspond to the geometry of F_4. Let λ_{ij} be the edge in σ opposite to the edge in σ of type ij for $i \neq j \in \{2, 6, 7, 8\}$. The 1-sphere in $\Sigma_{\lambda_{ij}} S$ spanned by the edge of type ij has geometry $I_2(m)$, where $\frac{\pi}{m}$ is the angle between the faces τ_i and τ_j. For example, the 1-sphere in $\Sigma_{\lambda_{26}} S$ (of type 

) spanned by an edge of type 26 has type 2626262. Thus, the angle between the faces τ_2 and τ_6 is $\frac{\pi}{3}$ and the vertices of the Dynkin diagram of S' corresponding to vertices of type 2 and 6 are joined by a simple edge. By doing the same argument with the other edges of σ, it is easy to verify that S' has F_4-geometry: 

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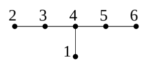
2.5 The Coxeter complex of type E_6

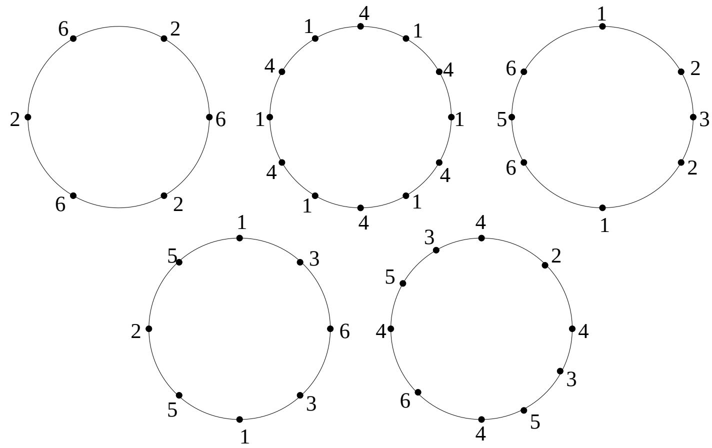
Let (S, W_{E_6}) be the spherical Coxeter complex of type E_6 with Dynkin diagram 

. It has dimension 5.

The 1-vertices are the vertices of *root type*. All hemispheres bounded by walls are centered at a 1-vertex.

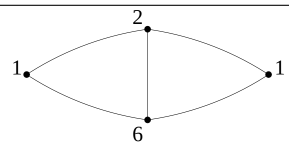
The Dynkin diagram has one symmetry, namely, the one that exchanges the vertices $2 \leftrightarrow 6$, $3 \leftrightarrow 5$ and fixes the 1- and 4-vertices. It corresponds to the canonical involution of the Weyl chamber $\triangle_{mod}^{E_6}$. Therefore, the properties of i- and $(8-i)$-vertices for $i = 2, 3, 5, 6$, are *dual* to each other.

These are the one dimensional singular spheres in (S, W_{E_6}):



We describe now the possible lengths and types (modulo the action of the Weyl group) of segments between vertices. We list only the ones that we will need later.

- Distances between two 1-vertices x and x':

Distance	Simplicial convex hull of segments xx'
$0, \pi$	
$\frac{\pi}{3}$	singular segment of type 141
$\frac{\pi}{2}$	
$\frac{2\pi}{3}$	singular segment of type 14141

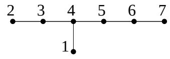
- Distances between two 2- (6)-vertices x and x':

Distance	Simplicial convex hull of segments xx'
0	
$\arccos(\frac{1}{4})$	singular segment of type 232 (656)
$\frac{2\pi}{3}$	singular segment of type 262 (626)

- Distances between a 2-vertex x and a 6-vertex y:

Distance	Simplicial convex hull of segments xx'
π	
$\arccos(-\frac{1}{4})$	singular segment of type 216
$\frac{\pi}{3}$	singular segment of type 26

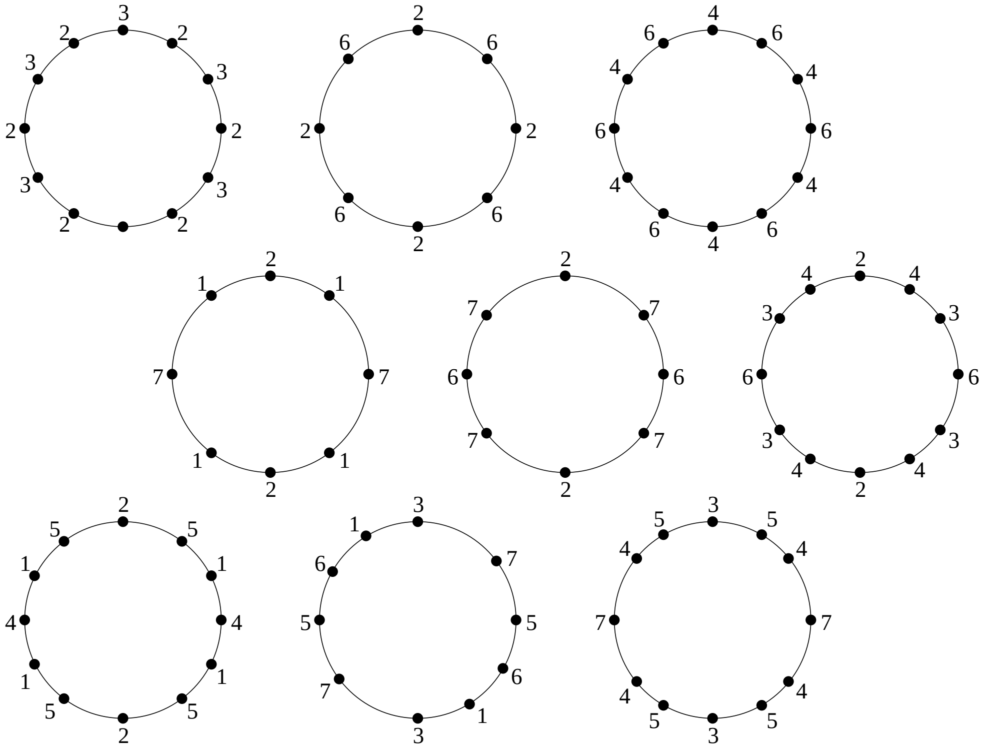
2.6 The Coxeter complex of type E_7

Let (S, W_{E_7}) be the spherical Coxeter complex of type E_7 with Dynkin diagram 

. It has dimension 6.

The Dynkin diagram for E_7 has no symmetries, therefore all automorphisms of (S, W_{E_7}) are *type preserving*.

These are the one dimensional singular spheres in (S, W_{E_7}):



The 2-vertices are the vertices of *root type*. All hemispheres bounded by walls are centered at a 2-vertex.

We describe now the possible lengths and types (modulo the action of the Weyl group) of segments between vertices. We list only the ones that we will need later.

- Distances between two 2-vertices x and x':

Distance	Simplicial convex hull of segments xx'
$0, \pi$	
$\frac{\pi}{3}$	singular segment of type 232
$\frac{\pi}{2}$	singular segment of type 262
$\frac{2\pi}{3}$	singular segment of type 23232

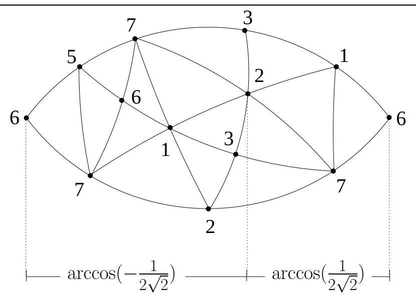
- Distances between two 7-vertices x and x':

Distance	Simplicial convex hull of segments xx'
$0, \pi$	
$\arccos(\frac{1}{3})$	singular segment of type 767
$\arccos(-\frac{1}{3})$	singular segment of type 727

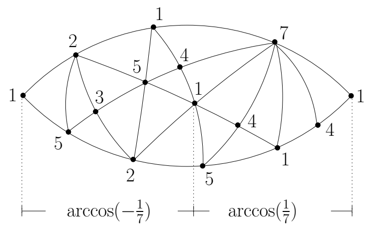
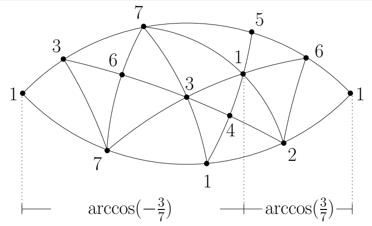
- Distances between a 2-vertex x and a 7-vertex y:

Distance	Simplicial convex hull of segments xy
$\arccos(\frac{1}{\sqrt{3}})$	singular segment of type 27
$\frac{\pi}{2}$	singular segment of type 217
$\arccos(-\frac{1}{\sqrt{3}})$	singular segment of type 2767

- Distances between a 2-vertex x and a 6-vertex y:

<table>
<tr><th>Distance</th><th>Simplicial convex hull of segments xy</th></tr>
<tr><td>$\frac{\pi}{4}$</td><td>singular segment of type 26</td></tr>
<tr><td>$\arccos(\frac{1}{2\sqrt{2}})$,
$\arccos(-\frac{1}{2\sqrt{2}})$</td><td>
</td></tr>
<tr><td>$\frac{\pi}{2}$</td><td>singular segment of type 276 /
singular segment of type 2436</td></tr>
<tr><td>$\frac{3\pi}{4}$</td><td>singular segment of type 2626</td></tr>
</table>

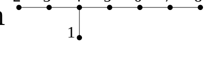
- Distances between two 1-vertices x and x':

Distance	Simplicial convex hull of segments xx'
$0, \pi$	
$\arccos(\frac{1}{7})$ $(\arccos(-\frac{1}{7}))$	singular segment of type 121 (171) / 
$\arccos(\frac{3}{7})$ $(\arccos(-\frac{3}{7}))$	
$\arccos(\frac{5}{7})$ $(\arccos(-\frac{5}{7}))$	singular segment of type 141 (15251)

- Distances between two 1-vertices x and x', such that the simplex containing $\overrightarrow{xx'}$ in its interior has no 1-, 2-, or 7-vertices:

Distance	Simplicial convex hull of segments xx'
$0, \pi$	
$\frac{\pi}{3}$	singular segment of type 646
$\frac{2\pi}{3}$	singular segment of type 64646

2.7 The Coxeter complex of type E_8

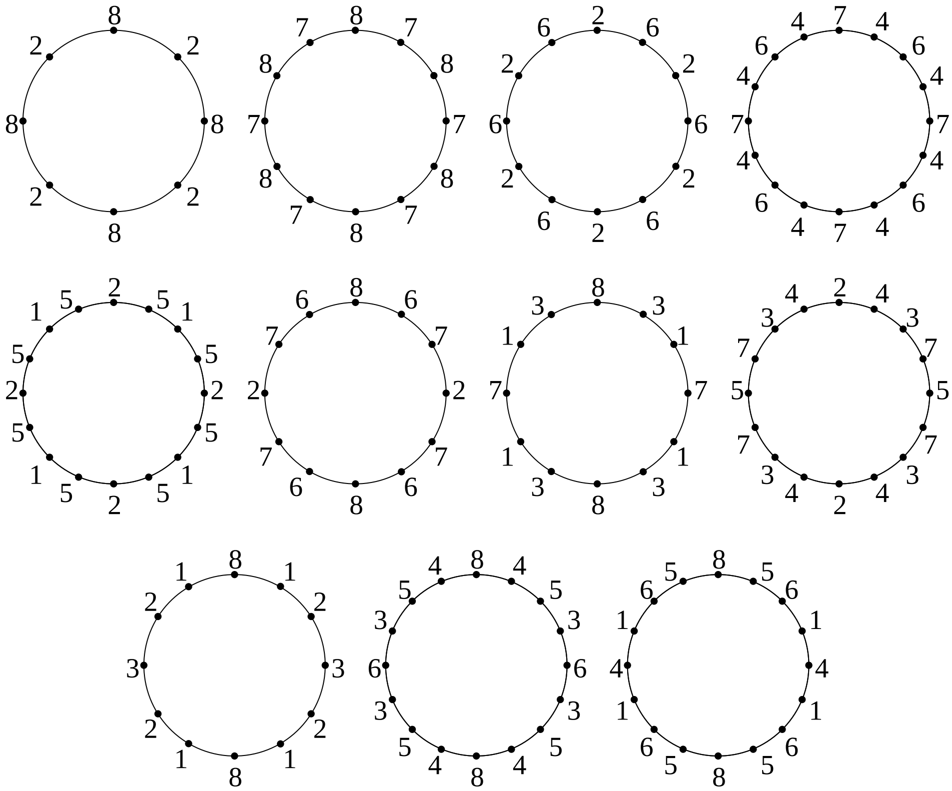
Let (S, W_{E_8}) be the spherical Coxeter complex of type E_8 with Dynkin diagram 

. It has dimension 7.

The Dynkin diagram for E_8 has no symmetries, therefore all automorphisms of (S, W_{E_8}) are *type preserving*.

The 8-vertices are the vertices of *root type*. All hemispheres bounded by walls are centered at an 8-vertex.

These are the one dimensional singular spheres in (S, W_{E_8}):

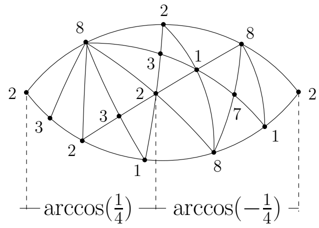


We describe now the possible lengths and types (modulo the action of the Weyl group) of segments between vertices. We list only the ones that we will need later.

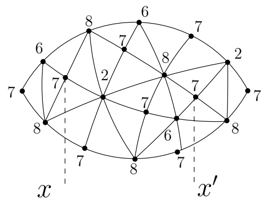
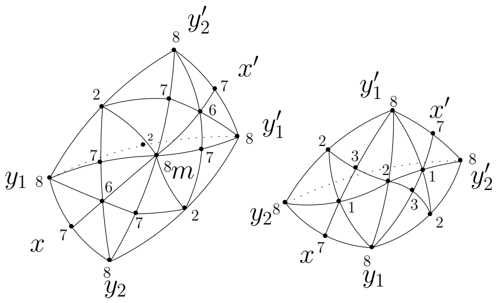
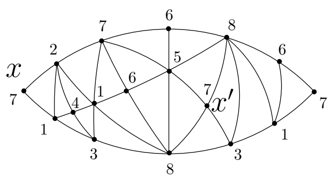
- Distances between two 8-vertices x and x':

Distance	Simplicial convex hull of segments xx'
$0, \pi$	
$\frac{\pi}{3}$	singular segment of type 878
$\frac{\pi}{2}$	singular segment of type 828
$\frac{2\pi}{3}$	singular segment of type 87878

- Distances between two 2-vertices x and x':

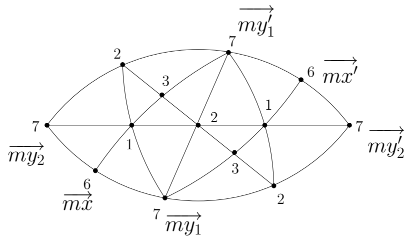
Distance	Simplicial convex hull of segments xx'
$0, \pi$	
$\arccos(\frac{3}{4})$	singular segment of type 232
$\frac{\pi}{3}$	singular segment of type 262
$\arccos(\frac{1}{4})$, $\arccos(-\frac{1}{4})$	
$\pi/2$	singular segment of type 282 / singular segment of type 25152
$\frac{2\pi}{3}$	singular segment of type 26262
$\arccos(-\frac{3}{4})$	singular segment of type 21812

- The possible distances between two 7-vertices x and x' are $\arccos(\frac{k}{6})$ for integer $-6 \leq k \leq 6$. Here we will just need to describe the following segments:

Distance	Simplicial convex hull of segments xx'	Comments
$\arccos(-\frac{1}{6})$	 	There are two types of segments xx'. The simplicial convex hull C of xx' is 2- or 3-dimensional, resp. For the case $dim(C) = 3$, we present two perspectives from the *front* and from *behind* of a larger polyhedron C'. It is the simplicial convex hull of $xx' \cup \{y_2, y_2'\}$. We describe $\Sigma_m C'$ below. (†)
$\arccos(-\frac{1}{3})$	singular segment of type 76867 / 	
$\arccos(-\frac{2}{3})$	singular segment of type 7342437	We present here only the segment xx', such that the simplex containing $\overrightarrow{xx'}$ does not contain 2- or 8-vertices.

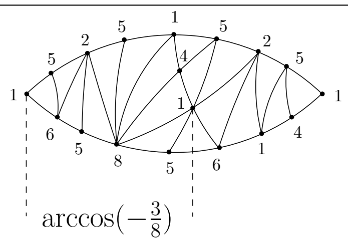
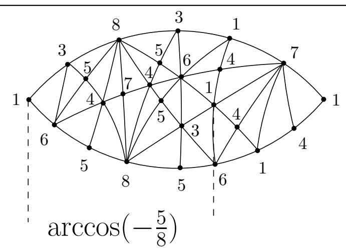
(†)
For a detailed description of the 3-dimensional spherical polyhedra C and C' we refer to Appendix A.7, p.93.

$\Sigma_m C'$:

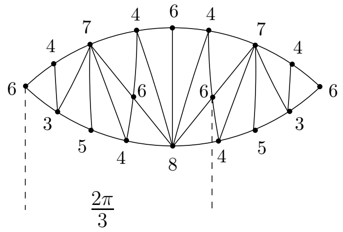
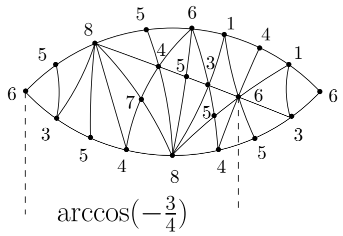


The possible lengths of segments xx', such that $\pi > d(x,x') > \frac{\pi}{2}$ and the simplex containing the direction $\overrightarrow{xx'}$ in its interior does not contain a 2- or 8-vertex, are only $\arccos(-\frac{1}{3})$ and $\arccos(-\frac{2}{3})$.

- Distances $> \frac{\pi}{2}$ and $< \pi$ between two 1-vertices x and x', such that the simplex containing $\overrightarrow{xx'}$ in its interior has no 2-, 7- or 8-vertex:

Distance	Simplicial convex hull of segments xx'
$\arccos(-\frac{3}{8})$	
$\frac{2\pi}{3}$	singular segment of type 13831
$\arccos(-\frac{5}{8})$	
$\arccos(-\frac{7}{8})$	singular segment of type 1658561

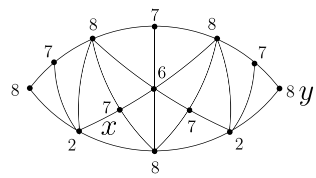
- Distances $> \frac{\pi}{2}$ and $< \pi$ between two 6-vertices x and x', such that the simplex containing $\overrightarrow{xx'}$ in its interior has no 1-, 2-, 7- or 8-vertices:

Distance	Simplicial convex hull of segments xx'
$\arccos(-\frac{1}{4})$	singular segment of type 65856
$\frac{2\pi}{3}$	
$\arccos(-\frac{3}{4})$	

- Distances between a 2-vertex x and an 8-vertex y:

Distance	Simplicial convex hull of segments xy
$\frac{\pi}{4}$	singular segment of type 28
$\arccos(\frac{1}{2\sqrt{2}})$	singular segment of type 218
$\frac{\pi}{2}$	singular segment of type 2768
$\arccos(-\frac{1}{2\sqrt{2}})$	singular segment of type 23218
$\frac{3\pi}{4}$	singular segment of type 2828

- Distances $> \frac{\pi}{2}$ between a 7-vertex x and an 8-vertex y:

Distance	Simplicial convex hull of segments xy
$\frac{5\pi}{6}$	singular segment of type 787878
$\arccos(-\frac{1}{\sqrt{3}})$	singular segment of type 72768
$\arccos(-\frac{1}{2\sqrt{3}})$	

Chapter 3

Convex subcomplexes

In this section we will describe some general properties of convex subcomplexes of buildings, as well as some results for buildings of specific types. These will be needed later in the proof of the Center Conjecture.

Let K be a convex subcomplex of a spherical building B.

Let $v \in \Sigma_x K$. We say that v is *d-extendable*, if there is a segment $xy \subset K$ of length d and so that $v = \overrightarrow{xy}$. We also say that v is *extendable to a segment xy*.

We say that a point $x \in K$ is *interior in K*, if the link $\Sigma_x K$ is a subbuilding of $\Sigma_x B$.

Lemma 3.0.1. *Let $x_1x_2 \subset K$ be a segment. Suppose z is a point in the interior of the simplicial convex hull of x_1x_2, which has an antipode $\hat{z} \in K$. Then x_i has also an antipode in K.*

Proof. Let C be the simplicial convex hull of x_1x_2. Notice that C is contained in an apartment and $\Sigma_z C$ is a sphere. Let $\gamma_i \subset K$ for $i = 1, 2$ be the geodesic connecting z and $\hat{z}$, such that the initial direction of γ_i at z is the antipode in $\Sigma_z C$ of $\overrightarrow{zx_i}$. Then $x_i z \cup \gamma_i$ is a geodesic of length $> \pi$. It is clear that γ_i contains an antipode of x_i. □

The following results give us conditions, under which K satisfies the conclusions of the Center Conjecture 1.

The next Lemma puts together the results [LR09, Prop. 2.4, Lemma 2.5]. Compare also [Se05, Thm. 2.2] and [KL98, Prop. 3.10.3].

Lemma 3.0.2. *The following assertions are equivalent:*

(i) K is a subbuilding of B,

(ii) every vertex *of K has an antipode in K,*

(iii) K contains a sphere of dimension equal to the dimension of K.

Proof. The implication $(i) \Rightarrow (ii)$ is clear.

$(ii) \Rightarrow (iii)$. If $dim(K) = 0$, then K is a set of pairwise antipodal vertices and it contains a 0-dimensional sphere. Suppose that the implication is true for subcomplexes of dimension k and let K be a convex subcomplex of dimension $k+1$. Let $x \in K$ be a vertex and let ξ be a vertex of $\Sigma_x K$. This implies that there is a vertex $y \in K$ adjacent to x, such that $\xi = \overrightarrow{xy}$. Let $\widehat{y} \in K$ be an antipode of y. It follows that $\overrightarrow{x\widehat{y}} \in \Sigma_x K$ is an antipode of ξ. Hence all vertices in $\Sigma_x K$ have antipodes in $\Sigma_x K$. Since $dim(\Sigma_x K) = k$, it follows by induction that $\Sigma_x K$ contains a sphere s of dimension k. Let $\widehat{x} \in K$ be an antipode of x. Then s is the link at x of a $(k+1)$-sphere $S \subset K$ through x and $\widehat{x}$.

$(iii) \Rightarrow (i)$. Let $S \subset K$ be a top-dimensional sphere. First we proof the following assertion: *Any point $x \in K$ has an antipode in S.* If $dim(K) = 0$ the assertion is clear. Suppose that the assertion is true for subcomplexes of dimension k and let K be a convex subcomplex of dimension $k+1$. Let $y \in S$ be any point. If y is antipodal to x, we are done. Otherwise, consider the segment yx. By induction, the direction $\overrightarrow{yx}$ has an antipode in the sphere $\Sigma_y S$. So we can extend the segment yx in S to a geodesic of length π, and we have found an antipode of x in S. Notice that the convex hull of a small neighborhood in S of an antipode of x in S and x is a top-dimensional sphere through x. Let now $x, y \in K$ be two arbitrary points. We know that there is a top-dimensional sphere $S_x \subset K$ containing x. The same argument as above shows that there is a geodesic γ of length π connecting y and an antipode $\widehat{y} \in S_x$ of y, and γ contains x. The convex hull of a small neighborhood of $\widehat{y}$ in S_x and y is a top-dimensional sphere in K containing γ and in particular it contains x and y. Hence K is a subbuilding. ☐

The following result was stated in [LR09, Cor. 2.10] for convex subcomplexes, but the proof works also for closed convex subsets. In [BL05] a more general result is shown, namely, for an arbitrary CAT(1) space C of finite dimension and the action $Isom(C) \curvearrowright C$. They also show, that under these hypothesis the set of circumcenters of C is nonempty.

Lemma 3.0.3. *Let $C \subset B$ be a closed convex subset. Suppose that $rad_C(C) \leq \frac{\pi}{2}$ and the set of circumcenters of C is nonempty, then the action $Stab_{Aut(B)}(C) \curvearrowright C$ has a fixed point.*

Proof. Let $Z \subset C$ be the set of circumcenters of C. It clearly has diameter $\leq \frac{\pi}{2}$. Let $z \in Z$ and let $A \subset Z$ be the $Stab_{Aut(B)}(C)$-orbit of z. It also has diameter $\leq \frac{\pi}{2}$. We need the following result.

Sublemma 3.0.4. *Let $Y \subset B$ be a subset containing points of only finitely many different types and suppose that $diam(Y) \leq \frac{\pi}{2}$. Then $rad_B(Y) < \frac{\pi}{2}$. In particular, $CH(Y)$ has a unique circumcenter.*

Proof. We use induction on the dimension of the building B. For $dim(B) = 0$ the assertion is clear. Suppose now that B has dimension $d > 0$. Let $y \in Y$. Notice that $d(y, y')$ takes only finitely many values for all $y' \in Y$ because Y contains points of finitely many different types. It follows that if $d(y, y') < \frac{\pi}{2}$ for all $y' \in Y$ then $rad(y, Y) < \frac{\pi}{2}$, so we are done.

Otherwise the set $Y' \subset \Sigma_y B$ of directions $\overrightarrow{yy'}$, where $y' \in Y$ has distance $\frac{\pi}{2}$ to y is nonempty. Observe that Y' contains points of only finitely many different types, and that $diam(Y) \leq \frac{\pi}{2}$ implies $diam(Y') \leq \frac{\pi}{2}$ by triangle comparison. It follows by induction that there is a direction $\xi \in \Sigma_y B$, such that $rad(\xi, Y') < \frac{\pi}{2}$. Again because $d(y, y')$ takes only finitely many values for all $y' \in Y$, we can choose an $\epsilon > 0$ small enough, so that for the point x in B at distance ϵ of y and $\overrightarrow{yx} = \xi$, it holds $rad(x, Y) < \frac{\pi}{2}$. □

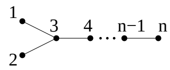
End of proof of Lemma 3.0.3. By the sublemma it follows that A has radius $< \frac{\pi}{2}$ and $Stab_{Aut(B)}(C)$ fixes the unique circumcenter of $CH(A) \subset C$. □

Lemma 3.0.5 ([LR09, Cor. 2.12]). *If K contains a singular sphere of dimension $dim(K) - 1$, then K is a subbuilding or $Stab_{Aut(B)}(K) \curvearrowright K$ has a fixed point.*

Proof. Let σ be a top-dimensional face of the singular sphere s of dimension $dim(K) - 1$ in K and let τ be a top-dimensional face of K containing σ. The convex hull of τ and s is a top-dimensional hemisphere $h \subset K$. Let $x \in h$ be the center of this hemisphere. If $rad(x, K) \leq \frac{\pi}{2}$, then by Lemma 3.0.3, $Stab_{Aut(B)}(K)$ fixes a point in K. Otherwise, there is a $y \in K$ with $d(x, y) > \frac{\pi}{2}$. By the same argument as in Lemma 3.0.2 ($(iii) \Rightarrow (i)$), we find an antipode $\widehat{y}$ of y in the interior of h. The convex hull of a small neighborhood of $\widehat{y}$ in h and y is a top-dimensional sphere in K, thus, K is a subbuilding by Lemma 3.0.2. □

Remark 3.0.6. The Lemmata 3.0.3 and 3.0.5 remain true if we consider the action $Aut_B(K) \curvearrowright K$ instead of $Stab_{Aut(B)}(K) \curvearrowright K$ (actually for the action $Isom(K) \curvearrowright K$ by [BL05]). The proofs are exactly the same.

3.1 Convex subcomplexes of buildings of type D_n

In this section let $L \subset B$ be a convex subcomplex of a building of type D_n for $n \geq 4$. We use the following labelling of the Dynkin diagram 

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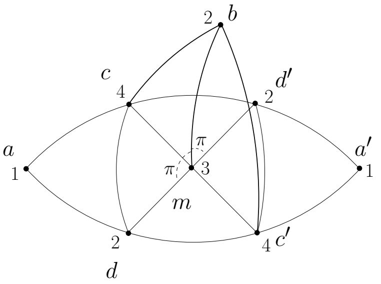
Lemma 3.1.1. *Let $n = 4$, i.e B is of type D_4 and suppose that L contains a pair of antipodal i-vertices and a pair of antipodal j-vertices for $i \neq j$ and $i, j \in \{1, 2, 4\}$. Then it contains a singular circle of type* 1241241.

Proof. By the symmetry of the Dynkin diagram of type D_4, we may assume w.l.o.g. that $i = 1$ and $j = 2$. Let $a, a' \in L$ be the antipodal 1-vertices and let $b, b' \in L$ be the antipodal 2-vertices. If b lies on a geodesic γ connecting a and a', then γ is of type 1421. The convex hull of b' and a small neighborhood of b in γ is the desired circle.



Let us suppose then, that $d(a,b)+d(b,a')>\pi$. The segments ba and ba' are of type 241. Let c, c' be the 4-vertices on the segments ba and ba', respectively. Let d, d' be the 2-vertices on the segments ac' and $a'c$, respectively. Since c, c' are adjacent to b, it follows that the segment cc' is of type 434. Let m be the 3-vertex $m(c,c')$, then the segment mb' is of type 3232. This implies that $\overrightarrow{mb'}$ must be antipodal to $\overrightarrow{md}$ or $\overrightarrow{md'}$. In particular b' is antipodal to d or d'. Either way, we find the desired circle, as above. □

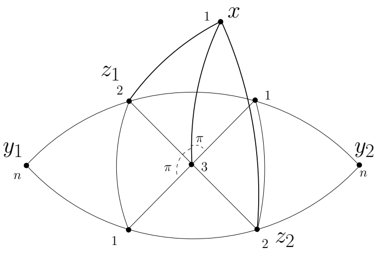
Remark 3.1.2. The proof of Lemma 3.1.1 shows that we can choose the circle in L to contain the two antipodal i-vertices or the two antipodal j-vertices.

Lemma 3.1.3. *If L contains a singular $(n-2)$-sphere S (i.e. S is a wall) and $x\in L$ is a 1-, 2- or n-vertex without antipodes in S, then $\Sigma_x L$ contains an apartment. In particular, x is an interior vertex in L.*

Proof. Let first x be an n-vertex. The sphere S contains $n-2$ pairwise orthogonal n-vertices and their antipodes. They span a singular $(n-3)$-sphere $S'\subset S$. Since x has no antipodes in S, then it must have distance $\frac{\pi}{2}$ to all these n-vertices, and $h:=CH(S',x)$ is a $(n-2)$-dimensional hemisphere centered at x. Put $D_3:=A_3$. The link $\Sigma_x B$ has type D_{n-1}. $\Sigma_x h$ is a $(n-3)$-sphere spanned by $n-2$ pairwise orthogonal $(n-1)$-vertices. This $(n-3)$-sphere is not a subcomplex, its simplicial convex hull is an apartment contained in $\Sigma_x L$.

We may now assume w.l.o.g. that x is a 1-vertex. We prove the assertion by induction on n. Let B be of type D_3 with Dynkin diagram 1—3—2. In this case the 1-dimensional sphere S contained in $L\subset B$ is a circle of type 1312321. Since the 1-vertex x has no antipodes in S, it must be adjacent to the 2-vertices in S and therefore it is also adjacent to the 3-vertex between them. It follows that the convex hull $CH(S,x)$ is a 2-dimensional hemisphere with x in its interior. $\Sigma_x CH(S,x)$ is an apartment in $\Sigma_x L$.

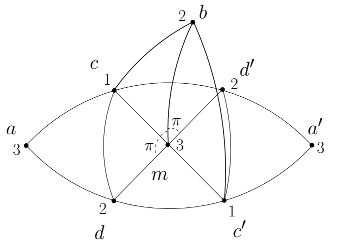
Let now B be of type D_n for $n\geq 4$. Let $y_1,y_2\in S$ be two antipodal n-vertices. If x lies on a geodesic of length π connecting y_1 and y_2, then the geodesic y_1xy_2 is of type $n21n$. The link $\Sigma_{y_1}L$ is of type D_{n-1}. By induction it follows that $\Sigma_{\overrightarrow{y_1x}}\Sigma_{y_1}L$ contains an apartment, and therefore, $\Sigma_x L$ contains also an apartment.



On the other hand, if $d(x,y_1)+d(x,y_2)>\pi$, then the segments xy_i are of type $12n$. Let z_i be the 2-vertex on the segment xy_i. Since z_i is adjacent to y_i we deduce that $z_1\neq z_2$. Since the link $\Sigma_x L$ has type A_{n-1}, it follows that the segment $\overrightarrow{xz_1}\overrightarrow{xz_2}\subset\Sigma_x L$ is of type 232. Again by the induction hypothesis, $\Sigma_{\overrightarrow{y_iz_i}}\Sigma_{y_i}L$ contains an apartment, which in turn implies that $\Sigma_{\overrightarrow{xz_i}}\Sigma_x L$ contains an apartment. In particular the 2-vertices $\overrightarrow{xz_i}$ are interior vertices in $\Sigma_x L$. Thus, we can extend the segment $\overrightarrow{xz_1}\overrightarrow{xz_2}$ to a geodesic in $\Sigma_x L$ of length π and type $232n$. The convex hull of a small neighborhood in $\Sigma_x L$ of the interior vertex $\overrightarrow{xz_1}$ and an antipode contains the desired apartment in $\Sigma_x L$. $\square$

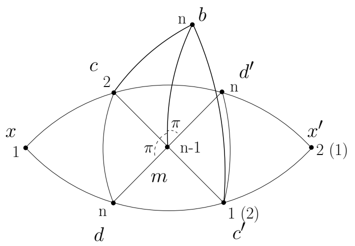
Lemma 3.1.4. *Let $n \geq k \geq 3$. Suppose that L contains a singular $(n-k)$-sphere S spanned by $n-k+1$ pairwise orthogonal n-vertices. Assume also that L contains a 1-vertex x and an antipode of x (of type 1 or 2 depending on the parity of n). Then L contains a singular $(n-k+1)$-sphere spanned by a simplex of type $1k(k+1)\dots(n-1)n$.*

Proof. We prove this again by induction on n. Let B be of type $D_3 := A_3$ with Dynkin diagram $\overset{1}{\bullet}\!-\!\overset{3}{\bullet}\!-\!\overset{2}{\bullet}$, that is $n = k = 3$.



The hypothesis in this case is that L contains a pair of antipodal 3-vertices a, a' and a pair of antipodal 1- and 2-vertices b', b, respectively. If b lies on a geodesic connecting a and a', then we find a circle of type 2321312 (compare with the proof of Lemma 3.1.1). Otherwise, $d(a,b) + d(b,a') > \pi$. The segments ba and ba' are of type 213. Let c, c' be the 1-vertices on these segments. It is clear that $c \neq c'$ and the segment connecting them must be of type 131. Let $m := m(c,c')$. Since c and c' are adjacent to b, it follows that m is also adjacent to b. Let d, d' be the 2-vertices in the segments of type 321 ac' and $a'c$. By considering the spherical triangles $CH(a,c,c')$ and $CH(a',c,c')$, we see that d and d' are adjacent to m. The segment mb' is of type 321. It follows that b' must be antipodal to d or d' (either $b'md$ or $b'md'$ is a geodesic of length π) and we find again a circle in L spanned by a simplex of type 13.

The argument for the induction step is very similar. Let $n \geq 4$. Let b, b' be a pair of antipodal n-vertices in the $(n-k)$-sphere $S \subset L$ and let x' be an antipode in L of the 1-vertex x. If b lies on a geodesic connecting x and x', then this geodesic is of type $1n21$, $1n12$, $12n2$ or $12n1$ depending on the parity of n and if b is adjacent to x or x'. It follows that $\Sigma_b L$ or $\Sigma_{b'} L$ contains a 1-vertex and an antipode of it.



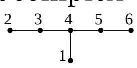
If $d(x,b) + d(b,x') > \pi$, then the segment bx is of type $n21$ and the segment bx' is of type $n12$ or $n21$. Let c, c' be the vertices in the interior of the segments bx, bx' and let d, d' be the n-vertices on the segments $c'x$ and cx'. Since c and d are adjacent to x, then they are adjacent or cxd is a segment. In this last case, c and c' must be antipodal, but this cannot happen, because they are adjacent to b. So c and d are adjacent. This implies that the segment cc' is of type $2(n-1)1$ or $2(n-1)2$. The $(n-1)$-vertex $m := m(d,d') = m(c,c')$ is adjacent to b. It follows that the segment mb' is of type $(n-1)n(n-1)n$. Again we conclude that b' is antipodal to d or d'. This implies that b' lies in a circle in L of type $n21n21n$ or $n21n12n$. In particular $\Sigma_{b'} L$ or $\Sigma_b L$ contains a 1-vertex and an antipode of it. Suppose w.l.o.g. that it holds for $\Sigma_b L$. It follows, that L contains a circle spanned by a simplex of type $1n$. So, if $k = n$, we are done. Suppose then, that $k \leq n-1$.

We have seen that the link $\Sigma_b L$ of type D_{n-1} contains a 1-vertex and an antipode of it. It also contains the singular $(n-1-k)$-sphere $\Sigma_b S$ spanned by $n-k$ pairwise orthogonal $(n-1)$-vertices. By the induction assumption, $\Sigma_b L$ contains a singular $(n-k)$-sphere spanned by a simplex of type $1k(k+1)\dots(n-1)$. Hence, L contains a singular $(n-k+1)$-sphere spanned by a simplex of type $1k(k+1)\dots(n-1)n$. □

Remark 3.1.5. Lemma 3.1.1 is just the special version of Lemma 3.1.4 where $n=k=4$. If $k=3$ in Lemma 3.1.4, then the conclusion is that L contains a wall.

Remark 3.1.6. The proof of Lemma 3.1.4 shows that if $n=k$, we can choose the 1-sphere in L to contain the 1-vertex x (this is true in general, but it is less obvious from the proof).

3.2 Convex subcomplexes of buildings of type E_6

In this section let $L \subset B$ be a convex subcomplex of a building of type E_6. We use the following labelling of the Dynkin diagram 

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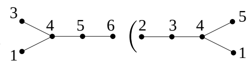
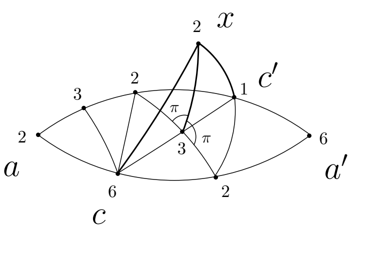
Lemma 3.2.1. *If L contains a singular 4-sphere S (i.e S is a wall) and $x \in L$ is a 2 or 6-vertex without antipodes in S, then $\Sigma_x L$ contains an apartment. In particular, x is an interior vertex in L.*

Proof. By the symmetry of the Dynkin diagram for E_6 it suffices to show it for a 2-vertex $x \in L$. The wall S contains a pair of antipodal 2- and 6-vertices a and a', respectively. The link $\Sigma_a B$ ($\Sigma_{a'} B$) is of type D_5 and Dynkin diagram 

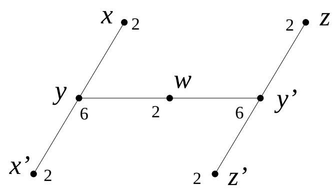
. $\Sigma_a L$ and $\Sigma_{a'} L$ contain a singular 3-sphere $\Sigma_a S$, respectively $\Sigma_{a'} S$. Suppose first that x lies on a geodesic γ connecting a and a'. γ is of type 23216 or 2626. Since x has no antipodes in S, the vertex $\overrightarrow{ax}$ of type 3 or 6 has no antipodes in $\Sigma_a S$. It follows from Lemma 3.1.3, that $\Sigma_{\overrightarrow{ax}} \Sigma_a L$ contains an apartment and this implies in turn, that $\Sigma_{\overrightarrow{xa}} \Sigma_x L$ contains also an apartment. Since $\overrightarrow{xa'} \in \Sigma_x L$ is antipodal to $\overrightarrow{xa}$, this implies that $\Sigma_x L$ contains an apartment.

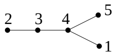


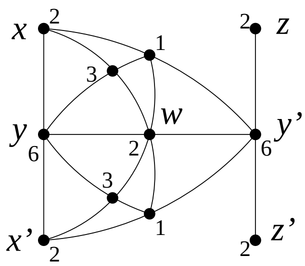
On the other hand, if $d(x,a) + d(x,a') > \pi$, then the segments xa and xa' are of type 262 and 216. Let c be the 6-vertex on xa and let c' be the 1-vertex on xa'. c is adjacent to a and c' is adjacent to a', therefore c and c' cannot be adjacent and since both are adjacent to x, it follows that the segment $\overrightarrow{xc}\overrightarrow{xc'}$ is of type 631. It follows again from Lemma 3.1.3, that $\Sigma_{\overrightarrow{ax}} \Sigma_a L$ and $\Sigma_{\overrightarrow{a'x}} \Sigma_{a'} L$ contain a 3-sphere. This implies that $\Sigma_{\overrightarrow{xc}} \Sigma_x L$ and $\Sigma_{\overrightarrow{xc'}} \Sigma_x L$ contain a 3-sphere, in particular, $\overrightarrow{xc}$ and $\overrightarrow{xc'}$ are interior vertices in $\Sigma_x L$. The segment $\overrightarrow{xc}\overrightarrow{xc'}$ is of type 631 and since $\overrightarrow{xc'}$ is interior, it can be extended in $\Sigma_x L$ to a segment of type 6316. This means that $\overrightarrow{xc}$ has an antipode in $\Sigma_x L$ implying that $\Sigma_x L$ contains a 4-sphere as desired. □

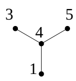
Lemma 3.2.2. *Suppose L contains 2-vertices $x, x', z, z', w \in L$ and 6-vertices $y, y' \in L$, such that xyx' and $zy'z'$ are segments of type 262 and ywy' is a segment of type 626. Assume further that y is not antipodal to z, z' and y' is not antipodal to x, x'. Then $\Sigma_w L$ contains a singular 2-sphere containing $\overrightarrow{wy}$ and $\overrightarrow{wy'}$ and spanned by a simplex of type 156. In particular, $\Sigma_w L$ contains a singular circle of type 656565656 containing $\overrightarrow{wy}$ and $\overrightarrow{wy'}$.*

Proof.

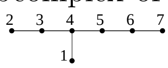


Notice that $\overrightarrow{yy'}$ cannot be antipodal to $\overrightarrow{yx}$ or $\overrightarrow{yx'}$ because y' is not antipodal to x, x' and a segment of type 6262 has length π. Since $\Sigma_y B$ is a building of type D_5 with Dynkin diagram  the distances between 2-vertices are $0, \frac{\pi}{2}, \pi$, it follows that $d(\overrightarrow{yy'}, \overrightarrow{yx}) = d(\overrightarrow{yy'}, \overrightarrow{yx'}) = \frac{\pi}{2}$. Analogously, it holds $d(\overrightarrow{y'y}, \overrightarrow{y'z}) = d(\overrightarrow{y'y}, \overrightarrow{y'z'}) = \frac{\pi}{2}$.

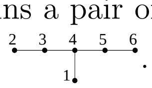


It follows that the convex hull $CH(x, x', y')$ is the union of the spherical triangles $CH(x, y, y')$ and $CH(x', y, y')$. Hence $CH(x, x', y')$ is an isosceles spherical triangle with sides of type 262, 216 and 216. The link $\Sigma_w CH(x', x, y')$ is a singular circle of type 6316136. This implies that the link $\Sigma_{\overrightarrow{wy}} \Sigma_w L$ contains a pair of antipodal 3-vertices and $\Sigma_{\overrightarrow{wy'}} \Sigma_w L$ contains a pair of antipodal 1-vertices. Analogously, considering the spherical triangle $CH(z, z', y)$ we deduce that $\Sigma_{\overrightarrow{wy}} \Sigma_w L$ also contains a pair of antipodal 1-vertices and $\Sigma_{\overrightarrow{wy'}} \Sigma_w L$ also contains a pair of antipodal 3-vertices. Recall that $\Sigma_{\overrightarrow{wy}} \Sigma_w B$ is a building of type D_4 with Dynkin diagram . We may apply Lemma 3.1.1 to conclude that $\Sigma_{\overrightarrow{wy}} \Sigma_w L$ contains a circle of type 1351351. This implies that $\Sigma_w L$ contains a singular sphere spanned by a simplex of type 156. □

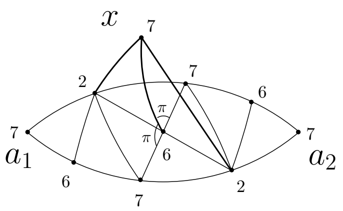
3.3 Convex subcomplexes of buildings of type E_7

In this section let $L \subset B$ be a convex subcomplex of a building of type E_7. We use the following labelling of the Dynkin diagram .

Lemma 3.3.1. *If L contains a singular 5-sphere S (i.e. S is a wall) and $x \in L$ is a 7-vertex without antipodes in S, then $\Sigma_x L$ contains an apartment. In particular, x is an interior vertex in L.*

Proof. The wall S contains a pair of antipodal 7-vertices a_1, a_2. The link $\Sigma_{a_i} B$ is of type E_6 with Dynkin diagram . $\Sigma_{a_i} L$ contains the wall $\Sigma_{a_i} S$.

Suppose w.l.o.g. that $d(x, a_1) = \arccos(-\frac{1}{3})$. Then the segment xa_1 is of type 727. Since x has no antipodes in S it follows that the 2-vertex $\overrightarrow{a_1x}$ has no antipodes in $\Sigma_{a_1}S$. We apply now Lemma 3.2.1 to deduce that $\Sigma_{\overrightarrow{a_1x}}\Sigma_{a_1}L$ contains an apartment. This implies in turn, that $\Sigma_{\overrightarrow{xa_1}}\Sigma_x L$ contains an apartment. Therefore, if we find an antipode in $\Sigma_x L$ of $\overrightarrow{xa_1}$, we are done. This is trivial if x lies on a geodesic connecting a_1 and a_2.



Otherwise also $d(x, a_2) = \arccos(-\frac{1}{3})$. We may argue as above and conclude that $\Sigma_{\overrightarrow{xa_2}}\Sigma_x L$ contains an apartment. In particular $\overrightarrow{xa_2}$ is an interior vertex in $\Sigma_x L$. Notice that the segment connecting $m(x, a_i)$ for $i = 1, 2$ cannot be of type 232, otherwise we find a curve of length $< \pi$ connecting a_1 and a_2. Therefore, the segment $\overrightarrow{xa_1}\overrightarrow{xa_2}$ is of type 262. Since $\overrightarrow{xa_2}$ is interior, we can extend the segment $\overrightarrow{xa_1}\overrightarrow{xa_2}$ to a segment of type 2626 and length π in $\Sigma_x L$. We have found an antipode of $\overrightarrow{xa_1}$. □

Chapter 4

The Center Conjecture

Let B be a spherical building and $K \subset B$ a convex subcomplex. We say that K is a *counterexample* to the Center Conjecture, if K is not a subbuilding and $G := Stab_{Aut(B)}(K)$ has no fixed points in K.

From the Lemmata 3.0.2, 3.0.3 and 3.0.5 we can deduce some general properties of convex subcomplexes $K \subset B$, which are counterexamples to the Center Conjecture:

1. If $x \in K$ and $y \in CH(G \cdot x)$, then there exists $x' \in G \cdot x$, such that $d(y, x') > \frac{\pi}{2}$. This is just Lemma 3.0.3 applied to $CH(G \cdot x)$. In particular, if $x \in K$, then there exists $x' \in G \cdot x$, such that $d(x, x') > \frac{\pi}{2}$.

   Another way to look at this is the following. If P is a property of vertices in K invariant under the action of G, then for every point y in the convex hull of the P-vertices, we can find a P-vertex x with $d(x, y) > \frac{\pi}{2}$.

2. K contains no sphere of dimension $dim(K) - 1$.

3. If K has dimension ≤ 1 and is not a subbuilding, then by Lemma 3.0.2, it contains no circles. It follows that K is a (bounded) tree and it has a unique circumcenter, which is fixed by $Isom(K)$. Hence, a counterexample K has dimension ≥ 2. By the main result in [BL05] mentioned in the introduction, a counterexample has actually dimension ≥ 3, but we do not use this fact in our proof.

Let A be the property of a point in K of not having antipodes in K. Let I be the property of a point in $x \in K$ of being interior, i.e. $\Sigma_x K$ is a subbuilding of $\Sigma_x B$, or equivalently, $\Sigma_x K$ contains a singular sphere of dimension $dim(K) - 1$.

Notice that an interior point in a counterexample K cannot have antipodes in K, that is, $I \Rightarrow A$. Otherwise K would contain a singular sphere of dimension $dim(K)$ and K would be a subbuilding.

4.1 The case of classical types

The Center Conjecture for buildings of classical types (A_n, B_n and D_n) was first proven by Mühlherr and Tits in [MT06] using combinatorial methods and the incidence geometries of the respective buildings. We present in this section a proof from the point of view of CAT(1) spaces using methods of comparison geometry.

4.1.1 The A_n-case

Theorem 4.1.1. *The Center Conjecture 1 holds for spherical buildings of type A_n.*

Proof. Let K be a convex subcomplex of a spherical building B of type A_n for $n \geq 2$ and suppose it is not a subbuilding. By Lemma 3.0.2, it follows that there are vertices in K without antipodes in K. Let $t_1 = \min\{\, i \mid \exists\ iA\text{-vertex in } K\}$ and $t_2 = \max\{\, i \mid \exists\ iA\text{-vertex in } K\}$. Let $x_i \in K$ be a t_iA-vertex for $i = 1, 2$.

Let $t < t_1$ and suppose that there exists a t-vertex $y \in K$ adjacent to x_1. The minimality of t_1, implies that y has an antipode $\widehat{y} \in K$. Notice that $\overrightarrow{x_1 y}$ is a t-vertex and the antipode $\overrightarrow{x_1 \widehat{y}}$ has type $t' < t_1$, because $\Sigma_{x_1} B$ is of type $A_{t_1 - 1} \circ A_{n - t_1}$ and the vertices of the Dynkin diagram of the A_{t_1-1}-factor have labels $1, \dots, t_1 - 1$. It follows that the segment $x_1 \widehat{y} \subset K$ has a t'-vertex z in its interior, and by Lemma 3.0.1 z cannot have antipodes in K, contradicting the minimality of t_1. Hence, x_1 has no vertices of type $t < t_1$ adjacent to it, and analogously, x_2 has no vertices of type $t > t_2$ adjacent to it.

Consider the segment $x_1 x_2$ embedded in the vector space realization of the Coxeter complex of type A_n presented in Appendix A (we use the notation introduced there). We may assume that $x_1 = v_{t_1}$ and $x_1 x_2 \subset \beta_{t_1}$. It follows from the observation above, that $x_1 x_2 \subset \beta_{t_1}(1, \dots, t_1 - 1)$. If $x_2 = (a_1, \dots, a_{n+1}) \in \mathbb{R}^{n+1}$, this implies that $a_1 = \cdots = a_{t_1}$ and $a_{t_1+1} \leq \cdots \leq a_{n+1}$. It follows that x_2 is adjacent to x_1 or

$$x_2 = (\underbrace{t_2, \dots, t_2}_{t_1}, \underbrace{-(n+1-t_2), \dots, -(n+1-t_2)}_{t_2}, \underbrace{t_2, \dots, t_2}_{n+1-t_1-t_2}).$$

Since there are exactly $n + 1 - t_2$ coordinates a_i such that $a_i = t_2$, it follows in particular that if x_1 and x_2 are not adjacent, then $n + 1 - t_2 \geq t_1$. And since x_1 is not antipodal to x_2, we have the strict inequality $n + 1 > t_1 + t_2$.

Consider now the embedding of $x_1 x_2$ such that $x_2 = v_{t_2}$ and $x_2 x_1 \subset \beta_{t_2}$. The observation above implies now, that $x_2 x_1 \subset \beta_{t_2}(t_2 + 1, \dots, n + 1)$. If $x_1 = (b_1, \dots, b_{n+1}) \in \mathbb{R}^{n+1}$, this implies that $b_1 \leq \cdots \leq b_{t_2}$ and $b_{t_2+1} = \cdots = b_{n+1}$. It follows that x_1 is adjacent to x_2 or

$$x_1 = (\underbrace{-(n+1-t_1), \dots, -(n+1-t_1)}_{t_1+t_2-(n+1)}, \underbrace{t_1, \dots, t_1}_{n+1-t_1}, \underbrace{-(n+1-t_1), \dots, -(n+1-t_1)}_{n+1-t_2}).$$

Since there are exactly t_1 coordinates b_i such that $b_i = -(n+1-t_1)$, this implies that x_1 is adjacent to x_2 or $t_1 \geq n+1-t_2$, but this inequality contradicts the inequality above. Hence,

x_1 and x_2 are adjacent and $d(x_1, x_2) < \frac{\pi}{2}$. It follows that $rad(x_i, \{\, t_{3-i}A\text{-vert. in } K \,\}) < \frac{\pi}{2}$ for $i = 1, 2$.

Let $G := Stab_{Aut(B)}(K)$ and $H := Stab_{Aut_0(B)}(K)$, where $Aut_0(B)$ are the type preserving automorphisms of B. If $G = H$, then the convex hull of the t_1A-vertices is a G-invariant subset of K with radius $< \frac{\pi}{2}$. It follows that G fixes a point in K. Otherwise there is an automorphism $\phi \in G - H$. Since the Dynkin diagram for A_n has only one symmetry, it follows that ϕ and H generate G and ϕ exchanges the vertices $i \leftrightarrow (n+1-i)$ for $i = 1, \dots, [\frac{n}{2}]$.

$\phi(x_1)$ is a $(n + 1 - t_1)A$-vertex in K, hence $n + 1 - t_1 \le t_2$ by the maximality of t_2. $\phi(x_2)$ is a $(n + 1 - t_2)A$-vertex in K, therefore $n + 1 - t_2 \ge t_1$ by the minimality of t_1. It follows that $t_1 + t_2 = n + 1$ and therefore $\phi(x_1)$ is a t_2A-vertex.

Notice that $G \cdot x_1 = H \cdot x_1 \cup H \cdot \phi(x_1)$ and $rad(y, H \cdot x_1) < \frac{\pi}{2}$ for all $y \in H \cdot \phi(x_1)$, because y is a t_2A-vertex. Let $c_1 \in CH(H \cdot x_1)$ be the unique circumcenter of the convex hull $CH(H \cdot x_1)$, in particular, H fixes c_1. Notice that $rad(c_1, H \cdot \phi(x_1)) < \frac{\pi}{2}$. It follows that $d(c_1, c_2) < \frac{\pi}{2}$ where $c_2 := \phi(c_1)$ is the circumcenter of $CH(H \cdot \phi(x_1))$. Observe that $\phi(c_2) = \phi^2(c_1) = c_1$ because $\phi^2 \in H$. This implies that ϕ preserves the segment c_1c_2 and H fixes it pointwise. In particular, H and ϕ fix the point $m(c_1, c_2)$. Hence G fixes the point $m(c_1, c_2) \in K$. □

4.1.2 The B_n-case

Theorem 4.1.2. *The Center Conjecture 1 holds for spherical buildings of type B_n.*

Proof. If $n = 2$, then the subcomplex has dimension ≤ 1 and we are done. So let K be a convex subcomplex of a spherical building B of type B_n for $n \ge 3$ and suppose it is not a subbuilding. By Lemma 3.0.2, it follows that there are vertices in K without antipodes in K. Let $t = \max\{\, i \mid \exists\, iA\text{-vertex in } K\}$.

Let $x \in K$ be a tA-vertex. Suppose there is a t'-vertex $y \in K$ adjacent to x for $t' > t$. It follows that y has an antipode $\widehat{y} \in K$. Notice that $\Sigma_x B$ is of type $B_{t-1} \circ A_{n-t}$ and the Dynkin diagram of the A_{n-t}-factor has labels $t + 1, \dots, n$. This implies that the direction $\overrightarrow{x\widehat{y}}$ has type $t'' > t$, in particular the segment $x\widehat{y}$ contains a t''-vertex z in its interior. By Lemma 3.0.1, z must be an A-vertex, contradicting the maximality of t. Hence there are no vertices of type $> t$ in K adjacent to x.

Let x' be another tA-vertex. Consider the segment xx' embedded in the vector space realization of the Coxeter complex of type B_n presented in Appendix A. We may choose the embedding, so that $x = v_t = (0, \dots, 0, \underbrace{1, \dots, 1}_{n+1-t})$ and $xx' \subset \beta_t$ The observation above implies that $xx' \subset \beta_t(t + 1, \dots, n)$. If $x' = (a_1, \dots, a_n)$, this means that $a_t = \cdots = a_n$. If $a_t = 1$, then $x = x'$; if $a_t = 0$, then $d(x, x') = \frac{\pi}{2}$; and if $a_t = -1$, then x and x' are antipodal. Hence, $d(x, x') \le \frac{\pi}{2}$. It follows that the convex hull of the tA-vertices in K is a G-invariant set with $rad \le \frac{\pi}{2}$. Therefore, G fixes a point in K by Lemma 3.0.3. □

4.1.3 The D_n-case

Theorem 4.1.3. *The Center Conjecture 1 holds for spherical buildings of type D_n.*

Proof. Let K be a convex subcomplex of a spherical building B of type D_n for $n \geq 5$. Since D_4 has more symmetries, this case will be treated separately. Suppose K is not a subbuilding. By Lemma 3.0.2, it follows that there are vertices in K without antipodes in K. Let $t = \max\{\, i \mid \exists\, iA\text{-vertex in } K\}$.

Suppose first that $t \geq 3$. Then the set of tA-vertices is a G-invariant subset of K.

Let $x \in K$ be a tA-vertex. Suppose there is a t'-vertex $y \in K$ adjacent to x for $t' > t$. It follows that y has an antipode $\widehat{y} \in K$. Notice that $\Sigma_x B$ splits a factor of type A_{n-t} and its Dynkin diagram has labels $t+1, \dots, n$. This implies that the direction $\overrightarrow{x\widehat{y}}$ has type $t'' > t$, in particular the segment $x\widehat{y}$ contains a t''-vertex z in its interior. By Lemma 3.0.1, z must be an A-vertex, contradicting the maximality of t. Hence there are no vertices of type $> t$ in K adjacent to x.

Let x' be another tA-vertex. Consider the segment xx' embedded in the vector space realization of the Coxeter complex of type D_n presented in Appendix A. Assume that $x = v_t = (0, \dots, 0, \underbrace{1, \dots, 1}_{n+1-t})$ and $xx' \subset \beta_t$ The observation above implies that $xx' \subset \beta_t(t+1, \dots, n)$. If $x' = (a_1, \dots, a_n)$, this means that $a_t = \dots = a_n$. If $a_t = 1$, then $x = x'$; if $a_t = 0$, then $d(x, x') = \frac{\pi}{2}$; and if $a_t = -1$, then x and x' are antipodal. Hence, $d(x, x') \leq \frac{\pi}{2}$. It follows that the convex hull of the tA-vertices in K is a G-invariant set with $rad \leq \frac{\pi}{2}$. Therefore, G fixes a point in K by Lemma 3.0.3.

Suppose now that $t \leq 2$. If $t = 1$, then by the same argument as above, a $1A$-vertex cannot have vertices in K adjacent to it of type > 1. Hence K is 0-dimensional and we are done in this case. Thus, $t = 2$. Let $x \in K$ be a $2A$-vertex. By the same argument, x is adjacent to vertices in K only of type 1 and n. Suppose $dim(K) > 1$, otherwise we are done. This implies that there are vertices y and z in K of type 1 and n, respectively, such that x, y, z are vertices of a simplex σ. There is also a n-vertex $\widehat{z} \in K$ antipodal to z. The convex hull $CH(\sigma, \widehat{z}) \subset K$ contains a 3-vertex adjacent to x. A contradiction.

Let K be a convex subcomplex of a spherical building B of type D_4 and suppose that K is a counterexample to the Center Conjecture. Suppose first, that K contains $3A$-vertices. Recall that the 3-vertices in D_4 are the vertices of root type. The midpoint of a segment connecting two 3-vertices at distance $\frac{\pi}{3}$ lies in the interior of a simplex of type 124 adjacent to both 3-vertices. Since K is a counterexample, we can find $x, x' \in K$ $3A$-vertices at distance $> \frac{\pi}{2}$, hence $d(x, x') = \frac{2\pi}{3}$. The convex hull of the segment xx' is 3-dimensional and the $3A$-vertex $y_1 = m(x, x')$ is an interior vertex. Let $y_2 \in G \cdot y_1$ be another $3I$-vertex at distance $\frac{2\pi}{3}$ to y_1. Since y_i is interior, we can find $z_i \in K$, with $d(z_i, y_i) = \frac{\pi}{6}$, such that $\overrightarrow{y_i z_i}$ is antipodal to $\overrightarrow{y_i y_{3-i}}$ in $\Sigma_{y_i} K$ for $i = 1, 2$. In particular $z_1 y_1 y_2 z_2$ is a geodesic of length π and z_1 and z_2 are antipodal. Notice that z_i lies in the interior of a simplex of type 124. It follows that K contains a 2-sphere, contradicting Lemma 3.0.5. Hence all 3-vertices in

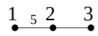
K have antipodes in K. Since K is a counterexample, there is a vertex $w \in K$ without antipodes in K. Suppose w.l.o.g. that w is of type 1. w cannot be adjacent to a 3-vertex in K, in particular, w is the only 1-vertex in K, because two distinct nonantipodal 1-vertices are joined by a segment of type 131. This implies that there are at most three A-vertices in K. Therefore, the convex hull of the A-vertices in K is just a vertex, an edge, a segment of type ijk or a simplex of type ijk for $\{i, j, k\} = \{1, 2, 4\}$. G fixes the unique circumcenter of this set. □

Remark 4.1.4. Our proof actually shows that in the case of classical types K is a subbuilding or the action of the group $Aut_B(K) \curvearrowright K$ fixes a point (see 1.3 for definitions).

4.2 The H_3-case

The Center Conjecture for buildings of type H_3 is a direct consequence of the main result of [BL05]. Nevertheless we give a direct proof as a preparation for the more complicated arguments that are used in the other cases.

Recall that a building of type H_3 is never thick ([Ti77]) and it is isometric to a suspension of a building of type $I_2(m)$ for $m = 3, 5$ or to a building of type $A_1 \circ A_1 \circ A_1$ ([Sch87]). However the H_3-case does not follow directly from the case of buildings of classical type, because a subcomplex of a building of type H_3 does not have to be a subcomplex in its thick structure.

We use following labelling of the Dynkin diagram of type H_3: 

.

The Weyl group of type H_3 can be identified with the symmetry group of the icosahedron. Thus, the polyhedral structure of (S^2, W_{H_3}) correspond to the barycentric subdivision of a spherical icosahedron. The vertices of the icosahedron correspond to the vertices of (S, W_{H_3}) of type 3, the midpoints of the edges of the icosahedron correspond to the vertices of type 2 and the centers of the faces correspond to vertices of type 1. For the vector space realization as in Appendix A we refer to [Co73, p. 53], where one can find vectors representing the vertices of the Coxeter complex.

Theorem 4.2.1. *The Center Conjecture 1 holds for spherical buildings of type H_3.*

Proof. Let K be a convex subcomplex of a building B of type H_3, which is a counterexample to the Center Conjecture. In particular, $dim(K) = 2$ and therefore K contains vertices of all types. First suppose that all 3-vertices in K have antipodes in K. Let $x \in K$ be a 1-vertex and $y \in K$ a 3-vertex adjacent to x. Let $\hat{y} \in K$ be an antipode of y and consider the geodesic γ of length π connecting y and $\hat{y}$ through x. γ is singular of type 3121323. The 3-vertex on the segment $x\hat{y}$ has an antipode in K and by Lemma 3.0.1 we conclude that x also has an antipode in K. Thus all 1-vertices in K have antipodes in K and by a similar argument the same holds for 2-vertices in K. This is a contradiction to the fact that K is not a subbuilding.

So K contains $3A$-vertices. Since K is a counterexample, it contains $3A$-vertices $x, x' \in K$ at distance $> \frac{\pi}{2}$. After a simple examination of the barycentric subdivision of the spherical icosahedron, we can conclude that the segment xx' is singular of type 31213. Let $y \in K$ be the $2A$-vertex $m(x, x')$. By the properties of a counterexample, there is another $3A$-vertex z at distance $> \frac{\pi}{2}$ to y. It follows that the segment yz is singular of type 23123 or 21323. Recall that $\Sigma_y B$ is a building of type $A_1 \circ A_1$. If yz is of type 21323, then $\overrightarrow{yz}$ must be antipodal to at least one of the directions $\overrightarrow{yx}$ and $\overrightarrow{yx'}$. This implies that z is antipodal to x or x', a contradiction. Hence yz is of type 23123. Let w be the 3-vertex on the segment yz adjacent to y. The direction $\overrightarrow{yz}$ is the midpoint of a geodesic of type 131 connecting $\overrightarrow{yx}$ and $\overrightarrow{yx'}$, in particular, $\overrightarrow{yz}$ is interior in $\Sigma_y K$. This implies that w is interior in K. Thus K contains $3I$-vertices.

Let $u_1, u_2 \in K$ be $3I$-vertices at distance $> \frac{\pi}{2}$, then as above, the segment u_1u_2 is singular of type 31213 (recall that in a counterexample $I \Rightarrow A$ holds). Since u_i is interior in K, we can find 2-vertices $v_i \in K$ for $i = 1, 2$, such that $v_1u_1u_2v_2$ is a segment of type 2312132 and length π. Again because u_1 is interior in K, there are two different chambers $\sigma, \sigma' \subset K$ containing the edge v_1u_1. The convex hull $CH(\sigma, \sigma', v_2) \subset K$ is a 2-dimensional hemisphere. This contradicts the properties of a counterexample. □

4.3 The F_4-case

A direct proof of the Center Conjecture for spherical buildings of type F_4 can be found in [LR09]. We present in this section basically the same proof with some minor changes. The proof is divided in two steps. Let K be a convex subcomplex of a spherical building B of type F_4. The first step is to verify that it suffices to prove that K is a subbuilding or the action $Stab_{Aut_0(B)}(K) \curvearrowright K$ has a fixed point, where $Aut_0(B)$ are the type preserving automorphisms of B (Lemma 4.3.1). In Section 4.6.1 we will see that the second step (to show that K is a subbuilding or the action $Stab_{Aut_0(B)}(K) \curvearrowright K$ has a fixed point) can also be deduced from the case of buildings of type E_8.

Lemma 4.3.1. *If the action $Stab_{Aut_0(B)}(K) \curvearrowright K$ has a fixed point, so does the action $Stab_{Aut(B)}(K) \curvearrowright K$.*

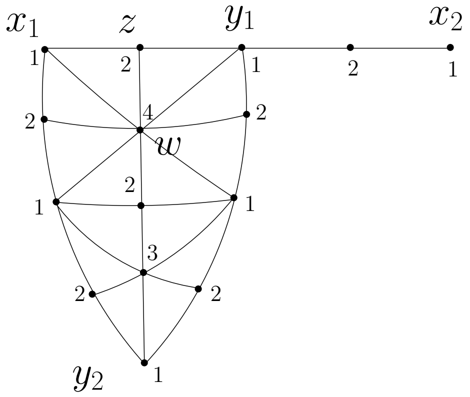
Proof. Suppose there is an element $\phi \in Stab_{Aut(B)}(K) - Stab_{Aut_0(B)}(K)$, otherwise there is nothing to prove. Recall that the Dynkin diagram of type F_4 1—2⇒3—4 has only one symmetry. It follows that $Aut(B)/Aut_0(B) \cong \mathbb{Z}_2$ and ϕ exchanges the vertices of type $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$.

Let $L = K \cap Fix(Stab_{Aut_0(B)}(K)) \neq \emptyset$. It is a convex subcomplex, because if a type preserving automorphism fixes a point, then it fixes the simplex spanned by it. Since $Aut_0(B)$ is normal in $Aut(B)$, it follows that L is $Stab_{Aut(B)}(K)$-invariant. ϕ acts on L as an involution because ϕ^2 is type preserving and therefore the identity in L.

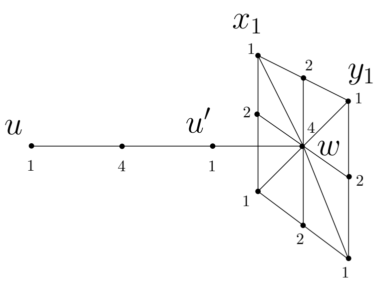
Let $v \in L$ be a vertex. The vertices $v, \phi(v) \in L$ have different type and therefore they cannot be antipodal. It follows that $m(v, \phi(v))$ is fixed by ϕ and $Stab_{Aut_0(B)}(K)$, hence it is fixed by $Stab_{Aut(B)}(K)$. □

Theorem 4.3.2. *The Center Conjecture 1 holds for spherical buildings of type F_4.*

Proof. Let K be a convex subcomplex of a building B of type F_4, which is a counterexample to the Center Conjecture. It follows from Lemma 4.3.1 that the action $Stab_{Aut_0(B)}(K) \curvearrowright K$ has no fixed points. We use the labelling 1 2 3 4 for the Dynkin diagram of type F_4.

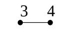


Suppose first that K contains $1A$-vertices and let $x_1 \in K$ be a $1A$-vertex. Since $1A$ is a $Stab_{Aut_0(B)}(K)$-invariant property and $Stab_{Aut_0(B)}(K)$ has no fixed points, it follows that there is another $1A$-vertex $x_2 \in K$ at distance $> \frac{\pi}{2}$ to x_1. Hence $d(x_1, x_2) = \frac{2\pi}{3}$. The midpoint $y_1 := m(x_1, x_2)$ is again a $1A$-vertex, by Lemma 3.0.1. Therefore we find again a $1A$-vertex $y_2 \in K$ at distance $\frac{2\pi}{3}$ to y_1. Notice that $\angle_{y_1}(x_i, y_2) < \pi$ for $i = 1, 2$ because y_2 cannot be antipodal to x_i. We may assume w.l.o.g. that $\angle_{y_1}(x_1, y_2) \geq \frac{\pi}{2}$. Since $\Sigma_{y_1} B$ is a building of type B_3 with Dynkin diagram 2 3 4, this implies that $\angle_{y_1}(x_1, y_2) = \arccos(-\frac{1}{3})$ and this angle is of type 242. Let $z := m(x_1, y_1)$. The convex hull $CH(z, y_1, y_2)$ is a spherical triangle because z and y_1 lie in a common Weyl chamber. The segment zy_2 is singular of type 24231. Let $w \in K$ be the 4-vertex on this segment. The convex hull $CH(z, x_1, y_2)$ is also a spherical triangle. Notice that $\Sigma_z B$ is a building of type $A_1 \circ A_2$ with Dynkin diagram 1 3 4. This implies that $\angle_z(x_1, y_2) = \angle_z(y_1, y_2) = \frac{\pi}{2}$. Hence the union of $CH(z, y_1, y_2)$ and $CH(z, x_1, y_2)$ is a convex subcomplex. It coincides with the convex hull $CH(x_1, y_1, y_2)$, it is an isosceles spherical triangle with sides of type 12121, 12121 and 121. The 4-vertex w lies in the interior of this triangle and $\Sigma_w CH(x_1, y_1, y_2)$ is a singular circle of type 121212121.



Since w lies in the convex hull of the $1A$-vertices in K, we can find another $1A$-vertex u at distance $> \frac{\pi}{2}$ to w. This implies that $d(w, u) = \frac{3\pi}{4}$ and the segment wu is of type 4141. Notice that $\overrightarrow{wu}$ cannot have antipodes in $\Sigma_w K$ otherwise we find an antipode of u in K. Recall that $\Sigma_w B$ is a building of type B_3 with Dynkin diagram 1 2 3. It follows that $\overrightarrow{wu}$ is orthogonal to the 1-vertices in $\Sigma_w CH(x_1, y_1, y_2)$. This implies that $d(\overrightarrow{wu}, \Sigma_w CH(x_1, y_1, y_2)) \equiv \frac{\pi}{2}$ and the convex hull $CH(\overrightarrow{wu}, \Sigma_w CH(x_1, y_1, y_2))$ is a 2-dimensional hemisphere h centered at $\overrightarrow{uw}$. In particular $\Sigma_{\overrightarrow{wu}} \Sigma_w K$ contains circle, i.e. an apartment. Let u' be the 1-vertex on the segment wu adjacent to w. It follows that $\Sigma_{\overrightarrow{u'w}} \Sigma_{u'} K$ and $\Sigma_{u'} K$ contain an apartment. In particular, u' is a $1I$-vertex.

Sublemma 4.3.3. *K contains no $1I$-vertices.*

Proof. Suppose the contrary. There are $1I$-vertices $x_1, x_2 \in K$ with distance $> \frac{\pi}{2}$. Clearly $I \Rightarrow A$, therefore, $d(x_1, x_2) = \frac{2\pi}{3}$ and the segment x_1x_2 is of type 12121. Since x_i are interior vertices. we can find 2-vertices $y_i \in K$ adjacent to x_i and such that $y_1x_1x_2y_2$ is a geodesic of length π and type 2121212. The direction $\overrightarrow{y_ix_i}$ is an interior 1-vertex in $\Sigma_{y_i}K$. Note that $\Sigma_{y_i}B$ is a building of type $A_1 \circ A_2$ and with Dynkin diagram 

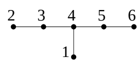
. It follows that $\Sigma_{y_i}K$ contains a top-dimensional hemisphere centered at $\overrightarrow{y_ix_i}$. This implies that K contains a hemisphere of dimension $dim(K)$. A contradiction to the properties of a counterexample. □

End of proof of Theorem 4.3.2. It follows from Sublemma 4.3.3 that there are no $1A$-vertices in K. By duality, we can use the same argument to show that K contains no $4A$-vertices. Observe that a $2A$-vertex cannot be adjacent to a 1-vertex in K. Otherwise, since all 1-vertex in K have antipodes, we find a geodesic in K of length π and of type 1212121 containing the $2A$-vertex in its interior, contradicting Lemma 3.0.1. By a similar argument, a $2A$-vertex cannot be adjacent to a 4-vertex in K. Hence if K contains $2A$-vertices, it must have dimension ≤ 1, a contradiction. By duality, we conclude that K contains no $3A$-vertices. Thus, all vertices in K have antipodes in K. A contradiction to Lemma 3.0.2. □

Remark 4.3.4. Our proof actually shows that K is a subbuilding or the action of the group $Aut_B(K) \curvearrowright K$ fixes a point (see 1.3 for definitions).

4.4 The E_6-case

The Center Conjecture for spherical buildings of type E_6 has been proven directly in [LR09]. We present here basically the same proof just for completeness of this work. Later, in Section 4.6.2, we give an alternative proof showing that the E_6-case follows from the case of buildings of type E_8.

Let K be a convex subcomplex of a building B of type E_6. Let $G := Stab_{Aut(B)}(K)$ and $H := Stab_{Aut_0(B)}(K)$. Recall that the Dynkin diagram of type E_6 

has only one symmetry. This symmetry exchanges the vertices $2 \leftrightarrow 6$ and $3 \leftrightarrow 5$ and fixes the vertices 1 and 4. It also follows that H is a normal subgroup of G of index ≤ 2.

Suppose K is a counterexample to the Center Conjecture.

Lemma 4.4.1. *Le P be a H-invariant property defined for 2-and 6-vertices in K implying A, $P \Rightarrow A$. Suppose K contains a $2P$- ($6P$-)vertex $x \in K$. Then there exists another $2P$- ($6P$-)vertex $x' \in H \cdot x$ with $d(x, x') = \frac{2\pi}{3}$.*

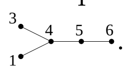
Proof. By the symmetry of the Dynkin diagram, it suffices to prove the case where x is a $2P$-vertex.

Since K is a counterexample there is a vertex $y \in G \cdot x$ at distance $> \frac{\pi}{2}$ to x. If y is a 2-vertex, then $d(x,y) = \frac{2\pi}{3}$ and we are done. So let us suppose that all 2-vertices in $H \cdot x$ are at distance $\arccos(\frac{1}{4})$ to x. Hence y is a 6-vertex and $d(x,y) = \arccos(-\frac{1}{4})$. The segment xy is of type 216. Let $m := m(x,y)$ be the 1-vertex between x and y. Notice that since $y \in G \cdot x$, it follows that all 6-vertices in $H \cdot y$ are at distance $\arccos(\frac{1}{4})$ to y.

Since m lies in the convex hull $CH(G \cdot x)$, we can find a vertex $z \in G \cdot x$ at distance $> \frac{\pi}{2}$ to m. By duality, we may assume w.l.o.g. that z is a 2-vertex. Consider the triangle (x,y,z) with side lengths $d(x,y) = \arccos(-\frac{1}{4})$, $d(x,z) = \arccos(\frac{1}{4})$ and $d(z,y) \leq \arccos(-\frac{1}{4})$. By triangle comparison with this triangle we conclude that $d(z, m(x,y)) \leq \frac{\pi}{2}$. That is, $d(z,m) \leq \frac{\pi}{2}$. A contradiction.

Lemma 4.4.2. *If K contains $2A$-vertices, it also contains $2I$-vertices.*

Proof. Let M be the property of a 2-vertex (6-vertex) of being the midpoint of a pair of $6A$-vertices ($2A$-vertices) at distance $\frac{2\pi}{3}$. By Lemma 3.0.1, $M \Rightarrow A$.

If K contains $2A$-vertices, then by Lemma 4.4.1, it contains $6M$-vertices and therefore also $2M$-vertices. Let x_1 be a $2M$-vertex between two $6M$-vertices at distance $\frac{2\pi}{3}$. It follows from Lemma 3.2.2 that $\Sigma_{x_1}K$ contains a circle c of type 656565656. Let x_2 be another $2M$-vertex at distance $\frac{2\pi}{3}$ to x_1. Let y_1 be the $6M$-vertex between x_1 and x_2. Notice that $\overrightarrow{x_1x_2}$ has no antipodes in $\Sigma_{x_1}K$, otherwise there would be antipodes of x_2 in K. Recall that $\Sigma_{x_1}B$ is a building of type D_5 with Dynkin diagram . It follows that $\overrightarrow{x_1x_2}$ has distance $\leq \frac{\pi}{2}$ to the 6-vertices in c and therefore $d(\overrightarrow{x_1x_2}, c) \equiv \frac{\pi}{2}$, because c is the convex hull of its 6-vertices. Hence the convex hull $CH(\overrightarrow{x_1x_2}, c)$ is a 2-dimensional hemisphere centered at $\overrightarrow{x_1x_2}$. In particular $\Sigma_{\overrightarrow{x_1x_2}}\Sigma_{x_1}K \cong \Sigma_{\overrightarrow{y_1x_1}}\Sigma_{y_1}K$ contains a singular circle of type 545454545. By Lemma 3.2.2 (and by duality of the vertices $2 \leftrightarrow 6$, $3 \leftrightarrow 5$), the link $\Sigma_{y_1}K$ contains a circle of type 232323232. And in particular, $\Sigma_{\overrightarrow{y_1x_1}}\Sigma_{y_1}K$ contains a pair of antipodal 3-vertices. We may apply now Lemma 3.1.4 to the building $\Sigma_{\overrightarrow{y_1x_1}}\Sigma_{y_1}B$ of type D_4 and the subcomplex $\Sigma_{\overrightarrow{y_1x_1}}\Sigma_{y_1}K$ to conclude that it contains a wall. This implies that $\Sigma_{y_1}K$ contains a wall. Let y_2 be another $6M$-vertex at distance $\frac{2\pi}{3}$ to y_1. Notice that $\overrightarrow{y_1y_2}$ has no antipodes in $\Sigma_{y_1}K$, otherwise there would be antipodes of y_2 in K. By Lemma 3.1.3 applied to $\Sigma_{y_1}K$ (of type D_5), it follows that $\overrightarrow{y_1y_2}$ is an interior vertex. This implies that the 2-vertex $m(y_1,y_2)$ is a $2I$-vertex in K. □

Lemma 4.4.3. *K contains no $2I$-vertices.*

Proof. Suppose K contains a $2I$-vertex x. Then since $I \Rightarrow A$, Lemma 4.4.1 implies that there is another $2I$-vertex $x' \in K$ at distance $\frac{2\pi}{3}$. Since x is interior in K, there is a 6-vertex $y \in K$ adjacent to x, such that $\overrightarrow{xy}$ is antipodal to $\overrightarrow{xx'}$. But this implies that y is antipodal to x', a contradiction. □

By duality, we have the corresponding results for 6-vertices in K. Thus combining the previous two Lemmata, we obtain:

Corollary 4.4.4. *All 2- and 6-vertices in K have antipodes in K.*

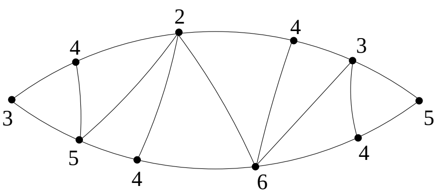
We can prove now that the other vertices in K also have antipodes in K

Lemma 4.4.5. *K contains no $1A$-vertices.*

Proof. Suppose K contains a $1A$-vertex $x_1 \in K$. Then there is another $1A$-vertex $x_2 \in K$ at distance $> \frac{\pi}{2}$ to x_1. Hence $d(x_1, x_2) = \frac{2\pi}{3}$ and the segment x_1x_2 is singular of type 14141. The midpoint $y_1 := m(x_1, x_2)$ is again a $1A$-vertex. Let $y_2 \in K$ be another $1A$-vertex at distance $\frac{2\pi}{3}$ to y_1. Observe that $\angle_{y_1}(x_i, y_2) < \pi$ for $i = 1, 2$ because y_2 cannot be antipodal to x_i. We may suppose w.l.o.g. that $\angle_{y_1}(x_1, y_2) \geq \frac{\pi}{2}$. Recall that $\Sigma_{y_1}B$ is a building of type A_5 with Dynkin diagram 2 3 4 5 6. It follows that $\angle_{y_1}(x_1, y_2) = \arccos(-\frac{1}{3})$ and the simplicial convex hull of the segment $\overrightarrow{y_1x_1}\overrightarrow{y_1y_2}$ is a rhombus with vertices of type 2, 4, 6 and 4. In particular, $\Sigma_{y_1}K$ contains 2-vertices. Let $w \in K$ be a 2-vertex adjacent to y_1. Since all 2-vertices in K have antipodes in K, we find a 6-vertex $\hat{w} \in K$ antipodal to w. The geodesic between $w, \hat{w}$ through y_1 is of type 21656. Thus there is a 6-vertex in the interior of the segment $y_1\hat{w} \subset K$. This 6-vertex also has an antipode in K, then by Lemma 3.0.1 y_1 must have an antipode in K, a contradiction. □

Lemma 4.4.6. *K contains no $3A$- or $5A$-vertices.*

Proof. By duality, it suffices to show that K contains no $3A$-vertices. Observe first that a $3A$-vertex x cannot be adjacent to a 2-vertex in K. Otherwise, since all 2-vertices in K have antipodes in K, we find a geodesic in K of length π and type 23216 containing x in its interior. This contradicts Lemma 3.0.1. A similar argument shows that a $3A$-vertex is not adjacent to vertices of type 1 or 6. Suppose that $x \in K$ is a $3A$-vertex and let $y \in G \cdot x$ be at distance $> \frac{\pi}{2}$ to x. Then y is a vertex of type 3 or 5.



By the observation above $\overrightarrow{xy}$ is contained in an edge in $\Sigma_x K$ of type 45. By considering this 2-dimensional spherical bigon connecting a pair of antipodal 3- and 5-vertices, we conclude that y must be a 3-vertex and xy is of type 34243. Since all 2-vertices in K have antipodes in K, this contradicts Lemma 3.0.1. □

Lemma 4.4.7. *K contains no $4A$-vertices.*

Proof. By a similar argument as in the beginning of the previous Lemma, we conclude that a $4A$-vertex in K cannot be adjacent to vertices in K of type 1, 2, 3, 5 or 6. It follows that if K contains $4A$-vertices, then it must have dimension 0. But this is not possible for a counterexample. □

We have shown so far that all vertices of a counterexample K have antipodes in K, by Lemma 3.0.2, this contradicts the fact that K is not a subbuilding. This proves:

Theorem 4.4.8. *The Center Conjecture 1 holds for spherical buildings of type E_6.*

Remark 4.4.9. Our proof actually shows that K is a subbuilding or the action of the group $Aut_B(K) \curvearrowright K$ fixes a point (see 1.3 for definitions).

4.5 The E_7-case

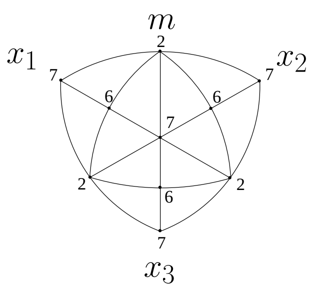
This section is devoted to give a direct proof of the Center Conjecture for buildings of type E_7. For a proof using the E_8-case see Section 4.6.3.

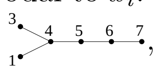
Let K be a convex subcomplex of a spherical building B of type E_7. Suppose that K is not a subbuilding and the action of $G := Stab_{Aut(B)}K \curvearrowright K$ has no fixed points, i.e it is a *counterexample.*

As in the previous cases, our strategy is to show that all the vertices of K have antipodes in K contradicting Lemma 3.0.2. First we focus our attention on the 7-vertices. The 7-vertices have the smallest orbits in the Coxeter complex of type E_7 under the action of the Weyl group, this implies that the types of segments between 7-vertices are very simple. Assuming that there are $7A$-vertex in K we conclude that K also contains $2I$-vertices (Lemma 4.5.2). Since the 2-vertices are the vertices of *root type* in E_7 it is easy to see that K cannot contain $2I$-vertices (Lemma 4.5.3). At this point it is quite simple to verify that the vertices of the other types also have antipodes.

Lemma 4.5.1. *Let P be a G-invariant property for 7-vertices implying A, $P \Rightarrow A$. Then if K contains $7P$-vertices, it also contains an equilateral spherical triangle with $7P$-vertices as vertices and side lengths* $\arccos(-\frac{1}{3})$.

Proof. Since K is a counterexample, for a $7P$-vertex $x_1 \in K$, there is another $7P$-vertex $x_2 \in K$ with distance $> \frac{\pi}{2}$, this implies $d(x_1, x_2) = \arccos(-\frac{1}{3})$. The segment x_1x_2 is of type 727.

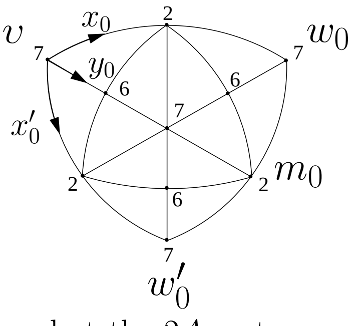


By the properties of a counterexample, if m is the 2-vertex in x_1x_2, then there must exist another $7P$-vertex $x_3 \in K$ with distance $> \frac{\pi}{2}$ to m. Thus, $d(m, x_3) = \arccos(-\frac{1}{\sqrt{3}})$ and the segment mx_3 is of type 2767. Note that for $i = 1, 2$ holds $0 < \angle_m(x_i, x_3) < \pi$, because x_3 is not antipodal to x_i. The building $\Sigma_m B$ is of type D_6 with Dynkin diagram , therefore $\angle_m(x_i, x_3) = \frac{\pi}{2}$. It follows that the union of the two spherical triangles $CH(x_i, m, x_3)$ is an equilateral, spherical triangle as wanted. ☐

Observe that the spherical triangle from Lemma 4.5.1 has a $7A$-vertex in its center. Let T be the property of *being center of such a triangle with $7A$-vertices as vertices.* There is the implication $T \Rightarrow A$.

Lemma 4.5.2. *If K contains $7A$-vertices, then it also contains $2I$-vertices.*

Proof. Let $v \in K$ be a $7T$-vertex (it exists, because of Lemma 4.5.1). Let $t \subset K$ be the equilateral, spherical triangle, whose center is v. Then $\Sigma_v t \subset \Sigma_v K$ is a singular 1-sphere of type 2626262. Let α_1, α_2 and α_3 be the 2-vertices in $\Sigma_v t$ and β_i the 6-vertex in $\Sigma_v t$ antipodal to α_i.

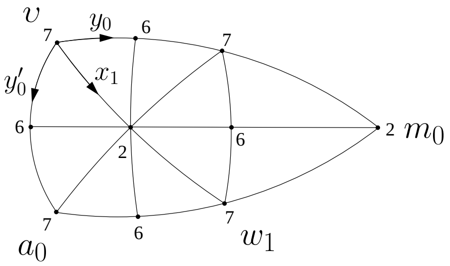


Since $T \Rightarrow A$, it follows from Lemma 4.5.1 that there are $7T$-vertices w_0 and w_0' in K such that the convex hull $CH(v, w_0, w_0')$ is an equilateral spherical triangle with 727-sides.

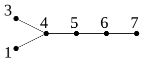
Set $x_0 := \overrightarrow{vw_0}$, $x_0' := \overrightarrow{vw_0'}$ and let y_0 be the midpoint of the segment x_0x_0' in $\Sigma_v K$. If m_0 is the $2A$-vertex between w_0 and w_0', then $y_0 = \overrightarrow{vm_0}$. Observe that the 6-vertex y_0 has no antipodes in $\Sigma_v K$. Otherwise, the segment $m_0 v$ of type 2767 could be extended in K to a segment of type 27672 and length π, but the $2A$-vertex m_0 has no antipodes in K.

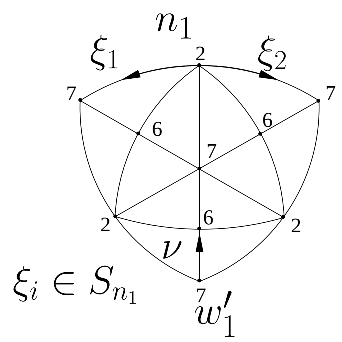
Note that the 6-vertex y_0 cannot have distance $< \frac{\pi}{2}$ to all the 6-vertices $\beta_i \in \Sigma_v t$, otherwise the circle $\Sigma_v t$ would be contained in a ball of radius $< \frac{\pi}{2}$ centered at y_0, but this cannot happen since $diam(\Sigma_v t) = \pi$. So there is a 6-vertex $y_0' \in \{\beta_1, \beta_2, \beta_3\}$ with $d(y_0, y_0') = \frac{2\pi}{3}$. Let z_0 and z_0' be the two 2-vertices in $\Sigma_v t$ adjacent to y_0'. The segment $y_0 y_0'$ is of type 626, let x_1 be the 2-vertex on this segment. Let a_0 be the vertex of the triangle $t \subset K$, such that $y_0' = \overrightarrow{va_0}$. The segment va_0 is of type 767. It is clear, that y_0' is not antipodal to $x_0 = \overrightarrow{vw_0}$ or $x_0' = \overrightarrow{vw_0'}$, otherwise, a_0 would be antipodal to w_0 or w_0'.

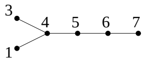
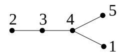
It follows from Lemma 3.2.2 (for the vertices $x_0, x_0', y_0, y_0', z_0, z_0', x_1$) that $\Sigma_{x_1}\Sigma_v K$ contains a singular circle of type 656565656.

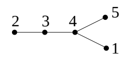
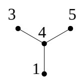


The convex hull $CH(a_0, v, m_0)$ is an isosceles, spherical triangle. It follows, that there is a $7A$-vertex $w_1 \in K$ on the segment $a_0 m_0$ of type 7672, such that vw_1 is a segment of type 727 and $x_1 = \overrightarrow{vw_1}$, i.e. $x_1 \in \Sigma_v K$ is extendable to a segment of type 727. The proof of Lemma 4.5.1 implies that there is another $7T$-vertex $w_1' \in K$ such that the convex hull $CH(v, w_1, w_1')$ is a spherical triangle with 727-sides.

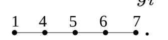
Let n_1 be the $2A$-vertex on the segment vw_1 and recall that $\Sigma_{n_1} B$ is a building of type D_5 with Dynkin diagram . The singular circle of type 656565656 in $\Sigma_{x_1}\Sigma_v K \cong \Sigma_{\overrightarrow{n_1 v}}\Sigma_{n_1} K$ implies that $\Sigma_{n_1} K$ contains a 2-sphere S_{n_1} spanned by three pairwise orthogonal 7-vertices.



Notice that $\overrightarrow{n_1 w_1'}$ has no antipodes in $\Sigma_{n_1} K$ because w_1' is a $7A$-vertex, in particular, $\overrightarrow{n_1 w_1'}$ has distance $\leq \frac{\pi}{2}$ to the 7-vertices in S_{n_1}. This implies that $d(\overrightarrow{n_1 w_1'}, S_{n_1}) \equiv \frac{\pi}{2}$, because S_{n_1} is the convex hull of its 7-vertices. Hence $CH(\overrightarrow{n_1 w_1'}, S_{n_1})$ is a 3-dimensional hemisphere centered at $\overrightarrow{n_1 w_1'}$. Recall that the segments between two orthogonal 7-vertices in a building with Dynkin diagram  are of type 767. It follows that $\Sigma_{\overrightarrow{n_1 w_1'}} CH(\overrightarrow{n_1 w_1'}, S_{n_1})$ is a 2-sphere spanned by three pairwise orthogonal 6-vertices. This implies in turn that $\Sigma_{\overrightarrow{w_1' n_1}} \Sigma_{w_1'} K$ (of type ) contains a 2-sphere s spanned by three pairwise orthogonal 2-vertices. Notice that the 2-vertices in the sphere s correspond to 2-vertices in $\Sigma_{w_1'} K$, which are extendable to segments of type 727. Let $\nu := \overrightarrow{w_1' n_1} \in \Sigma_{w_1'} K$.

We proceed now as above. Let $t' \subset K$ be the spherical triangle, whose center is w_1' (as described in the property T). Then $\Sigma_{w_1'} t \subset \Sigma_{w_1'} K$ is a singular 1-sphere of type 2626262. Observe that ν has no antipodes in $\Sigma_{w_1'} K$ because n_1 is a $2A$-vertex. Then we find as above a 6-vertex $\zeta \in \Sigma_{w_1'} t'$ at distance $\frac{2\pi}{3}$ to ν. Let μ be the 2-vertex in the segment $\nu\zeta$ of type 626. If the direction $\overrightarrow{\nu\zeta}$ has an antipode in the 2-sphere s, then ζ is antipodal to a 2-vertex in $\Sigma_{w_1'} K$, which is extendable in K to a segment of type 727. But this is not possible, since ζ is extendable to a segment of type 767 with final point a $7A$-vertex (a vertex of the triangle t'). Recall that $\Sigma_\nu \Sigma_{w_1'} B$ is a building of type D_5 with Dynkin diagram , therefore $\overrightarrow{\nu\zeta}$ is orthogonal to the 2-vertices in s and the segments between these 2-vertices and $\overrightarrow{\nu\zeta}$ are of type 232. It follows that $d(\overrightarrow{\nu\zeta}, s) \equiv \frac{\pi}{2}$ and the convex hull $CH(\overrightarrow{\nu\zeta}, s)$ is a 3-dimensional hemisphere centered at $\overrightarrow{\nu\zeta}$. This implies that $\Sigma_{\overrightarrow{\nu\zeta}} \Sigma_\nu \Sigma_{w_1'} K$ contains a 2-sphere spanned by three pairwise orthogonal 3-vertices. Since $\Sigma_{\overrightarrow{\nu\zeta}} \Sigma_\nu \Sigma_{w_1'} B$ is of type D_4 with Dynkin diagram , this 2-sphere is not simplicial, thus, its simplicial convex hull is an apartment. Hence $\Sigma_{\overrightarrow{\nu\zeta}} \Sigma_\nu \Sigma_{w_1'} K \cong \Sigma_{\overrightarrow{\mu\nu}} \Sigma_\mu \Sigma_{w_1'} K$ contains an apartment. This implies that $\Sigma_\mu \Sigma_{w_1'} K$ contains an apartment. We can argue as above (with $x_1 \in \Sigma_v K$) to see that μ is extendable in K to a segment of type 727. Hence the 2-vertex on this segment is interior in K. □

Lemma 4.5.3. *K contains no $2I$-vertices.*

Proof. Suppose the contrary. There are $2I$-vertices $x_1, x_2 \in K$ with distance $> \frac{\pi}{2}$. Clearly $I \Rightarrow A$, therefore, $d(x_1, x_2) = \frac{2\pi}{3}$ and the segment $x_1 x_2$ is of type 23232. Since x_i are interior vertices. we can find 3-vertices $y_i \in K$ adjacent to x_i and such that $y_1 x_1 x_2 y_2$ is a geodesic of length π and type 3232323. The direction $\overrightarrow{y_i x_i}$ is an interior 2-vertex in $\Sigma_{y_i} K$. Note that $\Sigma_{y_i} B$ is a building of type $A_1 \circ A_5$ and with Dynkin diagram . It follows that $\Sigma_{y_i} K$ contains a top-dimensional hemisphere centered at $\overrightarrow{y_i x_i}$. This implies that K contains a hemisphere of dimension $dim(K)$. A contradiction to the properties of a counterexample. □

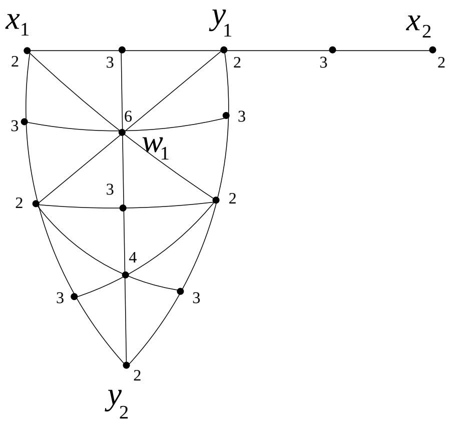
We conclude from Lemma 4.5.2 and Lemma 4.5.3:

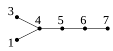
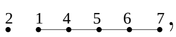
Corollary 4.5.4. *All 7-vertices in K have antipodes in K.*

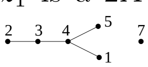
We can now address our attention to the other types of vertices in K.

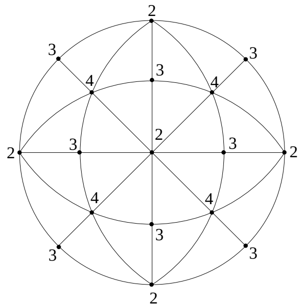
Lemma 4.5.5. *All 2-vertices in K have antipodes in K.*

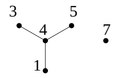
Proof. First observe that for a $2A$-vertex x, the link $\Sigma_x K$ contains no 7-vertices. Otherwise, suppose $y \in K$ is a 7-vertex adjacent to x. By Corollary 4.5.4, we find an antipode $\hat{y} \in K$ of y. The segment $x\hat{y}$ is of type 2767, the 7-vertex on its interior also has an antipode in K. A contradiction to Lemma 3.0.1.



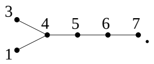
Assume K contains $2A$-vertices. Since K is a counterexample, there are $2A$-vertices $x_1, x_2 \in K$ at distance $\frac{2\pi}{3}$. The midpoint y_1 of the segment x_1x_2 is also a $2A$-vertex, hence, there exists another $2A$-vertex y_2 with $d(y_1, y_2) = \frac{2\pi}{3}$. The 3-vertex $\overrightarrow{y_1y_2}$ cannot be antipodal to the 3-vertices $\overrightarrow{y_1x_i}$. We may assume w.l.o.g. that $\angle_{y_1}(x_1, y_2) \geq \frac{\pi}{2}$. Note that $\Sigma_{y_1} B$ is a building of type D_6 and with Dynkin diagram . It follows that the segment $\overrightarrow{y_1y_2}\overrightarrow{y_1x_1}$ has length $\arccos(-\frac{1}{3})$ and is of type 363. The convex hulls $CH(y_1, y_2, m(y_1, x_1))$ and $CH(x_1, y_2, m(y_1, x_1))$ are spherical triangles. The segment $m(y_1, x_1)y_2$ is of type 36342. Since $\Sigma_{m(x_1,y_1)} B$ is of type $A_1 \circ A_5$ with Dynkin diagram , it follows that $\angle_{m(x_1,y_1)}(y_1, y_2) = \angle_{m(x_1,y_1)}(x_1, y_2) = \frac{\pi}{2}$. Hence, the convex hull $CH(x_1, y_1, y_2)$ is the union of the spherical triangles $CH(y_1, y_2, m(y_1, x_1))$ and $CH(x_1, y_2, m(y_1, x_1))$, and it is an isosceles spherical triangle with sides of type 232, 23232 and 23232.

Let w_1 be the 6-vertex on the interior of the triangle $CH(x_1, y_1, y_2)$. By Lemma 3.0.3, we can find a $2A$-vertex $z_1 \in K$ with distance $> \frac{\pi}{2}$ to w_1, hence, with distance $\arccos(-\frac{1}{2\sqrt{2}})$ or $\frac{3\pi}{4}$. But the link $\Sigma_{z_1} K$ contains no 7-vertices, therefore $d(w_1, z_1) = \frac{3\pi}{4}$ and the segment w_1z_1 is of type 6262. Let w_2 be the 6-vertex between w_1 and z_1. Let λ be the singular 1-sphere $\Sigma_{w_1} CH(x_1, y_1, y_2)$ of type 232323232. The 2-vertex $\overrightarrow{w_1z_1}$ has no antipodes in $\Sigma_{w_1} K$ because z_1 is a $2A$-vertex. Note that the building $\Sigma_{w_1} B$ has type $D_5 \circ A_1$ and Dynkin diagram . It follows that $\overrightarrow{w_1z_1}$ has distance $\frac{\pi}{2}$ to the 2-vertices in λ. Thus, the convex hull $CH(\lambda, \overrightarrow{w_1z_1})$ is a 2-dimensional hemisphere centered at $\overrightarrow{w_1z_1}$ and $\Sigma_{\overrightarrow{w_1z_1}} \Sigma_{w_1} K$ contains a singular 1-sphere of type 343434343. This in turn implies that $\Sigma_{w_2} K$ contains a 2-sphere s of type:



Let again $z_2 \in K$ be a $2A$-vertex with $d(w_2, z_2) = \frac{3\pi}{4}$. We see as above, that $d(\overrightarrow{w_2z_2}, \cdot)|_s \equiv \frac{\pi}{2}$ and $CH(\overrightarrow{w_2z_2}, s) =: h$ is a 3-dimensional hemisphere centered at $\overrightarrow{w_2z_2}$. The building $\Sigma_{\overrightarrow{w_2z_2}}\Sigma_{w_2}B$ is of type $D_4 \circ A_1$ and has Dynkin diagram 

. The 2-sphere $\Sigma_{\overrightarrow{w_2z_2}}h$ contains three pairwise orthogonal 3-vertices, hence, it is not a subcomplex. Its simplicial convex hull is a 3-sphere. This means that $\Sigma_{\overrightarrow{w_2z_2}}\Sigma_{w_2}K$ contains a wall (which is an apartment in the D_4-factor, compare with the end of the proof of Lemma 4.5.2).

Let u be the $2A$-vertex on the interior of the segment w_2z_2 of type 6262. It follows that $\Sigma_u K$ contains a wall. Note that the building $\Sigma_u K$ is of type D_6 and has Dynkin diagram 

. A wall in $\Sigma_u K$ must contain 7-vertices. A contradiction. □

Lemma 4.5.6. *All* 1*-vertices in* K *have antipodes in* K*.*

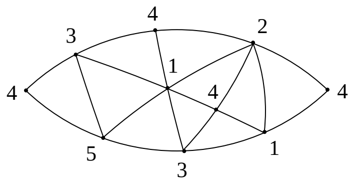
Proof. The same argument as at the beginning of the proof of Lemma 4.5.5 shows that $1A$-vertices are not adjacent to 2- or 7-vertices in K.

Suppose K contains $1A$-vertices. Since K is a counterexample, there exist $1A$-vertices $x, y \in K$ with distance $> \frac{\pi}{2}$. The interior of the segment xy cannot contain 2- or 7-vertices, the directions $\overrightarrow{xy}$ and $\overrightarrow{yx}$ do not span simplices with 2- or 7-vertices. It follows from the table of types of segments between 1-vertices that $d(x, y) = \arccos(\frac{5}{7})$. A contradiction. □

Lemma 4.5.7. *All* 6*-vertices in* K *have antipodes in* K*.*

Proof. First note again that a $6A$-vertex has no adjacent vertices of type 1, 2 or 7 in K.

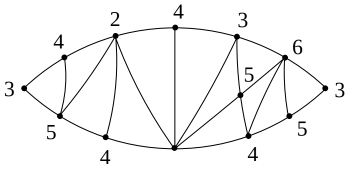
Suppose K contains $6A$-vertices, then we find $x_1, x_2 \in K$ $6A$-vertices with distance $> \frac{\pi}{2}$ and such that $\overrightarrow{x_1x_2} \in \Sigma_{x_1}K$ is contained in a simplex of type 345. This implies that $d(x_1, x_2) = \frac{2\pi}{3}$ and x_1, x_2 are joined by a singular segment of type 64646. The midpoint y of x_1x_2 is again a $6A$-vertex. Let z be another $6A$-vertex with $d(y, z) = \frac{2\pi}{3}$. Then $0 < \angle_y(x_i, z) < \pi$ for $i = 1, 2$, because z is not antipodal to x_i.



Since $\Sigma_y K$ contains no vertices of type 1,2 or 7; the segments connecting the 4-vertices $\overrightarrow{yx_i}$ and $\overrightarrow{yz}$ are contained in a 2-dimensional bigon. It follows that the segments $\overrightarrow{yx_i}\overrightarrow{yz}$ are of type 434 and $d(\overrightarrow{yx_i}, \overrightarrow{yz}) < \frac{\pi}{2}$ for $i = 1, 2$; but $d(\overrightarrow{yx_1}, \overrightarrow{yx_2}) = \pi$. This is a contradiction. □

Lemma 4.5.8. *All* 3*-vertices in* K *have antipodes in* K*.*

Proof. We can show again that a $3A$-vertex is not adjacent to vertices of type 1, 2, 6 or 7 in K.



Suppose K contains $3A$-vertices, then there exist two $3A$-vertices $x, y \in K$ at distance $> \frac{\pi}{2}$. The direction $\overrightarrow{xy}$ must be contained in an edge of type 35. This implies that the segment xy is contained in a 2-dimensional bigon. Then, this segment must be of type 34243, but it contains a 2-vertex on its interior and this 2-vertex has an antipode in K. A contradiction to Lemma 3.0.1. □

Lemma 4.5.9. *All* 4*- and* 5*-vertices in* K *have antipodes in* K*.*

Proof. A vertex in K of type 4 or 5 without antipodes in K cannot have vertices of type 1, 2, 3, 6 or 7 in K adjacent to it. It follows that, if K contains $4A$- or $5A$-vertices, then it has dimension ≤ 1. A contradiction. □

We have shown in the previous lemmata that all vertices of a counterexample K have antipodes, contradicting Lemma 3.0.2. This proves or main result:

Theorem 4.5.10. *The Center Conjecture 1 holds for spherical buildings of type* E_7*.*

Remark 4.5.11. Our proof actually shows that K is a subbuilding or the action of the group $Aut_B(K) \curvearrowright K$ fixes a point (see 1.3 for definitions).

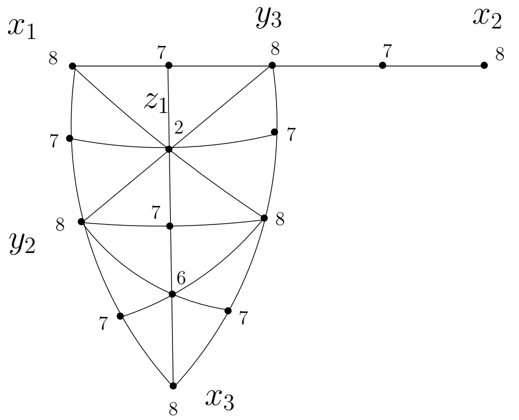
4.6 The E_8-case

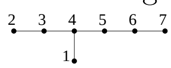
Let K be a convex subcomplex of a spherical building B of type E_8, which is a counterexample to the center conjecture.

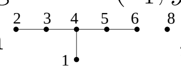
Our strategy is as follows. We focus our attention mainly on the vertices of type 2 and 8. The 8-vertices are the vertices of root type and there are few possibilities for the types of segments between 8-vertices. The 2-vertices have the second smallest orbit (after the 8-vertices) under the action of the Weyl group in the Coxeter complex of type E_8. This implies that the types of the segments between 2-vertices are still manageable. Another reason to consider 2-vertices is that their links have a relatively simple geometry, they are buildings of type D_7. In these buildings, there is only one type of segments between two distinct non-antipodal 8-vertices, namely 878, and it has length $\frac{\pi}{2}$. First we want to prove that K cannot contain 2- or 8-vertices, whose links contain spheres of large dimension. This is achieved in the Lemmata 4.6.1-4.6.9. Then under the assumption of existence of $8A$-vertices, we find 2- and 8-vertices in K, with links containing spheres of larger and larger dimensions. This allows us to conclude that all 8-vertices in K have antipodes in K (Corollary 4.6.17). At this point the hard work is already done. Finally we show that all other vertices in K must also have antipodes in K. This contradicts Lemma 3.0.2 and the assumption that K is not a subbuilding.

We describe first some configurations of points of K, which will be used several times during the argument.

Let P be a property of 8-vertices implying A (the property of not having antipodes in K) and suppose there are $8P$-vertices in K.

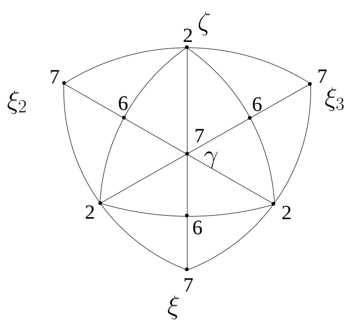


Since K is a counterexample, there are $8P$-vertices $x_1, x_2 \in K$ at distance $> \frac{\pi}{2}$. Since they do not have antipodes, it follows that $d(x_1, x_2) = \frac{2\pi}{3}$. Let $y_3 := m(x_1, x_2)$, it is an $8A$-vertex by Lemma 3.0.1. Again there is an $8P$-vertex $x_3 \in K$, such that $d(y_3, x_3) = \frac{2\pi}{3}$ because y_3 lies in the convex hull of the $8P$-vertices in K. Notice that, since x_i are $8A$-vertices, $0 < \angle_{y_3}(x_3, x_i) < \pi$ for $i = 1, 2$. We may assume w.l.o.g. that $\angle_{y_3}(x_3, x_1) \geq \frac{\pi}{2}$. The link $\Sigma_{y_3} B$ is a building of type E_7 and with Dynkin diagram 

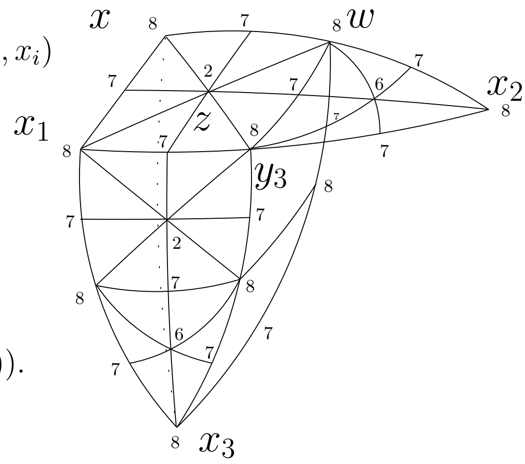
. It follows that $\angle_{y_3}(x_3, x_1) = \arccos(-\frac{1}{3})$ and this angle is of type 727, i.e. the segment $\overrightarrow{y_3x_1}\overrightarrow{y_3x_3} \subset \Sigma_{y_3} K$ is singular of type 727. The convex hulls $CH(x_3, y_3, m(x_1, y_3))$ and $CH(x_3, x_1, m(x_1, y_3))$ are spherical triangles, because y_3 and $m(x_1, y_3)$ (x_1 and $m(x_1, y_3)$, respectively) are contained in a common Weyl chamber and therefore x_3, y_3 and $m(x_1, y_3)$ (x_3, x_1 and $m(x_1, y_3)$, respectively) lie in a common apartment. The segment $m(x_1, y_3)x_3$ is of type 72768. Since $\Sigma_{m(x_1,y_3)} B$ is of type $E_6 \circ A_1$ with Dynkin diagram 

, it follows that $\angle_{m(x_1,y_3)}(x_1, x_3) = \angle_{m(x_1,y_3)}(x_1, x_3) = \frac{\pi}{2}$. Hence, the convex hull $CH(x_1, y_3, x_3)$ is the union of $CH(x_3, y_3, m(x_1, y_3))$ and $CH(x_3, x_1, m(x_1, y_3))$, and it is an isosceles spherical triangle with sides of type 878, 87878 and 87878. Let $y_2 := m(x_1, x_3)$ and $z_1 := (y_2, y_3)$.

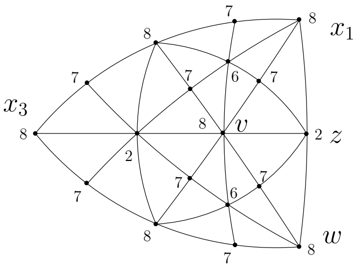
We refer to this configuration of $8P$-vertices as configuration $*$.



Let now $\xi_i := \overrightarrow{x_1x_i}$ for $i = 2, 3$ and $\zeta := \overrightarrow{x_1z_1}$. Suppose there is an 8-vertex x at distance $\frac{\pi}{3}$ to x_1, let $\xi := \overrightarrow{x_1x}$. Assume furthermore that $d(\zeta, \xi) = \arccos(-\frac{1}{\sqrt{3}})$, then the segment $\xi\zeta$ is of type 7672. Recall that ξ_i is $\frac{2\pi}{3}$-extendable to $8A$-vertices and ξ is $\frac{\pi}{3}$-extendable. Thus, $d(\xi, \xi_i) < \pi$ for $i = 2, 3$. It follows that $\angle_\zeta(\xi, \xi_i) = \frac{\pi}{2}$ and $d(\xi, \xi_i) = \arccos(-\frac{1}{3})$ for $i = 2, 3$. Hence, the convex hull $CH(\xi, \xi_2, \xi_3)$ is the union of the spherical triangles $CH(\xi, \zeta, \xi_i)$ for $i = 2, 3$. It is an equilateral spherical triangle with sides of type 727. Let γ be the 7-vertex at the center of this triangle.



From $d(\xi, \xi_i) = \arccos(-\frac{1}{3})$, it follows for $i = 2, 3$ that $d(x, x_i) = \frac{2\pi}{3}$ and the convex hulls $CH(x_1, x, x_i)$ are isosceles spherical triangles (compare with the spherical triangle $CH(x_1, y_3, x_3)$ above). Let $w := m(x, x_2)$ be the $8A$-vertex between x and x_2. Then by considering the triangle $CH(x_1, x, x_2)$, we see that $\omega := \overrightarrow{x_1 w} = m(\xi, \xi_2)$. Let $z := m(x_1, w)$ be the 2-vertex between x_1 and w, then $\overrightarrow{x_1 z} = m(\xi, \xi_2)$. The angle $\angle_{x_1}(z, x_3) = \arccos(-\frac{1}{\sqrt{3}})$ is of type 2767 (compare with the triangle $CH(\xi, \xi_2, \xi_3)$). Notice that $CH(z, x_1, x_3)$ is a spherical triangle, this implies that $d(z, x_3) = \frac{3\pi}{4}$.

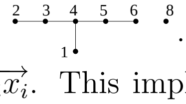


The segment zx_3 is of type 2828. Let v be the $8A$-vertex on the segment zx_3 adjacent to z. Recall that x_3 is an $8A$-vertex. Then x_3 cannot be antipodal to w, thus $d(x_3, w) = \frac{2\pi}{3}$ and $\angle_z(x_3, x_1) = \angle_z(x_3, w) = \frac{\pi}{2}$. Recall also that $d(x_3, y_3) = d(x_3, x) = \frac{2\pi}{3}$, therefore $\angle_z(x_3, y_3) = \angle_z(x_3, x) = \frac{\pi}{2}$. The convex hulls $CH(x_3, x_1, w)$ and $CH(x_3, y_3, x)$ are isosceles spherical triangles with sides of type 87878, 87878 and 828.

The convex hull in $\Sigma_z K$ of the 8-vertices $\overrightarrow{zx}$, $\overrightarrow{zx_1}$ $\overrightarrow{zy_3}$ $\overrightarrow{zw}$ and $\overrightarrow{zv}$ is a 2-dimensional singular hemisphere h centered at $\overrightarrow{zv}$. Let $s \subset \Sigma_z B$ be a singular 2-sphere containing h and let $\widehat{x_3}$ be an 8-vertex in B, such that it is adjacent to z and $\overrightarrow{z\widehat{x_3}}$ is the antipode of $\overrightarrow{zv}$ in s. It follows that $\widehat{x_3}$ is antipodal to x_3 in B. The convex hull in B of $x_3, \widehat{x_3}, x, x_1, y_3, w$ is a 3-dimensional spherical bigon connecting x_3 and $\widehat{x_3}$, with edges $x_3\alpha\widehat{x_3}$ for $\alpha \in \{x, x_1, y_3, w\}$ of type 8787878. It follows that the convex hull $CH(x_1, x, w, y_3, x_3)$ is a (3-dimensional) spherical convex polyhedron in K obtained by truncating this spherical bigon. Notice that the 7-vertex γ at the center of the triangle $CH(\xi, \xi_2, \xi_3) \subset \Sigma_{x_1} K$ is $\frac{2\pi}{3}$-extendable in K to the 8-vertex $m(x_3, w)$.

We refer to this configuration in K as configuration $**$.

Lemma 4.6.1. *K contains no $8I$-vertices.*

Proof. Suppose the contrary. There are $8I$-vertices $x_1, x_2 \in K$ with distance $> \frac{\pi}{2}$. Clearly $I \Rightarrow A$, therefore, $d(x_1, x_2) = \frac{2\pi}{3}$ and the segment x_1x_2 is of type 87878. Since x_i are interior vertices, we can find 7-vertices $y_i \in K$ adjacent to x_i and such that $y_1x_1x_2y_2$ is a geodesic of length π and type 7878787. The direction $\overrightarrow{y_i x_i}$ is an interior 8-vertex in $\Sigma_{y_i} K$. Note that $\Sigma_{y_i} B$ is a building of type $E_6 \circ A_1$ and with Dynkin diagram 

. It follows that $\Sigma_{y_i} K$ contains a top-dimensional hemisphere centered at $\overrightarrow{y_i x_i}$. This implies that K contains a hemisphere of dimension $dim(K)$. A contradiction to the properties of a counterexample. □

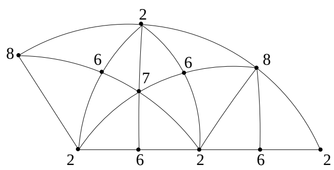
Lemma 4.6.2. *K contains no 8-vertices x, such that $\Sigma_x K$ contains a singular 5-sphere, i.e. a wall.*

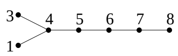
Proof. Let x_1 be an 8-vertex, such that $\Sigma_{x_1} K$ contains a singular 5-sphere S_1. Clearly, by Lemma 3.0.5, x_1 is an $8A$-vertex. Let $x_2 \in G \cdot x_1$ be at distance $\frac{2\pi}{3}$ to x_1. $\Sigma_{x_2} K$ contains a singular 5-sphere S_2. If $\overrightarrow{x_i x_{3-i}}$ has an antipode in S_i for $i = 1, 2$, then there are 7-vertices $y_i \in K$ adjacent to x_i, such that $y_1 x_1 x_2 y_2$ is a geodesic of length π. The midpoint $z := m(x_1, x_2)$ is again an $8A$-vertex and it is the center of a 6-dimensional hemisphere $h \subset K$ (cf. proof of Lemma 4.6.1). In particular, $\Sigma_z K$ contains the 5-sphere $\Sigma_z h$ and the 7-vertices in this sphere are all $\frac{\pi}{2}$-extendable. Let $z' \in G \cdot z$ be at distance $\frac{2\pi}{3}$ to z. Since z' is an $8A$-vertex and the 7-vertices in $\Sigma_z h$ are $\frac{\pi}{2}$-extendable, we deduce that $\overrightarrow{zz'}$ has no antipodes in $\Sigma_z h$. It follows from Lemma 3.3.1 that $\Sigma_{\overrightarrow{zz'}} \Sigma_z K$ contains an apartment and that $\Sigma_{\overrightarrow{wz}} \Sigma_w K$ contains an apartment for the 8-vertex $w := m(z, z')$. It follows that $\Sigma_w K$ contains also an apartment, contradicting Lemma 4.6.1. We may therefore assume w.l.o.g. that $\overrightarrow{x_1 x_2}$ has no antipodes in S_1. Using again Lemma 3.3.1 we conclude that $\Sigma_z K$ contains an apartment. Again a contradiction. □

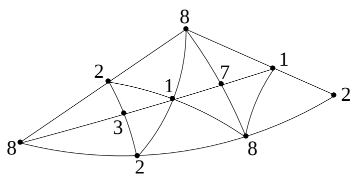
Lemma 4.6.3. *K contains no 2-vertices x, such that $\Sigma_x K$ contains an apartment.*

Proof. Let x be such a 2-vertex in K. Then there is another 2-vertex $x' \in G \cdot x$ at distance $> \frac{\pi}{2}$ to x. Notice that x, x' are interior vertices in K.

Case 1: $d(x, x') = \arccos(-\frac{3}{4})$. The segment xx' is of type 21812. Since x is interior, the direction $\overrightarrow{xx'}$ is also interior in $\Sigma_x K$. It follows that the 8-vertex $m(x, x')$ must be interior in K, contradicting Lemma 4.6.1.



Case 2: $d(x, x') = \frac{2\pi}{3}$. The segment xx' is of type 26262. Recall that $\Sigma_x B$ is of type D_7 with Dynkin diagram . Since x is interior and K is top-dimensional, then $\Sigma_x K$ is a building of type D_7 and we can find an 8-vertex $y \in K$ adjacent to x and such that $\angle_x(y, x') > \frac{\pi}{2}$. Then $\angle_x(y, x') = \arccos(-\frac{1}{\sqrt{3}})$ and it must be of type 8676. Since the triangle $CH(y, x, x')$ is spherical, it follows that $d(y, x') = \frac{3\pi}{4}$ and the segment yx' is of type 2828. The 8-vertex in the interior of this segment must be an interior vertex. A contradiction to Lemma 4.6.1.



Case 3: $d(x, x') = \arccos(-\frac{1}{4})$. The simplicial convex hull of the segment xx' is 2-dimensional and contains 8-vertices $y, y' \in K$ adjacent to x, x'. Let $z \in K$ be an 8-vertex adjacent to x, such that zxy is a segment of type 828. Then $d(z, x') = \frac{3\pi}{4}$. Again a contradiction as in Case 2 above. □

Lemma 4.6.4. *K contains no 7-vertices x, such that $\Sigma_x K$ contains an apartment.*

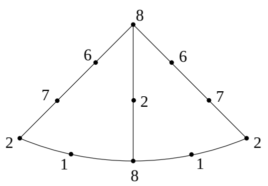
Proof. Suppose there is such a 7-vertex $x \in K$ and let $y \in K$ be an 8-vertex. If $d(x,y) = \frac{5\pi}{6}$, then the segment xy is of type 787878 and we would find interior 8-vertices in K. If $d(x,y) = \arccos(-\frac{1}{\sqrt{3}})$, then the segment xy is of type 72768 and we would find interior 2-vertices contradicting Lemma 4.6.3. So, $d(x,y) \le \arccos(-\frac{1}{2\sqrt{3}})$.

Let $y_1, y_2 \in K$ be 8-vertices adjacent to x, such that y_1xy_2 is a segment of type 878. Let $x' \in G\cdot x$ with $d(x,x') > \frac{\pi}{2}$. Then $d(x', y_i) \le \arccos(-\frac{1}{2\sqrt{3}})$ and triangle comparison with the triangle (x', y_1, y_2) implies that $d(x,x') \le \arccos(-\frac{1}{3})$.

Case 1: $d(x,x') = \arccos(-\frac{1}{3})$. If the segment xx' is singular of type 76867, then the 8-vertex $m(x,x')$ is interior, contradiction. If the segment xx' has 2-dimensional simplicial convex hull C, then there is an 8-vertex $y \in C$ adjacent to x or x'. Since x, x' are in the same G-orbit, we may suppose w.l.o.g. that y is adjacent to x. Let $y' \in K$ be another 8-vertex adjacent to x and such that yxy' is a segment of type 878. Then $d(x', y') = \arccos(-\frac{1}{\sqrt{3}})$ and this case cannot occur by the above.

Case 2: $d(x,x') = \arccos(-\frac{1}{6})$. Let C be the simplicial convex hull of the segment xx'. If C is 2-dimensional, there are 8-vertices $y, y' \in C \subset K$ adjacent to x and x' respectively. Let $z \in K$ be an 8-vertex adjacent to x and such that zxy is a segment of type 878. Define z' analogously. Then $d(x', z)$ or $d(x, z') = \arccos(-\frac{1}{\sqrt{3}})$, which is not possible.

If C is 3-dimensional, there is an 8-vertex $m \in C$, such that the segments mx and mx' are of type 867 and $\angle_m(x,x') = \arccos(-\frac{3}{4})$. Since x, x' are interior vertices, there exist 2-vertices $u, u' \in K$, such that mxu and $mx'u'$ are segments of length $\frac{\pi}{2}$ and of type 8672. $\angle_m(x,x') = \arccos(-\frac{3}{4})$ implies that $\pi > d(u,u') \ge \arccos(-\frac{3}{4})$. Hence $d(u,u') = \arccos(-\frac{3}{4})$.

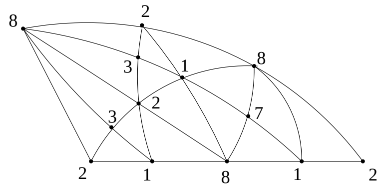


The segment uu' is of type 21812 and $CH(m,u,u')$ is a (non-simplicial) spherical triangle with a 2-vertex $u'' := m(m, m(u,u'))$ in its interior. This implies that the segment xu'' can be extended in K beyond u''. In particular u'' is an interior 2-vertex contradicting Lemma 4.6.3. ☐

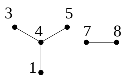
Lemma 4.6.5. *K contains no 2-vertices x, such that $\Sigma_x K$ contains a singular 5-sphere S, i.e. a wall.*

Proof. Suppose there is such an $x \in K$. Let $y \in K$ be an 8-vertex. If $d(x,y) = \frac{3\pi}{4}$, then the segment xy is of type 2828. Let y' be the 8-vertex between x and y. The link $\Sigma_x K$ is of type D_7 and contains a wall, then Lemma 3.1.3 implies that $\Sigma_{y'} K$ contains at least a singular 5-sphere, contradicting Lemma 4.6.2. So $d(x,y) \le \arccos(-\frac{1}{2\sqrt{2}})$ for all 8-vertices $y \in K$.

Let $x' \in G\cdot x$ with $d(x,x') > \frac{\pi}{2}$. It also holds $d(x', y) \le \arccos(-\frac{1}{2\sqrt{2}})$ for all 8-vertices $y \in K$.



Case 1: $d(x, x') = \arccos(-\frac{3}{4})$. The segment xx' is of type 21812. Let $y_1, y_2 \in K$ be 8-vertices adjacent to x, such that $y_1 x y_2$ is a segment of type 828. These vertices can be found, because $\Sigma_x K$ contains a wall. We may assume that $\angle_x(y_1, x') \geq \frac{\pi}{2}$. This implies that the angle $\angle_x(y_1, x') = \arccos(-\frac{1}{\sqrt{7}})$ and it is of type 831, because $\Sigma_x B$ is a building of type D_7. $CH(y, x, x')$ is a spherical triangle, therefore we can compute that $d(y_1, x') = \frac{3\pi}{4}$. A contradiction to the observation above.

Case 2: $d(x, x') = \frac{2\pi}{3}$. As in Lemma 4.6.3 (Case 2) we see that $d(\overrightarrow{xx'}, S') \equiv \frac{\pi}{2}$, where $S' \subset S$ is the 4-sphere spanned by the 8-vertices in S. Otherwise, there would be an 8-vertex y adjacent to x, such that $\overrightarrow{xy} \in S$ and $d(\overrightarrow{xx'}, \overrightarrow{xy}) > \frac{\pi}{2}$. This would imply that $d(x', y) = \frac{3\pi}{4}$.
The segments in $\Sigma_x K$ of length $\frac{\pi}{2}$ connecting the 6-vertex $\overrightarrow{xx'}$ and an 8-vertex $\in S'$ are of type 658. This implies that $\Sigma_{\overrightarrow{xx'}} \Sigma_x K$ contains a 4-sphere spanned by five pairwise orthogonal 5-vertices, but this is impossible in a building of type $D_4 \circ A_2$ with Dynkin diagram .

Case 3: $d(x, x') = \arccos(-\frac{1}{4})$. Let y be the 8-vertex adjacent to x contained in the simplicial convex hull of xx'. $\overrightarrow{xy}$ cannot have antipodes in $\Sigma_x K$. Otherwise there is an 8-vertex $z \in K$, such that zxy is a segment of type 828 and as in Lemma 4.6.3 (Case 3), we see that $d(x', z) = \frac{3\pi}{4}$. It follows from Lemma 3.1.3, that $\overrightarrow{xy}$ is interior in $\Sigma_x K$ (i.e. its link contains an apartment). Then the 7-vertex $m(x, x')$ must be interior (its link $\Sigma_{m(x,x')} K$ contains an apartment). A contradiction to Lemma 4.6.4. □

Lemma 4.6.6. *K contains no 7-vertices x, such that $\Sigma_x K$ contains a wall S of type 1, that is, a wall containing a pair of antipodal 8-vertices.*

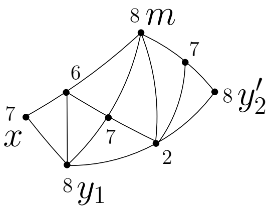
Proof. We proceed exactly as in the proof of Lemma 4.6.4. Recall that $\Sigma_x B$ is of type $E_6 \circ A_1$. Suppose there is such an $x \in K$ and let $y \in K$ be an 8-vertex. If $d(x, y) = \frac{5\pi}{6}$, then the segment xy is of type 787878. The direction $\overrightarrow{xy}$ has an antipode in S, therefore the link $\Sigma_{y'} K$ of the 8-vertex y' on the segment xy adjacent to x contains a wall, contradicting Lemma 4.6.2. If $d(x, y) = \arccos(-\frac{1}{\sqrt{3}})$, then the segment xy is of type 72768. Lemma 3.2.1 implies that $\overrightarrow{xy}$ has an antipode in S or $\Sigma_{\overrightarrow{xy}} \Sigma_x K$ contains an apartment. In both cases the link $\Sigma_z K$ of the 2-vertex z on the segment xy adjacent to x contains a wall. A contradiction to Lemma 4.6.5. So, $d(x, y) \leq \arccos(-\frac{1}{2\sqrt{3}})$.

Let $y_1, y_2 \in K$ be 8-vertices adjacent to x, such that $y_1 x y_2$ is a segment of type 878. Let $x' \in G \cdot x$ with $d(x, x') > \frac{\pi}{2}$. Then $d(x', y_i) \leq \arccos(-\frac{1}{2\sqrt{3}})$ and triangle comparison with the triangle (x', y_1, y_2) implies that $d(x, x') \leq \arccos(-\frac{1}{3})$.

Case 1: $d(x, x') = \arccos(-\frac{1}{3})$. If the segment xx' is singular of type 76867, then Lemma 3.2.1 implies that the 6-vertex $\overrightarrow{xx'}$ has an antipode in S or $\Sigma_{\overrightarrow{xx'}} \Sigma_x K$ contains an apartment. Either way, the link in K of the 8-vertex $m(x, x')$ contains a wall, which is not possible by Lemma 4.6.2. The case, where the segment xx' has 2-dimensional simplicial convex hull C, follows as in the proof of Lemma 4.6.4.

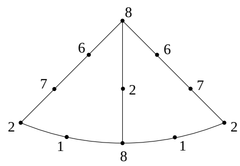
Case 2: $d(x,x') = \arccos(-\frac{1}{6})$. Let C be the simplicial convex hull of the segment xx'. If C is 2-dimensional, we argue as in the proof of Lemma 4.6.4.

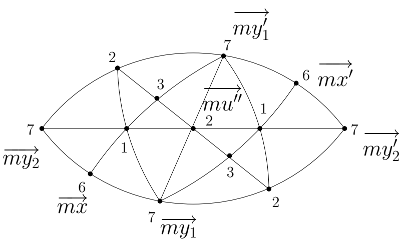
If C is 3-dimensional (see Section 2.7 for a description of C), there is an 8-vertex $m \in C$, such that the segments mx and mx' are of type 867 and $\angle_m(x,x') = \arccos(-\frac{3}{4})$. C contains also 8-vertices y_1, y_1' adjacent to x, x' respectively. Let $y_2 \in K$ be an 8-vertex adjacent to x and such that y_2xy_1 is a segment of type 878. Define y_2' analogously. Then the angle $\angle_m(x,y_2')$ is of type 6727 (compare with $\Sigma_m C'$ in Section 2.7). This implies that $d(x,y_2') = \arccos(-\frac{1}{2\sqrt{3}})$.



If the 6-vertex $\overrightarrow{xm}$ has no antipodes in S, then it follows from Lemma 3.2.1 that $\Sigma_{\overrightarrow{xm}}\Sigma_x K$ contains an apartment, i.e. $\overrightarrow{xm}$ is interior in $\Sigma_x K$. In particular the link $\Sigma_w K$ of the 7-vertex w in the interior of the simplicial convex hull of xy_2' contains an apartment. A contradiction to Lemma 4.6.4. It follows that $\overrightarrow{xm}$, $\overrightarrow{x'm}$ have antipodes in the walls $S \subset \Sigma_x K$, respectively $S' \subset \Sigma_{x'} K$. Therefore, there exist 2-vertices $u, u' \in K$, such that mxu and $mx'u'$ are segments of length $\frac{\pi}{2}$ and of type 8672. $\angle_m(x,x') = \arccos(-\frac{3}{4})$ implies that $\pi > d(u,u') \geq \arccos(-\frac{3}{4})$. Hence $d(u,u') = \arccos(-\frac{3}{4})$.

It follows that the segment uu' is of type 21812 and $CH(m,u,u')$ is a (non-simplicial) spherical triangle. The segment $m\,m(u,u')$ has length $\frac{\pi}{2}$ and therefore it has type 828. The 2-vertex $u'' := m(m, m(u,u'))$ lies in the interior of the spherical triangle $CH(m,u,u')$.





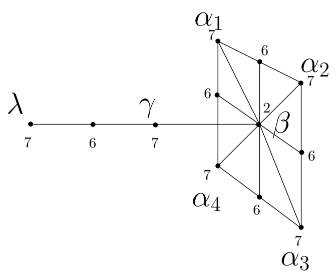
Consider the link of m. Since $\overrightarrow{xm}$ has an antipode in the wall $S \subset \Sigma_x K$, it follows that $\Sigma_{\overrightarrow{mx}}\Sigma_m K$ contains a wall. The link $\Sigma_{\overrightarrow{mx}}\Sigma_m B$ is of type $D_5 \circ A_1$. The wall in $\Sigma_{\overrightarrow{mx}}\Sigma_m K$ contains a wall in the D_5-factor. The direction $\xi := \overrightarrow{\overrightarrow{mx}\overrightarrow{my_1'}}$ is a 1-vertex in $\Sigma_{\overrightarrow{mx}}\Sigma_m K$. By Lemma 3.1.3 we conclude that the A_4-factor of $\Sigma_\xi \Sigma_{\overrightarrow{mx}}\Sigma_m K$ contains at least a wall. Taking spherical join with the directions to the 7-vertices $\overrightarrow{my_2}$ and $\overrightarrow{my_1}$ we find a wall in $\Sigma_\xi\Sigma_{\overrightarrow{mx}}\Sigma_m K$. This implies that $\Sigma_{\overrightarrow{mu''}}\Sigma_m K$ contains at least a wall. Since $\overrightarrow{mu''}$ is extendable, it follows that $\Sigma_{u''}K$ contains a wall. But this contradicts Lemma 4.6.5. □

In a special case we can also exclude 8-vertices, whose links contain a 3-sphere:

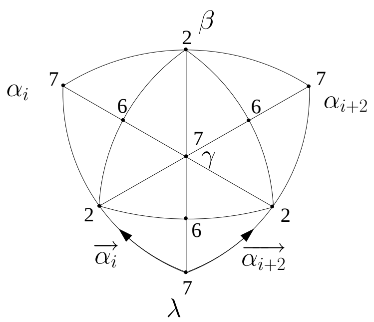
Lemma 4.6.7. *K contains no $8A$-vertices x, such that $\Sigma_x K$ contains a singular 3-sphere S with the following properties: S contains a pair of antipodal 2-vertices ξ_1, ξ_2, such that $\Sigma_{\xi_i} S$ is a singular 2-sphere spanned by three pairwise orthogonal 7-vertices. Furthermore, all 7-vertices in S are $\frac{\pi}{3}$-extendable to $8A$-vertices.*

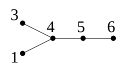
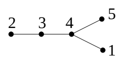
Notice that all 7-vertices in S are adjacent to ξ_i for some $i = 1, 2$. Indeed, a segment in $\Sigma_x K$ (of type E_7) connecting a 2- and a 7-vertex at distance $\leq \frac{\pi}{2}$ is of type 27 or 217. This last segment cannot occur between ξ_i and a 7-vertex in S because $\Sigma_{\xi_i} S$ does not contain 1-vertices. Observe also, that the link $\Sigma_\lambda S$ of a 7-vertex $\lambda \in S$ contains a singular circle of type 2626262: suppose w.l.o.g. that λ is adjacent to ξ_1, then $\overrightarrow{\xi_1\lambda}$ is contained in a circle in $\Sigma_{\xi_1} S$ of type 767676767. In particular $\Sigma_{\overrightarrow{\lambda\xi_1}}\Sigma_\lambda S$ contains a pair of antipodal 6-vertices. It follows that the antipodal directions $\overrightarrow{\lambda\xi_1}$ and $\overrightarrow{\lambda\xi_2}$ are contained in a singular circle in $\Sigma_\lambda S$ of type 2626262.

Proof of Lemma 4.6.7. Suppose there are such $8A$-vertices. Let $x_1, x_2, x_3 \in K$ be such $8A$-vertices as in configuration $*$, and let $S_{x_i} \subset \Sigma_{x_i} K$ denote the corresponding 3-spheres in their links. Let $y_3, z_1 \in K$ be as in the notation of the configuration $*$. Suppose that there is a 7-vertex $\xi \in S_{x_1} \subset \Sigma_{x_1} K$, such that $d(\xi, \zeta) = \arccos(-\frac{1}{\sqrt{3}})$ for $\zeta := \overrightarrow{x_1 z_1}$. The segment $\xi\zeta$ is of type 7672. By assumption, there exists an 8A-vertex $x \in K$, such that $d(x_1, x) = \frac{\pi}{3}$ and $\overrightarrow{x_1 x} = \xi$. Under these circumstances we obtain the configuration $**$. We use the same notation as in the configuration $**$. Let $\alpha_i \in \Sigma_{x_3} K$ for $i = 1, \ldots, 4$ be the directions $\overrightarrow{x_3 x_1}$, $\overrightarrow{x_3 x}$, $\overrightarrow{x_3 w}$ and $\overrightarrow{x_3 y_3}$. Let $\beta := \overrightarrow{x_3 z}$. Then the 7-vertices α_i are adjacent to the 2-vertex β. And the directions $\overrightarrow{\beta\alpha_i}$ lie on a circle κ of type 767676767 contained in $\Sigma_\beta \Sigma_{x_3} K$.



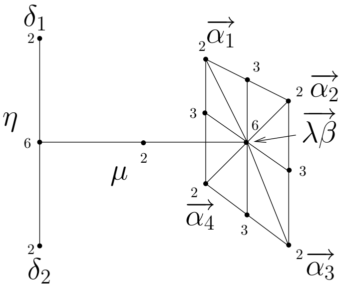
Suppose again that there is a 7-vertex λ in the 3-sphere $S_{x_3} \subset \Sigma_{x_3} K$, such that $d(\beta, \lambda) = \arccos(-\frac{1}{\sqrt{3}})$. So the segment $\beta\lambda$ is of type 2767. Recall that the 7-vertices α_i are $\frac{2\pi}{3}$-extendable and λ is $\frac{\pi}{3}$-extendable to an $8A$-vertex, so they cannot be antipodal. It follows that $\angle_\beta(\lambda, \alpha_i) = \frac{\pi}{2}$ and $d(\alpha_i, \lambda) = \arccos(-\frac{1}{3})$. The segments $\alpha_i\lambda$ are of type 727. Let $\gamma \in \Sigma_{x_3} K$ be the 7-vertex on the interior of the segment $\beta\lambda$. Then γ is the center of an equilateral spherical triangle $CH(\lambda, \alpha_1, \alpha_3)$ with sides of type 727. We are now in the situation of the configuration $**$ (compare with the triangle $CH(\xi, \xi_2, \xi_3)$ in the definition of the configuration $**$). It follows that γ is $\frac{2\pi}{3}$-extendable.



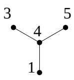
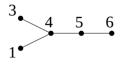
The convex hull $CH(\kappa, \overrightarrow{\beta\lambda})$ is a 2-dimensional hemisphere centered at $\overrightarrow{\beta\lambda}$. Hence, $\Sigma_{\overrightarrow{\beta\lambda}}\Sigma_\beta\Sigma_{x_3} K$ (of type ) contains a circle of type 656565656. This is equivalent to $\Sigma_{\overrightarrow{\lambda\beta}}\Sigma_\lambda\Sigma_{x_3} K$ (of type ) containing a circle of type 232323232. Note that the 2-vertices on this circle correspond to the 2-vertices $m(\lambda, \alpha_i) \in \Sigma_{x_3} K$ (consider the equilateral spherical triangles $CH(\lambda, \alpha_i, \alpha_{i+2})$ with sides of type 727). Let $\overrightarrow{\alpha_i} := \overrightarrow{\lambda\alpha_i} \in \Sigma_\lambda\Sigma_{x_3} K$.

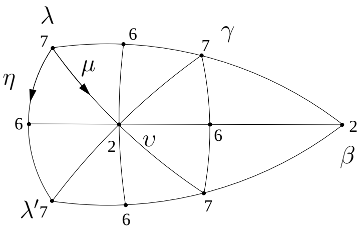
Recall that the link $\Sigma_\lambda S_{x_3}$ contains a circle c of type 2626262 and notice that $\overrightarrow{\lambda\beta}$ cannot be antipodal to any of the 2-vertices on this circle: otherwise, we find a 7-vertex in the

3-sphere S_{x_3} antipodal to γ. This cannot happen, because γ is $\frac{2\pi}{3}$-extendable and the 7-vertices in S_{x_3} are $\frac{\pi}{3}$-extendable to $8A$-vertices. It is also clear that $\overrightarrow{\lambda\beta}$ cannot have distance $< \frac{\pi}{2}$ to the three 6-vertices on the circle c, otherwise c would be contained in a ball centered at $\overrightarrow{\lambda\beta}$ with radius $< \frac{\pi}{2}$, but this is not possible since $diam(c) = \pi$.

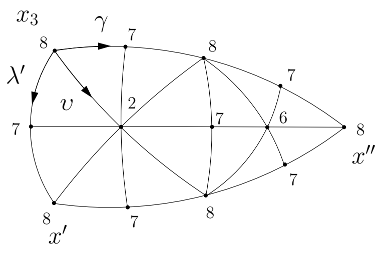


Therefore we can find a 6-vertex η on the circle $c \subset \Sigma_\lambda S_{x_3}$, such that $d(\eta, \overrightarrow{\lambda\beta}) \geq \frac{\pi}{2}$. Hence, $d(\eta, \overrightarrow{\lambda\beta}) = \frac{2\pi}{3}$ and the segment $\eta\overrightarrow{\lambda\beta}$ is of type 626. Let $\mu := m(\eta, \overrightarrow{\lambda\beta})$. Let also δ_1, δ_2 be the two 2-vertices in the circle $c \subset \Sigma_\lambda\Sigma_{x_3}K$ adjacent to η.

We have already seen, that $\overrightarrow{\lambda\beta}$ cannot be antipodal to δ_i. This implies that $\angle_\eta(\delta_i, \mu) = \frac{\pi}{2}$ and these angles are of type 232. It follows that $\Sigma_{\overrightarrow{\eta\mu}}\Sigma_\eta\Sigma_\lambda\Sigma_{x_3}K$ (of type D_4: ) contains a pair of antipodal 3-vertices. On the other hand, if η is antipodal to some $\overrightarrow{\alpha_i}$, then $\alpha_i \in \Sigma_{x_3}K$ has an antipode in S_{x_3}, but this cannot happen either, because α_i is $\frac{2\pi}{3}$-extendable in K. Therefore $\angle_{\overrightarrow{\lambda\beta}}(\mu, \overrightarrow{\alpha_i}) = \frac{\pi}{2}$ and these angles are of type 232. It follows that $\Sigma_{\overrightarrow{\lambda\beta\mu}}\Sigma_{\overrightarrow{\lambda\beta}}\Sigma_\lambda\Sigma_{x_3}K$ contains a singular circle of type 343434343. This in turn implies, that $\Sigma_{\overrightarrow{\eta\mu}}\Sigma_\eta\Sigma_\lambda\Sigma_{x_3}K$ contains a singular circle of type 141414141, because the antipode of a 3- (4)-vertex in $\Sigma_\mu\Sigma_\lambda\Sigma_{x_3}K$, of type , adjacent to $\overrightarrow{\mu\lambda\beta}$ is a 1- (4)-vertex adjacent to $\overrightarrow{\mu\eta}$. We apply now Lemma 3.1.4 to conclude that $\Sigma_{\overrightarrow{\eta\mu}}\Sigma_\eta\Sigma_\lambda\Sigma_{x_3}K$ contains a wall. Hence $\Sigma_\mu\Sigma_\lambda\Sigma_{x_3}K$ contains a wall.

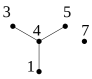


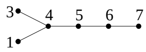
Let $\lambda' \in S_{x_3}$ be the 7-vertex at distance $\arccos(\frac{1}{3})$ to λ, so that $\overrightarrow{\lambda\lambda'} = \eta$. By considering the spherical triangle $CH(\lambda, \lambda', \beta) \subset \Sigma_{x_3}K$ we deduce that μ is $\arccos(-\frac{1}{3})$-extendable in $\Sigma_{x_3}K$. Let υ be the 2-vertex in $\Sigma_{x_3}K$ adjacent to λ with $\overrightarrow{\lambda\upsilon} = \mu$. It follows that $\Sigma_\upsilon\Sigma_{x_3}K$ contains a wall.

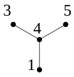


Recall that γ is $\frac{2\pi}{3}$-extendable and let $x'' \in K$ be an 8-vertex with $d(x_3, x'') = \frac{2\pi}{3}$ and $\overrightarrow{x_3x''} = \gamma$. Since $\lambda' \in S_{x_3}$, it is $\frac{\pi}{3}$-extendable. Let $x' \in K$ be an 8-vertex, so that $d(x_3, x') = \frac{\pi}{3}$ and $\overrightarrow{x_3x'} = \lambda'$. Consider the spherical triangle $CH(x_3, x'', x')$. One sees that υ is $\frac{\pi}{2}$-extendable in K, thus we have found a 2-vertex in K, whose link contains a wall, contradicting Lemma 4.6.5.

So it follows that $d(\beta, \lambda) \leq \frac{\pi}{2}$ for all 7-vertices $\lambda \in S_{x_3}$. Since S_{x_3} is the convex hull of the 7-vertices contained in it, this implies that $d(\beta, S_{x_3}) \equiv \frac{\pi}{2}$ and $s := \Sigma_\beta CH(\beta, S_{x_3})$ is a 3-sphere. Let $\theta \in S_{x_3} \subset \Sigma_{x_3}K$ be a 2-vertex, so that $\Sigma_\theta S_{x_3}$ is a 2-sphere spanned by three pairwise orthogonal 7-vertices (compare with the description of the 3-sphere S_{x_3}). The segment $\theta\beta$ is of type 262.

Notice that $d(\beta, S_{x_3}) \equiv \frac{\pi}{2}$ implies that $d(\overrightarrow{\theta\beta}, \Sigma_\theta S_{x_3}) \equiv \frac{\pi}{2}$. It follows that $\Sigma_{\overrightarrow{\theta\beta}}\Sigma_\theta CH(\beta, S_{x_3})$ (subset of a building of type 

) is a 2-sphere. Notice that in the building $\Sigma_\theta\Sigma_{x_3}B$ of type 

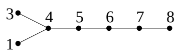
; two 7-, 6-vertices at distance $\frac{\pi}{2}$ are joined by a segment of type 756. This implies that $\Sigma_{\overrightarrow{\theta\beta}}\Sigma_\theta CH(\beta, S_{x_3})$ is spanned by three pairwise orthogonal 5-vertices. Such a 2-sphere in the Coxeter complex of type 

is not a subcomplex, thus, its simplicial convex hull is a 3-sphere. Therefore the 3-sphere $s \subset \Sigma_\beta\Sigma_{x_3}K$ is not a subcomplex and its simplicial convex hull is a wall. Recall that $\beta \in \Sigma_{x_3}K$ is $\frac{\pi}{2}$-extendable in K, hence, there are 2-vertices in K, with links containing a wall. We have now a contradiction to Lemma 4.6.5.

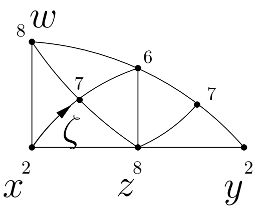
It follows that our first assumption, that there is a 7-vertex $\xi \in S_{x_1} \subset \Sigma_{x_1}K$, such that $d(\xi, \zeta) = \arccos(-\frac{1}{\sqrt{3}})$ cannot occur. Thus, $d(\zeta, S_{x_1}) \equiv \frac{\pi}{2}$ and repeating the previous argument, we can see that $\Sigma_\zeta\Sigma_{x_1}K$ contains a wall. Hence, $\Sigma_{z_1}K$ contains a wall, contradicting again Lemma 4.6.5. □

Lemma 4.6.8. *Let $x \in K$ be a 2-vertex, such that $\Sigma_x K$ contains a singular 4-sphere S of type 757 or $\frac{\pi}{3}$. Then, the 8-vertices in $S \subset \Sigma_x K$ are not $\frac{\pi}{2}$-extendable and there are no 8-vertices in K at distance $\frac{3\pi}{4}$ to x. In particular, x is a 2A-vertex, and all 8-vertices in S are directions to 8A-vertices in K adjacent to x.*

Proof. Suppose there is an 8-vertex in S that is $\frac{\pi}{2}$-extendable. This means that there is a 2-vertex $y \in K$ at distance $\frac{\pi}{2}$ to x, such that the segment xy is of type 282 and $\overrightarrow{xy} \in S$. In particular $\Sigma_{\overrightarrow{xy}}\Sigma_x S$ is a singular 3-sphere. This implies for the 8-vertex $z := m(x, y)$, that its link $\Sigma_z K$ contains a 4-sphere. By Lemma 4.6.1, $dim(K) \geq 6$. In particular, $\Sigma_x K$ contains a 5-dimensional hemisphere h bounded by S.

The hemisphere h is the intersection of a *wall* and a *root* in a building of type D_7 with Dynkin diagram 

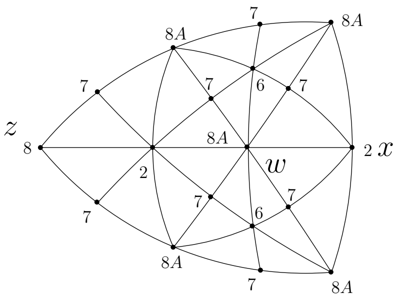
. Recall the description of hemispheres of codimension 1 in Section 2.3. If S is of type 757, then h is centered at a 7-vertex α and $\Sigma_\alpha h$ is a wall of type 5. In particular $\Sigma_\alpha h$ contains a pair of antipodal 8-vertices. If S is of type $\frac{\pi}{3}$, then h is centered at point contained in the interior of an edge of type 86. In particular, the 8-vertex of this edge is contained in h. In both cases h contains an 8-vertex η in its interior (notice that this is not true for a hemisphere bounded by a singular 4-sphere of type 787). It is clear that $d(\eta, \overrightarrow{xy}) = \frac{\pi}{2}$ and the segment is of type 878. The midpoint $\zeta := m(\eta, \overrightarrow{xy})$ is also in the interior of h, and in particular, $\Sigma_\zeta\Sigma_x K$ contains a wall of type 5, that is, a wall containing a pair of antipodal 8-vertices.



Let $w \in K$ be the 8-vertex in K adjacent to x with $\eta = \overrightarrow{xw}$. Then if we consider the spherical triangle $CH(x, y, w)$, we see that ζ is extendable to a segment of type 276 in K. Therefore we find 7-vertices in K, whose links in K contain a wall of type 1. A contradiction to Lemma 4.6.6.

So the 8-vertices in $S \subset \Sigma_x K$ are not $\frac{\pi}{2}$-extendable. In particular, x is a $2A$-vertex. Otherwise we find an antipode $\widehat{x} \in K$ of x and a segment connecting x and $\widehat{x}$ with initial direction an 8-vertex in S is of type 28282. It follows that the 8-vertices in S are $\frac{3\pi}{4}$-extendable. A contradiction.

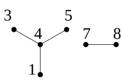
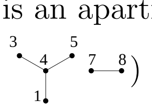
Let $u \in K$ be an 8-vertex adjacent to x, such that $\overrightarrow{xu} \in S$ and suppose that u has an antipode $\widehat{u} \in K$. Let c be the segment connecting u and $\widehat{u}$ through x. It is of type 82828. Since the direction $\overrightarrow{x\widehat{u}}$ has an antipode in S, namely $\overrightarrow{xu}$, it follows that the 8-vertex $\overrightarrow{x\widehat{u}}$ lies in a sphere $S' \subset \Sigma_x K$ of the same type as S. Hence $\overrightarrow{x\widehat{u}}$ cannot be $\frac{\pi}{2}$-extendable, but the segment $x\widehat{u}$ is of type 2828. A contradiction. Thus, all 8-vertices in S are directions to $8A$-vertices in K adjacent to x.



For the second assertion, suppose there is an 8-vertex $z \in K$ with $d(x, z) = \frac{3\pi}{4}$. Since all 8-vertices in S correspond to $8A$-vertices in K, the 8-vertex $\overrightarrow{xz}$ must be orthogonal to the 8-vertices in S. In both cases (of type 757 or $\frac{\pi}{3}$), S contains a singular 2-sphere spanned by three pairwise orthogonal 8-vertices (cf. Section 2.3). This implies that $\Sigma_{\overrightarrow{xz}}\Sigma_x K$ contains a 2-sphere spanned by three pairwise orthogonal 7-vertices. Let w be the 8-vertex in xz adjacent to x. Recall that x is a $2A$-vertex, therefore w is an $8A$-vertex. Then $\Sigma_w K$ contains a 3-sphere as described in the statement of Lemma 4.6.7. It also follows that the 7-vertices in this sphere are $\frac{\pi}{3}$-extendable to $8A$-vertices, contradicting Lemma 4.6.7. □

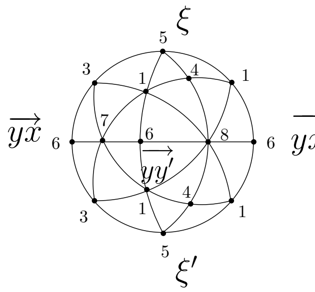
Lemma 4.6.9. *K contains no 2-vertices x, such that $\Sigma_x K$ contains a singular 4-sphere S of type $\frac{\pi}{3}$.*

Proof. Let x be such a 2-vertex. It follows from Lemma 4.6.8 that x is a $2A$-vertex and $rad(x, \text{8-vert. in } K) \leq \arccos(-\frac{1}{2\sqrt{2}})$. As in the proof of Lemma 4.6.5, we deduce that $diam(G \cdot x) \leq \frac{2\pi}{3}$. Let $x' \in G \cdot x$ with $d(x, x') = diam(G \cdot x)$.

Case 1: $diam(G \cdot x) = \frac{2\pi}{3}$. The segment xx' is of type 26262. As in the proof of Lemma 4.6.3 we deduce that the 6-vertex $\overrightarrow{xx'}$ has distance $\frac{\pi}{2}$ to the 8-vertices in S. If $S' \subset S$ is the 3-sphere spanned by the 8-vertices in S, then $d(\overrightarrow{xx'}, S') \equiv \frac{\pi}{2}$. It follows that $\Sigma_{\overrightarrow{xx'}}\Sigma_x K$ (of type ) contains a 3-sphere spanned by four pairwise orthogonal 5-vertices, this sphere is an apartment in the D_4-factor. Let $y := m(x, x')$. Then the link $\Sigma_{\overrightarrow{yx}}\Sigma_y K$ (again of type ) contains also an apartment in the D_4-factor. This is a 3-sphere spanned by a simplex of type 1345. This implies that the link $\Sigma_y K$ contains a singular 4-sphere S_y spanned by a simplex of type 13456. Hence S_y is of type $\frac{\pi}{3}$ and the 6-vertices $\overrightarrow{yx}$, $\overrightarrow{yx'}$ are orthogonal to the 3-sphere $S'_y \subset S_y$ spanned by the 8-vertices in S_y. To see this consider the vector space model of the Coxeter complex of type D_7 introduced in the Appendix A. The sphere S_y can be identified with the sphere $\{x_5 = x_6 = x_7\} \cap S^6 \subset \mathbb{R}^7$ and S'_y, with the sphere $\{x_5 = x_6 = x_7 = 0\} \cap S^6$. A 5-vertex in S'_y is of the form $(\pm 1, \dots, \pm 1, 0, 0, 0)$ and a 6-vertex orthogonal to this sphere must be of the form $(0, \dots, 0, \pm 1, \pm 1, \pm 1)$. Hence, a 5-vertex in S'_y and a 6-vertex orthogonal to S'_y are connected by a segment of type 536 or 516.

As in the beginning of the proof, we obtain that $rad(y, 8\text{-vert. in } K) \leq \arccos(-\frac{1}{2\sqrt{2}})$ and $diam(G \cdot y) \leq \frac{2\pi}{3}$. We assume again that $diam(G \cdot y) = \frac{2\pi}{3}$ and let $y' \in G \cdot y$ have distance $\frac{2\pi}{3}$ to y. It follows as above, that $\Sigma_{\overrightarrow{yy'}} \Sigma_y K$ contains an apartment in the D_4-factor.

Let $\xi, \xi' \in S'_y$ be antipodal 5-vertices. The vertices $\overrightarrow{yx}$, $\overrightarrow{yx'}$, ξ and ξ' lie on a singular circle of type 635161536 contained in S_y. The link $\Sigma_\xi \Sigma_y B$ is of type $A_3 \circ A_3$ and has Dynkin diagram $\overset{3}{\bullet}\!-\!\overset{4}{\bullet}\!-\!\overset{1}{\bullet}\ \ \overset{6}{\bullet}\!-\!\overset{7}{\bullet}\!-\!\overset{8}{\bullet}$. Notice that $\Sigma_\xi S'_y$ is an apartment in the second A_3-factor. Therefore the second factor in the spherical join splitting of $\Sigma_\xi \Sigma_y K$ is a subbuilding. Since $rad(y, 8\text{-vert. in } K) \leq \arccos(-\frac{1}{2\sqrt{2}})$, this implies as above that $d(\overrightarrow{yy'}, S'_y) \equiv \frac{\pi}{2}$. In particular, $d(\overrightarrow{yy'}, \xi) = \frac{\pi}{2}$ and the direction $\overrightarrow{\xi_{\overrightarrow{yy'}}}$ must be orthogonal to the 2-sphere $\Sigma_\xi S'_y$. Recall that this sphere is an apartment in the second A_3-factor. Thus $\overrightarrow{\xi_{\overrightarrow{yy'}}}$ must lie on the $\overset{3}{\bullet}\!-\!\overset{4}{\bullet}\!-\!\overset{1}{\bullet}$ -factor of $\Sigma_\xi \Sigma_y K$.



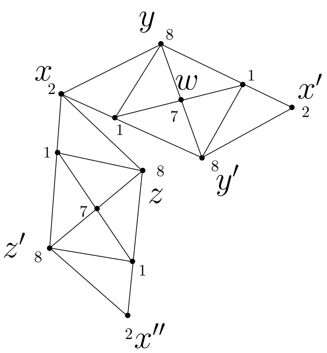
It follows from this that the segments $\xi\overrightarrow{yy'}$ and $\xi'\overrightarrow{yy'}$ must be of type 536 or 516. Further, since $d(\xi, \overrightarrow{yy'}) + d(\xi', \overrightarrow{yy'}) = d(\xi, \xi') = \pi$, the segments are of the same type. Observe also, that $\overrightarrow{yy'}$ cannot be antipodal to $\overrightarrow{yx}$ or $\overrightarrow{yx'}$, otherwise the $2A$-vertex y' would be antipodal to x or x'. Suppose w.l.o.g. that the segments $\xi\overrightarrow{yy'}\xi'$ and $\xi\overrightarrow{yx'}\xi'$ are of type 51615. This implies that the segment $\xi\overrightarrow{yx}\xi'$ is of type 53635. Since $\overrightarrow{yy'}$ is not antipodal to $\overrightarrow{yx}$, then the directions $\overrightarrow{\xi\overrightarrow{yx}}$ and $\overrightarrow{\xi\overrightarrow{yy'}}$ of type 3 and 1, respectively, cannot be antipodal, thus, they are adjacent (recall that these directions lie in a building of type $\overset{3}{\bullet}\!-\!\overset{4}{\bullet}\!-\!\overset{1}{\bullet}$). This implies that the segment $\overrightarrow{yx}\overrightarrow{yy'}$ has length $\arccos(\frac{1}{3})$ and is of type 676. It also follows that $\overrightarrow{yy'}$ lies on a segment of length π and type 67686 connecting $\overrightarrow{yx}$ and $\overrightarrow{yx'}$. Therefore, the segment $\overrightarrow{yx'}\overrightarrow{yy'}$ has length $\arccos(-\frac{1}{3})$ and is of type 686. Hence, $\Sigma_{\overrightarrow{yy'}} \Sigma_y K$ contains antipodal 7- and 8-vertices, that is, it contains a wall in the A_2-factor. Together with the apartment in the D_4-factor (compare with the beginning of Case 1), this implies that the link $\Sigma_{\overrightarrow{yy'}} \Sigma_y K$ contains a wall. It follows that the link in K of the 2-vertex $m(y, y')$ contains a wall, contradicting Lemma 4.6.5.

Thus, $diam(G \cdot y) = \arccos(-\frac{1}{4})$ and by relabeling y by x we have reduced the possibilities to the following case.

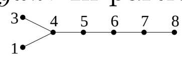
Case 2: $diam(G\cdot x) = \arccos(-\frac{1}{4})$. The simplicial convex hull C of xx' is 2-dimensional. Let $y, y' \in C$ be the 8-vertices adjacent to x and x', respectively. If $\overrightarrow{xy}$ has an antipode in $\Sigma_x K$, then there would be an 8-vertex in K at distance $\frac{3\pi}{4}$ to x', but this is not possible (cf. proof of Lemma 4.6.3). It follows that $d(\overrightarrow{xy}, S') \equiv \frac{\pi}{2}$, where $S' \subset S$ is the 3-sphere spanned by the 8-vertices in S. $\Sigma_{\overrightarrow{xy}} CH(\overrightarrow{xy}, S')$ is a 3-sphere spanned by four pairwise orthogonal 7-vertices.

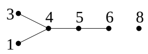
Let $w \in C$ be the 7-vertex $m(x, x')$ and let $x'' \in G\cdot x$ with $d(w, x'') > \frac{\pi}{2}$. The possible distances between 2- and 7-vertices in the Coxeter complex of type E_8 are of the form $\arccos(-\frac{k}{2\sqrt{6}})$ for k an integer (this can be deduced from the table of 2- and 7-vertices in Appendix A.7). Notice that $d(x, w) = d(w, x') = \arccos(\frac{3}{2\sqrt{6}})$. Triangle comparison for the triangle (x, x', x'') and $diam(G\cdot x) \leq \arccos(-\frac{1}{4})$ imply that $d(x'', w) = d(x'', m(x, x')) \leq \arccos(-\frac{1}{\sqrt{6}})$. If $d(w, x'') = \arccos(-\frac{1}{\sqrt{6}})$, then by rigidity, $CH(x, x', x'')$ is an equilateral spherical triangle with side lengths $\arccos(-\frac{1}{4})$. In particular $d(x, x'') = \arccos(-\frac{1}{4})$ and $\angle_x(x', x'') > \frac{\pi}{2}$.

If $d(w, x'') = \arccos(-\frac{1}{2\sqrt{6}})$, we may assume w.l.o.g. that $\angle_w(x, x'') \geq \frac{\pi}{2}$. This implies that $d(x, x'') \geq \arccos(-\frac{1}{8})$, i.e. $d(x, x'') = \arccos(-\frac{1}{4})$. Again by triangle comparison and $\angle_w(x, x'') \geq \frac{\pi}{2}$ we want to see that $CH(x, w, x'')$ must be a spherical triangle: let $\tilde{x}, \tilde{x''}$ be 2-vertices and let $\tilde{w}$ be a 7-vertex in the Coxeter complex of type E_8, such that $d(\tilde{x}, \tilde{w}) = d(x, w) = \arccos(\frac{3}{2\sqrt{6}})$, $d(\tilde{w}, \tilde{x''}) = d(w, x'') = \arccos(-\frac{1}{2\sqrt{6}})$ and $\angle_w(x, x'') = \angle_{\tilde{w}}(\tilde{x}, \tilde{x''})$. By triangle comparison, $d(\tilde{x}, \tilde{x''}) \leq d(x, x'') = \arccos(-\frac{1}{4})$, but since the angle $\angle_{\tilde{w}}(\tilde{x}, \tilde{x''}) = \angle_w(x, x'') \geq \frac{\pi}{2}$, then $d(\tilde{x}, \tilde{x''}) > \frac{\pi}{2}$. It follows that $d(\tilde{x}, \tilde{x''}) = \arccos(-\frac{1}{4}) = d(x, x'')$ and by rigidity $CH(x, w, x'')$ is a spherical triangle. We can now compute that $\angle_x(x', x'') = \arccos(-\frac{1}{15}) > \frac{\pi}{2}$.



Let C' be the 2-dimensional simplicial convex hull of xx'' and let $z, z' \in C'$ be the 8-vertices adjacent to x and x''. By considering the spherical triangle $CH(x, x', y)$, we can compute $\angle_x(y, x') = \arccos(\frac{3}{\sqrt{15}}) < \frac{\pi}{4}$. Then we can see that, if $\overrightarrow{xy} = \overrightarrow{xz}$, it follows $\angle_x(x', x'') < \frac{\pi}{2}$, thus $\overrightarrow{xy} \neq \overrightarrow{xz}$. They cannot be antipodal either, because $\overrightarrow{xy}$ has no antipodes in $\Sigma_x K$ (compare with the beginning of Case 2). Hence, the segment $\overrightarrow{xy}\overrightarrow{xz}$ has length $\frac{\pi}{2}$ and is of type 878.

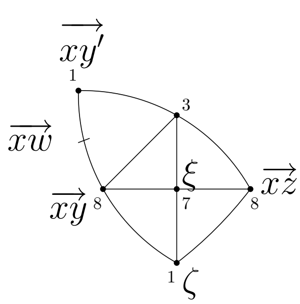
Let $\xi \in \Sigma_x K$ be the 7-vertex $m(\overrightarrow{xy}, \overrightarrow{xz})$. Notice that as for $\overrightarrow{xy}$, it also holds $d(\overrightarrow{xz}, S') \equiv \frac{\pi}{2}$. This implies that the convex hull of S' and the segment $\overrightarrow{xy}\overrightarrow{xz}$ is isometric to the spherical join $S' \circ \overrightarrow{xy}\overrightarrow{xz}$. In particular, $d(\xi, S') \equiv \frac{\pi}{2}$. Notice that in a building of type D_7 with Dynkin diagram 

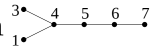
, a 7- and an 8-vertex at distance $\frac{\pi}{2}$ are joined by a segment of type 768. It follows that $\Sigma_\xi \Sigma_x K$ (of type 

) contains a 3-sphere spanned by four pairwise orthogonal 6-vertices. This 3-sphere is not simplicial, and its simplicial convex hull is an apartment in the D_5-

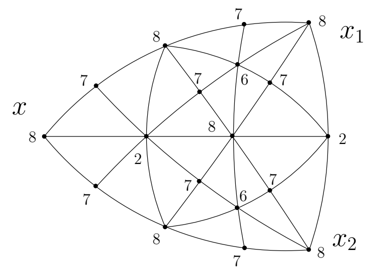
factor of $\Sigma_\xi \Sigma_x K$. Since $\{\overrightarrow{xy}, \overrightarrow{xz}\}$ is an apartment in the A_1-factor of $\Sigma_\xi \Sigma_x K$, it follows that $\Sigma_\xi \Sigma_x K$ contains an apartment. In particular ξ is an interior 7-vertex in $\Sigma_x K$.

We can also see, that if both 1-vertices $\overrightarrow{xy'}$ and $\overrightarrow{xz'}$ are adjacent to ξ, then $\angle_x(x', x'') < \frac{\pi}{2}$, because in this case $d(\xi, \overrightarrow{xw}) = d(\xi, \overrightarrow{xx''}) = \arccos(\frac{2\sqrt{2}}{\sqrt{15}}) < \frac{\pi}{4}$ (just consider the spherical triangle $CH(\overrightarrow{xy}, \overrightarrow{xw}, \xi)$ with sides $d(\overrightarrow{xy}, \overrightarrow{xw}) = \arccos(\frac{3}{\sqrt{15}})$, $d(\overrightarrow{xy}, \xi) = \frac{\pi}{4}$ and angle $\angle_{\overrightarrow{xy}}(\overrightarrow{xw}, \xi) = \arccos(\frac{1}{\sqrt{6}})$).



Therefore w.l.o.g. $\overrightarrow{xy'}$ is not adjacent to ξ, but since both are adjacent to $\overrightarrow{xy}$, the angle $\angle_{\overrightarrow{xy}}(\xi, \overrightarrow{xy'})$ must be of type 731, because $\Sigma_{\overrightarrow{xy}} \Sigma_x B$ is of type D_6 with Dynkin diagram . Now recall that ξ is an interior vertex in $\Sigma_x K$, this implies that we can find a 1-vertex $\zeta \in \Sigma_x K$, so that $\overrightarrow{xy'}\overrightarrow{xy}\zeta$ is a segment of type 181. Thus, the link $\Sigma_{\overrightarrow{xy}} \Sigma_x K$ (of type D_6) contains a pair of antipodal 1-vertices and a 3-sphere spanned by four pairwise orthogonal 7-vertices (compare with the beginning of Case 2). We can apply Lemma 3.1.4 to see that $\Sigma_{\overrightarrow{xy}} \Sigma_x K$ contains a wall. By Lemma 3.1.3, $\Sigma_{\overrightarrow{\overrightarrow{xy}\overrightarrow{xw}}} \Sigma_{\overrightarrow{xy}} \Sigma_x K$ contains at least a wall. This implies that $\Sigma_{\overrightarrow{xw}} \Sigma_x K$ contains a wall and $\Sigma_w K$ contains a wall of type 1, contradicting Lemma 4.6.6. □

Let $x \in K$ be an $8A$-vertex. We say that x has the *property* T, if there is no spherical triangle in K with $8A$-vertices x, x_1 and 8-vertex x_2, with side lengths $d(x, x_i) = \frac{2\pi}{3}$, $d(x_1, x_2) = \frac{\pi}{2}$, and such that the direction $\overrightarrow{xx_2}$ is $\frac{2\pi}{3}$-extendable to an $8A$-vertex in K. This last assumption is fulfilled if e.g. x_2 is also an $8A$-vertex.

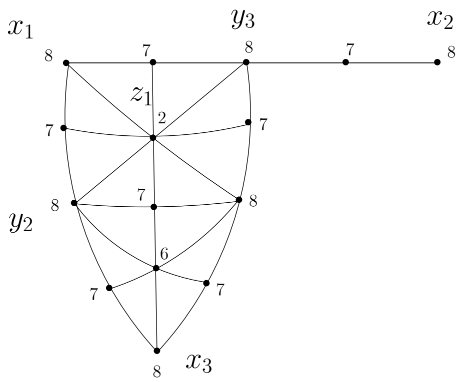


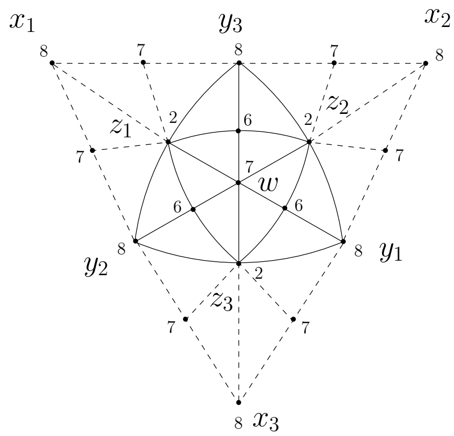
Let $x_1, x_2, x_3 \in K$ be $8T$-vertices as in configuration $*$. If $\angle_{y_3}(x_3, x_2) = \arccos(\frac{1}{3})$, then the simplicial convex hull of $y_3, x_3, m(y_3, x_2)$ is a spherical triangle with vertices x_3, y_3, x_2' and sides y_3x_3, x_3x_2' and $x_2'y_3$ of type 87878, 828 and 878, respectively, and $m(y_3, x_2) = m(y_3, x_2')$. It follows that the simplicial convex hull of $x_1, m(y_3, x_2), x_3$ is a spherical triangle in K as ruled out by the property T, hence the property T implies that $\angle_{y_3}(x_3, x_i) = \arccos(-\frac{1}{3})$ and $d(x_3, x_i) = \frac{2\pi}{3}$ for $i = 1, 2$. Thus, $\angle_{x_i}(x_{i-1}, x_{i+1}) = \arccos(-\frac{1}{3}))$ for $i = 1, 2$ (the indices to be understood modulo 3) and these angles are of type 727. Let $y_1 := m(x_2, x_3)$ and $y_2 := m(x_1, x_3)$. Then it also follows that $d(x_i, y_i) = \frac{2\pi}{3}$ for $i = 1, 2$. Consider the vertices x_1, x_3, x_2, y_2, then we are again in the situation of the configuration $*$ (just exchange the indices $2 \leftrightarrow 3$). It follows as above that $\angle_{y_2}(x_2, x_3) = \arccos(-\frac{1}{3})$ because x_1 is an $8T$-vertex. This implies that $\angle_{x_3}(x_1, x_2) = \arccos(-\frac{1}{3})$ as well, and this angle is of type 727.

The convex hulls $CH(x_i, y_j, x_j)$ for distinct $i, j = 1, 2, 3$ are isosceles spherical triangles with sides of type 87878, 87878 and 878. This implies $d(y_i, y_{i+1}) = \frac{\pi}{2}$ and the segments

$y_i y_{i+1}$ are of type 828. The intersection $CH(x_i, y_{i+1}, x_{i+1}) \cap CH(x_i, y_{i-1}, x_{i-1})$ is the spherical triangle $CH(x_i, y_{i-1}, y_{i+1})$ with sides of type 878, 878 and 828. In particular the 8-vertices $m(x_i, y_i)$ are pairwise distinct.

Observe that the 2-vertices $\overrightarrow{y_3y_2}, \overrightarrow{y_3y_1} \in \Sigma_{y_3}K$ are adjacent to the antipodal 7-vertices $\overrightarrow{y_3x_1}, \overrightarrow{y_3x_2}$, respectively. This implies that $d(\overrightarrow{y_3y_2}, \overrightarrow{y_3y_1}) \geq \arccos(\frac{1}{3}) > \frac{\pi}{3}$, thus $d(\overrightarrow{y_3y_2}, \overrightarrow{y_3y_1}) \geq \frac{\pi}{2}$. On the other hand, triangle comparison for the triangle (y_1, y_2, y_3) implies $d(\overrightarrow{y_3y_2}, \overrightarrow{y_3y_1}) \leq \frac{\pi}{2}$ and it follows that this triangle is rigid, i.e. the convex hull $CH(y_1, y_2, y_3)$ is an equilateral spherical triangle with sides of type 828. Let $z_i := m(y_i, y_{i-1})$. Notice that z_i does not lie on the segment $x_i y_i$ of type 87878. Let w be the 7-vertex at the center of the triangle $CH(y_1, y_2, y_3)$ and consider the spherical triangles $CH(x_i, z_i, y_i)$ for $i = 1, 2, 3$ with sides of type 82, 2768 and 87878. Notice that w is the 7-vertex on the segments $z_i y_i$. It follows that w is adjacent to the $8A$-vertices $m(x_i, y_i)$ for $i = 1, 2, 3$ and in particular, $\Sigma_w K$ contains three pairwise antipodal 8-vertices.





We say that an $8T$-vertex $x \in K$ has the *property* T', if $rad(z_i, \{\text{8-vert. in } K\}) \leq \arccos(-\frac{1}{2\sqrt{2}})$ for $i = 1, 2, 3$ and for any such configuration of vertices $x_1, x_2, x_3 \in G \cdot x$.

Lemma 4.6.10. *K contains no $8T'$-vertices.*

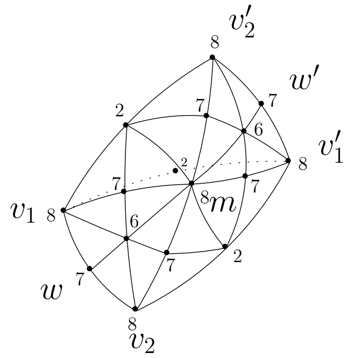
Proof. Suppose there are $8T'$-vertices. We use the notation as in the definition of the property T'. Let w be the center of the triangle $CH(y_1, y_2, y_3)$.

Let $u \in K$ be an 8-vertex. Then for some $i = 1, 2, 3$, $\angle_w(z_i, u) \geq \frac{\pi}{2}$. Suppose w.l.o.g. that it holds for $i = 1$. If $d(w, u) = \frac{5\pi}{6}$, then $\overrightarrow{wu}$ is an 8-vertex and $\angle_w(z_1, u) = \frac{\pi}{2}$. It follows that $d(u, z_1) = \frac{3\pi}{4}$, but this contradicts the definition of the property T'. If $d(w, u) = \arccos(-\frac{1}{\sqrt{3}})$, then $\overrightarrow{wu}$ is a 2-vertex and $\angle_w(z_1, u) = \frac{2\pi}{3}$. It follows again that $d(u, z_1) = \frac{3\pi}{4}$. Hence, $d(w, u) \leq \arccos(-\frac{1}{2\sqrt{3}})$ for all 8-vertices $u \in K$ and as in the beginning of the proof of Lemma 4.6.4 we deduce by triangle comparison that if $w' \in G \cdot w$, then $d(w, w') \leq \arccos(-\frac{1}{3})$. We may also choose w', so that $d(w, w') > \frac{\pi}{2}$.

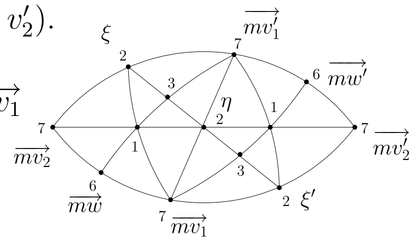
Case 1: $d(w, w') = \arccos(-\frac{1}{3})$. If the segment ww' is singular of type 76867, then for some $i = 1, 2, 3$, $\angle_w(y_i, w') = \frac{2\pi}{3}$ and this angle is of type 626. It follows that $d(w', y_i) =$

$\arccos(-\frac{1}{\sqrt{3}})$, a contradiction. If the simplicial convex hull of ww' is 2-dimensional, we can argue as in the proof of Lemma 4.6.4 (Case 1) to see that this case is not possible either.

Case 2: $d(w,w') = \arccos(-\frac{1}{6})$. The argument in the proof of Lemma 4.6.4 (Case 2) rules out the case where ww' has a 2-dimensional simplicial convex hull.

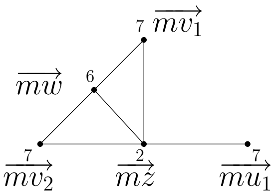


It remains to show that the case where the simplicial convex hull C of ww' is 3-dimensional is not possible either. Let $v_1, v_1' \in C$ be the 8-vertices adjacent to w and w', respectively. Notice that they are $8A$-vertices, otherwise an antipode of e.g. v_1 in K would have distance $\frac{5\pi}{6}$ to w; but this cannot happen. Recall that there is an 8-vertex $m \in C$, such that mw and mw' are segments of type 867 and $\angle_m(w,w') = \arccos(-\frac{3}{4})$. Let $v_2 \in K$ be an $8A$-vertex adjacent to w and so that v_1wv_2 is a segment of type 878. We can choose v_2 to be one of the $8A$-vertices $m(x_i,y_i)$. Define v_2' analogously. Then the convex hulls $CH(m,v_1,v_2)$ and $CH(m,v_1',v_2')$ are equilateral spherical triangles with sides of type 878.



We want now to consider the convex hull $C' := CH(C, v_2, v_2')$. The link $\Sigma_m C$ is a 2-dimensional spherical quadrilateral with vertices $\overrightarrow{mw}$, $\overrightarrow{mv_1}$, $\overrightarrow{mw'}$ and $\overrightarrow{mv_1'}$. Notice that $\overrightarrow{mv_2}\overrightarrow{mw}\overrightarrow{mv_1}$ and $\overrightarrow{mv_2'}\overrightarrow{mw'}\overrightarrow{mv_1'}$ are segments of type 767. It follows that $CH(\Sigma_m C, \overrightarrow{mv_2}, \overrightarrow{mv_2'})$ is a bigon connecting the antipodal 7-vertices $\overrightarrow{mv_2}$ and $\overrightarrow{mv_2'}$. Then $d(v_2,v_2') = \frac{2\pi}{3}$ and $m = m(v_2,v_2')$, in particular, m is an $8A$-vertex. Let $\xi, \xi' \in \Sigma_m C'$ be the 2-vertices $m(\overrightarrow{mv_1'}, \overrightarrow{mv_2})$ and $m(\overrightarrow{mv_1}, \overrightarrow{mv_2'})$. Let η be the 2-vertex $m(\overrightarrow{mv_1}, \overrightarrow{mv_1'})$. The convex hulls $CH(v_1,v_2,v_2')$ and $CH(v_1',v_2',v_2)$ are spherical triangles with sides of type 878, 87878 and 828.

Since $m \in K$ is contained in the convex hull of the $8T'$-vertices, it is also contained in the convex hull of the $8T$-vertices. We can find another $8T$-vertex $u_1 \in K$, such that $d(m,u_1) = \frac{2\pi}{3}$. Notice that the $8A$-vertex u_1 cannot be antipodal to v_2 or v_2', in particular, $\angle_m(u_1,v_2), \angle_m(u_1,v_2') < \pi$. Suppose w.l.o.g. that $\angle_m(u_1,v_2) \geq \frac{\pi}{2}$. Then $\angle_m(u_1,v_2) = \arccos(-\frac{1}{3})$ and $d(u_1,v_2) = \frac{2\pi}{3}$. $CH(v_2,m,u_1)$ is an isosceles spherical triangle (as in the configuration $*$) with a 2-vertex z in its interior. Recall that $d(w,u_1) \leq \arccos(-\frac{1}{2\sqrt{3}})$. This implies that $\angle_{v_2}(w,u_1) \leq \arccos(\frac{1}{3})$. This angle cannot be 0, because $\angle_{v_2}(m,w) = \arccos(\frac{1}{3})$ and $\angle_{v_2}(m,u_1) = \arccos(-\frac{1}{3})$. Thus $\angle_{v_2}(w,u_1) = \arccos(\frac{1}{3})$ and it is of type 767. $CH(\overrightarrow{v_2w}, \overrightarrow{v_2m}, \overrightarrow{v_2u_1})$ is then a spherical triangle with sides of type 767, 767 and 727. In particular w is adjacent to the 2-vertex z.

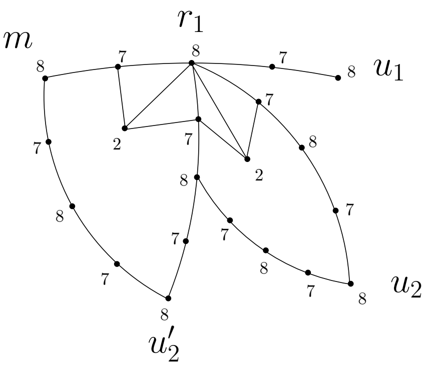


This consideration implies in the link $\Sigma_m K$ that $\overrightarrow{mz}$ and $\overrightarrow{mw}$ are adjacent. Suppose that the segment $\overrightarrow{mv_1}\overrightarrow{mu_1}$ is of type 727. This implies that the angle $\angle_{\overrightarrow{mv_1}}(\overrightarrow{mu_1}, \xi')$ is of type 262. It follows that the segment $\overrightarrow{mu_1}\xi'$ is of type 7672. Hence, $d(u_1, m(v_2', v_1)) = \frac{3\pi}{4}$ and $CH(v_1, v_2', u_1)$ is a spherical triangle with sides 87878, 87878 and 828. But this contradicts the definition of the property T for u_1. Therefore the segment $\overrightarrow{mv_1}\overrightarrow{mu_1}$ is of type 767.

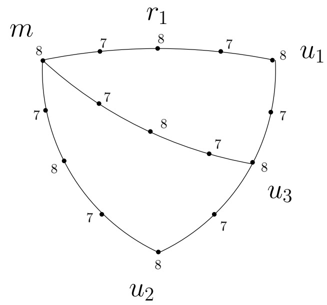
If $\angle_m(u_1, v_2') = \arccos(-\frac{1}{3})$ we argue analogously and conclude that the segment $\overrightarrow{mv_1'}\overrightarrow{mu_1}$ is of type 767. If $\angle_m(u_1, v_2') = \arccos(\frac{1}{3})$ we see as above that $d(\overrightarrow{mu_1}, \xi) \leq \frac{\pi}{2}$, otherwise we violate the property T for u_1. Using triangle comparison with the triangle $(\xi, \overrightarrow{mu_1}, \overrightarrow{mv_2'})$ (or using the convexity of the ball centered at $\overrightarrow{mu_1}$ with radius $\frac{\pi}{2}$) we see that $d(\overrightarrow{mv_1'}, \overrightarrow{mu_1}) \leq \arccos(\frac{1}{3})$. Since $\overrightarrow{mv_1}\overrightarrow{mu_1}$ is of type 767, then $\overrightarrow{mu_1} \neq \overrightarrow{mv_1'}$. Thus, $d(\overrightarrow{mv_1'}, \overrightarrow{mu_1}) = \arccos(\frac{1}{3})$ and the segment $\overrightarrow{mv_1'}\overrightarrow{mu_1}$ is of type 767 also in this case. It follows that $CH(\overrightarrow{mv_1'}, \overrightarrow{mv_1}, \overrightarrow{mu_1})$ is a spherical triangle with sides 767, 767 and 727. In particular $\overrightarrow{mu_1}$ is adjacent to η.

We have shown so far that any 7-vertex in $\Sigma_m K$ that is $\frac{2\pi}{3}$-extendable to an $8T$-vertex in K must be adjacent to η and the segments connecting it with $\overrightarrow{mv_1}$ and $\overrightarrow{mv_1'}$ are of type 767.

Let $r_1 := m(m, u_1) \in K$ and let $u_2' \in K$ be an $8T$-vertex with $d(r_1, u_2') = \frac{2\pi}{3}$. Since u_1 is an $8T$-vertex, the angle $\angle_{r_1}(m, u_2')$ cannot be of type 767. Hence, it is of type 727. If the angle $\angle_{r_1}(u_1, u_2')$ is also of type 727, then set $u_2 := u_2'$.

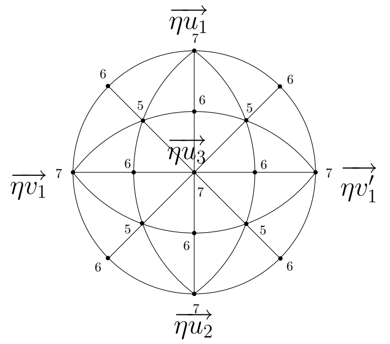


Otherwise, let $u_2 \in K$ be another $8T$-vertex, so that $d(u_2, m(r_1, u_2')) = \frac{2\pi}{3}$. Again, because u_2' is an $8T$-vertex, the angle $\angle_{m(r_1,u_2')}(r_1, u_2)$ is of type 727. In particular $d(r_1, u_2) = \frac{2\pi}{3}$ and again $\angle_{r_1}(m, u_2)$ is of type 727. We want to see now, that $\angle_{r_1}(u_1, u_2)$ is also of type 727. Suppose that $\angle_{r_1}(u_1, u_2)$ is of type 767. Then $CH(\overrightarrow{r_1u_2}, \overrightarrow{r_1u_1}, \overrightarrow{r_1u_2'})$ is a spherical triangle with sides of type 767, 767 and 727. In particular $\overrightarrow{r_1u_1}$ is adjacent to $\delta := m(\overrightarrow{r_1u_2}, \overrightarrow{r_1u_2'})$, this means that the segment $\delta\overrightarrow{r_1m}$ is of type 2767. Notice that this is the configuration $**$ for the vertices r_1, u_2', u_2, m. This implies that $CH(r_1, u_2, m(m, u_2'))$ is a spherical triangle with vertices of type $8A$ and sides of type 87878, 87878 and 828 and u_2 could not be an $8T$-vertex, a contradiction.



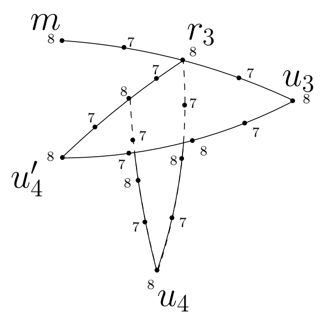
Thus $\angle_{r_1}(u_1, u_2)$ is of type 727. This implies that $d(u_1, u_2) = \frac{2\pi}{3}$ and $\angle_{u_1}(m, u_2)$ is of type 727. Let $u_3 \in K$ be the $8A$-vertex $m(u_1, u_2)$, then $\angle_{u_1}(m, u_3)$ is of type 727 and this implies that $d(m, u_3) = \frac{2\pi}{3}$. Observe that u_3 is not necessarily an $8T$-vertex. Notice that $\overrightarrow{mu_1}\overrightarrow{mu_2}$ is of type 727 and recall that $\overrightarrow{mu_i}$ is adjacent to η for $i = 1, 2$. It follows that $\eta = m(\overrightarrow{mu_1}, \overrightarrow{mu_2})$. In particular η is $\frac{\pi}{2}$-extendable in K. Consider the triangles (m, u_1, u_3) and (m, u_2, u_3), then by triangle comparison, it follows that $\angle_m(u_1, u_3), \angle_m(u_2, u_3) \leq \arccos(\frac{1}{3})$ and since $\angle_m(u_1, u_2) = \arccos(-\frac{1}{3})$, this implies that, $\angle_m(u_1, u_i) = \arccos(\frac{1}{3})$ and $\overrightarrow{mu_3}\overrightarrow{mu_i}$ is of type 767 for $i = 1, 2$. Hence, $CH(\overrightarrow{mu_1}, \overrightarrow{mu_2}, \overrightarrow{mu_3})$ is a spherical triangle with sides of type 767, 767 and 727. In particular, $\overrightarrow{mu_3}$ is adjacent to η as well.

Write $\overrightarrow{\eta\star} := \overrightarrow{\eta\overrightarrow{m\star}} \in \Sigma_\eta\Sigma_m K$, where $\star$ is any vertex in K adjacent to m, so that $\overrightarrow{m\star} \in \Sigma_m K$ is adjacent to η.



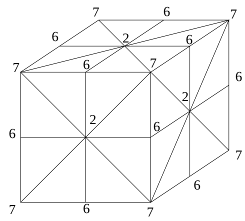
The 7-vertices $\overrightarrow{\eta v_1}$, $\overrightarrow{\eta v_1'}$, $\overrightarrow{\eta u_1}$ and $\overrightarrow{\eta u_2}$ are the 7-vertices of a circle $c \subset \Sigma_\eta\Sigma_m K$ of type 767676767, because as seen above, $\overrightarrow{\eta u_i}$ for $i = 1, 2$ is the midpoint of a geodesic of length π connecting $\overrightarrow{\eta v_1}$ and $\overrightarrow{\eta v_1'}$, and $\overrightarrow{\eta u_i}$ are antipodal for $i = 1, 2$. From the construction above we see that $d(\overrightarrow{\eta u_3}, \overrightarrow{\eta u_i}) = \frac{\pi}{2}$ for $i = 1, 2$ (the segments $\overrightarrow{\eta u_3}\overrightarrow{\eta u_i}$ are of type 767). Suppose $\overrightarrow{\eta u_3}$ is antipodal to $\overrightarrow{\eta v_1}$. This would imply that the segment $\overrightarrow{mu_3}\overrightarrow{mw} \subset \Sigma_m K$ is of type 7316 and therefore $d(\overrightarrow{mu_3}, \overrightarrow{mw}) > \frac{\pi}{2}$ (compare with the figure for $\Sigma_m C'$ above). Consider now the triangle (w, m, u_3), it has sides $d(m, w) = \arccos(\frac{1}{\sqrt{3}})$, $d(m, u_3) = \frac{2\pi}{3}$ and angle $\angle_m(w, u_3) > \frac{\pi}{2}$. It follows that $d(w, u_3) > \arccos(-\frac{1}{2\sqrt{3}})$, which is not possible. Hence, $d(\overrightarrow{\eta u_3}, \overrightarrow{\eta v_1}) = d(\overrightarrow{\eta u_3}, \overrightarrow{\eta v_1'}) = \frac{\pi}{2}$. Therefore $\overrightarrow{\eta u_3}$ is the center of a 2-dimensional hemisphere in $\Sigma_\eta\Sigma_m K$ bounded by c.

Let $r_3 := (m, u_3) \in K$ and let $u_4' \in K$ be another $8T$-vertex, so that $d(r_3, u_4') = \frac{2\pi}{3}$. Recall that u_3 is not necessarily an $8T$-vertex, therefore we cannot conclude directly that $\angle_{r_3}(m, u_4')$ is of type 727. If $\angle_{r_3}(m, u_4')$ is actually of type 727, then set $u_4 := u_4'$.



Otherwise (i.e. if $\angle_{r_3}(m, u_4')$ is of type 767), let $u_4 \in K$ be an $8T$-vertex, so that $d(u_4, m(r_3, u_4')) = \frac{2\pi}{3}$. Then, since u_4' is an $8T$-vertex, the angle $\angle_{m(r_3,u_4')}(r_3, u_4)$ must be of type 727. This implies that $d(r_3, u_4) = \frac{2\pi}{3}$ and the angle $\angle_{r_3}(u_4, u_4')$ is of type 727. It follows that $\angle_{r_3}(m, u_4)$ is of type 727, otherwise (as in the argument above for u_2) we find the configuration $**$ and $CH(u_3, u_4, m(r_3, u_4'))$ is a spherical triangle with sides of type 87878, 87878 and 828, contradicting the property T for u_4. From this we conclude that $d(m, u_4) = \frac{2\pi}{3}$ and $\angle_m(u_3, u_4)$ is of type 727. Recall that $\overrightarrow{mu_4}$ must be adjacent to η. This implies that $\overrightarrow{\eta u_4}$ is antipodal to $\overrightarrow{\eta u_3}$.

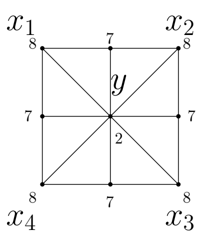
Thus, $\Sigma_\eta\Sigma_m K$ (of type D_6) contains a singular 2-sphere spanned by 3 pairwise orthogonal 7-vertices. Recall that it also contains a pair of antipodal 3-vertices $\overrightarrow{\eta\xi}$ and $\overrightarrow{\eta\xi'}$. Lemma 3.1.4 implies that $\Sigma_\eta\Sigma_m K$ contains a 3-sphere spanned by a simplex of type 1567. Since η is $\frac{\pi}{2}$-extendable in K, we have found a 2-vertex in K, whose link contains a 4-sphere spanned by a simplex of type 15678. This 4-sphere is of type $\frac{\pi}{3}$ (this can be easily seen in the vector space realization of the Coxeter complex of type D_n presented in Appendix A). A contradiction to Lemma 4.6.9. □

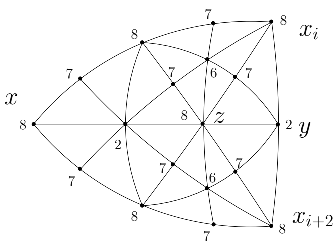


Let B_3 be the property of an $8A$-vertex $x \in K$, such that $\Sigma_x K$ contains a singular 2-sphere with B_3-geometry 7 6 2, and such that all the 7-vertices in this sphere are $\frac{\pi}{3}$-extendable.

Consider the configuration $**$ and notice that the 8-vertex v on the segment zx_3 (of type 2828) adjacent to z is an $8B_3$-vertex.

Another similar way of finding $8B_3$-vertices is the following. Let $x_1, x_2, x_3, x_4 \in K$ be $8A$-vertices adjacent to a 2-vertex y, so that $CH(x_i)$ is a 2-dimensional spherical quadrilateral with sides $x_i x_{i+1}$ of type 878. Let $x \in K$ be an 8-vertex at distance $\frac{3\pi}{4}$ to y. Since the x_i are $8A$-vertices, it follows that $\angle_y(x, x_i) = \frac{\pi}{2}$. This implies that $\Sigma_{\overrightarrow{yx}}\Sigma_y K$ contains a singular circle of type 767676767. Let z be the 8-vertex in yx adjacent to y. Then $\Sigma_z K$ contains a 2-sphere with B_3-geometry 7 6 2. Considering the spherical triangles $CH(x, x_i, x_{i+2})$, we see that the 7-vertices in this 2-sphere are $\frac{\pi}{3}$-extendable. Hence z is an $8B_3$-vertex.





Consider now the definition of the property T'. The 2-vertices z_i are centers of 2-dimensional spherical quadrilaterals as described above. In particular, if there are no $8B_3$-vertices in K, then it follows from the observation above, that $rad(z_i, \{\text{8-vert. in } K\}) \leq \arccos(-\frac{1}{2\sqrt{2}})$ for $i = 1, 2, 3$. Hence, if K contains no $8B_3$-vertices, it follows that the property T implies the property T'.

Recall that our strategy is to find spheres of large dimension in the links of vertices of type 2 or 8. Notice that we have made the first step in this direction:

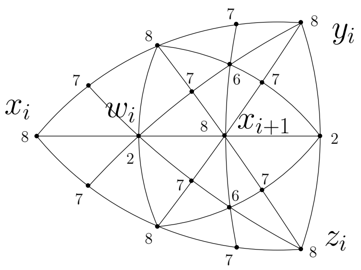
Corollary 4.6.11. *If K contains $8A$-vertices, then it contains 8-vertices, whose links in K contain a singular circle.*

Proof. If K contains $8B_3$-vertices, we are done. Otherwise, $8T \Rightarrow 8T'$, and Lemma 4.6.10 implies that there are no $8T$-vertices in K. In particular, we find a spherical triangle in K with sides of type 87878, 87878 and 828. The link in K of the 8-vertex in the interior of this triangle contains a singular circle. □

Now we find 8-vertices, such that their links contain singular 2-spheres.

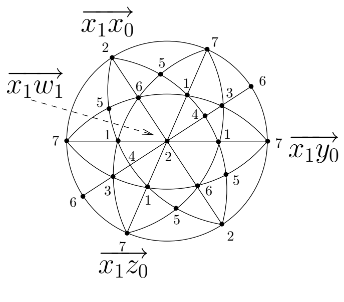
Lemma 4.6.12. *If K contains $8A$-vertices, then it also contains $8B_3$-vertices.*

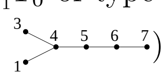
Proof. Suppose that K contains $8A$-vertices but no $8B_3$-vertices. Then, $8T \Rightarrow 8T'$ and Lemma 4.6.10 implies that there are no $8T$-vertices in K.



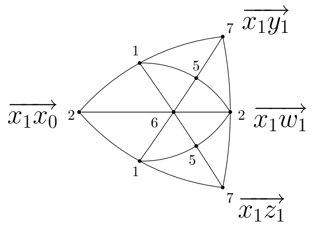
Hence, there are $8A$-vertices $x_0, y_0 \in K$ and an 8-vertex $z_0 \in K$, so that $T_0 := CH(x_0, y_0, z_0)$ is a spherical triangle with sides of type 87878, 87878 and 828; where $y_0 z_0$ is the side of type 828 (as in the definition of the property T). Let $x_1 \in K$ be the $8A$-vertex on the segment $x_0 m(y_0, z_0)$ (of type 8282) adjacent to the 2-vertex $m(y_0, z_0)$. Since x_1 is not an $8T$-vertex, we can find 8-vertices $y_1, z_1 \in K$ as vertices of a spherical triangle $T_1 := CH(x_1, y_1, z_1)$ as above. Define $x_i, y_i, z_i \in K$ and $T_i \subset K$ inductively. Let w_i be the $2A$-vertex $m(x_i, x_{i+1})$.

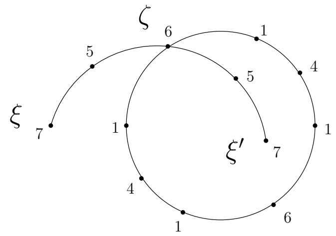
If $\xi \in \Sigma_{x_i} K$ is a $\frac{\pi}{3}$-extendable 7-vertex and $d(\xi, \overrightarrow{x_i x_{i+1}}) = \arccos(-\frac{1}{\sqrt{3}})$, then we are in the setting of the configuration $**$ because $\overrightarrow{x_i y_i}$ and $\overrightarrow{x_i z_i}$ are both $\frac{2\pi}{3}$-extendable to $8A$-vertices (definition of the property T). This implies that there are $8B_3$-vertices in K, contradicting our assumption. Hence, $\overrightarrow{x_i x_{i+1}}$ has distance $\leq \frac{\pi}{2}$ to all $\frac{\pi}{3}$-extendable 7-vertices in $\Sigma_{x_i} K$. Notice also that $d(\overrightarrow{x_i x_{i-1}}, \overrightarrow{x_i y_i})$ and $d(\overrightarrow{x_i x_{i-1}}, \overrightarrow{x_i z_i})$ are both $\leq \frac{\pi}{2}$, otherwise w_{i-1} would have distance $\frac{3\pi}{4}$ to the 8-vertex y_i or z_i and we would find an $8B_3$-vertex on the segment $w_{i-1} y_i$ ($w_{i-1} z_i$).



From these observations it follows, that $\overrightarrow{x_1 w_1}$ has distance $\equiv \frac{\pi}{2}$ to the circle $\Sigma_{x_1} T_0$ of type 727672767. This implies that $\Sigma_{\overrightarrow{x_1 w_1}} \Sigma_{x_1} K$ (of type ) contains a singular circle of type 161416141. It also contains the pair of antipodal 7-vertices $\xi := \overrightarrow{\overrightarrow{x_1 w_1} \overrightarrow{x_1 y_1}}$ and $\xi' := \overrightarrow{\overrightarrow{x_1 w_1} \overrightarrow{x_1 z_1}}$.

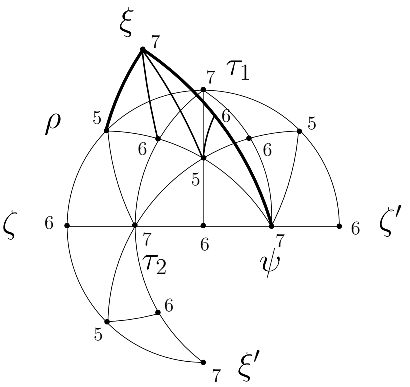
Since $d(\overrightarrow{x_1 x_0}, \overrightarrow{x_1 y_1})$, $d(\overrightarrow{x_1 x_0}, \overrightarrow{x_1 z_1}) \leq \frac{\pi}{2}$ and $d(\overrightarrow{x_1 x_0}, \overrightarrow{x_1 w_1}) = \frac{\pi}{2}$, it follows from triangle comparison that $d(\overrightarrow{x_1 x_0}, \overrightarrow{x_1 y_1}) = d(\overrightarrow{x_1 x_0}, \overrightarrow{x_1 z_1}) = \frac{\pi}{2}$, because the triangle $(\overrightarrow{x_1 x_0}, \overrightarrow{x_1 y_1}, \overrightarrow{x_1 z_1})$ must be rigid. Let $\zeta := \overrightarrow{\overrightarrow{x_1 w_1} \overrightarrow{x_1 x_0}}$. Then the segments $\zeta\xi$ and $\zeta\xi'$ have length $\frac{\pi}{2}$ and are of type 657.

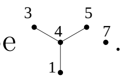




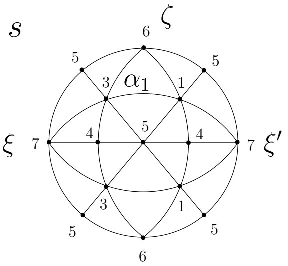
Sublemma 4.6.13. $\Sigma_{\overrightarrow{x_1w_1}}\Sigma_{x_1}K$ *contains a singular circle of type* 756575657. *This circle contains the vertices* ξ, ξ' *and* ζ.

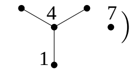
Proof. Let $\zeta' \in \Sigma_{\overrightarrow{x_1w_1}}\Sigma_{x_1}K$ be the 6-vertex in the circle of type 161416141 antipodal to ζ. If $d(\xi,\zeta') = \frac{\pi}{2}$, then $\zeta\xi\zeta'$ is a geodesic of type 65756. In particular, $\overrightarrow{\xi\zeta}$ has an antipode in $\Sigma_\xi\Sigma_{\overrightarrow{x_1w_1}}\Sigma_{x_1}K$ and we find the desired circle. If $d(\xi,\zeta') > \frac{\pi}{2}$, then the segment $\xi\zeta'$ is of type 7676.



Let ρ be the 5-vertex on the segment $\zeta\xi$ and let ψ be the 7-vertex on the segment $\xi\zeta'$ adjacent to ζ'. Consider the geodesics c_ρ and c_ψ of length π connecting ζ and ζ' through ρ and ψ. Let τ_1 be the 7-vertex at the center of c_ρ and τ_2 be the 7-vertex in c_ψ adjacent to ζ. Then ρ and τ_2 are adjacent because $\Sigma_\zeta\Sigma_{\overrightarrow{x_1w_1}}\Sigma_{x_1}K$ is of type . ξ cannot be adjacent to the 6-vertex at the center of c_ψ, otherwise it would have distance $\frac{3\pi}{4}$ to ζ. Thus, the intersection of the segments $\xi\zeta'$ and c_ψ is the segment $\psi\zeta'$. Considering the spherical triangle $CH(\rho,\xi,\psi)$ with sides of type 57, 767 and 7565, it follows that ξ is adjacent to the 6-vertex $m(\tau_1,\tau_2)$ on the segment $\rho\psi$. In particular, ξ' must be antipodal to at least one of τ_1 or τ_2. Since τ_2 is adjacent to ζ and $d(\zeta,\xi') = \frac{\pi}{2}$, then ξ' cannot be antipodal to τ_2. It follows that ξ' and τ_1 are antipodal. Let finally c be the geodesic connecting τ_1 and ξ', so that the initial direction coincides with $\overrightarrow{\tau_1\zeta'}$. Then the initial direction of c at ξ' is antipodal to $\overrightarrow{\xi'\zeta}$ and we can find the desired circle. □

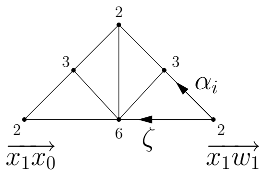
Continuation of proof of Lemma 4.6.12.

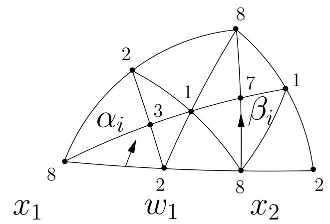


The link $\Sigma_\zeta\Sigma_{\overrightarrow{x_1w_1}}\Sigma_{x_1}K$ (of type ) contains a pair of antipodal 5-vertices $\overrightarrow{\zeta\xi}$ and $\overrightarrow{\zeta\xi'}$ and a pair of antipodal 1-vertices. We apply Lemma 3.1.1 and Remark 3.1.2 to conclude that $\Sigma_\zeta\Sigma_{\overrightarrow{x_1w_1}}\Sigma_{x_1}K$ contains a singular circle of type 5135135 with $\overrightarrow{\zeta\xi}$ and $\overrightarrow{\zeta\xi'}$ on it. It follows now from Sublemma 4.6.13 that $\Sigma_{\overrightarrow{x_1w_1}}\Sigma_{x_1}K$ contains a singular 2-sphere s containing the vertices ζ, ξ and ξ'. Therefore $\Sigma_{x_2}K$ contains a singular 3-sphere S containing the singular circle $\Sigma_{x_2}T_1$.

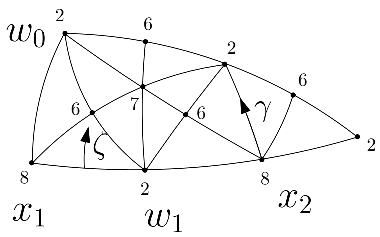
We investigate below which 7-vertices in S are $\frac{\pi}{3}$-extendable. Clearly the 7-vertices in $\Sigma_{x_2}T_1 \subset S$ are $\frac{\pi}{3}$-extendable.

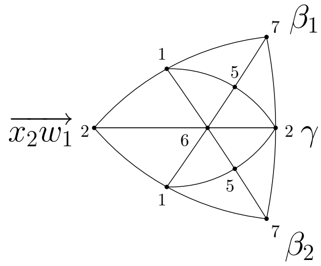
Let $\alpha_1, \alpha_2 \in s$ be the 3-vertices adjacent to ζ and recall that ζ is $\frac{\pi}{2}$-extendable (to $\overrightarrow{x_1x_0}$) in $\Sigma_{x_1}K$. This implies that α_i is $\frac{\pi}{3}$-extendable to a segment of type 232 in $\Sigma_{x_1}K$. Therefore, we find 7-vertices $\beta_1, \beta_2 \in S$ at distance $\frac{\pi}{2}$ to $\overrightarrow{x_2w_1}$ which are $\frac{\pi}{3}$-extendable in K (compare with the figure below).





The segment $\alpha_1\alpha_2 \subset \Sigma_{x_1w_1}\Sigma_{x_1}K$ is of type 363 with midpoint the 6-vertex ζ, this implies that the angle $\angle_{\overrightarrow{x_2w_1}}(\beta_1, \beta_2)$ is of type 161 and this implies in turn, that the segment $\beta_1\beta_2 \subset \Sigma_{x_2}K$ is of type 727. Let $\gamma \in S$ be the 2-vertex $m(\beta_1, \beta_2)$.

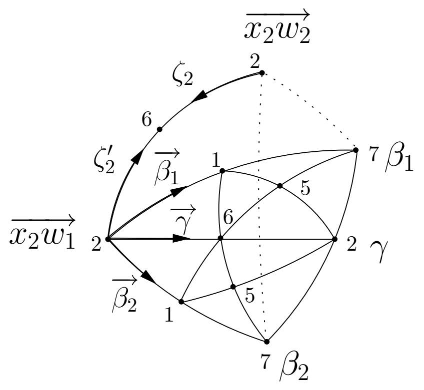




Let $\zeta_2 := \overrightarrow{\overrightarrow{x_2w_2}\overrightarrow{x_2x_1}}$. We can use the same argument as above to see that $\Sigma_{\zeta_2}\Sigma_{\overrightarrow{x_2w_2}}\Sigma_{x_2}K$ contains a singular circle of type 5135135. We want to prove next that it also contains a pair of antipodal 7-vertices.

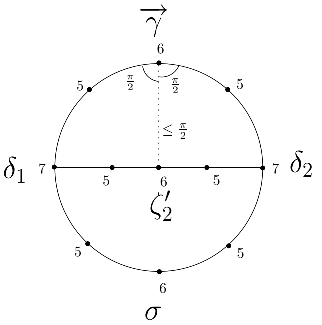
Sublemma 4.6.14. *The link $\Sigma_{\zeta_2}\Sigma_{\overrightarrow{x_2w_2}}\Sigma_{x_2}K$ contains a pair of antipodal 7-vertices.*

Proof. Notice again that $d(\overrightarrow{x_2w_2}, \Sigma_{x_2}T_1) \equiv \frac{\pi}{2}$, in particular, $d(\overrightarrow{x_2w_2}, \overrightarrow{x_2w_1}) = \frac{\pi}{2}$.

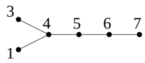
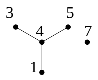


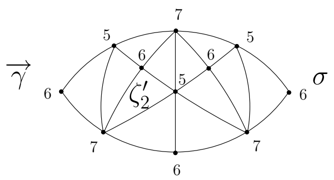
Recall also from the beginning of the proof Lemma 4.6.12 that $d(\overrightarrow{x_2w_2}, \beta_i) \leq \frac{\pi}{2}$, because β_i is $\frac{\pi}{3}$-extendable. This implies that $d(\overrightarrow{x_2w_2}, \gamma) \leq \frac{\pi}{2}$ and $\angle_{\overrightarrow{x_2w_1}}(\overrightarrow{x_2w_2}, \gamma) \leq \frac{\pi}{2}$. Let $\zeta_2' := \overrightarrow{\overrightarrow{x_2w_1}\overrightarrow{x_2w_2}}$, $\overrightarrow{\beta_i} := \overrightarrow{\overrightarrow{x_2w_1}\beta_i}$ and $\overrightarrow{\gamma} := \overrightarrow{\overrightarrow{x_2w_1}\gamma}$. We have already seen that $d(\zeta_2', \overrightarrow{\beta_i}) \leq \frac{\pi}{2}$ and $d(\zeta_2', \overrightarrow{\gamma}) \leq \frac{\pi}{2}$. Furthermore, it follows from triangle comparison, that if $d(\zeta_2', \overrightarrow{\gamma}) = \frac{\pi}{2}$, then $d(\zeta_2', \overrightarrow{\beta_i}) = \frac{\pi}{2}$ for $i = 1, 2$.

Notice that the link $\Sigma_{\zeta_2}\Sigma_{\overrightarrow{x_2w_2}}\Sigma_{x_2}K$ contains a pair of antipodal 7-vertices if and only if $\Sigma_{\zeta_2'}\Sigma_{\overrightarrow{x_2w_1}}\Sigma_{x_2}K$ contains a pair of antipodal 7-vertices. The latter is what we will show.

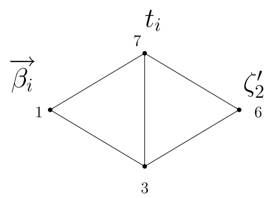


Let $\delta_1, \delta_2 \in \Sigma_{\overrightarrow{x_2w_1}}\Sigma_{x_2}K$ be the two 7-vertices in $\Sigma_{\overrightarrow{x_2w_1}}\Sigma_{x_2}T_1$ and recall that the 2-sphere $\Sigma_{\overrightarrow{x_2w_1}}S$ contains a singular circle of type 756575657 containing the vertices δ_1, δ_2 and $\overrightarrow{\gamma}$ (this is just the circle in $\Sigma_{\overrightarrow{x_2w_1}}\Sigma_{x_2}K$ corresponding to the circle in $\Sigma_{\overrightarrow{x_1w_1}}\Sigma_{x_1}K$ from the Sublemma 4.6.13 containing ξ, ξ' and ζ). Let σ be the 6-vertex in this circle antipodal to $\overrightarrow{\gamma}$. Further, we know that $d(\zeta_2', \delta_i) = \frac{\pi}{2}$, because $d(\overrightarrow{x_2w_2}, \Sigma_{x_2}T_1) \equiv \frac{\pi}{2}$. If ζ_2' has an antipode in the 2-sphere $\Sigma_{\overrightarrow{x_2w_1}}S$, then $\overrightarrow{x_2w_2}$ has an antipode in S. But this is impossible, since $\overrightarrow{x_2w_2} = \overrightarrow{x_2m(y_2, z_2)}$ and $m(y_2, z_2) \in K$ is a $2A$-vertex at distance $\frac{3\pi}{4}$ to x_2. Hence $\frac{\pi}{2} \geq d(\zeta_2', \overrightarrow{\gamma}) > 0$ and $d(\zeta_2', \sigma) < \pi$.

Notice that $\Sigma_{\overrightarrow{x_2w_1}}\Sigma_{x_2}B$ is a building of type D_6 and Dynkin diagram . The distances between 6-vertices are 0, $\frac{\pi}{3}$, $\frac{\pi}{2}$, $\frac{2\pi}{3}$ and π. The link $\Sigma_{\zeta_2'}\Sigma_{\overrightarrow{x_2w_1}}\Sigma_{x_2}K$ is of type , thus two distinct 7-vertices in this link must be antipodal.

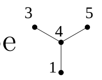


Case 1: $d(\zeta_2', \overrightarrow{\gamma}) = \frac{\pi}{3}$. Since $d(\zeta_2', \sigma) < \pi$, it follows that $\overrightarrow{\gamma}\zeta_2'\sigma$ is a geodesic of length π. Its simplicial convex hull is 2-dimensional and contains two 7-vertices adjacent to ζ_2'. It follows that $\Sigma_{\zeta_2'}\Sigma_{\overrightarrow{x_2w_1}}\Sigma_{x_2}K$ contains a pair of antipodal 7-vertices.



Case 2: $d(\zeta_2', \overrightarrow{\gamma}) = \frac{\pi}{2}$ and the segment $\zeta_2'\overrightarrow{\gamma}$ is of type 646. In this case, we know that $d(\zeta_2', \overrightarrow{\beta_i}) = \frac{\pi}{2}$ for $i = 1, 2$. Thus, $CH(\overrightarrow{\beta_1}, \overrightarrow{\beta_2}, \zeta_2')$ is an isosceles spherical triangle with side lengths $\frac{\pi}{2}$, $\frac{\pi}{2}$ and $\arccos(-\frac{1}{3})$. The simplicial convex hull of the segment $\zeta_2'\overrightarrow{\beta_i}$ contains a 7-vertex t_i adjacent to ζ_2' and to $\overrightarrow{\beta_i}$ for $i = 1, 2$. If $t_1 = t_2$, then t_1 is adjacent to $\overrightarrow{\beta_i}$ for $i = 1, 2$. It follows that t_1 is also adjacent to $\overrightarrow{\gamma} = m(\overrightarrow{\beta_1}, \overrightarrow{\beta_2})$. This means that $d(\overrightarrow{\gamma}, t_1) = d(t_1, \zeta_2') = \frac{\pi}{4}$. Since $d(\zeta_2', \overrightarrow{\gamma}) = \frac{\pi}{2}$, $\zeta_2' t_1 \overrightarrow{\gamma}$ must be a geodesic. This contradicts the fact that the segment $\zeta_2'\overrightarrow{\gamma}$ is of type 646. Hence, $t_1 \neq t_2$ and $\Sigma_{\zeta_2'}\Sigma_{\overrightarrow{x_2w_1}}\Sigma_{x_2}K$ contains a pair of antipodal 7-vertices.

Case 3: $d(\zeta_2', \overrightarrow{\gamma}) = \frac{\pi}{2}$ and the segment $\zeta_2'\overrightarrow{\gamma}$ is of type 676. If $d(\zeta_2', \sigma) = \frac{\pi}{2}$ then $\overrightarrow{\gamma}\zeta_2'\sigma$ is a geodesic of length π and of type 67676. If $d(\zeta_2', \sigma) = \frac{2\pi}{3}$, then the segment $\zeta_2'\overrightarrow{\gamma}$ contains a 7-vertex adjacent to ζ_2' at distance $\frac{\pi}{4}$ to $\overrightarrow{\gamma}$ and the simplicial convex hull of the segment $\zeta_2'\sigma$ contains a 7-vertex adjacent to ζ_2' at distance $\frac{\pi}{2}$ to σ. It follows that ζ_2' is adjacent to two different 7-vertices. Thus, $\Sigma_{\zeta_2'}\Sigma_{\overrightarrow{x_2w_1}}\Sigma_{x_2}K$ contains a pair of antipodal 7-vertices. □

End of proof of Lemma 4.6.12. We know now that $\Sigma_{\zeta_2}\Sigma_{\overrightarrow{x_2w_2}}\Sigma_{x_2}K$ (of type 

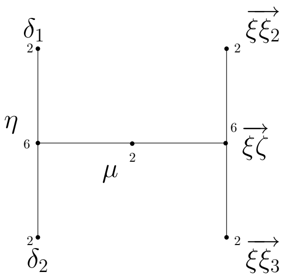
) contains a singular circle of type 5135135 and a pair of antipodal 7-vertices. Hence, it contains a singular 2-sphere (the spherical join of the singular circle and the pair of antipodal 7-vertices). Since ζ_2 has an antipode in $\Sigma_{\overrightarrow{x_2w_2}}\Sigma_{x_2}K$, this implies that $\Sigma_{\overrightarrow{x_2w_2}}\Sigma_{x_2}K$ contains a 3-sphere spanned by a simplex of type 1567. This in turn implies that $\Sigma_{w_2}K$ contains a singular 4-sphere spanned by a simplex of type 15678. This sphere is of type $\frac{\pi}{3}$ as can be verified by considering the vector space realization of the Coxeter complex of type D_n presented in Appendix A. We get a contradiction to Lemma 4.6.9 finishing the proof of the lemma. □

Lemma 4.6.15. *K contains no $8B_3$-vertices.*

Proof. We want to show first that an $8B_3$-vertex has the property T. Suppose $x_1 \in K$ is an $8B_3$-vertex and let $x_2, x_3 \in K$ be 8-vertices as in the configuration $*$. Suppose further, that x_3 is an $8A$-vertex and that $\overrightarrow{x_1x_2}$ is $\frac{2\pi}{3}$-extendable to an $8A$-vertex. To prove that x_1 has the property T, we have to show that $CH(x_1, x_2, x_3)$ is not a spherical triangle. Let $S \subset \Sigma_{x_1}K$ be the singular 2-sphere from the definition of the property B_3. Let $\zeta := \overrightarrow{x_1z_1}$ and $\xi_i := \overrightarrow{x_1x_i}$ for $i = 2, 3$, as in the notation of the configuration $*$.

Suppose there is a 7-vertex $\xi \in S$, such that $d(\zeta, \xi) = \arccos(-\frac{1}{\sqrt{3}})$. The segment $\zeta\xi$ is of type 2767. Since ξ is $\frac{\pi}{3}$-extendable in K and ξ_i is $\frac{2\pi}{3}$-extendable to an $8A$-vertex, ξ is not antipodal to ξ_i for $i = 1, 2$. It follows that $CH(\xi, \xi_2, \xi_3)$ is an equilateral spherical triangle sides of type 727. Let γ be the 7-vertex in $\zeta\xi$ adjacent to ζ. γ is the center of the spherical triangle $CH(\xi, \xi_2, \xi_3)$. It follows from the configuration $**$, that γ is $\frac{2\pi}{3}$-extendable to an $8A$-vertex in K.

$\Sigma_\xi S$ is a singular circle of type 2626262. Notice that $\overrightarrow{\xi\zeta} = \overrightarrow{\xi\gamma}$ is not antipodal to any 2-vertex in this circle, otherwise we could find in S an antipodal 7-vertex to γ, but this is not possible, since γ is $\frac{2\pi}{3}$-extendable to an $8A$-vertex in K. On the other hand, $\overrightarrow{\xi\zeta}$ cannot have distance $< \frac{\pi}{2}$ to all the 6-vertices in this circle, so let η be a 6-vertex in $\Sigma_\xi S$, so that $d(\eta, \overrightarrow{\xi\zeta}) = \frac{2\pi}{3}$ and let $\delta_i \in \Sigma_\xi S$ be the 2-vertices adjacent to η. Let $\mu := m(\eta, \overrightarrow{\xi\zeta})$. (Compare with the configuration in the proof of Lemma 4.6.7.)

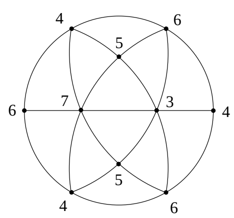


Since $\overrightarrow{\xi\zeta}$ is not antipodal to δ_i, it follows that $\angle_\eta(\delta_i, \overrightarrow{\xi\zeta}) = \frac{\pi}{2}$ and these angles are of type 232. Similarly, we see that η cannot be antipodal to $\overrightarrow{\xi\xi_i}$ because ξ_i has no antipodes in S. Thus, by Lemma 3.2.2 applied to this configuration in $\Sigma_\xi\Sigma_{x_1}K$ we conclude that $\Sigma_\mu\Sigma_\xi\Sigma_{x_1}K$ contains a singular 2-sphere spanned by a simplex of type 156. The same argument as in the proof of Lemma 4.6.7 (p. 58) shows that μ is extendable in $\Sigma_{x_1}K$ to a segment of type 727 and the 2-vertex on this segment is extendable in K to a segment of type 828 (this uses that $\gamma \in \Sigma_{x_1}K$ is $\frac{2\pi}{3}$-extendable and the 7-vertices in S are $\frac{\pi}{3}$-extendable). This produces a 2-vertex in K, whose link contains a 4-sphere spanned by a simplex of type 15678. This singular 4-sphere is of type $\frac{\pi}{3}$, a contradiction to Lemma 4.6.9.

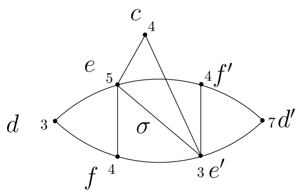
From this, it follows that ζ has distance $\leq \frac{\pi}{2}$ to all the 7-vertices in S. Since S is the convex hull of its 7-vertices, it follows that $d(\zeta, S) \equiv \frac{\pi}{2}$. Hence $\Sigma_\zeta\Sigma_{x_1}K$ contains the 2-sphere $s := \Sigma_\zeta CH(\zeta, S)$. The segments connecting ζ with the 2-vertices of S are of type 262, the segments connecting ζ with the 7-vertices of S are of type 217 and since the 6-vertices in S are midpoints of segments of type 767 in S, this implies that the segments connecting ζ with the 6-vertices of S are of type 2436. Since the sphere S has B_3-geometry 7 6 2, it follows that s has B_3-geometry 1 4 6. $\Sigma_\zeta\Sigma_{x_1}K$ also contains the two antipodal 7-vertices $\overrightarrow{\zeta x_1 x_i}$ for $i = 2, 3$.

Sublemma 4.6.16. *Let $L \subset B$ be a convex subcomplex of a building of type D_6 with Dynkin diagram* 3, 1, 4 5 6 7*. Suppose L contains a singular 2-sphere S with B_3-geometry* 1 4 6 *and also a pair of antipodal 7-vertices. Then L contains a 3-sphere spanned by a simplex of type* 1467.

Proof. Let $a, a' \in L$ be the antipodal 7-vertices and let b, b' be antipodal 1-vertices in $S \subset L$. By Lemma 3.1.4 and Remark 3.1.6, it follows that L contains a circle of type 7317317 through b and b'. In particular $\Sigma_b L$ contains a pair of antipodal 7- and 3-vertices. $\Sigma_b S$ is a singular circle of type 6464646. So, it will suffice to show that under these circumstances $\Sigma_b L$ contains a 2-sphere spanned by a simplex of type 467 (notice that such a sphere is also spanned by a simplex of type 346):



Let $d, d' \in \Sigma_b L$ be the antipodal 3- and 7-vertices, respectively. Let $c \in \Sigma_b S$ be a 4-vertex and let c' the 6-vertex in $\Sigma_b S$ antipodal to c. If c is adjacent to d, then dcd' is a geodesic of type 3437 and $\Sigma_c\Sigma_b L$ contains a pair of antipodal 3-vertices. If c is adjacent to d', then dcd' is a geodesic of type 3547 and $\Sigma_c\Sigma_b L$ contains a pair of antipodal 5- and 7-vertices.



Otherwise the segments cd and cd' are of type 453 and 437 respectively. Let e be the 5-vertex in cd and let e' be the 3-vertex in cd'. Let also f be the 4-vertex on the segment $e'd$ (of type 343) and let f' be the 4-vertex on the segment ed' (of type 547). Notice that since e, e' are adjacent to c, then e is adjacent to e'. It follows that e is adjacent to f and e' is adjacent to f'. Let σ be the edge ee'. The link $\Sigma_\sigma\Sigma_b B$ is of type $\overset{4}{\bullet}\ \overset{6}{\bullet}\!\!-\!\!\overset{7}{\bullet}$; and the direction $\overrightarrow{\sigma c'}$ is of type 4. It follows that c' is antipodal to f or f' and c' is contained in a circle in $\Sigma_b L$ of type 7673437 or 3657453. This implies that $\Sigma_{c'}\Sigma_b L$ contains a pair of antipodal 7-vertices or a pair of antipodal 3- and 5-vertices. This means for $\Sigma_c\Sigma_b L$, that it contains a pair of antipodal 3-vertices or a pair of antipodal 5- and 7-vertices.

Recall that $\Sigma_c\Sigma_b S$ consists of a pair of antipodal 6-vertices. $\Sigma_c\Sigma_b B$ is of type $A_1 \circ A_3$ with Dynkin diagram $\overset{3}{\bullet}\ \overset{5}{\bullet}\!\!-\!\!\overset{6}{\bullet}\!\!-\!\!\overset{7}{\bullet}$. If $\Sigma_c\Sigma_b L$ contains a pair of antipodal 3-vertices, then it contains a circle of type 63636. This implies that $\Sigma_b L$ contains a 2-sphere spanned by a simplex of type 364 as desired. If $\Sigma_c\Sigma_b L$ contains a pair of antipodal 5- and 7-vertices, then we apply Lemma 3.1.4 (for $n = k = 3$) to the A_3-factor of $\Sigma_c\Sigma_b B$ and conclude that $\Sigma_c\Sigma_b L$ contains a circle of type 7675657. We get again the 2-sphere in $\Sigma_b L$ as we wanted. □

End of proof of Lemma 4.6.15. Sublemma 4.6.16 implies that $\Sigma_\zeta\Sigma_{x_1} K$ contains a 3-sphere spanned by a simplex of type 1467. Recall the notation of the configuration $*$. Let u be the 8-vertex $m(x_3, y_3)$. $x_1 z_1 u$ is a segment of type 828. Then, it follows that $\Sigma_{z_1} K$ contains a singular 4-sphere spanned by a simplex of type 14678. This sphere is of type 757 (to verify this, one can consider the vector space realization of D_n in Appendix A). Lemma 4.6.8 implies that the segment $x_1 u$ cannot be extended beyond u in K. This implies in turn, that $CH(x_1, x_2, x_3)$ cannot be a spherical triangle. In particular x_1 must be an $8T$-vertex. i.e. $8B_3 \Rightarrow 8T$.

Let now x_1, x_2, x_3 be $8B_3$-vertices as in the definition of the property T'. Our argument above shows that $\Sigma_{z_i} K$ contains a 4-sphere of type 757 for $i = 1, 2, 3$. We apply again Lemma 4.6.8 and see that $rad(z_i, \{8\text{-vert. in } K\}) \leq \arccos(-\frac{1}{2\sqrt{2}})$ for $i = 1, 2, 3$. Hence, x_1 is an $8T'$-vertex. A contradiction to Lemma 4.6.10. □

If we combine the Lemmata 4.6.12 and 4.6.15 we obtain the following result, which is the main step towards the proof of Theorem 4.6.24.

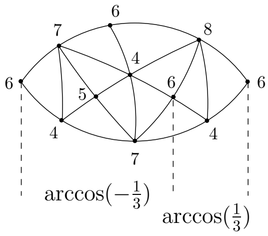
Corollary 4.6.17. *All 8-vertices in K have antipodes in K.*

Now we proceed to prove that the other vertices in K must also have antipodes in K. We use the information about types of segments between vertices in the Coxeter complex of type E_8 listed in Section 2.7.

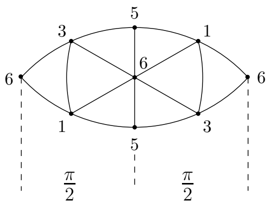
Lemma 4.6.18. *All 2-vertices in K have antipodes in K.*

Proof. First note that a $2A$-vertex $x \in K$ cannot be adjacent to an 8-vertex in K. Otherwise let $y \in K$ be an antipode of the 8-vertex adjacent to x. The segment xy is of type 2828. This in not possible due to Lemma 3.0.1 and 4.6.17.

Suppose there is a $2A$-vertex $x \in K$. There exists $x' \in G \cdot x$ with $d(x, x') > \frac{\pi}{2}$. From the observation above it follows, that $d(x, x') \neq \arccos(-\frac{1}{4})$. $d(x, x')$ cannot be $\arccos(-\frac{3}{4})$ either, because in this case the midpoint of the segment xx' is an 8-vertex. It follows that $d(x, x') = \frac{2\pi}{3}$ and the segment xx' is of type 26262. Let $y := m(x, x')$, it is also a $2A$-vertex. Therefore we can find $y' \in G \cdot y$ with $d(y, y') = \frac{2\pi}{3}$. Suppose w.l.o.g. that $\angle_y(x, y') \geq \frac{\pi}{2}$. Then $d(x, y') > \frac{\pi}{2}$, thus $d(x, y') = \frac{2\pi}{3}$. This implies by triangle comparison, that $\angle_y(x, y') \leq \arccos(-\frac{1}{3})$.



If $\angle_y(x, y') = \arccos(-\frac{1}{3})$, then either this angle is of type 686, which is not possible because K contains no 8-vertex adjacent to y; or the simplicial convex hull of the segment $\overrightarrow{yx}\overrightarrow{yy'}$ contains a 7-vertex adjacent to $\overrightarrow{yx}$. The segment connecting $\overrightarrow{yx}$ and $\overrightarrow{yx'}$ through this 7-vertex is of type 67686. This cannot happen either. Hence, $\angle_y(x, y') = \frac{\pi}{2}$. It follows that $\angle_y(x', y') \geq \frac{\pi}{2}$ and we conclude analogously that $\angle_y(x', y') = \frac{\pi}{2}$.



Let $\gamma \subset \Sigma_y K$ be the geodesic connecting $\overrightarrow{yx}$ and $\overrightarrow{yx'}$ through $\overrightarrow{yy'}$. The simplicial convex hull of γ is either 3-dimensional, in which case the direction $\overrightarrow{\overrightarrow{yx}\overrightarrow{yy'}}$ spans a simplex of type 578 and in particular, $\Sigma_y K$ contains 8-vertices, but this is not possible; or it is 2-dimensional and it contains a pair of 1-vertices adjacent to $\overrightarrow{yy'}$. Let $z := m(y, y')$ and let w be the 6-vertex $m(y, z)$. The segment joining $\overrightarrow{wy}$ and $\overrightarrow{wz}$ through the 1-vertex adjacent to $\overrightarrow{wy}$ is of type 2152. It follows that $\overrightarrow{zy}$ is adjacent to a 5-vertex. The geodesic connecting $\overrightarrow{zy}$ and $\overrightarrow{zy'}$ through this 5-vertex is of type 65856, but z is a $2A$-vertex and $\Sigma_z K$ cannot contain 8-vertices. □

Lemma 4.6.19. *All 7-vertices in K have antipodes in K.*

Proof. Considering the singular circles in E_8, we observe again that a $7A$-vertex cannot be adjacent to 2- or 8-vertices in K. Suppose K contains $7A$-vertices, then there exist $7A$-vertices $x_1, x_2 \in K$ at distance $> \frac{\pi}{2}$. There are two types of segments x_1x_2 of length $> \frac{\pi}{2}$ and so that the simplices containing $\overrightarrow{x_ix_{3-i}}$ in their interiors have no 2- or 8-vertices. They are of type 76867 and 7342437. These segments have a vertex of type 2 or 8 in their interiors, which yields a contradiction. □

Lemma 4.6.20. *All 1-vertices in K have antipodes in K.*

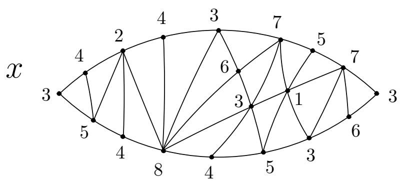
Proof. Suppose x is an $1A$-vertex in K. Then x cannot be adjacent to 2-, 7- or 8-vertices in K. Let $x' \in G \cdot x$ be another $1A$-vertex at distance $> \frac{\pi}{2}$ to x. It follows that the simplex spanned by the direction $\overrightarrow{xx'}$ has no 2-, 7- or 8-vertices. There are four possible types of segments xx'. If $d(x,x') = \arccos(-\frac{3}{8})$, then the simplicial convex hull of xx' contains an 8-vertex adjacent to x'. If $d(x,x') = \frac{2\pi}{3}$ or $\arccos(-\frac{7}{8})$, then the midpoint of xx' is an 8-vertex. If $d(x,x') = \arccos(-\frac{5}{8})$, then the midpoint of xx' is a 7-vertex. This is not possible by Lemma 3.0.1. Hence, there are no 1A-vertices in K. □

Lemma 4.6.21. *All* 6*-vertices in* K *have antipodes in* K.

Proof. Let x be a 6A-vertex. By the previous lemmata and according to the list of singular 1-spheres in the Coxeter complex of type E_8, x cannot be adjacent to vertices of type 1, 2, 7 or 8. There exists another 6A-vertex $x' \in K$ at distance $> \frac{\pi}{2}$ to x. It follows that the direction $\overrightarrow{xx'}$ span a simplex with no 1, 2, 7 or 8-vertices. Hence $d(x,x') \in \{\arccos(-\frac{1}{4}), \frac{2\pi}{3}, \arccos(-\frac{3}{4})\}$. In the first case the midpoint of xx' is an 8-vertex and in the third case, it is a 7-vertex. In the second case the simplicial convex hull of xx' contains an 8-vertex adjacent to x'. A contradiction. □

Lemma 4.6.22. *All* 3*-vertices in* K *have antipodes in* K.

Proof. Observe, that a 3A-vertex is not adjacent to a vertex of type 1, 2, 6, 7 or 8.



If K contains 3A-vertices, then it contains at least two distinct 3A-vertices x, x'. Then $\overrightarrow{xx'}$ is contained in an edge of type 45. Consider the bigon in the Coxeter complex of type E_8, which is the convex hull of a simplex of type 345 and the antipode of the 3-vertex of this simplex. We see that there are only three possibilities for the type of the segment xx'. In one of them, the midpoint of xx' is a 2-vertex; and in another possibility, it is an 8-vertex. The simplicial convex hull of xx' for the last possibility contains an 8-vertex adjacent to x'. We obtain again a contradiction to Lemma 3.0.1. □

Lemma 4.6.23. *All* 4*- and* 5*-vertices in* K *have antipodes in* K.

Proof. A vertex in K of type 4 or 5 without antipodes in K cannot have vertices of type 1, 2, 3, 6, 7 or 8 in K adjacent to it. It follows that, if K contains 4A- or 5A-vertices, then it has dimension ≤ 1. A contradiction. □

We have shown in the previous lemmata that all vertices of a counterexample K have antipodes in K, contradicting Lemma 3.0.2. This proves our main result:

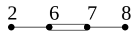
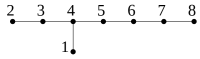
Theorem 4.6.24. *The Center Conjecture 1 holds for spherical buildings of type* E_8.

Remark 4.6.25. Our proof actually shows that K is a subbuilding or the action of the group $Aut_B(K) \curvearrowright K$ fixes a point (see 1.3 for definitions).

4.6.1 A proof for the F_4-case using the E_8-case

Theorem 4.6.26. *The Center Conjecture 1 holds for spherical buildings of type F_4.*

Proof. Let K be a convex subcomplex of a spherical building B of type F_4 and suppose it is not a subbuilding. By Lemma 4.3.1, we just have to show that $Stab_{Auto(B)}(K)$ has a fixed point in K.

For this proof we use following labelling of the Dynkin diagram of type F_4: . With this labelling the Coxeter complex of type F_4 can be considered as a subcomplex of the Coxeter complex (S, W_{E_8}) of type E_8 with Dynkin diagram  (cf. Section 2.4).

Let $\widehat{B} = B \circ S^3$, where S^3 denotes the unit sphere in $\mathbb{R}^4$. Then $\widehat{B}$ is a spherical building of dimension 7. From the observation above, it follows that $\widehat{B}$ carries a natural structure of a building of type E_8, and $B \subset \widehat{B}$ can be viewed as a subbuilding. The polyhedral structure of B (as a building of type F_4) coincides with the one induced by the polyhedral structure of $\widehat{B}$ (as a building of type E_8). In particular, K is a subcomplex of $\widehat{B}$.

Notice that $Aut_0(B) = Aut_{\widehat{B}}(B)$ (cf. Remark 4.6.25). Then by the Center Conjecture for buildings of type E_8 (Theorem 4.6.24) and the Remark 4.6.25, it follows that $Aut_{\widehat{B}}(K) \curvearrowright K$ has a fixed point. In particular, $Stab_{Auto(B)}(K) \subset Aut_{\widehat{B}}(K)$ also fixes a point in K. □

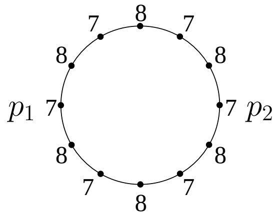
Remark 4.6.27. Notice that the subgroup $(Aut_B(K))_0$ of $Aut_B(K)$ of type preserving automorphisms is also a subgroup of $Aut_{\widehat{B}}(K)$. Thus our proof of Theorem 4.6.26 actually shows that K is a subbuilding or $(Aut_B(K))_0 \curvearrowright K$ has a fixed point. The proof of Lemma 4.3.1 can be used without changes to show that K is a subbuilding or $Aut_B(K) \curvearrowright K$ has a fixed point.

4.6.2 A proof for the E_6-case using the E_8-case

Theorem 4.6.28. *The Center Conjecture 1 holds for spherical buildings of type E_6.*

Proof. Let K be a convex subcomplex of a spherical building B of type E_6 and suppose that K is not a subbuilding.

Let κ be a circle of radius 1 with the structure of the spherical Coxeter complex of type $I_2(6)$ with labelling of its Dynkin diagram . Let p_1, p_2 be a pair of antipodal 7-vertices in κ.



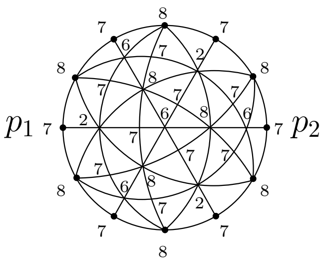
Consider the spherical join $\widehat{B} := B \circ \kappa$. There is a natural embedding $B \hookrightarrow \widehat{B}$ so we can regard B as a subset of $\widehat{B}$. Let $\widehat{K} := K \circ \kappa \subset \widehat{B}$.

Let $B_{p_i} := \Sigma_{p_i}(B \circ \{p_i\}) \subset \Sigma_{p_i}\widehat{B}$ for $i = 1, 2$. Then $\Sigma_{p_i}\widehat{B} = B_{p_i} \circ \Sigma_{p_i}\kappa$ and we have isometries $B \xrightarrow{\rho_i} B_{p_i}$ defined by $\rho_i(v) := \overrightarrow{p_i v}$. Let $K_{p_i} := \Sigma_{p_i}(K \circ \{p_i\}) \cong K$.

Let $B_1 \xrightarrow{\rho} B_2$ be the isometry that sends a direction $\xi \in \Sigma_{p_1}(B \circ \{p_i\})$ to the initial direction at p_2 of the geodesic connecting p_1 and p_2 with initial direction ξ at p_1. Then $\rho = \rho_1^{-1} \circ \rho_2$.

Recall that the link of a 7-vertex in the Coxeter complex of type E_8 is a Coxeter complex of type $E_6 \circ A_1$. We consider the building B_{p_1} of type E_6 with the labelling of vertices induced by the labelling of B and the isometry ρ_1. With this labelling, a chart $(S^5, W_{E_6}) \hookrightarrow B_{p_1}$ of the building B_{p_1} induces a chart $(S^7, W_{E_8}) \hookrightarrow \widehat{B}$, giving $\widehat{B}$ a structure of spherical building of type E_8, where the induced polyhedral structure of κ coincides with its structure as Coxeter complex of type $I_2(6)$. The labelling of the vertices of B_{p_2} induced by this building structure in $\widehat{B}$ can be obtained from the one induced by ρ_2 by exchanging the labels $2 \leftrightarrow 6$, $3 \leftrightarrow 5$ and fixing 1 and 4.

As an example, we present the E_8-structure of a 2-dimensional hemisphere $\{v\} \circ \kappa \subset \widehat{B}$, where v is a 2-vertex of B:



v of type 2

Notice that there is a natural isomorphism $Aut_0(B_{p_1}) \cong Fixator_{Aut(\widehat{B})}(\kappa)$ between the type preserving automorphisms of B_{p_1} and the automorphisms of $\widehat{B}$ fixing κ pointwise. It extends to an embedding $Aut(B_{p_1}) \overset{\iota}{\hookrightarrow} Stab_{Aut(\widehat{B})}(\kappa)$, where the image of a non type preserving automorphism of B_{p_1} restricted to κ is the antipodal involution ant_κ of κ. In particular, $\iota(\varphi)(p_1) = p_2$ for a non type preserving automorphism $\varphi \in Aut(B_{p_1})$, hence $\iota(\varphi)$ induces an isometry $B_{p_1} \to B_{p_2}$. This isometry is type preserving and coincides with $\rho \circ \varphi : B_{p_1} \to B_{p_2}$. This means that the image $\iota(Aut(B_{p_1}))$ acts on B_{p_1} via $\iota(\varphi)|_{B_{p_1}}$, if $\varphi \in Aut(B_{p_1})$ is type preserving; and via $\rho^{-1} \circ (\iota(\varphi)|_{B_{p_1}})$, if $\varphi \in Aut(B_{p_1})$ is not type preserving. The embedding ι restricts to an embedding $G := Stab_{Aut(B_{p_1})}(K_{p_1}) \overset{\iota}{\hookrightarrow} G' := Stab_{Aut(\widehat{B})}(\widehat{K})$.

There is an isometry ϕ_0 of $\widehat{B}$ that rotates κ an angle of $\frac{2\pi}{3}$ and preserves every 2-dimensional hemisphere bounded by κ. The restriction of ϕ_0 to an apartment $a \subset \widehat{B}$ is the composition of the reflections on two walls orthogonal to two 8-vertices in κ at distance $\frac{\pi}{3}$.

It follows that ϕ_0 is an automorphism of $\widehat{B}$ and $\phi_0 \in G'$.

We apply now the Center Conjecture for buildings of type E_8 (Theorem 4.6.24) to $\widehat{K} \subset \widehat{B}$. Since K is not a subbuilding, then $\widehat{K}$ cannot be a subbuilding. It follows that G' fixes a point $x \in \widehat{K}$. But since $\phi_0 \in G'$ and ϕ_0 has no fixed points in κ, it follows that $x \notin \kappa$. This implies that $\iota(G)$ preserves the 2-dimensional hemisphere $h \subset \widehat{K}$ bounded by κ and containing x. Hence, it preserves the geodesic γ connecting p_1 and p_2 contained in h. It follows that $\iota(G) \curvearrowright K_{p_1}$ fixes the initial direction of γ at p_1. □

Remark 4.6.29. Notice that the embedding $G \hookrightarrow G'$ in the proof of Theorem 4.6.28 extends to an embedding $Aut_{B_{p_1}}(K_{p_1}) \hookrightarrow Aut_{\widetilde{B}}(\widetilde{K})$. Then by Remark 4.6.25, the proof actually shows that K is a subbuilding or the action of the group $Aut_B(K) \curvearrowright K$ fixes a point.

4.6.3 A proof for the E_7-case using the E_8-case

Theorem 4.6.30. *The Center Conjecture 1 holds for spherical buildings of type E_7.*

Proof. It can be deduced from the E_8-case as follows: Let $K \subset B$ be a convex subcomplex of a spherical building of type E_7. Suppose that K is not a subbuilding. Let $\widetilde{B}$ be the suspension of B, i.e. the spherical join of B and a 0-sphere $\{p_1, p_2\}$. There is a natural embedding $B \hookrightarrow \widetilde{B}$, so we can consider B as a subset of $\widetilde{B}$. Notice that the map $v \mapsto \overrightarrow{p_i v}$ for $v \in B \subset \widetilde{B}$ is an isometry $B \cong B_{p_i} := \Sigma_{p_i}\widetilde{B}$. Let $\widetilde{K} \subset \widetilde{B}$ be the suspension of K and let $K_{p_i} := \Sigma_{p_i}\widetilde{K} \cong K$.

Recall that the link of an 8-vertex in the Coxeter complex of type E_8 is a Coxeter complex of type E_7. Hence a chart $(S^6, W_{E_7}) \hookrightarrow B_{p_1}$ of the building B_{p_1} induces a chart $(S^7, W_{E_8}) \hookrightarrow \widetilde{B}$, giving $\widetilde{B}$ a structure of spherical building of type E_8, where p_1 and p_2 are 8-vertices.

Observe that there is a natural isomorphism $Aut(B_{p_1}) \cong Stab_{Aut(\widetilde{B})}(p_1)$. The embedding $Aut(B_{p_1}) \hookrightarrow Aut(\widetilde{B})$ restricts to an embedding $G := Stab_{Aut(B_{p_1})}(K_{p_1}) \hookrightarrow \widetilde{G} := Stab_{Aut(\widetilde{B})}(\widetilde{K})$.

There is an isometry ϕ_0 of $\widetilde{B}$ that exchanges the points $p_1 \leftrightarrow p_2$ and preserves the geodesics connecting p_1 and p_2. The restriction of ϕ_0 to an apartment of $\widetilde{B}$ is the reflection on the wall orthogonal to p_1, p_2. Hence ϕ_0 is an automorphism of $\widetilde{B}$ and $\phi_0 \in \widetilde{G}$.

We apply now the Center Conjecture for buildings of type E_8 (Theorem 4.6.24) to the building $\widetilde{B}$ and the convex subcomplex $\widetilde{K}$. Since K is not a subbuilding, then $\widetilde{K}$ is not a subbuilding either. It follows that $\widetilde{G}$ fixes a point $x \in \widetilde{K}$ and since $\phi_0 \in \widetilde{G}$, this fixed point cannot be p_1 or p_2. The image of G in $\widetilde{G}$ fixes x, p_1 and p_2, hence it fixes pointwise the geodesic $\gamma \subset \widetilde{K}$ through x connecting p_1 and p_2. Therefore, the action $G \curvearrowright K_{p_1} \cong K$ has a fixed point $\overrightarrow{p_1 x} \in K_{p_1}$. □

Remark 4.6.31. Notice that the embedding $G \hookrightarrow \widetilde{G}$ in the proof of Theorem 4.6.30 extends to an embedding $Aut_{B_{p_1}}(K_{p_1}) \hookrightarrow Aut_{\widetilde{B}}(\widetilde{K})$. Then by Remark 4.6.25, the proof actually shows that K is a subbuilding or the action of the group $Aut_B(K) \curvearrowright K$ fixes a point.

Appendix A

Vector-space realizations of Coxeter complexes

In this appendix we present a vector space realization of the irreducible spherical Coxeter complexes. The information on the root systems can be found in [GB71, Ch. 5]. The orders of the irreducible Weyl groups can be found in [GB71, p. 80].

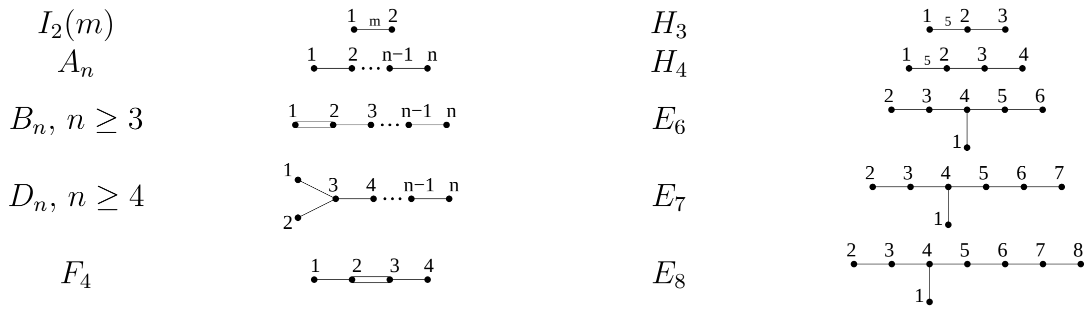
We consider the spherical Coxeter complex (S^{n-1}, W) embedded in $\mathbb{R}^n$ as the unit sphere. Let $\{e_i\}_{i=1}^n$ denote the canonical base of $\mathbb{R}^n$.

The *root system* of a Coxeter complex (S, W) is the set of (unit) vectors orthogonal to the hyperplanes inducing the reflections in W. The elements of the root system are called root vectors.

A subset F of the root system is called a base if there is a vector $v \in \mathbb{R}^n$ such that $\langle r, v\rangle \neq 0$ for all root vectors r, and F is minimal with respect to the property that any root vector r, such that $\langle r, v\rangle > 0$, can be written as a linear combination of elements in F with nonnegative coefficients. The *fundamental root vectors* are the elements of a given base of the root system. The *fundamental Weyl chamber* of (S, W) is $\triangle := \bar{\triangle} \cap S$, where $\bar{\triangle}$ is the intersection of the half spaces $\langle r_i, \cdot\rangle \geq 0$, where $r_1, \ldots, r_n$ are the fundamental root vectors. $\bar{\triangle}$ is a fundamental domain for the action of W in $\mathbb{R}^n$.

Let v_i be the vertex of $\triangle$ opposite to the face determined by $\langle r_i, \cdot\rangle = 0$. We say that a vertex of (S, W) is of type i, if it lies on the orbit $W \cdot v_i$.

We use the following labelling of the Dynkin diagrams of the irreducible spherical Coxeter complexes:



Recall, that the link $\Sigma_v S$ of a vertex $x \in S$ is a spherical Coxeter complex with Weyl group $Stab_W(v)$ and with Dynkin diagram obtained from the Dynkin diagram of (S, W) by deleting the vertex with label corresponding to the type of v.

The antipodal involution $v \mapsto -v$ is type preserving for the spherical Coxeter complexes of type $I_2(m)$ (m even), B_n ($n \geq 3$), D_{2n} ($n \geq 2$), H_3, H_4, F_4, E_7 and E_8. It exchanges the types $1 \leftrightarrow 2$ in $I_2(m)$ for m odd; the types $i \leftrightarrow (n+1-i)$, for $i = 1, \ldots, [\frac{n}{2}]$ in A_n; the types $1 \leftrightarrow 2$, in D_{2n+1}, $n \geq 2$ and the types $2 \leftrightarrow 6$ and $3 \leftrightarrow 5$ in E_6.

Suppose xy is an edge of S of type ij. By deleting the vertex with label j from the Dynkin diagram of (S, W), we obtain the Dynkin diagram of $(\Sigma_y S, Stab_W(y))$. We can easily read off this Dynkin diagram which type the antipode of $\overrightarrow{yx}$ in $\Sigma_y S$ has. Say it has type k, then the edge xy extends to a segment of type ijk. Repeating this procedure and taking into account the lengths of the different types of segments (which can be deduced from the description of the fundamental Weyl chamber), we can determine the different singular 1-spheres in S. A similar consideration can be used to determine the 2-dimensional singular bigons bounded by singular segments and with it the 2-dimensional singular spheres.

To determine the different types of segments modulo the action of the Weyl group connecting a vertex of type i with a vertex of type j, it suffices to compute the vertices of type j in the spherical bigon $\beta_i := CH(\triangle, \widehat{v_i}) \subset S$, where $\widehat{v_i}$ is the vertex antipodal to v_i.

The bigon β_i can be described by the set of inequalities $\{\langle r_l, \cdot \rangle \geq 0\}_{l \neq i}$.

More generally, suppose we want to determine the different types of segments connecting a vertex x of type i and a vertex y of type j, such that the vertices of the simplex in $\Sigma_x S$ spanned by the direction $\overrightarrow{xy}$ are not of type $i_1, \ldots, i_k \neq i$. Then, it suffices to compute the vertices of type j in the spherical bigon $\beta_i(i_1, \ldots, i_k) := CH(\triangle(i_1, \ldots, i_k), \widehat{v_i})$. Here, $\triangle(i_1, \ldots, i_k)$ denotes the face of the fundamental Weyl chamber $\triangle$, which does not contain the vertices $v_{i_1}, \ldots, v_{i_k}$.

The bigon $\beta_i(i_1, \ldots, i_k)$ can be described by the set of (in)equalities

$$\{\langle r_l, \cdot \rangle \geq 0\}_{l \neq i, i_1, \ldots, i_k}, \quad \{\langle r_l, \cdot \rangle = 0\}_{l = i_1, \ldots, i_k}.$$

Given a table listing the j-vertices in the bigon β_i, this list can be verified as follows. First, we have to check that the vertices listed indeed are of type j and are contained in β_i. Next we notice that β_i is a fundamental domain for the action $Stab_W(v_i) \curvearrowright S$. For a

j-vertex x in the list, let σ_x be the face of $\triangle$ spanned by the initial part of the segment $v_i x$. Then the orbit $Stab_W(v_i) \cdot x$ has cardinality $|Stab_w(v_i)|/|Stab_W(\sigma_x)|$. Since the stabilizers are again Weyl groups of spherical Coxeter complexes, their orders can be found in the table in [GB71, p. 80]. It remains to verify that the union of the orbits $Stab_W(v_i) \cdot x$ exhausts all the j-vertices in S, that is, we have to check that $\sum_{x \text{ in the list}} \frac{|Stab_W(v_i)|}{|Stab_W(\sigma_x)|} = \frac{|W|}{|Stab_W(v_i)|}$.

A.1 A_n

Let $n \geq 2$. The Weyl group W_{A_n} of type A_n is the finite group of isometries of $\mathbb{R}^n \cong \{x_1 + \cdots + x_{n+1} = 0\} \subset \mathbb{R}^{n+1}$ generated by the reflections at the hyperplanes orthogonal to the *fundamental root vectors*:

$$r_i = e_{i+1} - e_i \text{ for } 1 \leq i \leq n$$

The *fundamental Weyl chamber* $\triangle$ can be described by the inequalities:

$$x_1 \overset{(1)}{\leq} x_2 \overset{(2)}{\leq} \ldots \overset{(n)}{\leq} x_{n+1}.$$

Next we exhibit an element representing the vertices of the fundamental Weyl chamber $\triangle$, i.e. elements of $\mathbb{R}^+ \cdot v_i$:

$$\begin{array}{rlcrrrcrrrc}
\text{1-vertex:} & v_1 & (& -n, & 1, & 1, & \ldots, & 1, & 1, & 1 &) \\
\text{2-vertex:} & v_2 & (& -(n-1), & -(n-1), & 2, & \ldots, & 2, & 2, & 2 &) \\
\vdots & \vdots & & & & & \vdots & & & & \\
(n-1)\text{-vertex:} & v_{n-1} & (& -2, & -2, & -2, & \ldots, & -2, & n-1, & n-1 &) \\
n\text{-vertex:} & v_n & (& -1, & -1, & -1, & \ldots, & -1, & -1, & n &)
\end{array}$$

The Weyl group W_{A_n} acts on $\mathbb{R}^{n+1}$ by permutations of the coordinates.

A.2 B_n

Let $n \geq 2$. The Weyl group W_{B_n} of type B_n is the finite group of isometries of $\mathbb{R}^n$ generated by the reflections at the hyperplanes orthogonal to the *fundamental root vectors*:

$$r_1 = e_1, \quad r_i = e_i - e_{i-1} \text{ for } 2 \leq i \leq n$$

The *fundamental Weyl chamber* $\triangle$ can be described by the inequalities:

$$0 \overset{(1)}{\leq} x_1 \overset{(2)}{\leq} x_2 \overset{(3)}{\leq} \ldots \overset{(n)}{\leq} x_n.$$

Next we exhibit an element representing the vertices of the fundamental Weyl chamber $\triangle$, i.e. elements of $\mathbb{R}^+ \cdot v_i$:

1-vertex:	v_1	$(\ 1,1,1,\dots,1)$
2-vertex:	v_2	$(\ 0,1,1,\dots,1)$
3-vertex:	v_3	$(\ 0,0,1,\dots,1)$
$\vdots$	$\vdots$	$\vdots$
$(n-1)$-vertex:	v_{n-1}	$(\ 0,\dots,0,1,1)$
n-vertex:	v_n	$(\ 0,\dots,0,0,1)$

The Weyl group W_{B_n} acts on $\mathbb{R}^n$ by permutations of the coordinates and change of signs.

A.3 D_n

Let $n \geq 4$. The Weyl group W_{D_n} of type D_n is the finite group of isometries of $\mathbb{R}^n$ generated by the reflections at the hyperplanes orthogonal to the *fundamental root vectors*:

$$r_1 = e_1 + e_2, \quad r_i = e_i - e_{i-1} \text{ for } 2 \leq i \leq n$$

The *fundamental Weyl chamber* $\triangle$ can be described by the inequalities:

$$-x_2 \overset{(1)}{\leq} x_1 \overset{(2)}{\leq} x_2 \overset{(3)}{\leq} \dots \overset{(n)}{\leq} x_n.$$

Next we exhibit an element representing the vertices of the fundamental Weyl chamber $\triangle$, i.e. elements of $\mathbb{R}^+ \cdot v_i$:

1-vertex:	v_1	$(\ 1,1,1,\dots,1)$
2-vertex:	v_2	$(-1,1,1,\dots,1)$
3-vertex:	v_3	$(\ 0,0,1,\dots,1)$
$\vdots$	$\vdots$	$\vdots$
$(n-1)$-vertex:	v_{n-1}	$(\ 0,\dots,0,1,1)$
n-vertex:	v_n	$(\ 0,\dots,0,0,1)$

The Weyl group W_{D_n} acts on $\mathbb{R}^n$ by permutations of the coordinates and change of signs in an even number of places.

A.4 F_4

The Weyl group W_{F_4} of type F_4 is the finite group of isometries of $\mathbb{R}^4$ generated by the reflections at the hyperplanes orthogonal to the *fundamental root vectors*:

$$r_1 = -\frac{1}{2}(1,1,1,1), \quad r_2 = e_1, \quad r_3 = e_2 - e_1, \quad r_2 = e_3 - e_2.$$

The *fundamental Weyl chamber* $\triangle$ can be described by the inequalities:

$$x_1 + \cdots + x_4 \overset{(1)}{\leq} 0\,; \qquad 0 \overset{(2)}{\leq} x_1 \overset{(3)}{\leq} x_2 \overset{(4)}{\leq} x_3.$$

Next we exhibit an element representing the vertices of the fundamental Weyl chamber $\triangle$, i.e. elements of $\mathbb{R}^+ \cdot v_i$:

1-vertex: v_1	$(\ 0,\ 0,\ 0, -1)$
2-vertex: v_2	$(\ 1,\ 1,\ 1, -3)$
3-vertex: v_3	$(\ 0,\ 1,\ 1, -2)$
4-vertex: v_4	$(\ 0,\ 0,\ 1, -1)$

We list now the orbits of the vertices of $\triangle$ under the action of the Weyl group (modulo the following elements of the Weyl group: permutations and change of signs). We give representing vectors for the vertices.

1-vertices	$(\ 1,\ 0,\ 0,\ 0)$,	$\frac{1}{2}(\ 1,\ 1,\ 1,\ 1)$
2-vertices	$(\ 1,\ 1,\ 1, -3)$,	$(\ 2,\ 2,\ 2,\ 0)$
3-vertices	$(\ 0,\ 1,\ 1, -2)$	
4-vertices	$(\ 0,\ 0,\ 1, -1)$	

This list can be verified by checking that the vertices listed indeed lie on the orbit $W \cdot v_i$ and there are as many as $|W_{F_4}|/|Stab_{W_{F_4}}(v_i)|$.

We describe in the following table the 1- and 4-vertices x in β_1. Let σ be the face of $\Sigma_{v_1}\triangle$ containing $\overrightarrow{v_1 x}$ in its interior.

	x	$d(x, v_1)$	Type of σ
1-vertices $x \neq v_1, \widehat{v}_1$	$\frac{1}{2}(\ 1,\ 1,\ 1, -1)$	$\frac{\pi}{3}$	2
	$(\ 0,\ 0,\ 1,\ 0)$	$\frac{\pi}{2}$	4
	$\frac{1}{2}(\ 1,\ 1,\ 1,\ 1)$	$\frac{2\pi}{3}$	2
4-vertices x	$(\ 0,\ 0,\ 1, -1)$	$\frac{\pi}{4}$	4
	$(\ 0,\ 1,\ 1,\ 0)$	$\frac{\pi}{2}$	3
	$(\ 0,\ 0,\ 1,\ 1)$	$\frac{3\pi}{4}$	4

A.5 E_6

The Weyl group W_{E_6} of type E_6 is the finite group of isometries of $\mathbb{R}^6 \cong \{(x_1, \ldots, x_8) \in \mathbb{R}^8 \mid x_6 = x_7 = x_8\}$ generated by the reflections at the hyperplanes orthogonal to the *fundamental root vectors*:

$$r_1 = \frac{1}{2}(1,1,1,-1,-1,-1,-1,-1), \quad r_i = e_i - e_{i-1} \text{ for } 2 \le i \le 5;.$$

$$\text{and} \quad r_6 = \frac{1}{2}(1,1,1,1,-1,1,1,1).$$

The *fundamental Weyl chamber* $\triangle$ can be described by the inequalities:

$$x_4+x_5+\cdots+x_8 \overset{(1)}{\le} x_1+x_2+x_3\ ; \qquad x_1 \overset{(2)}{\le} x_2 \overset{(3)}{\le} \ldots \overset{(5)}{\le} x_5\ ; \qquad x_5 \overset{(6)}{\le} x_1+\cdots+x_4+x_6+x_7+x_8.$$

Next we exhibit an element representing the vertices of the fundamental Weyl chamber $\triangle$, i.e. elements of $\mathbb{R}^+ \cdot v_i$:

1-vertex: v_1	$(\ 1,\ 1,\ 1,\ 1,\ 1,-1,-1,-1)$
2-vertex: v_2	$(-3,\ 3,\ 3,\ 3,\ 3,-1,-1,-1)$
3-vertex: v_3	$(\ 0,\ 0,\ 3,\ 3,\ 3,-1,-1,-1)$
4-vertex: v_4	$(\ 1,\ 1,\ 1,\ 3,\ 3,-1,-1,-1)$
5-vertex: v_5	$(\ 3,\ 3,\ 3,\ 3,\ 9,-1,-1,-1)$
6-vertex: v_6	$(\ 3,\ 3,\ 3,\ 3,\ 3,\ 1,\ 1,\ 1)$

We list now the orbits of the 1- and 2-vertices of $\triangle$ under the action of the Weyl group (modulo the following elements of the Weyl group: permutations of the first five coordinates and change of sign in an even number of places in the first five coordinates). We give representing vectors for the vertices. The 6-vertices are just the antipodes of the 2-vertices.

1-vertices	$(\ 1,\ 1,\ 1,\ 1,\ 1,-1,-1,-1)$,	$(-1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1)$,
	$(\ 0,\ 0,\ 0,\ 2,\ 2,\ 0,\ 0,\ 0)$.	
2-vertices	$(-3,\ 3,\ 3,\ 3,\ 3,-1,-1,-1)$,	$(\ 0,\ 0,\ 0,\ 0,\ 3,\ 1,\ 1,\ 1)$,
	$(\ 0,\ 0,\ 0,\ 0,\ 0,-1,-1,-1)$.	

This list can be verified by checking that the vertices listed indeed lie on the orbit $W_{E_6} \cdot v_2$ and there are as many as $|W_{E_6}|/|W_{D_5}| = 3^3$.

We describe in the following table the 1-vertices x in β_1. Let σ be the face of $\Sigma_{v_1}\triangle$ containing $\overrightarrow{v_1 x}$ in its interior.

	x	$d(x, v_1)$	Type of σ
1-vertices $x \neq v_1, \widehat{v_1}$	$(\ 0,\ 0,\ 0,\ 2,\ 2,\ 0,\ 0,\ 0)$	$\frac{\pi}{3}$	4
	$(-1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1)$	$\frac{\pi}{2}$	26
	$(-1,-1,-1,\ 1,\ 1,\ 1,\ 1,\ 1)$	$\frac{2\pi}{3}$	4

We describe in the following table the 2- and 6-vertices x in β_2. Let σ be the face of $\Sigma_{v_2}\triangle$ containing $\overrightarrow{v_2x}$ in its interior.

	x	$d(x,v_2)$	Type of σ
2-vertices $x \neq v_2$	$(3,-3,3,3,3,-1,-1,-1)$	$\arccos(\frac{1}{4})$	3
	$(3,0,0,0,0,1,1,1)$	$\frac{2\pi}{3}$	6
6-vertices $x \neq \widehat{v_2}$	$(3,3,3,3,3,1,1,1)$	$\frac{\pi}{3}$	6
	$(3,0,0,0,0,-1,-1,-1)$	$\arccos(-\frac{1}{4})$	1

A.6 E_7

The Weyl group W_{E_7} of type E_7 is the finite group of isometries of $\mathbb{R}^7 \cong \{(x_1,\ldots,x_8) \in \mathbb{R}^8 \mid x_7 = x_8\}$ generated by the reflections at the hyperplanes orthogonal to the *fundamental root vectors*:

$$r_1 = \frac{1}{2}(1,1,1,-1,-1,-1,-1,-1), \quad r_i = e_i - e_{i-1} \text{ for } 2 \le i \le 6$$

$$\text{and} \quad r_7 = \frac{1}{2}(1,1,1,1,1,-1,1,1).$$

The *fundamental Weyl chamber* $\triangle$ can be described by the inequalities:

$$x_4+x_5+\cdots+x_8 \overset{(1)}{\le} x_1+x_2+x_3\ ; \quad x_1 \overset{(2)}{\le} x_2 \overset{(3)}{\le} \ldots \overset{(6)}{\le} x_6\ ; \quad x_6 \overset{(7)}{\le} x_1+\cdots+x_5+x_7+x_8.$$

Next we exhibit an element representing the vertices of the fundamental Weyl chamber $\triangle$, i.e. elements of $\mathbb{R}^+ \cdot v_i$:

1-vertex: v_1	$(1,1,1,1,1,1,-2,-2)$
2-vertex: v_2	$(-1,1,1,1,1,1,-1,-1)$
3-vertex: v_3	$(0,0,1,1,1,1,-1,-1)$
4-vertex: v_4	$(1,1,1,3,3,3,-3,-3)$
5-vertex: v_5	$(1,1,1,1,3,3,-2,-2)$
6-vertex: v_6	$(1,1,1,1,1,3,-1,-1)$
7-vertex: v_7	$(1,1,1,1,1,1,0,0)$

We list now the orbits of the 2- and 7-vertices of $\triangle$ under the action of the Weyl group (modulo the following elements of the Weyl group: permutations of the first six coordinates, change of sign in an even number of places in the first six coordinates and simultaneous change of sign of the last two coordinates). We give representing vectors for the vertices.

1-vertices	$(1,1,1,1,1,1,-2,-2)$,	$(0,0,0,2,2,2,1,1)$,
	$(1,1,1,1,1,-3,0,0)$.	
2-vertices	$(-1,1,1,1,1,1,-1,-1)$,	$(0,0,0,0,0,0,2,2)$,
	$(0,0,0,0,2,2,0,0)$.	
6-vertices	$(1,1,1,1,1,3,-1,-1)$,	$(0,0,0,0,2,2,2,2)$,
	$(0,0,0,0,0,4,0,0)$,	$(0,0,2,2,2,2,0,0)$.
7-vertices	$(1,1,1,1,1,1,0,0)$,	$(0,0,0,0,0,2,1,1)$.

This list can be verified by checking that the vertices listed indeed lie on the orbits $W_{E_7} \cdot v_i$ and there are as many as $|W_{E_7}|/|Stab_{W_{E_7}}(v_i)|$.

We describe in the following table the 2-vertices x in β_2. Let σ be the face of $\Sigma_{v_2}\triangle$ containing $\overrightarrow{v_2x}$ in its interior.

	x	$d(x, v_2)$	Type of σ
2-vertices $x \neq v_2, \widehat{v_2}$	$(1,-1,1,1,1,1,-1,-1)$	$\frac{\pi}{3}$	3
	$(2,0,0,0,0,2,0,0)$	$\frac{\pi}{2}$	6
	$(2,-2,0,0,0,0,0,0)$	$\frac{2\pi}{3}$	3

We describe in the following table the 2- and 7-vertices x in β_7. Let σ be the face of $\Sigma_{v_7}\triangle$ containing $\overrightarrow{v_7x}$ in its interior.

	x	$d(x, v_7)$	Type of σ
2-vertices x	$(-1,1,1,1,1,1,-1,-1)$	$\arccos(\frac{1}{\sqrt{3}})$	2
	$(0,0,0,0,0,0,-2,-2)$	$\frac{\pi}{2}$	1
	$(-1,-1,-1,-1,-1,1,-1,-1)$	$\arccos(-\frac{1}{\sqrt{3}})$	6
7-vertices $x \neq v_7, \widehat{v_7}$	$(0,0,0,0,0,2,1,1)$	$\arccos(\frac{1}{3})$	6
	$(-2,0,0,0,0,0,1,1)$	$\arccos(-\frac{1}{3})$	2

We describe in the following table the 1-vertices x in β_1. Let σ be the face of $\Sigma_{v_1}\triangle$ containing $\overrightarrow{v_1x}$ in its interior.

	x	$d(x, v_1)$	Type of σ
1-vertices $x \neq v_1, \widehat{v_1}$	$(0,0,0,2,2,2,-1,-1)$	$\arccos(\frac{5}{7})$	4
	$(0,0,0,2,2,2,1,1)$	$\arccos(\frac{1}{7})$	47
	$(-2,0,0,0,2,2,1,1)$	$\arccos(-\frac{1}{7})$	25
	$(1,1,1,1,1,1,2,2)$	$\arccos(-\frac{1}{7})$	7
	$(-1,-1,1,1,1,1,2,2)$	$\arccos(-\frac{3}{7})$	37
	$(-1,-1,-1,-1,1,1,2,2)$	$\arccos(-\frac{5}{7})$	5
	$(-1,1,1,1,1,3,0,0)$	$\arccos(\frac{3}{7})$	26
	$(-3,1,1,1,1,1,0,0)$	$\arccos(\frac{1}{7})$	2

We describe in the following table the 2- and 6-vertices x in β_6. Let σ be the face of $\Sigma_{v_6}\triangle$ containing $\overrightarrow{v_6 x}$ in its interior.

	x	$d(x, v_6)$	Type of σ
2-vertices x	$(-1,\ 1,\ 1,\ 1,\ 1,\ 1,-1,-1)$	$\frac{\pi}{4}$	2
	$(\ 1,\ 1,\ 1,\ 1,\ 1,-1,-1,-1)$	$\arccos(\frac{1}{2\sqrt{2}})$	17
	$(\ 1,\ 1,\ 1,\ 1,\ 1,-1,\ 1,\ 1)$	$\frac{\pi}{2}$	7
	$(-1,-1,\ 1,\ 1,\ 1,-1,-1,-1)$	$\frac{\pi}{2}$	3
	$(\ 0,\ 0,\ 0,\ 0,\ 1,-1,\ 0,\ 0)$	$\arccos(-\frac{1}{2\sqrt{2}})$	57
	$(-1,\ 0,\ 0,\ 0,\ 0,-1,\ 0,\ 0)$	$\frac{3\pi}{4}$	2
6-vertices $x \neq v_6, \widehat{v_6}$	$(\ 1,\ 1,\ 1,\ 1,\ 3,\ 1,-1,-1)$	$\arccos(\frac{3}{4})$	57
	$(-1,\ 1,\ 1,\ 1,\ 3,-1,-1,-1)$	$\arccos(\frac{1}{4})$	257
	$(\ 0,\ 0,\ 0,\ 2,\ 2,\ 0,-2,-2)$	$\frac{\pi}{3}$	4
	$(\ 0,\ 2,\ 2,\ 2,\ 2,\ 0,\ 0,\ 0)$	$\frac{\pi}{3}$	27
	$(-3,\ 1,\ 1,\ 1,\ 1,-1,-1,-1)$	$\frac{\pi}{2}$	2
	$(\ 0,\ 0,\ 0,\ 0,\ 2,-2,-2,-2)$	$\frac{\pi}{2}$	15
	$(\ 0,\ 0,\ 2,\ 2,\ 2,-2,\ 0,\ 0)$	$\frac{\pi}{2}$	37
	$(-1,\ 1,\ 1,\ 1,\ 1,-3,-1,-1)$	$\arccos(-\frac{1}{4})$	127
	$(-1,\ 1,\ 1,\ 1,\ 1,-3,\ 1,\ 1)$	$\frac{2\pi}{3}$	27
	$(-1,-1,-1,\ 1,\ 1,-3,-1,-1)$	$\frac{2\pi}{3}$	4
	$(\ 0,\ 0,\ 0,\ 0,\ 0,-4,\ 0,\ 0)$	$\arccos(-\frac{3}{4})$	17

A.7 E_8

The Weyl group W_{E_8} of type E_8 is the finite group of isometries of $\mathbb{R}^8$ generated by the reflections at the hyperplanes orthogonal to the *fundamental root vectors*:

$$r_1 = \frac{1}{2}(1, 1, 1, -1, -1, -1, -1, -1) \text{ and } r_i = e_i - e_{i-1} \text{ for } 2 \leq i \leq 8.$$

The *fundamental Weyl chamber* $\triangle$ can be described by the inequalities:

$$x_4 + x_5 + \cdots + x_8 \overset{(1)}{\leq} x_1 + x_2 + x_3 \ ; \qquad x_1 \overset{(2)}{\leq} x_2 \overset{(3)}{\leq} x_3 \overset{(4)}{\leq} \ldots \overset{(8)}{\leq} x_8.$$

Next we exhibit an element representing the vertices of the fundamental Weyl chamber $\triangle$, i.e. elements of $\mathbb{R}^+ \cdot v_i$:

1-vertex: v_1	$(-1,-1,-1,-1,-1,-1,-1,-1)$
2-vertex: v_2	$(-3,-1,-1,-1,-1,-1,-1,-1)$
3-vertex: v_3	$(-2,-2,-1,-1,-1,-1,-1,-1)$
4-vertex: v_4	$(-5,-5,-5,-3,-3,-3,-3,-3)$
5-vertex: v_5	$(-2,-2,-2,-2,-1,-1,-1,-1)$
6-vertex: v_6	$(-3,-3,-3,-3,-3,-1,-1,-1)$
7-vertex: v_7	$(-1,-1,-1,-1,-1,-1,0,0)$
8-vertex: v_8	$(-1,-1,-1,-1,-1,-1,-1,1)$

We list now (modulo the following elements of the Weyl group: permutations of the coordinates and change of sign in an even number of places) the orbits of the vertices of $\triangle$ of type 1, 2, 6, 7, 8 under the action of the Weyl group. We give representing vectors for the vertices.

1-vertices	$(-1,-1,-1,-1,-1,-1,-1,-1)$,	$\frac{1}{2}(-5,1,1,1,1,1,1,1)$,
	$\frac{1}{2}(3,3,3,1,1,1,1,1)$,	$(0,0,0,1,1,1,1,2)$.
2-vertices	$(-3,-1,-1,-1,-1,-1,-1,-1)$,	$(2,2,2,2,0,0,0,0)$,
	$(4,0,0,0,0,0,0,0)$.	
6-vertices	$(-3,-3,-3,-3,-3,-1,-1,-1)$,	$(6,2,2,2,0,0,0,0)$,
	$(4,4,4,0,0,0,0,0)$,	$(-5,3,3,1,1,1,1,1)$,
	$(4,4,2,2,2,2,0,0)$.	
7-vertices	$(-1,-1,-1,-1,-1,-1,0,0)$,	$(2,1,1,0,0,0,0,0)$,
	$\frac{1}{2}(-3,3,1,1,1,1,1,1)$.	
8-vertices	$(-1,-1,-1,-1,-1,-1,-1,1)$,	$(2,2,0,0,0,0,0,0)$.

This list can be verified by checking that the vertices listed indeed lie on the orbits $W_{E_8} \cdot v_i$ and there are as many as $|W_{E_8}|/|Stab_{W_{E_8}}(v_i)|$.

We describe in the following table the 2- and 8-vertices x in β_2. Let σ be the face of $\Sigma_{v_2}\triangle$ containing $\overrightarrow{v_2 x}$ in its interior.

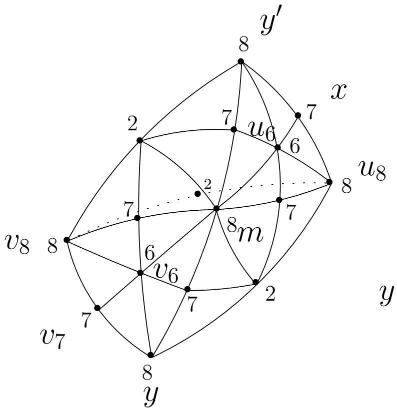
	x	$d(x, v_2)$	Type of σ
2-vertices $x \neq v_2, \widehat{v_2}$	$(\ 1, -3, -1, -1, -1, -1, -1,\ 1)$	$\arccos(\frac{3}{4})$	3
	$(\ 0, -2, -2, -2, -2,\ 0,\ 0,\ 0)$	$\frac{\pi}{3}$	6
	$(\ 1, -3, -1, -1, -1, -1, -1,\ 1)$	$\arccos(\frac{1}{4})$	38
	$(\ 1, -1, -1, -1, -1, -1, -1,\ 3)$	$\frac{\pi}{2}$	8
	$(\ 2, -2, -2, -2,\ 0,\ 0,\ 0,\ 0)$	$\frac{\pi}{2}$	5
	$(\ 3, -1, -1, -1, -1, -1, -1,\ 1)$	$\arccos(-\frac{1}{4})$	18
	$(\ 3, -1, -1, -1, -1,\ 1,\ 1,\ 1)$	$\frac{2\pi}{3}$	6
	$(\ 4,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0)$	$\arccos(-\frac{3}{4})$	1
8-vertices x	$(-1, -1, -1, -1, -1, -1, -1,\ 1)$	$\frac{\pi}{4}$	8
	$(\ 1, -1, -1, -1, -1, -1, -1, -1)$	$\arccos(\frac{1}{2\sqrt{2}})$	1
	$(\ 1, -1, -1, -1, -1, -1,\ 1,\ 1)$	$\frac{\pi}{2}$	7
	$(\ 2, -2,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0)$	$\arccos(-\frac{1}{2\sqrt{2}})$	3
	$(\ 2,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 2)$	$\arccos(\frac{3\pi}{4})$	8

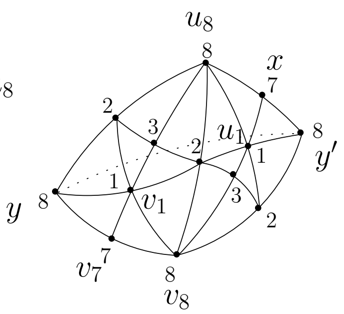
We describe in the following table the 7-vertices x in β_7, such that $d(x, v_7) = \arccos(-\frac{1}{3})$ or $\arccos(-\frac{1}{6})$, and the 8-vertices x in β_7, such that $d(x, v_7) > \frac{\pi}{2}$. Let σ be the face of $\Sigma_{v_7}\triangle$ containing $\overrightarrow{v_7 x}$ in its interior.

	x	$d(x, v_7)$	Type of σ
7-vertices x	$(\ 0,\ 0,\ 0,\ 0,\ 0,\ 2, -1, -1)$	$\arccos(-\frac{1}{3})$	6
	$(\ 0,\ 0,\ 0,\ 0,\ 1,\ 1, -2,\ 0)$	$\arccos(-\frac{1}{3})$	58
	$\frac{1}{2}(-1,\ 1,\ 1,\ 1,\ 1,\ 1, -3, -3)$	$\arccos(-\frac{1}{3})$	12
	$(\ 0,\ 0,\ 0,\ 0,\ 0,\ 1, -2,\ 1)$	$\arccos(-\frac{1}{6})$	68
	$\frac{1}{2}(-3,\ 1,\ 1,\ 1,\ 1,\ 1, -3, -1)$	$\arccos(-\frac{1}{6})$	28
	$(\ 0,\ 0,\ 0,\ 0,\ 0,\ 1, -2, -1)$	$\arccos(-\frac{1}{6})$	168
8-vertices x	$(\ 1,\ 1,\ 1,\ 1,\ 1,\ 1, -1,\ 1)$	$\arccos(-\frac{\sqrt{3}}{2})$	8
	$(-1,\ 1,\ 1,\ 1,\ 1,\ 1, -1, -1)$	$\arccos(-\frac{1}{\sqrt{3}})$	2
	$(\ 0,\ 0,\ 0,\ 0,\ 0,\ 2, -2,\ 0)$	$\arccos(-\frac{1}{2\sqrt{3}})$	68

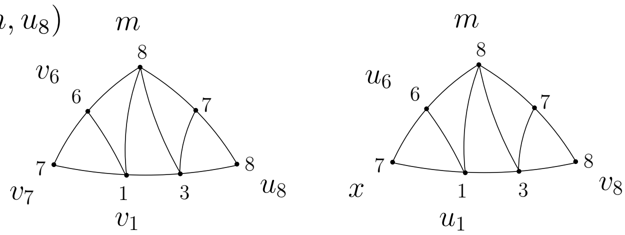
In order to make it easier to verify the table above, we present the complete table in Appendix B.

We want to describe the simplicial convex hull C of the segment v_7x for the 7-vertex $x = (0, 0, 0, 0, 0, 1, -2, -1)$, for this we present first a larger 3-dimensional spherical polyhedron, namely the tetrahedron $C' := CH(v_8, y, u_8, y')$, where $y = (-1, -1, -1, -1, -1, -1, 1, -1)$, $u_8 = (0, 0, 0, 0, 0, 0, -2, -2)$ and $y' = (0, 0, 0, 0, 0, 2, -2, 0)$. Notice that $v_7 = m(v_8, y)$ and $x = m(y', u_8)$. C' is a subcomplex with four 2-dimensional faces: the triangles $CH(v_8, y, y')$, $CH(z, y, y')$, $CH(y, u_8, v_8)$ and $CH(y', u_8, v_8)$. The figures illustrate the tetrahedron C' from the front and from behind.

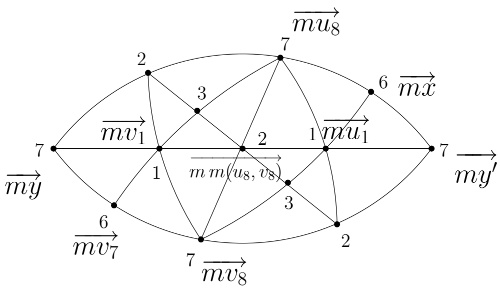




$x = (\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 1, -2, -1)$
$y = (-1, -1, -1, -1, -1, -1,\ \ 1, -1)$
$m = (-1, -1, -1, -1, -1,\ \ 1, -1, -1)$
$y' = (\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 2, -2,\ \ 0)$
$u_1 = (-1, -1, -1, -1, -1,\ \ 1, -5, -1)$
$u_6 = (-1, -1, -1, -1, -1,\ \ 3, -5, -3)$
$u_8 = (\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0, -2, -2)$
$m(u_8, v_8) =$
$(-1, -1, -1, -1, -1, -1, -3, -1)$



The triangles $CH(v_8, m, x)$ and $CH(v_7, m, u_8)$ are 2-dimensional subcomplexes. If we cut C' along these triangles, we obtain a convex subcomplex $C'' := CH(v_7, v_8, x, u_8, m)$. It has six 2-dimensional faces: the triangles $CH(m, v_7, u_8), CH(m, x, v_8), CH(m, v_7, v_8), CH(m, x, u_8)$, $CH(v_7, v_8, u_8)$ and $CH(x, u_8, v_8)$. Recall that the direction $\overrightarrow{v_7x}$ spans the 168-face in $\Sigma_{v_7}\triangle$, this implies that v_1, v_6 and v_8 are contained in the simplicial convex hull C of v_7x. We can also see that the direction $\overrightarrow{xv_7}$ spans the 168-face with vertices $\overrightarrow{xu_1}$, $\overrightarrow{xu_6}$ and $\overrightarrow{xu_8}$. In particular, $u_8 \in C$. Considering the triangle $CH(v_7, m, u_8)$ we deduce that also $m \in C$. It follows that $C = C''$. The next figure shows the link $\Sigma_m C'$.



We describe in the following table the 8-vertices x in β_8. Let σ be the face of $\Sigma_{v_8}\triangle$ containing $\overrightarrow{v_8x}$ in its interior.

	x	$d(x, v_8)$	Type of σ
8-vertices	$(-1, -1, -1, -1, -1, -1,\ \ 1, -1)$	$\frac{\pi}{3}$	7
$x \neq v_8, \widehat{v_8}$	$(-2,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0, -2)$	$\frac{\pi}{2}$	2
	$(\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 2, -2)$	$\frac{2\pi}{3}$	7

We describe in the following table the 7-vertices x in $\beta_7(2, 8)$ with $d(x, v_7) > \frac{\pi}{2}$. Let σ be the face of $\Sigma_{v_7}\triangle(2, 8)$ containing $\overrightarrow{v_7x}$ in its interior.

	x	$d(x, v_7)$	Type of σ
7-vertices	$(\ 0,\ 0,\ 1,\ 1,\ 1,\ 1, -1, -1)$	$\arccos(-\frac{2}{3})$	3
$x \neq v_7, \widehat{v_7}$	$(\ 0,\ 0,\ 0,\ 0,\ 0,\ 2, -1, -1)$	$\arccos(-\frac{1}{3})$	6

In order to make it easier to verify the table above, we present the complete table in Appendix B.

We describe in the following table the 1-vertices x in $\beta_1(2, 7, 8)$ with $d(x, v_1) > \frac{\pi}{2}$. Let σ be the face of $\Sigma_{v_1}\triangle(2, 7, 8)$ containing $\overrightarrow{v_1 x}$ in its interior.

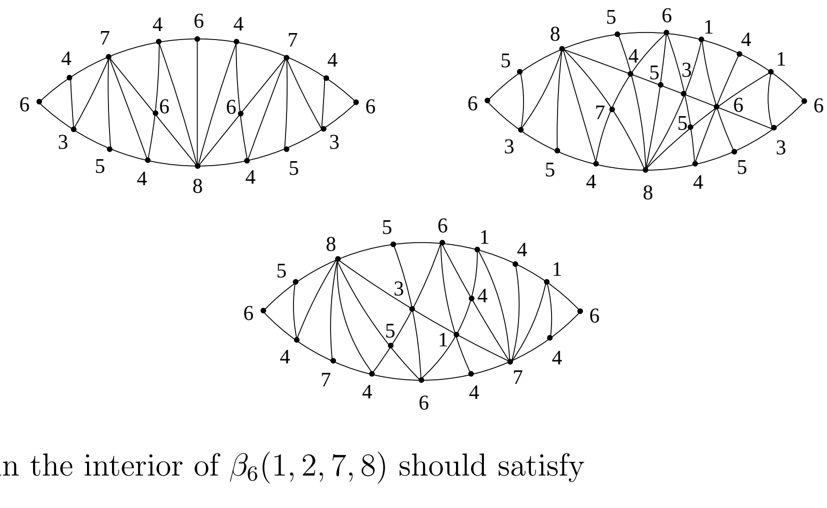
	x	$d(x, v_1)$	Type of σ
1-vertices	$\frac{1}{2}(-1, -1, -1, -1,\ 1,\ 3,\ 3,\ 3)$	$\arccos(-\frac{3}{8})$	56
$x \neq v_1, \widehat{v_1}$	$(-1, -1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1)$	$\frac{2\pi}{3}$	3
	$\frac{1}{2}(-1, -1,\ 1,\ 1,\ 1,\ 3,\ 3,\ 3)$	$\arccos(-\frac{5}{8})$	36
	$\frac{1}{2}(\ 1,\ 1,\ 1,\ 1,\ 1,\ 3,\ 3,\ 3)$	$\arccos(-\frac{7}{8})$	6

In order to make it easier to verify the table above, we present the complete table in Appendix B.

We describe in the following table the 6-vertices x in $\beta_6(1, 2, 7, 8)$ with $d(x, v_6) > \frac{\pi}{2}$. Let σ be the face of $\Sigma_{v_6}\triangle(1, 2, 7, 8)$ containing $\overrightarrow{v_6 x}$ in its interior.

	x	$d(x, v_6)$	Type of σ
6-vertices	$(\ 0,\ 0,\ 0,\ 0,\ 6, -2, -2, -2)$	$\arccos(-\frac{1}{4})$	5
$x \neq v_6, \widehat{v_6}$	$(\ 0,\ 0,\ 2,\ 4,\ 4, -2, -2, -2)$	$\frac{2\pi}{3}$	34
	$(\ 1,\ 1,\ 3,\ 3,\ 5, -1, -1, -1)$	$\arccos(-\frac{3}{4})$	35

Let us verify this last table. By considering the following 2-dimensional bigons, we can see that if there are 6-vertices missing in the table above, they must lie in the interior of $\beta_6(1, 2, 7, 8)$.



A 6-vertex x in the interior of $\beta_6(1, 2, 7, 8)$ should satisfy

$$x_4 + x_5 + \cdots + x_8 \overset{(1)}{=} x_1 + x_2 + x_3\ ; \qquad x_1 \overset{(2)}{=} x_2 \overset{(3)}{<} x_3 \overset{(4)}{<} x_4 \overset{(5)}{<} x_5\ ; \qquad x_5 > x_6 \overset{(7)}{=} x_7 \overset{(8)}{=} x_8.$$

In particular, we have four different values $x_2 < x_3 < x_4 < x_5$. Hence, x cannot be a permutation of $(\pm4, \pm4, \pm4, 0, 0, 0, 0, 0)$.

If x is obtained from $(-3, -3, -3, -3, -3, -1, -1, -1)$ by permutations of the coordinates and change of sign in an even number of places, then $x_1 = x_2 = -3$, $x_3 = -1$, $x_4 = 1$ and $x_5 = 3$. By the equalities (1), (2), (7) and (8), it follows that $x_6 = -\frac{11}{3}$, which is not possible.

If x is obtained from $(\pm6, \pm2, \pm2, \pm2, 0, 0, 0, 0)$ by permutations of the coordinates, then $(x_2, x_3, x_4, x_5) = (-6, -2, 0, 2)$, but $x_1 = x_2 = -6$ is not possible; or $(x_2, x_3, x_4, x_5) = (-2, 0, 2, 6)$. In this case the equalities (1), (2), (7) and (8) imply $x_6 = -4$, which is not possible.

If x is obtained from $(-5, 3, 3, 1, 1, 1, 1, 1)$ by permutations of the coordinates and change of sign in an even number of places, then $x_2 \in \{-5, -3, -1\}$. $x_1 = x_2 = -5$ is not possible. $x_2 = -3$ implies $(x_1, x_2, x_3, x_4, x_5) = (-3, -3, -1, 1, 5)$ and equalities (1), (2), (7) and (8) imply $x_6 = -\frac{13}{3}$. This is again impossible. $x_2 = -1$ implies $(x_1, x_2, x_3, x_4, x_5) = (-1, -1, 1, 3, 5)$ and equalities (1), (2), (7) and (8) imply $x_6 = x_7 = x_8 = -3$, which cannot happen.

If x is obtained from $(\pm4, \pm4, \pm2, \pm2, \pm2, \pm2, 0, 0)$ by permutations of the coordinates, then $(x_1, x_2, x_3, x_4, x_5) = (-4, -4, -2, 0, 2)$ or $(-2, -2, 0, 2, 4)$. In both cases the equalities $x_6 = x_7 = x_8$ cannot be satisfied.

So we have verified that $\beta_6(1, 2, 7, 8)$ contains no 6-vertices in its interior and therefore our table is complete.

Appendix B

More information about E_8

In this section, we complete some tables given in Appendix A.7. Although this information is not directly used in the proof of our main result, we present it here in order to make it easier to verify the tables in Appendix A.7.

The next table lists the 7-vertices x in β_7 with $d(x, v_7) \geq \frac{\pi}{2}$. The vertices marked with * are the ones at distance $= \frac{\pi}{2}$ to v_7. Let σ be the face of $\Sigma_{v_7}\triangle$ containing $\overrightarrow{v_7x}$ in its interior. Let σ_x be the face of $\triangle$ spanned by the initial part of the segment v_7x.

x	Type of σ	$\lvert Stab_{W_{E_8}}(v_7)\cdot x\rvert = \frac{\lvert Stab_{W_{E_8}}(v_7)\rvert}{\lvert Stab_{W_{E_8}}(\sigma_x)\rvert}$
$(1, 1, 1, 1, 1, 1, 0, 0)$		$\lvert W_{E_6}\rvert\lvert W_{A_1}\rvert/(\lvert W_{E_6}\rvert\lvert W_{A_1}\rvert) = 1$
$(0, 0, 1, 1, 1, 1, -1, -1)$	3	$\lvert W_{E_6}\rvert\lvert W_{A_1}\rvert/(\lvert W_{A_1}\rvert\lvert W_{A_4}\rvert\lvert W_{A_1}\rvert) = 216$
* $\frac{1}{2}(-1, -1, -1, 1, 1, 1, -3, -3)$	4	$\lvert W_{E_6}\rvert\lvert W_{A_1}\rvert/(\lvert W_{A_2}\rvert\lvert W_{A_1}\rvert\lvert W_{A_2}\rvert\lvert W_{A_1}\rvert) = 720$
$(0, 0, 0, 0, 0, 2, -1, -1)$	6	$\lvert W_{E_6}\rvert\lvert W_{A_1}\rvert/(\lvert W_{D_5}\rvert\lvert W_{A_1}\rvert) = 27$
$\frac{1}{2}(1, 1, 1, 1, 1, 1, -3, 3)$	8	$\lvert W_{E_6}\rvert\lvert W_{A_1}\rvert/\lvert W_{E_6}\rvert = 2$
$\frac{1}{2}(-1, 1, 1, 1, 1, 1, -3, -3)$	12	$\lvert W_{E_6}\rvert\lvert W_{A_1}\rvert/(\lvert W_{A_4}\rvert\lvert W_{A_1}\rvert) = 432$
$(0, 1, 1, 1, 1, 1, -1, 0)$	28	$\lvert W_{E_6}\rvert\lvert W_{A_1}\rvert/\lvert W_{D_5}\rvert = 54$
$\frac{1}{2}(-3, 1, 1, 1, 1, 1, -3, -1)$	28	$\lvert W_{E_6}\rvert\lvert W_{A_1}\rvert/\lvert W_{D_5}\rvert = 54$
$(0, 0, 0, 0, 1, 1, -2, 0)$	58	$\lvert W_{E_6}\rvert\lvert W_{A_1}\rvert/(\lvert W_{A_4}\rvert\lvert W_{A_1}\rvert) = 432$
$(0, 0, 0, 0, 0, 1, -2, 1)$	68	$\lvert W_{E_6}\rvert\lvert W_{A_1}\rvert/\lvert W_{D_5}\rvert = 54$
$\frac{1}{2}(1, 1, 1, 1, 1, 3, -3, 1)$	68	$\lvert W_{E_6}\rvert\lvert W_{A_1}\rvert/\lvert W_{D_5}\rvert = 54$
$(0, 0, 0, 0, 0, 1, -2, -1)$	168	$\lvert W_{E_6}\rvert\lvert W_{A_1}\rvert/\lvert W_{A_4}\rvert = 864$
* $(-1, 0, 0, 0, 0, 1, -2, 0)$	268	$\lvert W_{E_6}\rvert\lvert W_{A_1}\rvert/\lvert W_{D_4}\rvert = 540$
$\frac{1}{2}(-1, 1, 1, 1, 1, 3, -3, -1)$	268	$\lvert W_{E_6}\rvert\lvert W_{A_1}\rvert/\lvert W_{D_4}\rvert = 540$

Notice that since the antipode $\widehat{v}_7$ of v_7 is also a 7-vertex, then the number of 7-vertices in S at distance $\leq \frac{\pi}{2}$ to v_7 is the same as the number of 7-vertices in S at distance $\geq \frac{\pi}{2}$ to v_7. It follows that the number of 7-vertices in S is two times the number of 7-vertices in S at distance $\leq \frac{\pi}{2}$ to v_7 minus the number of 7-vertices at distance $= \frac{\pi}{2}$ to v_7. With this observation and the one at the end of the introductory section of Appendix A, we can verify the correctness of the list above: $2(1+216+720+27+2+432+54+54+432+54+54+864+540+540)-720-540 = 6720 = \frac{|W_{E_8}|}{|Stab_{W_{E_8}}(v_7)|} = \#\{$7- vertices in $S\}$.

The next table lists the 8-vertices x in β_7 with $d(x,v_7) \geq \frac{\pi}{2}$. The vertices marked with * are the ones at distance $= \frac{\pi}{2}$ to v_7. Let σ be the face of $\Sigma_{v_7}\triangle$ containing $\overrightarrow{v_7x}$ in its interior.

x	Type of σ
* $(0, 0, 0, 0, 0, 0, -2, -2)$	1
$(-1, 1, 1, 1, 1, 1, -1, -1)$	2
$(1, 1, 1, 1, 1, 1, -1, 1)$	8
* $(0, 0, 0, 0, 0, 0, -2, 2)$	8
$(0, 0, 0, 0, 0, 2, -2, 0)$	68

The next table lists the 1-vertices x in β_1 with $d(x,v_1) \geq \frac{\pi}{2}$. The vertices marked with * are the ones at distance $= \frac{\pi}{2}$ to v_1. Let σ be the face of $\Sigma_{v_1}\triangle$ containing $\overrightarrow{v_1x}$ in its interior.

x	Type of σ	$\lvert Stab_{W_{E_8}}(v_1)\cdot x\rvert = \frac{\lvert Stab_{W_{E_8}}(v_1)\rvert}{\lvert Stab_{W_{E_8}}(\sigma_x)\rvert}$
$(1, 1, 1, 1, 1, 1, 1, 1)$		$\lvert W_{A_7}\rvert/\lvert W_{A_7}\rvert = 1$
$\frac{1}{2}(-5, 1, 1, 1, 1, 1, 1, 1)$	2	$\lvert W_{A_7}\rvert/\lvert W_{A_6}\rvert = 8$
$(-1, -1, 1, 1, 1, 1, 1, 1)$	3	$\lvert W_{A_7}\rvert/(\lvert W_{A_1}\rvert\lvert W_{A_5}\rvert) = 28$
* $(-1, -1, -1, -1, 1, 1, 1, 1)$	5	$\lvert W_{A_7}\rvert/(\lvert W_{A_3}\rvert\lvert W_{A_3}\rvert) = 70$
$\frac{1}{2}(1, 1, 1, 1, 1, 3, 3, 3)$	6	$\lvert W_{A_7}\rvert/(\lvert W_{A_2}\rvert\lvert W_{A_4}\rvert) = 56$
$(-2, 0, 0, 0, 1, 1, 1, 1)$	25	$\lvert W_{A_7}\rvert/(\lvert W_{A_2}\rvert\lvert W_{A_3}\rvert) = 280$
$\frac{1}{2}(-1, 1, 1, 1, 1, 1, 1, 5)$	28	$\lvert W_{A_7}\rvert/\lvert W_{A_5}\rvert = 56$
$\frac{1}{2}(-1, -1, 1, 1, 1, 3, 3, 3)$	36	$\lvert W_{A_7}\rvert/(\lvert W_{A_1}\rvert\lvert W_{A_2}\rvert\lvert W_{A_2}\rvert) = 560$
$\frac{1}{2}(-3, -3, 1, 1, 1, 1, 1, 3)$	38	$\lvert W_{A_7}\rvert/(\lvert W_{A_1}\rvert\lvert W_{A_4}\rvert) = 168$
$(0, 0, 0, 1, 1, 1, 1, 2)$	48	$\lvert W_{A_7}\rvert/(\lvert W_{A_2}\rvert\lvert W_{A_3}\rvert) = 280$
$\frac{1}{2}(-1, -1, -1, 1, 1, 1, 1, 5)$	48	$\lvert W_{A_7}\rvert/(\lvert W_{A_2}\rvert\lvert W_{A_3}\rvert) = 280$
$\frac{1}{2}(-1, -1, -1, -1, 1, 3, 3, 3)$	56	$\lvert W_{A_7}\rvert/(\lvert W_{A_2}\rvert\lvert W_{A_3}\rvert) = 280$
$\frac{1}{2}(-1, -1, -1, -1, -1, 1, 1, 5)$	68	$\lvert W_{A_7}\rvert/(\lvert W_{A_4}\rvert\lvert W_{A_1}\rvert) = 168$
* $(-2, -1, 0, 0, 0, 1, 1, 1)$	236	$\lvert W_{A_7}\rvert/(\lvert W_{A_2}\rvert\lvert W_{A_2}\rvert) = 1120$
$\frac{1}{2}(-3, -1, 1, 1, 1, 1, 3, 3)$	237	$\lvert W_{A_7}\rvert/(\lvert W_{A_3}\rvert\lvert W_{A_1}\rvert) = 840$
$\frac{1}{2}(-3, -1, -1, -1, 1, 1, 3, 3)$	257	$\lvert W_{A_7}\rvert/(\lvert W_{A_2}\rvert\lvert W_{A_1}\rvert\lvert W_{A_1}\rvert) = 1680$
$(-1, 0, 0, 0, 1, 1, 1, 2)$	258	$\lvert W_{A_7}\rvert/(\lvert W_{A_2}\rvert\lvert W_{A_2}\rvert) = 1120$
$(-1, -1, 0, 0, 0, 1, 1, 2)$	368	$\lvert W_{A_7}\rvert/(\lvert W_{A_2}\rvert\lvert W_{A_1}\rvert\lvert W_{A_1}\rvert) = 1680$
* $(-1, -1, -1, 0, 0, 0, 1, 2)$	478	$\lvert W_{A_7}\rvert/(\lvert W_{A_2}\rvert\lvert W_{A_2}\rvert) = 1120$

We can verify this table as we did with the table above: $2(1+8+28+70+56+280+56+560+168+280+280+280+168+1120+840+1680+1120+1680+1120)-70-1120-1120 = 17280 = \frac{|W_{E_8}|}{|W_{A_7}|} = \#\{$1- vertices in $S\}$.

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