Some Axioms of Weak Determinacy

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Abstract

We consider two-player games of perfect information of length some cardinal κ. It is well-known that for $\kappa \geq \omega_1$ the full axiom of determinacy for these games fails, thus we investigate three weaker forms of it. We obtain the measurability of κ^+ under DC_κ-the axiom of dependent choices generalized to κ. We generalize the notions of perfect and meager sets and provide characterizations with some special kinds of games. We show that under an additional assumption one of our three axioms follows from the other two.

Zusammenfassung

Wir betrachten unendliche Spiele der Länge κ, wobei κ eine Kardinalzahl ist. Es ist bekannt, dass im Fall $\kappa \geq \omega_1$ für solche Spiele das übliche Determinierheitsaxiom inkonsistent ist. Aus diesem Grund betrachten wir drei schwächere Versionen hiervon. Mit hilfe von DC_κ zeigen wir die Messbarkeit von κ^+. Wir verallgemeinern die bekannten Begriffe der perfekten und mageren Mengen und geben Charakterisierungen durch spezielle Spielvarianten. Unter einer zusätzlichen Voraussetzung zeigen wir, dass eins unserer drei Axiomen aus den anderen beiden folgt.

Contents

1 Introduction

The Axiom of Determinacy is the following statement:

(AD) *Every two-person game of length ω where the players play ordinals smaller than ω is determined.*

A natural question to ask is whether the restriction on the length could be relaxed, i.e. whether an axiom of the following (stronger) kind is worth considering:

(AD_κ) *Every two-person game of length κ where the players play ordinals smaller than κ is determined.*

It turns out that, for $\kappa \geq \omega_1$, AD_κ is inconsistent with the axioms of ZF. The topic of this dissertation are three ways of weakening AD_κ which might be consistent with ZF, DC_κ and $\kappa = 2^{<\kappa}$. These three axioms are well-known and studied in the case $\kappa = \omega$ under the names Turing Determinacy (TD), $*$-Determinacy and $**$-Determinacy.

Informally our generalizations will be as follows. The axiom TD_κ says roughly that if we define Turing degrees on κ-sequences in a natural way, then the union of every set of Turing degrees is determined as a subset of the set of all κ-sequences. The axiom of $*$-Determinacy just states that if player I is allowed to play bounded subsets of κ, then the game is determined and $**$-determinacy states the same under the condition that both players are allowed to play non-empty bounded subsets.

We will study some consequences of these axioms in the presence of the axiom DC_κ which is a generalization of the usual axiom of dependent choices. With the help of Turing determinacy we present a generalization of the result of Martin that every set of Turing degrees contains a cone or is disjoint from a cone and thus show the measurability of the "next cardinal", while with the help of $*$ determinacy we provide a generalization of the perfect set property for the reals, whose role is played by κ-sequences in our context.

We will also generalize one result of the $\kappa = \omega$ case which gives a relation between the three axioms, namely that $*$-Determinacy follows from the other two. For this generalization however, we are going to need a stronger additional assumption. κ will always be a regular cardinal satisfying $\kappa = {}^{<\kappa}\kappa$. We will consider both the spaces of sequences of members of $\{0, 1\}$ of length κ and of sequences of ordinals smaller than κ of length κ. We will call the former space $\mathcal{C}_\kappa$ and the latter $\mathcal{B}_\kappa$ when they are endowed with the natural topology generalizing the usual topology of the Baire space.

It would also probably be meaningful to mention here some negative results which may lead one to suspect that a reasonable principle implying all these three axioms (other than their conjunction) may not exist. In the case $\kappa = \omega$ this role is played by AD itself as is well-known. Now since for $\kappa \geq \omega_1$ we cannot use AD_κ it seems natural to consider some of its weakenings.

One possibility is the following axiom which we now provisionally call AWD_κ. Consider the following situation. One of the players - X can consistently prevent the other - Y from reaching a winning position. Now for a game of length ω this is equivalent to Y having no winning strategy. So if we postulate that in such cases X has a winning strategy this would be equivalent to AD. With longer games however there is one big difference and that is the fact that the "partial games" are not necessarily determined. Here for a game of length κ a partial game is a game of length some ordinal $\gamma < \kappa$ where one of the players tries to reach a winning position and the other tries to prevent this from happening. Now if $\kappa = \omega$ the partial games are all finite and thus determined, but when $\kappa > \omega$ one cannot expect to have this property. Therefore the condition, that if one of the players prevents the other one from winning, then the first one wins, is strictly weaker than full determinacy and is thus not outright inconsistent.

One thing that could intuitively speak in favor of AWD_κ is the fact, that as we are going to show, under the additional requirement that κ is weakly compact, weak determinacy does hold for open sets, much like in the case of $\kappa = \omega$.

However it is not hard to see that AWD_κ contradicts AD. Let now B be an arbitrary subset of the Baire space. Consider the following game of length κ. The first move of I is ignored. Let δ be the first move of II. Then next δ moves of both players are also ignored. Then from δ onwards in the next ω moves the players' moves are interpreted as an ω-long game for B and whoever wins this game wins the original game of length κ. It is obvious that the game just described satisfies the assumptions for the application of AWD_κ, and so it is determined, but this means that B is determined as well. As a result we obtain that AWD_κ is inconsistent. A similar argument disproves the variant of AWD_κ where the game is played only with members of $\{0, 1\}$. If one tries to generalize determinacy via continuous maps as in [7], one gets an axiom equivalent to AWD_κ, which is then also inconsistent.

Another attempt may be to generalize a particularly useful property which follows from AD - the so-called "final segment determinacy". This is a full determinacy condition on a class of games which play an important role in some of the consequences of AD. A game depending on a final segment is a game where the beginning doesn't affect the outcome, or, in other words, a game depends on a final segment if either one of the players has a winning strategy from the start, or neither player can reach a winning position in a finite number of moves. Final segment determinacy in turn implies Turing determinacy and $*$- and $**$-Determinacy, but an argument similar to the one above shows that it cannot be generalized (at least directly) to games of length $\kappa > \omega$, if we want to have DC_κ, which is a too big constraint.

2 The spaces ${}^{\kappa}\kappa$ and ${}^{\kappa}2$

We introduce the notion of a κ-topology as a generalization of the usual notion of topology.

Definition 2.0.1 *Let κ be a cardinal, X be a set and $\mathcal{O} \subseteq \mathcal{P}(X)$. $\mathcal{O}$ is a* κ-topology on X *if the following conditions are satisfied:*

1) $\emptyset, X \in \mathcal{O}$.

2) If $\mathcal{F} \subseteq \mathcal{O}$, *then* $\bigcup \mathcal{F} \in \mathcal{O}$.

3) If $\mathcal{F} \subseteq \mathcal{O}$ *and* $|\mathcal{F}| < \kappa$, *then* $\bigcap \mathcal{F} \in \mathcal{O}$.

A base for the κ-topology $\mathcal{O}$ is a family $\mathcal{B} \subseteq \mathcal{O}$ such that any set in $\mathcal{O}$ can be written as a union of elements of $\mathcal{B}$. An open (or a κ-open) set is a member of $\mathcal{O}$, a closed (or a κ-closed) set is a member of $\{F \mid X - F \in \mathcal{O}\}$.

So $\mathcal{O}$ is a topology on X when $\mathcal{O}$ is an ω-topology on X.

In a similar fashion we can generalize topological notions to their corresponding "κ-notions". The intuition is that we just modify the respective definition by substituting according to following "dictionary":

"usual" notion	"κ-notion"
finite	having cardinality smaller than κ
countable	having cardinality smaller than or equal to κ
uncountable	having cardinality greater than κ
ω	κ
ω_1	κ^+
etc.	

So for example a κ-meager set is a union of at most κ many nowhere dense sets.

We now endow ${}^{\kappa}\kappa = \{f \mid f : \kappa \to \kappa\}$ and ${}^{\kappa}2 = \{f \mid f : \kappa \to 2\}$ with κ-topologies by generalizing the topologies of the Baire and the Cantor spaces in the obvious way.

Definition 2.0.2 *Let $s \in {}^{<\kappa}\kappa$, $t \in {}^{<\kappa}2$. Set $N_s := \{f \in {}^{\kappa}\kappa \mid s \subset f\}$ and $O_t := \{f \in {}^{\kappa}2 \mid s \subset f\}$. Denote by $\mathcal{B}_\kappa$ the κ-topology on ${}^{\kappa}\kappa$, generated by the base*

$$\bigcup_{s \in {}^{<\kappa}\kappa} \{N_s\}.$$

Denote by $\mathcal{C}_\kappa$ the κ-topology on ${}^{\kappa}2$, generated by the base

$$\bigcup_{t \in {}^{<\kappa}2} \{O_t\}.$$

By the observation that for every $s \in {}^{<\kappa}\kappa$,

$${}^{\kappa}\kappa - N_s = \bigcup_{r \not\subset s \wedge s \not\subset r} N_r,$$

we get that $\mathcal{B}_\kappa$ is zero-dimensional, and in the same way we can show this for $\mathcal{C}_\kappa$. As a further illustration of the terminology we observe that in case $\kappa = \kappa^{<\kappa}$, then $\mathcal{B}_\kappa$ is κ-separable and κ-second countable, and so is $\mathcal{C}_\kappa$ when $\kappa = 2^{<\kappa}$.

We give a characterization of κ-closed sets, which corresponds to the completeness property of the usual metric of the Baire space - a set is closed iff it contains all its limit points.

Proposition 2.0.3 *If a set $A \subseteq {}^{\kappa}\kappa$ is κ-closed, then for every sequence $\{f_\alpha\}_{\alpha<\kappa}$ of members of A, such that for each $\alpha < \kappa$ there is a $\gamma < \kappa$, so that for all $\xi, \eta > \gamma$, $f_\xi \restriction \alpha = f_\eta \restriction \alpha$, the 'limit' of the sequence is in A, in other words there exists an $f \in A$, such that for every $\alpha < \kappa$ there is a $\gamma < \kappa$, such that $f_\gamma \restriction \alpha = f \restriction \alpha$.*

Proof. Suppose A is κ-closed and let $\{f_\alpha\}_{\alpha<\kappa}$ be as required. That the 'limit' exists is obvious, just set for each γ,

$f(\gamma) =$ " that δ, which is the value at γ for cofinally many f_α's."

Assume now $f \in {}^{\kappa}\kappa - A$, then as ${}^{\kappa}\kappa - A$ is κ-open, there is a basic open neighbourhood of f, contained in it. Contradiction.
□

We leave the generalization of the perfect set notion for the next section, as the situation there is a bit trickier.

We will assume that we have a coding of ordinals below κ^+ by subsets of κ, or which is equivalent by members of $\mathcal{C}_\kappa$ or $\mathcal{B}_\kappa$ for every cardinal κ. This can be done uniformly in many ways of course and it doesn't matter which one we chose, so we will just assume that we are given a reasonable one.

3 Trees and perfect sets

The notion of a tree is standard in set theory. We fix now the notation for the most general case.

Definition 3.0.4 *A tree is an ordered pair* $\mathbf{T} = (T, <)$*, where* T *is a set,* $< \subseteq T \times T$*,* $<$ *is a partial order on* T *and for each* $x \in T$ *the set*

$$x_< = \{y \in T \mid y < x\}$$

of the $<$ *predecessors of* x *is well-ordered by* $<$*.*
If $\mathbf{T} = (T, <)$ *is a tree and* $x \in T$*, denote by*

$$ht(x, \mathbf{T}) = otp(x_<)$$

the height of x *in* $\mathbf{T}$*, where otp denotes order type for well-ordered sets.*
Define the height of $\mathbf{T}$ *as*

$$ht(\mathbf{T}) = sup\{ht(x, \mathbf{T}) + 1 \mid x \in T\},$$

for $\alpha < ht(\mathbf{T})$*, define the* α*-th level and the* α*-subtree of* $\mathbf{T}$*,*

$$T_\alpha = \{x \in T \mid ht(x, \mathbf{T}) = \alpha\},\ \mathbf{T}_{<\alpha} = \{x \in T \mid ht(x, \mathbf{T}) < \alpha\}.$$

Definition 3.0.5 *For a tree* $\mathbf{T} = (T, <_T)$ *a branch* B *of* $\mathbf{T}$ *is a maximal chain of* $<_T$*, i.e. a maximal linearly ordered (by* $<_T$*) subset of* T*. A branch* B *is cofinal if for every* $\alpha < ht(\mathbf{T})$ *there is an* $x \in T_\alpha \cap B$*.*

We state now several weak versions of the axiom of choice, the strongest of which we are going to assume for practical purposes, e.g. in order for our trees to behave as expected. The axiom of dependent κ-choices DC_κ was first introduced in [1].

(DC_κ) *For every set* $X \neq \emptyset$ *and every relation* $R \subseteq X^2$*, such that for all* $\alpha < \kappa$ *and every sequence* $s = \langle x_\delta \mid \delta < \alpha \rangle$ *of elements of* X *there exists a* $y \in X$ *with* sRy *(i.e. for every* $\delta < \alpha$ $x_\delta R y$*) there exists a function* $f : \kappa \to X$ *such that*

$$\forall \alpha < \kappa \forall \delta < \alpha f(\delta) R f(\alpha).$$

(AC_κ) *Every set* $\mathcal{X}$ *with* $|\mathcal{X}| \leq \kappa$ *has a choice function.*

(W_κ) *For every set* X *either* $|X| \leq \kappa$ *or* $\kappa \leq X$*.*

The axiom W_κ is to be read as "if there is no injection from X into κ, then there is an injection from κ into X".

Lemma 3.0.6 (DC_κ)

- *a)* AC_κ,
- *b)* W_κ,
- *c) If* $|\mathcal{X}| \leq \kappa$ *and for every* $X \in \mathcal{X}$ $X \leq \kappa$*, then* $|\bigcup \mathcal{X}| \leq \kappa$*,*
- *d)* κ^+ *is regular.*

Proof. Standard.
□

The notion of a weakly compact cardinal has many equivalent definitions. We give here the relevant one after introducing some terminology.

Definition 3.0.7 *For* κ *a cardinal and* $\mathbf{T}$ *a tree, we say that* $\mathbf{T}$ *is a* κ*-tree if* $ht(\mathbf{T}) = \kappa$ *and for each* $\alpha < \kappa$*,* $|T_\alpha| < \kappa$*. A* κ*-tree is* κ*-Aronszajn if it has no cofinal branch.*

Definition 3.0.8 *A cardinal* κ *has the tree property if there are no* κ*-Aronszajn trees.*

Definition 3.0.9 *A cardinal* κ *is weakly compact if* κ *has the tree property and* κ *is inaccessible, i.e.* $\forall \alpha < \kappa (2^\alpha < \kappa)$*.*

Note that the last definition is mostly interesting in the presence of at least as much choice as DC_κ provides, since otherwise the powerset of an ordinal smaller than κ may fail to be well-orderable.

We begin now the considerations leading to the generalization of the perfect set notion. First let us do a quick summary of some basic definitions and facts.

Definition 3.0.10 *Let $\langle X, \mathcal{O} \rangle$ be an ω-topological space. A subset A of X is called* perfect *if it is closed and consists exclusively of limit points (i.e. for every point x in A and every open set B with $x \in B$ there is a point $y \neq x$ with $y \in B$).*

Proposition 3.0.11 *(Cantor) For every perfect $A \subseteq \mathcal{C}_\omega$ $|A| = 2^\omega$.*

□

It is at this point that the generalization from ω-topologies to κ-topologies becomes a bit more subtle.

Let us start by directly generalizing the perfect set definition.

Definition 3.0.12 *Let $\langle X, \mathcal{O} \rangle$ be a κ-topological space. Call a subset A of X* perfect *(or κ-*perfect*) if it is closed and has no isolated points.*

Definition 3.0.13 *A tree $\mathbf{T} = \langle \mathbf{T}, < \rangle$ of height κ is* perfect *if for every $x \in T$ there are $y, z \in T$ with*

a) $x < y$ and $x < z$,

b) $z \not\leq y$ and $y \not\leq z$.

Notation 3.0.14 *For a tree $\mathbf{T} = \langle T, < \rangle$ of height κ let $[T]$ be the set of all cofinal branches of T.*

Proposition 3.0.15 *A subset A of $\mathcal{C}_\kappa$ is perfect iff $A = [T]$, where $\mathbf{T} = \langle T, < \rangle$ is a perfect tree of height κ, such that for every $x \in T$ $x \in {}^{<\kappa}2$ and $x < y \iff x \subseteq y$.*

Proof. Standard.
□

With the help of 3.0.15 we can easily construct a counterexample to the straight-forward generalization of the property 3.0.11, which might look appealing on a first glance. To this end consider some cardinals $\lambda < \kappa$ and the tree

$$\mathbf{T} = \langle \{x \in {}^{<\kappa}2 \mid |\{\alpha \mid x(\alpha) = 1\}| < \lambda\}, \subseteq \rangle.$$

To see that $\mathbf{T}$ is perfect is trivial and it is easy to see that $|[T]| = \kappa^\lambda$ which is usually less than 2^κ.

With the above in mind let us mention that there are at least two possible ways of generalizing the perfect sets, the most straight-forward being to just require that the size of such sets is big enough, i.e. 2^κ. While this generalization is sufficient for lifting some properties from ω to κ it will not be enough e.g. for the usual characterisation via $*$-games, so we take the other, more refined possibility.

Definition 3.0.16 *For A a subset of $\mathcal{C}_\kappa$ or $\mathcal{B}_\kappa$ call A strongly perfect iff A is closed and there is a homeomorphism*

$$h : \mathcal{C}_\kappa \to \langle A, \mathcal{O}_A \rangle,$$

where $\mathcal{O}_A$ is the topology on A induced by the topology on $\mathcal{C}_\kappa$.

Remark 3.0.17 *The above definition should formally require a κ-homeomorphism.*

The collection of sets with the perfect set property is as we have seen not so interesting in this case, so we modify the definition accordingly to get a meaningful generalization. Again we mention that this could have been done in other ways as well.

Definition 3.0.18 *A subset P of $\mathcal{C}_\kappa$ or $\mathcal{B}_\kappa$ is* strongly psp *if one of the following is satisfied.*

a) $|P| \leq \kappa$.

b) There is a subset B of P, such that B is strongly perfect.

We see now, that our generalization of perfect is stronger than the direct one.

Lemma 3.0.19 *If $P \subseteq \mathcal{B}_\kappa$ is strongly perfect, then P contains no isolated points and $|P| = 2^\kappa$.*

Proof. If f is an isolated point of P, then $\{f\}$ is an open set in $\langle X, \mathcal{O}_X \rangle$ contradicting the existence of a homeomorphism between X and $\mathcal{C}_\kappa$. For the second assertion let T be the tree on $\kappa^{<\kappa}$ with $P = [T]$ (=the set of infinite branches of T). We claim that T is $< \kappa$-closed. To see this assume the opposite and let $t_0 < \cdots < t_\alpha < \cdots$, $\alpha < \delta$ for some $\delta < \kappa$ be a counterexample. Then $N_{t_0} \restriction X \supset \cdots \supset N_{t_\alpha} \restriction X \supset \cdots$, $\alpha < \delta$ is a nested sequence of $< \kappa$ basic open sets with empty intersection again violating the homeomorphism requirement. Now T is a prefect $< \kappa$-closed tree, so it is easy to see that $|P| = |[T]| = 2^\kappa$.

□

We will give another characterisation of strongly perfect sets after we introduce the $*$-games for κ-sequences.

4 Turing reducibility for $\mathcal{C}_\kappa$ and $\mathcal{B}_\kappa$

In the case $\kappa = \omega$ the study of Turing reducibility and Turing degrees is one of the central subjects of recursion theory. The most common definition goes as follows.

Definition 4.0.20 *For $f, g \in {}^\omega\omega$, $f \leq_T g$ if there is an index e of an oracle Turing machine, so that for every $x \in \omega$, $f(x) = y$ iff $\{e\}^g = y$.*

In the context of our longer sequences however, if we want to generalize this definition directly, we would run into considerable technical difficulties, the most significant of which is having to deal with infinite computations. Instead of doing this, we consider a characterization of Turing reducibility, whose generalization turns out to be easier.

We state first the well known definition of the sets hereditarily of size less than a given cardinal.

Definition 4.0.21 *For λ a cardinal let*

$$H_\lambda = \{x \mid |TC(x)| < \lambda\},$$

the set of all sets of hereditary cardinality smaller than λ, where $TC(x)$ is the smallest transitive set, containing x as an element.

A nice feature of H_λ is that it is usually not too big.

Proposition 4.0.22 *(DC_κ) If $\kappa = 2^{<\kappa}$,*

$$|H_\kappa| = \kappa.$$

Proof. "$\geq$": Every ordinal smaller than κ is in $|H_\kappa|$, so we just take the identity here.
"$\leq$": Define the function $F : ON \to V$ by transitive recursion as follows:

$$F(0) = \emptyset,$$

$$F(\alpha+1) = \{x \subseteq F(\alpha) \mid |x| < \kappa\},$$
$$F(\lambda) = \bigcup_{\alpha<\lambda} F(\alpha) \text{ , for limit } \lambda.$$

We now have:

(1) $\bigcup_{\alpha \in On} F(\alpha) = \bigcup_{\alpha<\kappa} F(\alpha)$,

Proof. We first use the regularity of κ to show that $F(\kappa+1) = F(\kappa)$. Indeed if $x \in F(\kappa+1)$, then by definition $x \subseteq \bigcup_{\alpha<\kappa} F_\alpha$. Now since $|x| < \kappa$ and κ is regular there is a $\beta < \kappa$ with $x \in F(\beta)$, hence $x \in F(\kappa)$ and the equality follows.

Now immediately $F(\theta) = F(\kappa)$ for every $\theta > \kappa$.

(2) $H_\kappa = \bigcup_{\alpha<\kappa} F(\alpha)$,

Proof.

"$\supseteq$" by induction $F(\alpha) \subseteq H_\kappa$ for every α,

"$\subseteq$" by rank induction $x \in H_\kappa \rightarrow x \in \bigcup_{\alpha<\kappa} F(\alpha)$

(3) $|\bigcup F(\alpha)| \leq \kappa$.

Proof. By $\kappa = {}^{<\kappa}2$ it follows by induction that for every α $|F(\alpha)| \leq \kappa$. Now we use 3.0.6.

□

The characterization of Turing reducibility is now the following.

Proposition 4.0.23 *For f, $g \in {}^\omega\omega$*

$$f \leq_T g \textit{ iff } f \textit{ is } \Delta_1 \textit{ definable with parameters in } \langle H_\omega, \in, g\rangle.$$

Proof. Standard.

□

Our generalization now can be done by substituting κ for ω, just as our 'dictionary' above prescribes.

Definition 4.0.24 *If f, $g \in {}^\kappa\kappa$ (or f, $g \in {}^\kappa 2$), say that $f \leq_T g$ iff*

$$f \textit{ is } \Delta_1 \textit{ definable with parameters in } \langle H_\kappa, \in, g\rangle.$$

$$f \equiv_T g \textit{ iff } f \leq_T g \& g \leq_T f.$$

We first check that the definition is correct.

Proposition 4.0.25 *The relation $\equiv_T$ is an equivalence relation.*

Proof.

i) To see that $f \equiv_T f$, note that f is definable in $\langle H_\kappa, \in, f\rangle$ (even without parameters) via the Σ_0 formula

$$\varphi(\langle x, y\rangle) = \langle x, y\rangle \in f.$$

ii) To see the transitivity of $\leq_T$ suppose that $f \leq_T g$ via the Δ_1 formula φ with parameters $p_1, \cdots, p_n$ and $g \leq_T h$ via the Δ_1 formula ψ with parameters $q_1, \cdots, q_k$. Consider the following formula:

$$\chi(\langle x, y\rangle) = \text{"the formula } \varphi\text{, where each instance of the predicate}$$

$$g \text{ is substituted by the formula } \psi \text{ ."}$$

It is not hard to see that χ is a Δ_1 formula with parameters $p_1, \cdots, p_n$, $q_1, \cdots q_k$, and so f is definable in $\langle H_\kappa, \in, h\rangle$ via χ.
□

Remark 4.0.26 *In the proof of the last proposition we have tacitly used the convention that a formula which is true only on pairs is written with an explicit pair as a formal parameter, e.g. $\varphi(\langle x, y\rangle) = \langle x, y\rangle \in f$. Formally such a formula is to be understood as a formula with an arbitrary parameter which has in its definition an additional check that the argument is actually a pair. Thus the example formula just given is to be read as*

$$\varphi(z) = (z = \langle x, y\rangle) \wedge (\langle x, y\rangle \in f).$$

Note that there is no danger in confusing the complexity in this way, since the formula "$z = \langle x, y\rangle$" has a Σ_0 definition ($\exists a \in z \exists b \in z(a = \{x\} \wedge b = \{x, y\} \wedge \forall t \in z(t = a \vee t = b))$). We will keep using this convention.

Definition 4.0.27 *For $\mathbf{a} \subseteq \mathcal{B}_\kappa$, $\mathbf{a}$ is a* Turing degree *in $\mathcal{B}_\kappa$ iff for each $f, g \in \mathbf{a}$, $f \equiv_T g$ and for every $f \in \mathbf{a}$ and $h \in \mathcal{B}_\kappa - \mathbf{a}$ it is not true that $f \equiv_T h$. For $f \in \mathcal{B}_\kappa$ we write $[f]_\kappa$ for the unique Turing degree in $\mathcal{B}_\kappa$ to which f belongs.*

Definition 4.0.28 *For* $\mathbf{a} \subseteq \mathcal{C}_\kappa$*,* $\mathbf{a}$ *is a* Turing degree in $\mathcal{C}_\kappa$ *iff for each* $f, g \in \mathbf{a}$*,* $f \equiv_T g$ *and for every* $f \in \mathbf{a}$ *and* $h \in \mathcal{C}_\kappa - \mathbf{a}$ *it is not true that* $f \equiv_T h$*. For* $f \in \mathcal{C}_\kappa$ *we write* $[f]_2$ *for the unique Turing degree in* $\mathcal{B}_\kappa$ *to which* f *belongs.*

The correctness of these definitions follows, of course, by the fact that $\equiv_T$ is an equivalence relation, and so the ordering of members of $\mathcal{B}_\kappa$ and $\mathcal{C}_\kappa$ with respect to Turing reducibility lifts directly to orderings of the Turing degrees.

Definition 4.0.29 *For Turing degrees (in* $\mathcal{C}_\kappa$ *or* $\mathcal{B}_\kappa$*)* $\mathbf{a}$ *and* $\mathbf{b}$ *we say that* $\mathbf{a} \leq \mathbf{b}$ *if for some (all)* $f \in \mathbf{a}$ *and for some (all)* $g \in \mathbf{b}$ *we have* $f \leq_T g$*.*

We are not going to investigate the structure of these Turing degrees in too much detail here. Nevertheless we give some definitions and properties which will be relevant to us later.

First comes the observation that there is a one to one correspondence between the members of $\mathcal{B}_\kappa$ and $\mathcal{C}_\kappa$. This will help us later, when we are proving properties about the structure of one of these sets, to get the same properties for the other.

Lemma 4.0.30 *For every degree* $\mathbf{a}$ *in* $\mathcal{B}_\kappa$ *there is an* $h \in \mathcal{C}_\kappa$*, such that* $h \in \mathbf{a}$*.*

Proof. Pick a representative $f \in \mathbf{a}$ and consider $g \in \mathcal{C}_\kappa$, such that

$$g(\langle \alpha, \beta \rangle) = 0 \iff f(\alpha) = \beta.$$

It is not hard to see that $f \equiv_T g$.
□

Lemma 4.0.31 *For every* $f \in \mathcal{C}_\kappa$*,*

$$[f]_2 = \mathcal{C}_\kappa \cap [f]_\kappa.$$

Proof. By transitivity of $\equiv_T$.
□

Lemma 4.0.32 *There is a one to one correspondence between the degrees in* $\mathcal{C}_\kappa$ *and* $\mathcal{B}_\kappa$ *that preserves the relation* $\leq_T$*.*

Proof. By the previous two lemmas.

□

Next we turn to the generalization of the Turing jump operation known from the study of the usual Turing degrees. This will be our only way to obtain a Turing degree strictly higher than a given degree.

Lemma 4.0.33 (DC_κ), $(\kappa = {}^{<\kappa}2)$

For every degree $\mathbf{a} \in \mathcal{B}_\kappa$ *there is a degree* $\mathbf{b} \in \mathcal{B}_\kappa$*, such that* $\mathbf{a} < \mathbf{b}$.

Proof. Suppose $\mathbf{a} = [f]_\kappa$ for some $f \in \mathcal{B}_\kappa$. Consider now the set

$$A = \{\langle i, \alpha, p\rangle \mid \langle H_\kappa, \in, f\rangle \models \varphi_i(\alpha) \text{ with parameter } p\},$$

where $\langle \varphi_i \mid i \in \omega\rangle$ is some enumeration of all Σ_1 formulas. By lemma 4.0.22 $|A| \leq \kappa$ and so with some kind of appropriate coding we can regard A as a subset of κ. We now apply a standard diagonalization procedure to show that $\kappa - A$ is not Σ_1 definable in $\langle H_\kappa, \in, f\rangle$ with parameters. To this end assume $\kappa - A$ is definable, say via the formula φ_j with parameter q. Then for an arbitrary ordinal $\beta < \kappa$ we have

$$\langle j, \beta, q\rangle \in A \iff \langle H_\kappa, \in, f\rangle \models \varphi_j(\beta) \text{ with parameter q}$$
$$\iff \langle j, \beta, q\rangle \in \kappa - A.$$

As a corollary we have that A is not Π_1 definable in $\langle H_\kappa, \in, f\rangle$ with parameters, and hence not Δ_1 definable.

That f is Δ_1 definable in $\langle H_\kappa, \in, A\rangle$ is trivial, so we take $\mathbf{b} = [g]_\kappa$, where

$$g = \{\langle \alpha, 0\rangle \mid \alpha \in A\} \cup \{\langle \alpha, 1\rangle \mid \alpha \in \kappa - A\}.$$

□

Lemma 4.0.34 (DC_κ), $(\kappa = {}^{<\kappa}2)$

For every degree $\mathbf{a} \in \mathcal{B}_\kappa$ $|\mathbf{a}| \leq \kappa$.

Proof. Let $\mathbf{a} = [f]_\kappa$. Put

$$B = \{g \in \mathcal{B}_\kappa \mid g \text{ is definable with parameters in } \langle H_\kappa, \in, f\rangle\}.$$

Since the set of formulas is countable and by 4.0.22 $|H_\kappa| \leq \kappa$ we get $|B| \leq \kappa$. Now since obviously $[f]_\kappa \subseteq B$ the result follows.

□

Now we generalize the join operation from ordinary recursion theory.

Lemma 4.0.35 *For every sequence of Turing degrees* $\mathbf{a_1}, \mathbf{a_2}, \cdots, \mathbf{a}_\alpha, \cdots$ *with* $\alpha < \kappa$ *in* $\mathcal{B}_\kappa$*, there exists a Turing degree* $\mathbf{a}$ *in* $\mathcal{B}_\kappa$*, such that*

$$\forall \alpha < \kappa (\mathbf{a}_\alpha \leq \mathbf{a}).$$

Proof. Let for every $\alpha < \kappa$ $f_\alpha \in \mathbf{a}_\alpha$ be a representative. Consider the set

$$f = \oplus \{ f_\delta \mid \delta < \kappa \},$$

defined by $f(\langle \alpha, \beta \rangle) = f_\alpha(\beta)$. This is possible due to the existence of a bijection between κ and $\kappa \times \kappa$. To see that f_α is Δ_1 definable in f consider the formula

$$\varphi(\beta) = \langle \alpha, \beta \rangle \in f.$$

Obviously this is a defining formula for f_α with parameter α.

□

The last three properties hold in $\mathcal{C}_\kappa$ as well as in $\mathcal{B}_\kappa$ by the same proof, so we just state them in a lemma.

Lemma 4.0.36 *(*DC_κ*), (*$\kappa = {}^{<\kappa}2$*)*

a) For every degree $\mathbf{a} \in \mathcal{C}_\kappa$ *there is a degree* $\mathbf{b} \in \mathcal{C}_\kappa$*, such that* $\mathbf{a} < \mathbf{b}$*,*

b) for every degree $\mathbf{a} \in \mathcal{C}_\kappa$ $|\mathbf{a}| \leq \kappa$*,*

c) for every sequence of Turing degrees $\mathbf{a_1}, \mathbf{a_2}, \cdots, \mathbf{a}_\alpha, \cdots$ *with* $\alpha < \kappa$ *in* $\mathcal{C}_\kappa$*, there exists a Turing degree* $\mathbf{a}$ *in* $\mathcal{C}_\kappa$*, such that*

$$\forall \alpha < \kappa (\mathbf{a}_\alpha \leq \mathbf{a}).$$

Proof: Same as in lemma 4.0.33 , lemma 4.0.34 and lemma 4.0.35 .

□

The notion of a cone of Turing degrees is well-known from Martin's proof that AD implies the existence of a measurable cardinal. We are going to do a similar style proof later, and here we just give the definition.

Notation 4.0.37 *For* $f \in \mathcal{B}_\kappa$ *write*

$$K_\kappa(f) = \{ [g]_\kappa \mid f \leq_T g \}.$$

For $f \in \mathcal{C}_\kappa$ *write*

$$K_2(f) = \{ [g]_2 \mid f \leq_T g \}.$$

Definition 4.0.38 *A set $\mathcal{A}$ of Turing degrees in $\mathcal{B}_\kappa$ (resp. $\mathcal{C}_\kappa$) is a* cone *in $\mathcal{B}_\kappa$ ($\mathcal{C}_\kappa$) if there is an $f \in \mathcal{B}_\kappa$ (resp. $\mathcal{C}_\kappa$), such that $\mathcal{A} = K_\kappa(f)(resp. K_2(f))$.*

For $f \in \mathcal{B}_\kappa$ we call $K_\kappa(f)$ the cone determined by $[f]_\kappa$ and similarly for $f \in \mathcal{C}_\kappa$ and $K_2(f)$.

5 Games of length κ

In this section we define two-player games of length κ on ${}^{\kappa}\kappa$ and on ${}^{\kappa}2$ for a regular cardinal κ. After that we state the axioms we will be investigating. The context in which the usual AD is defined is easily obtained by substituting ω for κ everywhere.

5.1 Games and Strategies

5.1.1 The game $G_{{}^{\kappa}\kappa}(A)$

Let κ be a regular cardinal and A be a subset of ${}^{\kappa}\kappa$ which will be called *the payoff set*. The game $G_{{}^{\kappa}\kappa}(A)$ features two players which we call I and II, who cooperate to construct a sequence $f \in {}^{\kappa}\kappa$ step by step, where I plays at even steps and II at odd. Here step by step means, that first I plays some ordinal which is interpreted as $f(0)$, then II plays some ordinal which is interpreted as $f(1)$ and so on, until the order type of the constructed sequence reaches κ.

I	II
$f(0)$	
	$f(1)$
$f(2)$	
.	.
.	.
.	.
$f(\alpha)$	
	$f(\alpha+1)$
.	.
.	.
.	.

(Fig. 1)

$\alpha < \kappa$ even

At each point in time the players have complete information for the run of the game up to that point(e.g. before playing $f(\omega^2+43)$, II can see what I has played as $f(\omega+2)$, or what he himself has played as $f(17)$, etc.). We note that I plays first at limits (since limit ordinals are considered even). We will denote by f_I the moves of player I and by f_{II} the moves of player II:

$$f_I = \langle f(\alpha) \mid \alpha < \kappa,\ \alpha \text{ even}\rangle,$$

$$f_{II} = \langle f(\alpha) \mid \alpha < \kappa,\ \alpha \text{ odd}\rangle.$$

After the game is finished it is tested whether the resulting sequence f belongs to A. If this is the case, then I wins, otherwise II wins. So, intuitively the goal of I is to "get into A", while the goal of II is to "keep out of A" (or equivalently to "get into ${}^{\kappa}\kappa - A$"). Intuitively still, a strategy is an algorithm, which given the course of the game up to any point tells a player what move to play next.

Thus two ways to look at a strategy σ for I are:

A) As a function:

$$\sigma : \bigcup_{\alpha<\kappa,\ \alpha \text{ even}} \kappa^{\alpha} \to \kappa$$

B) As a tree on κ:

$$\sigma \subseteq {}^{<\kappa}\kappa,\ \sigma \text{ closed under initial segments}$$

with the additional properties:

1. if α is even and $\langle \gamma_0, \dots, \gamma_\alpha \rangle \in \sigma$, then for every $\eta < \kappa$,
$$\langle \gamma_0, \dots, \gamma_\alpha, \eta \rangle \in \sigma,$$
and
2. if α is odd and $\langle \gamma_0, \dots, \gamma_\alpha \rangle \in \sigma$, then there is exactly one $\eta < \kappa$, such that
$$\langle \gamma_0, \dots, \gamma_\alpha, \eta \rangle \in \sigma.$$

It is not hard to see that A) and B) are basically the same thing, and if ever for some technical purpose it is important which one we consider, we will state it explicitly. Also it is obvious from B) that $|\{\sigma | \sigma \text{ is a strategy for } I\}| \leq |\mathcal{P}({}^{<\kappa}\kappa)|$, so if there is a bijection between ${}^{<\kappa}\kappa$ and κ, we can code strategies by members of ${}^{\kappa}\kappa$. Further it is clear how to define the notion of a strategy for II.

Definition 5.1.1 *A* play *(or a run) of the game $G_{{}^\kappa\kappa}(A)$ is a member g of ${}^{\kappa}\kappa$. A* partial play *(or a partial run, or a valid position) of the game $G_{{}^\kappa\kappa}(A)$ is a member g of ${}^{<\kappa}\kappa$.*

Intuitively, a partial play $g \in {}^{\lambda}\kappa$ for some ordinal $\lambda < \kappa$ is just a sequence produced by the first λ moves the players have made, and is to be extended to an $f \in {}^{\kappa}\kappa$ in order to finish the game. Given a partial play $g \in {}^{\lambda}\kappa$ it is clear whose turn it is to play based on the parity of the ordinal λ.

Definition 5.1.2 *Let σ be a strategy for I in the game $G_{{}^\kappa\kappa}(A)$. A* play of the game (produced) according to σ *is a sequence $\langle g_\alpha \mid \alpha < \kappa \rangle$, where for every even ordinal $\delta < \kappa$,*
$$\sigma(\langle g_\alpha \mid \alpha < \delta \rangle) = g_\delta.$$
If $\langle g(\alpha) \mid \alpha < \kappa \rangle$ is such a play we denote it by $\sigma * y$, where $y = \langle g_\alpha \mid \alpha \text{ odd } \rangle$.

A partial play (or a partial run, or a valid position) of the game according to σ is a sequence $\langle g_\alpha \mid \alpha < \beta \rangle$, where $\beta < \kappa$ and for every even ordinal $\delta < \beta$
$$\sigma(\langle g_\alpha \mid \alpha < \delta \rangle) = g_\delta.$$
If $\langle g(\alpha) \mid \alpha < \beta \rangle$ is such a play we denote it by $\sigma * y$, where $y = \langle g_\alpha \mid \alpha \text{ odd } \rangle$. For τ a strategy for II, and $x \in {}^{<\kappa}\kappa \cup {}^{\kappa}\kappa$, $x * \tau$ has the natural corresponding meaning.

Next comes the definition of a winning strategy. The idea is obvious - if a player has a winning strategy and plays according to it, he always wins no matter what the other player plays.

Definition 5.1.3 *Let $A \subseteq {}^{\kappa}\kappa$. A* strategy *$\sigma$* for *$I$ in the game $G_{^{\kappa}\kappa}(A)$ is winning, if for every $g \in {}^{\kappa}\kappa$,*

$$\sigma * g \in A.$$

A strategy τ for II is winning, if for every $f \in {}^{\kappa}\kappa$

$$f * \tau \in {}^{\kappa}\kappa - A.$$

Now it is obviously impossible for the two players to simultaneously have winning strategies for one game: suppose σ is winning for I and τ for II in the game $G_{^{\kappa}\kappa}(A)$ for some $A \in {}^{\kappa}\kappa$, suppose $f = \sigma * g$, where g is a play according to τ. Then $f \in A \cap ({}^{\kappa}\kappa - A) = \emptyset$.
During the course of the game it might happen, that a position is reached, from where onwards one of the players has a winning strategy. The precise formulation follows.

Definition 5.1.4 *A* valid position *$g \in {}^{\lambda}\kappa$ in the game $G_{^{\kappa}\kappa}(A)$ is a winning position for I if:*

a) λ is even and I has a winning strategy in the game $G_{^{\kappa}\kappa}(B)$, where

$$B = \{f \in {}^{\kappa}\kappa \mid g^\frown f \in A\}$$

or

b) λ is odd and for every $\alpha < \kappa$, I has a winning strategy in the game $G_{^{\kappa}\kappa}(C)$, where

$$C = \{f \in {}^{\kappa}\kappa \mid g^\frown \langle\alpha\rangle^\frown f \in A\}.$$

A point to note in this definition is the following. If we were discussing games in the special situation of $\kappa = \omega$, then point b) above, could have been reformulated to "λ is odd (i.e. an odd number) and II has a winning strategy in the game $G_{^{\omega}\omega}({}^{\omega}\omega - B)$, where $B = \{f \in {}^{\omega}\omega \mid g^\frown f \in A\}$". That would have meant intuitively, that player I of the original game starts to act as player II in the game $G_{^{\omega}\omega}({}^{\omega}\omega - B)$, because now II (from the original game) has to make the first move. This is possible to do in the special case

$\kappa = \omega$ only because there are no limit cases to consider. If we want to generalize such a definition to arbitrary regular cardinals we run into the problem that the player who wants to get out of the payoff set (player II) plays first at limits, which contradicts our definition of a game.
We define now, based on a fixed game $G_{{}^\kappa\kappa}(A)$ the partial games $G_{{}^\lambda\kappa}(A)$ for ordinals $\lambda < \kappa$. This is the point where the asymmetry in the definition of a game (pointed out in the preceding paragraph) becomes even more obvious - the two players will have goals of different nature - I will try to win, and II will try not to lose.

Definition 5.1.5 *Let $A \subseteq {}^\kappa\kappa$ and λ be an ordinal smaller than κ. The game $G_{{}^\lambda\kappa}(A)$ is defined as follows: the two players cooperate in constructing a sequence $g \in {}^\lambda\kappa$. Player I wins if g is a winning position for him in the game $G_{{}^\kappa\kappa}(A)$, otherwise player II wins.*

Note that we could have defined winning positions for player II in a similar way as we did for player I. Also we could have defined respective partial games, where II tries to win and I tries not to lose. It will become clear, however, after we state the axiom of weak determinacy, that since I is privileged by the above asymmetry, that if we had stated a similar axiom with the roles of I and II interchanged, it would just have been a corollary of ours.

5.1.2 The game $G_{{}^\kappa 2}(A)$

Here again κ is a regular cardinal, this time the payoff set A is a subset of ${}^\kappa 2$. The entire previous section can be translated in the new context by just requiring the players to play numbers in the set $2 = \{0, 1\}$ instead of ordinals smaller than κ. Here we show that every such game can be seen as a game on ${}^\kappa\kappa$ and vice versa under mild requirements.

The idea is that we add the rule "whoever plays anything bigger than 1 loses". So let $A \subseteq {}^\kappa 2$. Pick $B \subseteq {}^\kappa\kappa$ with

$$B = A \cup \{f \in {}^\kappa\kappa \mid \exists \alpha < \kappa(\alpha \text{ odd} \wedge f(\alpha) > 1 \wedge \forall \delta < \alpha(f(\delta) \leq 1))\}.$$

Now if I has a winning strategy σ in $G_{{}^\kappa 2}(A)$, he can apply the same strategy to win $G_{{}^\kappa\kappa}(B)$: if II plays anything bigger than 1, I wins automatically, otherwise respond according to σ. Similarly for II - there we didn't need to modify the payoff set to favor II more, since if I plays outside of $\{0,1\}$ he loses automatically.

For the converse, if I has a winning strategy for the game $G_{^\kappa\kappa}(B)$, it must surely deal with the situation where II plays only zeroes and ones, so it is automatically contains a winning strategy for $G_{^\kappa 2}(A)$, as a subset, and similarly for a winning strategy for II.

For the other direction we require a bijection between ${}^{<\kappa}2$ and κ, so that bounded subsets of κ can be coded by ordinals in κ. The idea is that we can code the initial segments of an $f \in {}^\kappa\kappa$ in such a way that initial segments appear before their extensions and also initial segments of even length are coded by even ordinals and initial segments of odd length are coded by odd ordinals. We construct then a $g \in {}^\kappa 2$, corresponding to f, so that $g(\alpha) = 1$ if α codes an initial segment of f and $g(\alpha) = 0$ otherwise. We can now encode each element of a payoff set $A \subseteq {}^\kappa\kappa$ in this way; call the resulting subset of ${}^\kappa 2$ B. Let $G_{^\kappa 2}(B^*)$ be the game for B with the added requirement that whoever plays something which is not a code for a superset of the sequence thus far constructed loses automatically.
Now if I has a winning strategy for $G_{^\kappa\kappa}(A)$, he can just play $G_{^\kappa 2}(B^*)$ by decoding each of his prescribed moves, and putting 1 in the desired place - the properties of the coding guarantee that this would be a winning strategy for B.The case for II is the same. The converse follows in a similar way. We will not write everything down as explicitly as above, but it is clear that the following is true.

Proposition 5.1.6 *Assume $|{}^{<\kappa}2| = \kappa$. Then for every $A \subseteq {}^\kappa\kappa$, there is a $B \subseteq {}^\kappa 2$, such that I (resp. II) has a winning strategy in $G_{^\kappa\kappa}(A)$ if and only if I (resp. II) has a winning strategy in $G_{^\kappa 2}(B)$. And for every $B \subseteq {}^\kappa 2$, there is an $A \subseteq {}^\kappa\kappa$, such that I (resp. II) has a winning strategy in $G_{^\kappa 2}(B)$ if and only if I (resp. II) has a winning strategy in $G_{^\kappa\kappa}(A)$.*

Corollary 5.1.7 *If $|{}^{<\kappa}2| = \kappa$, then every game $G_{^\kappa\kappa}(A)$ is determined if and only if every game $G_{^\kappa 2}(B)$ is determined.*

Remark 5.1.8 *The above corollary holds for ω, but will be trivially true for cardinals bigger than ω once we show that it is inconsistent to generalize AD to bigger cardinals directly.*

Completely analogously to the previous section we define the partial games for subsets of ${}^{\kappa}2$.

Definition 5.1.9 *Let $A \subseteq {}^{\kappa}2$ and λ be an ordinal smaller than κ. The game $G_{{}^{\lambda}2}(A)$ is defined as follows: the two players cooperate in constructing a sequence $g \in {}^{\lambda}2$. Player I wins if g is a winning position for him in the game $G_{{}^{\kappa}2}(A)$, otherwise player II wins.*

Here the definition of a winning position is the same as in 5.1.4.

5.1.3 The games $G^{**}{}_{{}^{\kappa}\kappa}(A)$ and $G^{*}{}_{{}^{\kappa}\kappa}(A)$

For an $A \subseteq {}^{\kappa}\kappa$, the $**$-game $G^{**}{}_{{}^{\kappa}\kappa}(A)$ is played by the following rules. Instead of ordinals, the players play increasing bounded sequences of ordinals smaller than κ.

I	II
s_0	
	s_1
s_2	
.	.
.	.
.	.
s_α	
	$s_{\alpha+1}$
.	.
.	.
.	.

(Fig. 2)

$\alpha < \kappa$ even

For each $\alpha < \kappa$, $s_\alpha \in {}^{<\kappa}\kappa$, $s_\alpha \supsetneq \bigcup_{\delta<\alpha} s_\delta$ (no "pass"-moves are allowed). s_α is the α-th move in the game, it is played by I if α is even and by II if α is odd. Note that since κ is required to be regular $|\bigcup_{\delta<\alpha} s_\delta| < \kappa$ for all $\alpha < \kappa$. Put

$$s := \bigcup_{\alpha<\kappa} s_\alpha.$$

Obviously $s \in {}^{\kappa}\kappa$, the winning condition is the same: I wins if $s \in A$, II wins if $s \in {}^{\kappa}\kappa - A$.
The $*$-game $G^{*}{}_{{}^{\kappa}\kappa}(A)$ has 'hybrid' rules - I plays bounded sequences and II plays ordinals.

I	II
s_0	
	γ_1
s_2	
.	.
.	.
.	.
s_α	
	$\gamma_{\alpha+1}$
.	.
.	.
.	.

(Fig. 3)

$\alpha < \kappa$ even

For each odd $\alpha < \kappa$ $\gamma_\alpha \in \kappa$, for even $\alpha < \kappa$, $s_\alpha \in {}^{<\kappa}\kappa$, $s_\alpha \supseteq \bigcup_{\delta<\alpha, \delta \text{ even}} s_\delta \frown \alpha_{\delta+1}$ (we allow empty moves for player I). Put

$$s := \bigcup_{\alpha<\kappa, \alpha \text{ even}} s_\alpha.$$

I wins if $s \in A$, II wins if $s \in {}^{\kappa}\kappa - A$.

5.2 Some observations

5.2.1 Finite games

Fact 5.2.1 *Every finite game is determined.*

This almost does not require proof - if it is not true that there exists a move for I, such that for every move by II, there exists a move for I, ..., for every move by II (there exists a move for I such that) something holds, then for every move by I there exists a move for II such that for every move by I, ... , there exists a move for II such that (for every move by I) the reverse holds - where the dots stand for a finite amount of words - just the usual way in which quantified logical expressions are negated.

5.2.2 Games of length ω

We give here three well known facts ([3]):

Fact 5.2.2 *(AC) There exists an $A \subseteq ({}^{\omega}2)$, such that the game $G_{{}^{\omega}2}(A)$ is not determined .*

Fact 5.2.3 *(AC) For every closed or open $A \subseteq ({}^{\omega}2)$, the game $G_{{}^{\omega}2}(A)$ is determined .*

Fact 5.2.4 *(AC) For every closed or open $A \subseteq ({}^{\omega}\omega)$, the game $G_{{}^{\omega}\omega}(A)$ is determined .*

5.2.3 Games of length κ

We give a generalization of Fact 5.2.3 for arbitrary weakly compact cardinals. We will see why this is in fact a generalization in the next section.

Proposition 5.2.5 *(DC_κ) If κ is weakly compact, $\kappa = 2^{<\kappa}$, $A \subseteq ({}^{\kappa}2)$ is κ-open, then the following property holds: If one of the players has winning strategies for all games $G_{{}^{\alpha}2}(A)$ where α is an ordinal smaller than κ, then this player has a winning strategy in the whole game $G_{{}^{\kappa}2}(A)$.*

Proof. We consider the two possible cases:

Case I) Player I has winning strategies in the partial games. In this case we have actually allowed more than enough. A winning strategy for I for even one of the games $G_{^{\alpha}2}(A)$ is enough to guarantee him a winning strategy in the whole game (c.f. Definitions 5.1.5, 5.1.4)

Case II) Player II has winning strategies. For each α let τ_α be a winning strategy for for II for $G_{^{\alpha}2}(A)$, whose existence is assumed. Now for each $\delta < \alpha$ τ_α includes a winning strategy for II for the game $G_{^{\delta}2}(A)$, which we call τ_α^δ. Observe that

$$\tau_\alpha^\delta = \tau_\alpha \restriction \delta.$$

Now consider the space Φ of all winning strategies for the partial games that II has, equipped with the following relation: If $\delta < \alpha < \kappa$, τ is a winning strategy for $G_{^{\delta}2}(A)$, σ is a winning strategy for $G_{^{\alpha}2}(A)$ and $\tau \subset \sigma$ write $\tau \prec \sigma$. We will show that

(*) $(\Phi, \prec)$ is in fact a κ-tree:

Proof of (*): If τ_α is a winning strategy for $G_{^{\alpha}2}(A)$, then $\{\tau_\alpha^\delta \mid \delta < \alpha\}$ is obviously well-ordered by $\prec$. Φ certainly has a member on each level below κ. To see that each level is of cardinality smaller than κ proceed by induction as follows. Level 0 is obviously of cardinality smaller than κ, if α is even and τ_α is a winning strategy for $G_{^{\alpha}2}(A)$, then there are finitely many (four) possible ways to "prescribe an extension to level $\alpha + 2$" for each member of ${}^{\alpha}2$ reached via playing in accordance with τ_α - now the $\alpha + 2$'nd level of Φ will have size smaller than κ by the induction hypothesis, regularity and inaccessibility of κ. For α limit observe that $|\Phi_{<\alpha}| = \delta$ for some $\delta < \kappa$, now as each member of Φ_α is a union of members of $\Phi_{<\alpha}$ and the induction is finished once we compute that there are at most 2^δ such unions, which is smaller than κ by inaccessibility. $\square$(*)

Now we apply the weak compactness of κ to get a cofinal branch of Φ

$$B = \{\tau_\alpha \mid \alpha < \kappa\}.$$

Obviously $\sigma = \bigcup B$ is a strategy for II in $G_{^{\kappa}2}(A)$. To see that σ is winning consider the play $g = f * \sigma$, where $f = \langle f_\alpha \mid \alpha \text{ even}\rangle$. Now for every $\alpha < \kappa$ there is a $g_\alpha \in {}^{\kappa}2 - A$ with $g_\alpha \restriction \alpha = g \restriction \alpha$ - otherwise $g \restriction \alpha$ would be a winning position for I - in fact every strategy would be winning for him from there on. Now we apply 2.0.3 with ${}^{\kappa}2 - A$ and $\{g_\alpha\}_{\alpha<\kappa}$ to see that $g \notin A$.

$\square$

Note that we could get a similar result for closed A above only if reverse the roles of I and II - i.e. we should redefine the partial games so that II plays in them for a win in the whole game, while I plays not to lose, and suppose that I has winning strategies in all of them. Note also that we could not generalize 5.2.4 in this way if we tried, because the tree Φ from the proof could have big (of cardinality $\geq \kappa$) levels in this case.

6 The Axioms Of Weak Determinacy

Having gathered all the relevant definitions in the previous sections we are ready to proceed with the axioms.

6.1 Motivation

The usual axiom of determinacy has the following statement in our context.

(AD) *For every $A \subseteq {}^{\omega}\omega$ the game $G_{{}^{\omega}\omega}(A)$ is determined.*

Before we see that the outright generalization from ω to κ is inconsistent let us state it.

For κ a regular cardinal.
(AD_κ) *For every $A \subseteq {}^{\kappa}\kappa$ the game $G_{{}^{\kappa}\kappa}(A)$ is determined.*

The following is straight-forward.

Remark 6.1.1 *If $\lambda > \kappa$, AD_λ implies AD_κ.*

Proof. Let $A \subseteq {}^{\kappa}\kappa$. In order to prove that A is determined consider the set $B = \{f \in {}^{\lambda}\lambda \mid f \restriction \kappa \in A\} \cup \{f \in {}^{\lambda}\lambda \mid \exists \alpha < \kappa(\alpha \text{ odd } \wedge f(\alpha) \geq \kappa \wedge \forall \beta < \alpha(f(\beta) < \kappa))\}$. By AD_λ $G_{{}^{\lambda}\lambda}(B)$ is determined, so let σ be a winning strategy for one of the players. Then it is trivial to obtain from σ a winning strategy for the same player in the game $G_{{}^{\kappa}\kappa}(A)$.
□

By the above remark to show the inconsistency of AD_κ for all uncountable cardinals we only need to show the inconsistency of AD_{ω_1}. The following fact will help us establish this.

Proposition 6.1.2 (Mycielski) *If AD holds, then there is no injection from ω_1 into ${}^{\omega}\omega$.*

Proposition 6.1.3 (Mycielski) $\neg AD_{\omega_1}$

Proof. Put

$$A = \{f \in^{\omega_1} \omega_1 \mid \; f(0) \geq \omega \text{ and } f_{II} \restriction \omega \text{ is not a bijection with } f(0) \;\}.$$

Consider the game $G_{\omega_{1\omega_1}}(A)$. Obviously the moves after ω don't matter. Also obvious is that since countable ordinals are played I cannot have a winning strategy in the game. So assuming AD_{ω_1} II has a winning strategy σ. But now using σ it is easy to define an injection from ω_1 into ${}^{\omega}\omega$ - contradiction with 6.1.2!

□

Altogether we have.

Fact 6.1.4 *AD_κ is inconsistent for $\kappa > \omega$.*

So this obvious generalization of AD fails and we will have to consider weaker versions of it. We mentioned in the introduction, that a motivation inspired by Proposition 5.2.5 and based on winning conditions for the partial games does seem unlikely to be consistent with DC_κ. We will now discuss this point in some more detail.

The direct generalization based on 5.2.5 was called AWD_κ in the introduction. We will now show that final segment determinacy - a principle strictly weaker than AWD_κ implies $\neg AD$. As sketched in the introduction this will lead us to abandon the idea of using AWD_κ.

Definition 6.1.5 *For $A \subseteq {}^{\kappa}\kappa$ the game $G_{\kappa_\kappa}(A)$* depends on a final segment *if either I has a winning strategy for it or no valid position is a winning position for him (the 'or' is naturally exclusive, since if the first clause is true $\emptyset$ is a winning position for I).*

(FS_κ) *If the game $G_{\kappa_\kappa}(A)$ depends on a final segment, it is determined.*

It is easy to see that AWD_κ (even if it were consistent) would be stronger than FS_κ, since if the game depends on a final segment, every strategy would win the partial games for II, unless I had a strategy from the start.

Lemma 6.1.6 *If $\kappa > \omega$, then $FS_\kappa \Rightarrow \neg AD$.*

Proof. Consider the game G, which is defined as follows. If I plays on his first move an ordinal that is not in the set $\{\alpha \mid \omega \leq \alpha < \omega_1\}$, then he loses. Otherwise let δ be the first move of I. In order to win the game II now has to ensure that there is some ordinal γ, such that for every limit ordinal $\lambda > \gamma$ the next ω moves of II represent a bijection with δ. It is easy to see that I does not have a winning position in this game, so by FS_κ II must have a winning strategy but now we can proceed as in 6.1.3.
□

Remark 6.1.7 *We could try to reformulate FS_κ by switching the roles of I and II, i.e. by postulating that if II has no winning position, then I has a winning strategy. Another possible reformulation is to consider games played just with members of $\{0, 1\}$, i.e. producing a member of $\mathcal{C}_\kappa$ as a resuling sequence. It is not hard to see however that these two reformulations actually yield equivalent axioms.*

We turn now to the generalizations of the axioms of $*$, $**$ and Turing determinacy which we are actually going to use. As we are going to see, in the case $\kappa = \omega$ all of them follow from the axiom FS_ω.

6.2 $*$ and $**$ determinacy

The intuition behind the fact that final segment determinacy implies determinacy of the $*$ and $**$ games when $\kappa = \omega$ is that if I has any winning position in a game of one of these two kinds, he can just directly jump to it on his first move, which means that he has a winning strategy for the game, the other alternative is that he has no winning position at all and this is when FS_ω comes into play.

(AWD^*_κ) *For every $A \subseteq {}^\kappa\kappa$ game $G^*_{{}^\kappa\kappa}(A)$ is determined.*

(AWD^{**}_κ) *For every $A \subseteq {}^\kappa\kappa$ game $G^{**}_{{}^\kappa\kappa}(A)$ is determined.*

(AWD^{*2}_κ) *For every $A \subseteq {}^\kappa 2$ game $G^*_{{}^\kappa 2}(A)$ is determined.*

(AWD^{**2}_{κ}) *For every* $A \subseteq {}^{\kappa}2$ *game* $G^{**}_{\kappa 2}(A)$ *is determined.*

Let us point out here that if we try to derive one of these axiom from another one we run in the following trouble. The usual strategy that one employs in similar situations is to modify the payoff set by including a condition of the sort "whoever does something first loses", now the trouble with that sort of argument is that in these games we cannot reconstruct the run of the game based on the resulting sequence, because at least one of the players (I) can always make arbitrarily long moves, so given say an $f \in \mathcal{C}_\omega$ produced by a $*$-game the only certain thing is that $f(0)$ is chosen by I. We are later going to derive some implications, however these will require strong additional assumptions.

Proposition 6.2.1 $FS_\omega + DC$ *imply:*

i) AWD^*_ω,

ii) AWD^{**}_ω,

iii) AWD^{*2}_{ω},

iv) AWD^{**2}_{ω}.

Proof. Let g be a bijection between ω and ${}^{<\omega}\omega$, such that each member of ${}^{<\omega}\omega$ appears unboundedly many times in the enumeration g, and h a bijection between ω and ${}^{<\omega}2$ where each member of ${}^{<\omega}2$ appears unboundedly often. The existence of these functions is guaranteed by DC. The unboudedness requirements above serve to ensure that under appropriate rules of the game the players cannot "make an illegal move". The alternative would have been to add a rule that whoever plays a code of a set that doesnt adhere to the $*$ (respectively $**$) rules loses. Then however the game would not have depended on a final segment formally, so this is just a technical trick.
For i) suppose $A \subseteq {}^{\omega}\omega$. The function $* : {}^{\omega}\omega \to {}^{\omega}\omega$, which we define by recursion, maps a sequence of moves to the sequence resulting in translating the moves in terms of a $*$-game, i.e. applying g on the moves taking care that every move is a superset of the union of the preceding moves and that the moves of II increase the size by exactly 1 :

$$f^* := *(f) = \langle f^*(n) \mid n < \omega \rangle,$$

where for n even

$$f^*(n) = g(\beta(f(n), n)),$$

where for all i, j,

$$\beta(i,j) = \text{" the smallest } \beta \geq i \text{ with "} g(\beta) \supseteq \bigcup_{k<j} f^*(k)\text{"},$$

and for n odd

$$f^*(\alpha) = \bigcup_{k<n} f^*(k) \cup \{f(n)\}.$$

Now put

$$A^* = \{f^* \mid f \in A\}.$$

It is clear that player I has a winning strategy in $G_{\omega_\omega}(A^*)$ iff I has a winning strategy in $G^*_{\omega_\omega}(A)$. The same thing is clear also for II, so it only remains to check that $G_{\omega_\omega}(A^*)$ depends on a final segment. To this end suppose that I has no winning strategy in the full game and suppose that he has a winning position $p \in^n \omega$ with strategy σ. We have two cases:

A) $Even(n)$ then a winning strategy τ for I in $G_{\omega_\omega}(A^*)$ can be easily constructed, with

$$\tau(\emptyset) = \text{"a g-index for } \bigcup_{k\leq n} *(p(k))\text{"},$$

$$\tau(x) = \sigma(x), \text{ for } x \neq \emptyset.$$

B) $Odd(\alpha)$ Similar to A).

Contradiction! The other points are proved similarly.

□

6.3 Turing determinacy

The other axiom we are going to consider corresponds to the Turing determinacy axiom known for the case $\kappa = \omega$.

Let us first observe how the game for a single Turing degree is easily determined in ZFC with II having a winning strategy. To get the intuition we restrict for a second our attention to the case $\kappa = \omega$. Now whatever the Turing degree I is aiming for, II can just play a sequence of a higher degree and regardless of the moves of I, II wins.
DC_κ and an easy cardinality argument allow us to get a generalization.

Proposition 6.3.1 *(DC_κ) If $\kappa = 2^{<\kappa}$ and $f \in B_\kappa$, player II has a winning strategy in $G_{{}^\kappa\kappa}([f])$.*

Proof. We first prove a claim.

(*) There is a $g \in B_\kappa$, such that g is not Σ_1-definable in $\langle H_\kappa, \in, f\rangle$.

To see this note that the set of all formulas is countable and that the possible parameters come from H_κ, which by 4.0.22 has cardinality κ. By Cantor's theorem not every element of B_κ can be defined in this way.
By $(*)$ chose a $g \in B_\kappa$ with $g \not\leq_T f$. Define a strategy σ for II as follows:

$$\text{For } \alpha = \lambda + n,\ n \text{ odd},\ \lambda \text{ limit},\ x \in {}^\alpha\kappa,$$

$$\sigma(x) = g(\lambda + (n+1)/2 - 1).$$

It is now left to see that σ is indeed winning. To this end assume that for some $y \in {}^\kappa\kappa$ $h = y * \sigma \leq_T f$. This means that there is a Σ_1 formula φ with parameters from H_κ and a Π_1 formula ψ with parameters from H_κ, so that the following holds:

$$\langle H_\kappa, \in, f\rangle \models \varphi(\langle x, y\rangle)$$

iff

$$\langle H_\kappa, \in, f\rangle \models \psi(\langle x, y\rangle)$$

iff

$$\langle x, y\rangle \in h.$$

From here we can easily obtain a Δ_1 in $\langle H_\kappa, \in, f\rangle$ definition of g - consider say for Π_1 the formula

$$\chi(\langle x, y\rangle) = \text{ Ordinal}(x) \wedge x \in \kappa \wedge$$

$$\exists \lambda < x \exists n \in \omega (\text{Limit}(\lambda) \wedge x = \lambda + n \wedge \psi(\langle \lambda + 2n + 1, y\rangle)).$$

Contradiction!

□

Remark 6.3.2 *It is quite probable that the proposition above is not optimal with respect to minimality of assumptions. We are not going to be bothered by this however, as it is not a part of the main development.*

Remark 6.3.3 *Under suitable assumptions on κ the proposition also holds for boundedly many Turing degrees by taking a member of a degree higher than the join of all of them*

That determinacy holds for arbitrary sets of Turing degrees however is not provable under standard set theoretical assumptions. For the case $\kappa = \omega$ it is a known result of Woodin that in $L(\mathbb{R})$ it is equivalent to full determinacy. In our general setting we are not going to attempt a generalization of this result, but at least we state the axiom officially and prove that as for $*$ and $**$ determinacy FS_ω implies Turing determinacy.

(TD_κ) *For every set $\mathcal{A}$ of Turing degrees in ${}^\kappa\kappa$ the game $G_{{}^\kappa\kappa}(\bigcup\mathcal{A})$ is determined.*

(TD^2_κ) *For every set $\mathcal{A}$ of Turing degrees in ${}^\kappa 2$ the game $G_{{}^\kappa 2}(\bigcup\mathcal{A})$ is determined.*

We prove first a basic property of the $\mathcal{B}_\kappa$ and $\mathcal{C}_\kappa$ Turing degrees - every two sequences which agree on a final segment lie in the same degree.

Proposition 6.3.4 *a) If f, $g \in \mathcal{B}_\kappa$ and there exists an $\alpha < \kappa$, such that for every $\gamma > \alpha$, $f(\gamma) = g(\gamma)$, then $f \equiv_T g$.*

b) If f, $g \in \mathcal{C}_\kappa$ and there exists an $\alpha < \kappa$, such that for every $\gamma > \alpha$, $f(\gamma) = g(\gamma)$, then $f \equiv_T g$.

Proof. a) Let f and g be as in a). We are going to show $g \leq_T f$, the other direction being similar. Let α be such that for every $\gamma > \alpha$, $g(\alpha) = f(\alpha)$. Put

$$p = g \restriction (\alpha + 1).$$

The idea now is to use p as parameter in order to find a Δ_1 definition of g in $\langle H_\kappa, \in, f\rangle$. Of course for this to work p has to be a member of H_κ, but this is immediate as p is a set of pairs of ordinals, smaller than κ. We proceed now with the defining formula:

$$\varphi(\langle x,y\rangle) = \text{ Ordinal}(x) \land \text{ Ordinal}(y) \land$$
$$((x \le \alpha \land \langle x,y\rangle \in p) \lor (\alpha < x \land \langle x,y\rangle \in f)).$$

The parameters in this definition are α and p (and α can be defined from p in a "Σ_0 way") and obviously belong to H_κ. The formula φ is even Σ_0, so $g \le_T f$.

The proof of b) is similar.

□

Proposition 6.3.5 *(DC)* $FS_\omega \to TD$

Proof. Suppose $\mathcal{A}$ is a set of Turing degrees in the Baire space. Put $A = \bigcup \mathcal{A}$. What we are going to show is that the game $G_{^\omega\omega}(A)$ depends on a final segment and is thus determined by FS_ω. Consider the two cases:

a) I has a winning strategy in $G_{^\omega\omega}(A)$.

   In this case the game depends on a final segment by definition.

b) I has no winning strategy in $G_{^\omega\omega}(A)$.

   In this case what we have to show is that I has no winning position. Assume that $g \in {}^n\omega$ is a winning position for I. Put

$$B = \{f \in {}^\omega\omega \mid g^\frown f \in A\}$$

   and let τ be a winning strategy for I in the game $G_{^\omega\omega}(B)$. We are going to show that $\emptyset$ is a winning position for I thus reaching contradiction.

   The idea for constructing a winning strategy for I is simple - play the first n moves arbitrarily and then start following τ. The resulting sequence h will have the following property:

$$t = g^\frown(h - h \restriction n+1) \in \mathbf{a},$$

   for some degree $\mathbf{a}$ in $\mathcal{A}$. However now t and h agree on a final segment, so, according to 6.3.4, h is also in $\mathbf{a}$ and thus in $\bigcup \mathcal{A}$.

   Formally a strategy σ for I will look like this.

   For $m \le n$ odd and $q \in {}^m\omega$,

$$\sigma(q) = 0,$$

for $m > n$ odd and $q \in {}^m\omega$,

$$\sigma(q) = \tau(q - q \restriction n).$$

□

Later we are going to see that under standard assumptions TD_κ and TD^2_κ are actually equivalent for arbitrary κ.

7 Consequences

Both of the consequences that we present in this section will follow from FS_κ, together with the choice requirement DC_κ in the case $\kappa = \omega$.

7.1 Measurability of κ^+

We start with the generalization of the remarkable observation by Martin, that under AD every set of Turing degrees contains a cone, or is disjoint from a cone. We isolate this property as an axiom for both spaces $\mathcal{C}_\kappa$ and $\mathcal{B}_\kappa$ and then show the equivalence.

Notation 7.1.1 *Let $\mathcal{D}_T^\kappa$ denote the set of all Turing degrees in $\mathcal{B}_\kappa$. Let $\mathcal{D}_T^2$ denote the set of all Turing degrees in $\mathcal{C}_\kappa$.*

$(CC_{\mathcal{C}_\kappa})$ *For $\mathcal{A}$ a set of Turing degrees in $\mathcal{C}_\kappa$ either $\mathcal{A}$ or $\mathcal{D}_T^2 - \mathcal{A}$ contains a cone.*

$(CC_{\mathcal{B}_\kappa})$ *For $\mathcal{A}$ a set of Turing degrees in $\mathcal{B}_\kappa$ either $\mathcal{A}$ or $\mathcal{D}_T^\kappa - \mathcal{A}$ contains a cone.*

Lemma 7.1.2 $CC_{\mathcal{C}_\kappa} \iff CC_{\mathcal{B}_\kappa}$

Proof. Denote the isomorphism from lemma 4.0.32 by $\Theta : \mathcal{D}_T^\kappa \to \mathcal{D}_T^2$. Now for the direction from left to right suppose $\mathcal{A} \subseteq \mathcal{B}_\kappa$. Consider now the set

$$\mathcal{E} = \{\Theta(\mathbf{a}) \mid \mathbf{a} \in \mathcal{A}\}.$$

Now $\mathcal{E}$ is a set of Turing degrees in $\mathcal{C}_\kappa$ and by $CC_{\mathcal{C}_\kappa}$ either it or its complement contain a cone, and so when we apply Θ^{-1} we get that either $\mathcal{A}$ or its complement contain a cone.

□

Lemma 7.1.3 *($\kappa = {}^{<\kappa}2, DC_\kappa$)*

a) $TD_\kappa \iff CC_{\mathcal{B}_\kappa}$,

b) $TD^2_\kappa \iff CC_{\mathcal{C}_\kappa}$,

c) $TD_\kappa \iff TD^2_\kappa$.

Proof:

a) $\rightarrow$) Put $A = \bigcup \mathcal{A}$. Now by TD_κ the game $G_{^\kappa\kappa}(A)$ is determined. We consider the two possible cases.

Case 1: Player I has a winning strategy, say σ.

As we noted in section 5 we can regard strategies as members of $\mathcal{B}_\kappa$.

Now consider for an arbitrary $h \in \mathcal{B}_\kappa$ the composition $\sigma * h$. As σ is winning for I this composition is always in A. The only thing left to see now is that for every $h \in \mathcal{B}_\kappa$, such that $\sigma \leq_T h$, $h \leq_T \sigma * h$. Thus we have $K_\kappa(\sigma) \subseteq \mathcal{A}$.

Case 2: Player II has a winning strategy, say τ.

By a similar argument in this case $K_\kappa(\tau) \subseteq \mathcal{D}^\kappa_T - \mathcal{A}$.

$\leftarrow$) Let $\mathcal{A}$ be a set of Turing degrees in $\mathcal{B}_\kappa$. By $CC_{\mathcal{B}_\kappa}$ either $\mathcal{A}$ or $\mathcal{D}^\kappa_T - \mathcal{A}$ contains a cone. Suppose the former is true and say $\mathcal{K} = K_\kappa(f) \subseteq \mathcal{A}$. Now I has an easy winning strategy σ in $G_{^\kappa\kappa}(A)$ - namely he plays f. Whatever sequence y is played by II the result $\sigma * y$ is in $\bigcup \mathcal{K}$ and thus in $\bigcup \mathcal{A}$. The other case is similar.

b) the same as a).

c) by a), b) and lemma 7.1.2.

□

The next well-known definition is given for completeness.

Definition 7.1.4 *Let X be a set and λ a cardinal. A Filter $\mathcal{F}$ on X is* λ-complete, *if for every $\mathcal{O} \subseteq \mathcal{F}$, such that $|\mathcal{O}| < \lambda$, $\bigcap \mathcal{O} \in \mathcal{F}$.*

Martin's filter is defined in the same way as in the classical recursion theory.

Notation 7.1.5 *Let $M^\kappa_T = \{\mathcal{A} \subseteq D^\kappa_T \mid \mathcal{A}$ contains a cone $\}$ be Martin's filter over D^κ_T.*

Lemma 7.1.6 *($TD_\kappa, \kappa = 2^{<\kappa}, DC_\kappa$) M_T^κ is a κ^+-complete.*

Proof. That M_T^κ is a filter is easy to see. That it is an ultrafilter follows by lemma 7.1.3. The κ^+-completeness follows by lemma 4.0.35.
□

Theorem 7.1.7 *($TD_\kappa, \kappa = 2^{<\kappa}, DC_\kappa$) κ^+ is measurable.*

Proof. We are going to convert the κ^+-complete ultrafilter M_T^κ to a κ^+-complete ultrafilter over κ^+. To this end consider the function $g : D_T^\kappa \to \kappa^+$, defined by

$$g(\mathbf{a}) = sup\{\| f \| \mid f \in \mathbf{a}\},$$

where $\| . \|$ is any coding of ordinals below κ^+ by members of $\mathcal{B}_\kappa$. Now for each $\mathbf{a} \in D_T^\kappa$ $g(\mathbf{a}) < \kappa^+$ by lemma 4.0.34 and by the fact that κ^+ is regular. Finally consider

$$\mathcal{U} = \{U \subseteq \kappa^+ \mid \exists \mathcal{A} \in M_T^\kappa(U = g''\mathcal{A})\}.$$

To see that $\mathcal{U}$ is nontrivial we use lemma 4.0.33.
□

7.2 $*$-determinacy and the perfect set property

In this section we show that AWD^{*2}_κ is equivalent to every subset of $\mathcal{C}_\kappa$ and $\mathcal{B}_\kappa$ being strongly psp. The argument is similar to the argument for the case $\kappa = \omega$, where the conclusion is derived first for the Cantor space and then transferred to the Baire space via a coding map.

Lemma 7.2.1 *Every subset of $\mathcal{C}_\kappa$ is strongly psp iff every subset of $\mathcal{B}_\kappa$ is strongly psp.*

Proof. For the direction from right to left one just consults the definition and uses that $\mathcal{C}_\kappa$ is a subspace of $\mathcal{B}_\kappa$. For the other direction it is enough to see that every subset of $\mathcal{B}_\kappa$ is κ-homeomorphic to a subset of $\mathcal{C}_\kappa$. The idea from the case $\kappa = \omega$ can be used here as well.

For $f \in \mathcal{B}_\kappa$ define we $f^* \in \mathcal{C}_\kappa$ as follows. For every two ordinals α, β if α is even let $t^{\alpha,\beta} \in {}^{\beta+1}2$ with

i) for $\gamma < \beta$, $t^{\alpha,\beta}(\gamma) = 1$, and

ii) $t^{\alpha,\beta}(\beta) = 0$.

If α is odd we take $t^{\alpha,\beta} \in {}^{\beta+1}2$ dually with

i) for $\gamma < \beta$, $t^{\alpha,\beta}(\gamma) = 0$, and

ii) $t^{\alpha,\beta}(\beta) = 1$.

Finally put

$$f^* = t^{0,f(0)} \frown t^{1,f(1)} \cdots t^{\gamma,f(\gamma)} \frown \cdots, \text{ for } \gamma < \kappa.$$

It is easy to see that $* : \mathcal{B}_\kappa \to C$ is a homeomorphism, where $C \subseteq \mathcal{C}_\kappa$.
□
Now we provide the actual game characterization.

Lemma 7.2.2 *($\kappa = 2^{<\kappa}$) Let $A \subseteq {}^\kappa 2$.*

a) II has a winning strategy in $G^*_{^\kappa 2}(A)$ iff $|A| \leq \kappa$.

b) I has a winning strategy in $G^*_{^\kappa 2}(A)$ iff A has a strongly perfect subset.

Proof. a) Suppose σ is a winning strategy for II. Let for this proof a good position be a sequene $s = \langle s_\delta \mid \delta < \alpha \rangle$, where α is even, such that s is played according to the rules of the $*$-games and the moves of II are played according to σ. For s such a position let $s^* = \cup_{\delta<\alpha} s_\delta$.

(*) Let $f \in {}^\kappa 2$. If for every good position p, such that $p^* \subseteq f$ there exists a $t \supseteq p^*$ with $\sigma(p^\frown\langle t \rangle) \subseteq f$, then $f \notin A$.

Proof of (*). Since $\emptyset$ is a good position we can recursively construct good positions $p_0 \subseteq p_1 \subseteq \dots p_\alpha \subseteq \dots$ with $\alpha < \kappa$. Now we have $\cup_{\alpha<\kappa} p^*_\alpha = f$. Since all $p'_\alpha s$ are played according to σ and σ is winning for II we have $f \notin A$.

Now define for every good position $p = \langle p_\delta \mid \delta < \alpha \rangle$

$$F_p = \{ f \in {}^\kappa 2 \mid p^* \subseteq f \wedge \forall t \supseteq p^* (\sigma(p^\frown\langle t \rangle) \not\subseteq f) \}.$$

Now because of $(*)$ we have that $A \subseteq \bigcup_{p \text{ good}} F_p$. Since the number of good positions is at most κ by ($\kappa = 2^{<\kappa}$) we would be ready if we can show that

$|F_p| = 1$. But this is easy, since for each $f \in F_p$ and each $\alpha < \kappa$ $f(\alpha)$ is uniquely determined by p.

For the converse suppose that $A \subseteq \mathcal{C}_\kappa$ and $|A| \leq \kappa$. Let $\langle f_\alpha \mid \alpha < \kappa \rangle$ be an enumeration of A. Now a strategy σ for II can easily be constructed by making sure at step α that the resulting sequence will differ from f_α.

b) Let σ be a winning strategy for I. Consider the set $P = \{\sigma * y \mid y \in {}^\kappa 2\}$. $P \subseteq A$ because σ is winning, $|P| = 2^\kappa$ and P is easily seen to be strongly perfect.

Conversely let $P \subseteq X$ be strongly perfect. By the proof of lemma 3.0.19 $P = [T]$, where T is perfect and $< \kappa$-closed. Now at each successor move I can stay on the tree because the tree is perfect and limit stages go through with $< \kappa$-closedness.

□

Corollary 7.2.3 *($\kappa = 2^{<\kappa}$) $AWD^{*2}_\kappa \Rightarrow$ every subset of $\mathcal{C}_\kappa$ is strongly psp as is every subset of $\mathcal{B}_\kappa$.*

Proof. By 7.2.1 and 7.2.2.

□

7.3 $**$-determinacy and κ-meager sets

In this section we show how some results for $**$ games known from the case $\kappa = \omega$ can be generalized. In contrast with the situation for $*$ games and the perfect set property, it is not clear if one can define a corresponding Baire property for κ-topologies, which follows from $**$-determinacy.

Definition 7.3.1 *A subset of $\mathcal{C}_\kappa (\mathcal{B}_\kappa)$ is κ-meager if it is the union of at most κ-many nowhere dense sets.*

Lemma 7.3.2 *($\kappa = {}^{<\kappa}\kappa$)*

i) For $A \subseteq \mathcal{C}_\kappa$ we have A is κ-meager iff II has a winning stragegy in $G^{**}_{\mathcal{C}_\kappa}(A)$.

ii) For $A \subseteq \mathcal{B}_\kappa$ we have A is κ-meager iff II has a winning stragegy in $G^{**}_{\mathcal{B}_\kappa}(A)$.

Proof. *i)* Suppose first that A is κ-meager. Then $A \subseteq \bigcup_{\alpha<\kappa} C_\alpha$, where each C_α is closed nowhere dense. Then the complements of the C_α's are open dense, so at his α'th move II can get in the complement of C_α.

For the other direction suppose that σ is a winning strategy for II. Let for this proof a good position be a sequene $s = \langle s_\delta \mid \delta < \alpha \rangle$, where α is even, such that s is played according to the rules of the $**$-games and the moves of II are played according to σ. For s such a position let $s^* = \cup_{\delta<\alpha} s_\delta$.

(*) Let $f \in {}^\kappa 2$. If for every good position p, such that $p^* \subseteq f$ there exists a $t \supseteq p^*$ with $\sigma(p^\frown\langle t\rangle) \subseteq f$, then $f \notin A$.

Proof of (*). Since $\emptyset$ is a good position we can recursively construct good positions $p_0 \subseteq p_1 \subseteq \dots p_\alpha \subseteq \dots$ with $\alpha < \kappa$. Now we have $\cup_{\alpha<\kappa} p^*_\alpha = f$. Since all $p'_\alpha s$ are played according to σ and σ is winning for II we have $f \notin A$.

Now define for every good position $p = \langle p_\delta \mid \delta < \alpha \rangle$

$$F_p = \{f \in {}^\kappa 2 \mid p^* \subseteq f \wedge \forall t \supseteq p^* (\sigma(p^\frown\langle t\rangle) \not\subseteq f)\}.$$

Now because of $(*)$ we have that $A \subseteq \bigcup_{p \text{ good}} F_p$. One sees that for each good position p $N_{p^*} - F_p$ is open dense, so F_p is closed nowhere dense. Finally by $\kappa = {}^{<\kappa}2$ there are κ many good positions.
The proof of *ii)* is the same.
□

Corollary 7.3.3 *(Baire category theorem for $\mathcal{C}_\kappa$, $\mathcal{B}_\kappa$) $\mathcal{C}_\kappa$ and $\mathcal{B}_\kappa$ are not κ-meager (with their respective topologies).*

Proof. II obviously cannot have a winning strategy in $G^{**}_{\mathcal{C}_\kappa}(\mathcal{C}_\kappa)$ or in $G^{**}_{\mathcal{B}_\kappa}(\mathcal{B}_\kappa)$.
□

Looking at the above corollary one might think that the Baire property can be made to hold for all sets by an appropriate assumption (like $**$-determinacy). We will show now that if κ is regular this is impossible. First we define the κ-Baire property generalizing directly the $\kappa = \omega$ case.

Definition 7.3.4

i) A subset of $\mathcal{C}_\kappa$ has the κ-Baire property if there is an open set $B \subseteq \mathcal{C}_\kappa$, such that

$$A \triangle B = \{f \in \mathcal{C}_\kappa \mid f \in A \wedge f \notin B\} \cup \{f \in \mathcal{C}_\kappa \mid f \in B \wedge f \notin A\}$$

is κ-meager.

ii) A subset of $\mathcal{B}_\kappa$ has the κ-Baire property if there is an open set $B \subseteq \mathcal{B}_\kappa$, such that $A \triangle B$ is κ-meager.

Lemma 7.3.5 *If κ is a regular uncountable cardnial, then there are subsets of $\mathcal{C}_\kappa$ and $\mathcal{B}_\kappa$ that do not have the κ-Baire property.*

Proof. We will only treat the case of $\mathcal{C}_\kappa$ since the other case is similar. Consider the set

$$A = \{f \in \mathcal{C}_\kappa \mid \text{ there is a closed unbounded } C \subseteq \kappa,$$

$$\text{such that for every} \alpha \in C,\ f(\alpha) = 0\}.$$

To show that A does not have the κ-Baire property let B be an arbitrary open set. A winning strategy σ for I in the game $G^{**}_{\mathcal{C}_\kappa}(A \triangle B)$ can be described as follows. Let $s \in {}^{<\kappa}2$ be such that $N_s \subseteq B$. Now we define

$\sigma(\emptyset) = s$

$\sigma(t) = t^\frown\{1\}$ for $t \supsetneq \emptyset$.

Now since I moves first at limits, for each $h \in \mathcal{C}_\kappa$ produced according to σ the set $\{\alpha \mid h(\alpha) = 1\}$ is a closed unbounded subset of κ, so $f \in B - A$.

□

One can try to avoid the above difficulty by defining dual $**$-games where at limit stages II moves first and then define a subset of $\mathcal{C}_\kappa$ to be weakly κ-meager iff II wins the corresponding game. It turns out however that this notion has the drawback that the weakly κ-meager sets do not form an ideal.

We conclude with a lemma, which we will use in the next section.

Lemma 7.3.6 *($\kappa = {}^{<\kappa}\kappa$)*

i) AWD^{**2}_{κ} implies that every non-meager $A \subseteq \mathcal{C}_\kappa$ has a strongly perfect subset.

ii) AWD^{**}_{κ} implies that every non-meager $A \subseteq \mathcal{B}_\kappa$ has a strongly perfect subset.

Proof. For *i)* By $**$-determinacy I has a winning strategy σ in $G^{**}_{\mathcal{C}_\kappa}(A)$. Now the set $P = \{\sigma * \langle\{y(\alpha)\} \mid \alpha < \kappa\rangle \mid y \in {}^{\kappa}2\}$ is easily seen to be strongly perfect.

Similarly for *ii)*.

□

8 A proof of $*$-determinacy from Turing determinacy and $**$-determinacy

In this section we need to work with a strong assumption. The general idea will be to generalize Theorem 1-3 of [8] and then use the results of section 7.3. One of the main tools of [8] is the forcing notion for collapsing an inaccessible cardinal to ω_1. This forcing notion is used inside an inner model $L[r]$ where r is a real, so it is important that it is absolute for $L[r]$. If we were to try and generalize the proof by taking a member f of $\mathcal{C}_\kappa$ instead of a real, then in $L[f]$ the forcing notion for collapsing a cardinal to $\kappa^{+L[f]}$ would not be absolute for $L[f]$. The reason for this is that $L[f]$ is not necessarily closed under sequences of length smaller than κ. Therefore, instead of working with L we will use a model which is a "hybrid" between L and the model C^κ introduced in [9], the latter being the smallest inner model closed under sequences of length smaller than κ.

Unlike L however C^κ does not in general satisfy the axiom of choice (cf. [10]). In order to guarantee that our "hybrid" model does satisfy AC we will have to assume a sufficiently strong choice-like axiom.

(AWC_κ) *There is a well-ordering of the set* $\{X \subseteq \kappa^+ \mid |X| < \kappa\}$.

Note that the above principle asserts the existence of a well-ordering of the κ-finite subsets of κ^+ in V (the universe) and not in some other inner model.

In the entire section κ will be a regular cardinal and we will assume that DC_κ and $\kappa = {}^{<\kappa}2$ hold.

8.1 The model W^κ

In this section we introduce the model C^κ from [9] and its natural relativization to a member of $\mathcal{C}_\kappa$. Then we proceed with the definition of the model we are actually going to use. As a basis for comparison we first briefly summarize the relativized L-hierarchy.

Definition 8.1.1 *For a set A.*

$$L_0[A] = \emptyset;$$

$$L_{\alpha+1}[A] = \{x \subseteq L_\alpha[A] \mid x \textit{ is definable in } \langle L_\alpha[A], \in, A \cap L_\alpha[A]\rangle\};$$

$$L_\lambda[A] = \bigcup_{\alpha<\lambda} L_\alpha[A], \textit{ for } \lambda \textit{ limit.}$$

$L[A] = \bigcup_{\alpha \in Ord} L_\alpha[A]$; $L = L[\emptyset]$.

Remark 8.1.2 *By "definable" here is meant first order definable with parameters.*

To formally have this definition within our theory (ZF) we assume that we have formalized set theory within it. This involves a formalization of the language and of the satisfaction relation for sets (i.e. defining a formula $Sat(x_0, \varphi_0)$ which is true iff x_0 and φ_0 are the formalized versions of x and φ and $x \models \varphi$). After this has been done it is easy to define what the definable subsets of a given set are.

The definition of $C^\kappa[f]$ for $f \in \mathcal{C}_\kappa$ is based on a formalization of the infinitary language $L_{\kappa\kappa}$ and of the satisfaction relation for formulas of $L_{\kappa\kappa}$. Such a formalization can again be made within ZF (cf. [9]). We let f as a predicate in the formulas, so that we have a relation $Sat^\kappa(x, \varphi, A, f)$ which is true for a tuple (x, φ, A, f) iff $x \in {}^\kappa A$, x is eventually 0, φ is a formula of $L_{\kappa\kappa}$ with f as an additional predicate and $\langle A, \in, A \cap f\rangle$ satisfies φ where the free variables are interpreted as a corresponding initial segment the vector x.

Now we define the corresponding version of the definable subsets of a given set.

Definition 8.1.3 *Let $f \in \mathcal{C}_\kappa$ and let A be any set.*

$$D^\kappa(A, f) = \{B \subseteq A \mid \textit{ there is an eventually } 0 \textit{ sequence } x \in {}^\kappa A$$

$$\textit{and a formula } \varphi \textit{ of } L_{\kappa\kappa} \textit{ with } f \textit{ as an additional predicate,}$$

$$\textit{such that } B = \{b \in A \mid Sat^\kappa(b^\frown x, \varphi, A, f)\}\}.$$

Note that this definition runs completely parallel to the one in [9].

The model itself is defined now as follows.

Definition 8.1.4 *For $f \in \mathcal{C}_\kappa$.*

$C_0^\kappa[f] = \emptyset$;

$C_{\alpha+1}^\kappa[f] = D^\kappa(C_\alpha^\kappa, f)$;

$C_\lambda^\kappa[f] = \bigcup_{\alpha<\lambda} C_\alpha^\kappa[f]$, *for λ limit.*

$C^\kappa[f] = \bigcup_{\alpha \in Ord} C_\alpha^\kappa[f]$.

Remark 8.1.5 *Obviously $C^\omega[f] = L[f]$.*

We see now that a choice-like principle stronger than AWC_κ provides well-orderings for all the levels of C^κ.

Lemma 8.1.6 *Assume that for every ordinal α the set $\{X \subseteq \alpha \mid |X| < \kappa\}$ can be well-ordered. Then for every $f \in \mathcal{C}_\kappa$ and for every α there is a well-ordering (in V) of $C_\alpha^\kappa[f]$*

Proof. In the same way one proves AC in L. The regularity of κ assures that each formula has less than κ free variables. The proof itself proceeds by transfinite induction on α. By the induction hypothesis in the case "$\alpha \to \alpha + 1$" the model $C_\alpha^\kappa[f]$ is well-ordered, which means it is in a bijective correspondence with some cardinal and so one can use the assumption to well order all the possible formulas.

□

The next property is well-known.

Lemma 8.1.7 *Let W be an inner model of ZFC and suppose that λ is measurable (in V). Then $W \models$ " λ is inaccessible ".*

Proof. For every $\alpha < \lambda$ if $\langle X_\delta \mid \delta < \rho \rangle$ is a sequence of distinct subsets of α, then $\rho < \lambda$. Since in W AC holds it follows that in W λ is inaccessible.

□

With the above in mind we proceed with the definition of the model $W^\kappa[g, R]$, where $g \in \mathcal{C}_\kappa$ and R is the well-ordering of the κ-finite subsets of κ^+, whose existence is guranteed by AWC_κ. The idea is that up to level $\kappa^+ + 1$ the levels coincide with the levels of $C^\kappa[g, R]$ and the higher levels are formed as for L.

Definition 8.1.8 *For $f \in \mathcal{C}_\kappa$, R a well-ordering of* $\{X \subseteq \kappa^+ \mid |X| < \kappa\}$.

$$W^\kappa_\alpha[f, R] = C^\kappa_\alpha[f, R] \textit{ for } \alpha \leq \kappa^+;$$

$$W^\kappa_{\alpha+1}[f, R] = \{x \subseteq W_\alpha[f, R] \mid x \textit{ is definable in}$$

$$\langle W_\alpha[f, R], \in, f, R \cap W_\alpha[f, R]\rangle\} \textit{ for } \alpha \geq \kappa^+;$$

$W^\kappa[f, R] = \bigcup_{\alpha \in Ord} W^\kappa_\alpha[f, R]$, $W^\kappa[R] = W^\kappa[\mathbf{0}, R]$.

The important feature of the model is that it satisfies AC. This is analogous to lemma 8.1.6, and we are using a weaker assumption.

Lemma 8.1.9 *(AWC_κ) $W^\kappa[f, R]$ is an inner model of ZFC and $R, f \in W^\kappa$.*

Proof. All axioms except AC hold by the same arguments as for L and C^κ. AC holds as follows. Sets up to level $\kappa^+ + 1$ can be well-ordered because $R \in W^\kappa[f, R]$ (note that R comes at a higher level than $\kappa^+ + 1$ and for this reason we had to include R as a parameter in the second clause of 8.1.8). Sets in the upper levels can be well-ordered by the same argument as for L.
□

From now on we will work under AWC_κ with a fixed well-order R of $\{X \subseteq \kappa^+ \mid |X| < \kappa\}$. We will write $W^\kappa[f]$ for $W^\kappa[f, R]$.

Lemma 8.1.10 *(AWC_κ) For α an ordinal, such that $\kappa \leq \alpha < \kappa^+$ and $f \in \mathcal{C}_\kappa$*

$$|W^\kappa_\alpha[f]| \leq \kappa.$$

Proof. As for C^κ. Note that both κ^+ and the cardinality bound obtained are in the sense of V.
□

Lemma 8.1.11 *(AWC_κ) If κ^+ is inaccessible in $W^\kappa[f]$, then there exists an ordinal $\nu < \kappa^+$, such that ${}^\kappa 2 \cap W^\kappa[f] \subseteq W^\kappa_\nu[f]$*

Proof. Immediate, note that κ^+ is in the sense of V. □

8.2 The main proof

As indicated above this will be a modification of the proof of [8]. We start by giving some background notions.

Definition 8.2.1 *Let x be a set. Call a set S x-admissible if $\langle S, \in, x \cap S\rangle$ is a model of the Kriple-Platek set theory where for Σ_0-collection and Σ_0-separation we allow x as predicate for the atomic formulas.*

Note that if $x \in S$ we can replace the above requirement by $\langle S, \in\rangle \models \mathrm{KP}$.

Definition 8.2.2 *For $f \in \mathcal{C}_\kappa$ call the ordinal α f-admissible if $\alpha > \kappa$ and $W^\kappa_\alpha[f] \models KP$. Denote the smallest f-admissible ordinal by $\omega^f_{\kappa^+}$. ω_{κ^+} denotes the smallest ordinal β above κ, such that $W^\kappa_\beta \models KP$.*

The Levy Collapse is a well-known notion of forcing for collapsing cardinals to a given regular cardinal. We give some equally well-known properties in the three lemmata after the definition.

Definition 8.2.3 *For λ a regular cardinal and $\alpha > \lambda$ an ordinal.*

$$Col(\lambda, \alpha) = \{f \mid f \text{ is a function } \wedge |f| < \lambda \wedge dom(f) \subseteq \alpha \times \lambda \wedge rng(f) \subseteq \alpha \wedge$$

$$\forall \langle \beta, \xi\rangle \in dom(f)((\beta = 0 \rightarrow f(\langle \beta, \xi\rangle) = 0) \wedge (\beta > 0 \rightarrow f(\langle \beta, \xi\rangle) < \beta))\},$$

$\mathcal{P}_\lambda$ is the partial order for adjoining a Cohen generic subset of λ.

Lemma 8.2.4 *Let M be admissible and G be M-generic over $Col(\lambda, \alpha)$ and $\beta < \alpha$. Then*

$$\{\langle \delta, \gamma\rangle \mid \{\langle\langle \beta, \delta\rangle, \gamma\rangle\} \in G\}$$

is a surjective map of λ onto β.

Lemma 8.2.5 *$Col(\lambda, \alpha)$ is λ-closed and if $\alpha > \lambda$ and α is an inaccessible cardinal, then $Col(\lambda, \alpha)$ satisfies the α-c.c., thus it preserves cardinals $\leq \lambda$ and cardinals $\geq \alpha$.*

Lemma 8.2.6 *Let M be admissible and G be an M-generic filter over $Col(\lambda, \alpha)$, where $\alpha \geq \lambda^{+M}$. Then $\alpha = \lambda^{+M[G]}$.*

The following construction is a generalization of the construction in [6][III 1.10 - 1.11].

Suppose M, N are admissible, $M \subseteq N$, and ${}^{<\kappa}M \subseteq M$, λ is a regular cardinal of M and N and $\lambda < \alpha < o(M)$. Suppose further, that there is an enumeration in N with length λ of the dense subsets of $\mathrm{Col}(\lambda, \alpha) \times \mathrm{Col}(\lambda, \alpha)$ lying in M.

Then we can construct in N a funcion $H : {}^{<\lambda}2 \to \mathrm{Col}(\lambda, \alpha)$, such that:

H1) $s \subseteq t \Rightarrow H(s) \subseteq H(t)$ and $s \neq t \Rightarrow H(s) \neq H(t)$;

H2) for every $s \in 2^{<\lambda}$ $H(s^\frown\langle 0\rangle) \neq H(s^\frown\langle 1\rangle)$ and if $dom(s) = \delta$ is limit, then $H(s) = \bigcup_{\gamma<\delta} H(s \restriction \gamma)$;

H3) denoting for an $f \in {}^{\lambda}2$ the function $\bigcup\{H(f \restriction \gamma) \mid \gamma < \lambda\}$ from λ into α by f^H, if f_1, f_2 are distinct members of ${}^{\lambda}2$, then $\langle f_1^H, f_2^H\rangle$ is $\mathrm{Col}(\lambda, \alpha) \times \mathrm{Col}(\lambda, \alpha)$-generic over M,

H4) the function $H^* : {}^{\lambda}2 \to {}^{\lambda}\alpha$ defined by $H^*(f) = f^H$ is continuous.

There is a slight modification of the above construction in [8] which we are also going to generalize. If $\delta > \lambda$, $\delta < \lambda^{+W^\lambda}$, δ is admissible, $M = W^\lambda_\delta$ and in W^λ $|M| \leq \lambda$, then we can construct a function $E : {}^{<\lambda}2 \to \mathcal{P}_\lambda$, such that:

E1) $s \subseteq t \Rightarrow E(s) \subseteq E(t)$ and $s \neq t \Rightarrow E(s) \neq E(t)$;

E2) for every $s \in 2^{<\lambda}$ $E(s^\frown\langle 0\rangle) \neq E(s^\frown\langle 1\rangle)$ and if $dom(s) = \delta$ is limit, then $E(s) = \bigcup_{\gamma<\delta} E(s \restriction \gamma)$;

E3) if f_1, f_2 are distinct members of ${}^{\lambda}2$, then $\langle f_1^E, f_2^E\rangle$ is $\mathcal{P}_\lambda \times \mathcal{P}_\lambda$-generic over M, where f^E is defined as f^H in H3) above;

E4) $E \in W^\lambda$

We can get E with these properties as in [8].

From now on we are going to concentrate on the case $\lambda = \kappa$.

We will use the following general lemma about product forcing due to Solovay.

Lemma 8.2.7 *([6] pp.13-14) Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be notions of forcing in M and let $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$. Let $G = G_1 \times G_2$ be a generic filter on $\mathcal{P}$. Then $M[G_1] \cap M[G_2] = M$.*

Definition 8.2.8 *For notational convenience we introduce the following abbreviation. For $f, g \in {}^{\kappa}2$ call f big for g iff $f \in W^{\kappa}[g] - W^{\kappa}_{\omega^{g}_{\kappa^+}}[g]$.*

So f is big for g if f appears in $W^{\kappa}[g]$ but above the first admissible level. In the case $\kappa = \omega$ this is equivalent to saying that $f \in L[g]$, but f is not hyperarithmetic in g.

We proceed to the generalization of lemma 1.2 of [8].

Lemma 8.2.9 *(AWC_{κ}) For any $f, g \in {}^{\kappa}2$ there is an $h \geq_T g$ such that $f \in W^{\kappa}[h]$ and $\omega^{h}_{\kappa^+} = \omega^{g}_{\kappa^+}$, also if $f \notin W^{\kappa}_{\omega^{g}_{\kappa^+}}[g]$ we have that f is big for h.*

Proof. It is easy to see that we can work with $g = \mathbf{0}$ for simplicity. So we are looking for an h, such that $\omega^{h}_{\kappa^+} = \omega_{\kappa^+}$, $f \in W^{\kappa}[h]$ and such that if $f \notin W^{\kappa}_{\omega_{\kappa^+}}$ then f is big for h.

We construct the function E from above with $M = W^{\kappa}_{\omega_{\kappa^+}}$, $\lambda = \kappa$. We have $E \in W^{\kappa}$, so $f \in W^{\kappa}[f^E]$ by κ-constructibility. Since f^E is $\mathcal{P}_{\kappa}$-generic over $W^{\kappa}_{\omega_{\kappa^+}}$ we have $W^{\kappa}_{\omega_{\kappa^+}}[f^E]$ is admissible, so $\omega_{\kappa^+} = \omega^{f^E}_{\kappa^+}$ (since $\omega^{f^E}_{\kappa^+} \geq \omega_{\kappa^+}$ always).

To get the second property suppose that $f \notin W^{\kappa}_{\omega_{\kappa^+}}$. Take

$$h_1 = (\langle 0 \rangle ^\frown f)^E,$$

$$h_2 = (\langle 1 \rangle ^\frown f)^E.$$

Again $\omega_{\kappa^+} = \omega^{h_1}_{\kappa^+} = \omega^{h_2}_{\kappa^+}$ and $f \in W^{\kappa}[h_1] \cap W^{\kappa}[h_2]$. By 8.2.7 and condition (E3) we have that $W^{\kappa}_{\omega_{\kappa^+}} = W^{\kappa}_{\omega_{\kappa^+}}[h_1] \cap W^{\kappa}_{\omega_{\kappa^+}}[h_2]$ and since $f \notin W^{\kappa}_{\omega_{\kappa^+}}$ we get that for some $i \in \{1, 2\}$ f is big for h_i.

□

We adopt the coding of [8].

Definition 8.2.10 *Suppose $\iota : \kappa \times \kappa \to \kappa$ is "κ-recursive", i.e. Δ_1-definable in $\langle H_{\kappa}, \in \rangle$ without parameters. If γ is an ordinal between κ and κ^+ and $\Theta : \kappa \to \gamma$, the code of Θ is $\hat{\Theta} : \kappa \to 2$ defined by $\hat{\Theta}(\iota(\alpha, \beta)) = 1$ iff $\Theta(\alpha) < \Theta(\beta)$. As in [8] we have that for $f \in \mathcal{C}_{\kappa}$, $W^{\kappa}[f \oplus \hat{\Theta}] \subseteq W^{\kappa}[f][\Theta]$ and that if Θ is onto, then $W^{\kappa}[f \oplus \hat{\Theta}] = W^{\kappa}[f][\Theta]$.*

Lemma 8.2.11 *(TD_κ)(AWC_κ) If $A \subseteq \mathcal{C}_\kappa$ and $|A| > \kappa$ then the set*

$$S_A = \{f \in \mathcal{C}_\kappa \mid \exists g \in A(g \textit{ is big for } f)\}$$

contains a cone of Turing degrees.

Proof. This will follow by TD_κ once we show that S_A contains elements of arbitrarily big Turing degrees. To this end for $h \in \mathcal{C}_\kappa$ we can by $|A| > \kappa$ and 8.1.10 find an $f \in A$, such that $f \notin W^\kappa_{\omega^h_{\kappa^+}}[h]$. We apply now 8.2.9 to find a $g \geq_T h$, with g big for f. Now f is in S and as h was arbitrary the lemma is established.

□

Lemma 8.2.12 *(TD_κ)(AWC_κ) If $A \subseteq \mathcal{C}_\kappa$ and $|A| > \kappa$, then there is a bijection $F_A : {}^\kappa 2 \to A$.*

Proof. Let f be the vertex for a cone of degrees contained in S_A. By TD_κ κ^+ is measurable, and so inaccessible in $W^\kappa[f]$. Let ν be the ordinal from lemma 8.1.11, i.e. $\nu < \kappa^+$ and ${}^\kappa 2 \cap W^\kappa[f] \subseteq W^\kappa_\nu[f]$. Notice that for $\Theta : \kappa \to \nu$ we have $f \oplus \hat{\Theta} \in S_A$, so we can find $g \in A$ which is big for $f \oplus \hat{\Theta}$. Denote by g_Θ the first such g in the well-ordering of $W^\kappa[f \oplus \hat{\Theta}]$. For surjective Θ we immediately obtain $\nu \leq \omega^{\hat{\Theta}}_{\kappa^+} \leq \omega^{f \oplus \hat{\Theta}}_{\kappa^+}$ and so $W^\kappa[f] \cap {}^\kappa 2 \subseteq W^\kappa_\nu[f] \cap {}^\kappa 2 \subseteq W^\kappa_\nu[f \oplus \hat{\Theta}] \cap {}^\kappa 2 \subseteq W^\kappa_{\omega^{f \oplus \hat{\Theta}}_{\kappa^+}}[f \oplus \hat{\Theta}] \cap {}^\kappa 2$. As a result $g_\Theta \in W^\kappa[f][\Theta] - W^\kappa[f]$.

Since κ^+ is inaccessible in $W^\kappa[f]$ and $|\nu| = \kappa$ in V, we can construct a function $H : {}^{<\kappa}2 \to \mathrm{Col}(\kappa, \nu)$ as above where $M = W^\kappa[f]$.

Define

$$F_A(h) = g_{h^H} \text{ for every } h \in {}^\kappa 2.$$

For every h we have, since h^H is surjective, $F_A(h) \in (W^\kappa[f][h^H] - W^\kappa[f]) \cap A$. From lemma 8.2.7 we get that F_A is injective.

□

Corollary 8.2.13 *(TD_κ)(AWC_κ) If $A \subseteq \mathcal{C}_\kappa$ and $|A| > \kappa$, then there is a partition $\langle B_\alpha \mid \alpha < \kappa\rangle$ of ${}^\kappa 2$, such that $F_A \restriction B_\alpha$ is continuous for every $\alpha < \kappa$.*

Proof. Let f, ν be as in the proof of the lemma. A canonical name for a subset of κ with respect to $\mathrm{Col}(\kappa, \nu)$ is a name of the form $\{\langle \check{\alpha}, p\rangle \mid \alpha < \kappa \wedge p \in A_\alpha\}$, where A_α is an antichain in $\mathrm{Col}(\kappa, \nu)$. Since every antichain of

$\mathrm{Col}(\kappa,\nu)$ which is in $W^\kappa[f]$ has cardinality κ, we have that there are at most κ many canonical names for a subset of κ that lie in $W^\kappa[f]$. Let $\langle \dot{\tau}_\alpha \mid \alpha < \kappa \rangle$ be an enumeration of these names and put

$$E_\alpha = \{\Theta : \kappa \to \nu \mid \Theta \text{ is } \mathrm{Col}(\kappa,\nu)\text{-generic over } W^\kappa[f] \text{ and } g_\Theta = \dot{\tau}_\alpha{}^{W^\kappa[f][\Theta]}\},$$

where g_Θ is as in the lemma. Put

$$B_\alpha = \{h \in {}^\kappa 2 \mid H(h) \in E_\alpha\}.$$

Now since H is continuous the only thing left to prove is that the function $T : {}^\kappa\nu \to {}^\kappa 2$, defined by $T(\Theta) = g_\Theta$ is continuous when restricted to any E_α. This is however immediate by the Truth lemma for generic extensions of $W^\kappa[f]$ by the notion of forcing $\mathrm{Col}(\kappa,\nu)$.

□

The final result follows.

Theorem 8.2.14 *(TD_κ)(AWC_κ)(AWD^{**2}_κ) For every $A \subseteq \mathcal{C}_\kappa$ the game $G^*_{\kappa 2}(A)$ is determined.*

Proof.

What remains to be shown in addition to the facts in section 7.2 is that for $A \subseteq \mathcal{C}_\kappa$ with $|A| > \kappa$ there is a strongly perfect subset of A. By the Baire cathegory theorem one of the B_α's, from corollary 8.2.13 say B_{α_0} is non-meager, and so it has a strongly perfect subset P. Now since $F \restriction B_{\alpha_0}$ is continuous we get $F'' P$ is a strongly perfect subset of A.

□

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