Mirror Symmetry, Toric Branes and Topological String Amplitudes as Polynomials

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to all the close ones who haven't seen much of me because of this

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Zusammenfassung

Im Mittelpunkt dieser Arbeit stehen Erweiterungen und Anwendungen der Mirror Symmetrie der topologischen Stringtheorie. Mirror Symmetrie ist eine Äquivalenz zwischen dem A-Modell der topologischen Stringtheorie auf einer Mannigfaltigkeit X und dem B-Modell auf einer Mirror Mannigfaltigkeit Y, zusammen mit deren Deformationsräumen. Die Mannigfaltigkeit der A-Seite wird durch Kähler Deformationen verändert, die eine Änderung des Volumen bewirken. Die Deformationen auf der B-Seite bewirken eine Änderung der komplexen Struktur. Die Mächtigkeit der Mirror Symmetrie liegt darin, dass diese einen physikalischen einfachen Ursprung hat, jedoch die mathematische Übersetzung zwei äußerst unterschiedliche Bereiche verknüpft, die symplektische- und die komplexe Geometrie. Diese Verknüpfung hat weitreichende und ungeahnte Konsequenzen sowohl für die Physik als auch für die Mathematik. Physikalische Probleme können mittels der mathematischen Strukturen präzisiert und gelöst werden. In dieser Hinsicht werden in der Arbeit zum einen quantenkorrigierte Superpotenziale berechnet, zum anderen wird das mathematische Verständnis der Hintergrundabhängigkeit ausgenutzt, um eine Feynman Diagramm Entwicklung einer Zustandssumme durch eine effektivere polynomiale Entwicklung zu ersetzen. Der Beitrag dieser Arbeit zur mathematischen Seite ist gegeben durch die Übersetzung der Zustandssumme in eine generierende Funktion mathematischer Invarianten, welche für diverse Beispiele extrahiert werden.

Die Essenz der Mirror Symmetrie besteht darin, die Lösung einfacher Probleme auf die Lösung äquivalenter schwieriger Probleme abzubilden. Die dafür benötigte Mirror Abbildung ist das Herzstück der Mirror Symmetrie. Die Berechnung dieser wird ermöglicht durch eine physikalische Struktur, die in den mathematischen Realisierungen beider Modelle wiederzufinden ist. Diese Struktur ist das Vakuum-Bündel, zusammen mit einer Aufspaltung, die über dem Raum der Deformationen variiert. Im Kontext des B-Modells führt die Untersuchung dieser Variation auf Differenzialgleichungen, die es ermöglichen die Mirror Abbildung zu berechnen und weitere Größen zu bestimmen, deren Übersetzung im A-Modell die Quantengeometrie beschreibt. In dieser Arbeit wird die Erweiterung der Variation des Vakuumbündels untersucht, um zusätzlich D-Branen in kompakten Geometrien zu beschreiben. Basierend auf bestehenden Arbeiten für nicht-kompakte Geometrien werden Differenzialgleichungen hergeleitet, die es ermöglichen die Mirror Abbildung auf die Deformationsräume der D-Branen auszudehnen. Des Weiteren, ermöglichen diese Gleichungen die Berechnung der Superpotenziale, die durch die D-Branen induziert werden und alle Quantenkorrekturen beinhalten.

Basierend auf der holomorphen Anomalie Gleichung, die die Hintergrundabhängigkeit der topologischen Stringtheorie beschreibt und dabei rekursiv Schleifenamplituden in Verbindung setzt, wird in dieser Arbeit eine Polynomkonstruktion der Schleifenamplituden einer Mannigfaltigkeit mit eindimensionalem Deformationsraum auf beliebige Mannigfaltigkeiten verallgemeinert. Die Polynom-Generatoren werden allgemein in Abhängigkeit des Modells aufgestellt und es wird bewiesen, dass die Korrelationsfunktionen bei einer bestimmten Schleifenzahl Polynome eines bestimmten Grades in den Generatoren sind. Die Konstruktion wird zudem verwendet, um die Erweiterung der holomorphen Anomalie Gleichung für D-Branen ohne Deformationen zu lösen. Die Methode wird angewandt, um höhere Schleifenamplituden für mehrere Beispiele zu berechnen und um mathematische Invarianten daraus zu extrahieren.

Abstract

The central theme of this thesis is the extension and application of mirror symmetry of topological string theory. Mirror symmetry is an equivalence between the topological string A-model on a manifold X and the B-model on a mirror manifold Y, together with their deformation spaces. Deformations of the target space on the A-side are Kähler deformations which change the volume of the manifold. Deformations on the B-side change the complex structure. The power of mirror symmetry is due to its simple physical origin which connects two different areas of mathematics, symplectic- and complex geometry. This connection has far reaching and unexpected consequences both on the mathematical and on the physical side. Physical problems can be given a precise mathematical meaning and can be solved. In this regard, quantum corrected superpotentials are computed in this work on the one hand. On the other hand the mathematical understanding of the background dependence is used to reorganize a perturbative Feynman diagram expansion in terms of a more efficient polynomial expansion. The contribution of this work on the mathematical side is given by interpreting the calculated partition functions as generating functions for mathematical invariants which are extracted in various examples.

The main idea of mirror symmetry is to map the solution of simple problems to the solution of equivalent difficult problems. To do so, a mirror map is needed, which is at the heart of mirror symmetry. The computation of this map is possible thanks to a physical structure which occurs in both mathematical realizations. This structure is the vacuum bundle, together with a grading which varies over the space of deformations. In the context of the B-model the study of the variation of this grading leads to differential equations which allow the computation of the mirror map as well as other quantities which describe quantum geometry when translated to the A-side. In this thesis, the extension of the variation of the vacuum bundle to include D-branes on compact geometries is studied. Based on previous work for non-compact geometries a system of differential equations is derived which allows to extend the mirror map to the deformation spaces of the D-Branes. Furthermore, these equations allow the computation of the full quantum corrected superpotentials which are induced by the D-branes.

Based on the holomorphic anomaly equation, which describes the background dependence of topological string theory relating recursively loop amplitudes, this work generalizes a polynomial construction of the loop amplitudes, which was found for manifolds with a one dimensional space of deformations, to arbitrary target manifolds with arbitrary dimension of the deformation space. The polynomial generators are determined and it is proven that the higher loop amplitudes are polynomials of a certain degree in the generators. Furthermore, the polynomial construction is generalized to solve the extension of the holomorphic anomaly equation to D-branes without deformation space. This method is applied to calculate higher loop amplitudes in numerous examples and the mathematical invariants are extracted.

1 Introduction

This thesis contains work done within *mirror symmetry*, which is an equivalence of two versions of topological string theory called *A-model* and *B-model*. Mirror symmetry is perhaps the most interesting junction between mathematics and physics where the front row research is pursued on both sides. The mutual flow of knowledge has far reaching and unexpected implications. On the one hand the mathematical structure of mirror symmetry is used to address physical problems. The highlights on this side in the thesis are the computation of full quantum corrected superpotentials in physical theories and a surprising polynomial organization of a perturbative Feynman graph expansion together with a recursive procedure which allows to compute the partition function of topological string theory. On the other hand the mathematical translation of the structures involved allows to extract a tower of mathematical invariants from the physical correlation and partition functions, computations which are very hard from a purely mathematical point of view.

Spaces of theories

The conceptual novelty mirror symmetry introduces in physics is that the parameters of a field theory can be given a geometric meaning. One can vary the parameters by deforming the defining Lagrangian of the theory. Within the field theories underlying topological string theory the set of all parameters accessible by deformations are organized in geometrical spaces. One can thus speak of a space of deformations or space of theories which is called the moduli space of the theory. Mirror symmetry states the equivalence of the A-model and the B-model together with their moduli spaces, i.e., it is an equivalence of families of theories. Moreover a study of the dependence of physical quantities, such as the partition function, on the parameters used for the initial formulation of the theory is possible and gives thus a quantitative access to studying background dependence.

Background dependence

In field theories background dependence is the term employed for the fact that these theories depend on a choice of parameters and boundary conditions once the action of the theory is formulated. Examples would be the values of the couplings of the interactions as well as the space-time geometry within which the theory is formulated. Background dependence is an obstacle towards understanding quantum gravity. A field theory incorporating gravity is formulated in a certain geometry and the fluctuations around this reference geometry are considered to be the degrees of freedom of quantum gravity. Within string theory this approach is possible but not completely satisfactory. On the other hand also gauge couplings suffer from background dependence. This is manifest in the perturbative expansion of interactions which is only valid within a certain range of the parameters. A weak coupling expansion of a theory does for example not allow to understand the same theory at strong coupling.

The work of Seiberg and Witten

A breakthrough towards an understanding of the space of couplings, termed the moduli space of theories of a given interaction pattern was achieved by the work of Seiberg and Witten for field theories having $\mathcal{N} = 2$ supersymmetry [1]. Understanding the coupling dependence of these theories in terms of geometric quantities allowed for an exact description of the effective action at any coupling, i.e., at any given point in moduli space. This lead to excitement both on the physics and on the mathematics side. Physicists had at their disposal a method for computing the full quantum corrected effective action for a given theory at any value of the coupling. The mathematical understanding of the couplings paired with a physical understanding of the behavior of the theory at various special values opened up the understanding of mathematical invariants of the geometric spaces involved.

Superconformal field theories

Mirror Symmetry is the larger framework incorporating these ideas. $\mathcal{N} = 2$ superconformal field theories (SCFT) are the field theoretic origin of mirror symmetry. These are field theories in two dimensions having conformal symmetry and two copies of supersymmetry relating fermions and bosons. In the context of string theory the two dimensions parameterize the world sheet of a string, i.e., the spatial and time coordinate (σ, τ) of its propagation. A geometric realization of these theories are the nonlinear sigma models. In these models various copies of the fields are considered as coordinates in some target space X. The fields can be thought of as mappings from the two dimensional surface into X, which is interpreted as physical space-time

$$\phi^i(\sigma, \tau) : \Sigma_{g,h} \to X \,, \quad i = 1, \ldots, \dim(X) \,,$$

where $\Sigma_{g,h}$ denotes the two-dimensional surface. These are Riemann surfaces and they are classified by their genus g and by the number of holes h.

These SCFTs are at the heart of string theory and allow a precise description of what is meant by the space of theories. The rich structure carried by these theories allows an identification of the operators which cause deformations of the theory. The idea that the couplings of the two-dimensional theory are relevant space-time quantities can be appreciated by considering a typical action, where for simplicity only bosonic scalar fields are included

$$S \sim \int d\sigma d\tau \sqrt{-h} h^{\alpha\beta} \partial_\alpha \phi^i(\sigma,\tau) \partial_\beta \phi^j(\sigma,\tau) g_{ij} \,,$$

where $h^{\alpha\beta}$ is a two dimensional metric with determinant h, α and β take the values $0, 1$ corresponding to the directions σ and τ. The indices i, j run over the dimension of the target space of the sigma model X and g_{ij} is a metric on the target space. It is crucial that the target space metric is merely a coupling from the point of view of the two-dimensional field theory. The space of couplings of the field theory corresponds thus to the space of metrics of the target space.

A-model and B-model

The requirement of $\mathcal{N} = 2$ space-time supersymmetry in four dimensions puts restrictions on the target space X, these spaces have to be Kähler which refers to a rich structure within complex geometry. Conformal invariance restricts these Kähler spaces even more to be Calabi-Yau (CY). The advantage of this restrictive structure is that the spaces of deformations of CY metrics is mathematically well understood. This space splits generically into the product of two deformation spaces, one of them being the space of Kähler deformations which can be thought of as varying the size of the manifold. The other space is the space of complex structure deformations which can be pictured as changing the shape of the manifold. Each one of the two topological string theories which are related by mirror symmetry is affected by only one type of deformations while it is blind to the other type. These topological string theories are called the A-model and the B-model and were developed by Witten in references [2, 3].

To be able to separate the two types of deformations, the A- and the B-model are a truncation of the original SCFT to a finite subspace of states. Every state in a conformal field theory is created from the vacuum by an operator in the theory - this is the content of the operator-state correspondence. The $\mathcal{N} = 2$ SCFTs include a special class of operators which have a ring structure which means that the composition of two operators in the ring is expressible as the sum of single operators. Overall there are four versions of this ring structure, two copies come from an additional $U(1)$ charge of the SCFT and the whole is doubled by considering the separation of the fields into left-moving and right-moving fields which only depend on the combinations $(\sigma + \tau)$ and $(\sigma - \tau)$, respectively. The restriction to the states obtained from these operators in both the left and right moving sectors leads to four theories. Out of the four theories two are inequivalent, the A- and the B-model, and the other two are theories obtained from charge conjugation, the anti-A- and the anti-B-model.

Topological string theory

The operators forming the finite rings are defined to be the ones annihilated in the SCFT by certain charges $\mathcal{Q}_A$ and $\mathcal{Q}_B$ coming from the supersymmetry algebra, similarly for the charge conjugate ones. The restriction to the states created by these operators is achieved by considering only states as physical if they are annihilated by $\mathcal{Q}_A$ or $\mathcal{Q}_B$ for the A-model and the B-model, respectively. This notion is known in physics within the quantization of gauge theories using the BRST formalism where physical states are the ones annihilated by the BRST operator $\mathcal{Q}_{BRST}$ and are considered equivalent if they differ by a state which can be written as the BRST operator acting on another state. In mathematics this is the well known notion of cohomology. The truncated theories obtained in this manner are topological field theories of cohomological type. An important feature of topological field theories is that the correlation functions are not affected by a variation of the metric.

Topological string theories are obtained from the previous topological field theories by coupling to world-sheet gravity. This is achieved by integrating over all world-sheet metrics. As mentioned previously the world-sheets are Riemann surfaces $\Sigma_{g,h}$ organized by their genus and number of holes. Integration over all world-sheet metrics is achieved by summing over all genera and holes (g, h) and in addition integrating over the moduli space of the Riemann surfaces at each g and h. The partition function of topological string theory is obtained by assigning a measure to each g, h to obtain an amplitude $\mathcal{F}^{g,h}$. The full partition function is organized as

$$Z_{top} = \exp \sum_{g,h} \lambda^{2g-2} \mu^h \mathcal{F}^{g,h} \,,$$

where λ and μ are expansion parameters.

Mirror symmetry

Mirror symmetry can now be stated as the equivalence of the A-model on a target space X, together with the space of Kähler deformations which are parameterized locally by coordinates t^a, $a = 1, \ldots, h^{1,1}(X)$ [1] on the one hand and the B-model on a mirror target space Y, together with the space of complex structure deformations parameterized locally

[1] $h^{1,1}(X) = \dim H^{1,1}(X)$ where $H^{1,1}(X)$ is a space containing (1,1) forms with respect to a complex structure, this is the geometric space of infinitesimal Kähler deformations.

by coordinates z^i , $i = 1, \dots, h^{2,1}(Y)$[2] on the other hand. The matching is found by determining the *mirror map* $t(z)$. The two spaces which form a mirror pair are required to have $h^{1,1}(X) = h^{2,1}(Y)$. The idea of mirror symmetry evolved in a number of papers, starting from the equivalence of the finite rings of Lerche, Vafa and Warner [4], through the first explicit construction of mirror manifolds by Greene and Plesser [5] to the computation of Candelas, De La Ossa, Green and Parkes [6] where the mirror map for a mirror pair of quintics was found and used to predict mathematical invariants. This work generated a lot of excitement in the mathematical community. It exemplifies the power of mirror symmetry. Physical quantities were calculated on the B-model side and translated to the A-model side with the mirror map. On the A-model side the physical quantities provide generating functions for mathematical invariants which require otherwise hard mathematical computations.

The mirror map

The physical origin of the A- and the B-model allows the computation of the mirror map. To do so, one considers sub-rings of the finite rings introduced previously. The sub-rings are the span of operators that deform the theory. The variation of the states created by this ring over the space of deformations is then studied. There is a natural flat connection on this space of states which is considered as a bundle over the space of deformations, which is called the bundle of ground-states or the vacuum bundle. The connection has its origin in the ring structure. To find the mirror map one has to identify the mathematical problem which corresponds to the study of the variation of the vacuum bundle and find coordinates which describe the variation such that the connection of the SCFT is recovered. Since the SCFT is the common origin of both theories, it is natural to find the matching in this way.

Quantum geometry

The mathematical surprise is that the problem connected to the B-model is well understood. It is the study of variation of Hodge structure which is governed by differential equations, the *Picard-Fuchs* equations. Furthermore, the manipulation to obtain the *special* coordinates which mimic the SCFT is also understood. For the A-side however the mathematics had to be developed. It involves notions of quantum cohomology, which is the study of how string theory modifies the classical notions of geometry. Quantum corrections due to string instantons modify the notions of volume, intersections and distances. Even the moduli space of the A-model is in contrast to the B-model moduli space an intriguing space. Covering the whole deformation space of SCFT requires the consideration of topology changes and even singular target spaces to lie within the moduli space [7, 8, 9].

[2] $h^{2,1}(Y) = \dim H^{2,1}(Y)$ where $H^{2,1}(Y)$ is a space containing (2,1) forms with respect to a complex structure, this is the geometric space of infinitesimal complex structure deformations.

On the physics side these results are highly welcome as they provide some sought for concepts of quantum geometry. Indeed there are ideas of how to construct theories of Kähler gravity [10]. More recently these ideas were pushed forward, obtaining a foamy picture of space-time in the A-model in ref. [11].

Background (in)dependence

In their seminal work [12, 13], Bershadsky, Cecotti, Ooguri and Vafa (BCOV) derived a set of equations called the *holomorphic anomaly equations* describing the dependence of the topological string theory A(B)-model on deformations of the anti-A(anti-B)-model which are expected to decouple from a topological field theory reasoning. This is the sense in which they describe an anomaly. The importance of these equations lies in the fact that they relate recursively the genus g topological string amplitudes $\mathcal{F}^g$ to amplitudes of lower genus. The recursive solution of these equations was furthermore found to be organized in a Feynman-diagrammatic way. On the level of calculations, the equations are thus very useful for computing the topological string partition function. The information contained in the recursion has however to be supplemented by the so called holomorphic ambiguities. On a conceptual level these equations are at least equally important as they provide a quantitative study of the dependence of the full topological string partition function on the reference background. In the B-model language this is the reference complex structure which is chosen to initially formulate the theory. For a given complex structure, the partition function is naively expected to be holomorphic. The moduli space of the B-model is however the space of complex structures. After a deformation of complex structure, the notions of holomorphic and anti-holomorphic mix. The partition function, being an expansion in the moduli space coordinates, knows about this difference which is expressed in the anomaly equations. This could at first sight underline the background dependence of topological string theory but the equations allow a different interpretation outlined in the following.

Partition function as a wave-function

In a very elegant paper [14] Witten reinterpreted the anomaly equations as actually dictating the background *independence* of the partition function. To do so, the full topological string partition function has to be interpreted as representing a state in a Hilbert space which is obtained by geometric quantization of the vacuum bundle. This state is described by a wave-function. The notion of the partition function being holomorphic corresponds to a choice of polarization. Choosing a polarization is familiar in quantum mechanics, where wave functions depend on only half of the variables of phase space. For example choosing the quantum mechanical wave-function to depend only on position coordinates x or only on momenta p are two equivalent polarization choices related by the Fourier transform. The same reasoning can be employed for choosing a holomorphic polarization: the anomaly equation states that the partition function transforms by a Bogoliubov transformation once a different complex structure is chosen. This wave-function language turned out to be not just a modification of the understanding of background dependence, the quantum mechanical interpretation also appeared more recently in ref. [15] where the probability density associated to that wave-function is conjectured to capture the partition function of a Black hole. This is the Ooguri-Strominger-Vafa (OSV) conjecture.

Including D-branes

D-branes denote the locus in space-time to which the boundaries of the Riemann surfaces are mapped. Mirror symmetry up to this part was only discussed for the case of world-sheets without boundaries . The mathematical framework of mirror symmetry can be enlarged to take the boundaries into account. The mathematical picture of mirror symmetry is formulated in the language of categories and mirror symmetry represents the equivalence of two categories. This is denoted *Homological Mirror Symmetry* and is due to Kontsevich [16]. The equivalence is between

$$Fuk(X) \simeq D^b(\mathrm{coh}Y),$$

the Fukaya category on the A-side and the bounded derived category of coherent sheaves on the B-side. These categories should be thought of as the correct mathematical formalism to incorporate all possible boundary conditions. There is currently a lot of work both on the mathematical and on the physical side to understand the full details of this construction, an overview can be gained in [17].

Extending closed string formalism to D-branes

Following a work of Aganagic and Vafa [18] who calculated the superpotential induced by certain D-branes in non-compact topological string models, Lerche, Mayr and Warner [19, 20, 21] succeeded in finding the B-model formalism which extends the variation of Hodge structure of the closed string B-model. This is the variation of mixed Hodge structure which was found to be also governed by Picard-Fuchs equations, hence, allowing the calculation of superpotential and special coordinates for open-string mirror symmetry. More recently Walcher [22] derived extended holomorphic anomaly equations for D-branes which are fixed at the minima of their superpotential.

Contribution of this thesis

Yamaguchi and Yau found in ref.[23] that the Feynman graph expansion for solving the holomorphic anomaly equation for the quintic can be organized in terms of a finite number of generators. The amplitudes are polynomials of a certain degree in these generators. In two publications [24, 25] we generalized this result on the one hand to arbitrary CY target spaces with any number of moduli and on the other hand to also solve Walcher's extended holomorphic anomaly equation in terms of polynomials. This provides a significant enhancement for practical computations and carries very interesting mathematical structure. In [26], we extended the formalism of Lerche, Mayr and Warner to the case of compact target spaces, deriving Picard-Fuchs equations which allow a computation of the superpotentials in these geometries and to gain an understanding of open/closed mirror symmetry in moduli space.

Outline of the thesis

- In the second chapter the basic ingredients of mirror symmetry are introduced. $\mathcal{N} = 2$ superconformal field theories, their finite operator rings and the vacuum bundle are discussed. Next, the geometric realization of the vacuum bundle is presented. The topological string A- and B-model are introduced as a truncation of the SCFT to two possible choices of finite rings. Finally, mirror symmetry is explaed in the context of the A- and B- model, the calculation of the mirror map is exemplified and a glimpse of how mirror symmetry extends to include D-branes is given.

- In the third chapter the tt^* equations and their realization in the context of special geometry are discussed. Then, the holomorphic anomaly equations of BCOV and their extension by Walcher are presented. The reinterpretation of the equations to give the topological string partition function a background independent meaning is outlined.

- In the fourth chapter we extend the work of Lerche, Mayr and Warner to the case of compact CY target spaces. A system of Picard-Fuchs differential equations is derived governing open/closed mirror symmetry in these cases. Furthermore, the superpotentials are computed for examples of compact geometries. This work appeared in [26].

- In the fifth chapter we generalize the polynomial construction of Yamaguchi and Yau, which was found for the quintic, to arbitrary CY target spaces with any number of moduli. The polynomial generators are introduced and the polynomial structure is proven. Furthermore, it is discussed how boundary conditions allow to fix the holomorphic ambiguities. The method is applied to the quintic with a D-brane fixed at the real locus as well as to some non-compact CY target spaces. This work is published in [24, 25].

Publications

- Murad Alim and Jean Dominique Länge, "Polynomial Structure of the (Open) Topological String Partition Function," arXiv:0708.2886, *JHEP*, 10:045, (2007).
- Murad Alim, Jean Dominique Länge and Peter Mary, "Global Properties of Topological String Amplitudes and Orbifold Invariants," arXiv:0809.4253, (2008).
- Murad Alim, Michael Hecht, Peter Mary and Adrian Mertens, "Mirror Symmetry for Toric Branes on Compact Hypersurfaces," arXiv:0901.2937, (2009).

2 Mirror Symmetry

In the following some notions of $\mathcal{N} = 2$ SCFT will be introduced to pave the way towards topological string theory. Emphasis will be put on the ring structure carried by certain states of the SCFT known as chiral states. The SCFT underlying topological string theory is an $\mathcal{N} = (2, 2)$ SCFT, where the two copies refer to the right moving and left moving versions of the algebra, which can be called holomorphic and anti-holomorphic given an appropriate choice of coordinates on the worldsheet. A subset of the chiral states will also parameterize the deformations of the SCFT and hence of the topological string theory. Mirror symmetry will be the study of the interplay between two different combinations of chiral states in the left moving and right moving sectors. The following exposition is based on references [27, 28, 13].

2.1 $\mathcal{N} = 2$ superconformal field theory (SCFT)

2.1.1 $\mathcal{N} = 2$ superconformal algebra

The $\mathcal{N} = 2$ superconformal algebra is an extension of the Virasoro algebra of the energy momentum tensor $T(z)$, which has conformal weight $h = 2$, by two anti-commuting currents $G^{\pm}(z)$ of conformal weight $3/2$ and a $U(1)$ current $J(z)$ under which the $G^{\pm}(z)$ carry charge $\pm$.

$$\begin{array}{cccc} & & G^+(z) & \\ & T(z) & & J(z) \\ & & G^-(z) & \\ & & & \\ h = & 2 & 3/2 & 1 \end{array} \tag{2.1}$$

The boundary conditions which must be imposed for the currents $G^{\pm}(z)$ can be summarized as follows

$$G^{\pm}(e^{2\pi i} z) = -e^{\mp 2\pi i a} G^{\pm} \,, \tag{2.2}$$

with a continuous parameter a which lies in the range $0 \leq a < 1$. For integral and half integral a one recovers anti-periodic and periodic boundary conditions corresponding to the Ramond and Neveu-Schwarz sectors respectively. The currents can be expanded in

Fourier modes

$$T(z) = \sum_n \frac{L_n}{z^{n+2}}\,, \quad G^{\pm}(z) = \sum_n \frac{G^{\pm}_{n\pm a}}{z^{n\pm a+\frac{3}{2}}}\,, \quad J(z) = \sum_n \frac{J_n}{z^{n+1}}\,. \tag{2.3}$$

The $\mathcal{N} = 2$ superconformal algebra can be expressed in terms of the operator product expansion of the currents or by the commutation relations of its modes. In the following the latter are displayed for future reference

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}\,, \\
[J_m, J_n] &= \frac{c}{3}m\delta_{m+n,0}\,, \\
[L_n, J_m] &= -mJ_{m+n}\,, \\
\left[L_n, G^{\pm}_{m\pm a}\right] &= \left(\frac{n}{2} - (m\pm a)\right) G^{\pm}_{m+n\pm a}\,, \\
\left[J_n, G^{\pm}_{m\pm a}\right] &= \pm G^{\pm}_{n+m\pm a}\,, \\
\left\{G^{+}_{n+a}, G^{-}_{m-a}\right\} &= 2L_{m+n} + (n-m+2a)J_{n+m} + \frac{c}{3}\left((n+a)^2 - \frac{1}{4}\right)\delta_{r+s,0}\,.
\end{aligned} \tag{2.4}$$

The algebras obtained for every value of the continuous parameter a are isomorphic. This isomorphism induces an operation on the states which is called *spectral flow* and has far reaching implications. In particular the spectral flow operation shows the equivalence of the Ramond (R) and Neveu-Schwarz (NS) sectors as the states in each sector are continuously related by the flow. Moreover, as NS and R sectors give rise to space-time bosons and space-time fermions respectively, this isomorphism of the algebra induces space-time supersymmetry.

2.1.2 Chiral ring

The representation theory of the $\mathcal{N} = 2$ superconformal algebra is equipped with an interesting additional structure, namely a finite sub-sector of the operators creating the highest weight states carries an additional ring structure. Topological string theory is a truncation of the SCFT having as its physical states only those belonging to this subset. This ring structure will be exhibited in the following. The unitary irreducible representations of the algebra can be built up from highest weight states by acting on these with creation operators which are identified with the modes of negative indices. Similarly all the modes with positive indices can be thought of as annihilation operators which lower the L_0 eigenvalue of a state. A highest weight state is thus one which satisfies,

$$L_n|\phi\rangle = 0\,, \quad G^{\pm}_r|\phi\rangle = 0\,, \quad J_m|\phi\rangle = 0\,, \quad n, r, m > 0\,. \tag{2.5}$$

In the NS sector the zero index modes L_0 and J_0 modes can be used to label the states by their eigenvalues

$$L_0|\phi\rangle = h_\phi|\phi\rangle\,, \quad J_0|\phi\rangle = q_\phi|\phi\rangle\,. \tag{2.6}$$

In the Ramond sector there are furthermore the modes $G_0^{\pm}$. If a state is annihilated by these then it is called a Ramond ground state. A highest weight state is created by a primary field ϕ

$$\phi|0\rangle = |\phi\rangle\,. \tag{2.7}$$

The subset of primary fields which will be of interest is constituted of the *chiral primary* fields. States which are created by those satisfy furthermore

$$G^{+}_{-1/2}|\phi\rangle = 0\,. \tag{2.8}$$

The name anti-chiral primary will be used for the primary fields annihilated by $G^{-}_{-1/2}$. In combination with the representations of the anti-holomorphic currents $\overline{G}^{\pm}$ this leads to the notions of $(c,c), (a,c), (a,a)$ and (c,a) primary fields, where c and a stand for chiral and anti-chiral and the pair denotes the conditions in the holomorphic and anti-holomorphic sectors. Considering

$$\langle\phi|\{G^{-}_{1/2}, G^{+}_{-1/2}\}|\phi\rangle = \langle\phi|2L_0 - J_0|\phi\rangle\,, \tag{2.9}$$

for a chiral primary field, the left hand side vanishes and a relation between the conformal weight and the charge is obtained

$$h_\phi = \frac{q_\phi}{2}\,. \tag{2.10}$$

For a more general state $|\psi\rangle$ the left hand side is non-negative and the following inequality holds, $h_\psi \geq q_\psi/2$. This property of chiral primary states is an analogue of the BPS bound for physical states. Now looking at the operator product expansion of two chiral primary fields ϕ and χ

$$\phi(z)\chi(w) = \sum_i (z-w)^{h_{\psi_i} - h_\phi - h_\chi}\psi_i\,, \tag{2.11}$$

the $U(1)$ charges add $q_{\psi_i} = q_\phi + q_\chi$ and hence $h_{\psi_i} \geq h_\phi + h_\chi$. The operator product expansion has thus no singular terms and the only terms which survive in the expansion when $z \to w$ are the one for which ψ_i is itself chiral primary. It is thus shown that the chiral primary fields give a closed non-singular ring under operator product expansion. Furthermore, one can show the finiteness of this ring by considering

$$\langle\phi|\{G^{-}_{3/2}, G^{+}_{-3/2}\}|\phi\rangle = \langle\phi|2L_0 - 3J_0 + \frac{2}{3}c|\phi\rangle \geq 0\,, \tag{2.12}$$

to see that the conformal weight of a chiral primary is bounded by $c/6$. For the combinations with the anti-holomorphic sector $(c,c), (a,c), (a,a)$ and (c,a) one sees that the latter two are charge conjugates of the first two. The relation between charge and conformal weight for an anti-chiral primary becomes $h_\psi = -\frac{q_\psi}{2}$. Denoting the set of chiral primary fields by ϕ_i where the index i runs over all chiral primaries, the ring structure can be formulated as follows [1]

$$\phi_i\phi_j = C^k_{ij}\phi_k\,. \tag{2.13}$$

[1]Formulas for products of operators are understood to hold within correlation functions.

2.1.3 Deformation families

Mirror symmetry is a symmetry relating the deformation family of the (a, c) chiral ring with the deformation family of the (c, c) chiral ring. To make this statement more precise the deformation family of the (c, c) chiral ring will be discussed in the following. Deformations of a conformal field theory are achieved by adding marginal operators to the original action, these are operators having conformal weight $h + \overline{h} = 2$. In the following spinless operators will be studied which have $h = 1, \overline{h} = 1$. The operators which maintain their $h = \overline{h} = 1$ conformal weights after perturbation of the theory are called truly marginal operators. Such operators can be constructed from the chiral primary operators in two steps. For instance in the (c, c) ring, starting from an operator of charge $q = \overline{q} = 1$, $h = \overline{h} = 1/2$ one can first construct

$$\phi^{(1)}(w, \overline{w}) = \oint dz\, G^{-}(z)\phi(w, \overline{w})\,, \tag{2.14}$$

which now has $h = 1, q = 0$. In the next step

$$\phi^{(2)}(w, \overline{w}) = \oint d\overline{z}\, \overline{G}^{-}(\overline{z})\phi^{(1)}(w, \overline{w})\,, \tag{2.15}$$

which has $h = \overline{h} = 1$ and zero charge and is hence a truly marginal operator and can be used to perturb the action of the theory

$$\delta S = t^{i} \int \phi_{i}^{(2)} + \overline{t}^{\overline{i}} \int \phi_{\overline{i}}^{(2)}\,, \quad i = 1, \ldots, n\,, \tag{2.16}$$

where a priori also the deformations coming from the (a, a) operators are included and $n = \dim\mathcal{H}^{(1,1)}$ denotes the dimension of the subspace of the Hilbert space of the theory spanned by the states which are created by the charge $(1, 1)$ operators. A similar construction can be done for the (a, c) chiral ring. The superscript notation is borrowed from topological field theories where an analogous construction gives the two form descendants which can be used to perturb the topological theory. The deformations constructed in this fashion span a deformation space $\mathcal{M}$, the moduli space of the SCFT.

2.1.4 The vacuum bundle

For the study of the geometric realization of the equivalence of the SCFTs a further important construction has to be introduced, namely that of the vacuum bundle. This denotes a holomorphic vector bundle $V_{\mathbb{C}}$ over the deformation space $\mathcal{M}$, which corresponds to a subset of the ground-states of the theory. Its importance lies in the fact that the ground-states of the theory do not change over the space of deformations. However there is a certain way of splitting the bundle which varies smoothly over the moduli space. The vacuum bundle collects the states of the theory which are created by the sub-ring of the chiral ring spanned by the charge $(1, 1)$ operators. A basis for this sub-ring is denoted by $(\phi_0, \phi_i, \phi^i, \phi^0)$, $i = 1, \ldots, n$. ϕ_0 is the identity operator of charge $(0,0)$ and ϕ^i are the charge $(2,2)$ fields which are dual to ϕ_i with respect to the topological metric and ϕ^0 is the top element of charge $(3,3)$

$$\eta(\phi_a, \phi^b) = \delta_a^b\,, \quad a, b = 0, \ldots, n\,. \tag{2.17}$$

The ring structure can now be put in matrix form

$$\phi_i \begin{pmatrix} \phi_0 \\ \phi_j \\ \phi^j \\ \phi^0 \end{pmatrix} = \begin{pmatrix} 0 & \delta_i^k & 0 & 0 \\ 0 & 0 & C_{ijk} & 0 \\ 0 & 0 & 0 & \delta_i^j \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_k \\ \phi^k \\ \phi^0 \end{pmatrix}\,. \tag{2.18}$$

It should be noted that the topological metric and the ring structure constants are the two point and three point correlation functions on the sphere in the topological field theory. The states created by this sub-ring are organized in a vector bundle

$$\mathcal{V} = \mathcal{H}^{0,0} \oplus \mathcal{H}^{1,1} \oplus \mathcal{H}^{2,2} \oplus \mathcal{H}^{3,3}\,, \tag{2.19}$$

where $\mathcal{H}^{(i,i)}$ denotes the subspace of the Hilbert space of states created by charge (i,i) operators. The splitting of the bundle is hence given by the charge grading. The variation of this grading over the moduli space $\mathcal{M}$ and its geometric realization is going to be a central theme in the study of mirror symmetry. The operator-state correspondence can be used to obtain a basis of the vector bundle from the basis of the chiral ring operators. Denoting by $|e_0\rangle \in \mathcal{H}^{0,0}$ the unique ground state of charge $(0,0)$ up to scale, a basis for $\mathcal{V}$ can be obtained as follow

$$|e_i\rangle = \phi_i |e_0\rangle\,, \quad |e^i\rangle = \phi^i |e_0\rangle\,, \quad |e^0\rangle = \phi^0 |e_0\rangle\,. \tag{2.20}$$

The metric on the vacuum bundle is given by the topological metric

$$\langle e_a | e^b \rangle = \delta_a^b \quad a, b = 0, \ldots, n\,. \tag{2.21}$$

The representation of the chiral ring in this basis reads

$$\phi_i \begin{pmatrix} |e_0\rangle \\ |e_j\rangle \\ |e^j\rangle \\ |e^0\rangle \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \delta_i^k & 0 & 0 \\ 0 & 0 & C_{ijk} & 0 \\ 0 & 0 & 0 & \delta_i^j \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{:=C_i} \begin{pmatrix} |e_0\rangle \\ |e_k\rangle \\ |e^k\rangle \\ |e^0\rangle \end{pmatrix}\,. \tag{2.22}$$

This structure will turn out to be crucial for the understanding of mirror symmetry. In terms of coordinates on the moduli space $\mathcal{M}$ an insertion of the chiral field ϕ_i is equivalent to an infinitesimal displacement in moduli space and hence can be obtained by a derivative $\frac{\partial}{\partial t^i}$. Denoting the matrix on the right hand side by C_i, the whole equation can be read as a connection on the vacuum bundle $\nabla = \partial_i - C_i$ which is flat

$$[\nabla_i, \nabla_j] = 0\,.$$

2.1.5 Geometric realization of SCFT

A geometric realization of the $\mathcal{N}=(2,2)$ SCFT is provided by the nonlinear sigma model. This is discussed and reviewed in many places [29, 30, 31, 32, 33]. The goal here is to identify the geometric realization of the (a,c) and (c,c) rings. The nonlinear sigma model that will be considered is a field theory of bosons and fermions ϕ^i, ψ^i living on a Riemann surface Σ, being related by supersymmetry where the bosonic fields are considered as coordinates of some target space X, i.e., $\phi : \Sigma \to X$. In order for the theory to have $\mathcal{N}=(2,2)$ the target space spanned by the bosonic fields has to be a Kähler manifold, which allows a split of its tangent bundle $TX = T^{1,0}X \oplus T^{0,1}X$. The action of the theory is [29]

$$\begin{aligned} S &= \int_\Sigma d^2z\Big(\frac{1}{2}g_{i\bar{j}}\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{j}} + \frac{i}{2}g_{i\bar{j}}\psi^i D_z\psi^{\bar{j}} \\ &+ \frac{i}{2}g_{i\bar{j}}\chi^i D_{\bar{z}}\chi^{\bar{j}} + R_{i\bar{k}j\bar{l}}\psi^i\psi^{\bar{k}}\chi^j\chi^{\bar{l}}\Big) + \int_\Sigma \phi^*(B)\,, \end{aligned} \tag{2.23}$$

where the term involving the B-field is topological. Denoting by K the canonical bundle on Σ the fermions are section of

$$\begin{aligned} &\psi^i \in \Gamma(K^{1/2}\otimes\phi^*T^{1,0}X)\,, \qquad \psi^{\bar{i}} \in \Gamma(K^{1/2}\otimes\phi^*T^{0,1}X)\,, \\ &\chi^i \in \Gamma(\overline{K}^{1/2}\otimes\phi^*T^{1,0}X)\,, \qquad \chi^{\bar{i}} \in \Gamma(\overline{K}^{1/2}\otimes\phi^*T^{0,1}X)\,. \end{aligned} \tag{2.24}$$

The covariant derivative in the action is with respect to these bundles. The fermions ψ and χ correspond to the left and right moving sectors. The action is conformally invariant if the Ricci tensor is vanishing, together with the Kähler condition this yields a Calabi-Yau target space. The (a,c) and (c,c) rings stem from the representation of the zero-mode algebra of the fermions

$$\{\psi_0^i,\psi_0^j\} = 0 = \{\psi_0^{\bar{i}},\psi_0^{\bar{j}}\} \quad \{\psi_0^i,\psi_0^{\bar{j}}\} = g^{i\bar{j}}\,, \tag{2.25}$$

and similarly for the χs. The last anticommutator suggests considering the fermionic zero modes as creation and annihilation operators. This choice can be done for both ψ and χ algebras of the zero modes. The elements of the finite rings of the SCFT correspond to the Ramond ground states. In the present context these are the ones annihilated by the supercharges of the supersymmetry. The states created in either choice and the action of the supercharges on them can be translated into the correspondence[2]

$$\begin{aligned} \mathcal{R}^{(a,c)} &\simeq \bigoplus_{p,q} H_{\bar{\partial}}^{(p,q)}(X)\,, \\ \mathcal{R}^{(c,c)} &\simeq \bigoplus_{p,q} H_{\bar{\partial}}^{(3-p,q)}(X)\,. \end{aligned} \tag{2.26}$$

[2]For the intermediate steps of the derivation one can consult for example [28]

The indices i and $\bar{i}$ correspond to the $U(1)$ charges of the chiral ring, the two sectors correspond to the ψ and χ sectors. The sub-rings which appear in the construction in the vacuum bundle are those of equal absolute value of charges. This correspondence gives a first flavor of the geometric realization of mirror symmetry. On the geometric level mirror symmetry related even cohomology with the middle dimensional cohomology.

2.2 Topological string theory

The physical arena for the study of the geometric realization of mirror symmetry is topological string theory. The notions of a topological field theory, how it can be obtained from the previous SCFT discussion and the coupling to two dimensional gravity to obtain a topological string theory will be discussed in the following.

2.2.1 Topological field theory

In the following some features of topological sigma models of cohomological type are discussed. These are sigma models, i.e., theories of maps $x : \Sigma_g \to X$ into some target space X, the case of interest is where Σ_g denotes a Riemann surface of genus g. The maps are realized physically as bosons and fermions which are related by supersymmetry. The additional structure of cohomological topological field theories is the existence of a Grassmann, scalar symmetry operator $\mathcal{Q}$ with the following properties

- $\mathcal{Q}$ is nilpotent $\mathcal{Q}^2 = 0$.
- The sigma model action is exact [3] $S = \{\mathcal{Q}, V\}$.
- The energy momentum tensor is also $\mathcal{Q}$ exact,

$$T_{\mu\nu} = \{\mathcal{Q}, G_{\mu\nu}\}.$$

A direct consequence of the last property is that the correlation functions do not depend on the two dimensional metric. To see this one notices that a variation with respect to the metric is equivalent to inserting the energy momentum tensor in the correlators, this one being exact and $\mathcal{Q}$ being a symmetry of the theory shows the metric independence. Furthermore, physical states of the theory correspond to cohomology classes of the operator $\mathcal{Q}$ and the semi-classical approximation for evaluating the path integral is exact. Further background as well as derivations of these properties can be found in [34, 35, 30] .

[3] In topological string theory this condition is relaxed, the action is exact up to a topological term.

2.2.2 Topological strings

The special structure of the finite rings of marginal operators in superconformal field theories was emphasized in the last section. In particular it was shown that a sub-ring exists, out of which the vacuum bundle is constructed. Here it is shown that a topological field theory can be obtained by restricting to these finite rings. The way this restriction is achieved is by treating the annihilators of the states in question as BRST operators and considering only those states as physical states which are annihilated by the BRST operator. From the definition of a chiral state in eq. (2.8) and its anti-chiral counterpart it is clear that the operators in question are [4]

$$\underset{(a,c)}{G_0^- + \overline{G}_0^+}\,, \qquad \underset{(c,c)}{G_0^+ + \overline{G}_0^+}\,. \tag{2.27}$$

Topological string theory is obtained by coupling the resulting theory to $2d$ gravity which involves an integration over the space of all possible $2d$ metrics. The two dimensional spaces are the Riemann surfaces which are organized by their genus g. One has in particular to be able to define the BRST operator globally on every such Riemann surface. The charges corresponding to the SCFT G currents are however fermions. In order to have a global well defined charge on every Riemann surface the *topological twist* is introduced. Since the spin of a current is determined by its conformal weight, which appears in the operator product expansion with the energy momentum tensor, the idea of the twist is to modify the energy momentum tensor. The new energy momentum tensor is a combination of the old one and the $U(1)$ current J. Depending on which ring one wants to restrict to, this is achieved by

$$\begin{array}{c|c} A & B \\ \hline & \\ T \to T + \frac{1}{2}J & T \to T - \frac{1}{2}J \\ & \\ \overline{T} \to \overline{T} - \frac{1}{2}\overline{J} & \overline{T} \to \overline{T} - \frac{1}{2}\overline{J} \\ & \\ \hline (a,c) & (c,c) \end{array} \tag{2.28}$$

yielding Grassmann valued scalars Q_A and Q_B

$$Q_A = \underset{(a,c)}{G_0^- + \overline{G}_0^+}\,, \qquad Q_B = \underset{(c,c)}{G_0^+ + \overline{G}_0^+}\,, \tag{2.29}$$

these square to zero and provide hence the essential ingredient of a topological field theory of cohomological type. The cohomology of these operators gives the desired restriction to the finite rings. As the notation suggests restricting to the (a,c) ring gives the *A-model* and restricting to the (c,c) ring gives the *B-model*. The coupling to gravity is achieved by

[4] Here the operators from the NS sector with subscripts $-1/2$ are traded for the operators of the R sector with subscripts 0.

defining a correlator for every genus of a Riemann surface and integrating the correlator over the moduli space of Riemann surfaces. The moduli space of a Riemann surface at genus g is described by $3g-3$ so called Beltrami differentials $\mu_{\bar{z}}^{z}$, these are anti-holomorphic forms taking values in the holomorphic tangent bundle $\mu_a \in H^{0,1}(\Sigma_g, T\Sigma_g)$ and their complex conjugates $\overline{\mu}_{z}^{\bar{z}}$. These differentials are contracted with the two currents which get their conformal weight augmented to $h = 2\,(\bar{h} = 2)$. This construction is motivated from the bosonic string where the anti-ghosts are contracted with the Beltrami differentials in a similar manner. The genus g amplitudes or free energies are now defined by (here for the B-model and $g > 1$)[5]

$$\mathcal{F}^g = \int_{\mathcal{M}_g} [dm\, d\bar{m}] \langle \prod_{a=1}^{3g-3} (\int_{\Sigma} \mu_a G^-)(\int_{\Sigma} \mu_{\bar{a}} \overline{G}^-) \rangle_{\Sigma_g}\,, \tag{2.30}$$

where $\langle \ldots \rangle_{\Sigma_g}$ denotes the CFT correlator and $dmd\overline{m}$ are dual to the Beltrami differentials. For a more detailed exposition of the construction of topological string theory as well as for a thorough treatment of the A and B-model reference is given to the many reviews that exist by now on this vast subject [30, 36, 37, 38, 35, 39]. This work will focus instead on recovering the structure of the vacuum bundle discussed in the context of SCFT and highlight the structure that it carries which allows the matching between the A- and the B-model.

2.3 Mirror symmetry

This part is concerned with recovering in the context of the A-model and B-model the notions of the moduli space of theories, the vacuum bundle and its varying split. Using the common origin in SCFT mirror symmetry for the A and B-model predicts dramatic consequences relating different areas of mathematics. For the B-model, the notions of moduli space of theories, vacuum bundle and Yukawa coupling find their natural geometric counterparts in the study of the problem of variation of Hodge structures on the moduli space of complex structures of a family of CY manifolds. For the A-model however, the classical notions of geometry have to be modified. To begin with, the moduli space of SCFT which is smooth reflecting the smooth deformation of one SCFT to another does not have a smooth geometric counterpart of Kähler deformations. The moduli space of Kähler deformations of a given CY is encoded is the Kähler cone. What the moduli space of SCFT seems to suggest however is to consider an *enlarged* Kähler moduli space having for instance adjacent Kähler cones corresponding to manifolds which are related by topological transitions, the *flops* [9, 8]. The enlarged Kähler moduli space also includes loci corresponding to orbifold target spaces or singular geometries.

[5] The amplitudes at $g = 0, 1$ need separate treatment, in particular $\mathcal{F}^0$ denotes the prepotential and can be calculated from the geometry of the vacuum bundle.

2.3.1 Quantum cohomology

The geometric realization on the A-model side of the vacuum bundle construction and the study of its variation over the moduli space is hard and not understood in full generality. The reason for this is that string theory modifies the notions of classical geometry and defines quantum geometry. This is a statement that is commonly made in this context and the following text is intended to give a flavor of what is meant by that. One manifestation of the last statement is the aforementioned non-smoothness of the enlarged Kähler moduli space. Another manifestation lies in quantum cohomology which can be understood as introducing the notion of instanton effects into the classical intersection theory on a manifold. To avoid introducing the vast mathematical terminology needed to properly define what quantum cohomology is, reference is given to [40, 41, 42]. In the following a short exposition is given of what the vacuum bundle and what the corresponding Gauss-Manin connection on it are. This will follow closely [43]. The concept of quantum cohomology is only developed for the large radius regime of moduli space, it involves power series in a variable $q = e^{-t}$, where t is measuring the size of a two-cycle and hence the power series only converge for the *large radius* regime, where the value of t is large. There has however been progress recently on the mathematics side to develop mathematical notions for mirror symmetry also for the orbifold region in moduli space.[6] The vacuum bundle in this case corresponds geometrically to the even dimensional cohomology of the CY. The complexified Kähler classes are elements in $H^2(X,\mathbb{C})$, which constitutes the tangent space to the Kähler moduli space $\mathcal{M}$. The pairing is given by the product of forms. The discussion boils down to give an interpretation for the following diagram

$$H^0(X,\mathbb{C}) \xrightarrow{\nabla_A} H^2(X,\mathbb{C}) \xrightarrow{\nabla_A} H^4(X,\mathbb{C}) \xrightarrow{\nabla_A} H^6(X,\mathbb{C})\,. \tag{2.31}$$

To define the connection let η_0 be the generator of $H^0(X,\mathbb{Z})$, $\eta_i\,,\ i=1,\dots,n$ a basis of $H^2(X,\mathbb{Z})$, $\chi_i\,,\ i=1,\dots,n$ a basis of $H^4(X,\mathbb{Z})$ which is dual with respect to the symplectic pairing $\langle\eta_i,\chi_j\rangle = \delta_{ij}$ and finally let $\chi_0 \in H^6(X,\mathbb{Z})$ be the dual to η_0. The connection ∇_A is now defined by

$$\begin{aligned}
\nabla_A\eta_0 &= \sum_{i=1}^n \eta_i \otimes \frac{dq_i}{q_i}\,,\\
\nabla_A\eta_k &= \sum_{i,j=1}^n C_{ijk}\chi_j \otimes \frac{dq_i}{q_i}\,,\\
\nabla_A\chi_j &= \chi_0\frac{dq_j}{q_j}\,,\\
\nabla_A\chi_0 &= 0\,.
\end{aligned} \tag{2.32}$$

[6] Check for example [44] and references therein.

The coefficients C_{ijk} are power series in $q_1, \ldots, q_n$ defined by rational curves on X

$$C_{ijk} = \langle \eta_i, \eta_j, \eta_k \rangle + \sum_{C \subset X} n_{[C]} \langle C, \eta_i \rangle \langle C, \eta_j \rangle \langle C, \eta_k \rangle \frac{q^{[C]}}{1 - q^{[C]}} , \tag{2.33}$$

where $q^{[C]} = q^{c_1} \ldots q_n^{c_m}$, $c_n = \langle C, \eta_i \rangle$. And where

$$\Gamma_{[C]}(\eta_i, \eta_j, \eta_k) = n_{[C]} \langle C, \eta_i \rangle \langle C, \eta_j \rangle \langle C, \eta_k \rangle \tag{2.34}$$

are rational numbers called the Gromov-Witten invariants.

2.3.2 Variation of Hodge structures

In the following the geometric realization on the B-model side of the SCFT structures will be outlined. In particular the B-model has a well known realization of the vacuum bundle structure which does not vary over the moduli space but which is equipped with a split that does. This is the variation of Hodge structures. There exist many references for this subject, e. g. [40, 45], the exposition here will focus on the main ingredients needed for the discussion of mirror symmetry and will follow in some parts closely ref.[46]. The moduli space on the B-side corresponds to the moduli space of complex structures of the target space and in the case of Calabi-Yau (CY) threefolds the vacuum bundle corresponds to the Hodge bundle, a term which denotes the middle dimensional cohomology $H^3(Y, \mathbb{C})$. This space, which forms a vector bundle over the moduli space of complex structure has a natural splitting once a given complex structure is chosen, i.e., at a specific point in the complex structure moduli space. The split is

$$H^3(Y, \mathbb{C}) \simeq \bigoplus_{p+q=3} H^{p,q}(Y) . \tag{2.35}$$

This split has a natural notion of complex conjugation, namely $\overline{H^{p,q}(Y)} = H^{q,p}(Y)$. The term Hodge structure refers to $H^3(Y, \mathbb{C})$, together with the split (2.35) and with a lattice given by $H^3(Y, \mathbb{Z})$ which generates $H^3(Y, \mathbb{C})$ upon tensoring with $\mathbb{C}$. The split (2.35) does however not vary holomorphically when the complex structure moduli are varied. The non-holomorphicity of this split is in some sense at the origin of the holomorphic anomaly equations which will be discussed in the next chapter. There is however a different split of the bundle which varies holomorphically over the moduli space of complex structures. This split is given by the Hodge filtration $F^\bullet(Y) = \{F^p(Y)\}_{p=0}^3$, where the spaces in brackets are defined by

$$F^p(Y) = \bigoplus_{a \geq p} H^{a,3-a}(Y) . \tag{2.36}$$

To recover the splitting (2.35) one can intersect with the anti-holomorphic filtration

$$H^{p,q}(Y) = F^p(Y) \cap \overline{F^q(Y)} . \tag{2.37}$$

Moreover the filtration is equipped with a flat connection ∇ which is called the Gauss-Manin connection with the property $\nabla F^p \subset F^{p-1}$, which is called Griffiths transversality. Together with the symplectic pairing given by integrating the wedge product of two three forms over the CY, this transversality allows to define the Yukawa couplings in this language

$$C_{ijk} := -\int_Y \Omega \wedge \nabla_i \nabla_j \nabla_k \Omega \,, \tag{2.38}$$

where Ω denotes the unique up to scale holomorphic three-form of a CY manifold.
Having the mathematical framework at hand, it remains to show which computations can be done on the B-model side to find the mirror map connecting this to the A-model. The upshot of the following discussion is: given a patch in $\mathcal{M}$ described by local coordinates z^i, $i = 1, \ldots, h^{2,1}(Y)$, there exist coordinates $t^a(z^i)$, $a = 1, \ldots, h^{2,1}(Y) = h^{1,1}(X)$[7] in terms of which the Gauss-Manin connection takes the SCFT form, namely that a derivative corresponds to the chiral ring action (2.22) of the topological theory. As the topological theory constructed out of the SCFT is the common origin of both the A and the B-model, it is natural to identify the coordinate which mimics this structure on the B-model side studying the variation of Hodge structure of a CY Y with the coordinate describing quantum cohomology on the A-model side on a mirror manifold X with $h^{1,1}(X) = h^{2,1}(Y)$. On the A-model side the coordinates t^a are measuring the *sizes* of two-cycles in terms of the Kähler form.
Starting from the holomorphic 3-form Ω one can use the property of Griffiths transversality of the Gauss-Manin connection to construct a basis for $H^3(Y, \mathbb{C})$. Denote this basis with $2h^{2,1} + 2$ entries by

$$\begin{aligned} &\mathbf{\Omega} = (\Omega_\beta) = (\Omega^{3,0}, \Omega^{2,1}_i, \Omega^{i\,1,2}, \Omega^{0,3})^t \,, \\ &i = 1, \ldots, h^{2,1}(Y)\,, \quad \beta = 0, \ldots, 2h^{2,1} + 1 \,. \end{aligned} \tag{2.39}$$

The Gauss-Manin connection is expressed as

$$\nabla_i \mathbf{\Omega}(z) = (\partial_i - \mathcal{A}_i)\, \mathbf{\Omega}(z) = 0 \,, \tag{2.40}$$

this holds up to exact pieces. These drop out if one considers integrals over a fixed basis of three cycles $\gamma^\alpha \in H_3(Y, \mathbb{C})$. This defines the period matrix

$$\Pi^\alpha_\beta(z) = \int_{\gamma^\alpha} \Omega_\beta(z) \quad \alpha, \beta = 0, \ldots 2h^{2,1} + 1 \,. \tag{2.41}$$

which satisfies

$$\nabla_i \, \Pi^\alpha_\beta = 0 \,, \tag{2.42}$$

when the cycles have no boundary. The flatness of the Gauss-Manin connection $[\nabla_i, \nabla_j] = 0$ can be interpreted as the integrability condition for the matrix equation (2.42) . Bearing in mind that all elements of the vector $\mathbf{\Omega}$ are obtained by applying derivatives to the

[7]X and Y denote a mirror pair of CY 3-folds.

holomorphic 3-form, eq. (2.42) can be translated into an equivalent system of higher order differential equations [47] for the period vector $\Pi^\alpha = \int_{\gamma^\alpha} \Omega_0$

$$\mathcal{L}_a \Pi^\alpha = 0\,, \quad a = 1, \ldots, h^{2,1}\,, \tag{2.43}$$

which is called the Picard-Fuchs system. This system has $2h^{2,1} + 2$ solutions. In order to interpret these in the context of the A-model as measuring volumes of cycles it is further necessary to translate the result into the right coordinates. This is achieved by noticing that the Gauss-Manin connection decomposes in the following manner

$$\nabla_i = \partial_i - \mathcal{A}_i = \partial_i - \Gamma_i - C_i\,, \tag{2.44}$$

where the nonzero entries of the matrix Γ_i are on and below the diagonal while the matrix C_i has the form of the chiral ring action discussed in the context of the bundle of groundstates in eq. (2.22). Finding the coordinates for which the Γ_i part of the connection vanishes is termed finding the *flat* coordinates in the physics literature. The procedure to do so will be exemplified in the case of the quintic. Expressing the right linear combinations of the solutions of the Picard-Fuchs system in terms of the flat coordinates[8], the period vector of the holomorphic $(3,0)$ form becomes

$$\Pi^\alpha(z(t)) = (X_0, X_i, \mathcal{F}^i, \mathcal{F}^0) \simeq (1, t_i, \partial_i \mathcal{F}, 2\mathcal{F} - t^j \partial_j \mathcal{F})\,. \tag{2.45}$$

The period vector interpreted on the A-model side gives the quantum volumes of a point, 2-cycles, 4-cycles and of the 6-cycle. One finds moreover

$$C_{abc} = \partial_a \partial_b \partial_c \mathcal{F}\,. \tag{2.46}$$

$\mathcal{F}$ is called the *prepotential* of $\mathcal{N} = 2$ special geometry.

2.3.3 The Quintic

As an example of the geometric realization of mirror symmetry the mirror quintic will be discussed [6]. The goal is to exemplify the calculation of the mirror map and to give a flavor of the different phases of mirror. The derivation of the Picard-Fuchs equations is reviewed in [48, 30] which will be the guideline for the following exposition.
The quintic X denotes the CY manifold defined by

$$X := \{P(x) = 0\} \subset \mathbb{P}^4 \tag{2.47}$$

where P is a homogeneous polynomial of degree 5 in 5 variables $x_1, \ldots, x_5$. The mirror quintic Y can be constructed using the Greene-Plesser construction [5][9]. Equivalently it

[8]This is achieved by requiring integral monodromy around singular points in moduli space.
[9]See [49] for an outline of the idea of the construction.

may be constructed using Batyrev's dual polyhedra[50] in the toric geometry language[10]. In the Greene-Plesser construction the family of mirror quintics are the one parameter family of quintics defined by

$$\{p(Y) = \sum_{i=1}^{5} x_i^5 - \psi \prod_{i=1}^{5} x_i = 0\} \in \mathbb{P}^4 \tag{2.48}$$

after a $(\mathbb{Z}_5)^3$ quotient. This can be stated in Batyrev's formalism as imposing the following superpotential in the total space of the anti-canonical line bundle $\mathcal{O}(-5) \to \mathbb{P}^4$ over $\mathbb{P}^4$

$$p(Y) = \sum_{i=0}^{5} a_i y_i = 0\,, \tag{2.49}$$

where the coordinates y_i are subject to the relation

$$y_1 y_2 y_3 y_4 y_5 = y_0^5\,. \tag{2.50}$$

Changing the coordinates $y_i = x_i^5$, $i = 1, \dots, 5$ shows the equivalence of (2.48) and (2.49) with

$$\psi^{-5} = -\frac{a_1 a_2 a_3 a_4 a_5}{a_0^5} =: z\,. \tag{2.51}$$

The holomorphic $(3,0)$ form Ω can be written in every patch of $\mathbb{P}^4$ as a residue expression[52]

$$\Omega(z) = \int_\gamma \frac{a_0 \mu}{P}\,, \tag{2.52}$$

where the contour surrounds the pole $P = 0$ and the measure is given by

$$\mu = \sum_{k=1}^{5} (-1)^k x^k dx^1 \wedge \cdots \wedge \hat{dx^k} \wedge \cdots \wedge dx^5\,, \tag{2.53}$$

where $\hat{dx^k}$ is omitted from the product. The redundancy of the a_i to parameterize the complex structure of the mirror quintic can be seen from the gauge invariance of $\Omega(a_i)$

- It is invariant under $a_i \to \rho\, a_i\,, \rho \in \mathbb{C}^*$, which translates into

$$\sum_{i=0}^{5} \theta_{a_i} \Omega(a) = 0\,, \quad \theta_{a_i} = a_i \frac{\partial}{\partial a_i}\,.$$

- It is invariant under $(a_i, a_j) \to (\rho^{-5} a_i, \rho^5 a_j)\,, \quad i, j = 1, \dots, 5\, \rho \in \mathbb{C}^*$

$$(\theta_i - \theta_5)\Omega(a) = 0\,, \quad i = 1, \dots, 4$$

[10] For a review of toric geometry one can consult the references [28, 30, 51]

These equations dictate the dependence of $\Omega(a)$ on the combination $z = -\frac{a_1 a_2 a_3 a_4 a_5}{a_0^5}$. Furthermore one obtains from the definition (2.52) the equation

$$\left(\frac{\partial}{\partial a_0}\right)^5 \frac{\Omega(a)}{a_0} = \left(\prod_{i=1}^{5} \frac{\partial}{\partial a_i}\right) \frac{\Omega(a)}{a_0} . \tag{2.54}$$

Translating the last equation into the z variable one obtains the Picard-Fuchs equation

$$\mathcal{L}\Omega(z) = 0 , \quad \mathcal{L} = \theta^4 - 5z \prod_{i=1}^{4} (5\theta + i) , \quad \theta = z \frac{d}{dz} . \tag{2.55}$$

This differential equation has three regular singular points which correspond to points in the moduli space of the family of quintics where the defining equation becomes singular, these are the points [11]

- $z = 0$, the quintic at this value corresponds to the quotient of $\prod_{i=1}^{5} x_i = 0$ which is the most degenerate Calabi-Yau and corresponds to large radius when translated to the A-side.
- $z = 5^{-5}$ this corresponds to a discriminant locus of the differential equation (2.55) and also to the locus where the Jacobian of the defining equation vanishes. This type of singularity is called a *conifold* singularity. Its A-model interpretation is going to be very useful in the context of higher genus amplitudes in a later chapter of the thesis.
- $z = \infty$, this is known as the orbifold point in the moduli space of the quintic and it corresponds to a non-singular CY 3 fold with a large automorphism group. This will be reflected in a monodromy of order 5.

Finding flat coordinates

The GM connection matrix will first be analyzed at the large complex structure point, which is the term employed for the $z = 0$ point in the present case and it is termed large because is it mirror to large radius on the A-model side. Choosing as a section of the filtration

$$\mathbf{\Omega}_z(z) = (\Omega(z), \partial_z \Omega(z), \partial_z^2 \Omega(z), \partial_z^3 \Omega(z))^T , \tag{2.56}$$

one can use the Picard-Fuchs equation (2.55) to read off the following connection matrix

$$\partial_z \mathbf{\Omega} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{120}{z^3\Delta} & -\frac{1-15000z}{z^3\Delta} & -\frac{7-45000z}{z^2\Delta} & -\frac{6-25000z}{z\Delta} \end{pmatrix}}_{M_z} \mathbf{\Omega} , \tag{2.57}$$

[11] For more details on the mathematical meaning of these singular points one can consult reference [53], for the physical SCFT interpretation see [54]

where $\Delta = 1 - 3125z$. This can be written as

$$\nabla_z \mathbf{\Omega}_z = (\partial_z - M_z)\mathbf{\Omega}_z = 0\,. \tag{2.58}$$

To find the basis and the coordinate in terms of which the connection becomes the topological connection, the following ansatz is made

$$\mathbf{\Omega}_t(z(t)) = \left(\frac{\Omega(z(t))}{S_0(t)}, \partial_t\frac{\Omega(z(t))}{S_0(t)}, C(z(t))^{-1}\partial_t^2\frac{\Omega(z(t))}{S_0(t)}, \partial_t\left(C(z(t))^{-1}\partial_t^2\frac{\Omega(z(t))}{S_0(t)}\right)\right)\,, \tag{2.59}$$

the vector transforms as

$$\mathbf{\Omega}(z(t))_t = A(t)\mathbf{\Omega}_z\,.$$

Transforming the connection matrix M_z with

$$M_t = (\partial_t A(t) + \frac{\partial z(t)}{\partial t}A(t)\cdot M_z(z(t)))\cdot A^{-1}(t)\,,$$

the following matrix is found

$$M_t = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & C(z(t)) & 0 \\ 0 & 0 & 0 & 1 \\ * & * & * & 0 \end{pmatrix}\,, \tag{2.60}$$

where

$$C(z(t)) = \left(\frac{\partial z(t)}{\partial t}\right)^3 \frac{c}{z(t)^3(S_0(t))^2\Delta(z(t))}\,, \qquad \Delta(z(t)) = 1 - 3125z(t)\,.$$

The remaining $*$s in the matrix are differential equations for $S_0(t)$ and $z(t)$. The differential equations for the mirror map and for the normalization can be solved and yield

$$\begin{aligned} S_0(t) &= 1 + 120q + 21000q^2 + 14115000q^3 + \dots\,, \qquad q = e^{-t} \\ z(t) &= q - 770q^2 + 171525q^3 + \dots\,, \\ C_{ttt} &= \partial_t^3 F_0 = 5 + 2875q + 4876875q^2 + 8564575000q^3 + \dots\,, \end{aligned}$$

where the classical intersection number appearing in $C(t)$ has been fixed $c = 5$.

2.3.4 Including D-branes

This part contains an overview of how the mathematics involved in the variation of Hodge structures can be generalized to describe a certain class of D-branes. This is intended to be a short preparation for chapter 4 of the thesis and is based on work of Lerche, Mayr and Warner [19, 20, 21] which is excellently reviewed in [46]. To begin with, an overview of the description of D-branes in the topological string language is given.[12]

[12]More details can be found for example in [46, 31].

A-branes and B-branes

A-branes denote branes in the A-model. They wrap a special Lagrangian cycles γ^3, their deformation space is of dimension $b_1(\mathcal{M}_{\gamma^3})$ and a notion of special coordinates $\hat{t}^a$ can be introduced which measures disk sizes which end on the special Lagrangian. A superpotential is associated to the brane which is predicted to have an integral expansion in terms of the exponentiated special coordinate $\hat{q} = e^{-\hat{t}}$. The integral invariants are called Ooguri-Vafa invariants [55].

Branes in the B-model wrap holomorphic cycles and induce a World-volume superpotential which is given by the Chern-Simons action which reads

$$\mathcal{W} = \int \Omega^{3,0} \wedge \mathrm{Tr}(A \wedge \overline{\partial} A + \frac{2}{3} A \wedge A \wedge A) \,. \tag{2.61}$$

For 5-branes wrapping two-cycles in the internal space this can be reduced to

$$\mathcal{W}(z, \hat{z}) = \int_{\hat{\gamma^3}} \Omega^{3,0} \,, \tag{2.62}$$

where $\hat{\gamma^3}$ denotes a 3-*chain*, whose boundary are the two cycles.

Variation of mixed Hodge structure

This topic will be briefly reviewed in chapter 4 of this thesis which is concerned with the extension of the results of Lerche, Mayr and Warner which were derived and applied for non-compact CY spaces to the case of compact CY spaces. Here the idea of the extension of the variation of Hodge structures to describe certain D-branes will be given. The observation is that the superpotential stemming from background fluxes [56, 57] and the superpotential coming from the reduction of the Chern-Simons action

$$\mathcal{W} = \mathcal{W}_{flux}(z) + \mathcal{W}_{brane}(z, \hat{z}) \,, \tag{2.63}$$

can be described in terms of the third relative homology of the CY $H_3(Y, \mathcal{H}, \mathbb{Z})$ which combines the three cycles without boundaries from the closed string and the three chains with boundaries the two cycles wrapped by the branes. These two cycles are captured by a subset $\mathcal{H}$ which is embedded in the CY by $i : \mathcal{H} \to Y$. The variation of this embedding gives rise to the modulus $\hat{z}$ which is interpreted as the open string modulus. Also the notion of relative cohomology is introduced and a filtration is found. The variation of this filtration is described by the variation of mixed Hodge structure which allows in addition a separation of the variation into two variations of pure Hodge structures. From the variation Picard-Fuchs equations can be derived which govern the relative periods, schematically the variation diagram looks as follows

$$\begin{array}{ccccccc} F^3H^3(Z^*) & \stackrel{\delta_z}{\longrightarrow} & F^2H^3(Z^*) & \stackrel{\delta_z}{\longrightarrow} & F^1H^3(Z^*) & \stackrel{\delta_z}{\longrightarrow} & F^0H^3(Z^*) \\ & \searrow^{\delta_{\hat{z}}} & & \searrow^{\delta_{\hat{z}}} & & \searrow^{\delta_{\hat{z}}} & \downarrow^{\delta_{\hat{z}}} \\ & & F^2H^2(\mathcal{H}) & \xrightarrow{\delta_z,\delta_{\hat{z}}} & F^1H^2(\mathcal{H}) & \xrightarrow{\delta_z,\delta_{\hat{z}}} & F^0H^2(\mathcal{H}) \end{array} \qquad (2.64)$$

More details on this structure and the computations associated to it will be given in the fourth chapter of this thesis.

3 Holomorphic Anomaly

The central theme of this chapter are the holomorphic anomaly equations of Bershadsky, Cecotti, Ooguri and Vafa (BCOV)[12, 13]. These relate recursively the topological string amplitudes at genus g, $\mathcal{F}^g$ with amplitudes of lower genus and provide thus the central ingredient of the polynomial construction of the higher genus amplitudes in a later chapter of the thesis. The derivation of BCOV is based on a worldsheet analysis of the SCFT underlying topological string theory. It was previously shown that one can construct four topological string theories out of the SCFT depending on which finite chiral ring one wants to restrict the attention to. These were the A/*anti*-A-model and the B/anti-B-models corresponding to the $(a,c)/(c,a)$ and $(c,c)/(a,a)$ rings respectively. The key insight which allows the derivation of the equations is the failure of decoupling of deformations coming from the anti-ring. For instance, deformations coming from the (a,a) ring will have an effect on the amplitudes of the B-model which is supposed to be only deformed by marginal fields coming from the (c,c) ring. The starting point for the analysis are the tt^* equations of Cecotti and Vafa [58]. The setup for these is to reconsider the bundle of ground-states discussed previously and to replace half of the states with states coming from worldsheet CPT conjugate operators. In terms of the variation of Hodge structures the interplay of the two theories leads to a consideration of the non-holomorphic variation of the Hodge structure instead of the filtration which varies holomorphically as shown. The set of equations obtained from the variation are familiar in the language of supergravity where they reflect the *special geometry* of the manifold of the scalars of the vector multiplets. [1]

3.1 Special geometry

3.1.1 tt^* equations

To state the tt^* equations, half of the states in the vacuum bundle are replaced by states coming from the worldsheet CPT conjugate operators. The states created by the charge $(2,2)$ and $(3,3)$ operators that were introduced with upper indices with respect to the topological metric are replaced using the so called tt^* metric which is obtained using the

[1]See [59, 60] for an overview.

worldsheet CPT operator Θ [2],[3]

$$g_{a\bar{b}} = \langle e_a | \Theta e^b \rangle \,. \tag{3.1}$$

Using this metric a new basis for the dual states can be defined as follows

$$|e_{\bar{i}}\rangle = g_{k\bar{i}} |e^k\rangle \,, \quad |e_{\bar{0}}\rangle = g_{0\bar{0}} |e^0\rangle \,. \tag{3.2}$$

This tt^* metric induces a connection on the bundle $\mathcal{V}$ which is compatible with the holomorphic structure. It is given by

$$D_i(|e_a\rangle) = \partial_i(|e_a\rangle) - (A_i)_a^b(|e_b\rangle) \,, \tag{3.3}$$

where $(A_i)_b^a = g^{b\bar{c}}\partial_i g_{a\bar{c}}$ and $(|e_a\rangle)$ denotes the vector $(|e_0\rangle, |e_i\rangle, |e_{\bar{i}}\rangle, |e_{\bar{0}}\rangle)^T$, in matrix form the connection reads

$$A_i = \begin{pmatrix} g^{0\bar{0}}\partial_i g_{0\bar{0}} & 0 & 0 & 0 \\ 0 & g^{l\bar{j}}\partial_i g_{k\bar{j}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \,, \quad A_{\bar{i}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & g^{l\bar{j}}\partial_{\bar{i}} g_{k\bar{j}} & 0 \\ 0 & 0 & 0 & g^{0\bar{0}}\partial_{\bar{i}} g_{0\bar{0}} \end{pmatrix} \,. \tag{3.4}$$

Moreover the action of the chiral ring operators $\phi_i, \phi_{\bar{i}}$ becomes in this basis

$$\phi_i \begin{pmatrix} |e_0\rangle \\ |e_j\rangle \\ |e_{\bar{j}}\rangle \\ |e_{\bar{0}}\rangle \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \delta_i^k & 0 & 0 \\ 0 & 0 & C_{ijk}g^{k\bar{k}} & 0 \\ 0 & 0 & 0 & g_{i\bar{j}} \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{:=C_i} \begin{pmatrix} |e_0\rangle \\ |e_k\rangle \\ |e_{\bar{k}}\rangle \\ |e_{\bar{0}}\rangle \end{pmatrix} \,, \tag{3.5}$$

and

$$\phi_{\bar{i}} \begin{pmatrix} |e_0\rangle \\ |e_j\rangle \\ |e_{\bar{j}}\rangle \\ |e_{\bar{0}}\rangle \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ g_{\bar{i}j} & 0 & 0 & 0 \\ 0 & C_{\bar{i}\bar{j}\bar{k}}g^{\bar{k}k} & 0 & 0 \\ 0 & 0 & \delta_{\bar{i}}^{\bar{k}} & \end{pmatrix}}_{:=C_{\bar{i}}} \begin{pmatrix} |e_0\rangle \\ |e_k\rangle \\ |e_{\bar{k}}\rangle \\ |e_{\bar{0}}\rangle \end{pmatrix} \,. \tag{3.6}$$

As pointed out previously the ground-state $|e_0\rangle$ of the SCFT is unique up to scale. This means that it is a section of a line bundle $\mathcal{L}$ over the moduli space $\mathcal{M}$. Moreover, as the $|e_i\rangle$ states are created by the operators which also parameterize the deformations of the theory, these can be identified with sections of $\mathcal{L} \otimes T\mathcal{M}$. The reason they are also sections of the line bundle is because they inherit the arbitrary scale from the ground-state. The bundle of ground-states can hence be organized as

$$V_{\mathbb{C}} = \mathcal{L} \oplus \mathcal{L} \otimes T\mathcal{M} \oplus \overline{\mathcal{L} \otimes T\mathcal{M}} \oplus \overline{\mathcal{L}}. \tag{3.7}$$

[2]The notation here follows [22].

[3]Letters from the beginning of the alphabet are running from $0, \ldots, n$ while letter from the middle of the alphabet are only running over $1, \ldots, n$.

The tt^* equations [58][4] can now be spelled out

$$
\begin{array}{rclrcl}
[D_i, D_j] &=& 0, & [D_{\bar{i}}, D_{\bar{j}}] &=& 0, \\
[D_i, C_j] &=& [D_j, C_i], & [D_{\bar{i}}, C_{\bar{j}}] &=& [C_{\bar{i}}, D_{\bar{j}}], \\
\big[D_i, D_{\bar{j}}\big] &=& -[C_i, C_{\bar{j}}]\,. & & &
\end{array}
\tag{3.8}
$$

The last one of these equations is particularly interesting as it allows to define a modified connection which has vanishing curvature

$$
\nabla_i = D_i + C_i\,, \quad \nabla_{\bar{i}} = D_{\bar{i}} + C_{\bar{i}}\,, \quad [\nabla_i, \nabla_{\bar{i}}] = 0\,. \tag{3.9}
$$

This connection is the Gauss-Manin connection which was already encountered in the discussion of topological field theory and variation of Hodge structures.

3.1.2 Geometry of $\mathcal{M}$

The tt^* metric on $V_{\mathbb{C}}$ can further be used to define a metric on $\mathcal{M}$ by restricting to the $T\mathcal{M}$ valued part of the metric and its complex conjugate. The metric obtained in this way is the Zamolodchikov metric which is defined by

$$
G_{i\bar{j}} = \frac{g_{i\bar{j}}}{g_{0\bar{0}}}\,. \tag{3.10}
$$

Setting further

$$
g_{0\bar{0}} := e^{-K}\,,
$$

it is shown that $G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$ and hence the Zamolodchikov metric is Kähler. Furthermore, the last equation of (3.8) can be translated into the following statement about the curvature of $G_{i\bar{j}}$

$$
R_{i\bar{i}\,\,j}^{\,\,\,\,l} = [\bar{\partial}_{\bar{i}}, D_i]^l{}_j = \bar{\partial}_{\bar{i}} \Gamma^l_{ij} = \delta^l_i G_{j\bar{i}} + \delta^l_j G_{i\bar{i}} - C_{ijk} C^{kl}_{\bar{i}}, \tag{3.11}
$$

where D is the covariant derivative with connection parts which follow from the context

$$
\Gamma^k_{ij} = G^{k\bar{k}} \partial_i G_{\bar{k}j}, \quad \text{and} \quad K_i = \partial_i K,
$$

for the cotangent bundle and the line bundle respectively and

$$
\overline{C}^{jk}_{\bar{i}} := e^{2K} G^{k\bar{k}} G^{l\bar{l}} \overline{C}_{\bar{i}\bar{k}\bar{l}}.
$$

The Kähler condition together with the condition on the curvature makes the manifold $\mathcal{M}$ a *special* Kähler manifold, the geometry of which is called special geometry. Before revisiting the B-model realization of this whole structure some further notions will be

[4] For a pedagogical derivation see [30].

introduced on the abstract level. The Yukawa coupling in this setting is a holomorphic section of $\mathcal{L}^2 \otimes \mathrm{Sym}^3 T^*\mathcal{M}$. Also the filtration structure can be introduced

$$0 \subset F^3 V_{\mathbb{C}} \subset F^2 V_{\mathbb{C}} \subset F^1 V_{\mathbb{C}} \subset F^0 V_{\mathbb{C}} = V_{\mathbb{C}}, \tag{3.12}$$

where $F^k V_{\mathbb{C}}$ denotes the sum of the first 4-k summands in (3.7). $F^2 V_{\mathbb{C}}$ is a real sub-bundle. There exists further structure in $V_{\mathbb{C}}$ namely a lattice $V^*_{\mathbb{Z}} \subset V^*_{\mathbb{R}}$ which can be used to define the $2n+2$ *periods* given a local section Ω of $\mathcal{L}$:

$$X^I = \alpha^I(\Omega), \quad F_I = \beta_I(\Omega)\,. \tag{3.13}$$

It is also convenient at this stage to introduce a new ingredient needed for the extension of the holomorphic anomaly discussion to include D-branes which carry no open string moduli [22]. This is the normal function ν which is a section of [5]

$$(F^2 V_{\mathbb{C}})^* / V^*_{\mathbb{Z}},$$

this can be used to define the domainwall tension $\mathcal{T}$

$$\mathcal{T} = \nu(\Omega). \tag{3.14}$$

Now the open string analog of the Yukawa coupling, the disk two point function

$$\mathcal{F}^{(0,1)}_{ij} := \Delta_{ij}\ , \tag{3.15}$$

can be defined as

$$\Delta_{ij} = D_i D_j \mathcal{T} - C_{ijk} e^K G^{k\bar{k}} D_{\bar{k}} \bar{\mathcal{T}}. \tag{3.16}$$

It is not holomorphic but obeys the following equation

$$\bar{\partial}_{\bar{i}} \Delta_{ij} = -C_{ijk} \Delta^k_{\bar{i}}, \qquad \Delta^k_{\bar{i}} = \Delta_{\bar{i}\bar{j}} e^K G^{k\bar{j}}. \tag{3.17}$$

Here $\Delta_{\bar{i}\bar{j}}$ denotes the complex conjugate of Δ_{ij} .

3.1.3 Variation of Hodge structures revisited

Here it will be shown that the tt^* formulation of the geometry of the vacuum bundle corresponds on the B-model side to the non-holomorphic variation of the Hodge bundle

$$V_{\mathbb{C}} = H^3(X,\mathbb{C}) = H^{3,0}(X) \oplus H^{2,1}(X) \oplus H^{1,2}(X) \oplus H^{0,3}(X)\,. \tag{3.18}$$

[5] See [22] for more details.

Taking as section of $\mathcal{L}$ the unique up to scale holomorphic $(3,0)$ form $\Omega \in H^{3,0}(X)$ and the inner product which is given by taking wedge products of forms and integrating over X, the metric on the line bundle is

$$g_{0\bar{0}} = e^{-K} = -\int_X \Omega \wedge \overline{\Omega}\,. \tag{3.19}$$

One can define elements in $H^{2,1}$ and $H^{1,2}$ in the following way

$$\chi_i := D_i\Omega = (\partial_i + K_i)\,\Omega\,, \quad \overline{\chi_{\bar{i}}} := D_{\bar{i}}\overline{\Omega} = (\partial_{\bar{i}} + K_{\bar{i}})\,\overline{\Omega}\,. \tag{3.20}$$

With inner product

$$g_{i\bar{j}} = \int_X \chi_i \wedge \overline{\chi_{\bar{j}}} = e^{-K} G_{i\bar{j}} = e^{-K}\partial_i\partial_{\bar{j}} K\,. \tag{3.21}$$

Defining the Yukawa coupling

$$C_{ijk} := -\int_X \Omega \wedge D_i D_j D_k \Omega = -\int_X \Omega \wedge \partial_i\partial_j\partial_k\Omega\,, \tag{3.22}$$

it is easy to find

$$D_i D_j \Omega = D_i \chi_j = C_{ijk} e^K G^{k\bar{k}} \overline{\chi_{\bar{k}}}\,, \quad D_i \overline{\chi_{\bar{i}}} = G_{i\bar{i}}\overline{\Omega}\,. \tag{3.23}$$

Taking as a section of $V_{\mathbb{C}}$ the following vector

$$\mathbf{\Omega} = (\Omega, \chi_i, \overline{\chi_{\bar{i}}}, \overline{\Omega})\,, \tag{3.24}$$

one recovers the Gauss-Manin connection

$$D_i \begin{pmatrix} \Omega \\ \chi_j \\ \overline{\chi_{\bar{j}}} \\ \overline{\Omega} \end{pmatrix} = \begin{pmatrix} 0 & \delta_i^k & 0 & 0 \\ 0 & 0 & C_{ijk}e^K G^{k\bar{k}} & 0 \\ 0 & 0 & 0 & G_{i\bar{j}} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Omega \\ \chi_k \\ \overline{\chi_{\bar{k}}} \\ \overline{\Omega} \end{pmatrix}\,. \tag{3.25}$$

This matrix equation can be translated into the following fourth order differential equation for Ω

$$D_m D_n (C_i^{-1})^{lj} D_i D_j \Omega = 0 \tag{3.26}$$

which is the Picard-Fuchs equation. In ref.[61, 62] it is shown that this equation is equivalent to the holomorphic Picard-Fuchs equations encountered previously and that choosing the non-holomorphic variation of Hodge structures is merely a choice of gauge for the vector $\mathbf{\Omega}$.

To proceed, a realization of the additional structure of a normal function is given [22]. The basic example of normal function comes from a pair of homologically equivalent holomorphic curves C_+, C_- in X varying over $\mathcal{M}$. Over every point in $\mathcal{M}$ one can pick a three chain Γ in X such that $\partial\Gamma = C_+ - C_-$. An element ν of $(F^2V_{\mathbb{C}})^* \simeq (H^{3,0}\oplus H^{2,1})^*$ is given by associating to any $(3,0)$ or $(2,1)$ form ω the chain integral

$$\langle \omega, \nu \rangle = \int_\Gamma \omega\,, \tag{3.27}$$

this is known under the name of Abel-Jacobi map.

3.2 Holomorphic anomaly

3.2.1 Anomaly equations

It was argued that deformations of the topological string theory corresponds to the addition of terms of the form

$$z^i \int \phi_i^{(2)} + \bar{z}^{\bar{i}} \int \bar{\phi}_{\bar{i}}^{(2)},$$

to the action S. z^i are complex coordinates on $\mathcal{M}$, the notation change from t^i to z^i is to make manifest the difference between the *topological* coordinates t^i for which taking derivatives corresponds to insertions and the more general coordinates z^i for which insertions correspond to taking covariant derivatives. Hence amplitudes with insertions of n descendants of marginal (c,c) fields (for the B-model) can be defined by [13]

$$\mathcal{F}^{(g)}_{i_1 \dots i_n} = D_{i_n} \mathcal{F}^{(g)}_{i_1 \dots i_{n-1}} \,. \tag{3.28}$$

Taking a derivative with respect to $\bar{z}^{\bar{i}}$ on the other hand corresponds to inserting a descendant of marginal (a, a) operator

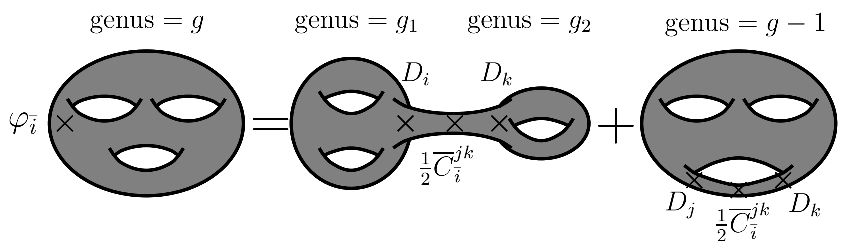
$$\phi_{\bar{i}}^{(2)} = dz d\bar{z} \{G^+, [\overline{G}^+, \phi_{\bar{i}}]\} = -\frac{1}{2} dz d\bar{z} \{G^+ + \overline{G}^+, [G^+ - \overline{G}^+, \phi_{\bar{i}}]\} \,. \tag{3.29}$$

Within topological *field* theory such an insertion in a correlator would vanish as the second part of the equation shows that the operator is Q_B exact. The field theory argument for this decoupling is that the operator Q_B is a symmetry of the theory and can be commuted through the correlator until it acts on a ground-state and gives zero. The failure of this argument in the case of topological string theory is outlined in the following. The formula for $\mathcal{F}^g$ is given again

$$\mathcal{F}^g = \int_{\mathcal{M}_g} [dm\, d\bar{m}] \langle \prod_{a=1}^{3g-3} (\int_\Sigma \mu_a G^-)(\int_\Sigma \mu_{\bar{a}} \overline{G}^-) \rangle_{\Sigma_g} \,, \tag{3.30}$$

one can see that while commuting G^+ and $\overline{G}^+$ through the correlation function defining $\mathcal{F}^g$ that these will hit the G^- and $\overline{G}^-$ which are contracted with the Beltrami differentials, this will give by the superconformal algebra the energy momentum tensor T. The energy momentum tensor in turn measures the response of the theory to a variation of the metric and can hence be traded for derivatives $\frac{\partial^2}{\partial m \partial \overline{m}}$on the moduli space of Riemann surfaces $\mathcal{M}_g$ as this latter space precisely parameterizes such changes. One is thus finally left with an integral of a derivative with respect to the moduli of Riemann surfaces over the moduli space of Riemann surfaces which is zero up to boundary contributions. It is exactly these contributions coming from the boundary of the moduli space of Riemann surfaces which is arranged in the holomorphic anomaly equations. These are pictured diagrammatically in the following, for the full derivation of the equations reference is given to the original work [12, 13]. The equations read

$$\bar{\partial}_{\bar{i}}\mathcal{F}^{(g)} = \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\left(\sum_{g_1+g_2=g} D_j\mathcal{F}^{(g_1)}D_k\mathcal{F}^{(g_2)} + D_jD_k\mathcal{F}^{(g-1)}\right), \tag{3.31}$$



where $g > 1$ and

$$\bar{C}_{\bar{k}}^{ij} = \bar{C}_{\bar{i}\bar{j}\bar{k}}G^{i\bar{i}}G^{j\bar{j}}\,\mathrm{e}^{2K}, \qquad \bar{C}_{\bar{i}\bar{j}\bar{k}} = \overline{C_{ijk}}. \tag{3.32}$$

For $g = 1$ the equation is

$$\bar{\partial}_{\bar{i}}\mathcal{F}_j^{(1)} \;=\; \frac{1}{2}C_{jkl}C_{\bar{i}}^{kl} + (1-\frac{\chi}{24})G_{j\bar{i}}\,. \tag{3.33}$$

3.2.2 Solving the anomaly equations

BCOV proposed together with the anomaly equation [13] a method to recursively solve for the $\mathcal{F}^{(g)}$ s. The idea is to partially integrate the anti-holomorphic derivative appearing in the anomaly equation until one ends up with an expression of the form

$$\partial_{\bar{i}}(\text{expression1}) = \partial_{\bar{i}}(\text{expression2}) \Rightarrow \text{expression1} = (\text{expression2}) + f(z),$$

where $f(z)$ is the holomorphic ambiguity. To be able to do this in practice they first note that the anti-holomorphic Yukawa coupling can locally be written as

$$\overline{C}_{\bar{i}\bar{j}\bar{k}} = e^{-2K}D_{\bar{i}}D_{\bar{j}}\overline{\partial}_{\bar{k}}S,$$

where S is a section of $\mathcal{L}^{-2}$. One can further introduce some objects S^{ij}, S^i which are sections of $\mathcal{L}^{-2}\otimes \mathrm{Sym}^2(T^*\mathcal{M})$ and of $\mathcal{L}^{-2}\otimes T^*(\mathcal{M})$ such that

$$\partial_{\bar{i}}S^{ij} = \bar{C}_{\bar{i}}^{ij}, \qquad \partial_{\bar{i}}S^j = G_{i\bar{i}}S^{ij}, \qquad \partial_{\bar{i}}S = G_{i\bar{i}}S^i. \tag{3.34}$$

Now the procedure is to successively partially integrate the anomaly equation as follows, first write

$$\begin{aligned}\bar{\partial}_{\bar{i}}\big(\mathcal{F}^{(g)} - \frac{1}{2}S^{jk}(\sum_{g_1+g_2=g} D_j\mathcal{F}^{(g_1)}D_k\mathcal{F}^{(g_2)} + D_jD_k\mathcal{F}^{(g-1)})\big) \\ = -\frac{1}{2}S^{jk}\bar{\partial}_{\bar{i}}(\sum_{g_1+g_2=g} D_j\mathcal{F}^{(g_1)}D_k\mathcal{F}^{(g_2)} + D_jD_k\mathcal{F}^{(g-1)}\big),\end{aligned} \tag{3.35}$$

then the repeated use of the expression found for the curvature $[\overline{\partial}_{\bar{i}}, D_j]$ and the knowledge of $\overline{\partial}_{\bar{i}}\mathcal{F}^{\tilde{g}}$ with $\tilde{g} < g$ yields the desired result. They further remarked that the expressions found this way resemble Feynman diagrams if one considers a quantum system with one more degree of freedom than the dimension of the moduli space $\mathcal{M}$, where the propagators are given by

$$K^{ij} = -S^{ij}, \qquad K^{i\phi} = -S^{i}, \quad \text{and} \quad K^{\phi\phi} = -2S,$$

and vertices are given by the correlation functions with insertions. Furthermore a proof was given for the expansion in terms of Feynman diagrams.
One of the practical shortcomings of this procedure is however that the number of iterations grows exponentially, expressions for the $\mathcal{F}^{(g)}$ become very long. On the other hand the recursive information contained in the anomaly equation needs to be supplemented by boundary conditions at every step in order to fix the holomorphic ambiguities.

Yamaguchi and Yau's proposal

Yamaguchi and Yau proposed in [23] that the higher genus amplitudes of the topological string for the mirror quintic which has a one dimensional space of deformations of complex structures can be written as polynomials of degree $3g-3+n$ where n refers to the number of insertions, in a finite number of generators. Hints of a polynomial structure of topological string amplitudes were already contained in [63, 64], where the polynomial building blocks are modular forms. As generators of the polynomials Yamaguchi and Yau used typical building blocks that appear on the r.h.s of the holomorphic anomaly equation namely multi-derivatives of the connections. The non-holomorphic generators consist of

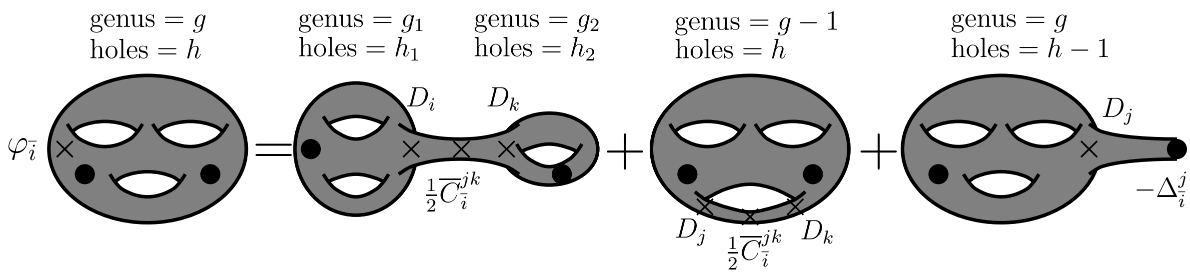
$$A_p = G^{z\bar{z}}(z\partial_z)^p G_{z\bar{z}} \quad \text{and} \quad B_p = e^{K}(z\partial_z)^p e^{-K}, \qquad p = 1, 2, 3, \ldots$$

As a holomorphic generator they take $X \sim \frac{1}{\Delta}$ with Δ discriminant.
In the next step they show relations between the generators such that the infinite number of non-holomorphic generators can be reduced to A_1, B_1, B_2 and B_3. Furthermore from the analysis of the holomorphic anomaly equation they further show that only special combinations of the generators appear in the topological string amplitudes and thus the number of non-holomorphic generators gets reduced by one.

3.2.3 Extension of the anomaly equation

The extension by Walcher [22] of the holomorphic anomaly equations to include D-branes which are fixed at the critical locus of a superpotential leads to the following modification of the anomaly equations

$$\bar{\partial}_{\bar{i}}\mathcal{F}^{(g,h)} = \frac{1}{2}\bar{C}_{\bar{i}}^{jk} \sum_{\substack{g_1+g_2=g \\ h_1+h_2=h}} D_j\mathcal{F}^{(g_1,h_1)} D_k\mathcal{F}^{(g_2,h_2)} + \frac{1}{2}\bar{C}_{\bar{i}}^{jk} D_j D_k\mathcal{F}^{(g-1,h)} - \Delta_{\bar{i}}^{j} D_j\mathcal{F}^{(g,h-1)}, \quad (3.36)$$



furthermore

$$\bar{\partial}_{\bar{i}}\mathcal{F}_j^{(0,2)} = -\Delta_{jk}\Delta_{\bar{i}}^k + \frac{N}{2}G_{j\bar{i}}\,. \tag{3.37}$$

The holomorphic anomaly equations were also derived [13] for the amplitudes with insertions $\mathcal{F}^{(g,h)}_{i_1\ldots i_n}$. Furthermore through the introduction of a generating function for all amplitudes, the holomorphic anomaly equations can be summarized into a master anomaly equation, here the conventions of [65] will be used which will be useful for later discussions. The generating function is

$$\Psi(t^i,\bar{t}^{\bar{i}};x^i,\lambda^{-1}) = \lambda^{\frac{\chi}{24}-1-\mu^2\frac{N}{2}}\exp\left[\sum_{g,h,n}\frac{\lambda^{2g+h+n-2}}{n!}\mu^h\mathcal{F}^{(g,h)}_{i_1\ldots i_n}x^{i_1}\ldots x^{i_n}\right], \tag{3.38}$$

the anomaly equation now reads

$$\left[\partial_{\bar{i}} - \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\frac{\partial^2}{\partial x^j\partial x^k} - G_{j\bar{i}}x^j\frac{\partial}{\partial\lambda^{-1}} - i\mu\bar{\Delta}_{\bar{i}}^j\frac{\partial}{\partial x^j}\right]\Psi = 0, \tag{3.39}$$

also, the equations $D_{i_n}\mathcal{F}^{(g,h)} = \mathcal{F}^{(g,h)}_{i_n}$ can be written as

$$\begin{aligned}&[\partial_i - \Gamma^k_{ij}x^j\frac{\partial}{\partial x^k} + \partial_i K(\lambda^{-1}\frac{\partial}{\partial\lambda^{-1}} + x^k\frac{\partial}{\partial x^k} + \frac{\chi}{24} - 1 - \mu^2\frac{N}{2})\\ &-\lambda^{-1}\frac{\partial}{\partial x^i} + \mathcal{F}_i^{(1,0)} + \mu^2\mathcal{F}_i^{(0,2)} + \frac{1}{2}C_{ijk}x^jx^k + \mu\Delta_{ij}x^j]\Psi = 0.\end{aligned} \tag{3.40}$$

Topological tadpole cancellation

In [66, 67] Gopakumar and Vafa conjectured that the partition function of the closed topological string can be interpreted as counting BPS states in M-theory compactified to five dimensions on a Calabi-Yau manifold. This conjecture was extended to cases with D-branes in ref. [55]. In ref. [68] evidence was found for the necessity of cancellation of total D-brane charge. Through localization computation it was shown that the integrality of some invariants associated to the given D-brane background could not be achieved worldsheet by worldsheet but by summing over world-sheets with a fixed Euler characteristic χ given by

$$\chi = 2g + h + c - 2, \tag{3.41}$$

where g, h, c refer to the genus, number of holes and number of crosscaps respectively. The perturbation series of the topological string is reorganized as follows

$$\mathcal{G} = \sum_{\chi} \lambda^{\chi} \mathcal{G}^{(\chi)}, \tag{3.42}$$

where $\mathcal{G}^{(\chi)}$ is given by

$$\mathcal{G}^{(\chi)} = \frac{1}{2^{\frac{\chi}{2}+1}} \left[\mathcal{F}^{(g_\chi)} + \sum_{2g+h-2=\chi} \mathcal{F}^{(g,h)} + \sum_{2g+h-1=\chi} \mathcal{R}^{(g,h)} + \sum_{2g+h-2=\chi} \mathcal{K}^{(g,h)} \right]. \tag{3.43}$$

It was further found that the $\mathcal{G}^{(\chi)}$ satisfy essentially the same equation as the proposed extension of the holomorphic anomaly equation.

In [69] a worldsheet derivation of the effects of a non-vanishing disc one-point function was worked out. It was already shown in [70] that disk one-point functions depend on moduli of the wrong model, i.e., on Kähler moduli in the B-model, as D-branes are described by holomorphic even cycles and complex structure moduli in the A-model as D-branes are described by Lagrangian sub-manifolds. It was found that it spoils the recursiveness of the anomaly equations is spoiled by a non-vanishing of the disk one-point functions.

3.3 Background independence

Background dependence is a longstanding problem of the formulation of string theory. It refers to an explicit choice of target space data which is needed to formulate perturbation theory like for example the background metric. On the level of two dimensional field theories these choices reflect different values of couplings.

Background dependence in the case of topological string theory translates into an explicit dependence of some reference expansion point in the moduli space of theories. This dependence is captured by the anomaly equations that were shown previously [12, 13]. However in a very elegant paper, Witten [14] reinterpreted the holomorphic anomaly equation as actually dictating background independence of the topological string, if one reinterprets the full partition function of the topological string as being a wave-function in some Hilbert space that is obtained by geometric quantization of $H^3(X, \mathbb{R})$. In this case the holomorphic anomaly equation can be interpreted as invariance under parallel transport between Hilbert spaces corresponding to different choices of background which are all identified with each other.

3.3.1 Wave-function property

Here the idea of Witten's paper [14] will be briefly reviewed. The idea is to examine geometric quantization[6] of $H^3(X,\mathbb{R})$ as a symplectic phase space $\mathcal{W}$. Quantization of the classical phase space requires a choice of polarization and there is no natural or background independent choice of polarization. Given a complex structure J on X, $\mathcal{W}$ gets a complex structure which depends on J. Then the Hilbert space $\mathcal{H}_J$ is constructed as a suitable space of holomorphic functions over $\mathcal{W}$. A wave function will be denoted by $\psi(t'^a;z^i)$ where z^i are complex coordinates on $\mathcal{W}$ and t'^a are coordinates parameterizing the choice of J, i.e., coordinates on $\mathcal{M}$. Now background independence does not hold in a naive sense, i.e. ψ cannot be independent of t'^a. Background independence can be formulated in another way. The $\mathcal{H}_J$ can be identified with each other projectively using a projectively flat connection over the space of J's. This connection ∇ is such that a covariantly constant wave function should have the following transformation property: as J changes, ψ changes by a Bogoliubov transformation. Using ∇ to identify various $\mathcal{H}_J$ one can speak of *the* quantum Hilbert space, background independence of ψ would now mean that ψ is covariantly under parallel transport by ∇. This can be written as an equation

$$0=\left(\frac{\partial}{\partial t'^a}-\frac{1}{4}\left(\frac{\partial\mathcal{J}}{\partial t'^a}\omega^{-1}\right)^{ij}\frac{D}{Dz^i}\frac{D}{Dz^j}\right)\psi\,, \tag{3.44}$$

which is essentially the same equation as (3.39) for the case of the closed string $\bar{\Delta}_{\bar{i}}^{j}=0$.

3.3.2 Large phase space anomaly

The coordinates that appear in the total holomorphic anomaly equation are $(t^i;x^i,\lambda^{(-1)})$, they parameterize $(\mathcal{L}\oplus\mathcal{L}\otimes T\mathcal{M})\to\mathcal{M}$. One can trade these coordinates for coordinates (X^I,z^I) on $T\tilde{\mathcal{M}}$ where $\tilde{\mathcal{M}}$ denotes the total space of the line bundle $\mathcal{L}$ minus the zero section. $X^I(t^i)$ are given by period maps and

$$z^I=2(\lambda^{-1}X^I+x^iD_iX^I).$$

Using these coordinates the relation of the holomorphic anomaly equation to a heat equation becomes manifest. The large phase space was introduced in [74, 75], the relation to heat equations was also studied in [76]. This section closely follows [65].

In the large phase space the holomorphic data becomes

$$F:=\frac{1}{2}X^IF_I,\quad F_I=\partial_IF,\quad \tau_{IJ}:=\partial_I\partial_JF,\quad C_{IJK}:=\partial_I\partial_J\partial_KF,\quad \partial_I:=\frac{\partial}{\partial X^I}. \tag{3.45}$$

[6] [71, 72, 73]are some references for background material on geometric quantization.

Open string data is built from the domain wall tension $\mathcal{T}$

$$\mathcal{T} = X^I \nu_I, \quad \nu_I = \partial_I \mathcal{T}, \quad \Delta_{IJ} = \nu_{IJ} - C_{IJ}^K \mathrm{Im}\nu_K. \tag{3.46}$$

The holomorphic anomaly equation now becomes in large phase space:

$$\left[\bar{\partial}_I - \frac{1}{2}\bar{C}_I^{JK}\frac{\partial^2}{\partial z^J \partial z^K} + i\mu\bar{\Delta}_I^J\frac{\partial}{\partial z^J}\right]\Psi = 0\,. \tag{3.47}$$

$$\begin{aligned}[\partial_I - \frac{1}{2}\partial_I \log\det \mathrm{Im}\tau_{IJ} + \frac{i}{2}C_{IJ}^K z^J \frac{\partial}{\partial z^K} + \frac{1}{8}C_{IJK}z^J z^K \\ +\frac{1}{2}\mu\Delta_{IJ}z^J - \frac{i}{2}\mu^2\Delta_{IJ}\Delta^J + \frac{1}{8}\mu^2 C_{IJK}\Delta^J\Delta^K]\Psi = 0\,.\end{aligned} \tag{3.48}$$

After absorbing some factors in the definition of Ψ and variable redefinition

$$\bar{y}_I = \mathrm{Im}\tau_{IJ}z^J\ , \tag{3.49}$$

going further to the conjugate $\Psi \to \overline{\Psi}$ the following relations are obtained

$$\left[\partial_I - \frac{1}{2}C_{IJK}\frac{\partial^2}{\partial y_J \partial y_K} - i\mu\nu_{IJ}\frac{\partial}{\partial y_J}\right]\Psi \;=\; 0\,, \tag{3.50}$$

$$\bar{\partial}_I\Psi \;=\; 0\,. \tag{3.51}$$

3.3.3 Shift relating open and closed strings

Rewriting the holomorphic anomaly in this may make explicit that by the following shift of variables

$$y_I \to y_I + i\mu\,\nu_I\,, \tag{3.52}$$

open string data can be removed from the equation. This shift maps the holomorphic anomaly equations to the ones of the closed string satisfying both equations.

The shifted open topological string partition function Ψ^ν satisfies the holomorphic anomaly equation of the closed string which means that it is a state in the same Hilbert space. However the conjecture of [65] is that it represents a different state and that the set of all possible D-branes give states that correspond to a basis of the Hilbert space.

4 Mirror Symmetry for Toric Branes on Compact Hypersurfaces

We use toric geometry to study open string mirror symmetry on compact Calabi–Yau manifolds. For a mirror pair of toric branes on a mirror pair of toric hypersurfaces we derive a canonical hypergeometric system of differential equations, whose solutions determine the open/closed string mirror maps and the partition functions for spheres and discs. We define a linear sigma model for the brane geometry and describe a correspondence between dual toric polyhedra and toric brane geometries. The method is applied to study examples with obstructed and classically unobstructed brane moduli at various points in the deformation space. Computing the instanton expansion at large volume in the flat coordinates on the open/closed deformation space we obtain predictions for enumerative invariants. This work is published in [26].

4.1 Introduction

Mirror symmetry has been the subject of intense research over many years and its study remains rewarding. Whereas the early works focused on the closed string sector and the Calabi–Yau (CY) geometry, the interest has shifted to the interpretation of mirror symmetry as a duality of D-brane categories and the associated open string sector [16]. One object of particular interest is the disc partition function $\mathcal{F}^{0,1}$ for an A brane on a compact CY 3-fold, which depends on the Kähler type deformations of the brane geometry and is an important datum for the definition of the category of A branes. If a modulus is classically unobstructed, the large volume expansion of the disc partition function captures an interesting enumerative problem of "counting" holomorphic discs that end on the A brane. In a certain parametrization motivated by physics, the coefficients of this instanton expansion in the A model are predicted to be the *integral* Ooguri-Vafa invariants [55].

One of the virtues of mirror symmetry, first demonstrated for the sphere partition function in [6] and for the disc partition functions in [18], is the ability to compute the instanton expansion of the A model partition function in the mirror B model. The disc partition function relates on the B model side to the holomorphic Chern-Simons functional on the CY Z^* [77]

$$S(Z^*, A) = \int_{Z^*} \mathrm{tr}(\frac{1}{2}A \wedge dA + \frac{1}{3}A \wedge A \wedge A) \wedge \Omega \ . \qquad (4.1)$$

In the physical string theory S represents a space-time superpotential obstructing some of the moduli of the brane geometry and the instanton expansion of the A model is, under certain conditions, the non-perturbative superpotential generated by space-time instantons [78, 55]. While the action of mirror symmetry on the moduli space and the computation of superpotentials is well understood for non-compact brane geometries[1], the physically interesting case of branes on compact CY 3-folds has been elusive. Starting with [79], superpotentials for a class of involution branes without open string moduli have been studied in [80, 68, 81, 82, 83]. The definition of the Lagrangian A brane geometry as the fixed point of an involution has various limitations: It allows to study only discrete brane moduli compatible with the involution and the instanton invariants computed by the superpotential are not generic disc invariants, but rather the number of real rational curves fixed by the involution [84].

The present lack of a systematic description of the geometric deformation space in the compact case is a serious obstacle to the general study of open string mirror symmetry on compact manifolds, in particular the computation of superpotentials and mirror maps for more general deformations including open string moduli. For the closed string case without branes, a powerful approach to study mirror symmetry is given in terms of gauged linear sigma models and toric geometry [7, 9], in particular if combined with Batyrev's construction of dual manifolds via toric polyhedra [50].[2] A similar description of open string mirror symmetry has been given for non-compact branes in [19, 21, 20], starting from the definition of toric branes of ref. [18]. A first important step to generalize these concepts to the compact case has been made in [83] by applying the $\mathcal{N} = 1$ special geometry defined in [21, 20] to involution branes.

The class of toric branes defined in [18] (see also [85]) is much larger than the class of involution branes and allows for relatively generic deformations. The purpose of this note is to describe a toric geometry approach to open string mirror symmetry for toric branes on compact manifolds. Specifically we consider mirror pairs (Z, L) and (Z^*, E), where Z and Z^* is a mirror pair of compact CY 3-folds described as hypersurfaces in toric varieties, and L and E is a mirror pair of branes on these manifolds with a simple toric description.[3] For these toric brane geometries we derive in sect. 4.2 a canonical system of differential equations that determines the open/closed string mirror maps and the partition functions for spheres and discs at any point in the moduli space. The B model geometry for this Picard-Fuchs system relates to a certain gauged linear sigma model, which may be associated with an "enhanced" toric polyhedron $\Delta_\flat$. A dual pair of enhanced polyhedra $(\Delta_\flat, \Delta_\flat^\star)$ encodes the mirror pair of compact CY manifolds (Z, Z^*) *and* the mirror pair (L, E) of A and B branes on it, extending in some sense Batyrev's [50] correspondence between toric polyhedra and CY manifolds to the open string sector. In sect. 4.3 we apply

[1]See e.g. [36, 46] for a summary.

[2]We refer to [40, 30] for background material and references.

[3]In the following, L will denote the A brane wrapped on a Lagrangian submanifold and E the holomorphic bundle corresponding to a B brane.

this method to study some compact toric brane geometries with obstructed and classically unobstructed moduli. The phase structure of the linear sigma model can be used to define and study large volume phases of the brane geometry, where the superpotential has an instanton expansion in the classically unobstructed moduli. We compute the mirror maps and the superpotentials and find agreement with the integrality predictions of [55, 79] for both closed and open string deformations.

4.2 Toric brane geometries and differential equations

4.2.1 Toric hypersurfaces and branes

Our starting point will be a mirror pair of compact CY 3-folds (Z, Z^*) defined as hypersurfaces in toric varieties (W, W^*). By the correspondence of ref. [50], one may associate to the pair of manifolds (Z, Z^*) a pair of integral polyhedra (Δ, Δ^*) in a four-dimensional integral lattice Λ_4 and its dual Λ_4^*. The k integral points $\nu_i(\Delta)$ of the polyhedron Δ correspond to homogeneous coordinates x_i on the toric ambient space W and satisfy $M = h^{1,1}(Z)$ linear relations[4]

$$\sum_i l_i^a \nu_i = 0, \quad a = 1, ..., M\,.$$

The integral entries of the vectors l^a for fixed a define the weights l_i^a of the coordinates x_i under the $\mathbf{C}^*$ action

$$x_i \to (\lambda_a)^{l_i^a} x_i, \qquad \lambda_a \in \mathbf{C}^*\,,$$

generalizing the idea of a weighted projective space. Equivalently, the l_i^a are the $U(1)_a$ charges of the fields in the gauged linear sigma model (GLSM) associated with the toric variety [7]. The toric variety W is defined as $\mathbf{C}^k$ divided by the $(\mathbf{C}^*)^M$ action and deleting a certain exceptional subset Ξ of degenerate orbits, $W \simeq (\mathbf{C}^k - \Xi)/(\mathbf{C}^*)^M$.

In the context of CY hypersurfaces, W will be the total space of the anti-canonical bundle over a toric variety with positive first Chern class. The compact manifold $Z \subset W$ is defined by introducing a superpotential $W_Z = x_0\, p(x_i)$ in the GLSM, where x_0 is the coordinate on the fiber and $p(x_i)$ a polynomial in the $x_{i>0}$ of degrees $-l_0^a$. At large Kähler volumes, the critical locus is at $x_0 = p(x_i) = 0$ and defines the compact CY as the hypersurface $Z:\ p(x_i) = 0$ [7]. To be concrete, we will later study A branes on the following examples

[4] For simplicity we neglect points on faces of codimension one of Δ and assume that $h^{1,1}(W) = h^{1,1}(Z)$.

of CY hypersurfaces:

$$\begin{array}{lrrrrrrrr}
\mathbf{X}_5^{(1,1,1,1,1)}: & & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & \\
 & (l^1) = & (-5 & 1 & 1 & 1 & 1 & 1\,) & \\
 & & & & & & & & \\
\mathbf{X}_{18}^{(1,1,1,6,9)}: & & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
 & (l^1) = & (-6 & 3 & 2 & 1 & 0 & 0 & 0\,) \\
 & (l^2) = & (\ 0 & 0 & 0 & -3 & 1 & 1 & 1\,) \\
 & & & & & & & & \\
\mathbf{X}_9^{(1,1,1,3,3)}: & & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
 & (l^1) = & (-3 & 1 & 1 & 1 & 0 & 0 & 0\,) \\
 & (l^2) = & (\ 0 & 0 & 0 & -3 & 1 & 1 & 1\,)
\end{array} \tag{4.2}$$

As indicated by the notation, this is the familiar quintic in projective space $\mathbf{P}^4 = \mathbf{WP}^4_{1,1,1,1,1}$ in the first case and a degree 18 (9) hypersurface in a blow up of a weighted projective space $\mathbf{WP}^4_{1,1,1,6,9}$ ($\mathbf{WP}^4_{1,1,1,3,3}$) in the other two cases.[5]

On these toric manifolds we consider a certain class of mirror pairs of branes, defined in [18] by another set of N charge vectors $\hat{l}^a$ for the fields x_i.[6] The Lagrangian submanifold wrapped by the A brane L is described in terms of the vectors $\hat{l}^a$ by the equations

$$\sum_i \hat{l}_i^a |x_i|^2 = c_a, \qquad \sum_i v_b^i \theta^i = 0, \qquad \sum_i \hat{l}_i^a v_b^i = 0\ , \tag{4.3}$$

where $a, b = M+1, ..., M+N$. The N real constants c_a parametrize the brane position and the integral vectors v_b^i may be defined as a linearly independent basis of solutions to the last equation. As in [18] we restrict to special Lagrangians which requires that the extra charges add up to zero as well, $\sum_i \hat{l}_i^a = 0$.

Applying mirror symmetry as in [86, 50], the mirror manifold Z^* is defined in the toric variety W^* by the equations

$$p(Z^*) = \sum_i y_i, \qquad \prod_i y_i^{l_i^a} = z_a\ , \quad a = 1, ..., M. \tag{4.4}$$

The parameters z_a are the complex moduli of the hypersurface Z^* and classically related to the complexified Kähler moduli t_a of Z by $z_a = e^{2\pi i t_a}$. The precise relation $z_a = z_a(t_b)$ is called the mirror map and is generically complicated. In the open string sector, the mirror transformation of [86] maps the A brane (4.3) to a B brane E defined by the holomorphic equations [18]

$$\mathcal{B}_a(E):\ \prod_i y_i^{\hat{l}_i^a} - \hat{z}_a = 0, \qquad \hat{z}_a = \epsilon_a e^{-c_a},\ a = M+1, ..., M+N. \tag{4.5}$$

[5]The deleted set is $\Xi = \{x_i = 0, \forall i > 0\}$ for $\mathbf{P}^4$ and $\Xi = \{\{x_1 = x_2 = x_3 = 0\} \cup \{x_4 = x_5 = x_6 = 0\}\}$ in the other two cases. The toric polyhedra will be given in sect. 3.

[6]A hat will be sometimes used to distinguish objects from the open string sector.

The (possibly obstructed) complex open string moduli $\hat{z}_a$ arise from the combination of the phases ϵ_a dual to the gauge field background on the A brane and the parameters c_a in (4.3) [87].

The class of toric branes defined above is quite general and describes many interesting cases, in particular involution branes with an obstructed modulus as well as branes with classically unobstructed moduli. It is instructive to consider the quintic $\mathbf{X}_5^{(1,1,1,1,1)}$, which will be one of the manifolds studied in sect. 4.3. The manifold Z for the A model is defined by a generic degree 5 polynomial in $\mathbf{P}^4$, while the mirror manifold Z^* is given in terms of eq. (4.4) by the superpotential and relation[7]

$$p(Z^*) = \sum_{i=0}^{5} a_i y_i = 0, \qquad y_1 y_2 y_3 y_4 y_5 = y_0^5. \tag{4.6}$$

A change of coordinates $y_i = x_i^5,\ i = 1, ..., 5$ and a rescaling leads to the more familiar form in $\mathbf{P}^4$

$$p(Z^*) = \sum_{i=1}^{5} x_i^5 - \psi\, x_1 x_2 x_3 x_4 x_5 = 0, \qquad \psi^{-5} = -\frac{a_1 a_2 a_3 a_4 a_5}{a_0^5} \equiv z_1\ . \tag{4.7}$$

The above definition of toric branes has an interesting overlap with more recent studies of B branes via matrix factorizations[8]. Consider the charge vectors

	x_0	x_1	x_2	x_3	x_4	x_5
$l^1 =$	-5	1	1	1	1	1
$\hat{l}^2 =$	0	1	-1	0	0	0
$\hat{l}^3 =$	0	0	0	1	-1	0

(4.8)

For the special values $c_a = 0$ the equation (4.3) for the Lagrangian submanifold can be rewritten as

$$x_1 = \overline{x}_2,\ x_3 = \overline{x}_4,\ x_5 = \overline{x}_5\ .$$

The above equation describes an involution brane on the quintic defined as the fixed set of the $\mathbf{Z}_2$ action $(x_1, x_2, x_3, x_4, x_5) \rightarrow (\overline{x}_2, \overline{x}_1, \overline{x}_4, \overline{x}_3, \overline{x}_5)$. The equation for the mirror B-brane follows from (4.5):

$$y_1 = \hat{z}_2\, y_2, \quad y_3 = \hat{z}_3\, y_4, \qquad \text{or} \qquad x_1^5 = \hat{z}_2\, x_2^5, \quad x_3^5 = \hat{z}_3\, x_4^5\,. \tag{4.9}$$

A naive match of the moduli of the A and B model together with a choice of phase leads to $\hat{z}_2 = \hat{z}_3 = -1$ and the above equations become

$$x_1^5 + x_2^5 = 0,\ x_3^5 + x_4^5 = 0,\ (x_5^2 - \psi^{1/2} x_1 x_3)\,(x_5^2 + \psi^{1/2} x_1 x_3)\, x_5 = 0\ . \tag{4.10}$$

[7] The coefficients a_i are homogeneous coordinates on the space of complex structure and related to the z_a in 4.4 by an rescaling of the variables y_i.

[8] See ref. [17, 33] for a summary.

These equations define a set of holomorphic 2-cycles in Z^* which may be wrapped by the D5 brane mirror to the A brane on the Lagrangian subset defined by (4.3).

Eq. (4.10) should be compared to the results of refs. [79, 80], where the 2-cycle wrapped by the B brane mirror to an involution brane has been determined in a much more involved way along the lines of [88], by proposing a matrix factorization and computing the second algebraic Chern class of the associated complex. The result agrees with the above result from a simple application of mirror symmetry for toric branes. A conclusive match of the toric brane defined by (4.5) and the matrix factorization brane studied in [79, 80] will given in sect. 4.3, where we compute the superpotential from the toric family and find agreement near a specific critical locus.

There are ambiguities in the above match between the A and the B model that need to be resolved by a careful study of boundary conditions. E.g. in (4.10), the last equation is the superpotential intersected with the two hypersurfaces (4.9), but one may permute the meaning of the three equations. The parametrization

	x_0	x_1	x_2	x_3	x_4	x_5
$l^1=$	-5	1	1	1	1	1
$\hat{l}^2=$	1	0	0	0	0	-1
$\hat{l}^3=$	0	1	-1	0	0	0

(4.11)

leads to the same equations (4.10) for the special values $\hat{z}_2 = \psi$, $\hat{z}_3 = -1$ of the (new) moduli. An important aspect in resolving these ambiguities is provided by the mirror map $z_a(t_b)$ on the open/closed moduli space, as it determines where a specific (family of) point(s) in the A moduli attaches in the moduli space of the B model and vice versa. In the above example we have simply used the classical version of the open string mirror maps $|\hat{z}_a| = e^{-c_a}$ to find agreement with the result from matrix factorizations. More seriously we will compute the exact mirror map – which may in principle deviate substantially from the classical expression – to determine the B brane configuration. Since some of the deformations will be fixed at the critical points of the superpotential it is in fact more natural to start with the computation of the B model superpotentials and to find its critical points. Computing the mirror map near these points determines a correlated set of points in the A model parameter space, which may or may not allow for a nice classical A brane interpretation.

4.2.2 $\mathcal{N} = 1$ special geometry of the open/closed deformation space

We proceed by discussing a general structure of the open/closed deformation space that will be central to the following approach to mirror symmetry for the toric branes defined above. In [21, 20, 83] it was shown, that the open/closed string deformation space for B-type D5-branes wrapping 2-cycles C in Z^* can be studied from the variation of mixed

Hodge structure on a deformation family of relative cohomology groups $H^3(Z^*, \mathcal{H})$ of Z^*, where $\mathcal{H}$ is a subset that captures the deformations of C.[9] In the simplest case, $\mathcal{H}$ is a single hypersurface and the action of the closed and open string variations is schematically

$$\begin{array}{ccccccc} F^3H^3(Z^*) & \xrightarrow{\delta_z} & F^2H^3(Z^*) & \xrightarrow{\delta_z} & F^1H^3(Z^*) & \xrightarrow{\delta_z} & F^0H^3(Z^*) \\ & \searrow^{\delta_{\hat{z}}} & & \searrow^{\delta_{\hat{z}}} & & \searrow^{\delta_{\hat{z}}} & \downarrow^{\delta_{\hat{z}}} \\ & & F^2H^2(\mathcal{H}) & \xrightarrow{\delta_z,\delta_{\hat{z}}} & F^1H^2(\mathcal{H}) & \xrightarrow{\delta_z,\delta_{\hat{z}}} & F^0H^2(\mathcal{H}) \end{array} \tag{4.12}$$

Here F^k is the Hodge filtration and δ_z and $\delta_{\hat{z}}$ denote the closed and open string variations, respectively. For more details on this structure we refer to refs. [21, 20, 83] (see also [89, 90]). The variations δ can be identified with the flat Gauss-Manin connection ∇, which captures the variation of mixed Hodge structure on the bundle with fibers the relative cohomology groups. The mathematical background is described in refs. [45, 91, 92, 93, 52, 94].

The flatness of the Gauss-Manin connection leads to a non-trivial "$\mathcal{N} = 1$ special geometry" of the combined open/closed field space, that governs the open/closed chiral ring of the topological string theory [21, 20]. This geometric structure leads to a Picard-Fuchs system of differential equations satisfied by the relative period integrals

$$\mathcal{L}_a \Pi_\Sigma = 0, \qquad \Pi_\Sigma(z, \hat{z}) = \int_{\gamma_\Sigma} \Omega, \qquad \gamma_\Sigma \in H_3(Z^*, \mathcal{H}) \ . \tag{4.13}$$

Here $\{\mathcal{L}_a\}$ is a system of linear differential operators, $z(\hat{z})$ stands collectively for the closed (open) string parameters and the holomorphic 3-form Ω and its period integrals are defined in relative cohomology. The relative periods $\Pi_\Sigma(z, \hat{z})$ determine the mirror map and the combined open/closed string superpotential, which can be written in a unified way as

$$\mathcal{W}_{\mathcal{N}=1}(z, \hat{z}) = \mathcal{W}_{closed}(z) + \mathcal{W}_{open}(z, \hat{z}) = \sum_{\gamma_\Sigma \in H^3(Z^*, \mathcal{H})} N_\Sigma \Pi_\Sigma(z, \hat{z}) \ . \tag{4.14}$$

Here $\mathcal{W}_{closed}(z)$ is the closed string superpotential proportional to the periods over cycles $\gamma_\Sigma \in H^3(Z^*)$ and $\mathcal{W}_{open}(z, \hat{z})$ the brane superpotential proportional to periods over chains γ_Σ with non-empty boundary $\partial\gamma_\Sigma$. The coefficients N_Σ are the corresponding "flux" and brane numbers.[10]

In the following we implement this general structure for the class of toric branes on compact manifolds defined in sect. 4.2.1. In the present context, the deformations of C are controlled by eq. (4.5) and the relative cohomology problem is naturally defined by the hypersurfaces $\mathcal{B}_a(E)$ in the B model. In [21, 20] this identification was used to set up the appropriate

[9]Physically, $\mathcal{H}$ may be interpreted as a D7-brane which contains the D5 brane world-volume.

[10]To obtain the physical superpotential, an appropriate choice of reference brane has to be made for the chain integrals, since a relative period more precisely computes the brane tension of a domain wall [95, 96, 97, 18]. This should be kept in mind in the following discussion where we simply refer to "the superpotential".

problem of mixed Hodge structure for branes in non-compact CY manifolds and to compute the Picard-Fuchs system of the $\mathcal{N}=1$ special geometry. This approach was extended to the compact case in [83] by relating $\mathcal{H}$ to the algebraic Chern class $c_2(E)$ of a B brane as obtained from a matrix factorization. As observed in sect. 4.2, these two definitions of $\mathcal{H}$ are closely related and it is straightforward to check that they coincide in concrete examples; in particular the hypersurfaces defined in [83] fit into the definition of $\mathcal{H}$ via (4.5) in [21, 20].[11]

4.2.3 GLSM and enhanced toric polyhedra

To make full use of the machinery of toric geometry we start with defining a GLSM for the CY/brane geometry. The GLSM puts the CY geometry and the brane geometry on equal footing and allows to study the phases of the combined system by standard methods of toric geometry. The GLSM thus provides valuable information on the *global* structure of the combined open/closed deformation space which will be important for identifying and investigating the various phases of the brane geometry, in particular large volume phases.

We will use the concept of toric polyhedra to define the GLSM for the mirror pairs of toric brane geometries. This approach has the advantage of giving a canonical construction of the B model mirror to a certain A brane geometry and provides a short-cut to derive the generalized hypergeometric system for the relative periods given in eq. 4.19 below. As discussed above, Batyrev's correspondence describes a mirror pair of toric hypersurfaces (Z, Z^*) by a pair of dual polyhedra $(\Delta, \Delta^\star)$. What we are proposing here is that there is a similar correspondence between "enhanced polyhedra" $(\Delta_\flat(Z,L), \Delta_\flat^\star(Z^*,E))$ and the pair (Z, Z^*) of mirror manifolds *together* with the pair of mirror branes (L, E) as defined before.

The enhanced polyhedron $\Delta_\flat(Z,L)$ has the following simple structure: The points $\nu_i(Z)$ of $\Delta(Z)$ defining the manifold Z are a subset of the points of $\Delta_\flat(Z,L)$ that lie on a hypersurface H in a five-dimensional lattice Λ_5. We choose an ordering of the points $\mu_i \in \Delta_\flat(Z,L)$ and coordinates on Λ_5 such that the points in H are given by

$$(\mu_i) = (\nu_i, 0), \ i = 1, ..., k \ ,$$

where k is the number of points of $\Delta(Z)$. The brane geometry is described by k' extra points ρ_i with $(\rho_i)_5 < 0$, where k' is related to the number $\hat{n}$ of (obstructed) moduli of the brane by $k' = \hat{n} + 1$. Thus $\Delta_\flat(Z,L)$ is defined as the convex hull of the points

$$\Delta_\flat(Z,L) = \mathrm{conv}\left(\{\mu_i(\Delta(Z))\} \cup \{\rho_i(L)\}\right), \qquad \{\mu_i(\Delta(Z))\} \subset \Delta_\flat(Z,L) \cap H \ , \qquad (4.15)$$

[11] As was stressed in sect. 3.6 of [80], the chain integrals, which define the *normal functions* associated with the superpotential, do not depend on the details of the infinite complexes constructed in [88]. Our results suggest that the relevant information or the superpotential is captured by the linear sigma model defined below.

For simplicity we assume that the polyhedron $\Delta_\flat^\star$ can be naively defined as the dual of $\Delta_\flat$ in the sense of [50].

To make contact between the definition of the toric branes in sect. 4.2 and the extra points ρ_i, consider the linear dependences between the points of $\Delta_\flat(Z,L)$

$$\sum_i \underline{l}_i^a(\Delta_\flat)\mu_i = 0 \ . \tag{4.16}$$

These relations may be split into two sets in an obvious way. There are $h^{1,1}(Z)$ relations, say

$$(\underline{l}^a(\Delta_\flat)) = (l^a(\Delta), 0^{k'}),\ a = 1, ..., h^{1,1}(Z) \ ,$$

which involve only the first k points and reflect the original relations $l^a(\Delta)$ between the points $\nu_i(Z)$ of $\Delta(Z)$; they correspond to Kähler classes of the manifold Z. The remaining relations $\underline{l}^a(\Delta_\flat)$, $a > h^{1,1}(Z)$ involve also the extra points ρ_i. To describe a brane as defined by the charge vectors $\hat{l}^a(L)$ we choose the points ρ_i such that the remaining relations are of the form

$$(\underline{l}^a(\Delta_\flat)) = (\hat{l}^a(L), ...),\ a > h^{1,1}(Z) \ .$$

The above prescription for the construction of the enhanced polyhedron $\Delta_\flat(Z,L)$ from the polyhedron $\Delta(Z)$ for a given manifold Z and the definition (4.3) of the A brane L in sect. 4.2 is well-defined if we require a minimal extension by $k' = \hat{n} + 1$ points.

4.2.4 Differential equations on the moduli space

The combined open/closed string deformation space of the brane geometries (Z,L) or (Z^*,E) can now be studied by standard methods of toric geometry. Let[12] $\{l_i^a\}$ denote a *specific* choice of basis for the generators of the relations (4.16) in the GLSM and a_i the coefficients of the hypersurface equation $p = \sum_i a_i y_i$ of the mirror B model. From the homogeneous coordinates a_i on the complex moduli space one may define local coordinates associated with the choice of a basis $\{l_i^a\}$ by[13]

$$z_a = (-)^{l_0^a} \prod_i a_i^{l_i^a} \ , \qquad a = 1, ..., M+N. \tag{4.17}$$

Our main tool will be a system of linear differential equations of the form

$$\mathcal{L}_a \Pi(z_b) = 0, \tag{4.18}$$

whose solutions are the relative periods (4.13). The relative periods determine not only the genus zero partition functions but also the mirror map $z_a(t_b)$ between the flat coordinates

[12] The underscore on $\underline{l}^a(\Delta_\flat)$ will be dropped again to simplify notation.

[13] The sign is a priori convention but receives a meaning if the classical limit of the mirror map is fixed as in [98].

t_a and the algebraic moduli z_a for the open/closed string deformation space [21, 20]. There are two ways to derive the system of differential operators $\{\mathcal{L}_a\}$: Either as the canonical generalized hypergeometric GKZ system associated with the enhanced polyhedron $\Delta_\flat(Z, L)$ [99, 50]. Or as the system of differential equations capturing the variation of mixed Hodge structure on the relative cohomology group $H^3(Z^*, \mathcal{H})$ as in refs. [21, 20, 83].

Here we use the short-cut of toric polyhedra and define the Picard-Fuchs system as the canonical GKZ system associated with $\Delta_\flat$.[14] The derivation of the Picard-Fuchs system from the variation of mixed Hodge structure on the relative cohomology group, which is similar to that in [21, 20], will be given in future work; the coincidence of the two definitions is non-trivial and reflects a string duality [100]. By the results of [99, 50], the generalized hypergeometric system associated to $(\Delta_\flat, \Delta_\flat^\star)$ leads to the following differential operators for $a = 1, ..., M + N$:

$$\mathcal{L}_a = \prod_{k=1}^{l_0^a}(\theta_{a_0} - k) \prod_{l_i^a>0} \prod_{k=0}^{l_i^a-1}(\theta_{a_i} - k) - (-1)^{l_0^a} z_a \prod_{k=1}^{-l_0^a}(\theta_{a_0} - k) \prod_{l_i^a<0} \prod_{k=0}^{-l_i^a-1}(\theta_{a_i} - k) \qquad (4.19)$$

Here θ_x denotes a logarithmic derivative $\theta_x = x\frac{\partial}{\partial x}$ and the derivatives of the homogeneous coordinates a_i on the complex structure moduli and the local coordinates (4.17) are related by $\theta_{a_i} = \sum_a l_i^a \theta_{z_a}$. The products are defined to run over non-negative k only so that the derivatives θ_{a_0} appear only in one of the two terms for given a. The solutions of the Picard-Fuchs system in eq. (4.19) have a nice expansion around $z_a = 0$; expansions around other points in the moduli space can be obtained from a change of variables.

Eqs. (4.18), (4.19) represent the homogeneous Picard-Fuchs system for the brane geometry (Z^*, E). These homogeneous Picard-Fuchs equations give rise to inhomogeneous Picard-Fuchs equations by splitting the operators $\mathcal{L}_a$ in a piece $\mathcal{L}_{a,bulk}$ that depends only on the moduli z of the manifold Z^* and essentially represent the Picard-Fuchs system of the CY geometry and a part $\mathcal{L}_{a,open}$ that governs the dependence on the open string deformations $\hat{z}$. Upon evaluation at a critical point w.r.t. the open string deformations, $\delta_{\hat{z}}\mathcal{W} = 0$, the split leads to an inhomogeneous term $f_a(z)$, if Π is a chain that depends non-trivially on the brane deformations $\hat{z}$.

$$\mathcal{L}_{a,bulk}\Pi(z, \hat{z}) = -\mathcal{L}_{a,open}\Pi(z, \hat{z}) \quad \overset{\delta_{\hat{z}}\mathcal{W}=0}{\longrightarrow} \quad \mathcal{L}_{a,bulk}\Pi(z) = f_a(z)\,. \qquad (4.20)$$

For the case of the quintic, the inhomogeneous term $f_a(z)$ has been computed by a careful application of the Dwork-Griffiths reduction method for the chain integrals in [80] and it is straightforward to check that this term agrees with the inhomogeneous term on the r.h.s of (4.20), see eq. (4.32) below.

In [90] it has been proposed that the problem of mixed hodge variations on the relative cohomology groups defined in [21, 20, 83] can be reinterpreted in terms of the deformations

[14] We are tacitly assuming that the GKZ system $\{\mathcal{L}_a\}$ is already a complete Picard-Fuchs system, which is possibly only true after a slight modification of the GKZ system.

of a certain non Ricci-flat Kähler blow up Y of the B model geometry. It has been further suggested that it should be possible to obtain a Picard-Fuchs system for the brane geometry by computing in the manifold Y and restricting the complex structure of Y in an appropriate way. At the moment the details appear to be unknown and it would be interesting to relate these ideas to the above results. It would also be interesting to understand a possible connection to the differential equations and superpotentials derived from matrix factorizations in [101, 82, 102].

4.2.5 Phases of the GLSM and structure of the solutions of 4.19

In the previous definitions we have used a specific choice of basis $\{l_i^a\}$ to define the local coordinates (4.17) and the differential operators (4.19). Different choices of coordinates correspond to different phases of the GLSM [7]. The extreme cases are on the one hand a large volume phase in all the Kähler parameters, where the GLSM describes a smooth classical geometry and on the other hand a pure Landau-Ginzburg phase. In between there are mixed phases, where only some of the moduli are at large volume and other moduli are fixed in a stringy regime of small volume. A nice instanton expansion can be expected a priori only for moduli at large volume.

Representing the GLSM by the toric polyhedron $\Delta_\flat$, the different phases of the GLSM may be studied by considering different triangulations of the polyhedron [9, 50]. Without going into the technical details of this procedure, let us outline the relevance of this phase structure in the present context. A given B brane configuration corresponds to a critical point of the superpotential which lies in a certain local patch of the parameter space. To study the critical points in a given patch and to give a nice local expansion of the superpotential it is necessary to work in the appropriate local coordinates. The different triangulations of $\Delta_\flat$ define different regimes in the parameter space, where the relative periods Π_Σ have a certain characteristic behavior depending on whether the brane moduli are at large or at small volume. To find an interesting instanton expansion we look for triangulations that correspond to patches where at least some of the moduli are at large volume.

From the interpretation of the system $\{\mathcal{L}_a\}$ of differential operators as the Picard-Fuchs system for the relative periods on Z^* we expect the solutions of the equations (4.18) to have the following structure:

a) There are $2M+2$ solutions $\Pi(z)$ that represent the periods of Z^* up to linear combination and depend only on the complex structure moduli z_a, $a = 1, ..., h^{1,1}(Z)$ of Z^*.

b) There are $2N$ further solutions $\hat{\Pi}(z, \hat{z})$ that do depend on all deformations and define the mirror map for the open string deformations and the superpotential (more precisely: brane tensions).

c)[15]For a maximal triangulation corresponding to a large complex structure point centered at $z_a = 0\ \forall a$, there will be a series solution $\omega_0(z_a) = 1 + \mathcal{O}(z_a)$ and $M + N$ solutions $\omega_c(z_a)$ with a single log behavior that define the open/closed mirror maps as (c is fixed in the following equation)

$$t_c(z_a) = \frac{\omega_c(z_a)}{\omega_0(z_a)} = \frac{1}{2\pi i}\ln(z_c) + S_c(z_a),$$

where $S_c(z_a)$ is a series in the coordinates z_a.

It follows from a) that the mirror map $t^{(cl)}(z)$ in the closed string sector does not involve the open string deformations, similarly as has been observed in [18, 103, 21, 20] in the non-compact case.[16] However the open string mirror map $t^{(op)}(z,\hat{z})$ depends on both types of moduli. For explicit computations of the mirror maps at various points in the moduli we refer to the examples.

The special solution $\Pi = \mathcal{W}_{open}(z,\hat{z})$ has the further property that its instanton expansion near a large volume/large complex structure point encodes the Ooguri-Vafa invariants of the brane geometry:

$$\mathcal{W}_{inst}(q_a) = \sum_\beta G_\beta q^\beta = \sum_\beta \sum_{k=1}^{\infty} N_\beta \frac{q^{k.\beta}}{k^2}\ . \tag{4.21}$$

Here β is the non-trivial homology class of a disc, $\beta \in H^2(Z,L)$, q^β a weight factor related to its appropriately defined Kähler volume, G_β the fractional Gromov-Witten type coefficients in the instanton expansion and N_β the integral Ooguri-Vafa invariants [55].

Below we study some illustrative examples and find agreement with the above expectations.

4.3 Applications

In the following we apply the above method to study some examples including involution branes with obstructed deformations as well as a class of branes with classically unobstructed moduli.

[15]The following holds for appropriate choices of normalization and the sign in (4.17) that have been made in (4.19), explaining the special appearance of the entry $i = 0$ corresponding to the fiber of the anti-canonical bundle.

[16]This statement holds at zero string coupling.

4.3.1 Branes on the quintic $\mathbf{X}_5^{(1,1,1,1,1)}$

Brane geometry

We first study a family of toric branes on the quintic that includes branes that have been studied before in [79, 80, 83] by different means. We recover these results for special choice of boundary conditions and study connected configurations. As in sect. 4.2. we consider a one parameter family of A branes defined by the two charge vectors

$$(l^1) = (-5,1,1,1,1,1), \qquad (\hat{l}^2) = (1,-1,0,0,0,0) \ . \tag{4.22}$$

As discussed in sect. 4.2.3 we may associate with this brane geometry a five-dimensional toric polyhedron $\Delta_\flat(Z,L)$ that contains the points of the polyhedron $\Delta(Z)$ of the quintic as a subset on the hypersurface $y_5 = 0$:

$\Delta(Z)$	$\nu_0 =$	(0,	0,	0,	0,	0)
	$\nu_1 =$	(−1,	0,	0,	0,	0)
	$\nu_2 =$	(0,	−1,	0,	0,	0)
	$\nu_3 =$	(0,	0,	−1,	0,	0)
	$\nu_4 =$	(0,	0,	0,	−1,	0)
	$\nu_5 =$	(1,	1,	1,	1,	0)
$\Delta_\flat(Z,L) = \Delta\cup$	$\rho_1 =$	(−1,	0,	0,	0,	−1)
	$\rho_2 =$	(0,	0,	0,	0,	−1)

Table 4.1: Points of the enhanced polyhedron $\Delta_\flat$ for the geometry (4.22) on $\mathbf{X}_5^{(1,1,1,1,1)}$.

Choosing a maximal triangulation of $\Delta_\flat(Z,L)$ determines the following basis of generators for the relations (4.16)[17]

$$l^1 = (-4,0,1,1,1,1;1,-1), \qquad l^2 = (-1,1,0,0,0,0;-1,1) \ , \tag{4.23}$$

where the last two entries correspond to the extra points. In the local variables[18]

$$z_1 = -\frac{a_2a_3a_4a_5a_6}{a_0^4a_7}, \qquad z_2 = -\frac{a_1a_7}{a_0a_6}, \tag{4.24}$$

the hypersurface equations for the B brane geometry (Z^*,E) read

$$\begin{aligned} p(Z^*) \ &: \ x_1^5+x_2^5+x_3^5+x_4^5+x_5^5 - x_1x_2x_3x_4x_5\, z^{-\frac{1}{5}} = 0, \\ \mathcal{B}(E) \ &: \ x_1^5 + x_1x_2x_3x_4x_5 \ z_2 z^{-\frac{1}{5}} = 0 \ . \end{aligned} \tag{4.25}$$

[17]The following computations have been performed using parts of existing computer codes: `Puntos`, http://www.math.ucdavis.edu/ deloera/RECENT_WORK/puntos2000; `Schubert`, http://math.uib.no/schubert; `Instanton`, http://www.math.uiuc.edu/ katz/software.html.

[18]We equipped z_1 with an additional minus sign compared to 4.17 for later convenience.

Here $z = -z_1z_2$ denotes the complex structure modulus of the CY geometry Z^*.

From eq. (4.23) one can immediately proceed and solve the toric Picard-Fuchs system (4.19) to derive the mirror maps and the superpotentials and we will do so momentarily. However, it is instructive to take a closer look at the geometry of the problem of mixed Hodge variations on the relative cohomology groups (4.12), which has the following intriguing structure. Rewriting the superpotential $p(Z^*)$ in the original variables y_i of the toric ambient space and restricting to the hypersurface $\mathcal{B}(E):\ y_1 = y_0$ in these variables (cpw. (4.5)) defines the following boundary superpotential $W_{\mathcal{H}} = p(Z^*)|_{y_1=y_0}$ for the relative cohomology problem on $\mathcal{H} = \mathcal{B}(E)$:

$$W_{\mathcal{H}} = (a_0 + a_1)y_0 + a_2y_2 + a_3y_3 + a_4y_4 + a_5y_5\ .$$

The boundary superpotential $W_{\mathcal{H}}$ describes a K3 surface defined as a quartic polynomial in $\mathbf{P}^3$ after the transformation of variables $y_i = x_i^4$, $i = 1, ..., 4$:

$$W_{\mathcal{H}} = x_1^4 + x_2^4 + x_3^4 + x_4^4 + z_{\mathcal{H}}^{-1/4}\ x_1x_2x_3x_4\ . \tag{4.26}$$

Thus the part of the Hodge variation associated with the lower row in (4.12), which can be properly defined as a subspace through the weight filtration [21, 20, 83], is the usual Hodge variation associated with the complex structure of the family of K3 manifolds defined by $W_{\mathcal{H}}$. The complex structure determined by the (2,0) form ω on the K3 is parametrized by the modulus

$$z_{\mathcal{H}} = \frac{z_1}{(1+z_2)^4} \overset{a_6/a_7=-1}{\longrightarrow} \frac{a_2a_3a_4a_5}{(a_0+a_1)^4}\ ,$$

which is a special combination of the closed and open string moduli. Since the dependence of the Hodge variation on the brane modulus z_2 localizes on $\mathcal{H}$, the open string mirror map and the brane tension will be directly related to periods on the K3 surface (4.26)! This observation is very useful in studying details of the critical points and generalizes to other brane geometries.

The differential operators (4.19) in the local variables z_1, z_2 defined by (4.23) read

$$\begin{aligned}\mathcal{L}_1 &= (\theta_1^4 - z_1\prod_{i=1}^{4}(4\theta_1+\theta_2+i))(\theta_1-\theta_2)\ ,\\ \mathcal{L}_2 &= (\theta_2 + z_2(4\theta_1+\theta_2+1))(\theta_1-\theta_2)\ .\end{aligned} \tag{4.27}$$

The above operators $\mathcal{L}_1$ and $\mathcal{L}_2$ reveal the relation of the variation of mixed Hodge structure to the family of K3 manifolds defined in (4.26). Indeed the combination $(\theta_1 - \theta_2)$ is the direction of the open string parameter that localizes on $\mathcal{H}$. The split

$$\mathcal{L}_a = \tilde{\mathcal{L}}_a(\theta_1-\theta_2)\,,$$

shows that the solutions π_σ of the equations $\tilde{\mathcal{L}}_a\pi_\sigma = 0$ are just the K3 periods. The operator $\tilde{\mathcal{L}}_2$ imposes that the periods depend non-trivially only on the variable $z_{\mathcal{H}}$[19]

$$\tilde{\mathcal{L}}_2\left((z_2+1)^{-1}f(z_{\mathcal{H}})\right) = 0\,,$$

[19]The z_2 dependent prefactor arises from the normalization of the holomorphic form.

whereas the operator $\tilde{\mathcal{L}}_1$ reduces to the Picard-Fuchs operator of the K3 surface in the new variable $z_{\mathcal{H}}$. It follows that the solutions of the K3 system are the first variations of the relative periods w.r.t. the open string deformation and a critical point $\hat{\delta} W = 0$ corresponds to a particular solution π of the K3 system that vanishes at that point. The solution that describes the involution brane is determined by requiring the right transformation property under the discrete symmetry of the moduli space as in [83].

Further differential operators can be obtained from linear combinations of the basis vectors l^a. E.g. the linear combination $l = l^1 + l^2$ defines the differential operator

$$\mathcal{L}_1' = \theta_2 \theta_1^4 + z_1 z_2 \prod_{i=1}^{5} (4\theta_1 + \theta_2 + i) \ ,$$

which also annihilates the relative periods.[20] The solutions of the complete system of differential operators have the expected structure described in sect. 4.2.5. The mirror maps can be computed to be

$$\begin{aligned} -z_1(t_1, t_2) &= q_1 + (24\, q_1^2 - q_1\, q_2) + (-396\, q_1^3 - 640\, q_1^2\, q_2) + \dots \ , \\ z_2(t_1, t_2) &= q_2 + (-24\, q_1\, q_2 + q_2^2) + (972\, q_1^2\, q_2 - 178\, q_1\, q_2^2 + q_2^3) + \dots \ , \end{aligned} \qquad (4.28)$$

with $q_a = \exp(2\pi i\, t_a)$. The deformation parameters t_1 and t_2 are the flat coordinates near the large complex structure point $z_1 = z_2 = 0$ associated with *open string* deformations [21, 20]. Their physical interpretation is the quantum volume of two homologically distinct discs as measured by the tension of D4 domain walls on the A model side [18, 103]. The other solutions of the differential operators (4.19) describe the brane tensions (4.14) of the domain walls in the family. We proceed with a study of various critical points of the superpotential.

Near the involution brane

To study brane configurations mirror to the involution brane we consider a critical point of the type (4.10), that is a D5 brane locus

$$x_2^5 + x_3^5 = 0, \qquad x_4^5 + x_5^5 = 0, \qquad x_1^5 - x_1 x_2 x_3 x_4 x_5\, z^{-\frac{1}{5}} = 0 \ .$$

Comparing with (4.25) we search for a superpotential with critical locus near $z_2 = -1$ and arbitrary z_1. Let us first look at the large volume phase $z_1 \sim 0$ of the mirror A brane, where one expects an instanton expansion with integral coefficients. The local variables

[20] One can further factorize the above operators to a degree four differential operator $\theta_1^4 + (5z_1z_2(4\theta_1 + \theta_2 + 4) - 4z_1(\theta_1 - \theta_2)) \prod_{i=1}^{3}(4\theta_1 + \theta_2 + i)$ which together with $\mathcal{L}_2$ represents a complete Picard-Fuchs system.

(4.24) are centered at $z_1 = z_2 = 0$, not $z_2 = -1$, however. To get a nice expansion of the superpotential near the locus $z_2 + 1 = 0$ we change variables to

$$u = z_1^{-1/4}(1 + z_2), \qquad v = z_1^{1/4} \; .$$

Examining the z_2-dependent solution of the GKZ system in these variables, we find the superpotential

$$c\,\mathcal{W}(u,v) = \frac{u^2}{8} + 15v^2 + \frac{5u^3v}{48} - \frac{15uv^3}{2} + \frac{u^6}{46080} + \frac{35v^2u^4}{384} - \frac{15v^4u^2}{8} + \frac{25025v^6}{3} + \dots \quad (4.29)$$

which has the expected critical locus $\hat{\delta}\mathcal{W} = 0$ at $u = 0$ for all values of v. Here c is a constant that can not be fixed from the consideration of the differential equations (4.19) alone.[21] At the critical locus $u = 0$ the above expression yields the critical value $\mathcal{W}_{crit}(z_1^{(cl)}) = \mathcal{W}(u = 0, v = z^{1/4})$

$$\mathcal{W}_{crit}(z) = 15\sqrt{z} + \frac{25025}{3}\, z^{3/2} + \frac{52055003}{5}\, z^{5/2} + \dots \; . \quad (4.30)$$

Here the constant has been fixed to $c = 1$ by comparing (4.30) with the result of [79] for $\mathcal{W}_{crit}(z)$.

As alluded to in sect. 4.2.5, the differential operators (4.27) have the special property that the periods of Z^* are amongst their solutions. One may check that the open string mirror maps (4.28) conspire such that the mirror map for the remaining modulus $z = -z_1 z_2$ at the critical point coincides with the closed string mirror map for the quintic. Using the multi-cover prescription of [55, 79] and expressing (4.30) in terms of the exponentials $q(z) = \exp(2\pi i\, t(z)) = z + \mathcal{O}(z^2)$ one obtains the integral instanton expansion of the A model

$$\begin{aligned} \frac{\mathcal{W}_{crit}(z(q))}{\omega_0(q)} &= 15\sqrt{q} + \frac{2300}{3}\, q^{3/2} + \frac{2720628}{5}\, q^{5/2} + \dots\,, \\ &= \sum_{k\ odd} \Big(\frac{15}{k^2} q^{k/2} + \frac{765}{k^2} q^{3k/2} + \frac{544125}{k^2} q^{5k/2} + \dots \Big)\,. \end{aligned}$$

To make contact with the inhomogeneous Picard-Fuchs equation of [80], we rewrite the differential operators above in terms of the bulk modulus z and the open string deformation z_2 and split off the z_2 dependent terms as in (4.20). In particular the operator $\mathcal{L}_1'$ leads to a non-trivial equation of the form $\theta\mathcal{L}_{bulk}\Pi = -\mathcal{L}_{open}\Pi$, where

$$\begin{aligned} \mathcal{L}_{bulk} &= \theta^4 - 5z\prod_{i=1}^{4}(5\theta + i)\,, \qquad \mathcal{L}_{open} = \mathcal{L}_1' - \theta\mathcal{L}_{bulk}\,, \\ \mathcal{L}_1' &= (\theta + \theta_2)\theta^4 - z\prod_{i=1}^{5}(5\theta + \theta_2 + i)\,, \end{aligned} \quad (4.31)$$

[21] The precise linear combination of the solutions of the Picard-Fuchs system that corresponds to a given geometric cycle can be determined by an intersection argument and possibly analytic continuation, similarly as in the closed string case [6]. Such an argument has been made in the present example already in [79], from which we will borrow the correct value for c.

and $\theta = \theta_z$. Setting $\Pi = \mathcal{W}(u, v)$ and restricting to the critical locus $z_2 = -1$ one obtains

$$\mathcal{L}_{bulk}\mathcal{W}_{crit} = \frac{15}{16}\sqrt{z}\ . \tag{4.32}$$

This identifies the inhomogeneous Picard-Fuchs equation of [79, 80] as the restriction of (4.19) to the critical locus.

While the result (4.30) had been previously obtained in [79], the above derivation gives some extra information. Since the definition of the toric branes holds off the involution locus, the superpotential $\mathcal{W}(u, v)$ describes more generally any member of the family of toric A branes defined by (4.3), not just the involution brane. It describes also the deformation of the large volume superpotential away from $z_2 = -1$. It is also possible to describe more general configurations with several deformations. It should also be noted that the use of the closed string mirror map in [79] was strictly speaking an assumption, as the closed string mirror map measures the quantum volume of fundamental sphere instantons, not the quantum tension of D4 domain walls wrapping discs, which is the appropriate coordinate for the integral expansion of [55]. It is neither obvious nor true in general that this D4 tension agrees with half the sphere volume of the fundamental string, in particular off the involution locus. In the present case it is not hard to justify this choice and to check it from the computation of the mirror map, but more generally there will be corrections to the D4 quantum volume that are not determined by the closed string mirror map, see eq. (4.28) and the examples below.

Small volume in the A model: $1/z_1 \sim 0$
Another interesting point in the moduli space is the Landau-Ginzburg point of the B model. This case has been studied previously in [83], so we will be very brief. The only non-trivial thing left to check is that the system of differential equations obtained in [83] from Dwork-Griffiths reduction is equivalent to the toric GKZ system (4.19) transformed to the local variables near the LG point. Choosing local variables

$$x_1 = \frac{a_0}{(a_2a_3a_4a_5)^{1/4}}\left(\frac{-a_7}{a_6}\right)^{1/4}, \qquad x_2 = \frac{a_1}{(a_2a_3a_4a_5)^{1/4}}\left(\frac{-a_7}{a_6}\right)^{5/4},$$

one obtains by a transformation of variables the differential operators

$$\begin{aligned} \mathcal{L}_1 &= (x_1^4(\theta_1+\theta_2)^4 - 4^4\prod_{i=1}^{4}(\theta_1 - i))(\theta_1 + 5\theta_2)\,, \\ \mathcal{L}_2 &= (x_2(\theta_1 - 1) - x_1\theta_2)\,(\theta_1 + 5\theta_2)\,, \\ \mathcal{L}_1' &= x_1^5(\theta_1+\theta_2)^4\theta_2 - 4^4 x_2 \prod_{i=1}^{5}(\theta_1 - i)\,, \end{aligned} \tag{4.33}$$

where θ_i denotes the logarithmic derivatives θ_{x_i}. The above operators agree with eqs.(5.14)-(5.16) of [83] up to a change of variables. The superpotential is

$$\mathcal{W} = -\frac{x_1^2}{2} - \frac{x_2x_1}{6} - \frac{x_1^6}{11520} - \frac{x_2x_1^5}{3840} - \frac{x_2^2x_1^4}{2688} - \frac{x_2^3x_1^3}{3456} - \frac{x_2^4x_1^2}{8448} - \frac{x_2^5x_1}{49920} + \dots\,,$$

which has its critical locus at $x_2 = -x_1$, which corresponds to $u = 0$ in these coordinates. In terms of the closed string variable $x_1^{(cl)} = -x_1 x_2^{-1/5}$ at the Landau Ginzburg point, the expansion at the critical locus reads

$$\mathcal{W}_{crit} = -\frac{x^{5/2}}{3} - \frac{x^{15/2}}{135135} - \frac{x^{25/2}}{1301375075} + \dots ,$$

which satisfies a similar equation as (4.32)

$$\mathcal{L}_{bulk}\mathcal{W}_{crit} = \frac{15}{16}\, x^{5/2} ,$$

where $\mathcal{L}_{bulk} = 5^{-4}x^5\theta_x^4 - 5\prod_{i=1}^4(\theta_x - i)$.

4.3.2 Branes on $\mathbf{X}_{18}^{(1,1,1,6,9)}$

As a second example we study branes on the two moduli CY $Z = \mathbf{X}_{18}^{(1,1,1,6,9)}$. Z is an elliptic fibration over $\mathbf{P}^2$ with the elliptic fiber and the base parametrized by the coordinates x_1, x_2, x_3 and x_4, x_5, x_6 in 4.2, respectively. In the decompactification limit of large fiber, the compact CY approximates the non-compact CY $\mathcal{O}(-3)_{\mathbf{P}^2}$ with coordinates x_3, x_4, x_5, x_6. This limit is interesting, as it makes contact to the previous studies of branes on $\mathcal{O}(-3)_{\mathbf{P}^2}$ in [103, 19].

Brane geometry

We consider a family of A branes parametrized by the relations

$$|x_4|^2 - |x_3|^2 = c^1, \qquad \hat{l} = (0,0,0,-1,1,0,0) \ . \tag{4.34}$$

This defines a family of D7-branes in the mirror parametrized by one complex modulus. To make contact with the non-compact branes we may add a second constraint $|x_5|^2 - |x_3|^2 = 0$ that selects a particular solution of the Picard-Fuchs system.[22] The brane geometry on the B model side is defined by the two equations

$$\begin{aligned} p(Z^*) &= \sum a_i y_i = a_0 x_1 x_2 x_3 x_4 x_5 + a_1 x_1^2 + a_2 x_2^3 + a_3 (x_3 x_4 x_5)^6 + a_4 x_3^{18} + a_5 x_4^{18} + a_6 x_5^{18}, \\ \mathcal{B}(E) &: \qquad y_3 = y_4 \qquad \text{or} \qquad (x_3 x_4 x_5)^6 = x_3^{18} \ . \end{aligned} \tag{4.35}$$

As in the previous case one observes that the complex deformations of the brane geometry are related to the periods of a K3 surface defined by

$$W_{\mathcal{H}} = a_0 x_1' x_2' x_3' x_4' + a_1 (x_1')^2 + a_2 (x_2')^3 + (a_3 + a_4)(x_3' x_4')^6 + a_5 (x_3')^{12} + a_6 (x_4')^{12}.$$

[22] Since the constant in this equation must be zero to get a non-zero superpotential [18], there is no new modulus.

The GLSM for the above brane geometry corresponds to the enhanced polyhedron.

$\Delta(Z)$	$\nu_0 =$	$(\ 0,$	$0,$	$0,$	$0,$	$0)$
	$\nu_1 =$	$(\ 0,$	$0,$	$0,$	$-1,$	$0)$
	$\nu_2 =$	$(\ 0,$	$0,$	$-1,$	$0,$	$0)$
	$\nu_3 =$	$(\ 0,$	$0,$	$2,$	$3,$	$0)$
	$\nu_4 =$	$(-1,$	$0,$	$2,$	$3,$	$0)$
	$\nu_5 =$	$(\ 0,$	$-1,$	$2,$	$3,$	$0)$
	$\nu_6 =$	$(\ 1,$	$1,$	$2,$	$3,$	$0)$
$\Delta_\flat(Z,L) = \Delta\cup$	$\rho_1 =$	$(\ 0,$	$0,$	$2,$	$3,$	$-1)$
	$\rho_2 =$	$(-1,$	$0,$	$2,$	$3,$	$-1)$

Table 4.2: Points of the enhanced polyhedron $\Delta_\flat$ for the geometry (4.34) on $\mathbf{X}_{18}$.

Choosing a triangulation of $\Delta_\flat$ that represents a large complex structure phase yields the following basis of the linear relations (4.16) between the points of $\Delta_\flat$:

$$\begin{aligned} l^1 = (-6,3,2,1,0,0,0,0,0), \quad l^2 &= (0,0,0,-2,0,1,1,-1,1), \\ l^3 &= (0,0,0,-1,1,0,0,1,-1). \end{aligned} \tag{4.36}$$

The last two charge vectors define a GLSM for the "inner phase" of the brane in the non-compact CY described in [19]. The differential operators (4.19) for the relative periods are given by

$$\begin{aligned} \mathcal{L}_1 &= \theta_1(\theta_1 - 2\theta_2 - \theta_3) - 12z_1(6\theta_1+5)(6\theta_1+1), \\ \mathcal{L}_2 &= \theta_2^2(\theta_2-\theta_3) + z_2(\theta_1 - 2\theta_2 - \theta_3)(\theta_1 - 2\theta_2 - 1 - \theta_3)(\theta_2-\theta_3), \\ \mathcal{L}_3 &= -\theta_3(\theta_2-\theta_3) - z_3(\theta_1 - 2\theta_2 - \theta_3)(\theta_2 - \theta_3). \end{aligned} \tag{4.37}$$

Large volume brane

The elliptic fiber compactifies the non-compact fiber direction x_3 of the non-compact CY $\mathcal{O}(-3)_{\mathbf{P}^2}$. In the limit of large elliptic fiber we therefore expect to find a deformation of the brane studied in [103, 19]. Large volume corresponds to $z_a = 0$ in the coordinates defined by eqs. (4.36), (4.17).

The mirror maps and the superpotential can be computed from (4.19). Expressing the superpotential in the flat coordinates t_a defines the Ooguri-Vafa invariants N_β in (4.21). The homology class β can be labelled by three integers (k,l,m) that determine the Kähler volume $kt_1 + lt_2 + mt_3$ of a curve in this class. Here t_1 is the volume of the elliptic fiber and t_2, t_3 are the (D4-)volumes of two homologically distinct discs in the brane geometry. The Kähler class of the section, which measures the volume of the fundamental sphere in $\mathbf{P}^2$, is $t_2 + t_3$.

For the discs that do not wrap the elliptic fiber we obtain the following invariants for $\beta = (0, l, m)$:

$l\backslash m$	0	1	2	3	4	5	6
0	*	1	0	0	0	0	0
1	1	*	-1	-1	-1	-1	-1
2	-1	-2	*	5	7	9	12
3	1	4	12	*	-40	-61	-93
4	-2	-10	-32	-104	*	399	648
5	5	28	102	326	1085	*	-4524
6	-13	-84	-344	-1160	-3708	-12660	*
7	35	264	1200	4360	14274	45722	159208
8	-100	-858	-4304	-16854	-57760	-185988	-598088
9	300	2860	15730	66222	239404	793502	2530946
10	-925	-9724	-58208	-262834	-1004386	-3460940	-11231776

Table 4.3: Invariants $N_{0,l,m}$ for the geometry (4.36).

The above result agrees with the results of [103, 19] for the disc invariants in the "inner phase" of the non-compact CY $\mathcal{O}(-3)_{\mathbf{P}^2}$. This result can be explained heuristically as follows. The holomorphic discs ending on the non-compact A brane in $\mathcal{O}(-3)_{\mathbf{P}^2}$ lie within the zero section of $\mathcal{O}(-3)_{\mathbf{P}^2}$. Similarly discs with $k = 0$ in $\mathbf{X}_{18}$ are holomorphic curves that must map to the section $x_3 = 0$ of the elliptic fibration. The moduli space of maps into the sections of the non-compact and compact manifolds, respectively, does not see the compactification in the fiber, explaining the agreement. The agreement of the two computations can be viewed as a statement of local mirror symmetry in the open string setup.

For world-sheets that wrap the fiber we obtain

$l\backslash m$	0	1	2	3	4	5
0	*	252	0	0	0	0
1	-240	*	300	300	300	300
2	240	780	*	-2280	-3180	-4380
3	-480	-2040	-6600	*	24900	39120
4	1200	6300	22080	74400	*	-315480
5	-3360	-21000	-82200	-276360	-957600	*
6	10080	73080	319200	1134000	3765000	13300560
7	-31680	-261360	-1265040	-4818240	-16380840	-54173880

Table 4.4: Invariants $N_{1,l,m}$ for the geometry (4.36).

$l\backslash m$	0	1	2	3	4
0	*	5130	-18504	0	0
1	-141444	*	-73170	-62910	-62910
2	-28200	-108180	*	544140	778560
3	85320	403560	1557000	*	-7639920
4	-285360	-1647540	-6485460	-24088680	*
5	1000440	6815160	29214540	106001100	392435460
6	-3606000	-28271880	-133294440	-505417320	-1773714840

Table 4.5: Invariants $N_{2,l,m}$ for the geometry (4.36).

It would be interesting to confirm some of these numbers by an independent computation.

Deformation of the non-compact involution brane

In [68] an involution brane in the local model $\mathcal{O}(-3)_{\mathbf{P}^2}$ has been studied. Similarly as in the previous case one expects to find a deformation of this brane by embedding it in the compact manifold and taking the limit of large elliptic fiber, $z_1 = 0$. In order to recover the involution brane of the local geometry we study the critical points near $z_3 = 1$ in the local coordinates

$$\tilde{z}_1 = z_1(-z_2)^{1/2}, \quad u = (-z_2)^{-1/2}(1-z_3)\,, \quad v = (-z_2)^{1/2}\,.$$

After transforming the Picard-Fuchs system to these variables, the solution corresponding to the superpotential has the following expansion

$$c\mathcal{W} = -v - \frac{35v^3}{9} + \frac{1}{2}uv^2 + \frac{200}{3}\tilde{z}_1 v^2 - \frac{u^2 v}{8} - 12320\tilde{z}_1^2 v - 60u\tilde{z}_1 v + \dots\ , \tag{4.38}$$

where c is a constant that will be fixed again by comparing the critical value with the results of [68]. In the decompactification limit $\tilde{z}_1 = 0$, the critical point of the superpotential is at $u = 0$, where we obtain the following expansion

$$c\mathcal{W}|_{crit} = -\sqrt{z_2} - \frac{35}{9}z_2^{3/2} - \frac{1001}{25}z_2^{5/2} + \dots\ , \tag{4.39}$$

The restricted superpotential satisfies the differential equation

$$\mathcal{L}_{bulk}\mathcal{W}|_{crit} = -\frac{\sqrt{z_2}}{8c}\,,$$

with $\mathcal{L}_{bulk}$ the Picard-Fuchs operator of the local geometry $\mathcal{O}(-3)_{\mathbf{P}^2}$. The above expressions at the critical point agree with the ones given in [68] for $c = 1$.

As might have been expected, the full superpotential (4.38) shows that the involution brane of the local model is non-trivially deformed in the compact CY manifold for $z_1 \neq 0$. It is not obvious that the modified multi-cover description of [79], which is adapted to real curves and differs from the original proposal of [55], can be generalized to obtain integral invariants for the deformations of the critical point in the z_1 direction. One suspects that an integral expansion in the sense of [79] exists only at critical points with an extra symmetry and for deformations that respect this symmetry. It will be interesting to study this further.

4.3.3 Branes on $\mathbf{X}_9^{(1,1,1,3,3)}$

As a third example we study branes on the two moduli CY $Z = \mathbf{X}_9^{(1,1,1,3,3)}$. Z is again an elliptic fibration over $\mathbf{P}^2$ and one can consider a similar compactification of the non-compact brane in $\mathcal{O}(-3)_{\mathbf{P}^2}$. The invariants for this geometry are reported in Appendix A.

Here we consider a different family of D7-branes which we expect to include a brane that exists at the Landau Ginzburg point of the two moduli Calabi–Yau. The mirror A brane is defined by

$$-|x_0|^2 + |x_1|^2 = c^1, \qquad \hat{l} = (-1, 1, 0, 0, 0, 0, 0) \ . \tag{4.40}$$

The polyhedron for the GLSM is

$\Delta(Z)$	$\nu_0 =$	(0,	0,	0,	0,	0)
	$\nu_1 =$	(0,	0,	0,	−1,	0)
	$\nu_2 =$	(0,	0,	−1,	0,	0)
	$\nu_3 =$	(0,	0,	1,	1,	0)
	$\nu_4 =$	(−1,	0,	1,	1,	0)
	$\nu_5 =$	(0,	−1,	1,	1,	0)
	$\nu_6 =$	(1,	1,	1,	1,	0)
$\Delta_\flat(Z, L) = \Delta \cup$	$\rho_1 =$	(0,	0,	0,	0,	−1)
	$\rho_2 =$	(0,	0,	0,	−1,	−1)

Table 4.6: Points of the enhanced polyhedron $\Delta_\flat$ for the geometry (4.40).

A suitable basis of relations for the charge vectors is

$$\begin{aligned} &l^1 = (-2, 0, 1, 1, 0, 0, 0, -1, 1), \quad l^2 = (0, 0, 0, -3, 1, 1, 1, 0, 0), \\ &l^3 = (-1, 1, 0, 0, 0, 0, 0, 1, -1), \end{aligned} \tag{4.41}$$

leading to the differential operators

$$\begin{aligned}\mathcal{L}_1 &= \theta_1(\theta_1-3\theta_2)(\theta_1-\theta_3)+z_1(\theta_1-\theta_3)(2\theta_1+1+\theta_3)(2\theta_1+2+\theta_3),\\ \mathcal{L}_2 &= \theta_2^3-z_2(\theta_1-3\theta_2)(\theta_1-3\theta_2-1)(\theta_1-3\theta_2-2),\\ \mathcal{L}_3 &= -\theta_3(\theta_1-\theta_3)-z_3(\theta_1-\theta_3)(2\theta_1+1+\theta_3)\ .\end{aligned} \tag{4.42}$$

The brane geometry on the B model side is defined by the two equations

$$\begin{aligned}p(Z^*) &= \sum a_i y_i = a_0x_1x_2x_3x_4x_5+a_1x_1^3+a_2x_2^3+a_3(x_3x_4x_5)^3+a_4x_3^9+a_5x_4^9+a_6x_5^9,\\ \mathcal{B}(E) &: \qquad y_0=y_1 \qquad \text{or} \qquad x_1x_2x_3x_4x_5=x_1^3\ .\end{aligned} \tag{4.43}$$

As in the previous cases, the deformations of the hypersurface $\mathcal{B}(E)$ are described by the periods on a K3 surface.

We are interested in a brane superpotential with critical point at $z_3=-1$. Choosing the following local coordinates centered around $z_3=-1$

$$u=(-z_1)^{-1/2}z_2^{-1/6}(z_3+1)\,,\quad v=(-z_1)^{1/2}z_2^{1/6}\quad x_2=z_2^{-1/3}\,,$$

we obtain the superpotential

$$c\,\mathcal{W}=-\frac{1}{2}ux_2+\frac{1}{24}u^3+210v^3+\frac{3}{4}vx_2^2-\frac{3}{8}u^2vx_2+\dots\ . \tag{4.44}$$

This superpotential has a critical point at $u=0$ and $x_2=0$. At the critical locus we have $v=z^{1/6}$, where z denotes the closed string modulus

$$z=-\frac{a_1^3a_2^3a_4a_5a_6}{a_0^9}\ .$$

The expansion of the superpotential at this critical locus reads

$$c\,\mathcal{W}|_{crit}=210\sqrt{z}+\frac{53117350}{3}z^{3/2}+\frac{18297568296042}{5}z^{5/2}+\frac{7182631458065952702}{7}z^{7/2}+\dots\ ,$$

As in the example of sect. 4.3.2 it is an interesting question to study the instanton expansion of the above expressions and its possible interpretation in terms of integral BPS invariants. We leave this for the future.

4.4 Summary and outlook

As proposed above, the open/closed string deformation space of the toric branes defined in [18] can be studied by mirror symmetry and toric geometry in a quite efficient way. The toric definition of the brane geometry in sect. 4.2 leads to the canonical Picard-Fuchs system (4.19), whose solutions determine the mirror maps and the superpotential.

The phase structure of the associated GLSM determines large volume regimes, where the superpotential has an disc instanton expansion with an interesting mathematical and physical interpretation.

Since the toric branes cover only a subset within the category of D-branes, e.g. matrix factorizations on the B model side, it is natural to ask for the precise relation between these two definitions. It is an interesting question to which extent it is possible to lift the machinery of toric geometry directly to the matrix factorization and to make contact with the works [101, 102]. On the positive side one notices that the class of toric branes is already rather large and not too special, as can be seen from the fact that the above framework covers all cases where explicit results have been obtained so far.

There are some other obvious questions left open by the above discussion, such as the geometric and physical interpretation of some of the objects appearing in the definition of the GLSM and the mirror B geometry, e.g. the appearance of the "enhanced polyhedra" $\Delta_\flat(Z,L)$ and K3 surfaces, which beg for an explanation. A discussion of these issues is beyond the scope of this work and will be given elsewhere, but here we outline some of the answers. As the reader may have noticed, the polyhedra $(\Delta_\flat(Z,L),\Delta_\flat^\star(Z^*,E))$ define Calabi-Yau fourfolds, which are the hallmark of F-theory compactifications with the same supersymmetry.[23] Another conclusive hint towards F-theory comes from the fact that we have effectively studied families of 7-branes on the B model side by intersecting a single equation with the Calabi–Yau hypersurface. In fact the "auxiliary geometry" defined in sect. 4.3 should be viewed as a physical 7-brane geometry and this interpretation suggests that the results of the GLSM determine also the Kähler metric on the open/closed deformation space.

[23] An M-theory interpretation of the 4-folds for local models has been given in [100].

5 Polynomial Structure of Topological String Amplitudes

We first show that the polynomial structure of the topological string partition function found by Yamaguchi and Yau for the quintic holds for an arbitrary Calabi-Yau manifold with any number of moduli. Furthermore, we generalize these results to the open topological string partition function as discussed recently by Walcher and reproduce his results for the real quintic. We then derive topological string amplitudes on local Calabi-Yau manifolds in terms of polynomials in finitely many generators of special functions. These objects are defined globally in the moduli space and lead to a description of mirror symmetry at any point in the moduli space. Holomorphic ambiguities of the anomaly equations are fixed by global information obtained from boundary conditions at few special divisors in the moduli space. As an illustration we compute higher genus orbifold Gromov-Witten invariants for $\mathbb{C}^3/\mathbb{Z}_3$ and $\mathbb{C}^3/\mathbb{Z}_4$. The work of this chapter appeared in [24, 25].

5.1 Introduction and summary

The holomorphic anomaly equation of the topological string [12, 13] relates the anti-holomorphic derivative of the genus g topological string partition function $\mathcal{F}^{(g)}$ with covariant derivatives of the partition functions of lower genus. This enables one to recursively determine the partition function at each genus up to a holomorphic ambiguity which has to be fixed by further information. A complete understanding of the holomorphic anomaly equation and its recursive procedure to determine the partition functions at every genus might lead to new insights in the understanding of the structure of the full topological string partition function $Z = \exp(\sum \lambda^{2g-2}\mathcal{F}^{(g)})$. For example in [14], Witten interpreted Z as a wave function for the quantization of the space $\mathrm{H}^3(X,\mathbb{R})$ of a Calabi-Yau X and the holomorphic anomaly equation as the background independence of this wave function. In [23], Yamaguchi and Yau discovered that the non-holomorphic part of the topological string partition function for the quintic can be written as a polynomial in a finite number of generators. This improves the method using Feynman rules proposed in [13]. This polynomial structure was used in [104] to solve the quintic up to genus 51 and was applied to other Calabi-Yau manifolds with one modulus in [104, 105].

The first aim of this work is to generalize the polynomial structure of the topological string partition function discovered in [23] to an arbitrary Calabi-Yau manifold with any number of moduli.[1] A related method for integrating the holomorphic anomaly equation using modular functions was presented in [107, 108].

Recently, an extension of the holomorphic anomaly equation which includes the open topological string was proposed by Walcher [22].[2] Its solution in terms of Feynman rules was proven soon after in [110]. The second task of this work is to extend Yamaguchi and Yau's polynomial construction to the open topological string. A similar generalization for the open topological string on the quintic appeared in [111].

The organization of this chapter is as follows. In the next section we briefly review the extended holomorphic anomaly equation and the initial correlation functions at low genus and number of holes which will be the starting point of the recursive procedure. Next we introduce the polynomial generators of the non-holomorphic part of the partition functions and show that holomorphic derivatives thereof can again be expressed in terms of these generators. As the initial correlation functions are expressions in these generators we will have thus shown that at every genus the partition functions will be again expressions in the generators. Afterwards we assign some grading to the generators and show that $\mathcal{F}^{(g,h)}_{i_1\ldots i_n}$, the partition function at genus g, with h holes and n insertions, will be a polynomial of degree $3g-3+3h/2+n$ in the generators. Finally, we determine the polynomial recursion relations and argue that, by a change of generators, the number of generators can be reduced by one. In order to solve the holomorphic anomaly equation it now suffices to make the most general Ansatz of the right degree in the generators for the partition function and use the recursion relation to match the coefficients. The deficiency of the recursive information contained in the anomaly equation lies in the additional purely holomorphic data which has to be determined by boundary conditions [12]. We discuss in detail the construction of the polynomial generators and analyze the freedom one has in choosing these. We identify the relevant boundary conditions at various expansion points and discuss how to use the fact that the polynomial expressions hold everywhere in moduli space to extract information.

We then apply the method to the real quintic and give the polynomial expressions for the partition functions and reproduce some recent results. We further apply the method to compute higher genus topological string amplitudes for local Calabi-Yau manifolds in terms of polynomials of a finite number of generating functions. The polynomial expression is globally defined and allows for an expansion of the topological string amplitudes at different points in moduli space. In particular we compute orbifold Gromov-Witten (GW) invariants which were recently studied on the physics and mathematics sides [107, 44, 112, 113][3] and make predictions for higher genus orbifold invariants for $\mathbb{C}^3/\mathbb{Z}_4$. This demonstrates the power of this framework for computations of mathematical invariants at various expansion

[1] This problem has independently solved in [106].
[2] A generalization of the holomorphic anomaly equations for the open topological string appeared in [109].
[3] We refer to [44] for a complete list of references.

points in moduli space.[4] The formalism is applied to some well studied local Calabi–Yau threefolds, namely local $\mathbb{P}_2$ which contains a $\mathbb{C}^3/\mathbb{Z}_3$ orbifold and local $\mathbb{P}_1 \times \mathbb{P}_1$ as well as local $\mathbb{F}_2$. The latter contains a $\mathbb{C}^3/\mathbb{Z}_4$ orbifold.
In a related work [115] the authors also use the polynomial construction to study local models.

5.2 Polynomial method

5.2.1 Holomorphic anomaly

In this work we consider the open topological string with branes as in [22]. The B-model on a Calabi–Yau manifold X depends on the space $\mathcal{M}$ of complex structures parameterized by coordinates z^i, $i = 1, ..., h^{1,2}(X)$. More precisely, the topological string partition function $\mathcal{F}^{(g,h)}$ at genus g with h boundaries is a section of a line bundle $\mathcal{L}^{2-2g-h}$ over $\mathcal{M}$ [22]. The line bundle $\mathcal{L}$ may be identified with the bundle of holomorphic $(3,0)$-forms Ω on X with first Chern class $G_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K$. Here K is the Kähler potential and $G_{i\bar{j}}$ the Kähler metric. Under Kähler transformations $K \to K(z^i, \bar{z}^{\bar{j}}) - \ln \phi(z^i) - \ln \bar{\phi}(\bar{z}^{\bar{j}})$, $\Omega \to \phi\Omega$ and more generally a section f of $\mathcal{L}^n \otimes \bar{\mathcal{L}}^{\bar{n}}$ transforms as $f \to \phi^n \bar{\phi}^{\bar{n}} f$. The fundamental objects of the topological string are the holomorphic three point couplings at genus zero C_{ijk} which can be integrated to the genus zero partition function F_0

$$C_{ijk} = D_i D_j D_k F_0, \qquad \bar{\partial}_{\bar{i}} C_{ijk} = 0, \tag{5.1}$$

and the disk amplitudes with two bulk insertions Δ_{ij} which are symmetric in the two indices but not holomorphic

$$\bar{\partial}_{\bar{i}} \Delta_{ij} = -C_{ijk} \Delta^k_{\bar{i}}, \qquad \Delta^k_{\bar{i}} = \Delta_{\bar{i}\bar{j}} \mathrm{e}^K G^{k\bar{j}}. \tag{5.2}$$

Here $\Delta_{\bar{i}\bar{j}}$ denotes the complex conjugate of Δ_{ij} and $D_i = \partial_i + \cdots = \frac{\partial}{\partial z_i} + \ldots$ denotes the covariant derivative on the bundle $\mathcal{L}^m \otimes \mathrm{Sym}^n T^*$ where m and n follow from the context. T^* is the cotangent bundle of $\mathcal{M}$ with the standard connection coefficients $\Gamma^i_{jk} = G^{i\bar{i}} \partial_j G_{k\bar{i}}$. The connection on the bundle $\mathcal{L}$ is given by the first derivatives of the Kähler potential $K_i = \partial_i K$.[5]

The correlation function at genus g with h boundaries and n insertions $\mathcal{F}^{(g,h)}_{i_1 \cdots i_n}$ is only non-vanishing for $(2g - 2 + h + n) > 0$. They are related by taking covariant derivatives as this represents insertions of chiral operators in the bulk, e.g. $D_i \mathcal{F}^{(g,h)}_{i_1 \cdots i_n} = \mathcal{F}^{(g,h)}_{i i_1 \cdots i_n}$. Furthermore, in [22] it is shown that the genus g partition function with h holes is recursively related to lower genus partition functions and to partition functions with less boundaries. This is

[4] See [108, 114] for other approaches addressing this problem.
[5] See section two of [13] for further background material.

expressed for $(2g-2+h)>0$ by an extension of the holomorphic anomaly equations of BCOV [13]

$$\bar{\partial}_{\bar{i}}\mathcal{F}^{(g,h)} = \frac{1}{2}\bar{C}^{jk}_{\bar{i}} \sum_{\substack{g_1+g_2=g\\ h_1+h_2=h}} D_j\mathcal{F}^{(g_1,h_1)}D_k\mathcal{F}^{(g_2,h_2)} + \frac{1}{2}\bar{C}^{jk}_{\bar{i}}D_jD_k\mathcal{F}^{(g-1,h)} - \Delta^j_{\bar{i}}D_j\mathcal{F}^{(g,h-1)}, \quad (5.3)$$

where

$$\bar{C}^{ij}_{\bar{k}} = \bar{C}_{\bar{i}\bar{j}\bar{k}}G^{i\bar{i}}G^{j\bar{j}}\,\mathrm{e}^{2K}, \qquad \bar{C}_{\bar{i}\bar{j}\bar{k}} = \overline{C_{ijk}}. \quad (5.4)$$

These equations, supplemented by

$$\bar{\partial}_{\bar{i}}\mathcal{F}^{(1,0)}_j = \frac{1}{2}C_{jkl}\bar{C}^{kl}_{\bar{i}} + (1-\frac{\chi}{24})G_{j\bar{i}}, \quad (5.5)$$

$$\bar{\partial}_{\bar{i}}\mathcal{F}^{(0,2)}_j = -\Delta_{jk}\Delta^k_{\bar{i}} + \frac{N}{2}G_{j\bar{i}}, \quad (5.6)$$

and special geometry, determine all correlation functions up to holomorhpic ambiguities. In (5.5), χ is the Euler character of the manifold and in (5.6) N is the rank of a bundle over $\mathcal{M}$ in which the charge zero ground states of the open string live. Similar to the closed topological string [13], a solution of the recursion equations is given in terms of Feynman rules. These Feynman rules have been proven for the open topological string in [110].

The propagators for these Feynman rules contain the ones already present for the closed topological string S, S^i, S^{ij} and new propagators Δ, Δ^i. Note that these are not the same as the Δ, with or without indices, that appear in [13] which there denote the inverses of the S propagators. S, S^i and S^{ij} are related to the three point couplings C_{ijk} as

$$\partial_{\bar{i}}S^{ij} = \bar{C}^{ij}_{\bar{i}}, \qquad \partial_{\bar{i}}S^j = G_{i\bar{i}}S^{ij}, \qquad \partial_{\bar{i}}S = G_{i\bar{i}}S^i. \quad (5.7)$$

By definition, the propagators S, S^i and S^{ij} are sections of the bundles $\mathcal{L}^{-2}\otimes\mathrm{Sym}^m T$ with $m=0,1,2$. Δ and Δ^i are related to the disk amplitudes with two insertions by

$$\bar{\partial}_{\bar{i}}\Delta^j = \Delta^j_{\bar{i}}, \qquad \bar{\partial}_{\bar{i}}\Delta = G_{i\bar{i}}\Delta^i. \quad (5.8)$$

They are sections of $\mathcal{L}^{-1}\otimes\mathrm{Sym}^m T$ with $m=0,1$. The vertices of the Feynman rules are given by the correlation functions $\mathcal{F}^{(g,h)}_{i_1\cdots i_n}$.

Note that the anomaly equation (5.3), as well as the definitions (5.7) and (5.8), leave the freedom of adding holomorphic functions under the $\overline{\partial}$ derivatives as integration constants. This freedom is referred to as holomorphic ambiguities.

5.2.2 Initial correlation functions

To be able to apply a recursive procedure for solving the holomorphic anomaly equation, we first need to have some initial data to start with. In this case the initial data consists of the first non-vanishing correlation functions. The first non-vanishing correlation functions at genus zero without any boundaries are the holomorphic three point couplings $\mathcal{F}^{(0,0)}_{ijk} \equiv C_{ijk}$. At genus zero with one boundary, the first non-vanishing correlation functions are the disk amplitudes with two insertions. The holomorhpic anomaly equation (5.2) is solved with (5.8) by

$$\mathcal{F}^{(0,1)}_{ij} \equiv \Delta_{ij} = -C_{ijk}\Delta^k + g_{ij}\,, \tag{5.9}$$

with some holomorphic functions g_{ij}. Finally we solve (5.5) and (5.6). (5.5) can be integrated with (5.7) to

$$\mathcal{F}^{(1,0)}_i = \frac{1}{2}C_{ijk}S^{jk} + (1-\frac{\chi}{24})K_i + f^{(1,0)}_i, \tag{5.10}$$

with ambiguity $f^{(1,0)}_i$. For the annulus we find

$$\begin{aligned} \bar{\partial}_{\bar{i}}\mathcal{F}^{(0,2)}_j &= C_{jkl}\Delta^l\bar{\partial}_{\bar{i}}\Delta^k + \bar{\partial}_{\bar{i}}(-g_{jk}\Delta^k + \frac{N}{2}K_j) \\ &= \bar{\partial}_{\bar{i}}(\frac{1}{2}C_{jkl}\Delta^k\Delta^l - g_{jk}\Delta^k + \frac{N}{2}K_j)\,, \end{aligned} \tag{5.11}$$

and therefore

$$\mathcal{F}^{(0,2)}_i = \frac{1}{2}C_{ijk}\Delta^j\Delta^k - g_{ij}\Delta^j + \frac{N}{2}K_i + f^{(0,2)}_i, \tag{5.12}$$

where $f^{(0,2)}_i$ are holomorhpic. As can be seen from these expressions, the non-holomorphicity of the correlation functions only comes from the propagators together with K_i. Indeed, we will now show that this holds for all partition functions $\mathcal{F}^{(g,h)}$.

5.2.3 Non-holomorphic generators

From the holomorphic anomaly equation and its Feynman rule solution it is clear that at every genus g with h boundaries the building blocks of the partition function $\mathcal{F}^{(g,h)}$ are the propagators S^{ij}, S^i, S, Δ, Δ^i and vertices $\mathcal{F}^{(g',h')}_{i_1\cdots i_n}$ with $g' < g$ or $h' < h$. Here it will be shown that all the non-holomorphic content of the partition functions $\mathcal{F}^{(g,h)}$ can be expressed in terms of a finite number of generators. The generators we consider are the propagators S^{ij}, S^i, S, Δ^i, Δ as well as K_i, the partial derivative of the Kähler potential. This construction is a generalization of Yamaguchi and Yau's polynomial construction for the quintic [23] where multi derivatives of the connections were used as generators. The propagators of the closed topological string as building blocks were also used recently by Grimm, Klemm, Marino and Weiss [108] for a direct integration of the topological string using modular properties of the big moduli space, where all propagators can be treated on equal footing.

In the following we prove that if the anti-holomorphic part of $\mathcal{F}^{(g,h)}$ is expressed in terms of the generators S^{ij}, S^i, S, Δ^i, Δ and K_i, then all covariant derivatives thereof are also expressed in terms of these generators. As the correlation functions for small genus and small number of boundaries are expressed in terms of the generators, it follows by induction, that all $\mathcal{F}^{(g,h)}$ are expressed in terms of the generators.

The covariant derivatives contain the Christoffel connection and the connection K_i of $\mathcal{L}$. By integrating the special geometry relation

$$\bar{\partial}_{\bar{i}}\Gamma^l_{ij} = \delta^l_i G_{j\bar{i}} + \delta^l_j G_{i\bar{i}} - C_{ijk}C^{kl}_{\bar{i}}\,, \tag{5.13}$$

to

$$\Gamma^l_{ij} = \delta^l_i K_j + \delta^l_j K_i - C_{ijk}S^{kl} + s^l_{ij}\,, \tag{5.14}$$

where s^l_{ij} denote holomorphic functions that are not fixed by the special geometry relation, we can express the Christoffel connection in terms of our generators. What remains is to show that the covariant derivatives of all generators are again expressed in terms of the generators. To obtain expressions for the covariant derivatives of the generators we first take the anti-holomorphic derivative of the expression, then use (5.13) and write the result as a total anti-holomorphic derivative again, for example

$$\partial_{\bar{i}}(D_i S^{jk}) = \partial_{\bar{i}}(\delta^j_i S^k + \delta^k_i S^j - C_{imn}S^{mj}S^{nk}). \tag{5.15}$$

This equation determines D_iS^{jk} up to a holomorphic term. In this manner we obtain the following relations

$$\begin{aligned}
D_iS^{jk} &= \delta^j_i S^k + \delta^k_i S^j - C_{imn}S^{mj}S^{nk} + h^{jk}_i, && (5.16)\\
D_iS^j &= 2\delta^j_i S - C_{imn}S^m S^{nj} + h^{jk}_i K_k + h^j_i, && (5.17)\\
D_iS &= -\frac{1}{2}C_{imn}S^m S^n + \frac{1}{2}h^{mn}_i K_m K_n + h^j_i K_j + h_i, && (5.18)\\
D_iK_j &= -K_iK_j - C_{ijk}S^k + C_{ijk}S^{kl}K_l + h_{ij}, && (5.19)\\
D_i\Delta^j &= \delta^j_i\Delta - g_{ik}S^{kj} + g^j_i, && (5.20)\\
D_i\Delta &= -g_{ij}S^j + g^j_i K_j + g_i, && (5.21)
\end{aligned}$$

where h^{jk}_i, h^j_i, h_i, h_{ij}, g^j_i and g_i denote holomorphic functions (ambiguities). This completes our proof that all non-holomorphic parts of $\mathcal{F}^{(g,h)}$ can be expressed in terms of the generators. Next, we will determine recursion relations, assign some grading to the generators and show that $\mathcal{F}^{(g,h)}_{i_1\cdots i_n}$ is a polynomial of degree $3g-3+3h/2+n$.

5.2.4 Polynomial recursion relation

Let us now determine some recursion relations from the holomorhpic anomaly equation. Computing the $\bar{\partial}_{\bar{i}}$ derivative of $\mathcal{F}^{(g,h)}$ expressed in terms of S^{ij}, S^i, S, Δ^i, Δ, K_i, and using

(5.3) one obtains

$$\bar{C}^{jk}_{\bar{i}} \frac{\partial \mathcal{F}^{(g,h)}}{\partial S^{jk}} + \Delta^{j}_{\bar{i}} \frac{\partial \mathcal{F}^{(g,h)}}{\partial \Delta^{j}} + G_{i\bar{i}} \left(\frac{\partial \mathcal{F}^{(g,h)}}{\partial K_i} + S^i \frac{\partial \mathcal{F}^{(g,h)}}{\partial S} + S^{ij} \frac{\partial \mathcal{F}^{(g,h)}}{\partial S^j} + \Delta^i \frac{\partial \mathcal{F}^{(g,h)}}{\partial \Delta} \right)$$
$$= \frac{1}{2} \bar{C}^{jk}_{\bar{i}} \sum_{\substack{g_1+g_2=g \\ h_1+h_2=h}} D_j \mathcal{F}^{(g_1,h_1)} D_k \mathcal{F}^{(g_2,h_2)} + \frac{1}{2} \bar{C}^{jk}_{\bar{i}} D_j D_k \mathcal{F}^{(g-1,h)} - \Delta^{j}_{\bar{i}} D_j \mathcal{F}^{(g,h-1)}. \quad (5.22)$$

Assuming linear independence of $\bar{C}^{jk}_{\bar{i}}$, $\Delta^{j}_{\bar{i}}$ and $G_{i\bar{i}}$ the equation splits into three equations

$$\frac{\partial \mathcal{F}^{(g,h)}}{\partial S^{ij}} = \frac{1}{2} \sum_{\substack{g_1+g_2=g \\ h_1+h_2=h}} D_i \mathcal{F}^{(g_1,h_1)} D_j \mathcal{F}^{(g_2,h_2)} + \frac{1}{2} D_i D_j \mathcal{F}^{(g-1,h)}, \quad (5.23)$$

$$\frac{\partial \mathcal{F}^{(g,h)}}{\partial \Delta^{i}} = -D_i \mathcal{F}^{(g,h-1)}, \quad (5.24)$$

$$0 = \frac{\partial \mathcal{F}^{(g,h)}}{\partial K_i} + S^i \frac{\partial \mathcal{F}^{(g,h)}}{\partial S} + S^{ij} \frac{\partial \mathcal{F}^{(g,h)}}{\partial S^j} + \Delta^i \frac{\partial \mathcal{F}^{(g,h)}}{\partial \Delta}. \quad (5.25)$$

The last equation (5.25) can be rephrased as the condition that $\mathcal{F}^{(g,h)}$ does not depend explicitly on K_i by making a suitable change of generators

$$\tilde{S}^{ij} = S^{ij}, \quad (5.26)$$
$$\tilde{S}^{i} = S^{i} - S^{ij} K_j, \quad (5.27)$$
$$\tilde{S} = S - S^i K_i + \frac{1}{2} S^{ij} K_i K_j, \quad (5.28)$$
$$\tilde{\Delta}^i = \Delta^i, \quad (5.29)$$
$$\tilde{\Delta} = \Delta - \Delta^i K_i, \quad (5.30)$$
$$\tilde{K}_i = K_i, \quad (5.31)$$

i.e. $\partial \mathcal{F}^{(g,h)}/\partial \tilde{K}_i = 0$ for $\mathcal{F}^{(g,h)}$ as a function of the tilded generators. Let us now assign a grading to the generators and covariant derivatives, which is naturally inherited from the $U(1)$ grading given by the background charge for the $U(1)$ current inside the twisted $\mathcal{N} = 2$ superconformal algebra. The covariant holomorphic derivatives D_i carry charge $+1$ as they represent the insertion of a chiral operator of $U(1)$ charge $+1$. As K_i is part of the connection, it is natural to assign charge $+1$ to K_i. From the definitions (5.7) and (5.8) one may assign the charges $1/2, 1, 3/2, 2, 3$ to the generators Δ^i, S^{ij}, Δ, S^i, S, respectively. The correlation functions $\mathcal{F}^{(g,h)}_{i_1 \cdots i_n}$ for small g and h are a polynomial of degree $3g - 3 + 3h/2 + n$ in the generators. By the recursion relations, it immediately follows that this holds for all g and h.

5.2.5 Constructing the propagators

The application of the polynomial method requires the construction of the propagators and determination of the holomorphic functions that appear in Eqs. (5.14-5.19). It should be noted that the discussion in the following holds in general and can be applied both for compact and local models. First one starts by defining the propagators S^{ij}, using the special geometry relation Eq. (5.14). We pick therefore a coordinate z_* for which C_{*jk} is invertible as a $n \times n$ matrix, this yields:

$$S^{kl} = (C_*^{-1})^{kj}(\delta_*^l K_j + \delta_j^l K_* - \Gamma_{*j}^l + s_{*j}^l) \ . \tag{5.32}$$

We begin by analyzing the freedom in the definition of the propagators which is related to a choice of holomorphic s_{ij}^l which have to satisfy constraints coming from the symmetry of S^{kl} and from the special geometry relations Eq. (5.14) with $i \neq *$. A counting of components of s_{ij}^l minus constraints gives

$$\underbrace{\frac{n^2(n+1)}{2}}_{\text{components of } s_{ij}^l} - \underbrace{\frac{n(n-1)}{2}}_{\text{symmetry of } S^{ij}} - \underbrace{\frac{n^2(n-1)}{2}}_{\text{remaining special geometry}} = \frac{n(n+1)}{2} \ . \tag{5.33}$$

This is equal to the number of components of a symmetric holomorphic $\mathcal{E}^{ij}$. This $\mathcal{E}^{ij}$ can be added to S^{ij} while still satisfying the defining requirement $\partial_{\bar{i}} S^{ij} = \bar{C}_{\bar{i}}^{ij}$. Two choices $\tilde{s}_{ij}^l$ and s_{ij}^l are related by:

$$\tilde{s}_{ij}^l = C_{ijk}\mathcal{E}^{kl} + s_{ij}^l \ . \tag{5.34}$$

In the next step we tackle Eq. (5.16) where it is obvious that h_i^{jk} for ($i \neq j$ and $i \neq k$) can already be computed with no freedom left, also the differences $h_i^{ii} - 2h_j^{ji}$ for $i \neq j$ can be computed. This leaves us with n free holomorphic h_i^{ii}. These are related to a freedom in S^i which we can define from Eq. (5.16)

$$S^i = \frac{1}{2}\left(D_i S^{ii} + C_{imn} S^{mi} S^{ni} - h_i^{ii}\right) \ . \tag{5.35}$$

Moving on to Eq. (5.17) we can now compute h_j^i with $i \neq j$ and we obtain $n-1$ relations $h_i^i - h_j^j$ for $i \neq j$. This leaves the freedom to choose just one holomorphic h_i^i. Again this is related to a freedom in S which can be defined from Eq. (5.17)

$$S = \frac{1}{2}\left(D_i S^i + C_{imn} S^m S^{ni} - h_i^{ik} K_k - h_i^i\right) \ . \tag{5.36}$$

The remaining h_{ij} in Eq. (5.19) can now be computed from the choices already made. This whole analysis of freedom in defining the propagators shows that given a set of propagators one can always add holomorphic pieces to each one. Of course holomorphic shifts in S^{ij} affect S^i and S.

Holomorphic freedom and simplification

Choosing a set of propagators amounts thus to choosing their holomorphic parts, their non-holomorphic part is fixed by the defining Eqs. (5.7). The freedom one has in the construction can be summarized as follows

$$\begin{aligned} S^{ij} &\to S^{ij} + \mathcal{E}^{ij} , & (5.37)\\ S^{i} &\to S^{i} + \mathcal{E}^{ij} K_j + \mathcal{E}^{i} , & (5.38)\\ S &\to S + \frac{1}{2}\mathcal{E}^{ij} K_i K_j + \mathcal{E}^{i} K_i + \mathcal{E} , & (5.39)\end{aligned}$$

all the holomorphic quantities in the polynomial setup change accordingly.

So $\mathcal{E}^{ij}, \mathcal{E}^{i}$ and $\mathcal{E}$ contain all the freedom in this polynomial setup. One special choice of propagators consistent with the Eqs. (5.16-5.19) is such that their holomorphic part vanishes. In that case, given the holomorphic limit of the connection parts as analyzed by BCOV [12]

$$K_{\text{hol}} = -\log X^0, \quad (\Gamma^k_{ij})_{\text{hol}} = \frac{\partial z^k}{\partial t^a}\frac{\partial^2 t^a}{\partial z^i \partial z^j} , \tag{5.40}$$

where t^a denote special[6] coordinates and X^0 the period used to define these. All the holomorphic quantities in the equations become trivial apart from s^l_{ij} of Eq. (5.14) and h_{ij} of Eq. (5.19) which are expressed by

$$\begin{aligned} s^l_{ij} &= \frac{\partial z^l}{\partial t^a}\frac{\partial^2 t^a}{\partial z^i \partial z^j} + \delta^l_i \partial_j \log X^0 + \delta^l_j \partial_i \log X^0, \\ h_{ij} &= -\partial_i \partial_j \log X^0 + \frac{\partial z^m}{\partial t^a}\frac{\partial^2 t^a}{\partial z^i \partial z^j}\,\partial_m \log X^0 + \partial_i \log X^0\,\partial_j \log X^0. & (5.41)\end{aligned}$$

This choice is of little use however as the polynomial part of the topological string amplitudes would be zero in the holomorphic limit and all the interesting information which allows to compute invariants would be encoded in the ambiguity. The goal is thus to use the insights of the polynomial structure coming from the non-holomorphic side of topological string theory to organize the amplitudes in a tractable form. In particular this means that we will try to absorb all the nontrivial series appearing in the holomorphic limit inside the holomorphic part of the propagators so that all remaining purely holomorphic quantities are simple closed expressions. Having done that the holomorphic ambiguity at every genus becomes a simple closed expression.
The periods giving t^a and X^0 are computed patch-wise in the moduli space of complex structures by solving the Picard-Fuchs system of differential equations. In general the solutions are given in terms of series which have a finite radius of convergence and hence s^l_{ij} and h_{ij} will not have a closed form as can be seen from Eq. (5.41). Our guideline for choosing the holomorphic part of the propagators is to keep these expressions simple. We note that there are cases where it is possible to both have propagators that vanish in the holomorphic limit *and* simple expressions for s^l_{ij} and h_{ij}, namely in the cases where one has a:

[6] Or, more generally, canonical [12].

- *Constant Period*
  In local models the constant period is a solution of the Picard-Fuchs system, accordingly the holomorphic limit of derivatives of the Kähler potential vanish. If we take Eq. (5.19) as a definition for the propagator S^i

$$S^k = (C_*^{-1})^{jk}(-D_*K_j - K_*K_j + C_{*jk}S^{kl}K_l + h_{*j}), \tag{5.42}$$

  we see from Eq. (5.40) that it is natural to choose its holomorphic limit to be zero which leads to vanishing h_{ij}. In this case the holomorphic limit of the remaining propagator S vanishes too. This fact is referred to in the literature as triviality of these propagators for local models. We emphasize however that this actually only means that it is natural to choose the *holomorphic limit* of these propagators to be zero. The propagators are not zero as they still are local potentials for the anti-holomorphic Yukawa couplings according to the definition Eq. (5.7). The full topological string amplitudes expressed in terms of polynomials would still contain these quantities. Doing the computations in order to fix the ambiguity and to extract A-model invariants however does not require keeping track of these propagators.

- *Special Mirror Map*
  In the two parameter local models that we study we encounter the situation where one of the mirror maps just depends on one of the two parameters. From Eq. (5.40) we see that in that case the holomorphic limit of the Christoffel connection with mixed lower indices vanishes. The latter appears in the definition of the propagators (5.32) if we pick as $*$ the coordinate on which the mirror map does not depend. Accordingly a simple s_{ij}^k leads to a trivial holomorphic limit for the corresponding propagator. In these cases we will later show that the corresponding propagators vanish identically and the models reduce effectively to one parameter models as noted before in [64] and in [107].

5.2.6 Holomorphic ambiguity and boundary conditions

To reconstruct the full topological string amplitudes we have to determine the purely holomorphic part of the polynomial. This holomorphic ambiguity can be fixed by imposing various boundary conditions at special points in the moduli space.

The leading behavior at large complex structure[7] was computed in [13, 12, 116, 117, 67, 118]. In particular the contribution from constant maps is

$$\mathcal{F}^{(g)}|_{q_a=0} = (-1)^g \frac{\chi}{2} \frac{|B_{2g}B_{2g-2}|}{2g\,(2g-2)\,(2g-2)!}\,, \quad g > 1, \tag{5.43}$$

[7] We will use the term "large complex structure" to denote the expansion point in the moduli space of the B-model which is mirror to large volume on the A-side.

where q_a denote the exponentiated mirror maps at large radius.

The leading singular behavior of the partition functions at a conifold locus has been determined in [13, 12, 119, 120, 117, 67]

$$\mathcal{F}^{(g)}(t_c) = \frac{B_{2g}}{2g(2g-2)t_c^{2g-2}} + \mathcal{O}(t_c^0), \qquad g > 1 \tag{5.44}$$

Here $t_c \sim \Delta$ is the canonical coordinate at the discriminant locus $\Delta = 0$ of a simple conifold. In particular the leading singularity in (5.44) as well as the absence of sub-leading singular terms follows from the Schwinger loop computation of [117, 67], which computes the effect of the extra massless hyper-multiplet in the space-time theory [121]. The singular structure and the "gap" of sub-leading singular terms have been also observed in the dual matrix model [122] and were first used in [123, 104] to fix the holomorphic ambiguity to very high genus.

Note that the space-time derivation of [117, 67] is not restricted to the conifold case and applies also to Calabi–Yau singularities which give rise to a different spectrum of extra massless vector and hyper-multiplets in space-time. So more generally one expects the singular structure

$$\mathcal{F}^{(g)}(t_c) = b\frac{B_{2g}}{2g(2g-2)t_c^{2g-2}} + \mathcal{O}(t_c^0), \qquad g > 1 \tag{5.45}$$

with $t_c \sim \Delta^\gamma$, $\gamma > 0$. The coefficient of the Schwinger loop integral is a weighted trace over the spin of the particles [121, 120] leading to the prediction $b = n_H - n_V$ for the coefficient of the leading singular term. In section 5.4.3 we will consider an example with a singularity that gives rise to a $SU(2)$ gauge theory in space-time and find agreement with the singular behavior and the generalized gap structure predicted by the Schwinger loop integral.

The singular behavior is taken into account by the local ansatz

$$\text{hol.ambiguity} \sim \frac{p(\tilde{z}_i)}{\Delta^{(2g-2)}}, \tag{5.46}$$

for the holomorphic ambiguity near $\Delta = 0$, where $p(\tilde{z}_i)$ is a priori a series in the local coordinates $\tilde{z}_i$ near the singularity. Patching together the local informations at all the singularities with the boundary divisors $z_i \to \infty$ for one or more i it follows however that the nominator $p(z_i)$ is generically a polynomial of low degree in the z_i. Here z_i denote the natural coordinates centered at large complex structure, $z_i = 0\ \forall i$. Generically the $\mathcal{F}^{(g)}, g > 1$ are regular at a boundary divisor $z_i \to \infty$ from which it follows that the degree of $p(z_i)$ in z_i is smaller or equal to the maximum power of z_i appearing in the discriminants in the denominator.[8]

[8]It can also happen, that a boundary divisor $z_i = \infty$ gives rise to a singularity of the type (5.44). For compact manifolds one has to take into account the effect of gauge transformations between different patches.

The finite number of coefficients in $p(z_i)$ is constrained by (5.45). In the computations for the local Calabi–Yau models considered below it turns out that the boundary conditions described above are sufficient to fix the holomorphic ambiguities.

5.3 Application to the real quintic

As an example of our polynomial construction of the partition functions $\mathcal{F}^{(g,h)}$ we consider the real quintic

$$X := \{P(x) = 0\} \subset \mathbb{P}^4,$$

where P is a homogeneous polynomial of degree 5 in 5 variables $x_1, \ldots, x_5$ with real coefficients. The real locus

$$L = \{x_i = \bar{x}_i\},$$

is a Lagrangian sub-manifold on which the boundary of the Riemann surface can be mapped.

For the closed topological string the polynomial construction was discovered by Yamaguchi and Yau in [23] and has been used in [104] to calculate $\mathcal{F}^{(g,0)}$ up to $g = 51$. The open string case was analyzed in [22, 79] where the real quintic is given as an example for solving the extended holomorphic anomaly equation. We will follow the notation of these two papers.

The mirror quintic has one complex structure modulus, which will be denoted by z. To parameterize the holomorphic ambiguities we introduce as a holomorphic generator the inverse of the discriminant

$$P = \frac{1}{1 - 5^5 z}. \tag{5.47}$$

The Yukawa coupling is given by

$$C_{zzz} = 5P/z^3. \tag{5.48}$$

For computational convenience we use instead of the generators S^{zz}, S^z, S, Δ^z and Δ the generators

$$T^{zz} = 5P\frac{S^{zz}}{z^2}, \quad T^z = 5P\frac{S^z}{z}, \quad T = 5PS, \quad \mathcal{E}^z = P^{1/2}\frac{\Delta^z}{z} \quad \text{and} \quad \mathcal{E} = P^{1/2}\Delta. \tag{5.49}$$

To obtain explicit forms of the generators we start with the integrated special geometry relation (5.14) and choose similar to [64]

$$s^z_{zz} = -1/z \tag{5.50}$$

in order to cancel the singular term in the holomorhpic limit of Γ^z_{zz}. In the language of [13] this corresponds to a gauge choice of $f = z^{-1/2}$ and $v = 1$. This choice of holomorhpic ambiguities fixes the propagators T^{zz}, T^z and T as

$$T^{zz} = 2\theta K - z\Gamma^z_{zz} - 1, \tag{5.51}$$
$$T^z = (\theta K)^2 - \theta^2 K - \frac{1}{4}, \tag{5.52}$$
$$T = \left(\frac{1}{5}P - \frac{9}{20}\right)\left(\theta K - \frac{1}{2}\right) + \frac{1}{2}\left(\theta T^z - (P-1)T^z\right), \tag{5.53}$$

with $\theta = z\frac{\partial}{\partial z}$. This choice of generators leads to the following ambiguities in the derivative relations of the generators (5.16)-(5.19)

$$5Ph^{zz}_z/z = -\frac{2}{5}P + \frac{9}{10}, \tag{5.54}$$
$$5Ph^z_z = \frac{1}{5}P - \frac{9}{20}, \tag{5.55}$$
$$5Pzh_z = -\frac{101}{1250}P + \frac{2241}{20000}, \tag{5.56}$$
$$z^2 h_{zz} = -\frac{1}{4}. \tag{5.57}$$

For the open string generators $\mathcal{E}^z$ and $\mathcal{E}$ we make the same choice as in [22] by setting

$$g_{zz} = 0 \qquad \text{and} \tag{5.58}$$
$$g^z_z = 0, \tag{5.59}$$

which leads to

$$\mathcal{E}^z = -\frac{1}{5}P^{-1/2}z^2\Delta_{zz}, \tag{5.60}$$
$$\mathcal{E} = -\frac{1}{2}(P-1)\mathcal{E}^z + \theta\mathcal{E}^z - T^{zz}\mathcal{E}^z + (\theta K)\mathcal{E}^z. \tag{5.61}$$

Finally, taking the holomorphic limit of (5.21) we obtain the last ambiguity in the derivative relations

$$zg_z = -\frac{3}{4}z^{1/2}. \tag{5.62}$$

Next, we fix the ambiguities for the initial correlation functions (5.9), (5.10) and (5.12) as in [22] and obtain

$$z^2\mathcal{F}^{(0,1)}_{zz} = -5P^{1/2}\mathcal{E}^z, \tag{5.63}$$
$$z\mathcal{F}^{(1,0)}_z = \frac{28}{3}\theta K + \frac{1}{2}T^{zz} + \frac{1}{12}P - \frac{13}{6}, \tag{5.64}$$
$$z\mathcal{F}^{(0,2)}_z = \frac{5(\mathcal{E}^z)^2}{2} + \frac{\theta K}{2} + \frac{3P}{250} - \frac{3}{250}. \tag{5.65}$$

It is now straightforward to use our method to determine higher $\mathcal{F}^{(g,h)}$ by writing the most general polynomial of degree $3g-3+3h/2$ in the generators $\tilde{T}^{zz}$, $\tilde{T}^{z}$, $\tilde{T}$, $\tilde{\mathcal{E}}^{z}$ and $\tilde{\mathcal{E}}$ and using the polynomial recursion relations. For $\mathcal{F}^{(2,0)}$ and $\mathcal{F}^{(3,0)}$ the gap condition at the conifold point [104] and the known expressions for the contribution of constant maps is enough to fix the holomorphic ambiguities and we give the explicit expressions in Appendix B.1. For $\mathcal{F}^{(1,1)}$ and $\mathcal{F}^{(0,3)}$ the vanishing of the first two instanton numbers fixes the ambiguities and read

$$\mathcal{F}^{(1,1)} = \frac{28\tilde{\mathcal{E}}}{3\sqrt{P}} + \frac{13\tilde{\mathcal{E}}^{z}}{6\sqrt{P}} - \frac{\tilde{\mathcal{E}}^{z}\sqrt{P}}{12} - \frac{\tilde{\mathcal{E}}^{z}\tilde{T}^{zz}}{2\sqrt{P}} - \frac{9\sqrt{z}P}{40} + \frac{211\sqrt{z}}{10}, \tag{5.66}$$

$$\mathcal{F}^{(0,3)} = \frac{1887\sqrt{z}}{2500} + \frac{\tilde{\mathcal{E}}}{2\sqrt{P}} + \frac{3\tilde{\mathcal{E}}^{z}}{250\sqrt{P}} - \frac{5(\tilde{\mathcal{E}}^{z})^{3}}{6\sqrt{P}} - \frac{3\tilde{\mathcal{E}}^{z}\sqrt{P}}{250} - \frac{3\sqrt{z}P}{625}. \tag{5.67}$$

In Appendix B.1 we also give the solution of $\mathcal{F}^{(1,2)}$ and $\mathcal{F}^{(2,1)}$ up to the holomorphic ambiguities. It would be interesting to fix this ambiguities by some further input.

5.4 Application to local mirror symmetry

Mirror symmetry in topological string theory refers to the equivalence of the A-model on a family of target spaces X_t which are related by deformations of Kähler structure on one side and the B-model on the family of target spaces Y_z which are related by deformations of complex structure on the other side. The mirror map $t(z)$, which represents the matching between the deformation spaces (moduli spaces) was first found at the large Kähler/Complex Structure expansion point [6]. This matching between the theories on both sides is believed to exist everywhere in the moduli space which has generically different phases [7, 9, 8]. The precise matching between the A-model and the B-model has to be found for each point in the moduli [12]. We will do this for special points in the case of some non-compact, i.e., local models. Local mirror symmetry has been developed in [124, 125, 126, 127, 128]. For reviews of many subsequent developments and a list of references see [38, 37].

The models we consider are described torically on the A-side. The A-side is most compactly described by giving the set of charge vectors $\mathbf{l}^{(a)}$ with $a = 1, \ldots, \dim h^{(1,1)}(X)$. The number of components of each vector corresponds to the number of homogeneous coordinates on the toric variety. The B-model moduli space is described by the secondary fan which is obtained from the columns of the matrix of charge vectors. This description is very useful for obtaining the right coordinates describing each phase. For the polynomial construction we analyze the information that can be obtained from each phase.

5.4.1 Local $\mathbb{P}^2$

Local $\mathbb{P}^2$ denotes the anti-canonical bundle over $\mathbb{P}^2$, $\mathcal{O}(-3) \to \mathbb{P}^2$, which can be obtained by taking one Kähler parameter of a two parameter compact Calabi-Yau to infinity. The compact CY is a torus fibration which is described by the charge vectors

$$l^{(a)} = \left(\begin{array}{c|cccccc} 0 & 0 & 0 & -3 & 1 & 1 & 1 \\ -6 & 3 & 2 & 1 & 0 & 0 & 0 \end{array} \right) . \tag{5.68}$$

We will denote by t_1 and t_2 the Kähler parameters of the base and the fiber respectively. The limit $t_2 \to i\infty$ corresponds to the decompactification. To take this limit we must find a linear combination of dual periods $\mathcal{F}_{t_i}$ which remains finite in this limit [127]. This combination is[9]

$$(\partial_{t_1} - \frac{1}{3}\partial_{t_2})\mathcal{F} = -\frac{1}{6}t_1^2 + \dots . \tag{5.69}$$

After the change of coordinates

$$t = t_1 \, , \quad s = t_2 + \frac{1}{3}t_1 \, , \tag{5.70}$$

this dual period is rephrased as $\partial_t \mathcal{F}^{(0)} = -\frac{1}{6}t^2 + \dots$, which can be integrated to give the prepotential of the local model. The "classical intersection" numbers in the basis given by t and s are then

$$C^{(0)}_{ttt} = -\frac{1}{3} \, , \quad C^{(0)}_{sss} = 9 \, , \quad \text{and} \quad C^{(0)}_{sst} = C^{(0)}_{stt} = 0 \, . \tag{5.71}$$

With the expression for the Kähler potential in special coordinates[10]

$$e^{-K} = i|X^0|^2 \left(2(\mathcal{F} - \overline{\mathcal{F}}) - (t_a - \bar{t}_{\bar{a}})(\mathcal{F}_a + \overline{\mathcal{F}}_{\bar{a}}) \right) , \tag{5.72}$$

we find for the inverse tt^* metric

$$g^{a\bar{b}} := e^K G^{a\bar{b}} = \left(\begin{array}{cc} g^{t\bar{t}} & g^{t\bar{s}} \\ g^{s\bar{t}} & g^{s\bar{s}} \end{array} \right) \overset{s \to i\infty}{\longrightarrow} \frac{1}{2\mathrm{Im}\tau} \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) , \tag{5.73}$$

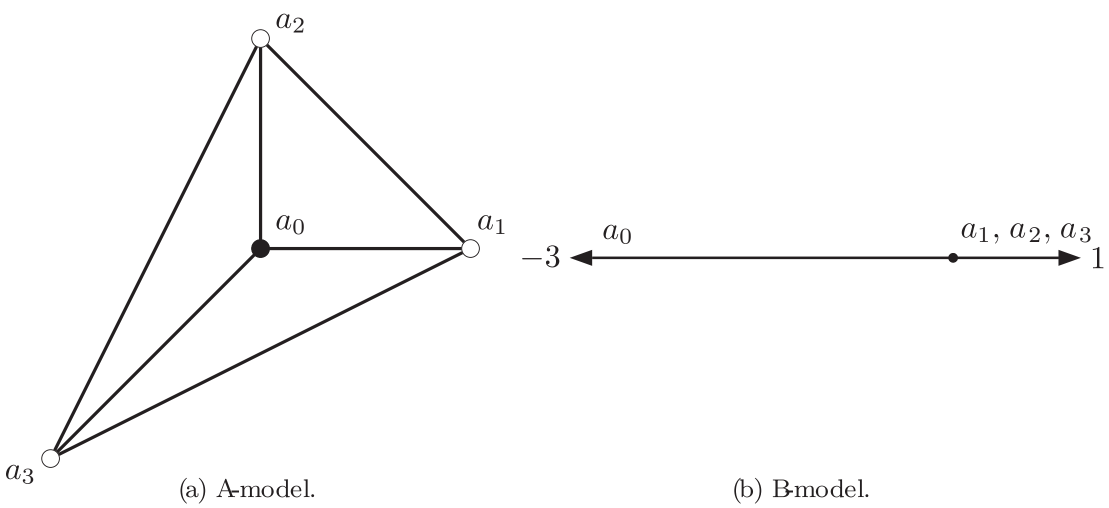
where $\tau := \mathcal{F}_{tt}$. The tt^* metric appears in the definition of the propagators, here in special coordinates

$$\partial_{\bar{a}} S^{bc} = \bar{C}^{bc}_{\bar{a}} = \overline{C}_{\bar{a}\bar{b}\bar{c}} g^{b\bar{b}} g^{c\bar{c}}, \tag{5.74}$$

which shows that in that limit the propagators S^{ts} and S^{ss} vanish. We can describe local $\mathbb{P}^2$ torically by the charge vector $l = (-3, 1, 1, 1)$.

[9] We analyzed the leading terms coming from classical intersection numbers which can be computed from the toric data.

[10] $\mathcal{F}_a := \frac{\partial \mathcal{F}}{\partial t_a}$.



(a) A-model. (b) B-model.

Figure 5.1: Fan and secondary fan for local $\mathbb{P}^2$.

Local mirror symmetry associates to this non-compact toric variety a one dimensional local geometry [125, 126] described by

$$P = a_0 u_1 u_2 u_3 + a_1 u_1^3 + a_2 u_2^3 + a_3 u_3^3 = 0 \, , \tag{5.75}$$

where u_1, u_2 and u_3 are projective coordinates. The rescaling $u_i \to \lambda_i u_i$ induces a $(\mathbb{C}^*)^3$ action on a_i

$$(a_0, a_1, a_2, a_3) \to (\lambda_1\lambda_2\lambda_3 a_0, \lambda_1^3 a_1, \lambda_2^3 a_2, \lambda_3^3 a_3) \, . \tag{5.76}$$

Invariant combinations of a_i under this action parameterize the moduli space of complex structures of the mirror geometry. Different phases on the B-side are encoded in the secondary fan which is one dimensional and shows that the moduli space of complex structures has two patches. These correspond on the A-model side to the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold phase and to the blown up phase. The Picard Fuchs system in homogeneous coordinates is given by[11]

$$\mathcal{L} = \theta_{a_1}\theta_{a_2}\theta_{a_3} - \frac{a_1 a_2 a_3}{a_0^3}\theta_{a_0}(\theta_{a_0} - 1)(\theta_{a_0} - 2) \, . \tag{5.77}$$

We refer to the literature for a complete discussion of solutions of the Picard-Fuchs system for local $\mathbb{P}^2$, see for example [129]. We will rather focus on the ingredients that we use for applying the polynomial method. We fix the choices of the holomorphic quantities in the polynomial procedure at large complex structure.

Large complex structure

We use the induced $(\mathbb{C}^*)^3$ action on a_i to set $a_2, a_3, a_0 \to 1, a_1 \to \frac{a_1 a_2 a_3}{a_0^3}$. The good coordinate in this patch is hence given by $z = \frac{a_1 a_2 a_3}{a_0^3}$. Our choice of data for Eqs. (5.14-

[11] $\theta_a := a\frac{\partial}{\partial a}$.

5.19) which completely fixes the polynomial construction is given by

$$s_{zz}^{z} = -\frac{4}{3z}\,, \quad h_{z}^{zz} = -\frac{z}{3}\,, \quad \text{and} \quad h_{z}^{z}, h_{z}, h_{zz} = 0\,. \tag{5.78}$$

We further need the starting amplitudes of the recursion

$$C_{zzz} = -\frac{1}{3z^3}P\,, \quad \text{where} \quad P := \frac{1}{\Delta}\,, \quad \Delta = 1 + 27z\,, \tag{5.79}$$

$$z\mathcal{F}_{z}^{1} = \frac{1}{12}(P-1) - \frac{1}{6}P\frac{S^{zz}}{z^2} + \frac{1}{12}\,. \tag{5.80}$$

We need to supplement the polynomial part with the holomorphic ambiguity at every genus, which we determine by moving to other patches in the moduli space of the B-model.

Orbifold

We use the $(\mathbb{C}^*)^3$ action to set $a_1, a_2, a_3 \to 1, a_0 \to \frac{a_0}{(a_1 a_2 a_3)^{1/3}}$. The good invariant coordinate to parameterize this patch is given by $x = \frac{a_0}{(a_1 a_2 a_3)^{(1/3)}}$, we find solutions for the Picard-Fuchs system

$$\begin{aligned} \omega_0 &= 1\,, \\ \omega_1 &= x - \frac{x^4}{648} + \frac{4\,x^7}{229635} - \frac{49\,x^{10}}{159432300} + \ldots\,, \\ \omega_2 &= x^2 - \frac{2\,x^5}{405} + \frac{25\,x^8}{367416} - \frac{160\,x^{11}}{122762871} + \ldots\,. \end{aligned} \tag{5.81}$$

Monodromy around large complex structure $z \to e^{2\pi i}z$ induces a transformation of the solutions as $(\omega_0, \omega_1, \omega_2) \to (\alpha^3\omega_0, \alpha\omega_1, \alpha^2\omega_2)$ with $\alpha = e^{\frac{-2\pi i}{3}}$.This describes an orbifold in the moduli space of the B-model and corresponds to a $\mathbb{C}^3/\mathbb{Z}_3$ target space on the A-side. Now we need to write these periods in the form $(1, t_o, \partial_{t_o}\mathcal{F})$ such that $\mathcal{F}$ is monodromy invariant. This is possible by identifying $t_o = \omega_1$ and $\partial_{t_o}\mathcal{F} = \frac{1}{6}\omega_2$. We introduced the factor 1/6 in order to reproduce the genus zero Orbifold Gromov-Witten invariants that appeared in [107, 44]. Higher genus Orbifold invariants are only sensitive to the normalization of t_o which we find to be 1. Our actual purpose for moving to this patch is to examine the behavior of the polynomial and to restrict the ansatz we have to make at large complex structure for the holomorphic ambiguity.

To do that we first note that the full expression for $\mathcal{F}^{(g)}$ does not transform under a coordinate change in the complex structure moduli space[12]. To clarify this we look at a typical expression appearing in the BCOV Feynman graph expansion at genus g which would have the form

$$\mathcal{F}^{(g)}_{i_1\ldots i_n} S^{i_1 i_2} \ldots S^{i_{n-1} i_n}\,,$$

[12] They transform as sections of $\mathcal{L}^{(2-2g)}$ but in local models this transformation is trivial due to the constant period.

where all indices are contracted. Hence, the whole expression does not transform. In the polynomial formalism all quantities in $\mathcal{F}^{(g)}_{i_1\ldots i_n}$ coming from the connections would also be expressed in terms of polynomial building blocks. We therefore only have to express the building blocks that we found at large complex structure in the new coordinates without worrying about tensor transformations of the indices. The only polynomial building block which survives when taking the holomorphic limit has the following expansion in t_o

$$\frac{S^{zz}}{z^2}(t_o) = \frac{1}{2} + \frac{1}{540}t_o^3 + \mathcal{O}(t_o^6)\,. \tag{5.82}$$

We further check that the polynomial part at every genus has a regular expansion in t_o. The regularity of the polynomial expression depends on the choice of s^z_{zz}. Being sure that we have made an appropriate choice we can make an ansatz for the holomorphic ambiguity at large complex structure which is also regular at the orbifold and has the right singular behavior at the conifold,

$$f^{(g)}(z) = \Delta^{2-2g}\sum_{i=0}^{2g-2} a_i z^i. \tag{5.83}$$

One of the $2g-1$ coefficients in the ansatz is determined by the contribution of constant maps at large radius Eq. (5.43), the other $2g-2$ are determined by the gap condition at the conifold. To implement the gap condition we need to go to a patch where the coordinate is the discriminant.

Conifold

The coordinate on complex structure moduli space in this patch is given by

$$y = 1 + 27z\,.$$

We solve the Picard-Fuchs system in terms of y and obtain the mirror map

$$t_c(y) = \frac{1}{\sqrt{3}}\left(y + \frac{11\,y^2}{18} + \frac{109\,y^3}{243} + \frac{9389\,y^4}{26244} + \ldots\right)\,. \tag{5.84}$$

The normalization is such that $\mathcal{F}^{(g)}(t_c)$ has the behavior described in Eq. (5.44). Once the normalization is fixed a simple counting of conditions vs. unknowns shows that the recursion is completely determined up to arbitrary genus. We list some low genus Orbifold Gromov-Witten invariants in the Appendix C.

5.4.2 Local $\mathbb{F}_0$

Local $\mathbb{F}_0$ denotes the anti-canonical bundle over $\mathbb{P}^1\times\mathbb{P}^1$, which is obtained by taking one Kähler parameter of a three parameter compact Calabi-Yau to infinity. The compact CY is a torus fibration which is described by the charge vectors

$$l^{(a)} = \left(\begin{array}{c|ccccccc} 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 \\ -6 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \end{array} \right) . \tag{5.85}$$

We denote by t_1 and t_2 the Kähler parameters of the base and by t_3 the Kähler parameter of the fiber. The limit $t_3 \to i\infty$ corresponds to the decompactification. Again, to take this limit we must find a linear combination of dual periods $\mathcal{F}_{t_i}$ which remains finite in this limit. This combination is

$$(\partial_{t_1} + \partial_{t_2} - \frac{1}{2}\partial_{t_3})\mathcal{F} = -\frac{1}{2}t_1 t_2 + \dots . \tag{5.86}$$

After the change of coordinates

$$t = t_1 , \quad u = t_2 - t_1 , \quad s = t_3 + \frac{1}{2}t_1 , \tag{5.87}$$

this dual period becomes

$$\partial_t \mathcal{F} = -\frac{1}{2}t(u+t) + \dots . \tag{5.88}$$

Integrating this we obtain for the prepotential

$$\mathcal{F} = -\frac{1}{6}t^3 - \frac{1}{4}t^2 u + \frac{a}{6}u^3 + \dots , \tag{5.89}$$

where a denotes an arbitrary constant which drops out in constructing the propagators but nevertheless affects the classical term when we determine the Yukawa couplings[13]. We set this constant to zero. The nonzero “classical intersections” in the new coordinates are

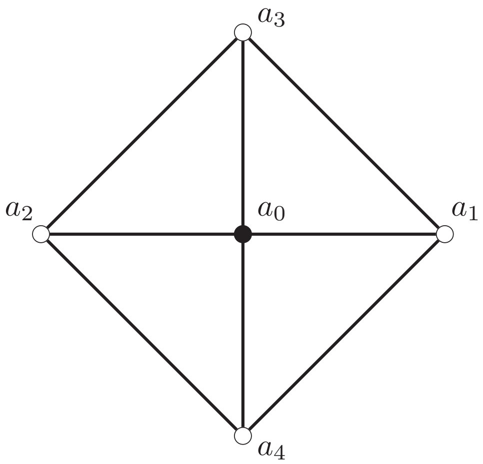
$$C^{(0)}_{ttt} = -1 , \quad C^{(0)}_{sss} = 8 , \quad C^{(0)}_{uss} = 2 , \quad \text{and} \quad C^{(0)}_{ttu} = -\frac{1}{2} . \tag{5.90}$$

We can redo the analysis of taking s to infinity. We find in this case the following inverse tt^* metric

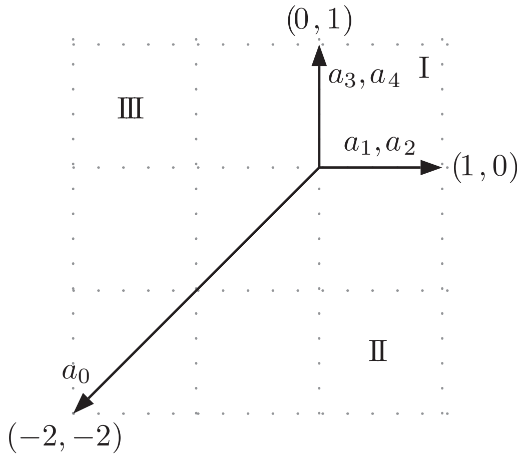
$$g^{a\bar{b}} = \begin{pmatrix} g^{t\bar{t}} & g^{t\bar{u}} & g^{t\bar{s}} \\ g^{u\bar{t}} & g^{u\bar{u}} & g^{u\bar{s}} \\ g^{s\bar{t}} & g^{s\bar{u}} & g^{s\bar{s}} \end{pmatrix} \stackrel{s\to i\infty}{\longrightarrow} \frac{1}{2\mathrm{Im}\mathcal{F}_{tt}} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{5.91}$$

This shows that all propagators containing s vanish in that limit. More interestingly the propagators S^{ut} and S^{uu} also vanish. It was already observed in [64] that local $\mathbb{F}_0$ is a two parameter problem which effectively reduces to a one parameter problem. In [107] it was argued that only the Kähler parameters that correspond to 2-cycle classes which are dual to 4-cycles which remain compact are true parameters of the theory. This limit shows this

[13]This explains why different classical data for local models lead to the same results.



(a) A-model.



(b) B-model.

Figure 5.2: Fan and secondary fan for local $\mathbb{F}_0$.

from a different point of view. Only one propagator S^{tt} survives the decompactification. It should be noted that we could still choose some non-zero holomorphic limit for the vanishing propagators as this analysis only shows the vanishing of their anti-holomorphic derivative, this would however be redundant.

Now we can examine which holomorphic anomaly equations survive the decompactification. We find that $\partial_{\bar{t}}\mathcal{F}^{(g)}$ and $\partial_{\bar{u}}\mathcal{F}^{(g)}$ give the same equation. As was proven in [24] the non-holomorphic dependence of $\mathcal{F}^{(g)}$ comes from the polynomial building blocks. The chain rule shows that the anomaly equation $\partial_{\bar{u}}\mathcal{F}^{(g)}$ reduces to $\partial_{\bar{t}}\mathcal{F}^{(g)}$. Only one non-trivial anomaly equation survives.

Local $\mathbb{F}_0$ is described by the charge vectors

$$l^{(a)} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 1 \end{pmatrix} . \tag{5.92}$$

The mirror geometry is given by,

$$P = a_0 u_1 u_2 u_3 u_4 + a_1 u_1^2 u_2^2 + a_2 u_3^2 u_4^2 + a_3 u_1^2 u_4^2 + a_4 u_2^2 u_3^2 = 0 . \tag{5.93}$$

The rescaling $u_i \to \lambda_i u_i$, $\lambda_i \in \mathbb{C}^*$ induces a $(\mathbb{C}^*)^3$-action

$$(a_0, a_1, a_2, a_3, a_4) \to (\lambda_1\lambda_2\lambda_3\lambda_4 a_0, \lambda_1^2\lambda_2^2 a_1, \lambda_3^2\lambda_4^2 a_2, \lambda_1^2\lambda_4^2 a_3, \lambda_2^2\lambda_3^2 a_4) , \tag{5.94}$$

as only three rescalings are independent. The Picard-Fuchs system is given by

$$\begin{aligned} \mathcal{L}_1 &= \theta_{a_1}\theta_{a_2} - \frac{a_1 a_2}{a_0^2}\theta_{a_0}(\theta_{a_0} - 1) , \\ \mathcal{L}_2 &= \theta_{a_3}\theta_{a_4} - \frac{a_3 a_4}{a_0^2}\theta_{a_0}(\theta_{a_0} - 1) . \end{aligned} \tag{5.95}$$

Large complex structure, region I

We use the induced $(\mathbb{C}^*)^3$ action to set $a_0, a_2, a_4 \to 1, a_1 \to \frac{a_1 a_2}{a_0^2}$ and $a_3 \to \frac{a_3 a_4}{a_0^2}$. Good coordinates are therefore $z_1 = \frac{a_1 a_2}{a_0^2}$, $z_2 = \frac{a_3 a_4}{a_0^2}$ with mirror maps[14]

$$t_1 = \log z_1 + 2z_1 + 2z_2 + 3z_1^2 + 12z_1 z_2 + 3z_2^2 + \cdots \,, \tag{5.96}$$

$$t_2 = \log z_2 + 2z_1 + 2z_2 + 3z_1^2 + 12z_1 z_2 + 3z_2^2 + \cdots \,. \tag{5.97}$$

To make contact with our previous discussion we change t_1 and t_2 to $t = t_1$ and $u = t_2 - t_1$ by an $SL(2,\mathbb{Z})$ transformation, which changes the coordinates on the complex structure moduli space to

$$y_1 = z_1 \,, \quad y_2 = \frac{z_2}{z_1} \,, \tag{5.98}$$

indeed we now find a mirror map which only depends on y_2

$$t(y_1, y_2) = t_1 = \log y_1 + 2y_1 + 3y_1^2 + 2y_1 y_2 + \cdots \,, \tag{5.99}$$

$$u(y_2) = t_2 - t_1 = \log y_2 \,. \tag{5.100}$$

Calculating the holomorphic limit of the Christoffel connections we find the following simple expressions[15]

$$\Gamma^2_{12}|_{\text{hol}} = \Gamma^2_{21}|_{\text{hol}} = \Gamma^2_{11}|_{\text{hol}} = 0 \quad \text{and} \quad \Gamma^2_{22}|_{\text{hol}} = -\frac{1}{y_2} \,. \tag{5.101}$$

The data we choose for constructing the polynomial building blocks is[16]

$$s^2_{22} = -\frac{1}{y_2} \,, \quad s^1_{11} = -\frac{3}{2y_1} \,, \quad s^1_{12} = s^1_{21} = -\frac{1}{4y_2} \,, \tag{5.102}$$

and all the other s^k_{ij} zero. With this data only the propagator S^{11} survives in the holomorphic limit but we know from the preceding discussion that also non-holomorphically there exists just one propagator with two indices. We further find for the data of Eqs. (5.16-5.19)

$$h^{11}_1 = -\frac{y_1}{4} \tag{5.103}$$

out of all the other holomorphic data only $h^{11}_2$is not zero in this setup but is not needed for the recursion. Now we still need the initial correlation functions to start the recursion. We only need one Yukawa coupling, namely

$$C_{111} = -(y_1^3 \Delta)^{-1} \,, \quad \text{where} \quad \Delta = 1 - 8y_1(1 + y_2) + 16y_1^2(1 - y_2)^2. \tag{5.104}$$

[14]We absorb factors of 2π in the definition of t_1, t_2.

[15]$\Gamma^k_{ij}|_{\text{hol}} = \frac{\partial y_k}{\partial t_a} \frac{\partial^2 t_a}{\partial y_i \partial y_j}$

[16]$s^k_{ij} := s^{y_k}_{y_i y_j}$.

And we further need $\mathcal{F}_1^{(1)}$ to completely determine all the polynomials

$$\mathcal{F}_1^{(1)} = \frac{1}{6y_1} - \frac{1}{2}\frac{S^{11}}{y_1^3\Delta} - \frac{1}{12\Delta}\left(32y_1(1-y_2)^2 - 8(1+y_2)\right). \tag{5.105}$$

To obtain a bound on the maximal powers of y_1 and y_2 that we have to allow in the ansatz of the ambiguity we move on to region II.

Region II

This is the orbifold expansion point in[122]. It is an orbifold in the moduli space of complex structures but does not correspond to $\mathbb{C}^3/\mathbb{Z}_n$ target space on the A-model side. We find the right invariant coordinates in this patch to be $x_1 = \frac{a_0}{\sqrt{a_1 a_2}} = \frac{1}{\sqrt{y_1}}$ and $x_2 = \frac{a_1 a_2}{a_3 a_4} = y_2$. The mirror maps are given by

$$\begin{aligned} t_1(x_1, x_2) &= x_1 + \frac{1}{4}x_1 x_2 + \left(\frac{x_1^3}{24} + \frac{9}{64}x_2^2 x_1\right) + \dots, \\ t_2(x_2) &= \log x_2 \,. \end{aligned} \tag{5.106}$$

We find that the propagator $S^{11}(t_1, t_2)$ is not regular on its own. We check however that the whole polynomial part of the amplitudes is regular. Here we show this for $\mathcal{F}^{(2)}$ which is expressed in terms of t_1 and $q_2 = e^{t_2}$

$$\mathcal{F}^{(2)}(t_1, t_2) = \frac{1}{360} + \frac{1}{480}q_2 + \left(\frac{31q_2^2}{3840} + \frac{t_1^2}{1920}\right) + \dots. \tag{5.107}$$

Now we have enough information to make an ansatz for the ambiguity in terms of large complex structure coordinates. The discussion is more transparent in terms of the right coordinates at large complex structure z_1 and z_2. The problem is obviously symmetric in these coordinates. From region II we learn that we can make an ansatz which has as its highest degree monomials of the form $z_1^i z_2^{(n-i)}$ where n refers to the highest degree of z_1 in the denominator in order to ensure regularity of the ansatz in x_1. In region III we get the same statement with the degree of z_2 in the denominator. So we can make a symmetric ansatz in the coordinates z_1 and z_2 of maximal joint degree $2(2g-2)$. To fix the coefficients of the ansatz we have to use the gap condition at the conifold locus.

Conifold

To parameterize the expansion around the conifold locus we pick the coordinates:

$$u_1 = \Delta\,, \quad \text{and} \quad u_2 = 1 + y_2\,, \tag{5.108}$$

the choice of the second coordinate is arbitrary, we only have to make sure that the coordinate is transverse to the discriminant. We solve the Picard-Fuchs equations and find the mirror maps

$$\begin{aligned} t_1(u_1,u_2) &= u_1 + \frac{5}{8}u_1^2 + \left(\frac{89u_1^3}{192} + \frac{3}{32}u_2^2u_1\right) + \ldots\,, \\ t_2(u_2) &= \log(1+u_2)\,. \end{aligned} \tag{5.109}$$

The propagator in the conifold coordinates now reads

$$\frac{S^{y_1y_1}}{y_1^2}(t_1,t_2) = \frac{1}{4}t_1 - \frac{1}{8}t_1^2 + \frac{1}{768}\left(43t_1^3 - 6t_1t_2^2\right) + \ldots\,. \tag{5.110}$$

A counting of independent conditions for the free parameters in the ambiguity ansatz is more involved in this case. One would think that there are infinitely many conditions as we can move in the u_2 direction. This fact has also another manifestation. Requiring the vanishing of all but the leading singularity in the conifold coordinate involves setting series in the other coordinates to zero. It turns out that the conditions are not unrelated. Once the correct normalization of the mirror map at the conifold is chosen we find that the gap conditions with the contribution from constant maps are enough to fix the ambiguity up to genus 4. We assume but cannot prove rigorously that this holds up to arbitrary genus.[17]

5.4.3 Local $\mathbb{F}_2$

Local $\mathbb{F}_2$ denotes the anti-canonical bundle over $\mathbb{F}_2$, which is obtained from

$$l^{(a)} = \left(\begin{array}{c|ccccccc} 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 & 1 \\ -6 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \end{array}\right). \tag{5.111}$$

by decompactification. Much of the discussion here will follow the last example. The finite dual period in this case is

$$(\partial_{t_1} - \frac{1}{2}\partial_{t_3})\mathcal{F} = -\frac{1}{2}t_1t_2 - \frac{1}{2}t_1^2 + \ldots\,. \tag{5.112}$$

After the change of coordinates

$$t = t_1\,, \quad u = t_2\,, \quad s = t_3 + \frac{1}{2}t_1\,, \tag{5.113}$$

the dual period becomes

$$\partial_t\mathcal{F} = -\frac{1}{2}t(u+t) + \ldots \tag{5.114}$$

[17] This was also found in [115].

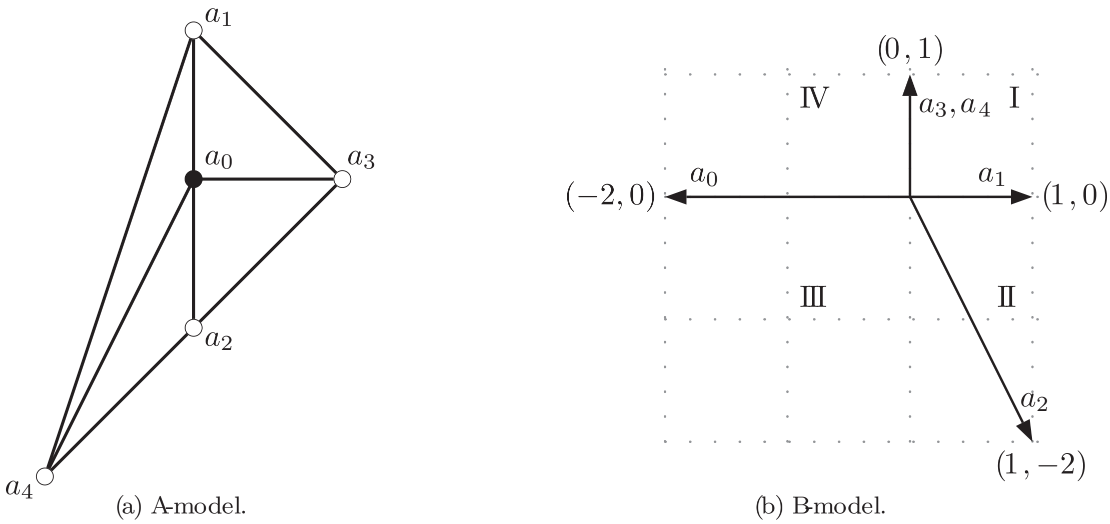


Figure 5.3: Fan and secondary fan for local $\mathbb{F}_2$.

Integrating this we obtain for the prepotential

$$\mathcal{F} = -\frac{1}{6}t^3 - \frac{1}{4}t^2 u + \frac{a}{6}u^3 + \dots\,, \tag{5.115}$$

with a the arbitrary constant that we will set to zero. The nonzero "classical intersections" are the same as in the last model when we changed the Kähler parameters

$$C^{(0)}_{ttt} = -1\,, \quad C^{(0)}_{sss} = 8\,, \quad C^{(0)}_{uss} = 2\,, \quad \text{and} \quad C^{(0)}_{ttu} = -\frac{1}{2}\,. \tag{5.116}$$

The analysis of taking s to infinity also gives the same result and again there exists only one non-trivial propagator S^{tt} and one holomorphic anomaly equation for the non-compact model.

Local $\mathbb{F}_2$ is described by the charge vectors

$$l^{(a)} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ -2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 1 \end{pmatrix}\,, \tag{5.117}$$

which represents a $\mathbb{P}^1$ fibered over $\mathbb{P}^1$. t_1 will denote the Kähler parameter corresponding to the fiber and t_2 corresponds to the base. The mirror geometry is given by,

$$P = a_0 u_1 u_2 u_3 + a_1 u_1^2 + a_2 u_2^2 u_3^2 + a_3 u_3^4 + a_4 u_2^4 = 0\,. \tag{5.118}$$

The rescaling $u_i \to \lambda_i u_i$, $\lambda_i \in \mathbb{C}^*$ induces a $(\mathbb{C}^*)^3$-action

$$(a_0, a_1, a_2, a_3, a_4) \to (\lambda_1\lambda_2\lambda_3 a_0, \lambda_1^2 a_1, \lambda_2^2\lambda_3^2 a_2, \lambda_3^4 a_3, \lambda_2^4 a_4)\,. \tag{5.119}$$

The Picard-Fuchs operators are

$$\mathcal{L}_1 = \theta_{a_1}\theta_{a_2} - \frac{a_1 a_2}{a_0^2}\theta_{a_0}(\theta_{a_0} - 1)\,, \tag{5.120}$$

$$\mathcal{L}_2 = \theta_{a_3}\theta_{a_4} - \frac{a_3 a_4}{a_2^2}\theta_{a_2}(\theta_{a_2} - 1)\,. \tag{5.121}$$

Large complex structure, region I

In this region of the secondary fan we use the $\mathbb{C}^{*3}$-action to set $a_0, a_2, a_4 \to 1$ and $a_1 \to a_0^{-2} a_1 a_2$, $a_3 \to a_2^{-2} a_3 a_4$ (alternatively $a_0, a_2, a_3 \to 1$ and $a_1 \to a_0^{-2} a_1 a_2$, $a_4 \to a_2^{-2} a_3 a_4$). In both cases the good $\mathbb{C}^{*3}$-invariant coordinates are given by

$$z_1 = \frac{a_1 a_2}{a_0^2}\,, \quad z_2 = \frac{a_3 a_4}{a_2^2}\,. \tag{5.122}$$

We find for the mirror maps

$$\begin{aligned} t_1(z_1, z_2) &= \log(z_1) + (2z_1 - z_2) + \left(3z_1^2 - \frac{3z_2^2}{2}\right) + \left(\frac{20z_1^3}{3} + 6z_2 z_1^2 - \frac{10z_2^3}{3}\right) + \ldots \\ t_2(z_2) &= \log(z_2) + 2z_2 + 3z_2^2 + \frac{20}{3}z_2^3 + \ldots\,. \end{aligned} \tag{5.123}$$

As the second mirror map only depends on z_2, the holomorphic limit of Christoffel connections with upper index 2 and mixed lower indices vanish. We choose as holomorphic data

$$s_{11}^1 = -\frac{3}{2z_1}, \quad s_{12}^1 = s_{21}^1 = -\frac{1}{4z_2}\,. \tag{5.124}$$

It is also possible to choose s_{22}^2 and s_{22}^1 such that only the propagator S^{11} survives. This is due to the special form of the mirror map. We will not need these for the recursion. All other s_{ij}^k are set to zero. We find for the other holomorphic quantities

$$h_1^{11} = -\frac{z_1}{4}, \tag{5.125}$$

also h_2^{11} is nonzero but again not needed in the following. All the other holomorphic quantities are zero. The only Yukawa coupling relevant for the setup is given by

$$C_{111} = -(z_1^3\Delta_1)^{-1}, \quad \Delta_1 = (1 - 4z_1)^2 - 64z_1^2 z_2. \tag{5.126}$$

There are however two discriminants, the second is given by $\Delta_2 = 1 - 4z_2$. We will examine the locus where $\Delta_2 = 0$ later on.

$$\mathcal{F}_1^1 = \frac{1}{6z_1} - \frac{S^{11}}{2\Delta_1 z_1^3} + \frac{8}{\Delta_1}\left((1 - 4z_1) + 16z_1 z_2\right). \tag{5.127}$$

From arguments in sect. 5.2.6 and equation (5.45) it follows that the holomorphic ambiguity has the form

$$\frac{f^{(g)}(z_1, z_2)}{\Delta_1^{2g-2}\Delta_2^{g-1}},$$

see below for a careful study of the local expansions at the singular loci. Furthermore the regularity of the polynomial part of the amplitudes at the orbifold expansion point in region III allows us again to put some bounds on the monomials that appear in $f^g(z_1, z_2)$. We find that the monomial of maximal degree at genus g has the form $z_1^{4(g-1)} z_2^{3(g-1)}$ moreover every monomial $z_1^n z_2^m$ has to satisfy $m \leq \frac{n}{2} + g - 1$.

Orbifold, region III

In region III (orbifold region) of the secondary fan, i.e. $a_1, a_3, a_4 \neq 0$ we use the $\mathbb{C}^{*3}$-action to set $a_1, a_3, a_4 \to 1$ and $a_0 \to a_0 a_1^{-1/2}(a_3 a_4)^{-1/4}$, $a_2 \to a_2(a_3 a_4)^{-1/2}$. Therefore the good $\mathbb{C}^{*3}$-invariant coordinates are given by

$$x_1 = \frac{a_0}{a_1^{1/2}(a_3a_4)^{1/4}} = \frac{1}{z_1^{1/2} z_2^{1/4}}, \quad x_2 = \frac{a_2}{(a_3a_4)^{1/2}} = \frac{1}{z_2^{1/2}}. \tag{5.128}$$

Under monodromy at large complex structure $z_1 \to \exp(2\pi \mathrm{i}) z_1$ we have $x_1 \to \alpha x_1$ and $x_2 \to x_2$ with $\alpha = \exp(-2\pi \mathrm{i}/2)$. Under $z_2 \to \exp(2\pi \mathrm{i}) z_2$ we have $x_1 \to \beta x_1$ and $x_2 \to \beta^2 x_2$ with $\beta = \exp(-2\pi \mathrm{i}/4)$. We fix the basis of solutions to the Picard-Fuchs equations

$$w_0 = 1\,, \tag{5.129}$$

$$w_1 = x_1 + \frac{1}{32}x_1x_2^2 - \frac{1}{192}x_1^3x_2 + \frac{1}{2560}x_1^5 + \frac{25}{6144}x_1x_2^4 + \cdots\,, \tag{5.130}$$

$$w_2 = x_2 + \frac{1}{24}x_2^3 + \frac{3}{640}x_2^5 + \cdots\,, \tag{5.131}$$

$$w_3 = x_1x_2 - \frac{1}{12}x_1^3 + \frac{3}{32}x_1x_2^3 - \frac{3}{128}x_1^3x_2^2 + \cdots\,, \tag{5.132}$$

such that they transform under the monodromy as

$$w_0 \to w_0\,, \tag{5.133}$$

$$w_1 \to \alpha\beta w_1\,, \tag{5.134}$$

$$w_2 \to \beta^2 w_2\,, \tag{5.135}$$

$$w_3 \to \alpha\beta^3 w_3\,. \tag{5.136}$$

We want to write this set of solutions in the form $(1, t_1, t_2, \partial_{t_1}\mathcal{F})$ such that $\mathcal{F}$ is monodromy invariant. The only possibility is given by $t_1 \propto w_1$, $t_2 \propto w_2$ and $\partial_{t_1}\mathcal{F} \propto w_3$. We normalize the mirror maps as

$$t_1 = w_1\,, \quad t_2 = w_2\,, \tag{5.137}$$

with inverse

$$x_1 = t_1 - \frac{1}{32}t_1t_2^2 + \frac{1}{192}t_1^3t_2 - \frac{1}{2560}t_1^5 - \frac{1}{2048}t_1t_2^4 + \cdots . \qquad (5.138)$$

$$x_2 = t_2 - \frac{1}{24}t_2^3 + \frac{1}{1920}t_2^5 + \cdots , \qquad (5.139)$$

and we normalize the prepotential as

$$\partial_{t_1}\mathcal{F} = \frac{1}{4}w_3 , \qquad (5.140)$$

to obtain the correct Orbifold Gromov-Witten invariants at genus zero which were computed in [113, 112]. We read off the orbifold Gromov-Witten invariants at genus g from the expansion of the topological string amplitudes on the A-model side in the two mirror maps t_1 and t_2 according to the formula

$$\mathcal{F}^{(g)} = \sum_{n_1n_2=0}^{\infty} \frac{1}{n_1!n_2!} N^{\text{orb}}_{g,(n_1,n_2)} t_1^{n_1} t_2^{n_2},$$

where $N^{\text{orb}}_{g,(n_1,n_2)}$ denote the orbifold Gromov-Witten invariants. The results agree for genus 0 and 1 with [112, 113]. The higher genus invariants are new results of our analysis. We list these invariants up to genus 4 in the Appendix C.

Conifold, $\Delta_1 = 0$

We take coordinates

$$y_1 = \Delta_1 \quad \text{and} \quad y_2 = z_2 - 1 \qquad (5.141)$$

and find the mirror maps

$$t_1(y_1, y_2) = y_1 + \left(\frac{87y_1^2}{128} - \frac{1}{4}y_1y_2\right) + \frac{(26217y_1^3 - 8736y_2y_1^2 + 9728y_2^2y_1)}{49152} + \ldots (5.142)$$

$$t_2(y_2) = y_2 - \frac{5}{6}y_2^2 + \frac{7}{9}y_2^3 + \ldots .$$

As we can again move in the y_2 direction we could implement the gap condition at infinitely many different points. In this model we furthermore have a second discriminant. We will examine the behavior of the amplitudes at the locus where that discriminant vanishes in the following.

Singularity at $\Delta_2 = 0$

We choose as coordinates

$$u_1 = z_1, \quad u_2 = \Delta_2 \qquad (5.143)$$

and find the mirror maps

$$\begin{aligned} t_1 &= \log(u_1) + \left(2u_1 - \frac{u_2}{2}\right) + \left(\frac{9u_1^2}{2} - \frac{u_2^2}{4}\right) + \frac{1}{6}\left(100u_1^3 - 9u_2u_1^2 - u_2^3\right) + \dots \\ t_2 &= \sqrt{u_2} + \frac{1}{3}u_2^{3/2} + \frac{1}{5}u_2^{5/2} + \dots . \end{aligned} \tag{5.144}$$

corresponding to $\gamma = 1/2$ in (5.45). At the singular locus $\Delta_2 = 0$ the space-time spectrum contains extra massless states from an enhanced $SU(2)$ gauge symmetry with one adjoint hyper-multiplet [130, 131]. According to eq.(5.45) the theory is regular due to the cancellation of the effects of the equal number of extra massless hyper and vector multiplets, $b = 0$. We find that the expansion of the polynomial part of the amplitudes is already regular in t_2; this allows us to further restrict the ansatz of the holomorphic ambiguity to be of the form

$$\frac{f^{(g)}(z_1, z_2)}{\Delta_1^{2g-2}}.$$

Regularity at the orbifold expansion point requires the monomials to be of the type $z_1^n z_2^m$ with $m \leq \frac{n}{2}$ and maximal degree of $n = 2(2g-2)$.
Once we have normalized the mirror map at the conifold locus to obtain the prefactor of Eq. (5.44) we find that the conditions obtained from implementing the gap are enough to supplement the polynomial part of the amplitudes with the correct holomorphic ambiguities. We refer to the Appendix C for the results. Since the gap conditions hold at infinitely many points, it is plausible that these boundary conditions might be sufficient for arbitrary genus. This finishes our analysis of the examples of the polynomial construction on local models.

5.5 Conclusion

In this chapter we have shown that the polynomial structure of the topological string partition function found by Yamaguchi and Yau for the quintic holds for an arbitrary Calabi-Yau manifold with any number of moduli. We have furthermore generalized these results to the open topological string partition function. We applied the method to the quintic with a D-brane fixed at the real locus and to local Calabi-Yau manifolds without D-branes. The polynomials provide expressions for the topological string amplitudes everywhere in moduli space. This can be used to study the topological string at different expansion points. In particular we analyzed the freedom in choosing the polynomial building blocks and how to exploit this freedom elaborately to maximize the information that can be obtained from various expansion points. We further clarified which simplifications can occur in the formalism on the computational side when local models are studied. As examples we studied local $\mathbb{P}^2$, local $\mathbb{F}_0$ and local $\mathbb{F}_2$. An immediate application of the construction is the possibility to extract Orbifold Gromov-Witten invariants which have been already computed for $\mathbb{C}^3/\mathbb{Z}_3$ and to make predictions for higher genus Orbifold GW invariants for $\mathbb{C}^3/\mathbb{Z}_4$.

A simple counting for local $\mathbb{P}^2$ shows that the information coming from the boundary conditions is enough to supplement the polynomials with the right holomorphic ambiguity at every genus. For the other two models we argue that the information coming from boundary conditions is enough for all genera but we cannot demonstrate this rigorously. However it appears that the boundary conditions at the various boundary divisors and divisors with extra massless states should provide enough information in general. Further information can be obtained by studying intersections of the singular divisors which are often described also by eq. (5.44), with extra massless states at the intersection point.

Having a powerful alternative computation method could help pushing forward the understanding in some directions which have been explored recently. As the polynomial construction of topological string amplitudes applies also to compactifications with background D-branes [22, 24, 111], it should be possible to work out the corresponding boundary conditions also for this case and fix the holomorphic ambiguity along the lines of the above arguments.

A Toric Branes

A.1 One parameter models

In the following we discuss the toric GKZ systems associated to brane families connected to the involution brane in one parameter compact models.[1] At the critical value of the superpotential we recover the results of [81, 82].

A.1.1 Sextic $\mathrm{X}_6^{(2,1,1,1,1)}$

We consider the charge vectors

$$l^1 = (-4,0,1,1,1,1;2,-2), \qquad l^2 = (-1,1,0,0,0,0;-1,1)\ .$$

Large volume

This region in moduli space is parameterized by local variables

$$z_1 = \frac{a_2 a_3 a_4 a_5 a_6^2}{a_0^4 a_7^2}\,, \quad z_2 = -\frac{a_1 a_7}{a_0 a_6}\,.$$

We obtain the differential operators

$$\begin{aligned}
\mathcal{L}_1 &= (\theta_1^4 - z_1 \prod_{i=1}^{4}(4\theta_1+\theta_2+i))(2\theta_1-\theta_2)\,,\\
\mathcal{L}_2 &= (\theta_2 + z_2(4\theta_1+\theta_2+1))(2\theta_1-\theta_2)\,,\\
\mathcal{L}_1' &= \theta_1^4 \prod_{i=0}^{1}(\theta_2 - i) - z_1 z_2^2 \prod_{i=1}^{6}(4\theta_1+\theta_2+i)\,.
\end{aligned}$$

Switching to coordinates which are centered around the critical point $z_2 = -1$ of the superpotential

$$u = z_1^{-1/4}(z_2+1)\,, \quad v = z_1^{1/4}\,,$$

[1] See [132] for a discussion of closed string mirror symmetry in these models.

we obtain the superpotential

$$c\mathcal{W}(u,v) = \frac{u^2}{24} + 24v^2 + \frac{u^3 v}{24} - 24uv^3 + \frac{u^6}{138240} + \frac{v^2u^4}{24} + \frac{143360v^6}{3} + \ldots \qquad \text{(A.1)}$$

At the critical point $u = 0$, we can express v in terms of the closed string modulus $z = z_1 z_2^2$ as

$$v|_{crit} = z^{1/4}\,.$$

We find for the superpotential at the minimum

$$c\mathcal{W}_{crit} = 24\sqrt{z} + \frac{143360}{3}z^{3/2} + \frac{5510529024}{25}z^{5/2} + \frac{334766662483968}{245}z^{7/2} + \ldots\,,$$

This expression satisfies the differential equation

$$\mathcal{L}_{bulk}\mathcal{W}_{crit} = \frac{3}{2c}\sqrt{z}\,,$$

where $\mathcal{L}_{bulk} = \theta^4 - 9z\prod_{i=1}^4(6\theta+i)$ denotes the Picard-Fuchs operator of the sextic. The above agrees with the results of [81] for the choice of constant $c = 1$.

Small volume

To study the Landau-Ginzburg phase of the B-model we change to the local coordinates

$$x_1 = \frac{a_0}{(a_2a_3a_4a_5)^{1/4}}\left(\frac{-a_7}{a_6}\right)^{1/2}, \quad x_2 = \frac{a_1}{(a_2a_3a_4a_5)^{1/4}}\left(\frac{-a_7}{a_6}\right)^{3/2}.$$

The differential operators obtained by a transformation of variables are ($\theta_i = \theta_{x_i}$)

$$\begin{aligned}
\mathcal{L}_1 &= (x_1^4(\theta_1+\theta_2)^4 - 4^4\prod_{i=1}^4(\theta_1 - i))(\theta_1+3\theta_2)\,,\\
\mathcal{L}_2 &= (x_2(\theta_1-1) - x_1\theta_2)(\theta_1+3\theta_2)\,,\\
\mathcal{L}_1' &= x_1^6(\theta_1+\theta_2)^4\theta_2(\theta_2-1) - 4^4x_2^2\prod_{i=1}^6(\theta_1 - i)\,.
\end{aligned}$$

We obtain the superpotential

$$\mathcal{W} = -\frac{1}{12}x_1^2 - \frac{1}{24}x_2x_1 - \frac{x_1^6}{69120} - \frac{x_2x_1^5}{18432} - \frac{x_2^2x_1^4}{11520} - \frac{x_2^3x_1^3}{13824} - \frac{x_2^4x_1^2}{32256} - \frac{x_2^5x_1}{184320} + \ldots \qquad \text{(A.2)}$$

which has its critical value at $x_2 = -x_1$. We can express x_1 in terms of the closed string variable $x = -x_1x_2^{-1/3}$ of the geometry in the Landau-Ginzburg phase as

$$x_1|_{crit} = -x^{3/2}$$

which gives the following critical value for the superpotential

$$\mathcal{W}_{crit} = -\frac{x^3}{24} - \frac{x^9}{3870720} - \frac{x^{15}}{137763225600} - \frac{5x^{21}}{16403566461714432} + \dots$$

This expression satisfies the equation

$$\mathcal{L}_{bulk}\mathcal{W}_{crit} = \frac{3}{2}x^3 \,,$$

with $\mathcal{L}_{bulk} = 6^{-4}x^6\theta^4 - 9(\theta-1)(\theta-2)(\theta-4)(\theta-5)$.

A.1.2 Octic

We consider the charge vectors

$$l^1 = (-4, 0, 1, 1, 1, 1; 4, -4), \qquad l^2 = (-1, 1, 0, 0, 0, 0; -1, 1) \,.$$

Large volume

This region in moduli space is parameterized by local variables

$$z_1 = \frac{a_2 a_3 a_4 a_5 a_6^4}{a_0^4 a_7^4} \,, \quad z_2 = -\frac{a_1 a_7}{a_0 a_6} \,.$$

The differential operators are

$$\begin{aligned} \mathcal{L}_1 &= (\theta_1^4 - z_1 \prod_{i=1}^{4}(4\theta_1 + \theta_2 + i))(4\theta_1 - \theta_2) \,, \\ \mathcal{L}_2 &= (\theta_2 + z_2(4\theta_1 + \theta_2 + 1))(4\theta_1 - \theta_2) \,, \\ \mathcal{L}_1' &= \theta_1^4 \prod_{i=0}^{3}(\theta_2 - i) - z_1 z_2^4 \prod_{i=1}^{8}(4\theta_1 + \theta_2 + i) \,. \end{aligned}$$

Switching to $u = z_1^{-1/4}(z_2+1)$ and $v = z_1^{1/4}$, we obtain

$$\mathcal{W}(u,v) = \frac{u^2}{16} + 48v^2 + \frac{u^3 v}{12} - 96uv^3 + \frac{u^6}{92160} + \frac{5v^2u^4}{48} + 48v^4u^2 + \frac{1576960v^6}{3} + \dots \quad \text{(A.3)}$$

At $u = 0$, we can express v in terms of the classical coordinate $z = z_1 z_2^4$ as $v|_{crit} = -z^{1/4}$. We find for the superpotential at the minimum

$$c\mathcal{W}_{crit} = 48\sqrt{z} + \frac{1576960}{3}z^{3/2} + \frac{339028738048}{25}z^{5/2} + \frac{230988997113937 92}{49}z^{7/2} + \dots \,,$$

which satisfies the differential equation

$$\mathcal{L}_{bulk}\mathcal{W}_{crit} = \frac{3}{c}\sqrt{z} \,,$$

where $\mathcal{L}_{bulk} = \theta^4 - 16z\prod_{i=1}^{4}(8\theta + 2i - 1)$ denotes the Picard-Fuchs operator of the octic. Setting $c = 1$ reproduces the disk invariants of [81, 82].

Small volume

We switch to local coordinates

$$x_1 = \frac{-a_0 a_7}{a_6(a_2a_3a_4a_5)^{1/4}}\,, \quad x_2 = \frac{a_1 a_7^2}{a_6^2(a_2a_3a_4a_5)^{1/4}}\,.$$

The differential operators are ($\theta_i = \theta_{x_i}$)

$$\begin{aligned}
\mathcal{L}_1 &= (x_1^4(\theta_1+\theta_2)^4 - 4^4 x_2^2 \prod_{i=1}^{4}(\theta_1 - i))(\theta_1+2\theta_2)\,,\\
\mathcal{L}_2 &= (x_2(\theta_1-1) - x_1\theta_2)(\theta_1+2\theta_2)\,,\\
\mathcal{L}_1' &= x_1^8(\theta_1+\theta_2)^4\prod_{i=0}^{3}(\theta_2 - i) - 4^4x_2^2\prod_{i=1}^{8}(\theta_1-i)\,.
\end{aligned}$$

We obtain the superpotential

$$\mathcal{W} = -\frac{1}{16}x_1^2 - \frac{1}{24}x_2x_1 - \frac{x_1^6}{92160} - \frac{x_2x_1^5}{21504} - \frac{x_2^2x_1^4}{12288} - \frac{x_2^3x_1^3}{13824} - \frac{x_2^4x_1^2}{30720} - \frac{x_2^5x_1}{168960} + \dots \quad \text{(A.4)}$$

At the critical value $x_2 = -x_1$, we have $x_1|_{crit} = -x^2$, where $x = -x_1x_2^{-1/2}$. This gives the following expansion for the superpotential

$$\mathcal{W}_{crit} = -\frac{x^4}{48} - \frac{x^{12}}{42577920} - \frac{x^{20}}{8475718451200} - \frac{x^{28}}{1131846085858295808} + \dots$$

which satisfies the equation

$$\mathcal{L}_{bulk}\mathcal{W}_{crit} = 3x^4\,,$$

with

$$\mathcal{L}_{bulk} = 8^{-4}x^8\theta^4 - 16(\theta-1)(\theta-3)(\theta-5)(\theta-7)\,.$$

These results are in agreement with [83], where this phase of the moduli space has been previously studied.

A.2 Invariants for $\mathbf{X}_9^{1,1,1,3,3}$

The compactification of the local brane in $\mathcal{O}(-3)_{\mathbf{P}^2}$ is described by the charge vectors

$$l^1 = (-3,1,1,1,0,0,0,0,0), l^2 = (0,0,0,-2,0,1,1,-1,1), l^3 = (0,0,0,-1,1,0,0,1,-1)\,. \quad \text{(A.5)}$$

Some invariants for this geometry are

$l\backslash m$	k=0 0	1	2	3	4	5
0	*	3	0	0	0	0
1	3	*	-3	-3	-3	-3
2	-3	-6	*	15	21	27
3	3	12	36	*	-120	-183
4	-6	-30	-96	-312	*	1197
5	15	84	306	978	3255	*
6	-39	-252	-1032	-3480	-11124	-37980
7	105	792	3600	13080	42822	137166

$l\backslash m$	k=1 0	1	2	3	k=2 0	1	2	3
0	*	27	0	0	*	81	-108	0
1	-72	*	90	90	-1269	*	-1539	-1377
2	72	234	*	-684	-684	-2808	*	13554
3	-144	-612	-1980	*	2268	11232	42336	
4	360	1890	6624	22320	-7848	-46656	-182916	-671922
5	-1008	-6300	-24660	-82908	27972	194832	835758	3020382
6	3024	21924	95760	340200	-102024	-813456	-3844512	-14554242
7	-9504	-78408	-379512	-1445472	377784	3390336	17598600	70975872

Table A.1: Invariants $N_{k,l,m}$ for the geometry (A.5).

The invariants for $k = 0$ are three times the invariants in table 4.3, where the overall factor comes from the three global sections of the elliptic fibration $\mathbf{X}_9$. It appears that the invariants for $k = 1,\ l \neq 0$ are generally 3/10 times the invariants in table 4.4.

Some invariants for the geometry (4.41) in the large volume phase are

$l\backslash m$	l=0 0	1	2	3	4	5
0	*	54	0	0	0	0
1	-36	*	54	-18	0	0
2	18	-54	*	36	0	0
3	0	0	-54	*	54	0
4	0	0	0	-36	*	54
5	0	0	0	18	-54	*
6	0	0	0	0	0	-54
7	0	0	0	0	0	0

$l\backslash m$	l=1 0	1	2	3	4	l=2 0	1	2	3
0	*	0	0	0	0	*	0	0	0
1	72	*	-108	36	0	-180	*	270	-90
2	-36	-1728	*	2772	-1026	108	7020	*	-11160
3	-1224	17280	-80460	*	243756	-108	-5832	-97686	*
4	5508	-64800	340092	-1075140	*	-10944	133488	-588276	2643372

Table A.2: Invariants $N_{k,l,m}$ for the geometry (4.41).

It would be interesting to check some of these predictions by an independent computation.

B The Real Quintic

B.1 The polynomials

Using the method described in this work we obtained polynomial expression for the topological string partition functions. In this appendix we give the explicit expressions of some of these polynomials in terms of the transformed generators.

$$\begin{aligned}
\mathcal{F}^{(2,0)} &= -\frac{1473}{2000}-\frac{139}{375P}-\frac{43P}{9000}+\frac{P^2}{1200}+\frac{140\tilde{T}}{9P}-\frac{5\tilde{T}^z}{36}+\frac{65\tilde{T}^z}{18P}-\frac{29\tilde{T}^{zz}}{450}\\
&+\frac{253\tilde{T}^{zz}}{900P}+\frac{13P\tilde{T}^{zz}}{1440}-\frac{5\tilde{T}^z\tilde{T}^{zz}}{6P}+\frac{(\tilde{T}^{zz})^2}{30}-\frac{29(\tilde{T}^{zz})^2}{120P}+\frac{(\tilde{T}^{zz})^3}{24P}, \qquad \text{(B.1)}\\
\mathcal{F}^{(3,0)} &= -\frac{2507719933}{22680000000}-\frac{1208767}{30000000P^2}-\frac{10405909}{90000000P}-\frac{1936909P}{2835000000}+\frac{4661P^2}{5040000}-\frac{29P^3}{90000}\\
&+\frac{P^4}{25200}+\frac{2021\tilde{T}}{67500}+\frac{13066\tilde{T}}{5625P^2}+\frac{23077\tilde{T}}{5000P}-\frac{47P\tilde{T}}{9000}-\frac{1316\tilde{T}^2}{27P^2}-\frac{12319\tilde{T}^z}{360000}+\frac{14437\tilde{T}^z}{45000P^2}\\
&+\frac{26201\tilde{T}^z}{27000P}+\frac{1067P\tilde{T}^z}{90000}-\frac{P^2\tilde{T}^z}{480}-\frac{611\tilde{T}\tilde{T}^z}{27P^2}+\frac{47\tilde{T}\tilde{T}^z}{54P}+\frac{1603(\tilde{T}^z)^2}{21600}-\frac{105539(\tilde{T}^z)^2}{27000P^2}\\
&-\frac{2621(\tilde{T}^z)^2}{27000P}-\frac{209(\tilde{T}^z)^3}{81P^2}-\frac{7573\tilde{T}^{zz}}{720000}-\frac{10231\tilde{T}^{zz}}{360000P^2}+\frac{118493\tilde{T}^{zz}}{2160000P}+\frac{48631P\tilde{T}^{zz}}{4320000}\\
&-\frac{4453P^2\tilde{T}^{zz}}{1080000}+\frac{19P^3\tilde{T}^{zz}}{36000}-\frac{611\tilde{T}\tilde{T}^{zz}}{10800}-\frac{11891\tilde{T}\tilde{T}^{zz}}{6750P^2}+\frac{1363\tilde{T}\tilde{T}^{zz}}{3375P}+\frac{2547\tilde{T}^z\tilde{T}^{zz}}{20000}\\
&-\frac{187013\tilde{T}^z\,\tilde{T}^{zz}}{135000P^2}-\frac{30983\tilde{T}^z\tilde{T}^{zz}}{135000P}-\frac{1613P\tilde{T}^z\tilde{T}^{zz}}{72000}+\frac{47\tilde{T}\tilde{T}^z\,\tilde{T}^{zz}}{9P^2}-\frac{3997(\tilde{T}^z)^2\tilde{T}^{zz}}{2700P^2}\\
&+\frac{2719\tilde{T}^z\tilde{T}^{zz}}{5400P}+\frac{61019(\tilde{T}^{zz})^2}{1080000}-\frac{385429(\tilde{T}^{zz})^2}{2160000P^2}-\frac{15577(\tilde{T}^{zz})^2}{360000P}-\frac{48557P(\tilde{T}^{zz})^2}{2160000}\\
&+\frac{1307P^2(\tilde{T}^{zz})^2}{432000}+\frac{1363\tilde{T}(\tilde{T}^{zz})^2}{900P^2}-\frac{47\tilde{T}(\tilde{T}^{zz})^2}{225P}-\frac{251\tilde{T}^z(\tilde{T}^{zz})^2}{2700}-\frac{14857\tilde{T}^z(\tilde{T}^{zz})^2}{54000P^2}\\
&+\frac{26227\tilde{T}^z(\tilde{T}^{zz})^2}{54000P}+\frac{293(\tilde{T}^z)^2(\tilde{T}^{zz})^2}{360P^2}-\frac{7123(\tilde{T}^{zz})^3}{108000}+\frac{29(\tilde{T}^{zz})^3}{8100P^2}+\frac{29761(\tilde{T}^{zz})^3}{216000P}\\
&+\frac{2539P(\tilde{T}^{zz})^3}{259200}-\frac{47\tilde{T}(\tilde{T}^{zz})^3}{180P^2}+\frac{19\tilde{T}^z(\tilde{T}^{zz})^3}{30P^2}-\frac{131\tilde{T}^z(\tilde{T}^{zz})^3}{720P}+\frac{7(\tilde{T}^{zz})^4}{360}+\frac{203(\tilde{T}^{zz})^4}{1500P^2}\\
&-\frac{3797(\tilde{T}^{zz})^4}{36000P}-\frac{3\tilde{T}^z(\tilde{T}^{zz})^4}{20P^2}-\frac{3(\tilde{T}^{zz})^5}{40P^2}+\frac{11(\tilde{T}^{zz})^5}{480P}+\frac{(\tilde{T}^{zz})^6}{80P^2}, \qquad \text{(B.2)}\\
\mathcal{F}^{(1,2)} &= \frac{(\tilde{\mathcal{E}})(\tilde{\mathcal{E}}^z)}{12}-\frac{17(\tilde{\mathcal{E}}^z)^2}{120}-\frac{14(\tilde{\mathcal{E}})^2}{3P}-\frac{13(\tilde{\mathcal{E}})(\tilde{\mathcal{E}}^z)}{6P}-\frac{113(\tilde{\mathcal{E}}^z)^2}{120P}-\frac{211(\tilde{\mathcal{E}})\sqrt{z}}{10\sqrt{P}}\\
&-\frac{71(\tilde{\mathcal{E}}^z)\sqrt{z}}{20\sqrt{P}}+\frac{9}{40}(\tilde{\mathcal{E}})\sqrt{z}\sqrt{P}-\frac{9}{80}(\tilde{\mathcal{E}}^z)\sqrt{z}\sqrt{P}+\frac{(\tilde{\mathcal{E}}^z)^2P}{24}+\frac{9}{40}(\tilde{\mathcal{E}}^z)\sqrt{z}P^{3/2}+\frac{53\tilde{T}}{30P}\\
&-\frac{17(\tilde{T}^z)}{600}+\frac{71(\tilde{T}^z)}{300P}-\frac{25(\tilde{\mathcal{E}}^z)^2(\tilde{T}^z)}{6P}-\frac{33(\tilde{T}^{zz})}{5000}+\frac{7(\tilde{\mathcal{E}}^z)^2(\tilde{T}^{zz})}{24}-\frac{73(\tilde{T}^{zz})}{10000P}+\frac{(\tilde{\mathcal{E}})(\tilde{\mathcal{E}}^z)(\tilde{T}^{zz})}{2P}\\
&-\frac{4(\tilde{\mathcal{E}}^z)^2(\tilde{T}^{zz})}{3P}+\frac{7P(\tilde{T}^{zz})}{5000}-\frac{(\tilde{T}^z)(\tilde{T}^{zz})}{20P}+\frac{3(\tilde{T}^{zz})^2}{2500}-\frac{3(\tilde{T}^{zz})^2}{2500P}+\frac{(\tilde{\mathcal{E}}^z)^2(\tilde{T}^{zz})^2}{2P}\\
&+a_{-1}^{(1,2)}P^{-1}+a_0^{(1,2)}+a_1^{(1,2)}P+a_2^{(1,2)}P^2. \qquad \text{(B.3)}
\end{aligned}$$

$$
\begin{aligned}
\mathcal{F}^{(2,1)} = {} & \frac{278(\tilde{\mathcal{E}})}{375P^{3/2}} - \frac{(\tilde{\mathcal{E}}^z)}{3000P^{3/2}} + \frac{1473(\tilde{\mathcal{E}})}{1000\sqrt{P}} + \frac{979(\tilde{\mathcal{E}}^z)}{3600\sqrt{P}} + \frac{43(\tilde{\mathcal{E}})\sqrt{P}}{4500} - \frac{157(\tilde{\mathcal{E}}^z)\sqrt{P}}{14400} - \frac{1}{600}(\tilde{\mathcal{E}})P^{3/2} \\
& + \frac{181(\tilde{\mathcal{E}}^z)P^{3/2}}{18000} - \frac{1}{600}(\tilde{\mathcal{E}}^z)P^{5/2} + \frac{3\sqrt{z}\tilde{T}}{4} - \frac{280(\tilde{\mathcal{E}})\tilde{T}}{9P^{3/2}} - \frac{65(\tilde{\mathcal{E}}^z)\tilde{T}}{9P^{3/2}} - \frac{211\sqrt{z}\tilde{T}}{3P} \\
& + \frac{5(\tilde{\mathcal{E}}^z)\tilde{T}}{18\sqrt{P}} + \frac{341\sqrt{z}(\tilde{T}^z)}{1200} - \frac{65(\tilde{\mathcal{E}})(\tilde{T}^z)}{9P^{3/2}} - \frac{3331(\tilde{\mathcal{E}}^z)(\tilde{T}^z)}{900P^{3/2}} - \frac{287\sqrt{z}(\tilde{T}^z)}{20P} + \frac{5(\tilde{\mathcal{E}})(\tilde{T}^z)}{18\sqrt{P}} \\
& - \frac{103(\tilde{\mathcal{E}}^z)(\tilde{T}^z)}{300\sqrt{P}} + \frac{29}{240}(\tilde{\mathcal{E}}^z)\sqrt{P}(\tilde{T}^z) + \frac{261\sqrt{z}P(\tilde{T}^z)}{800} - \frac{55(\tilde{\mathcal{E}}^z)(\tilde{T}^z)^2}{9P^{3/2}} + \frac{13\sqrt{z}(\tilde{T}^{zz})}{2400} \\
& - \frac{253(\tilde{\mathcal{E}})(\tilde{T}^{zz})}{450P^{3/2}} - \frac{1517(\tilde{\mathcal{E}}^z)(\tilde{T}^{zz})}{1800P^{3/2}} - \frac{239\sqrt{z}(\tilde{T}^{zz})}{240P} + \frac{29(\tilde{\mathcal{E}})(\tilde{T}^{zz})}{225\sqrt{P}} - \frac{419(\tilde{\mathcal{E}}^z)(\tilde{T}^{zz})}{3600\sqrt{P}} \\
& - \frac{13}{720}(\tilde{\mathcal{E}})\sqrt{P}(\tilde{T}^{zz}) + \frac{131(\tilde{\mathcal{E}}^z)\sqrt{P}(\tilde{T}^{zz})}{1200} + \frac{231\sqrt{z}P(\tilde{T}^{zz})}{1600} - \frac{13}{720}(\tilde{\mathcal{E}}^z)P^{3/2}(\tilde{T}^{zz}) \\
& - \frac{39}{800}\sqrt{z}P^2(\tilde{T}^{zz}) + \frac{5(\tilde{\mathcal{E}}^z)\tilde{T}(\tilde{T}^{zz})}{3P^{3/2}} - \frac{9\sqrt{z}(\tilde{T}^z)(\tilde{T}^{zz})}{400} + \frac{5(\tilde{\mathcal{E}})(\tilde{T}^z)(\tilde{T}^{zz})}{3P^{3/2}} - \frac{313(\tilde{\mathcal{E}}^z)(\tilde{T}^z)(\tilde{T}^{zz})}{90P^{3/2}} \\
& + \frac{211\sqrt{z}(\tilde{T}^z)(\tilde{T}^{zz})}{100P} + \frac{151(\tilde{\mathcal{E}}^z)(\tilde{T}^z)(\tilde{T}^{zz})}{180\sqrt{P}} + \frac{9\sqrt{z}(\tilde{T}^{zz})^2}{800} + \frac{29(\tilde{\mathcal{E}})(\tilde{T}^{zz})^2}{60P^{3/2}} - \frac{1537(\tilde{\mathcal{E}}^z)(\tilde{T}^{zz})^2}{3600P^{3/2}} \\
& + \frac{299\sqrt{z}(\tilde{T}^{zz})^2}{800P} - \frac{(\tilde{\mathcal{E}})(\tilde{T}^{zz})^2}{15\sqrt{P}} + \frac{761(\tilde{\mathcal{E}}^z)(\tilde{T}^{zz})^2}{1800\sqrt{P}} - \frac{109(\tilde{\mathcal{E}}^z)\sqrt{P}(\tilde{T}^{zz})^2}{1440} - \frac{9}{400}\sqrt{z}P(\tilde{T}^{zz})^2 \\
& + \frac{17(\tilde{\mathcal{E}}^z)(\tilde{T}^z)(\tilde{T}^{zz})^2}{12P^{3/2}} - \frac{(\tilde{\mathcal{E}})(\tilde{T}^{zz})^3}{12P^{3/2}} + \frac{17(\tilde{\mathcal{E}}^z)(\tilde{T}^{zz})^3}{30P^{3/2}} - \frac{3(\tilde{\mathcal{E}}^z)(\tilde{T}^{zz})^3}{20\sqrt{P}} - \frac{(\tilde{\mathcal{E}}^z)(\tilde{T}^{zz})^4}{8P^{3/2}} \\
& + \sqrt{z}\left(a_{-1}^{(2,1)}P^{-1} + a_0^{(2,1)} + a_1^{(2,1)}P + a_2^{(2,1)}P^2 + a_3^{(2,1)}P^3\right)
\end{aligned} \tag{B.4}
$$

B.2 Ooguri-Vafa invariants

Replacing the generators by their holomorphic limits we can extract the Ooguri-Vafa [55] invariants from the partition functions. We used for that the conjectured formula in [22]. It should be noted however that in our formalism the disk invariants $n_d^{(0,1)}$ are extracted from $\frac{1}{2}\mathcal{F}^{(0,1)}$ and the invariants $n_d^{(1,1)}$ are extracted from $2\mathcal{F}^{(1,1)}$ in order to reproduce the numbers given in [22]. The clarification of these factors and a better understanding of the multicover formula remains for future work.

d	$n_d^{(0,1)}$
1	30
3	1530
5	1088250
7	975996780
9	1073087762700
11	1329027103924410
13	1781966623841748930
15	2528247216911976589500
17	3742056692258356444651980
19	5723452081398475208950800270
21	8986460098015260183028517362890
23	14415044640432226873354788580437780
25	23538467987973866346057268850924917500

d	$n_d^{(0,2)}$
2	0
4	26700
6	38569640
8	58369278300
10	93028407124632
12	153664503936698600
14	260548631710304201400
16	450589019788320352336020
18	791322110332876233623166320
20	1406910190370608901650146628380
22	2526625340233528751485600411725000
24	4575532116961071429530804693412171800
26	8344559227219651245031796423390078968320

d	$n_d^{(1,1)}$
1	0
3	0
5	−2742710
7	−6048504690
9	−12856992579490
11	−26585948324529250
13	−54291611312718557630
15	−110080893552894679282680
17	−222191364375273687227005740
19	−447094506460510952531302800200
21	−897635279681074059801246576212490
23	−1799147979326007629352167081015835920
25	−3601314439974327136341483249650915239910

d	$n_d^{(0,3)}$
1	0
3	0
5	117240
7	230877000
9	462884815200
11	915855637274880
13	1804779141114184800
15	3550856539832617041600
17	6982400759593452862593000
19	13728998788327325796353771400
21	26997741895033909653348464555040
23	53102177883967748623102463313529200
25	104474620947846872117630548142256678000

C Local Mirror Symmetry

C.1 Ambiguities

Using the method described in this work we obtained polynomial expression for the topological string partition functions. In this appendix we give the explicit expressions of some of the holomorphic ambiguities fixed by the method discussed in the main part of this paper.

C.1.1 Local $\mathbb{P}^2$

$$
\begin{aligned}
f^{(2)} &= \frac{-216P^2+4P+1}{17280}\,, \\
f^{(3)} &= \frac{P^4}{112} - \frac{29P^3}{3360} + \frac{2263P^2}{1088640} - \frac{13P}{136080} - \frac{1}{4354560}\,, \\
f^{(4)} &= -\frac{3}{160}P^6 + \frac{7639P^5}{201600} - \frac{32957P^4}{1209600} + \frac{1211911P^3}{146966400} - \frac{252559P^2}{261273600} + \frac{3121P}{97977600} - \frac{311}{2351462400}\,.
\end{aligned}
$$

C.1.2 Local $\mathbb{F}_0$

In the following all $f^{(g)}$ are multiplied by Δ^{2-2g} to give the ambiguity at genus g.

Genus 2

$$
\begin{aligned}
f^{(2)} &= -\frac{1}{60} + \frac{121}{720}(1+y_2)y_1 + \frac{1}{180}(-75-338y_2-75y_2^2)y_1^2 + \frac{1}{15}(-7+71y_2+71y_2^2-7y_2^3)y_1^3 \\
&\quad + \frac{4}{45}(-1+y_2)^2(23+50y_2+23y_2^2)y_1^4\,.
\end{aligned}
$$

Genus 3

$$
\begin{aligned}
f^{(3)} &= \frac{10037}{2903040} - \frac{115}{2268}(1+y_2)y_1 + \frac{1}{22680}(1607 + 42269y_2 + 1607y_2^2)y_1^2 \\
&+\frac{1}{5670}(16699 - 91239y_2 - 91239y_2^2 + 16699y_2^3)y_1^3 \\
&+\frac{1}{5670}(-137695 + 142484y_2 + 1218774y_2^2 + 142484y_2^3 - 137695y_2^4)y_1^4 \\
&+\frac{16}{2835}(14653 + 42763y_2 - 147656y_2^2 - 147656y_2^3 + 42763y_2^4 + 14653y_2^5)y_1^5 \\
&-\frac{32}{2835}(11000 + 95349y_2 - 29772y_2^2 - 337474y_2^3 - 29772y_2^4 + 95349y_2^5 + 11000y_2^6)y_1^6 \\
&+\frac{128}{2835}(-1+y_2)^2(803 + 28023y_2 + 109414y_2^2 + 109414y_2^3 + 28023y_2^4 + 803y_2^5)y_1^7 \\
&+\frac{64}{2835}(-1+y_2)^4(2833 + 22172y_2 + 42150y_2^2 + 22172y_2^3 2833y_2^4)y_1^8\,.
\end{aligned}
$$

Genus 4

$$
\begin{aligned}
f^{(4)} &= -\frac{934993}{696729600} + \frac{1873}{82944}(1+y_2)y_1 + \frac{1}{21772800}(1805481 - 35643448y_2 + 1805481y_2^2)y_1^2 \\
&+\frac{1}{10886400}(-55395131 + 227680355y_2 + 227680355y_2^2 - 55395131y_2^3)y_1^3 \\
&+\frac{1}{226800}(12490827 + 343564y_2 - 139937742y_2^2 + 343564y_2^3 + 12490827y_2^4)y_1^4 \\
&+\frac{1}{170100}(-48921165 - 310982873y_2 + 864161766y_2^2 + 864161766y_2^3 - 310982873y_2^4 - 48921165y_2^5)y_1^5 \\
&+\frac{2}{42525}(14314083 + 327499585y_2 - 71267327y_2^2 - 1601136842y_2^3 - 71267327y_2^4 + 327499585y_2^5 + 14314083y_2^6)y_1^6 \\
&+\frac{2}{14175}(3603345 - 394817549y_2 - 1102751935y_2^2 + 2494798139y_2^3 + 2494798139y_2^4 - 1102751935y_2^5 \\
&-394817549y_2^6 + 3603345y_2^7)y_1^7 \\
&-\frac{4}{42525}(82804869 - 778119160y_2 - 8753666580y_2^2 + 1161603000y_2^3 + 26815280030y_2^4 + 1161603000y_2^5 \\
&-8753666580y_2^6 - 778119160y_2^7 + 82804869y_2^8)y_1^8 \\
&+\frac{128}{42525}(7059047 + 32956639y_2 - 493715972y_2^2 - 1075054388y_2^3 + 1776480754y_2^4 + 1776480754y_2^5 - 1075054388y_2^6 \\
&-493715972y_2^7 + 32956639y_2^8 + 7059047y_2^9)y_1^9 \\
&-\frac{128}{14175}(-1+y_2)^2(2818187 + 50363322y_2 + 78760776y_2^2 - 644284250y_2^3 - 1452576870y_2^4 - 644284250y_2^5 \\
&+78760776y_2^6 + 50363322y_2^7 + 2818187y_2^8)y_1^{10} \\
&+\frac{256}{4725}(-1+y_2)^4(168629 + 6546911y_2 + 54108709y_2^2 + 145614151y_2^3 + 145614151y_2^4 + 54108709y_2^5 \\
&+6546911y_2^6 + 168629y_2^7)y_1^{11} \\
&+\frac{2048}{42525}(-1+y_2)^6(85909 + 1579674y_2 + 7561563y_2^2 + 12511468y_2^3 + 7561563y_2^4 + 1579674y_2^5 + 85909y_2^6)y_1^{12}\,.
\end{aligned}
$$

C.1.3 Local $\mathbb{F}_2$

In the following all $f^{(g)}$ are multiplied by Δ_1^{2-2g} to give the ambiguity at genus g.

Genus 2

$$
f^{(2)} = -\frac{1}{60} + \frac{121}{720}z_1 - \frac{5}{12}z_1^2 - \frac{7}{15}z_1^3 - \frac{47}{45}z_2z_1^2 + \frac{92z_1^4}{45} + \frac{92}{15}z_2z_1^3 - \frac{352}{45}z_1^4z_2 - \frac{64}{45}z_1^4z_2^2\,.
$$

Genus 3

$$
\begin{aligned}
f^{(3)} \quad = \quad & \frac{10037}{2903040} - \frac{115 z_1}{2268} + \frac{1607 z_1^2}{22680} + \frac{16699 z_1^3}{5670} + \frac{7811 z_2 z_1^2}{4536} - \frac{27539 z_1^4}{1134} - \frac{7852}{315} z_2 z_1^3 + \frac{234448 z_1^5}{2835} + \\
& \frac{115544}{945} z_2 z_1^4 - \frac{70400}{567} z_1^6 - \frac{488032 z_2 z_1^5}{2835} + \frac{109736}{945} z_2^2 z_1^4 + \frac{102784 z_1^7}{2835} - \frac{34784}{105} z_2 z_1^6 - \frac{72064}{63} z_2^2 z_1^5 \\
& + \frac{181312 z_1^8}{2835} + \frac{887296}{945} z_2 z_1^7 + \frac{3329792}{945} z_2^2 z_1^6 - \frac{756736 z_2 z_1^8}{2835} - \frac{948224}{315} z_2^2 z_1^7 + \frac{3495424 z_2^3 z_1^6}{2835} - \frac{161792}{189} z_2^2 z_1^8 \\
& - \frac{3006464}{567} z_2^3 z_1^7 + \frac{9322496 z_1^8 z_2^3}{2835} + \frac{507904}{405} z_1^8 z_2^4 \, .
\end{aligned}
$$

Genus 4

$$
\begin{aligned}
f^{(4)} \quad = \quad & -\frac{934993}{696729600} + \frac{1873 z_1}{82944} + \frac{200609 z_1^2}{2419200} - \frac{55395131 z_1^3}{10886400} - \frac{3925441 z_2 z_1^2}{2177280} + \frac{4163609 z_1^4}{75600} + \frac{98466437 z_2 z_1^3}{2721600} \\
& - \frac{120793}{420} z_1^5 - \frac{3101234 z_2 z_1^4}{14175} + \frac{1363246 z_1^6}{2025} - \frac{16594262 z_2 z_1^5}{42525} - \frac{7227701 z_2^2 z_1^4}{14175} + \frac{480446 z_1^7}{945} + \frac{69032882 z_2 z_1^6}{6075} \\
& + \frac{25875076 z_2^2 z_1^5}{2835} - \frac{1752484}{225} z_1^8 - \frac{120011704 z_2 z_1^7}{2025} - \frac{55663952}{945} z_2^2 z_1^6 + \frac{903558016 z_1^9}{42525} + \frac{823176064 z_2 z_1^8}{6075} \\
& + \frac{122904352}{945} z_2^2 z_1^7 - \frac{1664462368 z_2^3 z_1^6}{42525} - \frac{360727936 z_1^{10}}{14175} - \frac{3913572352 z_2 z_1^9}{42525} + \frac{647694464 z_2^2 z_1^8}{2835} + \frac{7607485568 z_2^3 z_1^7}{14175} \\
& + \frac{43169024 z_1^{11}}{4725} - \frac{235307776 z_2 z_1^{10}}{1575} - \frac{22776242176 z_2^2 z_1^9}{14175} - \frac{12377030656 z_2^3 z_1^8}{4725} + \frac{175941632 z_1^{12}}{42525} + \frac{1028473856 z_2 z_1^{11}}{4725} \\
& + \frac{1017880576}{405} z_2^2 z_1^{10} + \frac{210322817024 z_2^3 z_1^9}{42525} - \frac{34826355712 z_2^4 z_1^8}{42525} + \frac{22740992 z_2 z_1^{12}}{14175} - \frac{281583616}{315} z_2^2 z_1^{11} \\
& - \frac{14985371648 z_2^3 z_1^{10}}{14175} + \frac{302835073024 z_2^4 z_1^9}{42525} - \frac{129335296}{405} z_2^2 z_1^{12} - \frac{22828384256 z_2^3 z_1^{11}}{4725} - \frac{89728712704 z_2^4 z_1^{10}}{4725} \\
& + \frac{8916041728 z_2^3 z_1^{12}}{8505} + \frac{66821619712 z_2^4 z_1^{11}}{4725} - \frac{52007469056 z_2^5 z_1^{10}}{14175} + \frac{3297247232 z_2^4 z_1^{12}}{2835} + \frac{20264517632 z_2^5 z_1^{11}}{1575} \\
& - \frac{75765907456 z_1^{12} z_2^5}{14175} - \frac{7038042112 z_1^{12} z_2^6}{6075} \, .
\end{aligned}
$$

C.2 Gopakumar-Vafa and orbifold Gromov-Witten invariants

Replacing the generators by their holomorphic limits we can extract the Gopakumar-Vafa invariants from the partition functions.

C.2.1 Local $\mathbb{P}^2$

Orbifold Gromov-Witten invariants

$g \backslash d$	0	1	2	3	4
0	0	$\frac{1}{3}$	$-\frac{1}{27}$	$\frac{1}{9}$	$-\frac{1093}{729}$
1	0	0	$\frac{1}{243}$	$-\frac{14}{243}$	$\frac{13007}{6561}$
2	$\frac{1}{17280}$	$\frac{1}{19440}$	$-\frac{13}{11664}$	$\frac{20693}{524880}$	$-\frac{12803923}{4723920}$
3	$-\frac{1}{4354560}$	$-\frac{31}{2449440}$	$\frac{11569}{22044960}$	$-\frac{2429003}{66134880}$	$\frac{871749323}{198404640}$
4	$-\frac{311}{2351462400}$	$\frac{313}{62985600}$	$-\frac{1889}{5038848}$	$\frac{115647179}{2550916800}$	$-\frac{29321809247}{3401222400}$

C.2.2 Local $\mathbb{F}_0$

Gopakumar-Vafa invariants

Genus 0

$d_1\backslash d_2$	0	1	2	3	4	5	6	7
0	0	-2	0	0	0	0	0	0
1	-2	-4	-6	-8	-10	-12	-14	-16
2	0	-6	-32	-110	-288	-644	-1280	-2340
3	0	-8	-110	-756	-3556	-13072	-40338	-109120
4	0	-10	-288	-3556	-27264	-153324	-690400	-2627482
5	0	-12	-644	-13072	-153324	-1252040	-7877210	-40635264
6	0	-14	-1280	-40338	-690400	-7877210	-67008672	-455426686
7	0	-16	-2340	-109120	-2627482	-40635264	-455426686	-3986927140

Genus 1

$d_1\backslash d_2$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	0	0	9	68	300	988	2698	6444
3	0	0	68	1016	7792	41376	172124	599856
4	0	0	300	7792	95313	760764	4552692	22056772
5	0	0	988	41376	760764	8695048	71859628	467274816
6	0	0	2698	172124	4552692	71859628	795165949	6755756732
7	0	0	6444	599856	22056772	467274816	6755756732	73400088512

Genus 2

$d_1\backslash d_2$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	0	0	0	-12	-116	-628	-2488	-8036
3	0	0	-12	-580	-8042	-64624	-371980	-1697704
4	0	0	-116	-8042	-167936	-1964440	-15913228	-99308018
5	0	0	-628	-64624	-1964440	-32242268	-355307838	-2940850912
6	0	0	-2488	-371980	-15913228	-355307838	-5182075136	-55512436778
7	0	0	-8036	-1697704	-99308018	-2940850912	-55512436778	-754509553664

Genus 3

$d_1\backslash d_2$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	0	0	0	0	15	176	1130	5232
3	0	0	0	156	4680	60840	501440	3059196
4	0	0	15	4680	184056	3288688	36882969	300668468
5	0	0	176	60840	3288688	80072160	1198255524	12771057936
6	0	0	1130	501440	36882969	1198255524	23409326968	319493171724
7	0	0	5232	3059196	300668468	12771057936	319493171724	5485514375644

Genus 4

$d_1\backslash d_2$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	-18	-248	-1842
3	0	0	0	-16	-1560	-36408	-450438	-3772316
4	0	0	0	-1560	-133464	-3839632	-61250176	-662920988
5	0	0	-18	-36408	-3839632	-144085372	-2989287812	-41557026816
6	0	0	-248	-450438	-61250176	-2989287812	-79635105296	-1400518786592
7	0	0	-1842	-3772316	-662920988	-41557026816	-1400518786592	-30697119068800

C.2.3 Local $\mathbb{F}_2$

Gopakumar-Vafa invariants

d_1 and d_2 denote the degrees of the fiber and base classes respectively.[1]

Genus 0

$d_1\backslash d_2$	0	1	2	3	4	5	6	7
0	0	**0**	0	0	0	0	0	0
1	-2	-2	0	0	0	0	0	0
2	0	-4	0	0	0	0	0	0
3	0	-6	-6	0	0	0	0	0
4	0	-8	-32	-8	0	0	0	0
5	0	-10	-110	-110	-10	0	0	0
6	0	-12	-288	-756	-288	-12	0	0
7	0	-14	-644	-3556	-3556	-644	-14	0
8	0	-16	-1280	-13072	-27264	-13072	-1280	-16

Genus 1

$d_1\backslash d_2$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0
4	0	0	9	0	0	0	0	0
5	0	0	68	68	0	0	0	0
6	0	0	300	1016	300	0	0	0
7	0	0	988	7792	7792	988	0	0
8	0	0	2698	41376	95313	41376	2698	0

[1]Note that the correct genus zero data gives a value $n^{(0)}_{0,1} = 0$ which is different from the naive result $n^{(0)}_{0,1} = -\frac{1}{2}$ obtained from local mirror symmetry in [127].

Genus 2

$d_1\backslash d_2$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0
5	0	0	-12	-12	0	0	0	0
6	0	0	-116	-580	-116	0	0	0
7	0	0	-628	-8042	-8042	-628	0	0
8	0	0	-2488	-64624	-167936	-64624	-2488	0

Genus 3

$d_1\backslash d_2$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0
6	0	0	15	156	15	0	0	0
7	0	0	176	4680	4680	176	0	0
8	0	0	1130	60840	184056	60840	1130	0

Genus 4

$d_1\backslash d_2$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0
6	0	0	0	-16	0	0	0	0
7	0	0	-18	-1560	-1560	-18	0	0
8	0	0	-248	-36408	-133464	-36408	-248	0

Orbifold Gromov-Witten invariants for $\mathbb{C}^3/\mathbb{Z}_4$

Genus 0

$n_2\backslash n_1$	2	4	6	8	10
0	0	$-\frac{1}{8}$	0	$-\frac{9}{64}$	0
1	$\frac{1}{4}$	0	$\frac{7}{128}$	0	$\frac{1083}{1024}$
2	0	$-\frac{1}{32}$	0	$-\frac{143}{512}$	0
3	$\frac{1}{32}$	0	$\frac{3}{32}$	0	$\frac{85383}{16384}$
4	0	$-\frac{11}{256}$	0	$-\frac{159}{128}$	0
5	$\frac{1}{32}$	0	$\frac{47}{128}$	0	$\frac{360819}{8192}$
6	0	$-\frac{147}{1024}$	0	$-\frac{157221}{16384}$	0
7	$\frac{87}{1024}$	0	$\frac{20913}{8192}$	0	$\frac{73893099}{131072}$

Genus 1

$n_2 \backslash n_1$	2	4	6	8	10
0	0	$\frac{1}{128}$	0	$\frac{441}{4096}$	0
1	$-\frac{1}{192}$	0	$-\frac{31}{1024}$	0	$-\frac{71291}{32768}$
2	0	$\frac{35}{3072}$	0	$\frac{235}{512}$	0
3	$-\frac{5}{768}$	0	$-\frac{485}{4096}$	0	$-\frac{2335165}{131072}$
4	0	$\frac{485}{12288}$	0	$\frac{458295}{131072}$	0
5	$-\frac{39}{2048}$	0	$-\frac{40603}{49152}$	0	$-\frac{58775443}{262144}$
6	0	$\frac{2025}{8192}$	0	$\frac{10768885}{262144}$	0
7	$-\frac{2555}{24576}$	0	$-\frac{293685}{32768}$	0	$-\frac{522517275}{131072}$

Genus 2

$n_2 \backslash n_1$	2	4	6	8	10
0	0	$-\frac{61}{30720}$	0	$-\frac{9023}{81920}$	0
1	$\frac{41}{46080}$	0	$\frac{6061}{245760}$	0	$\frac{36213661}{7864320}$
2	0	$-\frac{647}{92160}$	0	$-\frac{1066027}{1310720}$	0
3	$\frac{257}{92160}$	0	$\frac{168049}{983040}$	0	$\frac{887800477}{15728640}$
4	0	$-\frac{65819}{1474560}$	0	$-\frac{18530321}{1966080}$	0
5	$\frac{23227}{1474560}$	0	$\frac{43685551}{23592960}$	0	$\frac{62155559923}{62914560}$
6	0	$-\frac{437953}{983040}$	0	$-\frac{9817250341}{62914560}$	0
7	$\frac{418609}{2949120}$	0	$\frac{452348269}{15728640}$	0	$\frac{5851085490887}{251658240}$

Genus 3

$n_2 \backslash n_1$	2	4	6	8	10
0	0	$\frac{6439}{6193152}$	0	$\frac{123167}{786432}$	0
1	$-\frac{353}{1032192}$	0	$-\frac{724271}{24772608}$	0	$-\frac{342268673}{29360128}$
2	0	$\frac{82823}{12386304}$	0	$\frac{468858317}{264241152}$	0
3	$-\frac{2759}{1376256}$	0	$-\frac{41583137}{132120576}$	0	$-\frac{211129850593}{1056964608}$
4	0	$\frac{416779}{6193152}$	0	$\frac{15342735559}{528482304}$	0
5	$-\frac{914639}{49545216}$	0	$-\frac{3864359207}{792723456}$	0	$-\frac{2178379136683}{469762048}$
6	0	$\frac{257963189}{264241152}$	0	$\frac{2719587683017}{4227858432}$	0
7	$-\frac{48988931}{198180864}$	0	$-\frac{18042606251}{176160768}$	0	$-\frac{336935310613399}{2415919104}$

Genus 4

$n_2 \backslash n_1$	2	4	6	8	10
0	0	$-\frac{2244757}{2477260800}$	0	$-\frac{283653643}{943718400}$	0
1	$\frac{865427}{3715891200}$	0	$\frac{272614087}{5662310400}$	0	$\frac{11457822706721}{317089382400}$
2	0	$-\frac{91054037}{9909043200}$	0	$-\frac{10649523253}{2202009600}$	0
3	$\frac{9329603}{4246732800}$	0	$\frac{117628391911}{158544691200}$	0	$\frac{2091862017662453}{2536715059200}$
4	0	$-\frac{590227019}{4404019200}$	0	$-\frac{4806828087037}{45298483200}$	0
5	$\frac{1775895397}{59454259200}$	0	$\frac{4224848667521}{271790899200}$	0	$\frac{7773454487649391}{317089382400}$
6	0	$-\frac{421624177657}{158544691200}$	0	$-\frac{7687488828890201}{2536715059200}$	0
7	$\frac{116460407}{209715200}$	0	$\frac{76868168176019}{181193932800}$	0	$\frac{1772672261344760983}{1932735283200}$

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