Defects in Higher-Dimensional Quantum Field Theory

Relations to AdS/CFT-Correspondence and Kondo Lattices

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Zusammenfassung

Die vorliegende Arbeit befaßt sich vor dem Hintergrund der AdS/CFT-Korrespondenz mit Defekten beziehungsweise Rändern in der Quantenfeldtheorie.

Wir untersuchen die Wechselwirkungen von Fermionen mit auf diesen Defekten lokalisierten Spins. Dazu wird eine Methode weiterentwickelt, die die kanonische Quantisierungsvorschrift um Reflexions- und Transmissionsterme ergänzt und für Bosonen in zwei Raum-Zeit-Dimensionen bereits Anwendung fand. Wir erörtern die Möglichkeiten derartiger Reflexions-Transmissions-Algebren in zwei, drei und vier Dimensionen. Wir vergleichen mit Modellen aus der Festkörpertheorie und der Beschreibung des Kondo-Effektes mithilfe konformer Feldtheorie.

Wir diskutieren ferner Ansätze der Erweiterung auf Gitterstrukturen.

Abstract

The present work is addressed to defects and boundaries in quantum field theory considering the application to AdS/CFT correspondence.

We examine interactions of fermions with spins localised on these boundaries. Therefore, an algebra method is emphasised adding reflection and transmission terms to the canonical quantisation prescription. This method has already been applied to bosons in two space-time dimensions before. We show the possibilities of such reflection-transmission algebras in two, three, and four dimensions. We compare with models of solid state physics as well as with the conformal field theory approach to the Kondo effect.

Furthermore, we discuss ansatzes of extensions to lattice structures.

Dans la boucle de l'hirondelle un orage s'informe, un jardin se construit.

(In der Schleife des Schwalbenflugs fügt sich Gewitter, gestalten sich Gärten.)

René Char, A la santé du serpent

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Regarde l'image téméraire où se baigne ton pays, ce plaisir qui t'a longtemps fui.

(Sieh das verwegene Bild, worein deine Heimat getaucht ist. Weide dein Auge an ihm, das so lang dich gemieden.)

René Char, A la santé du serpent

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Il reste une profondeur mesurable là où le sable subjugue la destinée.

(Wo Schicksal von Sand unterjocht wird, bleibt eine lotbare Tiefe.)

René Char, A la santé du serpent

1

Introduction

The present thesis examines defects (and boundaries) in fermionic quantum field theories (QFT) using a reflection-transmission algebra (RT) technique and thus derives conserved quantities from boundary data.

For more than three decades, considerations of defects and especially boundaries in field theories have been playing an important role in solid state physics and statistical mechanics. Recently, they have enjoyed increasing research interest in particle physics as well. Notably, the description of the one-impurity Kondo effect by conformal field theory in two dimensions given by Affleck and Ludwig [1; 2] and the very lively research in AdS/CFT correspondence including description of flavour by adding D branes on AdS side [3] raised the importance of tools for dealing with boundaries and defects in field theories.

Main Ideas of AdS/CFT Correspondence

Considering unifying theories, at first glance it might appear surprising that one of the most intensely investigated fields in recent years in theory of elementary particles related to string theory is AdS/CFT correspondence. Originally this correspondence is a derivation from string theory mapping between a supergravity theory in a d-dimensional anti de Sitter space (the maximum symmetric solution of Einstein's equation with negative curvature) and a conformal field theory in $d-1$-dimensional Minkowski space. This is remarkable since string theory itself was not primarily intended to map gravitation to QFT and vice versa, but rather to merge both theories into a single one.

However, the attractivity of AdS/CFT theory consists precisely in consistency with a higher-dimensional unifying theory and at the same time the vicinity to "real", i.e. falsifiable theories. Such theories are general relativity and QFT which form today's understanding of elementary particles (i.e. the standard model). Certainly, to our knowledge, the universe has no negative curvature and the standard model is not conformal in general, however, both theories are well understood and are adaptable to more realistic scenarios as well. Moreover, IR and UV fixed points of ordinary QFT are conformal and thus directly accessible to AdS/CFT.

Supergravity is expected to be the low energy limit of full superstring theory where only gravitation remains relevant. In this way it is possible to derive type IIB supergravity from type IIB string theory. Within this procedure, an AdS geometry is generated by a stack of coincident D branes (manifolds that establish a Dirichlet boundary condition on the strings ending on them). Concretely, we are further interested in D3 brane stacks (a $1+3$-dimensional object) which induce an AdS_5 geometry in supergravity theory.

The original idea of AdS/CFT correspondence [4–6] was to connect Green functions with the boundary (radial coordinate $\rho \to \infty$). This boundary is a Minkowski space of co-dimension one; the related boundary two-point functions contain fields of a conformal field theory (CFT). A CFT is a QFT that is invariant not only under Poincaré transformations but also under conformal transformations: inversion on the unit circle and scaling. Conformal transformations are locally isogonal. The emergence of a $1+3$-dimensional CFT on the boundary of an AdS_5 space is a consequence of symmetry considerations. Firstly, the gauge zero mode of the D3 brane stack generates a $U(1)$ symmetry of the zero modes in AdS_5 which corresponds to the position symmetry of a unitary field theory, a so-called Yang-Mills (YM) theory. Secondly, the isometry group of an $\mathrm{AdS}_5 \times S^5$ space is $SO(2,4) \times SO(6)$ (the additional five spatial directions compactified as S^5 complete the ten-dimensional space-time wherein superstring theory is formulated). A CFT with an R symmetry according to $\mathcal{N} = 4$ has the conformal symmetry group $SO(2,4) \times SU(4) \simeq SO(2,4) \times SO(6)$. Hence $\mathcal{N} = 4$ SYM (super Yang-Mills theory) is considered the theory corresponding to $\mathrm{AdS}_5 \times S^5$ supergravity via [6]

$$\left\langle \exp\left[\int dx^4 \varphi_0 \mathcal{O}\right]\right\rangle_{\mathrm{CFT}} = \exp\{S_{\mathrm{sg}}[\varphi]\}\big|_{\varphi(\partial\mathrm{AdS})=\varphi_0}, \tag{1.1}$$

where $S_{\mathrm{sg}}[\varphi]$ is the supergravity action and thus the right hand side is the supergravity generating functional evaluated on the boundary with boundary field φ_0. The correspondence has been suggested also in *full* string theory, but only the supergravity/SYM relation has been studied extensively yet.

Relation (1.1) is dual (the coupling constant is inverted under the transformation) and suggests a dictionary between field theory operators and supergravity fields. Furthermore, the symmetry relation implies the possibility of correlating *asymptotical* AdS supergravity with CFT. It turned out that these non-AdS but asymptotical AdS geometries exhibit a singularity in the deep interior [7] at least in the case of completely broken supersymmetry. A solution with finite singularity horizon was given by [8].

Moreover, it is possible to add further D branes to these supergravity backgrounds as shown in [3; 9–12]. These additional branes were considered as probe branes, i.e., they do not deform the background and break some symmetry by adding open string states which could also end on the additional brane. Recently, some investigation in non-probe branes [13] has been published, however beyond the scope of basic probe brane properties we are interested in here. The D brane intersections are very useful to describe flavour since the open strings – which can end on the D brane stack as well as on the probe brane – become massive and show a CFT dual in the fundamental representation. Moreover, due to the duality property of AdS/CFT correspondence, a weak coupling theory on the AdS side describes a strong coupling one on the CFT side which then can be considered as QCD. In this way an indirect perturbational approach to QCD has been presented and induced intense investigation [9; 11; 14; 15].

Additionally (by adding lower-dimensional D brane probes on the AdS side), CFT shows lower-dimensional defects, for example the D3/D5 theory turns out to have a four-dimensional CFT dual with a three-dimensional defect. Such a defect is a δ-distribution-like hyperplane (in Minkowski space just a flat plane) that can contribute additional interaction terms to the theory describing the interaction of bulk fields with the defect. In fact, embedding a D5 probe brane into an $AdS_5 \times S^5$ space generated by a D3 brane stack exhibits an $AdS_4 \times S^2$ corresponding to a three-dimensional subspace on the CFT side [16; 17], preserving $SO(3,2)$ conformal symmetry to all orders in perturbation theory [18]. From the gravity perspective this field theory may be interpreted as the AdS/CFT dual of a four-dimensional AdS_4 subspace on which gravity is potentially localised. This localisation would correspond to the existence of a conserved three-dimensional energy-momentum tensor in dual field theory. This remains an open question. Similarly, by considering a D3 brane probe on $AdS_3 \times S^1$, a gravity dual of a four-dimensional conformal field theory with a two-dimensional interacting defect is obtained [11].

A further non-supersymmetric deformation of the $AdS_5 \times S^5$ background has been studied in [19; 20]. On the field theory side, the so-called Janus models exhibit a CFT with defect, but different couplings on both sides of the defect. Therefore, they show different “faces” to the defect depending on the side we look at. Such Janus deformations can be examined for lower dimensions [21] as well.

Application in Solid State Physics

Research on two-dimensional CFT led to applications in solid state physics as well. Especially influential was the description of the single-impurity Kondo effect – first stated in 1964 by Jun Kondo [22] – by an effective two-dimensional CFT with boundary given by Affleck and Ludwig [1]. The Kondo effect describes the low-temperature behaviour of resistivity in solid states including magnetic impurities. While an ideal solid state shows a decreasing resistivity for decreasing temperature T going to zero for $T = 0$, the existence of impurities generates a finite resistivity at $T = 0$ (for electrical charged impurities) or a resistivity minimum near $T = 0$ which rises to a finite value at $T = 0$. The latter is called Kondo effect. Since that time, a lot of modelling work has been undertaken in solid state physics, however failing to achieve an exact description of several impurities up to now. Recent works by Affleck and Ludwig [23] opened new prospects by incorporating two-dimensional CFT with boundary.

However, any field theoretic model considering the Kondo effect should be related to standard descriptions in solid state physics. According to standard textbooks (for instance [24]), there are different Hamiltonian operators for different band structures of solids. Essentially there are two such standard descriptions, the Heisenberg and the Hubbard model. The Heisenberg operator is commonly used to describe lattice spin interactions or interactions of fixed spins with conducting electrons, respectively, whereas the Hubbard model operator is used for spin interactions between transmission electrons (i.e. freely propagating, non-localised spins). Consequently, these applications are disjunct.

Heisenberg operator In detail, the most general Heisenberg operator is

$$H = -\sum_{i,j} J_{ij} \left[\alpha (S_i^x S_j^x + S_i^y S_j^y) + \beta (S_i^z S_j^z) \right] .$$

Depending on the formulation of the problem, α, β are chosen differently,

Heisenberg model	$(\alpha = \beta = 1)$	$H = -\sum_{i,j} J_{ij}\, \vec{S}_i \cdot \vec{S}_j,$
Ising model	$(\alpha = 0,\ \beta = 1)$	$H = -\sum_{i,j} J_{ij}\, S_i^z S_j^z,$
XY model	$(\alpha = 1,\ \beta = 0)$	$H = -\sum_{i,j} J_{ij}\, (S_i^x S_j^x + S_i^y S_j^y).$

For fixed spin defects interacting with conducting electrons (Kondo model), there exists a reformulation according to Rudermann-Kittel-Kasuya-Yosida (RKKY), transferring this "mixed form" of interaction to the Heisenberg form. The coupling constant J_{ij} is oscillating in correspondence with the distance between the localised defects. As a result, the Kondo effect is transformed into an *indirect* interaction of localised spins via interacting electrons.

The second simplification is the approximation of next neighbours, keeping J_{ij} = const. and considering only interactions of nearest neighbours. The simplifications then are as follows (i, j localised):

RKKY model $$H = -\sum_{i,j} J_{ij}^{\mathrm{RKKY}}\, \vec{S}_i \cdot \vec{S}_j,$$

NN approximation ($J_{ij} = J$) $$H = -J \sum_{i,j} \vec{S}_i \cdot \vec{S}_j.$$

Kondo model The Kondo model [22; 25] (for a review see [26]) describes the interaction of conduction electrons with fixed magnetic impurities; i.e. spins. In particular, this interaction is responsible for the low temperature resistivity behaviour that does not vanish (as would be the case for an ideal conductor) but shows a minimum near $T = 0$. Hence, the original operator for the Kondo model (with *two* localised spins, between which the indirect interaction is then mediated) reads

$$\begin{aligned} H_{sf} = -\frac{g\,\hbar}{2\,N} \sum_{i=1}^{2} \sum_{\vec{k},\vec{q}} e^{-i\,\vec{q}\vec{R}_i} \Big[& S_i^z (c^\dagger_{\vec{q}+\vec{k},\uparrow} c_{\vec{k},\uparrow} - c^\dagger_{\vec{q}+\vec{k},\downarrow} c_{\vec{k},\downarrow}) + \\ & + S_i^+ \, c^\dagger_{\vec{q}+\vec{k},\downarrow} c_{\vec{k},\uparrow} + S_i^- \, c^\dagger_{\vec{q}+\vec{k},\uparrow} c_{\vec{k},\downarrow} \Big]. \end{aligned} \tag{1.2}$$

Here, $c^\dagger$, c are creator and annihilator of the electrons with corresponding spin and momentum. $\vec{R}_i$ denotes the spatial position of spin S_i. In addition, the spin operators have been decomposed into components with common eigenstates,

$$\vec{S}_i \cdot \vec{S}_j = S_i^z \, S_j^z + \frac{1}{2} \left(S_i^+ \, S_j^- + S_i^- \, S_j^+ \right). \tag{1.3}$$

Thus the transformation simplifies the Hamiltonian operator significantly, but works only for *more than one* spin impurity. Moreover, in contrast to the present work, the generic point of view in solid states physics is considering the two-spin impurity case as a trap in which the electron freely propagates [27; 28].

Hubbard operator The Hubbard operator cannot be transformed into a Heisenberg form and is thus we are faced with a completely different problem,

$$\begin{aligned} H &= \sum_{ij\sigma} T_{ij} \, a^\dagger_{i\sigma} a_{j\sigma} + \frac{1}{2} \sum_{\substack{ijkl \\ \sigma\,\sigma'}} v(ij,kl) \, a^\dagger_{i\sigma} a^\dagger_{j\sigma'} a_{k\sigma'} a_{l\sigma} \\ &\approx \sum_{ij\sigma} T_{ij} \, a^\dagger_{i\sigma} a_{j\sigma} + \frac{1}{2} U \sum_{i,\sigma} n_{i,\sigma} n_{i,-\sigma}. \end{aligned} \tag{1.4}$$

Here, U actually is the matrix element $v(ii, ii)$ describing intraatomic exchange. Thus, interatomic interactions are not affected by the low overlap.

In the present work, we concentrate on a Heisenberg-like model. Using reflection-transmission formalism, we need fixed impurities. This is essential for description of the Kondo effect as well.

Dealing with Defects and Boundaries

For conformal field theories, pioneering work by John Cardy [29–32] raised the question of conformal boundary interactions in the 1980s. Inspired by conformal embeddings suggested by Altschuler et al. [33] breaking a part of the symmetry by introducing a boundary, he derived some fundamental results on fusion rules, a Verlinde formula for the boundary case, and a classification of conformal families [30; 31]. He also suggested an operator expansion on the boundary (BOE) influenced by the conformal bulk theory [32].

The proposed BOE was later developed in detail by Osborn and McAvity [34; 35] who gave the operator expectation values in dependence on the distance from the defect. In fact, they applied an elegant method of integrating the modes of the Fourier decomposition over hyperplanes parallel to the defect [36] and transforming the result back into position space. A generalisation to dynamical degrees of freedom was given in [37].

Strongly related to the infinite number of generators of two-dimensional conformal symmetry, the mathematical approach [30], generalising fusion rules and resulting correlation functions, was elaborated by Fuchs and Schweigert [38; 39] applying category theory and theorems about Kac-Moody algebras describing the defect [40–42]. Kac-Moody algebras (for review see [43]) have also been used for the above mentioned description of the Kondo effect by two-dimensional CFT with boundary. A Kac-Moody ansatz was implemented by Affleck and Ludwig [23] to derive the energy shiftings caused by one-impurity interactions. Moreover, they suggested a multi-channel ansatz [44] for the two-impurity Kondo problem.

Related to such spin interactions, models with additional $O(N)$ symmetry raised interest [45–48]. In particular, the $O(N)$ model is a generalised Ising model (where the vector multiplets are in the fundamental representation of

$O(N)$). This model shows a free UV-fixed point and a non-trivial fixed point in the infrared. At the fixed points the theory is conformal invariant. The avowed AdS dual is a gravity theory with infinite number of gauge fields of even spin [49]. In both energy regimes, both theories are dual to each other. Klebanov und Polyakov [49] argue that the Breitenlohner-Freedman bound is modified in this case and does not exclude the conformal dimension Δ_-,

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 L^2} \tag{1.5}$$

(d – dimension, m – mass, L – AdS radius) which is rather associated to the infrared (for a review see [50]).

Recently, this led to further investigations of deformed models in four dimensions [51] as well as for lower dimensions [52; 53].

Defects and Boundaries in RT Algebra Formalism

Leaving aside conformal symmetry and turning attention to integrable systems and considering possibilities of changing the canonical quantisation to more general algebras, a method was invented and proven [54–56] that correctly includes the boundary interaction terms right in the quantisation. Following former work of A. and A. Zamolodchikov [57] and L. D. Faddeev [58], such generalised FZ algebras (for a summary see [59]) were supplemented by δ distribution terms including the boundary position as well as boundary behaviour amplitudes, interpreted as reflection and transmission [56]. For integrable systems [60] this resulted directly in Yang-Baxter equations for the defect case.

Furthermore, it was shown by Bajnok et al. [61] that such a quantisation formalism (RT formalism) can describe both a defect and a boundary, because the defect case can be interpreted as gluing together two boundary theories at their respective boundaries. In this way, the somewhat unphysical defect type (it has measure zero) can simply be interpreted as boundary condition limit of some expanded defect. The effects in finite distance from the boundary or defect should then give the same picture as that provided by Osborn and McAvity [36].

Note that in the present thesis we are dealing only with theories that are free far from the defect; i.e., only scattered states are considered. Bound states can be easily added into RT algebras and will give additional terms for energy densities and conserved currents as well. However, doing so would have gone beyond the scope of this work.

Giving a general algebra framework for fermionic theories with boundaries and defects for arbitrary higher space-time dimensions, we derive a formalism to determine RT amplitudes in dependence on coupling parameters in the Lagrangean. We give two-point expectation values in terms of RT amplitudes and calculate explicitly conserved quantities in two, three and four dimensions. Furthermore, we extend this algebra approach to parallel defects of arbitrary number by means of a matrix optics ansatz. We thus provide a basis for introduction of further (for instance conformal) symmetry and discuss extensions to lower-dimensional defects.

Taken together, in the present thesis a complete description of fermionic defect theory in 2–4 space-time dimensions is given. In the second chapter the RT algorithm will be introduced in detail. In chapter 3 we show how the construction for fermions works and gives detailed results for energy densities and currents. The conclusion will summarise main aspects and point out future research. We add an appendix that states a technical completion of the longish derivation of RT coefficients from boundary data in arbitrary dimensions.

Les ténèbres que tu t'infuses sont régies par la luxure de ton ascendant solaire.

(Die Finsternisse, die du dir einflößt, durchwaltet die Wollust des Sonnenzeichens, dem du unterstehst.)

René Char, A la santé du serpent

2

Defects in Quantum Field Theory and RT Formalism

The idea of reflection-transmission (RT) algebras is based on the perception of defects as δ-distribution-like "walls", where particle wave functions obey a boundary condition. These boundary conditions should be consistent with all quantum mechanical requirements usually dealt with in quantum field theory, namely continuity of the absolute value of the wave function as well as integrability. Consequently, ideas of geometrical optics can be relied on, where transmission and reflection amplitudes for (light) waves have originally been defined. To consider similar amplitudes for particle waves means in particular "stepping back" from a perturbation theory of energy states and related scattering matrices to wave function scattering matrices that then become k-dependent (where k is the momentum), but not necessarily k^2-dependent, as one usually expects from energy-dependent scattering.

Actually, conventional quantum mechanics deals with two complementary concepts. On the one hand, according to the underlying Lie algebra, spin states of many particles are combined into common eigenstates. For instance, two spin-half particles couple to one singulett and one triplett state via Clebsch-Gordon; i.e. standard fusion rules. On the other hand, there is perturbation theory starting from an exactly solved model, considering perturbations εH_1 and ordering by ε, where $\varepsilon < 1$ is small enough to ensure the convergence of the method.

In addition, scattering theory deals with the idea that the perturbation is strongly localised. This implies that, far from the perturbation, the eigenstates are not affected by εH_1 and thus are asymptotically exact. Therefore, localised perturbation is understood as some operator – the scattering matrix – intermediating between different exact energy eigenstates $|E_n\rangle$ with eigenvalue E_n. Of course, this scattering matrix $S_{ij} = \lambda_{ij} |E_j\rangle \langle E_i|$ is of infinite rank.

We strengthen the scattering concept to δ-like defects that yield a boundary condition at the defect. In other words, we consider every defect theory that can be described by a boundary condition at a finite position x_0. Furthermore, we do not interpret the scattering process as a transition between eigenstates of different energy, but as scattering of eigenstates of the entire theory in order to fulfill the boundary condition. The advantage of such a theory becomes clear immediately when this S matrix (scattering matrix) is written down explicitly: it only mixes states of the same energy eigenvalue (that means it becomes diagonal). Therefore, the theory is exact – right from the beginning – and we do not have to take into account any perturbations.

Accordingly, the canonical algebra of creators and annihilators contains additional (transmission and reflection) terms affecting the two-point functions only in case that one of the particles interacts with the boundary. So we expect these reflection and transmission coefficients (RT coefficients) to enter the algebra as prefactors of δ distributions related to the defect. Such a very general algebra was established by Mintchev, Ragoucy, and Sorba [56; 62]. They gave a complete proof of the validity of the general formulation. Subsequently, two of these authors calculated the details for the bosonic case [63] and considered a φ^4-interaction term in the Lagrangean as well [64].

Furthermore, in geometric optics complicated composite structures can easily be calculated by multiplying matrices describing the single elements. Once the defect behaviour is formulated in terms of reflection and transmission, one might ask whether such defects act like ordinary reflecting and transmitting elements for light waves (i.e. lenses, mirrors etc.) and obey matrix optics rules as well.

In this chapter we give an overview over these techniques and state their bosonic results as examples. Furthermore, we show that the "light wave idea" is even more sustainable than assumed in the above mentioned papers: a matrix optics ansatz holds even for RT algebras.

2.1 Definition of an RT Algebra

According to the antecedent outline, the idea behind this chapter consists in interpreting the boundary condition of the quantum mechanical wave functions as δ defects that transmit part of the wave and reflect the rest. Consequently, in the second quantisation approach, these reflection and transmission amplitudes have to enter as operators in the creator-annihilator algebra. In a first step, such an algebra (without reflection and transmission) can be written in a general form known as Zamolodchikov-Fadeev algebra [57; 58; 65]:

$$a_{\alpha_1}(\chi_1)\, a_{\alpha_2}(\chi_2) = \mathcal{S}^{\beta_1\beta_2}_{\alpha_2\alpha_1}(\chi_2,\chi_1)\, a_{\beta_2}(\chi_2)\, a_{\beta_1}(\chi_1), \tag{2.1}$$

$$a^{\alpha_1}(\chi_1)\, a^{\alpha_2}(\chi_2) = \mathcal{S}^{\alpha_1\alpha_2}_{\beta_2\beta_1}(\chi_2,\chi_1)\, a^{\beta_2}(\chi_2)\, a^{\beta_1}(\chi_1) \tag{2.2}$$

$$a_{\alpha_1}(\chi_1)\, a^{\alpha_2}(\chi_2) = \mathcal{S}^{\alpha_2\beta_1}_{\alpha_1\beta_2}(\chi_1,\chi_2)\, a^{\beta_2}(\chi_2)\, a_{\beta_1}(\chi_1) + 2\pi\,\delta^{\alpha_2}_{\alpha_1}\,\delta(\chi_1-\chi_2) \tag{2.3}$$

Here we used a short form $a^\alpha := a^*_\alpha$ for the creators related to the annihilators by complex conjugation. The indices α_i, β_i label all degrees of freedom of the wave function in the Fourier decomposition. Due to the general formulation, the elements $\mathcal{S}^{\beta_1\beta_2}_{\alpha_2\alpha_1}(\chi_2,\chi_1)$ are some prefactor tensors that adjust the different creator indices and the difference in χ_i that shall be deemed to be the momentum. Of course, in canonical quantisation, the tensors $\mathcal{S}^{\beta_1\beta_2}_{\alpha_2\alpha_1}$ become just Kronecker δ functions with a sign depending on fermionic or bosonic requirements.

In order to describe defects, we add transmission and reflection elements $t^{\alpha_2}_{\alpha_1}$, $r^{\alpha_2}_{\alpha_1}$. It is not obvious at this stage how this can be done beneficially. However, we give a reasonable definition that will be justified in the following sections. We claim that the most general form of a second quantisation creator-annihilator algebra with elements $\{a_\alpha(\chi),\ a^\alpha(\chi),\ r^\beta_\alpha(\chi),\ t^\beta_\alpha(\chi)\}$ reads

[62]:

$$a_{\alpha_1}(\chi_1)\,a_{\alpha_2}(\chi_2) - S^{\beta_1\beta_2}_{\alpha_2\alpha_1}(\chi_2,\chi_1)\,a_{\beta_2}(\chi_2)\,a_{\beta_1}(\chi_1) = 0, \tag{2.4}$$

$$a^{\alpha_1}(\chi_1)\,a^{\alpha_2}(\chi_2) - a^{\beta_2}(\chi_2)\,a^{\beta_1}(\chi_1)\,S^{\alpha_1\alpha_2}_{\beta_2\beta_1}(\chi_2,\chi_1) = 0, \tag{2.5}$$

$$\begin{aligned} a_{\alpha_1}(\chi_1)\,a^{\alpha_2}(\chi_2) - a^{\beta_2}(\chi_2)\,S^{\alpha_2\beta_1}_{\alpha_1\beta_2}(\chi_1,\chi_2)\,a_{\beta_1}(\chi_1) &= \\ = 2\pi\,\delta(\chi_1-\chi_2)\,\left[\delta^{\alpha_2}_{\alpha_1}\mathbf{1} + t^{\alpha_2}_{\alpha_1}(\chi_1)\right] + 2\pi\,\delta(\chi_1+\chi_2)\,r^{\alpha_2}_{\alpha_1}(\chi_2). \end{aligned} \tag{2.6}$$

Obviously, the last equation (2.6) differs slightly from (2.3). The additional elements $t^{\alpha_2}_{\alpha_1}$, $r^{\alpha_2}_{\alpha_1}$ appear on the right hand side related to δ-distributions of $\chi_1 \pm \chi_2$ that we later on will interpret as outgoing momentum χ_1 and incoming one χ_2 while $t^{\alpha_2}_{\alpha_1}$ is related to $\chi_1 = \chi_2$ and $r^{\alpha_2}_{\alpha_1}$ to $\chi_1 = -\chi_2$. Hence we can interpret these as transmission and reflection operators, since transmission means that the outcoming momentum is the same as the incoming, while reflection is basically a momentum change of -2χ and χ the incoming momentum.

But the algebra is not yet complete. There are additional algebra relations called defect exchange relations,

$$\begin{aligned} S^{\gamma_2\gamma_1}_{\alpha_1\alpha_2}(\chi_1,\chi_2)\,r^{\delta_1}_{\gamma_1}(\chi_1)\,S^{\beta_1\delta_2}_{\gamma_2\delta_1}(\chi_2,-\chi_1)\,r^{\beta_2}_{\delta_2}(\chi_2) &= \\ = r^{\gamma_2}_{\alpha_2}(\chi_2)\,S^{\delta_2\delta_1}_{\alpha_1\gamma_2}(\chi_1,-\chi_2)\,r^{\gamma_1}_{\delta_1}(\chi_1)\,S^{\beta_1\beta_2}_{\delta_2\gamma_1}(-\chi_2,-\chi_1)& \end{aligned} \tag{2.7}$$

$$\begin{aligned} S^{\gamma_2\gamma_1}_{\alpha_1\alpha_2}(\chi_1,\chi_2)\,t^{\delta_1}_{\gamma_1}(\chi_1)\,S^{\beta_1\delta_2}_{\gamma_2\delta_1}(\chi_2,\chi_1)\,t^{\beta_2}_{\delta_2}(\chi_2) &= \\ = t^{\gamma_2}_{\alpha_2}(\chi_2)\,S^{\delta_2\delta_1}_{\alpha_1\gamma_2}(\chi_1,\chi_2)\,t^{\gamma_1}_{\delta_1}(\chi_1)\,S^{\beta_1\beta_2}_{\delta_2\gamma_1}(\chi_2,\chi_1)& \end{aligned} \tag{2.8}$$

$$\begin{aligned} S^{\gamma_2\gamma_1}_{\alpha_1\alpha_2}(\chi_1,\chi_2)\,t^{\delta_1}_{\gamma_1}(\chi_1)\,S^{\beta_1\delta_2}_{\gamma_2\delta_1}(\chi_2,\chi_1)\,r^{\beta_2}_{\delta_2}(\chi_2) &= \\ = r^{\gamma_2}_{\alpha_2}(\chi_2)\,S^{\delta_2\delta_1}_{\alpha_1\gamma_2}(\chi_1,-\chi_2)\,t^{\gamma_1}_{\delta_1}(\chi_1)\,S^{\beta_1\beta_2}_{\delta_2\gamma_1}(-\chi_2,\chi_1),& \end{aligned} \tag{2.9}$$

and mixed exchange relations,

$$a_{\alpha_1}(\chi_1)\,r^{\beta_2}_{\alpha_2}(\chi_2) = S^{\gamma_1\gamma_2}_{\alpha_2\alpha_1}(\chi_2,\chi_1)\,r^{\delta_2}_{\gamma_2}(\chi_2)\,S^{\beta_2\delta_1}_{\gamma_1\delta_2}(\chi_1,-\chi_2)\,a_{\delta_1}(\chi_1) \tag{2.10}$$

$$r^{\beta_1}_{\alpha_1}(\chi_1)\,a^{\alpha_2}(\chi_2) = a^{\delta_2}(\chi_2)\,S^{\gamma_2\delta_1}_{\alpha_1\delta_2}(\chi_1,\chi_2)\,r^{\gamma_1}_{\delta_1}(\chi_1)\,S^{\beta_1\alpha_2}_{\gamma_2\gamma_1}(\chi_2,-\chi_1) \tag{2.11}$$

$$a_{\alpha_1}(\chi_1)\,t^{\beta_2}_{\alpha_2}(\chi_2) = S^{\gamma_1\gamma_2}_{\alpha_2\alpha_1}(\chi_2,\chi_1)\,t^{\delta_2}_{\gamma_2}(\chi_2)\,S^{\beta_2\delta_1}_{\gamma_1\delta_2}(\chi_1,\chi_2)\,a_{\delta_1}(\chi_1) \tag{2.12}$$

$$t^{\beta_1}_{\alpha_1}(\chi_1)\,a^{\alpha_2}(\chi_2) = a^{\delta_2}(\chi_2)\,S^{\gamma_2\delta_1}_{\alpha_1\delta_2}(\chi_1,\chi_2)\,t^{\gamma_1}_{\delta_1}(\chi_1)\,S^{\beta_1\alpha_2}_{\gamma_2\gamma_1}(\chi_2,\chi_1). \tag{2.13}$$

Note that this algebra avoids an extra defect-related operator D due to addition of $t^{\alpha_2}_{\alpha_1}$ and $r^{\alpha_2}_{\alpha_1}$. Such a defect operator, previously suggested by Delfino et al. [54; 55], should describe the effect of the defect on the bulk states, but yields an additional relation

$$a^{\alpha_1}(\chi)\,D = r^{\alpha_1}_{\alpha_2}(\chi)\,a^{\alpha_2}(\chi)\,D + t^{\alpha_1}_{\alpha_2}(\chi)\,D\,a^{\alpha_2}(\chi), \tag{2.14}$$

$$D\,a^{\alpha_1}(\chi) = r^{\alpha_1}_{\alpha_2}(\chi)\,D\,a^{\alpha_2}(\chi) + t^{\alpha_1}_{\alpha_2}(\chi)\,a^{\alpha_2}(\chi)\,D, \tag{2.15}$$

eleminated by the present formulation. Moreover, Bajnok et al. [61] showed that this algebra can be used for describing boundary theories as well. They just set $t_{\alpha_1}^{\alpha_2} \equiv 0$ and $|r_{\alpha_1}^{\alpha_2}| = 1$. In this way, the algebra should be applicable to all relevant defect problems including boundaries. Furthermore, as already mentioned, it is not necessary to deal with general $\mathcal{S}_{\alpha_2\alpha_1}^{\beta_1\beta_2}$ as long as we investigate fermions or bosons, where they are substituted by

$$\mathcal{S}_{\alpha_1\alpha_2}^{\beta_1\beta_2} = \begin{cases} +\delta_{\alpha_1\alpha_2}^{\beta_1\beta_2} & \Leftrightarrow \quad \text{bosonic theory,} \\ -\delta_{\alpha_1\alpha_2}^{\beta_1\beta_2} & \Leftrightarrow \quad \text{fermionic theory.} \end{cases} \tag{2.16}$$

Both equations (2.4), (2.5) then induce vanishing (anti-)commutators. Additionally, $t_{\alpha_1}^{\alpha_2}$ and $r_{\alpha_1}^{\alpha_2}$ can be taken as transmission and reflection expectation values; i.e. transmission and reflection amplitudes $T_{\alpha_1}^{\alpha_2}, R_{\alpha_1}^{\alpha_2}$. In order to complete the algebra, we have to ensure that these amplitudes satisfy

$$T_{\alpha_1}^{\beta}(\chi)\, T_{\beta}^{\alpha_2}(\chi) + R_{\alpha_1}^{\beta}(-\chi)\, R_{\beta}^{\alpha_2}(\chi) = \delta_{\alpha_1}^{\alpha_2}, \tag{2.17}$$

$$T_{\alpha_1}^{\beta}(\chi)\, R_{\beta}^{\alpha_2}(-\chi) + R_{\alpha_1}^{\beta}(-\chi)\, T_{\beta}^{\alpha_2}(\chi) = 0. \tag{2.18}$$

These conditions now define the ultimate *RT algebra*, since they are in one-to-one correspondence with the unitarity of the scattering matrix,

$$\mathbf{S}^{\dagger}\mathbf{S} = \mathbb{1}. \tag{2.19}$$

Here the scattering matrix is defined as $S_{\alpha_1}^{\alpha_2} := T_{\alpha_1}^{\alpha_2} + R_{\alpha_1}^{\alpha_2}$.

After all these simplifications, it is now sufficient to consider

$$[a_{\alpha_1}(p_1),\, a_{\alpha_2}(p_2)]_{\pm} = 0, \tag{2.20}$$

$$[a^{\alpha_1}(p_1),\, a^{\alpha_2}(p_2)]_{\pm} = 0, \tag{2.21}$$

$$\begin{aligned}[a_{\alpha_1}(p_1),\, a^{\alpha_2}(p_2)]_{\pm} &= (2\pi)^{d-1}\delta(\hat{\mathbf{p}}_1 - \hat{\mathbf{p}}_2)\cdot \\ &\cdot \Big\{\delta(p_1 - p_2)\, \big[\delta_{\alpha_1}^{\alpha_2} + T_{\alpha_1}^{\alpha_2}(p_1)\big] + \delta(p_1 + p_2)\, R_{\alpha_1}^{\alpha_2}(p_2)\Big\}.\end{aligned} \tag{2.22}$$

The variables $\hat{\mathbf{p}}$ denote momenta parallel to the defects. In (2.20)–(2.22) they are just (constant) parameters. d denotes the space-time dimension. The mixed exchange relations are given by

$$a^{t,j}(p)\overline{T}_{t,j}^{s,i}(p) + a^{t,j}(-p)\overline{R}_{t,j}^{s,i}(-p) = a^{s,i}(p), \tag{2.23}$$

$$a_{t,j}(p)T_{s,i}^{t,j}(p) + a_{t,j}(-p)R_{s,i}^{t,j}(-p) = a_{s,i}(p), \tag{2.24}$$

where we split the indices $\alpha = (t, j)$. j is indicating by "±" the side of the defect $x \gtrless 0$ the operator refers to. t labels different solutions of the equations of motion; i.e. for example the solutions of different spin of the Dirac equation. Moreover, considering $T^{t,j}_{s,i}(p)$ as "transmission" and $R^{t,j}_{s,i}$ as "reflection", we claim a transmission to the same side of the defect $T^{t,\pm}_{s,\pm} \equiv 0$ as well as reflection to different sides $R^{t,\pm}_{s,\mp} \equiv 0$. This simplifies the exchange algebra (*without* summing over i) to

$$a^{t,-i}(p)\overline{T}^{s,i}_{t}(p) + a^{t,i}(-p)\overline{R}^{s,i}_{t}(-p) = a^{s,i}(p), \tag{2.25}$$

$$a_{t,-i}(p)T^{t,-i}_{s}(p) + a_{t,i}(-p)R^{t,i}_{s}(-p) = a_{s,i}(p), \tag{2.26}$$

with abbreviations $T^{t,\pm}_{s} :\equiv T^{t,\pm}_{s,\pm}$ and $R^{t,\pm}_{s} :\equiv R^{t,\pm}_{s,\mp}$ (and similar for the barred components).

2.2 RT Formalism in Bosonic Theory

We will now give a review of two-dimensional bosonic defect theory as derived in [63] in order to get an idea of the power of the formalism. Moreover, we will introduce our conventions that slightly differ from [63]. However, for purposes of legibility and in accordance with intuition, we believe our notation beneficial.

2.2.1 Properties of the RT Algebra

Lagrangean We start with the Lagrangean for the bosonic case. We consider a free d-dimensional theory interacting with a $(d-1)$-dimensional defect:

$$\mathcal{L} = T + V = \varphi^\dagger \left[\Box_{\text{bulk}} + \eta\, \delta(x)\right] \varphi. \tag{2.27}$$

Here $\Box_{\text{bulk}}$ is the operator that describes the theory in the bulk – it should be a free theory, not containing terms of higher order. There are more complicated interaction terms $\partial\mathcal{L}$ than $V = \eta\, \delta(x)$, but to simplify matters we will only handle this one here.

Boundary condition Due to integrability conditions for the operator ∂_x^2, we are able to give a general boundary condition [66; 67],

$$\begin{pmatrix} \varphi(t,+0) \\ \partial_x \varphi(t,+0) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \varphi(t,-0) \\ \partial_x \varphi(t,-0) \end{pmatrix}, \tag{2.28}$$

where $ad - bc = 1$ (the boundary matrix is an element of $SO(2)$). For the simple case of a δ impurity with $V = \eta\,\delta(x)$ we can integrate the equation of motion over the interval $x \in [-\varepsilon, +\varepsilon]$ and the take limit $\varepsilon \to 0$:

$$\begin{pmatrix} \varphi(t,+0) \\ \partial_x \varphi(t,+0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \eta & 1 \end{pmatrix} \begin{pmatrix} \varphi(t,-0) \\ \partial_x \varphi(t,-0) \end{pmatrix}. \tag{2.29}$$

Decomposition of the wave function Without impurity, a set of orthogonal wave functions is given by plane waves $e^{\pm ikx}$. With impurity, these incoming waves (from $\pm\infty$) are split into reflected and transmitted parts. This implies[1]

$$\psi_{x,\pm}(k) := \theta(\mp k)\left\{\theta(\mp x) T^{\pm}(k)\, e^{ikx} + \theta(\pm x)\left[e^{ikx} + R^{\pm}(k)\, e^{-ikx}\right]\right\}. \tag{2.30}$$

Here we already assumed that these amplitudes R and T – labelled with naïve intuition what reflection and transmission should be like – fit in the RT algebra picture defined above. We will prove this later (see for bosons equation (2.50) and for fermions (3.42)). Additionally, in contrast to [63] and the algebra definition, we employ a shorter notation of $T^{\pm}_{(\mp)}$ and $R^{\pm}_{(\pm)}$ respectively. The lower index is only necessary for the general algebra formulation.

Scattering matrix Since we deal with quantum mechanical functions, they obey $|T|^2 + |R|^2 = 1$. Due to their different parity, they have to be orthogonal, i.e. $\langle \varphi_+ | \varphi_- \rangle = 0$:

$$\overline{T}^{+}(k)\, T^{+}(k) + \overline{R}^{+}(k)\, R^{+}(k) = 1, \tag{2.31}$$

$$\overline{T}^{-}(k)\, T^{-}(k) + \overline{R}^{-}(k)\, R^{-}(k) = 1, \tag{2.32}$$

$$\overline{R}^{+}(k)\, T^{-}(k) + \overline{T}^{+}(k)\, R^{-}(k) = 0. \tag{2.33}$$

[1]Note that we believe it more suggestive to associate the reflection amplitude to the incoming momentum, not the outgoing one, as in [56], for instance (see figure 1 therein). Nevertheless, e.g. the amplitude R^+ describes the reflected particles from $-\infty$ that are scattered back to $-\infty$. We believe it delusive to denote R^+ as $R^+(-k)$ because it will never describe any particles from $+\infty$. Moreover, we could write more definitely $R^+(-|k|)$ which is a bit longish. Hence we define R as function of $+k$ in the decomposed waves and any inversions of k are matters of mathematics without physical meaning. Therefore, fitting conventions, one has to read $R^{\pm}(k)$ in the present thesis as $R^{\pm}_{\pm}(-k)$ in [56; 62; 63; 68].

We define composed RT matrices,

$$\mathbf{R}(k) := \begin{pmatrix} R^{+}(k) & \\ & R^{-}(k) \end{pmatrix}, \qquad \mathbf{T}(k) := \begin{pmatrix} & T^{-}(k) \\ T^{+}(k) & \end{pmatrix}, \tag{2.34}$$

and read equations (2.31)–(2.33) as

$$\mathbf{T}(k)^{\dagger}\,\mathbf{T}(k) + \mathbf{R}(k)^{\dagger}\,\mathbf{R}(k) = \mathbb{1}, \tag{2.35}$$

$$\mathbf{R}(k)^{\dagger}\,\mathbf{T}(k) + \mathbf{T}(k)^{\dagger}\,\mathbf{R}(k) = 0, \tag{2.36}$$

or the shorter one via the scattering matrix $\mathbf{S}$,

$$\mathbf{S}^{\dagger}\mathbf{S} = \mathbb{1}, \qquad \mathbf{S} := \mathbf{R} + \mathbf{T}, \tag{2.37}$$

where $\mathbf{S}$ is defined as usual, transforming the incoming into the outgoing states,

$$|\psi\rangle_{\text{out}} = \mathbf{S}\,|\psi\rangle_{\text{in}} = \mathbf{S}\begin{pmatrix}\psi_{+} \\ \psi_{-}\end{pmatrix}_{\text{in}}. \tag{2.38}$$

$\psi_{\pm\text{in}}$ are simply the incoming wave functions from left $(-)$ and right $(+)$.

Derivation of the RT coefficients Let us consider $\varphi_{k,\pm}$ the Fourier modes of the wave function φ defined in (2.30). They have to respect the boundary condition at $x = 0$ separately:

$$T^{-} = a(1 + R^{-}) + i\,k b(1 - R^{-}), \qquad 1 + R^{+} = T^{+}(a + i\,k b), \tag{2.39}$$

$$i\,k T^{-} = c(1 + R^{-}) + i\,k d(1 - R^{-}), \qquad i\,k(1 - R^{+}) = T^{+}(c + i\,k d). \tag{2.40}$$

This implies

$$R^{-}(k) = \frac{k^2 b - i\,k(a - d) + c}{k^2 b + i\,k(a + d) - c}, \qquad R^{+}(k) = \frac{k^2 b + i\,k(a - d) + c}{k^2 b - i\,k(a + d) - c}, \tag{2.41}$$

$$T^{-}(k) = \frac{2\,i\,k}{k^2 b + i\,k(a + d) - c}, \qquad T^{+}(k) = \frac{2\,i\,k}{k^2 b + i\,k(a + d) - c}. \tag{2.42}$$

Therefore, relation (2.37) is satisfied automatically by the boundary condition.[2] It has to be emphasised that this system of equations is unique even though the

[2]This is in one-to-one correspondence to [63], in case their definition (equation (2.13))

$$\psi_{+}(k) = \theta(-k)\left\{\theta(-x)T^{+}(k)\,e^{i\,kx} + \theta(+x)\left[e^{i\,kx} + R^{+}(-k)\,e^{-i\,kx}\right]\right\}$$

is read carefully for example as

$$\psi_{+}(k) = \theta(-k)\left\{\theta(-x)T^{+}(-k)\,e^{i\,kx} + \theta(+x)\left[e^{i\,kx} + R^{+}(k)\,e^{-i\,kx}\right]\right\}$$

as defined in [62], equation (2.3).

three equations plus the two boundary conditions over-determine the reflection and transmission coefficients (in contrast to the fermionic case). The condition $ad - bc = 1$ entails that one of the five equations is satisfied automatically.

Resulting algebra In fact, equation (2.30) follows from the RT algebra formulation given above following [62]. For the bosonic case we have the RT algebra of generator $a^{\pm}(k)$ and annihilators $a_{\pm}(k)$ defined as

$$[a_{\alpha_1}(p_1),\ a_{\alpha_2}(p_2)] = 0, \tag{2.43}$$

$$[a^{\alpha_1}(p_1),\ a^{\alpha_2}(p_2)] = 0, \tag{2.44}$$

$$\begin{aligned}[a_{\alpha_1}(p_1),\ a^{\alpha_2}(p_2)] &= 2\pi\,\delta(p_1 - p_2)\ \left[\delta_{\alpha_1}^{\alpha_2} + \delta_{\alpha_2}^{-\alpha_1} T^{\alpha_2}(p_1)\right] + \\ &+ 2\pi\,\delta(p_1 + p_2)\,\delta_{\alpha_2}^{\alpha_1} R^{\alpha_2}(p_2).\end{aligned} \tag{2.45}$$

Additional mixed exchange relations are given by

$$a^{j}(k)\overline{T}^{j}(k) + a^{-j}(-k)\overline{R}^{-j}(-k) = a^{-j}(k), \tag{2.46}$$

$$a_{j}(k)T^{j}(k) + a_{-j}(-k)R^{-j}(-k) = a_{-j}(k). \tag{2.47}$$

With

$$\varphi(x) = \sum_{j=\pm} \theta(jx)\Phi_j(x), \tag{2.48}$$

$$\Phi_j(x) := \int_{-\infty}^{+\infty} \frac{dk}{2\pi\sqrt{2\omega(k)}}\ \left[a^{j}(k)\ e^{i\,\omega(k)t - i\,kx} + a_j(k)\ e^{-i\,\omega(k)t + i\,kx}\right], \tag{2.49}$$

and via (2.46) and (2.47), we derive

$$\begin{aligned}\varphi(x) = \sum_{j=\pm}\int_{-\infty}^{+\infty} &\frac{dk}{2\pi\sqrt{2\omega(k)}}\cdot \\ \cdot \Big\{ &a^{j}(k)\theta(-jk)\overline{\{\theta(-jx)T^{j}(k)\ e^{i\,kx} + \theta(jx)\,[e^{i\,kx} + R^{j}(k)e^{-i\,kx}]\}}\,e^{i\,\omega(k)t} + \\ + &a_j(k)\theta(-jk)\,\left\{\theta(-jx)T^{j}(k)\ e^{i\,kx} + \theta(jx)\,\left[e^{i\,kx} + R^{j}(k)e^{-i\,kx}\right]\right\}\,e^{-i\,\omega(k)t}\Big\}.\end{aligned} \tag{2.50}$$

This implies

$$\varphi(x) = \varphi_+(x) + \varphi_-(x), \tag{2.51}$$

$$\varphi_\pm(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi\sqrt{2\omega(k)}}\ \left[a^{\pm}(k)\overline{\psi}_x^{\pm}(k)\,e^{i\,\omega(k)t} + a_\pm(k)\psi_x^{\pm}(k)\,e^{-i\,\omega(k)t}\right], \tag{2.52}$$

$$\psi_x^{\pm}(k) = \theta(\mp k)\,\left\{\theta(\mp x)T^{\pm}(k)\ e^{i\,kx} + \theta(\pm x)\,\left[e^{i\,kx} + R^{\pm}(k)e^{-i\,kx}\right]\right\}, \tag{2.53}$$

where the last equation is exactly (2.30). This indicates nothing else but the identity of the coefficients R and T defined in the algebra (2.43)–(2.44) and those we intuitively inserted into the wave function decomposition (2.30). This is a direct implication of the additional algebra relations (2.46) and (2.47).

2.2.2 Properties of the Theory in Terms of Reflection and Transmission

For non-vanishing chemical potential μ and temperature T (with number operator N), the Hamiltonian reads

$$\mathcal{H} = H - \mu N, \tag{2.54}$$

where

$$H = \sum_{j=\pm} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \omega(k) a^j(k) a_j(k), \qquad N = \sum_{j=\pm} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} a^j(k) a_j(k), \tag{2.55}$$

hence

$$\mathcal{H} = \sum_{j=\pm} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \left[\omega(k) - \mu\right] a^j(k) a_j(k). \tag{2.56}$$

This represents the standard many-particle description easily to be found in statistical physics textbooks. According to their Fourier mode of momentum k, the creators and annihilators have some eigenenergies $\omega(k)$ that are summed over all occupied states. The chemical potential μ enters with the particle number.

Normal ordering moves the creators to the left and the annihilators to the right:

$$:a_j(k_1) a^l(k_2): = a^l(k_2) a_j(k_1) = :a^l(k_2) a_j(k_1): . \tag{2.57}$$

In this (grand canonical) ensemble, the expectation value is defined as

$$\left\langle F(a^j, a_k) \right\rangle = \frac{\mathrm{tr}\, e^{-\beta\mathcal{H}} F(a^j, a_k)}{\mathrm{tr}\, e^{-\beta\mathcal{H}}}. \tag{2.58}$$

Using the identity

$$e^{-\beta\mathcal{H}} a^\alpha(p_1) = e^{-\beta[\omega(p_1)-\mu]} a^\alpha(p_1)\, e^{-\beta\mathcal{H}}, \tag{2.59}$$

we derive the non-vanishing Green functions,

$$\begin{aligned}\left\langle a^j(p_1)a_k(p_2)\right\rangle &= \frac{\mathrm{tr}\, e^{-\beta\mathcal{H}}\, a^j(p_1)a_k(p_2)}{\mathrm{tr}\, e^{-\beta\mathcal{H}}}\\ &= e^{-\beta[\omega(p_1)-\mu]}\,\frac{\mathrm{tr}\, a^j(p_1)\, e^{-\beta\mathcal{H}}\, a_k(p_2)}{\mathrm{tr}\, e^{-\beta\mathcal{H}}}\\ &= e^{-\beta[\omega(p_1)-\mu]}\,\frac{\mathrm{tr}\, e^{-\beta\mathcal{H}}\, a_k(p_2)a^j(p_1)}{\mathrm{tr}\, e^{-\beta\mathcal{H}}}\\ &= e^{-\beta[\omega(p_1)-\mu]}\left\{\left\langle a^j(p_1)a_k(p_2)\right\rangle + \left[a_k(p_2),\, a^j(p_1)\right]\right\}, \end{aligned} \tag{2.60}$$

$$\Rightarrow \quad \left\langle a^j(p_1)a_k(p_2)\right\rangle = \frac{e^{-\beta[\omega(p_1)-\mu]}}{1-e^{-\beta[\omega(p_1)-\mu]}}\,\left[a_k(p_2),\, a^j(p_1)\right], \tag{2.61}$$

$$\Rightarrow \quad \left\langle a_k(p_2)a^j(p_1)\right\rangle = \frac{1}{1-e^{-\beta[\omega(p_1)-\mu]}}\,\left[a_k(p_2),\, a^j(p_1)\right]. \tag{2.62}$$

The two-point expectation value simplifies in terms of R and T:

$$\begin{aligned}&\langle\varphi(x_1,t_1)\,\varphi(x_2,t_2)\rangle\\ &\quad = \Big\langle \iint_{-\infty}^{+\infty}\frac{\theta(\pm x_1)\theta(\pm x_2)\, dk_1\, dk_2}{8\pi^2\sqrt{\omega(k_1)\omega(k_2)}}\cdot\\ &\qquad\qquad \cdot\left[a^{\pm}(k_1)\, e^{i\,\omega(k_1)t_1-i\,k_1x_1} + a_{\pm}(k_1)\, e^{-i\,\omega(k_1)t_1+i\,k_1x_1}\right]\cdot\\ &\qquad\qquad \cdot\left[a^{\pm}(k_2)\, e^{i\,\omega(k_2)t_2-i\,k_2x_2} + a_{\pm}(k_2)\, e^{-i\,\omega(k_2)t_2+i\,k_2x_2}\right]\Big\rangle\\ &\quad = \sum_{j,l=\pm}\iint_{-\infty}^{+\infty}\frac{\theta(jx_1)\theta(lx_2)\, dk_1\, dk_2}{8\pi^2\sqrt{\omega(k_1)\omega(k_2)}}\cdot\\ &\qquad\qquad \cdot\Big[\left\langle a^j(k_1)a_l(k_2)\right\rangle e^{-i\,k_1x_1+i\,k_2x_2}\, e^{i\,\omega(k_1)t_1-i\,\omega(k_2)t_2} +\\ &\qquad\qquad\quad + \left\langle a_j(k_1)a^l(k_2)\right\rangle e^{i\,k_1x_1-i\,k_2x_2}\, e^{-i\,\omega(k_1)t_1+i\,\omega(k_2)t_2}\Big]\\ &\quad = \sum_{j=\pm}\int_{-\infty}^{+\infty}\frac{dk}{4\pi\omega(k)}\frac{1}{1-e^{-\beta[\omega(k)-\mu]}}\cdot\\ &\qquad \cdot\Big\{\theta(jx_1)\theta(jx_2)\left[e^{i\,k(x_1-x_2)} + R^j(k)\, e^{-i\,k(x_1+x_2)}\right]\cdot\\ &\qquad\qquad\qquad \cdot\left[e^{-\beta[\omega(k)-\mu]}\, e^{i\,\omega(k)(t_1-t_2)} + e^{-i\,\omega(k)(t_1-t_2)}\right] +\\ &\qquad\quad + \theta(jx_1)\theta(-jx_2)\Big[e^{-\beta[\omega(k)-\mu]}\, T^j(k)\, e^{i\,\omega(k)(t_1-t_2)-i\,k(x_1-x_2)} +\\ &\qquad\qquad\qquad + T^{-j}(k)\, e^{-i\,\omega(k)(t_1-t_2)+i\,k(x_1-x_2)}\Big]\Big\}. \end{aligned} \tag{2.63}$$

We can manipulate this two-point function slightly in order to derive the energy density and the conserved currents defined for the bosonic theory as

$$\left\langle T^{00}(x)\right\rangle = \frac{1}{2}\left\langle :\partial_t\varphi\partial_t\varphi:(t,x) - :\varphi\partial_x^2\varphi:(t,x) + m^2 :\varphi\varphi:(t,x)\right\rangle, \tag{2.64}$$

$$\langle J^\mu(x)\rangle = -i\,\langle :\partial_\mu\varphi^*\varphi:(t,x) - :\varphi^*\partial_\mu\varphi:(t,x)\rangle\,. \tag{2.65}$$

The derivatives act only on the exponentials and give some extra factors, and with $\mu = 0$,

$$\left\langle T^{00}(x)\right\rangle = \mathcal{E}_0 + \mathcal{E}_D, \tag{2.66}$$

where we decomposed into the part $\mathcal{E}_0$ without defect and the defect induced density $\mathcal{E}_D$,

$$\mathcal{E}_0 := \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\frac{\omega(k)}{e^{\beta\omega(k)}-1} = \frac{\pi}{6\beta^2}, \tag{2.67}$$

$$\mathcal{E}_D := \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\frac{\omega(k)}{e^{\beta\omega(k)}-1}\left[\theta(x)R^+(k)+\theta(-x)R^-(k)\right]e^{-2ikx} \tag{2.68}$$

$$= \int_0^{+\infty} \frac{dk}{\pi(e^{\beta k}-1)}\frac{2\eta k^2\sin(2k|x|) - \eta^2 k\cos(2k|x|)}{4k^2+\eta^2}. \tag{2.69}$$

The last equation (2.69) can be easily derived by carefully handling the terms $\omega(k) \approx |k|$ and inverting $(k \mapsto -k)$ half of the integral $k \in (-\infty, 0]$ to $k \in [0, +\infty)$. Unfortunately, this integral is not solvable analytically and the numerical behaviour is difficult to control. For this reason, we modify the reflection terms according to

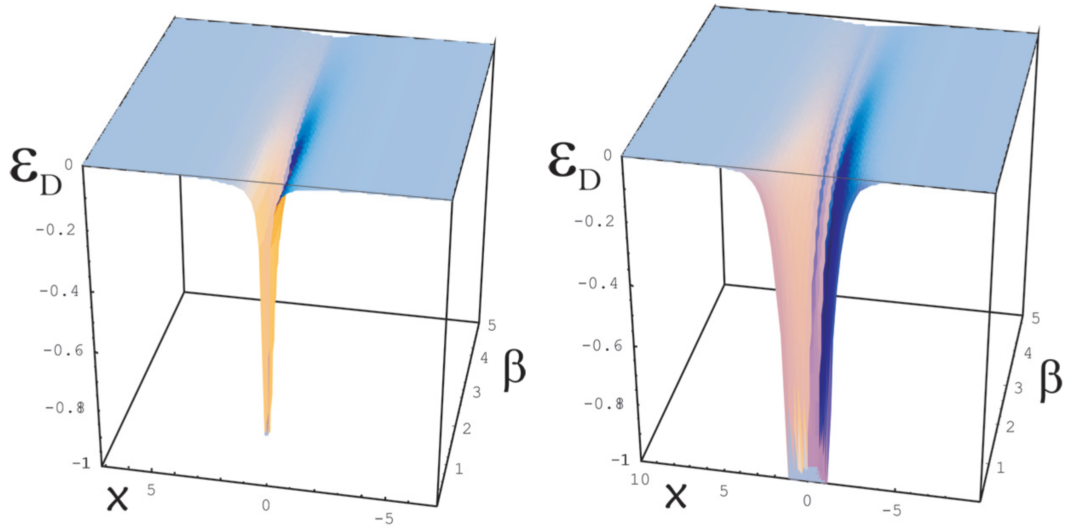
$$\frac{1}{a \pm i b} = \begin{cases} \int_0^\infty e^{-(a\pm ib)\alpha}\, d\alpha & \Leftrightarrow \quad \Re(a) > 0, \\ -\int_0^\infty e^{(a\pm ib)\alpha}\, d\alpha & \Leftrightarrow \quad \Re(a) < 0, \end{cases} \tag{2.70}$$

such that (for $a = d = 1$, $b = 0$, $c = \eta$),

$$R^\pm(k) = \frac{\eta}{\mp 2ik - \eta} = \begin{cases} -\int_0^\infty d\alpha\,\eta\, e^{-(\eta\pm 2ik)\alpha} & \Leftrightarrow \eta > 0, \\ \int_0^\infty d\alpha\,\eta\, e^{(\eta\pm 2ik)\alpha} & \Leftrightarrow \eta < 0. \end{cases} \tag{2.71}$$

The defect energy density $\mathcal{E}_D$ can be integrated over k and α (for $\eta > 0$) and thus becomes

$$\mathcal{E}_D = \frac{\eta}{2\pi}\left(\eta e^{2\eta|x|}\Gamma(2\eta|x|) + \frac{4\pi^2 e^{-\frac{4\pi|x|}{\beta}}\,{}_2F_1\left(2, \frac{\beta\eta}{2\pi}+1; \frac{\beta\eta}{2\pi}+2; e^{-\frac{4\pi|x|}{\beta}}\right)}{\beta(\beta\eta+2\pi)} - \frac{1}{2|x|}\right). \tag{2.72}$$



(a) $\mathcal{E}_D$ with $\eta = +1$ (b) $\mathcal{E}_D$ with $\eta = -1$

Figure 2.1: Bosonic energy density $\langle T^{00}(x)\rangle_\beta$ in two space-time dimensions for different parameters η plotted for varying distance from the defect x and inverse temperature β.

For $\eta < 0$ we derive

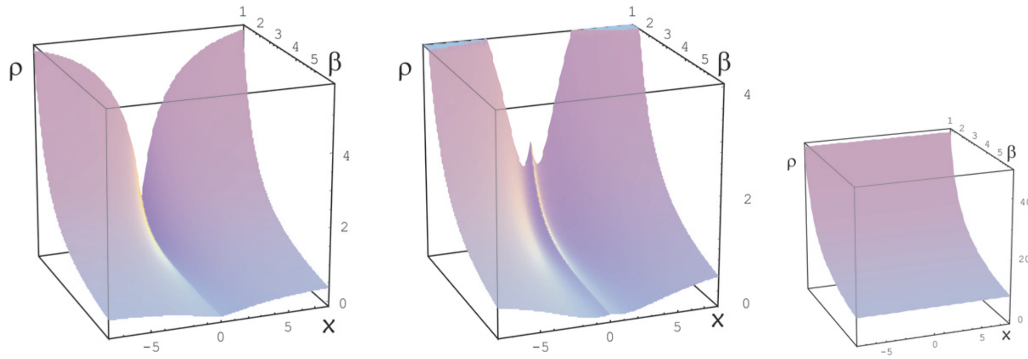
$$\begin{aligned}\mathcal{E}_D = \frac{\eta^2}{2\pi}\Big[& e^{\frac{4\pi|x|}{\beta}}\Gamma\left(-\frac{\beta\eta}{2\pi}\right) {}_2\widetilde{F}_1\left(2, 1-\frac{\beta\eta}{2\pi}; 2-\frac{\beta\eta}{2\pi}; e^{\frac{4\pi|x|}{\beta}}\right) + \\ & + e^{2\eta|x|}(\Gamma(2\eta|x|) - \log(-\eta) + \log(\eta) + i\pi) - \frac{1}{2\eta|x|}\Big]\end{aligned} \tag{2.73}$$

$\mathcal{E}_D$ is shown in figure 2.1. We should add that $\mathcal{E}_{D,\eta<0}$ obviously contains a divergent term $\log(\eta) - \log(-\eta)$ and does not look well defined. But in fact this is required because the ill-defined term cancels the terms of $\Gamma(2\eta|x|)$ that are not well-defined either and thereby gives the smooth plot shown in figure 2.1.

For completeness, we give the energy limit $T = 0 \Leftrightarrow \beta \to \infty$ as well. The Casimir energy,

$$\mathcal{E}_C = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{\omega(k)}{e^{\beta\omega(k)} - 1} \left[\theta(x)R^+(k) + \theta(-x)R^-(k)\right] e^{-2ikx}, \tag{2.74}$$

is the vacuum energy density that is subtracted ab initio by normal ordering.



(a) $J^0(x)$ with $\eta = 1$ (b) $J^0(x)$ with $\eta = -1$ (c) $J^0(x)$ with $\eta = 0$

Figure 2.2: Bosonic charge density $\rho(x)$ in two space-time dimensions for different defect masses η plotted for varying x and β. For numerical reasons, the background without defect (shown in subfigure (c)) was not subtracted from $\rho(x)$. Thus $\rho(x)$ diverges for high temperature (as β goes to zero). However, the effect induced by the defect is visible. The sign of the defect mass η determines whether there is a local minimum or maximum at $x = 0$. Globally seen, the defect decreases the value of the charge density $\rho(x)$ around $x = 0$ nearly to zero.

Hence the total energy density reads (for $\eta > 0$)

$$\mathcal{E}_{\text{tot}} = \mathcal{E}_{\text{C}} + \mathcal{E} = \mathcal{E}_{\text{C}} + \mathcal{E}_{\text{D}} + \mathcal{E}_0 \tag{2.75}$$

$$= \frac{\pi}{6\beta^2} + \frac{2\pi\eta\, e^{-\frac{4\pi|x|}{\beta}}}{\beta(\beta\eta + 2\pi)}\, {}_2F_1\left(2, \frac{\beta\eta}{2\pi} + 1; \frac{\beta\eta}{2\pi} + 2; e^{-\frac{4\pi|x|}{\beta}}\right). \tag{2.76}$$

Since this expression for the bosonic energy density is fully analytical, it is very demonstrative. The models we will investigate in the following do not show this property. Already the Noether currents defined in (2.65) have to be examined numerically. We deal with them a similar way:

$$\rho(x) = \left\langle J^0(x) \right\rangle = \int_{-\infty}^{+\infty} \frac{dk}{\pi(e^{\beta|k|} - 1)} \left\{1 + [\theta(x)R^+(k) + \theta(-x)R^-(k)]\, e^{-2ikx}\right\} \tag{2.77}$$

$$= \int_0^{+\infty} \frac{2\, dk}{\pi(e^{\beta k} - 1)} \frac{2\eta k \sin(2k|x|) - \eta^2 \cos(2k|x|)}{4k^2 + \eta^2}, \tag{2.78}$$

$$\left\langle J^1(x) \right\rangle = 0. \tag{2.79}$$

The charge density $\rho(x)$ for the δ defect is plotted in figure 2.2.

Note that, as stated before, bound states have not been considered. They should be added for negative defect mass $\eta < 0$ and give an additional energy density contribution.

2.3 Matrix Optics Construction for Bosonic Defect Theories

The idea of matrix optics, as presented in standard undergraduate textbooks, deals with incoming and outgoing directions of objects that propagate linearly and are diffracted at strongly localised objects. This diffraction is a linear map changing direction; i.e. a matrix multiplication. Due to the linearity of the propagation between the localised deflecting objects, it can be described by a matrix as well. The product of all such matrices of a complicated system will mediate between the incoming and outgoing states.

In this way matrix optics can be applied to geometrical optics as well as to particle accelerators, where batches of particles are diffracted by quadrupoles or sextupoles, for instance. For the case of RT formalism we have a certain boundary condition at each defect – which is simply a matrix – and we have to answer the question whether it is possible to transform such a boundary condition virtually by propagation from the real position of the defect to a distant position. In other words, we have to derive the lacking propagation matrix between two defects to complete the matrix optics description.

Single defect For a single defect described by Lagrangean (2.27), we have already deduced the boundary condition

$$\begin{pmatrix} \varphi(+0) \\ \partial_x \varphi(+0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \eta & 1 \end{pmatrix} \cdot \begin{pmatrix} \varphi(-0) \\ \partial_x \varphi(-0) \end{pmatrix}. \tag{2.80}$$

Hence, for a defect at $x = 0$ we obtain the transformation matrix $\begin{pmatrix} 1 & 0 \\ \eta & 1 \end{pmatrix}$.[3] However, a "propagation matrix" is still missing.

[3]This is analogous to a matrix $\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$ describing a lense of focal lenght f in geometrical optics. Consequently, the general bosonic boundary defect,

$$\begin{pmatrix} \varphi(+0) \\ \partial_x \varphi(+0) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \varphi(-0) \\ \partial_x \varphi(-0) \end{pmatrix}, \tag{2.81}$$

Two parallel defects For two parallel defects at the positions x_1, x_2, we have to rewrite the Lagrangean:

$$\mathcal{L} = \varphi^\dagger \left[\Box_{\text{bulk}} + \eta_1\, \delta(x - x_1) + \eta_2\, \delta(x - x_2)\right] \varphi. \tag{2.82}$$

Of course, boundary conditions at x_1 and x_2 can be derived the same way as before,

$$\begin{pmatrix} \varphi \\ \partial_x \varphi \end{pmatrix}\Big|_{x=x_i+0} = \begin{pmatrix} 1 & 0 \\ \eta_i & 1 \end{pmatrix} \cdot \begin{pmatrix} \varphi \\ \partial_x \varphi \end{pmatrix}\Big|_{x=x_i-0}. \tag{2.83}$$

A matrix P describing the propagation correctly has to fulfill the equivalent relations (where, without loss of generality, $x_1 \geqslant x_2$)

$$\begin{pmatrix} \varphi \\ \partial_x \varphi \end{pmatrix}\Big|_{x_1+0} = \begin{pmatrix} 1 & 0 \\ \eta_1 & 1 \end{pmatrix} \cdot P(x_1, x_2) \cdot \begin{pmatrix} 1 & 0 \\ \eta_2 & 1 \end{pmatrix} \cdot \begin{pmatrix} \varphi \\ \partial_x \varphi \end{pmatrix}\Big|_{x_2-0}, \tag{2.84}$$

$$\begin{pmatrix} \varphi \\ \partial_x \varphi \end{pmatrix}\Big|_{x_1-0} = P(x_1, x_2) \cdot \begin{pmatrix} \varphi \\ \partial_x \varphi \end{pmatrix}\Big|_{x_2+0}. \tag{2.85}$$

Obviously $\left(\begin{smallmatrix} \varphi \\ \partial_x \varphi \end{smallmatrix}\right)\Big|_{x_1-0}$ and $\left(\begin{smallmatrix} \varphi \\ \partial_x \varphi \end{smallmatrix}\right)\Big|_{x_2+0}$ are both values *between* the defects and therefore have the same Fourier modes (we label them "0" since for $x < x_2$ we indicate "−" and for $x > x_1$ we use "+"),

$$\varphi_0(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi\sqrt{2\omega(k)}} \left[a^0(k)\; e^{i\,\omega(k)t - i\,kx} + a_0(k)\; e^{-i\,\omega(k)t + i\,kx}\right]. \tag{2.86}$$

We expect the matrix P to be dependent on the distance of the defects $x_1 - x_2$ and k as well. Therefore, we evaluate (2.85) for the integrand of $\varphi_0(x)$ in (2.86):

$$\Psi(k) := \left[a^0(k)\; e^{i\,\omega(k)t - i\,kx} + a_0(k)\; e^{-i\,\omega(k)t + i\,kx}\right]. \tag{2.87}$$

This implies

$$P_{11} - i\,kP_{12} = \quad e^{-i\,k(x_1 - x_2)}, \tag{2.88}$$

$$P_{11} + i\,kP_{12} = \quad e^{i\,k(x_1 - x_2)}, \tag{2.89}$$

$$P_{21} - i\,kP_{22} = -i\,k\, e^{-i\,k(x_1 - x_2)}, \tag{2.90}$$

$$P_{21} + i\,kP_{22} = \quad i\,k\, e^{i\,k(x_1 - x_2)}, \tag{2.91}$$

---

can be already interpreted as a "complex lens construction".

and the propagation matrix for a particle with momentum k reads

$$P(k, x_1 - x_2) = \begin{pmatrix} \cos[k(x_1 - x_2)] & \frac{1}{k}\sin[k(x_1 - x_2)] \\ -k\sin[k(x_1 - x_2)] & \cos[k(x_1 - x_2)] \end{pmatrix}. \tag{2.92}$$

Due to $\det P = 1$, the absolute value of the wave function is conserved. This is a necessary requirement. Furthermore, P is equal to $\mathbb{1}$ for $x_1 = x_2$. This is important because in this case P should disappear in the product (2.84). It is straightforward to generalise this picture to $n + 1$ defects situated at $x_1 \geqslant x_2 \geqslant x_3 \geqslant \ldots \geqslant x_{n+1}$. The boundary condition yields

$$\begin{pmatrix} \varphi \\ \partial\varphi \end{pmatrix}_{x_1+0} = \int_{-\infty}^{+\infty} \frac{dk}{2\pi\sqrt{2\omega(k)}} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} P_1(k, x_1 - x_2) \cdot \ldots \cdot \tag{2.93}$$

$$\cdot \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} \varphi \\ \partial\varphi \end{pmatrix}\Big|_{x_{n+1}-0}. \tag{2.94}$$

As we give transmission and reflection for distinct k, they will read

$$R^{\pm}(k) = \frac{k^2B \pm i\,k(A - D) + C}{k^2B \mp i\,k(A + D) - C}, \quad T^{\pm}(k) = \frac{2\,i\,k}{k^2B + i\,k(A + D) - C}, \tag{2.95}$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} := \prod_{i=1}^{n} \left[\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} P(k, x_i - x_{i+1}) \right] \cdot \begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix}. \tag{2.96}$$

Hence, energy tensors and Noether currents will also change according to A, B, C, D that become k-dependent. However, we do not add any k dependence to T and R, since they have already been momentum-dependent before.

As the intention of this chapter was to illustrate RT techniques, we do not add any comments on higher-dimensional bosonic RT formalism. This extension is easily derived for the bosonic theory [63] and will be further clarified by considering the fermionic theory dealt with in the next chapter.

Pouvoir marcher, sans tromper l'oiseau, du cœur de l'arbre à l'extase du fruit.

(Gehen zu können vom Herzen des Baums bis zur Verzückung der Frucht, ohne dabei den Vogel zu hintergehen.)

René Char, A la santé du serpent

3

Fermionic δ Defects in the RT Picture

In this chapter the RT formalism for fermionic theory in two, three and four space-time dimensions will be developed. In principle, even considering higher dimensions, the formalism would be straightforward. But aside from the increase of technical difficulties, our major interest is to investigate theories that can be applied to spin interaction problems in solid state physics as well as to conformal field theories with boundaries and defects, as already pointed out in the introduction.

Here, firstly a brief discussion of possible interaction terms in the Lagrangean and the motivation of this choice will be given. Furthermore, we fix our conventions for this chapter. Subsequently, we state the explicit calculations and results of the RT formalism in different dimensions. In a further section will be shown how the matrix optics ansatz for fermions works.

$$H_1 = \lambda \vec{S} \cdot \psi^\dagger(t,\vec{x}) \frac{\vec{\sigma}}{2} \psi(t,\vec{x}) \quad \longrightarrow \quad H_2 = \lambda \vec{S} \cdot \psi^\dagger(t,\vec{x}) \frac{\vec{\gamma}}{2} \psi(t,\vec{x})$$

$$\searrow \qquad \swarrow$$

$$\begin{aligned} H_3 &= \lambda \sum_i \vec{S}_i \cdot \psi^\dagger(t,\vec{x}) \frac{\vec{\gamma}}{2} \psi(t,\vec{x}) \\ H_4 &= \sum_{i,j} \lambda_{ij} \vec{S}_i \cdot \psi^\dagger(t,\vec{x}) \frac{\vec{\gamma}_j}{2} \psi(t,\vec{x}) \\ H_5 &= \int dx_0 \, \lambda_{x_0} \vec{S}(x_0) \cdot \psi^\dagger(t,\vec{x}) \frac{\vec{\gamma}}{2} \psi(t,\vec{x}) \end{aligned}$$

Figure 3.1: Different possibilities of coupling spins to fermionic states

3.1 Interaction Terms and Conventions

Within the last decades, interactions between fermions and fixed spins have been a field of intense research at least in solid state physics, but also in the framework of investigations in integrable models [54; 69; 70]. While the interaction itself always couples the fixed spin to a scalar product of particle wave function, the models for more than one fixed interacting spin vary.

As shown schematically in figure 3.1, the simplest starting point is a two-dimensional single spin interaction with coupling constant λ. Herefrom, on the one hand, it is possible to increase space-time dimensions. Then the number of spin components increases as well while staying with a single spin and thus a single coupling constant. On the other hand, more spin impurities can be taken into account. Then there are different levels of considering coupling constants: the simplest way is of course to state a general unique constant λ. But also different coupling constants for every single spin are possible – which can be the indirect Ising interaction of spins for example. Moreover, the defect could be considered as a spin density, in case there are many defect spins that are strongly localised – dense – in a certain area. All these models are conceivable for special situations in solid state physics up to dimension four.

For a general RT formulation it is necessary to derive boundary conditions at the position of the defect (or at least to formulate the problem in terms of a δ-distribution-like impurity in order to derive such boundary conditions). The

	dimension of the defect	number of spin components: 1	2	3
$d = 2$	0	RT formalism		Kondo model via CFT one dim. Ising model
$d = 3$	0			two dim. Ising model
	1		RT formalism	
$d = 4$	0			Kondo model three dim. Ising model
	1			
	2			RT formalism

Figure 3.2: Different relations of space-time, defect, and spin dimensions depending on the theory. We should emphasise that the dimensionality of the spin is not as important as the defect dimension. The CFT approach to the Kondo effect is valid even for less spin components as well as the RT formalism will just change slightly for more spin components. Therefore, the theories in the same rows are comparable whereas those in the same columns are not.

reflection and transmission amplitudes are then nothing else but a reformulated boundary condition.

Up to now, for technical reasons, we only consider defects of co-dimension one that lead directly to such boundary conditions by integrating out the $\delta(x)$ distribution of the equations of motion, as will soon become clear. For higher co-dimension, we had to modify the Fourier decomposition of the wave functions and thus the interpretation of the reflection and transmission amplitudes. How to do this consistently is still an open question as we discuss in chapter 4.

Therefore, depending on the dimension, the RT description will start at H_1 in figure 3.1 but then turn to a density description H_5 (see figure 3.1) for higher dimensions as figure 3.2 suggests.

Moreover, we restrict ourselves to interactions with coupling constant $\lambda_{ij} := \lambda_i$ and involve it in the spin s_i,

$$S_i := \lambda_i s_i. \tag{3.1}$$

Therefore, $S_i \in \mathbb{R}$ is considered to be a continuous variable.

3.1.1 Lagrangean

Accordingly, we examine a Lagrangean of the form

$$\mathcal{L} = \overline{\Psi}\,(i\partial\!\!\!/ + i\,m)\,\Psi + \delta(x)\left[\,i\eta\overline{\Psi}\Psi + \vec{S}\,\overline{\Psi}\vec{\gamma}\Psi\right], \tag{3.2}$$

with $\partial\!\!\!/ = \gamma^\nu\partial_\nu$ and $\vec{\gamma} = (\gamma^1, \gamma^2, \ldots)$ and *mostly-plus metric.*

The interaction term consists of a direct product with impurity mass η and a spin product with fixed spin $\vec{S} = (S_1, \ldots)$. Contrary to the bosonic case, integrability does not give a "general" boundary condition like for ∂^2 operators [66]. Beside that our starting point is a theory already including an interaction term (whereas for the bosonic field the "pure" theory is done in [63] and thereafter in [64] the φ^4 term is treated). The term $\eta\,\delta(x)\,\overline{\Psi}\Psi$ was suggested by Delfino et al. [55], therein the RT algebra was derived in a slightly different way than in [62] which will guide us here. However, it seems natural to include this "impurity mass" term into the more general investigation of spin impurities. Moreover, it will turn out as necessary component in the general RT picture as we discuss in sections 3.2.6 and especially 3.4.5. On the other hand we should be aware that mass terms could break a solely left-handed theory.

We use here a mostly-plus metric. This implies for the Dirac equation in momentum space

$$(p^2 - m^2)\,\Psi = 0 \qquad \Leftrightarrow \qquad (p\!\!\!/ - i\,m)(p\!\!\!/ + i\,m)\,\Psi = 0. \tag{3.3}$$

For this reason the mass terms have imaginary prefactors in the Lagrangean (3.2).

3.1.2 Boundary Condition

There are several possibilities to introduce the spin matrices in arbitrary dimensions. For our purposes, we do not need any special requirements like Dirac or Majorana spinors at the moment. With the standard Pauli matrices

$$\begin{aligned} \sigma_0 &= \mathbb{1}_2, & \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \sigma_2 &= i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned} \tag{3.4}$$

we define according to [71],

$$\gamma_{2n-1} = \otimes_{k=1}^{n-1}\sigma_3 \otimes \sigma_1 \otimes_{l=1}^{o-n} \mathbb{1}_2, \qquad \Leftrightarrow \qquad d = 2o, 2o+1 \tag{3.5}$$

$$\gamma_{2n} \quad = \otimes_{k=1}^{n-1}\sigma_3 \otimes \sigma_2 \otimes_{l=1}^{o-n} \mathbb{1}_2, \qquad \Leftrightarrow \qquad d = 2o, 2o+1 \tag{3.6}$$

$$\gamma_{d=2o+1} = \otimes_{k=1}^{o}\sigma_3, \qquad \Leftrightarrow \qquad d = \quad 2o+1 \tag{3.7}$$

where $n \leqslant o$ and corresponding to the signature some matrices have additional factors $\pm\,i$. In our convention of mostly-plus metric, this implies $\gamma^0 = i\,\sigma_1 \otimes_{l=1}^{o-1} \mathbb{1}_2$. We named the spin matrices γ in order to distinguish them from the Pauli matrices. (Of course the convention of the γ matrices does not take effect neither on the reflection and transmission coefficients nor on the current behaviour in the further sections.)

We make a general statement about the wave function behaviour at the defect:

1. While for the non-defect case for physical reasons we would demand a continuous function everywhere, we now allow the wave function Ψ to jump at the origin $x = 0$, where the defect is located.

2. The physical condition, that holds in the defect case as well, is the continuity of the Lagrangean.

The second condition fixes the only possible multiplication constant for Ψ which is simply the shift with a constant phase,

$$\Psi \mapsto \alpha\Psi \quad \Rightarrow \quad \alpha = e^{i\,\theta},\ \theta \in \mathbb{R}. \tag{3.8}$$

This is a canonical requirement. The first condition tells us that the limits $\lim_{x\uparrow\downarrow 0}\Psi$ exist but are not necessarily equal ($x \downarrow 0$ is denoting the limit $x \to 0$ with $x > 0$, while $x \uparrow 0$ means the same limit but $x < 0$).

A generalised version of the Lagrangean (3.2) is

$$\mathcal{L} = \overline{\Psi}\,(i\not{\partial} + i\,m)\,\Psi + \delta(x)\overline{\Psi}\,U\,\Psi, \tag{3.9}$$

where U is acting on Ψ. The correct integration of the equation of motion over the interval $x \in [-\varepsilon, +\varepsilon]$ in the limit $\varepsilon \to 0$ leads to

$$i\,\gamma^x\,(\Psi_+ - \Psi_-)\Big|_{x=0} = -\frac{1}{2}\,U\,(\Psi_+ + \Psi_-)\Big|_{x=0} \tag{3.10}$$

$$\Psi_+\Big|_0 = \left(i\,\gamma^x + \frac{1}{2}\,U\right)^{-1}\left(i\,\gamma^x - \frac{1}{2}\,U\right)\Psi_-\Big|_0 \tag{3.11}$$

$$=: M_- \cdot \Psi_-\Big|_0, \tag{3.12}$$

where γ^x labels the γ matrix belonging to the direction perpendicular to the defect. For our special Lagrangean (3.2), the boundary matrix takes the form

$$M_- = \left[i\gamma^x + \frac{1}{2}(i\eta\mathbb{1} + \vec{S}\vec{\gamma}) \right]^{-1} \left[i\gamma^x - \frac{1}{2}(i\eta\mathbb{1} + \vec{S}\vec{\gamma}) \right]. \tag{3.13}$$

This allows us to derive the RT coefficients explicitly now.

3.1.3 Many-Particle Statistics

In order to give measurable values or conserved quantities we have to consider quantum mechanical many particle statistics. The Gibbs formulation for finite temperatures deals with

$$\mathcal{H} = H - \mu N, \tag{3.14}$$

where μ is the chemical potential, N the particle number operator, and H and N expressed in terms of the algebra elements via

$$H = \sum_\alpha \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \omega(k) a^\alpha(k) a_\alpha(k), \tag{3.15}$$

$$N = \sum_\alpha \int_{-\infty}^{+\infty} \frac{dk}{2\pi} a^\alpha(k) a_\alpha(k), \tag{3.16}$$

and $\alpha = (s, \pm)$ labels the different Fourier modes of the wave function as in chapter 2. Hence,

$$\mathcal{H} = \sum_\alpha \int_{-\infty}^{+\infty} \frac{dk}{2\pi} [\omega(k) - \mu] a^\alpha(k) a_\alpha(k). \tag{3.17}$$

The partition function reads

$$\Xi = \mathrm{tr}\, e^{-\beta\mathcal{H}}, \qquad \text{with} \qquad \beta = \frac{1}{k_B T}. \tag{3.18}$$

This implies for the expectation value

$$\langle F(a^{\alpha_1}, a_{\alpha_2}) \rangle = \frac{\mathrm{tr}\, e^{-\beta\mathcal{H}}\, F(a^{\alpha_1}, a_{\alpha_2})}{\mathrm{tr}\, e^{-\beta\mathcal{H}}}, \tag{3.19}$$

Using the identity (2.59),

$$e^{-\beta\mathcal{H}}\, a^\alpha(p_1) = e^{-\beta[\omega(p_1)-\mu]}\, a^\alpha(p_1)\, e^{-\beta\mathcal{H}}, \tag{3.20}$$

once more we derive similar to the bosonic case (2.61) and (2.62),

$$\langle a^{\alpha_1}(k_1)a_{\alpha_2}(k_2)\rangle = \frac{e^{-\beta[\omega(k_1)-\mu]}}{1+e^{-\beta[\omega(k_1)-\mu]}}\,\{a_{\alpha_2}(k_2),\, a^{\alpha_1}(k_1)\}, \tag{3.21}$$

$$\langle a_{\alpha_1}(k_1)a^{\alpha_2}(k_2)\rangle = \frac{1}{1+e^{-\beta[\omega(k_1)-\mu]}}\,\{a_{\alpha_1}(k_1),\, a^{\alpha_2}(k_2)\}. \tag{3.22}$$

Moreover, we should keep in mind the standard normal ordering for fermions:

$$\left\langle :a_j(k_1)a^l(k_2):\right\rangle = -\left\langle a^l(k_2)a_j(k_1)\right\rangle . \tag{3.23}$$

The knowledge of the creator-annihilator Green functions in terms of reflection and transmission will provide us with the possibility to calculate any other two-point function, in particular the energy density and the conserved Noether currents.

3.2 δ Defects in Two-Dimensional Fermionic Theory

In this section we derive the properties of a two-dimensional theory with fermionic δ defect via the RT formalism explicitly. However, we keep in mind the generalisation to higher dimensions. Therefore, we sometimes state results which could be simplified for purposes of two dimensions, but shorten in the given form the calculations for higher dimensions.

3.2.1 Boundary Condition

Considering the Lagrangean (3.2) in two dimensions with γ matrices according to (3.5)–(3.7),

$$\gamma^0 = i\,\sigma_1, \qquad \gamma^1 = \sigma_2, \tag{3.24}$$

we derive an equation of motion for Ψ that we integrate over a small interval $[-\varepsilon, +\varepsilon]$ assuming the defect at position $x = x_0$,

$$\begin{aligned}\int_{-\varepsilon}^{+\varepsilon} dx \left[i\partial\!\!\!/ + i\,m\mathbb{1} + \delta(x)\, i\eta\mathbb{1} + \delta(x)\, i \begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix} \right] \Psi(x) = \\ = \int_{-\varepsilon}^{+\varepsilon} dx \begin{pmatrix} i\,m + i\eta\,\delta(x) & -\partial_0 - \partial_1 + i\,S\delta(x) \\ -\partial_0 + \partial_1 - i\,S\delta(x) & i\,m + i\eta\,\delta(x) \end{pmatrix} \begin{pmatrix} \varphi(x) \\ \chi(x) \end{pmatrix} \xrightarrow[\varepsilon\to 0]{} 0.\end{aligned} \tag{3.25}$$

Of course, $\Psi = (\varphi, \chi)$ is the Weyl spinor notation and the spin $S = S_1$. We denote the limit from $x < 0$ with "−" and from $x > 0$ with "+". Both equations vanish independently. The integration over $\partial_0 \ldots$ drops out because of continuity along the line $x = 0$:

$$\frac{i}{2}\eta\,(\varphi_+ + \varphi_-) + \frac{i}{2}S(\chi_+ + \chi_-) = \chi_+ - \chi_-, \tag{3.26}$$

$$\frac{i}{2}\eta\,(\chi_+ + \chi_-) - \frac{i}{2}S(\varphi_+ + \varphi_-) = \varphi_- - \varphi_+. \tag{3.27}$$

For $S = 0$ this is in one-to-one correspondence to [55] (up to a sign change due to convention (3.24)). Equations (3.26) and (3.27) imply

$$\begin{pmatrix}\varphi_-\\ \chi_-\end{pmatrix} = \frac{1}{(2+i\,S)^2 - \eta^2}\begin{pmatrix} 4 + S^2 + \eta^2 & 4\,i\eta \\ -4\,i\eta & 4 + S^2 + \eta^2\end{pmatrix}\begin{pmatrix}\varphi_+\\ \chi_+\end{pmatrix}. \tag{3.28}$$

This is the boundary condition in two dimensions.

3.2.2 Quantisation with RT Algebra

The generator algebra (2.4)–(2.6) as defined in [62] reads

$$\{a_{\alpha_1}(k_1),\, a_{\alpha_2}(k_2)\} = 2\delta^{\alpha_1}_{\alpha_2}\delta(k_1 - k_2)\, a_{\alpha_1}, \tag{3.29}$$

$$\{a^{\alpha_1}(k_1),\, a^{\alpha_2}(k_2)\} = 2\delta^{\alpha_2}_{\alpha_1}\delta(k_1 - k_2)\, a^{\alpha_1}, \tag{3.30}$$

$$\{a_{\alpha_1}(k_1),\, a^{\alpha_2}(k_2)\} = 2\pi\,\delta(k_1 - k_2)\,\left[\delta^{\alpha_2}_{\alpha_1} + T^{\alpha_2}_{\alpha_1}(k_1)\right] + 2\pi\,\delta(k_1 + k_2)\, R^{\alpha_2}_{\alpha_1}(k_2). \tag{3.31}$$

Here $\alpha_s = (s, \pm 1)$ and s is the index of the spin solution which is suppressed in the further calculations for two dimensions since there is only one solution of the Dirac equation in momentum space. This means that

$$(\not{p} - i\,m)\, u_s(k) \equiv 0, \qquad (\not{p} + i\,m)\, v_s(k) \equiv 0 \tag{3.32}$$

have unique solutions $u(k)$ and $v(k)$. In this way, with the abbreviations

$$T^i := T^{s,i}_{s,-i} \quad \text{and} \quad R^i := R^{s,i}_{s,i}, \tag{3.33}$$

equation (3.31) reads

$$\left\{a_i(k_1),\, a^j(k_2)\right\} = 2\pi\,\delta(k_1 - k_2)\,\left[\delta^j_i + \delta^{-i}_j T^j(k_1)\right] + 2\pi\,\delta(k_1 + k_2)\,\delta^i_j R^j(k_2). \tag{3.34}$$

The additional exchange relations with reflection and transmission coefficient functions (2.10)–(2.13) are then

$$a^j(k)\overline{T}^j(k) + a^{-j}(-k)\overline{R}^{-j}(-k) = a^{-j}(k), \tag{3.35}$$

$$a_j(k)T^j(k) + a_{-j}(-k)R^{-j}(-k) = a_{-j}(k), \tag{3.36}$$

or, rewritten similar to Delfino et al. [55],

$$\begin{pmatrix} a^-(k) \\ a^+(-k) \end{pmatrix} = \begin{pmatrix} \overline{R}^-(-k) & \overline{T}^+(k) \\ \overline{T}^-(-k) & \overline{R}^+(k) \end{pmatrix} \cdot \begin{pmatrix} a^-(-k) \\ a^+(k) \end{pmatrix}, \tag{3.37}$$

$$\begin{pmatrix} a_-(k) \\ a_+(-k) \end{pmatrix} = \begin{pmatrix} R^-(-k) & T^+(k) \\ T^-(-k) & R^+(k) \end{pmatrix} \cdot \begin{pmatrix} a_-(-k) \\ a_+(k) \end{pmatrix}. \tag{3.38}$$

We decompose $\Psi(x) = \Psi_+ + \Psi_-$,

$$\Psi_\pm := \theta(\pm x) \int_{-\infty}^{+\infty} \frac{dk}{2\pi\sqrt{2\omega(k)}} \Big[u(k)\, a^\pm(k)\, e^{i\,\omega(k)t - i\,kx} + \\ + v(k)\, a_\pm(k)\, e^{-i\,\omega(k)t + i\,kx} \Big]. \tag{3.39}$$

Furthermore, we use equivalently $p_0 = \omega(k) = \sqrt{k^2 + p_i^2 + m^2}$, where p_i are the momenta parallel to the defect in higher dimensions.

In order to show the consistency of an interpretation as reflection and transmission amplitudes, we would like to rewrite $\Psi(x)$ in terms of R and T in analogy to the bosonic case in chapter 2. Therefore, we apply (3.37) and (3.38) and rearrange terms. In particular, we use extensively the possibility of inverting the integration variable $k \mapsto -k$,

$$\int_a^b F(k)\, dk = \int_b^a F(-k')\, dk'. \tag{3.40}$$

Hence,

$$\Psi(x) = \sum_{j=\pm} \theta(jx) \int_{-\infty}^{+\infty} \frac{dk}{2\pi\sqrt{2\omega(k)}} \Big[u(k)\, a^j(k)\, e^{i\,\omega(k)t - i\,kx} + v(k)\, a_j(k)\, e^{-i\,\omega(k)t + i\,kx} \Big] \tag{3.41}$$

$$= \theta(-x)\Bigg\{ \int_0^{+\infty} \frac{dk}{2\pi\sqrt{2\omega(k)}} \Big[u(k)\, a^-(k)\, e^{i\,\omega(k)t - i\,kx} + v(k)\, a_-(k)\, e^{-i\,\omega(k)t + i\,kx}\Big] +$$

$$+ \int_{-\infty}^0 \frac{dk}{2\pi\sqrt{2\omega(k)}} \Big[u(k)\, \Big(a^-(-k)\overline{R}^-(-k) + a^+(k)\overline{T}^+(k)\Big)\, e^{i\,\omega(k)t - i\,kx} +$$

$$+ v(k)\, \big(a_-(-k)R^-(-k) + a_+(k)T^+(k)\big)\, e^{-i\,\omega(k)t + i\,kx}\Big]\Bigg\}$$

$$+ \theta(+x)\Bigg\{ \int_0^{+\infty} \frac{dk}{2\pi\sqrt{2\omega(k)}} \Big[u(k)\, \Big(a^-(k)\overline{T}^-(k) + a^+(-k)\overline{R}^+(-k)\Big)\, e^{i\,\omega(k)t - i\,kx} +$$

$$+ v(k)\, \big(a_-(k)T^-(k) + a_+(-k)R^+(-k)\big)\, e^{-i\,\omega(k)t + i\,kx}\Big] +$$

$$+ \int_{-\infty}^0 \frac{dk}{2\pi\sqrt{2\omega(k)}}\, u(k)\, a^+(k)\, e^{i\,\omega(k)t - i\,kx} + v(k)\, a_+(k)\, e^{-i\,\omega(k)t + i\,kx}\Bigg\}$$

$$= \sum_{j=\pm} \int_{-\infty}^{+\infty} \frac{dk}{2\pi\sqrt{2\omega(k)}} \cdot \tag{3.42}$$

$$\cdot\Bigg\{ a^j(k)\theta(-jk)\Big\{\theta(-jx)\, u(k)\, \overline{T}^j(k)\; e^{-i\,kx} +$$

$$+ \theta(jx)\big[u(k)\, e^{-i\,kx} + u(-k)\, \overline{R}^j(k) e^{i\,kx}\big]\Big\}\, e^{i\,\omega(k)t} +$$

$$+ a_j(k)\theta(-jk)\Big\{\theta(-jx)\, v(k)\, T^j(k)\; e^{i\,kx} +$$

$$+ \theta(jx)\big[v(k)\, e^{i\,kx} + v(-k)\, R^j(k) e^{-i\,kx}\big]\Big\}\, e^{-i\,\omega(k)t}\Bigg\}$$

$$= \psi_+(x) + \psi_-(x). \tag{3.43}$$

Here we implied the definitions

$$\psi_\pm(x) := \int_{-\infty}^{+\infty} \frac{dk}{\sqrt{8\pi^2\omega(k)}} \Big[a^\pm(k)\overline{\varphi}^\pm_{x,s}(k)\, e^{i\,\omega t} + a_\pm(k)\varphi^\pm_{x,s}(k)\, e^{-i\,\omega t}\Big], \tag{3.44}$$

$$\varphi^\pm_{x,s}(k) := \theta(\mp k)\Big\{\theta(\mp x)\, v(k)\, T^\pm(k)\; e^{i\,kx} + \theta(\pm x)\, \Big[v(k)\, e^{i\,kx} + v(-k)\, R^\pm(k) e^{-i\,kx}\Big]\Big\}, \tag{3.45}$$

$$\overline{\varphi}^\pm_{x,s}(k) := \theta(\mp k)\Big\{\theta(\mp x)\, u(k)\, \overline{T}^\pm(k)\; e^{-i\,kx} + \theta(\pm x)\, \Big[u(k)\, e^{-i\,kx} + u(-k)\, \overline{R}^\pm(k) e^{i\,kx}\Big]\Big\}. \tag{3.46}$$

Obviously, the amplitudes $T^\pm(k), R^\pm(k)$, abstractly introduced by equations (3.37) and (3.38) (and originally by (2.20)–(2.22)), match via the canonical

Fourier decomposition (3.39) exactly with the naïve formulation (3.45) of reflection and transmission coefficients. We thus justify the interpretation of the exchange coefficients R, T as reflection and transmission amplitudes of fermionic wave functions with spinors $u(k)$ or $v(k)$ similar to the bosonic case.

3.2.3 RT Coefficients

We now derive these coefficients explicitly. The equation of motion yields a boundary condition of the form (3.12),

$$\Psi_+\big|_0 \quad = M_- \cdot \Psi_-\big|_0. \tag{3.47}$$

Due to (3.39),

$$\Psi_\pm = \theta(\pm x) \int_{-\infty}^{+\infty} \frac{dk}{2\pi\sqrt{2\omega(k)}} \Big[u_s(k)\, a^{s\pm}(k)\, e^{i\,\omega(k)t - i\,k x} + \\ + v_s(k)\, a^s_\pm(k)\, e^{-i\,\omega(k)t + i\,k x} \Big], \tag{3.48}$$

where $a^s_\pm(k) := a_{s\pm}(k)$ is just a notation to implement Einstein's sum convention, we have to examine the limit $x \to 0$ carefully and include the dependence on the spin solution index s during the next steps in order to apply these to higher dimensions as well. We find

$$\Psi_\pm\big|_0 = \lim_{\varepsilon \stackrel{>}{\to} 0} \int_{-\infty}^{+\infty} \frac{dk}{2\pi\sqrt{2\omega(k)}} \Big(u_s(k)\, a^{s\pm}(k)\, e^{i\,\omega(k)t - i\,k\varepsilon} + \\ + v_s(k)\, a^s_\pm(k)\, e^{-i\,\omega(k)t + i\,k\varepsilon} \Big) \tag{3.49}$$

$$= \int_0^{+\infty} \frac{dk}{\sqrt{8\pi^2\omega(k)}} \Big(\big[u_s(k)\, a^{s\pm}(k) + u_s(-k)\, a^{s\pm}(-k) \big] e^{i\,\omega(k)t} + \\ + \big[v_s(k)\, a^s_\pm(k) + v_s(-k)\, a^s_\pm(-k) \big] e^{-i\,\omega(k)t} \Big). \tag{3.50}$$

The boundary condition holds for every single mode $e^{\pm i\,\omega(k)t}$ on its own, since $k \in [0, \infty]$ and $\omega(k)$ is of order k^2. Furthermore, there is no decomposition of $a^\pm$ in terms of $a_\pm$ and vice versa. Hence, equation (3.47) splits into an annihilator and a creator equation,

$$v_s(k)\, a^s_+(k) + v_s(-k)\, a^s_+(-k) = M_- \, \left[v_s(-k)\, a^s_-(-k) + v_s(k)\, a^s_-(k) \right], \tag{3.51}$$

$$u_s(k)\, a^{s+}(k) + u_s(-k)\, a^{s+}(-k) = M_- \, \left[u_s(-k)\, a^{s-}(-k) + u_s(k)\, a^{s-}(k) \right]. \tag{3.52}$$

The reflection and transmission coefficients in two dimensions follow directly from the algebra formulation (3.37), (3.38),

$$\begin{aligned} R^-(-k) &= \frac{[M_- v(-k)] \cdot A \cdot v(-k)}{v(-k) \cdot A \cdot [M_- v(k)]}, & R^+(k) &= \frac{[M_- v(k)] \cdot A \cdot v(k)}{v(-k) \cdot A \cdot [M_- v(k)]}, \\ T^-(-k) &= \frac{[M_- v(-k)] \cdot A \cdot [M_- v(k)]}{v(-k) \cdot A \cdot [M_- v(k)]}, & T^+(k) &= \frac{v(-k) \cdot A \cdot v(k)}{v(-k) \cdot A \cdot [M_- v(k)]}, \end{aligned} \tag{3.53}$$

$$\begin{aligned} \overline{R}^-(-k) &= \frac{[M_- u(-k)] \cdot A \cdot u(-k)}{u(-k) \cdot A \cdot [M_- u(k)]}, & \overline{R}^+(k) &= \frac{[M_- u(k)] \cdot A \cdot u(k)}{u(-k) \cdot A \cdot [M_- u(k)]}, \\ \overline{T}^-(-k) &= \frac{[M_- u(-k)] \cdot A \cdot [M_- u(k)]}{u(-k) \cdot A \cdot [M_- u(k)]}, & \overline{T}^+(k) &= \frac{u(-k) \cdot A \cdot u(k)}{u(-k) \cdot A \cdot [M_- u(k)]}. \end{aligned} \tag{3.54}$$

In order to shorten the expressions, we used the matrix $A := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This is a technical tool which has no deeper meaning on its own. There is a more general formulation in terms of decomposed eigenstates of M_-. It is applicable in higher dimensions than three as well, and we derive it in appendix A.

The "mostly plus" convention (3.24), we use here, leads with (3.32) to[1]

$$u(\tau) = \frac{1}{\sqrt{e^{\tau} + e^{-\tau}}} \begin{pmatrix} e^{-\frac{\tau}{2}} \\ e^{\frac{\tau}{2}} \end{pmatrix} = \sqrt{\frac{k + p_0}{2\, p_0}} \begin{pmatrix} \frac{p_0 - k}{m} \\ 1 \end{pmatrix} = u(k), \tag{3.56}$$

$$v(\tau) = \frac{1}{\sqrt{e^{\tau} + e^{-\tau}}} \begin{pmatrix} -e^{-\frac{\tau}{2}} \\ e^{\frac{\tau}{2}} \end{pmatrix} = i \sqrt{\frac{k + p_0}{2\, p_0}} \begin{pmatrix} 1 \\ -\frac{p_0 - k}{m} \end{pmatrix} = v(k). \tag{3.57}$$

As we motivate in appendix A, the phase of $u(k)$ and $v(k)$ does not affect the coefficients R and T.

Concretely, we have with (3.53),

$$R^{\pm}(k) = -\frac{4\, i\, \eta \cosh(\tau)}{4\, i\, \eta \pm (g + 4) \sinh(\tau)} = \mp \frac{4\, i\, \eta\, \omega(k)}{4\, i\, m\eta \pm (g + 4) k}, \tag{3.58}$$

$$T^{\pm}(k) = \frac{(4 - g \mp 4\, i\, S) \sinh(\tau)}{(g + 4) \sinh(\tau) \pm 4\, i\, \eta} = -\frac{(g - 4 \pm 4\, i\, S)\, k}{(g + 4) k \pm 4\, i\, m\eta}, \tag{3.59}$$

[1] To facilitate the comparison with the work of Delfino et al. [55] where $\gamma^0 = \sigma_2$, $\gamma^1 = -i\,\sigma_1$, up to an $SU(2)$ transformation in terms of the rapidity τ ($\omega(k) := m \cosh \tau$, $k =: m \sinh \tau$), we have

$$u(\tau) = \frac{1}{\sqrt{e^{\tau} + e^{-\tau}}} \begin{pmatrix} -i\, e^{\frac{\tau}{2}} \\ e^{-\frac{\tau}{2}} \end{pmatrix}, \qquad v(\tau) = \frac{1}{\sqrt{e^{\tau} + e^{-\tau}}} \begin{pmatrix} i\, e^{\frac{\tau}{2}} \\ e^{-\frac{\tau}{2}} \end{pmatrix}, \tag{3.55}$$

where $[\not{p} - m]\, u \equiv 0$ and $[\not{p} + m]\, v \equiv 0$.

with $k = \frac{m}{2}(e^{\tau} - e^{-\tau})$ as defined above, the abbreviation $g = \eta^2 + s^2$ and the boundary matrix according to equation (3.28),

$$M_- = \frac{1}{4 - g - 4\,i\,S} \begin{pmatrix} g+4 & -4\,i\,\eta \\ 4\,i\,\eta & g+4 \end{pmatrix}. \tag{3.60}$$

This allows us now to evaluate expectation values in Gibbs statistics.

3.2.4 Gibbs States and Expectation Values

According to the creator and annihilator expectation values (3.21) and (3.22), the two-point integrals of the wave function $\Psi(x)$ read

$$\begin{aligned}
&\left\langle \overline{\Psi}(x_1, t_1)\,\Psi(x_2, t_2) \right\rangle = \\
&= \Big\langle \iint_{-\infty}^{+\infty} \frac{dk_1\, dk_2}{8\pi^2\sqrt{\omega(k_1)\omega(k_2)}} \cdot \\
&\quad \cdot \left[a^{s_1 \pm}(k_1) \overline{v}_{s_1}(k_1)\, e^{i\,\omega(k_1)t_1 - i\,k_1 x_1} + a^{s_1}_{\pm}(k_1) \overline{u}_{s_1}(k_1)\, e^{-i\,\omega(k_1)t_1 + i\,k_1 x_1} \right] \cdot \\
&\quad \cdot \left[a^{s_2 \pm}(k_2) u_{s_2}(k_2)\, e^{i\,\omega(k_2)t_2 - i\,k_2 x_2} + a^{s_2}_{\pm}(k_2) v_{s_2}(k_2)\, e^{-i\,\omega(k_2)t_2 + i\,k_2 x_2} \right] \Big\rangle \\
&= \sum_{j,l=\pm} \iint_{-\infty}^{+\infty} \frac{dk_1\, dk_2}{8\pi^2\sqrt{\omega(k_1)\omega(k_2)}} \theta(jx_1)\theta(lx_2) \cdot \\
&\quad \cdot \Big[\left\langle a^{s_1 j}(k_1) a^{s_2}_{l}(k_2) \right\rangle \overline{v}_{s_1}(k_1) \cdot v_{s_2}(k_2)\, e^{-i\,k_1 x_1 + i\,k_2 x_2}\, e^{i\,\omega(k_1)t_1 - i\,\omega(k_2)t_2} + \\
&\qquad + \left\langle a^{s_1}_{j}(k_1) a^{s_2 l}(k_2) \right\rangle \overline{u}_{s_1}(k_1) \cdot u_{s_2}(k_2)\, e^{i\,k_1 x_1 - i\,k_2 x_2}\, e^{-i\,\omega(k_1)t_1 + i\,\omega(k_2)t_2} \Big] \\
&= \sum_{j,l=\pm} \iint_{-\infty}^{+\infty} \frac{dk_1\, dk_2}{8\pi^2\sqrt{\omega(k_1)\omega(k_2)}} \frac{\theta(jx_1)\theta(lx_2)}{1 + e^{-\beta[\omega(k_1)-\mu]}} \cdot \\
&\quad \cdot \Big[e^{-\beta[\omega(k_1)-\mu]} \left\{ a^{s_2}_{l}(k_2),\, a^{s_1 j}(k_1) \right\} \cdot \\
&\qquad\qquad \cdot \overline{v}_{s_1}(k_1) \cdot v_{s_2}(k_2)\, e^{-i\,k_1 x_1 + i\,k_2 x_2}\, e^{i\,\omega(k_1)t_1 - i\,\omega(k_2)t_2} + \\
&\quad + \left\{ a^{s_1}_{j}(k_1),\, a^{s_2 l}(k_2) \right\} \overline{u}_{s_1}(k_1) \cdot u_{s_2}(k_2)\, e^{i\,k_1 x_1 - i\,k_2 x_2}\, e^{-i\,\omega(k_1)t_1 + i\,\omega(k_2)t_2} \Big].
\end{aligned} \tag{3.61}$$

We use the relation

$$\begin{aligned}
\sum_{j,l=\pm} \theta(jx_1)\theta(lx_2) \left\{ a_l(k_2),\, a^j(k_1) \right\} = \\
&= 2\pi\delta(k_2 - k_1)\, [\theta(x_1)\theta(x_2) + \theta(-x_1)\theta(-x_2)] + \\
&\quad + 2\pi\delta(k_2 - k_1) \left[\theta(x_1)\theta(-x_2)T^+(k_2) + \theta(-x_1)\theta(x_2)T^-(k_2) \right] + \\
&\quad + 2\pi\delta(k_2 + k_1) \left[\theta(x_1)\theta(x_2)R^+(k_1) + \theta(-x_1)\theta(-x_2)R^-(k_1) \right],
\end{aligned} \tag{3.62}$$

and drop the indices s_1, s_2 for the two-dimensional case,

$$
\begin{aligned}
&\left\langle \overline{\Psi}(x_1,t_1)\,\Psi(x_2,t_2)\right\rangle = \qquad (3.63)\\
&\quad = \int_{-\infty}^{+\infty} \frac{dk}{4\pi\omega(k)} \frac{1}{1+e^{-\beta[\omega(k)-\mu]}} \cdot \Bigg\{ e^{-\beta[\omega(k)-\mu]} \\
&\qquad \Big\{ \big[\theta(x_1)\theta(x_2)+\theta(-x_1)\theta(-x_2)\big]\overline{v}(k)\cdot v(k)\, e^{-ik(x_1-x_2)+i\omega(k)(t_1-t_2)} \\
&\qquad + \big[\theta(x_1)\theta(-x_2)T^+(k)+\theta(-x_1)\theta(x_2)T^-(k)\big]\,\overline{v}(k)\cdot v(k)\, e^{-ik(x_1-x_2)+i\omega(k)(t_1-t_2)} \\
&\qquad + \big[\theta(x_1)\theta(x_2)R^+(k)+\theta(-x_1)\theta(-x_2)R^-(k)\big]\,\overline{v}(k)\cdot v(-k)\, e^{-ik(x_1+x_2)+i\omega(k)(t_1-t_2)} \Big\} \\
&\qquad + \big[\theta(x_1)\theta(x_2)+\theta(-x_1)\theta(-x_2)\big]\overline{u}(k)\cdot u(k)\, e^{ik(x_1-x_2)-i\omega(k)(t_1-t_2)} \\
&\qquad + \big[\theta(x_1)\theta(-x_2)T^-(k)+\theta(-x_1)\theta(x_2)T^+(k)\big]\,\overline{u}(k)\cdot u(k)\, e^{ik(x_1-x_2)-i\omega(k)(t_1-t_2)} \\
&\qquad + \big[\theta(x_1)\theta(x_2)R^+(-k)+\theta(-x_1)\theta(-x_2)R^-(-k)\big]\,\overline{u}(k)\cdot u(-k)\, e^{ik(x_1+x_2)-i\omega(k)(t_1-t_2)} \Bigg\}.
\end{aligned}
$$

Dealing with fermionic operators, the normal ordering will change the expectation value according to (3.23),

$$\left\langle : a_j(k_1) a^l(k_2) : \right\rangle = -\left\langle a^l(k_2) a_j(k_1) \right\rangle . \qquad (3.64)$$

This implies for the two-point expectation values

$$
\begin{aligned}
&\left\langle : \overline{\Psi}(x_1,t_1)\,\Psi(x_2,t_2) : \right\rangle = \qquad (3.65)\\
&\quad = \sum \int_{-\infty}^{+\infty} \frac{dk}{4\pi\omega(k)} \frac{e^{-\beta[\omega(k)-\mu]}}{1+e^{-\beta[\omega(k)-\mu]}} \\
&\qquad \cdot \Big\{ [\theta(x_1)\theta(x_2)+\theta(-x_1)\theta(-x_2)]\,[\overline{v}(k)\cdot v(k)-\overline{u}(k)\cdot u(k)]\, e^{-ik(x_1-x_2)+i\omega(k)(t_1-t_2)} \\
&\qquad + [\theta(x_1)\theta(-x_2)T^+(k)+\theta(-x_1)\theta(x_2)T^-(k)]\,\overline{v}(k)\cdot v(k)\, e^{-ik(x_1-x_2)+i\omega(k)(t_1-t_2)} \\
&\qquad - [\theta(x_1)\theta(-x_2)T^-(k)+\theta(-x_1)\theta(x_2)T^+(k)]\,\overline{u}(k)\cdot u(k)\, e^{ik(x_1-x_2)-i\omega(k)(t_1-t_2)} \\
&\qquad + [\theta(x_1)\theta(x_2)R^+(k)+\theta(-x_1)\theta(-x_2)R^-(k)]\,[\overline{v}(k)\cdot v(-k)-\overline{u}(-k)\cdot u(k)]\cdot \\
&\qquad\qquad \cdot e^{-ik(x_1+x_2)+i\omega(k)(t_1-t_2)} \Big\}.
\end{aligned}
$$

3.2.5 Conserved Quantities

Energy density According to the mostly plus signature convention the stress tensor is given by

$$T^{\mu\nu} = -\frac{i}{2} : \overline{\Psi}\gamma^\mu \partial^\nu \Psi - \partial^\nu \overline{\Psi}\gamma^\mu \Psi : . \qquad (3.66)$$

We are interested in the energy density component T^{00}. The standard expression for the current operator (up to some coefficient) reads

$$J^{\mu}_{s}(x) \propto :\overline{\Psi}(x)\,\gamma^{\mu}\,\Psi(x):\,. \tag{3.67}$$

Since the γ matrices act only on $\nu(k)$ and $u(k)$, whereas the derivatives ∂_0 only affect the mode $e^{i\,\omega(k)t}$, we can directly read off the expectation values of $\langle\overline{\Psi}_1 \mathcal{O}\partial^0\Psi_2\rangle$ from (3.65) via

$$\overline{\nu}(k)\cdot\nu(k) \mapsto \overline{\nu}(k)\mathcal{O}\nu(k) \qquad \text{(and for } u(k) \text{ as well)}, \tag{3.68}$$

$$\partial_0\, e^{\pm i\,\omega(k)t_2} \mapsto \pm i\,\omega(k)\, e^{\pm i\,\omega(k)t_2}\,. \tag{3.69}$$

Here, $\mathcal{O}$ is an arbitrary product of γ matrices.

This implies for the energy density

$$\langle T^{00}(x)\rangle = \frac{i}{2}\,\langle:\overline{\Psi}(x)\,\gamma^0\partial^0\Psi(x) - \partial^0\overline{\Psi}\,\gamma^0\Psi:\rangle\Big|_{t_1=t_2} \tag{3.70}$$

$$= \int_{-\infty}^{+\infty}\frac{dk}{4\pi}\,\frac{e^{-\beta[\omega(k)-\mu]}}{1+e^{-\beta[\omega(k)-\mu]}}\cdot\Big\{\overline{\nu}(k)\gamma^0\nu(k)+\overline{u}(k)\gamma^0 u(k)+ \tag{3.71}$$

$$+\,e^{-2ikx}\,[\theta(x)R^+(k)+\theta(-x)R^-(k)]\,\big[\overline{\nu}(k)\gamma^0\nu(-k)+\overline{u}(-k)\gamma^0 u(k)\big]\Big\}$$

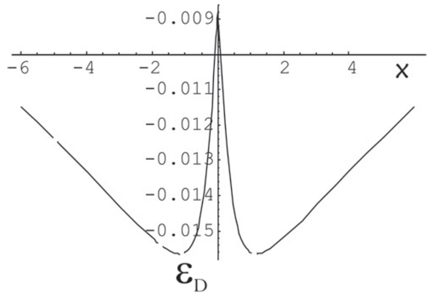
$$= -\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\,\frac{1}{1+e^{\beta[\omega(k)-\mu]}}\cdot \tag{3.72}$$

$$\cdot\Big\{1+\frac{m}{\omega(k)}\,e^{-2ikx}\,[\theta(x)R^+(k)+\theta(-x)R^-(k)]\Big\},$$

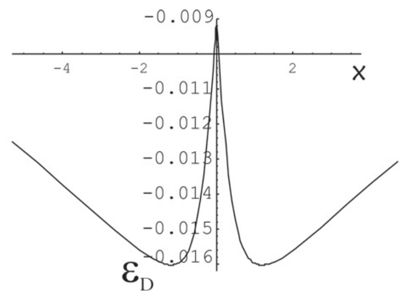
using the explicit forms (3.57) of u, ν in the last step. The result looks similar to the one of the bosonic theory (2.66)–(2.68). Due to Fermi and Bose statistics, the sign of the denominator varies as well as the prefactor of the reflection amplitudes, which is also affected by the Fourier decomposition. In both theories the transmission coefficients do not enter explicitly, since we consider a two-point function at a single point, while the transmission coefficients mediate the interaction between two points at different sides of the defect. Additionally, the reflection amplitudes occur always as sum,

$$e^{-2ikx}\,[\theta(+x)R^+(k)+\theta(-x)R^-(k)]\,,$$

which reflects the (anti-)symmetric properties of this RT picture: a parity transformation; i.e., sign change of x as well as of k should not change the theory. In other words, calculating two-point functions at positions $x_1 = x_2 =$



(a) $\mathcal{E}_D(x)$ exact



(b) $\mathcal{E}_D(x)_{m\approx 0}$ where $\omega(k) \approx |k|$

Figure 3.3: Defect energy density $\mathcal{E}_{D,\beta}(x)$ in two space-time dimensions with chemical potential $\mu = 0$, for mass parameter $m = 0.1$, defect mass $\eta = 10$, spin $S = 0$, and finite inverse temperature $\beta = 1$ plotted as function of the distance x: (a) numerical evaluation of the full density integral (3.74), (b) numerical evaluation of integral (3.75) where $m \approx 0$. Note that only the defect energy part is plotted here, since the Stefan-Boltzmann energy $\mathcal{E}_0$ (3.77) is not affected by the approximation.

x will never correlate a wave function with its transmitted one, but with the reflected one only.

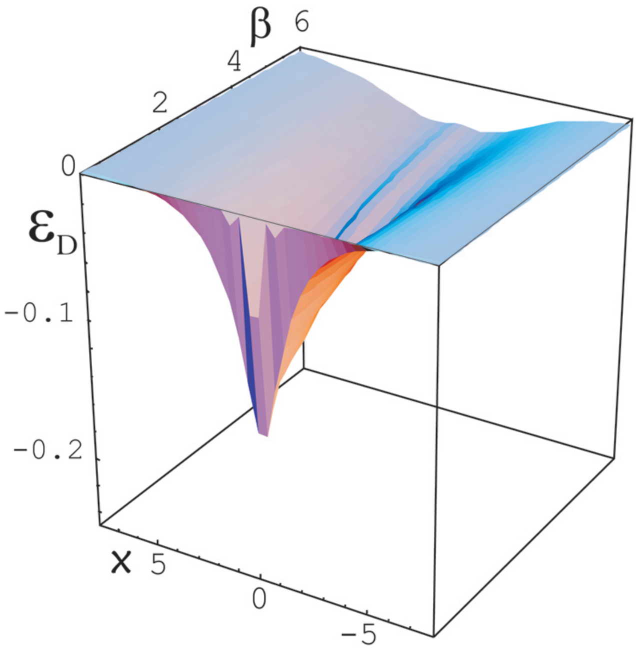
Considering the reflection amplitudes (3.58) for the energy density, this yields

$$\left\langle T^{00}(x)\right\rangle = \int_{-\infty}^{+\infty} \frac{dk}{2\pi(1+e^{\beta[\omega(k)-\mu]})} \left\{ 1 - (4\,i\,m\eta\, e^{-2ikx}) \cdot \right. \tag{3.73}$$

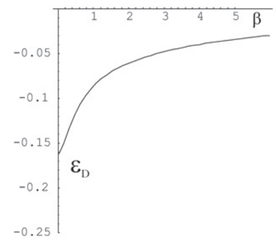
$$\left. \cdot \left[\frac{\theta(x)}{4\,i\,m\eta + (g+4)gk} + \frac{\theta(-x)}{4\,i\,m\eta - (g+4)gk} \right] \right\},$$

$$\left\langle T^{00}(\pm|x|)\right\rangle = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{1}{1+e^{\beta[\omega(k)-\mu]}} \left\{ 1 - \frac{4m\eta\, e^{\mp 2ik|x|}}{4m\eta \mp i(g+4)k} \right\}. \tag{3.74}$$

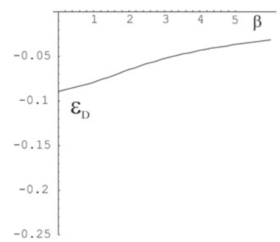
We implement the simplifications $\omega(k) \approx |k|$, $\mu \approx 0$, the abbreviation $\xi = \frac{4m\eta}{g+4}$ and integration of the first term in the sum. Note that $\omega(k) \approx |k|$ means $m \approx 0$ which seems antithetic to $\xi \neq 0$. Therefore, we have to be vigilant with this approximation. We assume m small enough but non-zero. The approximation is necessary for handling the numerics but does not spoil our results significantly as figure 3.3 shows: the correct numerical consideration of m leads to a "sharper" picture. The minima and maxima are slightly more pronounced here than in the approximation with $m \approx 0$. However, the



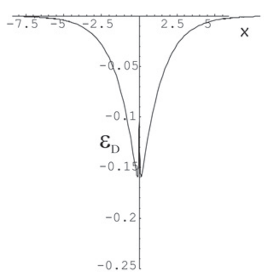
(a) $\mathcal{E}_D(x)$ with $m = 1$, $\eta = 10$, $S = 1$



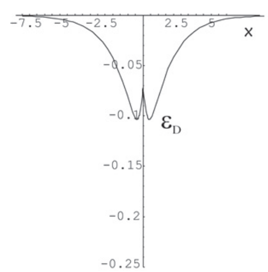
(b) $\mathcal{E}_D$ at $x = 0.2$ as function of β



(c) $\mathcal{E}_D$ at $x = 1$ as function of β



(d) $\mathcal{E}_D$ at $\beta = 0.001$ as function of x



(e) $\mathcal{E}_D$ at $\beta = 0.8$ as function of x

Figure 3.4: Defect energy density $\mathcal{E}_{D,\beta}(x)$ in two space-time dimensions for parameters $m = 1$, $\eta = 10$, $S = 1$ plotted as function of the distance from the defect x and the inverse temperature β. Subfigures (b)–(e) are slices of figure (a): (b), (c) in β direction for fixed x and (d), (e) in x direction with fixed β respectively. The latter are the functions one usually measures. They are added in this figure to illustrate the landscape plot. The colouring only illustrates the landscape behaviour of the energy density; any distinct physical meaning of the colours themselves is not intended here. The landscape plot shows that the UV limit turns the energy density into a nearly δ peak as β runs to zero; i.e., the temperature increases.

qualitative results do not change. We thus simplify the energy density to

$$\left\langle T^{00}(\pm|x|)\right\rangle = \frac{\ln 4}{2\pi\beta} - \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{1}{1+e^{\beta|k|}} \frac{\xi\, e^{\mp 2ik|x|}}{\xi \mp ik}. \tag{3.75}$$

Obviously, as in the bosonic case [63] we have the Stefan-Boltzmann energy density $\mathcal{E}_0$ of the system without any defect, plus an additional effect given by the defect energy density $\mathcal{E}_D$, hence

$$\left\langle T^{00}(x)\right\rangle = \mathcal{E}_0 + \mathcal{E}_D, \tag{3.76}$$

where

$$\mathcal{E}_0 = \frac{\ln 4}{2\pi\beta}, \qquad \mathcal{E}_D = -\int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{1}{1+e^{\beta|k|}} \frac{\xi\, e^{\mp 2ik|x|}}{\xi \mp ik}. \tag{3.77}$$

Due to equation (2.70),

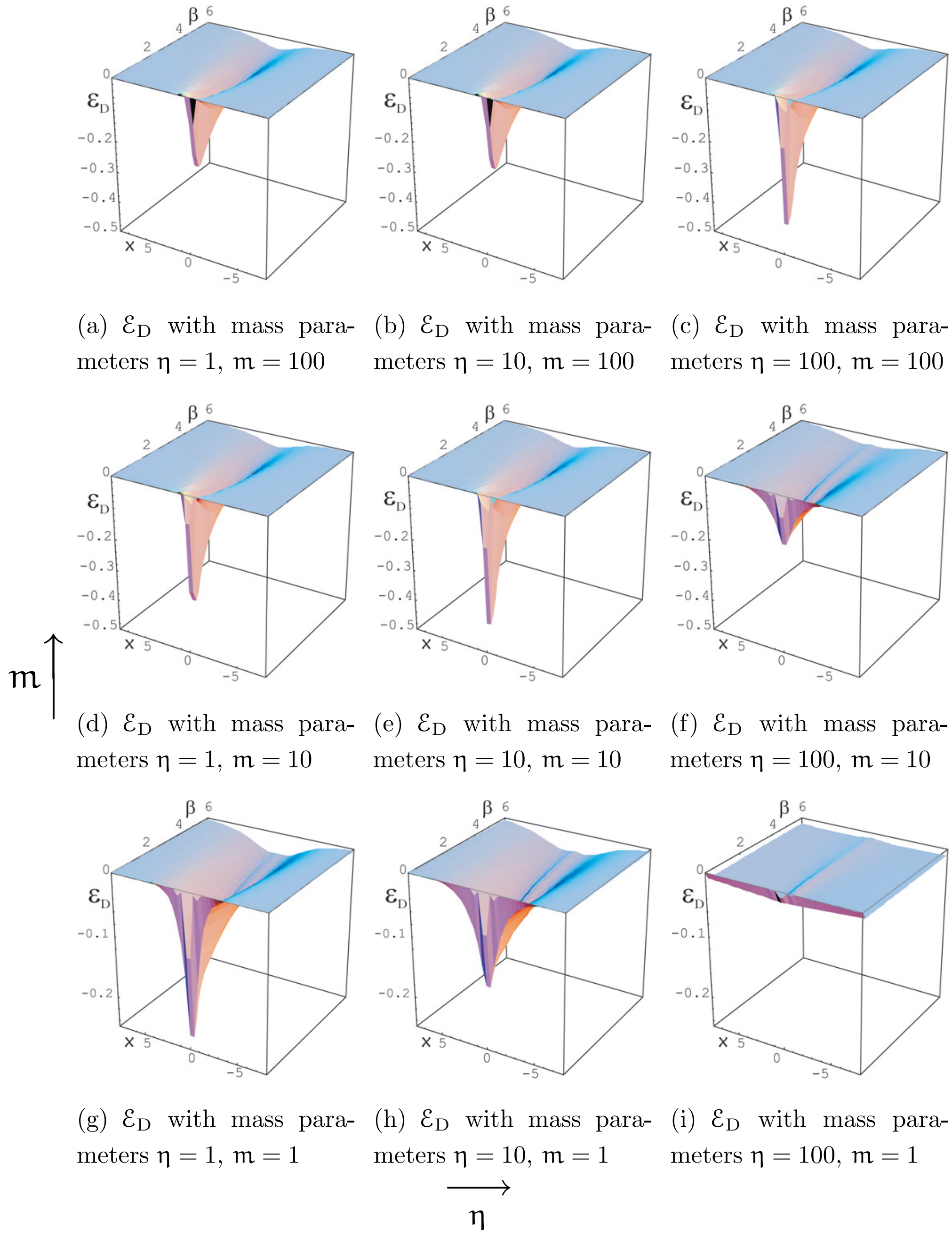
$$\int_0^\infty e^{-(a\pm ib)\alpha}\, d\alpha = \frac{1}{a\pm ib} \qquad \Leftrightarrow \qquad a>0, \tag{3.78}$$

$$-\int_0^\infty e^{(a\pm ib)\alpha}\, d\alpha = \frac{1}{a\pm ib} \qquad \Leftrightarrow \qquad a<0. \tag{3.79}$$

Thus, the complex denominators are replaced and the k integration is performed,

$$\begin{aligned}\mathcal{E}_{D,\eta\gtrless 0} = -\int_0^\infty d\alpha\, \frac{e^{-a|\xi|}|\xi|}{4\pi\beta} \Bigg[&\frac{\Gamma'\left(-\frac{i(\alpha\mp 2|x|)}{2\beta}\right)}{\Gamma\left(-\frac{i(\alpha\mp 2|x|)}{2\beta}\right)} + \frac{\Gamma'\left(\frac{i(\alpha\mp 2|x|)}{2\beta}\right)}{\Gamma\left(\frac{i(\alpha\mp 2|x|)}{2\beta}\right)} \\ &- \frac{\Gamma'\left(\frac{i\alpha+\beta\mp 2i|x|}{2\beta}\right)}{\Gamma\left(\frac{i\alpha+\beta\mp 2i|x|}{2\beta}\right)} - \frac{\Gamma'\left(\frac{-i\alpha+\beta\pm 2i|x|}{2\beta}\right)}{\Gamma\left(\frac{-i\alpha+\beta\pm 2i|x|}{2\beta}\right)}\Bigg].\end{aligned} \tag{3.80}$$

We are not able to solve the integral of $\mathcal{E}_D$ analytically, of course we can plot it as function of x and β, as shown in figures 3.4–3.8. We illustrate its properties and dependency on parameteres m, η, S in several plots: in figure 3.4 we give a clear idea of the landscape plots that show the energy density for varying inverse temperature β and distance from the defect x. Furthermore, this figure exhibits the usual numerical inaccuracies especially as β goes to zero. In this respect, the values of $\mathcal{E}_D(x)$ for $\beta \approx 0$ are not trustworthy anymore. However, we give evidence to their finiteness in figure 3.8.

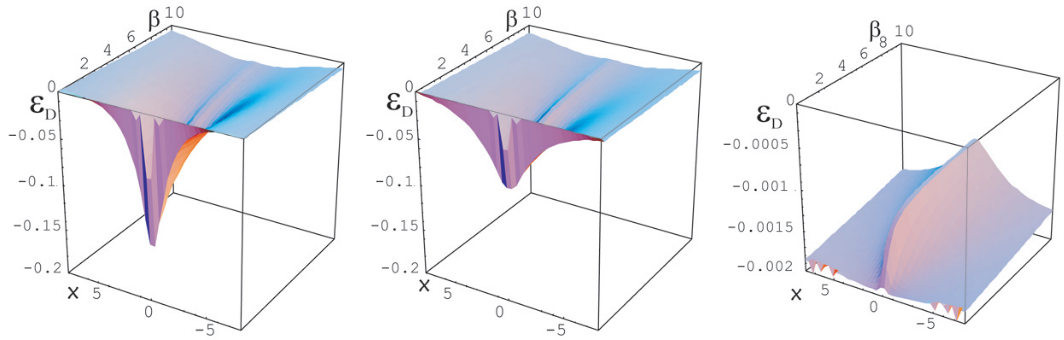


(a) $\mathcal{E}_{\mathrm{D}}$ with mass parameters $\eta = 1$, $m = 100$ (b) $\mathcal{E}_{\mathrm{D}}$ with mass parameters $\eta = 10$, $m = 100$ (c) $\mathcal{E}_{\mathrm{D}}$ with mass parameters $\eta = 100$, $m = 100$

(d) $\mathcal{E}_{\mathrm{D}}$ with mass parameters $\eta = 1$, $m = 10$ (e) $\mathcal{E}_{\mathrm{D}}$ with mass parameters $\eta = 10$, $m = 10$ (f) $\mathcal{E}_{\mathrm{D}}$ with mass parameters $\eta = 100$, $m = 10$

(g) $\mathcal{E}_{\mathrm{D}}$ with mass parameters $\eta = 1$, $m = 1$ (h) $\mathcal{E}_{\mathrm{D}}$ with mass parameters $\eta = 10$, $m = 1$ (i) $\mathcal{E}_{\mathrm{D}}$ with mass parameters $\eta = 100$, $m = 1$

Figure 3.5: Defect energy density $\mathcal{E}_{\mathrm{D},\beta}(x)$ in two space-time dimensions for different parameters $\eta > 0$ and $m > 0$ (with fixed spin $S = 1$) plotted as function of the distance from the defect x and the inverse temperature β. Apparently, the relation of the mass m and the defect mass η plays an important role, the density minimum near $\beta = 0$, i.e. $T \to \infty$, has a maximum depth at $m = \eta$.



(a) $\mathcal{E}_{\mathrm{D}}$ with $S = 0$ (b) $\mathcal{E}_{\mathrm{D}}$ with $S = 10$ (c) $\mathcal{E}_{\mathrm{D}}$ with $S = 100$

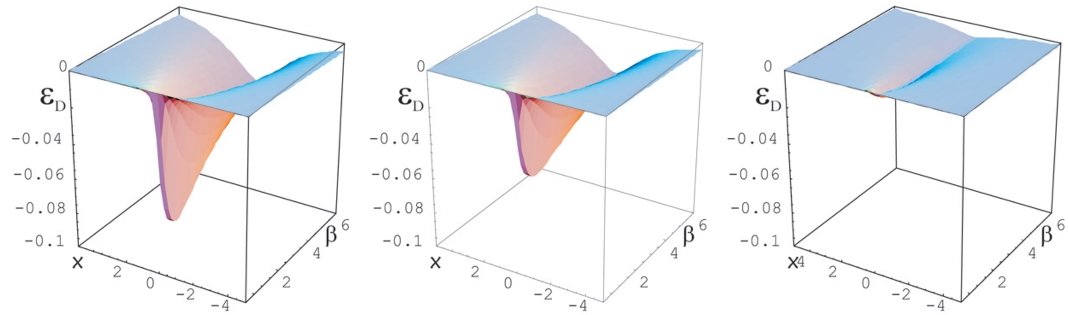
$\xrightarrow{\quad}$
S

Figure 3.6: Defect energy density $\mathcal{E}_{\mathrm{D},\beta}(x)$ in two space-time dimensions for different defect spin $S > 0$ (with fixed mass terms $m = 1$, $\eta = 10$) plotted as function of the distance from the defect x and the inverse temperature β. Note that $R^{\pm}$ depends on S^2 only. Therefore, we can restrict ourselves to positive defect spin. Furthermore, we should remember that $S \in \mathbb{R}$; i.e. a real spin multiplied by its coupling constant. In subfigure (c) the scale was enlarged in order to show that increasing S sharpens the peak at $x = 0$, but weakens the density minimum near $(x, \beta) = (0, 0)$.

In addition, figure 3.5 investigates the dependency on m in relation to η. Obviously, the ratio of m and η is important: for $m \approx \eta$, the absolute amount of the global minimum near $(x, \beta) = (0, 0)$ is maximal, while as the ratio turns away from one, the landscape flattens and the minimum might near vanish as for instance for $m = 1$, $\eta = 100$ – figure 3.5 (i).

Similarily, as the spin increases, the depth of the energy-landscape plot decreases – shown in figure 3.6. It seems that the spin has no other effects here. The local maximum at $x = 0$ – well identifiable for instance in figure 3.6 (c) – is actually an effect of the *sign of the defect mass* η, since this maximum vanishes for negative values given in figure 3.7. Moreover, these plots illustrate that there is a global minimum of the energy density for $\eta < 0$ at $(x, \beta) = (0, \varepsilon)$ with $\varepsilon > 0$. This indicates a phase transition as we discuss later.

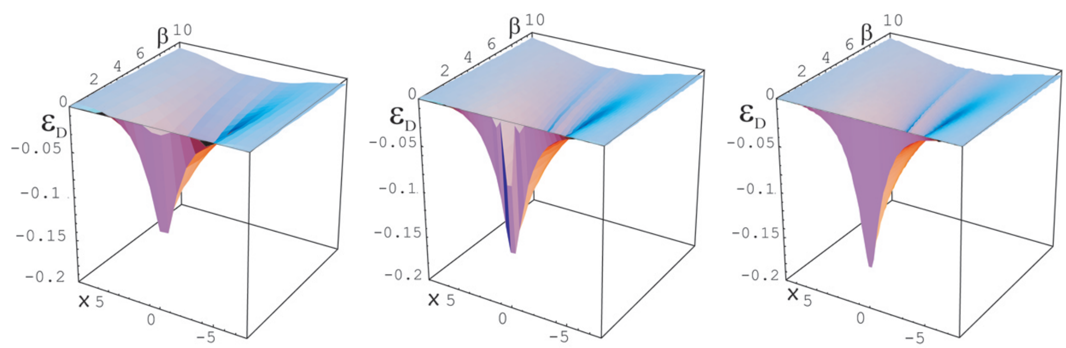
In comparison with the bosonic case we discussed in chapter 2, we notice that the dependence on the sign of η interchanges: for bosons there is a local maximum at $x = 0$ for $\eta < 0$, in contrast to the fermionic case:



(a) $\mathcal{E}_D$ with $\eta = -1$ (b) $\mathcal{E}_D$ with $\eta = -10$ (c) $\mathcal{E}_D$ with $\eta = -100$

$\longleftarrow$
η

Figure 3.7: Defect energy density $\mathcal{E}_{D,\beta}(x)$ in two space-time dimensions for negative defect mass $\eta < 0$ (with fixed mass term $m = 1$ and spin $S = 1$) plotted as function of the distance x and the inverse temperature β. We do not give the variation of the mass m since it is similar to the case $\eta > 0$. Moreover, for $\eta < 0$ there are also contributions of bound states we neglect here for reasons of simplicity. However, the minimum energy density has moved to non-zero β; i.e. finite temperature.



(a) 20 plot points per direction with $\beta \in [10^{-5}, 10]$ (b) 40 plot points per direction with $\beta \in [10^{-5}, 10]$ (c) 80 plot points per direction with $\beta \in [10^{-7}, 10]$

Figure 3.8: Defect energy density $\mathcal{E}_{D,\beta}(x)$ in two space-time dimensions with fixed mass terms $m = 1$, $\eta = 10$ and spin $S = 1$ for different plot resolutions. We believe it reasonable to regard the minima as finite since an increase of numerical accuracy does not change the plot dramatically, even if the starting point for β is moved towards zero – subfigure (c).

	local minimum	local maximum
bosonic	$\eta > 0$	$\eta < 0$
fermionic	$\eta < 0$	$\eta > 0$

Furthermore, for bosons, the energy density seems to turn into a near δ distribution as $\beta \to 0$, while the fermion energy density stays finite everywhere – according to numerics.

Noether Currents For the Noether currents we derive

$$\langle J^\mu(x)\rangle := \left\langle :\overline{\Psi}(x)\, A^\mu(\gamma)\, \Psi(x): \right\rangle \Big|_{t_1=t_2} \tag{3.81}$$

$$= \int_{-\infty}^{+\infty} \frac{dk}{4\pi\omega(k)} \frac{e^{-\beta[\omega(k)-\mu]}}{1+e^{-\beta[\omega(k)-\mu]}} \Big\{ \overline{v}(k) A\, v(k) - \overline{u}(k) A\, u(k) +$$

$$+ e^{-2ikx} \left[\theta(x) R^+(k) + \theta(-x) R^-(k)\right] \left[\overline{v}(k) A\, v(-k) - \overline{u}(-k) A\, u(k)\right] \Big\}$$

$$= \quad 0 \qquad \Leftrightarrow \qquad A = \gamma^0, \gamma^1, \tag{3.82}$$

where we did not specify the polonomial $A(\gamma)$ in γ^μ yet.

The vanishing of $\langle J^0(x)\rangle$ does not mean that there is no valid decomposition into non-zero parts that are conserved on their own. In particular, it should be possible to examine left and right movers separately. Moreover, the solutions with positive mass (i.e. the solutions with $(\not{p} - m)\psi = 0$) should separate from those with negative mass. Furthermore, it is not surprising that for example electron and positron currents sum up to zero.

We analyse such a possibility for left movers with positive mass, since we intend to compare our results with the single spin impurity formulation of Affleck and Ludwig [1; 23]. The approach by these authors is slightly different from ours because it deals with the rotational invariance of the single spin impurity. This leaves only the radial coordinate $r \in [0, \infty]$ and time t as space-time parameters. The reflected (outgoing) wave can be mapped on the negative radial axis. Therefore, the theory can be considered as effectively two-dimensional containing only left movers (or right movers respectively). We establish a similar left mover description in the RT picture.

Due to (3.58), the reflection terms vanish as soon as η goes to zero. In this way there are no right moving terms that are induced by reflection of left movers;

i.e., the left movers decouple completely, and we should therefore be able to compare the RT formalism with the CFT approach of [1]. Moreover, we have to restrict ourselves to solutions with positive mass only. For this reason we replace all terms $v(k)$ by zero.

The wave function Ψ decomposes via $\theta(\pm k)$ sign functions,

$$\Psi = \Psi_L + \Psi_R, \tag{3.83}$$

$$\Psi_R := \sum_{j=\pm} \theta(jx) \int_{-\infty}^{+\infty} \frac{dk}{2\pi\sqrt{2\omega(k)}} \Big[\theta(-k)\, u(k)\, a^j(k)\; e^{i\,\omega(k)t - i\,kx} + \tag{3.84}$$
$$+\, \theta(k) v(k)\, a_j(k)\; e^{-i\,\omega(k)t + i\,kx}\Big],$$

$$\Psi_L := \sum_{j=\pm} \theta(jx) \int_{-\infty}^{+\infty} \frac{dk}{2\pi\sqrt{2\omega(k)}} \Big[\theta(k)\, u(k)\, a^j(k)\; e^{i\,\omega(k)t - i\,kx} + \tag{3.85}$$
$$+\, \theta(-k) v(k)\, a_j(k)\; e^{-i\,\omega(k)t + i\,kx}\Big]$$

$$= \sum_{j=\pm} \theta(jx) \int_{-\infty}^{+\infty} \frac{dk}{2\pi\sqrt{2\omega(k)}}\, \theta(k)\, u(k)\, a^j(k)\; e^{i\,\omega(k)t - i\,kx}$$
$$+ \sum_{j=\pm} \theta(jx) \int_{-\infty}^{+\infty} \frac{dk}{2\pi\sqrt{2\omega(k)}}\, \theta(-k) v(k)\, a_j(k)\; e^{-i\,\omega(k)t + i\,kx} \tag{3.86}$$

$$=: \Psi_{L,e} + \Psi_{L,p}. \tag{3.87}$$

The results for the two point functions can easily be taken over from the full theory by the transformation

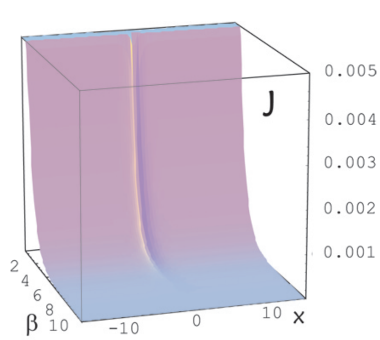
$$\left\langle :\overline{\Psi}(x_1,t_1)\,\Psi(x_2,t_2): \right\rangle \overset{f}{\mapsto} \left\langle :\overline{\Psi}_{L,e}(x_1,t_1)\,\Psi_{L,e}(x_2,t_2): \right\rangle, \tag{3.88}$$

$$\text{where} \qquad f: \qquad \overline{u}(k_1)u(k_2) \mapsto \quad \theta(k_1)\theta(k_2)\,\overline{u}(k_1)u(k_2). \tag{3.89}$$

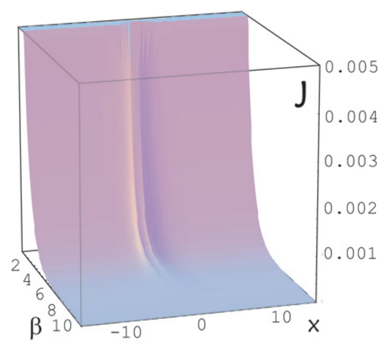
The left mover currents follow directly via (3.68) as before,

$$\overline{u}(k)\cdot u(k) \mapsto \overline{u}(k)\mathcal{O}u(k) \tag{3.90}$$

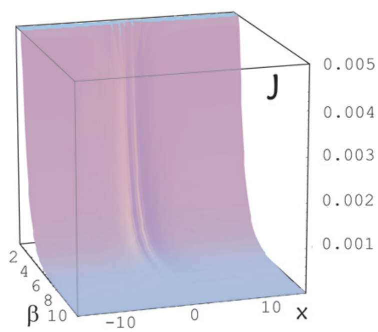
$$\partial_0\, e^{\pm i\,\omega(k)t_2} \mapsto \pm i\,\omega(k)\, e^{\pm i\,\omega(k)t_2}\,. \tag{3.91}$$



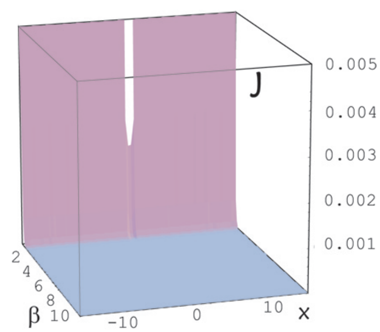
(a) $J^0_{L,e}$ with $m = 1, \eta = -1$



(b) $J^0_{L,e}$ with $m, \eta = 1$

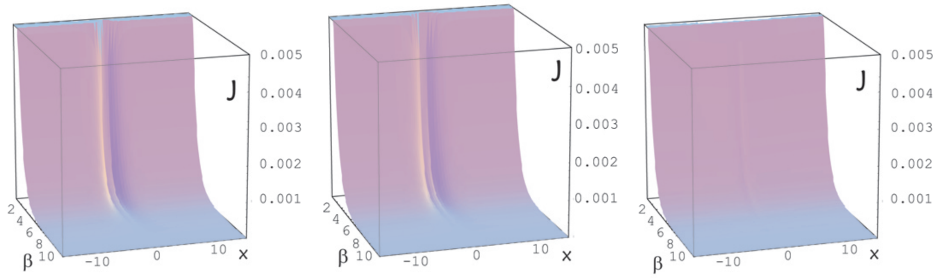


(c) $J^0_{L,e}$ with $m = 1, \eta = 10$



(d) $J^0_{L,e}$ with $m = 10, \eta = 1$

Figure 3.9: Fermionic charge density $\rho(x) \propto J^0_{L,e}$ in two space-time dimensions for different defect masses η plotted for varying x and β. The sign of the defect mass η determines the local minimum or maximum at $x = 0$. The "gap" in subfigure (d) results from the plot range $\beta \in [0.001, 10]$ which – only for numerical reasons – excludes in this graph values in a small vicinity of $(x, \beta) = (0, 0)$.



(a) $J^0_{\mathrm{L,e}}$ with $S = 0$ (b) $J^0_{\mathrm{L,e}}$ with $S = 1$ (c) $J^0_{\mathrm{L,e}}$ with $S = 10$

Figure 3.10: Fermionic charge density $\rho(x) \propto J^0_{\mathrm{L,e}}$ in two space-time dimensions for different spin values S plotted for varying x and β with fixed $m, \eta = 1$. As defect spin S increases the depression in the vicinity of $x = 0$ flattens out and vanishes for large S which means large coupling in particular.

This implies the vanishing of "half" of the upper integrals

$$\left\langle T^{00}_{\mathrm{L,e}}(x)\right\rangle = \frac{i}{2}\left\langle :\overline{\Psi}_{\mathrm{L,e}}(x)\,\gamma^0\partial^0\Psi_{\mathrm{L,e}}(x) - \partial^0\overline{\Psi}_{\mathrm{L,e}}(x)\,\gamma^0\Psi_{\mathrm{L,e}}(x):\right\rangle\Big|_{t_1=t_2} \tag{3.92}$$

$$= \int_{-\infty}^{+\infty}\frac{dk}{8\pi}\frac{e^{-\beta[\omega(k)-\mu]}}{1+e^{-\beta[\omega(k)-\mu]}}\cdot\Big\{\theta(k)\overline{u}(k)\gamma^0u(k)+ \tag{3.93}$$

$$+\,e^{-2ikx}\,\overline{u}(-k)\gamma^0u(k)\,\theta(-k)\,[\theta(x)R^+(k)+\theta(-x)R^-(k)]\Big\},$$

$$\left\langle J^{\mu}_{\mathrm{L,e}}(x)\right\rangle = \left\langle :\overline{\Psi}_{\mathrm{L,e}}(x)\,A^{\mu}(\gamma)\,\Psi_{\mathrm{L,e}}(x):\right\rangle\Big|_{t_1=t_2} \tag{3.94}$$

$$= -\int_{-\infty}^{+\infty}\frac{dk}{8\pi\omega(k)}\frac{e^{-\beta[\omega(k)-\mu]}}{1+e^{-\beta[\omega(k)-\mu]}}\left\{\theta(+k)\overline{u}(k)A(\gamma)u(k)+\right. \tag{3.95}$$

$$\left.+\,e^{-2ikx}\,\overline{u}(-k)A(\gamma)u(k)\,\theta(-k)\,[\theta(x)R^+(k)+\theta(-x)R^-(k)]\right\}$$

$$= -\int_{0}^{+\infty}\frac{dk}{8\pi\omega(k)}\frac{e^{-\beta[\omega(k)-\mu]}}{1+e^{-\beta[\omega(k)-\mu]}}\left\{\overline{u}(k)A(\gamma)u(k)+\right. \tag{3.96}$$

$$\left.+\,e^{2ikx}\,\overline{u}(k)A(\gamma)u(-k)\,[\theta(x)R^+(-k)+\theta(-x)R^-(-k)]\right\}.$$

Note that $\theta(\pm k_1)\theta(\pm k_2)$ became $\frac{1}{2}\,\theta(\pm k)$ due to the integration with respect to k_2, and because of the $\delta(k_1 \pm k_2)$ distribution (up to the renaming $k_1 \mapsto k$).

With the definitions (3.57) we end up with

$$\left\langle T^{00}_{L,e}(x)\right\rangle = \frac{1}{4}\mathcal{E}_0 + \frac{1}{4}\mathcal{E}_D = \frac{1}{4}\left\langle T^{00}(x)\right\rangle, \tag{3.97}$$

$$\left\langle J^0_L(x)\right\rangle = \int_0^{+\infty} \frac{dk}{8\pi p_0}\frac{1}{1+e^{\beta[p_0-\mu]}}\cdot \tag{3.98}$$
$$\cdot\left\{1+\frac{m}{p_0}e^{2ikx}\left[\theta(x)R^+(-k)+\theta(-x)R^-(-k)\right]\right\}$$

$$= \int_0^{+\infty} \frac{dk}{8\pi p_0}\frac{1}{1+e^{\beta[p_0-\mu]}}\cdot \tag{3.99}$$
$$\cdot\left\{1-4m\eta e^{2ikx}\left(\frac{\theta(-x)}{4m\eta-i(g+4)k}+\frac{\theta(x)}{i(g+4)k+4m\eta}\right)\right\},$$

$$\left\langle J^1_L(x)\right\rangle \equiv 0 \tag{3.100}$$

Equation (3.97) holds in higher dimensions as well. For this reason we will skip the discussion of left-mover energies and concentrate on their currents exclusively. Since we dropped any prefactor in the Noether current definition (3.81) the charge density $\rho(x)$ is just proportional to $\langle J^0_L(x)\rangle$ even if we sometimes call both charge density for the sake of brevity.

Given the similarities of two-point expectation values for the bosonic and the fermionic case, it is not very surprising that the currents – like the energy densities – differ in a general prefactor according to Bose and Fermi function and some prefactor of the reflection amplitudes according to the slightly different definitions of the Fourier modes; i.e. $a^{\pm}(k), a_{\pm}(k)$. Comparing figures 2.2 and 3.9 the similarities are obvious as well. The sign of η generates a local maximum or minimum at $x = 0$ respectively which stays non-zero as β runs to zero.

However, as already stated for the energy density, a negative defect mass $\eta < 0$ generates a local maximum for bosonic theory, but a local minimum for fermions and vice versa. Furthermore, the influence of the defect is much weaker for fermions than bosons where the defect shrinks the charge density nearly to zero at $x = 0$ regardless of the value of inverse temperature β.

Analogously to the Stefan-Boltzmann energy density $\mathcal{E}_0$ (3.77) the charge density diverges for $\beta \to 0$. This behaviour is *not* defect-induced.

3.2.6 Comparing with the One-Impurity CFT Approach

Our ansatz coincides with the complete description of the single spin Kondo effect in terms of CFT given by Affleck and Ludwig [1; 2]. As mentioned above, the idea consists in reducing the effectively considered space-time by the rotational symmetry of the problem. This leaves only the radial direction r (with the spin at the origin) and the time as parameters. Furthermore, the incoming particles from $r \to +\infty$ are reflected back to $r \to +\infty$ at the origin and it is straightforward to map the outgoing states to the negative real axis and thus produce a completely left moving theory.

In fact Bajnok et al. [61] use the same idea but invert the arguments to show that the RT formalism can be applied to boundaries as well. Besides this similarity in concepts, the approaches differ from each other (in particular the conformal symmetry is not applied to the RT theory). However, the results should be comparable and Affleck and Ludwig state a boundary condition

$$\Psi_+ = \Psi_- \qquad \Leftrightarrow \qquad \text{IR fixed point,} \tag{3.101}$$

$$\Psi_+ = -\Psi_- \qquad \Leftrightarrow \qquad \text{UV fixed point,} \tag{3.102}$$

which is in complete agreement with the boundary matrix (3.60) with parameter $\eta = 0$,

$$\Psi_+ = \frac{1}{2 - i S} \begin{pmatrix} 2 + i S & 0 \\ 0 & 2 + i S \end{pmatrix} \Psi_- \overset{S=0}{=} +\Psi_- \qquad \Leftrightarrow \qquad \text{IR fixed point,} \tag{3.103}$$

$$\Psi_+ = \frac{1}{2 - i S} \begin{pmatrix} 2 + i S & 0 \\ 0 & 2 + i S \end{pmatrix} \Psi_- \overset{S\to\infty}{=} -\Psi_- \qquad \Leftrightarrow \qquad \text{UV fixed point.} \tag{3.104}$$

Furthermore, we do not derive any energy shift here. However, since according to [23], this boundary condition directly implicates this energy shift, we use in our considerations this shift *implicitly*. We are only dealing with non-disturbed eigenwave functions and do not question the energy eigenvalue with or without defect.

Since we exclude bound states which are considered in the one-impurity CFT ansatz, we here give only the defect energy density for free particles. As the defect mass η is negative, the defect energy shows a minimum at finite temperature and vanishes for high temperatures, i.e. $\beta \to 0$. We interpret this minimum as the phase transition point where the energy levels of the bound states reach the edge of the potential pot induced by negative defect mass η. For increasing temperature, these bound states turn into free ones. We leave the exact phase transition behaviour for further studies.

3.3 RT Results in Three Dimensions

3.3.1 The Algebra

Let us now turn to the case of $d = 3$ with a $d = 2$ defect. As we still have unique solutions $u(k)$, $v(k)$ the RT formalism in the three-dimensional case looks similar to the two-dimensional one. It only has a further spin component S_2 and a momentum p_2.

The Lagrangean (3.2) contains γ matrices according to conventions (3.5)–(3.7),

$$\gamma^0 = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\,\sigma_1, \tag{3.105}$$

$$\gamma^1 = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \sigma_2, \tag{3.106}$$

$$\gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3. \tag{3.107}$$

Of course the algebra (2.4)–(2.6) shows additional δ distributions for the additional momenta $\hat{\mathbf{p}}_i$ parallel to the defect:

$$\{a_{\alpha_1}(k_1, \hat{\mathbf{p}}_1),\ a_{\alpha_2}(k_2, \hat{\mathbf{p}}_2)\} = 2\delta^{\alpha_1}_{\alpha_2}\delta(k_1 - k_2)\delta(\hat{\mathbf{p}}_1 - \hat{\mathbf{p}}_2)a_{\alpha_1}, \tag{3.108}$$

$$\{a^{\alpha_1}(k_1, \hat{\mathbf{p}}_1),\ a^{\alpha_2}(k_2, \hat{\mathbf{p}}_2)\} = 2\delta^{\alpha_2}_{\alpha_1}\delta(k_1 - k_2)\delta(\hat{\mathbf{p}}_1 - \hat{\mathbf{p}}_2)a^{\alpha_1}, \tag{3.109}$$

$$\begin{aligned}\{a_{\alpha_1}(k_1, \hat{\mathbf{p}}_1),\ a^{\alpha_2}(k_2, \hat{\mathbf{p}}_2)\} &= (2\pi)^{d-1}\delta\,(\hat{\mathbf{p}}_1 - \hat{\mathbf{p}}_2)\,\cdot \\ &\cdot \Big\{\delta\,(k_1 - k_2)\,\Big[\delta^{\alpha_2}_{\alpha_1} + \delta^{-\alpha_1}_{\alpha_2}T^{\alpha_2}(k_1, \hat{\mathbf{p}}_1)\Big] + \\ &+ \delta(k_1 + k_2)\,\delta^{\alpha_1}_{\alpha_2}R^{\alpha_2}(k_2, \hat{\mathbf{p}}_2)\Big\},\end{aligned} \tag{3.110}$$

and $\alpha_i = (s_i, \pm)$ – while we can still neglect s_i. Once again we have additional exchange relations (3.37) and (3.38),

$$\begin{pmatrix} a^-(k,\hat{\mathbf{p}}) \\ a^+(-k,\hat{\mathbf{p}}) \end{pmatrix} = \begin{pmatrix} \overline{R}^-(-k,\hat{\mathbf{p}}) & \overline{T}^+(k,\hat{\mathbf{p}}) \\ \overline{T}^-(-k,\hat{\mathbf{p}}) & \overline{R}^+(k,\hat{\mathbf{p}}) \end{pmatrix} \cdot \begin{pmatrix} a^-(-k,\hat{\mathbf{p}}) \\ a^+(k,\hat{\mathbf{p}}) \end{pmatrix}, \qquad (3.111)$$

$$\begin{pmatrix} a_-(k,\hat{\mathbf{p}}) \\ a_+(-k,\hat{\mathbf{p}}) \end{pmatrix} = \begin{pmatrix} R^-(-k,\hat{\mathbf{p}}) & T^+(k,\hat{\mathbf{p}}) \\ T^-(-k,\hat{\mathbf{p}}) & R^+(k,\hat{\mathbf{p}}) \end{pmatrix} \cdot \begin{pmatrix} a_-(-k,\hat{\mathbf{p}}) \\ a_+(k,\hat{\mathbf{p}}) \end{pmatrix}. \qquad (3.112)$$

3.3.2 Boundary Condition and RT Coefficients

The spin solutions are up to a phase

$$u(k,\hat{\mathbf{p}}) = -\frac{1}{\sqrt{2p_0(p_0-k)}} \begin{pmatrix} p_0 - k \\ m + i\,p_2 \end{pmatrix}, \qquad (3.113)$$

$$v(k,\hat{\mathbf{p}}) = -\frac{1}{\sqrt{2p_0(p_0-k)}} \begin{pmatrix} -p_0 + k \\ m - i\,p_2 \end{pmatrix}, \qquad (3.114)$$

where $p_0^2 = m^2 + k^2 + p_2^2 =: \omega(k)$.

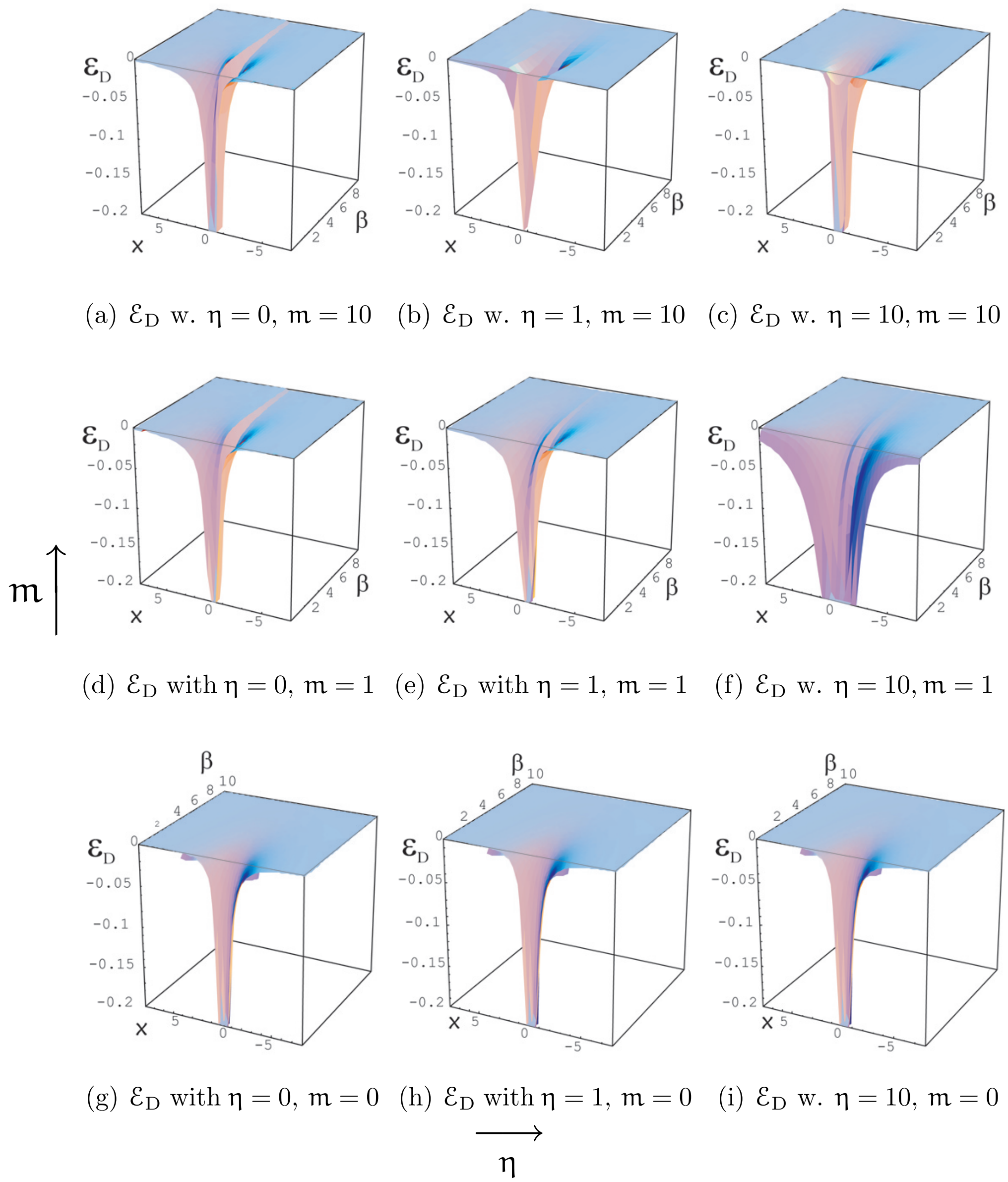
After integrating the equations of motion over the interval $x \in [-\varepsilon, +\varepsilon]$ with $\varepsilon \to 0$ once more, the boundary matrix M_- defined in (3.12) with $g := \eta^2 + S_1^2 + S_2^2$ reads,

$$M_- = -\frac{1}{g + 4\,i\,S_2 - 4} \begin{pmatrix} g - 4\eta + 4 & 4S_1 \\ 4S_1 & g + 4\eta + 4 \end{pmatrix}. \qquad (3.115)$$

Due to equations (3.53) the RT coefficients follow immediately,

$$R^\pm(k) = -\frac{4\sqrt{p_0+k}\,((S_2 - i\,\eta)p_2^2 + m(\eta + i\,S_2)p_2 + (k - p_0)(kS_2 - i\,\eta p_0))}{\sqrt{p_0 - k}(m - i\,p_2)(\mp(g+4)k + 4\,i(m\eta + S_2 p_2))}, \qquad (3.116)$$

$$T^\pm(k) = \frac{k(\mp g + 4\,i\,S_1 \pm 4)}{(g+4)k \mp 4\,i(m\eta + S_2 p_2)}. \qquad (3.117)$$



(a) $\mathcal{E}_\mathrm{D}$ w. $\eta = 0$, $m = 10$ (b) $\mathcal{E}_\mathrm{D}$ w. $\eta = 1$, $m = 10$ (c) $\mathcal{E}_\mathrm{D}$ w. $\eta = 10, m = 10$

(d) $\mathcal{E}_\mathrm{D}$ with $\eta = 0$, $m = 1$ (e) $\mathcal{E}_\mathrm{D}$ with $\eta = 1$, $m = 1$ (f) $\mathcal{E}_\mathrm{D}$ w. $\eta = 10, m = 1$

(g) $\mathcal{E}_\mathrm{D}$ with $\eta = 0$, $m = 0$ (h) $\mathcal{E}_\mathrm{D}$ with $\eta = 1$, $m = 0$ (i) $\mathcal{E}_\mathrm{D}$ w. $\eta = 10$, $m = 0$

Figure 3.11: Defect energy density $\mathcal{E}_{\mathrm{D},\beta}(x)$ in three space-time dimensions for different parameters $\eta > 0$ and $m > 0$ (with fixed spin $S_1, S_2 = 1$) plotted as function of the distance from the defect x and the inverse temperature β. For increasing η the defect energy density smears out as most apparent in subfigure (f). This does not take effect in case $m = 0$. Furthermore, in contrast to two dimensions as m vanishes, we are still left with a non-vanishing defect density $\mathcal{E}_\mathrm{D}(x)$ which is finite and non-zero even for $\eta = 0$.

3.3.3 Energy Density

The stress tensor zero component is given by

$$\left\langle T^{00}(x)\right\rangle = \frac{i}{2}\left\langle :\overline{\Psi}(x)\,\gamma^0\partial^0\Psi(x) - \partial^0\overline{\Psi}\,\gamma^0\Psi:\right\rangle\Big|_{t_1=t_2} \tag{3.118}$$

$$\begin{aligned} &= \sum_{j,l=\pm}\iint_{-\infty}^{+\infty}\frac{d\mathbf{p}_1\,d\mathbf{p}_2}{2^5\pi^4\sqrt{\omega(\mathbf{p}_1)\omega(\mathbf{p}_2)}}\frac{\theta(jx_1)\theta(lx_2)}{1+e^{\beta[\omega(\mathbf{p}_1)-\mu]}}[\omega(\mathbf{p}_1)+\omega(\mathbf{p}_2)] \\ &\cdot\Bigg[\left\{a_l(\mathbf{p}_2),\ a^j(\mathbf{p}_1)\right\}\overline{v}(\mathbf{p}_1)\gamma^0 v(\mathbf{p}_2)\,e^{-i\,\mathbf{p}_1x_1+i\,\mathbf{p}_2x_2}\,e^{i\,\omega(\mathbf{p}_1)t_1-i\,\omega(\mathbf{p}_2)t_2}+ \\ &+\left\{a^l(\mathbf{p}_2),\ a_j(\mathbf{p}_1)\right\}\overline{u}(\mathbf{p}_1)\gamma^0 u(\mathbf{p}_2)\,e^{i\,\mathbf{p}_1x_1-i\,\mathbf{p}_2x_2}\,e^{-i\,\omega(\mathbf{p}_1)t_1+i\,\omega(\mathbf{p}_2)t_2}\Bigg]. \end{aligned} \tag{3.119}$$

Using (3.62), setting $x_1 = x_2$, $t_1 = t_2$, integrating with respect to $\mathbf{p}_2$ and considering $\mathbf{p}_1 = (k, p)$ yields

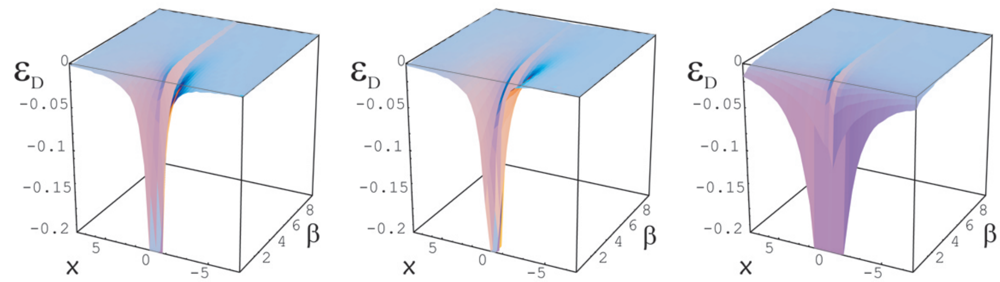
$$\begin{aligned} \left\langle T^{00}(x)\right\rangle &= \iint_{-\infty}^{+\infty}\frac{dk\,dp}{4\pi^2}\frac{e^{-\beta[\omega(k,p)-\mu]}}{1+e^{-\beta[\omega(k,p)-\mu]}}\cdot \\ &\cdot\Big\{\overline{v}(k,p)\gamma^0 v(k,p)+\overline{u}(k,p)\gamma^0 u(k,p)+ \\ &+e^{-2\,i\,kx}\left[\theta(x)R^+(k,p)+\theta(-x)R^-(k,p)\right]\cdot \\ &\cdot\left[\overline{v}(k,p)\gamma^0 v(-k,p)+\overline{u}(-k,p)\gamma^0 u(k,p)\right]\Big\} \end{aligned} \tag{3.120}$$

$$=: \mathcal{E}_0 + \mathcal{E}_D. \tag{3.121}$$

Taking $\mu = 0$ and $p_0 \approx \sqrt{k^2 + p_2^2 + m^2}$ yields

$$\mathcal{E}_0 = \frac{\pi}{12\beta^2} \tag{3.122}$$

$$\begin{aligned} \mathcal{E}_D(x) &= -\int_{-\infty}^{+\infty}\int_0^{2\pi}\frac{dr\,d\varphi}{\pi^2}\frac{e^{-2\,i\,rx\cos\varphi}r}{1+e^{\beta p_0}}\cdot \\ &\cdot\frac{2r\cos\varphi(\eta r\sin\varphi - mS_2)+2\,i\sqrt{m^2+r^2}(m\eta + rS_2\sin\varphi)}{\sqrt{m^2+r^2}(\mathrm{sgn}(x)(g+4)r\cos\varphi+4\,i(m\eta+rS_2\sin\varphi))} \end{aligned} \tag{3.123}$$



(a) $\mathcal{E}_D(x)$ with $S_1 = 1$, $S_2 = 10$ (b) $\mathcal{E}_D(x)$ with $S_1 = 1$, $S_2 = 1$ (c) $\mathcal{E}_D(x)$ with $S_1 = 10$, $S_2 = 1$

Figure 3.12: Defect energy density $\mathcal{E}_{D,\beta}(x)$ in three space-time dimensions for different parameters S_1, S_2 with $m, \eta = 1$ fixed. As S_1 increases, the defect density peak at $(x, \beta) = (0, 0)$ smears out. Increasing S_2 shows the same, but less intense influence. For $S_1 \approx S_2$ the peak at $(x, \beta) = (0, 0)$ is highly distinct.

For mass $m = 0$ the defect energy density simplifies,

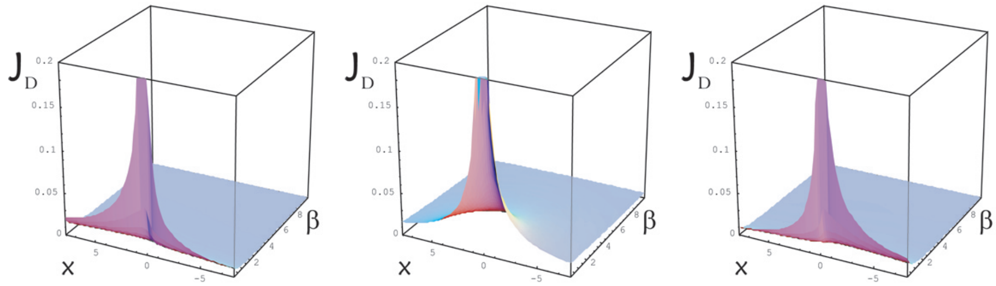
$$\mathcal{E}_{D,m=0}(x) = -\int_{-\infty}^{+\infty}\int_0^{2\pi} \frac{dr\, d\varphi}{\pi^2} \frac{2\, r\, e^{-2\, i\, r x \cos\varphi}}{1 + e^{\beta p_0}} \cdot \frac{\eta \cos\varphi + i\, S_2}{\mathrm{sgn}(x)(g+4)\cot\varphi + 4\, i\, S_2} \tag{3.124}$$

$$= -\int_0^{2\pi} \frac{d\varphi}{2\beta^2\pi^4} \frac{(S_2 - i\,\eta\cos\varphi)\left(\zeta\left(2, \frac{i\,x\cos\varphi}{\beta} + \frac{1}{2}\right) - \zeta\left(2, \frac{i\,x\cos\varphi}{\beta} + 1\right)\right)}{4S_2 - \mathrm{sgn}(x)\, i(g+4)\cot\varphi}. \tag{3.125}$$

The analytical integration with respect to φ fails, but the numerically integrated plots of $\mathcal{E}_D(x)$ for different values of S_1, S_2 are shown in figure 3.12. Figure 3.11 illustrates the behaviour of the defect energy density $\mathcal{E}_D(x)$ for different values of defect mass η.

Contrary to the two-dimensional case, for negative defect mass η, there is no energy density minimum at finite (inverse) temperature β in three dimensions. The landscape plots for negative η are similar to those for positive η (beside of turning the local maximum into a minumum). Thus we do not add them here.

The three-dimensional plots of the energy density show the same shape as the plots of the two-dimensional energy density $\mathcal{E}_D$. Its properties we discussed in the previous section 3.2.5. The most important difference is that in three



(a) J^0_D with $\eta = -10$ (b) J^0_D with $\eta = 0$ (c) J^0_D with $\eta = 10$

Figure 3.13: Fermionic charge density $\rho(x) \propto J^0_{L,e}$ in three space-time dimensions for different defect masses η plotted for varying x and β ($S_1, S_2 = 1$ fixed). For $\eta \neq 0$ the plots are not symmetrical. This means that the defect mass η breaks the chiral symmetry. The left and right moving theories mix. Aside from the impossibility to define a spin projector in three dimensions, the defect spins do not spoil this picture.

dimensions even for $m = 0$ *and* $\eta = 0$ the defect energy density does *not* vanish. Whereas in two dimensions we had to consider the approximation $m \approx 0$ very carefully and found a vanishing energy density $\mathcal{E}_D$ for $\eta = 0$, in three dimensions we observe a non-zero defect density fo a pure spin defect.

3.3.4 Noether Currents

As above, the conserved currents read

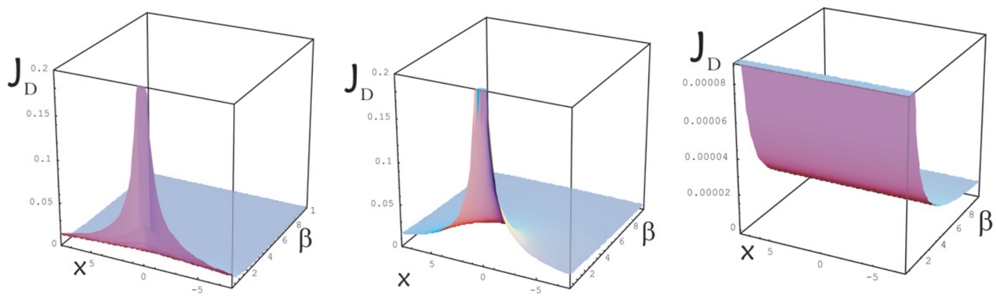
$$\langle J^\mu(x)\rangle = \left\langle :\overline{\Psi}(x)\, A^\mu(\gamma)\, \Psi(x): \right\rangle \Big|_{t_1=t_2} \tag{3.126}$$

$$= \iint_{-\infty}^{+\infty} \frac{dk\, dp_2}{8\pi p_0} \frac{1}{1+e^{\beta[\omega(k)-\mu]}} \Big\{ \overline{v}(k)A\, v(k) - \overline{u}(k)A\, u(k) +$$

$$+ e^{-2ikx} \left[\theta(x)R^+(k) + \theta(-x)R^-(k)\right] \left[\overline{v}(k)A\, v(-k) - \overline{u}(-k)A\, u(k)\right] \Big\}$$

$$= \quad 0 \qquad \Leftrightarrow \qquad A = \gamma^0, \gamma^1, \gamma^2. \tag{3.127}$$

We again analyse left movers with positive mass exclusively, comparing them with those in two dimensions. We should keep in mind here that the mass parameters m and η could break the symmetry of left and right movers.



(a) $\mathcal{E}_D(x)$ with $S_1 = 1$, $S_2 = 10$ (b) $\mathcal{E}_D(x)$ with $S_1 = 1$, $S_2 = 1$ (c) $\mathcal{E}_D(x)$ with $S_1 = 10$, $S_2 = 1$

Figure 3.14: Fermionic charge density $\rho(x) \propto J^0_{L,e}$ in three space-time dimensions for different spin values S_1, S_2 plotted for varying x and β (and $m, \eta = 0$ fixed). For S_2 very large compared to S_1 the plot becomes "flat".

In complete analogy to the two-dimensional case (3.92)–(3.96) this part of the full integral containing the spinors $v_s(k)$ vanishes,

$$\left\langle T^{00}_{L,e}(x)\right\rangle = \frac{i}{2}\left\langle :\overline{\Psi}_{L,e}(x)\,\gamma^0\partial^0\Psi_{L,e}(x) - \partial^0\overline{\Psi}_{L,e}(x)\,\gamma^0\Psi_{L,e}(x):\right\rangle\Big|_{t_1=t_2} \tag{3.128}$$

$$= \frac{1}{4}\left\langle T^{00}(x)\right\rangle \tag{3.129}$$

$$\left\langle J^{\mu}_{L,e}(x)\right\rangle = \left\langle :\overline{\Psi}_{L,e}(x)\,A^{\mu}(\gamma)\,\Psi_{L,e}(x):\right\rangle\Big|_{t_1=t_2} \tag{3.130}$$

$$= \iint_{-\infty}^{+\infty} \frac{dk\,dp_2}{8\pi p_0}\,\frac{1}{1+e^{\beta[\omega(k)-\mu]}}\,\Big\{\theta(+k)\overline{u}(k)A(\gamma)u(k)+ \tag{3.131}$$

$$+\,e^{-2ikx}\,\overline{u}(-k)A(\gamma)u(k)\,\theta(-k)\,[\theta(x)R^{s+}(k)+\theta(-x)R^{s-}(k)]\Big\}$$

$$= \quad 0 \qquad \Leftrightarrow \qquad A(\gamma) = \gamma^1, \gamma^2. \tag{3.132}$$

The last line is not obvious here, we rather have to examine the scalar products $\overline{u}(\pm k)A(\gamma)u(k)$ carefully. It turns out that the polar integration with respect to φ where $\tan\varphi := k/p_2$ of the non-vanishing terms $\overline{u}(\pm k)A(\gamma)u(k)$ vanishes for symmetry reasons. This is straightforward, thus we do not give any details.

Only for $A(\gamma) = \gamma^0$ we find a non-vanishing result (for $\mu = 0$),

$$\left\langle J^0_{L,e}(x) \right\rangle = \int_0^{+\infty} \int_0^{\pi} \frac{dr\, d\varphi}{8\pi\sqrt{m^2+r^2}} \frac{r}{1+e^{\beta\sqrt{m^2+r^2}}} \cdot \Bigg\{ 1 - 4e^{-i r x \cos\varphi} \cdot \tag{3.133}$$

$$\cdot \frac{r\cos\varphi(\eta r \sin\varphi - m S_2) + i\sqrt{m^2+r^2}(m\eta + r S_2 \sin\varphi)}{\sqrt{m^2+r^2}((g+4) r \cos\varphi\, \mathrm{sgn}(x) + 4i(m\eta + r S_2 \sin\varphi))} \Bigg\}$$

$$=: J_0 - J_D \tag{3.134}$$

In contrast to the two-dimensional case the component J_0 without defect can be integrated directly, i.e.

$$J_0 = \frac{\ln 2}{4\beta} \tag{3.135}$$

$$J_D = \int_0^{+\infty} \int_0^{\pi} \frac{dr\, d\varphi}{8\pi\sqrt{m^2+r^2}} \frac{r}{1+e^{\beta\sqrt{m^2+r^2}}} \cdot \Bigg\{ 4e^{-i r x \cos\varphi} \cdot \tag{3.136}$$

$$\cdot \frac{r\cos\varphi(\eta r \sin\varphi - m S_2) + i\sqrt{m^2+r^2}(m\eta + r S_2 \sin\varphi)}{\sqrt{m^2+r^2}((g+4) r \cos\varphi\, \mathrm{sgn}(x) + 4i(m\eta + r S_2 \sin\varphi))} \Bigg\}$$

$$J_{D,0} = \int_0^{+\infty} \int_0^{\pi} \frac{dr\, d\varphi}{8\pi} \frac{1}{1+e^{\beta r}} \cdot \frac{4e^{-i r x \cos\varphi}(i\, S_2 + \eta\cos\varphi)}{(g+4)\cot\varphi\, \mathrm{sgn}(x) + 4i\, S_2} \tag{3.137}$$

$$= -\int_0^{\pi} \frac{d\varphi}{4\beta\pi} \frac{i\, S_2 + \eta\cos\varphi}{(g+4)\,\mathrm{sgn}(x)\cot\varphi + 4i\, S_2} \cdot \tag{3.138}$$

$$\cdot \left(\psi^{(0)}\left(\frac{\beta + i x \cos\varphi}{2\beta}\right) - \psi^{(0)}\left(\frac{i x \cos\varphi}{2\beta} + 1\right) \right).$$

In the two-dimensional case there is no momentum component perpendicular to the defect, whereas in three dimensions, we have to take into account such a component p_2. Since we consider here a purely left-mover theory, we have to be aware of a mass term breaking the invariance of the left-moving spinors. On the other hand, a δ-potential mass term is a very specific case and a priori we cannot expect such a symmetry breaking. In addition with the impossibility of defining a γ^5 matrix in odd dimensions an thus projecting out spin directions by $(\mathbb{1} \pm \gamma^5)$, a non-zero defect mass obviously spoils the decomposition into purely left- and right-moving currents. Yet a detailed description is lacking. Since equation (3.138) is very close to the symmetric energy density (3.125), the symmetry breaking term including η is not obvious here.

3.3.5 Comparison with Two Dimensions

Once again, we are interested in the strong and weak coupling limits $S \to \infty$ and $S \to 0$. In three dimensions the spin has two components,

$$S = (S_1, S_2) = \lambda(s_1, s_2), \tag{3.139}$$

where $|s_i| = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ are quantised spins with a finite but non-zero value. This implies

$$\lambda s = S \to 0 \quad \Leftrightarrow \quad \lambda \to 0, \quad \text{with } \lambda s_i = S_i, \tag{3.140}$$

$$S \to \infty \quad \Leftrightarrow \quad \lambda \to \infty. \tag{3.141}$$

Since $\Psi_+ = M_- \Psi_-$ we have with $\eta = 0$

$$\Psi_+ = \frac{\begin{pmatrix} \lambda^2 s^2 + 4 & 4\lambda s_1 \\ 4\lambda s_1 & \lambda^2 s^2 + 4 \end{pmatrix} \Psi_-}{4 - \lambda^2 s^2 - 4\,i\,\lambda s_2} \overset{\lambda=0}{=} +\Psi_- \quad \Leftrightarrow \quad \text{IR fixed point,} \tag{3.142}$$

$$\Psi_+ = \frac{\begin{pmatrix} \lambda^2 s^2 + 4 & 4\lambda s_1 \\ 4\lambda s_1 & \lambda^2 s^2 + 4 \end{pmatrix} \Psi_-}{4 - \lambda^2 s^2 - 4\,i\,\lambda s_2} \overset{\lambda\to\infty}{=} -\Psi_- \quad \Leftrightarrow \quad \text{UV fixed point.} \tag{3.143}$$

We thus observe the same fixed point boundary conditions as in two dimensions. We discuss further possibilities for the four dimensional case.

3.4 Fermionic δ Defects in Four and Higher Dimensions

Since in more than three dimensions we have different solutions of the Dirac equation indicated by s, the whole formalism shows now its full power, but its full complexity as well. The RT terms in the algebra do not only carry an additional s index, there also appear mixing amplitudes $R_{s_i}^{s_j\,\pm}$, $T_{s_i}^{s_j\,\pm}$ between the different solutions s_i and s_j. Therefore, we give the general setup and the four-dimensional terms as an example.

3.4.1 The RT Algebra

The generator algebra in higher dimensions is given according to (2.20)–(2.22)

$$\{a_{\alpha_1}(k_1,\hat{\mathbf{p}}_1),\ a_{\alpha_2}(k_2,\hat{\mathbf{p}}_2)\} = 2\delta^{\alpha_1}_{\alpha_2}\delta(k_1-k_2)\delta(\hat{\mathbf{p}}_1-\hat{\mathbf{p}}_2)a_{\alpha_1}, \tag{3.144}$$

$$\{a^{\alpha_1}(k_1,\hat{\mathbf{p}}_1),\ a^{\alpha_2}(k_2,\hat{\mathbf{p}}_2)\} = 2\delta^{\alpha_2}_{\alpha_1}\delta(k_1-k_2)\delta(\hat{\mathbf{p}}_1-\hat{\mathbf{p}}_2)a^{\alpha_1}, \tag{3.145}$$

$$\begin{aligned}\{a_{\alpha_1}(k_1,\mathbf{p}_1),\ a^{\alpha_2}(k_2,\mathbf{p}_2)\} &= (2\pi)^{d-1}\delta\left(\mathbf{p}_1-\mathbf{p}_2\right)\cdot \\ &\cdot\left\{\delta\left(k_1-k_2\right)\left[\delta^{\alpha_2}_{\alpha_1}+T^{\alpha_2}_{\alpha_1}(k_1,\hat{\mathbf{p}}_1)\right]+\delta(k_1+k_2)\,R^{\alpha_2}_{\alpha_1}(k_2,\hat{\mathbf{p}}_2)\right\},\end{aligned} \tag{3.146}$$

where d is the space-time dimension, k_i the momenta perpendicular to the defect, $\mathbf{p}_i$ the full momenta, $\hat{\mathbf{p}} = (p_2, p_3, \ldots)$ the momenta parallel to the defect written in $d-2$ vector components, and $\alpha_i = (s_i, \pm)$ with s_i the spin solution index as before, especially $s_i = 1, 2$ in four dimensions.

Once again we have the additional exchange relations (3.37) and (3.38) which are modified for multiple spin solutions as well,

$$a^{s_l,j}(k,\hat{\mathbf{p}})\overline{T}^{s_i,-j}_{s_l,j}(k,\hat{\mathbf{p}}) + a^{s_l,-j}(-k,\hat{\mathbf{p}})\overline{R}^{s_i,-j}_{s_l,-j}(-k,\hat{\mathbf{p}}) = a^{s_i,-j}(k,\hat{\mathbf{p}}), \tag{3.147}$$

$$a_{s_l,j}(k,\hat{\mathbf{p}})T^{s_l,j}_{s_i,-j}(k,\hat{\mathbf{p}}) + a_{s_l,-j}(-k,\hat{\mathbf{p}})R^{s_l,-j}_{s_i,-j}(-k,\hat{\mathbf{p}}) = a_{s_i,-j}(k,\hat{\mathbf{p}}), \tag{3.148}$$

or rewritten as matrix equation for $s_i = 1, 2$,

$$\begin{pmatrix} a^{1-}(+k,\hat{\mathbf{p}}) \\ a^{1+}(-k,\hat{\mathbf{p}}) \\ a^{2-}(+k,\hat{\mathbf{p}}) \\ a^{2+}(-k,\hat{\mathbf{p}}) \end{pmatrix} = \begin{pmatrix} \overline{R}^{1-}(-k,\hat{\mathbf{p}}) & \overline{T}^{1+}(k,\hat{\mathbf{p}}) & \overline{R}^{1-}_2(-k,\hat{\mathbf{p}}) & \overline{T}^{1+}_2(k,\hat{\mathbf{p}}) \\ \overline{T}^{1-}(-k,\hat{\mathbf{p}}) & \overline{R}^{1+}(k,\hat{\mathbf{p}}) & \overline{T}^{1-}_2(-k,\hat{\mathbf{p}}) & \overline{R}^{1+}_2(k,\hat{\mathbf{p}}) \\ \overline{R}^{2-}_1(-k,\hat{\mathbf{p}}) & \overline{T}^{2+}_1(k,\hat{\mathbf{p}}) & \overline{R}^{2-}(-k,\hat{\mathbf{p}}) & \overline{T}^{2+}(k,\hat{\mathbf{p}}) \\ \overline{T}^{2-}_1(-k,\hat{\mathbf{p}}) & \overline{R}^{2+}_1(k,\hat{\mathbf{p}}) & \overline{T}^{2-}(-k,\hat{\mathbf{p}}) & \overline{R}^{2+}(k,\hat{\mathbf{p}}) \end{pmatrix} \cdot \begin{pmatrix} a^{1-}(-k,\hat{\mathbf{p}}) \\ a^{1+}(+k,\hat{\mathbf{p}}) \\ a^{2-}(-k,\hat{\mathbf{p}}) \\ a^{2+}(+k,\hat{\mathbf{p}}) \end{pmatrix}, \tag{3.149}$$

$$\begin{pmatrix} a_{1-}(+k,\hat{\mathbf{p}}) \\ a_{1+}(-k,\hat{\mathbf{p}}) \\ a_{2-}(+k,\hat{\mathbf{p}}) \\ a_{2+}(-k,\hat{\mathbf{p}}) \end{pmatrix} = \begin{pmatrix} R^{1-}(-k,\hat{\mathbf{p}}) & T^{1+}(k,\hat{\mathbf{p}}) & R^{2-}_1(-k,\hat{\mathbf{p}}) & T^{2+}_1(k,\hat{\mathbf{p}}) \\ T^{1-}(-k,\hat{\mathbf{p}}) & R^{1+}(k,\hat{\mathbf{p}}) & T^{2-}_1(-k,\hat{\mathbf{p}}) & R^{2+}_1(k,\hat{\mathbf{p}}) \\ R^{1-}_2(-k,\hat{\mathbf{p}}) & T^{1+}_2(k,\hat{\mathbf{p}}) & R^{2-}(-k,\hat{\mathbf{p}}) & T^{2+}(k,\hat{\mathbf{p}}) \\ T^{1-}_2(-k,\hat{\mathbf{p}}) & R^{1+}_2(k,\hat{\mathbf{p}}) & T^{2-}(-k,\hat{\mathbf{p}}) & R^{2+}(k,\hat{\mathbf{p}}) \end{pmatrix} \cdot \begin{pmatrix} a_{1-}(-k,\hat{\mathbf{p}}) \\ a_{1+}(+k,\hat{\mathbf{p}}) \\ a_{2-}(-k,\hat{\mathbf{p}}) \\ a_{2+}(+k,\hat{\mathbf{p}}) \end{pmatrix}. \tag{3.150}$$

Here some redundant lower indices have already been dropped. One can read similar to the abbreviations $R^{\pm}, T^{\pm}$ defined in (3.33)

$$R^{s_i,j} := R^{s_i,j}_{s_i,j}, \qquad R^{s_i,j}_{s_l} := R^{s_i,j}_{s_l,j}, \tag{3.151}$$

$$T^{s_i,j} := T^{s_i,j}_{s_i,-j}, \qquad T^{s_i,j}_{s_l} := T^{s_i,j}_{s_l,-j}, \tag{3.152}$$

and for the barred components as well. Of course it becomes more difficult and longish to show the interpretation of these RT terms similar to (3.43)–(3.46). Nevertheless, they still are reflection and transmission amplitudes between the different generator and annihilator functions.

3.4.2 Boundary Condition

According to (3.5)–(3.7), the γ matrices in four dimensions (with mostly plus signature) are

$$\gamma^0 = i\begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \quad \gamma^1 = i\begin{pmatrix} 0 & -\mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \\ \gamma^2 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}. \tag{3.153}$$

Obviously, this convention differs from standard notations like the Dirac one. Nevertheless, it will not have any effect on the RT formalism.

After integrating the equation of motion over $x \in [-\varepsilon, \varepsilon]$ as above, we derive the boundary matrix M_- in four dimensions,

$$M_- = -\frac{1}{g + 4\,i\,S_1 - 4}\begin{pmatrix} g+4 & 0 & 4\,i\,\eta & -4(S_2 - i\,S_3) \\ 0 & g+4 & -4(S_2 + i\,S_3) & 4\,i\,\eta \\ -4\,i\,\eta & -4(S_2 - i\,S_3) & g+4 & 0 \\ -4(S_2 + i\,S_3) & -4\,i\,\eta & 0 & g+4 \end{pmatrix}. \tag{3.154}$$

which satisfies $\Psi_+ = M_-\Psi_-$.

3.4.3 Spin Solutions in Four Dimensions and RT Coefficients

The γ matrices given in (3.153) lead with $[\not{p} - i\,m]\,u_s \equiv 0 \equiv [\not{p} + i\,m]\,v_s$, and the abbreviation $N = \sqrt{2\,p_0\,(p_0 + k)}$ (where $p_0^2 = m^2 + k^2 + p_2^2 + p_3^2$) to

$$u_1(\mathbf{p}) = \frac{1}{N}\begin{pmatrix} m \\ -i\,p_2 + p_3 \\ p_0 + k \\ 0 \end{pmatrix}, \qquad u_2(\mathbf{p}) = \frac{1}{N}\begin{pmatrix} -i\,p_2 - p_3 \\ m \\ 0 \\ p_0 + k \end{pmatrix}, \tag{3.155}$$

and

$$v_1(\mathbf{p}) = \frac{1}{N}\begin{pmatrix} -m \\ -i\,p_2 + p_3 \\ p_0 + k \\ 0 \end{pmatrix}, \qquad v_2(\mathbf{p}) = \frac{1}{N}\begin{pmatrix} -i\,p_2 - p_3 \\ -m \\ 0 \\ p_0 + k \end{pmatrix}. \tag{3.156}$$

With equations (3.51) and (3.52) and appendix A, we compute the RT amplitudes,

$$\mathcal{N}\,R^{a,i}(k,\hat{\mathbf{p}}) = -\frac{4\left((-1)^a k(S_3p_2 - S_2p_3) + i\,p_0X(\hat{\mathbf{p}})\right)}{\sqrt{p_0^2 - k^2}\,[i(g+4)k + 4\,i\,X(\hat{\mathbf{p}})]}, \tag{3.157}$$

$$\mathcal{N}\,T^{i\pm}_j(k,\hat{\mathbf{p}}) = -\frac{k(g + 4\,i\,S_1 - 4)}{i(g+4)k + 4\,i\,X(\hat{\mathbf{p}})}, \tag{3.158}$$

$$\mathcal{N}\,R^{1\pm}_2(k,\hat{\mathbf{p}}) = \frac{4k\left(m\,(S_2 + i(-1)^a S_3) - \eta\,(p_2 + i(-1)^a p_3)\right)}{\sqrt{p_0^2 - k^2}\,[i(g+4)k + 4\,i\,X(\hat{\mathbf{p}})]}. \tag{3.159}$$

Here, we used the abbreviations

$$g := \eta^2 + S_1^2 + S_2^2 + S_3^2 = \eta^2 + S^2, \tag{3.160}$$

$$\mathcal{N} := ((g+4)k + 4\,i\,X)\sqrt{p_0^2 - k^2}, \tag{3.161}$$

$$X(\hat{\mathbf{p}}) := m\eta + S_2p_2 + S_3p_3. \tag{3.162}$$

3.4.4 Gibbs States and Currents

Stress tensor and Noether currents are again defined in (3.66) and (3.67) respectively. For even dimensions we may add a $\gamma^P = \pm\,i\prod_{l=0}^{d-1}\gamma^l$ so that the axial current reads

$$J^\mu_s(x)_A = :\overline{\Psi}(x)\,\gamma^P\gamma^\mu\,\Psi(x):. \tag{3.163}$$

We can deal with the resulting integrals the same way as before, as long as one does not have to take into account the additional sums over s_i, s_j. Especially the transformation of the expectation values for additional operators (3.68) does not change:

$$\overline{v}_s(k)\cdot v_s(k) \mapsto \overline{v}_s(k)\mathcal{O}v_s(k) \quad \text{and for } u_s(k) \text{ as well}, \tag{3.164}$$

$$\partial_0\, e^{\pm i\,\omega(k)t_2} \mapsto \pm\,i\,\omega(k)\,e^{\pm i\,\omega(k)t_2}. \tag{3.165}$$

The enhanced algebra (3.62) changes to

$$\sum_{\substack{s,t=1,2\\ j,l=\pm}} \theta(jx_1)\theta(lx_2)\left\{a_{t,l}(k_2),\, a^{s,j}(k_1)\right\} =$$
$$\begin{aligned} &= (2\pi)^{d-1}\sum_s \Big\{\, \delta(k_2-k_1)\,[\theta(x_1)\theta(x_2)+\theta(-x_1)\theta(-x_2)] + \\ &+ \delta(k_2-k_1)\left[\theta(x_1)\theta(-x_2)T^{s,+}(k_2)+\theta(-x_1)\theta(x_2)T^{s,-}(k_2)\right] + \\ &+ \delta(k_2+k_1)\left[\theta(x_1)\theta(x_2)R^{s,+}(k_1)+\theta(-x_1)\theta(-x_2)R^{s,-}(k_1)\right] + \\ &+ \sum_{t\neq s}\Big\{\delta(k_2-k_1)\left[\theta(x_1)\theta(-x_2)T^{s,+}_t(k_2)+\theta(-x_1)\theta(x_2)T^{s,-}_t(k_2)\right] + \\ &+ \delta(k_2+k_1)\left[\theta(x_1)\theta(x_2)R^{s,+}_t(k_1)+\theta(-x_1)\theta(-x_2)R^{s,-}_t(k_1)\right]\Big\}\Big\}. \end{aligned} \tag{3.166}$$

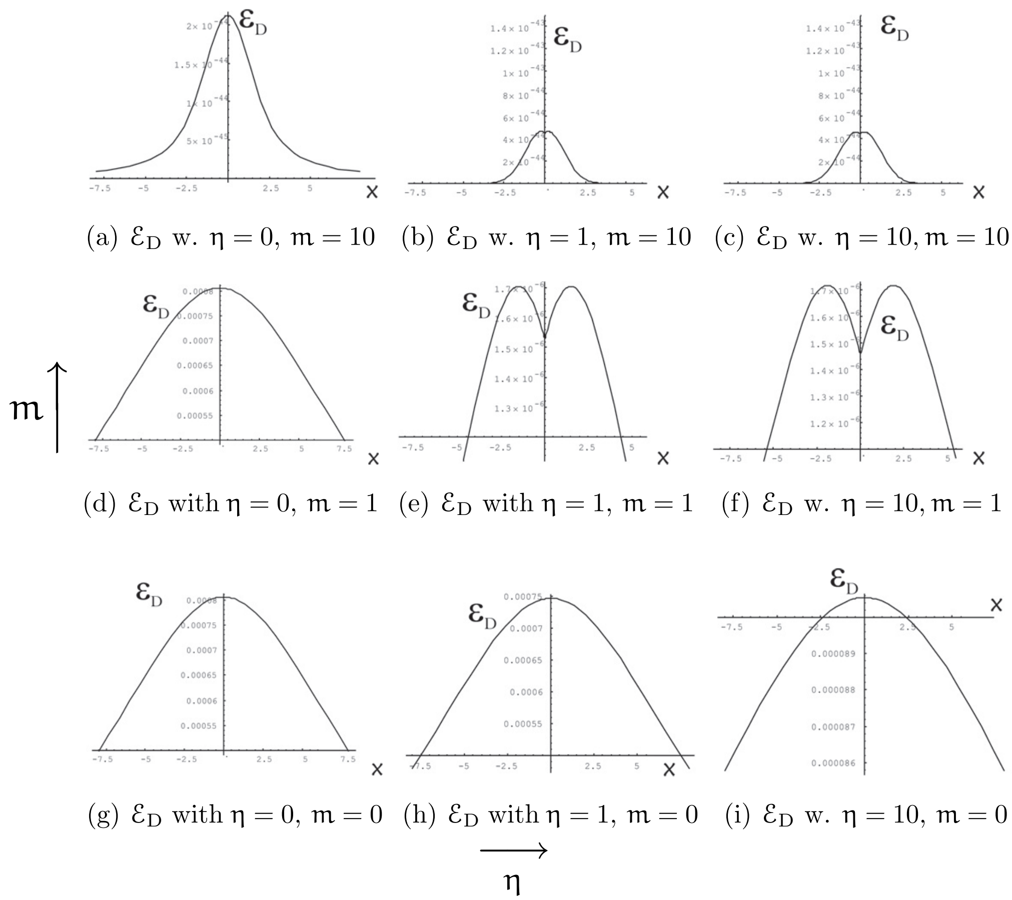
Energy density The stress tensor zero component is given by

$$\left\langle T^{00}(x)\right\rangle = \frac{i}{2}\left\langle :\overline{\Psi}(x)\,\gamma^0\partial^0\Psi(x) - \partial^0\overline{\Psi}\,\gamma^0\Psi:\right\rangle\Big|_{t_1=t_2} \tag{3.167}$$
$$= \sum_{s,t=1}^{2}\sum_{j,l=\pm}\iint_{-\infty}^{+\infty}\frac{d\mathbf{p}_1\,d\mathbf{p}_2\,[\omega(\mathbf{p}_1)+\omega(\mathbf{p}_2)]}{2^5\pi^4\sqrt{\omega(\mathbf{p}_1)\omega(\mathbf{p}_2)}}\,\frac{\theta(jx_1)\theta(lx_2)}{1+e^{\beta[\omega(\mathbf{p}_1)-\mu]}}\cdot$$
$$\cdot\Big[\left\{a_{t,l}(\mathbf{p}_2),\, a^{s,j}(\mathbf{p}_1)\right\}\overline{v}_s(\mathbf{p}_1)\gamma^0 v_t(\mathbf{p}_2)\,e^{-i\,\mathbf{p}_1x_1+i\,\mathbf{p}_2x_2}\,e^{i\,\omega(\mathbf{p}_1)t_1-i\,\omega(\mathbf{p}_2)t_2}+$$
$$+\left\{a^{t,l}(\mathbf{p}_2),\, a_{s,j}(\mathbf{p}_1)\right\}\overline{u}_s(\mathbf{p}_1)\gamma^0 u_t(\mathbf{p}_2)\,e^{i\,\mathbf{p}_1x_1-i\,\mathbf{p}_2x_2}\,e^{-i\,\omega(\mathbf{p}_1)t_1+i\,\omega(\mathbf{p}_2)t_2}\Big].$$

Using (3.166), setting $x_1 = x_2$, $t_1 = t_2$ and integrating with respect to $\mathbf{p}_2$ yields

$$\left\langle T^{00}(x)\right\rangle = \sum_s\iint_{-\infty}^{+\infty}\frac{dk\,d\hat{\mathbf{p}}}{4\pi^2}\,\frac{e^{-\beta[\omega(k,\hat{\mathbf{p}})-\mu]}}{1+e^{-\beta[\omega(k,\hat{\mathbf{p}})-\mu]}}\cdot\Big\{-2+e^{-2ikx}\cdot \tag{3.168}$$
$$\begin{aligned} &\cdot\Big\{\,[\theta(x)R^{s,+}(k,\hat{\mathbf{p}})+\theta(-x)R^{s,-}(k,\hat{\mathbf{p}})]\cdot\\ &\qquad\left[\overline{v}_s(k,\hat{\mathbf{p}})\gamma^0 v_s(-k,\hat{\mathbf{p}})+\overline{u}_s(-k,\hat{\mathbf{p}})\gamma^0 u_s(k,\hat{\mathbf{p}})\right]+\\ &+\sum_{t\neq s}\left[\theta(x)R^{s,+}_t(k,\hat{\mathbf{p}})+\theta(-x)R^{s,-}_t(k,\hat{\mathbf{p}})\right]\cdot\\ &\qquad\left[\overline{v}_s(k,\hat{\mathbf{p}})\gamma^0 v_t(-k,\hat{\mathbf{p}})+\overline{u}_t(-k,\hat{\mathbf{p}})\gamma^0 u_s(k,\hat{\mathbf{p}})\right]\Big\}\Big\}. \end{aligned}$$

Applying the upper expressions for R, u, v (3.157)–(3.159), (3.155)–(3.156), the sum over $t \neq s$ vanishes because the scalars of u and v sum up to zero.

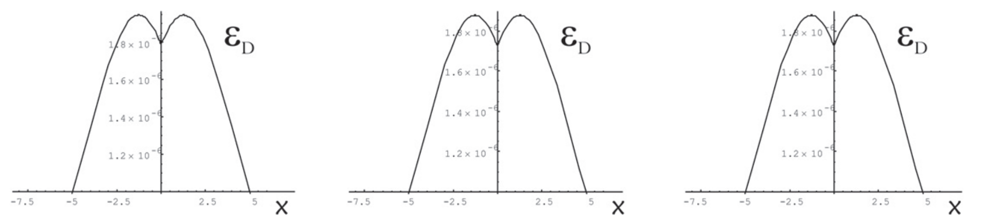


(a) $\mathcal{E}_D$ w. $\eta = 0$, $m = 10$ (b) $\mathcal{E}_D$ w. $\eta = 1$, $m = 10$ (c) $\mathcal{E}_D$ w. $\eta = 10, m = 10$

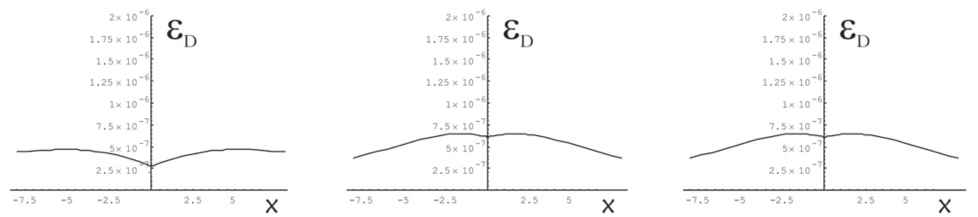
(d) $\mathcal{E}_D$ with $\eta = 0$, $m = 1$ (e) $\mathcal{E}_D$ with $\eta = 1$, $m = 1$ (f) $\mathcal{E}_D$ w. $\eta = 10, m = 1$

(g) $\mathcal{E}_D$ with $\eta = 0$, $m = 0$ (h) $\mathcal{E}_D$ with $\eta = 1$, $m = 0$ (i) $\mathcal{E}_D$ w. $\eta = 10$, $m = 0$

Figure 3.15: Defect energy density $\mathcal{E}_{D,\beta}(x)$ in four space-time dimensions for different parameters $\eta \geqslant 0$ and $m \geqslant 0$ (with fixed spin $S_1, S_2, S_3 = 1$) plotted as function of the distance from the defect x at the inverse temperature $\beta = 10$. In contrast to lower dimensions the numerical complexity increases so dramatically that we give an idea of the profile $\mathcal{E}_{D,\beta}(x)$ at $\beta = 10$. The behaviour for increasing temperature – i.e. decreasing β – is similar to the lower-dimensional cases and is illustrated in figures 3.5 and 3.11. In case mass m is very large against the defect mass η, the density $\mathcal{E}_{D,\beta=10}(x)$ vanishes numerically and thus we do not add such graphs here.

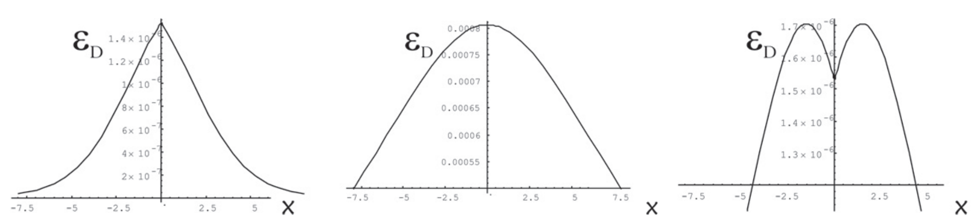


(a) $\mathcal{E}_D(x)$ with $S_1 = 1$, $S_2, S_3 = 0$
(b) $\mathcal{E}_D(x)$ with $S_2 = 1$, $S_1, S_3 = 0$
(c) $\mathcal{E}_D(x)$ with $S_3 = 1$, $S_1, S_2 = 0$



(d) $\mathcal{E}_D(x)$ with $S_1 = 10$, $S_2, S_3 = 0$
(e) $\mathcal{E}_D(x)$ with $S_2 = 10$, $S_1, S_3 = 0$
(f) $\mathcal{E}_D(x)$ with $S_3 = 10$, $S_1, S_2 = 0$

Figure 3.16: Defect energy density $\mathcal{E}_{D,\beta}(x)$ in four space-time dimensions for different parameters S_1, S_2, S_3, where $m, \eta = 1$ and $\beta = 10$ fixed. Since the spin should be a quantum object an exact adjustment into one direction will not be possible. However, setting components to zero is instructive in order to understand how changes influence the conserved quantities.
Obviously, the spin components S_2 and S_3 are equivalent whereas S_1 – the component perpendicular to the defect – plays a particular role.



(a) $\mathcal{E}_D(x)$ with $\eta = -1$
(b) $\mathcal{E}_D(x)$ with $\eta = 0$
(c) $\mathcal{E}_D(x)$ with $\eta = 1$

Figure 3.17: Defect energy density $\mathcal{E}_{D,\beta}(x)$ in four space-time dimensions for different parameters η, where $m = 1$, $S_i = 1$, and $\beta = 10$ fixed. Once more, the defect mass η is responsible for a local maximum ($\eta < 0$) or a local minumum ($\eta > 0$) at the postition of the defect $x = 0$. This indicates some bound states for $\eta < 0$, however, we mention once more that we did not consider any bound states explicitly. These could be added to the energy density in case $\eta < 0$.

Taking $\mu = 0$, we end up with

$$\left\langle T^{00}(x)\right\rangle =: \mathcal{E}_0 + \mathcal{E}_D, \tag{3.169}$$

$$\mathcal{E}_0 = \frac{4\pi}{\beta^3}\zeta(3), \tag{3.170}$$

$$\mathcal{E}_D(x) = \int_0^{+\infty}\iint_0^{\pi} \frac{dr\, d\varphi\, d\theta}{\pi}\frac{e^{-i r x \cos\varphi \sin\theta} r^2 \sin\theta}{1 + e^{\beta\sqrt{m^2+r^2}}}\cdot \tag{3.171}$$

$$\cdot \frac{m\eta + rS_3\cos\theta + rS_2\sin\varphi\sin\theta}{4rS_3\cos\theta - i\,\mathrm{sgn}(x)(g+4)r\cos\varphi\sin\theta + 4(m\eta + rS_2\sin\varphi\sin\theta)}. \tag{3.172}$$

If mass $m = 0$, the defect energy expression simplifies,

$$\begin{aligned}\mathcal{E}_{D,m=0}(x) &= \int_0^{+\infty}\iint_0^{\pi}\frac{dr\, d\varphi\, d\theta}{\pi(1+e^{\beta r})}\frac{e^{-i r x\cos\varphi\sin\theta} r^2(S_3\cos\theta + S_2\sin\varphi\sin\theta)}{4(S_3\cot\theta + S_2\sin\varphi) - i(g+4)\cos\varphi\,\mathrm{sgn}(x)} \\ &= \iint_0^{\pi}\frac{d\varphi\, d\theta}{4\beta^3\pi}\frac{i(S_3\cos\theta + S_2\sin\varphi\sin\theta)}{(g+4)\,\mathrm{sgn}(x)\cos\varphi + 4\,i(S_3\cot\theta + S_2\sin\varphi)}\cdot \\ &\quad\cdot\left(\zeta\left(3, \frac{i\,x\cos\varphi\sin\theta}{2\beta} + \frac{1}{2}\right) - \zeta\left(3, \frac{i\,x\cos\varphi\sin\theta}{2\beta} + 1\right)\right)\end{aligned} \tag{3.173}$$

The defect energy density $\mathcal{E}_D$ is evaluated numerically[2] in figures 3.15–3.17. Its properties are very similar the defect density in two dimensions.

Noether currents As above, the conserved Noether currents vanish in general,

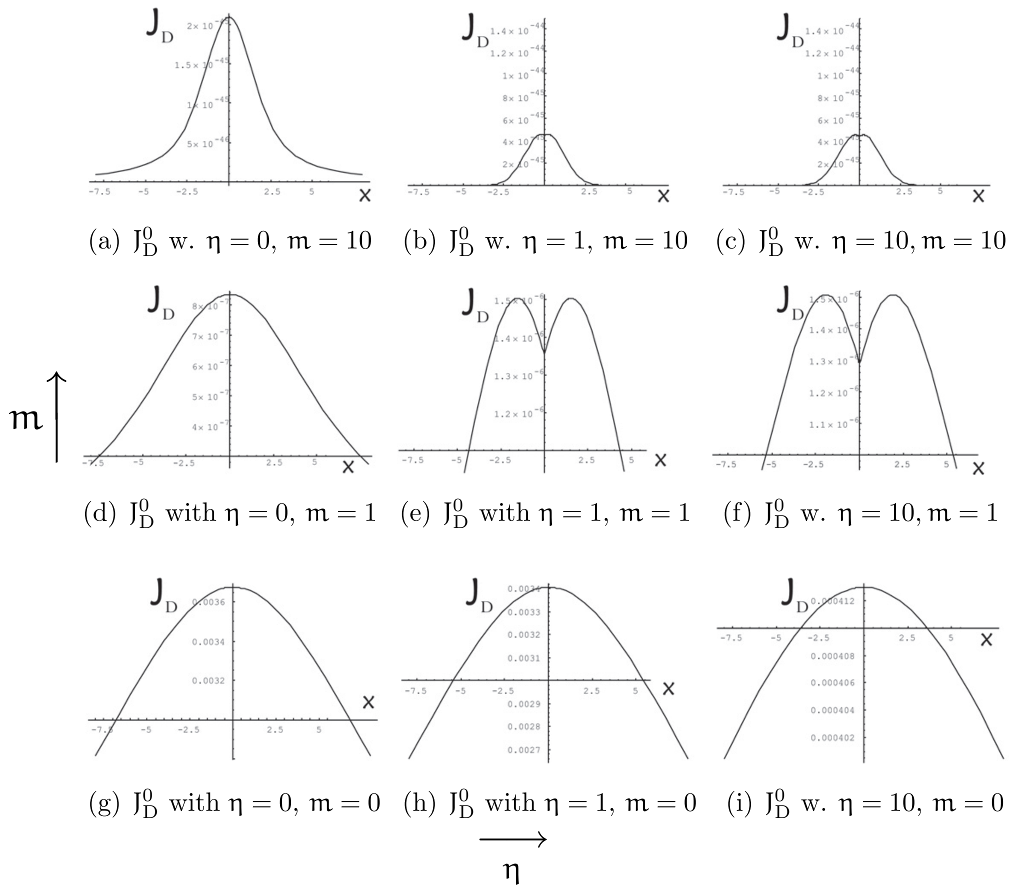
$$\langle J^{\mu}(x)\rangle = \left\langle :\overline{\Psi}(x)\, A^{\mu}(\gamma)\,\Psi(x):\right\rangle\Big|_{t_1=t_2} \tag{3.175}$$

$$= \quad 0 \qquad \Leftrightarrow \qquad A^{\mu} = \gamma^0, \gamma^1, \gamma^2, \gamma^3. \tag{3.176}$$

[2]Note that for numerical reasons, without explicit mention, some figures only show the real part of the generally imaginary integral. Since we are dealing here with measurable quantities, the imaginary parts of energy density as well as of the Noether currents should vanish automatically. In fact, this is what happens. But the numerical results for the imaginary parts by MATHEMATICA fluctuate in some cases so strongly that we had to ensure their vanishing by hand. Nevertheless, analytically and by plotting the imaginary parts of the integrand $F(\varphi)$ separately it is straightforward to check its antisymmetric behaviour around $\varphi_0 = 0, \pi$; i.e.

$$F(\varphi - \varphi_0) = -F(-\varphi - \varphi_0), \tag{3.174}$$

hence the integral over $\varphi \in [0, 2\pi]$ vanishes.



(a) J^0_D w. $\eta = 0$, $m = 10$ (b) J^0_D w. $\eta = 1$, $m = 10$ (c) J^0_D w. $\eta = 10, m = 10$

(d) J^0_D with $\eta = 0$, $m = 1$ (e) J^0_D with $\eta = 1$, $m = 1$ (f) J^0_D w. $\eta = 10, m = 1$

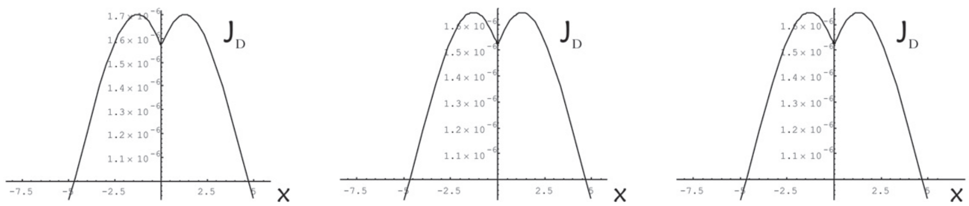
(g) J^0_D with $\eta = 0$, $m = 0$ (h) J^0_D with $\eta = 1$, $m = 0$ (i) J^0_D w. $\eta = 10$, $m = 0$

Figure 3.18: Fermionic charge density $\rho(x) \propto J^0_{L,e}$ in four space-time dimensions for different defect masses η plotted for varying x with $\beta = 10$ and $S_1, S_2, S_3 = 1$ fixed.

We again analyse left movers with positive mass exclusively for further comparison with those in two and three dimensions.

In analogy to the lower-dimensional cases, the parts containing $u(k)$ vanishes,

$$\begin{aligned}\left\langle J^{\mu}_{L,e}(x)\right\rangle &= \left\langle :\overline{\Psi}_{L,e}(x)\, A^{\mu}(\gamma)\, \Psi_{L,e}(x):\right\rangle \Big|_{t_1=t_2}\\ &= -\iint_{-\infty}^{+\infty} \frac{dk\, d\hat{p}}{8\pi\,\omega(k)} \frac{e^{-\beta[\omega(k)-\mu]}}{1+e^{-\beta[\omega(k)-\mu]}} \Big\{ \theta(+k)\overline{u}(k)A(\gamma)u(k) + \\ &\quad + e^{-2ikx}\,\overline{u}(-k)A(\gamma)u(k)\,\theta(-k)\,[\theta(x)R^{s+}(k) + \theta(-x)R^{s-}(k)] \Big\}.\end{aligned} \tag{3.177}$$



(a) J^0_D w. $S_1 = 1, S_2, S_3 = 0$ (b) J^0_D w. $S_2 = 1, S_1, S_3 = 0$ (c) J^0_D w. $S_3 = 1, S_1, S_2 = 0$

Figure 3.19: Fermionic charge density $\rho(x) \propto J^0_{L,e}$ in four space-time dimensions for different spin values S_1, S_2, S_3 plotted for varying x and β (and $m, \eta = 1$ fixed). The plots looks very similar, but as for the energy density in figure 3.16, the component S_1 perpendicular to the defect shows a slightly different influence.

With the definition (3.155), we derive

$$\left\langle T^{00}_{L,e}(x)\right\rangle = \frac{1}{4}\mathcal{E}_0 + \frac{1}{4}\mathcal{E}_D = \frac{1}{4}\left\langle T^{00}(x)\right\rangle , \tag{3.178}$$

$$\langle J^\mu_L(x)\rangle = J_0 - J_D \tag{3.179}$$

$$J_0 = \frac{\pi^2}{24\beta^2} \tag{3.180}$$

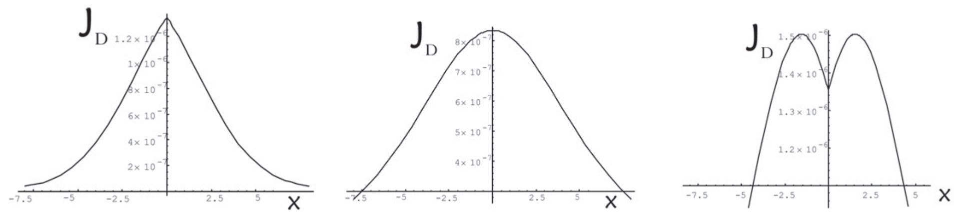
$$J_D = \int_0^{+\infty} \iint_0^{\pi} \frac{dr\, d\varphi\, d\theta}{\pi\sqrt{m^2+r^2}} \frac{e^{-i r x \cos\varphi \sin\theta} r^2 \sin\theta}{1+e^{\beta\sqrt{m^2+r^2}}} \cdot \\ \cdot \frac{m\eta + rS_3\cos\theta + rS_2\sin\varphi\sin\theta)}{(4rS_3\cos\theta - i(g+4)r\cos\varphi\,\mathrm{sgn}(x)\sin\theta + 4(m\eta + rS_2\sin\varphi\sin\theta))} . \tag{3.181}$$

As the mass m vanishes, the current satisfies

$$J_{D,m=0} = \int_0^{+\infty} \iint_0^{\pi} \frac{dr\, d\varphi\, d\theta}{\pi(1+e^{\beta r})} \frac{e^{-i r x \cos\varphi\sin\theta} r \sin\theta (S_3\cot\theta + S_2\sin\varphi)}{4(S_3\cot\theta + S_2\sin\varphi) - i(g+4)\cos\varphi\,\mathrm{sgn}(x)} \tag{3.182}$$

$$= \iint_0^{\pi} \frac{d\varphi\, d\theta}{4\beta^2\pi} \frac{S_3\cos\theta + S_2\sin\varphi\sin\theta}{4(S_3\cot\theta + S_2\sin\varphi) - i(g+4)\cos\varphi\,\mathrm{sgn}(x)} \cdot \\ \cdot \left[\psi^{(1)}\left(\frac{i x\cos\varphi\sin\theta}{2\beta} + \frac{1}{2}\right) - \zeta\left(2, \frac{i x\cos\varphi\sin\theta}{2\beta} + 1\right)\right] . \tag{3.183}$$

The four-dimensional defect energy and charge density exhibit analogous properties as those in two dimensions. Especially the influence of the defect mass



(a) $J^0_D(x)$ with $\eta = -1$ (b) $J^0_D(x)$ with $\eta = 0$ (c) $J^0_D(x)$ with $\eta = 1$

Figure 3.20: Fermionic charge density $\rho(x) \propto J^0_{L,e}$ in four space-time dimensions for different parameters η, where $m = 1$, $S_i = 1$, and $\beta = 10$ fixed. Once more, the defect mass η is responsible for a local maximum ($\eta < 0$) or a local minumum ($\eta > 0$) at the position of the defect $x = 0$. This is in complete similarity to the energy density (see figure 3.17) and indicates some bound states for $\eta < 0$.

sign on the local extrema is similar. Of course, as already in three dimensions, a complete vanishing of mass terms m and η does not force the defect densities to vanish. Thus, we can formulate an exclusively spin dependent RT theory in four dimensions. Furthermore, there is *no* symmetry breaking of the left-moving theory in contrast to the three-dimensional case.

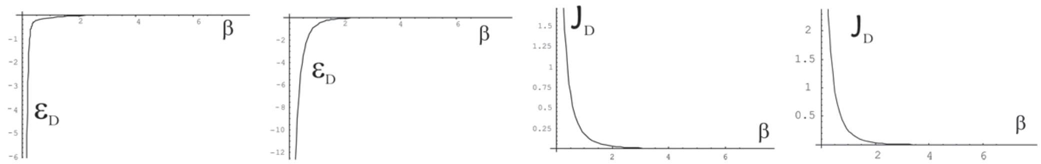
Comparing plots of defect energy densities $\mathcal{E}_D$ with charge densities J_D, the similarities are striking. However, they differ in their temperature dependence. For numerical reasons, we do not give the landscape plots here, but due to integrals (3.173) and (3.183), the energy density $\mathcal{E}_D$ diverges much faster than J_D as temperature increases, i.e. $\beta \to 0$ (see also figure 3.21).

Moreover, we should emphasise that the absolute value of the defect mass η does not affect the conserved quantities strongly. Apparently, the "direction" of the potential (the sign of η) denotes the fundamental quality.

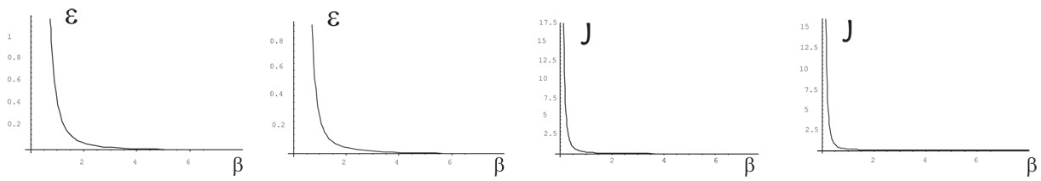
3.4.5 Comparison with Two and Three Dimensions

In four dimension as well, we are interested in the strong and weak coupling limits $S \to \infty$ and $S \to 0$. We here have to consider three spin components,

$$S = (S_1, S_2, S_3) = \lambda(s_1, s_2, s_3), \tag{3.184}$$



(a) $\mathcal{E}_{\mathrm{D}}(x)$ w. $\eta = -1$ (b) $\mathcal{E}_{\mathrm{D}}(x)$ w. $\eta = 1$ (c) $J^0_{\mathrm{D}}(x)$ w. $\eta = -1$ (d) $J^0_{\mathrm{D}}(x)$ w. $\eta = 1$



(e) $\mathcal{E}_{\beta}(x)$ w. $\eta = -1$ (f) $\mathcal{E}_{\beta}(x)$ w. $\eta = 1$ (g) $J^0_{\beta}(x)$ w. $\eta = -1$ (h) $J^0_{\beta}(x)$ w. $\eta = 1$

Figure 3.21: Energy and charge densities in four space-time dimensions for different sign of the defect mass η and varying inverse temperature β, but $S_1, S_2, S_3 = 1$ fixed: (a),(b) the pure defect part $\mathcal{E}_{\mathrm{D},\beta}(x)$; (c),(d) the pure defect part $\rho_{\mathrm{D}}(x) \propto J^0(x)$; (e),(f) the full energy density $\mathcal{E}(x) = \langle T^{00}(x)\rangle$ including the defect contribution $\mathcal{E}_{\mathrm{D}}$; (g),(h) the full charge density $\rho(x) \propto J^0(x)$ including the defect contribution J^0_{D}.

Note that according to our conventions, for deriving the full results, the defect energy density is added to the non-defect energy density whereas the defect charge density is subtracted from the non-defect charge density.

Obviously for the examined defect parameters $S_1, S_2, S_3 = 1$ and $\eta = \pm 1$, the defect parts $\mathcal{E}_{\mathrm{D}}$ and J_{D} – diverging as $\beta \to 0$ – are too small to induce a local density minimum for finite β in the full densities. It is still open whether for a special parameter set, such a minimum occurs.

where $|s_i| = \frac{1}{2}, 1, \frac{3}{2}, \dots$ are quantised spins with a finite but non-zero value. This implies

$$\lambda s = S \to 0 \quad \Leftrightarrow \quad \lambda \to 0, \quad \text{with } \lambda s_i = S_i, \tag{3.185}$$

$$S \to \infty \quad \Leftrightarrow \quad \lambda \to \infty. \tag{3.186}$$

Since $\Psi_+ = M_- \Psi_-$, we have with $\eta = 0$ (according to the interest in purely spin interactions related to [23]):

$$\underbrace{\Psi_+ = \frac{1}{\xi + \mathrm{i}\, S_1 - 2} \begin{pmatrix} -\xi & 0 & 0 & S_2 - \mathrm{i}\, S_3 \\ 0 & -\xi & S_2 + \mathrm{i}\, S_3 & 0 \\ 0 & S_2 - \mathrm{i}\, S_3 & -\xi & 0 \\ S_2 + \mathrm{i}\, S_3 & 0 & 0 & -\xi \end{pmatrix} \Psi_-}$$

$$\begin{array}{cc} \swarrow \lambda \to 0 & \searrow \lambda \to \infty \\ \Psi_- & -\Psi_- \\ \Updownarrow & \Updownarrow \\ \text{IR fixed point} & \text{UV fixed point} \end{array}$$

Here, we used $\xi := \frac{1}{4}(S^2 + 4) = \frac{\lambda^2}{4} s^2 + 1$.

Hence, we have the same boundary conditions in the UV and IR in all dimensions considered here. As already discussed for two dimensions we excluded bound states, whereas the one-impurity Kondo approach [23] deals with them as well. Nevertheless, after reducing this ansatz to an effectively two-dimensional theory, the energy shifts can be read off by comparing the current algebras with and without spin impurity. We claimed in section 3.2.6 that we implicitly considered these energy shifts by the RT ansatz since we supposed non-disturbed eigenstates of the defect system.

However, in view of the similarities of the conserved energy and charge densities in all dimensions, this suggests a generalisation of the one-impurity CFT approach [23]: by comparing current algebras for the defect and non-defect case it should be possible to derive energy shifts similar to the two-dimensional case for arbitrary dimensions. We leave this for future research.

3.5 Matrix Optics Construction for Fermionic Defect Theories

We already gave a brief idea of matrix optics in section 2.3. With the defect matrices we have a description of the linear deflection of some "optical elements". The unknown matrix in this picture is the propagation matrix that transforms the particle wave at the position x_1 into another position x_2 accounting the free propagating in between.

We emphasise once more that this ansatz stressing the analogy to geometrical optics is possible because the exact wave function propagates linearly and is decomposable into modes perpendicular and parallel to the defect. For lower-dimensional defects the latter is not necessarily true anymore. In these cases the matrix optics ansatz cannot be applied.

Single defect For a single defect described by Lagrangean (3.9) we already derived the boundary condition (3.12):

$$\Psi_+\Big|_0 = \left(i\gamma^x + \frac{1}{2}\,U\right)^{-1}\left(i\gamma^x - \frac{1}{2}\,U\right)\Psi_-\Big|_0 \tag{3.187}$$

$$=: M_- \cdot \Psi_-\Big|_0. \tag{3.188}$$

Like in the general bosonic case, for a single defect we have already a matrix depending on all the defect parameters and with non-zero entries at every position.

Two parallel defects For two parallel defects at positions x_1, x_2, we add a second defect term in the Lagrangean,

$$\mathcal{L} = \overline{\Psi}\,(i\partial\!\!\!/ + i\,m)\,\Psi + \delta(x_1)\overline{\Psi}\,U_1\,\Psi + \delta(x_2)\overline{\Psi}\,U_2\,\Psi, \tag{3.189}$$

where the different boundary conditions read (with $x_1 \geqslant x_2$ and the wave function index "0" denotes the interval $x \in [x_2, x_1]$ between the defects)

$$\Psi_+\Big|_{x=x_1} = \left(i\gamma^x + \frac{1}{2}\,U_1\right)^{-1}\left(i\gamma^x - \frac{1}{2}\,U_1\right)\Psi_-\Big|_{x=x_1} \tag{3.190}$$

$$=: M_{1-} \cdot \Psi_0\Big|_{x=x_1}, \tag{3.191}$$

$$\Psi_0\Big|_{x=x_2} = \left(i\gamma^x + \frac{1}{2}\,U_2\right)^{-1}\left(i\gamma^x - \frac{1}{2}\,U_2\right)\Psi_-\Big|_{x=x_2} \tag{3.192}$$

$$=: M_{2-}\cdot\Psi_-\Big|_{x=x_2}. \tag{3.193}$$

Similar to the bosonic case, the matrix P that describes the propagation has to fulfill the equivalent relations

$$\Psi_+\Big|_{x=x_1} = M_{1-}\cdot P(x_1,x_2)\cdot M_{2-}\cdot\Psi_-\Big|_{x=x_2}, \tag{3.194}$$

$$\Psi_0\Big|_{x=x_1} = P(x_1,x_2)\cdot\Psi_0\Big|_{x=x_2}. \tag{3.195}$$

We evaluate (3.195) where the wave function Ψ_0 is

$$\Psi_0(x) = \int_{-\infty}^{+\infty}\frac{dk}{2\pi\sqrt{2\omega(k)}}\;u_s(k)\,a^{s,0}(k)\;e^{i\,\omega(k)t-i\,kx} + v_s(k)\,a_{s,0}(k)\;e^{-i\,\omega(k)t+i\,kx}, \tag{3.196}$$

and $x\in[x_2,x_1]$ as stated above.

We allow the matrix P to be dependent not only on x_1-x_2, but on k as well and evaluate the integrand of $\Psi_0(x)$,

$$\varphi_0(k) = u_s(k)\,a^{s,0}(k)\;e^{i\,\omega(k)t-i\,kx} + v_s(k)\,a_{s,0}(k)\;e^{-i\,\omega(k)t+i\,kx}. \tag{3.197}$$

Since creator and annihilator separate, we derive eigenvalue equations for $P(x_1,x_2)$,

$$P(x_1,x_2)\cdot u_s(k) = e^{-i\,k(x_1-x_2)}\,u_s(k), \tag{3.198}$$

$$P(x_1,x_2)\cdot v_s(k) = e^{+i\,k(x_1-x_2)}\,v_s(k). \tag{3.199}$$

With the abbreviations $Y(k) := \sin[k(x_1-x_2)]$ and $Z(k) := \cos[k(x_1-x_2)]$ we solve directly for two dimensions,

$$P(k,x_1-x_2) = \begin{pmatrix} Z(k) & \frac{i(k-p_0)}{m}Y(k) \\ \frac{i(k+p_0)}{m}Y(k) & Z(k)\end{pmatrix}, \tag{3.200}$$

for three dimensions,

$$P(k,x_1-x_2) = \begin{pmatrix} Z(k)-\frac{p_2}{m}Y(k) & \frac{i(k-p_0)}{m}Y(k) \\ \frac{-i(k+p_0)}{m}Y(k) & Z(k)+\frac{p_2}{m}Y(k)\end{pmatrix}, \tag{3.201}$$

for four dimensions,

$$P(k, x_1 - x_2) = \begin{pmatrix} Z(k) & -\frac{p_2 - i p_3}{m} Y(k) & \frac{i(k-p_0)}{m} Y(k) & 0 \\ -\frac{p_2 + i p_3}{m} Y(k) & Z(k) & 0 & \frac{i(k-p_0)}{m} Y(k) \\ -\frac{i(k+p_0)}{m} Y(k) & 0 & Z(k) & \frac{p_2 - i p_3}{m} Y(k) \\ 0 & -\frac{i(k+p_0)}{m} Y(k) & \frac{p_2 + i p_3}{m} Y(k) & Z(k) \end{pmatrix}. \tag{3.202}$$

For the continuity of the wave function, it is necessary that

$$\det P = 1, \tag{3.203}$$

$$P = \mathbb{1} \qquad \Leftarrow \qquad x_1 = x_2. \tag{3.204}$$

Moreover, this ensures the "vanishing" of $P(k, x_1 - x_2)$ in the product (3.195) as soon as the defects merge (at $x_1 = x_2$) because in that case there is no propagation between them anymore.

The generalisation for $n + 1$ defects yields (where $x_1 \geqslant x_2 \geqslant x_3 \geqslant \ldots \geqslant x_{n+1}$) for the boundary condition

$$\Psi_+\Big|_{x=x_1} = \int_{-\infty}^{+\infty} \frac{dk}{2\pi\sqrt{2\omega(k)}} \left[\prod_{i=1}^{n} M_{i,-} P(k, x_i - x_{i+1})\right] M_{i+1,-}\, \varphi_-(k)\Big|_{x_{n+1}}. \tag{3.205}$$

Certainly, at this point we can not give the reflection and transmission amplitudes explicitly via a general formula for all dimensions. Nevertheless, it is not difficult but straightforward to calculate the composite boundary matrix,

$$\widetilde{M}_- = \left[\prod_{i=1}^{n} M_{i,-} P(k, x_i - x_{i+1})\right] M_{i+1,-} \overset{2D}{=}: \frac{1}{4 - \widetilde{g} - 4\,i\,\widetilde{S}} \begin{pmatrix} \widetilde{g} + 4 & 4\,i\,\widetilde{\eta} \\ -4\,i\,\widetilde{\eta} & \widetilde{g} + 4 \end{pmatrix}, \tag{3.206}$$

and compare the entries with those of the single defect boundary matrix and thus identify the values $\widetilde{S}_i, \widetilde{\eta}$ of the composite boundary matrix $\widetilde{M}$ (and still $\widetilde{g} = \widetilde{\eta}^2 + \widetilde{S}^2$). The reflection and transmission amplitudes are then simply the single defect amplitudes where the single defect parameters S_i, η were replaced by $\widetilde{S}_i$ and $\widetilde{\eta}$, respectively.

Hence with the RT formalism we are able to give all the properties of a composite defect system by determining the "single defect" parameters of the composite system and subsequently treating it like a single defect, as we have already described in detail.

Une rose pour qu'il pleuve. Au terme d'innombrables années, c'est ton souhait.

(Eine Rose, auf daß es regne. Nach zahllosen Jahren ist dieses dein Wunsch.)

René Char, A la santé du serpent

4
Summary and Outlook

In physics, boundaries and defects emerge in many areas: as spatial boundary, as defects in the lattice structure or as single impurities, they attract interest in condensed matter physics. Recently they also appeared in field theories describing the nature of elementary particles.

In particular, the magnetic impurities in solid states (i.e. the Kondo effect) are of great interest. One of its descriptions builds a bridge between both solid state physics and elementary particle theory by using two-dimensional conformal field theory (CFT). Investigation and characterisation of CFT is a most important topic in string theory and even more for the AdS/CFT correspondence. On the one hand, two-dimensional CFT with boundaries is very helpful in understanding the one-impurity Kondo effect. On the other hand, it draws attention as dual theory of a supergravity in an AdS background.

Of course, conformal symmetry is not a necessary characteristic of most solid state models. Accordingly, the RT algebra ansatz [62] gives a possible quantum field theory description of defects without requiring conformal symmetry.

However, most of the mentioned research was done in two dimensions, for several reasons: four dimensional theories are usually so complicated that it is difficult to calculate exact results, whereas in two dimensions many exact solutions are known. Furthermore, two-dimensional CFT exhibit an infinite number of conserved quantities (contrary to a finite number in higher dimensions) and thus nearly all physically relevant quantities can be given in dependence on few parameters (conformal dimensions and central charge). In addition, the two-dimensional world-sheet of a string is conformal.

For these reasons, very few of the previous models have been considered in higher dimensions. In the AdS/CFT correspondence, higher-dimensional defect theories have been introduced by adding an additional D5 brane [18]. Likewise, the RT ansatz [62; 63] has been developed for arbitrary dimensions.

In this work, we implemented fermionic degrees of freedom in the general RT framework dealing with higher-dimensional non-conformal (and thus more general) defect theories. In particular, we determined the algebra elements in general – namely the reflection and transmission amplitudes – from initial parameters which are the model-dependent boundary parameters. For all space-time dimensions up to four, we calculated the energy and charge densities, which dependent *only* on generic expressions of reflection and transmission amplitudes. We introduced a model Lagrangean which contained a defect mass η as well as interactions with spin components S_i. We studied the influence on these conserved quantities – namely energy and charge densities – for different values of our defect parameters η and S_i. By doing so, we showed that the sign of the defect mass η has specific importance and determines the local behaviour around the defect: a negative η induces a local minimum of the defect densities at the position of the defect $x = 0$ whereas positive values of η generate a local maximum (which is exactly opposite to the behaviour in the bosonic defect theory). We showed the similarity of the defect density plots in all dimensions and observed a symmetry breaking for left-moving currents in three dimensions. We gave evidence for phase transitions at finite temperature at least in two dimensions and showed consistency with the well-known two-dimensional model for the Kondo effect [23]. Moreover, we discussed an option deriving energy shifts in four dimensions by CFT.

For simplicity, we restricted our analysis by not taking into account conformal symmetry as well as bound states. Considering bound states – using the developed formalism – would be a next step towards the characterisation of Kondo lattices; implementing conformal theory will provide us with a prescription of AdS-dual CFT with defects. Furthermore, we examined exclusively defects of co-dimension one, i.e., $d-1$-dimensional defects in d-dimensional theories. We motivated this choice by the canonical Fourier decomposition into plane waves which move either perpendicular or parallel to the defect of co-dimension one. For co-dimension two (and higher dimensions analogously), there are two directions perpendicular to the defect. Therefore, the integration of the equation of motion over a small interval around the position of the defect has to be modified. A natural way to do this was to consider spherical waves and derive a boundary condition similar to that induced by defects with co-dimension one. In such a framework the absolute value of the reflection amplitude has to be one and thus will be fixed up to a phase. Nevertheless, a thus modified theory should be consistent. Unfortunately, the matrix optics ansatz for more than one impurity we discussed in sections 2.3 and 3.5 will not hold for such a construction anymore. It is not obvious to us how this ansatz could be modified and adapted to lower-dimensional defects, i.e. defects of higher co-dimension. Another possibility was to interpret some $d-2$-dimensional defects in a $d-1$-dimensional hyperplane as $d-1$-dimensional defect with varying wave function values on that defect; in other words, considering an additional induced potential on that $d-1$-dimensional defect. Yet a solution to this problem is not in sight, but non-constant potentials on the defect would also be very interesting for understanding correlations parallel to the defect. The currents $\left\langle J^{\mu}_{L,e}(x)\right\rangle$ for $\mu = 1, 2, \ldots$ which vanish if no potential is present (see sections 3.3.4 and 3.4.4) should exhibit a non-zero value for such an additional potential.

Examining the Kondo lattice and the CFT approach by [1], we expected a higher-dimensional CFT with less symmetry – hence additional central charges – to be in one-to-one correspondence to the additional parameters of the lattice. Thus, by considering the symmetry breaking from the one-impurity Kondo model to the lattice model, we hope to find a valid description of Kondo lattices by higher-dimensional conformal field theories. The consideration of ordinary field theories with defects is a first step into this direction. Based on the results of this thesis, we believe it possible to apply conformal symmetry to the presented RT formalism now. Doing so would restrict the non-CFT approach but also shed light on the proposed relation between Kondo lattice structure and CFT with defects in higher dimensions.

The most interesting question then is whether the two pictures match: on the one hand, the lower-dimensional defects interpreted as one higher-dimensional defect with additional potential, on the other hand, the additional conformal symmetry that is expected to map to the Kondo lattice.

Moreover, combining the RT approach with conformal symmetry leads to the possibility of further decribing defect CFT within the AdS/CFT correspondence. In particular, the calculation of the energy density at the defect is expected to lead to deeper insight into the localisation of gravity on the probe brane in the dual gravity theory.

Recently, another development within the framework of AdS/CFT – the so-called Janus models [19; 20] – attracted attention in relation to RT algebras. These Janus models are non-supersymmetric deformations of the AdS_5 geometry with AdS_4 defect that exhibit on the field theory side a CFT with defect, where both sides show different coupling constants. In fact, the coupling constants respect

$$\frac{1}{g^2} = \frac{\theta(x)}{g_+^2} + \frac{\theta(-x)}{g_-^2}, \tag{4.1}$$

where the action is

$$S = \int dx \, \frac{1}{g^2} \mathcal{L}'(x). \tag{4.2}$$

Moreover, the different coupling constants on different sides of the defect break the continuity of the Lagrangean at $x = 0$ explicitly. Obviously, this fits very well in the RT formalism on boundaries done by [61]. On the other hand, it is in contradiction to our continuity claim (ii) on page 47. However, we can reformulate the Lagrangean (3.9) by

$$\mathcal{L} = \overline{\Psi} \left(i \not{\partial} + i\, m \right) \Psi + \delta(x) \overline{\Psi} \left[U_+ \theta(x) + U_- \theta(-x) \right] \Psi, \tag{4.3}$$

and thus

$$\Psi_+\big|_0 = \left(i\gamma^x + \frac{1}{2}\, U_+ \right)^{-1} \left(i\gamma^x - \frac{1}{2}\, U_- \right) \Psi_-\big|_0. \tag{4.4}$$

Accordingly, the boundary matrix M_- will not necessarily be unitary. The consequences – especially regarding the influence of conformal symmetry – are now to be developed in detail.

A third model most interesting in relation to RT formulation including conformal symmetry is the D3-D3 brane intersection model on a $\mathbb{C}^2/Z_k$ orbifold in the infrared limit. The interaction can be described by M5 branes [11; 12]. This is analogous to NS5-NS5-D4 box models considered in [72] which denote M5-M5-M5 intersections in M theory and are described by three-cycles (the orbifold background in this case is $\mathbb{C}^3/\Gamma$). Such three-cycles are of interest since they encode the holographic information of an $\mathcal{N} = 1$ QFT, similarly to Seiberg-Witten curves [73] for $\mathcal{N} = 2$.

For all the future research outlined above, we hope to have provided a basis by the present work.

Les larmes méprisent leur confident.

(Verachtet wird von den Tränen, wer um sie mitweiß.)

René Char, A la santé du serpent

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The chapter quotations were taken from [74], German translation by Paul Celan.

Produis ce que la connaissance veut garder secret, la connaissance aux cent passages.

(Was die Erkenntnis verborgen zu halten bestrebt ist, bring es hervor. Sie, die Erkenntnis, die sich hundertfach Bahn bricht.)

René Char, A la santé du serpent

A

Explicit Determination of RT Coefficients From Boundary Matrices

In this appendix, we provide a short and thus efficient way to deduce RT amplitudes from boundary data; i.e. by the boundary matrix M_- defined in (3.12) as

$$M_- = \left(i\gamma^x + \frac{1}{2}\,U\right)^{-1}\left(i\gamma^x - \frac{1}{2}\,U\right), \tag{A.1}$$

where the general operator U mediates the defect interaction via (3.9),

$$\mathcal{L} = \overline{\Psi}\,(i\partial\!\!\!/ + i\,m)\,\Psi + \delta(x)\overline{\Psi}\,U\,\Psi. \tag{A.2}$$

To simplify matters, in our notation we drop the dependency on momenta $\hat{\mathbf{p}}$ parallel to the defect which stays unspoiled by our transformations.

The spin solutions $u_s(\mathbf{p}) \equiv u_s(k)$ and $v_s(\mathbf{p}) \equiv v_s(k)$ satisfy (3.32),

$$(p\!\!\!/ - i\,m)\,u_s(k) \equiv 0, \tag{A.3}$$

$$(p\!\!\!/ + i\,m)\,v_s(k) \equiv 0, \tag{A.4}$$

and are unique except for a phase. Furthermore, from the boundary condition (3.12),

$$\Psi_+ = M_- \Psi_-, \tag{A.5}$$

we derived decomposed boundary conditions (3.51), (3.52) for the Fourier modes $a^{s\,\pm}(k)$, $a^s_\pm(k) := a_{s,\pm}(k)$ (the index shift is implemented in order to sum according to Einstein's rule) of the wave function $\Psi(x)$,

$$v_s(k)\, a^s_+(k) + v_s(-k)\, a^s_+(-k) = M_- \left[v_s(-k)\, a^s_-(-k) + v_s(k)\, a^s_-(k)\right], \tag{A.6}$$

$$u_s(k)\, a^{s+}(k) + u_s(-k)\, a^{s+}(-k) = M_- \left[u_s(-k)\, a^{s-}(-k) + u_s(k)\, a^{s-}(k)\right]. \tag{A.7}$$

In addition, the exchange algebra conditions (2.25), (2.26) read

$$a^{t,-i}(p)\overline{T}^{s,i}_t(p) + a^{t,i}(-p)\overline{R}^{s,i}_t(-p) = a^{s,i}(p), \tag{A.8}$$

$$a_{t,-i}(p)T^{t,-i}_s(p) + a_{t,i}(-p)R^{t,i}_s(-p) = a_{s,i}(p). \tag{A.9}$$

We are exclusively interested in $R^{s,j}(\pm k)$ and $T^{s,j}(\pm k)$, hence the annihilator relations (A.6) and (A.9). The barred components are just complex conjugated components. Therefore, hereinafter equations (A.7) and (A.8) will be ignored.

Moreover, we know that all of the spinors $v_i(k), v_i(-k)$ are linearly independent from each other in general. Therefore, we can decompose the eigenstates of the boundary matrix M_- as

$$M_- v_s(k) \;= \lambda^{t-}_s(k)\, v_t(-k) + \lambda^{t+}_s(k)\, v_t(k), \tag{A.10}$$

and accordingly,

$$M_- v_s(-k) = \lambda^{t-}_s(-k)\, v_t(k) + \lambda^{t+}_s(-k)\, v_t(-k), \tag{A.11}$$

here using that the boundary matrix M_- is *not* dependent on momentum k. Equations (A.10), (A.11) are defining the linear coefficients $\lambda^{j\pm}_i$. As soon as U and thus M_- are defined explicitly, these coefficients $\lambda^{j\pm}_i$ can be computed directly.

Implementing equations (A.10) and (A.11), the boundary condition (A.6) reads

$$\begin{aligned} v_s(k)\, a^s_+(k) + v_s(-k)\, a^s_+(-k) = \; & \left[\lambda^{t-}_s(-k)\, v_t(k) + \lambda^{t+}_s(-k)\, v_t(-k)\right] a^s_-(-k) + \\ & + \left[\lambda^{t-}_s(k)\, v_t(-k) + \lambda^{t+}_s(k)\, v_t(k)\right] a^s_-(k) \end{aligned} \tag{A.12}$$

Since relation (A.9) defines reflection and transmission coefficients in terms of generators and annihilators, we just have to transform equation (A.12) to read off these RT coefficients purely in terms of linear coefficients $\lambda_i^{j\pm}$. This can be carried out directly by MATHEMATICA, for instance. In order to show that the result is unique, some further steps have to be taken.

Since the phase of the spin modes $\nu_s(k)$, $\nu_s(-k)$ should not affect reflection and transmission terms for reasons of consistency, we should not need their explicit form. Moreover, with $\widetilde{\nu}_s(k) := [\nu_s(k)^*]^T$, there holds the orthogonality relation

$$\widetilde{\nu}_s(k) \cdot \nu_t(k) = \delta_{st}. \tag{A.13}$$

Multiplying in relation (A.12) $\widetilde{\nu}_s(k)$ and $\widetilde{\nu}_s(-k)$ from the left yields two independent conditions for every single index s,

$$\begin{aligned} a_+^s(k) = & \left[\lambda_s^{s-}(-k) + \lambda_s^{t+}(-k)\,\widetilde{\nu}_s(k)\cdot\nu_t(-k)\right] a_-^s(-k)+, \qquad \text{(A.14)} \\ & + \left[\lambda_s^{t-}(k)\,\widetilde{\nu}_s(k)\cdot\nu_t(-k) + \lambda_s^{s+}(k)\right] a_-^s(k) \\ a_+^s(-k) = & \left[\lambda_s^{t-}(-k)\,\widetilde{\nu}_s(-k)\cdot\nu_t(k) + \lambda_s^{s+}(-k)\right] a_-^s(-k)+. \qquad \text{(A.15)} \\ & + \left[\lambda_s^{s-}(k) + \lambda_s^{t+}(k)\,\widetilde{\nu}_s(-k)\cdot\nu_t(k)\right] a_-^s(k) \end{aligned}$$

It is straightforward to check that the scalars $\widetilde{\nu}_s(k)\cdot\nu_t(-k)$ are not dependent on the phase of $\nu_s(k)$ anymore. Thus we are left with relations exclusively depending on the coefficients $\lambda_s^{t\pm}(k)$. We could extend this approach to arbitrary space-time dimensions, but only state the explicit results for four dimensions here. The lower-dimensional results just reproduce (3.58),(3.59) or (3.116), (3.117), respectively.

Four dimensions The matrix expression according to (3.150),

$$\begin{pmatrix} a_{1-}(+k) \\ a_{1+}(-k) \\ a_{2-}(+k) \\ a_{2+}(-k) \end{pmatrix} = \underbrace{\begin{pmatrix} R^{1-}(-k) & T^{1+}(k) & R_1^{2-}(-k) & T_1^{2+}(k) \\ T^{1-}(-k) & R^{1+}(k) & T_1^{2-}(-k) & R_1^{2+}(k) \\ R_2^{1-}(-k) & T_2^{1+}(k) & R^{2-}(-k) & T^{2+}(k) \\ T_2^{1-}(-k) & R_2^{1+}(k) & T^{2-}(-k) & R^{2+}(k) \end{pmatrix}}_{=:\,\mathcal{M}} \cdot \begin{pmatrix} a_{1-}(-k) \\ a_{1+}(+k) \\ a_{2-}(-k) \\ a_{2+}(+k) \end{pmatrix}, \tag{A.16}$$

already gets out of hand. We transform (A.14) and (A.15) to read off the RT coefficients which are very long in this form. But since

$$\lambda_1^{2+} = 0 = \lambda_2^{1+}, \tag{A.17}$$

these expressions simplify. The result is then

$$\mathcal{M} = \frac{1}{L}\begin{pmatrix} \lambda_1^{1-}(-k)\lambda_2^{2+}(k) & -\lambda_2^{2+}(k) & \lambda_2^{1-}(-k)\lambda_2^{2+}(k) & 0 \\ F_1(k) & -\lambda_1^{1-}(k)\lambda_2^{2+}(k) & F_3(k) & -\lambda_2^{1-}(k)\lambda_1^{1+}(k) \\ \lambda_1^{2-}(-k)\lambda_1^{1+}(k) & 0 & \lambda_2^{2-}(-k)\lambda_1^{1+}(k) & -\lambda_1^{1+}(k) \\ F_2(k) & -\lambda_1^{2-}(k)\lambda_2^{2+}(k) & F_4(k) & -\lambda_2^{2-}(k)\lambda_1^{1+}(k) \end{pmatrix}, \tag{A.18}$$

with

$$L = -\lambda_1^{1+}(k)\lambda_2^{2+}(k), \tag{A.19}$$

$$F_1(k) = L\lambda_1^{1+}(-k) + \lambda_1^{2-}(-k)\lambda_2^{1-}(k)\lambda_1^{1+}(k) + \lambda_1^{1-}(-k)\lambda_1^{1-}(k)\lambda_2^{2+}(k), \tag{A.20}$$

$$F_2(k) = \lambda_1^{2-}(-k)\lambda_2^{2-}(k)\lambda_1^{1+}(k) + \lambda_1^{1-}(-k)\lambda_1^{2-}(k)\lambda_2^{2+}(k), \tag{A.21}$$

$$F_3(k) = \lambda_2^{1-}(k)\lambda_2^{2-}(-k)\lambda_1^{1+}(k) + \lambda_1^{1-}(k)\lambda_2^{1-}(-k)\lambda_2^{2+}(k), \tag{A.22}$$

$$F_4(k) = \lambda_2^{2-}(-k)\lambda_2^{2-}(k)\lambda_1^{1+}(k) + \lambda_1^{2-}(k)\lambda_2^{1-}(-k) - L\lambda_2^{2+}(-k). \tag{A.23}$$

The non-vanishing linear coefficients $\lambda_s^{t\pm}(k)$ read

$$\lambda_1^{1+}(k) = -\frac{(g+4)k + 4\,i\,X(k)}{k(g+4\,i\,S_1-4)}, \tag{A.24}$$

$$\lambda_2^{2+}(k) = -\frac{(g+4)k + 4\,i\,X(k)}{k(g+4\,i\,S_1-4)}, \tag{A.25}$$

$$\lambda_1^{1-}(k) = \frac{4(i\,Xp_0 + kS_3p_2 - kS_2p_3)}{k(g+4\,i\,S_1-4)\sqrt{p_0^2-k^2}}, \tag{A.26}$$

$$\lambda_2^{2-}(k) = \frac{4(i\,Xp_0 - k(S_3p_2 - S_2p_3))}{k(g+4\,i\,S_1-4)\sqrt{p_0^2-k^2}}, \tag{A.27}$$

$$\lambda_2^{1-}(k) = \frac{4\eta(p_2 + i\,p_3) - 4m(S_2 + i\,S_3)}{(g+4\,i\,S_1-4)\sqrt{p_0^2-k^2}}, \tag{A.28}$$

$$\lambda_1^{2-}(k) = \frac{4\eta(p_2 - i\,p_3) - 4m(S_2 - i\,S_3)}{(g+4\,i\,S_1-4)\sqrt{p_0^2-k^2}}, \tag{A.29}$$

where $g = \eta^2 + S_1^2 + S_2^2 + S_3^2$ and $X(k) = m\eta + p_2S_2 + p_3S_3$ as defined in (3.160), (3.162). This gives us the RT amplitudes (3.157)–(3.159).

We would like to stress once more that this determination algorithm for RT amplitudes can be easily implemented into our numerical calculations whereas other approaches failed.

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René Char, A la santé du serpent

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