Characterization of intrinsically harmonic forms

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To

Olga Danilova

Zusammenfassung.

Wir beweisen, dass für eine geschlossene 1-Form auf einer geschlossenen, orientierten, zusammenhängenden differenzierbaren Mannigfaltigkeit, intrinsische Harmonizität gleichbedeutend ist mit Transitivität und lokaler intrinsischer Harmonizität. Wir untersuchen beide Eigenschaften getrennt. Wir betrachten Morse 1-Formen, die auf dem Rand des Inneren der Menge nicht-transitiver Formen bezüglich der C^1 Topologie liegen. Wir zeigen, dass die Kern-Blätterung einer solchen 1-Form mindestens ein singuläres geschlossenes Blatt hat, das mehr als eine Nullstelle der Form enthält. Für die Betrachtung der lokalen intrinsischen Harmonizität beschränken wir uns auf den Fall von Manifaltigkeiten der Dimension zwei. Es stellt sich heraus, dass die Frage nach lokaler intrinsischer Harmonizität dann gleichbedeutend ist mit einer Frage aus der Singularitäten-Theorie differenzierbarer Funktionen. Wir geben ein Kriterium dafür, dass eine differenzierbare Funktion auf $\mathbb{R}^2$ in der Nähe von $(0, 0)$ diffeomorph äquivalent zu ihrem Term höchster Ordnung ist, unter der Annahme, dass $(0, 0)$ ein isolierter kritischer Punkt der Funktion ist.

Summary.

We prove that for a closed 1-form on a closed oriented connected smooth manifold intrinsic harmonicity is equivalent to transitivity together with local intrinsic harmonicity. Then we study the two properties separately. We consider Morse 1-forms which lie on the boundary of the interior of the set of nontransitive forms with respect to the C^1 topology. We show that such a 1-form has at least 1 singular closed leaf of its kernel foliation containing more than one zero of the form. For local intrinsic harmonicity we restrict our attention to the case of dimension two. Then it turns out that the question of local intrinsic harmonicity is equivalent to a question from the singularity theory of smooth functions. We give a criterion for a smooth function on $\mathbb{R}^2$ to be diffeomorphically equivalent to its leading order term near $(0, 0)$, assuming that $(0, 0)$ is an isolated critical point for the function.

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Chapter 1

Introduction

1.1 History of intrinsic characterization of harmonic forms.

Let X be a connected orientable n-dimensional manifold without boundary, equipped with an orientation and a Riemannian metric g. Having these two structures at hand and given any $k = 0, 1, ..., n$ we can define the Hodge-star operator — a bundle map between the bundle of exterior k-forms and the bundle of exterior $(n-k)$-forms

$$\star_g : \Lambda^k(T^\star X) \longrightarrow \Lambda^{n-k}(T^\star X).$$

The precise definition will be given in Section 2.1. The Hodge-star operator on bundles induces a map on forms — sections of these bundles:

$$\Omega^k(X) \longrightarrow \Omega^{n-k}(X),$$

denoted by the same symbol $\star_g$. The Hodge-star operator on forms allows us to introduce the co-differential $d_g^\star : \Omega^k \longrightarrow \Omega^{k-1}$, by the formula $d_g^\star = (-1)^{kn+n+1} \star_g d \star_g$.

We define the Dirac operator $(d + d_g^\star) : \Omega^\star(X) \longrightarrow \Omega^\star(X)$. We can square the Dirac operator to get the Laplace operator $(d + d_g^\star)^2$.

If X is compact, then for any $k = 0, 1, ..., n$ we give the space $\Omega^k(X)$ an L^2-inner product $(,)_g$ by setting $(\alpha, \beta)_g = \int_M \alpha \wedge \star_g \beta$. The co-differential $d_g^\star$ happens to be the formal adjoint to the exterior derivative $d : \Omega^k \longrightarrow \Omega^{k+1}$ under this inner product.

We will be mainly interested in developing theory on manifolds that are closed, i.e. compact without boundary. In these cases our manifold will be usually denoted by M. But it is useful (e.g. for local analysis on M) to have some of the main concepts defined also on open manifolds, i.e. noncompact without boundary. In those cases, when the manifold under consideration is not necessarily assumed to be compact, it will be denoted by X.

Definition 1. *Let (M, g) be a closed oriented Riemannian manifold. A k-form α on it is called harmonic if it belongs to the kernel of the Laplace operator.*

Note that we define the notion of harmonicity only for compact manifolds. In this case the kernels of Laplace and the Dirac operators coincide. Indeed if $(d+d^\star)^2\alpha = 0$, then $(dd^\star\alpha+d^\star d\alpha, \alpha)_g = 0$ and hence $(d^\star\alpha, d^\star\alpha)_g+(d\alpha, d\alpha)_g = 0$, so $(d+d^\star)\alpha = 0$. Therefore harmonic forms α are exactly those which are simultaneously closed ($d\alpha = 0$) and co-closed ($d_g^\star\alpha = 0$).

It is crucial to note that the closedness of the form is a property which depends only on the smooth structure of the manifold M, whereas the co-closedness depends on the Riemannian metric — the additional structure we put on M. So a form only has a chance to be harmonic if it is closed and then it may or may not be harmonic depending on what Riemannian metric we put on M.

Note that if we change the orientation of X, then the Hodge-star $\star_g$ operator will change its sign, but the co-differential $d_g^\star = (-1)^{kn+n+1} \star_g d\star_g$ will not. This means that for the discussion of co-closedness or harmonicity the choice of the orientation on X is irrelevant, but it is important that X is orientable, for the construction of $\star_g$ to work. For the sake of determinacy we give M an orientation once and forever.

Definition 2. *A closed k-form α on a closed manifold M is called intrinsically harmonic if there exists a Riemannian metric g on M, which makes it harmonic.*

The following natural problem arises: to give an intrinsic characterization of intrinsically harmonic forms. For $k = 0$ these are constant functions and for $k = n$ these are exactly the volume forms. The question turns out to be more subtle in the intermediate degrees. The following definitions will help us to separate easier cases from more difficult ones.

Definition 3. *A k-form α on a manifold M is said to have nondegenerate zeros (or to be generic) if considered as a section of the bundle $\Lambda^k T^\star M$ it is transverse to the zero section.*

For functions to have nondegenerate zeros simply means to have nonzero differentials at their zeros. For forms of degree satisfying $1 < k < n-1$ to have nondegenerate zeros means to have no zeros at all.

Definition 4. *A closed 1-form α with nondegenerate zeros is called Morse.*

For a closed 1-form α to have a nondegenerate zero at p is the same as for its local primitive function f (usually normalized by $f(p) = 0$) to have a Morse-type singularity at this point. What makes life much easier in the case of Morse singularities is the Morse Lemma, which says that a function is diffeomorphically equivalent to a constant plus the algebraic sum of squares near a Morse singularity.

For closed forms with nondegenerate zeros a complete characterization of intrinsic harmonicity was given in degree 1 by Calabi in 1969 and in degree $n-1$ by Honda in 1996, cf. [5] and [11]. To give a unified formulation for 1 and $(n-1)$-forms, we need one more definition.

Definition 5. *A closed k-form α is called locally intrinsically harmonic if there exists a neighbourhood U of its zero set and a Riemannian metric g_U on U, such that $d\star_{g_U}\alpha = 0$.*

Theorem 1 (Calabi [5], Honda [11]). *Let $k \in \{1, n-1\}$. For a closed generic k-form α on a closed oriented connected n-manifold M to be intrinsically harmonic it is necessary and sufficient that*
(a) the form α is locally intrinsically harmonic and
(b) the form α is transitive.

Here transitive means that there exists a closed k dimensional submanifold N_p through every point p from the complement to the zero set of ω such that $\alpha|_{N_p}$ is a volume form on N_p. An immediate observation is that for 1-forms local intrinsic harmonicity is simply a condition on the Morse indices of the zeros: for local intrinsic harmonicity it is necessary and sufficient that there are no zeros of index 0 or n. Since transitivity implies the absence of zeros of index 0 and n, we get the following version of the above theorem for 1-forms.

Theorem 2 (Calabi [5]). *A closed Morse* 1*-form* ω *on a closed oriented connected* n*-manifold* M *is intrinsically harmonic if and only if it is transitive.*

For the discussion of local intrinsic harmonicity in the case of $(n-1)$-forms the reader is referred to the thesis of Honda cf. [11].

For a closed 1-form transitivity is a property of its kernel foliation. For a general account on both classical results and recent advances in the field see Farber's book [7].

Very little seems to be known about the question of intrinsic characterization in other degrees. Let us make a couple of remarks, illustrating potential difficulties. The simplest case of a form of a higher degree would be a 2-form on a 4-manifold. A generic 2-form on a 4-manifold does not have any zeros at all. So let α be a nowhere zero closed 2-form on a 4-manifold. Moreover assume α has constant rank. For dimension reasons we have only two possibilities for the rank of α — 2 or 4. In the last case the form α is symplectic, and therefore is harmonic for any metric g which is compatible with α. The question of intrinsic harmonicity is answered trivially and positively in this case. So the only potentially interesting case is when α has constant rank 2. It turns out that this case presents serious difficulties. The following example was suggested to the author by J. Latschev. This example shows, that transitivity is not sufficient for harmonicity.

Example 1. Let M be the total space of the nontrivial S^2-bundle $\xi = (S^2 \longrightarrow M \xrightarrow{\pi} S^2)$ over S^2. It is easy to see that there exists a section s of ξ through every point of M. Let $dvol_{S^2}$ be a volume form on the base S^2 and set $\alpha := \pi^\star dvol$. The form α is a closed 2-form of constant rank 2 on the 4-dimensional manifold M, where the fibers of ξ are the leaves of the 2-dimensional kernel foliation of α. Sections of ξ provide closed 2-dimensional submanifolds of M to which α restricts as a volume form, so α is transitive. But α is not (!) intrinsically harmonic. Assume by contradiction, that there exists a Riemannian metric g on M such that the form $\psi := \star_g \alpha$ is closed. The form ψ has constant rank 2 and the leaves of the kernel foliation of ψ are transverse to those of α, i.e. to the fibers of ξ. Take any leaf $\mathcal{L}$ of the kernel foliation of ψ. The restriction $\pi_{\mathcal{L}} : \mathcal{L} \longrightarrow S^2$ is a submersion and therefore for dimension reasons a covering map. So $\mathcal{L}$ is diffeomorphic to S^2. So the total space M of ξ admits a foliation by closed leaves transverse to the fibers with every leaf intersecting every fiber exactly once contradicting the nontriviality of ξ.

This tells us that for generic closed 2-forms of constant rank 2 on 4-manifolds transitivity does not imply intrinsic harmonicity. Whether or not intrinsic harmonicity implies transitivity for such forms is not clear at the moment. The relationship between transitivity and intrinsic harmonicity in this case is a subject for future work.

So from now on we restrict our attention to the case of 1-forms. A fairly straightforward argument shows that transitive closed 1-forms form an open set in the set of all Morse forms with respect to the C^1 topology. That is for a Morse form transitivity survives under C^1 small perturbations. As an immediate consequence of this observation and Theorem 2 we get

Theorem 3. *Intrinsically harmonic* 1*-forms on a closed manifold constitute an open set with respect to the* C^1 *topology in the set of Morse forms.*

We close this section by raising a question: how much of this remains true if we do not assume the zeros of ω to be nondegenerate? It will be answered in part later in this chapter.

1.2 Remarks on the notation.

Throughout the paper M denotes a closed smooth manifold of dimension n and ω a closed 1-form on it. We let S denote the zero set $\{p \in M | \omega(p) = 0\}$ of ω and $\mathcal{F}$ denote the restriction of the (singular) kernel foliation of ω to its regular set $M \setminus S$. So $\mathcal{F}$ is a regular foliation on a possibly noncompact manifold. Very often (e.g. when perturbing ω or considering sequences of forms converging to ω) we have to consider another closed form on the manifold M. It may be denoted $\tilde{\omega}$ or ω_m. Then its zero set will be denoted by $\tilde{S}$ respectively S_m and the regular part of the kernel foliation by $\tilde{\mathcal{F}}$ respectively $\mathcal{F}_m$.

When working globally on M we use the letter n only to denote the dimension of M. When discussing issues completely unrelated to M (say we discuss something happening near the origin in $\mathbb{R}^2$ and there is no M entering the discussion at all) we felt free to make an occasional use of n to denote things like the degree of a polynomial or an induction parameter. This should not cause a confusion.

We write $H_\star(M)$ to denote the singular homology of M with integer coefficients: $H_\star(M, \mathbb{Z})$. We tried not to overuse this convention and write

$H_\star(M,\mathbb{Z})$ explicitly in places requiring the usage of other coefficient rings as well.

We never explicitly use the Laplace operator $(d+d_g^\star)^2$, so we do not give it a special name. But we do use the following operator on functions:

$$\triangle_g : C^\infty(X) \longrightarrow \Omega^n(X),$$

which converts a smooth function f into a top degree form $\triangle_g f := d \star_g df$. The operator $\triangle_g$ will be called the Laplace-Beltrami operator.

If a function belongs to the kernel of the Laplace-Beltrami operator, then we call it harmonic. On compact manifolds this notion coincides with the previously defined (considering a function as a 0-form).

1.3 Transitivity versus nontransitivity.

In this section we work in the space of Morse forms with respect to the C^1 topology. As was remarked at the end of Section 1.1 for Morse forms transitivity survives under C^1 small perturbations. As a highlight for Chapter 4 we take up the following question: what happens to nontransitivity under C^1 small perturbations? Clearly, given a Morse 1-form on the boundary of the set of transitive forms (which is the same as the boundary of the set of nontransitive forms), there exists a small C^1-perturbation which makes the form transitive, i.e. destroys nontransitivity. It is tempting to assert that such boundary forms have some special properties concerning their kernel foliation. Assume for simplicity that a nontransitive 1-form ω_1 has integral cohomology class and assume that it is joined to a transitive 1-form ω_0 by a path $\{\omega_t\}_{t\in[0,1]}$ of closed Morse forms within its cohomology class. It is always possible to find such a path, see the paper by K. Honda [12]. The cohomology class being constant within the deformation crucially simplifies the subsequent discussion. For every $t \in [0,1]$ we consider the leaf space Γ_t for the (singular) kernel foliation of ω_t. That is Γ_t is obtained from M by collapsing the leaves (singular or nonsingular) of the kernel foliation of ω_t to points. The cohomology class $[\omega_t]$ being integral implies that Γ_t is a Hausdorff space. Moreover, Γ_t is a directed graph, where every edge is directed according to the increase of the local primitive function of ω. Let us see how many edges we have at every vertex. First, there are no zeros of ω_t of index 0 or n, because ω_0 does not have such zeros due to transitivity, and then as t increases from 0 to 1 to create such a zero we would have to leave the space of Morse forms. Assume now that for some $t \in [0,1]$ no two zeros of a form ω_t of index 1 or $n-1$ lie on one singular closed leave. Such a form ω_t is called non-heteroclinic (a more precise definition of non-heteroclinicity will be given in Section 4.2). Then every zero of ω_t of index 1 or $n-1$ when projected to Γ_t becomes a vertex where 3 edges come together. The zeros of ω_t of index greater than 1, but smaller than $n-1$ do not give rise to vertices of the graph Γ_t. We say that a directed graph Γ is Calabi if there exists a closed positive path through every edge. Here "positive path" means a path which goes along every edge in the positive direction. Note that ω_t being transitive/nontransitive exactly corresponds to the graph Γ_t being Calabi/non-Calabi. Deforming a nontransitive form ω_1 to a transitive form ω_0 comes down to deforming a non-Calabi graph Γ_1 to a Calabi graph Γ_0, i.e. changing the homotopy type of the graph. It means, that we can not perform such a homotopy through non-heteroclinic forms: there should exist a t_0 such that ω_{t_0} is heteroclinic (i.e. there is a pair of zeros of ω_{t_0} of index 1 or $n-1$ which lie on one singular closed leaf of the kernel foliation of ω_{t_0}). Passing through ω_{t_0} as our t increases means passing from transitivity to non-transitivity. This discussion suggests that a 1-form which lies on the boundary between transitive and nontransitive forms should be heteroclinic. Before we give a precise version of the theorem we were able to prove, we would like to discuss one subtle issue. In the formulation of this we need to consider the interior of the set of nontransitive forms in the space of Morse forms with respect to the C^1 topology. Those Morse forms which have zeros of index 0 or n trivially belong to the interior of the set of nontransitive forms, but it is not a priori clear that this interior contains at least one Morse form without zeros of index 0 or n. In Section 4.5 we give two examples of manifolds and Morse forms on them without zeros of index 0 or n which belong to the interior of the set of nontransitive forms. See also Section 6.4.

Theorem 4. *Let ω be a Morse form that belongs to the closure of the interior of the set of nontransitive forms in the space of Morse forms with respect to the C^1 topology. Assume that ω is non-heteroclinic. Then ω belongs to the interior of the set of nontransitive forms.*

We give the main ideas for the proof. It is useful to work with a slightly modified definition of transitivity: instead of asking to have a closed transversal through every point in the complement to the zero set of ω we ask that every two points from this complement can be joined by an ω-positive path. The two definitions turn out to be equivalent.

Since ω is a C^1-limit of nontransitive forms, it is also a nontransitive form. It is not very difficult to show (see characterization theorems in Chapter 4, more precisely Theorem 17), that nontransitivity is equivalent to the existence of a set of singular closed leaves $\mathcal{P}_1, ..., \mathcal{P}_l$ of the kernel foliation of ω, which represent the zero in $H_{n-1}(M)$. More precisely, $[\mathcal{P}_1] + ... + [\mathcal{P}_l] = 0$, where $[\mathcal{P}_i]$ denotes the image of $\mathcal{P}_i$ in $H_{n-1}(M)$ and the co-orientation of $\mathcal{P}_i$ is the direction of the decrease of the local primitive function of ω. A rough geometric idea why such a collection obstructs transitivity is that for homology reasons $\mathcal{P}_1, ..., \mathcal{P}_l$ separates M into two parts: "inside" and "outside". Now points from the "inside" can not be joined to the points in the "outside" by ω-positive paths, because such a path would have to cross the boundary $\mathcal{P}_1 \cup ... \cup \mathcal{P}_l$ in the "wrong" direction — recall the co-orientation of $\mathcal{P}_i$. Consider an open neighbourhood U_i of $\mathcal{P}_i$, which retracts to $\mathcal{P}_i$. Clearly, $\omega|_{U_i}$ is exact. Consider a closed Morse 1-form $\tilde{\omega}$ sufficiently C^1-close to ω.

Assume for the moment, that $\tilde{\omega}|_{U_i}$ is exact (this is not easy obtain). Because $\mathcal{P}_i$ contains not more than 1 zero of ω of index 1 or $n-1$, we find a singular closed leaf $\tilde{\mathcal{P}}_i$ of $\tilde{\omega}$ near $\mathcal{P}_i$ carrying the same element in homology as $\mathcal{P}_i$, i.e. $[\tilde{\mathcal{P}}_i] = [\mathcal{P}_i]$. This means that $[\tilde{\mathcal{P}}_1] + ... + [\tilde{\mathcal{P}}_l] = 0$, i.e. the set $\tilde{\mathcal{P}}_1, ..., \tilde{\mathcal{P}}_l$ by the characterization theorem (see Chapter 4 Theorem 17) obstructs transitivity of $\tilde{\omega}$. Once we are able to perform this for any closed Morse 1-form sufficiently C^1 close to ω, we get that ω belongs to the interior of the set of nontransitive forms.

The key problem here is that we have assumed $\tilde{\omega}|_{U_i}$ to be exact. One way to get the exactness of $\tilde{\omega}|_{U_i}$ would be to have proportionality between the cohomology classes of ω and $\tilde{\omega}$, that is $[\tilde{\omega}] = c[\omega]$ for some $c \in \mathbb{R}$. In the case of $b_1(M) = 1$ this is automatically true. In the case $b_1(M) > 1$ the above proportionality fails in general and we have to work harder. The idea is to ensure that the image of $\mathcal{P}_i$ in $H_1(M, \mathbb{Z})$ consists of torsion elements. For this we first assume for simplicity that $H_1(M, \mathbb{Z})$ has no torsion and then C^1-approximate ω by a sequence $\{\omega_m\}_{m \in \mathbb{N}}$ of nontransitive closed 1-forms, each ω_m lying in the interior of the set of nontransitive forms. The case when $H_1(M, \mathbb{Z})$ has torsion requires a bit more of notation and is dealt with in Section 4.4. We view ω_m as a homomorphism

$$[\omega_m] : H_1(M, \mathbb{Z}) \longrightarrow \mathbb{R},$$

or, tensoring everything with $\mathbb{R}$, as a homomorphism

$$[\omega_m]_{\mathbb{R}} : H_1(M, \mathbb{R}) \longrightarrow \mathbb{R}\,.$$

Consider the kernel $Ker[\omega_m]$ respectively $Ker[\omega_m]_{\mathbb{R}}$ of the homomorphism $[\omega_m]$ respectively $[\omega_m]_{\mathbb{R}}$. Clearly, $Ker[\omega_m]_{\mathbb{R}}$ is a codimension 1 linear subspace in the $\mathbb{R}$-linear space $H_1(M, \mathbb{R})$ and $Ker[\omega_m] = Ker[\omega_m]_{\mathbb{R}} \cap H_1(M, \mathbb{Z})$, where $H_1(M, \mathbb{Z})$ is viewed the integer lattice in $H_1(M, \mathbb{R})$. The set of codimension 1-subspaces missing the integer lattice completely is dense in the set of all codimension 1-subspaces, in our case it means that by perturbing ω_m slightly in the C^1 topology, we can achieve that

$$Ker[\omega_m] = Ker[\omega_m]_{\mathbb{R}} \cap H_1(M, \mathbb{Z}) = \{0\}.$$

Since ω_m was in the interior of the set of nontransitive forms, it remains nontransitive after a small perturbation. Let $\mathcal{P}_m$ be a singular closed leaf of ω_m. Now $Ker[\omega_m] = \{0\}$ implies that the image of $\mathcal{P}_m$ in $H_1(M, \mathbb{Z})$ is trivial. Let the set $\mathcal{P}_m^1, ..., \mathcal{P}_m^l$ of singular closed leaves of ω_m which obstruct transitivity of ω_m as in the characterization theorem. If we can establish that the singular closed leaf $\mathcal{P}_m^i$ of ω_m "converges" to a singular closed leaf $\mathcal{P}_i$ of ω as $m \to \infty$ in any reasonable sense, then we are done, because convergence of singular closed leaves implies stabilization of their images in $H_1(M, \mathbb{Z})$ and therefore we get a set $\mathcal{P}_1, ..., \mathcal{P}_l$ of singular closed leaves of ω obstructing transitivity of ω and with the property that the image of $\mathcal{P}_i$ in $H_1(M, \mathbb{Z})$ is trivial, finishing the argument as explained above. Unfortunately, justification of "convergence" is not so easy. In general closed leaves of ω_m do not have to converge to closed leaves of ω when $\omega_m \longrightarrow \omega$. For a baby example illustrating the difficulties consider the 2-torus with angular coordinates ϕ, θ on it and set $\omega = d\phi$ and $\omega_m = d\phi + r_m d\theta$, where $\{r_m\}_{m \in N}$ is a sequence of rational numbers, converging to zero. Closed leaves of ω_m in this example do not converge to closed leaves of ω in any reasonable sense and the reason is that the cohomology classes of ω_m vary. Establishing the convergence is the heart of the whole argument. This is explained in Section 4.3. The key idea is to establish certain upper bound on the diameters of singular closed leaves of ω_n which are of interest for us.

1.4 General zero sets.

So far we have assumed all the zeros of ω to be nondegenerate, in other words ω to be Morse. Things get much more exciting if we relax the nondegeneracy of zeros of ω. First of all transitivity alone does not imply harmonicity as we shall see later. So one can not hope for a direct analog of Theorem 2. Nevertheless, Theorem 1 repeated verbatim (see Theorem 7) remains true for closed 1-forms with arbitrary zero sets. This is the content of Chapter 3. The proof works exactly like Calabi's with two technical modifications that we briefly describe below, see Chapter 3 for details.

The first modification occurs when we go from intrinsic harmonicity to transitivity. We need to replace the geometric argument of Calabi (which implicitly uses nondegeneracy of the zeros of ω) by an argument closer to dynamical systems in spirit. Namely, the Hodge dual $\star_g\omega =: \psi$ of ω is a closed $(n-1)$-form on M transverse to ω, meaning that $\omega \wedge \psi > 0$ (with respect to a distinguished volume form *dvol* which orients our manifold) on the complement of the (common) zero set S of ω and ψ. Define a vector field X on M by the equation $i_X dvol = \psi$. Since X spans the kernel of ψ, it is transverse to the kernel foliation of ω away of S. By the Cartan formula the vector field X preserves the volume form *dvol*. Now an easy argument based on the Poincaré-recurrence theorem applied to X gives us closed transversals to the kernel foliation to ω, verifying transitivity.

The second modification occurs when we go from transitivity plus local intrinsic harmonicity to intrinsic harmonicity. At the last stage we end up with a closed $(n-1)$-form ψ'', which serves as a transversal to ω in some neighbourhood U of S (that is $\omega \wedge \psi|_{U\setminus S} > 0$) and a closed form ψ', which serves as a transversal to ω away of S and vanishes near S. Roughly speaking the proof is then concluded by gluing ψ'' and ψ' to give a global closed $(n-1)$-form ψ, serving as a transversal to ω. Then we define the desired Riemannian metric g by declaring it to make ω and ψ orthogonal to each other. The slight technical subtlety in gluing the forms ψ'' and ψ' is that we need the form ψ'' to be exact near S for the gluing to work out correctly. This is of course clear when S is a discrete set of points. For general S we use local intrinsic harmonicity of ω to say that S is the zero set of a solution to a first order elliptic partial differential equation. Then we apply the result by C. Bär [3] to conclude that S is contained in at most a countable union of submanifolds of codimension 2. The Countable sum Theorem tells us that the covering dimension of S is at most $n-2$. Just like de-Rham cohomology is zero above the dimension of a manifold, Čech cohomology is zero above the covering dimension. In our case it gives us that the $(n-1)$-st Čech cohomology of S is zero. The continuity property of Čech cohomology, cf. [4] (Section 14 "Continuity", Theorem 14.4) implies that after shrinking U if necessary, we can assume that ψ'' is exact.

1.5 Questions from singularity theory.

Chapter 5 is devoted to the study of local intrinsic harmonicity. Recall that with nondegenerate zeros, the question whether the form is locally intrinsically harmonic or not is answered in purely topological terms: the Morse index of ω at a nondegenerate zero p (determined by the principal part of ω at p) should be different from 0 and n. Allowing general zeros makes the question of local intrinsic harmonicity almost untractable. This is mainly because the answer starts depending not only on the principal part of ω near a (degenerate) zero p, but also on higher order terms and, unfortunately, in a complicated way. To make things easier we restrict to the case $n = 2$. The above cited theorem by Bär in this case amounts to the zero set S being discrete. So we can assume that we work on $\mathbb{R}^2$ near the origin — the unique zero of $\omega = df$. Normalize $f(0,0) = 0$. If ω is co-closed with respect to some Riemannian metric, then f is harmonic with respect to this metric and therefore is a real part of a holomorphic function, where holomorphic has to be understood with respect to the complex structure induced by the metric. So after a coordinate change, we can achieve that $f = Re(x+iy)^m$ for some integer $m \geq 2$. So we see that local intrinsic harmonicity for df is the same as for f to be equivalent to $Re(x+iy)^m$ under some coordinate change locally around the origin. Therefore, the question is reduced to the following problem from the theory of singularities of smooth functions: under what conditions can the function $f = Re(x+iy)^m + h.o.$ be brought to the form $Re(x+iy)^m$ by a smooth change of coordinates around zero? Here $h.o.$ denotes the terms of order higher than m. The answer is given by the following theorem, which is the highlight of Chapter 5.

Theorem 5. *Let $f = Re(x+iy)^m + h.o.$ be a function defined on an open ball around the origin in $\mathbb{R}^2$, where h.o. denotes the terms in the Taylor expansion around the origin of order higher than m. Then for $m = 1, 2, 3, 4$ the function f can always be brought to the form $Re(x+iy)^m$ by a smooth change of coordinates in some open ball around zero. For $m > 4$, the sufficient condition for such a coordinate transformation to exist is that h.o. starts with the order $2m-3$ or higher.*

We pause to give a little history. In late 50th early 60th J.Nash and

J. Moser developed a theory which enables one to write clever implicit function theorems in the smooth category. The theory grew out of the famous Nash embedding theorem from 1956. In 1968 Samoilenko (cf. [17]) used this theory to prove a fundamental result in singularity theory. Every smooth function near an isolated critical point of finite order is equivalent under some coordinate change to its truncated Taylor series at this point. This is a qualitative result: we know the function is equivalent to a polynomial and to give an upper bound on the degree of the polynomial is a separate question. In 1972 Arnold (cf. [1]) used a clever “Lie algebraic” trick, which together with the result by Samoilenko gives a very efficient criterion (Lemma 3.2 in [1]) to decide that a function is equivalent a given polynomial. In particular it allows us to decide when a function is equivalent to its leading term. Theorem 5 essentially follows from Arnold's criterion by means of an elementary algebra trick, which uses that we are in dimension 2.

We develop an alternative approach to Theorem 5 which does not rely on the work of Nash, Moser, Samoilenko and Arnold. It uses the equivalence between the above question from singularity theory and the harmonicity of f around zero, and proceeds as follows.

Step 1: we write out the equation

$$\triangle_g f = 0 \tag{1.1}$$

in coordinates and view it as a singular nonlinear first order partial differential equation for g.

Step2: we insert the formal Taylor power series for g in (1.1) and do a power series argument (Section 5.2). We arrive at a system of linear equations in every degree. Solvability of this linear system is a crucial point. In Section 5.3 we upgrade the elementary algebra trick by bringing Cauchy-Riemann operators into play to establish solvability of linear systems arising from the power series argument. This gives us the formal solution to Equation (1.1) near the origin. A classical result from analysis going back to the early 20-th century (see for example the paper by Mirkil [16]) tells us that there exists an actual Riemannian metric g having the Taylor power series as prescribed by the formal solution above. The metric g will satisfy Equation (1.1) up to an error exponentially small near the origin. This is done in Section 5.4.

Step 3: we correct the Riemannian metric g in an exponentially small fashion to make it satisfy (1.1) without any error at all. This is done in Section 5.5. Closer examining the power series argument for the case $m =$

2 leads us to the following result, which seems unaccessible by "classical" methods from singularity theory.

Theorem 6. *Let $f = Re(x + iy)^2 + h.o.$ and $h = Im(x + iy)^2 + h.o.$ be smooth functions on an open ball around $(0, 0)$ in $\mathbb{R}^2$ where the notation h.o. stands for the terms of order 3 and higher. Then there exists a Riemannian metric g which makes f harmonic on some open ball around zero and $\triangle_g h$ is exponentially small around zero. Moreover the Taylor expansion at zero for the conformal structure induced by g is uniquely determined.*

Of course, we would like to have $\triangle_g h$ equal to zero, not just exponentially small, but it is not clear how to make the final step (Section 5.5) work for both f and h simultaneously.

Chapter 6 is a logical continuation of the previous one. We consider a smooth function f on $\mathbb{R}^2$ with $f(0, 0) = 0$ and $df_{(0,0)} = 0$. Let $m \geq 2$ be the order of its leading power in Taylor expansion around the origin. We ask exactly the same question as we asked in Chapter 5: When can f be brought by a smooth change of variables to the normal form $f_0 = Re(x+iy)^m$ in some open neighbourhood of the origin? Theorem 5 suggests that this question should be "finite dimensional". In Section 6.1 we make the last sentence precise using the language of germs and jets of functions.

In brief, we introduce the following objects: $\mathcal{A}_m$ — the space of smooth functions on $\mathbb{R}^2$ of order m at the origin (the leading term in the Taylor expansion is of order m); $\mathcal{GA}_m$ — the space of germs at $(0, 0)$ of functions from $\mathcal{A}_m$; the space of jets $Jet^m_r := \mathcal{A}_m/\mathcal{A}_{r+1}$, for $r \geq m$ — truncated Taylor series, starting at the order m and going up to the order r; the group of diffeomorphisms of $\mathbb{R}^2$ fixing the origin — $Diff$ and the (finite dimensional!) Lie group $Diff_r$ which consists of truncations of the elements of $Diff$ at the origin neglecting the terms in the Taylor series of orders greater than r. The group $Diff$ acts on the spaces $\mathcal{GA}_m$ on the right by composition and the action descends to an action on the spaces of jets Jet^m_r. Moreover, there is a well-defined action of the group $Diff_r$ on the space Jet^m_{m+r-1}. The germ of a function f is denoted by $\{f\}$ and the r-th jet by $\{f\}_r$.

It turns out that the above question about the equivalence of f and the normal form f_0 under a coordinate change can be reformulated in the language of germs: does the germ $\{f\}$ of f lie on the orbit of the germ $\{f_0\}$ of $f_0 = Re(x + iy)^m$ under the action of the group $Diff$? The main result of Section 6.1 (Theorem 28) is the following: the germ $\{f\} \in \mathcal{GA}_m$ lies on the orbit of the germ $\{f_0\}$ in $\mathcal{GA}_m$ under the action of $Diff$ if and only if the jet

$\{f\}_{m+r-1} \in Jet^m_{m+r-1}$ belongs to the orbit of the jet $\{f_0\}_{m+r-1}$ in Jet^m_{m+r-1} under the action of the group $Diff_r$ for $r = max(1, m-3)$. Note that the latter is a finite dimensional representation of a finite dimensional group. Motivated by Theorem 28 we interpret the codimension of the orbit $\{f_0\}Diff$ of $\{f_0\}$ in $\mathcal{GA}_m$ under the action of $Diff$ as the codimension of the orbit $\{f_0\}_{m+r-1}Diff_r$ of $\{f_0\}_{m+r-1}$ in Jet^m_{m+r-1} for $r = max(1, m-3)$. And the latter codimension can be easily computed and is equal to $\frac{1}{2}(m-2)(m-3)$ (the first half of Proposition 17). The second half of this proposition computes the codimension of $\{f_0\}_5 Diff_1$ in Jet^5_5, which happens to be equal to 2. This little computation is used to give a smooth function $\tilde{f}$ on $\mathbb{R}^2$ arbitrarily C^∞ close to $Re(x+iy)^5$, having $Re(x+iy)^5$ as the leading term but not equivalent to $Re(x+iy)^5$ under a coordinate change in any open neighbourhood of $(0,0)$.

In Section 6.2 we consider intrinsic harmonicity for (not necessarily Morse) closed 1-forms on surfaces and go back to the question "how many terms in the Taylor expansion near a degenerate zero do we have to control to keep track of intrinsic harmonicity?" It turns out that the topology of the surface can give a certain upper bound. Indeed, let our surface have genus $g > 1$. Then the Poincare-Hopf theorem together with an easy Morse theoretic argument show that the highest singularity we can allow for ω in order to have a chance for intrinsic harmonicity is of order $g-1$ (that is the leading term should be of the form $dRe(x+iy)^g$). So if we control the first $max(g, 2g-4)$ Taylor coefficients at every zero of a closed 1-form on the surface of genus g that would be enough to ensure local intrinsic harmonicity.

In Section 6.3 we give an example of a closed intrinsically harmonic 1-form ω on the surface of genus 5 with the following two properties. It has exactly two zeros p and p' and it looks like $d(x+iy)^5$ in an appropriate coordinate system near each zero. There exists a closed not locally intrinsically harmonic transitive 1-form $\tilde{\omega}$ arbitrarily C^∞ close to ω with the same zero set $\{p, p'\}$ and with the principal parts at these zeros being the same as for the form ω. The construction of $\tilde{\omega}$ uses the function $\tilde{f}$ above as a local model near p and p'. This tells us that in the presence of degenerate zeros intrinsic harmonicity can not be detected by topological tools and that openness of the set of intrinsically harmonic 1-forms fails.

In Section 6.4 we outline possible directions for future work.

Chapter 2

Preliminaries

2.1 Hodge-star operator.

In this section we recall the definition of the Hodge-star operator as a bundle map and compute it explicitly for $n = 2$.

Take a point $x \in M$. Let $e_1, ..., e_n$ be an orthonormal basis for $T_x M$ defining the correct orientation. Let $f_1, f_2, ..., f_n$ be the dual basis for $T_x^\star M$. We equip $T_x^\star M$ with the inner product, by declaring the basis $f_1, f_2, ..., f_n$ to be orthonormal. The set $\{f_{j_1} \wedge f_{j_2} \wedge ... \wedge f_{j_k}\}_{j_1 < j_2 < ... < j_k}, j_l \in \{1, ..., n\}$ forms a basis for the vector space $\Lambda^k(T_x^\star M)$. We give this space an inner product by declaring this basis to be orthonormal. We define the linear map $\star_g$ (Hodge-star) form $\Lambda^k(T_x^\star M)$ to $\Lambda^{n-k}(T_x^\star M)$, by saying what it does to the above basis. By definition

$$\star_g(f_{j_1} \wedge f_{j_2} \wedge ... \wedge f_{j_k}) = sign(j_1, ..., j_k) f_{i_1} \wedge f_{i_2} \wedge ... \wedge f_{i_{n-k}},$$

where the integers $i_1 < i_2 < ... < i_{n-k}$ form the complementary set to $\{j_1, j_k, ..., j_n\}$ in $\{1, 2, ..., n\}$ and the $sign(j_1, ..., j_k)$ is chosen to be plus or minus so that

$$f_{j_1} \wedge f_{j_2} \wedge ... \wedge f_{j_k} \wedge \star_g(f_{j_1} \wedge f_{j_2} \wedge ... \wedge f_{j_k}) = f_1 \wedge ... \wedge f_n,$$

i.e. according to the orientation. Letting the point x run over the whole manifold M gives us the bundle map $\star_g$. Note that if the dimension n of the manifold M is even and $k = n/2$, then the Hodge-star operator does not change under rescalings of the metric, i.e. it depends only on the conformal structure. As a warm up and also a preparation for the future we write out explicit formulas for $\star_g$ in the case $M = \mathbb{R}^2$ (standard orientation), $k = 1$. Let the Riemannian metric g be defined by the matrix $\{g_{ij}\}_{i,j=1,2}$ in the standard coordinates (x, y), that is $g_{11} = g(\partial_x, \partial_x), g_{12} = g(\partial_x, \partial_y), g_{22} = g(\partial_y, \partial_y)$. Recall that the Riemannian metric g on $T\mathbb{R}^2$ induces the one on $T^\star\mathbb{R}^2$, we denote the induced Riemannian metric by the same letter g and consider $g^{11} = g(dx, dx), g^{12} = g(dx, dy), g^{22} = g(dy, dy)$. As it easily follows from how we introduced inner products on dual spaces, the matrix $\{g^{ij}\}^{i,j=1,2}$ is the inverse to the matrix $\{g_{ij}\}_{i,j=1,2}$. We want to compute the Hodge star on $\Lambda^1 T^\star\mathbb{R}^2$. Since we are in the middle dimension, we can rescale the Riemannian metric g as we like. It will be a standing convention throughout the paper to fix the rescaling (for the particular example of $\Lambda^1 T^\star\mathbb{R}^2$) in such a way that $det\{g_{ij}\}_{i,j=1,2} = det\{g^{ij}\}^{i,j=1,2} = 1$. It is an easy exercise that with the convention we have the formulas for the Hodge-star are

$$\star_g dx = -g^{12}dx + g^{11}dy$$

and

$$\star_g dy = -g^{22}dx + g^{12}dy,$$

so it brings $adx + bdy$ to $(-g^{12}a - g^{22}b)dx + (g^{11}a + g^{12}b)dy$, i.e. in standard coordinates (dx, dy) the Hodge-star operator is given by the following matrix: $\begin{pmatrix} -g^{12} & -g^{22} \\ g^{11} & g^{12} \end{pmatrix}$. In particular, the Laplace-Beltrami operator writes out as $\triangle_g f = [(g^{12}f_x + g^{22}f_y)_y + (g^{11}f_x + g^{12}f_y)_x]dx \wedge dy$.

2.2 Homogeneous polynomials.

In this section we recall some elementary facts about homogeneous polynomials in two variables. Let $P : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a homogeneous polynomial of the n-th degree ($degP = n$). Then there is a positive constant C such that $|P(x, y)| \leq C(x^2 + y^2)^{\frac{n}{2}}$ for all $(x, y) \in \mathbb{R}^2$. The zero set $KerP = \{(x, y) \in \mathbb{R}^2 | P(x, y) = 0\}$ of the polynomial P is either a 1-point set $\{(0, 0)\}$ or a finite union of lines through the origin (1-dimensional linear subspaces) (we leave off the trivial case of the zero polynomial). If $KerP = \{(0, 0)\}$, then there are positive constants c, C such that

$$c(x^2 + y^2)^{\frac{n}{2}} \leq |P(x, y)| \leq C(x^2 + y^2)^{\frac{n}{2}}. \tag{2.1}$$

Assume now that $KerP = \cup_{i\in I} l_i^1$, where $l_i^1 \subset \mathbb{R}^2$ is a linear subspace of dimension 1 and I is finite. We take a small positive δ and set

$$Cone_i^\delta(P) = \{(\xi,\eta) \in \mathbb{R}^2 \setminus \{(0,0)\} | \frac{dist((\xi,\eta), l_i^1)}{dist((\xi,\eta),(0,0))} < \delta\}$$

and

$$\Omega^\delta(P) = \mathbb{R}^2 \setminus \cup_{i\in I} Cone_i.$$

Then there are positive constants c_δ and C_δ such that

$$c_\delta(x^2+y^2)^{\frac{n}{2}} \leq |P|_{\Omega^\delta(P)}(x,y)| \leq C_\delta(x^2+y^2)^{\frac{n}{2}}. \tag{2.2}$$

Definition 6. *For a smooth function σ defined locally around the origin in $\mathbb{R}^2$, the expression $r^n[\sigma]$ will denote its n-th term in the Taylor expansion, which is a homogeneous polynomial in (x,y) of degree n. The expression $r^n[\cdot]$ has the same meaning for 2-forms which are then identified with functions by means of the fixed volume form $dx \wedge dy$.*

2.3 Exponentially small functions.

In this section we recall some facts about the ideal of functions exponentially small near a point. We denote the set of infinitely differentiable functions defined in some open neighbourhood of $(0,0) \in \mathbb{R}^2$ with vanishing Taylor series at $(0,0) \in \mathbb{R}^2$ by $O(exp)$. This is an ideal in the ring of all (locally defined) smooth functions, i.e. a multiplication of an element from $O(exp)$ with any smooth function gives us again a function of the class $O(exp)$. This class respects the operation of taking derivatives, i.e. partial derivatives of all orders taken from an $O(exp)$-function belong to $O(exp)$ (algebraically, $O(exp)$ is a differential ideal). The following criterion is a standard way to check that a given function belongs to the class $O(exp)$.

Lemma 1. *Let the function ϕ be smooth in a punctured neighbourhood of $(0,0)$. If ϕ decays at $(0,0)$ together with all its derivatives faster than any polynomial, then the continuation of ϕ across the origin by 0 belongs to the class $O(exp)$.*

Proof. It suffices to prove that all partial derivatives of ϕ at $(0,0)$ exist and vanish. Take for instance the first partial derivative with respect to x, namely

$$\partial_x \phi_{(0,0)} := \lim_{h\to 0} \frac{\phi(h,0)-\phi(0,0)}{h} = \lim_{h\to 0} \frac{\phi(h,0)}{h}.$$

The last limit exists and vanishes, since ϕ decays at $(0,0)$ faster than any polynomial. Similarly, $\partial_y \phi_{(0,0)} = 0$. For the partial derivatives of the second order similar procedure works. It uses that the partial derivatives of ϕ of the first order decay faster than any polynomial. Proceeding inductively one shows that all partial derivatives of ϕ at $(0,0)$ exist and vanish. □

This has an immediate application.

Lemma 2. *Let the function ϕ be of the class $O(exp)$ and σ be a (locally defined) smooth function which has zero of finite order at $(0,0)$ and does not vanish in a punctured neighbourhood of $(0,0)$. Then the ratio $\frac{\phi}{\sigma}$ is well defined locally around the origin and belongs to the class $O(exp)$.*

Proof. Writing out a partial derivative of some order of the fraction $\frac{\phi}{\sigma}$ gives us a fraction whose numerator decays faster than any polynomial and the denominator is equal to σ^n for some natural n. The function σ having isolated zero of finite order at $(0,0)$ implies that its leading term P in Taylor expansion at $(0,0)$ has the unique zero at $(0,0)$. Therefore (see Inequality (2.1)) P and hence σ^n can be estimated from below by $c(x^2+y^2)^{n/2}$ for some positive constant c. Altogether, the fraction, representing the partial derivative decays faster than any polynomial at $(0,0)$. Application of the previous lemma to $\frac{\phi}{\sigma}$ finishes the proof.□

There is one more technical remark that we will need in future. Let ϕ be of the class $O(exp)$ and σ be a (locally defined) smooth function which has a zero of finite order at $(0,0)$. Let the homogeneous polynomial P be the leading term in the Taylor series of σ at $(0,0)$. Assume that $KerP = \cup_{i\in I} l_i^1$, where $l_i^1 \subset \mathbb{R}^2$ is a linear subspace of dimension 1 with finite I and take a small positive δ to define $\Omega^\delta(P)$ as in the previous section. Then $\frac{\phi}{\sigma}|_{\Omega^\delta(P)}$ decays at $(0,0)$ together with all its derivatives faster than any polynomial (Inequality (2.2) is used for the proof).

These facts about the class $O(exp)$ will be used freely later on without special references. In calculations, by abuse of notation, we will sometimes denote an $O(exp)$-function by the symbol $O(exp)$. The $O(exp)$ notation for 2-forms defined in a neighbourhood of $(0,0) \in \mathbb{R}^2$ transfers by means of a fixed volume form.

Chapter 3

Characterization of intrinsically harmonic 1-forms

In this chapter we generalize Calabi's characterization of intrinsically harmonic 1-forms (cf. [5]) from Morse 1-forms to arbitrary closed 1-forms, i.e. we allow arbitrary zero sets. After we finish the proof of the characterization theorem we turn our attention to the Morse case. In the Morse case the characterization theorem is simply the equivalence between transitivity and intrinsic harmonicity. We use this to prove that the set of intrinsically harmonic 1-forms is open in the set of Morse forms with respect to the C^1 topology. The author would like to thank J. Latschev, who suggested the "global" part of the proof. For the local part we need a refinement of the Morse Lemma, essentially giving a certain lower bound on the size of the Morse neighbourhood of a critical point of a Morse function. This is done in the Appendix.

We are working on a smooth closed oriented n-dimensional manifold M with a closed 1-form ω on it; S denotes the zero set $\{p \in M | \omega(p) = 0\}$ of ω and $\mathcal{F}$ denotes the restriction of the (singular) kernel foliation of ω to its regular set $M \setminus S$. We begin by recalling the concept of transitivity in the specific situation of 1-forms.

Definition 7. *A closed* 1*-form* ω *is called transitive if for any point* $p \in M \setminus S$ *there is a closed (strictly)* ω*-positive smooth path* $\gamma : S^1 \longrightarrow M$ *through* p*. Here "*ω*-positive" means that* $\omega(\dot{\gamma}(t)) > 0$ *for all* $t \in S^1 = \mathbb{R}/\mathbb{Z}$*. That is to say that there exists a closed transversal to the kernel foliation of* ω *through every point of our manifold which does not lie in the zero set of* ω*.*

We also recall the concept of local intrinsic harmonicity.

Definition 8. *A closed* 1*-form* ω *is called locally intrinsically harmonic if there exists an open neighbourhood* U *of its zero set* S *and a Riemannian metric* g_U *on* U *which makes the restriction* $\omega|_U$ *co-closed.*

The following theorem goes back to the classical result by Calabi cf. [5].

Theorem 7. *For a closed* 1*-form* ω *on a closed oriented connected* n*-manifold* M *to be intrinsically harmonic it is necessary and sufficient that*
(a) the form ω *is locally intrinsically harmonic and*
(b) the form ω *is transitive.*

Proof. For necessity assume, there exists a Riemannian metric g which makes ω harmonic. Condition (a) is obviously satisfied. To show Condition (b) we recall a classical result from dynamical systems — the Poincaré-recurrence theorem.

Proposition 1. *Let* (Ω, Σ, μ) *be a probability space. Let* $\{\phi^t\}_{t\in\mathbb{R}}$ *be a measure preserving dynamical system on it. Assume that* A *is a* σ*-algebra element of positive measure. Then for any positive* N *there exists* n_0 *greater then* N *such that*

$$\mu(A \cap \phi^{n_0}(A)) > 0.$$

To apply this proposition in our situation we set Ω to be our manifold M, the σ-algebra Σ to be the usual borelian σ-algebra, and μ to be the probability measure defined by a distinguished volume form *dvol* on M with total volume equal to one. Furthermore, let the vector field X be defined by the following equation: $i_X dvol = \star_g \omega$. Note that X is transverse to the kernel foliation of ω outside S. By Cartan formula, we see that $L_X dvol = 0$. Let $\{\phi^t\}_{t\in\mathbb{R}}$ be the flow of X on M. In our setting $\{\phi^t\}_{t\in\mathbb{R}}$ becomes a measure preserving dynamical system on (Ω, Σ, μ). Let now p be a given point in $M \setminus S$. Let $(\bar{\xi}, \Phi)$ be a bi-foliated closed chart around p, i.e. $\bar{\xi}$ is a closed subset of M, containing an open neighbourhood of p and

$$\Phi : \bar{\xi} \longrightarrow B \times I,$$

is a diffeomorphism, where B is a closed ball in $\mathbb{R}^{n-1}$ and $I = [0, 1]$ is a unit time interval. Moreover, under the diffeomorphism Φ flowlines of $\{\phi^t\}_{t\in\mathbb{R}}$

correspond to the vertical leaves $b \times I$, $b \in B$ and integral submanifolds of the kernel foliation of ω correspond to the horizontal leaves $B \times t$, $t \in I$. In further considerations we identify $\bar{\xi}$ with its image under Φ. Since $\bar{\xi}$ is compact, all points of $\bar{\xi}$ will leave it by some time N, as you follow the flow $\{\phi^t\}_{t\in\mathbb{R}}$. We set $A := \bar{\xi}$ and apply Proposition 1 with the above choices of $\Omega, \Sigma, \mu, A, N$. This gives us a trajectory of $\{\phi^t\}_{t\in\mathbb{R}}$ which leaves $\bar{\xi}$ at some point $(b_1, 1)$ and then enters it again for the first time at some point $(b_0, 0)$. Let us denote the flowline between $(b_1, 1)$ and $(b_0, 0)$ by $\tilde{c}$. It is clear that except for its end points the path $\tilde{c}$ lies outside $\bar{\xi}$. Now we close up this flowline artificially inside the bifoliated chart $\bar{\xi}$, by connecting $(b_0, 0)$ and $(b_1, 1)$ with a smooth path $\hat{c}$ through p, transverse to the horizontal leaves $B \times t$, $t \in I$. Clearly, this can be done in such a way that the concatenation c of the paths $\tilde{c}$ and $\hat{c}$ is smooth. So as c is a smooth closed ω-positive path and the point p was arbitrary, we have that the form ω is transitive. This is Condition (b).

For sufficiency assume that conditions (a) and (b) hold true. Let U be a neighbourhood of S such that $\omega|_U$ is co-closed with respect to some Riemannian metric g_U on U. As it follows from the lemma below U can be chosen so small that the form $\star_{g_U}\omega_U$ is exact.

Lemma 3. *Let (X, g) be a smooth oriented n-dimensional Riemannian manifold without boundary. Let S be a compact zero set of a 1-form γ on X which is both closed and co-closed. There exists an open neighbourhood U of S, such that for any closed $(n-1)$-form ψ on X the restriction $\psi|_U$ is exact.*

Proof. The form γ is a solution to a first order linear elliptic equation

$$(d + d^\star)\gamma = 0, \tag{3.1}$$

where $d + d^\star = d + \star d \star$ is a Dirac operator on X. Locally (3.1) is equivalent to $\triangle_g f = 0$, where f is a local primitive function of γ and $\triangle_g$ denotes Laplace-Beltrami operator for the metric g. So, we can apply the result by Aronszajn, cf.[2], to get that the Dirac operator on 1-forms possesses the strong unique continuation property. Then we apply the theorem by C. Bär (cf. [3]) to find a sequence $\{L_k\}_{k\in\mathbb{N}}$ of submanifolds of X of codimension at least 2, with $S \subset \bigcup_{k\in\mathbb{N}} L_k$. Since every submanifold S_k can be countably exhausted by compact ones (possibly with boundary), we may without loss of generality assume that each L_k is compact, possibly with boundary. Set $Z_k = S \cap L_k$. Let *dim* denote the covering dimension of a topological space.

Then $dimZ_k \leq n-2$, since L_k is a compact manifold (possibly with boundary) of dimension at most $n-2$ and $Z_k \subset L_k$. Since $S = \bigcup_{k\in\mathbb{N}} Z_k$ and every Z_k is closed in S, the Countable sum Theorem (cf. [6] Theorem 7.2.1 on the page 394) implies that $dimS \leq n-2$. This, in turn, implies that $H^{n-1}_{\check{C}ech}(S) = 0$.

Take a sequence $\{U_j\}_{j\in\mathbb{N}}$ of open neighbourhoods U_j of S such that $U_{j+1} \subset U_j$ and $\cap_{j\in\mathbb{N}} U_j = S$ with $U_0 = X$. The continuity property of Čech cohomology, cf. [4] (section 14 "Continuity", Theorem 14.4) implies that $\varinjlim H^{n-1}_{\check{C}ech}(U_j) = 0$, but U_j is manifold, hence Čech cohomology of it is the same as de Rham and finite dimensional. So we have that a direct limit of a sequence of finite dimensional vector spaces vanishes. This implies that for j big enough the image of the 0-th vector space of the sequence in the j-th one vanishes. In other words if $i : U_j \longrightarrow X$ denotes the obvious inclusion, then $i^\star H^{n-1}(X)$ is the trivial subspace of $H^{n-1}(U_j)$. Take $U := U_j$.□

So, we can pick a primitive $(n-2)$-form α on U: $(d\alpha = \star_{g_U} \omega|_U)$. Using transitivity of the form ω, by a standard "thickening of a transversal argument" we obtain, that given a point $m \in M \setminus S$, there exists an open neighbourhood W_m of it, diffeomorphic to $S^1 \times B$, where B is an open ball in $\mathbb{R}^{n-1}$ centered at the origin. More precisely, let $\gamma : S^1 \longrightarrow M$ be a smooth ω-positive path through $m \in M \setminus S$, which we have be the transitivity of ω. Let $W_m \subset M \setminus S$ be a small tubular neighbourhood of $\gamma(S^1)$ in M. The neighbourhood W_m is the total space of a D^{n-1}-bundle ξ over S^1, where D^{n-1} is the closed unit disk in $\mathbb{R}^{n-1}$. Every fiber of ξ is a connected component of the intersection of a certain leaf of the kernel foliation of ω with W_m. Since W_m is a total space of a bundle over S^1, it can be realized as a mapping torus, i.e. W_m is diffeomorphic to $D^{n-1} \times [0,1]/\sim$, where the equivalence relation is given by $(x,0) \sim (\phi(x),1)$ and ϕ is diffeomorphism of D^{n-1}. Since W_m is orientable, the diffeomorphism ϕ is orientation preserving and therefore isotopic to the identity. Therefore, the bundle ξ is trivial, i.e. W_m is diffeomorphic to $P = D^{n-1} \times S^1$. For a moment we identify W_m with P via this diffeomorphism. Let $x_1, ..., x_{n-1}$ be the coordinates on D^{n-1} and let θ be the S^1 coordinate on P. In these coordinates the form $\omega|_{W_m}$ writes out as $fd\theta$, where f is a smooth function on P with $df \wedge d\theta = 0$. Let $\rho : [0,1] \longrightarrow \mathbb{R}$ be a smooth cut-off function: $\rho|_{[0,1/5]} = 1$, $\rho|_{[4/5,1]} = 0$. Set $\psi_m = \rho(x_1^2 + ... + x_{n-1}^2) dx_1 \wedge ... \wedge dx_{n-1}$. Clearly, the $(n-1)$-form ψ_m is closed, vanishes in a neighbourhood of the boundary of P and the top degree form $\Theta := \omega \wedge \psi_m$ satisfies the following properties: Θ is nonnegative everywhere and $\Theta > 0$ in some neighbourhood V_m of $\gamma(S^1)$. Vanishing of ψ_m near the boundary of P implies that ψ_m vanishes in some open neighbourhood U_m of S with $U_m \subset U$. This construction almost literally follows the one given by Calabi in [5].

Since $M \setminus U$ is compact it can be covered by $V_{m_1}, ..., V_{m_l}$ for some natural number l, where $m_1, ...m_l \in M \setminus U$. Set

$$U_0 := U_{m_1} \cap ... \cap U_{m_l},$$

$$V := V_{m_1} \cup ... \cup V_{m_l}$$

and

$$\psi^{'} := \Sigma_{i=1}^{l} \psi_{m_i}.$$

Note, that $U_0 \subset M \setminus V \subset U$ and $\psi^{'}|_{U_0} = 0$.

We pause for a moment to summarize what we have. We have an open neighbourhood U of S with an $(n-2)$-form α on U such that $d\alpha = \star_{g_U}\omega$; open sets U_0 and V with $U_0 \subset M \setminus V \subset U$ and an $(n-1)$-form $\psi^{'}$ with $\psi^{'} \wedge \omega$ bounded away form zero on V, nonnegative everywhere and satisfying $\psi^{'}|_{U_0} = 0$. This allows us to finish the proof with the standard gluing argument. We let ϕ be a smooth function with $\phi|_{M\setminus V} = 1$ and $\phi|_{M\setminus U} = 0$. Such a function ϕ exists since both sets $M \setminus V$ and $M \setminus U$ are closed and the first one is contained in the complement of the second. Set $\alpha^{''} = \phi\alpha$ and $\psi^{''} = d\alpha^{''}$. Note that $\psi^{''}|_{M\setminus V} = d\alpha|_{M\setminus V} = \star_{g_U}\omega|_{M\setminus V}$. Consider a closed form

$$\psi = K\psi^{'} + \psi^{''}$$

for sufficiently large positive constant K. We claim that the form ψ has the following properties:
(i) $\psi|_{U_0} = \star_{g_U}\omega|_{U_0}$,
$(ii) \omega \wedge \psi > 0$ everywhere on $M \setminus S$.
Indeed, since $\psi^{'}|_{U_0} = 0$, we have that

$$\psi|_{U_0} = \psi''|_{U_0} = \star_{g_U}\omega|_{U_0}.$$

This shows the first property. For the second one consider

$$\psi_{M\setminus V} = K\psi^{'}|_{M\setminus V} + \psi^{''}|_{M\setminus V} = K\psi^{'}|_{M\setminus V} + \star_{g_U}|_{M\setminus V},$$

multiplying with ω gives us

$$\omega \wedge \psi|_{M\setminus V} = K\omega \wedge \psi^{'}|_{M\setminus V} + \omega \wedge \star_{g_U}\omega|_{M\setminus V}.$$

The last expression is the sum of two nonnegative terms, the second one being strictly positive outside S. We are left the expression $\omega \wedge \psi$, restricted to V. Since $\omega \wedge \psi'|_V$ is bounded away from zero, we have that

$$\omega \wedge \psi|_V = K\omega \wedge \psi'|_V + \omega \wedge \psi''|_V > 0$$

for sufficiently large positive constant K.

Now, having the form ψ with the properties above we construct the desired metric g by gluing. Let ϕ_U, ϕ_V be a partition of unity, subordinate to the cover U, V. Let g'' be any metric on V, making ω and ψ orthogonal to each other. Consider the metric $\tilde{g} = \phi_U g_U + \phi_V g''$ on M. It makes ω and ψ orthogonal everywhere on M and $\star_{\tilde{g}}\omega|_{U_0} = \star_{g_U}\omega|_{U_0} = \psi|_{U_0}$. Consider the following orthogonal decomposition of the tangent bundle of $M \setminus S$:

$$\tilde{g} = \tilde{g}|_{Ker\omega} \oplus \tilde{g}|_{Ker\psi}.$$

There exists and unique smooth function $\tilde{f} : M \setminus S \longrightarrow \mathbb{R}$, such that for the metric

$$g = \tilde{f}\tilde{g}|_{Ker\omega} \oplus \tilde{g}|_{Ker\psi}$$

on $M \setminus S$ we have that $\star_g\omega|_{M\setminus S} = \psi|_{M\setminus S}$. Note, that $\tilde{f}|_{U_0} = 1$, therefore $g|_{U_0\setminus S} = \tilde{g}|_{U_0\setminus S}$, and hence g can be C^∞-regularly continued across points of S by just setting $g|_S = \tilde{g}|_S$. This means that the metric g is well-defined everywhere on M. The equation

$$\star_g\omega = \psi$$

holds on $M \setminus S$, by the choice of $\tilde{f}$ and it also holds on U_0, by the first property of the form ψ because $g|_{U_0} = \tilde{g}|_{U_0} = g_U|_{U_0}$. Thus, since, the form ψ is closed we obtain that the form ω is co-closed with respect to the metric g everywhere on M. $\square$

Next, we turn our attention to what happens in the nondegenerate case. We begin with two obvious remarks. The first remark is that for Morse forms local intrinsic harmonicity is equivalent to the absence zeros of index 0 or n. The second remark is that for Morse forms transitivity implies local intrinsic harmonicity. Note, that in the second remark the nondegeneracy assumption can not be weakened. There are examples of transitive forms with isolated but degenerate zeros which are not locally intrinsically harmonic, see Section 6.3. For Morse forms Theorem 7 takes the following form.

Theorem 8. *Let ω be a closed Morse 1-form on a closed oriented connected n-manifold M. Then ω is intrinsically harmonic if and only if it is transitive.*

We elaborate a little on the concept of transitivity. Note that to check transitivity for a Morse locally intrinsically harmonic form it is not necessary to look for a closed transversal to $\mathcal{F}$ through every point of $M \setminus S$. Firstly, it is enough to find a closed transversal through every leaf of $\mathcal{F}$, and secondly since every leaf of $\mathcal{F}$ intersects $M \setminus U$ (for U — an open neighbourhood of S small enough) it is enough to find a closed transversal through every leaf of the restriction $\mathcal{F}|_{M\setminus U}$. These observations lead to the following theorem

Theorem 9. *The set of transitive forms is C^1-open in the set of Morse forms.*

Proof. Let ω be a closed transitive Morse 1-form on M. According to Lemma 20 in the Appendix we can chose a C^1-open neighbourhood $\mathcal{U}$ of ω and a disjoint union U of Morse neighbourhoods of zeros of ω so small that for any $\tilde{\omega} \in \mathcal{U}$ any connected component $\hat{U}$ of U serves as a Morse neighbourhood for $\tilde{\omega}$, i.e. a primitive function of ω in $\hat{U}$, can be brought the canonical Morse form with the same Morse index. Therefore, there exists an $(n-2)$-form $\tilde{\alpha}$ on U with $d\tilde{\alpha} = \star_{g_U} \tilde{\omega}_U$.

We cover $M \setminus U$ be finitely many open neighbourhoods $V_{m_1}, ..., V_{m_l}$, as in the proof of Theorem 7. Every V_{m_j} is diffeomorphic to $S^1 \times B$. A closed transversal γ_j through m_j corresponds to $S^1 \times 0$. We set $V := V_{m_1} \cup ... \cup V_{m_l}$. Recall that $M \setminus U \subset V$. Note that now a finite number of closed transversals $\{\gamma_j\}_{j=1,...,1}$ intersect all the leaves of $\mathcal{F}$. Now we turn to our perturbed form $\tilde{\omega}$. By shrinking $\mathcal{U}$, if necessary, we can achieve that every γ_j is positive the for perturbed form $\tilde{\omega}$. Since every leaf of $\tilde{\mathcal{F}}$ (the kernel foliation of $\tilde{\omega}$ restricted to $M \setminus \tilde{S}$) intersects nonempty with $M \setminus U$ and therefore with V the old transversals $\{\gamma_j\}_{j=1,...,1}$ serve as transversals to the new foliation $\tilde{\mathcal{F}}$ intersecting all(!) the leaves of $\tilde{\mathcal{F}}$. Therefore the form $\tilde{\omega}$ is also transitive. □

Together with Theorem 8 this gives us the following

Theorem 10. *The set of intrinsically harmonic forms is C^1-open in the set of Morse forms.*

Chapter 4

Transitivity versus nontransitivity under small perturbations

The general setup for this chapter is the same as for the previous one: a closed oriented n-dimensional manifold M and a closed 1-form ω on it. In this chapter a slight change in terminology happens: what previously was called transitivity (existence of closed transversals through every nonsingular point of the kernel foliation of ω) will now be called "weak transitivity", because we want to reserve the word "transitivity" for a slightly more technical and seemingly stronger property. Eventually, the two properties turn out to be the same, so we can just drop "weakly" afterwards. This way our treatment of the concept of transitivity becomes a slight variation of that given by Calabi in [5]. The main technical tools developed in this chapter (upland, lowland, different versions of transitivity) appear in [5], maybe in a slightly different setups. We give a little summary of what happens in this chapter section by section. In Section 4.1 we discuss the properties of transitivity in the most general case possible — the zero set S of the 1-form ω is allowed to be arbitrary. The culmination of this technical section is Theorem 11 which essentially says that a form ω is nontransitive if and only if a certain set of closed leaves of the foliation $\mathcal{F}$ separates the (possibly noncompact) manifold $M \setminus S$. This theorem will be a key tool for everything else in this chapter. In Section 4.2 we assume that ω is a Morse form and see how Theorem 11 rewrites and what applications it has. Let us mention two of the applications right now. The first one is Theorem 15 which says that if a Morse 1-form does not have zeros of index $0, 1, n-1$ and n, then it is transitive. The second one, Theorem 16 is a purely topological statement about foliations, it says that if a Morse 1-form has at least one zero of index 0 or n and at least one leaf, which is not closed in $M \setminus S$, then it must have at least one zero of index 1 or $n-1$. The culmination of the section is Theorem 17 — a Morse version of Theorem 11. It says that nontransitivity is equivalent to the existence of a set of "singular closed leaves" $\mathcal{P}_1, ..., \mathcal{P}_l$ of $\mathcal{F}$, which induce zero in $H_{n-1}(M)$. An important tool for the main theorem is the "No Blow up" theorem which imposes a certain diameter bound on such singular closed leaves. In Section 4.3 we deal with convergent sequences of Morse forms ω_m converging to ω with respect to the C^1 topology. We give a criterion under which a sequence of "singular closed leaves" $\mathcal{P}_m$ of $\mathcal{F}_m$ converges to the "singular closed leaf" $\mathcal{P}$ of $\mathcal{F}$. An essential ingredient for this is the "No Blow up" theorem. In Section 4.4 we use all the technique we have worked out by then (especially the convergence theorem) to prove the main theorem of this chapter — Theorem 22. Its formulation was given and the proof was sketched in the introduction. In Section 4.5 we give some illustrating examples.

4.1 Generalities.

We begin by introducing the (modified) concept of transitivity.

Definition 9. *We say that a closed 1-form ω on M is transitive provided that for any two points $p, q \in M \setminus S$ there exists a smooth path γ joining p and q, such that $\omega(\dot{\gamma}) > 0$ all along ϕ including the endpoints p and q. In other words: for any two points from $M \setminus S$ one is reachable from the other by an ω-positive path.*

We recall basic facts and concepts related to transitivity.

Definition 10. *Given a point $p \in M \setminus S$ we define its upland U_p to be the set of points reachable from p be an ω-positive path and similarly L_p — the lowland to be the set of points reachable from p by an ω-negative path (with the obvious meaning of ω-negative).*

Note that by definition U_p is connected and that the relation that one point lies in the upland of the other is transitive, i.e. $p_2 \in U_{p_1}$ and $p_3 \in U_{p_2}$ imply that $p_3 \in U_{p_1}$.

Each of the following is equivalent to saying that ω is transitive:
i) $U_p = M \setminus S$ for all $p \in M \setminus S$,
ii)there exists $p \in M \setminus S$ with $U_p = L_p = M \setminus S$.
Now given a point $p \in M \setminus S$ we derive some topological consequences about the upland U_p and its boundary ∂U_p. Of course, similar discussions apply to the lowland L_p and its boundary ∂L_p.

Proposition 2. *Let $p \in M \setminus S$ be a point. Then its upland U_p is open.*

Proof. Consider a foliated chart U for the foliation $\mathcal{F}$ around a point $q \in U_p$ such that the leaves of the restricted foliation $\mathcal{F}|_U$ are connected. Let f be a suitably normalized local primitive function of ω on U, i.e. $df = \omega|_U$ and $f(q) = 0$. The leaves of $\mathcal{F}|_U$ are simply the level sets of f. Then it is easy to see that $\{f > 0\} \subset U_p$. In words: all the points which lie "above" q (have bigger values of f) also belong to U_p. Moreover, if $q \in U_p$ then we can consider an ω-positive path $\gamma : [0,1] \longrightarrow M$ with $\gamma(0) = p$, $\gamma(1) = q$ and take an intersection $\gamma[0,1] \cap U$ which is an open subset of $\gamma([0,1])$. Therefore there exists $t \in (0,1)$ (maybe very close to 1) such that $\gamma(t) \in U$. The point $\gamma(t)$ is by construction ω-positive reachable from p and lies below q in U. This amounts to saying that for $q \in U_p$ not only the points in U above q belong to U_p but also some of the points below q belong to U_p. From this it is clear that there is an open neighbourhood V of q in U with $V \subset U_p$ $\square$

The following (technical) lemma is a key tool for the sequence of propositions and lemmas below and most of the later assertions.

Lemma 4. *Let U be an open connected subset of M with the following properties:*

- $IB1$ *the restriction $\omega|_U$ is exact, i.e. $\omega|_U = df$ for some smooth function f on U,*
- $IB2$ *the boundary ∂U of U does not intersect S and consists of two parts: one is $\mathcal{F}$-saturated (consists of leaves of $\mathcal{F}$), the other is invariant under the local flow of some gradient-like vector field ξ for f (ξ-invariant). Moreover the image of the $\mathcal{F}$-saturated part of ∂U under the map f is contained in the boundary of $f(U)$.*
- $IB3$ *the leaves of $\mathcal{F}$ escape some small neighbourhood V of $S \cap U$,*
- $IB4$ *the level sets of f are connected.*

Then for any two points $q, \tilde{q} \in U \setminus S$ with $f(\tilde{q}) > f(q)$ we have that $\tilde{q} \in U_q$.

Definition 11. *The conditions $IB1$, $IB2$, $IB3$ and $IB4$ above will be abbreviated as IB conditions. An open connected subset U of M satisfying the IB conditions will be called an IB set.*

Proof. We start at q and follow the flowline of ξ upwards until one of the following three things happens:

1) we have reached q_1 with $f(q_1) = f(\tilde{q})$,

2) we have reached V,

3) we have reached ∂U.

It will be straightforward to finish the proof if we have 1), so we consider what happens if we are faced with 2) or 3). In the case of 2) we move along the leaf we have reached by then to escape V and start moving upwards along the flowlines of ξ again. In the case of 3) by the $IB2$ property of U, we have that ∂U consists of two parts: one is $\mathcal{F}$-saturated and the other ξ-invariant. Assume that we have arrived at the $\mathcal{F}$-saturated part. Clearly the value of the function f at the point we have arrived to is not the infimum of f. Then the second part of the $IB2$ property implies that the value of the function f at this point must be the supremum of f. This means that we should have stopped earlier, since we had 1) already. Therefore, we can assume that in the case of 3) we have arrived at the ξ invariant part of ∂U. It means that we can start to travel upwards ξ remaining on ∂U at the same time. It is easy to see that this program will terminate at finitely many steps, meaning we eventually arrive at 1). Then connect q_1 to $\tilde{q}$ inside the level set $\{f = f(q_1)\}$. This is possible since the level sets of f are connected. The last step is to modify this path to an ω-positive one inside a small open neighbourhood of those parts of the path which go along the leaves of $\mathcal{F}$. This shows that $\tilde{q} \in U_q$. □

Lemma 5. *Let U be an IB set and let $p \in M \setminus S$ be a point. Then there exists $s \in \mathbb{R}$, such that the following inclusions are true: $(\{f > s\} \setminus S) \subset U_p$ and $\{f \leq s\} \subset M \setminus U_p$. In particular $(\partial U_p \setminus S) \cap U = \{f = s\}$.*

Proof. Given any pair of points $q, \tilde{q} \in U \setminus S$ with $f(\tilde{q}) > f(q)$, Lemma 4 tells us that $q \in U_p$ implies that $\tilde{q} \in U_p$. This is so because "lying in the upland of" defines a transitive relation. Now since U is connected, its image I under

f is a bounded interval in $\mathbb{R}$. If $(M \setminus U_p) \cap (U \setminus S) = \emptyset$, then we take s to be any number to the left of I, else we set $s = sup\{f(q) | q \in (M \setminus U_p) \cap (U \setminus S)\}$. The inclusion $(\{f > s\} \setminus S) \subset U_p$ is clear since otherwise we have a point $q \in (M \setminus U_p) \cap (U \setminus S)$ with $f(q) > s$, contradicting the definition of s. It remains to show the inclusion $\{f \leq s\} \subset M \setminus U_p$. Assume by contradiction, that there is a point $q \in U_p$, with $f(q) \leq s$. Since U_p is open and q is a regular point of f, we can find $q_1 \in U_p$ with $f(q_1) = s_1 < s$. Consider the set of points $f_{s_1 s} := \{s_1 < f(\tilde{q}) \leq s\} \setminus S$. By the observation above we have that $f_{s_1 s} \subset U_p$, but this contradicts the definition of s. □

We suggest the following way of visualizing Lemma 5. The points of $U \setminus S$ are coloured in three different colours: red, black and blue depending on whether the value of the function f at a point is greater, equal or smaller than s respectively. The points of S which happen to lie in $\{f \leq s\}$ can also be coloured in black or blue accordingly. This allows us to think of red points as points in U_p, blue — as in the interior of $M \setminus U_p$, black — as in the boundary of the interior of $M \setminus U_p$, separating between red and blue. With this "red-black-blue" picture in mind the proofs of the following three statements are clear.

Proposition 3. *The sets U_p, $M \setminus U_p$ and ∂U_p are $\mathcal{F}$ saturated.*

Proposition 4. *Let $p \in M \setminus S$ be a point. Then $U_p \cup \partial U_p \setminus S$ is a n-dimensional submanifold of M with boundary $\partial U_p \setminus S$.*

Proposition 3 says in particular that the smooth part of ∂U_p consists of leaves of $\mathcal{F}$, but as we will now see not every leaf of $\mathcal{F}$ can serve as a part of ∂U_p. In fact, the topological behavior of such a leaf is very much constrained. The next definition is the first step in making the "very much constrained" precise.

Definition 12. *Let $\mathcal{L}$ be a leaf of $\mathcal{F}$. We say that the leaf is nonwinding, if for every IB foliated chart U, we have that $\mathcal{L}$ enters U at most once, i.e. $|\pi_0(U \cap \mathcal{L})| \leq 1$*

The following two lemmas are crucial. The first one being immediate from Lemma 5.

Lemma 6. *Let $\mathcal{L}$ be a leaf of $\mathcal{F}$. Assume that $\mathcal{L}$ is a part of ∂U_p. Then $\mathcal{L}$ is nonwinding.*

Lemma 7. *Let $\mathcal{L}$ be a nonwinding leaf of $\mathcal{F}$. Let p_0 be a nondegenerate zero of ω. Let U be a Morse neighbourhood of p_0 which satisfies IB1, IB2, IB3 and instead of IB4 one has that the level sets of f can have at most 2 connected components. Then $\mathcal{L}$ can enter U at most twice, i.e. $|\pi_0(U \cap \mathcal{L})| \leq 2$.*

Proof. Assume by contradiction, that $\mathcal{L}$ enters U at least three times. Then regardless of what the Morse index of p_0 is, we can find an IB-foliated chart $V \subset U$ near p_0, such that $\mathcal{L}$ enters V at least twice, contradicting Lemma 6. □

Note, that we can shrink any foliated chart to produce an IB one and any Morse neighbourhood to make it satisfy the assumptions of Lemma 7. In particular, if ω is a Morse form, then there exists an atlas for M with finitely many (M is compact) such foliated charts and Morse neighbourhoods.

The following assertion is a direct corollary of Lemmas 6 and 7.

Lemma 8. *Let $\mathcal{L}$ be a leaf of $\mathcal{F}$. Assume that $\mathcal{L}$ is a part of ∂U_p. Let p_0 be a nondegenerate zero of ω. Let U be a Morse neighbourhood of p_0 as in Lemma 7. Then $\mathcal{L}$ can enter U at most twice, i.e. $|\pi_0(U \cap \mathcal{L})| \leq 2$.*

Proposition 5. *Let $\mathcal{L}$ be a leaf of $\mathcal{F}$. Assume that $\mathcal{L} \subset \partial U_p$. Let $Cl\mathcal{L}$ denote the closure of $\mathcal{L}$. Then $Cl\mathcal{L} \setminus \mathcal{L} \subset S$. In other words, $\mathcal{L}$ is closed in $M \setminus S$.*

Proof. Assume by contradiction there is a point $s \in Cl\mathcal{L}$ with $s \notin \mathcal{L}$ and $s \notin S$. Consider an IB foliated chart around s. The leaf $\mathcal{L}$ enters this chart infinitely (!) many times, contradicting Lemma 6. □

Proposition 6. *Let $\mathcal{L}$ be a closed (in $M \setminus S$) leaf of $\mathcal{F}$ such that its closure (in M) $Cl\mathcal{L}$ has an open connected $\mathcal{F}$-saturated neighbourhood U satisfying IB conditions. Assume that $\mathcal{L} \subset \partial U_p$. Then $p \in \mathcal{L}$. In particular, there is at most one leaf closed (in $M \setminus S$) leaf $\mathcal{L}$ admitting a neighbourhood as above with $\mathcal{L} \subset \partial U_p$, e.g. ∂U_p can contain at most one regular closed leaf of $\mathcal{F}$.*

Proof. Let f a be primitive function of $\omega|_U$ with the normalization $f|_{Cl\mathcal{L}} = 0$. By Lemma 5 the level set $\{f = 0\}$ separates $U \setminus S$ into two parts:

$$U_+ = \{f > 0\} \setminus S \subset U_p$$

and

$$U_- = \{f < 0\} \subset M \setminus U_p.$$

Taking any point $p_+ \in U_+$ and following the ω-positive path joining p and p_+ backwards (from p_+ to p) we see that $p \in U$ and hence $p \in \mathcal{L}$. □

The above sequence of assertions (Propositions 3, 4, 5, 6 and Lemmas 6, 8) can be summarized as follows. The smooth part of ∂U_p is a finite union K_p of leaves of $\mathcal{F}$ closed in $M \setminus S$. Moreover at most one member of K_p admits a neighbourhood U as in Proposition 6. Furthermore, every member of K_p must satisfy conclusions of Lemmas 6, 8. This motivates the following discussions.

Definition 13. *The set of leaves of $\mathcal{F}$ which are closed in $M \setminus S$ will be denoted by K^0_ω. We define a characteristic set K_ω to be the set of leaves $\mathcal{L}$ of $\mathcal{F}$ satisfying the following three conditions:*
(i) The leaf $\mathcal{L}$ belongs to the set K^0_ω.
(ii) The leaf $\mathcal{L}$ is not (!) closed in M.
(iii)The closure $Cl\mathcal{L}$ (in M) of $\mathcal{L}$ does not (!) admit an open connected $\mathcal{F}$-saturated neighbourhood satisfying IB conditions.
Note, that (iii) implies (ii). We denote by K^w_ω the subset of K_ω, which consists of nonwinding leaves of $\mathcal{F}$. Where the upper w stands for "nonwinding".

We will also need to consider the closures of the leaves which constitute the sets K^0_ω, K_ω and K^w_ω.

Definition 14. *We introduce $C^0_\omega := \{\mathcal{P} = Cl\mathcal{L} | \mathcal{L} \in K^0_\omega\}$, $C_\omega := \{\mathcal{P} = Cl\mathcal{L} | \mathcal{L} \in K_\omega\}$ and $C^w_\omega := \{\mathcal{P} = Cl\mathcal{L} | \mathcal{L} \in K^w_\omega\}$.*

Definition 15. *A pair (M, ω) of a manifold and closed 1-form will be called a foliated manifold. The class of foliated manifolds (M, ω) such that there is a point $p \in M \setminus S$ with $\mathcal{L}_p$ not closed in $M \setminus S$ or $\mathcal{L}_p$ is closed in $M \setminus S$, but its closure $Cl\mathcal{L}_p$ in M does not admit an open connected $\mathcal{F}$-saturated neighbourhood satisfying IB conditions will be denoted by $\mathcal{C}$.*

Proposition 4 suggests that transitivity of ω must have something to do with the following question: is there a nonempty finite set K of leaves $\mathcal{L}$ of $\mathcal{F}$ such that $M \setminus (S \cup (\cup_{\mathcal{L} \in K} \mathcal{L}))$ is not connected? If we look at Proposition 4 more precisely, then we see that $\partial U_p \setminus S$ has a canonical orientation as a boundary of $U_p \cup \partial U_p \setminus S$ and this orientation is consistent with the co-orientation of $\partial U_p \setminus S$ defined by the direction of decrease of a local primitive function of ω. So before we can formulate a rigorous theorem, orientation questions have to be take care of. In what follows, M is given a definite choice of an orientation. The canonical orientation of a leaf $\mathcal{L}$ of $\mathcal{F}$ is defined to be consistent with the orientation of M and the co-orientation of $\mathcal{L}$ defined by the direction the decrease of a local primitive function f of ω.

Theorem 11. *Let ω be a closed form on a closed oriented manifold M. Then the following assertions are equivalent.*

1) *The form ω is nontransitive.*

2) *There is a nonempty subset K of K^0_ω, such that $\cup_{\mathcal{L}\in K}\mathcal{L}$ bounds an n-dimensional submanifold M_n in $M \setminus S$ and the boundary orientation of any $\mathcal{L} \in K$ coincides with the canonical one.*

Assume in addition that $(M, \omega) \in \mathcal{C}$. Then each of the above assertions is equivalent to the following assertion.

3) *There is a nonempty subset K of K^w_ω, such that $\cup_{\mathcal{L}\in K}\mathcal{L}$ bounds an n-dimensional submanifold M_n in $M \setminus S$ and the boundary orientation of any $\mathcal{L} \in K$ coincides with the canonical one.*

Proof. For the implication from 2) to 1) consider a pair (p, q) of points in $M \setminus S$ with $p \in M_n$ and $q \in (M \setminus S) \setminus M_n$. Note that q is not reachable from p by an ω-positive path, since if it were so the path would have to intersect ∂M_n outwards and at the intersection point $\tilde{q}$ the ω-positivity of the path would contradict the fact that the boundary orientation of $\mathcal{L}_{\tilde{q}}$ coincides with the canonical one. For the implication form 1) to 2) take any point $p \in M \setminus S$. Since ω is nontransitive either $U_p \neq M \setminus S$ or $L_p \neq M \setminus S$ is true. We can assume without loss of generality that $U_p \neq M \setminus S$. Now $\partial U_p \setminus S$ is nonempty. By Proposition 3 and we have that $\partial U_p \setminus S$ is a union of closed (in $M \setminus S$) leaves of $\mathcal{F}$. Moreover, it is clear that the boundary orientation of $\partial U_p \setminus S$ coincides with the canonical one. Assume now that $(M, \omega) \in \mathcal{C}$. Then we could have chosen p as in the definition of $\mathcal{C}$. By Propositions 3 - 6 and the choice of p it is clear that any leaf $\mathcal{L} \subset \partial U_p$ belongs to the set K_ω. Moreover, by Lemma 6, the leaf $\mathcal{L}$ is nonwinding. This proves the implication from 1) to 3). The implication from 3) to 2) is clear, because $K^w_\omega \subset K_\omega \subset K^0_\omega$. □

The first application of this theorem exploiting the equivalence of 1) and 2) is a statement that transitivity is the same that weak transitivity.

Proposition 7. *A closed 1-form ω is transitive if and only if it is weakly transitive.*

Proof. Assume that ω is transitive. Consider a point $p \in M \setminus S$. Since $U_p = M \setminus S$, we have in particular that $p \in U_p$. This gives us a closed transversal through p, possibly not smooth at p. We perturb the transversal near p, if necessary to obtain smoothness.

For the converse assume that ω is not transitive, then by Theorem 11 there exists is a nonempty set K of closed (in $M \setminus S$) leaves of $\mathcal{F}$ such that $\partial M_n = \cup_{\mathcal{L}\in K}\mathcal{L}$ bounds an n-dimensional submanifold M_n in $M \setminus S$ and the boundary orientation of ∂M_n coincides with the canonical one. Take any point $p \in \partial M_n$. Assume by contradiction that we can find a closed transversal γ to $\mathcal{F}$ through p. This leads to two observations eventually contradicting each other. Note that geometric intersection number of γ with ∂M_n is positive ($p \in \gamma \cap \partial M_n$). Furthermore algebraic contributions to the algebraic intersection number $i_{\gamma,\partial M_n}$ of γ with ∂M_n at all the points $q \in \gamma \cap \partial M_n$ are negative. This is so because the boundary orientation of ∂M_n coincides with the canonical one and γ is ω-positive. This leads to the first observation: $i_{\gamma,\partial M_n} \neq 0$. The second observation is that $i_{\gamma,\partial M_n} = 0$, because ∂M_n separates $M \setminus S$. This is a contradiction.□

At first glance, the part of Theorem 11, which states equivalence between 1) and 3) is hardly ever applicable, since the definition of the set K_ω looks somewhat complicated. This is to some extent true, in this generality it is hard to say much more than we have said already, but in the next section we proceed to the generic case of ω being a Morse form. Then K_ω will become more transparent and more applications will appear.

We close this section by making a remark about what happens if $(M, \omega) \notin \mathcal{C}$. Then, in particular, every leaf $\mathcal{L}$ of $\mathcal{F}$ is closed in $M \setminus S$. The subset $S_{\mathcal{L}} := Cl\mathcal{L} \setminus \mathcal{L}$ of the zero set S consists of points which can be approximated by points in $\mathcal{L}$. Set

$$S_{appr} = \cup_{p\in M\setminus S} S_{\mathcal{L}_p}.$$

Theorem 12. *Assume that $(M, \omega) \notin \mathcal{C}$ and the set $S \setminus S_{appr}$ is closed in M. Then ω is transitive if and only $S \setminus S_{appr} = \emptyset$.*

Proof. Let $\mathcal{L}$ be a leaf of $\mathcal{F}$. Since (M, ω) is not from the class $\mathcal{C}$, the closure (in M) $Cl\mathcal{L}$ of $\mathcal{L}$ admits an open connected $\mathcal{F}$-saturated neighbourhood $U_{\mathcal{L}}$ satisfying IB conditions. Let $f_{\mathcal{L}} : U_{\mathcal{L}} \longrightarrow \mathbb{R}$ be a primitive function for $\omega|_{U_{\mathcal{L}}}$. Consider the following equivalence relation on M. Two points p and q are equivalent provided $f_{\mathcal{L}}(p) = f_{\mathcal{L}}(q)$ for some $\mathcal{L}$ and that is well-defined since any two local primitive functions for the same 1-form differ by a constant. Let the quotient space be denoted by X and let $\pi : M \longrightarrow X$ be the natural projection. Since for every $x \in X$, the preimage $\pi^{-1}(x)$ is a level set of some local primitive function of ω and therefore is closed, we have that X is Hausdorff. The space X is second countable since it is a quotient of M, which is second countable. We set

$$X_S := \pi(M \setminus (S \setminus S_{appr}))$$

and

$$\pi_S = \pi|_{M\setminus(S\setminus S_{appr})}.$$

The next step is to give X_S a structure of a smooth oriented 1-manifold, such that π_s becomes a smooth map. Since for any $\mathcal{L}$ the function $f_{\mathcal{L}}$ factors through π, we obtain a continuous one to one map $\pi(f_{\mathcal{L}}) : \pi(U_{\mathcal{L}}) \longrightarrow I \subset \mathbb{R}$ with $f_{\mathcal{L}} = \pi(f_{\mathcal{L}}) \circ \pi$. Note that $f_{\mathcal{L}}$ satisfies maximum principle, since for every critical value can be attained at a regular point. This implies that $f_{\mathcal{L}}$ is open. Thus the map $\pi(f_{\mathcal{L}})$ is open. Therefore, $\pi(f_{\mathcal{L}})$ is a homeomorphism of $\pi(U_{\mathcal{L}})$ with an open interval I of the real line. This gives X_S the structure of a topological 1-manifold. Moreover, transition maps are of the form $x \mapsto x + const$, therefore, X_S gets a structure of an oriented C^∞ manifold with the canonical projection π_S being a smooth map. Clearly, ω-positive paths on $M \setminus S$ descend to positive paths on X_S. The next step is to show that $X_S = X \setminus \pi(S \setminus S_{appr})$. Indeed, consider any level set

$$\{f_{\mathcal{L}} = c\} = (\{f_{\mathcal{L}} = c\} \cap Cl\mathcal{L}) \cup (\{f_{\mathcal{L}} = c\} \cap (S \setminus S_{appr})).$$

The union is disjoint, both of $Cl\mathcal{L}$ and $S \setminus S_{appr}$ are closed, therefore one of $\{f_{\mathcal{L}} = c\} \cap Cl\mathcal{L})$ and $\{f_{\mathcal{L}} = c\} \cap (S \setminus S_{appr})$ is empty, since level sets of $f_{\mathcal{L}}$ are connected. This shows that $S \setminus S_{appr}$ is saturated with respect to the equivalence relation above. Thus,

$$X_S = \pi(M \setminus (S \setminus S_{appr})) = \pi(M) \setminus \pi(S \setminus S_{appr}) = X \setminus \pi(S \setminus S_{appr}).$$

This allows us to finish the argument as follows. Assume that $(S \setminus S_{appr}) \neq \emptyset$, then X_S is a proper open subset of X. This says that X_S is noncompact (as a proper open subset of a Hausdorff space) and, therefore, topologically is a union of several open intervals. Thus, not every point in X_S admits a closed positive path through it (in fact no point admits). This shows that ω is nontransitive.

For the converse assume that $(S \setminus S_{appr}) = \emptyset$, then $X_S = X = \pi(M)$ is compact. Now X is a compact connected smooth 1-manifold, so it is a circle. Every two points of X can be joined by a positive path. Given any two points

p and $q \in \mathcal{L}$ in $M \setminus S$ we construct an ω-positive path joining p to some point $p_1 \in U_{\mathcal{L}}$ by lifting a positive path in X between $\pi(p)$ and $\pi(p_1)$. Then we join p_1 to q by an ω-positive path inside $U_{\mathcal{L}}$. This, by the transitivity of the relation "can be joined by an ω-positive path" gives us that p and q can be joined by an ω-positive path. Since p and q were arbitrary, this completes the proof.□

The following theorem is an easy corollary.

Theorem 13. *Assume that $M \setminus S$ is foliated by regular closed leaves of $\mathcal{F}$. Then ω is transitive if and only $S = \emptyset$.*

Proof. Note that $(M, \omega) \notin \mathcal{C}$ and the set $S \setminus S_{appr} = S$ is closed. Therefore Theorem 12 applies.□

4.2 Nondegenerate zeros.

In this section we assume all the zeros of ω to be nondegenerate. That is ω is a Morse form. The local picture of $\mathcal{P} = Cl\mathcal{L} \in C^0_\omega$ around a singular point is given by the Morse Lemma. There is a local coordinate system $(U; x_1, x_2, ...x_n)$ around p such that $\mathcal{L} \cap U$ is a union of several connected components of the set $\{f = 0\} \setminus \{p\}$, where $f = -x_1^2 - ... - x_\lambda^2 + x_{\lambda+1}^2 + ... + x_n^2$ is a local primitive function for ω and λ is the Morse index of the point p ($n = dimM$).

Definition 16. *The set of zeros of ω of index 0 or n will be denoted by S_{top} (top zeros). The set of zeros of ω of index 1 or $n-1$ will be denoted by S_{ess} (essential zeros), the set $S \setminus (S_{top} \cup S_{ess})$ (intermediate value zeros) will be denoted by S_{int}.*

First of all we see what Theorem 12 transfers to in the Morse case.

Theorem 14. *Let ω be a closed Morse form on a closed n-manifold M. Assume that $(M, \omega) \notin \mathcal{C}$. Then ω is transitive if and only $S_{top} = \emptyset$.*

Proof. Clearly, $S - S_{appr} = S_{top}$. Since the set S_{top} is closed, we are done by Theorem 12. □

The characteristic set K_ω is now finite. A potential candidate $\mathcal{P} \in C^0_\omega$ for being a member of C_ω should meet some zeros of ω. First of all, we show that if $\mathcal{P}$ meets only intermediate value zeros, then $\mathcal{P}$ actually is not a member of C_ω.

Lemma 9. *Let $\mathcal{L}$ be a leaf of $\mathcal{F}$ closed in $M \setminus S$. Assume that $Cl\mathcal{L} \cap S_{ess} = \emptyset$. Then there exists an open connected $\mathcal{F}$-saturated neighbourhood of $\mathcal{L}$ which satisfies IB conditions.*

Proof. Consider an open connected neighbourhood U of $\mathcal{P} = Cl\mathcal{L}$ which can be retracted to $\mathcal{P}$. This gives us a primitive function f for $\omega|_U$ (this is IB1). For any $p \in \mathcal{P} \cap S$, the set $Cl\mathcal{L}$ separates a Morse neighbourhood around p into two parts: $\{f > 0\}$ and $\{f < 0\}$. This crucial observation is due to the Morse index of p having intermediate value. Since every foliated chart around regular points of $\mathcal{P}$ is also separated in a similar way ($\{f > 0\}$ and $\{f < 0\}$), we have the whole U is separated by $\mathcal{P}$ into $\{f > 0\}$ and $\{f < 0\}$. Therefore $\partial U \cap \{f = 0\} = \emptyset$ and hence we can assume U to be $\mathcal{F}$-saturated (by making it smaller, if necessary). We can also assume that $\partial U \cap S = \mathcal{P} \cap S$. This gives us IB2. Note that IB3 is satisfied by hyperbolicity of the zero set $\mathcal{P} \cap S$. To see IB4 we cover U be finitely many Morse neighbourhoods and foliated charts, note that within every such a model neighbourhood (a foliated chart or a Morse neighbourhood) all the level sets $\{f = c\}$ are connected and then we can force the connectedness of global level sets by traveling from one model neighbourhood to the other using the fact that U is $\mathcal{F}$ saturated. □

An immediate corollary of Lemma 9 is that if $\mathcal{P} \in C^0_\omega$ does not contain essential zeros, then it is not a member of the set C_ω. Note, however, that even if $\mathcal{P} \in C^0_\omega$ does contain essential zeros, it does not necessarily belong to C_ω. The following theorem is a remark in a paper [5] by Calabi. E. Calabi states it there without a proof only saying "it can be shown by a method of continuity that..." Here this statement comes as an immediate corollary of the technique we worked out by now.

Theorem 15. *Let ω be a closed Morse form on a closed n-dimensional manifold M, such that there are no zeros of index 1 or $n-1$. Then it is transitive if and only if it has no zeros of indices 0 and n.*

Proof. One direction is clear, indeed, if a form has at least one zero of index 0 or n, then it is automatically nontransitive. So we assume $S_{top} = \emptyset$ and show transitivity. Note that since $S_{ess} = \emptyset$, we have that the characteristic set K_ω is also empty. If there there exists a point $p \in M \setminus S$ with $\mathcal{L}_p$ not closed in $M \setminus S$, then $(M, \omega) \in \mathcal{C}$ and the form ω is transitive by Theorem 11. If, alternatively, such a point p does not exists, then, as it follows from Lemma 9, the foliated manifold (M, ω) does not belong to the class $\mathcal{C}$ and we apply Theorem 14.□

We give one result, similar in spirit, which may be viewed as an application of the concept of transitivity to a purely topological statement about foliations.

Theorem 16. *Let ω be a closed Morse form on a closed n-dimensional manifold M, such that there is at least one zero of ω of index 0 or n and at least one leaf of $\mathcal{F}$ not closed in $M \setminus S$, then there is at least one zero of ω of index 1 or $n-1$.*

Proof. Presence of zeros of index 0 or n implies immediately that ω is nontransitive. A nonclosed (in $M \setminus S$) leaf of $\mathcal{F}$ tells us that $(M, \omega) \in \mathcal{C}$. As it follows from Theorem 11 the set K_ω is nonempty, therefore C_ω is nonempty and hence the set S_{ess} is nonempty.□

The case $n = 2$ is special for dimension reasons and requires a little different definitions and notation to get later theorems correct. So we assume that $n > 2$. Let $\mathcal{P} \in C^0_\omega$. Let $V_{sing} \subset U_{sing}$ be two open sets one compactly contained in the other, both of them being disjoint unions of Morse neighbourhoods. Then there exists a smooth submanifold $\widehat{\mathcal{P}}$ of M such that the following holds true:
i) $\widehat{\mathcal{P}} \setminus U_{sing} = \mathcal{P} \setminus U_{sing}$,
ii) For a connected component V of V_{sing} and a Morse neighbourhood of $p \in \mathcal{P}$ at the same time, the intersection $\widehat{\mathcal{P}} \cap V$ is a nonsingular level set of a primitive function of $\omega|_V$.
Let $i_0 : \widehat{\mathcal{P}} \longrightarrow M$ be the obvious inclusion. Then there exists a homotopy $\{i_t : \widehat{\mathcal{P}} \longrightarrow M\}_{t\in[0,1]}$ such that $i_1(\widehat{\mathcal{P}}) = \mathcal{P}$ and every $i_t(\widehat{\mathcal{P}}), t \in [0,1)$ is a smooth submanifold of M satisfying i) and ii). Inside every connected component of V_{sing} this homotopy is just pushing a regular level set of a local primitive function of ω along the flowlines of some gradient-like vector field, eventually reaching the singular level set. We can assume U_{sing} to be small enough so that for all $t \in [0,1)$ the submanifolds $i_t(\widehat{\mathcal{P}})$ and $i_t(\widehat{\mathcal{P}}_j)$ are disjoint whenever $\mathcal{P}_i$ and $\mathcal{P}_j$ are different elements of C^0_ω. By property ii), outside U_{sing} any $\mathcal{P} \in C^0_\omega$ is a leaf of the kernel foliation of ω, so it bears the canonical co-orientation — the direction of decrease of a local primitive function of ω. This induces the canonical orientation for $\widehat{\mathcal{P}}$. The image of the fundamental class $\widehat{\mathcal{P}}_{or}$ of $\widehat{\mathcal{P}}$ in $H_{n-1}(M)$ under i_0 will be denoted by $[\widehat{\mathcal{P}}]$. This orientation procedure is the same that was used for elements of K^0_ω. We use the map i_1 to make the following definition:

Definition 17. *Consider $\mathcal{P} \in C^0_\omega$. The image in $H_{n-1}(M)$ of the fundamental class $\widehat{\mathcal{P}}_{or}$ under the induced map $i_{1\star} : H_{n-1}(\widehat{\mathcal{P}}) \longrightarrow H_{n-1}(M)$ will be denoted by $[\mathcal{P}]$. In brief: $[\mathcal{P}] = i_{1\star}(\widehat{\mathcal{P}}_{or})$.*

Philosophically speaking, the element $[\mathcal{P}]$ in $H_{n-1}(M)$ should be thought of as the image of the "fundamental class of $\mathcal{P}$" in homology under the obvious inclusion. Of course, $\mathcal{P}$ is not a manifold, so defining the notion of fundamental class requires some additional effort, but we do not care, since we have a rigorous definition of $[\mathcal{P}]$, which does not rely on any slippery concept. We apply the homotopy axiom to the continuous family $\{i_t\}_{t\in[0,1]}$ of maps: $\widehat{\mathcal{P}} \longrightarrow M$ to get that $i_{0\star} = i_{1\star}$ and therefore

$$[\mathcal{P}] = [\widehat{\mathcal{P}}]. \tag{4.1}$$

We also make following notation $[\mathcal{P}]_1$ — the image of $H_1(\mathcal{P})$ in $H_1(M)$ under the map induced by the canonical inclusion of $\mathcal{P}$ in M. Note that whereas $[\mathcal{P}]$ is an element in $H_{n-1}(M)$, the above defined $[\mathcal{P}]_1$ is a subset of $H_1(M)$. Consider $\mathcal{P}_1, ..., \mathcal{P}_l \in C^0_\omega$ and let $\mathcal{K} = \mathcal{P}_1 \cup ... \cup \mathcal{P}_l \in C^0_\omega$. Since the homotopy bringing $\mathcal{P}_{i(sm)}$ to $\mathcal{P}_i$ is happening only near zeros of ω and near a nondegenerate zero of ω we can always choose a pair of points locally separated by all $i_t(\mathcal{P}_{sm})$, $t \in [0, 1]$, it is easy to see that $\mathcal{K}$ separates M, such that the boundary orientation of $\mathcal{K} \setminus S$ coincides with the canonical one if and only if the disjoint union $\mathcal{K}_{sm} := \mathcal{P}_{1(sm)} \cup ... \cup \mathcal{P}_{l(sm)} \in C^0_\omega$ of smooth submanifolds of M does so. The later is equivalent to

$$[\widehat{\mathcal{P}}_1] + ... + [\widehat{\mathcal{P}}_l] = 0$$

and this, in view of (4.1) is equivalent to

$$[\mathcal{P}_1] + ... + [\mathcal{P}_l] = 0.$$

Observe that $\mathcal{L}_1 \cup ... \cup \mathcal{L}_l$, for $\mathcal{L}_i \in K^0_\omega, i = 1, ..., l$ separates $M \setminus S$ if and only if $\mathcal{P}_1 \cup ... \cup \mathcal{P}_l$ for $\mathcal{P}_i = Cl\mathcal{L}_i \in C^0_\omega, i = 1, ..., l$ separates M.

This leads us to the following "Morse" version of Theorem 11, sometimes referred to as a characterization theorem.

Theorem 17. *Let ω be a closed Morse 1-form on closed connected oriented manifold M. Then the following assertions are equivalent:*
1) The form ω is nontransitive.
2) There exist $\mathcal{P}_1, \mathcal{P}_2, ..., \mathcal{P}_l \in C^0_\omega$ with

$$[\mathcal{P}_1] + [\mathcal{P}_2] + ... + [\mathcal{P}_l] = 0. \tag{4.2}$$

Assume in addition that $(M, \omega) \in \mathcal{C}$, then each of the above is equivalent to the following:
3) There exist $\mathcal{P}_1, \mathcal{P}_2, ..., \mathcal{P}_l \in C_\omega^w$ with (4.2).

In the discussion above we have treated the case $n = dimM > 2$. Now we set up the definitions and state the characterization theorem for the case $n = 2$.

Definition 18. *A homoclinic orbit of ω is an element $\mathcal{P} \in C_\omega^0$, such that $\mathcal{P} \cap S = \mathcal{P} \cap S_{ess} = \{p\}$ - a zero of Morse index 1 (a hyperbolic zero).*

Definition 19. *A heteroclinic orbit of ω is an element $\mathcal{P} \in C_\omega^0$, such that $\mathcal{P} \cap S = \{p, q\}$ — a pair of distinct zeros of Morse index 1.*

Definition 20. *A heteroclinic pair of ω is a pair $(\mathcal{P}_1, \mathcal{P}_2)$ of distinct elements of C_ω^0, such that $\mathcal{P}_1 \cap S = \mathcal{P}_2 \cap S = \{p, q\}$ — a pair of distinct zeros of Morse index 1.*

Every homoclinic orbit $\mathcal{P}$ of ω can be given a canonical co-orientation — the direction of decrease of the local primitive function of ω. This together with the orientation of the ambient manifold M gives us the canonical orientation of $\mathcal{P}$. The image of the canonical orientation class of $\mathcal{P}$ in homology of M will be denoted by $[\mathcal{P}]$. Note that this orientation procedure is not that easy with heteroclinic pair, since it consists of two pieces and the two orientations may mismatch (say both $\mathcal{P}_1$ and $\mathcal{P}_2$ are oriented "from p to q"). This leads to the following definition.

Definition 21. *Let $(\mathcal{P}_1, \mathcal{P}_2)$ be a heteroclinic pair. Both $\mathcal{P}_1$ and $\mathcal{P}_2$ get a canonical orientation as explained above. If the two orientations piece together to give an orientation on $(\mathcal{P}_1 \cup \mathcal{P}_2)$, then the union $(\mathcal{P}_1 \cup \mathcal{P}_2)$ is called a true heteroclinic pair. Otherwise (if the two orientations mismatch)it is called a virtual heteroclinic pair.*

Let $\mathcal{P}$ be a true heteroclinic pair, then the image of the canonical orientation class of $\mathcal{P}$ in the homology of M will be denoted by $[\mathcal{P}]$.

Definition 22. *The set of all regular closed leaves of $\mathcal{F}$, homoclinic orbits and true heteroclinic pairs of ω will be denoted by D_ω^0.*

Remark on the notation. For $n = 2$ we have $H_1 = H_{n-1}$, so we do not introduce "$[\mathcal{P}]_1$" since that would be redundant.

Now we are in a position to formulate the 2-dimensional characterization theorem.

Theorem 18. *Let ω be a closed Morse 1-form on closed connected oriented 2-dimensional manifold M. Then ω is nontransitive if and only there exist $\mathcal{P}_1, \mathcal{P}_2, ..., \mathcal{P}_l \in D^0_\omega$ with*

$$[\mathcal{P}_1] + [\mathcal{P}_2] + ... + [\mathcal{P}_l] = 0. \tag{4.3}$$

The proof is completely analogous to that of Theorem 17, it is even easier in the 2-dimensional case, since all the elements of D^0_ω are topological manifolds. Note that if there are no heteroclinic orbits at all, then $D^0_\omega = C^0_\omega$ and Theorem 17 applies in 2-dimensional case without any changes in terminology.

Now we are back to the general case $n \geq 2$. It is useful to remark at this point that although the relation (4.2) formed by regular closed leaves obstructs transitivity, as regular closed leaves belong to C^0_ω, it is not true, that if the form is nontransitive, then we can always find this homological obstruction presented by regular closed leaves of the kernel foliation. However in one important ("generic") case this hope is true.

Definition 23. *Let $\mathcal{P} \in C^0_\omega$ contain 2 or more essential zeros of ω. Then $\mathcal{P}$ will be called heteroclinic.*

Definition 24. *Let ω be a closed Morse form on M such that there exists a heteroclinic element of C^0_ω. Then ω will be called heteroclinic. The set all heteroclinic forms will be denoted by Het.*

Heteroclinic forms are not generic — given a heteroclinic form ω, it can be removed from the set Het by a small perturbation within its cohomology class.

Proposition 8. *Let ω be a non-heteroclinic closed Morse form. Then for every $\mathcal{P} \in C^0_\omega$ there is a closed regular leaf $\mathcal{Q}$ of the kernel foliation of ω inducing the same image in homology: $[\mathcal{Q}] = [\mathcal{P}]$, $[\mathcal{Q}]_1 = [\mathcal{P}]_1$.*

Proof Let $\mathcal{P} \in C_\omega$. Let U be an open contractible neighbourhood of $\mathcal{P}$. Let f be a primitive function for ω in this neighbourhood, normalized such that $f|_\mathcal{P} = 0$. Since ω is not heteroclinic, we have only one essential zero of ω lying on $\mathcal{P}$, and consequently at least one of the two parts into which $\mathcal{P}$ separates U does not contain preimages of zero under f. We denote this part of U by U_- and without loss of generality assume that $f|_{U_-} \in (-\infty, 0)$. The boundary ∂U_- of U_- consists of two connected components: one is $\mathcal{P}$ and the other $\partial U_- \setminus \mathcal{P}$ with $f_{\partial U_- \setminus \mathcal{P}}$ being strictly negative and bounded away from zero. Consider $sup f|_{\partial U_- \setminus \mathcal{P}} = \rho < 0$. Since $f_{U_-}^{-1}(0.33\rho) \cap \partial U_- = \emptyset$, we have that $\mathcal{Q} := f_{U_-}^{-1}(0.33\rho)$ is compactly contained in U_-. Therefore, $\mathcal{Q}$ is a regular closed leaf of $\mathcal{F}$. Let $h_0 : \mathcal{Q} \longrightarrow M$ be the obvious inclusion. Then there exists a homotopy $\{h_t : \mathcal{Q} \longrightarrow U\}_{t \in [0,1]}$, such that $h_1(\mathcal{Q}) = \mathcal{P}_{sm}$. This homotopy may be described as pushing a regular level set of a local primitive function of ω along the flowlines of some gradient-like vector field. By the homotopy axiom we have: $[\mathcal{Q}] = [\mathcal{P}]$ and $[\mathcal{Q}]_1 = [\mathcal{P}]_1$.

This gives us the following version of Theorem 17 for non-heteroclinic forms.

Theorem 19. *Let ω be a closed Morse 1-form on a closed connected oriented manifold M. Assume that $\omega \notin Het$. Then ω is nontransitive if and only if one can find $\mathcal{Q}_1, \mathcal{Q}_2, ..., \mathcal{Q}_l$ - regular closed leaves of $\mathcal{F}$, with*

$$[\mathcal{Q}_1] + [\mathcal{Q}_2] + ... + [\mathcal{Q}_l] = 0.$$

The following "No Blow up" theorem plays a crucial role in the proof of the main theorem of Section 4.3. The proof of it follows from the fact that elements of C^w_ω can not enter an IB foliation chart more than once and a Morse neighbourhood satisfying the conditions of Lemma 7 more than twice.

Theorem 20. *Let $\{\omega_m\}_{m \in \mathbb{N}}$ be a C^1-convergent sequence of nontransitive closed Morse 1-forms. Set $d_m := max\{diam\mathcal{P} | \mathcal{P} \in C^w_{\omega_m}\}$, where diameter is understood to be the intrinsic diameter, taken with respect to some fixed Riemannian metric. Then the sequence $\{d_m\}_{m \in \mathbb{N}}$ is bounded.*

Informally speaking this theorem states that elements of C^w_ω do not blow up when one takes C^1-convergent sequences of forms.

Definition 25. *Let p be a zero of ω of index $n-1$. Consider a primitive function f for ω in a Morse neighbourhood of p with a normalization $f(p) = 0$. In Morse coordinates: $f = x_1^2 - x_2^2 - ... - x_n^2$. Define*

$$\mathcal{L}_{loc}(p) := \{f = 0\} \setminus \{p\}.$$

This set consists of two (or four if $n = 2$) connected components:

$$\mathcal{L}_{loc}(p)^+ := \mathcal{L}_{loc}(p) \cap \{x_1 > 0\}$$

and

$$\mathcal{L}_{loc}(p)^- := \mathcal{L}_{loc}(p) \cap \{x_1 < 0\}$$

with $\mathcal{L}_{loc}(p)^+$ and $\mathcal{L}_{loc}(p)^-$ being connected if $n > 2$ and having 2 connected components otherwise. If $n > 2$ and p a zero of ω of index 1, then we define $\mathcal{L}_{loc}(p)$, $\mathcal{L}_{loc}(p)^+$ and $\mathcal{L}_{loc}(p)^-$ be replacing ω with $-\omega$.

Definition 26. *Let p be a zero of ω of index 1 or $n-1$. A leaf $\mathcal{L}$ of $\mathcal{F}$ is called a top attached to p if $\mathcal{L}_{loc}(p)^+ \subset \mathcal{L}$. Analogously, a leaf $\mathcal{L}$ of $\mathcal{F}$ is called a bottom attached to p if $\mathcal{L}_{loc}(p)^- \subset \mathcal{L}$.*

Note that it may happen that a leaf is both top and bottom attached to the same essential zero of ω. Let $\mathcal{L} \in K^0_\omega$, then we say that $\mathcal{P} = Cl\mathcal{L} \in C^0_\omega$ is top (bottom) attached to $p \in S$ if $\mathcal{L}$ is. Assume now that $\mathcal{P}$ is not heteroclinic. Consider an open neighbourhood U of $\mathcal{P}$, which can be retracted to $\mathcal{P}$. We integrate $\omega|_U$ to get a primitive function f ($df = \omega|_U$) with the normalization $f|_\mathcal{P} = 0$. Note that if $\mathcal{L}$ is top and bottom attached to p, then $\{f = 0\} = \mathcal{P}$, else $\{f = 0\} = \mathcal{P} \cup \mathcal{R}$, where $\mathcal{R}$ is contained in a Morse neighbourhood of p and is defined by the following equations: $\mathcal{R} = \mathcal{L}_{loc}(p)^+$ if $\mathcal{L}_{loc}(p)^- \subset \mathcal{P}$ and $\mathcal{R} = \mathcal{L}_{loc}(p)^-$ if $\mathcal{L}_{loc}(p)^+ \subset \mathcal{P}$.

Lemma 10. *Assume that $\mathcal{P} \in C^0_\omega$ contains exactly one essential zero p of ω. According to the above notation $\mathcal{P}$ is top (bottom) attached to p. Consider an open neighbourhood U of $\mathcal{P}$ as above. Let $\tilde{\omega}$ be a closed 1-form sufficiently C^1 close to ω. Assume furthermore that $\tilde{\omega}|_U$ is exact. Let $\tilde{p}$ be a unique essential zero of $\tilde{\omega}$ in U. Let $\tilde{\mathcal{L}}$ be a top (bottom) attached to $\tilde{p}$ (with respect to the kernel foliation $\tilde{\mathcal{F}}$ of $\tilde{\omega}$). Then $\tilde{\mathcal{L}}$ is contained in U and closed in $M \setminus \tilde{S}$.*

Proof. Let f a primitive function for the form $\omega|_U$ normalized so that $f|_\mathcal{P} = 0$. Consider a primitive function $\tilde{f}$ of $\tilde{\omega}$ and normalize it such that $\tilde{f}(p) = 0$. With this normalization $\tilde{f}$ is C^2-close to f. We assume that p has index $n-1$, the case of index 1 is completely analogous. We also assume that $\mathcal{P}$ is top attached to p (the "bottom" case is analogous). Let U_M be a common Morse neighbourhood for p and $\tilde{p}$. That is we have to sets of coordinates on U_M: $x_1, ..., x_n$ and $\tilde{x}_1, ..., \tilde{x}_n$. With f looking like $x_1^2 - ... - x_n^2$ and $\tilde{f}$ looking like $c + \tilde{x}_1^2 - ... - \tilde{x}_n^2$, for some $c \in \mathbb{R}$, where $x_1 = ... = x_n = 0$ corresponds to p and $\tilde{x}_1 = ... = \tilde{x}_n = 0$ corresponds to $\tilde{p}$. Moreover the diffeomorphism bringing one set of coordinates to the other is close to identity. See Lemma 20 in the Appendix. In particular, we have that $\tilde{\mathcal{L}}_{loc}(\tilde{p})^+$ is (Hausdorff-)close to $\mathcal{L}_{loc}(p)^+$ and $\tilde{\mathcal{L}}_{loc}(\tilde{p})^-$ is (Hausdorff-)close to $\mathcal{L}_{loc}(p)^-$. Since $\tilde{\mathcal{L}}$ is top attached to $\tilde{p}$, we have that $\tilde{\mathcal{L}}_{loc}(\tilde{p})^+ \subset \tilde{\mathcal{L}}$. This motivates us to consider the connected component of $\{\tilde{f} = \tilde{f}(\tilde{p})\} \setminus \tilde{S}$ containing $\tilde{\mathcal{L}}_{loc}(\tilde{p})^+$. Let such connected component be denoted by $C^+_{\tilde{f}}$. Assume for a moment that we have that $C^+_{\tilde{f}} \cap \partial U = \emptyset$. Then $C^+_{\tilde{f}}$ is a leaf of $\tilde{\mathcal{F}}$ itself and therefore coincides with $\tilde{\mathcal{L}}$. Since $C^+_{\tilde{f}}$ is contained in U by construction and is closed in $M \setminus \tilde{S}$ as a part of a level set, we are done modulo our assumption that $C^+_{\tilde{f}} \cap \partial U = \emptyset$. To see this we would like to say that $C^+_{\tilde{f}}$ and $\mathcal{L}$ are connected components of level sets of $\tilde{f}$ respectively f, so they must be close to each other, and hence $\mathcal{L} \cap \partial U = \emptyset$ should give us that $C^+_{\tilde{f}} \cap \partial U = \emptyset$. Unfortunately, the above mentioned level sets are singular, so the idea is to subtract something from U, so that on the complement both of our connected components of the relevant level sets remain connected and become regular. First, we take a small open neighbourhood U_p of $\{p, \tilde{p}\}$, serving as a Morse neighbourhood for both p and $\tilde{p}$. We can take U_p to be literally a ball in Morse coordinates for f near p. Then in Morse coordinates for $\tilde{f}$ it will look like a slightly deformed ball (recall that the diffeomorphism bringing one set of coordinates to the other is close to identity). The sets $\mathcal{L} \setminus U_p$ and $C^+_{\tilde{f}} \setminus U_p$ are connected. This is so because $\mathcal{L}$ and $C^+_{\tilde{f}}$ are singular near p respectively $\tilde{p}$. Let $q_1, ..., q_l$ be possible intermediate value (!) zeros of ω on $\mathcal{L}$ and let $\tilde{q}_1, ..., \tilde{q}_l$ be the corresponding set of zeros of $\tilde{\omega}$ that lie close by.

We consider small balls U_i around q_i, $i = 1, ..., l$ containing $\tilde{q}_i$. They will play a role in our analysis near $q_i, \tilde{q}_i$ analogous to that played by U_p near $p, \tilde{p}$. Let us look at how $\mathcal{L} \setminus U_p$ respectively $C^+_{\tilde{f}} \setminus U_p$ passes through U_1. Both of them fall into the same pattern: a level set (singular or regular) of a function near its nondegenerate critical point with the index of intermediate value. Such level sets remain connected after removing a small open neighbourhood containing the critical point. (The open neighbourhood is not necessarily the round ball around the critical point, but a maybe a slightly deformed one). So the sets $\mathcal{L} \setminus (U_p \cup U_1)$ and $C^+_{\tilde{f}} \setminus (U_p \cup U_1)$ are connected. Proceeding inductively, we get that the sets $\widehat{\mathcal{L}} := \mathcal{L} \setminus (U_p \cup U_1 \cup ... \cup U_l)$ and $\widehat{C}^+ := C^+_{\tilde{f}} \setminus (U_p \cup U_1 \cup ... \cup U_l)$ are connected. Now $\widehat{\mathcal{L}}$ and $\widehat{C}^+$ are two connected level sets of smooth functions $f|_{U\setminus(U_p\cup U_1\ldots\cup U_l)}$ respectively $\tilde{f}|_{U\setminus(U_p\cup U_1\ldots\cup U_l)}$. The two functions are C^2-close to each other both with nonvanishing differentials up to the boundary of their common domain of definition. Moreover, the set $\widehat{\mathcal{L}}$ contains $\mathcal{L}_{loc}(p)^+ \setminus U_p$, which is close to $\tilde{\mathcal{L}}_{loc}(\tilde{p})^+ \setminus U_p$, contained in $\widehat{C}^+$. So the two sets $\widehat{\mathcal{L}}$ and $\widehat{C}^+$ must be close to each other. The first one does not intersect ∂U, so the second one also does not, i.e $\widehat{C}^+ \cap \partial U = \emptyset$. Hence $C^+_{\tilde{f}} \cap \partial U$ is also empty. This completes the proof.□

We would like to make a remark of a psychological nature. This lemma does not deserve a proof of this length. One even may want to remove the whole proof replacing it with an expression like “obvious”. The purpose of writing so detailed proof is to help the reader to get closer to the objects we are playing with.

Lemma 11. *Assume we are in the situation of the last lemma. Let* $\tilde{\mathcal{P}} = Cl\tilde{\mathcal{L}}$ *denote the closure of* $\mathcal{L}$*. Then* $[\tilde{\mathcal{P}}] = [\mathcal{P}]$ *and* $[\tilde{\mathcal{P}}]_1 = [\mathcal{P}]_1$*.*

Proof. Let U_{sing} be a disjoint union of Morse neighbourhoods of zeros of ω lying in $\mathcal{P}$ such that the complement $U_{tub} = U \setminus U_{sing}$ of U_{sing} in U is diffeomorphic to $Y \times (-a, b)$ for a compact $(n-1)$-manifold Y (with boundary) with f corresponding to the projection to the second factor and $\mathcal{P} \cap U_{tub}$ corresponding to $Y \times \{0\}$. We identify U_{tub} with $Y \times (-a, b)$ via this diffeomorphism.

First, we look at the image in $H_{n-1}(M)$. Let $[\gamma]$ be a homology class in $H_1(M)$. We can assume that its representative γ does not intersect U_{sing}. Since algebraic intersection number of any line segment $y \times (-a, b)$ with $\mathcal{P} \cap U_{tub}$ and $\tilde{\mathcal{P}} \cap U_{tub}$ is the same (plus one), we have that homology intersection number of $([\tilde{\mathcal{P}}] - [\mathcal{P}])$ with $[\gamma]$ is equal to zero. Let $\alpha := PD([\tilde{\mathcal{P}}] - [\mathcal{P}]) \in H^1(M, \mathbb{Z})$ be the Poincaré dual of $[\tilde{\mathcal{P}}] - [\mathcal{P}]$. We identify α with an element of $Hom(H_1(M, \mathbb{Z}))$ via the isomorphism given by the Universal Coefficient Theorem. The intersection number of $([\tilde{\mathcal{P}}] - [\mathcal{P}])$ with $[\gamma]$ being equal to zero translates to the evaluation of α on $[\gamma]$ being equal to zero. Since γ was arbitrary this gives us that $\alpha = 0$ and hence $[\tilde{\mathcal{P}}] = [\mathcal{P}]$.

Next, we look at the image in $H_1(M)$. Note that if $n = 2$, we are done already since $1 = n - 1$. So we can assume that $n > 2$. In this case we can smoothly deform any closed loop γ in $\mathcal{P}$ representing a homology class in $H_1(\mathcal{P})$ such that $\gamma \subset U_{tub}$ and then move along the line segments $y \times (-a, b)$ to get a closed loop $\tilde{\gamma}$ in $\tilde{\mathcal{P}}$ representing the same class in $H_1(M)$ as γ. Conversely, starting from any closed loop $\tilde{\gamma}$ in $\tilde{\mathcal{P}}$ we can smoothly perturb it to a loop γ in $\mathcal{P}$ representing the same homology class in $H_1(M)$ as $\tilde{\gamma}$. This proves that $[\tilde{\mathcal{P}}]_1 = [\mathcal{P}]_1$.$\square$

4.3 Convergence.

The following definition is a precise version of saying that "a sequence of singular closed leaves $\mathcal{P}_m$ of ω_m converges to a singular closed leaf $\mathcal{P}$ of ω".

Definition 27. *Let $\{\omega_m\}_{m\in\mathbb{N}}$, ω be closed Morse forms and $\omega_m \xrightarrow{C^1} \omega$. Let $\mathcal{P}_m \in C^0_{\omega_m}$ and $\mathcal{P} \in C^0_{\omega}$. Moreover, assume that $\mathcal{P}_m$ contains exactly one essential zero of ω_m and $\mathcal{P}$ contains exactly one essential zero of ω. We say that the sequence $\{\mathcal{P}_m\}_{m\in\mathbb{N}}$ converges with respect to the C^1 topology to $\mathcal{P}$ and write $\mathcal{P}_m \xrightarrow{C^1} \mathcal{P}$ if there exist an open neighbourhood U of $\mathcal{P}$ retractable to $\mathcal{P}$ and a natural number N, such that for all $m > N$ we have the following two properties:*
i) $\omega_m|_U$ is exact,
ii)$\mathcal{P}_m \subset U$.

The following assertion, which is an immediate corollary of Lemma 11 states what we really need from the concept of convergence.

Proposition 9. *Let a sequence of closed Morse forms $\{\omega_m\}_{m\in\mathbb{N}}$ converges with respect to the C^1 topology to a closed Morse form ω. If $\mathcal{P}_m \in C^0_{\omega_m}$ and $\mathcal{P}_m \xrightarrow{C^1} \mathcal{P}$, where $\mathcal{P} \in C^0_{\omega}$, then there exists $N \in \mathbb{N}$, such that for all $m > N$ we have a stabilization in homology: $[\mathcal{P}_m] = [\mathcal{P}], [\mathcal{P}_m]_1 = [\mathcal{P}]_1$.*

And now we give a (crucial) criterion saying under what conditions we actually have C^1 convergence.

Theorem 21. *Let a sequence of closed Morse forms $\{\omega_m\}_{m\in\mathbb{N}}$ converge with respect to the C^1 topology to a closed Morse form ω. Let $p_m \to p$, where p_m is an essential zero of ω_m and p is an essential zero of ω. Let $\mathcal{P}_m \in C^w_{\omega_m}$ be top attached to p_m. Assume that ω is not a heteroclinic form. Then there exists a top attached element $\mathcal{P} \in C^0_{\omega}$ to p with $\mathcal{P}_m \xrightarrow{C^1} \mathcal{P}$. Analogous statement holds if we replace "top" with "bottom".*

Proof. We may assume without loss of generality that $ind(p_m) = ind(p) = n-1$ (otherwise replace ω_m by $-\omega_m$). Consider a leaf $\mathcal{L}$ of $\mathcal{F}$ which contains

$\mathcal{L}^+_{loc_p}$. We show that $\mathcal{L}$ is closed in $M \setminus S$. Assume by contradiction that it is not, then there exists a point in $M \setminus S$ to which $\mathcal{L}$ accumulates, and therefore an IB foliated chart U for the kernel foliation of ω through which it passes twice (at least). In particular there exists a closed path γ in $\mathcal{L}$ which starts near p and then passes through U twice: $U \cap \gamma$ is union of two connected components γ_1 and γ_2. Choose a small neighbourhood $V \subset M \setminus S$ around γ, retractable to γ. Assuming that V is small enough we can achieve that $V \cap U$ consist of two connected components: $V_1 \supset \gamma_1$ and $V_2 \supset \gamma_2$. Since V is contractible, we can find smooth functions f, f_m on V, such that $df = \omega|_V, df_n = \omega_n|_V$. By making m big enough we can assume that $\mathcal{P}_m$ passes near the starting point of γ, say some point $\widehat{p_m} \in \mathcal{L}_m \cap V$ is close to the starting point of γ. Then (assuming m is big enough) we start the path γ_m at $\widehat{p_m}$ and draw it "along" γ inside the level set $\{f_m = f_m(p_m)\}$, so that γ_m passes through U twice: $U \cap \gamma_m$ consists of two connected components: $\gamma_{m1} \subset V_1$ and $\gamma_{m2} \subset V_2$. It means that $\mathcal{P}_m$ passes through U twice: $\mathcal{P}_m \cap U$ consists of two connected components: $\mathcal{P}_{m1} \subset V_1$ and $\mathcal{P}_{m2} \subset V_2$. Since for m big enough kernel foliations of ω and ω_m become arbitrarily close to each other, we can find an IB foliated chart U_m for the kernel foliation of ω_m inside U which is met by $\mathcal{P}_m$ twice. This is forbidden since $\mathcal{P}_m$ is nonwinding as a member of $C^w_{\omega_m}$. This contradiction shows that $\mathcal{L}$ is closed in $M \setminus S$. Set $\mathcal{P} := Cl\mathcal{L}$.

For the rest of the proof $length(\cdot)$ stands to denote the length of a path with respect to some reference Riemannian metric. We show that $\mathcal{P}_m \xrightarrow{C^1} \mathcal{P}$. Let U_{sing} be a disjoint union of Morse neighbourhoods of zeros of ω lying in $\mathcal{P}$ such that the complement $U_{tub} = U \setminus U_{sing}$ of U_{sing} in U is diffeomorphic to $Y \times (-a, a)$ for a compact $(n-1)$-manifold Y (with boundary) with f corresponding to the projection to the second factor and $\mathcal{P} \cap U_{tub}$ corresponding to $Y \times \{0\}$. We identify U_{tub} with $Y \times (-a, a)$ via this diffeomorphism.

The first step is to show that for m big enough $\mathcal{P}_m \cap \partial U \cap \partial U_{tub} = \emptyset$. Indeed, if not, we join a point $q_m \in \mathcal{P}_m \cap \partial U \cap \partial U_{tub}$ to some point $\widehat{p_m} \in \mathcal{L}^+_{loc_{p_m}}$ by a path γ_m within $\mathcal{P}_m \cap U$ with $length(\gamma_m) \leq d_m \leq C$, where $d_m = diam(\mathcal{P}_m)$ and the existence of a uniform constant C is guaranteed by "No Blow up" Lemma. On the one hand, since $|f(\widehat{q_m})|$ is equal to a and $f(\widehat{p_m})$ is close to zero, we have that the absolute value of the integral $\int_{\widehat{p_m}}^{q_m} \omega$ should be close to a. On the other hand,

$$|\int_{\widehat{p_m}}^{q_m} \omega| = |\int_{\widehat{p_m}}^{q_m} \omega - \omega_m| \leq length(\gamma_m)\epsilon_m \leq C\epsilon_m$$

with ϵ_m converging to zero as $\omega-\omega_m$ converges to zero with respect to the C^1 topology. It means that $\int_{\widehat{p_m}}^{q_m}\omega$ becomes arbitrarily close to zero as $m\to\infty$. This is a contradiction.

The next step is to show that $\omega_m|_U$ is exact. Indeed, if $n>2$, then given any homology class in $H_1(U)=H_1(\mathcal{P})$, we can find a closed loop $\gamma\subset\mathcal{P}$ representing this class, such that γ misses U_{sing}, i.e. $\gamma\subset U_{tub}$. Consider the holonomy of $\mathcal{P}_m$ along γ. First, the holonomy exists, because $\mathcal{P}_m$ can not "run away" through $\partial U\cap\partial U_{tub}$. Second, the holonomy is trivial, otherwise the leaf $\mathcal{P}_m$ meets some IB foliated chart of $\mathcal{F}$ twice, and therefore ($\mathcal{F}$ and $\mathcal{F}_m$ are close to each other) it comes through some IB foliated chart of $\mathcal{F}_m$ twice, but this is forbidden. The triviality of this holonomy implies that the form ω_m acts trivially on the homology class $[\gamma]$ of γ. Therefore, since $[\gamma]$ was arbitrary, we see that $\omega_m|_U$ is exact.

If $n=2$, then $\mathcal{P}$ is simply a homoclinic orbit and it is clear that not only $\mathcal{P}_m\cap\partial U\cap\partial U_{tub}=\emptyset$, but also $\mathcal{P}_m\cap\partial U\cap U_{sing}=\emptyset$ and therefore $\mathcal{P}_m\cap\partial U=\emptyset$. So we consider the holonomy of $\mathcal{P}_m$ along $\mathcal{P}$, which is well-defined and trivial, and hence we also get that $\omega_m|_U$ is exact.

The inclusion $\mathcal{P}_m\subset U$ follows from Lemma 10. □

In order to show the existence of $\mathcal{P}$ and the convergence $\mathcal{P}_m\xrightarrow{C^1}\mathcal{P}$ the nonwinding property of the elements $\mathcal{P}_m$ and the diameter bound it implies were used essentially. It seems plausible that both the existence of $\mathcal{P}$ and the convergence can be obtained just using the diameter bound without using the nonwinding property itself.

Lemma 12. *Let $\{\omega_m\}_{m\in\mathbb{N}}$ be a sequence of closed nontransitive Morse forms, such that $(M,\omega_m)\in\mathcal{C}$ for all m. Assume that $\omega_n\xrightarrow{C^1}\omega$, where ω is a closed (nontransitive) Morse form. Assume that ω is not heteroclinic. Then there exist $\mathcal{P}^1_m,...,\mathcal{P}^l_m\in C^w_{\omega_m}$ for each m, a natural number N and $\mathcal{P}^1,...,\mathcal{P}^l\in C^0_\omega$ with $[\mathcal{P}^i_m]=[\mathcal{P}^i]$, $[\mathcal{P}^i_m]_1=[\mathcal{P}^i]$ for all $m>N$ and $[\mathcal{P}^1]+...+[\mathcal{P}^l]=0$.*

Proof. Let $p^i_m, i=1,...,N$ be the zeros of ω_m, and $p^i, i=1,...,N$ — zeros of ω. Assume we have set the enumeration of zeros such that $p^i_m\to p^i$ as $m\to\infty$. Since ω_m is nontransitive, by Theorem 17 we get $\mathcal{P}^1_m,...,\mathcal{P}^l_m\in C^w_{\omega_m}$ with $[\mathcal{P}^1_m]+...+[\mathcal{P}^l_m]=0$ (relation (4.3)). By passing to a subsequence of $\{\omega_m\}_{m\in\mathbb{N}}$ if necessary (and we call it again $\{\omega_m\}_{m\in\mathbb{N}}$) and to a subset $\{1,2,...,l\}$ of $\{1,2,...,N\}$ we can achieve that for every ω_m and every zero $p^i_m, i=1,...,l$ of ω_m there exists $\mathcal{P}^i_m$ top (bottom) attached to p^i_m, such that $[\mathcal{P}^1_m]+...+[\mathcal{P}^l_m]=0$. Apply Theorem 21 and then Proposition 9.□

4.4 Main Theorem.

We denote the space of all closed Morse forms with the C^1 topology by $\mathcal{M}$, the subspace of transitive ones by $\mathcal{T}$. Nontransitive forms will be denoted by $\bar{\mathcal{T}}$ (as a complement). The cohomology class $[\omega]$ of ω may be viewed as an Abelian group homomorphism

$$[\omega] : H_1(M, \mathbb{Z}) \longrightarrow \mathbb{R}.$$

Since the target group $\mathbb{R}$ has no torsion we get that the torsion subgroup Tor of $H_1(M, \mathbb{Z})$ is automatically in the kernel $Ker[\omega]$ of $[\omega]$.

Definition 28. *Let ω be a closed 1-form. We say it is totally irrational, provided the kernel of the homomorphism $[\omega]$ coincides with Tor.*

Tensoring everything with $\mathbb{R}$ gives us an $\mathbb{R}$-linear map

$$[\omega]_{\mathbb{R}} : H_1(M, \mathbb{R}) \longrightarrow \mathbb{R},$$

whose kernel $Ker[\omega]_{\mathbb{R}}$ is a codimension 1 linear subspace of $H_1(M, \mathbb{R})$. Now $Ker[\omega]$ being equal to Tor is the same as $Ker[\omega]_{\mathbb{R}}$ missing integer lattice of $H_1(M, \mathbb{R})$ completely, i.e. $Ker[\omega]_{\mathbb{R}} \cap (H_1(M, \mathbb{Z})/Tor) = 0$. The subspace $Ker[\omega]_{\mathbb{R}}$ is positioned "totally irrationally" in $H_1(M, \mathbb{R})$. If $b_1(M) = dim H^1(M) > 1$, then the set of totally irrational forms is C^1 dense in the set of all closed forms ("the set of totally irrational positions is dense" and the map sending a form to its real cohomology class is continuous and open). The main property of totally irrational forms we need is that for every totally irrational closed 1-form $\widehat{\omega}$ and every $\mathcal{P} \in C^0_{\widehat{\omega}}$ we have $[\mathcal{P}]_1 \in Tor$. Indeed, every element of $[\mathcal{P}]_1$ trivially belongs to the kernel of $[\omega]$, since it is contained in a leaf of the kernel foliation of ω. Here is the main theorem of this chapter.

Theorem 22. *Let ω be a closed Morse form on M, such that $\omega \in Cl(Int\bar{\mathcal{T}})$. Assume that ω is not heteroclinic. Then $\omega \in Int\bar{\mathcal{T}}$.*

Proof. Assume that $b_1(M) > 1$. In this case we approximate ω by $\omega_m \in Int\bar{\mathcal{T}}$ in C^1. Since the set of totally irrational forms is dense in the set of all closed forms we may assume that ω_m is totally irrational. By passing to a subset we can achieve that either $\{(M, \omega_m)\}_{m\in\mathbb{N}} \subset \mathcal{C}$ or $(M, \omega_m) \notin C$ for all $m \in \mathbb{N}$. In the second case we are easily done. Indeed, $(M, \omega_m) \notin C$ together with the nontransitivity of ω according to Theorem 14 imply that ω_m has elliptic zeros (i.e. of index 0 or n), therefore ω does, and hence as elliptic zeros are stable under small C^1 perturbations we get that $\omega \in Int\bar{\mathcal{T}}$. So we stick to the first one: $\{(M, \omega_m)\}_{m \in \mathbb{N}} \subset \mathcal{C}$. We apply Lemma 12 to get $\mathcal{P}^1, ..., \mathcal{P}^l \in C^0_\omega$ with

$$(i)\ [\mathcal{P}^i]_1 \subset Tor$$

and

$$(ii)\ [\mathcal{P}^1] + ... + [\mathcal{P}^l] = 0.$$

Let U^i be a small open neighbourhood of $\mathcal{P}^i$ retractable to $\mathcal{P}^i$. Relation (i) implies that for any (!) closed 1-form $\tilde{\omega}$ the restrictions $\tilde{\omega}|_{U^i}$ are exact. In particular they are exact for Morse forms $\tilde{\omega} \in \mathcal{U}$, where $\mathcal{U}$ is a small C^1 neihghbourhood of ω in $\mathcal{M}$. Since ω is not heteroclinic, to any such a form $\tilde{\omega} \in \mathcal{U}$ Lemma 10 and Lemma 11 apply to give $\tilde{\mathcal{P}}^1, ..., \tilde{\mathcal{P}}^l \in C^0_{\tilde{\omega}}$ with $[\tilde{\mathcal{P}}^1] + ... + [\tilde{\mathcal{P}}^l] = 0$. Now we apply Theorem 17 to deduce nontransitivity of $\tilde{\omega}$. This shows that ω belongs to the set of nontransitive forms $\bar{\mathcal{T}}$ together with an open neighbourhood, i.e. ω belongs to the interior of $\bar{\mathcal{T}}$.

In the remaining case $b_1(M) = 1$ the proof is even easier. We apply Theorem 17 to get $\mathcal{P}^1, ..., \mathcal{P}^l \in C^0_\omega$ with (ii) and Let U^i be a small open neighbourhood of $\mathcal{P}^i$ retractable to $\mathcal{P}^i$. Now $b_1(M) = 1$ implies that for any closed 1-form $\tilde{\omega}$ the restrictions $\tilde{\omega}|_{U^i}$ are exact and we conclude the proof as above. □

4.5 Examples.

In this section we give two examples illustrating the results of the previous section.

Example 2. We take our manifold M to be the double torus Σ_2 — an oriented surface of genus 2. Theorem 18 now takes the following form.

Theorem 23. *Let ω be a closed Morse 1-form on Σ_2. Assume that all zeros of ω are hyperbolic. Then ω is nontransitive if and only if one can find $\mathcal{P} \in D^0_\omega$ with*

$$[\mathcal{P}] = 0. \tag{4.4}$$

If, in addition ω is not heteroclinic, then there exists a regular closed leaf $\mathcal{Q}$ with

$$[\mathcal{Q}] = 0. \tag{4.5}$$

Proof. Since we already have Theorem 18, the only nontrivial part to do here is to show that existence of $\mathcal{P}_1, \mathcal{P}_2, ..., \mathcal{P}_l \in D^0_\omega$ with (4.3) implies that $l = 1$. Assume that ω is not heteroclinic. Then by Theorem 19 we get l regular closed leaves of $\mathcal{F}$ whose union separates M on M_1 and M_2. Now both M_1 and M_2 are punctured Riemannian surfaces (say with genera g_1 and g_2 respectively) with l punctures. Then we can write out the genus of Σ_2 in terms of g_1, g_2 and l, that is $g_1 + g_2 + l - 1$. On the other hand we know that the last expression must be equal to 2. Note that both g_1 and g_2 are positive, otherwise the restriction of ω on either M_1 or M_2 is exact and we have an elliptic zero, which is forbidden. It means that l can only be equal to 1. If ω is heteroclinic, then the proof is a little longer, but the crucial point is the same, so we leave it out.□

Let ω be a nontransitive closed Morse form on the double torus without elliptic zeros. We take up the main question for this chapter: "when $\omega \in Int\bar{\mathcal{T}}$?"

Theorem 24. *Let ω be a closed non-heteroclinic Morse 1-form on Σ_2. Let all the zeros of ω be hyperbolic. Assume that $\omega \in \bar{\mathcal{T}}$. Then $\omega \in Int\bar{\mathcal{T}}$*

Proof. We apply Theorem 23 to get $\mathcal{Q}$ — a regular closed leaf of the kernel foliation with $[\mathcal{Q}] = 0$. Let U be a small neighbourhood of $\mathcal{Q}$ such that $\mathcal{Q}$ is a deformation retract of U. We integrate $\omega|_U$ to give a primitive function f, normalized such that $\mathcal{Q} = \{f = 0\}$. Let ω_1 be sufficiently C^1-close to ω. Since the image of $\mathcal{Q}$ in $H_1(M)$ is trivial (here it plays a role that for dimension reasons $1 = n-1$), we can integrate ω_1 to give a primitive function f_1, normalized such that $f_1(p) = 0$, for some $p \in \mathcal{Q}$. Since $f|_{\partial U}$ is bounded away from zero, we have that $\{f_1 = 0\} \cap \partial U = \emptyset$, and hence $\{f_1 = 0\} \subset U$. Moreover, by joining f_1 to f within smooth functions without critical points (say linearly), we get a continuous deformation of $\{f_1 = 0\}$ to $\mathcal{Q}$. Thus, by the homotopy axiom we see that $\{f_1 = 0\}$ is a regular closed leaf of ω_1, whose image in homology is zero, so ω_1 is nontransitive. In other words, ω belongs to $\bar{\mathcal{T}}$ together with an open neighbourhood, i.e. $\omega \in Int\bar{\mathcal{T}}$.□

So we see that in the case of a double torus we can "by hand" get results even stronger than the main theorem suggests. All nontransitive Morse forms which are non-heteroclinic belong to the interior of the set of nontransitive forms.

If all the leaves of the kernel foliation of the above form ω are closed, then the leaf space Γ is a non-Calabi graph, first mentioned by Calabi in his paper [5]. As an illustration, we give one more example of a 1-form having the same graph as the leaf space of the kernel foliation.

Example 3. Let B be the standard unit ball in $\mathbb{R}^3$. Let B_1 and B_2 be the balls centered at the points $p_1, p_2 \in B$, so small that $B_1 \subset B$, $B_2 \subset B$ and $B_1 \cap B_2 = \emptyset$. Let $S := \partial B$, $S_1 := \partial B_1$, $S_2 := \partial B_2$ be the three copies of the to 2-sphere S^2. Let C be the cobordism from S to $S_1 \dot{\cup} S_2$ given by $B \setminus (B_1 \dot{\cup} B_2)$. This cobordism can be given a Morse function f with $f|_S = 1$, $f|_{S_1 \cup S_2} = 2$ and one critical point of index 2 at the level 3/2. Take S_1 and S_2 the copies of the 2-sphere at the level 1. Apply C to S_1 to get S_3 and S_4 at the level 2 and a trivial cobordism to S_2 to get S_2 at the level 2. Now apply C^{-1} to S_2 and S_3 to get S_5 at the level 3 and a trivial cobordism to S_4 to get S_4 at the level 3. This gives a cobordism $\tilde{M}$ from $S_1 \dot{\cup} S_2$ to $S_4 \dot{\cup} S_5$ together with a Morse function h on this cobordism with $h|_{S_1 \dot{\cup} S_2} = 1$, $h|_{S_4 \dot{\cup} S_5} = 3$ and two critical points, one at the level 3/2 of index 2 (disconnecting) and the other at the level 5/2 of index 1 (connecting). We glue S_1 to S_4 and S_2 to S_5. Under this gluing the cobordism $\tilde{M}$ transforms to a closed 3-manifold M, the 1-form dh on $\tilde{M}$ descends to a closed 1-form ω on M, whose leaf space is the graph Γ. Since the graph Γ is non-Calabi, we have that $\omega \in \bar{\mathcal{T}}$. But now we ask a finer question: is it true that $\omega \in Int\bar{\mathcal{T}}$?. The answer is yes. Indeed, since the leaf space of ω is the same as in the first example, there exists a regular closed leaf $\mathcal{Q}$ with $[\mathcal{Q}] = 0$. Note that $\mathcal{Q}$ must be a copy a the 2-sphere, so $[\mathcal{Q}]_1 = \{0\}$, because the 2-sphere is simply connected. The rest is standard: we take a small open neighbourhood U, retractable to $\mathcal{Q}$, argue that any closed 1-form $\tilde{\omega}$ is exact, when restricted to U, because the image of U in first homology of M is trivial e.t.c.

Again, we see that in this particular example we get more information about the above constructed form ω than the main theorem gives us without applying the theorem. We were able to deduce that a nontransitive non-heteroclinic form ω belongs to the interior of the set of nontransitive forms directly.

Chapter 5

Smooth functions near isolated critical points: reduction to normal forms

Consider a C^∞ function defined on an open ball around $(0,0) \in \mathbb{R}^2$ with the following Taylor expansion around zero:

$$f = Re(x+iy)^m + h.o.,$$

where $Re(x+iy)^m$ is the leading term of f and $h.o.$ stands for the terms of order higher than m. In this chapter we would like to address the following question. When can f be brought by a smooth change of variables to the normal form $f_0 = Re(x+iy)^m$ in some open neighbourhood of the origin? In other words, when does there exist an open neighbourhood U around $(0,0) \in \mathbb{R}^2$ and a diffeomorphism $\phi : U \longrightarrow \phi(U)$ fixing the origin with $f|_U = f_0 \circ \phi|_U$? The notation f, f_0, $h.o.$ is fixed throughout this chapter. In Section 5.1 we give the answer to this question, this is Theorem 25 — the main theorem of this chapter. The first proof of this theorem is based on the work of Arnold [1] and forms the core of Section 5.1. The second independent proof of this theorem occupies the rest of the chapter. The later sections correspond exactly to the steps of the proof as outlined in the introduction.

5.1 Main results.

Theorem 25. *Let $f = Re(x+iy)^m + h.o.$ be a function defined on an open ball around the origin in $\mathbb{R}^2$, where h.o. denotes the terms in the Taylor expansion around the origin of order higher than m. Then for $m = 1, 2, 3, 4$ the function f can always be brought to $Re(x+iy)^m$ by a smooth change of coordinates in some open ball around zero. For $m > 4$, the sufficient condition for such a coordinate transformation to exist is that h.o. starts with the order $2m-3$ or higher.*

Proof. First, note that for $m = 1$ the theorem follows from the flowbox theorem for vector fields, for $m = 2$ it is the Morse Lemma, the case $m = 3$ is covered by Arnold in [1], so we assume $m \geq 4$. For this we apply Lemma 3.2 in [1] with $r = 2m - 4$. This lemma says that for the desired coordinate transformation to exist it suffices that for each function of the form $\chi = Q_{r+1} + Q_{r+2} + ...$, there exist $h_1 = Q_l^1 + Q_{l+1}^1 + ...$ and $h_2 = Q_l^2 + Q_{l+1}^2 + ...$ such that

$$\chi = h_1 f_x + h_2 f_y \mod (r+2). \tag{5.1}$$

Here Q_j, Q_j^1, Q_j^2 denote homogeneous polynomials of order j in x, y and mod $(r+2)$ means modulo functions whose Taylor expansion begins with something of order $r+2$. On the one hand the leading terms in Equation (5.1) give us

$$Q_{r+1} = Q_l^1 f_{0x} + Q_l^2 f_{0y}. \tag{5.2}$$

So we should take $l = r - m + 2$ and solve the above equation with respect to Q_l^1 and Q_l^2. On the other hand once (5.2) is satisfied, Equation (5.1) is also satisfied regardless of what is happening with higher order terms since (5.1) is understood mod $(r+2)$ anyway. So we are left with (5.2) which is a linear system of equations when we write out the homogeneous polynomials involved as sums of monomials. In this setup we are given the coefficients of Q_{r+1} and we are looking for coefficients of Q_l^1 and Q_l^2. So the number of unknowns is $2(l+1) = 2(r-m+2+1) = 2(r-m+3)$ and the number of equations is $r+2$. Now we substitute $r = 2m-4$. The number of unknowns becomes $2(r-m+3) = 2(2m-4-m+3) = 2(m-1)$ and the number of equations $r+2 = 2m-4+2 = 2(m-1)$. Since the number of equations is equal to the number of unknowns, we are done provided that the $2(m-1) \times 2(m-1)$ matrix of the above system determined by the leading term of f_x and f_y is nonsingular. To prove that the matrix is, indeed, nonsingular we rewrite

Equation (5.2) in a more convenient form. Note that $f_{0x} = Re(x+iy)^m_x = mRe(x+iy)^{m-1}$, $f_{0y} = Re(x+iy)^m_y = mRei(x+iy)^{m-1} = -mIm(x+iy)^{m-1}$. So if we divide both parts of (5.2) by m we can write it as

$$Q_{r+1} = Re(Q_l z^{m-1}),$$

where $Q_l = Q_l^1 + iQ_l^2$. Let $Hom_\star$ denote the graded algebra of (real valued) polynomials in (x, y) and let $Hom_\star \times Hom_\star$ denote the bi-graded algebra of pairs (P_1, P_2) of polynomials. We write (P_1, P_2) as $P = (P_1 + iP_2)$ viewing a pair of real-valued polynomials as one complex valued polynomial in (x, y). The notation $P = (P_1 + iP_2)$ suggests what the algebra structure on $Hom_\star \times Hom_\star$ should be. This way $Hom_\star \times Hom_\star$ also gets a structure of $\mathbb{C}$-vector space. To emphasize complex issues we also introduce the notation $Hom_{\mathbb{C}}$ for $Hom_\star \times Hom_\star$ and use it synonymously. Note that $P = P_1 + iP_2$ is a complex valued, but not necessarily a holomorphic function of the complex variable $z = x + iy$. In this setup the solvability of Equation (5.2) may be expressed as follows. Let the map F be a composition of multiplication with z^{m-1} and then taking the real part. Solvability of (5.2) is equivalent to this map being surjective.

The following action ρ of S^1 (considered as the unit circle in $\mathbb{C}$) on $Hom_{\mathbb{C}}$ will be of fundamental importance for us. For the general discussion of this action we allow a slightly wider range for m, namely $m \geq 2$. Let

$$P \in Hom_{\mathbb{C}},$$

then for $s \in S^1$ we define

$$\rho(s)P(z) := P(sz).$$

We make several important remarks about this action. The first is that this action respects the algebra structure, i.e. for $P, Q \in Hom_{\mathbb{C}}$ and $s \in S^1$ we have $\rho(s)(PQ) = \rho(s)(P)\rho(s)(Q)$. The second is that the action of S^1 on $\mathbb{C} = Hom_0 \times Hom_0 \subset Hom_\star \times Hom_\star$ is trivial and more generally on a holomorphic polynomial $z^n \in Hom_n \times Hom_n$ the group S^1 acts as follows: $\rho(s)z^n = (sz)^n = s^n z^n$. For every $s \in S^1$ consider the map

$$F_s : Hom_{\mathbb{C}} \longrightarrow Hom_{\mathbb{C}}$$

defined by

$$F_s(P) := Psz^{m-1}, \tag{5.3}$$

where on the righthandside Psz^{m-1} denotes the polynomial which takes value $P(z)sz^{m-1}$ at the point $z = x + iy$. Note that $F = ReF_1$. Now we formulate three lemmas illustrating the "equivariant" behavior of the family $\{F_s\}_{s\in S^1}$ of linear maps $Hom_{\mathbb{C}} \longrightarrow Hom_{\mathbb{C}}$.

Lemma 13. *For any $s_1, s_2 \in S^1$ we have the following relation*

$$F_{s_2} = F_{s_1} s_1^{-1} s_2. \tag{5.4}$$

Here the right hand side is understood as first multiplying the polynomial by $s_1^{-1}s_2$ and the applying the map F_{s_1}.

Proof. Apply the left hand side of (5.4) to a polynomial $P \in Hom_{\mathbb{C}}$: $F_{s_2}(P) = Ps_2z^{m-1} = Ps_1^{-1}s_2s_1z^{m-1} = F_{s_1}(Ps_1^{-1}s_2)$. Since we have ended up with the right hand side, we are done.□

A consequence of Lemma 13 is that the image of F_s does not depend on $s \in S^1$.

Lemma 14. *For any $s_1, s \in S^1$ we have the following relation*

$$\rho(s)F_{s_1} = F_{s_1 s^{m-1}}\rho(s). \tag{5.5}$$

Proof. Apply the left hand side of (5.5) to a polynomial $P \in Hom_{\mathbb{C}}$ and evaluate it at a point $z \in \mathbb{C}$:

$$(\rho(s)F_{s_1}(P))(z) = P(z_1)s_1z_1^{m-1}|_{z_1=sz} = P(sz)s_1s^{m-1}z^{m-1} = F_{s_1s^{m-1}}(\rho(s)(P))(z).$$

□

The immediate consequence of the above lemmas is the following "equivariance" lemma.

Lemma 15. *For any $s_1, s \in S^1$ we have the following relation*

$$\rho(s)F_{s_1} = F_{s_1}\rho(s)s^{m-1}. \tag{5.6}$$

The following fact is a useful consequence of the last lemma

Lemma 16. *The image of the map F_s is invariant under the ρ-action of S^1. In particular the image of $F = ReF_1$ is invariant under this action.*

Consider Hom_N, the space of homogeneous polynomials of degree N, as a ρ-representation of S^1 via restriction from $Hom_\star$. We recall the decomposition of Hom_N into the irreducible summands:

$$Hom_N = \Sigma_{q=0}^{[N/2]} Irr_N^q, \tag{5.7}$$

where $Irr_N^q = (x^2 + y^2)^q Span\{Re(x + iy)^p, Im(x + iy)^p\}$, $p + 2q = N$, and square brackets $[\cdot]$ denote "the biggest integer not greater than". To finish the proof of Theorem 25 we restrict our attention back to

$$F : Hom_l \times Hom_l \longrightarrow Hom_{r+1}$$

and $m \geq 4$, where $r = 2m - 4$ and

$$l = r - m + 2 = 2m - 4 - m + 2 = m - 2.$$

The following trick is suggested by the last lemma. We view $Hom_{r+1} = Hom_{2m-3}$ as the ρ-representation of S^1 and decompose it into irreducible summands:

$$Hom_{2m-3} = \Sigma_{q=0}^{m-2} Irr_{2m-3}^q,$$

where

$$Irr_{2m-3}^q = (x^2 + y^2)^q Span\{Re(x + iy)^p, Im(x + iy)^p\},$$

$p + 2q = 2m - 3$. Note that the last equality allows us to write out

$$p + q - (m - 1) = 2m - 3 - q - (m - 1) = m - 2 - q \geq 0$$

and therefore makes the following calculation possible:

$$(z\bar{z})^q z^p = \bar{z}^q z^{p+q} = (\bar{z}^q z^{m-2-q}) z^{m-1},$$

therefore

$$(x^2 + y^2)^q Re(x + iy)^p = F(\bar{z}^q z^{m-2-q}),$$

which means that every irreducible summand Irr_{2m-3}^q is present in the image of F, implying surjectivity of F. This finishes the proof of Theorem 25.□

We would like to reformulate the above question from the theory of singularities of differentiable functions in terms of intrinsic harmonicity. To do this we need a preliminary statement. From now on (until the end of this chapter) we allow $m \geq 2$.

Proposition 10. *Let w be a C^∞ function defined an open ball B around $(0,0) \in \mathbb{R}^2$. Then the following two assertions are equivalent:*
1. There exists a diffeomorphism $\phi : B \longrightarrow \phi(B)$ fixing the origin, which brings w to the form $f_0 = Re(x+iy)^m$ for some nonnegative integer m, i.e. $w = f_0 \circ \phi$.
2. There exists a Riemannian metric g on B which makes w harmonic.

Proof. The implication from 1. to 2. is obvious, since the function $Re(x+iy)^m$ is harmonic with respect to the standard (Euclidean) metric on $\mathbb{R}^2$. For the converse, take a Riemannian metric g which makes w harmonic and consider an almost complex structure J induced by g — a rotation by 90 degrees counterclockwise. For dimension reasons any almost complex structure on B is integrable. This means that locally around a point $q \in B$ there is a complex coordinate $z = x+iy$ such that $J\partial_x = \partial_y$, i.e. ∂_y is obtained from ∂_x by means of the rotation by 90 degrees (counterclockwise). This implies that g looks like a multiple of identity in the coordinate system (x,y). We pause for a moment to make a historical remark: existence of such a coordinate system ("isothermal coordinates") essentially goes back to Gauss. Since for dimension reasons the Hodge-star operator and therefore the Laplace(-Beltrami) operator on B does not change if we re-scale the metric conformally, we have that w is harmonic with respect to the metric represented by the identity matrix in the coordinates (x,y). This means that w in coordinates (x,y) is a real part of some complex-valued function F, which depends holomorphically on $z = x+iy$. The last step is to bring $F = a_m z^m + a_{m+1}z^{m+1} + ...$, $a_j \in \mathbb{C}$ to its leading power z^m by a biholomorphic change of coordinates. So finally $w = ReF = Rez^m = Re(x+iy)^m$. Moreover, we can assume that all our coordinate changes preserved the origin.□

Now we are ready to reformulate Theorem 25.

Theorem 26. *Consider a smooth function $f = Re(x+iy)^m + h.o.$ on an open ball around $(0,0)$ in $\mathbb{R}^2$ as in the beginning of this section. Let $h.o.$ begin with the order $max(m+1, 2m-3)$. Then f is harmonic with respect to some smooth Riemannian metric on some open ball around zero, possibly smaller than the original one.*

By Proposition 10 we know already, that Theorem 26 is true. Still we would like to give an independent proof of it and the method we employ will also give the following result.

Theorem 27. *Let $f = Re(x+iy)^2 + h.o.$ and $h = Im(x+iy)^2 + h.o.$ be smooth functions on an open ball around $(0,0)$ in $\mathbb{R}^2$ where the notation $h.o.$ stands for the terms of order 3 and higher. Then there exists a Riemannian metric g which makes f harmonic on some open ball around zero and $\triangle_g h$ is exponentially small around zero. Moreover the Taylor expansion at zero for the conformal structure induced by g is uniquely determined.*

The remainder of this chapter is devoted to proving the two theorems above.

5.2 Inductive setup.

In this section we consider a pair of smooth functions f, h, defined on some open ball around the origin in $\mathbb{R}^2$ with the following Taylor expansions at the origin: $f = f_0 + h.o.$, $h = h_0 + h.o.$, where $f_0 = Re(x+iy)^m$, $h_0 = Im(x+iy)^m$. The higher order terms in the Taylor expansions of f and h may of course be different, but we denote them by the same symbol "*h.o.*" to simplify the notation. Assume that the higher order terms $h.o.$ begin with the order $max(2m-3, m+1)$. This assumption on $h.o.$ will be a standing assumption from now on unless otherwise specified. We use the following notation: $\alpha := df$, $\beta := dh$, $\alpha_0 := df_0$, $\beta_0 := dh_0$, $\gamma \in \{\alpha, \beta\}$, $\gamma_0 \in \{\alpha_0, \beta_0\}$. Finally, let us denote as $\tilde{h.o.}$ the higher oder terms for 1-forms (to distinguish from higher order terms for functions) and recall from Section 2.2, Definition 6 that $r^n[\cdot]$ stands for taking the n-th term in the Taylor expansion around $(0,0)$.

Proposition 11. *There exists a sequence $\{T_k\}_{k=0,1,...}$ of traceless $C^\infty(\mathbb{R}^2)$-linear operators*

$$T_k : \Omega^1(\mathbb{R}^2) \longrightarrow \Omega^1(\mathbb{R}^2),$$

with $T_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ such that

(i) $r^n[dT_k\alpha] = 0$ for $n = 0, 1, 2, ..., k+m-2$;

(ii) $r^0[T_k] = T_0$;

(iii) $r^n[detT_k] = 0$ for $n = 1, 2, ..., k$;

(iv) the entries of T_k are polynomials in (x, y) of order at most k;

(v) $r^n[T_k] = r^n[T_{k-1}]$ *for* $n = 0, 2, ..., k-1$ *and* $1 \leq k$.

If $m = 2$, *then we can achieve that* $\{T_k\}_{k=0,1,...}$ *in addition satisfies*

(i)' $r^n[dT_k\beta] = 0$ *for* $n = 0, 1, 2, ..., k+m-2$.

Moreover, the Taylor expansion at the origin of T_k *up to order* k *is uniquely determined by (i),(ii),(iii),(iv),(v) and (i)'.*

Proof. We proceed by induction on k. For the basic step we consider $k = 0, 1, ..., max(m-4, 0)$. For these values of k we can take T_k to be equal to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Conditions (ii)-(v) are satisfied automatically, so we have to check (i),(i)'. Note that $r^n[dT_k\gamma] = r^n[dT_k\gamma_0] + r^n[dT_k\tilde{h.o.}]$. The first summand is automatically zero, since

$$dT_k\gamma_0 \in \{\triangle_{st}Re(x+iy)^m, \triangle_{st}Im(x+iy)^m\} = \{0\},$$

where $\triangle_{st}$ denotes the standard Laplace-Beltrami operator in $\mathbb{R}^2$. In the second summand the leading power of $\tilde{h.o.}$ is $max(2m-4, m)$, so the leading power of $dT_k\tilde{h.o.}$ is $max(2m-5, m-1)$, which is strictly greater than $max(2m-6, m-2) \geq k+m-2 \geq n$, so the $n-th$ order of $dT_k\tilde{h.o.}$ is zero anyway.

For inductive step assume, that the statement is true for $k-1$, where $k \geq max(m-3, 1)$, this gives us the operator T_{k-1} with (i), (ii), (iii), (iv) and (i)' if $m = 2$. Let T_{k-1} be represented by the following matrix: $\begin{pmatrix} -T^{12}_{k-1} & -T^{22}_{k-1} \\ T^{11}_{k-1} & T^{12}_{k-1} \end{pmatrix}$. We are looking for the operator T_k in the following form: $T_k = T_{k-1} + G_k$, where $G_k = \begin{pmatrix} -G^{12}_k & -G^{22}_k \\ G^{11}_k & G^{12}_k \end{pmatrix}$, here G^{ij}_k is a homogeneous polynomial in (x, y) of order k. Clearly,

$$dT_k\gamma = d(T_{k-1} + G_k)\gamma = dT_{k-1}\gamma + dG_k\gamma.$$

Consider $n = 0, 1, ..., k+m-3 = (k-1)+m-2$. Since the leading term of $dG_k\gamma$ is of order $k+m-2$ we have that $r^n[dG_k\gamma] = 0$. By induction hypothesis $r^n[dT_{k-1}\alpha] = 0$ (and $r^n[dT_{k-1}\beta] = 0$ if $m = 2$). It means that for the above values of n Equation (i) (and also Equation (i)' if $m = 2$) is satisfied. It remains to check Equation (i) and Equation (i)' for $n = k+m-2$. That is $r^{k+m-2}[dT_k\gamma] = 0$ which is the same as

$$r^{k+m-2}[dG_k\gamma] = -r^{k+m-2}[dT_{k-1}\gamma].$$

Since the leading power of $\tilde{h.o.}$ is greater than or equal to m the last equation is equivalent to

$$r^{k+m-2}[dG_k\gamma_0] = -r^{k+m-2}[dT_{k-1}\gamma], \tag{5.8}$$

where we know the right hand side and the unknowns are G_k^{11}, G_k^{12}, G_k^{22}. Equation (ii) is clear by the choice of T_0. Now we consider Equation (iii). Since it is automatically true for $n = 1, ..., k-1$ by induction hypothesis, this equation will follow if

$$r^k[detT_k] = 0. \tag{5.9}$$

Clearly, $r^k[detT_k] = G_k^{11} + G_k^{22} + r^k[detT_{k-1}]$, therefore Equation (5.9) is equivalent to $G_k^{22} = r^k[detT_{k-1}] - G_k^{11}$. Now we substitute this in Equation (5.8), obtaining the following equation:

$$d\tilde{G}_k\gamma_0 = \phi_{k-1}dx \wedge dy, \tag{5.10}$$

where ϕ_{k-1} is a homogeneous polynomial in (x, y) of degree $k+m-2$ which depends only on the components of T_{k-1}, i.e. on the data we know already by induction hypothesis and $\tilde{G}_k$ is the operator represented by the matrix $\begin{pmatrix} -G_k^{12} & G_k^{11} \\ G_k^{11} & G_k^{12} \end{pmatrix}$.

The left hand side of (5.10) is the volume form $dx \wedge dy$ times a homogeneous polynomial of order $k+m-2$. So we if specify $\gamma = \alpha$ in (5.10), then we have $k+m-1$ equations and $2(k+1)$ unknowns. Since $k \geq m-3$, the balance between the two is "correct": $2(k+1) \geq k+m-1$ — the number of unknowns is at least the number of equations. So we expect solvability and therefore the first part of the statement to be true. For the second part we specialize to $m = 2$ and try to solve (5.10) for both cases $\gamma = \alpha$ and $\gamma = \beta$ simultaneously. The number of unknowns is still $2(k+1)$, but the number of equations doubles — it becomes $2(k+m-1)$. Fortunately, we can use that $m = 2$, rewriting the number of unknowns as $2(k+m-1) = 2(k+2-1) = 2(k+1)$. We see that the number of unknowns is equal (!) to the number of equations. So we expect the unique (!) solvability. This finishes the proof of Proposition 11 modulo the two expectations above. These will be treated in the next section.

We close this section by making a remark of heuristic nature. In the inductive setup above we have taken the function f to be of the form $Re(x+iy)^m + h.o.$, where $h.o.$ starts from the order $max(2m-3, m+1)$. Assume for a moment, that we allow more general form for the function f, i.e. $\Sigma_{j=m}^{\infty}P_j$, where P_j is a homogeneous polynomial in (x, y) of order j. Then we are faced with a system of equations for every k as above, but now k ranges from 0 to ∞, not just from $max(m-3,1)$ to ∞. For $k \geq max(m-3,1)$ we argue with the number of unknowns versus the number of equations exactly as above, but for $k = 0, ..., max(m-4,0)$ the balance between the two is not in our favour: the number of equations minus the number of unknowns is equal to $k+m-1-2(k+1) = m-k-3$. So the system is $(m-k-3)$-overdetermined. In other words for a solution to exist, the right hand side has to satisfy $m-k-3$ conditions. Summing these conditions up while k runs from 0 to $max(m-4,0)$ gives us $\frac{1}{2}(m-2)(m-3)$. Philosophically speaking these are $\frac{1}{2}(m-2)(m-3)$ conditions, that polynomials P_j, $j = 0, ..., max(2m-4, m)$ have to satisfy in order for df to be co-closed near the origin, that is for f to look like f_0 in an appropriate coordinate system near the origin. We remark that only finitely many of $P_j's$ decide the question and denote the number $\frac{1}{2}(m-2)(m-3)$ by $L(m)$. These heuristic considerations will be given a precise formalism in Chapter 6.

5.3 Key algebraic trick: Cauchy-Riemann operator.

Note that the operator $\begin{pmatrix} -G_k^{12} & G_k^{11} \\ G_k^{11} & G_k^{12} \end{pmatrix}$ applies to the form $\gamma_0 = \gamma_{01}dx + \gamma_{02}dy$ to give $(-G_k^{12}\gamma_{01} + G_k^{11}\gamma_{02})dx + (G_k^{11}\gamma_{01} + G_k^{12}\gamma_{02})dy$. Taking exterior derivative from the last expression gives us:

$$((G_k^{12}\gamma_{01} - G_k^{11}\gamma_{02})_y + (G_k^{11}\gamma_{01} + G_k^{12}\gamma_{02})_x)dx \wedge dy. \tag{5.11}$$

Recall that $\gamma_0 \in \{\alpha_0, \beta_0\} = \{df_0, dh_0\}$. More precisely: $f_0 = Re(x+iy)^m$,
$df_0 = m\{Re(x+iy)^{m-1}dx - Im(x+iy)^{m-1}dy\}$,
$h_0 = Im(x+iy)^m$,
$dh_0 = m\{Im(x+iy)^{m-1}dx + Re(x+iy)^{m-1}dy\}$.

This suggests to introduce complex notation: $z = x+iy$, $G_k = G_k^{11} + iG_k^{12}$. It should not be confused with the G_k in the preveous section, denoting a certain matrix. We also recall the Cauchy-Riemann operator: $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$. For a complex valued function $a+ib$ of z one has

$$\partial_z(a+ib) = \frac{1}{2}(\partial_x - i\partial_y)(a+ib) = \frac{1}{2}((a_x + b_y) + i(-a_y + b_x)).$$

This notation, together with (5.11) allows us to rewrite the lefthandside of (5.10) as follows. For $\gamma = \alpha$:

$$d\tilde{G}_k\alpha_0 = ((G_k^{12}f_{0x} - G_k^{11}f_{0y})_y + (G_k^{11}f_{0x} + G_k^{12}f_{0y})_x)dx \wedge dy =$$

$$= \{((G_k^{12}Rez^{m-1} + G_k^{11}Imz^{m-1})_y + (G_k^{11}Rez^{m-1} - G_k^{12}Imz^{m-1})_x)\}dx \wedge dy =$$

$$= m\{Re(G_kz^{m-1})_x + Im(G_kz^{m-1})_y\}dx \wedge dy = 2mRe\partial_z(G_kz^{m-1})dx \wedge dy$$

and for $\gamma = \beta$:

$$d\tilde{G}_k\beta_0 = ((G_k^{12}h_{0x} - G_k^{11}h_{0y})_y + (G_k^{11}h_{0x} + G_k^{12}h_{0y})_x)dx \wedge dy =$$

$$= m\{((G_k^{12}Imz^{m-1} - G_k^{11}Rez^{m-1})_y + (G_k^{11}Imz^{m-1} + G_k^{12}Rez^{m-1})_x)\}dx \wedge dy =$$

$$= m\{-Re(G_kz^{m-1})_y + Im(G_kz^{m-1})_x\}dx \wedge dy = 2mIm\partial_z(G_kz^{m-1})dx \wedge dy.$$

In short, viewing γ_0 as $\alpha_0 + i\beta_0$ one gets:

$$d\tilde{G}_k\gamma_0 = 2m\partial_z(G_kz^{m-1})dx \wedge dy = 2m\partial_z \circ F_1(G_k)dx \wedge dy.$$

Now we turn to the second part of the Proposition 11, which came down to the unique solvability of (5.10) for $m = 2$. Since the number of unknowns is equal to the number of equations, this is the same as injectivity of the linear operator $G_k \mapsto \partial_z(G_kz)$ acting on the space $Hom_k \times Hom_k$. Let a G_k be the element of the kernel of this operator. This tells us that G_kz is in the kernel of ∂_z. So G_kz should write out as a $c\bar{z}^{k+1}$ for some $c \in \mathbb{C}$, but this immidiatelly implies that $c = 0$ since G_kz is a multiple of z.

For the first part of the Proposition 11, which came down to the (not necessarily unique) solvability of (5.10) for any $m \geq 2$, but only for $\gamma = \alpha$ we consider the family $\{D_s\}_{s\in S^1}$ of maps

$$D_s = \partial_z \circ F_s : Hom_k \times Hom_k \longrightarrow Hom_{k+m-2} \times Hom_{k+m-2}$$

which takes G_k to $(G_ksz^{m-1})_z$ for every $s \in S^1$. Note that $\partial_{sz} = \frac{1}{s}\partial_z$. This together with Lemmas 13, 14 and 15 gives us the following

Lemma 17. *We have the following "equivariance" relations for the family* $\{D_s\}_{s\in S^1}$.

$$D_{s_2} = D_{s_1}s_1^{-1}s_2, \tag{5.12}$$

$$\rho(s)D_{s_1} = D_{s_1}\rho(s)s^{m-2} = D_{s_1s^{m-2}}\rho(s). \tag{5.13}$$

This, of course, implies that the image of D_s does not depend on s and is invariant under the ρ-action of S^1. Consequently, the image of of $Re \circ \partial_z \circ F_1$ is invariant under this action of S^1. This suggests to decompose the target space Hom_{k+m-2} of $Re \circ \partial_z \circ F_1$ into irreducible representations (recall (5.7)):

$$Hom_{k+m-2} = \Sigma_{q=0}^{[(k+m-2)/2]} Irr^q_{k+m-2},$$

where

$$Irr^q_{k+m-2} = (x^2+y^2)^q Span\{Re(x+iy)^p, Im(x+iy)^p\},$$

$p + 2q = k + m - 2$. For the next calculation it is useful note that since $k \geq m - 3$ we have that

$$n := k - q = \frac{1}{2}(2k - 2q) \geq \frac{1}{2}(2k - (k + m - 2)) = \frac{1}{2}(k - (m - 2)) \geq 0.$$

Now for the nonnegative integer n we have

$$p = n + m - 2 - q = k + m - 2 - 2q \geq 0.$$

Finally,

$$Re\partial_z(F_1(\bar{z}^q z^n)) = Re(\bar{z}^q z^n z^{m-1})_z = Re(z^{n+m-1}\bar{z}^q)_z =$$

$$= Re(n+m-1)z^{n+m-2}\bar{z}^q = Re(n+m-1)z^p z^q \bar{z}^q = (n+m-1)(x^2+y^2)^q Re(x+iy)^p,$$

which means that every irreducible summand Irr^q_{k+m-2} is present in the image of $Re \circ \partial_z \circ F_1$, implying surjectivity of this map. This completes the proof of Proposition 11. $\square$

5.4 Approximate solution.

Proposition 12. *There exists an open ball W around the origin and a traceless $C^\infty(W)$-linear operator $T : \Omega^1(W) \longrightarrow \Omega^1(W)$ with the following properties:*

(i) $T_{(0,0)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,

(ii) $dT\alpha = O(exp)$,

(iii) $detT = 1$,

where $O(exp)$ stands for exponentially small functions or top degree differential forms (Section 2.3). Moreover, if $m = 2$, then in addition

(ii)' $dT\beta = O(exp)$,

holds and the Taylor power series of T at the origin is uniquely determined by (i), (ii), (iii) and (ii)'.

Proof. Since "the inductive structure (v)" (see condition (v) of Proposition 11) is true, we have that the sequence of operators $\{T_k\}_{k=0,1,\ldots}$ constitute a formal power series T_{formal} at the origin. By the result of Mirkil cf. [16], there exists an operator $\tilde{T} : \Omega^1(W) \longrightarrow \Omega^1(W)$, where W is a small ball around the origin, such that its (formal) Taylor power series is exactly T_{formal}. This allows us to make the following computations. Fix a positive integer k, then $r^k[d\tilde{T}\alpha] = r^k[dT_k\alpha] = 0$ by Equation (i) of Proposition 11; $r^k[Trace\tilde{T}] = r^k[TraceT_k] = 0$; $r^0[det\tilde{T}] = r^0[detT_k] = 1$, by Equation (ii) of Proposition 11 and if $k > 0$, then $r^k[det\tilde{T}] = r^k[detT_k] = 0$, by Equation (iii) of Proposition 11. Therefore the following three equalities hold true:
(a) $d\tilde{T}\alpha = O(exp)$,
(b) $Trace\tilde{T} = O(exp)$,
(c) $det\tilde{T} = 1 + O(exp)$.
Now we are going to correct the operator $\tilde{T}$ a little bit. The property (b) says that the sum of the diagonal elements of the operator $\tilde{T}$ is of the class $O(exp)$, therefore by changing the lower right element of $\tilde{T}$, we can achieve that the new operator (called again $\tilde{T}$) is now traceless and the properties (a) and (c) still hold true. The last step is to set $T = (det\tilde{T})^{-1/2}\tilde{T}$ (we shrink the neighbourhood W if necessary to insure that $det\tilde{T}|_W > 0$). Clearly, the operator T is traceless and $detT = 1$. We check whether it satisfies property (a). Indeed,

$$dT\alpha = d((det\tilde{T})^{-1/2}\tilde{T}\alpha) = d(det\tilde{T})^{-1/2} \wedge \tilde{T}\alpha + (det\tilde{T})^{-1/2}d\tilde{T}\alpha.$$

Working out the two terms one by one gives us:

$$d(det\tilde{T})^{-1/2} \wedge \tilde{T}\alpha = d(1 + O(exp))^{-1/2} \wedge \tilde{T}\alpha =$$

$$= d(1 + O(exp)) \wedge \tilde{T}\alpha = d(O(exp)) \wedge \tilde{T}\alpha = O(exp);$$

$$(det\tilde{T})^{-1/2}d\tilde{T}\alpha = (1 + O(exp))O(exp) = O(exp).$$

Altogether, $dT\alpha = O(exp)$ and of course $T_{(0,0)} = \tilde{T}_{(0,0)} = T_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.□

Proposition 13. *There exists an open ball W around the origin and a Riemannian metric $\tilde{g}$ on it such that*

$$\triangle_{\tilde{g}} f = O(exp) \tag{5.14}$$

and in the case $m = 2$ we have in addition

$$\triangle_{\tilde{g}} h = O(exp). \tag{5.15}$$

Moreover, the Taylor expansion for the conformal structure induced by g at the origin is uniquely determined by 5.14 and 5.15.

Proof. Let $\begin{pmatrix} -\tilde{g}^{12} & -\tilde{g}^{22} \\ \tilde{g}^{11} & \tilde{g}^{12} \end{pmatrix}$ be the matrix representing T in standard coordinates (dx, dy). We set the Riemann metric $\tilde{g}$ to be defined by the matrix $\begin{pmatrix} \tilde{g}^{11} & \tilde{g}^{12} \\ \tilde{g}^{12} & \tilde{g}^{22} \end{pmatrix}$.

We recall the explicit computation for the Hodge-star operator from Section 2.1 and apply Proposition 12. □

This means that we are almost done. Recall that we would like to have zero in the right hand side of (5.14) instead of $O(exp)$. The next step is to correct $\tilde{g}$ in an exponentially small fashion to achieve zero on the right hand side of (5.14). In the case $m = 2$ such a modification (being exponentially small) will not spoil (5.15). Of course we would be more happy to also improve (5.15) to achieve zero on the right hand side. Unfortunately, it is not clear whether this is possible or not.

5.5 Technical analysis around zero.

We set up the machinery which starts with a C^∞-metric making f harmonic "up to order l" at the origin and produces a C^l-metric out of it making f honestly harmonic, $l = 0, 1, \ldots\infty$. Doing this for $l = \infty$ would obviously finish the job. It turns out, however, that it is convenient to start out slowly with $l = 0$, postponing the case $l = \infty$ until later. Note that the most naive metric — the standard one in coordinates (x, y) — already makes f harmonic "up to order 0" at the origin. So for the next proposition we do not need any assumptions on the higher order terms $h.o.$ of f at all.

Proposition 14. *Let $f = Re(x+iy)^m + h.o.$ There exists a continuous Riemannian metric g on some open neighbourhood U around zero in $\mathbb{R}^2$, which makes $f = Re(x+iy)^m + h.o.$ harmonic. Moreover the metric is smooth in the punctured open neighbourhood $U \setminus \{(0,0)\}$. The higher order terms h.o. here are allowed to begin with the order $m+1$.*

Proof. In standard coordinates (x, y) we have: $\alpha_0 = f_{0x}dx + f_{0y}dy$. Clearly, $Ker f_{0x} \cap Ker f_{0y} = \{(0,0)\}$. Let the desired Riemannian metric g be represented by the matrix $\{g^{ij}\}^{i,j=1,2}$. With the convention that $det\{g^{ij}\} = 1$ the equation $\triangle_g f = 0$ for g reads as

$$(g^{12}f_x + g^{22}f_y)_y + (g^{11}f_x + g^{12}f_y)_x = 0. \tag{5.16}$$

Assume for the moment, that Equation (5.16) is solved by a Riemannian metric g with regularity we want. Then the combination

$$A = g^{12}f_x + g^{22}f_y, \tag{5.17}$$

gives us a function on R^2 with a-priori the same regularity as g has. Analogously

$$B = g^{11}f_x + g^{12}f_y. \tag{5.18}$$

Note that $A_y + B_x = 0$. Recall, that according to our convention the determinant of the matrix $\{g^{ij}\}^{i,j=1,2}$ must be equal to 1, i.e.

$$g^{11}g^{22} - (g^{12})^2 = 1. \tag{5.19}$$

The set of equations (5.17), (5.18) and (5.19) can be viewed as a system of equations on our matrix elements $\{g^{ij}\}^{i,j=1,2}$. To solve this system we express g^{11} and g^{22} in terms of g^{12}, A and B using (5.17) and (5.18):

$$g^{11} = \frac{B - g^{12}f_y}{f_x}, \tag{5.20}$$

$$g^{22} = \frac{A - g^{12}f_x}{f_y}. \tag{5.21}$$

and substitute these expressions in (5.19):

$$\frac{B - g^{12}f_y}{f_x}\frac{A - g^{12}f_x}{f_y} - (g^{12})^2 = 1,$$

and hence,

$$g^{12} = \frac{AB - f_x f_y}{Af_y + Bf_x} \tag{5.22}$$

Formulas (5.20), (5.21) and (5.22) can be viewed as an expression of our matrix elements g^{11}, g^{22}, g^{12} through the functions A and B. Of course, while writing out these formulas we have divided by zero in several places, but it does not hurt to do this at the moment, since our computations were done under the assumption that $\{g^{ij}\}^{i,j=1,2}$ is well defined on the whole of U a-priory.

Now we change the direction of the logic. We want to solve Equation (5.16) together with (5.19), thus obtaining the desired Riemannian metric. For this we give ourselves smooth functions A and B, defined in some open neighbourhood U around the origin with $A_y + B_x = 0$. The freedom of this choice will be exploited later. We insert the functions A and B in the system (5.17), (5.18) and (5.19) as a right hand side and note that the solution to this system will automatically satisfy (5.16) and (5.19), thus giving the Riemannian metric we want provided that regularity questions are taken care of. Unfortunately, the direct usage of the formulas (5.20), (5.21) and (5.22) in order to solve the system (5.17), (5.18), (5.19) will run into problems like division by zero. Therefore, we will do the following. First, we exploit the freedom in the choice of the functions A and B by fixing their principal parts properly. The higher order terms remain arbitrary. Next, we will see, that Formula (5.22) does not have problems in a small neighbourhood of the origin, and hence defines a function g^{12} in this small neighbourhood. The function g^{11} will be defined in two steps. First, we use Formula (5.20) to defined it away from the set where the corresponding denominator is small and then we use Equation (5.19) to extend it over the problematic set. The function g^{22} is defined analogously. The last step is to show that the so defined functions g^{12}, g^{11} and g^{22} do satisfy the system (5.17), (5.18), (5.19), which is not automatic, because the formulas (5.20) and (5.21) do not apply everywhere in the domain of definition of the functions $\{g^{ij}\}^{i,j=1,2}$.

Now we carry out this plan. We set $A_{m-1} = f_{0y}$, $B_{m-1} = f_{0x}$ to be the principal parts of $A = A_{m-1} + A_r$ and $B = B_{m-1} + B_r$ respectively, where A_r and B_r are left to be arbitrary smooth functions of the order higher than $m-1$, subject to the relation $A_{ry} + B_{rx} = 0$. (the lower r in "A_r" and "B_r"

stands for "rest") First, we analyze Formula (5.22):

$$g^{12} = \frac{AB - f_x f_y}{Af_y + Bf_x} = \frac{(A_{m-1} + A_r)(B_{m-1} + B_r) - (f_{0x} + \phi_1)(f_{0y} + \phi_2)}{(A_{m-1} + A_r)(f_{0y} + \phi_2) + (B_{m-1} + B_r)(f_{0x} + \phi_1)} =$$

$$= \frac{A_{m-1}B_{m-1} - f_{0x}f_{0y} + r_n^{12}}{A_{m-1}f_{0y} + B_{m-1}f_{0x} + r_d^{12}} = \frac{r_n^{12}}{f_{0y}^2 + f_{0x}^2 + r_d^{12}},$$

where

$$r_n^{12} = A_{m-1}B_r + A_rB_{m-1} - f_{0x}\phi_2 - f_{0y}\phi_1 + A_rB_r - \phi_1\phi_2$$

and

$$r_d^{12} = A_{m-1}\phi_2 + A_rf_{0y} + B_{m-1}\phi_1 + B_rf_{0x} + A_r\phi_2 + B_r\phi_1,$$

where ϕ_1 and ϕ_2 are partial derivatives of $h.o.$ with respect to x and y respectively. Note that $f_{0x}^2 + f_{0y}^2$ is a nowhere zero homogeneous polynomial of order $2(m-1)$, so the estimate (2.1) applied to $f_{0x}^2 + f_{0y}^2$ implies that

$$\frac{r_n^{12}}{f_{0y}^2 + f_{0x}^2} = o(1)$$

at $(0,0)$ and

$$\frac{r_d^{12}}{f_{0y}^2 + f_{0x}^2} = o(1)$$

at $(0,0)$. Therefore, the function g^{12} is well-defined in some neighbourhood U of the origin, belongs to the class $C^\infty(U \setminus (0,0)) \cap C^0(U)$ and the relation

$$\lim_{(x,y)\to(0,0)} g^{12} = g^{12}|_{(x,y)=(0,0)} = 0$$

holds true. Now, we analyze the formulas (5.20) and (5.21) and define the functions g^{11} and g^{22}. The idea is that for each formula we cut out "problematic" sectors and work on those parts of $\mathbb{R}^2$ where we are guaranteed from small or vanishing denominators. Since $f_{0x} = Re(x+iy)^{m-1}$ is a homogeneous polynomial of order $m-1$, not identically zero, we fix a small positive δ and set $\Omega_{11} = \Omega^\delta(f_{0x})$. Next, we rewrite (5.20) in a more convenient way:

$$g^{11} = \frac{B - g^{12}f_y}{f_x} = \frac{B_{m-1} + B_r - g^{12}(f_{0y} + \phi_2)}{f_{0x} + \phi_1} = \frac{f_{0x} + r_n^{11}}{f_{0x} + \phi_1} = \frac{f_{0x}(1 + \frac{r_n^{11}}{f_{0x}})}{f_{0x}(1 + \frac{\phi_1}{f_{0x}})},$$

where $r_n^{11} = B_r - g^{12} f_{0y} - g^{12}\phi_2$. To take a more precise look at Formula (5.20) we restrict ourselves to $U \cap \Omega_{11}$. Now Estimate (2.2) applied to f_{0x} implies that

$$\frac{r_n^{11}}{f_{0x}}|_{\Omega_{11}} = o(1)$$

at $(0,0)$ and

$$\frac{\phi_1}{f_{0x}}|_{\Omega_{11}} = o(1)$$

at $(0,0)$. Therefore, $f_x|_{U\cap\Omega_{11}}$ has an isolated zero at the origin, and the right hand side of (5.20) is well-defined on $U \cap \Omega_{11}$ (we shrink the neighbourhood U if necessary). At this point we set the function g^{11} to be defined on $U \cap \Omega_{11}$ by Formula (5.20). The so defined function g^{11} (only on $U \cap \Omega_{11}$ so far) exhibits the following regularity: $g^{11}|_{U\cap\Omega_{11}} \in C^\infty((U \cap \Omega_{11}) \setminus (0,0)) \cap C^0(U \cap \Omega_{11})$ and the relation

$$\lim_{(x,y)\to(0,0)} g^{11}|_{U\cap\Omega_{11}} = g^{11}|_{(x,y)=(0,0)} = 1$$

holds true. The latter allows us to assume (by shrinking U further if necessary) that $g^{11}|_{U\cap\Omega_{11}}$ is nowhere zero. Similar discussions apply to Formula (5.21). In brief, $\Omega_{22} = \Omega^\delta(f_{0y})$,

$$g^{22} = \frac{f_{0y} + r_n^{22}}{f_{0y} + \phi_2},$$

for r_n^{22} being a function with the faster decay at $(0,0)$ than $(x^2+y^2)^{(m-1)/2}$. By the same token as before,

$$\frac{r_n^{22}}{f_{0y}}|_{\Omega_{22}} = o(1)$$

and

$$\frac{\phi_2}{f_{0y}}|_{\Omega_{22}} = o(1)$$

at $(0,0)$. Therefore, the function g^{22} is well-defined on $U \cap \Omega_{22}$ (the neighbourhood U can be shrunk further if needed). Moreover, we have $g^{22}|_{U\cap\Omega_{22}} \in C^\infty((U \cap \Omega_{22}) \setminus (0,0)) \cap C^0(U \cap \Omega_{22})$ and the relation

$$\lim_{(x,y)\to(0,0)} g^{22}|_{U\cap\Omega_{22}} = g^{22}|_{(x,y)=(0,0)} = 1$$

holds true. Therefore $g^{22}|_{U\cap\Omega_{22}}$ is nowhere zero. Since we know that

$$Ker(f_{0x}) \cap Ker(f_{0y}) = \{(0,0)\},$$

we can choose δ small enough and achieve that

$$\Omega_{11} \cup \Omega_{22} = \mathbb{R}^2$$

and

$$Int(\Omega_{11} \cap \Omega_{22}) \neq \emptyset.$$

Now comes a crucial moment. We are to extend the functions g^{11} and g^{22} to the whole of U. Equation (5.19)

$$g^{11}g^{22} - (g^{12})^2 = 1$$

holds true on the triple intersection $U \cap \Omega_{11} \cap \Omega_{22}$ by the formulas (5.20) and (5.21). Since $g^{22}|_{U\cap\Omega_{22}}$ is nowhere zero this equation equivalently reads as

$$g^{11} = \frac{1 + (g^{12})^2}{g^{22}}. \tag{5.23}$$

The right hand side of this equation makes perfect sense and has the regularity required for the function g^{11} on $U \cap \Omega_{22}$. This allows us to define the function g^{11} on $U \cap \Omega_{22}$ by (5.23). So now we have defined the function g^{11} on $U \cap \Omega_{11}$ via (5.20) and on $U \cap \Omega_{22}$ via (5.23). The two definitions overlap on $U \cap \Omega_{11} \cap \Omega_{22}$ and clearly agree there, since Equation (5.19), where the second definition has come from, holds true on $U \cap \Omega_{11} \cap \Omega_{22}$ with g^{11} defined in the first way. Altogether, we have that the function g^{11} is defined and has the regularity we need on both $U \cap \Omega_{11}$ and $U \cap \Omega_{22}$ and hence on U — their union. Note that Equation (5.19), after we have made this extension, holds true not only on $U \cap \Omega_{11} \cap \Omega_{22}$, but on the large set $U \cap \Omega_{22}$. Analogously we extend the function g^{22} from $U \cap \Omega_{22}$ to the whole of U.

We remark that Equation (5.19) now holds true not only on $U \cap \Omega_{22}$, but on the whole of U. Now we have come to the last step, i.e. we are to show that the so defined functions g^{12}, g^{11} and g^{22} do actually satisfy the system (5.17), (5.18), (5.19) and hence both (5.16) and (5.19), therefore, giving us the Riemannian metric g which makes α co-closed and has the $C^\infty(U \setminus (0,0)) \cap C^0(U)$ regularity.

Equation (5.19) is satisfied automatically by the remark above. For (5.18) we start we a point $(x, y) \in U$ and consider $g^{11} f_x$ at this point. Here we

distinguish between the following two cases:
1) $(x, y) \in \Omega_{11}$ and
2) $(x, y) \in \Omega_{22}$.
In the first case we are done by Formula (5.20). In the second case Formula (5.20) does not apply, but fortunately (5.21) does apply. For this we carry out an easy computation:

$$g^{11} f_x|_{(x,y)} = g^{11} g^{22} f_x f_y \frac{1}{g^{22} f_y}|_{(x,y)} = (1 + (g^{12})^2) f_x f_y \frac{1}{g^{22} f_y}|_{(x,y)} =$$

$$= (AB - g^{12}(Af_y + Bf_x) + (g^{12})^2 f_x f_y) \frac{1}{g^{22} f_y}|_{(x,y)} =$$

$$= \frac{(B - g^{12} f_y)(A - g^{12} f_x)}{g^{22} f_y}|_{(x,y)} = (B - g^{12} f_y)|_{(x,y)}.$$

The first equality sign is valid, because g^{22} is nowhere zero and $f_y|_{U \cap \Omega_{22}}$ has a unique zero at the origin. The second one is valid by (5.19). The third one easily follows from the definition of g^{12}. The fourth one is just an elementary algebra. The fifth one follows from (5.21). This shows (5.18). It can be shown completely analogously that Equation (5.17) is also satisfied. □

Next proposition is the final step. We take up the case $l = \infty$. That is we find a smooth Riemannian metric g making f harmonic with $g - \tilde{g} = O(exp)$.

Proposition 15. *There exists an open neighbourhood U of the origin, possibly smaller than W and a Riemannian metric $g \in C^\infty(U)$, which makes $\alpha = df$ co-closed. Moreover, $g - \tilde{g} = O(exp)$.*

Proof. Recall from Proposition 14 that Riemannian metric g on some neighbourhood U of the origin making α co-closed. The only problem is the regularity of this metric at $(0, 0)$. To take care of these questions we are going to exploit the freedom in the choice of functions A and B. Recall that Proposition 12 gave us an open neighbourhood W of the origin and an operator $T = \begin{pmatrix} -\tilde{g}^{12} & -\tilde{g}^{22} \\ \tilde{g}^{11} & \tilde{g}^{12} \end{pmatrix}$ on $\Omega^1(W)$ with $dT\alpha = \triangle_{\tilde{g}} f$ exponentially small. Set $\kappa = \triangle_{\tilde{g}} f$ and recall that the last expression is equal to $(\tilde{g}^{12} f_x + \tilde{g}^{22} f_y)_y + (\tilde{g}^{11} f_x + \tilde{g}^{12} f_y)_x$. We shrink U if necessary, so that U is convex and contained in W. We introduce a smooth function ξ, defined on U, by the formula:

$$\xi(x, y) = \int_0^x \kappa(\tilde{x}, y) d\tilde{x}.$$

Clearly, the function ξ is of the class $O(exp)$. We set

$$A_\infty = \tilde{g}^{12} f_x + \tilde{g}^{22} f_y,$$

$$B_\infty = \tilde{g}^{11} f_x + \tilde{g}^{12} f_y$$

and pick some function $\phi \in O(exp)$ arbitrarily. Set $A = A_\infty + \phi_x$, $B = B_\infty - \xi - \phi_y$. First, we check, that the principal parts A_{m-1} and B_{m-1} of A and B coincide with those chosen previously. Indeed,

$$A_{m-1} = \tilde{g}^{12}_{(0,0)} f_{0x} + \tilde{g}^{22}_{(0,0)} f_{0y} = f_{0y}$$

and

$$B_{m-1} = \tilde{g}^{11}_{(0,0)} f_{0x} + \tilde{g}^{12}_{(0,0)} f_{0y} = f_{0x}$$

as before. Next, we check, that A and B satisfy the condition $A_y + B_x = 0$. Indeed,

$$A_y + B_x = A_{\infty y} + \phi_{xy} + B_{\infty x} - \phi_{yx} - \xi_x = \kappa - \kappa + \phi_{xy} - \phi_{yx} = 0.$$

Now we analyze formula (5.22) for the off diagonal element of the metric deeper than previously. Basically, it follows the same pattern as before, but now we want infinite differentiability of g^{12} at the origin instead of just continuity.

$$g^{12} = \frac{AB - f_x f_y}{A f_y + B f_x} = \frac{(A_\infty + \phi_x)(B_\infty - \xi - \phi_y) - f_x f_y}{(A_\infty + \phi_x) f_y + (B_\infty - \xi - \phi_y) f_x} =$$

$$= \frac{A_\infty B_\infty - f_x f_y + r_n^{12}}{A_\infty f_y + B_\infty f_x + r_d^{12}} = \frac{\frac{A_\infty B_\infty - f_x f_y}{A_\infty f_y + B_\infty f_x} + \tilde{r}_n^{12}}{1 + \tilde{r}_d^{12}} = \frac{A_\infty B_\infty - f_x f_y}{A_\infty f_y + B_\infty f_x} + r,$$

where the functions r_n^{12}, r_d^{12}, $\tilde{r}_n^{12}$, $\tilde{r}_d^{12}$, r are all of the class $O(exp)$ and the neighbourhood U around the origin we are working at is taken to be small enough for $A_\infty f_y + B_\infty f_x$ to be nonzero in the punctured neighbourhood. By the choice of A_∞ and B_∞, we have that

$$\frac{A_\infty B_\infty - f_x f_y}{A_\infty f_y + B_\infty f_x} = \tilde{g}^{12},$$

therefore $g^{12} = \tilde{g}^{12} + r$, in particular g^{12} is smooth. To work out the desired regularity for diagonal elements is a little harder. First, we consider the difference $g^{11} - \tilde{g}^{11}$ restricted to the set $U \cap \Omega_{11}$, where the formula (5.20) works:

$$(g^{11} - \tilde{g}^{11})|_{U\cap\Omega_{11}} = \frac{B - g^{12} f_y}{f_x} - \tilde{g}^{11} = \frac{B_\infty - \xi - \phi_y - (\tilde{g}^{12} + r) f_y}{f_x} - \tilde{g}^{11} =$$

$$= \frac{B_\infty - \tilde{g}^{12} f_y + r_n^{11}}{f_x} = \tilde{g}^{11} + \tilde{r}^{11} - \tilde{g}^{11} = \tilde{r}^{11},$$

where the function r_n^{11} is of the class $O(exp)$ and we have to be a little more careful about the function $\tilde{r}^{11}$. It is smooth on the set $U \cap \Omega_{11} \setminus \{(0,0)\}$ and decays at $(0,0)$ together with all its derivatives faster than any polynomial. Analogously, $g^{22}|_{U\cap\Omega_{22}} = \tilde{g}^{22} + \tilde{r}^{22}$, where the function $\tilde{r}^{22}$ is smooth on the set $U \cap \Omega_{22} \setminus \{(0,0)\}$ and decays at $(0,0)$ together with all its derivatives faster than any polynomial. This allows us to write out the difference $g^{11} - \tilde{g}^{11}$ restricted to the set $U \cap \Omega_{22}$:

$$(g^{11} - \tilde{g}^{11})|_{U\cap\Omega_{22}} = \frac{1 + (g^{12})^2}{g^{22}} - \tilde{g}^{11} = \frac{1 + (\tilde{g}^{12})^2 + 2\tilde{g}^{12} r + r^2}{\tilde{g}^{22} + \tilde{r}^{22}} - \tilde{g}^{11} =$$

$$= \frac{1 + (\tilde{g}^{12})^2}{\tilde{g}^{22}} + \hat{r}^{11} - \tilde{g}^{11} = \tilde{g}^{11} + \hat{r}^{11} - \tilde{g}^{11} = \hat{r}^{11},$$

where the function $\hat{r}^{11}$ is smooth on the set $(U \cap \Omega_{22}) \setminus \{(0,0)\}$ and decays at $(0,0)$ together with all its derivatives faster than any polynomial. Altogether, we have that the difference $g^{11} - \tilde{g}^{11}$ is smooth in a punctured neighbourhood of $(0,0)$ and decays at $(0,0)$ together with all its derivatives faster than any polynomial (in the above calculations we shrink the neighbourhood U of the origin whenever necessary, to keep track of the denominators). Consequently, the difference $g^{11} - \tilde{g}^{11}$ is of the class $O(exp)$. In particular, the upper left element g^{11} of the metric is smooth. The lower right element g^{22} can be treated analogously. The above calculations show that the metric g is smooth and moreover, $g - \tilde{g} = O(exp)$. $\square$

This finishes the proof of Theorem 26 and Theorem 27.

Chapter 6

Applications

This chapter is the logical continuation of the previous one. In Section 6.1 with the hard analytical work of proving Theorem 25 behind us we introduce the algebraic formalism of jets (following the spirit of [1]) to derive some applications of Theorem 25. In Section 6.2 we apply the results of Section 6.1 to give a smooth characterization of intrinsically harmonic forms on surfaces with arbitrary zeros. In Section 6.3 we give an illustrating example. Section 6.4 is an "epilogue".

6.1 Finite dimensional reduction.

Let $Diff$ denote the (infinite dimensional Lie) group of diffeomorphisms of $\mathbb{R}^2$ fixing the origin. It acts linearly on the space of smooth functions $C^\infty(\mathbb{R}^2)$ on $\mathbb{R}^2$ by composition on the right. Let $C^\infty(\mathbb{R}^2)_{(0,0)}$ denote the subspace of $C^\infty(\mathbb{R}^2)$ consisting of functions $f \in C^\infty(\mathbb{R}^2)$ with $f(0,0) = 0$ and $df_{(0,0)} = 0$. Let $\mathcal{G}C^\infty(\mathbb{R}^2)_{(0,0)}$ denote the space of germs of functions $f \in C^\infty(\mathbb{R}^2)_{(0,0)}$ at $(0,0)$. More generally, for any function space the corresponding space of germs will be denoted by adding $\mathcal{G}$ in front. The germ at $(0,0)$ of a function $f \in C^\infty(\mathbb{R}^2)_{(0,0)}$ will be denoted by $\{f\}$. The map sending a function to its germ gives the space of germs a target topology — the strongest topology in which this map is continuous. The action of the group $Diff$ on $f \in C^\infty(\mathbb{R}^2)$ descends to the space of germs $\mathcal{G}C^\infty(\mathbb{R}^2)_{(0,0)}$. Indeed, assume that $\{f_1\} = \{f_2\} \in \mathcal{G}C^\infty(\mathbb{R}^2)_{(0,0)}$ for $f_1, f_2 \in C^\infty(\mathbb{R}^2)_{(0,0)}$, that is $f_1|_U = f_2|_U$ for some open neighbourhood U of $(0,0)$. Then $f_1 \circ \phi|_{\phi^{-1}(U)} = f_2 \circ \phi|_{\phi^{-1}(U)}$, i.e. $\{f_1 \circ \phi\} = \{f_2 \circ \phi\} \in \mathcal{G}C^\infty(\mathbb{R}^2)_{(0,0)}$. The action of $\phi \in Diff$ on a germ

$\{h\} \in \mathcal{G}C^\infty(\mathbb{R}^2)_{(0,0)}$ will be denoted by writing ϕ to the right of $\{h\}$, i.e.$\{h\}\phi$, remembering the obvious formula $\{h\}\phi = \{h \circ \phi\}$.

Now pick $f \in C^\infty_{(0,0)}$. Let $m \geq 2$ be the order of its leading power in the Taylor expansion around the origin. We ask exactly the same question as we asked in Chapter 5: When can f be brought by a smooth change of variables to the normal form $f_0 = Re(x + iy)^m$ in some open neighbourhood of the origin? In other words, when does there exist an open neighbourhood U around $(0, 0) \in \mathbb{R}^2$ and a diffeomorphism $\phi : U \longrightarrow \phi(U)$ fixing the origin with $f|_U = f_0 \circ \phi|_U$? This question has a very transparent reformulation in the language of germs. Indeed, assume such a diffeomorphism $\phi : U \longrightarrow \phi(U)$ exists. Then the restriction of ϕ to a sufficiently small ball B around zero can be extended to a diffeomorphism $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. In the language of germs the equation $f|_B = f_0 \circ \Phi|_B$ translates to $\{f\} = \{f_0\}\Phi$, i.e. the germ $\{f\}$ of the function f belongs to the orbit of the germ $\{f_0\}$ of the normal form f_0 under the action of the group $Diff$, in formulas: $\{f\} \in \{f_0\}Diff$. Assume conversely that $\{f\} \in \{f_0\}Diff$. Then there exists a neighbourhood U of the origin and a diffeomorphism $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ fixing the origin with $f|_U = f_0 \circ \phi|_U$, answering positively the question of Chapter 5. Therefore the “germ” version of this question is: does the germ $\{f\}$ of the function f belongs to the orbit of the germ $\{f_0\}$ of the normal form f_0 under the action of the group $Diff$? In formulas: is it true that $\{f\} \in \{f_0\}Diff$? In quantitative terms: what is the co-dimension of the orbit of $\{f_0\}$ under the action of the group $Diff$? Theorem 25 suggests that the question is “finite dimensional” and the above mentioned co-dimension is finite. In order to make this precise we need some machinery, essentially borrowed from [1].

For $n \geq 1$ let $\mathcal{A}_n$ denote the algebra of smooth functions on $\mathbb{R}^2$ which vanish at $(0, 0)$ together with $n - 1$ derivatives. With this notation $\mathcal{A}_2 = C^\infty(\mathbb{R}^2)_{(0,0)}$. Let $r \geq n$ be a natural number. Note that $\mathcal{A}_{r+1} \subset \mathcal{A}_n$ is an ideal. The finite dimensional quotient-algebra $\mathcal{A}_n/\mathcal{A}_{r+1}$ will be denoted by Jet^n_r; it is usually referred to as an algebra of jets of smooth functions. Note that since for $r_2 \geq r_1 \geq n$ we have the following inclusion of ideals: $\mathcal{A}_{r_2+1} \subset \mathcal{A}_{r_1+1} \subset \mathcal{A}_n$ and hence also a natural “forgetful” map $Jet^n_{r_2} \longrightarrow Jet^n_{r_1}$ that will be referred to as a projection. Sometimes it is useful to fix some coordinate system and think of Jet^n_r as a space of truncated Taylor series which start at order n and go up to order r. Since in coordinates a vector field on $\mathbb{R}^2$ is simply a pair of functions, we can define the space $VJet^n_r$ of jets of vector fields.

An analogous constructions also work for diffeomorphisms. Consider $r \geq$

1 and let D_r be a normal subgroup of $Diff$ consisting of elements ϕ such that the function $(x, y) \mapsto \phi(x, y) - (x, y)$ vanishes at $(0, 0)$ together with its derivatives up to order r. The quotient group $Diff_r := Diff/D_r$ is a finite dimensional Lie group. Indeed the algebraic structure has just been explained and the manifold structure is given by viewing a diffeomorphism $\phi \in Diff$ as pair (ϕ_1, ϕ_2) of real valued functions on $\mathbb{R}^2$ such that $\phi_1 \in \mathcal{A}_1$ and $\phi_2 \in \mathcal{A}_1$. This way an element $j \in Diff_r$ is viewed as a point in the vector space $Jet^1_r \times Jet^1_r$ and is characterized by the condition of having an inverse, which is an open condition. So, as a smooth manifold, $Diff_r$ is just an open subset of $Jet^1_r \times Jet^1_r$ and this smooth structure is compatible with the algebraic one. As with functions, the truncated Taylor series interpretation for $Diff_r$ is useful to keep in mind. For instance, $Diff_1$ consists of linear maps, $Diff_2$ allows for quadratic terms etc. It is easy to see that our construction of Jet^m_r and $Diff_r$ is independent of the choice of coordinates. Let $\{\cdot\}_r$ denote the operation of taking the r-th jet of a function or a diffemorphism. The operation $\{\cdot\}_r$ will sometimes be called truncation. Since the ideal of functions vanishing at $(0, 0)$ up to order r is invariant under the action of the group $Diff$, we get the induced action of $Diff$ on Jet^m_{m+r-1} by the formula: $\{h\}_{m+r-1}\phi := \{h\phi\}_{m+r-1}$. It is easy to see that the normal subgroup D_r acts trivially on Jet^m_{m+r-1}, so we get an action of $Diff_r$ on Jet^m_{r+m-1} by the formula: $\{h\}_{m+r-1}\{\phi\}_r := \{h\}_{m+r-1}\phi$, i.e.

$$\{h\}_{m+r-1}\{\phi\}_r = \{h\phi\}_{m+r-1}. \tag{6.1}$$

The latter is a linear action of a finite dimensional Lie group on a finite dimensional $\mathbb{R}$-linear space. Moreover the corresponding map

$$Diff_r \longrightarrow Aut(Jet^m_{m+r-1})$$

is smooth. Note that the map producing the r-th jet out of a function factors through the map producing a germ out of a function, so we can take jets of germs. The quotient map producing jets out of germs will also be denoted by $\{\cdot\}_r$ and called truncation. In the theorem below we introduce another notation for this map coming from the first letter of the word “truncation”.

Theorem 28. *Let $f \in \mathcal{A}_m$. Set $r := max(1, m - 3)$. Let*

$$t : Diff \longrightarrow Diff_r$$

and

$$T : \mathcal{GA}_m \longrightarrow Jet^m_{m+r-1}$$

be maps sending diffeomorphisms respectively germs of functions to their jets in $Diff_r$ *respectively* Jet^m_{r+m-1}. *Then the following statements are true.*

First: the map T *is* $(Diff - Diff_r)$*-equivariant, in other words*

$$T(\{h\}\phi) = T(\{h\})t(\phi), \tag{6.2}$$

for all $\{h\} \in \mathcal{GA}_m$ *and* $\phi \in Diff$.

Second: a germ $\{f\} \in \mathcal{GA}_m$ *belongs to the orbit* $\{f_0\}Diff$ *of* $\{f_0\}$ *under the action of* $Diff$ *if and only if its truncation* $T\{f\}$ *belongs to the orbit* $T\{f_0\}Diff_r$ *of* $T\{f_0\}$ *under the action of* $Diff_r$. *In other words*

$$\{f_0\}Diff = T^{-1}(T\{f_0\}Diff_r). \tag{6.3}$$

Proof. Equation (6.2) is simply the Formula (6.1). For Equation (6.3) take any $\{f\} \in \{f_0\}Diff$, that is $\{f\} = \{f_0\}\phi$ for some diffeomorphism $\phi \in Diff$, then Equation (6.2) applied to $\{f_0\}$ tells us that $T\{f\} = T(\{f_0\}\phi) = T\{f_0\}t(\phi)$, i.e. $\{f\} \in T^{-1}(T\{f_0\}Diff_r)$. For the converse inclusion take $\{f\} \in T^{-1}(T\{f_0\}Diff_r)$, i.e. $T\{f\} \in T\{f_0\}Diff_r$, therefore there exists a diffeomorphism $\tilde{\phi} \in Diff$ such that $\{f\}_{m+r-1} = \{f_0\tilde{\phi}\}_{m+r-1}$. The last equation means that in some coordinate system (call it (x, y)) the function f looks like $Re(x + iy)^m + h.o.$ with $h.o.$ starting from the order $m + r = m + max(1, m - 3) = max(m + 1, 2m - 3)$. Now we apply Theorem 25 to get that f can be brought to the form $Re(x + iy)^m$ by a coordinate change around the fixing the origin. That is for some open neighbourhood U of the origin and some diffeomorphism $\phi : U \longrightarrow \phi(U)$ fixing the origin we have $f|_U = f_0\phi|_U$. The last equation gives us that $\{f\} \in \{f_0\}Diff$.□

Now we compute the codimension the orbit $\{f_0\}_{m+r-1}Diff_r$ in Jet^m_{m+r-1} for the values of r we are interested in. As $\{f_0\}_{m+r-1}Diff_r$ is an immersed manifold in $\mathcal{A}_m$, its codimension is understood as a difference between the dimension of Jet^m_{m+r-1} and the dimension of $\{f_0\}_{m+r-1}Diff_r$ as a manifold. The dimension of the orbit $\{f_0\}_{m+r-1}Diff_r$ of $\{f_0\}_{m+r-1} \in Jet^m_{m+r-1}$ under the action of the group $Diff_r$ is equal to the difference between the dimension of the group $Diff_r$ and the the dimension of the stabilizer of the element $\{f_0\}_{m+r-1}$ under the action of the group. The dimension of the stabilizer is equal to the dimension of the kernel of the derivative of the evaluation map induced by the group action. To carry out this program we fix $\{f\}_{m+r-1} \in Jet^m_{m+r-1}$ and consider the evaluation map

$$ev_{\{f\}_{m+r-1}} : Diff_r \longrightarrow Jet^m_{m+r-1}$$

given by the action:

$$ev_{\{f\}_{m+r-1}}(\phi) = \{f\}_{m+r-1}\phi.$$

Note that $id \in Diff_r$ maps to $\{f\}_{m+r-1}$ under the map $ev_{\{f\}_{m+r-1}}$. Taking the derivative of $ev_{\{f\}_{m+r-1}}$ at $id \in Diff_r$ gives us

$$D_{id}ev_{\{f\}_{m+r-1}} : T_{id}Diff_r \longrightarrow T_{\{f\}_{m+r-1}}Jet^m_{m+r-1}$$

Since Jet^m_{m+r-1} is a linear space, its tangent space at every point can be canonically identified with itself. The tangent space to $Diff_r$ at id is identified with $VJet^1_r$. In coordinates we view jets as truncated Taylor series, i.e. certain sums of homogeneous polynomials and the map

$$D_{id}ev_{\{f\}_{m+r-1}} : VJet^1_r \longrightarrow Jet^m_{r+m-1}$$

writes out as

$$P\partial_x + Q\partial_y \mapsto P\{f_x\}_{m+r-2} + Q\{f_y\}_{m+r-2}.$$

The Lie subalgebra $Ker(D_{id}ev_{\{f\}_{m+r-1}})$ of $VJet^1_r$ is tangent to the stabilizer

$$Stab_{\{f\}_{m+r-1}} := \{g \in Diff_r | \{f\}_{m+r-1}\phi = \{f\}_{m+r-1}\}$$

of the element $\{f\}_{m+r-1}$. This allows us to compute the dimension of the stabilizer via the dimension of this Lie subalgebra and once we know the dimension of a stabilizer of an element this allows us to compute the dimension and therefore the codimension of the orbit of the element.

Lemma 18. *Let $1 \leq r \leq max(1, m-3)$ and consider the jet $\{f_0\}_{m+r-1} \in Jet^m_{m+r-1}$ of the function $f_0 = Re(x+iy)^m$. Let $Diff_r$ act on Jet^m_{m+r-1} as above. Then*
$Ker(D_{id}ev_{\{f_0\}_{m+r-1}}) = 1$ for $m = 2$ and
$Ker(D_{id}ev_{\{f_0\}_{m+r-1}}) = 0$ for $m > 2$.

Proof. Let $P = P_1\partial_x + P_2\partial_y \in Ker(D_{id}ev_{\{f_0\}_{m+r-1}})$ that is

$$P_1\{f_{0x}\}_{m+r-2} + P_2\{f_{0y}\}_{m+r-2} = 0. \tag{6.4}$$

We use the complex notation of Section 5.1 to view P as

$$P_1 + iP_2 \in Hom_\star \times Hom_\star = Hom_{\mathbb{C}}$$

and say that Equation (6.4) is equivalent to

$$RePz^{m-1} = 0.$$

We write P as a sum of homogeneous polynomials $P_k \in Hom_k \times Hom_k$ as follows: $P = \Sigma_{k=1}^r P_k$. This brings the last equation to the form

$$ReP_kz^{m-1} = 0 \tag{6.5}$$

for all $k = 1, ..., r$. Let us take up some k from this range and express P_k in terms of the basis $\{z^j\bar{z}^{k-j}\}_{j=0,1,...,k}$ for $Hom_k \times Hom_k$ as $\mathbb{C}$-linear space:

$$P_k = \Sigma_{i=0}^k a_j z^j \bar{z}^{k-j},$$

$a_j \in \mathbb{C}$. We compute:

$$ReP_kz^{m-1} = \Sigma_{j=0}^k Rea_j z^{j+m-1}\bar{z}^{k-j} = \Sigma_{j=0}^k (x^2+y^2)^q Rea_j z^p,$$

where $q = k - j \geq 0$ and

$$p = (j+m-1)-(k-j) = m-k+2j-1 \geq m-r-1 = m-1-max(1, m-3) \geq 0.$$

Assume that $m = 2$. Then $r = k = 1$. So the Equation (6.5) transforms to

$$(x^2+y^2)Rea_0 + Rea_1z^2 = 0,$$

or writing $a_1 = a + ib$, for $a, b \in \mathbb{R}$, to

$$(x^2+y^2)Rea_0 + a(x^2-y^2) - 2bxy = 0,$$

which immediately leads to the solution space $a = b = 0, a_0 \in i\mathbb{R}$ of dimension 1, so for $m = 2$ we get $Ker(D_{id}ev_{f_{0_{m+r-1}}}) = 1$.

Assume now that $m > 2$. Then $p \geq m - 1 - max(1, m-3) > 0$. So p is strictly(!) positive. We write $a_j = a_{j1} + ia_{j2}$, with this Equation (6.5) becomes

$$\Sigma_{j=0}^{k}(x^2+y^2)^q(a_{1j}Re(x+iy)^p - a_{2j}Im(x+iy)^p) = 0.$$

With the notation for the irreducible representations for the ρ-action of S^1 (see (5.7)) we have

$$(x^2+y^2)^q(a_{1j}Re(x+iy)^p - a_{2j}Im(x+iy)^p) \in Irr^q_{m+k-1}.$$

As a check for the lower $m+k-1$ in Irr^q_{m+k-1} we compute: $2q+p = 2(k-j)+m-k+2j-1 = m+k-1$. Since irreducible representations form a direct sum, we get

$$(a_{1j}Re(x+iy)^p - a_{2j}Im(x+iy)^p) = 0$$

for all $j = 0, ..., k$. Fix some j form this range. The last equation asks a homogeneous polynomial of order $p > 0$ to vanish. The x^p coefficient of this polynomial is equal to a_{1j}, so we get $a_{1j} = 0$ and hence $a_{2j}Im(x+iy)^p = 0$, i.e. $a_{2j} = 0$. So for $m > 2$ we get $Ker(D_{id}ev_{\{f_0\}_{k+m-1}}) = 0$.□

The following is an immediate corollary

Lemma 19. *Let $1 \leq r \leq max(1, m-3)$ and $\{f_0\}_{m+r-1} \in Jet^m_{m+r-1}$. Consider $Diff_r$ acting on Jet^m_{m+r-1} as above. Then $dimStab_{\{f_0\}_{m+r-1}} = 1$ for $m = 2$ and $dimStab_{\{f_0\}_{m+r-1}} = 0$ for $m > 2$.*

This allows us to finish the computation:

$$codim(\{f_0\}_{m+r-1}Diff_r) = dimJet^m_{m+r-1} - dim(\{f_0\}_{m+r-1}Diff_r) =$$

$$= dimJet^m_{m+r-1} - dimDiff_r + dimStab_{\{f_0\}_{m+r-1}}.$$

It easy to compute that

$$dimJet^m_{m+r-1} = \Sigma_m^{j=m+r-1}(j+1) = \Sigma_{m+1}^{j=m+r} j = \frac{1}{2}[(r+m)(r+m+1) - m(m+1)]$$

and

$$dimDiff_r = dimVJet^1_r = 2\Sigma_{j=1}^{r}(j+1) = (r+1)(r+2) - 2 = r(r+3).$$

This gives us the following

Proposition 16. *Let $s(m)$ denote 0 for $m > 2$ and 1 for $m = 2$. Then under the conditions of Lemma 19 we have the following formula*

$$codim(\{f_0\}_{m+r-1} Diff_r) =$$

$$= \frac{1}{2}[(r+m)(r+m+1) - m(m+1)] - r(r+3) + s(m) \qquad (6.6)$$

Now we use this proposition to compute some interesting partial cases. For $m = 2$ (and then $r = 1$, $s(m) = 1$) we have

$$codim(\{f_0\}_{m+r-1} Diff_r) = \frac{1}{2}[(1+2)(1+2+1) - 2(2+1)] - 4 + 1 = 0.$$

For $m = 3, 4$ (and then $r = 1$, $s(m) = 0$) we have

$$codim(\{f_0\}_{m+r-1} Diff_r) = \frac{1}{2}[(1+m)(2+m) - m(m+1)] - 4 = (m+1) - 4 = m - 3.$$

For $m = 5$ and $r = 1$ (and then $s(m) = 0$) we have

$$codim(\{f_0\}_5 Diff_1) = \frac{1}{2}[(1+5)(1+5+1) - 5(5+1)] - 1(1+3) = 2.$$

For $m \geq 4$ and $r = max(1, m-3) = m - 3$ (and then $s(m) = 0$) we have

$$codim(\{f_0\}_{m+r-1} Diff_r) = \frac{1}{2}[(2m-3)(2m-2) - m(m+1)] - (m-3)m =$$

$$= \frac{1}{2}(m^2 - 5m + 6) = \frac{1}{2}(m-2)(m-3).$$

Note that the last formula also makes sense for $m = 3, 4$ and gives correct numbers for these values of m. Altogether we have just proved the following

Proposition 17. *For $m \geq 2$ and $r = max(1, m-3)$ we have*

$$codim(\{f_0\}_{m+r-1} Diff_r) = \frac{1}{2}(m-2)(m-3).$$

For $m = 5$ and $r = 1$ we have

$$codim(\{f_0\}_5 Diff_1) = 2.$$

Now we use this proposition to deduce the following

Proposition 18. *Set* $j_0 := \{Re(x+iy)^5\}_6 \in Jet_6^5$. *Then there exists a jet* $j \in Jet_6^5$ *arbitrarily close to* j_0*, such that the projection of* $j - j_0 \in Jet_6^5$ *to* Jet_5^5 *vanishes (that is* j_0 *and* j *have the same leading terms) and* $j \notin j_0 Diff_2$.

Proof. Assume by contradiction that all the jets in some neighbourhood U of j_0 in Jet_6^5 with the same leading term as j_0 were on the orbit $j_0 Diff_2$ of j_0 under the action of $Diff_2$. Let the projection of j_0 to Jet_5^5 be denoted by j_{05} Let the affine subspace of Jet_6^5 consisting of jets j with leading term $\{f_0\}_5$ by denoted by L. By our assumption

$$L \cap U \subset j_0 Diff_2.$$

Therefore

$$(L \cap U) Diff_2 \subset j_0 Diff_2.$$

Consider the direct sum decomposition

$$Jet_6^5 = Jet_5^5 \oplus Jet_6^6.$$

The tangent space $T_{j_0}((L\cap U)Diff_2)$ to $(L\cap U)Diff_2$ at j_0 can be canonically identified with Jet_6^5 and contains both Jet_6^6 and $T_{j_{05}}(j_{05}Diff_1)$ and therefore their direct sum, altogether

$$T_{j_{05}}(j_{05}Diff_1) \oplus Jet_6^6 \subset T_{j_0}((L \cap U)Diff_2) \subset T_{j_0}(j_0 Diff_2) \subset Jet_5^5 \oplus Jet_6^6.$$

On the one hand this implies that the codimension of $j_0 Diff_2$ in Jet_6^5, is less or equal than the codimension of $j_{05} Diff_1$ in Jet_5^5 and the latter was computed in Proposition 17 to be equal to 2. On the other hand, the codimension of $j_0 Diff_2$ in Jet_6^5 can be computed using the first half of the same proposition to be equal to 3. This is a contradiction. □

As a corollary we get the following

Theorem 29. *There exists a smooth function* $\tilde{f}$ *on* $\mathbb{R}^2$ *arbitrarily* C^∞ *close to* $f_0 = Re(x+iy)^5$ *with the properties*
1) The leading term of the Taylor expansion of the function $\tilde{f}$ *at* $(0,0)$ *is* $Re(x+iy)^5$.
2) The function $\tilde{f}$ *is not equivalent to* f_0 *under any coordinate change in any open neighbourhood of* $(0,0)$. *In the language of germs: the germ* $\{\tilde{f}\}$ *of the function* $\tilde{f}$ *does not lie on the orbit of the germ* $\{f_0\}$ *of the function* f_0 *under the action of the group* $Diff$.

Proof. We write the jet j (see Proposition 18) in coordinates (x, y) as a truncated Taylor series: $f_0 + f_1$, where f_1 is a homogeneous polynomial in (x, y) of order 6. We take the function $\tilde{f}$ to be equal to $f_0 + \sigma f_1$ on $\mathbb{R}^2$, where σ is cut-off function, which is constantly 1 in some fixed small neighbourhood of $(0, 0)$ and vanishes outside the unit ball in $\mathbb{R}^2$. Then clearly $\{\tilde{f}\}_6 = j$. Now $\tilde{f}$ is C^∞ close to $f_0 = Re(x + iy)^5$ because j is close to j_0 in Jet_6^5. Property 1) is satisfied by construction. Property 2) follows from Proposition 18 and Theorem 28.□

Morally speaking Theorem 28 allows us to reduce the question of having a nice normal form near a critical point to working with a finite dimensional representation of a finite dimensional Lie group. Moreover, as we have just seen, certain codimension computations in this finite dimensional space lead to existence results on the level of smooth functions. This motivates the following definition.

Definition 29. *Let $m \geq 2$ and consider the action of $Diff$ on $\mathcal{GA}_m$. Consider the function $f_0 = Re(x + iy)^m$. The codimension of the orbit of the germ $\{f_0\}$ of f_0 in $\mathcal{GA}_m$ under the action of the group $Diff$ is defined to be the codimension of the orbit of $\{f_0\}_{m+r-1}$ in Jet^m_{m+r-1} under the action of the group $Diff_r$ for $r = max(1, m - 3)$.*

With this definition we can rewrite Proposition 18 as follows:

Theorem 30. *Let $m \geq 2$, $f_0 = Re(x + iy)^m$ and consider the action of $Diff$ on $\mathcal{GA}_m$. Then the codimension of $\{f_0\}Diff$ in $\mathcal{G}C^\infty(\mathbb{R}^2)_{(0,0)}$ is equal to $\frac{1}{2}(m - 2)(m - 3)$.*

We close this section by recalling the number $L(m) = \frac{1}{2}(m - 2)(m - 3)$ from the end of Section 5.2 and leaving it for the reader to think about the miraculous coincidence that occurred.

6.2 Characterization of intrinsically harmonic 1-forms on surfaces.

Since we already have from see Chapter 3 a characterization of intrinsically harmonic 1-forms as those which are simultaneously transitive and locally intrinsically harmonic, we concentrate on local intrinsic harmonicity here.

Since we are allowing the zeros of our 1-form to be of arbitrary order, the characterization of local intrinsic harmonicity stops being "topological" and becomes "smooth". As we already mentioned in the introduction, the answer to the question whether or not a given closed 1-form on a surface of genus g is locally intrinsically harmonic near its zero p starts depending on *higher order terms* in the Taylor expansion of ω near p. The main consequence of the previous section (in fact the previous chapter) is that we will be able to say *how many terms* at most we have to control in order to ensure local intrinsic harmonicity. The upper bound is given in terms of the genus g of the underlying Riemann surface.

Let ω be a locally intrinsically harmonic 1-form on a connected oriented surface Σ_g of genus $g \geq 2$. Then ω has isolated zeros and near every zero p_j it looks like $dRe(x+iy)^{m_j}$. We say that ω has degree $d_j = m_j - 1$ near p_j. The index of ω at p_j (as a section of the cotangent bundle) is equal to $-p_j$. So, by the Poincare-Hopf Theorem the sum of all of the d_j is equal to $2g-2$, which gives an obvious upper bound of $2g-2$ for the value of d_j. We now perform a Morse-theoretic trick, which allows us to give a twice better upper bound: $g-1$. We approximate ω by a closed Morse form $\tilde{\omega}$, which foliates Σ_g by closed (singular and nonsingular) leaves. Zeros of $\tilde{\omega}$ all have Morse index 1, some of them being connecting, some disconnecting. Clearly, the number of connecting zeros must be equal to the number of disconnecting, i.e. $g-1$. This has the following implications for the zeros of ω. The zero set S of ω can be decomposed as a disjoint union $S = P\dot{\cup}Q$, where $P = \{p_j\}_{j=1,\dots,k_P}$, $Q = \{q_i\}_{i=1,\dots,k_Q}$. Let d_j^p denote the degree of p_j and d_i^q denote the degree of q_i. Under the perturbation $\omega \to \tilde{\omega}$ the set P gives rise to connecting zeros of $\tilde{\omega}$, Q — to disconnecting. Therefore, the degrees of zeros of ω satisfy the following relation: $d_1^p + \dots + d_{k_P}^p = d_1^q + \dots + d_{k_Q}^q = g-1$. Now we give a characterization for local intrinsic harmonicity for closed 1-forms on surfaces. The proof of it follows from the discussion above and Theorem 28.

Theorem 31. *A closed 1-form ω on Σ_g is locally intrinsically harmonic if and only if it has isolated zeros $p_1, \dots, p_{K_P}, q_1, \dots, q_{K_Q}$ with the positive integers $d_1^p, \dots, d_{K_P}^p, d_1^q, \dots, d_{K_Q}^q$ being the corresponding orders such that the following holds true:*
(i) $d_1^p + \dots + d_{k_P}^p = d_1^q + \dots + d_{k_Q}^q = g-1$.
(ii)Near every zero of ω the jet (at this zero) $\{f\}_{m+r-1} \in Jet_{m+r-1}^m$ of the local primitive function f of ω lies on the orbit $\{f_0\}_{m+r-1}Diff_r$ of the jet $\{f_0\}_{m+r-1} \in Jet_{m+r-1}^m$ under the action of the group $Diff_r$ for $r =$

$max(1, m-3)$.

Note that by (i) the order $m+r-1$ of the jet space Jet^m_{m+r-1} has a definite upper bound. Indeed, near a zero $p(q)_j$ the order m_j of the local primitive function f of ω satisfies: $m_j = d_j^{p(q)} + 1$, hence $m_j - 1 = d_j^{p(q)} \leq g-1$, so $m_j \leq g$ and $m_j+r-1 \leq g-1+max(1, g-3) = max(g, 2g-4)$. Moreover, we can give an upper bound on the codimension of the orbit $\{f_0\}_{m_j+r-1} Diff_r$ in $Jet^m_{m_j+r-1}$. By Proposition 17 that is $\frac{1}{2}(g-2)(g-3)$.

6.3 An example.

Example 4. Consider the cobordism $C = (W, V = S^1, V' = \cup_{j=1}^5 S^1)$, that is S^1 is cobordant to the disjoint union of five S^1's via W, where W is the surface of genus zero with boundary 6 components. Take the function $f_0 = Re(x+iy)^5$ on $\mathbb{R}^2$. Using this function as a local model for an isolated singularity it is possible to construct a function f on the cobordism C which has a unique critical point p and near p it looks like f_0. Moreover $f|_V = -1$, $f|_{V'} = 1$. By doubling the cobordism along V' we obtain a cobordism of S^1 to S^1 and after gluing these S^1's we obtain the surface Σ_5. The form df on W gives rise to a closed form ω on Σ_5 with two critical points p and p' where it looks like df_0 and $-df_0$ respectively. Note that whereas df is exact on W, the form ω it gives rise to on Σ_5 is not exact. Clearly, the form ω is transitive and locally intrinsically harmonic and hence harmonic. We perturb ω near p to give $\tilde{\omega}$ by perturbing the local model f_0 near $(0,0)$ to $\tilde{f}$. The form $\tilde{\omega}$ so constructed is C^∞ close to ω, has the same zeros and the same principle parts at zeros. But $\tilde{\omega}$ is not locally intrinsically harmonic and therefore not harmonic.

Philosophically speaking, this example shows two things happening if we allow closed 1-forms with isolated zeros of finite order instead of just Morse forms: intrinsic harmonicity can not be detected by topological tools, and openness of the set of intrinsically harmonic 1-forms fails.

A recent result by M. Farber and D. Schütz (cf. [8]) stating that in any nonzero cohomology class $\xi \in H^1(M, \mathbb{R})$ there always exists a closed 1-form ω having at most one zero motivates the second example. Consider such a 1-form on the oriented surface Σ_g of genus g. In this case it is clear that ω is not intrinsically harmonic, since otherwise by Theorem 31 it would have at least two zeros, contradicting the construction of ω.

6.4 Concluding remarks.

This section addresses possible "what is next?" questions.

Chapter 3 suggests to try to decide the question of intrinsic harmonicity in concrete examples. The form constructed by M. Farber and D. Schütz in [8] seems to be challenging if one looks at it on manifolds of dimensions higher than two. The following consideration illustrates the possible complexity of its behavior. Let M be an odd-dimensional manifold and $\xi \in H^1(M, \mathbb{Q})$ a rational cohomology class. Assume furthermore, that M does not fiber over the circle. Let ω be a closed 1-form representing ξ with at most one zero. Let this zero be denoted by p. We take a small ball B around p and consider the primitive function $f : B \longrightarrow \mathbb{R}$ for ω in this ball: $df = \omega|_B$. This function displays a complicated behavior near its critical point p. Let X be a vector field dual to df with respect to some Riemannian metric. By the Poincaré-Hopf Theorem the index of X at p is equal to zero. But it is not possible to construct a function $\tilde{f} : B \longrightarrow \mathbb{R}$ being equal to f near ∂B and without critical points. Indeed, that would give us a closed 1-form $\tilde{\omega}$ on M without zeros representing the same cohomology class as ω, i.e. $[\tilde{\omega}] = \xi$. But the class ξ is rational so M fibers over the circle contradicting our assumption.

Chapter 4 leaves open the following question. Let $\mathcal{M}_{0,n}$ denote the subspace of Morse forms without zeros of index 0 or n. Is it true that the intersection of the interior $Int\bar{\mathcal{T}}$ of the set $\bar{\mathcal{T}}$ of nontransitive forms with $\mathcal{M}_{0,n}$ is nonempty? On manifolds for which the answer to this question is negative Theorem 22 — the main theorem of Chapter 4 is empty (meaning trivially true). So it seems interesting to find a large class of manifolds where the answer to the question is positive. So far we know two examples with the positive answer to this question — these are the examples from Section 4.5 Potential examples of manifolds on which the answer is negative look exotic and counterintuitive. So it is interesting to find one.

Chapter 5 contains to main theorems — Theorem 25 and Theorem 27. The second one raises the obvious question: can we improve $\triangle_g h$ from being exponentially small to being exactly zero? Concerning Theorem 25, one would like to have some analogs in higher dimensions. So what about a function $f = f_0 + h.o.$ defined on an open ball around the origin in $\mathbb{R}^n$? Here f_0 is a homogeneous polynomial of order m, which is harmonic with respect to the standard metric on $\mathbb{R}^n$ and $h.o.$ stands for the terms of higher order.

Chapter 7

Appendix

Lemma 20. *Consider the following smooth function on* $B_R(0) \subset \mathbb{R}^n$*:* $f_0(x_1, ..., x_n) = -x_1^2 - ... - x_\lambda + x_{\lambda+1}^2 + ... + x_n^2$. *Then there exists a* C^2*-open neighbourhood* $\mathcal{U}$ *of* f_0 *in the space of smooth functions, such that every* $\tilde{f} \in \mathcal{U}$ *can be brought to the canonical Morse form with* λ *minus signs in it, throughout* $B_{R/2}(0)$ *by a smooth change of coordinates. Moreover, the diffeomorphism which brings one set of coordinates to the other is close to the identity.*

Proof. We closely follow Milnor cf. [15]. Let δ be a positive real number and consider all smooth functions $\tilde{f}$ on $B_R(0)$ with $||\tilde{f} - f_0||_{C^2} < \delta$. Fix a smooth function $\chi : \mathbb{R}^n \longrightarrow \mathbb{R}$ with the following cut-off properties: $\chi|_{B_{R/2}}(0) = 1$, $\chi|_{\mathbb{R}^n \setminus B_R(0)} = 0$. The function $f = f_0 + \chi(\tilde{f} - f_0)$ is a real valued function defined on the whole of $\mathbb{R}^n$. Moreover, the function f coincides $\tilde{f}$ in $B_{R/2}(0)$. Note that f inherits nice properties of $\tilde{f}$, namely for any $\varepsilon > 0$ we can choose δ to be so small that $||f - f_0||_{C^2} < \varepsilon$, but the advantage is that both f_0 and f are defined on the whole of $\mathbb{R}^n$. Another nice (although not necessary for us) remark is that $(f - f_0)|_{\mathbb{R}^n \setminus B_R(0)} = 0$. By taking δ small enough we can achieve that there exists a unique critical point a of the function f and a is close to the origin. We set $c_a := f(a)$ and $y = x - a$ (expressions like x and y are shortcuts for $(x_1, ..., x_n)$, $(y_1, ..., y_n)$ etc). The function f can now be expressed in the following form: $f(x) = c_a + k(y)$, where $k(y) = \Sigma_{i,j=1}^n y_i y_j h_{ij}(y)$, where the matrices $\{h_{ij}(y)\}_{i,j=1,...,n}$ are symmetric and the map producing h_{ij} out of f is continuous if we equip the domain with the C^2 topology and the target with the the C^1 topology. In particular $|h_{ij} - (\pm 1)\delta_{ij}|$ is C^1 small provided δ is small enough. Here $\pm$ should be taken as a minus if $i \leq \lambda$ and as a plus otherwise, δ_{ij} is the usual Kronecker delta. The smallness of $|h_{ij} - (\pm)\delta_{ij}|$ guarantees that the Morse index of f at a coincides with λ — the Morse index of f_0 at 0.

Suppose by induction that there exist coordinates $u_1, ..., u_n$ in $\mathbb{R}^n$ so that

$$k = \pm u_1^2 \pm u_2^2 \pm ... \pm u_{r-1}^2 + \Sigma_{i,j\geq r} u_i u_j H_{ij}(u)$$

throughout $\mathbb{R}^n$, where the matrices $\{H_{ij}(u)\}_{i,j=1,...,n}$ are symmetric and $|H_{ij} - (\pm 1)\delta_{ij}|$ is C^1 small. Here $\pm$ should be taken as a minus if $i \leq \lambda$ and as a plus otherwise. Moreover the diffeomorphism taking y coordinates to u coordinates is close to identity. Let $g(u)$ denote the square root of $|H_{rr}(u)|$. This will be a smooth non-zero function throughout $\mathbb{R}^n$. Now we introduce new coordinates v by $v_i = u_i$ for $i \neq r$ and

$$v_r(u) = g(u)[u_r + \Sigma_{i>r} u_i H_{ir}(u)/H_{rr}(u)].$$

Since the smooth map producing v out of u does not have critical points, and moreover the norm of the inverse of the derivative is bounded, the global inverse function theorem by Hadamard cf. [13] Theorem 6.2.4. page 125 (see also [9] and [10]) applies to guarantee that $v_1, ..., v_n$ will serve as coordinate functions within $\mathbb{R}^n$. It is easily verified that k can be expressed as

$$k = \Sigma_{i\leq r}(\pm) v_i^2 + \Sigma_{i,j>r} v_i v_j H'_{ij}(v)$$

throughout $\mathbb{R}^n$, where

$$H'(v) = H_{ij}(u) + \frac{H_{ri}(u)H_{rj}(u)}{H_{rr}(u)}.$$

Therefore the matrices $\{H'_{ij}(v)\}_{i,j=1,...,n}$ are symmetric, $|H'_{ij} - (\pm 1)\delta_{ij}|$ is C^1 small and the diffeomorphism taking u coordinates to v coordinates is close to identity. This completes the induction step. Therefore f can be brought to the canonical form on the whole of $\mathbb{R}^n$. Recall that f coincides with $\tilde{f}$ on $B_{R/2}(0)$.$\square$

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