Construction of Quantum Symmetries for Realistic Field Theories on Noncommutative Spaces

Doctoral Thesis

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Zusammenfassung

Die nichtkommutative Geometrie stellt den ältesten Zugang zur Regularisierung von Ultraviolettdivergenzen der Punktwechselwirkungen in der Stöhrungstheorie dar. Dieser Zugang ist eine Verallgemeinerung der Quantenmechanik. Die Regularisierung erfolgt durch nichtverschwindende Unschärferelationen, die sich aus der neu eingeführten Nichtkommutativität der Ortsoperatoren ergibt. Zusätzlich ist das Ortseigenwertspektrum quantisiert - der messbare Raum erhält eine diskrete Struktur. Diese wird physikalisch als gravitativer Hochenergieeffekt auf der Planck-Skala verstanden. Der Bruch der Poincaré-Symmetrie durch nichkommutative Ortsoperatoren stellt die zentrale technische Problematik der nichtkommutativen Geometrie dar. Die mathematische Handhabung dieser Problemstellung ist aufwendig und wird im mathematischen Fachgebiet der Quantengruppen behandelt. Die mathematische Entwicklung hat sich dabei teilweise von den Bedürfnissen der Physik entfernt. Diese Doktorarbeit leistet einen Betrag dazu, Quantengruppen für die Anforderungen der Quantenfeldtheorie besser zugänglich zu machen. Zu diesem Zweck wird im Rahmen dieser Arbeit die Quantisierung der Poincaré-Algebra für nichtkommutative Räume mit kanonischen Kommutatorrelationen berechnet. Diese Räume sind äußerst populär unter Feldtheoretikern und verfügten bisher nur über Translationsinvarianz. Die Deformationen werden über einen notwendigen Satz von Bedingungen und einem allgemeinen Ansatz für die Lorentz-Generatoren bestimmt. Es wird eine zweiparametrige Schar von äquivalenten aber nichttrivialen Deformationen der Poincaré-Algebra erhalten. Die vollständige Hopf-Struktur wird berechnet und bewiesen. Casimir-Operatoren und Raumzeitinvarianten werden bestimmt. Desweiteren wird ein allgemeines Quantisierungsverfahren entwickelt, in dem die universelle Einhüllende von Matrix-Darstellungen von Lie-Algebren in eine eigens konstruierte Hopf-Algebra von Vektorfeldern als Unteralgebra eingebettet wird. Die unter Physikern populären Sternprodukte können damit generell zur Twist-Quantisierung von Lie-Algebren verwendet werden. Da die Hopf-Algebra der Vektorfelder gößer ist als die universelle Einhüllende der Lie-Algebra, sind allgemeinere Deformationen möglich als bisher. Dieses Verfahren wird weiterhin auf die Heisenbergalgebra mit Minkowski-Signatur angewendet. Dadurch erhält man eine fundamentale Verallgemeinerung der Quantenmechanik, motiviert als gravitativer Hochenergieeffekt. Nichtkommutativitat wird dadurch in Abhängigkeit von Energie und Impuls gesetzt. Technisch wird dazu das Quantisierungsverfahren von Weyl und Moyal formalisiert. Die Mehrfachanwendung von Twists wird eingeführt.

In diesen heil'gen Hallen
Kennt man die Rache nicht,
Und ist ein Mensch gefallen,
Führt Liebe ihn zur Pflicht.
Dann wandelt er an Freundes Hand
Vergnügt und froh ins bess're Land.

In diesen heil'gen Mauern
Wo Mensch den Menschen liebt,
Kann kein Verräter lauern,
Weil man dem Feind vergibt.
Wen solche Lehren nicht erfreun,
Verdienet nicht, ein Mensch zu sein.

(Aria No. 15, The Magic Flute, E. Schikaneder)

To Whom It May Concern

Contents

1 Introduction

"Die Welt ist meine Vorstellung"[1]
"Der Stil ist die Physiognomie des Geistes"[2]

(Arthur Schopenhauer)

1.1 Noncommutative Geometry: A brief Status Report

Noncommutative geometry represents the oldest and most abstract approach towards regularization of ultraviolet divergencies in quantum field theory. Its roots can be found as early as the time quantum mechanics obtained its final state of development in 1925. Among the first, in 1930, Heisenberg considered the generalization of the scheme of quantization towards a noncommutative algebra of coordinates. In order to solve the problem of diverging electron self-energy, Heisenberg already at this early stage of research pursued such ideas to regularize his computations [37]. Equal time commutation relations of the Heisenberg algebra exhibit well known noncommutativity among pairs of coordinates and momenta along a common axis. Represented on a Hilbert space, this gives rise to uncertainty relations that provide a lower bound for the precision of equal time measurements of such pairs of observables. Enhancing the Heisenberg algebra to a noncommutative algebra of coordinates would equally endow the theory with uncertainty relations for measurements of points in spacetime and moreover result in a degeneration of the spacetime continuum to a discrete structure, as we know it for angular momentum in standard quantum mechanics. Noncommutativity thus results in nonlocality within quantum field theory and, similar to crystaline structures of condensed

[1]"The world is my perception", Arthur Schopenhauer, Die Welt als Wille und Vorstellung I, Erstes Buch: Der Welt als Vorstellung, Erste Betrachtung: Die Vorstellung unterworfen dem Satze vom Grunde: Das Objekt der Erfahrung und Wissenschaft [75]

[2]"Style is the physiognomy of mind"

matter, it evokes a natural upper bound for momenta. Via Fourier transformation we dualy understand the latter as a finite lower bound in the length scale. The theory thus obtains a finite minimal length that can possibly be measured. It is thus expected that by the introduction of noncummutative geometry, divergencies, caused within the ultrviolet regime, cancel out of perturbation series. This represents the aspect of noncommutative geometry as a renormalization procedure for effective quantum field theories. On the other hand, the introduction of a finite length scale into the theory also rises more fundamental questions, such as the origin of such scales, its magnitude as well as its dependence on physical constants. But such conceptually most interesting issues were drown by severe complications arising from the breakdown of Poincaré covariance due to the newly introduced noncommutativity. To this day, this represents the major obstacle noncommutative geometry has to struggle with. Nevertheless, already in 1947 a first successful approach to a Lorentz-covariant formulation of noncommutative geometry, was performed by Snyder [80]. He introduced a quantum spacetime using the symmetry group $\mathrm{SO}(1,4)$ of five dimensional de Sitter space. While the zero components of the group are interpreted as spacetime coordinates, the remaining subgroup $\mathrm{SO}(1,3)$ provides conventional boosts and space rotations that are graded by a finite length scale parameter. The independent de Sitter coordinates themselves represent the energy-momentum operators of spacetime displacement. In contrast to the commutative case, momentum space thus carries the topology of a de Sitter space. In order to obtain proper translational invariance of the theory, Yang modified the setup towards a de Sitter spacetime in the limit of large radii [90]. Within a second publication [79], Snyder introduces electrodynamics into his framwork of quantum spacetime and experiences, as one of the first, the most characteristic conceptual issues of noncommutative geometry, such as a proper definition of functions depending on noncommutative variables and their multiplication or such as a neat introduction of partial derivatives into a spacetime that actually is discrete. While Snyder intended his constructions as an approach to renormalization, he nevertheless as a first also addressed the fundamental aspect of noncommutative geometry. Within his specific construction, the algebra of coordinates relates energy and momentum to geometry as an effect of the high energy regime. However, the lack of a suitable mathematical framework isolated this single example of noncommutative geometry in physics for decades. Till the days Snyder's construction has often been reconsidered, as for example in the works of Gol'fand in 1960 and 1963 [33, 34, 32], who incorporated Snyders momentum space of constant curvature into the setup of quantum field theory, but, as indicated above, in general severe complications such as the breakdown of Lorentz-covariance remained unsolvable and quantum field theory developed alternative schemes of renormalization. The requirement for a renormalization procedure in quantum field theory first appeared in perturbative computations of closed loop diagrams over virtual particles in quantum electrodynamics. These diagrams incorporated vacuum polarizations of bare pointlike charges and thus turned the problem into a multiparticle setup. While Snyder released his article on the quantum space construction, the problem of ultraviolet divergencies in quantum electrodynamics had been solved by Tamonaga, Schwinger and Feynman. They formulated as the first the modern prototype of a renormalization procedure: virtual particles were collectively associated to the bare charge and mass of particles. For these contributions, they received the Nobel prize in 1965. Since this time, the development of renormalization procedures has been accomodated within the research of particle theories that needed their specific treatment of ultraviolet divergencies. At that time the production of experimental data preceded the development of theoretical models and thus renormalization theory mostly oriented itself to actual requirements than conceptual issues. Thus a broad variety of renormalization procedures has been developed along the research of abelian gauge theories and Yang-Mills theories. And while methods such as Wilson's lattice regularization were based on ideas most similar to the concept of noncommutative geometry and thus carried a deeper conceptual footing, alternative methods such as Pauli-Villars regularization or dimensional regularization carried through. When 't Hooft and Veltman finally published their proof on the renormalizability of gauge theories in 1972 [83, 82], they thus took the final step to the formulation of the standard model of particles between 1970 and 1973. Within their framework, they moreover correctly predicted renormalized particle properties such as the top quark mass and through the verification in LEP at CERN got awarded with the Nobel prize in 1999. The success of the standard model thus is a success of quantum field theory, renormalization theory and accelerator experiments. Since gravity remained nonrenormalizable in this framework and thus could not be incorporated into the standard model, research obtained a more fundamental orientation towards gravity motivated Planck scale physics. The Planck length

$$\lambda_p = (\frac{G\hbar}{c^3})^{\frac{1}{2}} \simeq 1.6 \times 10^{-33}\mathrm{cm}.$$

thus marks the finite fundamental length scale where noncommutative geometry has to be incorporated as well. However, physicists who developed renormalization procedures, such as Feynman, understood that the requirement of such methods represents an actual lack within the understanding of fundamental physics. The phenomenon that renormalization at all applies to the standard model characterizes it as an effective theory, whose mechanics are decoupled from a more fundamental footing in the high energy regime. While this generation of physicists, that actually invented renormalization, remained quite uneasy and suspicious about it, younger scientists, probably because of its success, began to change their attitudes towards renormalization and accepted it as a procedure such as quantization itself. In this spirit superstring theory was developed as a first serious attempt to Planck scale physics. On the basis of quantum field theory and renormalization theory, string theory represents a new model of extended one dimensional objects that regularize Feynman diagramms. String excitations thereby evoke a discrete spectrum of particle masses and spins. Due to the existence of a spin two particle that features fluctuations of the metric tensor of general relativity in the low energy regime, it is also considered as a viable approach to unification of fundamental interactions. However, in the 1980s and early 1990s, noncommutative spaces and their symmetries were investigated more systematically within the context of quantum groups, which arose from the work of Faddeev on the inverse scattering method [31]. The first objects studied in quantum groups were deformed Lie algebras and groups such as $U_q(sl_2)$ of Kulish and Reshetikhin [55] or compact quantum matrix groups such as $SU_q(2)$ of Woronowicz [89]. These quantum groups were identified to be Hopf algebras as Sklyanin showed for example for $U_q(sl_2)$ in [78]. Moreover Drinfeld and Jimbo found a whole class of one parameter deformations of semisimple Lie algebras [39, 30] being Hopf algebras of quantum universal enveloping algebra type. The study of representations always kept the contact to physical aspects. At the beginning of 1990s q-deformations of the Lorentz and Poincaré algebra, represented on a q-Minkowski space, were obtained [15, 71, 16, 73, 70, 87]. Despite their elegance and their mathematically rigorous construction these noncommutative spaces turn out to be far too complicated to construct field theories on them with a reasonable amount of effort. This is mainly due to the fact that the commutation relations of the corresponding quantum spaces are fully quadratic in the coordinates. This makes it impossible to define Moyal-Weyl star products in terms of exponential expressions. Prominent exception is the κ-Minkowski space [58, 56, 66] that allows a study of field theoretic aspects as for example in [23, 24, 25, 57, 54]. Apart from this, several toy model constructions, such as the the fuzzy sphere of Madore [61, 62], provided a complete framework for covariant noncommutative spaces. Parallel to the development of quantum groups, string theory became the most popular approach to a unified theory of quantum gravity. In the last years open strings with homogenous magnetic background field [20, 74] gave rise to so called brane world scenarios where the effective field theories live on noncommutative spaces with canonical commutation relations[3]. Seiberg-Witten map [77] and deformation quantization [11, 53]

[3] A similar result was received at the beginning of the 1970s where the effective theory

opened the doorway to gauge theories on noncommutative spaces that even lead to noncommutative versions of the standard model of particles and the grand unified theory [63, 41, 42, 40, 14, 9]. The main drawback in this latest approach is the absence of a scheme of quantization and of spacetime symmetries other than translation invariance. Note that noncommutative spaces with canonical commutation relations were also obtained by introducing the nonlocality by general relativistic arguments, where the involved constructions become covariant under Lorentz symmetries by imposing additional quantum conditions on the antisymmetric constant tensor [27, 28]. Apart form this, string theory evoked a new discussion of matrix models in respect to noncommutative geometry [6, 22]. In parallel, conceptual problems were discussed such as the unitarity problem arising from a noncommutative time-coordinate [35, 1, 21, 10] and the IR-UV mixing effects, that correlate short and long-distance terms in perturbation theory [67]. These issues are best reviewed in [29, 81]. In order to summarize this short review on noncommutative geometry one might conclude that the situation at the beginning of this thesis is mostly characterized by technical obstacles that devide the field in two sections. From one hand, there are sophisticated gauge field theory constructions on noncommutative spaces that lack symmetry and on the other hand, there are realistic noncommutative spaces, such as q-Minkowski space, that provide the required deformation of Lorentz symmetry, but turn out to be too complex for field theories to be considered. These more complex noncommutative spaces are required to solve such problems as the IR-UV mixing that directly depends on the noncommutative structure. Merely toy model constructions allow for a combinaion of these two basic directions. But a realistic setup still was missing. And thus there were two basic opportunities at hand. Either suitable starproducts had to be found for realistic quantum spaces that are already endowed with a deformed version of Lorentz symmetry, or those spaces that allowed for field theories had to be enhanced by a deformed symmetry setup. We are thus still confronted with the most characteristic obstacle of noncommutative geometry, being the problem of missing covariance of quantum spaces. The most significant moments in the research on noncommutative geometry have always been those, where a relevant step could be made in respect to this problem. In such a light appears the invention of quantum groups, that evoked an active developement until the mid 1990s, and especially the construction of κ-Minkowski space. On the other side, for the first time a most significant progress was made within the construction of gauge field theories on noncommutative spaces. Thus the task at hand was to join these two directions in order to obtain a realistic setup for noncommutative field theories.

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of charged particles in a homogenous electric field lead to the same noncommutative space [76].

1.2 Thesis Objective

This thesis is intended to be a contribution to those efforts that pave the way towards realistic gauge field theories on noncommutative spacetime. The notion of a *realistic* field theory is further specified. First of all, we understand this to be a theory formulated on a four dimensional deformation of Minkowski space, that is the representation space of an accordingly deformed Poincaré algebra. We also consider supersymmetric versions of noncommutative geometry among realistic setups. Independent of this, the formulation of a gauge theory might be performed using the popular Seiberg-Witten approach - but this has not necessarily to be the case. However the approach to gauge field theories is chosen, it is a topic on its own to incorporate the gauge symmetries of the standard model into the Lorentz-covariant noncommutative setup. As *realistic* gauge symmetries, we consider those of the standard model or of the grand unified theory. Beyond this perspective on realistic field theories, there is the fundamental approach, since after all, a solution to correctly grasp the high energy regime of nature might not be performed in terms of effective theories but instead together with quantum effects of gravity. As such, extensions of noncommutative geometry, being dependend on energy and momentum densities of the field theory, might turn out as the only alternative to renormalization. Due to its locality, this fundamental setup might moreover avoid effects such as IR-UV-mixing in general and enhance quantum mechanics with finite in-principle precision of measurements in spacetime. Moreover, only the fundamental approach can possibly pave the way towards a deeper understanding of the origin of gauge symmetries - at least in long terms. At the beginning of work on this thesis most publications on noncommutative geometry in physics concentrated on the topic of gauge field theories, using the Seiberg-Witten approach. A considerable amount of these were written by field theorists and string theorists. On the other hand there has been a smaller community of mostly mathematiciens, working on quantum groups. As already mentioned above, quantum groups barely developed along the lines that are required by physicists. The Munich group finds itself in a quite unique situation, since it pursues research in both of these directions. Taking this advantage into account, the primary objective of this thesis had been set to the junction of these two directions in order to provide physicists with quantum group techniques that are suited to their specific requirements in realistic gauge field theories. It is thus the intend of this thesis to develop quantum symmetries for noncommutative spaces that yet miss any notion of symmetry and moreover develop a scheme that provides quantum symmetries to, in principle any, given noncommutative space - also in the fundamental approach. A contribution to this aim is vital for the progress of noncommutative geometry, because its central needs unfortunately are located within this specific vaccuum between mathematical interests and physical applications.

1.3 Outline and Results

We give a short outline of results obtained within this thesis. The most popular noncommutative space in physics is that ruled by canonical commutation relations. For a long period of time this space merely exhibited translational covariance. A quantized version of the Lorentz symmetry was missing. In the third chapter we construct such deformations of the Poincaré-algebra as representations on a noncommutative spacetime with canonical commutation relations. These deformations are obtained by solving a set of conditions by an appropriate ansatz for the deformed Lorentz generators. They turn out to be equivalent Hopf algebras of quantum universal enveloping algebra type with nontrivial antipodes. In order to present a notion of θ-deformed Minkowski space $\mathcal{M}_\theta$, we introduce Casimir operators and a spacetime invariant. In the fourth chapter we consider a general scheme to quantize symmetry algebras as matrix representations by means of starproducts. In quantum groups, coproducts of Lie-algebras are twisted in terms of generators of the corresponding universal enveloping algebra. If representations are considered, twists also serve as starproducts that accordingly quantize representation spaces. In physics, the situation turns out to be the other way around. Physics comes up with noncommutative spaces in terms of starproducts that miss a suiting quantum symmetry. In general the classical limit is known, i.e. there exists a representation of the Lie-algebra on a corresponding finitely generated commutative space. In this setup quantization can be considered independently from any representation theoretic issue. We construct an algebra of vector fields from a left cross-product algebra of the representation space and its Hopf-algebra of momenta. The latter can always be defined. The suitingly devided cross-product algebra is then lifted to a Hopf-algebra that carries the required genuine structure to accomodate a matrix representation of the universal enveloping algebra as a subalgebra. We twist the Hopf-algebra of vector fields and thereby obtain the desired twisting of the Lie-algebra. Since we twist with vector fields and not with generators of the Lie-algebra, this is the most general twisting that can possibly be obtained. In other words, we push starproducts to twists of the desired symmetry algebra and to this purpose solve the problem of turning vector fields into a Hopf-algebra. We give some genuine examples. In the fifth and last chapter we use a Hopf-algebra of vector fields defined on Minkowskian Heisenberg-algebra to deform its algebraic relations. Such deformations are found in discussions of high energy motivated minimal uncertainty theories. This is thus an application with respect to the fundamental approach to noncommutative geometry. We push these vector fields in terms of twists to deformations of the Lorentz-algebra. The original formalizm of Weyl and Moyal is applied in order to induce the commutation relations of the Heisenberg-algebra. Such a setup of starproducts is the closest to physical applications. We then once more use a twists of vector fields to deform the algebra-sector of the Heisenberg-algebra and the coalgebra-sector of the Lorentz-algebra. We thus introduce a double application of twists using the fact that the products of twists are twists as well. We give some basic example.

1.4 Aftermath and Acknowledgement

Aesthetics and mathematics can be regarded as two basic prototypes of nature implemented within the human mind. While studies serve as a bare acquirement of knowledge and general techniques, the doctoral thesis provides the first and only opportunity to shape these *a priorie* prototypes and *a posteriorie* resources to a fundament for scientific studies. As an artist, a theoretical physicist has thus to master technical as well as aesthetical skills in order to come down with substancial new insights. These skills dependently evolve in closed cycles and together with profound knowledge give rise to an increasingly coherent picture. Curiosity, nourished on these grounds, then finds its very own way to truth in research. In other words, a human being necessarily lives and acts according to the individual picture it made itself of its environment. Or as Arthur Schopenhauer says, "Die Welt ist meine Vorstellung". Thus scientists honestly have to engage into their quest, in order to achieve an objective picture that guides them towards actual discoveries. Nature unveils its mysteries to those that endeavour on her paths - that necessarily are those of aesthetics, logic and knowledge. Thus also artistic studies, such as literature, classical music and visual arts, train a scientist's symbolic perception and link it to that of his aestetical prototypes. But in contrast to most artists, techniques as well are confined to the mind of a theoretical physicist. He has to develop his very own tool kit and access to the matter. This cannot be taught. This can only evolve by his very own initiative. A theoretical physi-

cist thus endeavours two puzzles at the same time - his very own perception and techniques as well as his actual research on a specific subject. And as every puzzle, both begin with questions, go on with basic hypothesis that cluster to first answers and thus in long terms, order by order, develop a profound expertise. These puzzles evolve hand in hand and thus a theoretical physicist, in his first years, requires time to spend on his very own thoughts and trails through his perception until he is ready to share his ideas. "Ein geistreicher Mensch hat in gänzlicher Einsamkeit an seinen eigenen Gedanken und Phantasien vortreffliche Unterhaltung."[4] Fundamental research at the complexity of today's theoretical physics demands a daily compromise of scientists whose funding dictates a tight schedule. Within such objective limits, abilities, creativity, courage and confidence determine, whether a scientist is rather guided by opportunities or by ambition. And real life, however, once more decides for a compromise between these two. While the amount of knowledge increases in powers of time, students obtain as much time to do their studies as their fellow colleagues a hundred years ago. The knowledge acquired, broadens and gets shallow. A healthy equilibrium between these two states is a key for scientific success, because only within these bounds the human mind is able to constitute a coherent picture. If students do not obtain the required time and thus do not dare to get involved into the various questions that have to be answered over years and shape their individual picture, their knowledge and abilities remain fractionated and thus nourish belief. Understanding remains to be superficial and judgements begin to solely rely on formal criteria that boost some sort of activism. Curiosity and substancial ideas, that are the root of any invention, become drown by a hunt for attention and the formal satisfaction of expectations. Unfortunately such developments can be observed and endanger long term progress and trust in research. Not only in this concern I am very grateful for the honor and privilege that I could write this thesis under the wise guidance of my supervisor Prof. Dr. Julius Wess. His carefull support as well as his faith, his patience and his respect for my individual needs, that probably were not always easy to understand, have been vital ingredients for the success of this thesis. He never tried to put me into any corset - he gave me the freedom that I needed without getting me out of sight. Only this specific support enabled me to build up a substancial foundation for my future work. And thus today, if once more I had to decide about a doctoral project, I am glad to say that without any hesitation my choice would turn out to be the same. And in this respect I would like to express my loving gratitude to Prof. Dr. Julius Wess for all he has done for me. In this concern I also would like to thank Prof. Dr. Herman Schulz from the University of

[4]Arthur Schopenhauer, "In wholly solitude, a witty human being finds splendid entertainment in his own thoughts and phantasies"

Hannover for many rough and controversal discussions as well as his faith in me and his respect for my opinion, that actually almost never matched his own. I very much required this harsh exchange to find my orientation and this thesis would never have been written without this experience and, in the mean time, never without his recommendation and trust. My loving gratitude also goes to him. I would also like to thank Jan-Mark Iniotakis for his friendship and, once more, many years of vivid discussions, shared thoughts and a lot of new ideas. May our adventures continue ! In the same way I would like to thank Murad Alim for his great friendship, countless creazy ideas, that I hope one day we find the time for to realize, his great humor and fine instinct for physics. I moreover want to thank Efrossini Tsouchnika for a nice and fruitfull collaboration, as well as for her charming and inspiring greek point of view on life. Then of course my gratitude goes to Branislav Jurčo for many discussions, his sparkeling humor and for sharing the same passion for J.S. Bach. Then all the guys of our group, Michael Wohlgenannt, Hartmut Wachter, Alexander Schmidt, Fabian Bachmeier and Lutz Möller - what would I have done without your friendship as well as our countless discussions about physics and philosophy. My gratitude also goes to the members of the Arnold-Sommerfeld Center for Theoretical Physics at the Ludwig-Maximilians-University of Munich, for the excellent working atmosphere. Especially I would like to thank Prof. Dr. Dieter Lüst for hosting our group of noncommutative geometry and integrating us into his chair. We always felt welcome and at home. In this respect I am glad to thank also members of his group for their friendship and great collaboration, especially these are Viviane Grass and Maren Stein as well as Johannes Oberreuther, Fernando Izaurieta and Eduardo Rodríguez. Since this thesis had to start without any official funding in the first year, I am very grateful to the heads of administration of the faculty of physics at Ludwig-Maximilians-University, Susanne Weiß and her predecessor Kai Wede, who kept me alive with countless contracts, until I obtained my first studentship. This supportive attitude is part of that great and unique spirit that settles the success of our faculty. In this respect I also want to express my gatitude to the Ludwig-Maximilians-University for funding my project within the program "Gesetz zur Förderung des wissenschaftlichen und künstlerischen Nachwuchses". I am moreover gratefull to the Max-Planck Institut für Physik, Munich for integrating me into their program of doctoral fellows and thus financing me for the last period of this thesis. Last but not least I of course thank my parents Anke and Jan as well as my brother Oliver for their year-long support, trust and help.

1.5 Outlook

During the last years, noncommutative geometry once more made a significant progress towards the implementation of quantum symmetries and gauge field theories. Thus the fundamental approach of noncommutative geometry, i.e. the quantum gravity motivated attempt, technically obtained a first mature setup. For example, the introduction of a deformed Poincaré-symmetry to noncommutative spaces with canonical commutation relations paved the way for a first consideration of general relativity on a noncommutative space [7]. Its formulation on a noncommutative space provides the chance to obtain a consistent theory, where standard procedures of renormalization do not apply. In the last years enhancements of noncommutative geometry to a full phase space deformation were considered by discussions of modified quantum mechanics with gravity induced minimal uncertaint properties [3, 2, 5], [36], [49, 48, 47, 46, 45, 44]. However, most of these considerations yet do not enclose quantum symmetries. The mid-eigthies mark a significant turnabount in fundamental physics. Green and Schwarz formulated a first anomaly free supersymmetric open string theory with gauge group SO(32) that launched the tremendeous development of string theory. Meanwhile the invention of quantum groups and their accomodation within the Hopf-algebraic setup signify the crucial technical step, that had to be made in noncommutative geometry to push any progress. In parallel the youngest approach to fundamental physics, being canonical quantization of gravity, began its development. While string theory very much stands in the tradition of quantum field theory and renormalization, noncommutative geometry and canonical quantization of gravity, i.e. loop quantum gravity, represent two complementary fields of research. Up to now these two attempts required time to develop their own specific mathematical framework and it seems that the time has come that these technical issues can be overcome. However, while noncommutative geometry merely accomodates gauge field theories, loop quantum gravity merely considers the quantization of the background - it had first been formulated without any matter content. In the last years it had been shown that loop quantum gravity does not really restrict on particle physics. Virtually any kind of matter and gauge fields, even in a supersymmetric setting, can be accomodated. In this perspective a quantum theory of general relativity does not provide enough structure, to imply properties of the particle and gauge sector - as a unification of interactions within a single theory of quantum gravity would be expected to do. Nevertheless, while loop quantum gravity yet struggles with conceptual as well as technical problems, such as consistency of the quantization setup or solving Hamiltonian constraints, it implements the principles of general relativity in the best conceptual way we have. It exhibits Planck-scale behaviour by quantization of area and volume to elements of finite size. In the last years however there appeared strong indications that noncommutative geometry and loop quantum gravity not only are complementary topics but share the same conceptual footing. There is a well-known relation between curved spacetime and noncommutativity. As such noncommutative geometry easily arises by quantizing a theory over curved backgrounds due to constraints that have to be imposed. However, topological models such as lower dimensional Chern-Simons theory, that arise in the discussion of loop quantum gravity, give rise to quantized spacetime such as κ-Minkowski space. This is not an accidental coincidence, since Wilson loops introduce specific topologies, especially knot topologies, to the background that in turn are covered by braid groups, used to quantize Lie-algebras and theit quantum spaces. The newly enhanced setup of Drinfeld-twists especially covers quasitriangular deformations that are common tools in respect to knot invariants in mathematics. These twists might thus be closely related to Wilson loops. In this thesis it has been shown [52], that there are equivalence classes of θ-quantized Poincaré algebras. These equivalence classes are parametrized by real constants that in turn can be regarded as global U(1) gauge-parameters. Embedded into the twist approach of θ-quantized spacetime [17, 18], we obtain that gauge-invariance equivalently appears as the independence of the model from a specific choice of defomation. Thus intensifying research to twists and the question of how these might be enhanced to accomodate the gauge symmetries of the standard model, thus provides a fully new appoach to unification. Through relating twists to Wilson loops, we thus would obtain a direct connection of the gauge sector of particle physics and the topology of a quantized background. However, these new insights have yet to be worked out. But we now do have the technical opportunities to pursue these indications, that might point towards a new and more fundamental approach to high energy physics that moreover introduces knot topology as a new principle.

2 Mathematical Introduction

"When we have the Hamiltonian, we can apply a standard method which gives us a first approximation to a quantum theory, and if we are lucky we might be able to go on and get an accurate quantum theory."

(Paul A. M. Dirac[1])

This chapter introduces the concept of quantum groups and noncommutative geometry. Its scope is to provide a commentary to standard textbooks [19, 43, 50, 64] oriented towards the actual requirements of physicists. The discussion thus mostly restricts itself to quantum universal enveloping algebras of Lie-algebras and their dual algebras of functions over group manifolds. Most proofs in quantum groups are bare and straight computations that can be found in most textbooks - we thus omit these in order to focus on basic ideas. The specific view on the matter presented here, had been obtained during the work on this thesis. Most physicists consider quantum groups to be of rather exotic interest. Although invented in high energy physics, quantum groups quickly developed into a mathematical topic on their own. The mathematical framework required to accomodate quantum groups and their representations, is a slim and elegant setup of Hopf-algebras and monoidal categories that, however, does not belong to the standard education of a field theorist. The mathematical development quickly pursued its own interests and thus left quite a vacuum concerning physical applicability. However, the basic principles of quantum groups that come into account for most physicists do not require the full setup and are easy to grasp. We thus stick to these in order to provide a guideline for field theorists and intentionally keep an informal style. The chapter is organized as follows. In order to introduce the mathematical concept of *quantization*, we embed the scheme of canonical quantization, as it is known to every physicist, as deformations of Poisson-mainfolds. This lo-

[1] Paul A. M. Dirac, Lectures on Quantum Mechanics, Lecture No. 1: The Hamiltonian Method [26]

cates and formalizes canonical quantization within quantum groups and thus gives some orientation how its generalization immediately calls for a Hopf-algebraic setup, when it comes to representations. The first section thus starts with operator algebras in physics, that are represented on a Hilbert-space and through generalization of this scheme draws the lines towards the basic concepts of quantum groups. While the first section thus comes from physics to mathematics and motivates the basic constructs, we enter the actual matter in the following two sections by a more axiomatic approach. To this purpose the second part first introduces Hopf-algebras and their representations. This is the required preparation to introduce quantization of Lie-algebras and their covariant module spaces by quasitriangular structures $\mathcal{R}$ or their dual R-matrices in the third section. This final one closes, with the consideration of Drinfeld-twists and their relation to quasitriangular structures and starproducts. Unfortunately we do not have the space to give comments in respect to monoidal categories and cohomology that are required to rigorously perform representation theory and classify deformations, i.e. to study the existence of nontrivial quantizations of algebras.

2.1 Quantum Groups from Physics Perspective

The present section is basically divided into two parts. Within the first part we shortly recall canonical quantization as it is known to every physicists. We discuss this scheme and show how it correctly formalizes into a neat mathematical setup. We thus give *quantization* a precise mathematical meaning that is required for its generalization. Along the example of canonical quantization we introduce Poisson manifolds as well as their quantization. We further reduce quantum mechanics to the bare consideration of operator algebras and their representation on a Hilbert space. We thus recall some basic requirements of representation theory - this, however, might turn out a little sketchy. In the second part we show how generalization of the scheme of quantization requires for a Hopf-algebraic setup. Actually already textbook quantum mechanics would need such a framework, if the Heisenberg algebra would exhibit a more complex algebraic structure. More precisely this would be the case if bosonic or fermionic statistics would endow tensor-products of Hilbert spaces with a noncommutative structure. The Heisenberg algebra after all is nothing but the algebra of creation and annihilation operators. Noncommutative geometry faces exactly this kind of enhancement, required to neatly accomodate such modified quantum mechanics.

2.1.1 Quantum Mechanics within the Setup of Quantum Groups

Kinematics of a nonrelativistic Hamiltonian system of n degrees of freedom are formulated in terms of a $2n$-dimensional phase space $\Pi \subset \mathbf{R}^{2n}$. The Hamiltonian function $H(q_i, p_j)$ determines the time evolution of classical states $(q_i, p_j)_{i,j\in\{1,\ldots,n\}} \in \Pi$ along trajectories $\Gamma(t),\ t \in G \subset \mathbf{R}$ in Π. Dynamics are performed according to Hamiltonian equations of motion

$$\dot{q}_k = \frac{\partial H(q_i, p_j)}{\partial p_k} \qquad \dot{p}_l = -\frac{\partial H(q_i, p_j)}{\partial q_l}. \tag{2.1}$$

As initial condition, every state $(q_i, p_j) \in \mathbf{R}^{2n}$ fully determines the dynamics of the physical system in this set of first order differential equations.
We further want to focus our considerations on complex-valued functions $\varphi \in \mathcal{F}(\Pi) \subset C^{\infty}(\Pi, \mathbb{C})$ over Π that can locally be expanded in terms of power series

$$\varphi(q_i, p_j) = \sum_{\mathbf{i},\mathbf{j}} C_{\mathbf{ij}} \cdot (q_1)^{i_1} \cdot \ldots \cdot (q_n)^{i_n} \cdot (p_1)^{j_1} \cdot \ldots \cdot (p_n)^{j_n}, \quad C_{\mathbf{ij}} \in \mathbb{C}, \quad \mathbf{i}, \mathbf{j} \in \mathbb{N}_0^n$$

Up to initial conditions, the time evolution of $\varphi(\Gamma(t))$ as well is determined by Hamiltonian equations of motion (2.1)

$$\begin{aligned}
\dot{\varphi}(q_k, p_l) &= \sum_{i=1}^{n} \frac{\partial \varphi(q_k, p_l)}{\partial q_i} \cdot \dot{q}_i + \frac{\partial \varphi(q_k, p_l)}{\partial p_i} \cdot \dot{p}_i \\
&= \sum_{i=1}^{n} \frac{\partial \varphi(q_k, p_l)}{\partial q_i} \cdot \frac{\partial H(q_k, p_l)}{\partial p_i} - \frac{\partial \varphi(q_k, p_l)}{\partial p_i} \cdot \frac{\partial H(q_k, p_l)}{\partial q_i} \\
&= \{\varphi, H\}.
\end{aligned}$$

Here we introduced the Poisson bracket of two arbitrary phase space functions $\omega, \varphi \in \mathcal{F}(\Pi)$ by

$$\{\omega, \varphi\} := \sum_{i=1}^{n} \frac{\partial \omega}{\partial q_i} \cdot \frac{\partial \varphi}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \cdot \frac{\partial \varphi}{\partial q_i}. \tag{2.2}$$

Hamiltonian equations of motion thus simplify to

$$\dot{q}_k = \{q_k, H\}, \qquad \dot{p}_l = -\{p_l, H\} \tag{2.3}$$

and in particular, phase space coordinates of $\Gamma(t)$ themselves are regarded as elements of $\mathcal{F}(\Pi)$, we thus especially obtain

$$\{q_i, q_j\} = 0, \qquad \{p_i, p_j\} = 0, \qquad \{q_i, p_j\} = \delta_{ij}. \tag{2.4}$$

This is the Heisenberg Lie-algebra $\mathfrak{h}_{2n}$ in a canonical basis. More abstractly it is introduced as the complex vector space $\mathbf{R}^n \oplus \mathbf{R}^n \oplus i\mathbf{R}$, endowed with a bracket

$$[(\mathbf{p}_1, \mathbf{q}_1, c_1), (\mathbf{p}_2, \mathbf{q}_2, c_2)] = (0, 0, i\,(\,<\mathbf{p}_1, \mathbf{q}_2>\;-\;<\mathbf{q}_1, \mathbf{p}_2>\,))\,.$$

The scalar product $<\mathbf{p}, \mathbf{q}>$ is that of $\mathbf{R}^n$. From the complex numbers $\mathbb{C}$, a pointwise multiplication is induced on the set of functions $\mathcal{F}(\Pi)$

$$\forall\, \omega, \phi \in \mathcal{F}(\Pi) : (\omega \cdot_{\mathcal{F}} \phi)(q_i, p_j) = \omega(q_i, p_j) \cdot_{\mathbb{C}} \phi(q_i, p_j).$$

This turns $\mathcal{F}(\Pi)$ into an algebra. The Poisson bracket (2.2) moreover makes the phase space Π into an example of a Poisson manifold.

2.1.1 Definition (Poisson Manifold) *Let $\mathcal{M}$ be a $2n$-dimensional manifold and $C^\infty(\mathcal{M}, \mathbb{C})$ be the set of complex-valued smooth functions on $\mathcal{M}$. Then $\mathcal{M}$ is called* a Poisson Manifold, *if there exists a bracket $\{\cdot, \cdot\}$*

$$\{\cdot, \cdot\} : C^\infty(\mathcal{M}, \mathbb{C}) \times C^\infty(\mathcal{M}, \mathbb{C}) \to C^\infty(\mathcal{M}, \mathbb{C}),$$

such that the following properties (Antisymmetry, Leibniz-rule, Jacobi-Identity) hold:

$$\begin{array}{rcl} \forall \omega, \varphi, \psi \in C^\infty(\mathcal{M}, \mathbb{C}) & : & \{\varphi, \omega\} = -\{\omega, \varphi\} \\ & & \{\varphi \cdot \omega, \psi\} = \varphi \cdot \{\omega, \psi\} + \{\varphi, \psi\} \cdot \omega \\ & & \{\{\varphi, \omega\}, \psi\} + \{\{\omega, \psi\}, \varphi\} + \{\{\psi, \varphi\}, \omega\} = 0 \end{array}$$

Canonical quantization as we know it from textbooks comprises two basic steps. At first the algebra of functions $\mathcal{F}(\Pi)$ is *associated* to the universal enveloping algebra $U(\mathfrak{h}_{2n})$ of the Heisenberg-algebra according to the following scheme

$$\begin{array}{rcl} q_i & \to & Q_i \\ p_j & \to & P_j \\ \lambda \in \mathbf{K} & \to & \lambda \cdot \mathbf{1} \\ \{\cdot, \cdot\} & \to & \dfrac{\imath}{\hbar}\,[\cdot, \cdot]\,. \end{array} \tag{2.5}$$

The operators Q_i and P_j are the generators of $U(\mathfrak{h}_{2n})$ and according to (2.2), the Poisson bracket $\{\omega, \varphi\}$ translates to the commutator $[\Omega, \Phi] := \Omega \cdot \Phi - \Phi \cdot \Omega$. The scheme (2.5) yet is not sufficient to associate any function $\varphi \in \mathcal{F}(\Pi)$ to an object $\Phi \in U(\mathfrak{h}_{2n})$. We come to this subtle point later.

Using the association table (2.5), relations (2.4) become the generating relations of $U(\mathfrak{h}_{2n})$

$$[Q_i, Q_j] = 0, \qquad [P_i, P_j] = 0, \qquad [P_i, Q_j] = i\hbar\delta_{ij}\mathbf{1}. \tag{2.6}$$

The universal enveloping algebra $U(\mathfrak{g})$ of a Lie-algebra is precisely defined in the next section of this chapter. In the case of $U(\mathfrak{h}_{2n})$ it is basically a free multiplicative and additive algebra, generated by Q_i and P_j being subject to relations (2.6). In fact $U(\mathfrak{h}_{2n})$ is exactly that type of Lie-algebra, physicists are used to in quantum mechanics and thus should not bother about the specific notation used here. However, for further generalization we have to be more precise in this respect and thus already here should get in touch with basic notions.
As a second step the universal enveloping algebra $U(\mathfrak{h}_{2n})$ is represented on a complex and separable Hilbert space $\mathcal{H}$. To this purpose the generators $Q_i, P_j \in U(\mathfrak{h}_{2n})$ are mapped into the endomorphisms $\mathrm{End}(\mathcal{H})$ by an algebra homomorphism ρ, i.e. a $\mathbb{C}$-linear map that satisfies

$$\forall\, A, B \in U(\mathfrak{h}_{2n}),\ \rho \in \mathrm{End}(\mathcal{H}):\ \rho([A,B]) = [\rho(A), \rho(B)]\,.$$

In order to neatly represent $U(\mathfrak{h}_{2n})$ on $\mathcal{H}$, specifically its generating relations

$$[\rho(Q_i), \rho(Q_j)] = 0, \quad [\rho(P_i), \rho(P_j)] = 0, \quad [\rho(P_i), \rho(Q_j)] = i\hbar\delta_{ij}\rho(\mathbf{1}). \tag{2.7}$$

have to be represented, such that, with $\rho(\mathbf{1}) = \mathrm{id}_{\mathcal{H}}$, we explicitly require for states $|\Psi\rangle \in \mathcal{H}$ that

$$\begin{aligned}
&[\rho(Q_i), \rho(Q_j)]\ |\Psi\rangle = 0\\
&[\rho(P_i), \rho(P_j)]\ |\Psi\rangle = 0\\
&([\rho(P_i), \rho(Q_j)] - \imath\hbar\delta_{ij}\mathrm{id}_{\mathcal{H}})\ |\Psi\rangle = 0.
\end{aligned} \tag{2.8}$$

Since $q_i(t)$ and $p_j(t)$ are real-valued functions, their representations on $\mathcal{H}$ become hermitean self-adjoint operators $\rho(Q_i), \rho(P_j) \in \mathrm{End}(\mathcal{H})$. Each of them provides an eigenbasis $(\ |\mathbf{q}\rangle\ )_{q\in\mathbf{R}^n}$ and $(\ |\mathbf{p}\rangle\ )_{p\in\mathbf{R}^n}$ of $\mathcal{H}$ respectively, that possesses a corresponding real-valued eigenspectrum. We thus have two distinct representations of $U(\mathfrak{h}_{2n})$ being

$$\begin{aligned}
\rho_q(Q_i)\,|\mathbf{q}\rangle &= q_i\,|\mathbf{q}\rangle\,, \qquad q_i \in \mathbf{R}\\
\rho_p(P_j)\,|\mathbf{p}\rangle &= p_j\,|\mathbf{p}\rangle\,, \qquad p_j \in \mathbf{R}.
\end{aligned} \tag{2.9}$$

Introducing the wave function $\langle\mathbf{q}|\mathbf{p}\rangle = e^{\frac{\imath}{\hbar}p_i q_i}$ and completeness relations

$$\mathrm{id}_{\mathcal{H}_i} = \frac{1}{(2\pi)^n}\int d^n q\ |\mathbf{q}\rangle\langle\mathbf{q}|\,, \qquad \mathrm{id}_{\mathcal{H}_j} = \int d^n p\ |\mathbf{p}\rangle\langle\mathbf{p}| \tag{2.10}$$

we obtain for $\rho_q(P_i)$

$$\begin{aligned}
\rho_q(P_i)\ |\mathbf{q}{>} &= \int d^n p\ |\mathbf{p}{>}{<}\mathbf{p}|\ \rho_Q(P_i)\ |q_i{>} = \int d^n p\ |\mathbf{p}{>}\ p_i\ {<}\mathbf{p}|\mathbf{q}{>} \\
&= \int d^n p\ |\mathbf{p}{>}\ p_i e^{-\frac{\imath}{\hbar} p_j q_j} = \int d^n p\ |\mathbf{p}{>}\ -\frac{\hbar}{\imath}\frac{\partial}{\partial q_i} e^{-\frac{\imath}{\hbar} p_j q_j} \\
&= -\frac{\hbar}{i}\frac{\partial}{\partial q_i}\ |\mathbf{q}{>}\quad .
\end{aligned}$$

Analogously we compute

$$\rho_p(Q_j) = \frac{\hbar}{i}\frac{\partial}{\partial p_j}$$

and thus obtain two equivalent representations ρ_q and ρ_p that satisfy (2.8) and are bijected via Fourier transformation. By the use of completeness relations (2.10), conditions (2.8) are thus satisfied for any $|\Psi{>} \in \mathcal{H}$.
So much for the essentials of canonical quantization in textbook physics. We are now formalizing this procedure in order to obtain a mathematical term of *quantization*. To this purpose we have to discuss some difficulty that arises from the scheme (2.5). The question is, how this procedure might be put into a mathematical term, or in other words, what this procedure is at all.
At the first glance table (2.5) might suggest the existence of a map Γ that assigns to every component of the state vector its corresponding operator on $\mathcal{H}$, i.e. its corresponding generator in $U(\mathfrak{h}_{2n})$, such that the commutator relations (2.7) are satisfied. With $\Gamma(q_i) = \rho(Q_i)$ and $\Gamma(p_j) = \rho(P_j)$ we thus in particular require that

$$\Gamma([p_j, q_i]) = [\Gamma(p_j), \Gamma(q_i)] = [\rho(P_j), \rho(Q_i)] = \imath\hbar\delta_{ji}\mathrm{id}_{\mathcal{H}}$$

The problem arises form the fact that phase space functions $\mathcal{F}(\Pi)$ constitute a commutative algebra that cannot be mapped to $U(\mathfrak{h}_{2n})$ by an algebra homomorphism, which Γ actually is. Thus in order to obtain such a homomorphism, we have to deform or *quantize* the multiplication within the algebra of functions $\mathcal{F}(\Pi)$ in such a way that the commutator of the resulting algebra reflects the properties of the Poisson bracket.
This gives the way free to a precise definition of a mathematical notion of *quantization*. In particular, canonical quantization is considered as an example of a quantized Poisson manifold in quantum groups, that we want to define now

2.1.2 Definition (Quantization of Poisson Manifolds) *Let a Poisson manifold $(\mathcal{M}, \{\cdot,\cdot\}, \mathbf{K})$ over the field $\mathbf{K}$ be given.* A quantization *of $\mathcal{M}$ with deformation parameter $h \in \mathbf{K}$ is a manifold $\mathcal{M}_h = (\mathcal{M}, [\cdot \stackrel{*_h}{,} \cdot], \mathbf{K})$, such* *that to first order in the deformation parameter h the commutator* $[\cdot \overset{*_h}{,} \cdot]$ *satisfies the following property:*

$$\forall f_1, f_2 \in \mathcal{F}(\mathcal{M}) : \ \frac{[f_1 \overset{*_h}{,} f_2]}{h} = \frac{f_1 *_h f_2 - f_2 *_h f_1}{h} = \{f_1, f_2\} \quad mod(h)$$

One might ask why this actually is called a deformation of the Poisson manifold $\mathcal{M}$, since it is rather the multiplication map of the algebra of functions $\mathcal{F}(\mathcal{M})$ that has been deformed. In fact this is a subtle and crucial point in quantum groups. It turns out that there actually is no difference in these two points of view. This is due to the *duality* between the algebra of functions $\mathcal{F}(\mathcal{M})$ and the manifold $\mathcal{M}$ it is defined on. Duality relates the algebraic properties of the manifold $\mathcal{M}$ to those of the set of functions $\mathcal{F}(\mathcal{M})$ over it. Duality survives the process of quantization. It thus is a central notion within quantum groups that characterises their central idea: *The deformation of a manifold $\mathcal{M}$ is described by the deformation of its algebra of functions $\mathcal{F}(\mathcal{M})$* such that

$$\mathcal{F}(\mathcal{M}_h) \equiv \mathcal{F}_h(\mathcal{M}).$$

If a manifold provides more algebraic structure, such as a Hopf-algebra $\mathcal{H}$, we then moreover see that its coproduct is dual to the product of the algebra of functions over $\mathcal{H}$. In the next subsection we have a closer look at coproducts and explain what they are and what they can be used for. However, the coproduct is required to consider deformations of tensor products of representation spaces, that for example constitute an algebra of coordinates and thus give the desired link to noncommutative geometry. In our specific example of Hamiltonian mechanics, the phase space Π, at least in the way we were treating it, does not possess an algebraic structure. In fact there is a dual coalgebra structure on Π induced from the product on $\mathcal{F}(\Pi)$ that we simply ignored. The key to understand in this respect is that we do not represent so much the Heisenberg algebra $\mathfrak{h}_{2n}$ on the states $|\Psi\rangle \in \mathcal{H}$ than a deformation of the algebra of functions $\mathcal{F}(\Pi)$, that is deformed in such a way that to first order it corresponds to the Poisson bracket of the phase space Π. In this light we understand that quantum mechanics, as we know it from physics, is only one very specific choice of quantization of this specific Poisson manifold. In fact it is one of the most simple possible. However, we further elucidate this point in the next subsection. Duality is thoroughly discussed in the next two sections of this chapter, where we give a more precise definition of it. Before that we have to consider one more subtlety that is hidden in the scheme of quantization. The definition above is that of the quantization of a Poisson manifold. We will see further definitions of quantizations of other objects. But these merely represent a generalization of this specific definition. Since we learned that quantization has to be considered as a deformation of the product of the algebra of functions $\mathcal{F}(\Pi)$, it is now the question how this is usually performed from the technical side within quantum groups. Moreover we are barely able to actually map functions $\omega \in \mathcal{F}(\Pi)$ to the corresponding operator $\Omega \in U(\mathfrak{h}_{2n})$. Thus our scheme of quantization yet is uncomplete and we are filling that gap with the following discussion. In quantum groups the quantization itself is always performed in terms of a bilinear operator that satisfies certain conditions. In every case - be it the quasitriangular structure $\mathcal{R}$, the closely related Drinfeld-twist $\mathcal{F}$ or the R-matrix $\mathbf{R}$ being the dual object to $\mathcal{R}$ - the basic principle is always the same: A bilinear operator, that can be expressed as a formal power series, i.e. in terms of powers of the deformation parameter h, acts separately on the two components of the product such that for $h \to 0$ the undeformed product is recovered. In the case of quantized Poisson manifold, as in (2.1.2), we additionally require that the bilinear operator respects the structure implied by the Poisson structure in first order of the deformation parameter. In the case of quantum mechanics, as we consider it here, Weyl and Moyal in 1949 developed the starproduct in order to deform the algebra of functions $\mathcal{F}(\Pi)$. This requires a little preparation and in the mean time gives us the required tool to map one to one any function $\omega \in \mathcal{F}(\Pi)$ to the corresponding object $\Omega \in U(\mathfrak{h}_{2n})$.:

2.1.3 THEOREM (POINCARÉ-BIRKHOFF-WITT) *Let $\mathfrak{g}$ be an n-dimensional Lie-algebra with basis $(g_i)_{i\in\{1\ldots n\}}$ over the field $\mathbf{K}$. Furthermore let*

$$\begin{aligned} \pi : \{1\ldots n\} \subset \mathbb{N} &\to \{1\ldots n\} \\ k &\mapsto i_k \end{aligned}$$

be any permutation, then the ordered monomials

$$(g_{i_1})^{m_{i_1}} \ldots (g_{i_k})^{m_{i_k}} \ldots (g_{i_n})^{m_{i_n}} \in U(\mathfrak{g}), \quad m_{i_k} \in \mathbb{N}$$

constitute a basis of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ and there exists an isomorphism W of vector spaces

$$\begin{aligned} W \;:\; & U(\mathfrak{g}) \to U(\mathbf{R}^n) \\ & (g_{i_1})^{m_{i_1}} \ldots (g_{i_k})^{m_{i_k}} \ldots (g_{i_n})^{m_{i_n}} \mapsto (x_{i_1})^{m_{i_1}} \ldots (x_{i_k})^{m_{i_k}} \ldots (x_{i_n})^{m_{i_n}}. \end{aligned}$$

The universal enveloping algebra $U(\mathbf{R}^{2n})$ is the commutative algebra generated by the real vector space $\mathbf{R}^{2n}$ with basis $(q_i, p_j)_{i,j\in\{1,\ldots,n\}}$. The exponentiation of a Lie-algebra is a Lie-group. Thus comming back to the Heisenberg Lie-algebra $\mathfrak{h}_{2n}$ with generators Q_i, P_i, the exponentiation constitutes the basis of a Lie group. In the mean time the exponentiation also consitutes, in the sense of (2.1.3), a basis of the universal enveloping algebra $U(\mathfrak{h}_{2n})$ of $\mathfrak{h}_{2n}$. With the isomorphism W we can now map a basis of ordered monomials of $U(\mathfrak{h}_{2n})$ to a basis of monomials of $\mathcal{F}(\Pi)$ and vice versa. In quantum mechanics this is known as the ordering procedure, that our scheme of quantization lacked up to now. In this context we are now able to meaningfully enhance the quantization procedure to entire functions $\omega \in \mathcal{F}(\Pi)$. Since exponentiation of $\mathfrak{h}_{2n}$ constitutes a basis of $U(\mathfrak{h}_{2n})$, there exists an isomorphism W of vector spaces, such that we can perform the following mapping of basis elements using $\eta_i, \xi_j \in \mathbf{K}$

$$\begin{array}{rcl} W : U(\mathfrak{h}_{2n}) & \rightarrow & \mathcal{F}(\Pi) \\ e^{i(\eta_i Q_i + \xi_j P_j)} & \mapsto & e^{i(\eta_i q_i + \xi_j p_j)}. \end{array}$$

We can thus develop functions over Π in terms of this basis and by the use of the Fourier transformation

$$\begin{array}{rcl} \varphi(q_i, p_j) & = & \displaystyle\int d^n\eta \; d^n\xi \; \hat{\varphi}(\eta_i, \xi_j) \; e^{-i(\eta_i q_i + \xi_j p_j)}, \\ \hat{\varphi}(\eta_i, \xi_j) & = & \displaystyle\frac{1}{(2\pi)^{2n}} \int d^n \; q d^n p \; \varphi(q_i, p_j) \; e^{+i(\eta_i q_i + \xi_j p_j)}. \end{array}$$

By application of the inverse map W^{-1} we obtain for two functions $\varphi, \omega \in \mathcal{F}(\Pi)$ the corresponding objects of $U(\mathfrak{h}_{2n})$, by

$$\begin{array}{rcl} W^{-1}(\varphi)(Q_i, P_j) & = & \displaystyle\int d^n\eta \; d^n\xi \; \hat{\varphi}(\eta_i, \xi_j) \; e^{-i(\eta_i Q_i + \xi_j P_j)}, \\ W^{-1}(\omega)(Q_i, P_j) & = & \displaystyle\int d^n\eta \; d^n\xi \; \hat{\omega}(\eta_i, \xi_j) \; e^{-i(\eta_i Q_i + \xi_j P_j)}. \end{array}$$

Endowing our vector spaces with a multiplication map $*_h$ we thus require that

$$\begin{array}{rcl} W^{-1}(\varphi *_\hbar \omega) & := & W^{-1}(\varphi) \cdot W^{-1}(\omega) \\ & = & \displaystyle\int d^n\eta \; d^n\xi \; d^n\kappa \; d^n\lambda \; \hat{\varphi}(\eta_i, \xi_j)\hat{\omega}(\kappa_i, \lambda_j) \\ & & \qquad \times e^{-i(\eta_i Q_i + \xi_j P_j)} \; e^{-i(\kappa_i Q_i + \lambda_j P_j)} \\ & = & \displaystyle\int d^n\eta \; d^n\xi \; d^n\kappa \; d^n\lambda \; \hat{\varphi}(\eta_i, \xi_j)\hat{\omega}(\kappa_i, \lambda_j) \; \times \\ & & \qquad \times e^{-i((\eta_i + \kappa_i)Q_i + (\xi_j + \lambda_j)P_j) + i\frac{\hbar}{2}(\eta_i \lambda_i - \xi_j \kappa_j)\mathbf{1}}. \end{array}$$

The last step we performed by the use of the Baker-Campbell-Hausdorff formula

$$e^A \; e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}([A,[A,B]]-[B,[A,B]])+\frac{1}{48}([A,[B,[B,A]]]-[B,[A,[A,B]]])+\ldots)}.$$

Now we transform back by the use of the algebra isomorphism W and thus obtain

$$\begin{aligned}(\varphi *_\hbar \omega)(q_k, p_l) &= \int d^n\eta\, d^n\xi\, d^n\kappa\, d^n\lambda\, \hat{\varphi}(\eta_i, \xi_j)\hat{\omega}(\kappa_i, \lambda_j) \\ &\qquad \times e^{-i((\eta_i+\kappa_i)q_i+(\xi_j+\lambda_j)p_j)+i\frac{\hbar}{2}(\eta_i\lambda_i-\xi_j\kappa_j))} \\ &= \int d^n\eta\, d^n\xi\, d^n\kappa\, d^n\lambda\, \hat{\varphi}(\eta_i, \xi_j)\, e^{-i(\eta_i q_i+\xi_j p_j)} \\ &\qquad \times\hat{\omega}(\kappa_i, \lambda_j)\, e^{-i(\kappa_i q_i+\lambda_j p_j)}\, e^{+i\frac{\hbar}{2}(\eta_i\lambda_i-\xi_j\kappa_j)}\end{aligned}$$

Here we sum over double indices. Setting now $\eta_i \to i\frac{\partial}{\partial q_i}, \xi_j \to i\frac{\partial}{\partial p_j}$ and $\kappa_j \to i\frac{\partial}{\partial \hat{q}_j}, \lambda_i \to i\frac{\partial}{\partial \hat{p}_i}$ we moreover obtain the deformed product in terms of a bilinear operator being the *starproduct*

$$(\varphi *_\hbar \omega)(q_k, p_l) = e^{-i\frac{\hbar}{2}(\frac{\partial}{\partial q_i}\frac{\partial}{\partial \hat{p}_i}-\frac{\partial}{\partial p_j}\frac{\partial}{\partial \hat{q}_j})}\, \varphi(q_k, p_l)\, \omega(\hat{q}_k, \hat{p}_l)|_{(\hat{q}_k,\hat{p}_l)\to(q_k,p_l)}.$$

We thus obtained a bilinear operator that describes the deformation of the product. In order to verify, that we actually obtained a quantization of a Poisson manifold in the sense of (2.1.2), we look what happens to first order in $\hbar$. Thus for the special choice of $\varphi(q_k, p_l) = p_l$ and $\omega(q_k, p_l) = q_k$ we recover the relations (2.6), generating the universal enveloping algebra $U(\mathfrak{h}_{2n})$ of the Heisenberg algebra $\mathfrak{h}_{2n}$.

$$\begin{aligned}[p_l \overset{*_\hbar}{,} q_k] &= p_l *_\hbar q_k - q_k *_\hbar p_l \\ &= e^{-i\frac{\hbar}{2}(\frac{\partial}{\partial q_i}\frac{\partial}{\partial \hat{p}_i}-\frac{\partial}{\partial p_j}\frac{\partial}{\partial \hat{q}_j})} p_l \cdot \hat{q}_k|_{\hat{q}_k\to q_k} - e^{-i\frac{\hbar}{2}(\frac{\partial}{\partial q_i}\frac{\partial}{\partial \hat{p}_i}-\frac{\partial}{\partial p_j}\frac{\partial}{\partial \hat{q}_j})} q_k \cdot \hat{p}_l|_{\hat{p}_l\to p_l} \\ &= p_l \cdot q_k + \frac{\hbar}{2}\,\delta_{jl}\delta_{jk} - q_k \cdot p_l + \frac{\hbar}{2}\,\delta_{jk}\delta_{jl} \\ &= +i\hbar\,\delta_{kl}\end{aligned}$$

Many other starproducts thus potentially exist and in this sense, there exist many different quantizations of Hamiltonian mechanics. In order to incorporate, as an example, high energy effects, the scheme of canonical quantization can thus be modified along these lines.

2.1.2 Quantization of Lie-algebras and their Representation

In this subsection we proceed one step closer to the actual issue of quantum groups. To this purpose we discuss the prominent example of $U_q(sl_2)$ as a deformation of the universal enveloping algebra of the Lie-algebra sl_2. Such deformations represent one of the most genuine types of quantization. Duality however requires some more preparation, thus about a second genuine type of quantum groups, being the quantum matrix groups, we refere to at the end of this chapter. For our example, the corresponding quantum matrix group is $\mathrm{SL}_q(2)$. It is our intend to give a basic motivation for the introduction of Hopf-algebras in this context. We thus are introducing several notions in this respect without defining them. Here once more we merely want to focus on basic ideas. Everything discussed here is introduced more generally in the next two sections of this chapter. For now the reader, if new to the topic, should not expect to grasp every step in its specifics. He should rather take this subsection as a reference example for the theory that is introduced in the following sections. There is also a pedagogical introduction of this example in [85]. The Lie-algebra sl_2 of the Lie-group $\mathrm{SL}(2,\mathbb{C})$ is identical to that of SU(2) and SO(3). The basis of sl_2 comprises three generators $(\sigma_i)_{i\in 1,2,3}$ with the bracket relation

$$[\sigma_i, \sigma_j] = \epsilon_{ijk}\sigma_k. \tag{2.11}$$

Representation theory of sl_2 is well known to physicists. For any $j \in 0, \frac{1}{2}, 1, \ldots$ there exists an irriducible $(2j+1)$-dimensional representation of sl_2 on a complex Hilbert space $\mathcal{H}_j$. Diagonalized on σ_3, we thus obtain for states $|j,m\rangle \in \mathcal{H}_j$

$$\begin{aligned}(\sigma_1^2+\sigma_2^2+\sigma_3^2)\,|j,m\rangle &= j(j+1)\,|j,m\rangle\,,\\ \sigma_3\,|j,m\rangle &= m\,|j,m\rangle\,,\end{aligned}$$

with $m = j, j-1, \ldots, 0, \ldots, -j+1, j$. In particular we have creation an annihilation operators

$$\sigma_\pm = \sigma_1 \pm i\sigma_2,$$

such that for each j the spectrum of eigenstates can be exhausted by relations

$$\sigma_\pm\,|j,m\rangle = \sqrt{j(j+1)-m(m\pm 1)}\,|j,m\pm 1\rangle\ .$$

For simplicity we further consider the case of $j=\frac{1}{2}$, such that the generators of sl_2 can be represented in terms of Pauli-matrices. With $|j=\frac{1}{2}, m=\pm\frac{1}{2}\rangle \equiv |\pm\frac{1}{2}\rangle \in \mathcal{H}_2$ we thus obtain

$$\sigma_1\,|\pm\frac{1}{2}\rangle = \frac{1}{2}\,|\mp\frac{1}{2}\rangle\,,\quad \sigma_2\,|\pm\frac{1}{2}\rangle = \pm\frac{i}{2}\,|\mp\frac{1}{2}\rangle\,,\quad \sigma_3\,|\pm\frac{1}{2}\rangle = \pm\frac{1}{2}\,|\pm\frac{1}{2}\rangle\ .$$

Regarding $U(sl_2)$ we can verify, as in the last subsection, that the bracket relation (2.11) is once more realized as a commutator relation on $\mathcal{H}_2$. However, since this is well known to every physicist, we immediately turn to tensor representations that help us to motivate our need for Hopf-algebras. The tensor product $\mathcal{H}_2 \otimes \mathcal{H}_2$ is a direct sum of the singlet $\mathcal{H}_{j=0}$ and triplet $\mathcal{H}_{j=3}$ space, i.e.

$$\mathcal{H}_2 \otimes \mathcal{H}_2 = \mathcal{H}_{j=3} \oplus \mathcal{H}_{j=0}.$$

The tensor product $\mathcal{H}_2 \otimes \mathcal{H}_2$ is thus a reducible representation of sl_2 and as such we have a diagonal operator $\hat{\sigma}_3$ that is acting on it. For the tensor components we have $\sigma_3 \otimes \mathbf{1}$ for the first copy of $\mathcal{H}_2$ and $\mathbf{1} \otimes \sigma_3$ for the second one. For a state $|m> \otimes |m'> \in \mathcal{H}_2 \otimes \mathcal{H}_2$ the eigenvalue relation reads

$$\hat{\sigma}_3 \, |m> \otimes |m'> = (m + m') \, |m> \otimes |m'> ,$$

such that the operator $\hat{\sigma}_3$ of the tensor-product representation can be written in terms of the operators of the tensor components, i. e.

$$\hat{\sigma}_3 = \sigma_3 \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_3.$$

From the representation theoretic point of view it is important to understand that instead of interpreting $|+\frac{1}{2}>$ and $|-\frac{1}{2}>$ as states of a specific Hilbert space $\mathcal{H}_2$, we can also treat them as a set of generators $x \equiv |-\frac{1}{2}>$ and $y \equiv |+\frac{1}{2}>$ that can be regarded as a two dimensional plane that transforms covariantly under the isometry sl_2. If we want the representation space $\mathrm{span}(x, y)$ to be more than a complex vector space, as $\mathcal{H}_2$, i.e. if we want it to be enhanced to an algebra, then this is performed by enhancing to a free tensor algebra, that is suitably devided by some ideal that relates tensor products of $x \otimes y$ to $y \otimes x$. Tensor products and direct sums of vector spaces again are vector spaces. Thus in order to be more specific, the free or tensor algebra $T(\mathcal{H}_2)$ is the vector space

$$T(\mathcal{H}_2) = \mathbb{C} \oplus \mathcal{H}_2 \oplus \mathcal{H}_2 \otimes \mathcal{H}_2 \oplus \mathcal{H}_2^{\otimes 3} \ldots \oplus \mathcal{H}_2^{\otimes n} \oplus \ldots.$$

We can now divide it by an ideal $\mathcal{I} \subset T(\mathcal{H}_2)$ that is generated by relation

$$x \otimes y - y \otimes x = 0. \tag{2.12}$$

Thus the ideal $\mathcal{I}$ consists of all $\Phi \in T(\mathcal{H}_2)$ to which a $\varphi \in T(\mathcal{H}_2)$ exists such that

$$\Phi = \varphi \otimes (x \otimes y - y \otimes x).$$

With (2.12) we thus identify objects

$$\varphi \otimes x \otimes y = \varphi \otimes y \otimes x.$$

If the ideal $\mathcal{I}$ is two-sided, then it can as well be generated by multiplying (2.12) from the left instead of the right. This is the case in our example. This represents the standard procedure to enhance a vector space with an algebraic structure. We can loosely call it an *algebra of coordinates*

$$\mathfrak{X} = \frac{T(\mathcal{H}_2)}{\mathcal{I}},$$

that, if we omit the tensor product and treat it as a multiplication, can be considered as a space of formal power-series in x and y over the complex numbers $\mathbb{C}$. Returning to our operator $\hat{\sigma}_3$ of the tensor representation of sl_2, we now as well obtain a law how products such as $x \cdot y$ covariantly transform under the action of sl_2. Of course $\mathfrak{X}$ also accomodates all powers of monomials in x and y, we thus have to go on and find those operators, that are represented on higher tensor products. Moreover, if the algebra of coordinates $\mathfrak{X}$ is associative - and it is in our case - then we can moreover associate tensor products of vector spaces, i.e. the action of a corresponding operator to σ_3 on monomials

$$x \otimes (y \otimes z) = (x \otimes y) \otimes z, \tag{2.13}$$

has to return the same result. Thus we require an operator that maps σ_3 to the corresponding operator on the tensor product vector space, such that it respects the associativity of $\mathfrak{X}$. Although we considered the representation of sl_2 on $\mathcal{H}_2$ it is more precisely that of $U(sl_2)$, since we use the commutator bracket in the representation. The universal enveloping algebra $U(sl_2)$ is - as its name suggests - an algebra. If we enhance it by a map that delivers us the representation on tensor products of $\mathcal{H}_2$, then we add another *dual* structure to $U(sl_2)$. It is yet not obvious that this actually is an algebraic enhancement of $U(sl_2)$ - but we will understand this on the next pages and moreover throughout the whole chapter. This structure is the coproduct

$$\Delta : \; U(sl_2) \to U(sl_2) \otimes U(sl_2),$$

that maps operators of $U(sl_2)$ to the tensor representation and if we want it to respect associativity of the representation space $\mathfrak{X}$ according to (2.13), then we require that

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta.$$

This is called *coassociativity*. Up to now, we do not see that Δ is an intrinsic operation of $U(sl_2)$ as the multiplication is. But this is the case. We thus in particular obtain that

$$\Delta(\sigma_3) = \hat{\sigma}_3 = \sigma_3 \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_3.$$

Coproduct and multiplication of $U(sl_2)$ moreover have to be made compatible. More specifically we will require that comultiplication and the multiplication can be exchanged, such that computations are independent whether the product or the coproduct have been applied at first. If we enhance $U(sl_2)$ moreover by a *counit* element ϵ, then $U(sl_2)$ is called a bialgebra. If we once more enhance it by a map called *antipode*, we finally receive a Hopf-algebra. All this is considered in detail within the next two sections.
In the last subsection however, we discussed quantization as a deformation of the multiplication of, for example, $\mathfrak{X}$. We could thus once more apply a starproduct with deformation-parameter h on $\mathfrak{X}$, such that we obtain a noncommutative space $\mathfrak{X}_h$. In this case the coprocuct Δ on $U(sl_2)$ has to be modified as well in order to maintain its properties. If we would succeed to reformulate the starproduct in terms of generators of $U(sl_2)$, we would obtain a *twist* that correspondingly deformes the coproduct. However, it can be shown that all deformations of the algebraic sector of $U(sl_2)$ are equivalent - they are related by algebra isomorphisms. But the deformation of the coproduct can actually lead to a nontrivial deformation of $U(sl_2)$, that cannot be mapped by a bialgebra-isomorphism. And this we want to do now. Actually not by applying a twist - but a quasitriangular structure $\mathcal{R} \in U(sl_2) \otimes U(sl_2)$ that possesses similar properties as the twist. The invertible quasitriangular structure $\mathcal{R}$ determines the modification of cocommutativity of the coproduct. If the two tensor components of the coproduct Δ can be exchanged - as in the case of $U(sl_2)$ - then we speak of a cocommutativity. Note that cocommutativity is closely related of the commutativity of $\mathfrak{X}$. In formulas, cocommutativity is expressed by $\Delta = \sigma \circ \Delta$, where σ exchanges the tensor components. In order to obtain a deformed coproduct, the use of the quasitriangular structure $\mathcal{R} \in U(sl_2) \otimes U(sl_2)$ breaks cocommutativity through conjugation with $\mathcal{R}$ according to

$$\sigma \circ \Delta(\zeta) = \mathcal{R}\, \Delta(\zeta)\, \mathcal{R}^{-1}, \tag{2.14}$$

with $\zeta \in U(sl_2)$. In general it is sufficient to formulate this relation for a basis of $U(sl_2)$, such as $(\sigma_i)_{i\in 1,2,3}$ from above, and from these to deduce those relations for arbitrary $\zeta \in U(sl_2)$. It actually is an art to find nontrivial deformation of $U(sl_2)$ in terms of a quasitriangular structure $\mathcal{R}$. There are no standard procedures to obtain such solutions - similar to the art of finding solutions to differential equations. As such, much trickery is in order to find solutions. The deformation we present here, makes use of the fact that the algebra sector of $U(sl_2)$ is always isomorphic as an algebra. The crucial point is, whether the coproduct can be mapped by the same isomorphism such that we obtain a bialgebra-isomorphism. Only in this case the deformation is trivial. This fact is used to turn to another set of generators for $U_q(sl_2)$. This is only a different basis for the same algebraic object, but not necessarily of the same coalgebraic object as just mentioned. But changing the basis helps to find an appropriate coproduct that finally makes the deformation nontrivial. The deformation parameter of $U_q(sl_2)$ is $q \neq 1 \in \mathbb{C}$. We consider it not to be a root of unity. We introduce $U_q(sl_2)$ with generators $\tau_+, \tau_-, q^{\pm\frac{\tau_3}{2}}$ with commutation relations

$$q^{\frac{\tau_3}{2}}\tau_\pm q^{-\frac{\tau_3}{2}} = q^{\pm 1}\tau_\pm, \quad [\tau_+, \tau_-] = \frac{q^{\tau_3} - q^{-\tau_3}}{q - q^{-1}}.$$

Moreover there are coproducts formulated by

$$\Delta(q^{\pm\frac{\tau_3}{2}}) = q^{\pm\frac{\tau_3}{2}} \otimes q^{\pm\frac{\tau_3}{2}}, \quad \Delta(\tau_\pm) = \tau_\pm \otimes q^{\frac{\tau_3}{2}} + q^{-\frac{\tau_3}{2}} \otimes \tau_\pm. \tag{2.15}$$

The *almost cocommutativity*, according to (2.14), is now governed by the quasitriangular structure

$$\mathcal{R} = q^{\frac{\tau_3 \otimes \tau_3}{2}} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]!} (q^{\frac{\tau_3}{2}}\tau_+ \otimes q^{-\frac{\tau_3}{2}}\tau_-) q^{\frac{n(n-1)}{2}}. \tag{2.16}$$

Here we used the q-numbers $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$ and $[n]! = [n][n-1]\ldots[1]$. We additionally have counits and antipodes

$$\epsilon(q^{\pm\frac{\tau_3}{2}}) = 1, \;\; \epsilon(\tau_\pm) = 0, \;\; \mathrm{S}(\tau_\pm) = -q^{\pm 1}\tau_\pm, \;\; \mathrm{S}(q^{\pm\frac{\tau_3}{2}}) = q^{\mp\frac{\tau_3}{2}}$$

that make $U_q(sl_2)$ into an actual Hopf-algebra. However, we do not need to care about this additional structure here. We see that for $q \to 1$ we obtain the former universal enveloping algebra $U(sl_2)$ in terms of creation- and annihilation operators σ^+, σ_- as well as σ_3. As for $U(sl_2)$ we can once more consider the representation of $U_q(sl_2)$. This turns out to be very similar to that of $U_q(sl_2)$, with the difference that now the quantum numbers are replaced by q-numbers $[n]$, as we defined them above. Thus once more to every $J = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ we obtain a $2J+1$-dimensional irreducible representation of $U_q(sl_2)$. In particular the action of the generators on states $|J, M>$ is given by

$$\tau_\pm \; |J, M> = \sqrt{[J \mp M][J \pm M + 1]} \; |J, M \pm 1> , \quad q^{\frac{\tau_3}{2}} \; |J, M> = q^M \; |J, M> \; .$$

Since $U_q(sl_2)$ is isomorphic to $U(sl_2)$ as an algebra, we understand that the generators $\tau_+, \tau_-, q^{\pm\frac{\tau_3}{2}}$ can be expressed in terms of $\sigma_1, \sigma_2, \sigma_3$ of $U(sl_2)$ and thus also the states $|J, M>$ are linear combinations of $|j, m>$. Once more, the actual quantization is caused by a nontrivial deformation of the coproduct in $U_q(sl_2)$. The algebra-isomorphism does not map the coproduct of $U(sl_2)$ to that of $U_q(sl_2)$. We are thus once more interested in tensor-representations. In particular for $J = \frac{1}{2}$ we obtain for states $|M = \pm\frac{1}{2}>$

$$\tau_+ \; |+\frac{1}{2}> = 0, \;\; \tau_- \; |+\frac{1}{2}> = \; |-\frac{1}{2}> , \;\; \tau_+ \; |-\frac{1}{2}> = \; |+\frac{1}{2}> , \;\; \tau_- \; |-\frac{1}{2}> = 0. \tag{2.17}$$

If once more we make the identification $x \equiv \ |-\frac{1}{2}>$, $y \equiv \ |+\frac{1}{2}>$ and apply our newly achived coproducts (2.15), that provide us with the tensor operation of the generators of $U_q(sl_2)$ on $x \otimes y$ and $y \otimes x$, we see that the commutation relation $x \otimes y - y \otimes x = 0$ is not preserved anymore. Instead we find that setting $x \otimes y - q\, y \otimes x = 0$ is now covariantly transformed according to

$$\begin{aligned} q^{\pm\frac{\tau_3}{2}} \otimes q^{\pm\frac{\tau_3}{2}} \triangleright (x \otimes y - q\, y \otimes x) &= 0 \\ (\tau_\pm \otimes q^{\frac{\tau_3}{2}} + q^{-\frac{\tau_3}{2}} \otimes \tau_\pm) \triangleright (x \otimes y - q\, y \otimes x) &= 0. \end{aligned}$$

Here we already used the symbol '$\triangleright$' of a *left action*. It is another notation for a representation, introduced in the next section. Thus as we defined the algebra of coordinates $\mathfrak{X}$, covariant under the action of $U(sl_2)$, we can now proceed and in the same way define the quantized algebra of coordinates $\mathfrak{X}_q$ that is once more defined as the free or tensor algebra $T(\mathcal{H}_2)$ devided by the ideal $\mathcal{I}_q$ that is generated by relations $xy - q\, yx$, i.e.

$$\mathfrak{X}_q = \frac{T(\mathcal{H}_2)}{\mathcal{I}_q}.$$

While here we considered a two dimensional plane, we can set $J = 1$ and receive a q-deformed version of Euclidean three dimensional space with a suitably deformed version of the universal enveloping algebra $U(so_3)$. In principle we can then once more build our construction of nonrelativistic quantum mechanics upon this setup. But now with the modification of a noncommutative space.

2.1.3 Discussion

This section contains the very essentials of quantum groups that are relevant for physicists. We saw several types of deformation-quantizations and understood how canonical quantization is accomodated in the framework of quantum groups. Throughout the chapter we put the emphasize on quantization itself. In the last subsection we saw how tensor representations play a vital part in quantum groups. This is the actual reason why quantum groups in general have to be treated in terms of tensor categories. But the actual information that physicist require do not really take this into account. One can do without it. And thus we introduce the quantum groups in the next two sections in such a way, that we can override category theory. More information on the case of $U_q(sl_2)$ can be found in most textbooks on quantum groups, such as [19, 50, 64]. In the last section of this chapter we learn about a third genuine class of deformations, being dual to that of universal enveloping algebras, called quantum matrix groups. We take a short opportunity there to construct the dual **R**-Matrix from the quasitriangular structure $\mathcal{R}$ of $U_q(sl_2)$ that determines the deformation of the product of $\mathrm{SL}(2,\mathbb{C})$. Once more we see the duality-property - the deformation of the coproduct of a universal enveloping algebra gives rise to a deformation of the product of the corresponding quantum matrix group.

2.2 Hopf Algebras: A Conceptual Introduction

In the last section we showed how generalizing the scheme of canonical quantization naturaly leads to a mathematical notion of deformation within the framework of Hopf algebras and tensor categories. We saw how this scheme itself is reduced to an example within a broader framework. After this physical motivation and outline of the topic in the last section, we are now prepared to proceed the other way around and develop the essential mathematical point of view from its roots. We begin with Hopf-algebras and their representations in the present section and then turn to quantization and duality in the next one. Duality is the last basic notion required within quantum groups, that we introduce her. Duality is an intrinsic and vital notion of Hopf-algebras. We thus could not introduce it within our previous motivation section.

2.2.1 Hopf Algebras

Algebras and especially Hopf-algebras are build upon linear spaces. If not indicated otherwise we consider these to be of finite dimension. Specifically we consider vector spaces over a field $\mathbf{K}$. For simplicity we restrict our choice of fields to real and complex numbers. Apart from this, we also take modules into account, where the role of the scalars is taken over by a ring. We recall that a *vector space* $(\mathfrak{V},+;\mathbf{K})$ is an Abelian group $(\mathfrak{V},+)$ that is endowed with a compatible multiplication of elements $\lambda \in \mathbf{K}$. Compatibility is given by distributivity, such that

$$\begin{aligned}\forall v,w \in \mathfrak{V},\ \lambda,\mu \in \mathbf{K} \quad : \quad & \lambda\cdot(v+w) = \lambda\cdot v + \lambda\cdot w \\ & (\lambda+\mu)\cdot v = \lambda\cdot v + \mu\cdot v \\ & 0\cdot v = \lambda\cdot\mathbf{0} = \mathbf{0}.\end{aligned}$$

As just mentioned, replacing the field $\mathbf{K}$ by a ring $\mathfrak{R}$ provides us with a linear space called *module*. All the statements and definitions given within this section are likewise valid for vector spaces and modules respectively. If we thus refer to a vetor space the reader may think in terms of a module as well. On the cartesian product of two vector spaces $(\mathfrak{V}, +; \mathbf{K})$ and $(\mathfrak{W}, +; \mathbf{K})$

$$\mathfrak{V} \times \mathfrak{W},$$

there exist two other vector spaces, being the *direct sum* vector space $\mathfrak{V} \oplus \mathfrak{W}$ and the *tensor product* vector space $\mathfrak{V} \otimes \mathfrak{W}$. The direct sum vector space $\mathfrak{V} \oplus \mathfrak{W}$ is given by relations

$$\begin{aligned} \forall\, (v, w), (\hat{v}, \hat{w}) \in \mathfrak{V} \oplus \mathfrak{W},\ \lambda \in \mathbf{K} \quad : \quad & (v, w) + (\hat{v}, \hat{w}) = (v + \hat{v}, w + \hat{w}) \\ & \lambda \cdot (v, w) = (\lambda \cdot v, \lambda \cdot w). \end{aligned}$$

The tensor product vector space $\mathfrak{V} \otimes \mathfrak{W}$ is given by relations

$$\begin{aligned} \forall (v, w), (\hat{v}, \hat{w}) \in \mathfrak{V} \otimes \mathfrak{W},\ \lambda \in \mathbf{K} \quad : \quad & (v, w) + (\hat{v}, w) = (v + \hat{v}, w) \\ & (v, w) + (v, \hat{w}) = (v, w + \hat{w}) \\ & \lambda \cdot (v, w) = (\lambda \cdot v, w) = (v, \lambda \cdot w). \end{aligned}$$

We denote elements of these spaces as $v \otimes w$ and $v \oplus w$ for a direct sum or a tensor product of two vectors respectively. For future consideration we require the following isomorphic spaces

$$\mathbf{K} \otimes \mathfrak{V} \cong \mathfrak{V} \cong \mathfrak{V} \otimes \mathbf{K}. \tag{2.18}$$

Algebras are vector spaces $(\mathfrak{V}, +; \mathbf{K})$ enhanced by multiplication map

$$\mu : \mathfrak{V} \otimes \mathfrak{V} \to \mathfrak{V}, \tag{2.19}$$

that through its bilinearity respects the vector space structure of $\mathfrak{V}$. For future applications we define it on the tensor product $\mathfrak{V} \otimes \mathfrak{V}$ rather than on the cartesian product $\mathfrak{V} \times \mathfrak{V}$, because we treat μ as a homomorphism from $\mathfrak{V}$ to $\mathfrak{V} \otimes \mathfrak{V}$. Future compatibility conditions are expressed in terms of algebra homomorphisms. If the multiplication $\cdot_\mu$ is associative it satisfies

$$\forall v_1, v_2, v_3 \in \mathfrak{V} : \quad (v_1 \cdot_\mu (v_2 \cdot_\mu v_3)) = ((v_1 \cdot_\mu v_2) \cdot_\mu v_3).$$

In terms of the multiplication map (2.19) and the identity id on $\mathfrak{V}$ this becomes

$$\mu \circ (\mathrm{id} \otimes \mu) = \mu \circ (\mu \otimes \mathrm{id}). \tag{2.20}$$

Algebras to be considered here are moreover endowed with a two-sided unit element $\mathbf{1} \in \mathfrak{V}$, i.e.

$$\forall\, v \in \mathfrak{V} :\ \mathbf{1} \cdot_\mu v = v \cdot_\mu \mathbf{1} = v.$$

As the multiplication (2.19), the unit element as well is to be treated as a map

$$\begin{aligned} \eta : \mathbf{K} &\to \mathfrak{V} \\ \lambda &\mapsto \lambda \cdot \mathbf{1}, \end{aligned} \tag{2.21}$$

such that the properties of the unit element can be absorbed as

$$\mu \circ (\mathrm{id} \otimes \eta) = \mathrm{id} = \mu \circ (\eta \otimes \mathrm{id}). \tag{2.22}$$

We use this formalization in order to introduce the notion of an algebra by the following definition

2.2.1 Definition (Algebra) *An* algebra $(A, \mu, \eta; \mathbf{K})$ *is a* $\mathbf{K}$*-linear vector space* $(A, +; \mathbf{K})$ *endowed with two bilinear maps* μ *and* η *as given by (2.19) and (2.21) such that the axioms of associativity (2.20) and of the unit element (2.22) are satisfied.*

As an algebra homomorphism $\chi : \mathfrak{A} \to \mathfrak{B}$ between algebras $\mathfrak{A}$ and $\mathfrak{B}$, we regard a $\mathbf{K}$-linear map with the property

$$\forall\, a, b \in \mathfrak{A} : \ \chi(a \cdot_{\mathfrak{A}} b) = \chi(a) \cdot_{\mathfrak{B}} \chi(b).$$

In the last section we encountered an algebra of functions $\mathcal{F}(\mathcal{M})$ over a manifold $\mathcal{M}$ as a first example of an algebra. Its pointwise operations are given by

$$\begin{aligned} \forall\, f, g \in \mathcal{F}(\mathcal{M}) \quad : \quad & (f + g)(m) = f(m) +_{\mathbf{K}} g(m), \\ & (f \cdot_\mu g)(m) = f(m) \cdot_{\mathbf{K}} g(m). \end{aligned} \tag{2.23}$$

This algebra possesses a unit element $I \in \mathcal{F}(\mathcal{M})$ in terms of the function $I(m) \equiv 1 \in \mathbf{K}$. As another example we introduce the *tensor product algebra* of a vector space $(\mathfrak{V}, +; \mathbf{K})$ by

$$T(\mathfrak{V}) := \mathbf{K} \oplus \mathfrak{V} \oplus \mathfrak{V} \otimes \mathfrak{V} \oplus \ldots \oplus \mathfrak{V}^{\otimes k} \oplus \ldots. \tag{2.24}$$

As tensor products and direct sums are vector spaces themselves, $T(\mathfrak{V})$ again is a $\mathbf{K}$-linear vector space. Treating the tensor product as a multiplication, we turn $T(\mathfrak{V})$ into an algebra. We thus omit any explicit notion of the tensor product '$\otimes$', i.e. $v \otimes w$ simplifies to $v \cdot w$.
The Lie-algebra however does not fit into definition (2.2.1). Instead of an associative product, it is endowed by a bracket, such as for the Poisson manifold in the last section. We now give its precise definition by

2.2.2 Definition (Lie-Algebra) *Let* $\mathfrak{g}$ *be a p-dimensional* $\mathbf{K}$*-linear vector space with basis* $(g_i)_{i\in\{1,\ldots,p\}}$. *Then* $(\mathfrak{g}, +, [\cdot,\cdot]_{\mathfrak{g}}; \mathbf{K})$ *is called a* Lie-algebra, *if the bracket*

$$[\cdot,\cdot]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

is antisymmetric, bilinear and satisfies the Jacobi-Identity:

$$\begin{aligned} \forall\, g, h, k \in \mathfrak{g} \quad : \quad & [g, h]_{\mathfrak{g}} = -[h, g]_{\mathfrak{g}} \\ & [g + h, k]_{\mathfrak{g}} = [g, k]_{\mathfrak{g}} + [h, k]_{\mathfrak{g}} \\ & [[g, h]_{\mathfrak{g}}, k]_{\mathfrak{g}} + [[h, k]_{\mathfrak{g}}, g]_{\mathfrak{g}} + [[k, g]_{\mathfrak{g}}, h]_{\mathfrak{g}} = 0 \end{aligned}$$

In analogy to algebra homomorphisms, a Lie-algebra homomorphism $\psi : \mathfrak{g} \to \hat{\mathfrak{g}}$ is a $\mathbf{K}$-linear map that preserves the bracket, i.e.

$$\forall\, g, h \in \mathfrak{g} :\; \psi([g, h]_{\mathfrak{g}}) = [\psi(g), \psi(h)]_{\hat{\mathfrak{g}}}$$

In the first section we already encountered the universal enveloping algebra $U(\mathfrak{h}_{2n})$ of the Heisenberg Lie-algebra $\mathfrak{h}_{2n}$. In contrast to the Lie algebra itself, its universal enveloping algebra, as it posesses an associative product as well as a unit element, satisfies definition (2.2.1). We now once more give a precise definition.

2.2.3 Definition (Universal Enveloping Algebra) *Let there be given a p-dimensional Lie-algebra* $(\mathfrak{g}, +, [\cdot,\cdot]_{\mathfrak{g}}; \mathbf{K})$ *with basis* $(g_i)_{i\in\{1,\ldots,p\}}$ *and* $T(\mathfrak{g})$ *its tensor algebra. Then the* universal enveloping algebra $U(\mathfrak{g})$ *is defined to be the tensor algebra*

$$U(\mathfrak{g}) = \frac{T(\mathfrak{g})}{\mathcal{I}_{\mathfrak{g}}},$$

that is divided by the two-sided ideal $\mathcal{I}_{\mathfrak{g}} \subset T(\mathfrak{g})$*, generated by relations*

$$\forall\, g_i, g_j \in \mathfrak{g} :\; g_i \otimes g_j - g_j \otimes g_i - [g_i, g_j]_{\mathfrak{g}} = 0.$$

For $\varphi, \omega \in U(\mathfrak{g})$ *the bracket* $[\varphi, \omega] = \varphi \cdot \omega - \omega \cdot \varphi$ *is called the* commutator.

Due to the Poincaré-Birkhoff-Witt theorem (2.1.3) there exists an injective algebra homomorphism $\chi : \mathfrak{g} \to U(\mathfrak{g})$ such that the Lie algebra $\mathfrak{g}$ is contained in its larger universal enveloping algebra $U(\mathfrak{g})$. Associativity of $U(\mathfrak{g})$ is equivalent to the Jacobi-identity of the commutator bracket, as can be verified by performing the necessary commutations along the two alternative ways $g, h, k \in U(\mathfrak{g}) :\;\; ghk \to hgk \to hkg \to khg$ and $ghk \to gkh \to kgh \to khg$. As the dual objects to algebras, we now introduce *coalgebras*. The notion of duality for algebras and coalgebras is an enhancement of that used for vector spaces. Before we can discuss this point in the next subsection, we first require a neat understanding of coalgebras. As we will see, bialgebras and Hopf-algebras carry both structures, that of an algebra and that of a coalgebra - in a compatible way. Duality then relates the algebra sector of a bi- or Hopf-algebra to the coalgebra sector of a dual bi- or Hopf-algebra and vice versa. Instead of endowing a vector space $(\mathfrak{V}, +; \mathbf{K})$ with a bilinear map μ, as in (2.19), we can alternatively enhance it by a $\mathbf{K}$-linear coproduct structure

$$\Delta : \mathfrak{V} \to \mathfrak{V} \otimes \mathfrak{V}. \tag{2.25}$$

Such an operation of course has to be specified. In order to do some general manipulations, we introduce the Sweedler notation for the coproduct, such that for $v \in \mathfrak{V}$ we have

$$\Delta v := \sum v_{(1)} \otimes v_{(2)}.$$

The indices in parenthesis (1) and (2) denote the sets of indices the sum is covering. We thus often omit the explicit symbol of the sum and instead shortly write $\Delta v = v_{(1)} \otimes v_{(2)}$. As the product of an algebra is associative according to (2.20), we introduce the dual notion of *coassociativity* for the coproduct to be

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta. \tag{2.26}$$

In terms of Sweedler notation this can also be expressed as

$$v_{(1)} \otimes v_{(2)(1)} \otimes v_{(2)(2)} = v_{(1)(1)} \otimes v_{(1)(2)} \otimes v_{(2)}.$$

As associative algebras make no difference for the hierarchy of brackets fixing the sequence of multiplication, coassociativity makes no difference in what order the coproduct is applied to tensor components - the result is independent from the sequence the coproduct was applied to these components. In this sense we understand the summation indices of the Sweedler notation as completely independent, since they merely reflect exactly that sequence, the coproduct was applied - and due to coassociativity gives the same result. In formulas this means

$$v_{(1)} \otimes v_{(2)(1)} \otimes v_{(2)(2)} \equiv v_{(1)} \otimes v_{(2)} \otimes v_{(3)} \equiv v_{(1)(1)} \otimes v_{(1)(2)} \otimes v_{(2)}.$$

Analogously to the unit element as a neutral element of the multiplication, we define the *counit*, for the coproduct. This is once more defined as a linear map, being

$$\epsilon : \mathfrak{V} \to \mathbf{K}, \tag{2.27}$$

with the property

$$(\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id} = (\epsilon \otimes \mathrm{id}) \circ \Delta. \tag{2.28}$$

Expressed in terms of the Sweedler notation, we obtain

$$v_{(1)} \cdot \epsilon(v_{(2)}) = v = \epsilon(v_{(1)}) \cdot v_{(2)}.$$

We thus define a coalgebra as follows.

2.2.4 DEFINITION (COALGEBRA) *A* coalgebra $(\mathfrak{C}, \Delta, \epsilon; \mathbf{K})$ *is a vector space* $(\mathfrak{C}, +; \mathbf{K})$ *that is endowed with linear maps* Δ *and* ϵ *given by (2.25) and (2.27) such that the axioms of coassociativity (2.26) and of the counit (2.28) are satisfied.*

We remark that a coalgebra $\mathfrak{C}$ is called *cocommutative* if the following condition for its coproduct holds:

$$\Delta = \sigma \circ \Delta. \tag{2.29}$$

Here we define the transposition map by

$$\begin{aligned} \sigma : \mathfrak{C} \otimes \mathfrak{C} \quad &\rightarrow \quad \mathfrak{C} \otimes \mathfrak{C} \\ b \otimes c \quad &\mapsto \quad c \otimes b. \end{aligned} \tag{2.30}$$

As already mentioned, bialgebras are vector spaces endowed with both, an algebra structure and a coalgebra structure that are made compatible. Compatibility is achieved by demanding that the coproduct Δ and the counit ϵ shall be algebra homomorphisms. This property turns out to be equivalent to that the product μ and the unit η are coalgebra homomorphisms. For two coalgebra-homomorphic coalgebras $\mathfrak{C}$ and $\mathfrak{B}$ this is a $\mathbf{K}$-linear map $\Lambda: \ \mathfrak{C} \rightarrow \mathfrak{B}$ with the property that

$$\Lambda(\Delta_{\mathfrak{C}}(c)) = \Delta_{\mathfrak{B}}(\Lambda(c)).$$

This little preparation is sufficient, to introduce bialgebras by the following definition.

2.2.5 DEFINITION (BIALGEBRA) *A* bialgebra $(\mathcal{B}, \mu, \eta, \Delta, \epsilon; \mathbf{K})$ *is vector space* $(\mathcal{B}, +; \mathbf{K})$ *that is both, an algebra and a coalgebra that are made compatible by relations*

$$\begin{aligned} \forall\, b, c \in \mathcal{B} \quad : \quad & \Delta(b \cdot c) = \Delta(b) \cdot \Delta(c), \qquad \Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}, \\ & \epsilon(b \cdot c) = \epsilon(b) \cdot \epsilon(c), \qquad \epsilon(\mathbf{1}) = 1. \end{aligned}$$

Using the transposition map (2.30), above relations can also be expressed in terms of maps

$$\begin{aligned} \Delta \circ \mu \quad &= \quad (\mu \otimes \mu) \circ (\mathrm{id} \otimes \sigma \otimes \mathrm{id}) \circ (\Delta \otimes \Delta), \\ \epsilon \circ \mu \quad &= \quad \mu \circ (\epsilon \otimes \epsilon). \end{aligned}$$

The bialgebra once more can be enhanced to a Hopf-algebra by introducing some sort of inverse element, called the *antipode*, that is given by a $\mathbf{K}$-linear map

$$\mathrm{S} : \mathcal{B} \to \mathcal{B}. \tag{2.31}$$

Its properties are given by the antipode axiom

$$\mu \circ (\mathrm{S} \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon = \mu \circ (\mathrm{id} \otimes \mathrm{S}) \circ \Delta. \tag{2.32}$$

The square S^2 of the antipode is not necessarily the identity map. Moreover, an inverse map S^{-1} of the antipode as well does not necessarily exist. We can thus give the definition of a Hopf-algebra.

2.2.6 Definition (Hopf-Algebra) *A* Hopf-algebra *is a* $\mathbf{K}$*-linear bialgebra* $(\mathcal{H}, \mu, \eta, \Delta, \epsilon; \mathbf{K})$ *that is endowed with an antipode map (2.31) that satisfies the antipode axiom (2.32).*

A Hopf-algebra homomorphism is a $\mathbf{K}$-linear map that is both an algebra homomorphism and a coalgebra homomorphism. The universal enveloping algebra (2.2.3) is enhanced to a Hopf-algebra by introducting the following coproduct, counit and antipode on the generators:

$$\begin{aligned} \forall\, g \in U(\mathfrak{g}) \quad : \quad & \Delta(g) = g \otimes \mathbf{1} + \mathbf{1} \otimes g \\ & \epsilon(g) = 0 \\ & \mathrm{S}(g) = -g \end{aligned}$$

It is quite easy to verify that the Hopf algebra axioms are satisfied for this case. This specific cocommutative coproduct defined here for the universal enveloping algebra is said to be of *primitive* type. As a next step we are of course interested to obtain a notion of a Lie group in the Hopf algebra setting. Universal enveloping algebras already carry the structure of a Lie group. But here we are interested in some dual object that will turn out to be an algebra of functions over a compact group manifold. We postpone this specific point to the next subsection but nevertheless present the applied scheme on the example of a finite group. As we learned in the first section, the algebra of functions carries the algebraic information of the space it is defined on, such that it is possible to study its properties by investigating the algebra of functions on it. This is known as the Gelfand-Naimark Theorem. Since it is also our aim to put Lie groups into a dual notion to the universal enveloping algebras $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$, we absorb them, as just mentioned, as the algebra of functions over the specific group. Due to duality one can see that the coproduct of the algebra of functions carries the group multiplication. We thus consider the algebra of functions $\mathcal{F}(\mathcal{G})$ over a finite group $(\mathcal{G}, \cdot_{\mathcal{G}})$ with neutral element

$\mathbf{1}_{\mathcal{G}}$. It is again defined by pointwise addition and multiplication as well as by multiplication with scalars of the field $\mathbf{K}$ as for (2.23). Additionally, since there is now more algebraic structure than provided by the mainfold $\mathcal{M}$, we additionally consider the coalgebra sector and an antipode map to be given by

$$\begin{aligned}\forall\, \Phi \in \mathcal{F}(\mathcal{G}),\ g,h\in\mathcal{G} \quad : \quad & \Delta(\Phi)(g,h) = \Phi_{(1)}(g)\cdot\Phi_{(2)}(h) := \Phi(g\cdot_{\mathcal{G}} h),\\ & \epsilon(\Phi) = \Phi(\mathbf{1}_{\mathcal{G}}),\\ & S(\Phi)(g) = \Phi(g^{-1}).\end{aligned}$$

It is now interesting to see how the group properties are transfered to the coproduct. As such we find coassociativity induced by the associativity of the group multiplication $\cdot_{\mathcal{G}}$ because

$$\begin{aligned}\left((\Delta\otimes\mathrm{id})\circ\Delta\right)(\Phi)(g,h,k) &= \Phi((g\cdot_{\mathcal{G}} h)\cdot_{\mathcal{G}} k)\\ &= \Phi(g\cdot_{\mathcal{G}}(h\cdot_{\mathcal{G}} k)) = \left((\mathrm{id}\otimes\Delta)\circ\Delta\right)(\Phi)(g,h,k).\end{aligned}$$

Moreover the neutral element $\mathbf{1}_{\mathcal{G}}$ rules the properties of the counit, i.e.

$$\begin{aligned}\left((\mathrm{id}\otimes\epsilon)\circ\Delta\right)(\Phi)(g,h) &= \Delta(\Phi)(g,\mathbf{1}_{\mathcal{G}}) = \Delta(\Phi)(g)\\ &= \Delta(\Phi)(\mathbf{1}_{\mathcal{G}},g) = \left((\epsilon\otimes\mathrm{id})\circ\Delta\right)(\Phi)(h,g).\end{aligned}$$

Here we used the isomorphy property (2.18) to do this manipulation. The group properties finally are covered by the introduction of the inverse element by the antipode map

$$\begin{aligned}\left(\mu\circ(\mathrm{S}\otimes\mathrm{id})\circ\Delta\right)(\Phi)(g) &= \mathrm{S}(\Phi_{(1)})(g)\cdot\Phi_{(2)}(g)\\ &= \Phi_{(1)}(g^{-1})\cdot\Phi_{(2)}(g) = \Phi(g^{-1}g) = \Phi(\mathbf{1}_{\mathcal{G}}) = \epsilon(\Phi).\end{aligned}$$

The most crucial point in this example is the fact that we identified the algebras $\mathcal{F}(\mathcal{G}\times\mathcal{G})$ and $\mathcal{F}(\mathcal{G})\otimes\mathcal{F}(\mathcal{G})$, i.e. we assumed that there exists an algebra isomorphism between them. In fact they are isomorphic for finite groups. But in the infinite case however this is not generally true anymore as we see in the next subsection. Because of this reason we also postponed our consideration of Lie groups until we discussed duality. After all, this final example is a first example of that what we consider to be a duality in future. However, for this purpose, the formalism will be enhanced by dual pairing that can roughly be considered as a generalization of scalar products of vector spaces to our requirements. In fact the finite group $\mathcal{G}$ and its algebra of functions $\mathcal{F}(\mathcal{G})$ are dual objects in this sense. As we saw in this latest example, the group operation $\cdot_{\mathcal{G}}$ and unit element $\mathbf{1}_{\mathcal{G}}$ are dual to the coproduct and counit of $\mathcal{F}(\mathcal{G})$. This is an important point we discuss in the next subsection and the basic reason, why we want to consider Lie groups in terms of its algebra of function: The noncommutative product in $U(\mathfrak{g})$ becomes dual to the coproduct of $\mathcal{F}(\mathfrak{G})$, that carries the former group multiplication. The duality relation between them of course accounts for the exponentiation that anyway relates them. On the other hand the coproduct of $U(\mathfrak{g})$ remains cocommutative and thus the product of $\mathcal{F}(\mathfrak{g})$ remains commutative on the dual side. It turns out that nontrivial deformations of $U(\mathfrak{g})$ are always Hopf-algebra-isomorphic to such deformations where only the coproduct is deformed and the product remains as in the undeformed case. The other way around we find that the coproduct in $\mathcal{F}(\mathfrak{g})$ remains undeformed, while the product is deformed in a manner that is dual to the deformation of the coproduct in $U(\mathfrak{g})$.

2.2.2 Duality

To any finite dimensional $\mathbf{K}$-linear vector space $\mathfrak{V}$ there exists the dual vector space $\mathfrak{V}^*$ of linear functionals

$$\begin{array}{rcl} \Phi \in \mathfrak{V}^* :\ \Phi : \mathfrak{V} & \longrightarrow & \mathbf{K} \\ v & \mapsto & < \Phi, v > . \end{array}$$

As we mentioned already in the last subsection, an additional coalgebra structure on $\mathfrak{V}$ induces an algebra structure on $\mathfrak{V}^*$, thus for a coalgebra $\mathfrak{C}$ with $\Delta : \mathfrak{C} \to \mathfrak{C} \otimes \mathfrak{C}$ we obtain the dual operation on $\mathfrak{C}^*$

$$\Delta^* :\ \mathfrak{C}^* \otimes \mathfrak{C}^* \to \mathfrak{C}^*$$

for the dual vectorspace. It thus obtains an associative algebra structure with unit element by means of the coalgebra axioms (2.26) and (2.28). In contrast to this, the dual $\mathfrak{A}^*$ of an algebra $\mathfrak{A}$ with multiplication $\mu : \mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{A}$ obtains a coalgebra structure by

$$\mu^* :\ \mathfrak{A}^* \to (\mathfrak{A} \otimes \mathfrak{A})^*.$$

As mentioned above, the crucial point is that for the finite dimensional case we have isomorphy $\mathfrak{A}^* \otimes \mathfrak{A}^* \simeq (\mathfrak{A} \otimes \mathfrak{A})^*$ but in the infinit dimensional case $\mathfrak{A}^* \otimes \mathfrak{A}^*$ is a true subset of $(\mathfrak{A} \otimes \mathfrak{A})^*$. The latter is the completion of the former. Moreover μ^* is not restricted on $\mathfrak{A}^* \otimes \mathfrak{A}^*$ in general. We will see later how this problem is easily solved at least for those cases that are of most interesting for physics. For instance we stick to the finite dimensional case. Thus in this regime bialgebras $\mathcal{B}$ induce bialgebras on the dual $\mathcal{B}^*$. More generally we define the pairing for general Hopf-algebras by

2.2.7 DEFINITION (DUAL PAIRING) *Let $\mathcal{H}, \mathcal{B}$ be two Hopf algebras. $\mathcal{H}, \mathcal{B}$ are called* dually paired *if there exists a bilinear map*

$$\begin{aligned} <\,,\,>: \mathcal{H} \otimes \mathcal{B} &\longrightarrow \mathbf{K} \\ \Phi \otimes b &\mapsto\ <\Phi, b>, \end{aligned}$$

that satisfies the following relations

$$\begin{aligned} <\Phi \cdot \Psi, b> &= <\Phi \otimes \Psi, \Delta b> = <\Phi, b_{(1)}> \cdot <\Psi, b_{(2)}>, \\ <\mathbf{1}, b> &= \epsilon(b), \\ <\Phi, b \cdot c> &= <\Delta\Phi, b \otimes c> = <\Phi_{(1)}, b> \cdot <\Psi_{(2)}, c>, \\ <\Phi, \mathbf{1}> &= \epsilon(b), \\ <S(\Phi), b> &= <\Phi, S(b)>. \end{aligned}$$

Such a pairing may be degenerate, i.e. there exist nonzero elements either $h \in \mathcal{H}$ or $\Phi \in \mathcal{H}^*$ such that $<\Phi, h>= 0$. Since the dual pairing is after all a $\mathbf{K}$-linear homomorphism of vector spaces, we can divide either $\mathcal{H}$ or $\mathcal{H}^*$ by the corresponding kernel, such that the pairing becomes nondegenerate on the received quotient. Before we discuss the problem of the infinit dimensional case, we consider our example from above for the finite dimensional case. It already gives some insight into the structure required later. It is possible to enhance the algebra of functions $\mathcal{F}(\mathcal{G})$ on a finite dimensional group $\mathcal{G}$ by linearity that is induced from the field $\mathbf{K}$. This provides a $\mathbf{K}$-linear vector space structure on $\mathcal{G}$ such that we denote the received vector space by $\mathcal{G}_{\mathbf{K}}$. Thus for linear $\Phi \in \mathcal{F}(\mathcal{G})$ and $g_i, g_j \in \mathcal{G}, \lambda, \mu \in \mathbf{K}$ we have

$$\lambda \cdot \Phi(g_i) + \mu \cdot \Phi(g_j) =: \Phi(\lambda \cdot g_i + \mu \cdot g_j),$$

where $\lambda \cdot g_i + \mu \cdot g_j \in \mathcal{G}_{\mathbf{K}}$. The group multiplication and vector space structure make $\mathcal{G}_{\mathbf{K}}$ into a *group algebra*. This enhances to a Hopf-algebra by

$$\forall\, g \in \mathcal{G}_{\mathbf{K}}: \ \Delta g = g \otimes g, \qquad \epsilon(g) = \mathbf{1}_{\mathcal{G}}, \qquad \mathrm{S}(g) = g^{-1}.$$

This specific cocommutative coproduct is called a *group-like* coproduct. The group algebra is dual to its algebra of linear functions. The corresponding pairing for $\Phi \in \mathcal{F}(\mathcal{G})$ and $h = \sum_i h(g_i) \cdot g_i \in \mathcal{G}_{\mathbf{K}}$ is given by

$$<\Phi, h> := \Phi(h) = \Phi(\sum_i h(g_i) \cdot g_i) = \sum_i h(g_i) \cdot \Phi(g_i),$$

where $h \in \mathcal{F}(\mathcal{G})$. Since the dimension of $\mathcal{G}_{\mathbf{K}}$ and $\mathcal{F}(\mathcal{G}))$ are equal, we can choose a basis $(\varphi_i)_i$ and $(g_i)_i$ for each vector space and normalize the pairing by

$$<\varphi_i, g_j>= \delta_{ij}.$$

Since the pairing is nondegenerate, we obtain

$$\mathbf{1}_{\mathcal{G}} = \sum_i g_i \cdot \varphi_i(.)$$

as a completeness relation for $\mathcal{G}_{\mathbf{K}}$. We now discuss the infinit dimensional case using the example of the universal enveloping algebra $U(\mathfrak{g})$ of an Abelian p-dimensional Lie-algebra. As we recall from the beginning of this subsection, the coproduct Δ on $U(\mathfrak{g})$ implies a product Δ^* on the dual Hopf-algebra that we further denote by $U(\mathfrak{g})^*$. Thus with the cocommutative coproduct

$$\Delta g_i = g_i \otimes \mathbf{1} + \mathbf{1} \otimes g_i$$

for the generators $g_i \in U(\mathfrak{g})$, we obtain the coproduct for the ordered monomials

$$g_{\mathbf{n}} = \frac{g_1^{n_1}}{n_1!} \cdot \frac{g_2^{n_2}}{n_2!} \cdot \ldots \cdot \frac{g_k^{n_k}}{n_k!} \cdot \ldots \cdot \frac{g_p^{n_p}}{n_p!}$$

of $U(\mathfrak{g})$ that according to (2.1.3) form a basis $(g_{\mathbf{n}})_{\mathbf{n} \in \mathbb{N}^p}$. We thus obtain

$$\Delta g_{\mathbf{n}} = \sum_{\mathbf{k}=0}^{\mathbf{n}} g_{\mathbf{k}} \otimes g_{(\mathbf{n}-\mathbf{k})}.$$

We denote the basis of $U(\mathfrak{g})^*$ by $(\varphi^{\mathbf{m}})_{\mathbf{m} \in \mathbb{N}^p}$ and normalize the pairing in these basis by

$$< \varphi^{\mathbf{m}}, g_{\mathbf{n}} >= \delta_{\mathbf{n}}^{\mathbf{m}}.$$

We thus obtain the commutative product of two elements $\varphi^{\mathbf{l}}, \varphi^{\mathbf{m}} \in U(\mathfrak{g})^*$ by

$$\begin{aligned} < \varphi^{\mathbf{l}} \cdot_* \varphi^{\mathbf{m}}, \Delta g_{\mathbf{n}} > &= \sum_{\mathbf{k}=0}^{\mathbf{n}} < \varphi^{\mathbf{l}}, g_{\mathbf{k}} > \cdot < \varphi^{\mathbf{m}}, g_{(\mathbf{n}-\mathbf{k})} > \\ &= \sum_{\mathbf{k}=0}^{\mathbf{n}} \delta_{\mathbf{k}}^{\mathbf{l}} \, \delta_{(\mathbf{n}-\mathbf{k})}^{\mathbf{m}} = \delta_{(\mathbf{n}-\mathbf{l})}^{\mathbf{m}} = \delta_{\mathbf{n}}^{(\mathbf{m}+\mathbf{l})} =< \varphi^{(\mathbf{m}+\mathbf{l})}, g_{\mathbf{n}} > \end{aligned}$$

and in particular, we receive

$$\varphi^{\mathbf{l}} \cdot_* \varphi^{\mathbf{m}} = \varphi^{(\mathbf{m}+\mathbf{l})}$$

on $U(\mathfrak{g})^*$. We go ahead in order to obtain the coproduct for $\varphi^{\mathbf{n}} \in U(\mathfrak{g})^*$ in an analogous way. The Abelian product among two monomials $g_{\mathbf{l}}$ and $g_{\mathbf{m}}$ is given by

$$g_{\mathbf{l}} \cdot g_{\mathbf{m}} = \left(\prod_k^p \frac{(l_k + m_k)!}{l_k! m_k!} \right) g_{(\mathbf{l}+\mathbf{m})} =: \binom{\mathbf{l}+\mathbf{m}}{\mathbf{m}} g_{(\mathbf{l}+\mathbf{m})}.$$

We thus make the ansatz for the coproduct by

$$\begin{aligned}
< \varphi^{\mathbf{n}}, g_{\mathbf{l}} \cdot g_{\mathbf{m}} > &= \binom{\mathbf{l}+\mathbf{m}}{\mathbf{m}} < \varphi^{\mathbf{n}}, g_{(\mathbf{l}+\mathbf{m})} >= \binom{\mathbf{l}+\mathbf{m}}{\mathbf{m}} \delta^{\mathbf{n}}_{(\mathbf{l}+\mathbf{m})} \\
&= \sum_{\mathbf{k}=0}^{\mathbf{n}} \binom{\mathbf{l}+\mathbf{k}}{\mathbf{k}} \delta^{\mathbf{n}}_{(\mathbf{l}+\mathbf{k})} \, \delta^{\mathbf{k}}_{\mathbf{m}} = \sum_{\mathbf{k}=0}^{\mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} \delta^{(\mathbf{n}-\mathbf{k})}_{\mathbf{l}} \, \delta^{\mathbf{k}}_{\mathbf{m}} \\
&= \sum_{\mathbf{k}=0}^{\mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} < \varphi^{(\mathbf{n}-\mathbf{k})}, g_{\mathbf{l}} > \cdot < \varphi^{\mathbf{k}}, g_{\mathbf{m}} > .
\end{aligned}$$

We thus obtain

$$\Delta\varphi^{\mathbf{n}} = \sum_{\mathbf{k}=0}^{\mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} \varphi^{(\mathbf{n}-\mathbf{k})} \otimes \varphi^{\mathbf{k}}.$$

We obviously encounter the problem that $\Delta\varphi^{\mathbf{n}} \in (U(\mathfrak{g}) \otimes U(\mathfrak{g}))^*$ and that it does not belong to the subalgebra $U(\mathfrak{g})^* \otimes U(\mathfrak{g})^*$. This problem is solved by choosing, as the dual Hopf-algebra of $U(\mathfrak{g})$, not $U(\mathfrak{g})^*$ but a subalgebra

$$U(\mathfrak{g})^0 = \{\varphi^{\mathbf{n}} \mid \Delta\varphi^{\mathbf{n}} \in \; U(\mathfrak{g})^* \otimes U(\mathfrak{g})^*\} \;\subset\; U(\mathfrak{g})^*$$

Since $U(\mathfrak{g})^* \otimes U(\mathfrak{g})^*$ is a subalgebra, the set $U(\mathfrak{g})^0 \subset U(\mathfrak{g})^*$ becomes a subalgebra as well. It can be shown that the multiplication μ_0, the coproduct Δ_0 and the antipode S_0 map into $U(\mathfrak{g})^0$, $U(\mathfrak{g})^0 \otimes U(\mathfrak{g})^0$ and $U(\mathfrak{g})^0$ respectively. This procedure works not only for the case of $U(\mathfrak{g})^*$ but for any infinit dimensional Hopf-algebra. The important point in this respect is that it can be shown that $U(\mathfrak{g})^0$ is isomorphic as a Hopf-algebra to finite dimensional representations of $U(\mathfrak{g})^*$. This is also true for the general case just mentioned. Thus if we want to consider the algebra of functions on Lie-groups as the dual to $U(\mathfrak{g})$, we can do this by sticking to functions on matrix representations of $\mathfrak{G}$. And this is also the way how it is actually performed and thus provides the required setup for later deformations, since the object $\mathcal{R}$ that rules the deformation of $U(\mathfrak{g})$ obtains its dual counterpart, the **R**-Matrix, that rules the deformation of the functions on the matrix representation of the Lie-group $\mathfrak{G}$. Before we come to this point we thus have to study representations and more specifically finite dimensional representations of $\mathcal{F}(\mathfrak{G})$ and $U(\mathfrak{g})$. This is covered in the next subsection that will also teach us how these representations are related by duality.

2.2.3 Representations

A central application of algebras, such as universal enveloping algebras $U(\mathfrak{g})$ or Lie groups $\mathfrak{G}$, in physics is given by their representation on a vector space.

The latter might constitute such linear spaces as configuration spaces, phase spaces, wave functions or more generally Hilbert spaces and so forth. Our particular interest of course lies in representation spaces that exhibit more structure than a vector space. If we enhance a vector space to an algebra or coalgebra, we immediately require more structure on the symmetry algebra that has to be represented as well. This is provided by the coproduct. There are several variants of representations that we are dealing with in this subsection. However, the central idea concerning representations of Hopf algebras again lies in a sort of duality. The structure of the coproduct in the symmetry algebra is reflected by the algebra or coalgebra sector of representation space, depending on the specific sort of representation. This will play an important role concerning later deformation. As an example, we already mentioned that a deformation of a universal enveloping algebra is actually performed by deforming its coproduct - that is a dual reflection of the algebra structure of the representation space. Thus the noncommutativity of a representation space directly communicates with the coproduct structure of the symmetry algebra. We thus also understand that a commutative algebra of coordinates, as it is common in physics, does not require specific attention concerning a coproduct, we apply it as a derivation, i.e. by the use of the standard Leibniz-rule. This is reflected by the *primitive* coproduct structure of the universal enveloping algebra $U(\mathfrak{g})$. We come to these details in this subsection. Apart from this we learn about the relation of dual Hopf algebras that are represented on the same space. Moreover we show how symmetry algebras and representation spaces can be joined together into a single algebra as cross-product algebras. In contrast to this we introduce adjoint and coadjoint representations that kind of reverse this process. The latter representations play a crucial role in representation theory. The present section will end by the discussion of our examples of $U(\mathfrak{g})$ and $\mathcal{F}(\mathfrak{G})$, where the latter, as we discussed in the last subsection, is investigated as the algebra of functions on the matrix representation of the Lie group $\mathfrak{G}$. The theory we develop here again has its root in the original notion of a representation of a symmetry algebra on a vector space. We thus begin by recalling the notion of a representation of an algebra $\mathfrak{A}$ on a vector space $\mathfrak{V}$.

2.2.8 Definition (Representation) *A* left representation *of an algebra* $\mathfrak{A}$ *is a pair* $(\rho_{\mathfrak{A}}, \mathfrak{V})$ *consisting of a vector space* $(\mathfrak{V}, +; \mathbf{K})$ *and a* $\mathbf{K}$*-linear map*

$$\begin{aligned} \rho_{\mathfrak{A}} : \mathfrak{A} \otimes \mathfrak{V} &\rightarrow \mathfrak{V} \\ a \otimes v &\mapsto a \triangleright v, \end{aligned}$$

such that the action '$\triangleright$' *of* $a \in \mathfrak{A}$ *on* $v \in \mathfrak{V}$ *satisfies the following relations:*

$$\begin{aligned} \forall\, a, b \in \mathfrak{A},\ v \in \mathfrak{V} \quad : \quad & (a \cdot b) \triangleright v = a \triangleright (b \triangleright v) \\ & \mathbf{1}_{\mathfrak{A}} \triangleright v = v. \end{aligned}$$

Once more we can now enhance the vector space $\mathfrak{V}$ with an algebraic structure, by introducing a suitable product and unit element. In this case we have to clearify how we have to deal with the action of an element $a \in \mathfrak{A}$ on the product of two elements $v, w \in \mathfrak{V}$

$$a \triangleright (v \cdot_{\mathfrak{V}} w),$$

such that the action of $\mathfrak{A}$ on the algebra $\mathfrak{V}$ becomes an algebra homomorphism. The central idea that solves this task, lies in the fact that it can be transferred to the tensor product representation of $\mathfrak{A} \otimes \mathfrak{A}$ by the use of the coproduct $\Delta_{\mathfrak{A}}$. We thus again obtain a notion of an algebra represented on a vector space. In particular we define for the representation of a bialgebra $\mathcal{H}$ on an algebra $\mathfrak{A}$

$$\begin{aligned} \forall\, h \in \mathcal{H},\ a, b \in \mathfrak{A} \quad : \quad & h \triangleright (a \cdot_{\mathfrak{A}} b) = \sum (h_{(1)} \triangleright a) \cdot_{\mathfrak{A}} (h_{(2)} \triangleright b) \\ & h \triangleright \mathbf{1}_{\mathfrak{A}} = \epsilon(h) \end{aligned} \tag{2.33}$$

Then $\mathfrak{A}$ is called a *left* $\mathcal{H}$*-module algebra.* This is a genuine case in physics. If we represent $U(\mathfrak{g})$ on a d-dimensional configuration space $\mathfrak{X}$ in a given coordinate system with basis $(x_i)_{i \in \{1, \ldots d\}}$, we then have also to consider the action of $g_i \in U(\mathfrak{g})$ on products $x_i \cdot x_j$, since the coordinates represent real numbers. In physics textbooks the action is very often given in terms of a commutator such that

$$g_i \triangleright x_j \equiv [g_i, x_j]$$

This is quite a sloppy notation, since it assumes that g_i and x_j are elements of a common algebra. The commutator suggests the existence of a product of elements $g_i \in U(\mathfrak{g})$ and $x_j \in \mathfrak{X}$ that has nowhere been defined yet. Moreover it is an actual luck that the commutator $[g_i, x_j]$ 'maps' into the 'subalgebra' $\mathfrak{X}$ as it is required to be a representation. The above notation is futile in the case of deformations, because then the commutator constitutes no representation on $\mathfrak{X}$ anymore. The answer to this issue lies in *cross-product algebras* that are joint algebras related by a representation. However, we want to use this specific example in order to consider the action on products $x_i \cdot x_j$ of coordinates. Thus using the well known properties of the commutator bracket we obtain for $g_i \in U(\mathfrak{g}),\ x_m, x_n \in \mathfrak{X}$:

$$\begin{aligned} g_i \triangleright (x_m \cdot x_n) \quad &= \quad [g_i, x_m \cdot x_n] = x_m \cdot [g_i, x_n] + [g_i, x_m] \cdot x_n \\ & \qquad x_m \cdot (g_i \triangleright x_n) + (g_i \triangleright x_m) \cdot x_n = \Delta(g_i) \triangleright (x_m \cdot x_n) \\ \\ g_i \triangleright \mathbf{1}_{\mathfrak{X}} \quad &= \quad [g_i, \mathbf{1}_{\mathfrak{X}}] = 0 = \epsilon(g_i). \end{aligned}$$

We thus see how the product in $\mathfrak{X}$ is reflected in the coproduct of $U(\mathfrak{g})$. In the context of Hopf-algebras this well known scheme is generalized. This is presented now. First of all we consider the case that the former vector space $\mathfrak{V}$ is enhanced by coproduct structure. Then we confront ourselves with the question, how we have to deal with the action of an element $a \in \mathfrak{A}$ on the coproduct of an element $v \in \mathfrak{V}$

$$a \triangleright \Delta_{\mathfrak{V}}(v)$$

such that the action of $\mathfrak{A}$ on the coalgebra $\mathfrak{V}$ becomes an coalgebra homomorphism. On the other hand the Hopf-structure provides a new type of representations that are called *corepresentations*. In this latter case we can also distinguish between a *coaction* on an algebra and on the other hand on the coaction on a coalgebra. But before we discuss corepresentations, we first come back to the problem just mentioned, that a bialgebra $\mathcal{B}$ acts on a coalgebra $\mathfrak{C}$. This is as well ruled by the coproduct structure on $\mathcal{B}$. We define

$$\begin{aligned} \forall\, h \in \mathcal{H},\ c \in \mathfrak{C} \quad : \quad \Delta(h \triangleright c) &= \sum (h_{(1)} \triangleright c_{(1)}) \otimes (h_{(2)} \triangleright c_{(2)}) \\ \epsilon(h \triangleright c) &= \epsilon(h) \cdot \epsilon(c). \end{aligned} \tag{2.34}$$

Then $\mathfrak{C}$ is called a *left $\mathcal{H}$-module coalgebra*. Corepresentations at the first glance appear quite unusual, since they require some dual formulation. Instead of representing an algebra $\mathfrak{A}$ on a vector space $\mathfrak{V}$, we now consider the representation of a coalgebra, more specifiacally, we define

2.2.9 Definition (Corepresentation) *A* right corepresentation *of a coalgebra $\mathfrak{C}$ is a pair $(\omega_{\mathfrak{C}}, \mathfrak{V})$ consisting of a vector space $(\mathfrak{V}, +; \mathbf{K})$ and a $\mathbf{K}$-linear map*

$$\begin{aligned} \omega_{\mathfrak{C}} : \mathfrak{V} \quad &\rightarrow \quad \mathfrak{V} \otimes \mathfrak{C} \\ v \quad &\mapsto \quad \sum v^{(1)} \otimes v^{(2)}, \end{aligned}$$

such that the coaction *$\omega_{\mathfrak{C}}$ satisfies the following relations:*

$$\begin{aligned} (\omega_{\mathfrak{C}} \otimes id) \circ \omega_{\mathfrak{C}} \quad &= \quad (id \otimes \Delta) \circ \omega_{\mathfrak{C}} \\ (id \otimes \epsilon) \circ \omega_{\mathfrak{C}} \quad &= \quad id \end{aligned}$$

The structure of the coaction very much reminds the structure of the coproduct and the counit as the action corresponds to the structure of an algebra. These observations reflect the duality between symmetry bialgebra and representation space that we commented at the beginning of this subsection. Moreover representations and corepresentations are dual operations in the analogous sense as discussed in the last subsection. We come to this point later. As we see soon, the structure of actions and coactions also provides the basis to join algebras and coalgebras to cross-product algebras, coalgebras and cross-coproduct algebras and coalgebras. Above relations can also be writen as

$$\begin{aligned} \forall v \in \mathfrak{V} \quad : \quad & \sum v^{(1)(1)} \otimes v^{(1)(2)} \otimes v^{(2)} = \sum v^{(1)} \otimes {v^{(2)}}_{(1)} \otimes {v^{(2)}}_{(2)} \\ & \sum v^{(1)} \epsilon(v^{(2)}) = v. \end{aligned}$$

In future computations we will omit the explicit symbol of summation. Now again, we can ask about enhancements of a corepresentation of a bialgebra $\mathcal{H}$ if the vector space $\mathfrak{V}$ is extended to an algebra $\mathfrak{A}$ or a coalgebra $\mathfrak{C}$, such that the corepresentation becomes an algebra- or coalgebra-homomorphism. This provides the notion of a *right $\mathcal{H}$-comodule algebra* that is given by relations

$$\begin{aligned} \forall\ a, b \in \mathfrak{A} \quad : \quad & \omega_{\mathcal{H}}(a \cdot b) = \omega_{\mathcal{H}}(a) \cdot \omega_{\mathcal{H}}(b) \\ & \omega_{\mathcal{H}}(\mathbf{1}_{\mathfrak{A}}) = \mathbf{1}_{\mathfrak{A}} \otimes \mathbf{1}_{\mathcal{H}}, \end{aligned}$$

and in turn that of a *right $\mathcal{H}$-comodule coalgebra* that is given by

$$\begin{aligned} (\Delta \otimes \mathrm{id}) \circ \omega_{\mathcal{H}} &= (\mathrm{id} \otimes \mathrm{id} \otimes \mu) \circ (\mathrm{id} \otimes \sigma \otimes \mathrm{id}) \circ (\omega_{\mathcal{H}} \otimes \omega_{\mathcal{H}}) \circ \Delta_{\mathfrak{C}} \\ (\epsilon \otimes \mathrm{id}) \circ \omega_{\mathcal{H}} &= \epsilon \cdot \mathbf{1}_{\mathcal{H}}. \end{aligned}$$

These relations can also be written as

$$\begin{aligned} \forall\ c \in \mathfrak{C} \quad : \quad & \sum {c^{(1)}}_{(1)} \otimes {c^{(1)}}_{(2)} \otimes c^{(2)} = {c_{(1)}}^{(1)} \otimes {c_{(2)}}^{(1)} \otimes {c_{(1)}}^{(2)} {c_{(2)}}^{(2)} \\ & \sum \epsilon(c^{(1)}) \cdot c^{(2)} = \epsilon(c) \cdot \mathbf{1}_{\mathcal{H}}. \end{aligned}$$

The definitions for representations and corepresentations were given as their left and right versions respectively. In a fully analogous way, the right action and the left coaction are defined. With this comment we present the following lemma.

2.2.10 Lemma *Let $\mathfrak{V}$ be a $\mathbf{K}$-linear left module of a Hopf-algebra $\mathcal{H}$, then the dual $\mathbf{K}$-linear space $\mathfrak{V}^*$ becomes a right module of $\mathcal{H}$, related by the following relation*

$$\forall\ \Phi \in \mathfrak{V}^*,\ v \in \mathfrak{V},\ h \in \mathcal{H} :\ (\Phi \triangleleft h)(v) = \Phi(h \triangleright v)$$

The dual of a finite dimensional left $\mathcal{H}$-module algebra $\mathfrak{A}$ *becomes a* right $\mathcal{H}$-module coalgebra $\mathfrak{A}^*$. *The dual of any* left $\mathcal{H}$-module coalgebra $\mathfrak{C}$ *becomes a* right $\mathcal{H}$-module algebra $\mathfrak{C}^*$.

On the other hand we can also observe, what happens, if we consider the dual of the acting Hopf-algebra $\mathcal{H}$.

2.2.11 Lemma *Let $\mathcal{H}$ be a finite dimensional Hopf algebra. Then a right coaction $\beta(v) = \sum v^{(1)} \otimes v^{(2)} \in \mathfrak{V} \otimes \mathcal{H}^*$ of the dual $\mathcal{H}^*$ on some linear space $\mathfrak{V}$ is equivalent to a left action of $\mathcal{H}$ on $\mathfrak{V}$ given by*

$$h \triangleright v = \sum v^{(1)} < v^{(2)}, h > .$$

Moreover, a left $\mathcal{H}$-modul algebra is equivalent to a right $\mathcal{H}^*$-comodule algebra and a left $\mathcal{H}$-modul coalgebra is equivalent to a right $\mathcal{H}^*$-comodule coalgebra.

Due to the pairing we had to restrict this lemma to the finite dimensional case. It is also valid if the dual hopfalgebra $\mathcal{H}^*$ can be defined appropriately. Among the most important examples of representations and corepresentations for physicists are the adjoint and coadjoint actions and coactions. Closely related to these examples we find the cross-product algebras and cross coproduct coalgebras. For a Hopf-algebra $\mathcal{H}$ we define the left adjoint action on $\mathcal{H}$ by

$$\forall\, h, g \in \mathcal{H} :\ \mathrm{ad}_h(g) = \sum h_{(1)} g\, \mathrm{S}(h_{(2)}). \tag{2.35}$$

It is easy to see that this is a left action, since $\mathrm{ad}_{(h\cdot k)}(g) = \sum (h \cdot k)_{(1)} g\ \mathrm{S}((h \cdot k)_{(2)}) = \sum h_{(1)} \cdot k_{(1)} g\ \mathrm{S}(k_{(2)}) \cdot \mathrm{S}(h_{(2)}) = \sum h_{(1)} \mathrm{ad}_k(g)\ \mathrm{S}(h_{(2)}) = \mathrm{ad}_h(\mathrm{ad}_k(g))$. We also find that $\mathrm{ad}_{\mathbf{1}_\mathcal{H}}(g) = g$. Here we used that the antipode is an antialgebra map, i. e. $\mathrm{S}(h \cdot k) = \mathrm{S}(k) \cdot \mathrm{S}(h)$. For the adjoint action on products we find that $\mathrm{ad}_h(k \cdot g) = \sum h_{(1)} k \cdot g\ \mathrm{S}(h_{(2)}) = \sum h_{(1)} k \epsilon(h_{(2)}) g\ \mathrm{S}(h_{(3)}) = \sum h_{(1)} k \mathrm{S}(h_{(2)}) h_{(3)} g\ \mathrm{S}(h_{(4)}) = \mathrm{ad}_{h_{(1)}}(k) \cdot \mathrm{ad}_{h_{(2)}}(g)$. Finally we obtain $\mathrm{ad}_h(\mathbf{1}_\mathcal{H}) = \sum h_{(1)} \mathrm{S}(h_{(2)}) = \epsilon(h) \cdot \mathbf{1}_\mathcal{H}$. The Hopf-algebra $\mathcal{H}$ thus becomes a *left $\mathcal{H}$-module algebra.*

If we have left action of a Hopf-algebra $\mathcal{H}$ on an algebra $\mathfrak{A}$ we can define the *left cross-product algebra* $\mathfrak{A} >\!\!\!\lhd \mathcal{H}$ on the tensor product $\mathfrak{A} \otimes \mathcal{H}$ by introducing the associative product and the unit element by

$$\begin{aligned} \forall\, g, h \in \mathcal{H},\ a, b \in \mathfrak{A} \quad : \quad & (a \otimes g) \odot (b \otimes h) = \sum a \cdot (g_{(1)} \triangleright b) \otimes g_{(2)} \cdot h \\ & \mathbf{1} = \mathbf{1} \otimes \mathbf{1}. \end{aligned}$$

The left cross-product algebra $\mathfrak{A} >\!\!\!\lhd \mathcal{H}$ contains the former Hopf-algebra $\mathbf{1} \otimes \mathcal{H}$ and algebra $\mathfrak{A} \otimes \mathbf{1}$ as subalgebras. The left action can now be written as $(\mathbf{1} \otimes h) \triangleright (a \otimes \mathbf{1}) = (h \triangleright a) \otimes \mathbf{1}$. The former left action is thus recovered as a left adjoint action of the subalgebra $\mathbf{1} \otimes \mathcal{H}$ on the subalgebra $\mathfrak{A} \otimes \mathbf{1}$. We thus obtain

$$\mathrm{ad}_h(a) = \sum (\mathbf{1} \otimes h_{(1)}) \odot (a \otimes \mathbf{1}) \odot (\mathbf{1} \otimes \mathrm{S}(h_{(2)})) = (h \triangleright a) \otimes \mathbf{1}.$$

If the coproduct on elements $h \in \mathcal{H}$ is *primitive*, i.e. $\Delta(h) = h \otimes \mathbf{1} + \mathbf{1} \otimes h$ as it is the case for $g \in U(\mathfrak{g})$ the left adjoint action is equal to the commutator in the cross-product algebra $\mathfrak{A} \rtimes \mathcal{H}$. Thus setting $(\mathbf{1} \otimes h) \equiv h$ and $(\mathbf{1} \otimes a) \equiv a$ we obtain

$$\mathrm{ad}_h(a) = [h \,\overset{\circ}{,}\, a]$$

As mentioned above, the deformation of the coproduct of $U(\mathfrak{g})$ leads to a more complex structure of the coproduct of the generators g_i but in order to have an undeformed limit also incorporates the original primitive coalgebra structure. Thus in the deformed case the adjoint action is not equal anymore to the commutator. For a finite dimensional Hopf-algebra $\mathcal{H}$ we also define the *left coadjoint action* of a Hopf-algebra $\mathcal{H}$ on its dual $\mathcal{H}^*$ by

$$\mathrm{ad}^*_h(\Phi) = \sum \Phi_{(2)} \cdot < h, \mathrm{S}(\Phi_{(1)})\Phi_{(3)} > . \tag{2.36}$$

The dual $\mathcal{H}^*$ then becomes a left $\mathcal{H}$-module coalgebra. In the same way we obtain two corepresentations being the *right adjoint coaction* of $\mathcal{H}$ on itself

$$\forall\, h \in \mathcal{H} :\ \mathrm{Ad}(h) = \sum h_{(2)} \otimes \mathrm{S}(h_{(1)})h_{(3)}, \tag{2.37}$$

that makes $\mathcal{H}$ into a right $\mathcal{H}$-comodule coalgebra. On the other hand we obtain for finite dimensional Hopf-algebras $\mathcal{H}$ the *right coadjoint coaction* on $\mathcal{H}^*$

$$\begin{aligned} \forall\, h \in \mathcal{H}, \Phi \in \mathcal{H}^* :\ \mathrm{Ad}^*_h(\Phi) &= \sum h_{(1)}\mathrm{S}(h_{(3)}) < h_{(2)}, \Phi > \\ &= \sum < h, \Phi^{(1)} > \Phi^{(2)}, \end{aligned} \tag{2.38}$$

that makes $\mathcal{H}^*$ into a right $\mathcal{H}$-comodule algebra. The left adjoint action and the right coadjoint coaction as well as the left coadjoint action and the right adjoint coaction are dual in the sense of lemma (2.2.11). As announced above we close this section by the discussion of duality and representation of $U(\mathfrak{g})$ and $\mathcal{F}(\mathfrak{G})$. Because of the problems concerning duality that we discussed above we have to treat the latter as a matrix representation. Before we come to that point, we want to recall the definition of a Lie-group

2.2.12 Definition (Lie-group) *A* Lie-group $\mathfrak{G}$ *is a finite dimensional C^∞-manifold, that is endowed with a compatible group structure $(\mathfrak{G}, \mu_{\mathfrak{G}})$ such that the multiplication map*

$$\begin{aligned} \mu_{\mathfrak{G}} : \mathfrak{G} \times \mathfrak{G} &\to \mathfrak{G} \\ (g, h^{-1}) &\mapsto g \cdot_{\mathfrak{G}} h^{-1} \end{aligned}$$

is smooth (C^∞). There exists a countable open covering of $\mathfrak{G}$.

We consider $U(\mathfrak{g})$ to be the universal enveloping algebra of an n-dimensional Lie-algebra with a basis of generators $(g_i)_{i\in\{1\ldots n\}}$. We assume that there exists a representation of $U(\mathfrak{g})$ on the tensor algebra of coordinates $\mathfrak{X}$ being generated by a d-dimensional basis $(x_a)_{a\in\{1\ldots d\}}$. Since we do not want to discuss specifics of finite dimensional representations of Lie-algebras, we assume that everything is well behaved such that we can emphasise on duality here. Since $\mathfrak{X}$ is an algebra we consider it to be a *left $U(\mathfrak{g})$-module algebra.* Thus for a finite dimensional representation we most generally write

$$g_i \triangleright x_a = \sum_b x_b \cdot (g_i)^b{}_a \;, \quad (g_i)^k{}_j \in \mathbf{K}. \tag{2.39}$$

If everything is well-behaved the coefficients $(g_i)^k{}_j$ constitute a matrix representation of the generators $(g_i)_{i\in\{1\ldots n\}}$ on $\mathfrak{X}$. As we learned above, the dual of a Hopf-algebra can be considered as the algebra of linear functionals on it. More abstractly we express this as a pairing between a Hopf-algebra and its dual. We dualize $U(\mathfrak{g})$ in a specific manner. On a d-dimensional representation space, the Lie-group $\mathfrak{G}$ is represented by invertible $d\times d$-Matrices. We thus define the following d^2 maps $t^i{}_j$, that send an element g of a Lie-group or Lie-algebra to its matrix entry $(g)^a{}_b \in \mathbf{K}$, i.e. we have

$$\begin{array}{rcl} t^a{}_b : \mathfrak{g} & \rightarrow & \mathbf{K} \\ g & \mapsto & t^a{}_b(g) = (g)^a{}_b \end{array}$$

We now specify $\mathcal{F}(\mathfrak{G})$ as the algebra of functions that is generated by the elements $t^a{}_b$. More abstractly, we introduce a pairing between elements $t^a{}_b \in \mathcal{F}(\mathfrak{G})$ and $g_k \in U(\mathfrak{g})$ by

$$\forall\; t^a{}_b \in \mathcal{F}(\mathfrak{G}),\; g_k \in U(\mathfrak{g}) : < t^a{}_b, g_k >= t^a{}_b(g_k) = (g_k)^a{}_b$$

We can thus write the left action (2.39) expressed in terms of this pairing by

$$g_i \triangleright x_a = \sum_b x_b \cdot < t^b{}_a, g_i > . \tag{2.40}$$

As we recall from the first section, exponentiating the Lie-algebra $\mathfrak{G}$ generates a corresponding Lie-group $\mathfrak{G}$. Of course the relation between Lie-groups and their algebras is more subtle than that, but concerning the scheme we want to present here, this does not play any role. Thus with $\xi = \xi^i \in \mathbf{K}^n$ we obtain a matrix representation of a corresponding element $\mathfrak{G}_\xi$ of the Lie-group $\mathfrak{G}$ by

$$(\mathfrak{G}_\xi)^a{}_b = < t^a{}_b, e^{i\sum_i \xi^i \cdot g_i} >$$

As we know from above, $U(\mathfrak{g})$ enhances to a Hopf-algebra with

$$\Delta(g_i) = g_i \otimes \mathbf{1} + \mathbf{1} \otimes g_i \;,\;\; \epsilon(g_i) = 0 \;,\;\; \mathrm{S}(g_i) = -g_i$$

such that $\mathfrak{X}$ becomes a *left $U(\mathfrak{g})$-module algebra.* As we know from lemma (2.2.10), we thus make $\mathfrak{X}$ a *right $\mathcal{F}(\mathfrak{G})$-comodule algebra* by means of relation (2.40). We use this relation to enhance $\mathcal{F}(\mathfrak{G})$ to a Hopf-algebra. We first define the coproduct and counit on $\mathcal{F}(\mathfrak{G})$ to be

$$\Delta(t^a{}_b) = t^a{}_c \otimes t^c{}_b\,, \quad \epsilon(t^a{}_b) = t^a{}_b(\mathbf{1}) = \delta^a{}_b. \tag{2.41}$$

In contrast to the coproduct of an element $g_i \in \mathfrak{g}$ of the Lie-algebra that is said to be of *primitive* type, (2.41) is called a *group-like* coproduct. Since the multiplication of matrices is associative we thus have ensured for coassociativity:

$$\begin{aligned}(\Delta \otimes \mathrm{id}) \circ \Delta(t^a{}_b) &= (t^a{}_c \otimes t^c{}_d) \otimes t^d{}_b \\ &= t^a{}_c \otimes (t^c{}_d \otimes t^d{}_b) = (\mathrm{id} \otimes \Delta) \circ \Delta(t^a{}_b).\end{aligned}$$

It is also no obstacle that the counit axiom is satisfied as well, since

$$(\epsilon \otimes \mathrm{id}) \circ \Delta(t^a{}_b) = \delta^a{}_c \otimes t^c{}_b \equiv t^a{}_b \equiv t^a{}_c \otimes \delta^c{}_b = (\mathrm{id} \otimes \epsilon) \circ \Delta(t^a{}_b).$$

Since we consider invertible matrices, there also exists an antipode, given by

$$\mathrm{S}(t^a{}_b)(\mathfrak{G}_\xi) = (\mathfrak{G}_\xi^{-1})^a{}_b =: (t^{-1})^a{}_b(\mathfrak{G}_\xi). \tag{2.42}$$

Here we see that the construction can only be performed on the Lie-group $\mathfrak{G}$ and not on its Lie-algebra since the generators $(g_i)_{i\in 1\ldots n}$ do not have an inverse. Introducing the unit element $\eta(g_i) = \mathbf{1}$ we check for the antipode axiom

$$\begin{aligned}\mu \circ (\mathrm{S} \otimes \mathrm{id}) \circ \Delta(t^a{}_b) &= (t^{-1})^a{}_c \cdot t^c{}_b = \delta^a{}_b \cdot \mathbf{1} \\ &= t^a{}_c \cdot (t^{-1})^c{}_b = \mu \circ (\mathrm{id} \otimes \mathrm{S}) \circ \Delta(t^a{}_b).\end{aligned}$$

With this little preparation we can now deduce the right coaction from (2.40), thus with the right coaction $\omega(x_a) = x_b \otimes t^b{}_a$ we induce from the left action

$$\begin{aligned}(g_i \cdot g_j) \triangleright x_a &= \sum_b x_b \cdot < t^b{}_a, g_i \cdot g_j > \\ &= \sum_b x_b \cdot < \Delta(t^b{}_a), g_i \otimes g_j > \\ &= \sum_c (\sum_b x_b \cdot < t^b{}_c, g_i >) \cdot < t^c{}_a, g_j > \\ &= g_j \triangleright (g_i \triangleright x_a).\end{aligned}$$

We thus obtain that

$$(\omega \otimes \mathrm{id}) \circ \omega(x_a) = (x_c \otimes t^c{}_b) \otimes t^b{}_a = x_c \otimes (t^c{}_b \otimes t^b{}_a) = (\mathrm{id} \otimes \Delta) \circ \omega(x_a)$$

is satisfied. Moreover we find that

$$x_a = \mathbf{1} \triangleright x_a = \sum_b x_b \cdot < t^b{}_a, \mathbf{1} >= \sum_b x_b \cdot \epsilon(t^b{}_a) = \sum_b x_b \cdot \delta^b{}_a,$$

such that with

$$\mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \omega,$$

we obtained a valid right coaction of $\mathcal{F}(\mathfrak{G})$ on the vector space sector of $\mathfrak{X}$. In order to incorporate the algebraic sector of $\mathfrak{X}$ into the coaction of $\mathcal{F}(\mathfrak{G})$ we consider the given relations

$$\begin{aligned} g_i \triangleright (x_a \cdot x_b) &= \sum (g_{i\,(1)} \triangleright x_a) \cdot (g_{i\,(2)} \triangleright x_b) \\ &= (g_i \triangleright x_a) \cdot x_b + x_a \cdot (g_i \triangleright x_a) \\ g_i \triangleright \mathbf{1} &= \epsilon(g_i) = 0. \end{aligned}$$

These relations then translate via (2.40) to

$$\begin{aligned} &\sum_c \sum_d (x_c \cdot x_d) \cdot < t^c{}_a \cdot t^d{}_b, g_i > \\ &\quad = \sum_c \sum_d (x_c \cdot x_d) \cdot < t^c{}_a \cdot t^d{}_b, \Delta(g_i) > \\ &\quad = \sum (\sum_c x_c < t^c{}_a, g_{i\,(1)} >) \cdot (\sum_d x_d < t^d{}_b, g_{i\,(2)} >) \\ &\mathbf{1} \cdot < \mathbf{1}, g_i >= \mathbf{1} \cdot \epsilon(g_i). \end{aligned}$$

We obtain the required property that

$$\begin{aligned} \omega(x_a \cdot x_b) &= (x_c \cdot x_d) \otimes (t^c{}_a \cdot t^d{}_b) = (x_c \otimes t^c{}_a) \cdot (x_d \otimes t^d{}_b) \\ &= \omega(x_a) \cdot \omega(x_b) \\ \omega(\mathbf{1}) &= \mathbf{1} \otimes \mathbf{1}. \end{aligned}$$

And hence we thus obtained the dual formulation of the left action of $U(\mathfrak{g})$. We emphasize that the elements $t^a{}_b \in \mathcal{F}(\mathfrak{G})$ are nothing but functions that we found to be corepresented on $\mathfrak{X}$. It is the algebra of functions itself and not the actual matrix-entries that is corepresented. One might ask why we need this dual construction at all. Why do we have to consider the functions $t^a{}_b \in \mathcal{F}(\mathfrak{G})$ at all, since we would also be satisfied to stick to the well-known conventional matrix representation. The solution to this secret comes together with quantization in the next chapter.

2.3 Quantization

As seen in the first section of this chapter, a quantization cannot be performed in terms of an algebra homomorphism, but by deforming the product by the use of a bilinear operator. We also understood that the structure of an algebra is not enough to cover the implications of a deformation. We thus introduced the most basic ingredients of Hopf-algebras in the last section that are required to perform deformations and to discuss their implications for corresponding representations of the deformed Hopf-algebras. In the present section we finally come to quantization and thus introduce the formalism, required for physically most relevant examples of $U(\mathfrak{g})$ and $\mathcal{F}(\mathfrak{G})$. There are two basic kinds of bilinear operators that rule the deformation of a Hopf-algebra, that are moreover closely related to another. These are the *quasitriangular structure* $\mathcal{R} \in \mathcal{H} \otimes \mathcal{H}$ and the *twist* $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$ that had been introduced by Drinfeld. In the context of these two objects, we do not consider the deformation of the multiplication but that of the coproduct. By conjugation with these operators we obtain, what is called an *almost cocommutative* Hopf-algebra. In the case of the twist $\mathcal{F}$ we not only modify the coproduct but also the antipode by a conjugation with suiting objects that depend on the specific twist used. It is easy to verify for the undeformed case that with $\mathcal{H}$ being a Hopf-algebra also $\mathcal{H}^{\text{cop}}$ with the modified coproduct $\Delta^{\text{cop}} = \sigma \circ \Delta$ is a Hopf-algebra as well. We recall that cocommutativity of a Hopf-algebra $\mathcal{H}$ is given by the property that $\Delta = \Delta^{\text{cop}}$. Thus in order to obtain an *almost commutative* Hopf-algebra we have to relax that property in a suitable way. The principle that guides us through these constructions is the requirement that $\mathcal{H}^{\text{cop}}$ has to remain a Hopf-algebra. In order to deform the multiplication rule of a Hopf-algebra we have to consider the *dual quasitriangular structure* $\mathbf{R}$ that is the dual object of $\mathcal{R}$. We recall that the dual of a coalgebra is an algebra and, under suiting circumstances, the dual of an algebra becomes a coalgebra. In this sense the dual of $\mathcal{H}^{\text{cop}}$ is the Hopf-algebra $(\mathcal{H}^*)^{\text{op}}$ with the opposit product $\mu^{op} = \mu \circ \sigma$. Thus if we relax cocommutativity, we also relax commutativity on the dual, that is ruled by $\mathbf{R}$. This also is one reason why we begin by relaxing cocommutativity, because we can also consider its dual under any circumstances. In the context of our guiding examples we already mentioned that quantizations of universal enveloping algebras $U(\mathfrak{g})$ reduce to deformations of their coproduct. Thus in the dual picture we obtain a noncommutative version of $\mathcal{F}(\mathfrak{G})$. For physical applications we are mostly interested in noncommutative representation spaces. Since these shall continue to be *left-module algebras* of deformations of $U(\mathfrak{g})$, the quasitriangular structure $\mathcal{R}$ also rules the deformation and thus the noncommutativity of the representation space. So does the twist $\mathcal{F}$ that moreover provides a starproduct as we got to know it in the first section. As we saw in the last section, left-module algebras are right-comodule algebras of the dual. Thus we also obtain the noncommutativity of the representation space in terms of the dual quasitriangular structure $\mathbf{R}$. Actually, in order to consider deformations of representations on has to perform this in terms of *quasi-tensor categrories*. We are not doing this here since the basic principles can also be understood without this machinary.

2.3.1 Quasitriangular Hopf Algebras and their Duals

We begin with the simple remark that for $\mathcal{H}$ being a Hopf-algebra with coproduct Δ implies that the opposit object $\mathcal{H}^{\mathrm{cop}}$ with coproduct $\Delta^{\mathrm{cop}} = \sigma \circ \Delta$ is a Hopf-algebra too, thus

$$\mathcal{H}: \text{ Hopf-algebra } \Longrightarrow \mathcal{H}^{\mathrm{cop}}: \text{ Hopf-algebra}$$

This is very easily verified by checking the Hopf-algebra axioms and the homomorphy property of the modified coproduct. In order to obtain an *almost cocommutative* Hopf algebra $\mathcal{H}$ with coproduct $\Delta_{\mathcal{R}}$, we set the *opposite* coproduct $\Delta^{\mathrm{cop}}_{\mathcal{R}}(h) = \sigma \circ \Delta(h)$ of $h \in \mathcal{H}$ to be cocommutative *up to* conjugation by an invertible element $\mathcal{R} \in \mathcal{H} \otimes \mathcal{H}$ such that

$$\begin{aligned} \Delta_{\mathcal{R}}(h) &= \Delta(h) \\ \Delta^{\mathrm{cop}}_{\mathcal{R}}(h) &= \sigma \circ \Delta_{\mathcal{R}}(h) = \mathcal{R} \cdot \Delta(h)\mathcal{R}^{-1}. \end{aligned} \tag{2.43}$$

In order to ensure that $\mathcal{H}_{\mathcal{R}}$ endowed with the almost cocommutative coproduct $\Delta_{\mathcal{R}}$ remains to be a Hopf-algebra we invert the above statement to

$$\mathcal{H}^{\mathrm{cop}}_{\mathcal{R}}: \text{ not a Hopf-algebra } \Longrightarrow \mathcal{H}_{\mathcal{R}}: \text{ not a Hopf-algebra}$$

and as a consequence we have to require that $\mathcal{H}^{\mathrm{cop}}_{\mathcal{R}} = (\mathcal{H}_{\mathcal{R}}, \mu, \eta, \Delta^{\mathrm{cop}}_{\mathcal{R}}, \epsilon; \mathbf{K})$ becomes a Hopf-algebra. Of course this is only a necessary and not a sufficient argument, but it is all we have to require for our purpose. Thus coassociativity

$$(\Delta^{\mathrm{cop}}_{\mathcal{R}} \otimes \mathrm{id}) \circ \Delta^{\mathrm{cop}}_{\mathcal{R}} = (\mathrm{id} \otimes \Delta^{\mathrm{cop}}_{\mathcal{R}}) \circ \Delta^{\mathrm{cop}}_{\mathcal{R}}$$

leads to the condition

$$\begin{aligned} \mathcal{R}_{12} \cdot (\Delta \otimes \mathrm{id})(\mathcal{R}) &= \mathcal{R}_{23} \cdot (\mathrm{id} \otimes \Delta)(\mathcal{R}) \\ (\Delta \otimes \mathrm{id})(\mathcal{R}^{-1}) \cdot \mathcal{R}^{-1}_{12} &= (\mathrm{id} \otimes \Delta)(\mathcal{R}^{-1}) \cdot \mathcal{R}^{-1}_{23} \end{aligned} \tag{2.44}$$

on the quasitriangular structure $\mathcal{R}$. With $\mathcal{R} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$ the structures $\mathcal{R}_{12}$ and $\mathcal{R}_{23}$ are given by

$$\begin{aligned} \mathcal{R}_{12} &= \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \otimes \mathrm{id} \\ \mathcal{R}_{23} &= \sum \mathrm{id} \otimes \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}. \end{aligned}$$

In a similar way we obtain from the counit axiom

$$(\epsilon \otimes \mathrm{id}) \circ \Delta_{\mathcal{R}}^{\mathrm{cop}} = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta_{\mathcal{R}}^{\mathrm{cop}}$$

additional conditions on the quasitriangular structure $\mathcal{R}$ to be

$$(\epsilon \otimes \mathrm{id})(\mathcal{R}) = (\epsilon \otimes \mathrm{id})(\mathcal{R}^{-1}) = 1 = (\mathrm{id} \otimes \epsilon)(\mathcal{R}) = (\mathrm{id} \otimes \epsilon)(\mathcal{R}^{-1}). \tag{2.45}$$

In contrast to this, the antipode axiom

$$\mu \circ (\mathrm{S} \otimes \mathrm{id}) \circ \Delta_{\mathcal{R}}^{\mathrm{cop}} = \eta \circ \epsilon = \mu \circ (\mathrm{id} \otimes \mathrm{S}) \circ \Delta_{\mathcal{R}}^{\mathrm{cop}}$$

does not provide any further requirement for $\mathcal{R}$ since

$$\begin{aligned} \mu \circ (\mathrm{S} \otimes \mathrm{id}) \circ \Delta_{\mathcal{R}}^{\mathrm{cop}}(h) &= \mu \circ (\mathrm{S} \otimes \mathrm{id})(\mathcal{R}\Delta(h)\mathcal{R}^{-1}) \\ &= \mathrm{S}(\mathcal{R}^{(1)} \cdot h_{(1)} \cdot \mathcal{R}^{-1\,(1)}) \cdot \mathcal{R}^{(2)} \cdot h_{(2)} \cdot \mathcal{R}^{-1\,(2)} \\ &= \mathrm{S}(h_{(1)} \cdot \mathcal{R}^{(1)} \cdot \mathcal{R}^{-1\,(1)}) \cdot h_{(2)} \cdot \mathcal{R}^{(2)} \cdot \mathcal{R}^{-1\,(2)} \\ &= \mathrm{S}(\mathcal{R}^{(1)} \cdot \mathcal{R}^{-1\,(1)}) \cdot \mathrm{S}(h_{(1)}) \cdot h_{(2)} \cdot \mathcal{R}^{(2)} \cdot \mathcal{R}^{-1\,(2)} \\ &= \epsilon(h). \end{aligned}$$

And in the same way we derive the right hand side of the antipode axiom. Here we used the property of S to be an antialgebra homomorphism, i.e. $\mathrm{S}(a \cdot b) = \mathrm{S}(b) \cdot \mathrm{S}(a)$ and used relation (2.43) as $(\sigma \circ \Delta(h)) \cdot \mathcal{R} = \mathcal{R} \cdot \Delta(h)$. Also the homomorphy property does not add any new conditions on $\mathcal{R}$:

$$\begin{aligned} \Delta_{\mathcal{R}}^{\mathrm{cop}}(h \cdot k) &= \mathcal{R}\Delta(h \cdot k)\mathcal{R}^{-1} = \mathcal{R}\Delta(h) \cdot \Delta(k)\mathcal{R}^{-1} \\ &= \mathcal{R}\Delta(h)\mathcal{R}^{-1} \cdot \mathcal{R}\Delta(k)\mathcal{R}^{-1} = \Delta_{\mathcal{R}}^{\mathrm{cop}}(h) \cdot \Delta_{\mathcal{R}}^{\mathrm{cop}}(k) \end{aligned}$$

We thus could take (2.44) and (2.45) as the defining conditions for the quasitriangular structure. And indeed these are the defining relations for the twist $\mathcal{F}$ that was introduced by Drinfeld and we discuss in the next subsections. But the quasitriangular structure $\mathcal{R}$ we define in a slightly stronger way.

2.3.1 Definition (quasitriangular Structure) *Let $\mathcal{H}$ be a Hopf-algebra over the field $\mathbf{K}$ and $\mathcal{R} \in \mathcal{H} \otimes \mathcal{H}$ be an invertible element, then $\mathcal{R}$ is called* quasitriangular *if*

$$\begin{aligned} (\Delta \otimes id)(\mathcal{R}) &= \mathcal{R}_{13}\mathcal{R}_{23} \\ (id \otimes \Delta)(\mathcal{R}) &= \mathcal{R}_{13}\mathcal{R}_{12} \end{aligned} \tag{2.46}$$

and it is called triangular *if it is quasitriangular and additionally satisfies the relation*

$$\mathcal{R}_{21} = \mathcal{R}^{-1}. \tag{2.47}$$

A tuple $(\mathcal{H}, \mathcal{R})$ *of a Hopf-algebra and quasitriangular structure is called a* quasitriangular Hopf-algebra.

With this definition, coassociativity of $\Delta_{\mathcal{R}}$ becomes a property that does not depend on the former coproduct Δ anymore. In order to discuss this point we first present the following lemma.

2.3.2 LEMMA *Let* $(\mathcal{H}, \mathcal{R})$ *be a quasitriangular Hopf-algebra. Then the following relations hold:*

$$\begin{array}{rcl} \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} & = & \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \\ (\epsilon \otimes id)(\mathcal{R}) = 1 & , & (id \otimes \epsilon)(\mathcal{R}) = 1 \\ (S \otimes id)(\mathcal{R}) = \mathcal{R}^{-1} & , & (id \otimes S)(\mathcal{R}^{-1}) = \mathcal{R} \end{array}$$

We thus assert that a quasitriangular structure $\mathcal{R}$ with the property (2.43) and (2.46) satisfies the requirements (2.44) and (2.45), knowing that with $\mathcal{R}$ being a quasitriangular structure of $\mathcal{H}$ as defined above also $\mathcal{R}_{21}^{-1}$ is a quasitrianguar structure, as well as $\mathcal{R}_{21}$ and $\mathcal{R}^{-1}$ are for $\mathcal{H}^{\mathrm{op}}$ and $\mathcal{H}^{\mathrm{cop}}$ respectively. The first relation of lemma (2.3.2) is called the *quantum Yang-Baxter equation*, that is most often refered to by the mnemonic *QYBE* in the literature. If the QYBE is satisfied, our bilinear operator $\mathcal{R}$ also satisfies the coassociativity condition (2.44) independently of the coproduct Δ in $\mathcal{H}$ - it is merely a property of $\mathcal{R}$ itself. As mentioned above the coassociativity of the coproduct assures associativity of the product on the dual space and, depending on the sort of representation or corepresentation, as well assures the associativity or coassociativity of the deformed representation space. This provides the central meaning of the QYBE in the field of quantum groups. It is a matter on its own to study all the possible solutions of this equation. And there are of course nontrivial examples. As we announced already earlier in this section the quasitriangular structure $\mathcal{R}$ rules the deformation of the coproduct. We further introduce the *dual quasitriangular structure* $\mathbf{R}$ that in turn deformes the product on the dual. Since we are specifically interested in the duality of deformations of $U(\mathfrak{g})$ and $\mathcal{F}(\mathfrak{G})$ as finite dimensional representations, we are introducting the dual quasitriangular structure via the pairing between $\mathcal{H}$ and $\mathcal{H}^*$ that is already defined for the undeformed case. Again we introduce the guiding principle that will carry through the constructions. However we are putting these results to axioms such that they can be applied to arbitrary Hopf-algebras, not caring whether or not there exists a dual Hopf-algebra. This is important since, as stressed many times, an algebra does not in general have a dual coalgebra. Thus deforming the product of a Hopf-algebra by a dual quasitriangular structure $\mathbf{R}$ has to be independent of such considerations. But in the mean time has to be defined correctly for the case that there exists a dual Hopf-algebra. We begin with our guiding principle that, again, is closely related to the fact that $\mathcal{H}^{\mathrm{cop}}$ has to be kept a Hopf-algebra after deformation. We now dualise this condition and thus assume to have a Hopf-algebra $\mathcal{H}$ and an existing dual object $\mathcal{H}^*$ that is related via pairing. We alrady know that $\mathcal{H}^{\mathrm{cop}}$ is a Hopf-algebra. Then $\mathcal{H}^{*\,\mathrm{op}}$ with $\mu_{\mathrm{op}} = \sigma \circ \mu$ is a Hopf-algebra too, since for $\varphi, \psi \in \mathcal{H}^*$ and $h \in \mathcal{H}$ we obtain

$$\begin{aligned} < \varphi \cdot_{\mathrm{op}} \psi, \Delta^{\mathrm{cop}}(h) > &= < \psi \cdot \varphi, h_{(2)} \otimes h_{(1)} >=< \psi, h_{(2)} > \cdot < \varphi, h_{(1)} > \\ &= < \varphi, h_{(1)} > \cdot < \psi, h_{(2)} > \\ &= < \varphi \cdot \psi, \Delta(h) > . \end{aligned}$$

We now introduce the the dual quasitriangular structure $\mathbf{R}$ in such a way that this property is conserved for the case that we replace the coproduct by $\Delta_{\mathcal{R}}$. The quasitriangular structure $\mathcal{R}$ can be regarded as a map

$$\mathcal{R}: \ \mathbf{K} \rightarrow \mathcal{H} \otimes \mathcal{H}$$

We understand this as deformation parameter $h \in \mathbf{K}$ that is mapped in such a way to the object $\mathcal{R}$. The dual quasitriangular structure $\mathbf{R}$ in turn is a map given by

$$\mathbf{R}: \ \mathcal{H}^* \otimes \mathcal{H}^* \rightarrow \mathbf{K}.$$

We introduce the dual quasitriangular structure $\mathbf{R}$ for a given quasitriangular structure $\mathcal{R}$ by

$$\begin{aligned} \forall\, \varphi, \psi \in \mathcal{H}^*: \quad \mathbf{R}(\varphi \otimes \psi) &= < \varphi \otimes \psi, \mathcal{R} > \\ &= \sum < \varphi, \mathcal{R}^{(1)} > \cdot < \psi, \mathcal{R}^{(2)} > \\ \mathbf{R}^{-1}(\varphi \otimes \psi) &= < \varphi \otimes \psi, \mathcal{R}^{-1} > \\ &= \sum < \varphi, \mathcal{R}^{-1\,(1)} > \cdot < \psi, \mathcal{R}^{-1\,(2)} > \end{aligned}$$

Using these relations we find the actual meaning of $\mathbf{R}^{-1}$ in relation to $\mathbf{R}$. With $\Delta(\varphi) = \sum \varphi_{(1)} \otimes \varphi_{(2)}$ and $\Delta(\psi) = \sum \psi_{(1)} \otimes \psi_{(2)}$ we obtain

$$\begin{aligned} &\mathbf{R}^{-1}(\varphi_{(1)} \otimes \psi_{(1)}) \cdot \mathbf{R}(\varphi_{(2)} \otimes \psi_{(2)}) \\ &\quad =< \varphi_{(1)}, \mathcal{R}^{-1\,(1)} > \cdot < \psi_{(1)}, \mathcal{R}^{-1\,(2)} > \cdot < \varphi_{(2)}, \mathcal{R}^{(1)} > \cdot < \psi_{(2)}, \mathcal{R}^{(2)} > \\ &\quad =< \varphi_{(1)}, \mathcal{R}^{-1\,(1)} > \cdot < \varphi_{(2)}, \mathcal{R}^{(1)} > \cdot < \psi_{(1)}, \mathcal{R}^{-1\,(2)} > \cdot < \psi_{(2)}, \mathcal{R}^{(2)} > \\ &\quad =< \varphi, \mathcal{R}^{-1\,(1)} \cdot \mathcal{R}^{(1)} > \cdot < \psi, \mathcal{R}^{-1\,(2)} \cdot \mathcal{R}^{(2)} > \\ &\quad =< \varphi \otimes \psi, \mathcal{R}^{-1} \cdot \mathcal{R}^{(1)} > \\ &\quad = \epsilon(\varphi) \cdot \epsilon(\psi). \end{aligned} \tag{2.48}$$

For simplicity we omitted symbols of summation. This property is called the *convolution invertibility* of $\mathbf{R}$. In the same way we obtain

$$\mathbf{R}(\varphi_{(1)} \otimes \psi_{(1)}) \cdot \mathbf{R}^{-1}(\varphi_{(2)} \otimes \psi_{(2)}) = \epsilon(\varphi) \cdot \epsilon(\psi)$$

We can now derive the dualized relation of (2.43), we obtain

$$\begin{aligned} \varphi \cdot_{\mathbf{R}} \psi &= \mathbf{R}(\psi_{(1)} \otimes \varphi_{(1)}) \cdot \psi_{(2)} \cdot_{\mathbf{R}} \varphi_{(2)} \cdot \mathbf{R}^{-1}(\psi_{(3)} \otimes \varphi_{(3)}) \\ \psi \cdot_{\mathbf{R}} \varphi &= \mathbf{R}^{-1}(\varphi_{(1)} \otimes \psi_{(1)}) \cdot \varphi_{(2)} \cdot_{\mathbf{R}} \psi_{(2)} \cdot \mathbf{R}(\varphi_{(3)} \otimes \psi_{(3)}). \end{aligned}$$

With this little preparation we now find that our guiding principle is satisfied

$$\begin{aligned} &< \varphi \cdot_{\mathrm{op}_{\mathbf{R}}} \psi, \Delta_{\mathcal{R}}^{\mathrm{cop}}(h) > \\ &\qquad =< \mathbf{R}^{-1}(\psi_{(1)} \otimes \varphi_{(1)}) \cdot \psi_{(2)} \cdot \varphi_{(2)} \cdot \mathbf{R}(\psi_{(3)} \otimes \varphi_{(3)}), \mathcal{R}\Delta(h)\mathcal{R}^{-1} > \\ &\qquad = \mathbf{R}^{-1}(\psi_{(1)} \otimes \varphi_{(1)}) \cdot < \psi_{(2)} \otimes \varphi_{(2)}, \mathcal{R} > \cdot < \psi_{(3)} \otimes \varphi_{(3)}, \Delta(h) > \\ &\qquad\qquad \times < \psi_{(4)} \otimes \varphi_{(4)}, \mathcal{R}^{-1} > \cdot \mathbf{R}(\psi_{(5)} \otimes \varphi_{(5)}) \\ &=< \varphi \cdot \psi, \Delta(h) > . \end{aligned}$$

In a similar way we obtain the dualization of the defining relations (2.46) of the quasitriangular structure $\mathcal{R}$. We thus obtain

$$\begin{aligned} \mathbf{R}(\varphi \cdot \psi \otimes \chi) &= \sum \mathbf{R}(\varphi \otimes \chi_{(1)}) \cdot \mathbf{R}(\psi \otimes \chi_{(2)}) \\ \mathbf{R}(\varphi \otimes \psi \cdot \chi) &= \sum \mathbf{R}(\varphi_{(1)} \otimes \chi) \cdot \mathbf{R}(\varphi_{(2)} \otimes \psi). \end{aligned}$$

These relations make the dual quasitriangular structure $\mathbf{R}$ a *bialgebra bicharacter*. Before we continue dualizing also the results we obtained as a concequense of defining the quasitriangular struture $\mathcal{R}$, we now want to define the dual quasitriangular structure $\mathbf{R}$ independently of $\mathcal{R}$. This definition then provides the required framework to deform the product of any Hopf-algebra, whether the dual $\mathcal{H}^*$ exists or not, by conjugation with $\mathbf{R}$.

2.3.3 DEFINITION (DUAL QUASITRIANGULAR STRUCTURE) *Let there be a Hopf-algebra $\mathcal{H}$ over the field $\mathbf{K}$ and $\mathbf{R}$ be a map $\mathcal{H} \otimes \mathcal{H} \to \mathbf{K}$. Let $\mathbf{R}$ be* convolution invertible, *i.e for $h, k \in \mathcal{H}$*

$$\begin{aligned} \sum \mathbf{R}^{-1}(h_{(1)} \otimes k_{(1)}) \cdot \mathbf{R}(h_{(2)} \otimes k_{(2)}) &= \epsilon(h) \cdot \epsilon(k) \\ &= \sum \mathbf{R}(h_{(1)} \otimes k_{(1)}) \mathbf{R}^{-1}(h_{(2)} \otimes k_{(2)}) \end{aligned}$$

and a bialgebra bicharacter, *i.e. $h, k, l \in \mathcal{H}$*

$$\begin{aligned} \mathbf{R}(h \cdot k \otimes l) &= \sum \mathbf{R}(h \otimes l_{(1)}) \cdot \mathbf{R}(k \otimes l_{(2)}) \\ \mathbf{R}(h \otimes k \cdot l) &= \sum \mathbf{R}(h_{(1)} \otimes l) \cdot \mathbf{R}(h_{(2)} \otimes k) \end{aligned}$$

with the property

$$\sum h_{(1)} \cdot k_{(1)} \cdot \mathbf{R}(h_{(2)} \otimes k_{(2)}) = \sum \mathbf{R}(k_{(1)} \otimes h_{(1)}) \cdot h_{(2)} \cdot k_{(2)}$$

then we call $\mathbf{R}$ *a* dual quasitriangular structure. *A tuple* $(\mathcal{H}, \mathbf{R})$ *is called a* dual quasitriangular Hopf-algebra.

This definition now implies the dual version of lemma (2.3.2)

2.3.4 LEMMA *If* $(\mathcal{H}, \mathbf{R})$ *is a dual quasitriangular Hopf-algebra then the following relations hold for* $h, k, l \in \mathcal{H}$

$$\begin{aligned} &\sum \mathbf{R}(h_{(1)} \otimes k_{(1)}) \cdot \mathbf{R}(h_{(2)} \otimes l_{(1)}) \cdot \mathbf{R}(k_{(2)} \otimes l_{(2)}) \\ &\quad = \sum \mathbf{R}(k_{(1)} \otimes l_{(1)}) \cdot \mathbf{R}(h_{(1)} \otimes l_{(2)}) \cdot \mathbf{R}(h_{(2)} \otimes k_{(2)}) \end{aligned}$$

$$\begin{aligned} \mathbf{R}(h \otimes \mathbf{1}) = \epsilon(h) \quad &, \quad \mathbf{R}(\mathbf{1} \otimes h) = \epsilon(h) \\ \mathbf{R}(S(h) \otimes k) = \mathbf{R}^{-1}(h \otimes k) \quad &, \quad \mathbf{R}^{-1}(h \otimes S(k)) = \mathbf{R}(h \otimes k) \end{aligned}$$

The first relation is thus a dual version of the quantum Yang-Baxter equation that again provides the required associativity of the products.

2.3.2 DEFORMATION OF $U(\mathfrak{g})$ AND $\mathcal{F}(\mathfrak{G})$ AND THEIR REPRESENTATIONS

The present subsection is probably the most central part of the whole introduction to the matter. All preparations we made so far are coming into account, when we now consider the quantization of $U(\mathfrak{g})$ and $\mathcal{F}(\mathfrak{G})$. In the last section we already mentioned that $U(\mathfrak{g})$ is usually represented on algebras that for simplicity will be thought of as algebras of coordinates $\mathfrak{X}$. We also already observed that the structure of the coproduct on $U(\mathfrak{g})$ is in direct correspondence with the product in $\mathfrak{X}$. In the dual formulation, concerning the corepresentation of $\mathcal{F}(\mathfrak{G})$ on $\mathfrak{X}$, the product of $\mathcal{F}(\mathfrak{G})$ of course carries the corresponding structure. Since our objective is very much motivated from physics, we are of course interested in noncommutative deformations of $\mathfrak{X}$ and moreover obtain a suitable deformation of $U(\mathfrak{g})$ or $\mathcal{F}(\mathfrak{G})$ such that covariant transformation is preserved. Since these notions are yet rather vaguely formulated, we first have to concretise our notions before we come to actual quantizations. To this purpose we first of all have to give a sharp definition of the algebra $\mathfrak{X}$ itself. Yet we have used this algebra only intuitively. On this footing we have to say what we understand to be a covariant transformation of $\mathfrak{X}$ as an algebra and as in the very first section of this introduction we discuss how $\mathfrak{X}$ is deformed in terms of a bilinear operator $\mathcal{R}$. We do this in such a way that $\mathcal{R}$ of course turns out to be the quasitriangular structure that deformes the coproduct on $U(\mathfrak{g})$. The algebra $\mathfrak{X}$ is defined similarly as $U(\mathfrak{g})$. We consider it to be generated by a d-dimensional basis $(x_a)_{a\in\{1,\dots d\}}$ that containes the coordinates of a corresponding $\mathbf{K}$-linear space $\mathfrak{V}$ in a specific coordinate system Σ. We thus consider its tensor algebra in Σ to be

$$T(\mathfrak{V}) = \mathbf{K} \oplus \mathfrak{V} \oplus \mathfrak{V} \otimes \mathfrak{V} \oplus \ldots \oplus \mathfrak{V}^{\otimes n} \oplus \ldots,$$

such that elements $\Phi(x_a) \in T(\mathfrak{V})$ can be considered as formal power series over $\mathbf{K}$ being

$$\Phi(x_a) = \sum_\lambda \Phi_\lambda \cdot \frac{x_1^{\lambda_1}}{\lambda_1!} \cdot \frac{x_2^{\lambda_2}}{\lambda_2!} \cdot \ldots \cdot \frac{x_d^{\lambda_d}}{\lambda_d!}.$$

We once more omitted the explicit notion of the tensor product and used $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{N}^d$. This algebra is too large for most applications. We have to devide it by an ideal $\mathcal{I}$, i.e. for the commutative case we can commute any product $x_a \cdot x_b \in T(\mathfrak{V})$ of a pair of any two coordinates according to

$$\forall\, a, b \in 1, \ldots, d \;:\; x_a \cdot x_b - x_b \cdot x_a = 0 \tag{2.49}$$

These are $d^2 - d$ relations that are said to be generating the ideal $\mathcal{I}$ as we explained it for the case of $U(\mathfrak{g})$ in the last section. We can thus identify power series $\Phi(x_a)$ for the case that there exists a $\varphi(x_c) \in T(\mathfrak{V})$ such that

$$\Phi(x_c) = \varphi(x_c) \cdot x_a \cdot x_b = \varphi(x_c) \cdot x_b \cdot x_a.$$

The algebra $\mathfrak{X}$ is then defined to be the quotient of $T(\mathfrak{V})$, devided by the two-sided ideal $\mathcal{I}$ that is generated by the $d^2 - d$ relations (2.49), i.e. we obtain

$$\mathfrak{X} = \frac{T(\mathfrak{V})}{\mathcal{I}}.$$

We now generalise this to the noncomutative case. We consider the noncommutativity to be ruled by a deformation parameter $h \in \mathbf{K}$. In order to define a general noncommutative algebra $\mathfrak{X}_h$ we first introduce the function

$$\omega_h : \mathfrak{X}_h \otimes \mathfrak{X}_h \to \mathfrak{X}_h \tag{2.50}$$

that can be considered as the right hand side of the commutator

$$\forall\, a, b \in 1, \ldots, d \;: [x_a, x_b] = x_a \cdot x_b - x_b \cdot x_a \equiv \omega_h(x_a, x_b). \tag{2.51}$$

As for the case of the commutative algebra $\mathfrak{X}$ we define $\mathfrak{X}_h$ to be the quotient

$$\mathfrak{X}_h = \frac{T(\mathfrak{V})}{\mathcal{I}_h},$$

where the ideal $\mathcal{I}_h$ according to (2.51) is generated by the $d^2 - d$ relations

$$\forall\, a, b \in 1, \ldots, d \;: x_a \cdot x_b - x_b \cdot x_a - \omega_h(x_a, x_b) = 0. \tag{2.52}$$

The choice of the map ω_h is of course restricted by the requirement that the resulting algebra is associative. Moreover it depends on ω_h whether the algebra $\mathfrak{X}_h$ possesses the Poincaré-Birkhoff-Witt property (2.1.3) that is indispensable for most applications - at least those of physical purpose. As for $U(\mathfrak{g})$, associativity is ensured by the fulfillment of the Jacobi-Identity that is evaluated by the use of (2.52), we thus obtain

$$[x_a, \omega_h(x_b, x_c)] + [x_b, \omega_h(x_c, x_a)] + [x_c, \omega_h(x_a, x_b)] = 0. \tag{2.53}$$

If we have a Lie-group $\mathfrak{G}$ that is represented on the vector space $\mathfrak{V}$, we have to clearify what we consider to be the covariance of $\mathfrak{X}_h$ under the action of $\mathfrak{G}$. This means that the algebra shall remain the same after the action of an element $(g_\xi)^a{}_b$ on $\mathfrak{X}_h$. Since the algebra is fullly characterized by the ideal $\mathcal{I}_h$, we thus have to require that the $d^2 - d$ relations (2.52), that generate $\mathcal{I}_h$ have to transform covariantly, i.e. with $x'_b = x_a \cdot (g_\xi)^a{}_b$, we require that

$$\begin{aligned} &x_a \cdot x_b - x_b \cdot x_a - \omega_h(x_a, x_b) = 0 \\ &\qquad\qquad \Longrightarrow \quad x'_a \cdot x'_b - x'_b \cdot x'_a - \omega_h(x'_a, x'_b) = 0. \end{aligned} \tag{2.54}$$

The crucial point here is that the function ω_h has of course not to be transformed itself - only its arguments are subjet to the action of $\mathfrak{G}$. Exactly this reason causes the product of $\mathcal{F}(\mathfrak{G})$ and the coproduct of $U(\mathfrak{g})$ to be deformed for the noncommutative cases that we consider in this section. For the commutative case however the undeformed product of $\mathcal{F}(\mathfrak{G})$ and undeformed coproduct of $U(\mathfrak{g})$ satisfy the requirements. For the functions $t^a{}_b$ we thus satisfy (2.54) by the use of $\omega(x_a) = x_b \otimes t^b{}_a$ and $\omega(x_a \cdot x_b) = x_c \cdot x_d \otimes t^c{}_a \cdot t^d{}_b$, i.e. we obtain

$$\begin{aligned} x_a \cdot x_b - x_b \cdot x_a = 0 \;\; \Longrightarrow \;\; & \omega(x_a \cdot x_b) - \omega(x_b \cdot x_a) \\ & = x_c \cdot x_d \otimes t^c{}_a \cdot t^d{}_b - x_d \cdot x_c \otimes t^d{}_b \cdot t^c{}_a \\ & = (x_c \cdot x_d - x_d \cdot x_c) \otimes t^c{}_a \cdot t^d{}_b \\ & = 0 \end{aligned}$$

since to product among $t^b{}_a \in \mathcal{F}(\mathfrak{G})$ is commutative. On the dual side, i.e. for $g_i \in U(\mathfrak{g})$ and $g_i \triangleright (x_a \cdot x_b) = \sum((g_{i\ (1)} \triangleright x_a) \cdot ((g_{i\ (2)} \triangleright x_a)$ we thus require

$$\begin{aligned} x_a \cdot x_b - x_b \cdot x_a = 0 \;\; \Longrightarrow \;\; & g_i \triangleright (x_a \cdot x_b - x_b \cdot x_a) \\ & = (g_i \triangleright x_a) \cdot x_b + x_a \cdot (g_i \triangleright x_b) \\ & \quad - (g_i \triangleright x_b) \cdot x_a - x_b \cdot (g_i \triangleright x_a) \\ & = 0, \end{aligned}$$

because of the commutativity of $\mathfrak{X}$. We now want to introduce noncommutativity to $\mathfrak{X}$ in a very specific way that deformes $\mathcal{F}(\mathfrak{G})$ and $U(\mathfrak{g})$ as dual quasitriangular and quasitriangular Hopf-algebras respectively. To that purpose we replace relations (2.52) by

$$\forall\, a, b \in 1, \ldots, d\ : x_a \cdot x_b - x_c \cdot x_d \cdot \mathbf{R}^{cd}{}_{ba} = 0 \tag{2.55}$$

where $\mathbf{R}^{cd}{}_{ba} \in \mathrm{Mat}(2d \times 2d; \mathbf{K})$ will turn out to be the famous $\mathbf{R}$-*Matrix* that represents the dual quasitriangular structure on the deformation of $\mathcal{F}(\mathfrak{G})$. We thus identify

$$\omega_h(x_a, x_b) = x_c \cdot x_d \cdot \mathbf{R}^{cd}{}_{ba} - x_b \cdot x_a \tag{2.56}$$

This represents a very specific choice for the functions $\omega_h(x_a, x_b)$ that are now restricted to be quadratic in the coordinates. Moreover we check for associativity using relation (2.53) and plugging in (2.56) such that we obtain

$$x_d \cdot x_e \cdot x_g(\mathbf{R}^{de}{}_{h_2h_1}\mathbf{R}^{h_2g}{}_{ch_3}\mathbf{R}^{h_1h_3}{}_{ba} - \mathbf{R}^{eg}{}_{h_1h_2}\mathbf{R}^{dh_2}{}_{h_3a}\mathbf{R}^{h_3h_1}{}_{cb}) = 0.$$

Thus satisfying associativity equally means that the matrix $\mathbf{R}^{cd}{}_{ba}$ has to be a solution of the *matrix quantum Yang-Baxter equation*

$$\mathbf{R}^{de}{}_{h_2h_1}\mathbf{R}^{h_2g}{}_{ch_3}\mathbf{R}^{h_1h_3}{}_{ba} = \mathbf{R}^{eg}{}_{h_1h_2}\mathbf{R}^{dh_2}{}_{h_3a}\mathbf{R}^{h_3h_1}{}_{cb}. \tag{2.57}$$

Thus before we continue to discuss covariance of the algebra $\mathfrak{X}_h$ that is generated by the relations (2.55) we introduce $\mathbf{R}^{cd}{}_{ba}$ as dual quasitriangular structure for $\mathcal{F}(\mathfrak{G})$ such that $\mathcal{F}_h(\mathfrak{G}) = (\mathcal{F}(\mathfrak{G}), \mathbf{R}^{cd}{}_{ba})$ becomes a dual quasitriangular Hopf-algebra. We introduce the $\mathbf{R}$-matrix by

$$\mathbf{R}^{cd}{}_{ba} = \mathbf{R}(t^c{}_b \otimes t^d{}_a) = < t^c{}_b \otimes t^d{}_a, \mathcal{R} > . \tag{2.58}$$

With $\mathbf{R}^{-1}{}^{cd}{}_{ba} = \mathbf{R}^{-1}(t^c{}_b \otimes t^d{}_a)$ we satisfy the conditions for the dual quasitriangular structure given by definition (2.3.3), by the use of the coproduct and the counit (2.41) in $\mathcal{F}(\mathfrak{G})$. We can now shortly return to our example of $U_q(sl_2)$ of the first section and consider the dual of the quasitriangular structure (2.16). For the specific case of $j = \frac{1}{2}$, we found the representation of generators $\tau_\pm, q^{\pm\frac{\tau_3}{2}}$ on states $|\pm\frac{1}{2}>$ in (2.17). We thus have a matrix representation of our generators given by

$$t^a{}_b(\tau_3) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \ t^a{}_b(\tau_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \ t^a{}_b(\tau_-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Inserting this into the expression for the quasitriangular structure $\mathcal{R}$ of (2.16), we obtain the $\mathbf{R}$-matrix of the dual quantum matrix group $\mathrm{SL}_q(2)$ of $U_q(sl_2)$

according to (2.58)

$$\mathbf{R} = q^{-\frac{1}{2}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

The RTT-relations

$$\mathbf{R}^{ab}{}_{cd}\, t^c{}_f\, t^d{}_g = t^a{}_c\, t^b{}_d\, \mathbf{R}^{cd}{}_{fg}$$

then specifically determine the deformation of the multiplication of quantum group elements. In particular we obtain the deformation of the product of $\mathrm{SL}_q(2)$. For given $\Omega \in \mathrm{SL}_q(2)$, we then obtain the follwing commutation relations for the matrix entries

$$\begin{aligned}
&\Omega_{11}\Omega_{21} = q^{-1}\Omega_{21}\Omega_{11}, \;\; \Omega_{12}\Omega_{22} = q^{-1}\Omega_{22}\Omega_{12}, \\
&\Omega_{11}\Omega_{12} = q^{-1}\Omega_{12}\Omega_{11}, \;\; \Omega_{21}\Omega_{22} = q^{-1}\Omega_{22}\Omega_{21}, \\
&\Omega_{12}\Omega_{21} = \Omega_{21}\Omega_{12}, \;\; \Omega_{11}\Omega_{22} - \Omega_{22}\Omega_{11} = (q^{-1} - q)\Omega_{12}\Omega_{21}
\end{aligned}$$

Note that noncommutativity of the classical group $\mathrm{SL}_q(2)$ comes from the matrix multiplication - here we have an additional noncommutativity of the matrix entries that thus are no elements of the field $\mathbf{K}$ anymore. This new noncommutativity is independent of the group properties of $\mathrm{SL}_q(2)$. As a closing remark to this section and maybe for the whole introduction to quantum groups, we would like to emphasize, that the deformation of objects $U(\mathfrak{g})$ and $\mathcal{F}(\mathfrak{G})$ as well as their representation on a deformed version of $\mathfrak{X}$ certainly is the most interesting example for physicists. Moreover we see that as long the representation in the undeformed case already exists, the deformation is nothing but an algebraic manipulation. In general this discussion is performed in terms of *quasi-tensor categories*. This is an exciting topic, since all algebraic manipulations we considered so far can be understood as morphisms within these categories, whose objects are the tensor products of the various vector spaces we considered. Deformation then becomes a functor from one category to another. This is an elegant and efficient rigging if one wants to consider quantum groups and their representations for the most general case. Nevertheless this is a time consuming issue, that phyisicist barely require for their everyday work. And we saw that the matter actually comming into account can also be understood without these constructions.

2.3.3 Drinfeld-Twist and quasitriangular Structure

This subsection briefly discusses the *twist* $\mathcal{F}$ as another structure that is closely related to the quasitriangular structure $\mathcal{R}$, discussed in the last two sections. If we have a Hopf-algebra $\mathcal{H}$, we again can relax cocommutativity of the coproduct Δ by an alternative way. While we used $\mathcal{R}$ to explicitly represent the noncocommutativity by relation (2.43), i.e. for $h \in \mathcal{H}$ where $(\mathcal{H}, \mathcal{R})$ is a quasitriangular Hopf-algebra we have

$$\sigma \circ \Delta_{\mathcal{R}}(h) = \mathcal{R}\Delta_{\mathcal{R}}(h)\mathcal{R}^{-1}.$$

The twist as well is an element $\mathcal{F} \in \mathcal{H}\otimes\mathcal{H}$ but we now conjugate the coproduct in a similar way that does not immediately exhibit the noncocommutativity - we merely set

$$\Delta_{\mathcal{F}}(h) = \mathcal{F}\Delta(h)\mathcal{F}^{-1}.$$

Moreover the former Hopf-algebra $\mathcal{H}$ has not necessarily to be cocommutative itself. Again we have to ensure that the coproduct twisted in such a way is still the coproduct of a twisted Hopf-algebra $\mathcal{H}_{\mathcal{F}}$. This results in conditions on $\mathcal{F}$ that are similar to that of $\mathcal{R}$. But in contrast to the quasitriangular Hopf-algebra $(\mathcal{H}, \mathcal{R})$, where only the coproduct gets modified, we also have to find a new antipode $\mathrm{S}_{\mathcal{F}}$ in order to satisfy the Hopf-algebra axioms for $\mathcal{H}_{\mathcal{F}}$. We show that $\mathrm{S}_{\mathcal{F}}$ can be derived from the undeformed antipode S and the twist $\mathcal{F}$. In contrast to $\mathcal{R}$, the twist has the advantage that it can be used without further modification as a starproduct for the deformed representation space. It thus provides a highly convenient tool for physical applications as we discussed them in the very first section.

2.3.5 Definition (Drinfeld-Twist) *Let $\mathcal{H}$ be a Hopf-algebra over the field $\mathbf{K}$. Then an invertible element $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$ is called* a twist *if it satisfies the relations*

$$\mathcal{F}_{12}(\Delta \otimes id)(\mathcal{F}) = \mathcal{F}_{23}(id \otimes \Delta)(\mathcal{F}) \tag{2.59}$$

and

$$(\epsilon \otimes id)(\mathcal{F}) = \mathbf{1} = (id \otimes \epsilon)(\mathcal{F}). \tag{2.60}$$

We directly present the lemma that provides us with the required conditions in order to make the twist of a Hopf-algebra $\mathcal{H}_{\mathcal{F}}$ into a Hopf-algebra itself.

2.3.6 Lemma *Let $(\mathcal{H}, \mu, \eta, \Delta, \epsilon, S;\ \mathbf{K})$ be a Hopf-algebra. We define $v = \mu\circ(id\otimes S)(\mathcal{F})$ with inverse $v^{-1} = \mu\circ(S\otimes id)(\mathcal{F}^{-1})$. Then $(\mathcal{H}, \mu, \eta, \Delta_{\mathcal{F}}, \epsilon, S_{\mathcal{F}};\ \mathbf{K})$ with*

$$\Delta_{\mathcal{F}}(h) = \mathcal{F}\Delta(h)\mathcal{F}^{-1}\ ,\ S_{\mathcal{F}}(h) = v \cdot S(h) \cdot v^{-1} \tag{2.61}$$

is a Hopf-algebra and is called the twist *of* $\mathcal{H}$ *by* $\mathcal{F}$*.*

The proof is a direct computation and argumentation that is very similar to that of $\mathcal{H}_{\mathcal{R}}^{\text{cop}}$. The conditions (2.59) and (2.60) defining the twist are very similar to those we first encountert for $\mathcal{R}$ to ensure coassociativity and the counit axiom for $\mathcal{H}_{\mathcal{R}}^{\text{cop}}$. We recall that the were no further conditions arising from the antipode axiom and the homomorphy property. Indeed if we withdraw condition (2.59), the Hopf-algebra $\mathcal{H}_{\mathcal{F}}$ lacks coassociativity and is called a *quasi Hopf-algebra*. In contrast to $\mathcal{H}_{\mathcal{R}}^{\text{cop}}$ we now need a modification of S to satisfy the antipode axiom. Thus there are many parallels to quasitrangular Hopf-algebras. In fact if $\mathcal{H}$ is cocommutative then its twist turns out to be triangular, i.e. we have as an additional property of the quasitriangular structure that $\mathcal{R}_{21} = \mathcal{R}^{-1}$.

2.3.7 LEMMA *Let* $\mathcal{H}$ *be a cocommutative Hopf-algebra over the field* $\mathbf{K}$ *and* $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$ *a twist according to definition (2.3.5) then the twist* $\mathcal{H}_{\mathcal{F}}$ *of* $\mathcal{H}$ *is triangular with*

$$\mathcal{R} = \mathcal{F}_{21} \cdot \mathcal{F}^{-1}.$$

Moreover, as mentioned above, we can twist also Hopf-algebras that are non-cocommutative, thus in particular we can apply the twisting procedure to quasitriangular Hopf-algebras $(\mathcal{H}, \mathcal{R})$. In particular we obtain the following lemma in this concern.

2.3.8 LEMMA *Let* $(\mathcal{H}, \mathcal{R})$ *be a quasitriangular Hopf-algebra and* $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$ *be a twist according to definition (2.3.5) that additionally satisfies two more conditions being*

$$\begin{aligned} \mathcal{F}_{21} &= \mathcal{F}^{-1} \\ \mathcal{F}_{12} \cdot \mathcal{F}_{13} \cdot \mathcal{F}_{23} &= \mathcal{F}_{23} \cdot \mathcal{F}_{13} \cdot \mathcal{F}_{12} \end{aligned}$$

then $(\mathcal{H}_{\mathcal{F}}, \mathcal{R}_{\mathcal{F}}$ *is a quasitriangular Hopf-algebra with quasitriangular structure*

$$\mathcal{R}_{\mathcal{F}} = \mathcal{F}^{-1} \mathcal{R} \mathcal{F}^{-1}.$$

We hereby close our brief introduction to basic principles of quantum groups.

3 Construction of θ-Poincaré Algebra and its Invariants on $\mathcal{M}_\theta$

In the present chapter we construct deformations of the Poincaré-algebra as representations on a noncommutative spacetime with canonical commutation relations. These deformations are obtained by solving a set of conditions by an appropriate ansatz for the deformed Lorentz generators. They turn out to be equivalent Hopf algebras of quantum universal enveloping algebra type with nontrivial antipodes. In order to present a notion of θ-deformed Minkowski space $\mathcal{M}_\theta$, we introduce Casimir operators and a spacetime invariant.

3.1 Introduction

In the traditional approach of quantum groups, the algebraic properties of quantum spaces are determined by the deformation applied to the symmetry algebra. Here we fairly follow the opposite way, since we rather push noncommutativity of the representation space to a deformation of the Poincaré-algebra. This procedure does not uniquely determine a single deformation, since there merely is a set of conditions to be solved that is necessary but not sufficient. This whole procedure applies, because the representation theory of the Poincaré-algebra is known for the commutative case. Deformation of the symmetry algebra is then fully decoupled from representation theoretic questions. However, one has to discuss the relation among the set of solutions, we obtained by solving the necessary conditions. In fact we find continously many solutions that all turn out to be equivalent Hopf algebras. Thus our solutions are equivalent to those found independently by alternative considerations as Wess et al. in the attempt to study concepts of derivatives on deformed coordinate spaces [86] and furthermore the solution obtained by the authors of [17] using a suitable Drinfeld-twist. The chapter is organised as follows. In the first section we collect all requirements that any deformation of a symmetry algebra to any given noncommutative space has to obey. These are expressed in terms of three conditions, that have to be solved. For the specific case of θ - Poincaré algebras we restrict ourselves to the case of quantizations that are linear in the deformation parameters. After that we show how nontrivial deformations arise by the choice of an appropriate generating ansatz for the deformed Lorentz operator. As already mentioned above, these solutions are then further discussed, concerning their equivalence. In the second part we finally work out explicit expressions for the Pauli-Lubanski vector and the configuration space invariant. Thus we obtain a notion of θ-Minkowski space $\mathcal{M}_\theta$. In general the field and quantum group theoretic aspects of our considerations orient themselves to the references [72], [84] and [19], [43]. We recommend the latter for a review of all mathematical aspects that are encountered throughout this chapter. Our considerations incorporate the following notations and conventions. We use latin and greek letters for indices of space and spacetime coordinates respectively

$$\begin{aligned} i, j, k, \ldots &\in \{1, 2, 3\} \\ \mu, \nu, \ldots &\in \{0, 1, 2, 3\}. \end{aligned}$$

The matter presented in this chapter is independent of any specific choice of the signature for the metric tensor $\eta^{\mu\nu}$ in commutative Minkowski space $\mathcal{M}$.

3.2 THE POINCARÉ ALGEBRA AND ITS θ - DEFORMATIONS $U_\theta^\lambda(\mathfrak{p})$

In this section we derive θ-deformations of the Poincaré algebra $\mathfrak{p}$ represented on noncommutative spacetime algebras $\mathfrak{X}_\theta$ with canonical commutation relations

$$[x^\mu, x^\nu] = i\theta^{\mu\nu},$$

where $\theta^{\mu\nu}$ is a constant real antisymmetric matrix. We find continuously many solutions of quantum universal enveloping algebra type, $U_\theta^\lambda(\mathfrak{p})$, that are parametrized by real parameters $\lambda = (\lambda_1, \lambda_2)$.

The section contains four parts. In the first subsection we collect a set of three conditions that any deformation of the type $U_\theta(\mathfrak{p})$ has to satisfy. In the second part we present an ansatz for the operators $M^{\mu\nu}$ of the deformed Lorentz algebra that generates the desired solutions $U_\theta^\lambda(\mathfrak{p})$. After that we explore the Hopf structure of $U_\theta^\lambda(\mathfrak{p})$, i.e. we give explicit formulas for counits, coproducts and antipodes for all solutions that are considered here and give the proof of

the Hopf algebra axioms. Finally in the fourth subsection we investigate, how the solutions derived by our method are related one to another.
In parallel, as mentioned in the introduction, we sketch a first scheme of a general method that shall provide the opportunity to derive deformations $U_h(\mathfrak{g})$ to any given noncommutative spacetime algebra $\mathfrak{X}_h$ with deformation parameter h. Hence the line of our arguments is drawn in terms of a general Lie algebra $\mathfrak{g}$ and we treat $U_\theta^\lambda(\mathfrak{p})$ on $\mathfrak{X}_\theta$ as an example. The reader may best familiarize with mathematical notions in this section by the use of the excellent textbooks [19], [43].
We emphasize that the development of this method is still a work in progress. Here we merely want to draw the outline of our idea and show that already at this stage it can be applied successfully, as the solutions that we present here, $U_\theta^{(\frac{1}{2},0)}(\mathfrak{p})$, were also achieved recently by alternative approaches [86], [17]. Thus, many aspects that touch on to the presented scheme will be treated independently in our subsequent work.

3.2.1 Conditions for Deformations $U_\theta^\lambda(\mathfrak{p})$ as Actions on $\mathfrak{X}_\theta$

Since deformations $U_\theta^\lambda(\mathfrak{p})$ of the Poincaré algebra $\mathfrak{p}$ are of quantum universal enveloping algebra type we first clarify such notions as that of $U_h(\mathfrak{g})$ and that of representations on a given spacetime algebra $\mathfrak{X}_h$ in terms of a general Lie algebra $\mathfrak{g}$.

3.2.1 Definition *A p-dimensional* Lie algebra *over the field* $\mathbf{K}$ *is a* $\mathbf{K}$*-linear vector space endowed with a map*

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

called bracket *with the following properties:*

$$\begin{aligned} \forall g, h, k \in \mathfrak{g} : \qquad & [g, h] = -[h, g] \ \textit{(Antisymmetry)} \\ & [g + h, k] = [g, k] + [h, k] \ \textit{(Linearity)} \\ & 0 = [[g, h], k] + [[h, k], g] + [[k, g], h] \ \textit{(Jacobi-Identity)} \end{aligned}$$

Linearity holds for both components.

Since it is a vector space, the Lie algebra $\mathfrak{g}$ has a p-dimensional basis $(g)_{i\in I}$ with $I = \{1 \dots p\}$. Hence the bracket $[\ ,\ ]$ can be expressed as a linear combination of the basis elements in terms of the Lie algebra's structure constant $c_{ij}^k \in \mathbf{K}$. For all $i, j, k \in I$ we have then

$$[g_i, g_j] = ic_{ij}^k g_k.$$

Since direct sums and tensor products of vector spaces are vector spaces themselves, to any Lie algebra $\mathfrak{g}$ there exists the tensor or free algebra $\mathcal{T}(\mathfrak{g})$

$$\mathcal{T}(\mathfrak{g}) = \mathbf{K} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \ldots \oplus \mathfrak{g}^{j\otimes} \oplus \ldots .$$

that leads us directly to the constructive definition of a universal enveloping algebra.

3.2.2 Definition *If $\mathfrak{g}$ is a Lie algebra with bracket $[\ ,\ ]$ then the* universal enveloping algebra $U(\mathfrak{g})$ *of $\mathfrak{g}$ is the tensor algebra $\mathcal{T}(\mathfrak{g})$ divided by the two-sided ideal $\mathcal{I}$*

$$U(\mathfrak{g}) = \frac{\mathcal{T}(\mathfrak{g})}{\mathcal{I}}$$

that is generated by the relations

$$g_i \otimes g_j - g_j \otimes g_i - ic_{ij}^k g_k = 0.$$

The deformation $U_h(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is performed by quantizing its universal enveloping algebra $U(\mathfrak{g})$. This is because the commutator bracket $[G_i \overset{\otimes}{,} G_j] = G_i \otimes G_j - G_j \otimes G_i$ for generators $(G_i)_{i\in I}$ of $U_h(\mathfrak{g})$ maps within $U_h(\mathfrak{g})$, i.e. the commutator in general is a linear combination in terms of the infinit dimensional basis of $U_h(\mathfrak{g})$ generated by $(G_i)_{i\in I}$. Thus $U_h(\mathfrak{g})$ becomes a Lie algebra with

$$\begin{array}{rcl} [\ ,\ ] : U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g}) & \longrightarrow & U_h(\mathfrak{g}) \\ {[G_i, G_j]} & \mapsto & iC_{ij}(G_k, h). \end{array} \tag{3.1}$$

In the further consideration, we omit the symbol of tensor multiplication. The quantum universal enveloping algebra is thus defined to be the free associative algebra generated by $(G_i)_{i\in I}$ that is divided by the ideal $\mathcal{I}_h$ generated by (3.1) such that for $h \to 0 :\ \mathcal{I}_h \to \mathcal{I}$ and consequently

$$U_h(\mathfrak{g}) \to U(\mathfrak{g}).$$

We adjourn the discussion concerning the potential change of the number of generators $(G_i)_{i\in I}$ under deformations and exclude such solutions for $U_h(\mathfrak{g})$.
To any quantum universal enveloping algebra $U_h(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$ there exists a $\mathbf{K}[\![h]\!]$-linear algebra-isomorphism to $U(\mathfrak{g})[\![h]\!]$. The latter denotes the algebra of power series in the deformation parameter h with coefficients in $U(\mathfrak{g})$.
Thus in particular any generator G_i can be mapped to $g_i + O(h) \in U(\mathfrak{g})[\![h]\!]$ such that the algebra structure map of $U_h(\mathfrak{g})$, generating the ideal $\mathcal{I}_h$, is conveyed to the structure map of $\mathfrak{g}$. In other words, in order to quantize a universal

enveloping algebra $U(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$ the algebra itself has not necessarily to be deformed. To be well defined, the multiplication in $U_h(\mathfrak{g})$ has to satisfy closure and associativity. This is the first condition expressed by the Jacobi-Identity for the functions $C_{ij}(G_k, h)$.

CONDITION 1

$$\begin{aligned} 0 &= \left[\left[G_i, G_j\right], G_k\right] + \left[\left[G_j, G_k\right], G_i\right] + \left[\left[G_k, G_i\right], G_j\right] \\ &= i\left[C_{ij}(G_l, h), G_k\right] + i\left[C_{jk}(G_l, h), G_i\right] + i\left[C_{ki}(G_l, h), G_j\right]. \end{aligned}$$

We now apply Condition 1 to the example of $U_\theta(\mathfrak{p})$. The commutation relations of the Poincaré algebra $\mathfrak{p}$ are given by

$$\begin{aligned} [p^\mu, p^\nu] &= 0 \\ [m^{\mu\nu}, p^\rho] &= i\eta^{\mu\rho}p^\nu - i\eta^{\nu\rho}p^\mu \\ [m^{\mu\nu}, m^{\rho\sigma}] &= i\eta^{\mu\rho}m^{\nu\sigma} - i\eta^{\nu\rho}m^{\mu\sigma} + i\eta^{\nu\sigma}m^{\mu\rho} - i\eta^{\mu\sigma}m^{\nu\rho}. \end{aligned} \tag{3.2}$$

For the case of canonical commutation relations the deformation is actually limited to the Lorentz algebra itself, such that the first relation of (3.2) is preserved. Since the Lorentz algebra is simple, the algebraic sector, as we just explained, has not necessarily to be deformed.
However, since we aim at a development of a general scheme of quantization from representations, we also include deformations of the algebra itself. In the final subsection of this chapter we are coming back to this issue and discuss how the various deformations, that are derived throughout this chapter, are related to one another.
As generators for $U_\theta(\mathfrak{p})$ we use momentum operators P^μ and Lorentz operators $M^{\mu\nu}$. We make the following ansatz for the commutation relations of the deformed Poincaré algebra

$$\begin{aligned} [P^\mu, P^\nu] &= 0 \\ [M^{\mu\nu}, P^\rho] &= i(\eta^{\mu\rho}P^\nu - \eta^{\nu\rho}P^\mu) + i\chi_\theta^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda}) \\ [M^{\mu\nu}, M^{\rho\sigma}] &= +i\eta^{\mu\rho}M^{\nu\sigma} - i\eta^{\nu\rho}M^{\mu\sigma} + i\eta^{\nu\sigma}M^{\mu\rho} - i\eta^{\mu\sigma}M^{\nu\rho} \\ &\quad +i\phi_\theta^{\mu\nu\rho\sigma}(P^\gamma, M^{\kappa\lambda}). \end{aligned} \tag{3.3}$$

The function $\phi_\theta^{\mu\nu\rho\sigma}(P^\gamma, M^{\kappa\lambda})$ is antisymmetric in the first and second pair of indices and has physical dimension 1. The function $\chi_\theta^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda})$ is antisymmetric in the first pair of indices and has physical dimension length^{-1}. Inserting the commutation relations (3.3) into the Jacobi-Identities corresponding to Condition 1

$$\begin{aligned} 0 &= \left[\left[P^\mu, P^\nu\right], M^{\rho\sigma}\right] + \left[\left[P^\nu, M^{\rho\sigma}\right], P^\mu\right] + \left[\left[M^{\rho\sigma}, P^\mu\right], P^\nu\right] \\ 0 &= \left[\left[M^{\mu\nu}, M^{\rho\sigma}\right], P^\lambda\right] + \left[\left[M^{\rho\sigma}, P^\lambda\right], M^{\mu\nu}\right] + \left[\left[P^\lambda, M^{\mu\nu}\right], M^{\rho\sigma}\right] \\ 0 &= \left[\left[M^{\mu\nu}, M^{\rho\sigma}\right], M^{\kappa\lambda}\right] + \left[\left[M^{\rho\sigma}, M^{\kappa\lambda}\right], M^{\mu\nu}\right] + \left[\left[M^{\kappa\lambda}, M^{\mu\nu}\right], M^{\rho\sigma}\right] \end{aligned}$$

gives the following relations for functions $\phi_\theta^{\mu\nu\rho\sigma}(P^\gamma, M^{\kappa\lambda})$ and $\chi_\theta^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda})$

$$\begin{aligned}
0 &= [P^\mu, \chi_\theta^{\rho\sigma\nu}] - [P^\nu, \chi_\theta^{\rho\sigma\mu}] \\
0 &= \left[P^\lambda, \phi_\theta^{\mu\nu\rho\sigma}\right] + [M^{\mu\nu}, \chi_\theta^{\rho\sigma\lambda}] - [M^{\rho\sigma}, \chi_\theta^{\mu\nu\lambda}] \\
0 &= i\left[\phi_\theta^{\mu\nu\rho\sigma}, M^{\kappa\lambda}\right] + i\left[\phi_\theta^{\rho\sigma\kappa\lambda}, M^{\mu\nu}\right] + i\left[\phi_\theta^{\kappa\lambda\mu\nu}, M^{\rho\sigma}\right] \\
&\quad + \eta^{\nu\rho}\phi_\theta^{\mu\sigma\kappa\lambda} - \eta^{\mu\rho}\phi_\theta^{\nu\sigma\kappa\lambda} + \eta^{\mu\sigma}\phi_\theta^{\nu\rho\kappa\lambda} - \eta^{\nu\sigma}\phi_\theta^{\mu\rho\kappa\lambda} \\
&\quad - \eta^{\sigma\kappa}\phi_\theta^{\mu\nu\rho\lambda} + \eta^{\rho\kappa}\phi_\theta^{\mu\nu\sigma\lambda} - \eta^{\rho\lambda}\phi_\theta^{\mu\nu\sigma\kappa} + \eta^{\sigma\lambda}\phi_\theta^{\mu\nu\rho\kappa} \\
&\quad + \eta^{\lambda\mu}\phi_\theta^{\kappa\nu\rho\sigma} - \eta^{\kappa\mu}\phi_\theta^{\lambda\nu\rho\sigma} + \eta^{\kappa\nu}\phi_\theta^{\lambda\mu\rho\sigma} - \eta^{\lambda\nu}\phi_\theta^{\kappa\mu\rho\sigma}.
\end{aligned} \tag{3.4}$$

Any pair of functions $\phi_\theta^{\mu\nu\rho\sigma}(P^\gamma, M^{\kappa\lambda})$ and $\chi_\theta^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda})$ solving these equations leads to well defined algebraic properties of $U_\theta(\mathfrak{p})$.
Since $U_\theta(\mathfrak{p})$ shall be represented on $\mathfrak{X}_\theta$, we now consider the action of $G_i \in U_h(\mathfrak{g})$ on coordinates $x^\mu \in \mathfrak{X}_h$. To this purpose the symmetry algebra has to be enhanced by a coalgebra structure.

3.2.3 Definition *The* coalgebra structure *on the* $\mathbf{K}$*-vector space* $U_h(\mathfrak{g})$ *is given by the two linear operations counit* ϵ *and coproduct* Δ*. These are the maps*

$$\begin{aligned}
\Delta : U_h(\mathfrak{g}) &\rightarrow U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g}) \\
G_i &\mapsto \Delta(G_i) = \sum G_{i(1)} \otimes G_{i(2)} \\
\epsilon : U_h(\mathfrak{g}) &\rightarrow K \\
G_i &\mapsto \epsilon(G_i)
\end{aligned}$$

that obey the two coalgebra axioms of counit and coassociativity

$$\begin{aligned}
(\epsilon \otimes id) \circ \Delta = \; & id \; = (id \otimes \epsilon) \circ \Delta \\
(\Delta \otimes id) \circ \Delta \; &= \; (id \otimes \Delta) \circ \Delta.
\end{aligned}$$

3.2.4 Definition *A* bialgebra *is* $\mathbf{K}$*-vector space with algebra and coalgebra structure made compatible by demanding that the coproduct and counit are algebra homomorphisms*

$$\begin{aligned}
\Delta(G_i G_j) = (\Delta G_i)(\Delta G_j), &\qquad \Delta\mathbf{1} = \mathbf{1} \otimes \mathbf{1} \\
\epsilon(G_i G_j) = \epsilon(G_i)\epsilon(G_j), &\qquad \epsilon(\mathbf{1}) = 1.
\end{aligned}$$

We now assume that $U_h(\mathfrak{g})$ is a bialgebra and represent it on $\mathfrak{X}_\mathfrak{h}$. Its associative multiplication for the coordinates $x^\mu, x^\nu \in \mathfrak{X}_h$ is given by the commutator

$$[x^\mu, x^\nu] = x^\mu x^\nu - x^\nu x^\mu = i\omega_h^{\mu\nu}(x^\rho).$$

Since the coordinates of $\mathfrak{X}_\mathfrak{h}$ are hermitean operators, the antisymmetric function $\omega_h^{\mu\nu}(x^\mu)$ is real valued.

3.2.5 DEFINITION *A* representation *is a pair $(\rho, \mathfrak{X}_h)$ containing a vector space $\mathfrak{X}_h$ and a homomorphism*

$$\begin{aligned} \rho:\ U_h(\mathfrak{g}) &\ \rightarrow\ gl(\mathfrak{X}_h) \\ G_i &\ \mapsto\ \rho(G_i), \end{aligned}$$

such that for $G_i, G_j \in U_h(\mathfrak{g})$ and $x^\mu, x^\nu \in \mathfrak{X}_h$

$$\rho(G_iG_j - G_jG_i - iC_{ij}(G_k, h))x^\mu = 0 \tag{3.5}$$

and

$$\rho(G_i)(x^\mu x^\nu - x^\nu x^\mu - i\omega_h^{\mu\nu}(x^\rho)) = 0 \tag{3.6}$$

are satisfied.

In other terms the algebraic structure of $U_h(\mathfrak{g})$ shall be represented in $gl(\mathfrak{X}_h)$ and these act as endomorphisms on $\mathfrak{X}_h$.
Introducing the left-action of $G_i \in U_h(\mathfrak{g})$ on coordinates $x^\mu \in \mathfrak{X}_h$ by

$$G_i \triangleright x^\mu = \rho(G_i)x^\mu,$$

products of coordinates are mapped according to

$$\begin{aligned} G_i \triangleright (x^\mu x^\nu) &= \sum m\left(\left(G_{i(1)} \triangleright x^\mu\right) \otimes \left(G_{i(2)} \triangleright x^\nu\right)\right) \\ G_i \triangleright \mathbf{1} &= \epsilon(G_i)\mathbf{1}. \end{aligned}$$

The multiplication m is that of the coordinate algebra $\mathfrak{X}_h$. In order to incorporate these properties of the representation, we define the action of $G_i \in U_h(\mathfrak{g})$ on the coordinates $x^\mu, x^\nu \in \mathfrak{X}_h$ with $\mathbf{1} \in \mathfrak{X}_h$ by

$$\rho\,(G_i)\,x^\mu = G_i \triangleright x^\mu := [G_i, x^\mu] \triangleright \mathbf{1}.$$

This way relation (3.5) now reads

$$\begin{aligned} (G_iG_j - G_jG_i - iC_{ij}(G_k, h)) \triangleright x^\mu &= 0 \\ \Leftrightarrow ([G_i,[G_j,x^\mu]] - [G_j,[G_i,x^\mu]] - i\,[C_{ij}(G_k,h), x^\mu]) \triangleright \mathbf{1} &= 0 \end{aligned}$$

and thus we obtain

CONDITION 2

$$[G_i,[G_j,x^\mu]] - [G_j,[G_i,x^\mu]] - i\,[C_{ij}(G_k,h), x^\mu] = 0.$$

Turning to relation (3.6) we compute

$$\begin{aligned} G_i \triangleright (x^\mu x^\nu - x^\nu x^\mu - i\omega_h^{\mu\nu}(x^\rho)) &= 0 \\ \Leftrightarrow ([[G_i,x^\mu]\,,x^\nu] - [[G_i,x^\nu]\,,x^\mu] - i\,[G_i, \omega_h^{\mu\nu}(x^\rho)]) \triangleright \mathbf{1} &= 0 \end{aligned} \tag{3.7}$$

and obtain

CONDITION 3

$$[[G_i, x^\mu], x^\nu] + [[x^\nu, G_i], x^\mu] + i\,[\omega_h^{\mu\nu}(x^\rho), G_i] = 0.$$

To read off the bialgebra structure of $U_h(\mathfrak{g})$ we assume that the coproduct is of the general form

$$\Delta(G_i) = G_i \otimes \mathbf{1} + \mathbf{1} \otimes G_i + \sum \xi_{i(1)} \otimes \xi_{i(2)}.$$

We apply this again to the relation (3.6)

$$\begin{aligned} 0 \;=\; & (G_i \triangleright x^\mu)x^\nu + x^\mu(G_i \triangleright x^\nu) + \sum(\xi_{i(1)} \triangleright x^\mu)(\xi_{i(2)} \triangleright x^\nu) \\ & -(G_i \triangleright x^\nu)x^\mu - x^\nu(G_i \triangleright x^\mu) - \sum(\xi_{i(1)} \triangleright x^\nu)(\xi_{i(2)} \triangleright x^\mu) \\ & -iG_i \triangleright \omega_h^{\mu\nu}(x^\rho) \end{aligned} \tag{3.8}$$

and compare with the computation (3.7) from above. We obtain for the coproduct

$$\begin{aligned} \sum(\xi_{i(1)} \triangleright x^\mu)(\xi_{i(2)} \triangleright x^\nu) &= ([G_i, x^\mu]\, x^\nu) \triangleright \mathbf{1} - ([G_i, x^\mu] \triangleright \mathbf{1})\, x^\nu \\ &= ([[G_i, x^\mu], x^\nu] + x^\nu\, [G_i, x^\mu]) \triangleright \mathbf{1} \\ &\quad - ([G_i, x^\mu] \triangleright \mathbf{1})\, x^\nu. \end{aligned} \tag{3.9}$$

This formula will be used in the next subsection to compute the coproduct for the deformed Lorentz generators $M^{\mu\nu}$. For instance we turn again to the example $U_\theta(\mathfrak{p})$. We make an ansatz for the commutator of $M^{\mu\nu}$ and coordinates $x^\rho \in \mathfrak{X}_\theta$. The corresponding relation for P^μ is of the classical form such that we have

$$\begin{aligned} [P^\mu, x^\rho] &= -i\eta^{\mu\rho} \\ [M^{\mu\nu}, x^\rho] &= i(x^\nu\eta^{\rho\mu} - x^\mu\eta^{\rho\nu}) + i\psi_\theta^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda}). \end{aligned} \tag{3.10}$$

The function $\psi_\theta^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda})$ has the physical dimension of length and is antisymmetric in the first two indices. Inserting this ansatz into Condition 2

$$\begin{aligned} 0 &= \left[[P^\mu, P^\nu], x^\lambda\right] + \left[\left[P^\nu, x^\lambda\right], P^\mu\right] + \left[\left[x^\lambda, P^\mu\right], P^\nu\right] \\ 0 &= \left[[M^{\mu\nu}, P^\rho], x^\lambda\right] + \left[\left[P^\rho, x^\lambda\right], M^{\mu\nu}\right] + \left[\left[x^\lambda, M^{\mu\nu}\right], P^\rho\right] \\ 0 &= \left[[M^{\mu\nu}, M^{\rho\sigma}], x^\lambda\right] + \left[\left[M^{\rho\sigma}, x^\lambda\right], M^{\mu\nu}\right] + \left[\left[x^\lambda, M^{\mu\nu}\right], M^{\rho\sigma}\right] \end{aligned}$$

and replacing the commutators $[P^\mu, P^\nu], [M^{\mu\nu}, P^\rho]$ and $[M^{\mu\nu}, M^{\rho\sigma}]$ by their right hand sides, we obtain

$$\begin{aligned} 0 &= [\psi_\theta^{\mu\nu\lambda}, P^\rho] - [\chi_\theta^{\mu\nu\rho}, x^\lambda] \\ 0 &= i[M^{\rho\sigma}, \psi_\theta^{\mu\nu\lambda}] - i[M^{\mu\nu}, \psi_\theta^{\rho\sigma\lambda}] + i[\phi_\theta^{\mu\nu\rho\sigma}, x^\lambda] \\ &\quad -\eta^{\mu\rho}\psi_\theta^{\nu\sigma\lambda} + \eta^{\nu\rho}\psi_\theta^{\mu\sigma\lambda} - \eta^{\nu\sigma}\psi_\theta^{\mu\rho\lambda} + \eta^{\mu\sigma}\psi_\theta^{\nu\rho\lambda} \\ &\quad -\eta^{\sigma\lambda}\psi_\theta^{\mu\nu\rho} + \eta^{\rho\lambda}\psi_\theta^{\mu\nu\sigma} - \eta^{\mu\lambda}\psi_\theta^{\rho\sigma\nu} + \eta^{\nu\lambda}\psi_\theta^{\rho\sigma\mu}. \end{aligned} \tag{3.11}$$

Turning finally to Condition 3

$$\begin{aligned} 0 &= \left[\left[P^\lambda, x^\mu\right], x^\nu\right] + \left[\left[x^\mu, x^\nu\right], P^\lambda\right] + \left[\left[x^\nu, P^\lambda\right], x^\mu\right] \\ 0 &= \left[\left[M^{\rho\sigma}, x^\mu\right], x^\nu\right] + \left[\left[x^\mu, x^\nu\right], M^{\rho\sigma}\right] + \left[\left[x^\nu, M^{\rho\sigma}\right], x^\mu\right] \end{aligned}$$

and replacing again by the corresponding right hand sides we obtain the single equation

$$0 = i\left[\psi_\theta^{\mu\nu\rho}, x^\sigma\right] - i\left[\psi_\theta^{\mu\nu\sigma}, x^\rho\right] - \eta^{\mu\rho}\theta^{\nu\sigma} + \eta^{\nu\rho}\theta^{\mu\sigma} + \eta^{\mu\sigma}\theta^{\nu\rho} - \eta^{\nu\sigma}\theta^{\mu\rho}. \tag{3.12}$$

This final relation shows that the ansatz for $\psi^{\mu\nu\rho}$ can never be chosen to be zero or constant and such the coproduct of $M^{\mu\nu}$ is necessarily deformed. In the classical limit $\theta^{\mu\nu} \to 0$ we have $\phi_\theta^{\mu\nu\rho\sigma}(P^\gamma, M^{\kappa\lambda}) \to 0$, $\psi_\theta^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda}) \to 0$ and $\chi_\theta^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda}) \to 0$.
Now we have obtained all conditions for $U_\theta^\lambda(\mathfrak{p})$ as a representation on $\mathfrak{X}_\theta$. In the next subsection we find solutions $U_\theta^\lambda(\mathfrak{p})$ by making an appropriate ansatz for the deformed Lorentz generator $M^{\mu\nu}$.

3.2.2 The Computation of Explicit Solutions $U_\theta^{(\lambda_1,\lambda_2)}(\mathfrak{p})$

In general the functions $\phi_\theta^{\mu\nu\rho\sigma}(P^\gamma, M^{\kappa\lambda})$, $\psi_\theta^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda})$ and $\chi_\theta^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda})$ can be considered as power series in $\theta^{\mu\nu}$, given by

$$\begin{aligned} \phi_\theta^{\mu\nu\rho\sigma}(P^\gamma, M^{\kappa\lambda}) &= \sum_{k=1}^{\infty} \phi_{\theta,k}^{\mu\nu\rho\sigma}(P^\gamma, M^{\kappa\lambda}) \\ \psi_\theta^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda}) &= \sum_{k=1}^{\infty} \psi_{\theta,k}^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda}) \\ \chi_\theta^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda}) &= \sum_{k=1}^{\infty} \chi_{\theta,k}^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda}). \end{aligned}$$

The index of summation k denotes the power of $\theta^{\mu\nu}$ in $\phi_{\theta,k}^{\mu\nu\rho\sigma}(P^\gamma, M^{\kappa\lambda})$ as well as in $\psi_{\theta,k}^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda})$ and $\chi_{\theta,k}^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda})$ respectively. We merely want to consider the most simple solutions to the set of equations (3.4), (3.11) and (3.12) and thus we restrict ourselves to the case that is linear in $\theta^{\mu\nu}$. If we account for the physical unities, we find that

$$\begin{aligned} \phi_\theta(P^\gamma, M^{\mu\nu}) = \phi_\theta(P^\gamma) &\sim \theta P P \\ \psi_\theta(P^\gamma, M^{\mu\nu}) = \psi_\theta(P^\gamma) &\sim \theta P \\ \chi_\theta(P^\gamma, M^{\mu\nu}) = \chi_\theta(P^\gamma) &\sim \theta P P P. \end{aligned}$$

Inserting this ansatz into the three conditions from the previous section generates the set of solutions in first order in θ. An alternative method that gives the same results is assuming the deformed Lorentz generator $M^{\mu\nu}$ to be of the following general form

$$M^{\mu\nu} = x^\mu P^\nu - x^\nu P^\mu + \Lambda^{\mu\nu}. \tag{3.13}$$

The function $\Lambda^{\mu\nu}$ has physical dimension 1 and is antisymmetric in its indices. It turns out in the next steps that any choice of $\Lambda^{\mu\nu}$ with these properties generates a valid solution of $U_\theta^\lambda(\mathfrak{p})$. We now express the functions $\psi_\theta^{\mu\nu\rho}(P^\gamma)$, $\phi_\theta^{\mu\nu\rho\sigma}(P^\gamma)$ and $\chi_\theta^{\mu\nu\rho}(P^\gamma)$ in terms of $\Lambda^{\mu\nu}$

$$\begin{aligned}
\chi_\theta^{\mu\nu\rho} &= -i[\Lambda^{\mu\nu}, P^\rho] \\
\psi_\theta^{\mu\nu\rho} &= \theta^{\mu\rho}P^\nu - \theta^{\nu\rho}P^\mu - i\,[\Lambda^{\mu\nu}, x^\rho] \\
\phi_\theta^{\mu\nu\rho\sigma} &= -\eta^{\mu\sigma}\Lambda^{\rho\nu} - \eta^{\nu\sigma}\Lambda^{\mu\rho} + \eta^{\mu\rho}\Lambda^{\sigma\nu} + \eta^{\nu\rho}\Lambda^{\mu\sigma} \\
&\quad +\theta^{\mu\rho}P^\nu P^\sigma - \theta^{\nu\rho}P^\mu P^\sigma - \theta^{\mu\sigma}P^\nu P^\rho + \theta^{\nu\sigma}P^\mu P^\rho \\
&\quad -i\,[\Lambda^{\mu\nu}, x^\rho]\,P^\sigma + i\,[\Lambda^{\mu\nu}, x^\sigma]\,P^\rho - i\,[x^\mu, \Lambda^{\rho\sigma}]\,P^\nu + i\,[x^\nu, \Lambda^{\rho\sigma}]\,P^\mu \\
&\quad +ix^\mu[\Lambda^{\rho\sigma}, P^\nu] - ix^\nu[\Lambda^{\rho\sigma}, P^\mu] + ix^\sigma[\Lambda^{\mu\nu}, P^\rho] - ix^\rho[\Lambda^{\mu\nu}, P^\sigma] \\
&\quad -i\,[\Lambda^{\mu\nu}, \Lambda^{\rho\sigma}]\,.
\end{aligned} \tag{3.14}$$

Inserting these expressions into the conditions (3.4), (3.11) and (3.12) results in

$$\begin{aligned}
0 &= \left[[\Lambda^{\mu\nu}, \Lambda^{\rho\sigma}], P^\lambda\right] + \left[[\Lambda^{\rho\sigma}, P^\lambda], \Lambda^{\mu\nu}\right] + \left[[P^\lambda, \Lambda^{\mu\nu}], \Lambda^{\rho\sigma}\right] \\
0 &= \left[[\Lambda^{\mu\nu}, \Lambda^{\rho\sigma}], \Lambda^{\kappa\lambda}\right] + \left[[\Lambda^{\rho\sigma}, \Lambda^{\kappa\lambda}], \Lambda^{\mu\nu}\right] + \left[[\Lambda^{\kappa\lambda}, \Lambda^{\mu\nu}], \Lambda^{\rho\sigma}\right] \\
\\
0 &= \left[[\Lambda^{\mu\nu}, x^\sigma], P^\rho\right] \\
&= \left[[\Lambda^{\mu\nu}, x^\sigma], P^\rho\right] + \left[[x^\sigma, P^\rho], \Lambda^{\mu\nu}\right] + \left[[P^\rho, \Lambda^{\mu\nu}], x^\sigma\right] \\
0 &= \left[[\Lambda^{\mu\nu}, \Lambda^{\rho\sigma}], x^\lambda\right] + \left[[\Lambda^{\rho\sigma}, x^\lambda], \Lambda^{\mu\nu}\right] + \left[[x^\lambda, \Lambda^{\mu\nu}], \Lambda^{\rho\sigma}\right] \\
\\
0 &= \left[[\Lambda^{\mu\nu}, x^\rho], x^\sigma\right] + \left[[x^\sigma, \Lambda^{\mu\nu}], x^\rho\right] \\
&= \left[[\Lambda^{\mu\nu}, x^\rho], x^\sigma\right] + \left[[x^\sigma, \Lambda^{\mu\nu}], x^\rho\right] + \left[[x^\rho, x^\sigma], \Lambda^{\mu\nu}\right].
\end{aligned} \tag{3.15}$$

Due to its physical dimension, the most simple structure of $\Lambda^{\mu\nu}$ is of the form

$$\Lambda^{\mu\nu} \sim \theta P P.$$

Obviously any choice of $\Lambda^{\mu\nu}$ of this kind leads to a solution for $U_\theta^\lambda(\mathfrak{p})$. Omitting a possible constant, we choose $\Lambda^{\mu\nu}$ to be

$$\Lambda^{\mu\nu} := \lambda_1 P_\alpha \left(\theta^{\mu\alpha}P^\nu - \theta^{\nu\alpha}P^\mu\right) + \lambda_2 \eta_{\alpha\beta}P^\alpha P^\beta \theta^{\mu\nu}, \tag{3.16}$$

with real parameters λ_1, λ_2. Computing now the functions $\psi^{\mu\nu\rho}(P^\gamma)$, $\phi^{\mu\nu\rho\sigma}(P^\gamma)$ and $\chi^{\mu\nu\rho\sigma}(P^\gamma)$ with this expression, we obtain

$$\begin{aligned}
\chi_\theta^{\mu\nu\rho} &= 0 \\
\psi_\theta^{\mu\nu\rho} &= (1-\lambda_1)\left(\theta^{\mu\rho}P^\nu - \theta^{\nu\rho}P^\mu\right) + \lambda_1 P_\alpha \left(\eta^{\mu\rho}\theta^{\nu\alpha} - \eta^{\nu\rho}\theta^{\mu\alpha}\right) - 2\lambda_2\theta^{\mu\nu}P^\rho \\
\phi_\theta^{\mu\nu\rho\sigma} &= (1-2\lambda_1)\left(\theta^{\mu\rho}P^\nu P^\sigma - \theta^{\nu\rho}P^\mu P^\sigma - \theta^{\mu\sigma}P^\nu P^\rho + \theta^{\nu\sigma}P^\mu P^\rho\right) \\
&\quad -\lambda_2\left(\theta^{\mu\rho}\eta^{\nu\sigma} - \theta^{\nu\rho}\eta^{\mu\sigma} - \theta^{\mu\sigma}\eta^{\nu\rho} + \theta^{\nu\sigma}\eta^{\mu\rho}\right)\eta_{\alpha\beta}P^\alpha P^\beta
\end{aligned} \tag{3.17}$$

and by this we have finally found all solutions $U_\theta^{(\lambda_1,\lambda_2)}(\mathfrak{p})$. Moreover the solution $U_\theta^{(\frac{1}{2},0)}(\mathfrak{p})$ gives the classical relations for the Lorentz algebra but with deformed coproduct - as we shall see in the next section. This result was also obtained by alternative considerations [17], [86].

3.2.3 The Hopf Algebra Structure of $U_\theta^{(\lambda_1,\lambda_2)}(\mathfrak{p})$

In this part we discuss the Hopf algebra structure of $U_\theta^{(\lambda_1,\lambda_2)}(\mathfrak{p})$. Since the subalgebra of momentum operators P^μ is undeformed

$$\Delta(P^\mu) = P^\mu \otimes \mathbf{1} + \mathbf{1} \otimes P^\mu,\ \epsilon(P^\mu) = 0,\ S(P^\mu) = -P^\mu,$$

we merely focus on the corresponding properties for the deformed Lorentz generators $M^{\mu\nu}$. The necessary computations to prove the Hopf algebra axioms for $M^{\mu\nu}$ are straight forward, such that we limit ourselves to present the results and step through the necessary points without going into computational details. To ensure that counit and coproduct are algebra homomorphisms, they have to map the unit operator $\mathbf{1} \in U_\theta^{(\lambda_1,\lambda_2)}(\mathfrak{p})$ according to

$$\begin{aligned}
\epsilon(\mathbf{1}) &= 1 \\
\Delta(\mathbf{1}) &= \mathbf{1} \otimes \mathbf{1}.
\end{aligned}$$

From relation (3.9) we read off the coproduct of $M^{\mu\nu}$ to be

$$\begin{aligned}
\Delta(M^{\mu\nu}) &= M^{\mu\nu} \otimes \mathbf{1} + \mathbf{1} \otimes M^{\mu\nu} - (1-\lambda_1)P_\alpha \otimes \left(\theta^{\mu\alpha}P^\nu - \theta^{\nu\alpha}P^\mu\right) \\
&\quad +\lambda_1\left(\theta^{\mu\alpha}P^\nu - \theta^{\nu\alpha}P^\mu\right) \otimes P_\alpha + 2\lambda_2\theta^{\mu\nu}\eta^{\alpha\beta}P_\alpha \otimes P_\beta.
\end{aligned} \tag{3.18}$$

Choosing the counit of $M^{\mu\nu}$ by

$$\epsilon(M^{\mu\nu}) = 0,$$

it is easy to see that $U_\theta^{(\lambda_1,\lambda_2)}(\mathfrak{p})$ satisfies the axioms of a bialgebra by proving the coalgebra axioms presented in the first subsection, such as for counit and coproduct

$$
\begin{aligned}
(\epsilon \otimes \mathrm{id}) \circ \Delta(M^{\mu\nu}) = \quad & \mathrm{id} \quad = (\mathrm{id} \otimes \epsilon) \circ \Delta(M^{\mu\nu}) \\
(\Delta \otimes \mathrm{id}) \circ \Delta(M^{\mu\nu}) \quad = \quad & (\mathrm{id} \otimes \Delta) \circ \Delta(M^{\mu\nu}).
\end{aligned}
$$

To finally make $U_\theta^{(\lambda_1,\lambda_2)}(\mathfrak{p})$ a Hopf algebra, we need an antipode S for $M^{\mu\nu}$. We find that

$$
\mathrm{S}(M^{\mu\nu}) = -M^{\mu\nu} - (1-2\lambda_1)(\theta^{\mu\alpha}P^\nu - \theta^{\nu\alpha}P^\mu)P_\alpha + 2\lambda_2\theta^{\mu\nu}\eta_{\alpha\beta}P^\alpha P^\beta \tag{3.19}
$$

satisfies the axiom for the antipode

$$
m \circ (\mathrm{id} \otimes \mathrm{S}) \circ \Delta(M^{\mu\nu}) = \epsilon(M^{\mu\nu})\mathbf{1} = m \circ (\mathrm{S} \otimes \mathrm{id}) \circ \Delta(M^{\mu\nu}),
$$

where m represents the multiplication within $U_\theta^{(\lambda_1,\lambda_2)}(\mathfrak{p})$. We remark that the double application of the antipode map S on $M^{\mu\nu}$ is the identity operator

$$
\mathrm{S}^2 = \mathrm{id}.
$$

The coproduct $\Delta(M^{\mu\nu})$ and the antipode $S(M^{\mu\nu})$ for $\theta^{\mu\nu} \to 0$ converge to the undeformed case

$$
\Delta(m^{\mu\nu}) = m^{\mu\nu} \otimes \mathbf{1} + \mathbf{1} \otimes m^{\mu\nu},\ S(m^{\mu\nu}) = -m^{\mu\nu},
$$

with $m^{\mu\nu} \in U(\mathfrak{p})$. Finally we have to ensure that coproduct and counit are algebra homomorphisms. Since the counit is trivial, the task reduces itself to satisfy the relations

$$
\begin{aligned}
[\Delta(M^{\mu\nu}), \Delta(P^\rho)] \quad &= \quad i\eta^{\mu\rho}\Delta(P^\mu) - i\eta^{\nu\rho}\Delta(P^\nu) \\
[\Delta(M^{\mu\nu}), \Delta(M^{\rho\sigma})] \quad &= \quad i\eta^{\mu\rho}\Delta(M^{\nu\sigma}) - i\eta^{\nu\rho}\Delta(M^{\mu\sigma}) + i\eta^{\nu\sigma}\Delta(M^{\mu\rho}) \\
& \quad\ -i\eta^{\mu\sigma}\Delta(M^{\nu\rho}) + i\Delta(\phi^{\mu\nu\rho\sigma})
\end{aligned}
$$

An easy computation shows that this is the case for all solutions $U_\theta^{(\lambda_1,\lambda_2)}(\mathfrak{p})$ that we have presented here.

3.2.4 Equivalence among derived Solutions $U_\theta^{(\lambda_1,\lambda_2)}(\mathfrak{p})$

In this final subsection we discuss how the derived deformations $U_\theta^{(\lambda_1,\lambda_2)}(\mathfrak{p})$ are related to one among another. To this purpose we shortly review some basic definitions and properties concerning the equivalence of deformed Hopf algebras.

The kind of deformations we obtained so far are more generally known as topological Hopf algebras $U_h(\mathfrak{g})$ over the ring of formal power series in the deformation parameter $\mathbf{K}\llbracket h \rrbracket$ with coefficients in the field $\mathbf{K}$.
Such a deformation, as a $\mathbf{K}\llbracket h \rrbracket$-module, is isomorphic to the set of formal power series $U(\mathfrak{g})\llbracket h \rrbracket$ with coefficients in $U(\mathfrak{g})$.

3.2.6 DEFINITION *Two deformations $U_h(\mathfrak{g})$ and $\hat{U}_h(\mathfrak{g})$ are* equivalent *if there exists a $\mathbf{K}\llbracket h \rrbracket$-module isomorphism $\varphi : U_h(\mathfrak{g}) \to \hat{U}_h(\mathfrak{g})$ with properties*

$$
\begin{aligned}
&i.) \quad \hat{\Delta} \circ \varphi = (\varphi \otimes \varphi) \circ \Delta \\
&ii.) \quad \hat{m} \circ (\varphi \otimes \varphi) = \varphi \circ m.
\end{aligned}
$$

With coproducts being the maps $\Delta : U_h(\mathfrak{g}) \to U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$ and $\hat{\Delta} : \hat{U}_h(\mathfrak{g}) \to \hat{U}_h(\mathfrak{g}) \otimes \hat{U}_h(\mathfrak{g})$ and the multiplication maps m and $\hat{m}$ respectively.

In other words, for an equivalence it is required that $\mathbf{K}\llbracket h \rrbracket$-module isomorphisms extend to bialgebra-isomorphisms. If a Hopf algebra A_h is equivalent to a bialgebra B_h, the latter extends to a Hopf algebra by means of the Hopf algebra structure of A_h and the bialgebra-isomorphism φ. Because of this reason the definition only accounts for the coproduct and not for counit and antipode.
A deformation A_h of a Hopf algebra A is called trivial if it is equivalent to $A\llbracket h \rrbracket$.
For semisimple Lie algebras $\mathfrak{g}$ all deformations of the universal enveloping algebra $U_h(\mathfrak{g})$ are isomorphic to $U(\mathfrak{g})\llbracket h \rrbracket$ as algebras such that property *ii.*) of the above definition is always fulfilled. Since the Lorentz algebra is simple, we thus can always find an algebra-isomorphism between $U(\mathfrak{p})$ and $U_\theta^{(\lambda_1,\lambda_2)}(\mathfrak{p})$. In particular this is the map

$$
\begin{aligned}
\varphi_{(\lambda_1,\lambda_2)} : U(\mathfrak{p}) \quad &\to \quad U_\theta^{(\lambda_1,\lambda_2)}(\mathfrak{p}) \\
m^{\mu\nu} \quad &\mapsto \quad M^{\mu\nu} - \frac{1}{2}(2\lambda_1 - 1)\left(\theta^{\mu\alpha} P^\nu P_\alpha - \theta^{\nu\alpha} P^\mu P_\alpha\right) \\
&\qquad -\lambda_2 \theta^{\mu\nu} \eta_{\alpha\beta} P^\alpha P^\beta.
\end{aligned}
$$

Since we only account for deformations of first order in the deformation parameter $\theta^{\mu\nu}$, i.e. for infinitesimal deformations, the inverse map is given by a change of sign

$$
\varphi^{-1}_{(\lambda_1,\lambda_2)}(M^{\mu\nu}) \;=\; m^{\mu\nu} + \frac{1}{2}(2\lambda_1 - 1)\left(\theta^{\mu\alpha} p^\nu p_\alpha - \theta^{\nu\alpha} p^\mu p_\alpha\right) + \lambda_2 \theta^{\mu\nu} \eta_{\alpha\beta}\, p^\alpha p^\beta.
$$

The momenta p^μ, P^μ are mapped by identity. It is easy to verify that the map $\varphi_{(\lambda_1,\lambda_2)}$ transfers the corresponding algebra structure maps into another.

Now we have to discuss whether these algebra-isomorphisms extend to bialgebra-isomorphisms. In this respect it is important to notice that due to the fixed basis of generators and the order of the deformation parameter, the presented algebra-isomorphisms are unique.

It turns out that the isomorphism $\varphi_{(\lambda_1,\lambda_2)}$ does not extend to a bialgebra-isomorphism such that none of our deformations is trivial.

In order to further elucidate the relation among the deformations themselves, we consider now the following algebra-isomorphism

$$
\begin{array}{rcl}
\hat{\varphi} : U_\theta^{(\lambda_1,\lambda_2)}(\mathfrak{p}) & \to & U_\theta^{(\mu_1,\mu_2)}(\mathfrak{p}) \\
M^{\mu\nu} & \mapsto & M^{\mu\nu} - (\mu_1 - \lambda_1)\,(\theta^{\mu\alpha} P^\nu P_\alpha - \theta^{\nu\alpha} P^\mu P_\alpha) \\
& & - (\mu_2 - \lambda_2)\,\theta^{\mu\nu} \eta_{\alpha\beta} P^\alpha P^\beta ,
\end{array}
$$

with $\hat{\varphi} = \varphi_{(\mu_1,\mu_2)} \circ \varphi^{-1}_{(\lambda_1,\lambda_2)}$. This finally extends to a bialgebra-isomorphism such that we can conclude that the parameters (λ_1, λ_2) in $U_\theta^{(\lambda_1,\lambda_2)}(\mathfrak{p})$ parametrize equivalent deformations of $U(\mathfrak{p})$. Therefore our final result states that, up to isomorphisms, there exists a single non-trivial deformation of $U(\mathfrak{p})$ in first order of the deformation parameter $\theta^{\mu\nu}$.

3.3 Casimir Operators and Space Invariants

In order to study field theoretical properties of the presented deformation $U^{(\lambda_1,\lambda_2)}(\mathfrak{p})$ we consider its central elements and spacetime invariants in this final section.

Since the algebra of momenta P^μ is undeformed, the d'Alembert operator $\Box = \partial_\mu \partial^\mu = -P_\nu P^\nu$ and thus the Klein-Gordon operator are those of the classical case.

Concerning the Pauli-Lubanski vector and the spacetime invariant the situation is changed.

We present a deformed Pauli-Lubanski vector W^λ that transforms as a classical vector under the action of the Lorentz operators $M^{\mu\nu}$, such that its square is invariant under these operations. In order to obtain a spacetime invariant we find that the parameters λ_1 and λ_2 become dependent.

3.3.1 Pauli-Lubanski vector

Since the commutation relations of the Lorentz generators $[M^{\mu\nu}, M^{\alpha\beta}]$ are deformed in general, the classical Pauli-Lubanski vector $\epsilon^{\lambda\kappa\rho\sigma}P_\kappa M_{\rho\sigma}$ does not transform as a classical vector anymore

$$\begin{aligned}[M^{\mu\nu}, \epsilon^{\lambda\kappa\rho\sigma}P_\kappa M_{\rho\sigma}] &= i\eta^{\mu\lambda}\epsilon^{\nu\kappa\rho\sigma}P_\kappa M_{\rho\sigma} - i\eta^{\nu\lambda}\epsilon^{\mu\kappa\rho\sigma}P_\kappa M_{\rho\sigma} \\ &\quad +2i\lambda_2(\theta^\mu_\alpha\epsilon^{\nu\alpha\lambda\rho} - \theta^\nu_\alpha\epsilon^{\mu\alpha\lambda\rho})P_\rho P_\beta P^\beta.\end{aligned}$$

Moreover its square $W^\lambda W_\lambda$ turns out not to be invariant as well. We define the deformed Pauli-Lubanski vector by the following properties

$$\begin{aligned}\left[M^{\mu\nu}, W^\lambda\right] &= i\eta^{\mu\lambda}W^\nu - i\eta^{\nu\lambda}W^\mu \\ \left[M^{\mu\nu}, W_\lambda W^\lambda\right] &= 0 \\ \left[P^\mu, W_\lambda W^\lambda\right] &= 0,\end{aligned} \tag{3.20}$$

and make an ansatz of the form $\epsilon^{\lambda\kappa\rho\sigma}P_\kappa M_{\rho\sigma} + (\epsilon\theta PPP)^\lambda$. We thus obtain the deformed Pauli-Lubanski vector to be

$$W^\lambda = \epsilon^{\lambda\kappa\rho\sigma}P_\kappa M_{\rho\sigma} + \lambda_2\epsilon^{\lambda\kappa\rho\sigma}\theta_{\kappa\rho}P_\sigma P_\alpha P^\alpha. \tag{3.21}$$

It is remarkable that W^λ is independent of λ_1.

3.3.2 Spacetime Invariants

Concerning the spacetime invariant I we demand that it is merely an element of $\mathfrak{X}_\theta$. This is a strong requirement, since it becomes impossible to deform I in any way. On the other hand the coproduct of $M^{\mu\nu}$ and thus its action on $I = x^\rho x_\rho$ is necessarily deformed, as we stated in reference to relation (3.12). We obtain for the action of $M^{\mu\nu}$ on $I = x^\rho x_\rho$

$$\begin{aligned}M^{\mu\nu} \triangleright (x^\rho x_\rho) &= ([M^{\mu\nu}, x^\rho x_\rho]) \triangleright \mathbf{1} \\ &= (-\theta^{\mu\nu}(2\lambda_2 n + 4\lambda_1 - 2) - 4i\lambda_2\theta^{\mu\nu}x^\rho P_\rho \\ &\quad -2i\lambda_1(\theta^{\mu\rho}x^\nu P_\rho - \theta^{\nu\rho}x^\mu P_\rho) \\ &\quad -2i(\lambda_1 - 1)(\theta^{\mu\rho}x_\rho P_\nu - \theta^{\nu\rho}x_\rho P_\mu)) \triangleright \mathbf{1} \\ &= -\theta^{\mu\nu}(2\lambda_2 n + 4\lambda_1 - 2),\end{aligned} \tag{3.22}$$

where n denotes the dimension of spacetime. To ensure that

$$M^{\mu\nu} \triangleright I = 0$$

we thus have to require that

$$\lambda_1 = \frac{1}{2}(1 - n\lambda_2), \tag{3.23}$$

and such the parameters λ_1 and λ_2 become dependent. In order to do the contraction of indices in I we used the metric tensor of the undeformed theory. In general the metric tensor is not invariant under deformations. Here we make use of the equivalence of our deformations $U_\theta^{(\lambda_1,\lambda_2)}(\mathfrak{p})$ to do this specific computation.
The latter relation thus determines the pair of parameters λ_1 and λ_2 such that the undeformed metric tensor can be applied.

3.4 Conclusion

In this chapter we have derived a set of deformations $U_\theta^{(\lambda_1,\lambda_2)}(\mathfrak{p})$ represented on noncommutative spacetime algebras $\mathfrak{X}_\theta$. We have furthermore indicated how such deformations could possibly be constructed for arbitrarily given spacetime algebras.
Moreover we showed how these deformations are related one among the other and found that in first order of the deformation parameter there exists, up to isomorphisms, a single non-trivial deformation of $U(\mathfrak{p})$. With this result we find ourselves in accordance with [86] and [17].
Concerning the sketched method to derive deformations from representations on given noncommutative spaces, we emphasize that by introducing a generating function, we were able to *compute* all existing deformations in first order of the deformation parameter. Usually such deformations have to be guessed. In order to generalize this process for arbitrary noncommutative spaces the set of three conditions and the choice of a generating function should be enhanced by additional conditions that divide out equivalent solutions. Moreover the Hopf structure should be derived in the same step as the conditions are solved. However, from the mathematical point of view it might be desirable to develop a method that only produces non-trivial and non-equivalent solutions. Some reader might wonder whether there is at all any interest in a set of equivalent deformations. But mathematical rigorous implementation should not obscure the original physical motivation.
From the physical point of view there is indeed an important interest in these equivalent deformations.

When the algebras $U_\theta^{(\lambda_1,0)}(\mathfrak{p})$ are represented on particle states the function $\phi^{\mu\nu\rho\sigma}$ can be treated as a constant that becomes a global $U(1)$ - phase factor. It is a nontrivial result that the $U(1)$-phase factor could be absorbed into the deformation itself. The reader should be aware that quantized spacetime or a generalized scheme of quantization should always be considered as the direct result of a quantum theory of gravity. Thus a deformation is mediated from such theories to particle physics and by this equivalent deformations of this kind might be a suitable junction between gauge-degrees of freedom and a quantized background.

4 Vector Field Twisting of Lie-Algebras

In quantum groups coproducts of Lie-algebras are twisted in terms of generators of the corresponding universal enveloping algebra. If representations are considered, twists also serve as starproducts that accordingly quantize representation spaces. In physics, requirements turn out to be the other way around. Physics comes up with noncommutative spaces in terms of starproducts that miss a suiting quantum symmetry. In general the classical limit is known, i.e. there exists a representation of the Lie-algebra on a corresponding finitely generated commutative space. In this setup quantization can be considered independently from any representation theoretic issue. We construct an algebra of vector fields from a left cross-product algebra of the representation space and its Hopf-algebra of momenta. The latter can always be defined. The suitingly devided cross-product algebra is then lifted to a Hopf-algebra that carries the required genuine structure to accomodate a matrix representation of the universal enveloping algebra as a subalgebra. We twist the Hopf-algebra of vector fields and thereby obtain the desired twisting of the Lie-algebra. Since we twist with vector fields and not with generators of the Lie-algebra, this is the most general twisting that can possibly be obtained. In other words, we push starproducts to twists of the desired symmetry algebra and to this purpose solve the problem of turning vector fields into a Hopf-algebra. We give some genuine examples.

4.1 Introduction

Studies of quantum groups require for a considerable mathematical framework that historically caused the topic to be turned into a mathematical field on its own. As a consequence it then naturally followed its own mathematical interests - apart from actual physical requirements. In quantum groups deformations of a Lie-algebra $\mathfrak{g}$ are considered in terms of its universal enveloping algebra $U(\mathfrak{g})$. Coproducts of $U(\mathfrak{g})$ are deformed by conjugation with quasitriangular structures $\mathcal{R} \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ or twists $\mathcal{F} \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$. The noncocommutative coproduct of the deformed version of the universal enveloping algebra $U(\mathfrak{g})$ dually implies a noncommutative structure on representation space. As an example see [12]. Thus within the standard workflow of quantum groups, symmetry algebras are first deformed and represented afterwards. Physics, however, requires for the opposit procedure. Theories and models come down with noncommutative spaces, as canonical spacetime in [20, 74, 77], that miss the corresponding quantum symmetry. In most cases the classical limit exists, i.e. there exists a representation of $\mathfrak{g}$ on a finitely generated commutative space. The task at hand is to find the corresponding deformation of the symmetry algebra. But quantum groups do not provide the required techniques. It thus takes quite a time until such quantizations are found - if they are found at all. For the case of canonical commutation relations these were constructed in [69, 17, 86, 52]. While twists can be used as starproducts, the opposit only holds for some specific exceptions. This is the standard situation in physics. Quite often it has been observed that quantization requires for some enhancement of the symmetry algebra [91]. For example, the well-known κ-deformation of the Poincaré algebra [57, 56, 58] cannot be reduced to that of the Lorentz-algebra alone. The algebra of momenta is a vital component of this deformation. The mathematical setup to this example had been provided by [66]. The same holds for the mentioned θ-deformation of the Poincaré algebra for canonical commutation relations. Obviously only those very specific deformations can merely be performed *within* the symmetry algebra, that are ruled by a quasitriangular structure $\mathcal{R}$. But these only provide quantum spaces with quadratic commutation relations. We can thus observe the *physical reason* why κ- and θ-deformations required for some algebraic enhancement: The deformation parameter carries a physical dimension. Thus while the mathematical workflow restricted to a single version of quantum spaces, that turned out quite unhandy for physical applications, physics itself came up with deformations beyond this setup. And mathematics, as often, delivered an explanation afterwards. The universal enveloping algebra of a Lie-algebra is obviously not large enough in order to perform most general quantizations of its coproducts. The authors of [60, 59], [13] incorporated this idea and used the Poincaré algebra as a whole in order to obtain more general twistings. They receive quantum spaces with quadratic as well as Lie-algebra valued commutation relations. Here we want to push this a little further. Within another example of physics, phase space deformations were considered in order to obtain high energy motivated minimal uncertainty models [47, 46, 45, 44]. The author speculates that the deformation of a corresponding Poincaré-algebra might be obtained by the use of the phase space algebra itself. In contrast to this, the authors of [38] formulate starproducts in terms of vector fields. Vector fields are most fundamental objects of differential geometry and Lie-algebras themselves describe nothing else than the currents on curved manifolds. Apart from this, there is a close relation between noncommutative geometry and quantization over curved spaces. In this respect vector fields also played a crucial role for noncommutative gravity [8, 7]. Vector fields might thus provide the actual and most genuine structure underlying any deformation-quantization. But in order to consider such twist-deformations, an algebra of vector fields would have to be enhanced to a Hopf-algebra. The actual question is, how this is possibly done. A very elegant solution to this problem was provided by the authors of [65]. But they already incorporated a physical interpretation into their setup that we want to avoid here. To any representation space we can formaly define an action of a Hopf-algebra of momenta. These can be joined to a left cross-product algebra that we devide in such a way, that we can lift it to an actual Hopf-algebra. In fact this construction provides a very clear and genuine structure that we further denote as a Hopf-algebra of vector fields. This Hopf-algebra is large enough to accomodate any matrix representation of the universal enveloping algebra $U(g)$ as a subalgebra. This is the commutative limit that is well-known in physics and has to be fed into this setup. By twisting the Hopf-algebra of vector fields we thus twist its subalgebra as well - but more general than the generators of $U(g)$ could possibly do. In the mean time the twist is nothing else than the starproduct, that comes with the noncommutative associative space. We thus achieve several goals. Starproducts directly can be used as twists in order to obtain a quantization of the desired symmetry and in parallel we open the formalizm for most general quantizations and thus stay as close as possible to the actual requirements of physics. The chapter is organised as follows. In the first section we formulate the classical limit that we have to feed as input into our procedure. We take the opportunity to recall basic definitions and properties of required notions in order to be self-contained. In the following section we construct the Hopf-algebra of vector fields and the actual twists will be considered in the third section. We close with the basic example of a deformation of the two-dimensional representation of $U(sl_2)$. The chapter orients itself to the textbooks [19, 64].

4.2 Representation of $U(\mathfrak{g})$ on $U(\mathfrak{X})$

As outlined in the introduction, the deformation of a universal enveloping algebra $U(\mathfrak{g})$ of a Lie-algebra $\mathfrak{g}$ and its accordingly deformed representation space

$\mathfrak{X}$ is actually independent of any representation theoretic issues, presupposing that the non-quantized limit exists and is well defined.
In this section we concretize this specific undeformed setup and in order to be self-contained we take the opportunity to recall basic definitions and properties of Lie-algebras and their representations.
It is our aim to represent $\mathfrak{g}$ on a finite dimensional $\mathbf{K}$-linear vector space $\mathfrak{X}$. As fields $\mathbf{K}$ we consider complex or real numbers. Let us shortly recall the definition of a Lie-algebra before we continue.

4.2.1 Definition (Lie-algebra) *Let $\mathfrak{g}$ be a p - dimensional vector space over the field $\mathbf{K}$. The vector space $\mathfrak{g}$ is called a* Lie-algebra *if the there exists a bracket*

$$[\cdot,\cdot]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

that holds the following properties:

$$\begin{aligned} \forall g, h, k \in \mathfrak{g}: \quad & [g,h]_{\mathfrak{g}} = -[h,g]_{\mathfrak{g}} && \text{(Antisymmetry)} \\ & [g+h,k]_{\mathfrak{g}} = [g,k]_{\mathfrak{g}} + [h,k]_{\mathfrak{g}} && \text{(Bilinearity)} \\ & [g,[h,k]_{\mathfrak{g}}]_{\mathfrak{g}} + [h,[k,g]_{\mathfrak{g}}]_{\mathfrak{g}} + [k,[g,h]_{\mathfrak{g}}]_{\mathfrak{g}} = 0 && \text{(Jacobi-Identity)} \end{aligned}$$

As an element of the Lie-algebra $\mathfrak{g}$, the bracket can be expressed as a linear combination in terms of basis elements $(g_a)_{a\in\{1,\dots,p\}}$, i. e.

$$[g_a, g_b]_{\mathfrak{g}} = i \sum_{c=1}^{p} f_{abc}\, g_c, \qquad f_{abc} \in \mathbf{K}.$$

Formally a representation of $\mathfrak{g}$ on $\mathfrak{X}$ is much more the representation of its universal enveloping algebra $U(\mathfrak{g})$ on $\mathfrak{X}$, that we define as follows.

4.2.2 Definition (Universal Enveloping Algebra) *Let $\mathfrak{g}$ be a Lie-algebra over the field $\mathbf{K}$ with p-dimensional basis $(g_a)_{a\in\{1,\dots,p\}}$ and bracket $[\cdot,\cdot]_{\mathfrak{g}}$. Then the* universal enveloping algebra *$U(\mathfrak{g})$ is defined to be the quotient of the tensor algebra $T(\mathfrak{g})$ and the two-sided ideal $\mathcal{I}_{\mathfrak{g}} \subset T(\mathfrak{g})$*

$$U(\mathfrak{g}) = \frac{T(\mathfrak{g})}{\mathcal{I}_{\mathfrak{g}}}.$$

The two-sided ideal $\mathcal{I}_{\mathfrak{g}}$ is generated by relations

$$\forall\, g_a, g_b \in \mathfrak{g}: \; g_a \otimes g_b - g_b \otimes g_a - i \sum_{c=1}^{p} f_{abc}\, g_c = 0 \tag{4.1}$$

For $\varphi(g_a), \omega(g_b) \in U(\mathfrak{g})$ the bracket $[\varphi(g_a), \omega(g_b)] := \varphi(g_a) \otimes \omega(g_b) - \omega(g_a) \otimes \varphi(g_b)$ is called the commutator.

Before we continue to discuss our specific case let us also recall the definition of the representation of an algebra on a **K**-linear vector space.

4.2.3 DEFINITION (REPRESENTATION) *Let* $(\mathfrak{A}, \mu, \eta, +; \mathbf{K})$ *be an algebra over the field* **K** *and let* $(\mathfrak{V}, +; \mathbf{K})$ *be a vector space. A* left representation *of* $\mathfrak{A}$ *on* $\mathfrak{V}$ *is a pair* $(\rho, \mathfrak{V})$ *consisting of a map*

$$\begin{aligned} \rho : \mathfrak{A} \otimes \mathfrak{V} &\longrightarrow \mathfrak{V} \\ a \otimes v &\mapsto \rho(a \otimes v) = \rho_a(v) = a \triangleright v \end{aligned}$$

such that for all $a \in \mathfrak{A}$ *the maps* ρ_a *realize the algebra* $\mathfrak{A}$ *within the endomorphism of* $\mathfrak{V}$, *i.e.*

$$\forall\, a, b, \mathbf{1} \in \mathfrak{A},\ v \in \mathfrak{V} \ : \ \begin{aligned} &(a \cdot b) \triangleright v = a \triangleright (b \triangleright v) \\ &\mathbf{1} \triangleright v = v \end{aligned}$$

The representation ρ *is also called a* left action "$\triangleright$".

With this little preparation we understand that a representation ρ of $U(\mathfrak{g})$ on the finite dimensional vector space $\mathfrak{X}$ is more specifically defined in terms of a matrix representation, i.e. for basis elements $g_a \in U(\mathfrak{g})$ and $x_i \in \mathfrak{X}$ we obtain

$$\rho(g_a \otimes x_i)_j = (g_a \triangleright x_i)_j = \sum_{i=1}^{n} (g_a)_{j\,i} x_i, \tag{4.2}$$

where $(g_a)_{j\,i} \in GL(n, \mathbf{K}) \subset \mathrm{Mat}(n, \mathbf{K})$. Moreover, the generating relations of $U(\mathfrak{g})$ have to be represented on $\mathfrak{X}$ by

$$\begin{aligned} \forall\, g_a, g_b \in \mathfrak{g} \ : \ & (g_a \cdot g_b - g_b \cdot g_a - [g_a, g_b]_{\mathfrak{g}}) \triangleright x_i \\ & = g_a \triangleright (g_b \triangleright x_i) - g_b \triangleright (g_a \triangleright x_i) - i \sum_{c=1}^{p} f_{abc} (g_c \triangleright x_i) = 0. \end{aligned}$$

Here we replaced the tensor product "$\otimes$" by conventional multiplication "$\cdot$". In terms of matrix representations (4.2) these relations then read

$$\begin{aligned} \forall\, g_a, g_b \in \mathfrak{g} \ : \ & \sum_{i=1}^{n} (\sum_{j=1}^{n} (g_a)_{k\,j} (g_b)_{j\,i} - \sum_{j=1}^{n} (g_b)_{k\,j} (g_a)_{j\,i} - ([g_a, g_b]_{\mathfrak{g}})_{k\,i}) x_i \\ & = \sum_{j=1}^{n} (g_a)_{k\,j} (\sum_{i=1}^{n} (g_b)_{j\,i} x_i) - \sum_{j=1}^{n} (g_b)_{k\,j} (\sum_{i=1}^{n} (g_a)_{j\,i} x_i) \\ & \quad - i \sum_{c=1}^{p} f_{abc} \sum_{i=1}^{n} (g_c)_{k\,i} x_i = 0 \end{aligned} \tag{4.3}$$

Up to this point we consider the Lie-algebra $\mathfrak{g}$ and the vector space $\mathfrak{X}$ to be given and moreover that the representation ρ exists and is well behaved. This setup represents the actual input from outside that we require for our considerations. Of course we want more structure than that. For our purpose we have to enhance $\mathfrak{X}$ to an algebra and thus extend $U(\mathfrak{g})$ to a Hopf-algebra. Enhancing $\mathfrak{X}$ to an algebra is usually performed in several blends of one and the same idea: enhancing to the tensor algebra of $\mathfrak{X}$ and then deviding by a suitable two-sided ideal. In order to get things straight, we first turn $\mathfrak{X}$ into a Lie-algebra and then as well consider it as a universal enveloping algebra.
We thus fix an n-dimensional basis for $\mathfrak{X}$ to be $(x_i)_{i\in 1,2,\dots,n}$. Enhancing $\mathfrak{X}$ to a Lie-algebra is easily performed by introducing a $\mathbf{K}$-bilinear bracket

$$[\cdot,\cdot] \;:\; \mathfrak{X}\times\mathfrak{X}\longrightarrow\mathfrak{X}.$$

The easiest choice for a bracket $[\cdot,\cdot]$, that satisfies the requirements of a Lie-algebra and later as well delivers the required commutative algebra of coordinates, is the vanishing bracket

$$\forall\, x_i,x_j\in\mathfrak{X} \;:\; [x_i,x_j]=0.$$

We thus have turned $\mathfrak{X}$ into a Lie-algebra. As we did for the Lie-algebra $\mathfrak{g}$, we can now consider the universal enveloping algebra $U(\mathfrak{X})$ of $\mathfrak{X}$ and thus enhanced the vector space to a *commutative* and associative algebra that is generated by relations

$$\forall\, x_i,x_j\in U(\mathfrak{X}) :\; x_i\otimes x_j - x_j\otimes x_i = 0. \tag{4.4}$$

We once more replace the tensor product "$\otimes$" by a multiplication "$\cdot$". In order to transfer the action of $U(\mathfrak{g})$ on the vector space $\mathfrak{X}$ to an action on the algebra $U(\mathfrak{X})$ we have to enhance $U(\mathfrak{g})$ to a Hopf-algebra by introducing a coproduct, counit and antipode by

$$\forall\, g_a\in U(\mathfrak{g}) :\; \Delta(g_a)=g_a\otimes\mathbf{1}+\mathbf{1}\otimes g_a,\;\; \epsilon(g_a)=0,\;\; \mathrm{S}(g_a)=-g_a.$$

It is quickly verified that this definition of the Hopf-algebra $U(\mathfrak{g})$ satisfies all axioms and requirements of a Hopf-algebra. The following definition then tells us how the representation ρ on $\mathfrak{X}$ is enhanced to that of $U(\mathfrak{X})$.

4.2.4 Definition *Let $(\mathcal{H},\mu,\eta,\Delta,\epsilon,S;\mathbf{K})$ be a Hopf-algebra over the field $\mathbf{K}$. Let $(\mathfrak{A},\mu,\eta,+;\mathbf{K})$ be an algebra. The left representation of $\mathcal{H}$ on $\mathfrak{A}$ is a left action that additionally satisfies*

$$\begin{aligned}\forall\, h\in\mathcal{H},\; a,b,\mathbf{1}\in\mathfrak{A} \;\;:\;\; & h\triangleright(a\cdot b)=\sum(h_{(1)}\triangleright a)\cdot(h_{(2)}\triangleright b)\\ & h\triangleright\mathbf{1}=\epsilon(h)\end{aligned} \tag{4.5}$$

with $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$. *The algebra* $\mathfrak{A}$ *then becomes a* left $\mathcal{H}$-module algebra.

Since the multiplication of $U(\mathfrak{X})$ is defined by the generating relations $\forall\ x_i, x_j \in U(x_i) : [x_i, x_j] = 0$, we have to verify that the action of $U(\mathfrak{g})$ respects this, i.e. for $g_a \in U(\mathfrak{g})$

$$\begin{aligned} & g_a \triangleright (x_i \cdot x_j - x_j \cdot x_i) \\ & \quad = \Delta(g_a) \triangleright (x_i \cdot x_j - x_j \cdot x_i) \\ & \quad = (g_a \triangleright x_i)x_j + x_i(g_a \triangleright x_j) - (g_a \triangleright x_i)x_j - x_i(g_a \triangleright x_j) \\ & \quad = (g_a \triangleright x_i)x_j - (g_a \triangleright x_i)x_j + x_i(g_a \triangleright x_j) - x_i(g_a \triangleright x_j) = 0, \end{aligned}$$

since any $g_a \triangleright x_i \in U(\mathfrak{X})$ once more commutes with an $x_j \in U(\mathfrak{X})$. Thus the commutation relations of $U(\mathfrak{X})$ have to be compatible with the coalgebra sector of $U(\mathfrak{g})$. We thus have completed our setup that from now on is denoted by the *commutative limit*. Note that we do *not* enhance $U(\mathfrak{X})$ to a Hopf-algebra as well. In the next section we continue with basic constructions that pave the way to deformations of this setup.

4.3 A Hopf-Algebra of Vector Fields $\mathfrak{W}(\Pi, \mathfrak{X})$

In this section we construct the Hopf-algebra of vector fields $\mathfrak{W}(\Pi, \mathfrak{X})$ that we require for general deformations of $U(\mathfrak{g})$ and $U(\mathfrak{X})$. To this purpose we first introduce a Hopf-algebra of momenta $U(\Pi)$ that is represented as a left action on $U(\mathfrak{X})$. We continue with the construction of a *left cross-product algebra* $U(\mathfrak{X}) \rtimes U(\Pi)$ that we further devide in order to lift it to the Hopf-algebra of vector fields $\mathfrak{W}(\Pi, \mathfrak{X})$. In the last subsection we further more define the left action of $\mathfrak{W}(\Pi, \mathfrak{X})$ on $U(\mathfrak{X})$.

4.3.1 A Hopf-Algebra $U(\Pi)$ of Momenta

We begin this section with one more Hopf-algebra $U(\Pi)$ that we loosely denote as the *algebra of momenta*. As long $U(\mathfrak{X})$ is actually considered to be an algebra of coordinates, $U(\Pi)$ can actually be considered to be nothing than that. We introduce $U(\Pi)$ as a copy of $U(\mathfrak{X})$, with the exception that in contrast to $U(\mathfrak{X})$ it is enhanced by coalgebra structure and an antipode. We thus understand $U(\Pi)$ to be generated by a n-dimensional basis $(\pi_i)_{i \in 1,2,\ldots,n}$ with commutation relations

$$\pi_i\pi_j - \pi_j\pi_i = [\pi_i, \pi_j] = 0, \tag{4.6}$$

and a primitive coalgebra structure for all $\pi_i \in U(\Pi)$ as well as a standard antipode

$$\Delta(\pi_i) = \pi_i \otimes \mathbf{1} + \mathbf{1} \otimes \pi_i, \quad \epsilon(\pi_i) = 0, \quad \mathrm{S}(\pi_i) = -\pi_i. \tag{4.7}$$

We define the left action of $U(\Pi)$ on $U(\mathfrak{X})$ by

$$\forall\ \pi_i, \mathbf{1} \in U(\Pi) \wedge x_j, \mathbf{1} \in U(\mathfrak{X}) : \pi_i \triangleright x_j = -i\delta_{ij},\ \mathbf{1}\triangleright x_j = x_j,\ \pi_i \triangleright \mathbf{1} = \epsilon(\pi_i) \tag{4.8}$$

We could also have omitted the imaginary unit here, but since we are interested in physical applications, we stick as close as possible to physical notions. It is evident that (4.8) is a well defined action, since the relations (4.6) are realized on $U(\mathfrak{X})$ by

$$\begin{aligned}(\pi_i\pi_j - \pi_j\pi_i) \triangleright x_k &= \pi_i \triangleright (\pi_j \triangleright x_k) - \pi_j \triangleright (\pi_i \triangleright x_k) \\ &= \pi_i \triangleright (-i\delta_{jk}\mathbf{1}) - \pi_j \triangleright (-i\delta_{ik}\mathbf{1}) = 0\end{aligned} \tag{4.9}$$

and in turn, $U(\Pi)$ respects the algebra relations (4.4) of $U(\mathfrak{X})$ by means of the coalgebra structure (4.7) of $U(\Pi)$ by

$$\begin{aligned}\pi_i \triangleright (x_k x_l - x_l x_k) &= \Delta(\pi_i) \triangleright (x_k x_l - x_l x_k) \\ &= (\pi_i \triangleright x_k)x_l + x_k(\pi_i \triangleright x_l) - (\pi_i \triangleright x_l)x_k - x_l(\pi_i \triangleright x_k) \\ &= -i\delta_{ik}x_l - ix_k\delta_{il} + i\delta_{il}x_k + ix_l\delta_{ik} = 0.\end{aligned}$$

4.3.2 The Left Cross-Product $U(\mathfrak{X}) \rtimes U(\Pi)$

Within the next step towards a Hopf-algebra of vector fields, we join the algebra $U(\mathfrak{X})$ and the Hopf-algebra $U(\Pi)$ to a single left cross-product algebra. Before we do so, we shortly recall its definition-proposition, that can be found in the literature.

4.3.1 Definition-Proposition *Let $\mathcal{H}$ be a Hopf-algebra and let $\mathfrak{A}$ be a left $\mathcal{H}$-module algebra. Then there exists a left cross-product algebra $\mathfrak{A} \rtimes \mathcal{H}$ on $\mathfrak{A} \otimes \mathcal{H}$ with the associative product*

$$\forall\ a, b \in \mathfrak{A},\ h, k \in \mathcal{H} :\ (a \otimes h) \odot (b \otimes k) = \sum a(h_{(1)} \triangleright b) \otimes h_{(2)}k$$

and unit element $\mathbf{1} \otimes \mathbf{1}$.

Thus for the algebraic relations of $U(\mathfrak{X}) \rtimes U(\Pi)$, by the use of (4.7) and (4.8), we obtain for $x_i \otimes \pi_r, x_j \otimes \pi_s \in U(\mathfrak{X}) \otimes U(\Pi)$

$$\begin{aligned}(x_i \otimes \pi_r) \odot (x_j \otimes \pi_s) &= x_i(\pi_r \triangleright x_j) \otimes \pi_s + x_i x_j \otimes \pi_r \pi_s \\ &= -i\delta_{rj} x_i \otimes \pi_s + x_i x_j \otimes \pi_r \pi_s\end{aligned}$$

In particular we compute that with $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$ we obtain

$$\begin{aligned}(x_i \otimes \mathbf{1}) \odot (x_j \otimes \mathbf{1}) &= x_i x_j \otimes \mathbf{1} \\ (\mathbf{1} \otimes \pi_r) \odot (\mathbf{1} \otimes \pi_s) &= \mathbf{1} \otimes \pi_r \pi_s,\end{aligned}$$

such that $U(\mathfrak{X}) \equiv U(\mathfrak{X}) \otimes \mathbf{1}$ and $U(\Pi) \equiv \mathbf{1} \otimes U(\Pi)$ are contained as subalgebras within $U(\mathfrak{X}) \rtimes U(\Pi)$. We thus also find that

$$\begin{aligned}[x_i \otimes \pi_r, x_j \otimes \pi_s]_\odot &= (x_i \otimes \pi_r) \odot (x_j \otimes \pi_s) - (x_j \otimes \pi_s) \odot (x_i \otimes \pi_r) \\ &= -i\delta_{rj} x_i \otimes \pi_s + i\delta_{si} x_j \otimes \pi_r.\end{aligned}$$

Moreover, we find in particular that

$$\begin{aligned}[x_i \otimes \pi_r, x_j \otimes \mathbf{1}]_\odot &= x_i(\pi_r \triangleright x_j) \otimes \mathbf{1} = -i\delta_{rj}(x_i \otimes \mathbf{1}) \\ [x_i \otimes \pi_r, \mathbf{1} \otimes \pi_s]_\odot &= -(\pi_s \triangleright x_i) \otimes \pi_r = i\delta_{si}(\mathbf{1} \otimes \pi_r) \\ [\mathbf{1} \otimes \pi_r, x_j \otimes \mathbf{1}]_\odot &= (\pi_r \triangleright x_j) \otimes \mathbf{1} = -i\delta_{rj}(\mathbf{1} \otimes \mathbf{1})\end{aligned}$$

As $U(\mathfrak{X}) \rtimes U(\Pi)$ provides the algebraic structure on $U(\mathfrak{X}) \otimes U(\Pi)$, that is a vector space, we can thus once more understand $U(\mathfrak{X}) \rtimes U(\Pi)$ to be the tensor algebra $T(U(\mathfrak{X}) \otimes U(\Pi))$ that is devided by a suitable two-sided ideal. Making thus the identification

$$\begin{aligned}\mathfrak{w}^0_{ir} \equiv x_i \otimes \pi_r, &\qquad \mathfrak{w}^+_r \equiv \mathbf{1} \otimes \pi_r, \\ \mathfrak{w}^-_i \equiv x_i \otimes \mathbf{1}, &\qquad \mathbf{1} \equiv \mathbf{1} \otimes \mathbf{1},\end{aligned}$$

we regard $\mathfrak{w}^\circ, \mathfrak{w}^\pm$ as the generators of $U(\mathfrak{X}) \rtimes U(\Pi)$ that by relations

$$\begin{aligned}\left[\mathfrak{w}^0_{ir}, \mathfrak{w}^0_{js}\right]_\odot = -i\delta_{rj}\mathfrak{w}^0_{is} + i\delta_{si}\mathfrak{w}^0_{jr}, &\qquad \left[\mathfrak{w}^+_r, \mathfrak{w}^-_j\right]_\odot = -i\delta_{rj}\mathbf{1} \\ \left[\mathfrak{w}^0_{ir}, \mathfrak{w}^-_j\right]_\odot = -i\delta_{rj}\mathfrak{w}^-_i, &\qquad \left[\mathfrak{w}^0_{ir}, \mathfrak{w}^+_s\right]_\odot = i\delta_{si}\mathfrak{w}^+_r, \\ \left[\mathfrak{w}^+_r, \mathfrak{w}^+_s\right]_\odot = 0, &\qquad \left[\mathfrak{w}^-_i, \mathfrak{w}^-_j\right]_\odot = 0,\end{aligned} \tag{4.10}$$

constitute the required two-sided ideal $\mathcal{I}_{\mathfrak{X},\Pi}$. We can thus set

$$U(\mathfrak{X}) \rtimes U(\Pi) = \frac{T(U(\mathfrak{X}) \otimes U(\Pi))}{\mathcal{I}_{\mathfrak{X},\Pi}},$$

as for any universal enveloping algebra.

4.3.3 The Hopf-algebra $\mathfrak{W}(\Pi, \mathfrak{X})$ of vector fields

The Relations (4.10) exhibit a nice structure of subalgebras within the cross-product algebra $U(\mathfrak{X}) \rtimes U(\Pi)$, that already indicates into the desired direction of our purpose. However, since we would like to lift our construction to a Hopf-algebra, such that we can represent it once more on an algebra, we have to perform further modifications. The second relation of (4.10) does not allow for a Hopf-algebra enhancement, since it would not confirm for the homomorphy property of the coproduct. Moreover we do not really have a use for a coproduct on $\mathfrak{w}_i^-$, i.e. a coproduct on a coordinate. The authors of [65] found an elegant way to deal with a similar issue by a specific bicross-product construcion. However, they had to introduce a physical interpretation as well that we avoid here by the pursuing another direction.
We reach our goal by further deviding our algebra $U(\mathfrak{X}) \rtimes U(\Pi)$ by relation

$$\mathfrak{w}_i^- = 0,$$

such that we define our algebra of vector fields by

$$\mathfrak{W}(\Pi, \mathfrak{X}) = \frac{T(U(\mathfrak{X}) \otimes U(\Pi))}{\mathcal{I}_{\mathfrak{W}}}.$$

The two-sided ideal $\mathcal{I}_{\mathfrak{W}}$ is generated by relations

$$\begin{aligned} \left[\mathfrak{w}^0_{ir}, \mathfrak{w}^0_{js}\right]_\odot = -i\delta_{rj}\mathfrak{w}^0_{is} + i\delta_{si}\mathfrak{w}^0_{jr}, \qquad \left[\mathfrak{w}^0_{ir}, \mathfrak{w}^+_{s}\right]_\odot = i\delta_{si}\mathfrak{w}^+_{r}, \\ \left[\mathfrak{w}^+_{r}, \mathfrak{w}^+_{s}\right]_\odot = 0, \qquad \mathfrak{w}^-_i = 0. \end{aligned} \tag{4.11}$$

We already see that this is very similar to the structure that we, for example, expect from a Poincaré-algebra. But it is much more general in its foundations. And we see how this applies to any desired setup based on the commutative limit we discussed above. It is easily checked that these relations induce a closed algebra, i.e. that the Jacobi-Identities

$$\begin{aligned}
\left[\left[\mathfrak{w}^0_{ir}, \mathfrak{w}^0_{js}\right]_\odot, \mathfrak{w}^0_{kt}\right]_\odot + \left[\left[\mathfrak{w}^0_{js}, \mathfrak{w}^0_{kt}\right]_\odot, \mathfrak{w}^0_{ir}\right]_\odot + \left[\left[\mathfrak{w}^0_{kt}, \mathfrak{w}^0_{ir}\right]_\odot, \mathfrak{w}^0_{js}\right]_\odot &= 0 \\
\left[\left[\mathfrak{w}^0_{ir}, \mathfrak{w}^+_{s}\right]_\odot, \mathfrak{w}^0_{jt}\right]_\odot + \left[\left[\mathfrak{w}^+_{s}, \mathfrak{w}^0_{jt}\right]_\odot, \mathfrak{w}^0_{ir}\right]_\odot + \left[\left[\mathfrak{w}^0_{jt}, \mathfrak{w}^0_{ir}\right]_\odot, \mathfrak{w}^+_{s}\right]_\odot &= 0 \\
\left[\left[\mathfrak{w}^0_{ir}, \mathfrak{w}^+_{s}\right]_\odot, \mathfrak{w}^+_{t}\right]_\odot + \left[\left[\mathfrak{w}^+_{s}, \mathfrak{w}^+_{t}\right]_\odot, \mathfrak{w}^0_{ir}\right]_\odot + \left[\left[\mathfrak{w}^+_{t}, \mathfrak{w}^0_{ir}\right]_\odot, \mathfrak{w}^+_{s}\right]_\odot &= 0 \\
\left[\left[\mathfrak{w}^+_{r}, \mathfrak{w}^+_{s}\right]_\odot, \mathfrak{w}^+_{t}\right]_\odot + \left[\left[\mathfrak{w}^+_{s}, \mathfrak{w}^+_{t}\right]_\odot, \mathfrak{w}^+_{r}\right]_\odot + \left[\left[\mathfrak{w}^+_{t}, \mathfrak{w}^+_{r}\right]_\odot, \mathfrak{w}^+_{s}\right]_\odot &= 0
\end{aligned}$$

are satisfied, as it should for an associative algebra of this kind.
We proceed by the following definition-proposition to enhance $\mathfrak{W}(\Pi, \mathfrak{X})$ to a Hopf-algebra.

4.3.2 Definition-Proposition *Let $\mathfrak{W}(\Pi, \mathfrak{X})$ be an algebra with the two-sided ideal $\mathcal{I}_{\mathfrak{W}}$, defined as above. Then $\mathfrak{W}(\Pi, \mathfrak{X})$ is a Hopf-algebra with the following coproduct, counit and antipode*

$$\begin{aligned} \forall\, i, r \in 1, 2, \dots n \quad : \quad & \Delta(\mathfrak{w}^0_{ir}) = \mathfrak{w}^0_{ir} \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{w}^0_{ir}, \quad \epsilon(\mathfrak{w}^0_{ir}) = 0, \\ & S(\mathfrak{w}^0_{ir}) = -\mathfrak{w}^0_{ir}, \\ & \Delta(\mathfrak{w}^+_r) = \mathfrak{w}^+_r \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{w}^+_r, \quad \epsilon(\mathfrak{w}^+_r) = 0, \\ & S(\mathfrak{w}^+_r) = -\mathfrak{w}^+_r. \end{aligned} \tag{4.12}$$

Proof: It is evident that the axioms of coassociativity $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ and that of the counit $(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta$ are satisfied for both, $\mathfrak{w}^0_{ir}$ and $\mathfrak{w}^+_r$. Moreover the antipode axiom $\mu \circ (\mathrm{S} \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon = \mu \circ (\mathrm{id} \otimes \mathrm{S}) \circ \Delta$ is fulfilled as well. Here μ is the multiplication within $\mathfrak{W}(\Pi, \mathfrak{X})$ and η is the unit element, being the map

$$\begin{aligned} \eta : \mathbf{K} &\longrightarrow \mathfrak{W}(\Pi, \mathfrak{X}) \\ \lambda &\mapsto \lambda \cdot \mathbf{1} \end{aligned}$$

Since it is an important issue here, we explicitly check on the homomorphy property of the coproduct. Thus we check that

$$\begin{aligned} \left[\Delta(\mathfrak{w}^0_{ir}), \Delta(\mathfrak{w}^0_{js})\right]_\odot &= \left[\mathfrak{w}^0_{ir}, \mathfrak{w}^0_{js}\right]_\odot \otimes \mathbf{1} + \mathbf{1} \otimes \left[\mathfrak{w}^0_{ir}, \mathfrak{w}^0_{js}\right]_\odot \\ &= -i\delta_{rj}\Delta(\mathfrak{w}^0_{is}) + i\delta_{si}\Delta(\mathfrak{w}^0_{jr}) \end{aligned}$$

and

$$\begin{aligned} \left[\Delta(\mathfrak{w}^0_{ir}), \Delta(\mathfrak{w}^+_s)\right]_\odot &= \left[\mathfrak{w}^0_{ir}, \mathfrak{w}^+_s\right]_\odot \otimes \mathbf{1} + \mathbf{1} \otimes \left[\mathfrak{w}^0_{ir}, \mathfrak{w}^+_s\right]_\odot \\ &= i\delta_{si}\Delta(\mathfrak{w}^+_r). \end{aligned}$$

The same trivially holds for the counit. The antipode obviously is an anti-algebra homomorphism, as it should, since

$$\begin{aligned} -\left[\mathrm{S}(\mathfrak{w}^0_{ir}), \mathrm{S}(\mathfrak{w}^0_{js})\right]_\odot &= -i\delta_{rj}\mathrm{S}(\mathfrak{w}^0_{is}) + i\delta_{si}\mathrm{S}(\mathfrak{w}^0_{jr}) \\ -\left[\mathrm{S}(\mathfrak{w}^0_{ir}), \mathrm{S}(\mathfrak{w}^+_s)\right]_\odot &= i\delta_{si}\mathrm{S}(\mathfrak{w}^+_r), \end{aligned}$$

□

We are now prepared to consider representations of $\mathfrak{W}(\Pi, \mathfrak{X})$ on algebras.

4.3.4 Representation of $\mathfrak{W}(\Pi, \mathfrak{X})$ on $U(\mathfrak{X})$

It is our aim within this subsection to represent $\mathfrak{W}(\Pi, \mathfrak{X})$ on the algebra of coordinates $U(\mathfrak{X})$. Remember that we do not treat $U(\mathfrak{X})$ as a Hopf-algebra.

As vector fields, we introduce the *left action* of $\mathfrak{w}^0_{ir}, \mathfrak{w}^+_s \in \mathfrak{W}(\Pi, \mathfrak{X})$ on $U(\mathfrak{X})$ by

$$\begin{aligned} \mathfrak{w}^0_{ir} \triangleright x_j = x_i(\pi_r \triangleright x_j) = -i\delta_{rj}x_i, \qquad & \mathfrak{w}^0_{ir} \triangleright \mathbf{1} = \epsilon(\mathfrak{w}^0_{ir}), \\ \mathfrak{w}^+_r \triangleright x_j = \pi_r \triangleright x_j = -i\delta_{rj}\mathbf{1}, \qquad & \mathfrak{w}^+_r \triangleright \mathbf{1} = \epsilon(\mathfrak{w}^+_r). \end{aligned} \tag{4.13}$$

We have thus to verify that the Hopf-algebra of vector fields $\mathfrak{W}(\Pi, \mathfrak{X})$ is realized as vector space endomorphisms on $U(\mathfrak{X})$. In particular this means that the first two relations of (4.11) have to be realized by means of (4.13), i.e. we obtain

$$\begin{aligned} &\left(\left[\mathfrak{w}^0_{ir}, \mathfrak{w}^0_{js}\right]_\odot + i\delta_{rj}\mathfrak{w}^0_{is} - i\delta_{si}\mathfrak{w}^0_{jr} \right) \triangleright x_k \\ &\quad = \mathfrak{w}^0_{ir} \triangleright \left(\mathfrak{w}^0_{js} \triangleright x_k\right) - \mathfrak{w}^0_{js} \triangleright \left(\mathfrak{w}^0_{ir} \triangleright x_k\right) + i\delta_{rj}\left(\mathfrak{w}^0_{is} \triangleright x_k\right) - i\delta_{si}\left(\mathfrak{w}^0_{jr} \triangleright x_k\right) \\ &\quad = \mathfrak{w}^0_{ir} \triangleright (-i\delta_{sk}x_j) - \mathfrak{w}^0_{js} \triangleright (-i\delta_{rk}x_i) + i\delta_{rj}\left(-i\delta_{sk}x_i\right) - i\delta_{si}\left(-i\delta_{rk}x_j\right) \\ &\quad = -\delta_{sk}\delta_{rj}x_i + \delta_{rk}\delta_{si}x_j + \delta_{jr}\delta_{sk}x_i - \delta_{si}\delta_{rk}x_j = 0 \end{aligned}$$

and

$$\begin{aligned} &\left(\left[\mathfrak{w}^0_{ir}, \mathfrak{w}^+_s\right]_\odot - i\delta_{si}\mathfrak{w}^+_r \right) \triangleright x_j \\ &\quad = \mathfrak{w}^0_{ir} \triangleright \left(\mathfrak{w}^+_s \triangleright x_j\right) - \mathfrak{w}^+_s \triangleright \left(\mathfrak{w}^0_{ir} \triangleright x_j\right) - i\delta_{si}\left(\mathfrak{w}^+_r \triangleright x_j\right) \\ &\quad = \mathfrak{w}^0_{ir} \triangleright (-i\delta_{sj}\mathbf{1}) - \mathfrak{w}^+_s \triangleright (-i\delta_{rj}x_i) - i\delta_{si}\left(-i\delta_{rj}\mathbf{1}\right) \\ &\quad = \delta_{rj}\delta_{si} - \delta_{si}\delta_{rj} = 0. \end{aligned}$$

The third relation is already represented on $U(\mathfrak{X})$ given by (4.9). We further more have to check whether the representation (4.13) respects the algebra relations (4.4) of $U(\mathfrak{X})$, i.e. we have

$$\begin{aligned} &\mathfrak{w}^0_{ir} \triangleright (x_jx_k - x_kx_j) = \Delta(\mathfrak{w}^0_{ir}) \triangleright (x_jx_k - x_kx_j) \\ &\quad = (\mathfrak{w}^0_{ir} \triangleright x_j)x_k + x_j(\mathfrak{w}^0_{ir} \triangleright x_k) - (\mathfrak{w}^0_{ir} \triangleright x_k)x_j - x_k(\mathfrak{w}^0_{ir} \triangleright x_j) \\ &\quad = (-i\delta_{rj}x_i)x_k + x_j(-i\delta_{rk}x_i) - (-i\delta_{rk}x_i)x_j - x_k(-i\delta_{rj}x_i) = 0 \end{aligned}$$

and

$$\begin{aligned} &\mathfrak{w}^+_r \triangleright (x_jx_k - x_kx_j) = \Delta(\mathfrak{w}^+_r) \triangleright (x_jx_k - x_kx_j) \\ &\quad = (\mathfrak{w}^+_r \triangleright x_j)x_k + x_j(\mathfrak{w}^+_r \triangleright x_k) - (\mathfrak{w}^+_r \triangleright x_k)x_j - x_k(\mathfrak{w}^+_r \triangleright x_j) \\ &\quad = (-i\delta_{rj})x_k + x_j(-i\delta_{rk}) - (-i\delta_{rk})x_j - x_k(-i\delta_{rj}) = 0. \end{aligned}$$

We thus made all necessary preparations to attack the actual interesting step in the next section.

4.4 Representation of $U(\mathfrak{g})$ in $\mathfrak{W}(\Pi, \mathfrak{X})$

In this section we map $U(\mathfrak{g})$ as a subalgebra within $\mathfrak{W}(\Pi, \mathfrak{X})$ by means of its matrix representation (4.2) and the Hopf-algebra homomorphism

$$\begin{array}{rcl} \rho: \ U(\mathfrak{g}) & \longrightarrow & \mathfrak{W}(\Pi, \mathfrak{X}) \\ g_a & \mapsto & i(g_a)_{ri}\mathfrak{w}^0_{ir}. \end{array}$$

We verify that the generating relations (4.1) of $U(\mathfrak{g})$ are realized in terms of (4.11). In particular, we obtain for basis elements $g_a, g_b \in U(\mathfrak{g})$

$$\begin{aligned} [g_a, g_b]_\odot &= \left[(g_a)_{ri}\mathfrak{w}^0_{ir}, (g_b)_{sj}\mathfrak{w}^0_{js}\right]_\odot = (g_a)_{ri}(g_b)_{sj}\left[\mathfrak{w}^0_{ir}, \mathfrak{w}^0_{js}\right]_\odot \\ &= (g_a)_{ri}(g_b)_{sj}\left(-i\delta_{rj}\mathfrak{w}^0_{is} + i\delta_{si}\mathfrak{w}^0_{jr}\right) \\ &= -i(g_b)_{sk}(g_a)_{ki}\mathfrak{w}^0_{is} + i(g_a)_{rk}(g_b)_{kj}\mathfrak{w}^0_{jr} \\ &= i((g_a)_{sk}(g_b)_{ki} - (g_b)_{sk}(g_a)_{ki})\mathfrak{w}^0_{is} = if_{abc}i(g_c)_{si}\mathfrak{w}^0_{is} = if_{abc}g_c \end{aligned}$$

Here we use summation convention for any pair of equal indices. The Hopf structure (4.12) $\mathfrak{W}(\Pi, \mathfrak{X})$ corresponds to that of $U(\mathfrak{g})$, i. e.

$$\begin{aligned} \Delta(g_a) &= \Delta(i(g_a)_{ri}\mathfrak{w}^0_{ir}) = i(g_a)_{ri}\Delta(\mathfrak{w}^0_{ir}) = i(g_a)_{ri}\left(\mathfrak{w}^0_{ir} \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{w}^0_{ir}\right) \\ &= g_a \otimes \mathbf{1} + \mathbf{1} \otimes g_a \\ \epsilon(g_a) &= \epsilon(i(g_a)_{ri}\mathfrak{w}^0_{ir}) = i(g_a)_{ri}\epsilon(\mathfrak{w}^0_{ir}) = 0 \\ \mathrm{S}(g_a) &= \mathrm{S}(i(g_a)_{ri}\mathfrak{w}^0_{ir}) = i(g_a)_{ri}\mathrm{S}(\mathfrak{w}^0_{ir}) = -i(g_a)_{ri}\mathfrak{w}^0_{ir} = -g_a \end{aligned}$$

We verify that the representation of $U(\mathfrak{g})$ in $\mathfrak{W}(\Pi, \mathfrak{X})$ also accomodates the correct representation on $U(\mathfrak{X})$. The representation of $\mathfrak{W}(\Pi, \mathfrak{X})$ on $U(\mathfrak{X})$ implies that

$$\begin{aligned} (g_a \triangleright x_i)_k &= \left(\left(i(g_a)_{sj}\mathfrak{w}^0_{js}\right) \triangleright x_i\right)_k = \left(i(g_a)_{sj}\left(\mathfrak{w}^0_{js} \triangleright x_i\right)\right)_k \\ &= \left(i(g_a)_{sj}\left(-i\delta_{si}x_j\right)\right)_k = \left((g_a)_{ij}x_j\right)_k = (g_a)_{kj}x_j \end{aligned}$$

This neatly corresponds to the matrix representation (4.2). We obtain double applications of the represented generators of $U(\mathfrak{g})$ according to

$$\begin{aligned} ((g_a g_b) \triangleright x_i)_k &= \left(i(g_b)_{sj}\mathfrak{w}^0_{js} \triangleright \left(i(g_a)_{rl}\mathfrak{w}^0_{lr} \triangleright x_i\right)\right)_k \\ &= \left(-(g_b)_{sj}(g_a)_{rl}\mathfrak{w}^0_{js} \triangleright \left(-i\delta_{ir}x_l\right)\right)_k \\ &= \left(-(g_b)_{sj}(g_a)_{rl}(-i\delta_{ir})\left(\mathfrak{w}^0_{js} \triangleright x_l\right)\right)_k \\ &= \left(-(g_b)_{sj}(g_a)_{rl}(-i\delta_{ir})(-i\delta_{ls})x_j\right)_k \\ &= \left((g_a)_{il}(g_b)_{lj}x_j\right)_k = (g_a)_{kl}(g_b)_{lj}x_j \end{aligned}$$

Note that the formal reversal of the order of generators $\mathfrak{w}^0$ is only applied to get indices straight. The actual order of application of generators remains unchanged as one can see from the last equation. We once more verify that this actually realizes the generating relations (4.1) of $U(\mathfrak{g})$ on $U(\mathfrak{X})$ via matrix representation according to (4.3), i.e.

$$\begin{aligned}((g_a g_b - g_b g_a) \triangleright x_i)_k &= (((g_a)_{il}(g_b)_{lj} - (g_b)_{il}(g_a)_{lj})\, x_j)_k \\ &= ((if_{abc}(g_c)_{ij})\, x_j)_k = \left(if_{abc}\left(i(g_c)_{sj}\mathfrak{w}^0_{js}\right) \triangleright x_i\right)_k \\ &= (if_{abc}(g_c \triangleright x_i))_k\end{aligned}$$

Through the coproduct in $\mathfrak{W}(\Pi, \mathfrak{X})$ it is clear that our realization of $U(\mathfrak{g})$ in $\mathfrak{W}(\Pi, \mathfrak{X})$ respects the generating relations of $U(\mathfrak{X})$. We thus have received a left action of the Hopf-algebra $U(\mathfrak{g})$ on $U(\mathfrak{X})$ via its matrix representation within $\mathfrak{W}(\Pi, \mathfrak{X})$. We can now proceed to twist $\mathfrak{W}(\Pi, \mathfrak{X})$ and thus to most generally twist its subalgebra $U(\mathfrak{g})$ as well.

4.5 Twisting

In order to obtain deformations $\mathfrak{W}(\Pi, \mathfrak{X})$, we introduce twists in this section. To this purpose we recall some basic properties of twists. Since we want to consider the twists of vector fields to be starproducts of associative algebras of coordinates $U(\mathfrak{X})$ at the same time, it is our intend to cleatify that the definition of twists incorporates this demand. We then proceed and give some examples of twists for $\mathfrak{W}(\Pi, \mathfrak{X})$ that we apply to a two-dimensional representation of $U(sl_2)$ in the next section. For this section we recommend [19] as a textbook for reference. We begin by recalling the definition of a twist.

4.5.1 Definition *Let $(\mathcal{H}, \mu, \eta, \Delta, \epsilon, S; \mathbf{K})$ be a Hopf-algebra over the field $\mathbf{K}$. Then an invertible object $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$ is called a* twist*, if the following two conditions hold*

$$\mathcal{F}_{12}\,(\Delta \otimes id)\,(\mathcal{F}) = \mathcal{F}_{23}\,(id \otimes \Delta)\,(\mathcal{F}) \tag{4.14}$$

$$(\epsilon \otimes id)\,(\mathcal{F}) = 1 = (id \otimes \epsilon)\,(\mathcal{F}). \tag{4.15}$$

For $\mathcal{F} = \sum \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)}$ the objects $\mathcal{F}_{12}$ and $\mathcal{F}_{23}$ are defined by

$$\begin{aligned}\mathcal{F}_{12} &= \sum \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)} \otimes \mathbf{1} \\ \mathcal{F}_{23} &= \sum \mathbf{1} \otimes \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)}.\end{aligned}$$

Using this definition, we can now recall the required proposition stating how a twist is used to deform the corresponding Hopf-algebra.

4.5.2 PROPOSITION *Let $(\mathcal{H}, \mu, \eta, \Delta, \epsilon, S; \mathbf{K})$ be a Hopf-algebra and let furthermore the objects $\eta, \eta^{-1} \in \mathcal{H}$ be given by*

$$\begin{aligned} \eta &= \mu\,(id \otimes S)\,(\mathcal{F}) \\ \eta^{-1} &= \mu\,(S \otimes id)\,(\mathcal{F}). \end{aligned}$$

Then $(\mathcal{H}, \mu, \eta, \Delta_{\mathcal{F}}, \epsilon, S_{\mathcal{F}}; \mathbf{K})$ with

$$\begin{aligned} \Delta_{\mathcal{F}}(h) &= \mathcal{F}\Delta(h)\mathcal{F}^{-1} \\ S_{\mathcal{F}}(h) &= \eta S(h)\eta^{-1} \end{aligned}$$

and $h \in \mathcal{H}$ is the Hopf-algebra $\mathcal{H}_{\mathcal{F}}$ that is called the twist of *$\mathcal{H}$.*

Note that the Hopf-algebra $\mathcal{H}$ not necessarily has to be cocommutative. We further elucidate some consequences and properties of the defined twist before we come to specific examples for $\mathfrak{W}(\Pi, \mathfrak{X})$. If the Hopf-algebra $\mathcal{H}$ is represented on $U(\mathfrak{X})$ by a left action, then the generating relations (4.4) of $U(\mathfrak{X})$ are preserved under the action of $\mathcal{H}$, i.e. for $h \in \mathcal{H}$ we have

$$\begin{aligned} x_i x_j - x_j x_i = 0 \quad \Rightarrow \quad & h \triangleright (x_i x_j - x_j x_i) \\ & = \Delta(h) \triangleright (x_i x_j - x_j x_i) \\ & = \sum (h_{(1)} \triangleright x_i)(h_{(2)} \triangleright x_j) - (h_{(1)} \triangleright x_j)(h_{(2)} \triangleright x_i) = 0. \end{aligned}$$

Within the representation of $\mathcal{H}$ on $U(\mathfrak{X})$ we can consider a twist $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$ to deform the product μ of $U(\mathfrak{X})$ to a noncommutative product $\mu_{\mathcal{F}}$ by

$$\mu_{\mathcal{F}}(x_i, x_j) = x_i *_{\mathcal{F}} x_j = \mathcal{F}^{-1} \triangleright (x_i \cdot x_j) = \mu\left((\mathcal{F}^{-1\ (1)} \triangleright x_i), (\mathcal{F}^{-1\ (2)} \triangleright x_j)\right).$$

This implies new generating relations for a deformation of $U(\mathfrak{X})$, that we further denote by $U(\mathfrak{X}_{\mathcal{F}})$, being

$$x_i *_{\mathcal{F}} x_j - x_j *_{\mathcal{F}} x_i - [x_i \overset{*_{\mathcal{F}}}{,} x_j] = 0, \tag{4.16}$$

where the commutator $[x_i \overset{*_{\mathcal{F}}}{,} x_j]$ has to be replaced by a corresponding right hand side. This nonvanishing commutator reflects the noncocommutativity of the twisted coproduct $\Delta_{\mathcal{F}}$ in $\mathcal{H}_{\mathcal{F}}$. The defining relations (4.14) and (4.15) of the twist $\mathcal{F}$ thereby ensure that the axiom of coassociativity and the counit axiom of the coproduct $\Delta_{\mathcal{F}}$ are satisfied, i.e. that

$$\begin{aligned} (\Delta_{\mathcal{F}} \otimes \mathrm{id}) \circ \Delta_{\mathcal{F}} &= (\mathrm{id} \otimes \Delta_{\mathcal{F}}) \circ \Delta_{\mathcal{F}} \\ (\epsilon \otimes \mathrm{id}) \circ \Delta_{\mathcal{F}} &= (\mathrm{id} \otimes \epsilon) \circ \Delta_{\mathcal{F}} \end{aligned}$$

Covariance of of the generating relations (4.16) of $U(\mathfrak{X}_\mathcal{F})$ under the action of $\mathcal{H}_\mathcal{F}$ is then given by

$$\begin{aligned} & h \triangleright (x_i *_\mathcal{F} x_j - x_j *_\mathcal{F} x_i - [x_i \stackrel{*}{,}_\mathcal{F} x_j]) \\ &\quad = h \triangleright (\mathcal{F}^{-1} \triangleright (x_i \cdot x_j)) - h \triangleright (\mathcal{F}^{-1} \triangleright (x_j \cdot x_i)) - h \triangleright [x_i \stackrel{*}{,}_\mathcal{F} x_j] \\ &\quad = \mathcal{F}^{-1} \triangleright \left(\mathcal{F}\Delta(h)\mathcal{F}^{-1}\right) \triangleright (x_i \cdot x_j) \\ &\qquad -\mathcal{F}^{-1} \triangleright \left(\mathcal{F}\Delta(h)\mathcal{F}^{-1}\right) \triangleright (x_j \cdot x_i) - h \triangleright [x_i \stackrel{*}{,}_\mathcal{F} x_j] \\ &\quad = \mathcal{F}^{-1} \triangleright (\Delta_\mathcal{F}(h) \triangleright (x_i \cdot x_j) - \Delta_\mathcal{F}(h) \triangleright (x_j \cdot x_i)) - h \triangleright [x_i \stackrel{*}{,}_\mathcal{F} x_j] \end{aligned}$$

Thus transformation and deformation commute, such that noncommutativity of $U(\mathfrak{X}_\mathcal{F})$ is preserved under the left action of $\mathcal{H}_\mathcal{F}$. Coassociativity of $\Delta_\mathcal{F}$ implies the associativity of the starproduct $*_\mathcal{F}$, i.e. we have

$$\begin{aligned} \mathcal{F} \triangleright (h \triangleright (x_i *_\mathcal{F} (x_j *_\mathcal{F} x_i))) &= (\mathrm{id} \otimes \Delta_\mathcal{F}) \circ \Delta_\mathcal{F}(h) \triangleright (x_i \cdot x_j \cdot x_i)) \\ &= (\Delta_\mathcal{F} \otimes \mathrm{id}) \circ \Delta_\mathcal{F}(h) \triangleright (x_i \cdot x_j \cdot x_i)) \\ &= \mathcal{F} \triangleright (h \triangleright ((x_i *_\mathcal{F} x_j) *_\mathcal{F} x_i)) \end{aligned}$$

In the following we consider specific twistings of $\mathfrak{W}(\Pi, \mathfrak{X})$. It is our intend to merely outline the application of the formalism. We thus stick to some simple but nontrivial and genuine examples. We encourage the reader to derive more sophisticated twists for his very own purpose and use the following consideration as an examplary guiding line. Our first example is given by

$$\mathcal{F}_\theta := e^{\frac{i}{2}\theta_{rs}\mathfrak{w}_r^+ \otimes \mathfrak{w}_s^+}, \quad \theta_{rs} = -\theta_{sr} \in \mathbf{R} \tag{4.17}$$

Since $\epsilon(\mathfrak{w}_r^+) = 0$ relation (4.15) is satisfied. Relation (4.14) is satisfied as well since

$$\begin{aligned} & e^{\frac{i}{2}\theta_{rs}\mathfrak{w}_r^+ \otimes \mathfrak{w}_s^+ \otimes \mathbf{1}} (\Delta \otimes \mathrm{id})(e^{\frac{i}{2}\theta_{rs}\mathfrak{w}_r^+ \otimes \mathfrak{w}_s^+}) \\ &\quad = e^{\frac{i}{2}\theta_{rs}(\mathfrak{w}_r^+ \otimes \mathfrak{w}_s^+ \otimes \mathbf{1} + \mathfrak{w}_r^+ \otimes \mathbf{1} \otimes \mathfrak{w}_s^+ + \mathbf{1} \otimes \mathfrak{w}_r^+ \otimes \mathfrak{w}_s^+)} \\ &\quad = e^{\frac{i}{2}\theta_{rs}\mathbf{1} \otimes \mathfrak{w}_r^+ \otimes \mathfrak{w}_s^+} (\mathrm{id} \otimes \Delta)(e^{\frac{i}{2}\theta_{rs}\mathfrak{w}_r^+ \otimes \mathfrak{w}_s^+}), \end{aligned}$$

due to the vanishing commutator $[\mathfrak{w}_r^+, \mathfrak{w}_s^+] = 0$. In general these computations are performed using the Baker-Campbell-Hausdorff formula

$$e^A\, e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}([A,[A,B]]-[B,[A,B]])+\frac{1}{48}([A,[B,[B,A]]]-[B,[A,[A,B]]])+\cdots}.$$

Using the formula

$$e^A\, B\, e^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} [A, [A, [A, \ldots [A, B]]]] \tag{4.18}$$

we can now compute the twisted coproducts of $\mathfrak{w}_r^+$ and $\mathfrak{w}_{ir}^0$ to be

$$\begin{aligned}\Delta_{\mathcal{F}}(\mathfrak{w}_r^+) &= \mathfrak{w}_r^+ \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{w}_r^+ \\ \Delta_{\mathcal{F}}(\mathfrak{w}_{ir}^0) &= \mathfrak{w}_{ir}^0 \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{w}_{ir}^0 \\ & \quad -\frac{1}{2}\theta_{is}\left(\mathfrak{w}_s^+ \otimes \mathfrak{w}_r^+ - \mathfrak{w}_r^+ \otimes \mathfrak{w}_s^+\right).\end{aligned}$$

These of course correspond to the results of [17], but now this twist can be applied to any representation of a universal enveloping algebra $U(\mathfrak{g})$. We obtain the generating relations of $U(\mathfrak{X}_{\mathcal{F}_\theta})$ by

$$\begin{aligned}& x_i *_{\mathcal{F}_\theta} x_j - x_j *_{\mathcal{F}_\theta} x_i \\ &\quad = e^{-\frac{i}{2}\theta_{rs}\mathfrak{w}_r^+\otimes\mathfrak{w}_s^+} \triangleright (x_i x_j) - e^{-\frac{i}{2}\theta_{rs}\mathfrak{w}_r^+\otimes\mathfrak{w}_s^+} \triangleright (x_j x_i) \\ &\quad = (1 - \frac{i}{2}\theta_{rs}\mathfrak{w}_r^+ \otimes \mathfrak{w}_s^+) \triangleright (x_i x_j) - (1 - \frac{i}{2}\theta_{rs}\mathfrak{w}_r^+ \otimes \mathfrak{w}_s^+) \triangleright (x_j x_i) \\ &\quad = x_i x_j - \frac{i}{2}\theta_{rs}(-i\delta_{ri})(-i\delta_{sj}) - x_j x_i + \frac{i}{2}\theta_{rs}(-i\delta_{rj})(-i\delta_{si}) \\ &\quad = i\,\theta_{ij}.\end{aligned}$$

We come now to a more genuine example taken from [38]. We introduce the twist

$$\mathcal{F}_h := e^{i\,h\,\mathfrak{w}_{11}^0\otimes\mathfrak{w}_2^+}. \tag{4.19}$$

The generators $\mathfrak{w}_{11}^0$ and $\mathfrak{w}_2^+$ commute according to (4.11), i.e. $[\mathfrak{w}_{11}^0, \mathfrak{w}_2^+] = 0$. Relation (4.15) once more is trivially satisfied. We check for (4.14), i.e.

$$\begin{aligned}& e^{i\,h\,\mathfrak{w}_{11}^0\otimes\mathfrak{w}_2^+\otimes\mathbf{1}}\,(\Delta\otimes\mathrm{id})\,(e^{i\,h\,\mathfrak{w}_{11}^0\otimes\mathfrak{w}_2^+}) \\ &\quad = e^{i\,h\,(\mathfrak{w}_{11}^0\otimes\mathfrak{w}_2^+\otimes\mathbf{1}+\mathfrak{w}_{11}^0\otimes\mathbf{1}\otimes\mathfrak{w}_2^+ + \mathbf{1}\otimes\mathfrak{w}_{11}^0\otimes\mathfrak{w}_2^+)} \\ &\quad = e^{i\,h\,\mathbf{1}\otimes\mathfrak{w}_{11}^0\otimes\mathfrak{w}_2^+}\,(\mathrm{id}\otimes\Delta)\,(e^{i\,h\,\mathfrak{w}_{11}^0\otimes\mathfrak{w}_2^+}).\end{aligned}$$

We once more derive the twisted coproducts using formula (4.18). The coproducts of $\mathfrak{w}_s^+$ remain undeformed for $s \neq 1$. For the coproduct of $\mathfrak{w}_1^+$, we obtain

$$\Delta_{\mathcal{F}_h}(\mathfrak{w}_1^+) = \mathfrak{w}_1^+ \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{w}_1^+ + \mathfrak{w}_1^+ \otimes \left(e^{-h\mathfrak{w}_2^+} - 1\right). \tag{4.20}$$

The twisted coproduct of $\mathfrak{w}_{ir}^0$ also remains undeformed apart from four specific cases, that are

$$\begin{aligned} r \neq 1 &\;:\; \Delta_{\mathcal{F}_h}(\mathfrak{w}_{1r}^0) = \mathfrak{w}_{1r}^0 \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{w}_{1r}^0 + \mathfrak{w}_{1r}^0 \otimes \left(e^{+h\mathfrak{w}_2^+} - 1\right), \\ i \neq 1,2 &\;:\; \Delta_{\mathcal{F}_h}(\mathfrak{w}_{i1}^0) = \mathfrak{w}_{i1}^0 \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{w}_{i1}^0 + \mathfrak{w}_{i1}^0 \otimes \left(e^{-h\mathfrak{w}_2^+} - 1\right), \\ i = 2, r = 1 &\;:\; \Delta_{\mathcal{F}_h}(\mathfrak{w}_{21}^0) = \mathfrak{w}_{21}^0 \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{w}_{21}^0 + h\,\mathfrak{w}_{11}^0 \otimes \mathfrak{w}_1^+ \\ &\qquad + \mathfrak{w}_{21}^0 \otimes \left(e^{-h\,\mathfrak{w}_2^+} - 1\right), \\ r \neq 1 &\;:\; \Delta_{\mathcal{F}_h}(\mathfrak{w}_{2r}^0) = \mathfrak{w}_{2r}^0 \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{w}_{2r}^0 + h\,\mathfrak{w}_{11}^0 \otimes \mathfrak{w}_r^+.\end{aligned}$$

The generating relations of $U(\mathfrak{X}_{\mathcal{F}_h})$ are then given by

$$\begin{aligned}
&x_i *_{\mathcal{F}_h} x_j - x_j *_{\mathcal{F}_h} x_i \\
&\quad = e^{-i\,h\,\mathfrak{w}^0_{11}\otimes\mathfrak{w}^+_2} \triangleright (x_i x_j) - e^{-i\,h\,\mathfrak{w}^0_{11}\otimes\mathfrak{w}^+_2} \triangleright (x_j x_i) \\
&\quad = (1 - i\,h\,\mathfrak{w}^0_{11} \otimes \mathfrak{w}^+_2) \triangleright (x_i x_j) - (1 - i\,h\,\mathfrak{w}^0_{11} \otimes \mathfrak{w}^+_2) \triangleright (x_j x_i) \\
&\quad = x_i x_j + i\,h\delta_{i1}\delta_{j2}x_1 - x_j x_i - i\,h\delta_{j1}\delta_{i2}x_1 \\
&\quad = i\,h(\delta_{i1}\delta_{j2} - \delta_{j1}\delta_{i2})x_1.
\end{aligned}$$

We thus see in this final example how the introduced formalism of vector fields $\mathfrak{W}(\Pi, \mathfrak{X})$ unfolds its impact. The twist $\mathcal{F}_h$ cannot be expressed in terms of generators of $U(\mathfrak{g})$ but through the representation of $U(\mathfrak{g})$ in $\mathfrak{W}(\Pi, \mathfrak{X})$ we now, nevertheless, use it to twist its coproduct and thus obtain the desired deformation of the symmetry algebra. This is sketched in the next section at the example of $U(sl_2)$.

4.6 Deformation of a two-dimensional Representation of $U(sl_2)$

In this section we shortly consider the two-dimensional representation of $U(sl_2)$ that we want to twist by means of (4.19). To this purpose we directly consider the corresponding matrix representation of $U(sl_2)$ given in terms of Pauli-matrices and a canonical basis for the representation space. The Hopf-algebra of $U(sl_2)$ can thus be considered to be generated by the basis $(\sigma_i)_{i\in 1,2,3}$ with the Hopf-structure

$$\Delta(\sigma_i) = \sigma_i \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_i, \quad \epsilon(\sigma_i) = 0, \quad \mathrm{S}(\sigma_i) = -\sigma_i,$$

In the two-dimensional representation we then identify with the well-known Pauli-matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Making the identification

$$x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

we obtain the explicit left action of the two-dimensional representation of $U(sl_2)$ by

$$\begin{aligned}
&\sigma_1 \triangleright x_1 = x_2, \quad \sigma_2 \triangleright x_1 = i\,x_2, \quad \sigma_3 \triangleright x_1 = x_1, \\
&\sigma_1 \triangleright x_2 = x_1, \quad \sigma_2 \triangleright x_2 = -i\,x_1, \quad \sigma_3 \triangleright x_2 = -x_2
\end{aligned}$$

The Hopf-algebra $U(sl_2)$ thus gets represented in the accordingly dimensioned Hopf-algebra of vector fields $\mathfrak{W}(\Pi, \mathfrak{X})$ by

$$\begin{aligned} \sigma_1 &= i(\mathfrak{w}^0_{21} + \mathfrak{w}^0_{12}) \\ \sigma_2 &= \mathfrak{w}^0_{12} - \mathfrak{w}^0_{21} \\ \sigma_3 &= i(\mathfrak{w}^0_{11} - \mathfrak{w}^0_{22}) \end{aligned}$$

For the twist-deformation of these coproducts we now merely have to insert these expressions in those for the coproducts of σ_i from above and afterwards insert the twisted expressions for the vector fields from the last section. In particular for the twist (4.19) we obtain in two dimensions the following explicit expressions for the twisted coproducts of $\mathfrak{w}^+_1$ and $\mathfrak{w}^+_2$ to be

$$\begin{aligned} \Delta_{\mathcal{F}_h}(\mathfrak{w}^+_1) &= \mathfrak{w}^+_1 \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{w}^+_1 + \mathfrak{w}^+_1 \otimes (e^{-h\mathfrak{w}^+_2} - 1) \\ \Delta_{\mathcal{F}_h}(\mathfrak{w}^+_2) &= \mathfrak{w}^+_2 \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{w}^+_2 . \end{aligned}$$

We as well obtain the twisted coproducts of $\mathfrak{w}^0_{11}$, $\mathfrak{w}^0_{12}$, $\mathfrak{w}^0_{21}$ and $\mathfrak{w}^0_{21}$ to be given by

$$\begin{aligned} \Delta_{\mathcal{F}_h}(\mathfrak{w}^0_{11}) &= \mathfrak{w}^0_{11} \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{w}^0_{11} \\ \Delta_{\mathcal{F}_h}(\mathfrak{w}^0_{12}) &= \mathfrak{w}^0_{12} \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{w}^0_{12} + \mathfrak{w}^0_{12} \otimes (e^{+h\mathfrak{w}^+_2} - 1) \\ \Delta_{\mathcal{F}_h}(\mathfrak{w}^0_{21}) &= \mathfrak{w}^0_{21} \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{w}^0_{21} + h\, \mathfrak{w}^0_{11} \otimes \mathfrak{w}^+_1 + \mathfrak{w}^0_{21} \otimes (e^{-h\mathfrak{w}^+_2} - 1) \\ \Delta_{\mathcal{F}_h}(\mathfrak{w}^0_{22}) &= \mathfrak{w}^0_{22} \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{w}^0_{22} + h\, \mathfrak{w}^0_{11} \otimes \mathfrak{w}^+_2 . \end{aligned}$$

The generating relations of $U(\mathfrak{X}_{\mathcal{F}_h})$ then read

$$x_1 *_{\mathcal{F}_h} x_2 - x_2 *_{\mathcal{F}_h} x_1 = i\, h\, x_1 .$$

The twisted coproducts of the generators σ_i of $U_{\mathcal{F}_h}(sl_2)$ are then given by

$$\begin{aligned} \Delta_{\mathcal{F}_h}(\sigma_1) &= i(\Delta_{\mathcal{F}_h}(\mathfrak{w}^0_{21}) + \Delta_{\mathcal{F}_h}(\mathfrak{w}^0_{12})) \\ \Delta_{\mathcal{F}_h}(\sigma_2) &= \Delta_{\mathcal{F}_h}(\mathfrak{w}^0_{12}) - \Delta_{\mathcal{F}_h}(\mathfrak{w}^0_{21}) \\ \Delta_{\mathcal{F}_h}(\sigma_3) &= i(\Delta_{\mathcal{F}_h}(\mathfrak{w}^0_{11}) - \Delta_{\mathcal{F}_h}(\mathfrak{w}^0_{22})). \end{aligned}$$

4.7 Closing Remarks

In this chapter we introduced a general construction that allows for an introduction of a Hopf-algebra of vector fields on a finitely generated representation space of universal enveloping algebra type. Existing representations of $U(\mathfrak{g})$ can be embedded into the vector fields. Since the latter is larger than $U(\mathfrak{g})$, twisting of the vector fields provides a larger varity of deformations for $U(\mathfrak{g})$ that could not be obtained within $U(\mathfrak{g})$ alone. In the mean time the twists of our vector fields are nothing else than starproducts. In the last section we presented some examples that outline applicability of our construction. However, we emphasize that this setup is of course not restricted to commutative vector fields as the examples might suggest.

5 Twist-Deformed Lorentzian Heisenberg-Algebras

The Moyal-Weyl quantization procedure is embedded into the twist formalism of vector fields on phase space. Double application of twists provide most general deformations of Minkowskian Heisenberg-algebras and corresponding quantizations of the Lorentz-algebra. Such deformations deliver high-energy extensions of standard relativistic quantum mechanics. These are required to obtain minimal uncertainty properties for high-energy spacetime measurements that standard quantum mechanics lacks. The procedure of double twist application is outlined. We give an instructive and genuine example.

5.1 Introduction

The scheme of canonical quantization, presented in textbooks of quantum mechanics, is the most simple quantization one might perform. Noncommutative geometry is considered as some enhancement of this scheme. There are two basic ideas of how noncommutative geometry can be interpreted in physics. From the side of effective theories, we hope for some alternatives to standard perturbative treatment of field theories and their renormalization. Such alternatives would be required by quantum chromodynamics and gravity such as [8, 7] already suggests. On the other hand one might stick to a more fundamental point of view. Noncommutative geometry is then regarded as a gravity effect itself. Such approaches can be found in gravity motivated canonical noncommutative geometry [28, 27], but also within discussions of minimal uncertainty theories such as in [47, 46, 45, 44]. Moreover there are close relations of noncommutative geometry as well as of doubly special relativity to loop quantum gravity [3, 2, 4, 5]. Within such a fundamental approach, noncommutative geometry should not be expected as a static noncommutative background for field theories anymore. Instead, noncommutative geometry itself should become subject to gravity by making it dependent on energy and momentum.

After all we expect, Planck scale effects at high energy-momentum densities and thus a grainy structure of spacetime, obtained from noncommutative geometry, can only be mediated by operators of energy and momentum. This is nothing else than a more general deformation of phase space than obtained by canonical quantization. Moreover in such an approach, noncommutative geometry should become localized to those space volumes, where densities of energy and momentum enter the actual high energy regime. Standard problems such as IR-UV-Mixing effects should thus not occure in such a setup. A first and actually most prominent example of such a *general quantization* is the well known Quantum-Spacetime of Snyder [33, 34, 32, 80, 79, 90]. Canonical quantization can be understood as a deformation-quantization of the phase space towards the Heisenberg-algebra. Weyl and Moyal [68, 88] performed this deformation by means of starproducts. In this chapter we formalise this setup by introducing a Hopf-algebra of vector fields on phase space. We use these vector fields to twist the phase space to the standard Heisenberg-algebra. In a second step we further apply twists to deform the Heisenberg-algebra itself. These two twists can be merged to a single one. The chapter is organized as follows. In the first section we introduce the $2n$-dimensional Heisenberg-algebra $\mathfrak{h}_{2n}$ and its universal enveloping algebra $U(\mathfrak{h}_{2n})$. We then recall how this algebra is obtained by deformation-quantization of a commutative phase space algebra. This is due to Weyl and Moyal. We formalise and introduce a Hopf-algebra of vector fields on the phase space. In the following we discuss twisting by means of these vector fields. To this purpose we show that the product of two twist once more is a twist. We further present basic examples and discuss results in a conlusion.

Before we actually come to general matters, we first have to do some remarks that clearify and motivate the directions pursued in the following constructions and that indeed go hand in hand with the formalism chosen by Weyl and Moyal.

In textbooks on field theory, we often find the representation of the Lorentz-algebra in terms of generators of $U(\mathfrak{h}_{2n})$. In particular the generators $m^{\mu\nu}$ of the Lorentz-algebra are represented in $U(\mathfrak{h}_{2n})$ by

$$m^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu.$$

Using the commutation relation

$$[p^\mu, x^\rho] = -i\eta^{\mu\rho}, \tag{5.1}$$

the *action* of $m^{\mu\nu}$ on basis elements x^ρ and p^σ of $U(\mathfrak{h}_{2n})$ is then evaluated by commutators

$$\begin{aligned}
[m^{\mu\nu}, x^\rho] &= [x^\mu p^\nu - x^\nu p^\mu, x^\rho] = x^\mu [p^\nu, x^\rho] - x^\nu [p^\mu, x^\rho] \\
&= -i\eta^{\nu\rho} x^\mu + i\eta^{\mu\rho} x^\nu \qquad (5.2) \\
[m^{\mu\nu}, p^\sigma] &= [x^\mu p^\nu - x^\nu p^\mu, p^\sigma] = [x^\mu, p^\sigma] p^\nu - [x^\nu, p^\sigma] p^\mu \\
&= i\eta^{\mu\sigma} p^\nu - i\eta^{\nu\sigma} p^\mu. \qquad (5.3)
\end{aligned}$$

There are several pictures how this setup can be interpreted in physics. At first we can stick to the Poincaré-algebra, generated by $m^{\mu\nu}$ and p^ρ, that is represented on Minkowski-space. In this scheme we do not consider the Lorentz-algebra to be represented in terms of generators of $U(\mathfrak{h}_{2n})$, as we did above - but nevertheless consider the "representation" of the Lorentz-algebra in terms of commutators $[m^{\mu\nu}, x^\rho]$ or $[p^\nu, x^\rho]$ althought this already incorporates a multiplicative structure between the symmetry algebra and its representation space. For the commutative case this is alright - but deformations to noncommutative geometry modify the commutation relations in such a way that they do not close on the representation space anymore. There is actually a mixing of the symmetry algebra and the representation space. This phenomenon is also discribed in [91]. To fix this problem we might thus argue that we have actually to stay within the Heisenberg-algebra $U(\mathfrak{h}_{2n})$. Then, with $m^{\mu\nu}$ represented in $U(\mathfrak{h}_{2n})$ as performed above, we do not care anymore if a mixing occures. In this case the commutator $[p^\nu, x^\rho]$ manages everything that is represented on Minkowski-space. At first this argumentation makes perfect sense and in the case of deformations of Minkowskian $U(\mathfrak{h}_{2n})$ it has been reasoned a long such a way [47, 46, 45, 44]. Algebraically the subalgebra of momenta in $U(\mathfrak{h}_{2n})$ does not differ from that of coordinates and thus if the commutator $[p^\nu, x^\rho]$ is considered to represent the subalgebra of momenta on the coordinates, we might as well argue that in turn $[x^\mu, p^\sigma]$ is some sort of representation of coordinates on the momenta as also performed in our computation in (5.3) from above. But this as well rises the question how a coordinate would possibly act on products of generators of momenta. Or in other words, what is the coproduct of a coordinate ? This argumentation is of course too naive and these issues actually do not become a question for the commutative case - but if we are to consider deformations, we have to know about such coproducts, at least in principle. We have to have a neat bialgebra or Hopf-algebra as a framework to consider any deformation. In fact it is not possible to endow the coordinates with the same *primitive type* of coproduct as we use it for the momenta. Such an introduction of a coproduct contradicts the property of the coproduct to be an algebra-homomorphism. Nevertheless there are examples that neatly and quite elegantly endow a phase space with proper coproducts on momenta and coordinates [65]. However these also incorporate some specific structure that already accomodates some physics. The solution to this dilemma can be found in the introduction of vector fields on the entire phase space that we are presenting here. This had been performed first by Moyal and Weyl in [68, 88]. We thus first concentrate on their work in a Minkowkian setting and formalize this to our requirements. In particular we lift these vector fields to a Hopf-algebra as presented in [51]. We are then able to fit in the Lorentz-symmetry and consider further deformations.

5.2 Quantum Mechanics according to Weyl and Moyal

This section is intended as a basic review and outline that constitutes the actual input and fundaments of our constructions. The section is divided in two parts. In the first subsection we introduce n-dimensional Minkowski-space and the corresponding representation of the Poincaré-algebra. This is the only input we require for all of our considerations in this chapter. Based on this we build the $2n$-dimensional Minkowskian phase space and the Heisenberg Lie-algebra by taking direct sums of copies of Minkowski-space. These three vector spaces are further more enhanced to algebras of universal enveloping algebra type. The second subsection then reviews the deformation-quantization of Minkowskian phase space towards the Heisenberg-algebra according to Weyl and Moyal using the starproduct. In mathematical terms this is a deformation-quantization of a Poisson-Manifold. For completeness we shortly review this latter notion. We thereby obtain the required setup for further deformations with the double application of twists that is discussed in the next sections. As a textbook we recommend [19] as reference for this section.

5.2.1 The Minkowskian Heisenberg-Algebra

The n-dimensional Minkowski-space $\mathbf{R}^{(1,n-1)}$ is a vector space with scalar product

$$< \mathbf{x}, \mathbf{y} >= \eta_{\mu\nu}\, x^\mu y^\nu, \quad \mathbf{x}, \mathbf{y} \in \mathbf{R}^{(1,n-1)}, \tag{5.4}$$

that is left invariant under the action of the Lorentz-group $\mathrm{SO}(1, n-1)$. Within a specific coordinate system, the invariance of (5.4) under matrix representations of transformations $\Lambda \in \mathrm{SO}(1, n-1)$ is given by

$$\eta_{\rho\sigma} = \eta_{\mu\nu}\, \Lambda^\mu{}_\rho\, \Lambda^\nu{}_\sigma, \quad \mu, \nu, \rho, \sigma \in 0, \dots (n-1).$$

The signature of the metric tensor $\eta_{\mu\nu}$ has not to be specified within our consideration. We consider Minkowski space to be generated by a basis $(x^\mu)_{\mu\in 0,1,\ldots n-1}$. Apart from isotropy of spacetime, homogeneity of $\mathbf{R}^{(1,n-1)}$ is generated by the action of the n-dimensional translational group T_n. The Poincaré group $\mathfrak{P}$ is the semi-direct product $\mathrm{SO}(1,n-1) \rtimes T_n$. The Lie groups $\mathrm{SO}(1,n-1)$ and T_n are generated by Lie-algebras $\mathrm{so}_{1,n-1}$ and t_n respectively to constitute the Poincaré-algebra $\mathfrak{p}$. In particular for representations we actually consider the universal enveloping algebras $U(\mathfrak{p})$, $U(\mathrm{so}_{1,n-1})$ and $U(t_n)$. In order to endow Minkowski-space with a commutative algebraic structure, we enhance it to a Lie-algebra by the introduction of a trivial bracket

$$[\ ,\ ]\ :\ \mathbf{R}^{(1,n-1)} \times \mathbf{R}^{(1,n-1)} \longrightarrow \mathbf{R}^{(1,n-1)},$$

that for $x^\rho, x^\sigma \in \mathbf{R}^{(1,n-1)}$ is given by

$$[x^\rho, x^\sigma] = 0. \tag{5.5}$$

On this basis we consider the universal enveloping algebra $U(\mathbf{R}^{(1,n-1)})$. The generators $m^{\mu\nu} \in U(\mathrm{so}_{1,n-1})$ and $\pi^\rho \in U(t_n)$ of the Poincaré-algebra $U(\mathfrak{p})$ are subject to commutation relations

$$\begin{aligned} [m^{\mu\nu}, m^{\rho\sigma}] &= i\eta^{\mu\rho}m^{\nu\sigma} - i\eta^{\nu\rho}m^{\mu\sigma} + i\eta^{\nu\sigma}m^{\mu\rho} - i\eta^{\mu\sigma}m^{\nu\rho}, \\ [m^{\mu\nu}, \pi^\rho] &= i\eta^{\mu\rho}\pi^\nu - i\eta^{\nu\rho}\pi^\mu, \\ [\pi^\rho, \pi^\sigma] &= 0. \end{aligned} \tag{5.6}$$

that generate its two-sided ideal. The Poincaré-algebra $U(\mathfrak{p})$ becomes a Hopf-algebra with the following coproduct, counit and antipode:

$$\begin{aligned} &\Delta(m^{\mu\nu}) = m^{\mu\nu} \otimes \mathbf{1} + \mathbf{1} \otimes m^{\mu\nu}, \quad \epsilon(m^{\mu\nu}) = 0, \quad \mathrm{S}(m^{\mu\nu}) = -m^{\mu\nu}, \\ &\Delta(\pi^\rho) = \pi^\rho \otimes \mathbf{1} + \mathbf{1} \otimes \pi^\rho, \quad \epsilon(\pi^\rho) = 0, \quad \mathrm{S}(\pi^\rho) = -\pi^\rho. \end{aligned} \tag{5.7}$$

The Hopf-algebra $U(\mathfrak{p})$ is represented on $U(\mathbf{R}^{(1,n-1)})$ as a *left action* by

$$\begin{aligned} m^{\mu\nu} \triangleright x^\rho &= -i\eta^{\nu\rho}x^\mu + i\eta^{\mu\rho}x^\nu \\ \pi^\mu \triangleright x^\rho &= -i\eta^{\mu\rho}, \\ \mathbf{1}_\mathfrak{p} \triangleright x^\rho &= x^\rho, \end{aligned} \tag{5.8}$$

such that relations (5.6) are realized on the vector space $\mathbf{R}^{(1,n-1)}$, i.e.

$$\begin{aligned} &(m^{\mu\nu}m^{\rho\sigma} - m^{\rho\sigma}m^{\mu\nu} - i\eta^{\mu\rho}m^{\nu\sigma} + i\eta^{\nu\rho}m^{\mu\sigma} - i\eta^{\nu\sigma}m^{\mu\rho} + i\eta^{\mu\sigma}m^{\nu\rho}) \triangleright x^\lambda = 0, \\ &(m^{\mu\nu}\pi^\rho - \pi^\rho m^{\mu\nu} - i\eta^{\mu\rho}\pi^\nu + i\eta^{\nu\rho}\pi^\mu) \triangleright x^\lambda = 0, \\ &(\pi^\rho\pi^\sigma - \pi^\sigma\pi^\rho) \triangleright x^\lambda = 0. \end{aligned} \tag{5.9}$$

The action of the generators $m^{\mu\nu}, \pi^\mu \in U(\mathfrak{p})$ on products of coordinates in $U(\mathbf{R}^{(1,n-1)})$ is given by

$$\begin{aligned}
m^{\mu\nu} \triangleright (x^\rho x^\sigma) &= \Delta(m^{\mu\nu}) \triangleright (x^\rho x^\sigma) = (m^{\mu\nu} \triangleright x^\rho) x^\sigma + x^\rho (m^{\mu\nu} \triangleright x^\sigma), \\
\pi^\mu \triangleright (x^\rho x^\sigma) &= \Delta(\pi^\mu) \triangleright (x^\rho x^\sigma) = (\pi^\mu \triangleright x^\rho) x^\sigma + x^\rho (\pi^\mu \triangleright x^\sigma), \\
m^{\mu\nu} \triangleright \mathbf{1} &= \epsilon(m^{\mu\nu}), \quad \pi^\mu \triangleright \mathbf{1} \ = \ \epsilon(p^\mu),
\end{aligned} \tag{5.10}$$

such that the generating relations (5.5) of $U(\mathbf{R}^{(1,n-1)})$ are respected by their action according to

$$\begin{aligned}
m^{\mu\nu} \triangleright (x^\rho x^\sigma - x^\sigma x^\rho - [x^\rho, x^\sigma]) &= 0, \\
\pi^\mu \triangleright (x^\rho x^\sigma - x^\sigma x^\rho - [x^\rho, x^\sigma]) &= 0.
\end{aligned} \tag{5.11}$$

As a next step we introduce Minkowskian phase space Γ as the direct sum of two copies of Minkowski-space $\mathbf{R}^{(1,n-1)}$, i.e. we obtain

$$\Gamma = \mathbf{R}^{(1,n-1)} \oplus \mathbf{R}^{(1,n-1)}. \tag{5.12}$$

As for Minkowski-space, we enhance Γ with a commutative Lie-algebraic structure. Within a specific coordinate system we thus take $(x^\mu, p^\nu)_{\mu,\nu \in 0,1,\ldots n-1}$ as a basis and introduce the brackets

$$\begin{aligned}
[x^\mu, x^\nu] &= 0, \\
[x^\mu, p^\nu] &= 0, \\
[p^\mu, p^\nu] &= 0,
\end{aligned} \tag{5.13}$$

We then obtain the universal enveloping algebra $U(\Gamma)$ by once more taking these brackets as the generating relations for the corresponding two-sided ideal of $U(\Gamma)$. Concerning covariance under the action of $U(\mathfrak{p})$, we can replace coordinates x by momenta p in conditions (5.8) and (5.9), i.e. on the vector space $\Gamma = \mathbf{R}^{(1,n-1)} \oplus \mathbf{R}^{(1,n-1)}$ the Lorentz group $\mathrm{SO}(1, n-1)$ is represented by block-diagonal matrices

$$\Lambda_{\mathcal{P}} = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}. \tag{5.14}$$

With respect to the covariance of the algebraic structure of $U(\Gamma)$ we can replace products of coordinates $x^\rho x^\sigma$ in (5.10) and (5.10) by products of coordinates and momenta $x^\rho p^\sigma$ and products of momenta $p^\rho p^\sigma$. We thereby obtained a left action of $U(\mathfrak{p})$ on $U(\Gamma)$.

In a similar manner, as for Γ, we obtain the Minkowskian Heisenberg-algebra $\mathfrak{h}_{2n}$ by taking the direct sum of two copies of $\mathbf{R}^{(1,n-1)}$ and the real numbers, i.e.

$$\mathfrak{h}_{2n} = \mathbf{R}^{(1,n-1)} \oplus \mathbf{R}^{(1,n-1)} \oplus i\mathbf{R}. \tag{5.15}$$

This vector space becomes a Lie-algebra by introducing a bracket

$$[\ ,\ ]\ :\ \mathfrak{h}_{2n} \times \mathfrak{h}_{2n} \longrightarrow \mathfrak{h}_{2n}$$

that for $\mathbf{X}_1, \mathbf{Y}_1, \mathbf{X}_2, \mathbf{Y}_2 \in \mathbf{R}^{(1,n-1)}$ and $c_1, c_2 \in \mathbf{R}$ is defined by

$$[(\mathbf{X}_1, \mathbf{Y}_1, c_1), (\mathbf{X}_2, \mathbf{Y}_2, c_2)] = (0, 0, i \cdot (< \mathbf{X}_1, \mathbf{Y}_2 > - < \mathbf{Y}_1, \mathbf{X}_2 >))\,. \quad (5.16)$$

Through the scalar product (5.4) used in this definition we obtain $\mathfrak{h}_{2n}$ to be covariant under the action of $U(\mathfrak{p})$. Besides this, Lorentz-covariance is equally intruduced as for the phase space Γ. By the identification

$$X^\mu \equiv\ (e^\mu, 0, 0)\ \in\ \mathfrak{h}_{2n}, \quad P_\nu \equiv (0, e_\nu, 0)\ \in \mathfrak{h}_{2n},$$

we obtain the bracket-relations between coordinates X^μ and momenta P^ν

$$[X^\mu, X^\nu] = 0\ , \quad [X^\mu, P^\nu] = i\hbar\eta^{\mu\nu}\ , \quad [P^\mu, P^\nu] = 0. \quad (5.17)$$

These relations once more generate the two-sided ideal that is required to formulate the universal enveloping algebra $U(\mathfrak{h}_{2n})$ of the Heisenberg-algebra $\mathfrak{h}_{2n}$. We are now prepared to consider deformation-quantization of $U(\Gamma)$ towards $U(\mathfrak{h}_{2n})$ as it had been introduced by Moyal.

5.2.2 Phase Space Quantization with Starproducts

In the last subsection we considered the phase space algebra as the universal enveloping algebra $U(\Gamma)$. Dually we have the algebra of complex-valued functions $\mathcal{F}(\Gamma)$ on Γ. Defining the Poisson-bracket on functions $\mathcal{F}(\Gamma)$, we turn Γ into a Poisson-manifold. As such we deform it to $U(\mathfrak{h}_{2n})$ according to the quantization procedure applied by Moyal [68]. This is more generally known as a deformation of Poisson manifolds. We recall these notions here. In order to perform this quantization we switch between the dual pictures of $U(\Gamma)$ and $\mathcal{F}(\Gamma)$. We begin by introducing the algebra of functions $\mathcal{F}(\Gamma)$ on Γ.
On the vector space $\Gamma = \mathbf{R}^{(1,n-1)} \oplus \mathbf{R}^{(1,n-1)}$ we consider the subset $\mathcal{F}(\Gamma) \subset C^\infty(\Gamma, \mathbb{C})$ of smooth complex-valued functions, that we endow with a Poisson-bracket

$$\{\ ,\ \}\ :\ \mathcal{F}(\Gamma) \times \mathcal{F}(\Gamma) \longrightarrow \mathcal{F}(\Gamma),$$

that in particular is defined for $\omega, \varphi \in \mathcal{F}(\Gamma)$ by

$$\{\omega, \varphi\} := \frac{\partial\omega}{\partial p_\mu} \cdot \frac{\partial\varphi}{\partial x^\mu} - \frac{\partial\omega}{\partial x^\mu} \cdot \frac{\partial\varphi}{\partial p_\mu}. \quad (5.18)$$

In addition to this bracket, the vector space of functions $\mathcal{F}(\Gamma)$ is endowed with pointwise multiplication that is induced from the product within the complex numbers, i.e. for $\omega, \varphi \in \mathcal{F}(\Gamma)$ we have

$$(\omega \cdot_{\mathcal{F}} \varphi)(x^\mu, p_\nu) = \omega(x^\mu, p_\nu) \cdot_{\mathbb{C}} \varphi(x^\mu, p_\nu)$$

By the introduction of the Poisson-bracket (5.18), we turn the vector space Γ into what is called a Poisson manifold that is more generally defined as follows.

5.2.1 DEFINITION *Let $\mathcal{M}$ be a d-dimensional manifold and $C^\infty(\mathcal{M}, \mathbb{C})$ be the set of complex-valued smooth functions on $\mathcal{M}$. Then $\mathcal{M}$ is called a* Poisson Manifold*, if there exists a bracket $\{\cdot, \cdot\}$*

$$\{\cdot, \cdot\} : C^\infty(\mathcal{M}, \mathbb{C}) \times C^\infty(\mathcal{M}, \mathbb{C}) \to C^\infty(\mathcal{M}, \mathbb{C}),$$

such that the following properties hold:

$$\begin{aligned} \forall\, \omega, \varphi, \psi \in C^\infty(\mathcal{M}, \mathbb{C}) \quad : \quad & \{\varphi, \omega\} = -\{\omega, \varphi\} \\ & \{\varphi \cdot \omega, \psi\} = \varphi \cdot \{\omega, \psi\} + \{\varphi, \psi\} \cdot \omega \\ & \{\{\varphi, \omega\}, \psi\} + \{\{\omega, \psi\}, \varphi\} + \{\{\psi, \varphi\}, \omega\} = 0 \end{aligned}$$

We thus have two distinct algebraic structures on Γ, i.e. on $\mathcal{F}(\Gamma)$. The original problem considered by Weyl and Moyal in [68, 88] had been to grasp the procedure of quantization as mathematical term. The procedure of quantization in particular sends the Poisson-bracket of $\mathcal{F}(\Gamma)$ to the commutator of $U(\mathfrak{h}_{2n})$ according to

$$\{\ ,\ \} \longrightarrow \frac{i}{\hbar}[\ ,\ ].$$

This procedure agitates the former algebraic structures of Γ. It "maps" the commutative algebra of functions $\mathcal{F}(\Gamma)$ to the noncommutative $U(\mathfrak{h}_{2n})$. The solution is to consider quantization to be the deformation of the product of the algebra of functions $\mathcal{F}(\Gamma)$ performed in such a way that the commutator of the deformed algebra of functions corresponds to the structure implied by the Poisson-bracket. More generally this is known to be a quantization of a Poisson-manifold that more precisely is defined as follows.

5.2.2 DEFINITION *Let a Poisson manifold $(\mathcal{M}, \{\cdot, \cdot\}, \mathbf{K})$ over the field $\mathbf{K}$ be given. A* quantization *of $\mathcal{M}$ with deformation parameter $h \in \mathbf{K}$ is a manifold $\mathcal{M}_h = (\mathcal{M}, [\cdot \stackrel{*_h}{,} \cdot], \mathbf{K})$, such that to first order in the deformation parameter h the commutator $[\cdot \stackrel{*_h}{,} \cdot]$ satisfies the following property:*

$$\forall f_1, f_2 \in \mathcal{F}(\mathcal{M}) : \; \frac{[f_1 \stackrel{*_h}{,} f_2]}{h} = \frac{f_1 *_h f_2 - f_2 *_h f_1}{h} = \{f_1, f_2\} \quad mod(h)$$

The quantization of the algebra of functions is typically performed in terms of starproducts. To this purpose it is convenient to consider $U(\Gamma)$ instead of $\mathcal{F}(\Gamma)$. Since $\mathcal{F}(\Gamma) \subset C^\infty(\Gamma, \mathbb{C})$ and $\mathcal{F}(\Gamma)$ is commutative, this duality merely means that functions $\varphi \in \mathcal{F}(\Gamma)$ can be represented in terms of formal power series in $U(\Gamma)$ that moreover can be regarded as power series of a real parameters and thus can converge locally. We thus express functions $\varphi \in \mathcal{F}(\Gamma)$ as power series

$$\begin{aligned} \varphi(x^\mu, p_\nu) &= \sum_{\mathbf{r},\mathbf{s}} C_{\mathbf{r},\mathbf{s}} \cdot (x^0)^{r_0} \cdot \ldots \cdot (x^{(n-1)})^{r_{(n-1)}} \cdot (p_0)^{s_0} \cdot \ldots \cdot (p_{(n-1)})^{s_{(n-1)}} \\ & \quad C_{\mathbf{r},\mathbf{s}} \in \mathbb{C}; \quad \mathbf{r}, \mathbf{s} \in \mathbb{N}_0^n. \end{aligned}$$

With exponential functions

$$e^{i(\eta_\mu x^\mu + \xi^\nu p_\nu)}, \quad \eta_\mu, \xi^\nu \in \mathbf{R}^{(1,n-1)}$$

as a basis for $\mathcal{F}(\Gamma)$ we can also consider $\varphi \in \mathcal{F}(\Gamma)$ as a linear combination in terms of its Fourier-transformation

$$\begin{aligned} \varphi(x^\mu, p_\nu) &= \int d^n\eta \; d^n\xi \; \hat{\varphi}(\eta_\mu, \xi^\nu) \; e^{-i(\eta_\mu x^\mu + \xi^\nu p_\nu)} \\ \hat{\varphi}(\eta_\mu, \xi^\nu) &= \frac{1}{(2\pi)^{2n}} \int d^n x \; d^n p \; \varphi(x^\mu, p_\nu) \; e^{+i(\eta_\mu x^\mu + \xi^\nu p_\nu)}. \end{aligned}$$

Of course also for $X^\mu, P_\nu \in U(\mathfrak{h}_{2n})$ exponential functions

$$e^{i(\eta_\mu X^\mu + \xi^\nu P_\nu)}, \quad \eta_\mu, \xi^\nu \in \mathbf{R}^{(1,n-1)}$$

constitute a basis for $U(\mathfrak{h}_{2n})$. In particular these exponentails are group elements of the corresponding Heisenberg Lie-group. Note that $U(\mathfrak{h}_{2n})$ is dual to a corresponding algebra of functions over the Heisenberg Lie-group. The Poincaré-Brikhoff-Witt theorem enables us to map $U(\mathfrak{h}_{2n})$ to $\mathcal{F}(\Gamma)$ by an isomorphism W of vector spaces. In particular this statement reads as follows.

5.2.3 Theorem *Let $\mathfrak{g}$ be an n-dimensional Lie-algebra with basis $(g_i)_{i\in\{1\ldots n\}}$ over the field $\mathbf{K}$. Furthermore let*

$$\begin{aligned} \pi : \{1 \ldots n\} \subset \mathbb{N} &\rightarrow \{1 \ldots n\} \\ k &\mapsto i_k \end{aligned}$$

be any permutation, then the ordered monomials

$$(g_{i_1})^{m_{i_1}} \ldots (g_{i_k})^{m_{i_k}} \ldots (g_{i_n})^{m_{i_n}} \in U(\mathfrak{g}), \;\; m_{i_k} \in \mathbb{N}$$

constitute a basis of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ and there exists an isomorphism W of vector spaces

$$\begin{aligned} W \;\; : \;\; & U(\mathfrak{g}) \rightarrow U(\mathbf{R}^n) \\ & (g_{i_1})^{m_{i_1}} \ldots (g_{i_k})^{m_{i_k}} \ldots (g_{i_n})^{m_{i_n}} \mapsto (x_{i_1})^{m_{i_1}} \ldots (x_{i_k})^{m_{i_k}} \ldots (x_{i_n})^{m_{i_n}}. \end{aligned}$$

Introducing a starproduct on $\mathcal{F}(\Gamma)$, i.e. performing the quantization of the Poisson-manifold as described, actually enhances the isomorphism W of vector spaces to an isomorphism of corresponding algebras. In particular we therefore consider how basis elments are mapped, i.e. we obtain

$$\begin{array}{rcl} W : U(\mathfrak{h}_{2n}) & \rightarrow & \mathcal{F}(\Gamma) \\ e^{i(\eta_\mu X^\mu + \xi^\nu P_\nu)} & \mapsto & e^{i(\eta_\mu x^\mu + \xi^\nu p_\nu)}. \end{array}$$

By application of the inverse map W^{-1} we receive for two functions $\varphi, \omega \in \mathcal{F}(\Gamma)$ the corresponding objects within $U(\mathfrak{h}_{2n})$. In particular we obtain

$$\begin{array}{rcl} W^{-1}(\varphi)(X^\mu, P_\nu) & = & \displaystyle\int d^n\eta \; d^n\xi \;\; \hat{\varphi}(\eta_\mu, \xi^\nu) \; e^{-i(\eta_\mu X^\mu + \xi^\nu P_\nu)} \\ W^{-1}(\omega)(X^\mu, P_\nu) & = & \displaystyle\int d^n\eta \; d^n\xi \;\; \hat{\omega}(\eta_\mu, \xi^\nu) \; e^{-i(\eta_\mu X^\mu + \xi^\nu P_\nu)}. \end{array}$$

In order to endow the vector space Γ with a deformed multiplication map $*_\hbar$ we require that

$$\begin{array}{rcl} W^{-1}(\varphi *_\hbar \omega)(X^\mu, P_\nu) & := & W^{-1}(\varphi)(X^\mu, P_\nu) \cdot W^{-1}(\omega)(X^\mu, P_\nu) \\ & = & \displaystyle\int d^n\eta \; d^n\xi \; d^n\kappa \; d^n\lambda \; \hat{\varphi}(\eta_\mu, \xi^\nu)\hat{\omega}(\kappa_\mu, \lambda^\nu) \\ & & \qquad \times \; e^{-i(\eta_\mu X^\mu + \xi^\nu P_\nu)} \; e^{-i(\kappa_\mu X^\mu + \lambda^\nu P_\nu)} \\ & = & \displaystyle\int d^n\eta \; d^n\xi \; d^n\kappa \; d^n\lambda \; \hat{\varphi}(\eta_\mu, \xi^\nu)\hat{\omega}(\kappa_\mu, \lambda^\nu) \\ & & \qquad \times e^{-i((\eta_\mu + \kappa_\mu)X^\mu + (\xi^\nu + \lambda^\nu)P_\nu) - i\frac{\hbar}{2}\eta^{\mu\nu}(\eta_\mu\lambda_\nu - \xi_\nu\kappa_\mu)\mathbf{1}}. \end{array}$$

The final step we performed by the use of the Baker-Campbell-Hausdorff formula

$$e^A \; e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}([A,[A,B]]-[B,[A,B]])+\frac{1}{48}([A,[B,[B,A]]]-[B,[A,[A,B]]])+\ldots}.$$

We transform back by the use of the isomorphism W and thus obtain

$$\begin{array}{rcl} (\varphi *_\hbar \omega)(x^\mu, p_\nu) & = & \displaystyle\int d^n\eta \; d^n\xi \; d^n\kappa \; d^n\lambda \; \hat{\varphi}(\eta_\mu, \xi^\nu)\hat{\omega}(\kappa_\mu, \lambda^\nu) \\ & & \qquad \times e^{-i((\eta_\mu + \kappa_\mu)X^\mu + (\xi^\nu + \lambda^\nu)P_\nu) - i\frac{\hbar}{2}\eta^{\mu\nu}(\eta_\mu\lambda_\nu - \xi_\nu\kappa_\mu)} \\ & = & \displaystyle\int d^n\eta \; d^n\xi \; d^n\kappa \; d^n\lambda \; \hat{\varphi}(\eta_\mu, \xi^\nu) \; e^{-i(\eta_\mu x^\mu + \xi^\nu p_\nu)} \\ & & \qquad \times \hat{\omega}(\kappa_\mu, \lambda^\nu) \; e^{-i(\kappa_\mu x^\mu + \lambda^\nu p_\nu)} \; e^{-i\frac{\hbar}{2} \; \eta^{\mu\nu}(\eta_\mu\lambda_\nu - \xi_\nu\kappa_\mu)} \end{array}$$

Replacing $\eta_\mu \rightarrow i\frac{\partial}{\partial x^\mu}, \xi_\nu \rightarrow i\frac{\partial}{\partial p^\nu}$ and $\kappa_\mu \rightarrow i\frac{\partial}{\partial \hat{q}^\mu}, \lambda_\nu \rightarrow i\frac{\partial}{\partial \hat{p}^\nu}$, we finally received the starproduct

$$(\varphi *_\hbar \omega)(x^\mu, p_\nu) \;\; = \;\; e^{+i\frac{\hbar}{2} \; \eta^{\mu\nu}(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial \hat{p}^\nu} - \frac{\partial}{\partial p^\nu}\frac{\partial}{\partial \hat{x}_\mu})} \; \varphi(x^\mu, p_\nu) \; \omega(\hat{x}^\mu, \hat{p}_\nu)|_{(\hat{x}^\mu, \hat{p}_\nu) \rightarrow (x^\mu, p_\nu)}. \tag{5.19}$$

In particular for $\varphi(x^\rho, p_\sigma) = x^\rho$ and $\omega(x^\rho, p_\sigma) = p^\sigma$ we recover the second relation of (5.17), distinguishing the generating relations of $U(\mathfrak{h}_{2n})$ from those of $U(\Gamma)$.

$$
\begin{aligned}
[x^\rho \stackrel{*_\hbar}{,} p^\sigma] &= x^\rho *_\hbar p^\sigma - p^\sigma *_\hbar x^\rho \\
&= e^{+i\frac{\hbar}{2}\, \eta^{\mu\nu}(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial \hat{p}^\nu} - \frac{\partial}{\partial p^\nu}\frac{\partial}{\partial \hat{x}_\mu})} x^\rho \cdot \hat{p}^\sigma|_{\hat{p}^\sigma \to p^\sigma} \\
&\qquad - e^{+i\frac{\hbar}{2}\, \eta^{\mu\nu}(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial \hat{p}^\nu} - \frac{\partial}{\partial p^\nu}\frac{\partial}{\partial \hat{x}_\mu})} p^\sigma \cdot \hat{x}^\rho|_{\hat{x}^\rho \to x^\rho} \\
&= x^\rho \cdot p^\sigma + i\,\frac{\hbar}{2}\,\eta^{\rho\sigma} - p^\sigma \cdot x^\rho + i\,\frac{\hbar}{2}\,\eta^{\rho\sigma} \\
&= i\,\eta^{\rho\sigma}.
\end{aligned}
$$

This final computation closes our short review of Weyl-Moyal deformation-quantization. We are now prepared to formalise this procedure.

5.3 Vector Fields $\mathfrak{W}(\Pi, \Gamma)$ on Minkowskian Phase Space

Beginning with this section we formalise the presented constructions of Weyl and Moyal. In particular we intend to absorb the starproduct (5.19) into the modern setup of twists of vector fields, as presented in [51]. We thus make a step beyond mere quantizations of Poisson manifolds because the twist formalism also enables us to further deform the Heisenberg-algebra $U(\mathfrak{h}_{2n})$ itself. Furthermore the twist formalism also provides us with the opportunity to make required deformations of the Poincaré-algebra such that we can preserve spacetime covariance under deformations. In this section we therefore introduce the required Hopf-algebra of vector fields $\mathfrak{W}(\Pi, \Gamma)$ on $U(\Gamma)$ that provides us with the necessary tools to express the starproduct (5.19) as a twist within $\mathfrak{W}(\Pi, \Gamma) \otimes \mathfrak{W}(\Pi, \Gamma)$. In the next section we accomodate $U(\mathfrak{p})$ within $\mathfrak{W}(\Pi, \Gamma)$ as a subalgebra. In this way, the starproduct turned into a twist thus also manages the deformation of $U(\mathfrak{p})$. In the mean time twists in $\mathfrak{W}(\Pi, \Gamma) \otimes \mathfrak{W}(\Pi, \Gamma)$ enable us, as already mentioned, to go beyond the quantization of Poisson manifolds. As already announced, double application of such twists then provides us with desired deformations of the Heisenberg-algebra $U(\mathfrak{h}_{2n})$, covariant under corresponding deformations of $U(\mathfrak{p})$. In order to undertake this step of formalisation, we first consolidate our formulation of $U(\Gamma)$ by setting

$$
\xi^R = \begin{cases} x^\rho & : \quad \rho = R \ \wedge \ R = 0, \ldots, (n-1) \\ p^\mu & : \quad \mu = R - n \ \wedge \ R = n, \ldots, (2n-1) \end{cases}
$$

$$
R \in 0, \ldots, (n-1), n, \ldots, (2n-1). \tag{5.20}
$$

The generating relations (5.13) of $U(\Gamma)$ are then reduced to the single equation

$$\xi^R \xi^S - \xi^S \xi^R = 0. \tag{5.21}$$

As a first step towards the Hopf-algebra of vector fields $\mathfrak{W}(\Pi, \Gamma)$, we introduce an algebra of momenta $U(\Pi)$ in the following subsection.

5.3.1 The Algebra of Momenta $U(\Pi)$ represented on $U(\Gamma)$

In order to obtain a $2n$-dimensional Hopf-algebra of momenta $U(\Pi)$, we take a copy of $U(\Gamma)$ and enhance it to a Hopf-algebra. In praticular we consider $(\pi_N)_{N \in 0, \ldots 2n-1}$ as a basis for $U(\Pi)$. The generating relations, analogous to (5.13), are then given by

$$\pi_M \cdot \pi_N - \pi_N \cdot \pi_M = 0, \quad M, N \in 0, \ldots, (n-1), n, \ldots, (2n-1).$$

The Hopf-structure on $U(\Pi)$ is given by the following coproduct, counit and antipode

$$\Delta(\pi_M) = \pi_M \otimes \mathbf{1}_\pi + \mathbf{1}_\pi \otimes \pi_M, \quad \epsilon(\pi_M) = 0, \quad \mathrm{S}(\pi_M) = -\pi_M.$$

The Hopf-algebra axioms are easily verified. The Hopf-algebra of momenta $U(\Pi)$ is represented by a left action on $U(\Gamma)$, as follows.

$$\begin{array}{rcl} \pi^M \triangleright \xi^R &=& -iE^{MR}, \\ \pi^M \triangleright \mathbf{1} &=& \epsilon(\pi^M), \\ \mathbf{1} \triangleright \xi^R &=& \xi^R, \end{array} \tag{5.22}$$

To this purpose we introduce the $2n$-dimensional tensor

$$E^{MR} = \begin{cases} \eta^{MR} & : \ M = 0, \ldots, (n-1) \ \wedge \ R = 0, \ldots, (n-1) \\ 0 & : \ M = 0, \ldots, (n-1) \ \wedge \ R = n, \ldots, (2n-1) \\ 0 & : \ M = n, \ldots, (2n-1) \ \wedge \ R = 0, \ldots, (n-1) \\ \eta^{(M-n)(R-n)} & : \ M = n, \ldots, (2n-1) \ \wedge \ R = n, \ldots, (2n-1) \end{cases}$$

Alternatively we can also formulate (5.22) in the form

$$\pi_M \triangleright \xi^R = -i\Delta_M^R,$$

with

$$\Delta_M^R = \begin{cases} \delta_M^R & : \ M = 0, \ldots, (n-1) \ \wedge \ R = 0, \ldots, (n-1) \\ 0 & : \ M = 0, \ldots, (n-1) \ \wedge \ R = n, \ldots, (2n-1) \\ 0 & : \ M = d, \ldots, (2n-1) \ \wedge \ R = 0, \ldots, (n-1) \\ \delta_{M-n}^{R-n} & : \ M = d, \ldots, (2n-1) \ \wedge \ R = d, \ldots, (2n-1) \end{cases}$$

We further verify that $U(\Pi)$ is realized on the vector space Γ by

$$\begin{aligned}(\pi_M \cdot \pi_N - \pi_N \cdot \pi_M) \triangleright \xi^R &= \pi_M \triangleright (\pi_N \triangleright \xi^R) - \pi_N \triangleright (\pi_M \triangleright \xi^R) \\ &= -i\Delta_N^R \, \epsilon(\pi_M) + i\Delta_M^R \, \epsilon(\pi_N) = 0. \end{aligned} \tag{5.23}$$

Moreover the action of $U(\Pi)$ respects the algebraic structure (5.21) of $U(\Gamma)$, i.e. we have

$$\begin{aligned}\pi_M \triangleright (\xi^R \cdot \xi^S - \xi^S \cdot \xi^R) &= \Delta(\pi_M) \triangleright (\xi^R \cdot \xi^S) - \Delta(\pi_M) \triangleright (\xi^S \cdot \xi^R) \\ &= (\pi_M \triangleright \xi^R)\xi^S + \xi^R(\pi_M \triangleright \xi^S) \\ &\qquad -(\pi_M \triangleright \xi^S)\xi^R - \xi^S(\pi_M \triangleright \xi^R) \\ &= -i\Delta_M^R \, \xi^S - i\Delta_M^S \, \xi^R + i\Delta_M^S \, \xi^R + i\Delta_M^R \, \xi^S = 0. \end{aligned} \tag{5.24}$$

We thus obtained a valid representation of $U(\Pi)$ on $U(\Gamma)$ and can join them now to a single cross-product algebra.

5.3.2 The Hopf-Algebra $\mathfrak{W}(\Pi, \Gamma)$ of Vector Fields

In order to obtain the Hopf-algebra of vector fields $\mathfrak{W}(\Pi, \Gamma)$ on $U(\Gamma)$, we have to consider the associative left cross-product algebra $U(\Gamma) \rtimes U(\Pi)$ that is build on the tensor product $U(\Gamma) \otimes U(\Pi)$. Additional division of this cross-product enables us to lift $\mathfrak{W}(\Pi, \Gamma)$ itself to a Hopf-algebra that is once more represented on $U(\Gamma)$. The left cross-product in $U(\Gamma) \otimes U(\Pi)$ is given by

$$\begin{aligned}(\xi^R \otimes \pi^M) \odot (\xi^S \otimes \pi^N) &= \sum \xi^R(\pi^{M(1)} \triangleright \xi^S) \otimes \pi^{M(2)}\pi^N \\ &= \xi^R(\pi^M \triangleright \xi^S) \otimes \pi^N + \xi^R(\mathbf{1} \triangleright \xi^S) \otimes \pi^M\pi^N \\ &= -iE^{MS}(\xi^R \otimes \pi^N) + \xi^R\xi^S \otimes \pi^M\pi^N \end{aligned}$$

$$\Delta(\pi^M) = \sum \pi^{M(1)} \otimes \pi^{M(2)}. \tag{5.25}$$

Within $U(\Gamma) \rtimes U(\Pi)$ the former subalgebras $U(\Gamma)$ and $U(\Pi)$ are also accomodated. They are identified by elements $\xi^R \equiv \xi^R \otimes \mathbf{1}$ and $\pi^M \equiv \mathbf{1} \otimes \pi^M$ respectively. We introduce the following elements

$$\begin{aligned}\mathfrak{w}_0^{RM} &:= \xi^R \otimes \pi^M, \qquad \mathfrak{w}_+^M := \mathbf{1} \otimes \pi^M, \\ \mathfrak{w}_-^R &:= \xi^R \otimes \mathbf{1}, \qquad \mathbf{1} = \mathbf{1} \otimes \mathbf{1}, \end{aligned} \tag{5.26}$$

that generate $U(\Gamma) \rtimes U(\Pi)$, i.e. we obtain

$$U(\Gamma) \rtimes U(\Pi) = \frac{T(U(\Gamma) \otimes U(\Pi))}{\mathcal{I}_{\Gamma,\Pi}},$$

where $T(U(\Gamma) \otimes U(\Pi))$ is the tensor algebra of $U(\Gamma) \otimes U(\Pi)$ and $\mathcal{I}_{\Gamma,\Pi}$ is the two-sided ideal generated by relations

$$\begin{aligned} \left[\mathfrak{w}_0^{RM}, \mathfrak{w}_0^{SN}\right]_\odot = -iE^{MS}\mathfrak{w}_0^{RN} + iE^{NR}\mathfrak{w}_0^{SM}, \quad & \left[\mathfrak{w}_+^{M}, \mathfrak{w}_-^{R}\right]_\odot = -iE^{RM}\mathbf{1} \\ \left[\mathfrak{w}_0^{RM}, \mathfrak{w}_-^{S}\right]_\odot = -iE^{SM}\mathfrak{w}_-^{R}, \quad & \left[\mathfrak{w}_0^{RM}, \mathfrak{w}_+^{N}\right]_\odot = iE^{RN}\mathfrak{w}_+^{M}, \\ \left[\mathfrak{w}_+^{M}, \mathfrak{w}_+^{N}\right]_\odot = 0, \quad & \left[\mathfrak{w}_-^{R}, \mathfrak{w}_-^{S}\right]_\odot = 0, \end{aligned} \tag{5.27}$$

These are induced by (5.25) and (5.26). We further enhance the ideal $\mathcal{I}_{\Gamma,\Pi}$ by setting $\mathfrak{w}_-^R = 0$ such that we receive a new two-sided ideal $\mathcal{I}_{\mathfrak{W}}$ that is generated by relations

$$\begin{aligned} \left[\mathfrak{w}_0^{RM}, \mathfrak{w}_0^{SN}\right]_\odot = -iE^{MS}\mathfrak{w}_0^{RN} + iE^{NR}\mathfrak{w}_0^{SM}, \quad & \left[\mathfrak{w}_0^{RM}, \mathfrak{w}_+^{N}\right]_\odot = iE^{RN}\mathfrak{w}_+^{M}, \\ \left[\mathfrak{w}_+^{M}, \mathfrak{w}_+^{N}\right]_\odot = 0, \quad & \end{aligned} \tag{5.28}$$

such that we finally obtain the algebra of vector fields $\mathfrak{W}(\Pi, \Gamma)$ by

$$\mathfrak{W}(\Pi, \Gamma) = \frac{T(U(\Gamma) \otimes U(\Pi))}{\mathcal{I}_{\mathfrak{W}}}.$$

The algebra $\mathfrak{W}(\Pi, \Gamma)$ is lifted to a Hopf-algebra by introducing coproducts, counits and antipodes on its generators $\mathfrak{w}_0^{RM}$ and $\mathfrak{w}_+^N$ according to

$$\begin{aligned} &\Delta(\mathfrak{w}_0^{RM}) = \mathfrak{w}_0^{RM} \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{w}_0^{RM}, \quad \epsilon(\mathfrak{w}_0^{RM}) = 0, \quad \mathrm{S}(\mathfrak{w}_0^{RM}) = -\mathfrak{w}_0^{RM}, \\ &\Delta(\mathfrak{w}_+^{M}) = \mathfrak{w}_+^{M} \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{w}_+^{M}, \quad \epsilon(\mathfrak{w}_+^{M}) = 0, \quad \mathrm{S}(\mathfrak{w}_+^{M}) = -\mathfrak{w}_+^{M}. \end{aligned}$$

It is easy to verify the axioms of Hopf-algebras and homomorphy. A detailed proof can be found in [51]. The Hopf-algebra of vector fields $\mathfrak{W}(\Pi, \Gamma)$ is represented by a left action on $U(\Gamma)$ according to

$$\begin{aligned} \mathfrak{w}_0^{RM} \triangleright \xi^S &= -iE^{SM}\xi^R \\ \mathfrak{w}_+^{M} \triangleright \xi^R &= -iE^{RM}\mathbf{1}. \end{aligned}$$

We verify that the generating relations of $\mathfrak{W}(\Pi, \Gamma)$ are realized on Γ, i.e. for the first relation in (5.28) we obtain that

$$\begin{aligned} &(\mathfrak{w}_0^{RM} \cdot \mathfrak{w}_0^{SN} - \mathfrak{w}_0^{SN} \cdot \mathfrak{w}_0^{RM} + iE^{MS}\mathfrak{w}_0^{RN} - iE^{NR}\mathfrak{w}_0^{SM}) \triangleright \xi^V \\ &\quad = \mathfrak{w}_0^{RM} \triangleright (\mathfrak{w}_0^{SN} \triangleright \xi^V) - \mathfrak{w}_0^{SN} \triangleright (\mathfrak{w}_0^{RM} \triangleright \xi^V) \\ &\qquad + iE^{MS}(\mathfrak{w}_0^{RN} \triangleright \xi^V) - iE^{NR}(\mathfrak{w}_0^{SM} \triangleright \xi^V) \\ &\quad = -iE^{VN}(\mathfrak{w}_0^{RM} \triangleright \xi^S) + iE^{VM}(\mathfrak{w}_0^{SN} \triangleright \xi^R) \\ &\qquad + E^{MS}E^{VN}\xi^R - E^{NR}E^{VM}\xi^S \\ &\quad = -E^{VN}E^{SM}\xi^R + E^{VM}E^{RN}\xi^S + E^{MS}E^{VN}\xi^R - E^{NR}E^{VM}\xi^S = 0. \end{aligned}$$

For the second relation we further compute that

$$
\begin{aligned}
&(\mathfrak{w}_0^{RM} \cdot \mathfrak{w}_+^{N} - \mathfrak{w}_+^{N} \cdot \mathfrak{w}_0^{RM} - iE^{RN}\mathfrak{w}_+^{M}) \triangleright \xi^V \\
&\quad = \mathfrak{w}_0^{RM} \triangleright (\mathfrak{w}_+^{N} \triangleright \xi^V) - \mathfrak{w}_+^{N} \triangleright (\mathfrak{w}_0^{RM} \triangleright \xi^V) - iE^{RN}(\mathfrak{w}_+^{M} \triangleright \xi^V) \\
&\quad = -iE^{VN}(\mathfrak{w}_0^{RM} \triangleright \mathbf{1}) + iE^{VM}(\mathfrak{w}_+^{N} \triangleright \xi^R) - E^{RN}E^{VM} \\
&\quad = E^{VM}E^{RN} - E^{RN}E^{VM} = 0.
\end{aligned}
$$

The third relation is already satisfied by (5.23). We further check that $\mathfrak{W}(\Pi, \Gamma)$ respects the generating relations of $U(\Gamma)$. For $\mathfrak{w}_+^M$ this is already verified by (5.24). We thus consider

$$
\begin{aligned}
&\mathfrak{w}_0^{RM} \triangleright (\xi^V \xi^W - \xi^W \xi^V) \\
&\quad = \Delta(\mathfrak{w}_0^{RM}) \triangleright (\xi^V \xi^W) - \Delta(\mathfrak{w}_0^{RM}) \triangleright (\xi^W \xi^V) \\
&\quad = (\mathfrak{w}_0^{RM} \triangleright \xi^V)\xi^W + \xi^V(\mathfrak{w}_0^{RM} \triangleright \xi^W) \\
&\qquad -(\mathfrak{w}_0^{RM} \triangleright \xi^W)\xi^V - \xi^W(\mathfrak{w}_0^{RM} \triangleright \xi^V) \\
&\quad = -iE^{VM}\xi^R\xi^W - iE^{WM}\xi^V\xi^R + iE^{WM}\xi^R\xi^V + iE^{VM}\xi^W\xi^R = 0.
\end{aligned}
$$

We are now prepared to take the next step that embeds the Poincaré-algebra $U(\mathfrak{p})$ within $\mathfrak{W}(\Pi, \Gamma)$.

5.4 The Vector Field Representation of the Lorentz-Algebra

The previous preparations of the last sections enable us to represent the Poincaré-algebra $U(\mathfrak{p})$ within $\mathfrak{W}(\Pi, \Gamma)$. As a corresponding representation of the Lorentz-generators $M^{LN} \in U(so_{1,n-1})$ we introduce

$$
M^{LN} \quad = \quad \begin{cases} \mathfrak{w}_0^{LN} - \mathfrak{w}_0^{NL} & : \ L = 0, \ldots, (n-1) \quad \wedge \quad N = 0, \ldots, (n-1) \\ 0 & : \ L = 0, \ldots, (n-1) \quad \wedge \quad N = n, \ldots, (2n-1) \\ 0 & : \ L = n, \ldots, (2n-1) \quad \wedge \quad N = 0, \ldots, (n-1) \\ \mathfrak{w}_0^{LN} - \mathfrak{w}_0^{NL} & : \ l = n, \ldots, (2n-1) \quad \wedge \quad N = n, \ldots, (2n-1) \end{cases} \tag{5.29}
$$

Translational operators are already given by the algebra of momenta $U(\Pi)$, i.e. we have

$$
P^N := \mathfrak{w}_+^N. \tag{5.30}
$$

With relations (5.28) we compute the generating relations (5.6) of $U(\mathfrak{p})$ in their block-diagonal form (5.14) to be

$$
\begin{aligned}
\left[M^{LN}, M^{IP}\right] &= -iE^{NI}M^{LP} + iE^{PL}M^{IN} + iE^{NP}M^{LI} - iE^{IL}M^{PN} \\
\left[M^{LN}, P^{M}\right] &= iE^{LM}P^N - iE^{NM}P^L
\end{aligned}
$$

Due to the linaer combination of the Lorentz generators (5.29) in terms of generators of $\mathfrak{W}(\Pi,\Gamma)$, the Hopf-structure of the vector fields is carried over to the expected Hopf-structure in this representation of $U(\mathfrak{p})$, i.e. we have

$$\Delta(M^{LN}) = M^{LN}\otimes\mathbf{1}+\mathbf{1}\otimes M^{LN},\quad \epsilon(M^{LN})=0,\quad \mathrm{S}(M^{LN})=-M^{LN}.$$

The representation of $\mathfrak{W}(\Pi,\Gamma)$ on $U(\Gamma)$ determines that of $U(\mathfrak{p})$, i.e.

$$M^{LN}\triangleright\xi^R = iE^{NR}\xi^L - iE^{LR}\xi^N.$$

According to (5.20), we receive the corresponding $n+n$-decomposition being

$$\begin{array}{rcl} m^{\mu\nu}\triangleright x^\rho &=& -i\eta^{\nu\rho}x^\mu + i\eta^{\mu\rho}x^\nu, \\ m^{\mu\nu}\triangleright p^\sigma &=& -i\eta^{\nu\rho}p^\mu + i\eta^{\mu\rho}p^\nu, \end{array}$$

that is in accordance with (5.8). Since $U(\mathfrak{p})$ is a sub-Hopf-algebra of $\mathfrak{W}(\Pi,\Gamma)$, we do not require to further verify properties of the representation of $U(\mathfrak{p})$ on $U(\Gamma)$. Before we turn to actual twist-deformations of $U(\Gamma)$ and $U(\mathfrak{h}_{2n})$, we have to consider the twist formalism as such. In particular we have to discuss now double application of twists.

5.5 Twisting

In this section we first shortly review basic definitions and properties of twists. Our primary aim however is to show that a double application of twists in turn can be treated as a twist as well. This comes in handy when we first deform the $2n$-dimensional commutative phase space algebra $U(\Gamma)$ to the Heisenberg-algebra $U(\mathfrak{h}_{2n})$ and in a second step twist once more in order to obtain some deformation of $U(\mathfrak{h}_{2n})$ itself. These two twists of course can be merged to a single expression by the use of the Baker-Campbell-Hausdorff formula. But application of this formula might turn out to be complicated by the computation of higher order terms in the exponent. It might thus be a better choice not to evaluate this product of twists, althought the application then becomes a little bulky. Thus, up to the double application of twists, the first subsection of this section is rather a review to keep everything clear. The second subsection further embeds the starproduct of Weyl and Moyal (5.19) into the vector field formalism.

5.5.1 Double Twisting

As announced, we begin with a little review of the definition of twists and basic properties. We thus define twists for a general Hopf-algebra $\mathcal{H}$ to be given by

5.5.1 Definition *Let $(\mathcal{H}, \mu, \eta, \Delta, \epsilon, S; \mathbf{K})$ be a Hopf-algebra over the field $\mathbf{K}$. Then an invertible object $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$ is called a* twist, *if the following two conditions hold*

$$\begin{aligned} \mathcal{F}_{12}\,(\Delta \otimes id)\,(\mathcal{F}) \quad &= \quad \mathcal{F}_{23}\,(id \otimes \Delta)\,(\mathcal{F}) && (5.31) \\ (\epsilon \otimes id)\,(\mathcal{F}) \quad &= 1 = \quad (id \otimes \epsilon)\,(\mathcal{F}). && (5.32) \end{aligned}$$

For $\mathcal{F} = \sum \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)}$ the objects $\mathcal{F}_{12}$ and $\mathcal{F}_{23}$ are defined by

$$\begin{aligned} \mathcal{F}_{12} &= \sum \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)} \otimes \mathbf{1} \\ \mathcal{F}_{23} &= \sum \mathbf{1} \otimes \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)}. \end{aligned}$$

This definition is the basic ingredient to perform deformations. That these twists in turn provide the desired deformations of Hopf-algebras $\mathcal{H}_{\mathcal{F}}$ is stated within the following proposition.

5.5.2 Proposition *Let $(\mathcal{H}, \mu, \eta, \Delta, \epsilon, S; \mathbf{K})$ be a Hopf-algebra and let furthermore the objects $\eta, \eta^{-1} \in \mathcal{H}$ be given by*

$$\begin{aligned} \eta &= \mu\,(id \otimes S)\,(\mathcal{F}) \\ \eta^{-1} &= \mu\,(S \otimes id)\,(\mathcal{F}). \end{aligned}$$

Then $(\mathcal{H}, \mu, \eta, \Delta_{\mathcal{F}}, \epsilon, S_{\mathcal{F}}; \mathbf{K})$ with

$$\begin{aligned} \Delta_{\mathcal{F}}(h) &= \mathcal{F}\Delta(h)\mathcal{F}^{-1} \\ S_{\mathcal{F}}(h) &= \eta S(h)\eta^{-1} \end{aligned}$$

and $h \in \mathcal{H}$ is the Hopf-algebra $\mathcal{H}_{\mathcal{F}}$ that is called the twist of *$\mathcal{H}$.*

The crucial point we have to emphasize within the next step is that the Hopf-algebra $\mathcal{H}$ is not necessarily cocommutative. And in this respect $\mathcal{H}$ might already be the outcome of a preceding twist, applied to a Hopf-algebra that actually might have been cocommutative. Lets thus assume that we have a twist $\mathcal{J} \in \mathcal{H} \otimes \mathcal{H}$ in the tensor product of a Hopf-algebra $\mathcal{H}$. In particular it satisfies conditions (5.31) and (5.32) of above definition, i.e.

$$\begin{aligned} \mathcal{J}_{12}\,(\Delta \otimes \mathrm{id})\,(\mathcal{J}) \quad &= \quad \mathcal{J}_{23}\,(\mathrm{id} \otimes \Delta)\,(\mathcal{J}) && (5.33) \\ (\epsilon \otimes \mathrm{id})\,(\mathcal{J}) \quad &= 1 = \quad (\mathrm{id} \otimes \epsilon)\,(\mathcal{J}). \end{aligned}$$

We then receive a Hopf-algebra $\mathcal{H}_{\mathcal{J}}$ according to above proposition. We can now go ahead and twist once more. Thus let $\mathcal{G} \in \mathcal{H} \otimes \mathcal{H}$ be a twist of $\mathcal{H}_{\mathcal{J}}$, i.e. we have

$$\begin{aligned} \mathcal{G}_{12}\left(\Delta_{\mathcal{J}} \otimes \mathrm{id}\right)(\mathcal{G}) &= \mathcal{G}_{23}\left(\mathrm{id} \otimes \Delta_{\mathcal{J}}\right)(\mathcal{G}) \\ (\epsilon \otimes \mathrm{id})(\mathcal{G}) = 1 &= (\mathrm{id} \otimes \epsilon)(\mathcal{G}). \end{aligned}$$

With $\Delta_{\mathcal{J}}(h) = \mathcal{J}\Delta(h)\mathcal{J}^{-1}$ for $h \in \mathcal{H}$ the first of these two relations can be writen as

$$\mathcal{G}_{12}\mathcal{J}_{12}\left(\Delta \otimes \mathrm{id}\right)(\mathcal{G})\mathcal{J}_{12}^{-1} = \mathcal{G}_{23}\mathcal{J}_{23}\left(\mathrm{id} \otimes \Delta\right)(\mathcal{G})\mathcal{J}_{23}^{-1}. \tag{5.34}$$

We thus claim that $\mathcal{F} = \mathcal{G} \cdot \mathcal{J}$ is a twist of $\mathcal{H}$ as well. Relation (5.32) is directly satisfied by the homomorphy property of the counit ϵ. Relation (5.31) in turn is verified by direct computation. In particular we obtain by the use of (5.34) and (5.33) that

$$\begin{aligned} \mathcal{F}_{12}\left(\Delta \otimes \mathrm{id}\right)(\mathcal{F}) &= \mathcal{G}_{12} \cdot \mathcal{J}_{12}\left(\Delta \otimes \mathrm{id}\right)(\mathcal{G} \cdot \mathcal{J}) \\ &= \mathcal{G}_{12} \cdot \mathcal{J}_{12}\left(\Delta \otimes \mathrm{id}\right)(\mathcal{G})\left(\Delta \otimes \mathrm{id}\right)(\mathcal{J}) \\ &= \mathcal{G}_{12} \cdot \mathcal{J}_{12}\left(\Delta \otimes \mathrm{id}\right)(\mathcal{G})\mathcal{J}_{12}^{-1}\mathcal{J}_{23}\left(\mathrm{id} \otimes \Delta\right)(\mathcal{J}) \\ &= \mathcal{G}_{23}\mathcal{J}_{23}\left(\mathrm{id} \otimes \Delta\right)(\mathcal{G})\mathcal{J}_{23}^{-1}\mathcal{J}_{23}\left(\mathrm{id} \otimes \Delta\right)(\mathcal{J}) \\ &= \mathcal{G}_{23}\mathcal{J}_{23}\left(\mathrm{id} \otimes \Delta\right)(\mathcal{G})\left(\mathrm{id} \otimes \Delta\right)(\mathcal{J}) \\ &= \mathcal{F}_{23}\left(\mathrm{id} \otimes \Delta\right)(\mathcal{F}). \end{aligned}$$

We thus collected all required ingredients to proceed to actual deformations of $U(\mathfrak{h}_{2n})$.

5.5.2 Twists, Starproducts and Vector Fields

The Hopf-algebra of vector fields $\mathfrak{W}(\Pi, \Gamma)$ enables us to express the starproduct (5.19) as the inverse of a twist $\mathcal{G} \in \mathfrak{W}(\Pi, \Gamma) \otimes \mathfrak{W}(\Pi, \Gamma)$ that in the mean time is capable to deform the Poincaré-algebra $U(\mathfrak{p})$. The twist $\mathcal{G}$ corresponding to starproduct (5.19) is given by

$$\mathcal{G} = e^{+i\frac{\hbar}{2}\,\Xi_{MN}\,\mathfrak{w}_+^M \otimes \mathfrak{w}_+^N}, \tag{5.35}$$

where we define the antisymmetric tensor Ξ_{MN} by

$$\Xi_{MN} = \begin{cases} 0 & : \; M = 0, \ldots, (n-1) \quad \wedge \quad N = 0, \ldots, (n-1) \\ \eta_{M,(N-n)} & : \; M = 0, \ldots, (n-1) \quad \wedge \quad N = n, \ldots, (2n-1) \\ -\eta_{(M-n),N} & : \; M = n, \ldots, (2n-1) \quad \wedge \quad N = 0, \ldots, (n-1) \\ 0 & : \; M = n, \ldots, (2n-1) \quad \wedge \quad N = n, \ldots, (2n-1) \end{cases}$$

The defining conditions (5.31) and (5.32) for twists are easily checked. It is also easily verified that the generating relations (5.17) of $U(\mathfrak{h}_{2n})$ are reproduced by the inverse $\mathcal{G}^{-1}$ of (5.35). We can thus use (5.35) in order to deform the algebraic sector of $U(\Gamma)$ to that of $U(\mathfrak{h}_{2n})$. We further concentrate on the deformation of coproducts (5.7) in $U(\mathfrak{p})$ within the representation (5.29). Due to commutativity between P^M and $\mathcal{G}$ we only expect possible deformations for the coproduct of M^{LN}. With the undeformed coproduct

$$\Delta(M^{LN}) = \Delta(\mathfrak{w}_0^{LN} - \mathfrak{w}_0^{NL}) = (\mathfrak{w}_0^{LN} - \mathfrak{w}_0^{NL}) \otimes \mathbf{1} + \mathbf{1} \otimes (\mathfrak{w}_0^{LN} - \mathfrak{w}_0^{NL})$$

and with the help of the formula

$$e^A\ B\ e^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!}\,[A,[A,[A,\dots[A,B]]]]\,,$$

we compute the deformed coproduct to be

$$\begin{aligned}\Delta(M^{LN}) \;&=\; M^{LN} \otimes \mathbf{1} + \mathbf{1} \otimes M^{LN} \\ &\qquad + \frac{\hbar}{2}\,\Xi_{RS}\,(E^{RL}\mathfrak{w}_+^N - E^{RN}\mathfrak{w}_+^L) \otimes \mathfrak{w}_+^S \\ &\qquad + \frac{\hbar}{2}\,\Xi_{RS}\,\mathfrak{w}_+^R \otimes (E^{SL}\mathfrak{w}_+^N - E^{SN}\mathfrak{w}_+^L)\end{aligned}$$

This corresponds to results presented in [17, 52, 69, 86]. However, we should give some comments to this particular deformed coproduct in respect to the discussion of the introduction. Textbooks on field theory of course never mention the existence of a deformed coproduct of the Poincaré-algebra $U(\mathfrak{p})$ in order to respect the commutation relations of the Heisenberg-algebra $U(\mathfrak{h}_{2n})$. Within the introduction we argued that $U(\mathfrak{p})$ could be embedded in $U(\mathfrak{h}_{2n})$ - without the requirement to explicitly introduce any deformed coproducts. In fact the coproducts of $U(\mathfrak{p})$ actually are deformed without being manifest. This can be seen as follows. We freely choose the upper part of the block-diagonal generator M^{LN} and consider its coproduct, i.e.

$$\begin{aligned}\Delta(M^{\lambda\nu}) \;&=\; M^{\lambda\nu} \otimes \mathbf{1} + \mathbf{1} \otimes M^{\lambda\nu} + \frac{\hbar}{2}\,\Xi_{RS}\,(E^{R\lambda}\mathfrak{w}_+^\nu - E^{R\nu}\mathfrak{w}_+^\lambda) \otimes \mathfrak{w}_+^S \\ &\qquad + \frac{\hbar}{2}\,\Xi_{RS}\,\mathfrak{w}_+^R \otimes (E^{S\lambda}\mathfrak{w}_+^\nu - E^{S\nu}\mathfrak{w}_+^\lambda) \\ &=\; M^{\lambda\nu} \otimes \mathbf{1} + \mathbf{1} \otimes M^{\lambda\nu} + \frac{\hbar}{2}\,\eta_{\rho\sigma}\,(\eta^{\rho\lambda}\mathfrak{w}_+^\nu - \eta^{\rho\nu}\mathfrak{w}_+^\lambda) \otimes \mathfrak{w}_+^{(\sigma+n)} \\ &\qquad - \frac{\hbar}{2}\,\eta_{\rho\sigma}\,\mathfrak{w}_+^{(\sigma+n)} \otimes (\eta^{\rho\lambda}\mathfrak{w}_+^\nu - \eta^{\rho\nu}\mathfrak{w}_+^\lambda)\end{aligned}$$

We see that the coproduct of $M^{\lambda\nu}$ is nearly cocommutative - up to a minus sign in the deformed part. A cocommutative deformation would be trivial and thus we have a true but hidden deformation for the case we embed $U(\mathfrak{p})$ in $U(\mathfrak{h}_{2n})$ as we did in the introduction. This particular minus sign distinguishes the naive "action" of the momentum on a coordinate $[p^\mu, x^\rho]$ from the "action" of a coordinate on momentum $[x^\mu, p^\rho]$. This comes into account when we determine the representation of $m^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$ on a coordinate or a momentum operator by the commutator $[\ ,\ ]$ as in (5.2) and (5.3).

5.6 An Example for a Twisted Heisenberg-Algebra

In this final section we intend to outline the presented construction for a specific example. In particular we concentrate on how the Heisenberg-algebra $U(\mathfrak{h}_{2n})$ is further deformed by a an additional twist $\mathcal{I}$. This corresponds to a second deformation of $U(\Gamma)$. In this section we merely whish to give some guidance to the presented apperatus and thus stick to a very simple but nontrivial example. We leave it to the reader to find more interesting or even more realistic deformations. We sketch an example that corresponds to a twist presented earlier in [51] and [38] and adapt it to our context. This specific twist is given by

$$\mathcal{I} = e^{i\ a\ \mathfrak{w}_0^{(2n-1)(2n-1)} \otimes \mathfrak{w}_+^{(n-1)}}, \quad a \in \mathbf{R}.$$

With the total twist

$$\mathcal{F} = \mathcal{G} \cdot \mathcal{I} = e^{+i\frac{\hbar}{2}\ \Xi_{MN}\ \mathfrak{w}_+^M \otimes \mathfrak{w}_+^N} \cdot e^{+i\ a\ \mathfrak{w}_0^{(2n-1)(2n-1)} \otimes \mathfrak{w}_+^{(n-1)}},$$

we obtain the starproduct

$$\mathcal{F}^{-1} = \mathcal{I}^{-1} \cdot \mathcal{G}^{-1} = e^{-i\ a\ \mathfrak{w}_0^{(2n-1)(2n-1)} \otimes \mathfrak{w}_+^{(n-1)}} \cdot e^{-i\frac{\hbar}{2}\ \Xi_{MN}\ \mathfrak{w}_+^M \otimes \mathfrak{w}_+^N},$$

that provides us with a deformation of $U(\mathfrak{h}_{2n})$. With the starproduct $\mathcal{I}^{-1}$ only some of the generating relations of $U(\mathfrak{h}_{2n})$ actually become deformed. We first generally consider the starproduct of the product of two generators $\xi^R, \xi^S \in U(\Gamma)$, i.e.

$$\xi^R *_{\mathcal{F}} \xi^S = \xi^R \xi^S + i\ a\ E^{(2n-1)R} E^{(n-1)S} \xi^{(2n-1)} + i\ \frac{\hbar}{2}\ \Xi^{RS}.$$

In particular we thus obtain for the choice $R \to 2n-1$ and $S \to n-1$ that

$$\xi^{(2n-1)} *_{\mathcal{F}} \xi^{(n-1)} = \xi^{(2n-1)} \xi^{(n-1)} + i\ a\ \xi^{(2n-1)} - i\frac{\hbar}{2} \eta^{(n-1)(n-1)},$$

such that within the $n+n$-separation we obtain the commutator

$$\left[x^{(n-1)} \stackrel{*_{\mathcal{F}}}{,} p^{(n-1)}\right] = i\,\hbar\,\eta^{(n-1)(n-1)} - i\,a\,p^{(n-1)}.$$

We thus obtained the expected deformation of the Heisenberg-algebra $U(\mathfrak{h}_{2n})$ for one of its most characteristic relations. We further compute an example for a deformation of the coproduct of M^{LN} such that we obtain manifest covariance with respect to deformed $U(\mathfrak{p})$. In particular we choose the coproduct $\Delta_{\mathcal{F}}(M^{(2n-1)n})$ and to this purpose we first compute the corresponding twisted coproducts of $\mathfrak{w}_0^{(2n-1)n}$ and $\mathfrak{w}_0^{n(2n-1)}$, i.e. we have

$$\begin{aligned}
\Delta_{\mathcal{F}}(\mathfrak{w}_0^{(2n-1)n}) &= \mathcal{G}\cdot\mathcal{I}\left(\mathfrak{w}_0^{(2n-1)n}\otimes\mathbf{1}+\mathbf{1}\otimes\mathfrak{w}_0^{(2n-1)n}\right)\mathcal{I}^{-1}\cdot\mathcal{G}^{-1}\\
&= \mathcal{G}\Big(\mathfrak{w}_0^{(2n-1)n}\otimes\mathbf{1}+\mathbf{1}\otimes\mathfrak{w}_0^{(2n-1)n}\\
&\quad +\mathfrak{w}_0^{(2n-1)n}\otimes(e^{+a\,\eta^{(n-1)(n-1)}\,\mathfrak{w}_+^{(n-1)}}-1)\Big)\,\mathcal{G}^{-1}\\
&= \mathfrak{w}_0^{(2n-1)n}\otimes e^{+a\,\eta^{(n-1)(n-1)}\,\mathfrak{w}_+^{(n-1)}}+\mathbf{1}\otimes\mathfrak{w}_0^{(2n-1)n}\\
&\quad -\frac{\hbar}{2}\,\eta^{(n-1)(n-1)}\,\mathfrak{w}_+^{n}\otimes\mathfrak{w}_+^{(2n-1)}e^{+a\,\eta^{(n-1)(n-1)}\,\mathfrak{w}_+^{(n-1)}}\\
&\quad +\frac{\hbar}{2}\,\eta^{(n-1)(n-1)}\,\mathfrak{w}_+^{(2n-1)}\otimes\mathfrak{w}_+^{n}
\end{aligned}$$

and

$$\begin{aligned}
\Delta_{\mathcal{F}}(\mathfrak{w}_0^{n(2n-1)}) &= \mathcal{G}\cdot\mathcal{I}\left(\mathfrak{w}_0^{n(2n-1)}\otimes\mathbf{1}+\mathbf{1}\otimes\mathfrak{w}_0^{n(2n-1)}\right)\mathcal{I}^{-1}\cdot\mathcal{G}^{-1}\\
&= \mathcal{G}\Big(\mathfrak{w}_0^{n(2n-1)}\otimes\mathbf{1}+\mathbf{1}\otimes\mathfrak{w}_0^{n(2n-1)}\\
&\quad +\mathfrak{w}_0^{n(2n-1)}\otimes(e^{-a\,\eta^{(n-1)(n-1)}\,\mathfrak{w}_+^{(n-1)}}-1)\Big)\,\mathcal{G}^{-1}\\
&= \mathfrak{w}_0^{n(2n-1)}\otimes e^{-a\,\eta^{(n-1)(n-1)}\,\mathfrak{w}_+^{(n-1)}}+\mathbf{1}\otimes\mathfrak{w}_0^{n(2n-1)}\\
&\quad +\frac{\hbar}{2}\,\eta^{(n-1)(n-1)}\,\mathfrak{w}_+^{(2n-1)}\otimes\mathfrak{w}_+^{n}e^{-a\,\eta^{(n-1)(n-1)}\,\mathfrak{w}_+^{(n-1)}}\\
&\quad -\frac{\hbar}{2}\,\eta^{(n-1)(n-1)}\,\mathfrak{w}_+^{n}\otimes\mathfrak{w}_+^{(2n-1)}.
\end{aligned}$$

We thus obtain that

$$\begin{aligned}
&\Delta_{\mathcal{F}}(M^{(2n-1)n})\\
&\quad = M^{(2n-1)n}\otimes e^{+a\,\eta^{(n-1)(n-1)}\,P^{(n-1)}}+\mathbf{1}\otimes M^{(2n-1)n}\\
&\quad +2\,\mathfrak{w}_0^{n(2n-1)}\otimes\sinh(+a\,\eta^{(n-1)(n-1)}\,P^{(n-1)})\\
&\quad -\frac{\hbar}{2}\,\eta^{(n-1)(n-1)}\,P^{n}\otimes\left(P^{(2n-1)}e^{+a\,\eta^{(n-1)(n-1)}\,P^{(n-1)}}-P^{(2n-1)}\right)\\
&\quad +\frac{\hbar}{2}\,\eta^{(n-1)(n-1)}\,P^{(2n-1)}\otimes\left(P^{n}-P^{n}e^{-a\,\eta^{(n-1)(n-1)}\,P^{(n-1)}}\right)
\end{aligned}$$

Note that this deformation can only be performed within the Hopf-algebra of vector fields $\mathfrak{W}(\Pi, \Gamma)$ and not solely within the subalgebra $U(\mathfrak{p})$. There are of course several more deformed coproducts for this specific example of deformation. However, since we merely whish to give some idea of how the constructions in this chapter are applied to specific examples, we close our considerations at this point.

5.7 Conclusion

Providing the formalism to perform deformations of the Heisenberg-algebra and the corresponding twists of the Poincaré-algebra is certainly only one step of several that have to be mastered in order to obtain some enhanced version of relativistic quantum mechanics. In order to receive useful representations of the deformed Heisenberg-algebra on states of a Hilbert-space, it is for example a crucial point to discuss hermiticity and self-adjointness of the generators in deformed $U(\mathfrak{h}_{2n})$. It is moreover not yet clear what further implications for the interpretation of quantum mechanics might result form such algebraic mixture of coordinates and momenta. However, quantum mechanics as we apply it in field theories, does not discribe high-energy measurements in such a way as we would expect them from Planck-scale physics. Thus, regarding noncommutative geometry as a high-energy approach, we should als take into account that gravity might not only provide a static form of noncommutativity - but one that is caused by the properties of matter itself that exists within such backgrounds.

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