Matrix Formulation of Fractional Supersymmetry

and

q-Deformation

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Abstract

Supersymmetry, which is the only non-trivial Z_2 extension of the Poincaré algebra, can be generalized to fractional supersymmetry, when the space time is smaller than 3. Since symmetries play an important role in physics; the principal task of quantum groups consist in extanding these standard symmetries to the deformed ones, which might be used in physics as well. This two aspects will be the main focus of this thesis. In this work, we discuss the matrix formulation of fractional supersymmetry, the q-deformation of KdV hierarchy systems and noncommutative geometry.

In the first part fractional supersymmetry generated by more than one charge operator and those which can be described as a matrix model are studied. Using parafermionic field-theoretical methods, the fundamentals of two-dimensional fractional supersymmetry $Q^k = P$ are set up. Known difficulties induced by methods based on the $U_q(sl(2))$ quantum group representations and noncommutative geometry are avoided in the parafermionic approach. Moreover, we find that fractional supersymmetric algebras are naturally realized as matrix models. The $k = 3$ case is studied in detail.

In the second part we will study the q-deformed algebra and the q-analogues of the generalised KdV hierarchy. We construct in this part the algebra of q-deformed pseudo-differential operators, shown to be an essential step toward setting up a q-deformed integrability program. In fact, using the results of this q-deformed algebra, we derive the q-analogues of the generalised KdV hierarchy. We focus in particular on the first leading orders of this q-deformed hierarchy, namely the q-KdV and q-Boussinesq integrable systems. We also present the q-generalisation of the conformal transformations of the currents u_n, $n \geq 2$, and discuss the primary condition of the fields w_n, $n \geq 2$, by using the Volterra gauge group transformations for the q-covariant Lax operators.

In the last part we will discuss quantum groups and noncommutative space. All studies in this part are based on the idea of replacing the ordinary coordinates with non commuting operators. We will also formulate some aspects of noncommutative geometry mathematically and we will be mainly concerned with quantum algebra and quantum spaces.

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Chapter 1

Introduction

Supersymmetry has been a popular and fruitful area of research for at least twenty years. It is a symmetry between bosons and fermions [1], [2], where the fermions pick up a minus sign (or a phase factor $e^{i\pi}$) each time two of them commute.
Study of supersymmetry in space time of one dimension, time, has given rise to the important topic of supersymmetric quantum mechanics [3], [4]. The most primitive version of supersymmetric quantum mechanics is one that uses a single real Grassmann number θ such that:

$$\begin{aligned} \theta &= \bar{\theta} \\ \theta^2 &= 0. \end{aligned} \tag{1.1}$$

As a results the theory possesses a natural Z_2-grading and a single generator Q of its supersymmetry transformations which obeys:

$$Q^2 = -\partial_t. \tag{1.2}$$

The distinctive features of supersymmetric theories which possesses such a Z_2-grading can be seen by reference to various works [5], [6].
Recently, two methods of generalizing ordinary supersymmetry have received considerable attention. The generalization to parasupersymmetry [7], [8] involves the replacement of the usual bilinear supersymmetry algebra with a trilinear algebra in analogy to the way in which the ordinary bosonic and fermionic algebras are generalized to those associated with parabosons and parafermions [9], [10]. Such generalizations involve the introduction of a parasuperfield and parasuperspace, and the natural variables to use in working with these are the paragrassmann variables. On the other hand, the second is the generalization to what is known as fractional supersymmetry [11], [12], which seeks to replace the Z_2-grading associated with the supersymmetric algebra with a Z_n-graded algebra in such a way that the fractional supersymmetry transformations mix elements of all degrees.

This involves the introduction of fractional superfields and fractional superspace. This term fractional supersymmetry is currently being applied to a class of generalisation of supersymmetry in one dimension. The work on fractional supersymmetry can be presented most straightforwardly by creating theories with Z_n-grading by generalisation of theories with Z_2 grading. Thus one considers theories involving a single real generalised Grassmann number θ which obeys:

$$\begin{aligned} \theta &= \bar{\theta} \\ \theta^n &= 0, \quad n = 2, 3, 4, \ldots \quad , \end{aligned} \tag{1.3}$$

in which the generator Q of the generalised fractional supersymmetry transformations that leave such a theory invariant obeys:

$$Q^n = -\partial_t. \tag{1.4}$$

The last results accounts loosly for the use of the term fractional as an identifier of the theory.

The generalisation from ordinary to fractional supersymmetry not only has intrinsic interest but may also be expected to produce interesting new models in classical and quantum mechanics. Indeed, there have been a large number of studies of fractional supersymmetry [13], Some of these deal with a complex Grassmann variable θ obeying equation (1.3). Others employ N different copies of θ which satisfy eq(1.1), thus developing N-extended fractional supersymmetry. Fractional supersymmetry is then contrasted below with a distinct class of generalisations of basic, or Z_2 -graded, supersymmetry, those which possess parasupersymmetry. There has been a great deal of attention given recently to work in this field [14], [15], and often these studies contain thoughts relevant also to fractional supersymmetry. We believe that the whole area promises both activity and progress in the feature. Two areas need attention: One concerns differentiation with respect to θ; the other is the situation surrounding families of multiplicative rules of the type:

$$\begin{aligned} \epsilon\theta &= q^{-1}\theta\epsilon \\ q &= exp(2i\pi/n), \end{aligned} \tag{1.5}$$

involving a Grassmann number θ, its associated transformation parameter ϵ and dynamical variables of Grassmann type.

Fractional supersymmetry which aims at a generalisation of supersymmetry, has been investigated in dimensions one and two. In one dimensional spaces, where no rotation is available, this symmetry is generated by one generator which can be seen as the F^{th} root of the time translation $(Q_t)^F = \partial_t$. $F = 2$ corresponds to the usual supersymmetry. A group theoretical justification was then given in

[16], [17] and this symmetry was applied in the world-line formalism [17]. The second particuliar cases, are the two-dimensional spaces where, by use of conformal transformations, the antiholomorphic part of the fields transforms independently [18]. In this case the fractional supersymmetric algebra is extended by two Q_z and $Q_{\bar{z}}$ generators satisfying: $(Q_z)^F = \partial_z$ and $(Q_{\bar{z}})^F = \partial_{\bar{z}}$.
Fractional supersymmetry and quantum groups are emerged on one hand as a symmetric generalisation of groups of some integrable systems and as an algebraic structure associated to noncommutative geometry on the other hand. Just as Lie groups and their associated homogenous spaces provided definitive examples of classical geometry even before Riemann formulated their intrinsic structure as a theory of manifolds, so quantum groups and their associated quantum homogenous spaces, quantum planes etc., provide large classes of examples of proven mathematical and physical worth and clear geometrical content on which to build and develop noncommutative geometry. They are noncommutative spaces in the sense that they have generators or coordinates like the noncommuting operators $\hat{x}$ and $\hat{p}$ in quantum mechanics but with much richer and more geometric algebraic structure than the Heisenberg algebra.
Noncommutative geometry is a relatively new field of mathematics which is now becoming one of important tools, or rather ways of thinking, in many areas of mathematics and theoretical physics. It takes roots in quantum mechanics and representation theory. But the main motivations come from relatively recent amazing developments in mathematical physics (quantum groups and related quantized spaces) and from physics [19], [20].
Noncommutative geometry in simplest terms is the idea that space coordinate (say, x and y) do not have to commute with each other. This means that:

$$xy - yx = [x, y] \neq 0. \tag{1.6}$$

Since commutation relations are used in quantum mechanics to express uncertainly, noncommuting coordinates were proposed to quantize space at small distances. Until recently, such proposals were not taken very seriously, since they require an a priori violation of locality and Lorentz invariance. The idea of noncommutative geometry was first put on a solid mathematical footing by Alain Connes [21] in 1980, where it was applied to noncommutative tori.
If one uses Weyl-quantization then classical functions f and g are mapped to noncommutative one $\hat{f}$ and $\hat{g}$ with the property that the product $\hat{f}\hat{g}$ is given by a deformed product called star product($\star$) such that:

$$f \star g = f \cdot g + \theta^{ij} \partial_i f \partial_j g + \mathcal{O}(\theta^2), \tag{1.7}$$

where the noncommutativity is controlled by the deformation parameter θ^{ij}.

The first part of this thesis will be devoted to study fractional supersymmetry generated by more than one charge operator and which can be described as a matrix model. Using conformal parafermionic field theoretical methods, we have studied realisations of the fractional supersymmetric algebras extending the usual $2D$ supersymmetries. We will find in this part of thesis that the well known difficulties present in models based on $U_q(sl(2))$ and non commutative geometry representations are overpassed. One of the consequences of the new approach is that instead of one generator $Q_{-1/k}$, fractional supersymmetric algebras are generated by many basic charge generators. The poliferation of the fractional supersymmetric generators is the price one should pay in order to build a $2D$ local fractional supersymmetric quantum field theory.

In chapter 2, we will first discuss fractional supersymmetry in one dimension [22], [23], [24]. In this chapter we particularize the case $k = 3$. In the first step we define, in analogy with the superspace, the fractional superspace as some kind of coset space reobtaining all what has been done in the framework of fractional supersymmetry [25], [26]. The second step is to construct a representation of the fractional supersymmetric algebra acting on the field Φ, as well as a covariant derivative to establish the fractional action S. Then we study the $2D$- fractional supersymmetry, this means that we extend the results already obtained in one dimension to build the $2D$ fractional supersymmetric lagrangian, introducing an adapted fractional superspace by help of generalized Grassmann variables and its differential structure. In the third step, we will introduce the main lines of RdTS (the Rauch de Traubenberg-Slupinski algebra) analysis [27], [28]. Non-trivial extensions of the three dimensional Poincaré algebra, beyond the supersymmetric one, are explicitly built. These algebraic structures are the natural three dimensional generalisations of fractional supersymmetry of order k already considered in one and two dimensions. Representations of these algebra are exhibited. It is shown that these extensions generate symmetries which connect fractional spin states or anyons.

In chapter 3, using parafermionic field theoretical methods, the fundamentals of 2D fractional supersymetry $Q^k = P$ are set up. Known difficulties induced by methods based on the $U_q(sl(2))$ quantum group representations and non commutative geometry are overpassed in the parafermionic approach. Moreover we find that fractional supersymmetric algebras are realized as a matrix model [29]. So we start by studying the matrix realisation of 2D supersymmetry. We work out the links between fractional supersymmetry and parafermions. Then we discuss the matrix realisation of fractional supersymmetry. The case $k = 3$ is studied in details. One of the consequences of the new approach is that instead of one generator $Q_{-1/k}$, fractional supersymmetric algebras are generated by many basic

charge operators $Q_{-x}, x = 1/k, 2/k, \ldots$ [29], [30]. For $K = 3$ for example, the $2D(\frac{1}{3}, 0)$ fractional supersymmetric algebra is generated by two main charge operators $Q_{-1/3}$ and $Q_{-2/3}$ satisfying equations (3.32) and the $2D((\frac{1}{3})^2, 0)$ fractional supersymmetric algebra generated by two doublets $Q^{\pm}_{-1/3}$ and $Q^{\pm}_{-2/3}$ verifying equations (3.34-35).
The results of the chapter 3 have already been published in [29] together with E.H. Saidi and in [30] together with E.H. Saidi and A. El Rhalami. See also the work [31] done again together with E.H. Saidi and A. El Rhalami.

The second part of this thesis will be devoted to the study of q-deformed algebra and the q-analogues of the generalised KdV hierarchy. Motivated by the relevance of both the generalised integrable KdV hierarchies and quantum deformations, we focus in this part of this thesis to present a systematic study of bidimensional q-deformed non linear integrable models by building the algebra of q-deformed pseudo-differential operators, shown to be an essential step toward setting up a q-deformed integrability program [32]. In fact, using the results of this q-deformed algebra, we derive the q-analogues of the generalised KdV hierarchy.

In chapter 4, we represent the algebra of q-deformed pseudo-differential operators. This provides the basic ingredients, which we need in the q-deformed integrability study. We start in this chapter from the well known q-deformed derivation law, $\partial z = 1 + qz\partial$ [33], [34], and derive the q-analogue of the Leibnitz rule for both local and non local differential operators. This result, which gives naturally the algebra of q-deformed pseudo-differential operators, will provide a way for generating a hierarchy of q-deformed Lax evolution equations.

In chapter 5, we will build up the q-analogue of the generalised KdV hierarchy. We will concentrate in particular on the first leading orders of this hierarchy, namely q-KdV and q-Boussinesq integrable systems. Then we present the q-generalisation of the conformal transformations of the currents $u_n, n \geq 2$ by taking the two particular examples discussed previously, namely the q-KdV and q-Boussinesq integrable models described respectively by $L_2(u)$ and $L_3(u)$. Having given explicitly the conformal transformation of the currents u_2 and u_3 of conformal spin 2 and 3, we generalise these results to higher conformal spin currents $u_n(z)$, with $n = 2, 3, \ldots$.

In chapter 6, we will discuss the primary condition of the fields $W_n, n \geq 2$ by using the Volterra gauge group transformations for the q-covariant Lax operators which is associated to an orbit in which no anomalous terms appear. Our aim after is to make an appropriate choice on the Volterra parameters a_i such that w_i become primary conformal currents satisfying the last conditions. In the classical limit, the analytic field u_2 behaves as spin 2-field of 2D conformal field theory which coincide with the w_2 current. Similarly in the deformed case; we will require for w_2 to be proportional to u_2.

In chapter 7, we will discuss the field theoritical models describing the self couplings of the matter multiplets $(0^2, (\frac{1}{3})^2, (\frac{2}{3})^2)$ and $(0^4, (\frac{1}{3})^4, (\frac{2}{3})^4)$. More precisely, we will describe brefly the superfield theory of the matter couplings of $(\frac{1}{3}, \frac{1}{3})$ and $((\frac{1}{3})^2, (\frac{1}{3})^2)$ fractional superalgebras. We start first by describing the superfield theory of the $2D(\frac{1}{3}, \frac{1}{3})$, fractional supersymmetry equation already studied in chapter 3, especially the matter coupling of the on shell scalar representation $(\varphi, \Psi_{\pm 1/3}, \Psi_{\pm 2/3})$, using the formal analogy between equations (3.32) and those of $2D$ $N = 2$ $U(1)$ supersymmetry. After these results we will give the action $S[\Phi]$ describing the dynamics and the couplings of the superfields Φ which is similar to that of $2D$ $((\frac{1}{2})^4, 0)$ $su(2)$ harmonic superspace. More generally the matter couplings of the $((\frac{1}{3})^2, (\frac{1}{3})^2) su(2)$ fractional supersymmetry, extending equations (7.4-5) by adjoining the analytic part, give us the action S by help of the harmonic superspace formulation [35].
The end of this chapter, is devoted to the Toda field theory construction using the results obtained in chapters 4,5 and 6. We will present here the $su(n)$-Toda ($su(2)$-Liouville) field theory construction by building the q-analogue of the $su(2)$- Liouville and $su(n)$- Toda conformal field theories.
All the results of the second part of this thesis have already been published in [32] together with M.Hssaini, M. Kessabi, B. Maroufi and M.B. Sedra and in [29] together with E.H. Saidi.

The last part of this thesis will be devoted to the discussion of quantum groups and noncommutative space. All studies in this part are based on the idea of replacing the ordinary coordinates with non commuting operators. We will also formulate some aspects of noncommutative geometry mathematically and we will be mainly concerned with quantum algebra and quantum spaces.

In chapter 8, we start by studying quantum planes and quantum groups and their differentials calculus on the noncommutative space. We will give some differential relations and as an example we will discuss the Manin plane. In the second step, we will treat the star product of functions and as examples we will take the three type of noncommutative structure namely:
• Canonical structure:

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}, \quad \theta^{ij} \in C.$$

- Lie algebra structure:

$$[\hat{x}^i, \hat{x}^j] = i\lambda^{ij}_k \hat{x}^k, \quad \lambda^{ij}_k \in C.$$

- Quantum space structure:

$$\hat{x}^i \hat{x}^j = q^{-1} \hat{R}^{ij}_{kl} \hat{x}^k \hat{x}^l.$$

We end this chapter by formulating gauge theory on noncommutative space. We will see that this gauge theorie is based on the idea that multiplication of a field by a coordinate or a function is not covariant only if that function does not commute with gauge transformation. This can be resolved by adding an appropriate noncommutative gauge potentials and thus introducing covariant coordinate in analogy to the covariant derivative of ordinary gauge theory.

Chapter 2

Fractional supersymmetry in 1D and 2D

The spin statistics theorem and the Haag, Lopuszanski and Sohnius no-go theorem [36] tell us that supersymmetry is the most non trivial symmetry that one can consider. If we are in a $D \leq 3$ dimensional space time one can find statistics that are neither fermions nor bosons, but anyons [37] or particles which admit fractional statistics.
In this chapter we will study fractional supersymmetry in one and two dimensions. Following the way which leads from 1D supersymmetry to the Dirac equation applied in the context of fractional supersymmetry, we get a new equation acting on states which are in the representation of the braid group. This equation can be seen as an extension of the Dirac equation in the sense that the K-th power of the field operator is equal to the Klein-Gordon one. In this chapter we particularize the case $k = 3$.
In the first step we define, in analogy with the superspace, the fractional superspace as some kind of coset space reobtaining all what has been done in the framework of fractional supersymmetry. In the second step we construct explicitly the 1D and 2D fractional supersymmetric actions, introducing an adapted fractional superspace by help of generalized Grassmann variables and its differential structure. Then we will introduce the main lines of RdTS (the Rauch de Traubenberg-Slupinski algebra) analysis.

2.1 $1D$-Fractional supersymmetry

Supersymmetry, which is the only non-trivial Z_2 extension of the Poincaré algebra [38], can be generalized to fractional supersymmetry [39], when the space time is smaller than 3. The fractional supersymmetric algebra, possesses a Z_k structure

whose basic fields will be of graduation $0 \dots k-1$, generalizing the concept of boson and Fermion. In this section we will consider only the case $k=3$.
Fractional supersymmetry is generated by H, the hamiltonian or the generator of time translation and Q, the generator of the fractional supersymmetry transformations. The algebra is given by:

$$\begin{aligned}
[Q, H] &= 0 \\
Q^3 &= -H,
\end{aligned} \tag{2.1}$$

it is important to note that the algebra (2.1) is neither a Lie algebra nor a superalgebra. To developp a field theory which is invariant under the fractional transformation, we introduce a fractional superspace. The time t is then extented to (t, θ) where θ is a real generalized Grassmann variable, satisfying the relation $\theta^3 = 0$ [40].

2.1.1 Transformation of t and θ

In the usual superspace $\theta^2 = 0$, the point (t, θ) is parametrized by:

$$exp(tH + \theta Q). \tag{2.2}$$

Using the supersymmetric algebra:

$$\begin{aligned}
[Q, H] &= 0 \\
\{Q, Q\} &= -2H,
\end{aligned} \tag{2.3}$$

and the definition of a supersymmetry transformation with parameter ϵ, we get the transformation law:

$$\begin{aligned}
exp(t' H + \theta' Q) &= exp(\epsilon Q) exp(tH + \theta Q) \\
&= exp[(t + i\epsilon\theta)H + (\theta + \epsilon)Q].
\end{aligned} \tag{2.4}$$

Let now Q the generator of fractional supersymmetry satisfying the condition:

$$Q^3 = -H, \tag{2.5}$$

and define a point in the fractional superspace (t, θ), with $(\theta^3 = 0)$ by its parametrization:

$$\begin{aligned}
exp_{gr}(tH + \theta Q) &= exp_{gr}(tH) exp_{gr}(\theta Q) \\
&= exp(tH) exp_{gr}(\theta Q)
\end{aligned} \tag{2.6}$$

where exp_{gr} is the graded exponential and q is a primitive cubic root of unity that we can take equal to $exp(2i\pi/3)$ and:

$$exp_{q^a}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\{n\}_{a!}} \quad , \tag{2.7}$$

where:

$$\{n\}_{a!} = \{n\}_a \{n-1\}_a \ldots \{1\}_a$$

is the q-factorial.

$$\{k\}_a = \frac{1-q^{ak}}{1-q^a}$$

is the q-number, with $q = exp(\frac{2i\pi}{3})$.
This series exactly stops with its $(n-1)$-th power because $\theta^n = 0$, in the general case. For $n = 2$ we have only two terms and in this case the usual exponential coincides exactly with the $q(=-1)$-exponential. Going back to $n = 3$ we get:

$$exp_q(\theta Q) = 1 + \theta Q - q(\theta Q)^2. \tag{2.8}$$

Introducing ϵ the real parameter of the fractional supersymmetry ($\epsilon^3 = 0$), and using the following commutation relations:

$$\begin{aligned} Q\theta &= q^2\theta Q \\ Q\epsilon &= q^2\epsilon Q, \end{aligned} \tag{2.9}$$

we get the fractional supersymmetry transformation in the superspace:

$$\begin{aligned} exp(t'H)exp_q(\theta' Q) &= exp_q(\epsilon Q)exp(tH)exp_q(\theta Q) \\ &= exp[(t + q(\epsilon^2\theta + \epsilon\theta^2))H]exp_q((\theta+\epsilon)Q). \end{aligned} \tag{2.10}$$

From this transformations we obtain:

$$\begin{aligned} t' &= t + q(\epsilon^2\theta + \epsilon\theta^2) \\ \theta' &= \theta + \epsilon. \end{aligned} \tag{2.11}$$

The parameter of the fractional supersymmetry transformation ϵ verifying $\epsilon^3 = 0$ satisfies the following commutation law:

$$\theta\epsilon = q\epsilon\theta,$$

this commutation relation between the two variables ϵ and θ has the following

consequences:
• It ensures that if $\epsilon^3 = \theta^3 = 0$ then $(\epsilon + \theta)^3 = 0$ [41].
• The time remains real after a fractional supersymmetry transformation.
• The fractional supersymmetry transformations commute with the covariant derivatives.
• The fractional supersymmetry transformations ϵQ satisfy the Leibnitz rule [24].

2.1.2 Construction of the action in $\mathbf{D = 1}$

In this part we construct a representation of the fractional supersymmetric algebra acting on Φ, as well as a covariant derivative to establish the action. We first need to recall some basic features of the derivation acting on generalized Grassmann variables θ ($\theta^3 = 0$). This Z_k structure admits in general $(k-1)$ derivatives, noted ∂_θ and δ_θ in the case of $k = 3$ and which satisfy:

$$\begin{aligned} \partial_\theta \theta - q\theta\partial_\theta &= 1 \\ \delta_\theta \theta - q^2\theta\delta_\theta &= 1 \\ \partial_\theta^3 &= 0 \\ \delta_\theta^3 &= 0 \\ \partial_\theta\delta_\theta &= q^2\delta_\theta\partial_\theta. \end{aligned} \tag{2.12}$$

In the general case this two derivatives act as:

$$\begin{aligned} \partial_\theta(\theta^k) &= (1 + q + q^2 + \ldots + q^{k-1})\theta^{k-1} + q^k\theta^k\partial_\theta \\ \delta_\theta(\theta^k) &= (1 + \bar{q} + \bar{q}^2 + \ldots + \bar{q}^{k-1})\theta^{k-1} + \bar{q}^k\theta^k\delta_\theta, \end{aligned} \tag{2.13}$$

where $\bar{q} = q^{-1}$.
Let us consider the two basic objects Q and D, which represent respectively the fractional supersymmetry generator and the covariant derivative [24], [25], [42]:

$$\begin{aligned} Q &= \partial_\theta + q\theta^2\partial_t \\ D &= \delta_\theta + q^2\theta^2\partial_t. \end{aligned} \tag{2.14}$$

Using the equations (2.14) and (2.5), we can check explicitly that:

$$D^3 = Q^3 = -\partial_t = -H. \tag{2.15}$$

There is an other method to verify that $Q^3 = -\partial_t$ using the matrix representation of ∂_θ and δ_θ as follows:

$$\partial_\theta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1+q \\ 0 & 0 & 0 \end{pmatrix}$$

$$\theta = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad (2.16)$$

from this two matrices we find that:

$$\begin{aligned} Q^3 &= \begin{pmatrix} -\partial_t & 0 & 0 \\ 0 & -\partial_t & 0 \\ 0 & 0 & -\partial_t \end{pmatrix} \\ &= -\partial_t \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \qquad (2.17)$$

which implies the relation:

$$Q^3 = -\partial_t. \qquad (2.18)$$

The two representations D and Q satisfy the commutation law given by:

$$QD = q^2 DQ. \qquad (2.19)$$

We consider now a real fractional superfield Φ belonging to the fractional superspace. The Taylor expansion of $\Phi(t,\theta)$ gives:

$$\Phi(t,\theta) = x(t) + q^2\theta\Psi_2(t) + q^2\theta^2\Psi_1(t), \qquad (2.20)$$

where $x(t)$, $\Psi_1(t)$, $\Psi_2(t)$ are three real fields respectively of grade $0,1,2$ such that $\Psi_1^3 = \Psi_2^3 = 0$ and are submitted to the commutation relations:

$$\begin{aligned} \theta x &= x\theta \\ \theta\Psi_1 &= q\Psi_1\theta \\ \theta\Psi_2 &= q^2\Psi_2\theta \\ \Psi_2\Psi_1 &= q\Psi_1\Psi_2. \end{aligned} \qquad (2.21)$$

It should be stressed that these relations are the only ones which are arbitrary, all the other follow naturally [17]. It is easy to obtain the transformations on the fractional superfield induced by fractional supersymmetry transformations $\Phi(t,\theta) \longrightarrow \Phi(t',\theta')$ as folows:

$$\Phi(t,\theta) \to \Phi(t',\theta') = x(t') + q^2\theta'\Psi_2(t') + q^2\theta'^2\Psi_1(t'). \qquad (2.22)$$

Inserting the values of t' and θ' obtained previously in equation (2.11), we get the transformed fields:

$$\Phi(t',\theta') = x(t) + \epsilon\Psi_2(t) + q^2\epsilon^2\Psi_1(t) + q^2\theta(\Psi_2(t) + \epsilon\dot{x}(t) - q\epsilon\Psi_1(t)) + q^2\theta^2(\Psi_1(t) + \epsilon\dot{x}(t) - q\epsilon^2\dot{\Psi}_2(t)). \quad (2.23)$$

We obtain then the fractional supersymmetry transformations of the fields x(t), $\Psi_1(t)$ and $\Psi_2(t)$:

$$\begin{aligned} \delta_\epsilon x &= q^2\epsilon\Psi_2 \\ \delta_\epsilon\Psi_2 &= -q\epsilon\Psi_1 \\ \delta_\epsilon\Psi_1 &= \epsilon\dot{x}. \end{aligned} \quad (2.24)$$

A direct calculation proves on one hand:

$$\Phi(t',\theta') = exp_{q^2}(\epsilon Q)\Phi(t,\theta),$$

and on the other hand that:

$$\delta_\epsilon\Phi = \epsilon Q\Phi(t,\theta).$$

Using the relation (2.19), we get:

$$\delta_\epsilon D\Phi = D\delta_\epsilon\Phi. \quad (2.25)$$

Finally arguing that the θ^2 component of Φ transforms like a total derivative, we can take the opportunity to construct the action by taking the θ^2 part of the action built in the fractional superspace. In other words, using the results on integration upon generalized Grassmann variables, which is interpreted as a derivation of order $(k-1)$ over θ as [43]:

$$\int d\theta = (\frac{d}{d\theta})^{k-1}, \quad (2.26)$$

we construct the fractional real action in one dimension:

$$\begin{aligned} S &= -\frac{q^2}{2}\int dtd\theta\dot{\Phi}D\Phi \\ &= \int dt(\frac{\dot{x}^2}{2} + \frac{q^2}{2}\dot{\Psi}_1\Psi_2 - \frac{q}{2}\dot{\Psi}_2\Psi_1). \end{aligned} \quad (2.27)$$

This action represent the fractional supersymmetry action in one dimension. In the next section we will extend this results found in one dimension to construct an action in the case of fractional supersymmetry in two dimensions.

2.2 2D- Fractional supersymmetry

In this section, we study the fractional supersymmetry in two dimensions. We construct explicitly the 2D fractional supersymmetry action, introducing an adapted fractional superspace by help of generalized Grassmann variables and its differential structure.
We want to extend all those results of fractional supersymmetry in one dimension to build an action in the complex plane.
Let θ and $\bar{\theta}$ two generalized Grassmann variables satisfying the following commutation law:

$$\begin{aligned} \theta^3 &= 0, \\ \bar{\theta}^3 &= 0 \\ \theta\bar{\theta} &= q\bar{\theta}\theta. \end{aligned} \tag{2.28}$$

It is clear that if $\bar{\theta} = \theta^*$, the equation (2.28) is not satisfied. In fact, if we take $\bar{\theta}^2$ instead of $\bar{\theta}$ in this equation, the complex conjugation lead to a contradiction ($q^2 = 1$). To solve this problem, we will proceed like in heterotic string [44], where z and $\bar{z}$ are extended differently $z \rightarrow (z, \theta)$ and $\bar{z}$ remains unaffected. We associate to z and $\bar{z}$ two real generalized Grassmann variables θ_L and θ_R acting respectively from the right and left.
Consedering the generalized Grassmann variables θ_1 and θ_2 and their two derivatives ∂_1 and δ_1 and ∂_2 and δ_2 respectively; let:

$$\begin{aligned} \theta_1 &= \theta_L \\ \theta_2 &= \theta_R \\ \partial_1 &= \partial_L \\ \partial_2 &= \partial_R. \end{aligned} \tag{2.29}$$

From :

$$\theta_L\theta_R = q\theta_R\theta_L, \tag{2.30}$$

the consistency of the algebra leads to the following relations:

$$\begin{aligned} \partial_L\partial_R &= q\partial_R\partial_L \\ \partial_L\theta_R &= q^2\theta_R\partial_L \\ \partial_R\theta_L &= q\theta_L\partial_R. \end{aligned} \tag{2.31}$$

In general case and in presence of k variables θ_i (i runing from 1 to k), we have:

$$\theta_i\theta_j = q\theta_j\theta_i, \quad i < j. \tag{2.32}$$

From which we get:

$$\begin{aligned}\partial_i \partial_j &= q \partial_j \partial_i, \quad i < j \\ \partial_i \theta_j &= q^{-1} \theta_j \partial_i, \quad i < j \\ \partial_j \theta_i &= q \theta_i \partial_j, \quad i < j.\end{aligned} \tag{2.33}$$

Returning to our heterotic extension of the complex plane, we can define an automorphism of the algebra exchanging (z, θ_L) and $(\bar{z}, \theta_R)$. The algebra defined in relations (2.33) is neither stable under complex conjugation $(*)$, nor under the permutation σ of variables θ. However it is stable under the composition of both $(* \circ \sigma)$:

$$(AB)^{*\circ\sigma} = A^{*\circ\sigma} B^{*\circ\sigma}. \tag{2.34}$$

Under this automorphism, $(z, \theta_L, \partial_L, \delta_L)$ is mapped onto $(\bar{z}, \theta_R, \delta_R, \partial_R)$ and vice-versa. Remark that $(\partial \theta^a)^* = \theta^a \delta$, where ∂ acts from the right and δ from the left and where $\partial^* = \delta$ et $\theta^* = \theta$.
If we set:

$$\begin{aligned}D_L &= \delta_L + q^2 \theta_L^2 \partial_z \\ Q_L &= \partial_L + q \theta_L^2 \partial_z,\end{aligned} \tag{2.35}$$

where D_L and Q_L are respectively the covariant derivative and the fractional supersymmetry generator, associated to z and acting from the left. Using the equation (2.35), we prove that:

$$D_L^3 = Q_L^3 = -\partial_z. \tag{2.36}$$

Under the $(* \circ \sigma)$, conjugation, we obtain the covariant derivative and the fractional supersymmetry generator of the $\bar{z}$ modes

$$\begin{aligned}D_R &= \partial_R + q \theta_R^2 \partial_{\bar{z}} \\ Q_R &= \delta_R + q^2 \theta_R^2 \partial_{\bar{z}}.\end{aligned} \tag{2.37}$$

A direct calculation gives:

$$D_R^3 = Q_R^3 = -\partial_{\bar{z}}, \tag{2.38}$$

where D_R and Q_R act from the right. The two representations D_L and Q_L satisfy the commutation relation:

$$Q_L D_L = q^2 D_L Q_L. \tag{2.39}$$

Remark that the equations (2.36) and (2.38) are the same case as this discussed in one dimension. The relation of commutation between Q_R and D_R acting from right can be written in the form:

$$Q_R D_R = q D_R Q_R. \tag{2.40}$$

Introduce the fractional superfield,

$$\begin{aligned}\Phi(z,\theta_L,\bar{z},\theta_R) = X(z,\bar{z}) + q^2\theta_L\Psi_{20}(z,\bar{z}) + q^2\theta_L^2\Psi_{10}(z,\bar{z}) \\ +q^2\theta_R\Psi_{02}(z,\bar{z}) + \theta_L\theta_R\Psi_{22}(z,\bar{z}) + q^2\theta_L^2\theta_R\Psi_{12}(z,\bar{z}) + \\ q^2\theta_R^2\Psi_{01}(z,\bar{z}) + q^2\theta_L\theta_R^2\Psi_{21}(z,\bar{z}) + \theta_L^2\theta_R^2\Psi_{11}(z,\bar{z}).\end{aligned} \tag{2.41}$$

The components Ψ_{ab} with $X = \Psi_{00}$, are of grade $(a+b)$ and satisfy the commutation relations,

$$\begin{aligned}\theta_L\Psi_{ab} &= q^{a+b}\Psi_{ab}\theta_L \\ \theta_R\Psi_{ab} &= q^{a+b}\Psi_{ab}\theta_R.\end{aligned} \tag{2.42}$$

Let ϵ_L and ϵ_R, the parameters of the fractional supersymmetry transformations. Using the structure of the algebra for the left and right sectors ($Q_L D_L = q^2 D_L Q_L$ et $Q_R D_R = q D_R Q_R$), the fact that the covariant derivative has to commute with the fractional supersymmetry transformations and the $(* \circ \sigma)$ automorphism, we get the following commutation relations:

$$\begin{aligned}\epsilon_L\epsilon_R &= q\epsilon_R\epsilon_L \\ \epsilon_L\theta_R &= q\theta_R\epsilon_L \\ \epsilon_L\theta_L &= q^2\theta_L\epsilon_L \\ \epsilon_R\theta_L &= q^2\theta_L\epsilon_R \\ \epsilon_R\theta_R &= q\theta_R\epsilon_R.\end{aligned} \tag{2.43}$$

Then the fractional supersymmetry transformations of the field Φ are in the form:

$$\delta_\epsilon\Phi = \epsilon_L Q_L\Phi + \Phi Q_R\epsilon_R, \tag{2.44}$$

using this results we find:

$$\begin{aligned}\delta_\epsilon X &= q^2\epsilon_L\Psi_{20} + q\Psi_{02}\epsilon_R \\ \delta_\epsilon\Psi_{20} &= -q\epsilon_L\Psi_{10} + q^2\Psi_{22}\epsilon_R \\ \delta_\epsilon\Psi_{10} &= \epsilon_L\partial_z X + \Psi_{12}\epsilon_R \\ \delta_\epsilon\Psi_{02} &= q^2\epsilon_L\Psi_{22} - q^2\Psi_{01}\epsilon_R \\ \delta_\epsilon\Psi_{01} &= q^2\epsilon_L\Psi_{21} + \partial_{\bar{z}}X\epsilon_R \\ \delta_\epsilon\Psi_{22} &= -q\epsilon_L\Psi_{12} - \Psi_{21}\epsilon_R \\ \delta_\epsilon\Psi_{11} &= \epsilon_L\partial_z\Psi_{01} + q^2\partial_{\bar{z}}\Psi_{10}\epsilon_R \\ \delta_\epsilon\Psi_{12} &= \epsilon_L\partial_z\Psi_{02} - q\Psi_{11}\epsilon_R \\ \delta_\epsilon\Psi_{21} &= -q\epsilon_L\Psi_{11} + q\partial_{\bar{z}}\Psi_{20}\epsilon_R.\end{aligned} \tag{2.45}$$

With similar arguments as those used in $1D$, and with D_L (D_R) acting from the left (right), we build the $2D$ action S. This action is given by the formula [45]:

$$\begin{aligned} S = q \int dz d\bar{z} d\theta_R d\theta_L [D_L \Phi(z, \theta_L, \bar{z}, \theta_R) \Phi(z, \theta_L, \bar{z}, \theta_R) D_R] \\ = \int dz d\bar{z} [\partial_z X(z,\bar{z}) \partial_{\bar{z}} X(z,\bar{z}) - q \partial_z \Psi_{02}(z,\bar{z}) \Psi_{01}(z,\bar{z}) \\ + q^2 \partial_z \Psi_{01}(z,\bar{z}) \Psi_{02}(z,\bar{z}) + q \Psi_{20}(z,\bar{z}) \partial_{\bar{z}} \Psi_{10}(z,\bar{z}) \\ - q^2 \Psi_{10}(z,\bar{z}) \partial_{\bar{z}} \Psi_{20}(z,\bar{z}) - q \Psi_{11}(z,\bar{z}) \Psi_{22}(z,\bar{z}) \\ - q^2 \Psi_{22}(z,\bar{z}) \Psi_{11}(z,\bar{z}) + \Psi_{12}(z,\bar{z}) \Psi_{21}(z,\bar{z}) \\ + \Psi_{21}(z,\bar{z}) \Psi_{12}(z,\bar{z})]. \end{aligned} \tag{2.46}$$

Note that This action S is a grade null number. If we choose $\Psi^*_{ab} = \Psi_{ba}$ and also with the appropriate choice of the power of q, in the definition of Φ, we ensure that the lagrangian is real. Solving the equations of motion, we see that:

- X has a holomorphic and an antiholomorphic part.
- Ψ_{10} and Ψ_{20} are holomorphic.
- Ψ_{01} and Ψ_{02} are antiholomorphic.
- Ψ_{12},Ψ_{21}, Ψ_{11} and Ψ_{22} are auxilliary fields that vanish on-shell.

2.3 RdTS Fractional supersymmetry

A non trivial generalisation of the (1+2) dimensional Poincaré algebra going beyond the standard supersymmetric extension has been obtained in [46]. In addition to the usual Poincaré generators, this extension refered herebelow to as the Rauch de Traubenberg-Slupinski algebra (RdTS algebra for short), involves two kinds of conserved charges $Q^{\pm}_s$ transforming as $so(1,2)$ Verma modules of spin $s = \pm\frac{1}{k}$, $k \geq 2$. This consruction is interesting first because it goes beyond standard 2D-fractional supersymmetry based on considering k-th roots of the $so(2)$ vector and second because it gives a new algebraic structure which a priori is valid for higher rank Lie algebra g where $so(2)$ and $so(1,2)$ appear just as two special examples. In one-dimensional spaces, where no rotation is available, the fractional supersymmetry is generated by one generator which can be seen as the k^{th} root of the time translation $(Q_t)^k = \partial_t$. The case $k = 2$ corresponds to the usual supersymmetry. In two-dimensional spaces, the fractional supersymmetric algebra was extended by two generators satisfying $(Q_z)^k = \partial_z$ and $(Q_{\bar{z}})^k = \partial_{\bar{z}}$. In this section, we will study the case of $(1+2)$ dimensions which have also arbitrary spin and statistics where its particles are called anyons. In fact, studying the representations of the (1+2)D- Poincaré algebra $P_{1,2}$ the unitary irreducible representations divide into two classes: massive and massless.

In this section we will build the non-trivial extensions of the Poincaré algebra $P_{1,2}$ which contains the fractional supersymmetry then we will give the representations of the RdTS fractional supersymmetry.

2.3.1 RdTS Fractional supersymmetry

To start consider the Poincaré symmetry in (1+2) dimensions generated by the space time translations P_μ and the Lorentz rotations J_α satisfying altogether the following closed commutations:

$$\begin{aligned}
[P^\alpha, P^\beta] &= 0 \\
[J^\alpha, P^\beta] &= i\eta^{\alpha\gamma}\eta^{\beta\delta}\epsilon_{\gamma\delta\eta}P^\eta \\
[J^\alpha, J^\beta] &= \eta^{\alpha\gamma}\eta^{\beta\delta}\epsilon_{\gamma\delta\eta}J^\eta .
\end{aligned} \tag{2.47}$$

In this equations, $\eta^{\alpha\beta} = diag(1,-1,-1)$ is the (1+2) Minkowski metric and $\epsilon_{\alpha\beta\gamma}$ is the completly antisymmetric Levi-Civita tensor such that $\epsilon_{012} = 1$.
Particles are then classified according to the values of the Casimir operators of the Poincaré algebra. This means for a mass m particle of positive or negative energy, the unitary irreducible representations are obtained by studying the little group leaving the momentum $P^\alpha = (m, 0, 0)$ invariant. The stability group in $\overline{SO(1,2)}$ is the abelian sub-group $SO(2)$ generated by J^0 ($J_\pm = 0$). One of the remarkable property is that the transformation $J^0 \to J^0 + s$ leaves the $\overline{SO(2)}$ part invariant.This property of $\overline{SO(1,2)}$ shows that the translation of J^i which is given by $J^i \to J^i + \frac{sP^i}{P^0+m}$, leaves the algebraic structure eq(2.47) unchanged.

$$\begin{aligned}
J_s^0 &= J^0 + s \\
J_s^1 &= J^1 + \frac{sp^1}{p^0+m} \\
J_s^2 &= J^2 + \frac{sp^2}{p^0+m},
\end{aligned} \tag{2.48}$$

with

$$\begin{aligned}
J^0 &= i(p^1\frac{\partial}{\partial p^2} - p^2\frac{\partial}{\partial p^1}) \\
J^1 &= -i(p^2\frac{\partial}{\partial p^0} - p^0\frac{\partial}{\partial p^2}) \\
J^2 &= -i(p^0\frac{\partial}{\partial p^1} - p^1\frac{\partial}{\partial p^0}),
\end{aligned}$$

where p^α, $\alpha = 0, 1, 2$, are the eigenvalues of the operators P^α.
The two Casimir operators of $P_{1,2}$ are the two scalars $P.P$ and $P.J$. When acting

on highest weight states of mass m and spin s, The eigenvalues of these operators are respectively m^2 and ms. The equations of motion are then:

$$\begin{aligned}(P^2 - m^2)\psi &= 0\\ (P.J - sm)\psi &= 0.\end{aligned} \tag{2.49}$$

A convinient way to handle eqs (2.47) is to work with an equivalent formulation using the following Cartan basis of generators $P_\mp = P_1 \pm iP_2$ and $J_\mp = J_1 \pm iJ_2$. In this basis eqs (2.47) read as:

$$\begin{aligned}[J_+, J_-] &= -2J_0\\ [J_0, J_\pm] &= \pm J_\pm\\ [J_\pm, P_\mp] &= \pm P_0\\ [J_+, P_+] &= [J_-, P_-] = 0\\ [J_0, P_0] &= [P_\pm, P_\mp] = 0,\end{aligned} \tag{2.50}$$

where the Casimir operators P^2 and $P.J$ are given by:

$$P^2 = P_0^2 - \frac{1}{2}(P_+P_- + P_-P_+)$$

$$P.J = P_0J_0 - \frac{1}{2}(P_+J_- + P_-J_+).$$

For a given s, one distinguishes two classes of irreducible representations: massive and massless representations. To build the $so(1,2)$ massive representations, it is convinient to go to the first frame where the momentum vector P_μ is $(m, 0, 0)$ and the $SO(1,2)$ group reduces to its abelian $SO(2)$ little subgroup generated by J_0. In this case, massive irreducible representations are one dimensional and are parametrized by a real parameter. For the full $SO(1,2)$ group however, the representations are either finite dimensional for $|s| \in \mathbf{Z}^+/2$ or infinite dimensional for the remaining values of s.

Given a primary state $|s>$ of spin s, and using the abovementioned $SO(1,2)$ group theoretical properties, one may construct in general two representations HWR(I) and HWR(II) out of this state $|s>$.

The first representation HWR(I) is a highest weight representation given by:

$$\begin{aligned}J^0|s> &= s|s>\\ J_-|s> &= 0\\ |s,n> &= \sqrt{\frac{\Gamma(2s)}{\Gamma(2s+n)\Gamma(n+1)}}(J_+)^n|s>, \quad n \geq 1\\ J_0|s,n> &= (s+n)|s,n>\\ J_+|s,n> &= \sqrt{(2s+n)(n+1)}|s,n+1>\\ J_-|s,n> &= \sqrt{(2s+n-1)n}|s,n-1>.\end{aligned} \tag{2.51}$$

The second representation is a lowest weight representation which we refer to denote as HWR(II) is defined as:

$$\begin{aligned}
\bar{J}^0|\bar{s}> &= -s|\bar{s}> \\
\bar{J}_-|\bar{s}> &= 0 \\
|\bar{s},n> &= (-1)^n\sqrt{\frac{\Gamma(2s)}{\Gamma(2s+n)\Gamma(n+1)}}(\bar{J}_+)^n|\bar{s}>, \quad n\geq 1 \\
\bar{J}_0|\bar{s},n> &= -(s+n)|\bar{s},n> \\
\bar{J}_+|\bar{s},n> &= -\sqrt{(2s+n-1)n}|\bar{s},n-1> \\
\bar{J}_-|\bar{s},n> &= -\sqrt{(2s+n)(n+1)}|\bar{s},n+1> .
\end{aligned} \tag{2.52}$$

Note in passing that the second module we have supplemented the generators and the representations states with a bar index. This convention of notation will be justified later on. Note moreover that both HWR(I) and HWR(II) representations have the same $so(1,2)$ Casimir $C_s = s(s-1), \quad s<0$. For $s \in \mathbf{Z}^-/\mathbf{2}$, these representations are finite dimensional and their dimension is $(2|s|+1)$. For generic real values of s, the dimension of the representations is however infinite. If one chooses a fractional value of s say $s=-\frac{1}{k}$; each of the two representations (2.51-52) splits a priori into two isomorphic representations respectively denoted as $D^+_{\pm\frac{1}{k}}$ and $D^-_{\pm\frac{1}{k}}$. This degeneracy is due to the redundancy in choosing the spin structure of $\sqrt{-2/k}$ which can be taken either as $+i\sqrt{-2/k}$ or $-i\sqrt{-2/k}$. These representation are not independent since they are related by conjugations, this why we shall use hereafter the choice of [46] by considering only $D^+_{-\frac{1}{k}}$ and $D^-_{-\frac{1}{k}}$. In this case the two representations generators $J_{0,\pm}$ and $\bar{J}_{0,\pm}$ are related as:

$$\bar{J}_{0,\mp} = (J_{0,\pm})^*. \tag{2.53}$$

Furthermore taking the tensor product of the primary states $|s>$ and $|\bar{s}>$ of the two $so(1,2)$ modules HWR(I) and HWR(II) and using eqs(2.51-52), it is straightforward to check that it behaves like a scalar under the full charge operator $J_0 \times 1_d + 1_d \times \bar{J}_0$ which we denote simply as $J_0 + \bar{J}_0$ [30]:

$$(J_0+\bar{J}_0)|s>\otimes|\bar{s}>=0. \tag{2.54}$$

The equation (2.54) is familiar relation in the study of primary states of Virasoro algebra. This equation together with the mode operators J^n_- and $\bar{J}^m_+$ which act on $|s>\otimes|\bar{s}>$ as:

$$\begin{aligned}
J^n_-|s>\otimes|\bar{s}> &= 0, \quad n\geq 1 \\
(\bar{J}_+)^m|s>\otimes|\bar{s}> &= 0, \quad m\geq 1
\end{aligned} \tag{2.55}$$

define a highest weight state which looks like a Virasoro primary state of spin $2s$ and scale dimension $\Delta = 0$.
If we respectively associate to HWR(I) and HWR(II) the mode operators $Q^{+}_{s+n} = Q_{s+n}$ and $Q^{-}_{-s-n} = \bar{Q}_{s+n}$ and using $SO(1,2)$ tensor product properties one may build under some assumptions, an extension $\mathbf{S}$ of the $so(1,2)$ algebra going beyond the standard supersymmetric one. To do so, note first that the system J_0, J_+, J_- and Q_{s+n} obey the following commutation relations for $s = -1/k$.

$$\begin{aligned} [J_0, Q_{s+n}] &= (s+n)Q_{s+n} \\ [J_+, Q_{s+n}] &= \sqrt{(2s+n)(n+1)}Q_{s+n+1} \\ [J_-, Q_{s+n}] &= \sqrt{(2s+n-1)n}Q_{s+n-1}. \end{aligned} \tag{2.56}$$

Similarly we have for the antiholomorphic sector:

$$\begin{aligned} [\bar{J}_0, \bar{Q}_{s+n}] &= -(s+n)\bar{Q}_{s+n} \\ [\bar{J}_+, \bar{Q}_{s+n}] &= -\sqrt{(2s+n-1)n}\bar{Q}_{s+n-1} \\ [\bar{J}_-, \bar{Q}_{s+n}] &= -\sqrt{(2s+n)(n+1)}\bar{Q}_{s+n+1}. \end{aligned} \tag{2.57}$$

To close these commutations relations with the Q_s's through a k-th order product one should fulfill some constraints:

1. The generalised algebra $\mathbf{S}$, we are looking for should be a generalisation of what is known in two dimensions which means a generalisation of fractional supersymmetry.

2. When the charge operator Q_{s+n} goes arround an other, say Q_{s+m}, it picks a phase $\Phi = 2i\pi/k$, i.e:

$$Q_{s+n}Q_{s+m} = e^{\pm 2i\pi s}Q_{s+m}Q_{s+n} + \ldots; \quad s = -\frac{1}{k}, \tag{2.58}$$

where the dots refer for possible extra charge operators of total J_0 eigenvalue $(2s+n+m)$.
The equation (2.58) shows also that the algebra we are looking for has a $\mathbf{Z_k}$ graduation. Under this discret symmetry, Q_{s+n} carries $a+1(modk)$ charge while the $P_{0,\pm}$ energy momentum components have a zero charge mod k.

3. The generalised algebra $\mathbf{S}$ should split into a bosonic B part and an anyonic A and may be written as:

$$\mathbf{S} = \bigoplus_{r=0}^{k-1} A_r = B \bigoplus_{r=1}^{k-1} A_r.$$

Since $A_n A_m \subset A_{(n+m)(mod k)}$, one has:

$$\begin{aligned} \{A_r, \ldots, A_r\} &\subset B \\ [B, A] &\subset A \\ [B, B] &\subset B. \end{aligned} \tag{2.59}$$

In this equations, $\{A_r, \ldots, A_r\}$ means the complete symmetrisation of the k anyonic operators A_r and is defined as:

$$\{A_{s_r}, \ldots, A_{s_r}\} = \frac{1}{k!} \sum_{\sigma \in \Sigma} (A_{s_{\sigma(1)}} \ldots A_{s_{\sigma(k)}}), \tag{2.60}$$

where the sum is carried over the k elements of the permutation group $\{1, \ldots, k\}$.

4. The algebra **S** should obey generalised Jacobi identities. In particular we should have:

$$adB\{A_{s_1} \ldots A_{s_k}\} = 0, \tag{2.61}$$

where B stands for the bosonic generators $J_{0,\pm}$ or $P_{0,\pm}$ of the Poincaré algebra. Using eq(2.59) to write $\{A_r, \ldots A_r\}_k$ as $\alpha_\mu P^\mu + \beta_\mu J^\mu$ where α and β are real constants; then putting back into the above relation we find that $\{A_r, \ldots A_r\}_k$ is proportional to P_μ only. In other words, β_μ should be equal to zero; a property which is easily seen by taking $B = P_\mu$ in eq(2.61). Put differently the symmetric product of the $D_s^\pm$, denoted hereafter as $S^k[D_s^\pm]$ contains the space time vector representation D_1 of $so(1,2)$ and so the primitive charge operators $Q_{-1/k}$ and $\bar{Q}_{1/k}$ obey:

$$\begin{aligned} [J_0, (Q_{-1/k})^k] &= -(Q_{-1/k})^k \sim P_- \\ [J_-, (Q_{-1/k})^k] &= 0. \end{aligned} \tag{2.62}$$

Similarly we have:

$$\begin{aligned} [\bar{J}_0, (\bar{Q}_{1/k})^k] &= (\bar{Q}_{1/k})^k \sim P_+ \\ [\bar{J}_+, (\bar{Q}_{1/k})^k] &= 0. \end{aligned} \tag{2.63}$$

Moreover acting on $(Q_{-1/k})^k$ by adJ_+^n and on $(\bar{Q}_{1/k})^k$ by $ad\bar{J}_-^n$, we obtain:

$$\begin{aligned} adJ_+(Q_{-1/k})^k &\sim P_0 \\ ad\bar{J}_-(\bar{Q}_{1/k})^k &\sim P_0 \\ ad^2 J_+(Q_{-1/k})^k &\sim P_- \\ ad^2 \bar{J}_-(\bar{Q}_{1/k})^k &\sim P_+. \end{aligned} \tag{2.64}$$

In summary, starting from $so(1,2)$ Lorentz algebra (2.47-48) and the two Verma modules HWR(I) and HWR(II) (2.51-52), one may build the following new extended symmetry:

$$\begin{aligned}
\{Q^{\pm}_{-\frac{1}{k}},\ldots,Q^{\pm}_{-\frac{1}{k}}\}_k &= P_{\mp} = P_1 \pm iP_2 \\
\{Q^{\pm}_{-\frac{1}{k}},\ldots,Q^{\pm}_{-\frac{1}{k}},Q^{\pm}_{1-\frac{1}{k}}\}_k &= \pm i\sqrt{\frac{2}{k}}P_0 - (k-1)\{Q^{\pm}_{-\frac{1}{k}},\ldots,Q^{\pm}_{-\frac{1}{k}},Q^{\pm}_{1-\frac{1}{k}},Q^{\pm}_{1-\frac{1}{k}}\}_k \\
&\pm i\sqrt{k-2}\{Q^{\pm}_{-\frac{1}{k}},\ldots,Q^{\pm}_{-\frac{1}{k}},Q^{\pm}_{1-\frac{1}{k}},Q^{\pm}_{2-\frac{1}{k}}\}_k \qquad (2.65) \\
[J^{\pm},[J^{\pm},[J^{\pm},(Q^{\pm}_{-\frac{1}{k}})^k]]] &= 0.
\end{aligned}$$

with

$$k\{Q^{\pm}_{-\frac{1}{k}},\ldots,Q^{\pm}_{-\frac{1}{k}},Q^{\pm}_{-\frac{1}{k}}\} = (Q^{\pm}_{-\frac{1}{k}})^{k-1}Q^{\pm}_{1-\frac{1}{k}} + \ldots + Q^{\pm}_{1-\frac{1}{k}}(Q^{\pm}_{-\frac{1}{k}})^{k-1}.$$

The equation (2.65) defines what we have been refering to as RdTS algebra. Then RdTS fractional supersymmetry is a special generalisation of fractional supersymmetry living in two dimensions and considered in many occasions in the past in connection with integrable deformation of conformal invariance and representations of the universal enveloping $U_q(sl(2))$ quantum ordinary and affine symmetries [29], [47]. Like for fractional supersymmetry, highest weight representations of RdTS algebra carry fractional values of the spin and obey more a less quite similar fractional supersymmetry equations.

2.3.2 Representations of the RdTS algebra

In this family of RdTS algebra, if we take $k=2$ we are in an exceptional situation because instead of having an infinite number of charges we have only two. In this case and with one series of supercharges Q, we obtain the well known supersymmetric extension of the Poincaré algebra. Then the equations (2.56-57) and (2.65) can be easily rewritten with the Pauli matrices. For more details, one can see the book of Wess and Bagger [48]. The RdTS algebra obtained is then a direct generalisation of the super-Poincaré one. The supersymmetric algebra which is constructed for one and two dimensional spaces, cann also be considered in (1+2) dimensions which allows to define states with fractional statistics or anyons. Before starting to study the representations of the RdTS algebra, let first address some general properties.

1. P^2 commutes with all the generators because it is a Casimir operator. So all states in an irreducible representation have the same mass.

2. If we define an anyonic number operator by $exp(2i\pi N_A)$ which gives the phase $exp(2i\pi s)$, we have then $trexp(2i\pi N_A) = 0$. This formula shows that in each irreducible representation there are k possible statistics namely $(s, s-\frac{1}{k}, \ldots, s-\frac{k-1}{k})$, where s is an anyonic spin. Then, we have the following identities:

$$
\begin{aligned}
Tr(exp(2i\pi N_A)\{Q^+_{-\frac{1}{k}}, \ldots, Q^+_{-\frac{1}{k}}, Q^+_{1-\frac{1}{k}}\}_k) &= \frac{1}{k}Tr(\sum_{a=0}^{k-1} exp(2i\pi N_A)(Q^+_{-\frac{1}{k}})^a \\
&\quad Q^+_{1-\frac{1}{k}} \ (Q^+_{-\frac{1}{k}})^{k-a-1}) = 0 \qquad (2.66)\\
exp(2i\pi N_A)Q_s &= exp(2i\pi s)Q_s exp(2i\pi N_A).
\end{aligned}
$$

The unitarity of the representation force us to consider both supercharges Q^+ and Q^-. For the Poincaré and its supersymmetric extension, the massive representations $P^\alpha P_\alpha = m^2$ are builded by studying the sub-algebra leaving the momentum $P^\alpha = (m, 0, 0)$ invariant. Using the framework of the fractional supersymmetric algebra, all the representations are obtained by studying the sub-algebra when $P_\pm$ and $J_\pm$ are suposed to be zero. Looking the equations (2.65), with $P_\pm = J_\pm = 0$, we find that only one fundamental bracket does not vanish, the one involving $(k-1)$ times the charge $Q_{-\frac{1}{k}}$(see the second equality of (2.65)). So all brackets involving $Q_{\frac{n-1}{k}}$, with $n > 0$ are represented by zero. An appropriate normalization of the RdTS algebra eq(2.65) give:

$$
\begin{aligned}
\{Q^+_{1-\frac{1}{k}}, \ldots, Q^+_{1-\frac{1}{k}}, Q^+_{1-\frac{1}{k}}\} &= \frac{1}{k} \\
\{Q^+_{s_{i_1}}, \ldots, Q^+_{s_{i_k}}\} &= 0 \qquad (2.67)
\end{aligned}
$$

where

$$i_1, \ldots, i_k = -\frac{1}{k}, 1-\frac{1}{k}$$

and

$$i_1 + \ldots + i_k \neq 0.$$

To obtain the irreducible representations for an arbitrary k, we first observe that the k^{th} power of $Q_{-\frac{1}{k}}$ vanish; $(Q_{-\frac{1}{k}})^k = 0$. In other words the rank of $Q_{-\frac{1}{k}}$ is $k-1$. Indeed, if we suppose that $Q^{k-b}_{-\frac{1}{k}} = 0; \quad b > 1$, and multiplying the eqs(2.67) by $Q_{-\frac{1}{k}}$, on the left and by $(Q_{-\frac{1}{k}})^{k-b-2} = 0$ on the right, we get a contradiction. Using the Jordan decomposition and the fact that all the eigenvalues of $Q_{-\frac{1}{k}}$ are zero, we write $Q^+_{-\frac{1}{k}}$ and $Q^+_{1-\frac{1}{k}}$ as:

$$Q^+_{-\frac{1}{k}} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \sqrt{1} & \cdots & \cdots & \cdots \\ \vdots & \sqrt{2} & \vdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \sqrt{k-1} & 0 \end{pmatrix}_{k\times k}$$

$$Q^+_{1-\frac{1}{k}} = \begin{pmatrix} 0 & 0 & \cdots & \sqrt{[k-1]!} \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}_{k\times k}. \quad (2.68)$$

An other representation is valable and is given by:

$$Q^+_{-\frac{1}{k}} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \sqrt{1(k-1)} & \cdots & \cdots & \cdots \\ \vdots & \sqrt{2(k-2)} & \vdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \sqrt{(k-1)1} & 0 \end{pmatrix}_{k\times k}$$

$$Q^+_{1-\frac{1}{k}} = \begin{pmatrix} 0 & 0 & \cdots & \frac{1}{(k-1)!} \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}_{k\times k}, \quad (2.69)$$

with $[a] = \frac{q^{-\frac{a}{2}}-q^{\frac{a}{2}}}{q^{-\frac{1}{2}}-q^{\frac{1}{2}}}$; $[k-1]! = [k-1][k-2]\ldots[2][1]$ and $q = exp(2i\pi/k)$.
The two representations Q^+ and Q^- are related by the conjugation:

$$\begin{aligned} Q^-_{-\frac{1}{k}} &= (Q^+_{-\frac{1}{k}})^+ \\ Q^-_{1-\frac{1}{k}} &= (Q^+_{1-\frac{1}{k}})^+. \end{aligned} \quad (2.70)$$

This two representations $Q^+_{-\frac{1}{k}}$ and $Q^-_{-\frac{1}{k}}$ satisfy the quadratic relations [30], [31]. In the case of the first serie representations we obtain:

$$\begin{aligned} Q^-_{-\frac{1}{k}}Q^+_{-\frac{1}{k}} - q^{\pm\frac{1}{2}}Q^+_{-\frac{1}{k}}Q^-_{-\frac{1}{k}} &= q^{\mp\frac{N}{2}} \\ [N, Q^+_{-\frac{1}{k}}] &= Q^+_{-\frac{1}{k}} \\ [N, Q^-_{-\frac{1}{k}}] &= Q^-_{-\frac{1}{k}}, \end{aligned} \quad (2.71)$$

with $N = diag(0,1,\ldots,k-1)$ is the scalar operator.
For the second choice we have:

$$\begin{aligned} [Q^-_{-\frac{1}{k}}, Q^+_{-\frac{1}{k}}] &= N = diag(k-1, k-2, \ldots, 1-k) \\ [N, Q^\pm_{-\frac{1}{k}}] &= \mp 2 Q^\pm_{-\frac{1}{k}}, \end{aligned} \tag{2.72}$$

which show that the Q generate the k-dimensional representation of $sl(2,R)$. The representation built with the Q's is unitary and indeed the quadratic relations (2.71) and (2.72)prove that $(Q^+_{-\frac{1}{k}})^n$, with $n = 0, \ldots, k-1$, constructed from the $Q^+_{-\frac{1}{k}}$ representation is positive.

Chapter 3

Fractional supersymmetry as a matrix model

In the few last years, there has been attempts to develop a superspace formulation of 2D quantum field theory that is invariant under fractional supersymmetry(FSS) defined by: $Q^k = P$, $\bar{Q}^k = \bar{P}$, $k > 2$, together with relations involving both Q and $\bar{Q}$. In these equations $P = P_{-1}$ and $\bar{P} = P_{+1}$ are the two heterotic components of 2D energy momentum vector operator P_m; $Q \equiv Q_{-1/k}$ and $\bar{Q} \equiv Q_{1/k}$ are the basic generators of FSS which carry a 2D spin $s = 1/k$. Most of the studies of this special class of 2D massive quantum field theory(QFT) are essentially based on methods of $U_q(sl(2))$ quantum group representations and noncommutative geometry [49], [50]. This approach however has met many difficulties and has lead to partial results only. Among these difficulties we quote the two following ones associated with the heterotic $K = 3$ case ($Q^3 = P$). The first one deals with the computation of the two points correlation function $< \Psi_{-1/3}(z_1), \Psi_{-2/3}(z_2) >$ of the two partners $\Psi_{-1/3}(z_1)$ and $\Psi_{-2/3}(z_2)$ of the bosonic field $\varphi(z)$ of the scalar representation $(\varphi, \Psi_{-1/3}, \Psi_{-2/3})$ of the $Q^3 = P$ algebra. From the view point of the 2D conformal field theory, one expects from the braiding feature $z_{12}\Psi_{-1/3}(z_1)\Psi_{-2/3}(z_2) = z_{21}\Psi_{-2/3}(z_2)\Psi_{-1/3}(z_1)$, that the field operators $\Psi_{-1/3}$ and $\Psi_{-2/3}$ should anticommute. This result however is not fulfilled in the approach based on the $U_q(sl(2))$ quantum groups and non commutative geometry where the $\Psi_{-1/3}$ and $\Psi_{-2/3}$ fields obey a generalised commutation rule [51] leading to a two point correlation function $< \Psi_{-1/3}(z_1), \Psi_{-2/3}(z_2) >_{U_q(sl(2))}$ which violate locality.

The second thing we want to quote concerns the construction of the generalised superspace and superfields. In this context, one uses generalised Grassmann variables $\theta_1 = \theta_{1/3}$ and $\theta_2 = \theta_{-1/3}$ satisfying a third oder nilpotency condition $\theta^3_{\pm 1/3} = 0$, together with $\theta^2_{\pm 1/3} \neq 0$ and $\theta_1\theta_2 = \omega\theta_2\theta_1$, where ω is a C-number such that $\omega^3 = 1$. The condition $\theta^3_{\pm 1/3} = 0$, which generalizes the usual condition

of Grassmann variables of spins 1/2, is necessary in order to describe off shell representations of the fractional supersymmetric algebra in terms of superfields $\Phi(z,\theta,\bar{z},\bar{\theta})$. In this language, FSS is generated by translations along the θ and $\bar{\theta}$ directions, i.e. $\theta \to \theta+\epsilon$ and $\bar{\theta} \to \bar{\theta}+\bar{\epsilon}$, where ϵ and $\bar{\epsilon}$ are generalized Grassmann variables of same nature as θ and $\bar{\theta}$. Until now things are quite similar to the superspace formulation of ordinary supersymmetry associated with $k=2$. However there are weak points in the construction of the generalised superspace for $k>2$. One of these weakness deals with the treatement of θ^2, $\bar{\theta}^2$ and the ϵ and $\bar{\epsilon}$ parameters. In trying to establish a given relation by following two equivalent paths, we get different results. On one hand ,$\theta(\bar{\theta})$ should commute with itself as required by the identity $(1-x)\theta^2=0$, which implies that either $\theta^2=0$ and $x\neq 0$ or $x=1$ and $\theta^2\neq 0$. On the other hand $\theta(\bar{\theta})$ and $\epsilon(\bar{\epsilon})$ are required to satisfy generalized commutation rules type $\theta\epsilon=\omega\epsilon\theta$ $(\bar{\theta}\bar{\epsilon}=\omega\bar{\epsilon}\bar{\theta})$; $\omega^3=1$.
This is a contradiction since $\theta(\theta+\epsilon)\neq\omega(\theta+\epsilon)\theta$. This difficulty may be also viewed from the generalised commutation rule $\theta\eta=\omega\eta\theta$. Taking the limit $\eta=\theta$, one obtains $\theta^2=\omega\theta^2$ which implies that θ^2 should be zero.
The aim of this chapter is to set up the basis of 2D fractional supersymmetric QFT by using parafermionic field theoretical methods. These methods , which were considered recently in [52], seem to be the right language to develop a local fractional supersymetric QFT. This believe is also supported by the fact that after all fractional supersymmetry is nothing but a residual symmetry of massive perturbations of parafermionic critical models. From this point of view, the generators of fractional supersymmetry are just remanant constants of motion surviving after deformations of parafermionic conformal invariance. Thinking of 2D fractional supersymmetry as a finite dimensional subsymmetry of the Z_k conformal invariance [53] for instance, one discovers that all the known difficulties of the abovementioned approach disappear. Moreover we find that fractional supersymmetric algebras are generated by more than one charge operator and can be described in a natural way as a matrix model. The charge operators Q_{-x}, $x=0,1/k,...$, form altogether a $k\times k$ matrix operator allowing to define the fractional supersymmetric algebra as: $tr\mathbf{Q}^k=P_{-1}$.
This chapter is organised as follows. In the first section, we study the matrix realisations of 2D supersymmetry. In the second section, we work out the links between fractional supersymmetry and parafermions. Matrix realisations of fractional supersymmetry are analysed in sections 3 and 4.

3.1 2D Supersymmetry as a matrix model

In this paragraph we consider the 2D heterotic supersymmetric algebras respectively defined as:

$$
\begin{aligned}
\{Q_{-1/2}, Q_{-1/2}\} &= P_{-1} \\
[P_{-1}, Q_{-1/2}] &= 0
\end{aligned}
\tag{3.1}
$$

and

$$
\begin{aligned}
\{Q^{+}_{-1/2}, Q^{-}_{-1/2}\} &= P_{-1} \\
\{Q^{\pm}_{-1/2}, Q^{\pm}_{-1/2}\} &= 0 \\
[P_{-1}, Q^{\pm}_{-1/2}] &= 0,
\end{aligned}
\tag{3.2}
$$

and show that they can be represented as 2×2 matrix models. A generalisation to higher dimensional spaces of these representations turns out to offer a natural framework to study fractional supersymmetry. It shows, moreover, that methods of standard supersymmetry can be also used to deal with exotic supersymmetries. To start consider the following off-diagonal symmetric 2×2 matrix $\mathbf{Q}$ whose entries are given by the 2D$(\frac{1}{2}, 0)$ supersymmetric generator $Q_{-1/2}$ [29]:

$$
\mathbf{Q} = \begin{pmatrix} 0 & Q_{-1/2} \\ Q_{-1/2} & 0 \end{pmatrix}, \tag{3.3}
$$

This matrix operator acts on the 2D space of quantum states $|B>= B(0)|0>$ and $|F>= F(0)|0>$, where the letters B and F stand for Bose and Fermi fields. Taking the square product of eq(3.3), it is not difficult to see that the $(\frac{1}{2}, 0)$ supersymmetric algebra (3.1) may be defined as:

$$
\begin{aligned}
tr[\mathbf{Q}^2] &= P_{-1} \\
tr[\mathbf{Q}] &= 0
\end{aligned}
\tag{3.4}
$$

where the suffix tr means the usual matrix trace operation. There are several remarkable features of the matrix definition Eqs(3.3-4) of the 2D$(\frac{1}{2}, 0)$ supersymmetric algebra; some of them are manifest at the level of the $k \times k = 2 \times 2$ matrix model, others are hidden and emerge for $k \geq 3$ representations.
Moreover, combining the usual constraints of 2D supersymmetry, in particular Hermiticity of the energy-momentum vector P_{-1}, $P^{+}_{-1} = P_{-1}$, with the features of the matrix model (3.4), one sees that the realization of $\mathbf{Q}$, eq (3.3), is not the unique one. Indeed, decomposing $\mathbf{Q}$ als $\mathbf{Q} = \mathbf{Q}^{+} + \mathbf{Q}^{-}$, such that:

$$
\mathbf{Q}^{+} = \begin{pmatrix} 0 & Q^{+}_{-x} \\ 0 & 0 \end{pmatrix},
$$

$$
\mathbf{Q}^{-} = \begin{pmatrix} 0 & 0 \\ Q^{-}_{-y} & 0 \end{pmatrix} \tag{3.5}
$$

one sees that the spins x and y carried by the $Q^{\pm}$ charge operators are related as:

$$x + y = 1 \tag{3.6}$$

as required by the first relation of Eq(3.4). Abstraction done from equations (3.1) and (3.2), there are infinitely many solutions of equations (3.5) and (3.6); we shall first consider the two solutions $x = y = \frac{1}{2}$ and $x = 0$ and $y = +1$ related with the leading zero-mode operators of the NS and Ramond sectors of 2D superconformal invariance. Later on we shall explore the interesting cases $x = \frac{1}{k}$ and $y = \frac{(k-1)}{k}$, $k = 3, 4, ...$, and their link with Z_k parafermionic invariance. Now using equations (3.5) instead of Eqs(3.3) one finds that the algebra Eqs(3.4) reads in general as:

$$\begin{aligned} tr[\mathbf{Q}^+\mathbf{Q}^-] &= P_{-1} \\ tr[\mathbf{Q}^+\mathbf{Q}^+] &= tr[\mathbf{Q}^-\mathbf{Q}^-] = 0 \\ tr[\mathbf{Q}^+] &= tr[\mathbf{Q}^-] = 0 \end{aligned} \tag{3.7}$$

The equation (3.7) has a $U(1)$ automorphism symmetry which breaks down to Z_2 in the case of Eqs(3.4), this is why we shall refer to Eqs(3.4) and (3.7) as 2D$(\frac{1}{2}, 0)$ and $((\frac{1}{2})^2, 0)$ supersymmetric algebras respectively. Moreover introducing the $E_{ij} = |i> \otimes <j|$, $i, j = B, F$ generators of the $gl(2)$ Lie algebra, we have the following 2×2 matrix representations:

(**i**) The NS-like superalgebra.

In this case the supersymmetric 2×2 matrix generators $Q^{\pm}$ read as:

$$\begin{aligned} \mathbf{Q}^+ &= Q_{-1/2}E_{12} \\ \mathbf{Q}^- &= Q_{-1/2}E_{21} \end{aligned} \tag{3.8}$$

for $(\frac{1}{2}, 0)$ supersymmetry and

$$\begin{aligned} \mathbf{Q}^+ &= Q^+_{-1/2}E_{12} \\ \mathbf{Q}^- &= Q^-_{-1/2}E_{21} \end{aligned} \tag{3.9}$$

for $((\frac{1}{2})^2, 0)$ supersymmetry.

(**ii**) The Ramond like algebra.

Here the matrix generators are realized as:

$$\begin{aligned} \mathbf{Q}^+ &= Q_0E_{12} \\ \mathbf{Q}^- &= Q_{-1}E_{21} \end{aligned} \tag{3.10}$$

for the Hermitian case.
and

$$\begin{aligned}\mathbf{Q}^+ &= Q_0^+ E_{12} \\ \mathbf{Q}^- &= Q_{-1}^- E_{21},\end{aligned} \tag{3.11}$$

For the complex one.
Note that compatibility between the usual hermiticity (+) of the energy momentum vector P_{-1} and the adjoint conjugation (+) of the $Q^\pm$ matrix generators requires the following identifications:

$$\begin{aligned}[Q_{-x}^\pm]^+ &= Q_{-1+x}^\mp \\ [Q_{-x}]^+ &= Q_{-1+x}\end{aligned} \tag{3.12}$$

equations (3.12) show that $Q_{-1/2}$ is self-adjoint whereas Q_0 and Q_{-1} are interchanged under the (+) conjugation. However knowing that Q_0 should satisfy a 2D Clifford algebra $\{Q_0, Q_0\} = 1$, and using the current modes of the Ramond superconformal algebra [54], one may set

$$2Q_{-1} = \{Q_0, P_{-1}\} = 2Q_0 P_{-1}.$$

This identification reduces the number of generators of the algebra Eq(3.10). Likewise, one may also set $Q_{-1}^- = Q_0^- P_{-1}$ where Q_0^- together with Q_0^+ satisfy:

$$\begin{aligned}\{Q_0^+, Q_0^-\} &= 1 \\ \{Q_0^\pm, Q_0^\pm\} &= 0,\end{aligned} \tag{3.13}$$

and where $Q_0 = Q_0^+ + Q_0^-$. Before going ahead note that the 2D NS-like superalgebra generated by $Q_{-1/2}$ ($Q_{-1/2}^\pm$) exchanges bosons B and fermions F wheras the Ramond like one generated by Q_0 ($Q_0^\pm$) preserves the statistics. In the 2D quantum field language, the Ramond like algebra acts on the field doublet $(\Phi_{1/2}(z), \Phi_{-1/2}(z))$, with:

$$Q_0^\pm \Phi_{\pm 1/2} = 0,$$

and

$$Q_0^\pm \Phi_{\mp 1/2} = \Phi_{\pm 1/2},$$

as follows:

$$\Phi_{\pm 1/2} \xrightarrow{Q_0^\mp} \Phi_{\mp 1/2} \xrightarrow{Q_{-1}^\pm} \partial_z \Phi_{\pm 1/2} \tag{3.14}$$

Note, moreover, that the NS- and R-like algebras can be realized with the help of Grassmann variables $\theta^{\pm}_{\pm 1/2}$ and $\theta^{\pm}_0$ as:

$$\begin{aligned} Q^{\pm}_{-1/2} &= \frac{\partial}{\partial\theta^{\mp}_{1/2}} + \frac{1}{2}\theta^{\pm}_{1/2}\partial_{-1}, \\ Q^{\pm}_0 &= \frac{\partial}{\partial\theta^{\mp}_0} + \frac{1}{2}\theta^{\pm}_0. \end{aligned} \tag{3.15}$$

Similar realisations may also be written down for the Hermitian charge operators $Q_{-1/2}$ and Q_0.
In what follows, we want to extend the above matrix formulation to the case $k = 3$. At first sight this should be possible if one succeeds in relating fractional supersymmetry to an infinite dimentional fractional superconformal invariance in the same manner as usual 2D supersymmetry is related to the superconformal invariances. To that end we start by establishing the link between fractional supersymmetry and the Z_k parafermionic invariance. Then we study its matrix realisation.

3.2 Fractional supersymmetry and parafermions

In this section we want to show that from the point of view of 2D Z_k parafermionic invariance, the definition of fractional supersymmetry say

$$(Q_{-1/k})^k = P, \quad k > 2,$$

is just a formal one. The right way to define it is as $tr\mathbf{Q}^k = P_{-1}$, where $P = P_{-1}$ is heterotic component of the 2D energy momentum vector operator P_m. The latter show that fractional supersymmetry is, in fact, generated by many charge operators which we denote as $Q_{-x}, \quad x = 0, 1/k....$
More precisely fractional supersymmetry is generated by a $k \times k$ matrix $\mathbf{Q}$ whose entries are the charge operators Q_{-x} carrying various values of fractional spins. For the $k = 3$ case we are interested in here, there are in total three pairs of charge operators $Q^{\pm}_{-x} \quad x = 0, \frac{1}{3}, \frac{2}{3}$, together with P_{-1}. Altogether, these operators generate the formal cubic nonlinear algebra $Q^3 = P_{-1}$, which as we shall see, is correctly defined as

$$\begin{aligned} tr[\mathbf{Q}^+\mathbf{Q}^-] &= P_{-1} \\ tr[\mathbf{Q}^{+2}] &= tr[\mathbf{Q}^{-2}] = 0 \\ tr[\mathbf{Q}^+] &= tr[\mathbf{Q}^-] = 0 \end{aligned} \tag{3.16}$$

where $\mathbf{Q}^{\pm}$ are 3×3 matrices given by:

$$\mathbf{Q}^{-} = \begin{pmatrix} 0 & 0 & Q^{-}_{-1/3} \\ Q^{-}_{-2/3} & 0 & 0 \\ 0 & Q^{-}_{-1} & 0 \end{pmatrix},$$

$$\mathbf{Q}^{+} = \begin{pmatrix} 0 & Q^{+}_{-1/3} & 0 \\ 0 & 0 & Q^{+}_{0} \\ Q^{+}_{-2/3} & 0 & 0 \end{pmatrix}. \qquad (3.17)$$

Note by the way that here also Q^{+}_{-x} and Q^{-}_{-y} with $Q^{-}_{-1} = Q^{-}_{0}P_{-1}$ obey the equations (3.12). A way to derive equations (3.16) and (3.17) is to consider the Z_3 parafermionic invariance of Zamolodchikov and Fateev [53], [55] given by:

$$\begin{aligned} \Psi^{\pm}(z_1)\Psi^{\pm}(z_2) &\approx -z_{12}^{-2/3}\Psi^{\pm}(z_2) \\ \Psi^{+}(z_1)\Psi^{-}(z_2) &\approx z_{12}^{-4/3}[1+5/3z_{12}^{2}T(z_2)] \\ T(z_1)\Psi^{\pm}(z_2) &\approx \frac{2/3}{z_{12}^{2}}\Psi^{\pm}(z_2)+\frac{1}{z_{12}}\partial_z\Psi^{\pm}(z_2) \\ T(z_1)T(z_2) &= \frac{k-1}{k+2}z_{12}^{-4}+2z_{12}^{-2}T(z_2)+z_{12}^{-1}\partial T(z_2). \end{aligned} \qquad (3.18)$$

In these equations $T(z)$ is the usual energy-momentum current and $\Psi^{\pm}(z)$ ($[\Psi^{\pm}(z)]^{+} = \Psi^{\mp}(z)$) are the Zamolodchikov and Fateev parafermionic currents of spin $\frac{2}{3}$. The algebra (3.18) has three parafermionic highest weight representations $[\Phi^{q}_{q}]$; $q = 0,1,2$, namely the identity family $I = [\Phi^{0}_{0}]$ of highest-weight $h_0 = 0$ and two degenerate families $[\Phi^{1}_{1}]$ and $[\Phi^{2}_{2}]$ of weights $h_1 = h_2 = \frac{1}{15}$. Each one of these parafermionic highest-weight representations $[\Phi^{q}_{q}]$ is reducible into three Virasoro highest-weight representations $[\Phi^{p}_{q}]$; $p = q, p = q \pm 2mod6$. For more details on the representation theory of equations (3.18) see [53], [56]. All we need in the achievement of this study is the algebra equations (3.18), the identities

$$\begin{aligned} \Psi^{\pm}\times\Phi^{p}_{q} &= \Phi^{p\pm 2}_{q} \\ \Phi^{p\pm 6}_{q} &= \Phi^{p}_{q} \end{aligned} \qquad (3.19)$$

as well as the following braiding property of the conformal field operators $\Phi_1(z_1)$ and $\Phi_2(z_2)$:

$$\begin{aligned} \Phi(z) &= z_{12}^{\Delta}\Phi_1(z_1)\Phi_2(z_2) \\ &= z_{21}^{\Delta}\Phi_2(z_2)\Phi_1(z_1), \end{aligned} \qquad (3.20)$$

where $\Delta = \Delta_1+\Delta_2-\Delta_3$, Δ_i, $i=1,2,3$, are the weights of the Φ_i's. Note that the first equation of (3.19) is the formal equation which should be understood as:

$$\Psi^{\pm}(z_1)\Phi_q^p(z_2) = \sum z_{12}^{n-1\mp p/3} Q^{\pm}_{-n\pm(p\pm1)/3}\Phi_q^p(z_2), \tag{3.21}$$

where $Q^{\pm}_{-n\pm(p\pm1)/3}$ are the mode operators of $\Psi^{\pm}$ which in turn may be defined as

$$Q^{\pm}_{-n\pm(p\pm1)/3}|\Phi_q^p>= \int dz z^{n\pm p/3}\Psi^{\pm}(z)\Phi_q^p(0)|0> . \tag{3.22}$$

In addition to equations (3.19) which predict the existance of three doublets of the charge operators $Q^{\pm}_{-x}$, $x=0,\frac{1}{3},\frac{2}{3}$, one also has the braiding feature (3.20) playing a crucial role in the building of fractional supersymmetry. For example, the deformation parameter ω (often denoted by q in the literature) of quantum groups and noncommutative geometry dealing with fractional supersymmetry should be related with the braiding property of the conformal field blocks $\Phi_1(z_1)$ and $\Phi_2(z_2)$. The equation (3.20) tells us that the parameter ω is equal to $exp(\pm i\pi\Delta)$. For $\Delta_3=0$ for instance, ω reduces to $exp(\pm i[\pi(\Delta_1+\Delta_2)])$, so that for $\Delta_1+\Delta_2=1$, the field operators Φ_1 and Φ_2 anticommute and then should be treated as fermions. Choosing $\Delta_1=1/3$ and $\Delta_2=2/3$ for example, the two-point function $<\Phi_{1/3}(z_1),\Phi_{2/3}(z_2)>$ should be equal to $-<\Phi_{2/3}(z_2),\Phi_{1/3}(z_1)>= 1/z_{12}$. This result, derived from parafermionic conformal field methods, solves the difficulty of references [45], [51] according to which the two-point function $<\Phi_{1/3}(z_1),\Phi_{2/3}(z_2)>_{U_q(sl(2))}$ computed in a model of fractional supersymmetry based on quantum groups and noncommutative geometry methods; i.e:

$$\begin{aligned} <\Phi_{1/3}(z_1),\Phi_{2/3}(z_2)>_{U_q(sl(2))} &= e^{2i\pi/3}<\Phi_{2/3}(z_2),\Phi_{1/3}(z_1)>_{U_q(sl(2))} \\ &= \frac{1}{z_{12}} \end{aligned} \tag{3.23}$$

Equation (3.23) shows that the model based on the $U_q(sl(2))$ methods is non local. In summary local 2D field theoretical realisation of fractional supersymmetry cannot be generated by only one charge operator. The number of generators may be obtained by analysing the mode operators $Q^{\pm}_{-n\pm(p\pm1)/3}$, n integer. The $Q^{\pm}_{-n\pm(p\pm1)/3}$'s depend on the p charge of the conformal representation $|\Phi_q^p>$ on which they act. For $q=0$ for example, the non vanishing actions of $Q^{\pm}_{-x}, x= 0,\frac{1}{3},\frac{2}{3}$ on the leading states $|s,p>$ of spin s, $0\leq s\leq 1$ and charge p read as:

$$\begin{aligned} Q^{\pm}_{-2/3}|0,0> &= |\frac{2}{3},0> \\ Q^{+}_{0}|\frac{2}{3},+2> &= |\frac{2}{3},-2> \end{aligned}$$

$$\begin{aligned} Q_0^-|\frac{2}{3},-2> &= |\frac{2}{3},+2> \\ Q_{-1/3}^+|\frac{2}{3},-2> &= |1,0> \\ Q_{-1/3}^-|\frac{2}{3},+2> &= |1,0> . \end{aligned} \quad (3.24)$$

From these equations and equations (3.21) and (3.22), one sees that $Q^{\pm}_{-1/3}$ and $Q_0^{\pm}$ cannot act directly on the state $|0,0>$ similarly $Q^{\pm}_{-2/3}$ cannot operate directly on $|\frac{2}{3},\pm 2>$. This feature gives another indication that fractional supersymmetry should be generated by more than one Q operator as it is currently used in the literature based on quantum groups and noncommutative geometry approaches. Recall that the first indication we have mentioned in the begining of this chapter refers to the inconsistencies induced by the introduction of the generalized Grassmann variable θ satisfying a higher order nilpotency condition $\theta^k = 0$, $k > 2$ with $\theta^{k-1} \neq 0$ and where the problem of locality raised above is just one of the manifestation of the limit of the methods used. In the approach based on parafermionic conformal field theoretical techniques we are considering here, these problems are avoided. Locality is restored since all fields carrying fractional spins obey anticommuting statistics and the higher-order Grassmann nilpotency necessary for the description of off shell representations of fractional supersymmetry is ensured by the presence of more than one charge operator. Having discussed the link between fractional supersymmetry and parafermions, we turn now to study its relation with matrix theory.

3.3 Fractional supersymmetry as a matrix model

Starting from Eqs(3.24) and denoting by $\Pi_r, r = 0, \pm 1$, the projectors along the states $|s,p>=|i>$ $(p = 2i)$ and by $E_{ij} = |i> \otimes <j|$ the generators of the $gl(3)$ Lie algebra rotating the state $|i>$ into the state $|j>$, $\quad j = 0, \pm 1$, (the indices $\pm$ refer to the two fractional supersymmetric partners of the bosonic state indexed by $i = 0$), one sees that the component P_{-1} of the energy momentum vector reads in terms of the $Q^{\pm}_{-x}$'s, $x = 0, \frac{1}{3}, \frac{2}{3}$ and the projectors as:

$$\begin{aligned} P_{-1} &\approx tr\mathbf{P} \\ \mathbf{P} &= Q^+_{-1/3}Q^+_0Q^+_{-2/3}\Pi_0 + Q^+_{-2/3}Q^+_{-1/3}Q^+_0\Pi_1 + Q^+_0Q^+_{-2/3}Q^+_{-1/3}\Pi_{-1} \end{aligned} \quad (3.25)$$

Note that a similar relation to this equation using Q^-_{-x} instead of Q^+_{-x} is also valid. Moreover, using equations (3.24), one may rewrite equations (3.25) and its Hermetic conjugate in the following form.

$$
\begin{aligned}
2P &= [Q^{+}_{-1/3}Q^{-}_{-2/3} + Q^{-}_{-1/3}Q^{+}_{-2/3}]\Pi_0 + [Q^{+}_{-2/3}Q^{-}_{-1/3} + Q^{-}_{-1}Q^{+}_{0}]\Pi_1 \\
&+ [Q^{-}_{-2/3}Q^{+}_{-1/3} + Q^{+}_{-1}Q^{-}_{0}]\Pi_{-1}.
\end{aligned} \tag{3.26}
$$

Comparing equations (3.25) and (3.26), one finds the following constraint relations:

$$
\begin{aligned}
Q^{-}_{-1/3} &= Q^{+}_{-1/3}Q^{+}_{0} \\
Q^{+}_{-1/3} &= Q^{-}_{-1/3}Q^{-}_{0} \\
Q^{-}_{-2/3} &= Q^{+}_{0}Q^{+}_{-2/3} \\
Q^{+}_{-2/3} &= Q^{-}_{0}Q^{-}_{-2/3} \\
Q^{-}_{-1} &= Q^{+}_{-2/3}Q^{+}_{-1/3} \\
Q^{+}_{-1} &= Q^{-}_{-2/3}Q^{-}_{-1/3}
\end{aligned} \tag{3.27}
$$

Equations (3.27) and consequently equations (3.25) and (3.26) may be satisfied identically by introducing the following 3×3 matrix operator $\mathbf{Q} = \mathbf{Q}^{+} + \mathbf{Q}^{-}$ whose entries are $Q^{\pm}_{-x}$, $x = 0, \frac{1}{3}, \frac{2}{3}$ are the generators of fractional supersymmetry. This matrix representation is in agreement with the Zamolodchikov and Fateev parafermionic invariance property (3.21) and (3.22) and the constraint equations (3.27). Using the $gl(3)$ generators $E_{ij} = |i > \otimes |j >, i, j = 0, \pm 1$, one may write down the matrix realisations of $(\frac{1}{3}, 0)$ real fractional supersymmetry and $((\frac{1}{3})^2, 0)$ fractional supersymmetric algebra for the $\mathbf{Q}^{\pm}_{-x}$'s [29].

3.3.1 $(\frac{1}{3}, 0)$ Real fractional supersymmetry

The $(\frac{1}{3}, 0)$ real fractional supersymmetric algebra, to which we shall refer hereafter to as 2D$(\frac{1}{3}, 0)$ fractional supersymmetry is the analogue of equations (3.1) and (3.4). The matrix representation of the $Q^{\pm}_{-x}$'s reads as:

$$
\begin{aligned}
Q^{+}_{-2/3} &= Q_{-2/3}E_{1,0} \\
Q^{-}_{-2/3} &= Q_{-2/3}E_{-1,0} \\
Q^{+}_{0} &= Q_{0}E_{-1,1} \\
Q^{-}_{0} &= Q_{0}E_{1,-1} \\
Q^{+}_{-1/3} &= Q_{-1/3}E_{0,-1} \\
Q^{-}_{-1/3} &= Q_{-1/3}E_{0,1}
\end{aligned} \tag{3.28}
$$

The charge carried by the $Q^{\pm}_{-x}$ of equations (3.28) is the Z_3 charge of the automorphism symmetry of the matrix operator equation.

$$(E^3)_{ii} = \Pi_i, \quad i = 0, \pm 1. \tag{3.29}$$

In the orthonormal basis $\{|i>, \quad i = 0, \pm 1\}$, the matrix representation of the $\mathbf{Q}^{\pm}_{-x}$'s reads as [29]:

$$\mathbf{Q}^{+} = \begin{pmatrix} 0 & Q_{-2/3} & 0 \\ 0 & 0 & Q_0 \\ Q_{-1/3} & 0 & 0 \end{pmatrix};$$

$$\mathbf{Q}^{-} = \begin{pmatrix} 0 & 0 & Q_{-2/3} \\ Q_{-1/3} & 0 & 0 \\ 0 & Q_{-1} & 0 \end{pmatrix}. \tag{3.30}$$

Using equations (3.30) it is not difficult to check that the following relations hold:

$$\begin{aligned} 2\mathbf{P} &= \mathbf{Q}^{+}\mathbf{Q}^{-} + \mathbf{Q}^{-}\mathbf{Q}^{+} \\ \mathbf{Q}^{-} &= \mathbf{Q}^{+}\mathbf{Q}^{+} \\ \mathbf{Q}^{-2} &= \mathbf{P}.\mathbf{Q}^{+} \end{aligned} \tag{3.31}$$

Taking the traces of both sides of these matrix equations, one discovers the algebra equations(3.16) which reads in terms of $Q_{-1/3}$ and $Q_{-2/3}$ as:

$$\begin{aligned} P_{-1} &= Q_{-1/3}Q_{-2/3} + Q_{-2/3}Q_{-1/3} \\ 0 &= \{Q_{-1/3}, Q_{-1/3}\} \\ &= \{Q_{-2/3}, Q_{-2/3}\}. \end{aligned} \tag{3.32}$$

Note that equation(3.32) was expected from the constraint equation (3.6). It was suggested in ref [57] as a linearized form of the non linear operator equation $[Q_{-1/3}]^3 = P_{-1}$. The relation between equation (3.32) and 2D$((\frac{1}{2})^2, 0)$ supersymmetry suspected in [57] will be considered latter on.

3.3.2 $((\frac{1}{3})^2, 0)$ Fractional supersymmetric algebra

The $((\frac{1}{3})^2, 0)$ fractional supersymmetric algebra is a complex solution for which the matrix representation of the $\mathbf{Q}^{\pm}_{-x}$'s reads as:

$$\mathbf{Q}^{+} = \begin{pmatrix} 0 & Q^{+}_{-2/3} & 0 \\ 0 & 0 & Q^{+}_0 \\ Q^{+}_{-1/3} & 0 & 0 \end{pmatrix};$$

$$\mathbf{Q}^- = \begin{pmatrix} 0 & 0 & Q^-_{-2/3} \\ Q^-_{-1/3} & 0 & 0 \\ 0 & Q^-_{-1} & 0 \end{pmatrix}. \tag{3.33}$$

Here also the charges carried by the $Q^\pm_{-x}$ are Z_3 charges. Similar calculations as for the $(\frac{1}{3}, 0)$ algebra show that equations (3.31) are again fulfilled for the representation (3.33). Using equations (3.33), and solving equations (3.31) we find the following relations:

$$\begin{aligned} 0 &= \{Q^\pm_{-1/3}, Q^\pm_{-1/3}\} = \{Q^\pm_{-2/3}, Q^\pm_{-2/3}\} \\ 2P_{-1} &= \{Q^+_{-2/3}, Q^-_{-1/3}\} + \{Q^+_{-1/3}, Q^-_{-2/3}\} \end{aligned} \tag{3.34}$$

and

$$\begin{aligned} Q^-_{-1/3} &= Q^+_{-1/3} Q^+_0; \\ Q^+_{-1/3} &= Q^-_0 Q^-_{-1/3} \\ Q^-_{-2/3} &= Q^+_0 Q^+_{-2/3}; \\ Q^+_{-2/3} &= Q^-_{-2/3} Q^-_0. \end{aligned} \tag{3.35}$$

Equations (3.34) and (3.35) define the $((\frac{1}{3})^2, 0)$ fractional supersymmetric algebra. It is generated by two Z_3 doublets of anticommuting charges operators $Q^\pm_{-1/3}$ and $Q^\pm_{-2/3}$. The components of each doublet are related to each another by the $Q^\pm_{-x}$'s as shown in equations (3.35). Note that the algebra (3.34-35) is stable under the three following conjugations:

$$\begin{aligned} (Q^\pm_{-x})^* &= Q^\mp_{-x} \\ (Q^\pm_{-x})^+ &= Q^\mp_{-1+x} \\ (Q^\pm_{-x})^{+*} &= Q^\pm_{-1+x}, \end{aligned}$$

suggesting a link with 2D$N = ((\frac{1}{2})^4, 0) su(2)$ supersymmetry formulated in harmonic superspace [58]. In what follows we shall explore this relation in order to use it for the construction of off shell representations of the 2D$((\frac{1}{3})^2, 0)$ fractional supersymmetry.

3.4 More on the $((\frac{1}{3})^2, 0)$ fractional supersymmetric algebra

Because of the periodicity ($p \equiv p \pm 6$) of the representations of the Zamolodchikov and Fateev parafermionic algebra (3.18) which allow to identify the field operators

$\Phi^{\pm}$ as $\Phi^{\pm} \equiv \Phi^{\pm 2}$ and $\Phi^{\pm\pm} \equiv \Phi^{\pm 4}$; see the second equation of (3.19), one may rewrite the charge operators Q_0^+, Q_0^- as Q_0^{--}, Q_0^{++} respectively.
Moreover denoting by Q_0^0 the commutator of the Q_0^+ and Q_0^- charge operators, one sees from equations (3.28) that Q_0^0 and $Q_0^{\pm}$ generated altogether an su(2) algebra,

$$\begin{aligned} [Q_0^-, Q_0^+] &= Q_0^0 \\ [Q_0^0, Q_0^{\pm}] &= \mp Q_0^{\pm} \end{aligned} \tag{3.36}$$

acting on the $Q_{-x}^{\pm}, x = \frac{1}{3}, \frac{2}{3}$ charges as:

$$\begin{aligned} [Q_0^0, Q_{-1/3}^{\pm}] &= \pm Q_{-1/3}^{\pm} \\ [Q_0^0, Q_{-2/3}^{\pm}] &= \pm Q_{-2/3}^{\pm} \\ [Q_0^{\pm}, Q_{-1/3}^{\pm}[&= \mp Q_{-1/3}^{\mp} \\ [Q_0^{\pm}, Q_{-2/3}^{\pm}] &= \pm Q_{-2/3}^{\mp} \\ [Q_0^-, Q_{-x}^+] &= 0 \\ [Q_0^+, Q_{-x}^-] &= 0 \end{aligned} \tag{3.37}$$

Equations (3.36) correspond just to the zero mode subalgebra of the level 3 of the $su_3(2)$ Kac Moody symmetry. The latter is known to be homomorphic to the Z_3 parafermionic invariance (3.18) [59]. Now using the identification $Q_0^{\mp} \equiv Q_0^{\pm\pm}$ and the Z_3 periodicity $q = qmod(3)$, one may rewrite the algebra (3.36) as:

$$\begin{aligned} [Q_0^{++}, Q_0^{--}] &= Q_0^0 \\ [Q_0^0, Q_0^{\pm\pm}] &= \pm 2 Q_0^0 \end{aligned} \tag{3.38}$$

Substituting $Q_0^{\mp}$ by $Q_0^{\pm\pm}$ in equations (3.37), one gets the following relations which look like the corresponding ones in 2D$N = 4su(2)$ supersymmetry [58]:

$$\begin{aligned} [Q_0^{++}, Q_{-x}^+] &= [Q_0^{--}, Q_{-x}^-] = 0 \\ [Q_0^{++}, Q_{-1/3}^-] &= a Q_{-1/3}^+ \\ [Q_0^{++}, Q_{-2/3}^-] &= b Q_{-2/3}^+ \\ [Q_0^{--}, Q_{-1/3}^+] &= b Q_{-1/3}^- \\ [Q_0^{--}, Q_{-2/3}^+] &= a Q_{-2/3}^- \\ [Q_0^0, Q_{-2/3}^{\pm}] &= \pm a Q_{-2/3}^{\pm} \\ [Q_0^0, Q_{-1/3}^{\pm}] &= \pm a Q_{-1/3}^{\pm} \end{aligned} \tag{3.39}$$

where $a = -b = 1$. Recall that in $2D$ $((\frac{1}{2})^4, 0)$ $su(2)$ supersymmetry the coefficients a and b are equal to one, $a = b = 1$, and the analogues of equations (3.34) read as:

$$
\begin{aligned}
2P_{-1} &= \{Q^+_{-1/2}, \bar{Q}^-_{-1/2}\} - \{Q^-_{-1/2}, \bar{Q}^+_{-1/2}\} \\
0 &= \{Q^\pm_{-1/2}, Q^\pm_{-1/2}\} \\
&= \{\bar{Q}^\pm_{-1/2}, \bar{Q}^\pm_{-1/2}\}.
\end{aligned}
\tag{3.40}
$$

Observe by the way that a part from the spin of the charge operators in the first equation of (3.38) differs from the second equation of (3.34) by the presence of the minus sign which is required by invariance under the $su(2)$ automorphism group of the $2D$ $N=4$ $su(2)$ supersymmetric algebra. Nevertheless, the similarity between equation (3.34-35) and (3.37-38) allows us to build an off shell superspace formulation of $2D((\frac{1}{3})^2, 0)$ supersymmetry by mimicking the harmonic superspace formalism [60].

Chapter 4

On the algebra of q-deformed pseudo differential operators

An interesting subject which have been studied recently from different point of views deals with the field of non-linear integrable systems and their various higher and lower spin extensions [61], [62], [63],[64]. These are exactly solvable models exhibiting a very rich structure in lower dimensions and are involved in many areas of mathematical physics. One recall for instance the two dimensional Toda (Liouville) fields theories [62], [65] and the KdV hierarchy models [61], [62], both in the bosonic as well as in the supersymmetry case.
Non linear integrable models are associated to systems of non-linear differential equations, which we can solve exactly. Mathematically these models have become more fascinating by introducing some new concepts such as the infinite dimensional Lie (super) algebras [66], Kac-Moody algebras [67], W-algebras [63], [64], quantum groups [68] and the theory of formal pseudo-differential operators [61], [62]. Note by the way that techniques developped for the analysis of non-linear integrable systems and quantum groups can be used to understand many features appearing in various problems of theoretical physics [69], [70].
Recall that, since symmetries play an important role in physics; the principal task of quantum groups consist in extanding these standard symmetries to the deformed ones, which might be used in physics as well.
We start in this chapter from the well known q-deformed derivation law, $\partial z = 1 + qz\partial$ [68] and derive the q-analogue of the Leibnitz rule for both local and nonlocal differential operators. This result, which gives naturally the algebra of q-deformed pseudo-differential operators, will provide a way for generating a hierarchy of q-deformed Lax evolution equations.

4.1 The ring of q-analytic currents

Let us precise that the deformation parameter q we consider in this study is assumed to be a non vanishing positive number this means that $q \in R^*$. However if we suppose that $q \in C$, then we shall impose for q to differ from the k-th root of unity i.e $q^k \neq 1$. This requirement is justified by our need of consistency when we go to the standard limit $q = 1$.
Consider then the following q-deformed derivation rule [68]

$$\partial z = 1 + qz\partial \tag{4.1}$$

where the symbol ∂ stand for the q-derivative $\partial \equiv \partial_q = (\frac{\partial}{\partial z})_q$. The conserved currents are ingredients that we need highly in the programs of non linear integrable models and two-dimensional conformal field theory building. As we are interested in the present study to set-up the basic tools towards extending such programs to q-analogue ones, we will try to describe first the ring of arbitrary q-analytic fields which we denote by R. Following the analysis developed in [71], this space describes a tensor algebra of fields of arbitrary conformal spin. This is a completely reducible infinite dimensional $SO(2)$ Lorentz representation(module) that can be decomposed as:

$$R = \bigoplus_{k\in Z} R_k^{(0,0)} \tag{4.2}$$

where $R_k^{(0,0)} = R_k$ are one dimensional spin k-irreducible modules generated by the q-analytic fields $u_k(z)$ of conformal spin $k \in Z$. The upper indices $(0,0)$ carried by R and that we shall drop whenever no confusion can arise, are special values of general indices (r,s) introduced in [71] and referring to the lowest and highest degrees of some pseudo-differential operators.
Inspering from the derivation law Eq(4.1), we introduce in this ring a q-deformed derivative $\partial \equiv \partial_q$ satisfying:

$$\partial u_k(z) = u_k^{'}(z) + \bar{q}^k u_k(z)\partial \tag{4.3}$$

with $\bar{q} = q^{-1}$ and $u_k^{'} = (\frac{\partial u_k}{\partial z})_q$ stands for the standard prime derivative. Note by the way, the important fact, that we have to distinguish between the prime derivative $u_k^{'} = \partial u_k$ and the operator derivative $\partial u_k = (\partial u_k) + \bar{q}^k u_k \partial$ given by the equation (4.3). To illustrate what does it means, consider the following two examples.

Example 1: Let $u_{-k}(z) = z^k, k \geq 0$.
For this choice of the field $u_{-k}(z)$, we derive the following expression:

$$u_{-k}^{'}(z) = (\sum_{i=0}^{k-1} q^i) z^{k-1}, \tag{4.4}$$

which we can easily check by proceeding with the first leading terms $k = 0, 1, 2, \ldots$ Indeed for $k = 0$, $u_0' = 0$ and for $k = 1$, we have $u_{-1} \equiv z$ and by virtue of Eq(4.1) we have

$$\begin{aligned} u_{-1}'(z) \equiv (\partial u_{-1}) &= \partial u_{-1} - \bar{q}^{-1} u_{-1} \partial \\ &= \partial z - \bar{q}^{-1} z \partial \\ &= 1 \end{aligned} \tag{4.5}$$

which we can derive also from Eq(4.4), with $\bar{q}^{-1} = q$. The non trivial case is given by $k = 2$, such that $u_{-2} \equiv z^2$, we have

$$\begin{aligned} u_{-2}'(z) \equiv (\partial u_{-2}) &= \partial z^2 - \bar{q}^{-2} u_{-2} \partial \\ &= (1 + q) z + q^2 z^2 \partial - \bar{q}^{-2} z^2 \partial \\ &= (1 + q) z \end{aligned} \tag{4.6}$$

which can also easily seen from eq(4.4). These first leading cases, show then clearly from where the prime derivative formula eq(4.4) comes from.
The total Leibnitz applied to $u_{-k}(z) = z^k, k \geq 0$, is simply derived using successive action of the deformed q-derivative $\partial \equiv \partial_q$. We find

$$\partial z^k = (\sum_{i=0}^{k-1} q^i) z^{k-1} + q^k z^k \partial, \tag{4.7}$$

which justify, in some sense, the consistency of eq(4.4) in describing the conformal spin content of the analytic fields $u_k(z)$. Setting $k = 1$, one recovers in a natural way, the standard relation eq(4.1). The second example we consider is the following:

Example 2: $u_k(z) = z^{-k}, k \geq 1$,
Corresponding relations are computed in the same way. We find

$$\partial z^{-k} = -(\sum_{i=1}^{k} \bar{q}^i) z^{-k-1} + \bar{q}^k z^{-k} \partial, \tag{4.8}$$

which reduces to:

$$\partial z^{-1} = -\bar{q} z^{-2} + \bar{q} z^{-1} \partial, \tag{4.9}$$

upon setting $k = 1$.
Now having introduced the ring R, of analytic q-deformed currents, and show how the q-deformed derivative act on, we are now in position to introduce the space of q-deformed pseudo-differential operators.

4.2 The space of q-deformed Lax operators

Let $\Xi_m^{(r,s)}$ denote the space of q-deformed local differential operators, labelled by three quantum numbers m, r and s defining respectively the conformal spin, the lowest and highest degrees. Typical elements of this space, are given by:

$$L_m = \sum_{i=r}^{s} u_{m-i}(z)\partial^i, \quad r, s, m \in Z. \tag{4.10}$$

The symbol ∂ stands for the q-derivative and $u_{m-i}(z)$ are analytic fields of conformal spin $(m-i)$. The space $\Xi_m^{(r,s)}$ behaves then as a $(1+s-r)$ dimensional space generated by $L_m^{(r,s)} \equiv L_m$ and whose space decomposition is given by the linear sum:

$$\Xi_m^{(r,s)} = \bigoplus_{i=r}^{s} \Xi_m^{(i,i)}, \tag{4.11}$$

with

$$\Xi_m^{(i,i)} = R_m \otimes \partial^i \tag{4.12}$$

A special example of the space $\Xi_m^{(r,s)}$ is given by $R_m \equiv \Xi_m^{(0,0)}$ eq(4.2), the set of analytic fields $u_m(z)$ introduced previously and $\partial^i \equiv \partial_q^i$ is the i-th q-derivative. A natural example of eq(4.10), is given by the q-deformed Hill operator,

$$L_2 = \partial^2 + u_2(z), \tag{4.13}$$

which will play an important role in the study of the q-deformed KdV equation and the associated conformal q-Liouville field theory.
A result concerning the algebra $\Xi_m^{(r,s)}$ is the derivation of the q-Leibnitz rule for local q-differential operators. Focusing to derive the general formula, let us start first by examining the first leading orders. Iteration processing applied to eq(4.3) gives the following relations

$$\begin{aligned} \partial u_k(z) &= u_k'(z) + \bar{q}^k u_k(z)\partial \\ \partial^2 u_k(z) &= u_k''(z) + \bar{q}^k(1+\bar{q})u_k'(z)\partial + \bar{q}^{2k}u_k(z)\partial^2 \\ \partial^3 u_k(z) &= u_k'''(z) + \bar{q}^k(1+\bar{q}+\bar{q}^2)u_k''(z)\partial + \bar{q}^{2k}(1+\bar{q}+\bar{q}^2)u_k'(z)\partial \\ &+ \bar{q}^{3k}u_k(z)\partial^3 \\ &\vdots \end{aligned} \tag{4.14}$$

The crucial point was the observation that, these higher first order derivations formulas can be summarised into the following general Leibnitz rule [32]

$$\partial^p u_k(z) = \sum_{j=0}^{p} \bar{q}^{(p-j)k} \chi_p^j(q) u_k^{(j)}(z)\partial^{p-j}, \quad p \geq 0, \tag{4.15}$$

where $\chi_p^j(q)$ are q-coefficient functions that we have introduced such that:

$$\chi_p^0(q) = \chi_p^p(q) = 1, \tag{4.16}$$

and

$$\begin{aligned} \chi_p^j(q) &= 1 + \bar{q}^j \sum_{m_1=0}^{j-1} q^{m_1} + \bar{q}^{2j} \sum_{m_1=0}^{j-1} \sum_{m_2=0}^{j-1-m_1} q^{2m_1+m_2} \\ &+ \bar{q}^{3j} \sum_{m_1=0}^{j-1} \sum_{m_2=0}^{j-1-m_1} \sum_{m_3=0}^{j-1-(m_1+m_2)} q^{3m_1+2m_2+m_3} \\ &+ \dots \\ &+ \bar{q}^{(p-j)j} \sum_{m_1=0}^{j-1} \sum_{m_2=0}^{j-1-m_1} \dots \sum_{m_{p-j}=0}^{j-1-\sum_{i=1}^{p-j-1} m_i} q^{\sum_{\beta=0}^{p-j-1}(p-j-1-\beta)m_{\beta+1}} \end{aligned} \tag{4.17}$$

for $1 \leq j \leq p-1$.
Some remarks are in order:

1. From the Leibnitz rule eq(4.15); we can deduce the q-analogue of the standard binomial coefficients c_p^j as follows:
For $j = 0$ and $j = p$, we have:

$$\begin{aligned} c_p^0 &\longrightarrow \bar{q}^{pk}\chi_p^0(q) \equiv \bar{q}^{pk} \\ c_p^p &\longrightarrow \chi_p^p(q) = 1 \end{aligned}$$

and for $1 \leq j \leq p-1$, we have:

$$c_p^j \longrightarrow \bar{q}^{(p-j)k}\chi_p^j(q) \tag{4.18}$$

2. Setting $q = 1$, the local Leibnitz rule (4.15) reduces naturally to the standard derivation law:

$$\partial^p u_k(z) = \sum_{j=0}^{p} c_p^j u_k^{(j)}(z)\partial^{p-j}, \quad p \geq 0 \tag{4.19}$$

giving rise to the following useful relations

$$\begin{aligned} \chi_p^0(1) &= c_p^0 = 1 \\ \chi_p^p(1) &= c_p^p = 1 \end{aligned} \tag{4.20}$$

and for $1 \leq j \leq p-1$

$$
\begin{aligned}
c_p^j &= \chi_p^j(1) = 1 + j + \frac{j(j+1)}{2} \\
&+ \sum_{m_1=0}^{j-1} \sum_{m_2=0}^{j-1-m_1} \sum_{m_3=0}^{j-1-(m_1+m_2)} 1 \\
&+ \ldots + \sum_{m_1=0}^{j-1} \sum_{m_2=0}^{j-1-m_1} \cdots \sum_{m_{p-j}=0}^{j-1-\sum_{i=1}^{p-j-1}}
\end{aligned}
\tag{4.21}
$$

3. As we can easily check, eq(4.17) is a sum of $(p-j+1)$ objects starting from the value 1 which corresponds to set $(j=p)$ with zero summation. In each term of the remaining $(p-j)$ objects, we have a product of (n) summation $\sum_{m_1=0}\sum_{m_2=0}\ldots\sum_{m_n=0}$ with $1 \leq n \leq p-j$. This structure is useful in the standard limit $q=1$, recovering then the explicit form eq(4.21) of the well known binomial coefficient $c_p^j = \frac{p!}{(p-j)!j!}$.

Moreover eq(4.10) which is well defined for local differential operators with $s \geq r \geq 0$, may be extended the negative integers(non local ones) by introducing q-deformed pseudo-differential operators of the type $\partial_q^{-p}, \quad p \geq 1$, whose action on the fields $u_k(z)$ of conformal spin $k \in Z$ is constrained to satisfy:

$$
\partial^p \partial^{-p} u_k(z) = \partial^{-p} \partial^p u_k(z) = u_k(z). \tag{4.22}
$$

Following the same analysis developed previously, we derive the following formulas

$$
\begin{aligned}
\partial^{-1} u_k(z) &= \sum_{i=0}^{\infty} (-1)^i q^{(k(i+1))+\frac{i(i+1)}{2}} u_k^{(i)}(z) \partial^{-1-i} \\
\partial^{-2} u_k(z) &= \sum_{i=0}^{\infty} (-1)^i q^{(k(i+2))+\frac{i(i+1)}{2}} (\sum_{j=0}^{i} q^j) u_k^{(i)}(z) \partial^{-2-i} \\
\partial^{-3} u_k(z) &= \sum_{i=0}^{\infty} (-1)^i q^{(k(i+3))+\frac{i(i+1)}{2}} (\sum_{j_1=0}^{i} \sum_{j_2=0}^{j_1} q^{j_1+j_2}) u_k^{(i)}(z) \partial^{-3-i} \\
&\vdots
\end{aligned}
\tag{4.23}
$$

From these first leading formulas, we extract the following nonlocal Leibnitz rule [32]:

$$
\partial^{-p} u_k(z) = \sum_{i=0}^{\infty} (-1)^i q^{(k(i+p))+\frac{i(i+1)}{2}} [\sum_{j_1=0}^{i} \sum_{j_2=0}^{j_1} \cdots \sum_{j_{p-1}=0}^{j_{p-2}} q^{\sum_{m=1}^{p-1} j_m}] u_k^{(i)}(z) \partial^{-p-i}
\tag{4.24}
$$

Here we also remark that, for a fixed value of $p \geq 1$, we have a q-deformed binomial coefficient given by a product of $(p-1)$-summation $\sum_{m_1=0} \dots \sum_{m_{p-1}=0}$. Taking $q=1$ we find the standard Leibnitz rule for nonlocal differential operators namely:

$$\partial^{-p} u_k(z) = \sum_{i=0}^{p} (-1)^i C^i_{i+p-1} u_k^{(i)}(z) \partial^{-p-i}, \tag{4.25}$$

where C^i_{i+p-1} is the identity relation given by:

$$C^i_{i+p-1} = \sum_{j_1=0}^{i} \sum_{j_2=0}^{j_1} \dots \sum_{j_{p-1}=0}^{j_{p-2}} 1, \quad p \geq 1 \tag{4.26}$$

this relation coincide exactly with $\chi^i_{i+p-1}(1)$, given by eq(4.20).

4.3 q-Generalized formulas

Let $f(z)$ be an arbitrary analytic function of conformal spin $\Delta f = \widetilde{f}$. Using eq(4.3) and iterative action of the q-deformed derivative we find:

$$\partial f^n(z) = (1 + \bar{q}^{\widetilde{f}} + \bar{q}^{2\widetilde{f}} + \dots + \bar{q}^{(n-1)\widetilde{f}}) f' f^{n-1} + \bar{q}^{n\widetilde{f}} f^n \partial, \tag{4.27}$$

where n is a positive integer number. Setting $q=1$ one discovers, once again the following ordinary derivation rule,

$$\partial f^n(z) = n f' f^{n-1} + f^n \partial. \tag{4.28}$$

A special choice of $f(z)$ in eq(4.27) is given by $f(z) = z$ with $\widetilde{z} = -1$, we have:

$$\partial z^n = (1 + q + q^2 + \dots + q^{n-1}) z^{n-1} + q^n z^n \partial, \tag{4.29}$$

which reduces to eq(4.1) upon setting $n=1$. For negative integer numbers we easily find,

$$\partial f^{-n}(z) = -(q^{\widetilde{f}} + q^{2\widetilde{f}} + \dots + q^{n\widetilde{f}}) f' f^{-n-1} + q^{n\widetilde{f}} f^{-n} \partial, \tag{4.30}$$

which becomes upon setting $q=1$,

$$\partial f^{-n}(z) = -n f' f^{-n-1} + f^{-n} \partial \tag{4.31}$$

As before, setting $f(z) = z$ we obtain:

$$\partial z^{-n}(z) = -(\bar{q} + \bar{q}^2 + \dots + \bar{q}^n) z^{-n-1} + \bar{q}^n z^{-n} \partial. \tag{4.32}$$

Furthermore, we note that for half integer powers of $f(z)$ we can obtain general formulas. The method to do this starts from setting

$$\partial f^{1/2} = \alpha(q) f' f^{-1/2} + \beta(q) f^{1/2}\partial, \tag{4.33}$$

where $\alpha(q)$ and $\beta(q)$ are two arbitrary q-dependent functions that we can determine explicitly by the following trivial property:

$$\partial(f^{1/2}f^{1/2}) \equiv \partial(f). \tag{4.34}$$

General formulas are given by:

$$\begin{aligned} \partial f^{\frac{2n+1}{2}}(z) &= \frac{(1+\bar{q}^{\frac{\tilde{f}}{2}}+\bar{q}^{\frac{2\tilde{f}}{2}}+\ldots+\bar{q}^{\frac{2n\tilde{f}}{2}})}{(1+\bar{q}^{\frac{\tilde{f}}{2}})} f' f^{\frac{2n-1}{2}} \\ &+ \bar{q}^{\frac{(2n+1)\tilde{f}}{2}} f^{\frac{2n+1}{2}}\partial, \end{aligned} \tag{4.35}$$

and

$$\begin{aligned} \partial f^{\frac{-(2n+1)}{2}}(z) &= \frac{-q^{\frac{\tilde{f}}{2}}(1+q^{\frac{\tilde{f}}{2}}+q^{2\tilde{f}}+\ldots+q^{\frac{(2n+1)\tilde{f}}{2}})}{(1+q^{\frac{\tilde{f}}{2}})} f' f^{\frac{-(2n+3)}{2}} \\ &+ q^{\frac{(2n+1)\tilde{f}}{2}} f^{\frac{-(2n+1)}{2}}\partial. \end{aligned} \tag{4.36}$$

These q-generalized results are important in discussing the q-deformed Lax evolution equations and the covariantisation of q-differential Lax operators.
Note that the ring $R = \bigoplus_{k\in Z} R_k$ defined in eq(4.2) is a commutative ring, which means that for each $u_k(z)$ and $u_l(z)$ belonging to R we have $u_k(z)u_l(z) = u_l(z)u_k(z)$.
However, applying the q-Leibnitz rule eq(4.3), we can easily show the existance of a non commutative structure in the space $\Xi_m^{(r,s)}$ of local and non local q-differential operators. Indeed, let f and g be two arbitrary functions of conformal spin $\tilde{f}$ and $\tilde{g}$, with $fg = gf$, we have

$$\begin{aligned} (\partial f)g &= f' g + \bar{q}^{\tilde{f}} f g' + \bar{q}^{(\tilde{f}+\tilde{g})} f g\partial \\ (\partial g)f &= g' f + \bar{q}^{\tilde{g}} g f' + \bar{q}^{(\tilde{f}+\tilde{g})} g f\partial, \end{aligned} \tag{4.37}$$

which clearly shows that $(\partial f)g \neq (\partial g)f$ for $\widetilde{f} \neq \widetilde{g}$. Note that this noncommutativity property of f and g, with respect to the action of the q-derivative ∂_q, arises naturaly from eq(4.3). Note also the important fact that when the function g is for example, the n-th power of the function f with $n \in R$, we can set $g = f^n$ which yields $\widetilde{g} = n\widetilde{f}$ and then:

$$(\partial f)g = (\partial g)f, \tag{4.38}$$

with $f'f^n = f^n f'$. We can deduce that the two subspaces $R_{\widetilde{f}}$ and $R_{\widetilde{g}}$ of analytic functions $f(z)$ and $g(z)$ of conformal spin $\widetilde{f}$ and $\widetilde{g}$ respectively do not commute under the action of the q-derivative ∂q unless if there exist a relative integer $n \in Z$, such that $g = f^n$.

Up to now, we have introduced the ring R of analytic functions and constructed the space of arbitrary q-deformed Lax operators by deriving the generalized q-Leibnitz rules. In the next chapter we will see how we can apply the obtained results to study some important features of nonlinear integrable systems and conformal symmetry. Special examples, namely the Liouville field theory and the KdV equation as well as their extensions will be considered.

Chapter 5

q-Generalized conformal transformation

In this chapter we will use the backgrounds found in chapter 4 to construct the q-analogues of the KdV-hierarchy systems. We will concentrate in particular on the first leading orders of this hierarchy, namely the q-KdV and q-Boussinesq integrable systems. Then we present the conformal transformation of the q-KdV and q-Boussinesq integrable models described respectively by $L_2(u)$ and $L_3(u)$.

5.1 Generalised q-deformed KdV hierarchy

Let L_2 be the q-deformed KdV Lax operator defined by:

$$L_2 = \partial^2 + u_2, \tag{5.1}$$

which belongs to the coset space $\frac{\Xi_2^{(0,2)}}{\Xi_2^{(1,1)}}$, for which we have

$$u_0 = 1$$

and

$$u_1 = 0.$$

5.1.1 Half power of the q-KdV Lax operator

Referring to the references in non-linear integrable models [61], [62] we can define by analogy the n-th evolution of the q-deformed KdV -hierarchy as follows:

$$\frac{\partial L_2}{\partial t_{2n+1}} = [H_{2n+1}, L_2]_q. \tag{5.2}$$

The bracket introduced in this equation is the q-deformed commutator (see appendix B), which we define as:

$$[f\partial^n, g\partial^m]_q = \bar{q}^{m\widetilde{f}} f\partial^n g\partial^m - \bar{q}^{n\widetilde{g}} g\partial^m f\partial^n \tag{5.3}$$

here f and g are two arbitrary functions of conformal spin $\widetilde{f}$ and $\widetilde{g}$.
In equation (5.2), H_{2n+1} is given by:

$$H_{2n+1} = (L_2^{\frac{2n+1}{2}})_+, \tag{5.4}$$

where the index "+" in this equation stands for the local part of the deformed pseudo differential operator.
The deformed pseudo-differential operator $L_2^{\frac{2n+1}{2}}$ is defined as:

$$L_2^{\frac{2n+1}{2}} = L_2^{1/2} L_2^n, \tag{5.5}$$

here $L_2^{1/2}$ represent the half power of the q-KdV Lax operator given in eq(5.1). It describes a q-deformed pseudo-differential operator of dimension $2 \times \frac{1}{2} = 1$.
Note that the nonlinear q-deformed pseudo differential operator $L_2^{\frac{2n+1}{2}}$ is the $(2n+1)$-th power of $L_2^{1/2}$. Using dimensional arguments, we assume that $L_2^{1/2}$ takes the following form:

$$L_2^{1/2} = \partial + A(q)u_2\partial^{-1} + B(q)u_2'\partial^{-2} + (C(q)u_2'' - D(q)u_2^2)\partial^{-3} + \dots \tag{5.6}$$

where the coefficients A(q),B(q), C(q) and D(q) are required to satisfy

$$L_2 = L_2^{1/2} L_2^{1/2}. \tag{5.7}$$

Using this requirement, we obtain explicitly:

$$\begin{aligned} A(q) &= \frac{1}{1+\bar{q}^2} \\ B(q) &= \frac{-1}{(1+\bar{q}^3)(1+\bar{q}^2)} \\ C(q) &= \frac{1}{(1+\bar{q}^2)(1+\bar{q}^3)(1+\bar{q}^4)} \\ D(q) &= \frac{q^2}{(1+\bar{q}^2)^2(1+\bar{q}^4)} \end{aligned} \tag{5.8}$$

5.1.2 q-Deformed KdV integrable system

If we take $n = 0$, the equation (5.2) becomes:

$$\frac{\partial L_2}{\partial t_1} = [H_1, L_2]_q, \tag{5.9}$$

where $H_1 = (L_2^{\frac{1}{2}})_+ = \partial$. The equation (5.9) corresponds simply to the chiral wave equation,

$$\dot{u}_2 = u_2^{'} \tag{5.10}$$

in this formula we have introduce the dot on the analytic fields u_2 to describe the derivation with respect to time coordinates while the prime derivative is already used in eq(4.3) to denote the derivation with respect to the space variable z.
The equation (5.10) means the equality of the dimensions $[t_1] = [z]$. For $n = 1$, we have

$$\frac{\partial L_2}{\partial t_3} = [(L_2^{3/2})_+, L_2]_q, \tag{5.11}$$

where $(L_2^{3/2})_+$, explicitly given by:

$$(L_2^{3/2})_+ = \partial^3 + (\bar{q}^2 + A(q))u_2\partial + (1 + B(q))u_2^{'} \tag{5.12}$$

is the q-deformed Hamiltonian operator associated with the q-Virasoro algebra. Injecting this expression into eq(5.11), we can extract a non linear differential equation giving the evolution of the q-spin two current u_2. Now identifying the right hand side(r.h.s) and the left hand side(l.h.s) of (5.11), we shall impose some terms of the r.h.s to vanish. We find then the following nonlinear differential equation representing the q-deformed KdV system:

$$\dot{u}_2 = a(q)u_2u_2^{'} + b(q)u_2^{'''}, \tag{5.13}$$

a(q) and b(q) are two constrained, q-dependent coefficients functions, which can be determined by a required of consistency. Simple computations then lead to:

$$\begin{aligned} a(q) &= \frac{1 + \bar{q} + \bar{q}^4}{1 + \bar{q}^2} \\ b(q) &= -\frac{1 + \bar{q} + \bar{q}^2}{(1 + \bar{q})^2} \end{aligned} \tag{5.14}$$

then the q-deformed KdV system is [32]:

$$\dot{u}_2 = \frac{1 + \bar{q} + \bar{q}^4}{1 + \bar{q}^2}u_2u_2^{'} - \frac{1 + \bar{q} + \bar{q}^2}{(1 + \bar{q})^2}u_2^{'''} \tag{5.15}$$

In the classical limit ($q = 1$) eq(5.15) is just the well known KdV integrable system [61], [72], given by:

$$\dot{u}_2 = \frac{3}{2}u_2 u_2^{'} - \frac{3}{4}u_2^{'''} \tag{5.16}$$

which is associated to the Hamiltonian differential operator:

$$(L_2^{3/2})_+ = \partial^3 + \frac{3}{2}u_2\partial + \frac{3}{4}u_2^{'}. \tag{5.17}$$

5.1.3 q-Deformed Boussinesq equation

In this section we use the same technique developed in the case of q-deformed KdV system to present a q-generalisation of the Boussinesq integrable hierarchy [61].
Let L_3 be the Lax operator associated with the q-Boussinesq hierarchy as follows:

$$L_3 = \partial^3 + u_2\partial + u_3 \tag{5.18}$$

where u_2 and u_3 are two currents of conformal spin 2 and 3 respectively.
Knowing that $(L_3^{1/3})^3 = L_3$ and the fact that $L_3^{1/3}$ is an object of conformal spin 1, we can set:

$$L_3^{1/3} = \partial + Au_2\partial^{-1} + (Bu_3 - Cu_2^{'})\partial^{-2} + (Du_2^{''} - Eu_2^2 - Fu_3^{'})\partial^{-3} + \dots \tag{5.19}$$

where the coefficients A, B, C, D, E and F are given explicitly by:

$$\begin{aligned}
A &= \frac{1}{1+\bar{q}^2+\bar{q}^4}\\
B &= \frac{1}{1+\bar{q}^3+\bar{q}^6}\\
C &= \frac{1+\bar{q}^2+\bar{q}^3}{(1+\bar{q}^2+\bar{q}^4)(1+\bar{q}^3+\bar{q}^6)}\\
D &= \frac{1}{(1+\bar{q}^2+\bar{q}^4)(1+\bar{q}^4+\bar{q}^8)}\left(\frac{(1+\bar{q}^2+\bar{q}^3)(1+\bar{q}^3+\bar{q}^4)}{1+\bar{q}^3+\bar{q}^6} - 1\right)\\
E &= \frac{1+\bar{q}+q^2}{(1+\bar{q}^2+\bar{q}^4)^2}\\
F &= \frac{1+\bar{q}^3+\bar{q}^4}{1+\bar{q}^3+\bar{q}^6},
\end{aligned} \tag{5.20}$$

for which we have

$$(L_3^{1/3})_+ = \partial \tag{5.21}$$

Comparing the r.h.s and l.h.s of the following expression:

$$\frac{\partial L_3}{\partial t_1} = [(L_3^{1/3})_+, L_3]_q, \tag{5.22}$$

we obtain

$$\begin{aligned} u_2' &= \dot{u}_2 \\ u_3' &= \dot{u}_3, \end{aligned} \tag{5.23}$$

which give the chiral wave equations for the Boussinesq hierarchy.
Similarly the identification of

$$\frac{\partial L_3}{\partial t_2} = [(L_3^{2/3})_+, L_3]_q$$

with

$$(L_3^{2/3})_+ = \partial^2 + A(\bar{q}^2 + 1)u_2$$

gives

$$\begin{aligned} \dot{u}_3 &= u_3'' + A(1+\bar{q}^2)(\alpha u_2''' + \beta u_2 u_2') \\ \dot{u}_2 &= u_2''(1 + \alpha A\bar{q}^2(1+\bar{q}^2)(1+\bar{q}+\bar{q}^2)) + \bar{q}^3(1+\bar{q})u_3' \\ \bar{q}^3 u_3'' &= (1+\bar{q})(1+\alpha A\bar{q}^2(1+\bar{q}^2))u_2''' \end{aligned} \tag{5.24}$$

α and β are two arbitrary functions of the parameter q, which can be conveniently fixed in such a way that $\alpha = \beta = -1$ in the classical limit.
Combining the first and third equations of (5.24) we find:

$$\dot{u}_3 = -q^2(1 + q + A\alpha\bar{q}(1+\bar{q}^2))u_2''' + A\beta(1+\bar{q}^2)u_2 u_2'. \tag{5.25}$$

Note that we can write the third equation of (5.24) as:

$$u_2''' = \frac{\bar{q}^3}{(1+\bar{q})(1+A\alpha\bar{q}^2(1+\bar{q}^2))}u_3'' \tag{5.26}$$

which imply using the second equation of (5.24) that

$$\dot{u}_2 = b(q,\alpha)u_3', \tag{5.27}$$

with:

$$b(q,\alpha) = \bar{q}^3(1+\bar{q}) + \frac{\bar{q}^3(1+A\alpha\bar{q}^2(1+\bar{q}^2)(1+\bar{q}+\bar{q}^2))}{(1+\bar{q})(1+A\alpha\bar{q}^2(1+\bar{q}^2))} \tag{5.28}$$

The two equations (5.25) and (5.27) represent the q-deformed Boussinesq equations. Setting $q = 1$ we recover the classical Boussinesq equation namely [61]

$$\begin{aligned} \dot{u}_2 &= \frac{7}{2}u_3^{'} \\ \dot{u}_3 &= -\frac{4}{3}u_2^{'''} - \frac{2}{3}u_2 u_2^{'} \end{aligned} \tag{5.29}$$

Then the q-deformed Boussinesq equations (5.25) and (5.27) can be written in the simple form using the algebraic computations (5.25) and (5.29) as:

$$\ddot{u}_2 = b(q,\alpha)(x_1 u_2^{''} + \frac{x_2}{1+\bar{q}^2}(u_2^2))^{''}, \tag{5.30}$$

where x_1 and x_2 are given by:

$$\begin{aligned} x_1 &= -\bar{q}^2(1 + q + A\alpha\bar{q}(1+\bar{q}^2)) \\ x_2 &= A\beta(1+\bar{q}^2) \end{aligned} \tag{5.31}$$

For $q = 1$, we obtain the standard Boussinesq equation namely:

$$\ddot{u}_2 = 2u_2^{''''} + \frac{1}{2}(u_2^2)^{''}. \tag{5.32}$$

5.2 q-Generalised conformal transformations

In this section, we consider two particular examples discussed previously, namely the q-KdV and q-Boussinesq integrable models described respectively by $L_2(u)$ and $L_3(u)$.

5.2.1 Conformal transformation of q-KdV hierarchy

Here we consider the Lax operator of the KdV hierarchy discussed in section 1:

$$L_2 = \partial^2 + u_2, \tag{5.33}$$

to show how the spin 2-conformal current $u_2(z)$ transforms under a conformal transformation:

$$z \longrightarrow \widetilde{z} = f(z). \tag{5.34}$$

Under such a transformation, we assume that the q-deformed KdV Lax operator defined in eq(5.33), transforms as [62], [73]:

$$L_2(u(z)) \longrightarrow \widetilde{L}_2(\widetilde{u}_2(z)) = \psi^{3/2} L_2(u_2(z))\psi^{1/2} \tag{5.35}$$

where $\psi = \frac{\partial z}{\partial \widetilde{z}}$. The ψ-powers given in this equation is dictated by the fact that $L_2(u_2(z))$ maps from the densities of degree $(\frac{1}{2})$ to densities of degree $(+\frac{3}{2})$.
The transformation law of the derivative ∂ is as:

$$\partial \longrightarrow \widetilde{\partial} = \psi \partial, \tag{5.36}$$

which gives for $\widetilde{\partial}^2$:

$$\widetilde{\partial}^2 = \psi\psi'\partial + \psi^2\partial^2. \tag{5.37}$$

Using straightforward computations, we find:

$$\psi^{3/2}L_2(u_2(z))\psi^{1/2} = \psi^2\partial^2 + \frac{1}{2}(1+\bar{q})\psi'\psi\partial + \psi^2 u_2 + \frac{1}{2}(\psi''\psi - \frac{1}{2}\bar{q}(\psi')^2), \tag{5.38}$$

from which we can derive the following result

$$\widetilde{L}_2(\widetilde{u}_2(\widetilde{z})) = \widetilde{\partial}^2 + \frac{\bar{q}-1}{2}\psi'\widetilde{\partial} + \widetilde{u}_2 \tag{5.39}$$

This results show how the conformal transformation violates the standard covariantisation property in the case of the q-Lax operators. This property, we recover in the classical case by taking $q = 1$, since the coefficient term of ∂ in eq(5.39) vanishes as is proportional to $\frac{\bar{q}-1}{2}$.
Identifying the two equations (5.38)and (5.39), we obtain the following conformal transformation of the field $u_2(z)$:

$$u_2(z) = \psi^{-2}\widetilde{u}_2(\widetilde{z}) - \frac{1}{2}S^{(2)}_{u_2}(q,\psi), \tag{5.40}$$

where $S^{(2)}_{u_2}(q,\psi)$ denote the q-Shwarzian derivative associated with the current u_2 and defined as:

$$S^{(2)}_{u_{(2)}}(q,\psi) = \frac{\psi''}{\psi} - \frac{1}{2}\bar{q}(\frac{\psi'}{\psi})^2, \tag{5.41}$$

here the upper index in $S^{(2)}_{u_2}$ stands for the order of the q-KdV hierarchy.
From the equation (5.40), we can see that $u_2(z)$ transform as a field of conformal spin two, up to an anomalous term $S^{(2)}_{u_2}(q,\psi)$ exactly like the energy momentum tensor of two dimensional conformal fields theories.

5.2.2 Conformal transformation of Boussinesq hierarchy

Now we consider the q-Boussinesq hierarchy associated with the q-deformed Lax operator:

$$L_3(u_2,u_3) = \partial^3 + u_2\partial + u_3. \tag{5.42}$$

Similarly, the conformal transformation eq(5.34) implies in this case:

$$L_3(u_2,u_3) \longrightarrow \widetilde{L}_3(\widetilde{u}_2,\widetilde{u}_3) = \psi^2 L_3(u_2,u_3)\psi. \tag{5.43}$$

Using strightforward computation, we find:

$$\begin{aligned}\psi^2 L_3(u_2,u_3)\psi &= \psi^3\partial^3 + (1+\bar{q}+\bar{q}^2)\psi^2\psi'\partial^2 + ((1+\bar{q}+\bar{q}^2)\psi^2\psi'' + u_2\psi^3)\partial \\ &+ u_3\psi^3 + u_2\psi^2\psi' + \psi^2\psi'''\end{aligned} \tag{5.44}$$

which means that:

$$\widetilde{L}_3(\widetilde{u}_2,\widetilde{u}_3) = \widetilde{\partial}^3 + (\bar{q}^2-1)\psi'\widetilde{\partial}^2 + \widetilde{u}_2\widetilde{\partial} + \widetilde{u}_3. \tag{5.45}$$

Comparing the equations (5.44) and (5.45), we obtain the following expression of u_2 and u_3:

$$\begin{aligned}u_2 &= \psi^{-2}\widetilde{u}_2 - S^{(3)}_{u_2}(q,\psi) \\ u_3 &= \psi^{-3}\widetilde{u}_3 - \frac{\psi'}{\psi}\widetilde{u}_2 - S^{(3)}_{u_3}(q,\psi)\end{aligned} \tag{5.46}$$

where $S^{(3)}_{u_2}(q,\psi)$ and $S^{(3)}_{u_3}(q,\psi)$ are the q-Shwarzian derivatives associated respectively to the currents u_2 and u_3 such that:

$$\begin{aligned}S^{(3)}_{u_2}(q,\psi) &= \bar{q}^2(\frac{\psi'}{\psi})^2 - \bar{q}(\bar{q}+1)\frac{\psi''}{\psi} \\ S^{(3)}_{u_3}(q,\psi) &= \frac{\psi'''}{\psi} + \frac{\psi'}{\psi}S^{(3)}_{u_2}(q,\psi).\end{aligned} \tag{5.47}$$

Note that $S^{(3)}_{u_2}(q,\psi)$ and $S^{(3)}_{u_3}(q,\psi)$ are related by the following formula:

$$\partial S^{(3)}_{u_2} + \bar{q}(\bar{q}+1)S^{(3)}_{u_3} = 0 \tag{5.48}$$

Putting $q=1$, we obtain the standad expressions given by [62], [73]:

$$\begin{aligned}u_2 &= \psi^{-2}\widetilde{u}_2 - S^{(3)}_{u_2}(1,\psi) \\ u_3 &= \psi^{-3}\widetilde{u}_3 - \frac{\psi'}{\psi}\widetilde{u}_2 - S^{(3)}_{u_3}(1,\psi) \\ S^{(3)}_{u_2}(1,\psi) &= (\frac{\psi'}{\psi})^2 - 2\frac{\psi''}{\psi} \\ S^{(3)}_{u_3}(1,\psi) &= \frac{\psi'''}{\psi} + \frac{\psi'}{\psi}S^{(3)}_{u_2}(1,\psi) \\ 0 &= \partial S^{(3)}_{u_2} + 2S^{(3)}_{u_3}\end{aligned} \tag{5.49}$$

5.2.3 Conformal transformation of $u_n(z)$

Having given explicitly the conformal transformation of the currents u_2 and u_3 of conformal spin 2 and 3 respectively, we focus now to generalise these results to higher conformal spin currents $u_n(z)$, with $n = 2, 3, \ldots$.
Let $L_n(u)$ be the higher order Lax operator involving $(n-1)$ conformal currents defined by:

$$L_n(u) = \partial^n + \sum_{i=0}^{n-2} u_{n-i}\partial^i, \tag{5.50}$$

with $u_0 = 1$ and $u_1 = 0$ and where $\partial = \partial_q$.
Under the conformal transformation eq(5.34), this Lax operator is assumed to transform as:

$$L_n(u) \longrightarrow \widetilde{L}_n(\widetilde{u}) = \psi^{\frac{n+1}{2}} L_n(u) \psi^{\frac{n-1}{2}}. \tag{5.51}$$

Similarly to the previous study, the structure of the Lax operator $L_n(u)$ takes the following generale conformal transformation:

$$\widetilde{L}_n(\widetilde{u}) = \widetilde{\partial}^n + A\psi'\widetilde{\partial}^{n-1} + \sum_{i=0}^{n-2} \widetilde{u}_{n-i}\widetilde{\partial}^i, \tag{5.52}$$

where A is an arbitrary Lorentz scalar function which we will precise.
Starting from eq(5.36) and using simple algebraic manipulation, we find that $\widetilde{\partial}^n$ can be written in the form:

$$\widetilde{\partial}^n = \sum_{i=1}^{n} M_i^n \partial^i, \tag{5.53}$$

where M_i^n are functions of conformal spin $(n-i)$, which we can summarize as:

$$\begin{aligned} M_n^n &= \psi^n \\ M_1^n &= \psi\partial M_1^{n-1} \\ M_i^n &= \psi[M_{i-1}^{n-1}\bar{q}^{n-i} + \partial M_i^{n-1}]. \qquad 2 \le i \le n-1 \end{aligned} \tag{5.54}$$

Substituting these relations into eq(5.52) we find:

$$\widetilde{L}_n = \sum_{i=0}^{n} X_i(A, M, \psi)\partial^{n-i}, \tag{5.55}$$

where

$$X_i(A, M, \psi) = \sum_{j=0}^{i} \widetilde{u}_j M_{n-i}^{n-j} \tag{5.56}$$

Simple algebraic calculation lead to

$$\psi^{\frac{n+1}{2}} L_n(u)\psi^{\frac{n-1}{2}} = \psi^{\frac{n+1}{2}} \sum_{i=0}^{n}(\sum_{j=0}^{i} u_{i-j} X^{j}_{n-j+i}(q)(\psi^{\frac{n-1}{2}}))\partial^{n-i}. \tag{5.57}$$

Comparing the two relations (5.52) and (5.57) we obtain:

$$A(q,\psi) = \frac{\psi^{1-n}}{\psi'}[\chi^1_n(q)\psi^{\frac{n+1}{2}}(\psi^{\frac{n-1}{2}})' - M^n_{n-1}], \tag{5.58}$$

with

$$\begin{aligned} \chi^1_n(q) &= \sum_{i=0}^{n-1} \bar{q}^i \\ \chi^0_n &= \chi^n_n = 1, \end{aligned} \tag{5.59}$$

and

$$u_i = \psi^{-n}(M^n_{n-i} + \sum_{j=1}^{i}[\widetilde{u}_j M^{n-j}_{n-i} - \psi^{\frac{n+1}{2}} u_{i-j}\chi^j_{n-i+j}(q)(\psi^{\frac{n-1}{2}})^{(j)}]), 0 \leq i \leq n \tag{5.60}$$

The equation (5.60) represent the general transformation of the conformal currents $u_i \quad i \geq 2$.
To illustrate the obtained results, we consider the two particular q-KdV and q-Boussinesq hierarchy discussed previously and described respectively by $L_2(u)$ and $L_3(u)$.
The former is easily obtained by setting $n = 2$ into the equations (5.58) and (5.60), which recover the relations (5.40) and (5.41) with

$$A = \frac{\bar{q}-1}{2}.$$

Similarly eqs(5.46) and (5.47) are obtained by taking $n = 3$ in eqs (5.57) and (5.58) with

$$A = \bar{q}^2 - 1.$$

Chapter 6

q-w-Currents

In the last chapter we have generalised the conformal transformation to the q-deformed case, we found in addition to new features, the presence of anomalous terms at the level of the conformal currents $u_3, u_4, \ldots, u_n$.
Our idea in this chapter, is to consider the Volterra gauge group transformation associated to an orbit in which no such anomalous terms can appear. Our aim is then to make an appropriate choice on the Volterra parameters a_i such that w_i become primary conformal currents satisfying the last conditions given in chapter 5. In the classical limit, the analytic field u_2 behaves as spin 2-field of 2D conformal field theory which coincide with the w_2 current. Similarly in the q-deformed case; we will require for w_2 to be proportional to u_2.

6.1 Volterra gauge group transformation

The Volterra gauge group symmetry is a symmetry group whose typical elements are given by the Lorentz scalar q-pseudo-differential operators [74]

$$K[a] = 1 + \sum_{i\geq 1} a_i(z)\partial^i, \tag{6.1}$$

where $a_i(z)$ are arbitrary analytic functions of conformal spin $i = 1, 2, 3, \ldots$.
These functions, which are the Volterra gauge parameters can be expressed in term of the residue operation as:

$$a_i(z) = Res(K(a)\partial^{i-1}), \tag{6.2}$$

with

$$Res\partial^i = \delta_{i+1,0} \tag{6.3}$$

Now, we apply this Volterra gauge group symmetry to the algebra of q-Lax operators given by the relation:

$$L_n(u) \longrightarrow L_n(w) = K^{-1}(a)L_n(u)K(a) \tag{6.4}$$

$L_n(w)$ is the transform of $L_n(u)$ under the Volterra group action, with $w = w(a, u)$ is a function depending on the Volterra parameter a_i and the u-fields.
Note that from eq(6.4), we can see that the u-currents may be expressed in terms of the Volterra gauge parameters a_i and their kth derivatives. Solving also this equation we find that the new fields w_i are plynomials of u-fields, the Volterra parameters and their derivatives.
For an appropriate choices of the Volterra parameters dictated by the primary condition, the w-fields can then be written in terms of the u fields exactly as do the primary w-currents which satisfy [73]:

$$w_s(z) = \psi^{-s}\widetilde{w}_s(\widetilde{z}). \tag{6.5}$$

Next, we solve the equation (6.4) for the Lax operator of the q-Boussinesq integrable system

$$L_3(u) = \partial^3 + u_2\partial + u_3, \tag{6.6}$$

which represent the special case $n = 3$. Applying the Volterra gauge group symmetry eq(6.4) to eq(6.6), we have:

$$K(a)L_3(w) = L_3(u)K(a). \tag{6.7}$$

The algebraic calculation lead to the following formulas of the first parameters a_1, a_2, a_3, a_4:

$$\begin{aligned}
\bar{q}^3 a_1 &= a_1 \\
a_2 + w_2 &= u_2 + \bar{q}^6 a_2 + \bar{q}^2(1+\bar{q}+\bar{q}^2)a_1^{'} \\
a_3 + w_3 + q^2 a_1 w_2 &= u_3 + \bar{q}^9 a_3 + \bar{q} a_1 u_2 + \bar{q}^4(1+\bar{q}+\bar{q}^2)a_2^{'} \\
&+ \bar{q}(1+\bar{q}+\bar{q}^2)a_1^{''} \\
a_4 + q^3 a_1 w_3 - q^5 a_1 w_2^{'} + q^4 a_2 w_3 &= a_1 u_3 + \bar{q}^2 a_2 u_2 + \bar{q}^{12} a_4 + \bar{q}^2(1+\bar{q}+\bar{q}^2)a_2^{''} \\
&+ \bar{q}^6(1+\bar{q}+\bar{q}^2)a_3^{'} + a_1^{'''} + a_1^{'} u_2.
\end{aligned} \tag{6.8}$$

This equation show that the spin-1 Volterra gauge parameter a_1 vanishes naturally for an arbitrary values of the parameter q. This allow us to set:

$$\begin{aligned}
a_1 &= 0 \\
(1-\bar{q}^6)a_2 &= u_2 - w_2 \\
(1-\bar{q}^9)a_3 &= u_3 - w_3 + \bar{q}^4(1+\bar{q}+\bar{q}^2)a_2^{'} \\
(\bar{q}^9 - 1)a_4 &= q^4 a_2 w_3 - \bar{q}^2(1+\bar{q}+\bar{q}^2)a_2^{''} - \bar{q}^6(1+\bar{q}+\bar{q}^2)a_3^{'} - (a_1^{'} + \bar{q}^2 a_2)u_2.
\end{aligned} \tag{6.9}$$

Note that when $q = 1$, we recover from this relations the Volterra gauge orbit $K_{q=1}\{a_i\}$ in which the w_i- fields are seen as primary currents [74].
In the general case the Volterra parameters $a_j, \quad j \geq 2$ satisfy the formula:

$$\begin{aligned}
a_{j+3}(\bar{q}^{3(j+3)} - 1) &= a_1(-1)^{j-1}q^{3j+j(\frac{j-1}{2})}w_3^{(j-1)} + a_1(-1)^j q^{2(j+1)+j(\frac{j+1}{2})}w_2^{(j)} \\
&+ \sum_{i=0}^{\infty} a_{j-i}q^{3j+i(\frac{i+1}{2})}(\sum_{k_1=0}^{i}\cdots\sum_{k_{j-i}=0}^{k_{j-i-1}} q^{\sum_{m=1}^{j-1}k_m})w_3^{(i)} \\
&+ \sum_{i=0}^{\infty} a_{j-i+1}q^{2(j+1)+i(\frac{i+1}{2})}(\sum_{k_1=0}^{i}\cdots\sum_{k_{j-i}=0}^{k_{j-i-1}} q^{\sum_{m=1}^{j-1}k_m})w_2^{(i)} \qquad (6.10) \\
&- \bar{q}^{j+1}(1+\bar{q}+\bar{q}^2)a''_{j+1} - \bar{q}^{j+1}a_{j+1}u_2 - a'''_j \\
&- \bar{q}^{2(j+2)}(1+\bar{q}+\bar{q}^2)a'_{j+2} - a'_j u_2 - a_j u_3.
\end{aligned}$$

6.2 q-w-Currents

Our aim in this section is to make an appropriate choice of the Volterra parameters a_i such that w_i become primary conformal currents satisfying eq(6.5).
Recall that in the classical limit, the analytic field u_2 behaves as spin 2-field of 2D conformal field theory which coincide with the w_2 current. Similarly in the deformed case; we can require for w_2 to be proportional to u_2.
Using the second equation of (6.9), we set:

$$a_2 = \delta u_2 \qquad (6.11)$$

where δ is an arbitrary constant. Then, we have:

$$w_2 = u_2(1 - \delta(1 - \bar{q}^6)) \qquad (6.12)$$

substituting this equation into the third equation of (6.9), we obtain:

$$a_3 = \beta_1 u_3 + \beta_2 u'_2 \qquad (6.13)$$

here β_1 and β_2 are arbitrary constant which can be fixed.
The resulting expression for the q-deformed w-current of spin 3 is

$$w_3 = u_3[1 + (\bar{q}^9 - 1)\beta_1] + u'_2[\bar{q}^4(1 + \bar{q} + \bar{q}^2)\delta + \beta_2(\bar{q}^9 - 1)], \qquad (6.14)$$

with the constraints equation (6.10) giving the remaining Volterra parameters $a_j, \quad j \geq 5$.

$$
\begin{aligned}
a_4(\bar{q}^9 - 1) &= q^4 a_2 w_3 - \bar{q}^2(1+\bar{q}+\bar{q}^2)a_2^{''} - \bar{q}^6(1+\bar{q}+\bar{q}^2)a_3^{'} - (a_1^{'} + \bar{q}^2 a_2)u_2 \\
a_{j+3}(\bar{q}^{3(j+3)} - 1) &= \sum_{i=0}^{3} a_{j-i} q^{3j+i(\frac{i+1}{2})} (\sum_{k_1=0}^{i} \cdots \sum_{k_{j-i}=0}^{k_{j-i-1}} q^{\sum_{m=1}^{j-1} k_m}) w_3^{(i)} \\
&+ \sum_{i=0}^{3} a_{j-i+1} q^{2(j+1)+i(\frac{i+1}{2})} (\sum_{k_1=0}^{i} \cdots \sum_{k_{j-i}=0}^{k_{j-i-1}} q^{\sum_{m=1}^{j-1} k_m}) w_2^{(i)} \qquad (6.15) \\
&- \bar{q}^{j+1}(1+\bar{q}+\bar{q}^2)a_{j+1}^{''} - \bar{q}^{j+1} a_{j+1} u_2 - a_j^{'''} \\
&- \bar{q}^{2(j+2)}(1+\bar{q}+\bar{q}^2)a_{j+2}^{'} - a_j^{'} u_2 - a_j u_3.
\end{aligned}
$$

We consider now the conformal transformation of the spin-3 w-current (6.14) such that:

$$\widetilde{w}_3 = \psi^3 w_3 + y_3 \qquad (6.16)$$

where y_3 is a function of conformal spin 3 given by:

$$
\begin{aligned}
y_3 &= \psi^2 \psi^{'} \{1 + 2\bar{q}^4(1+\bar{q}+\bar{q}^2)\delta + (\bar{q}^9 - 1)\beta_1 + 2(\bar{q}^9 - 1)\beta_2\} u_2 \\
&+ \psi^3 \{(S_{u_3}^{(3)} - \frac{\psi^{'}}{\psi} S_{u_2}^{(3)}) - (2\frac{\psi^{'}}{\psi} S_{u_2}^{(3)} + \partial S_{u_2}^{(3)})\bar{q}^4(1+\bar{q}+\bar{q}^2)\delta \qquad (6.17) \\
&+ (\bar{q}^9 - 1)(S_{u_3}^{(3)} - \frac{\psi^{'}}{\psi} S_{u_2}^{(3)})\beta_1 - (\bar{q}^9 - 1)(2\frac{\psi^{'}}{\psi} S_{u_2}^{(3)} + \partial S_{u_2}^{(3)})\beta_2\}
\end{aligned}
$$

Imposing the primarity conditions (6.5) imply the vanishing of y_3. This results can derive a solution for the constant $\delta(q), \beta_1(q)$ and $\beta_2(q)$ which are required to coincide in the classical limit with $\delta(1) = -1/6, \beta_1(1) = 0$ and $\beta_2(1) = 1/6$ respectively.

Chapter 7

Field and superfield theory

In this chapter, we will discuss the field theoretical models describing the self couplings of the matter multiplets $(0^2, (\frac{1}{3})^2, (\frac{2}{3})^2)$ and $(0^4, (\frac{1}{3})^4, (\frac{2}{3})^4)$. More precisely, we will describe brefly the superfield theory of the matter couplings of $(\frac{1}{3}, \frac{1}{3})$ and $((\frac{1}{3})^2, (\frac{1}{3})^2)$ fractional superalgebras studied in chapter 3. We start first by describing the superfield theory of the $2D(\frac{1}{3}, \frac{1}{3})$, fractional supersymmetry equation already studied in chapter 3, especially the matter coupling of the on shell scalar representation $(\varphi, \Psi_{\pm 1/3}, \Psi_{\pm 2/3})$, using the formal analogy between equations (3.32) and those of $2DN = 2U(1)$ supersymmetry. In the second step we will give The action $S[\Phi]$ describing the dynamics and the couplings of the superfields Φ which is similar to that of $2D((\frac{1}{2})^4, 0)su(2)$ harmonic superspace. More generally the matter couplings of the $((\frac{1}{3})^2, (\frac{1}{3})^2)su(2)$ fractional supersymmetry, extending equations (3.32) by adjoining the analytic part, give us the action S by help of the harmonic formulation.
The end of this chapter, is devoted to the construction of the q-deformed Euler Lagrange equations using the results obtained in chapters 4,5 and 6. We will present here also the $su(n)$-Toda ($su(2)$- Liouville) field theory construction by building the q-analogue of the $su(2)$- Liouville and $su(n)$- Toda conformal field theories.

7.1 Superfield theory

We start first by describing the superfield theory of the $2D(\frac{1}{3}, \frac{1}{3})$, fractional supersymmetry equation (3.32) especially the matter coupling of the on shell scalar representation $(\varphi, \Psi_{\pm 1/3}, \Psi_{\pm 2/3})$. Using the formal analogy between equations (3.32) and those of $2D$ $N = 2$ $U(1)$ supersymmetry, namely:

$$\begin{array}{rcl} Q_{\mp 1/3} & \longrightarrow & Q^{+}_{\mp 1/2} \\ \theta_{\pm 1/3} & \longrightarrow & \theta^{+}_{\pm 1/2} \\ Q_{\mp 2/3} & \longrightarrow & Q^{-}_{\mp 1/2} \\ \theta_{\pm 2/3} & \longrightarrow & \theta^{-}_{\pm 1/2} \\ \varphi & \longrightarrow & \varphi(y) \\ \Psi_{\mp 1/3} & \longrightarrow & \Psi^{+}_{\mp 1/2} \\ \Psi_{\mp 2/3} & \longrightarrow & \Psi^{-}_{\mp 1/2} \end{array} \tag{7.1}$$

and following the same lines used in the building of $2D$ $N = 2$ $U(1)$ supersymmetric matter coupling [75], one sees that the superfield action $S[\Phi, \bar{\Phi}]$ invariant under the $2D(\frac{1}{3}, \frac{1}{3})$ fractional supersymmetry reads as:

$$S[\Phi, \bar{\Phi}] = \int d^2 z d\theta_{1/3} d\theta_{-1/3} d\theta_{2/3} d\theta_{-2/3} K[\Phi, \bar{\Phi}]. \tag{7.2}$$

In this equation K is the kahler potential depending on the chiral superfields Φ and $\bar{\Phi}$ given by:

$$\begin{array}{rcl} \Phi & = & \varphi + \theta_{1/3}\Psi_{-1/3} + \theta_{-1/3}\Psi_{1/3} + \theta_{1/3}\theta_{-1/3}F \\ \bar{\Phi} & = & \bar{\varphi} + \theta_{2/3}\Psi_{-2/3} + \theta_{-2/3}\Psi_{2/3} + \theta_{2/3}\theta_{-2/3}F \end{array} \tag{7.3}$$

Φ and $\bar{\Phi}$ describe a complex scalar representation of the algebra (3.32); each bosonic degree of freedom has two partners of spin $\frac{1}{3}$ and $\frac{2}{3}$. Note that using equation (7.2) and the matter superfield representation equation (7.3), we get the right two point free correlation functions, i.e: $< \Psi_{-1/3}(z_1), \Psi_{-2/3}(z_2) >$ and $< \Psi_{-1/3}(z_1), \Psi_{-1/3}(z_2) >=< \Psi_{-2/3}(z_1), \Psi_{-2/3}(z_2) >= 0$.
Concerning the $((\frac{1}{3})^2, 0)$ fractional supersymmetric algebra (3.34-35) generated by the four generators $Q^{\pm}_{-1/3}$ and $Q^{\pm}_{-2/3}$, one may use here also the similarity with $2D$ $N = 4$ extended supersymmetry. Thus extending equations (3.34-35) as:

$$\begin{array}{rcl} P_{-1} & = & \{D^{+}_{-2/3}, D^{-}_{-1/3}\} \\ & = & -\{D^{+}_{-1/3}, D^{-}_{-2/3}\} \\ 0 & = & \{D^{\pm}_{-x}, D^{\pm}_{-y}\}, \end{array} \tag{7.4}$$

together with

$$\begin{array}{rcl} [D^{++}, D^{+}_{-x}] & = & [D^{--}, D^{-}_{-x}] = 0; \quad x = \frac{1}{3}, \frac{2}{3} \\ {[D^{++}, D^{-}_{-x}]} & = & D^{+}_{-x} \\ {[D^{++}, D^{--}]} & = & D^{0} \\ {[D^{0}, D^{\pm}_{-x}]} & = & \pm D^{\pm}_{-x}, \end{array} \tag{7.5}$$

and using the Grassmann variables $\theta^{\pm}_{1/3}$ and $\theta^{\pm}_{2/3}$ of spin $\frac{1}{3}$ and $\frac{2}{3}$ respectively, but still satisfying ${\theta^{\pm}_x}^2 = 0$ one can build a superspace realisation of equations (7.4-5) and consequently a 2D quantum superfield theory. A remarkable realisation of the algebra (7.4-5), stable under the combined conjugation $(+*)$ and using the covariant superderivatives $D^{\pm}_x$ and $D^{\pm\pm}_x$, D^0_0 instead of the $Q^{\pm}_{-x}$ generators is given by:

$$\begin{aligned} D^{+}_{-1/3} &= -\frac{\partial}{\partial\theta^{-}_{-1/3}} \\ D^{+}_{-2/3} &= -\frac{\partial}{\partial\theta^{-}_{-2/3}} \\ D^{-}_{-2/3} &= \frac{\partial}{\partial\theta^{+}_{2/3}} - \theta^{-}_{1/3}P_{-1} \\ D^{-}_{-1/3} &= \frac{\partial}{\partial\theta^{+}_{1/3}} - \theta^{-}_{2/3}P_{-1} \\ P_{-1} &= \frac{\partial}{\partial_y}, \end{aligned} \tag{7.6}$$

where

$$y = z - \frac{1}{2}(\theta^{-}_{1/3}\theta^{+}_{2/3} + \theta^{+}_{1/3}\theta^{-}_{2/3})$$

and where:

$$\begin{aligned} D^{++} &= [u^{+i}\frac{\partial}{\partial u^{-i}} - \theta^{+}_{1/3}\theta^{+}_{2/3}P_{-1}] \\ D^{--} &= [u^{-i}\frac{\partial}{\partial u^{+i}} - \theta^{-}_{1/3}\theta^{-}_{2/3}P_{-1}] \\ D^{0} &= [D^{++}, D^{--}]. \end{aligned} \tag{7.7}$$

In this realisation $u^{\pm}_i$ are the well known harmonic variable stisfying $u^{+i}u^{-}_i = 1$ and $u^{\pm i}u^{\pm}_i = 0$. Using the realisation (7.6-7), the on shell matter multiplet $(0^4, (\frac{1}{3})^4, (\frac{2}{3})^4)$ of the fractional supersymmetric algebra (7.4-5) is given by a Hermitian superfield satisfying the analytic conditions:

$$D^{+}_{-1/3}\Phi = D^{+}_{-2/3}\Phi = 0, \tag{7.8}$$

and the equation of motion

$${D^{++}}^2\Phi = 0 \tag{7.9}$$

The θ^{+}_x expansion of the superfield Φ, stable under the combined conjugation $(+*)$ reads as:

$$\Phi = \varphi^{ij}u^{+}_i u^{-}_j + \theta^{+}_{1/3}\Psi^{i}_{-1/3}u^{-}_i + \theta^{+}_{2/3}\Psi^{i}_{-2/3}u^{-}_i + \theta^{+}_{1/3}\theta^{+}_{2/3}F^{ij}u^{-}_i u^{-}_j. \tag{7.10}$$

The action of $S[\Phi]$ describing the dynamics and the couplings of the superfields Φ is similar to that of $2D((\frac{1}{2})^4, 0)su(2)$ harmonic superspace. More generally the matter couplings of the $((\frac{1}{3})^2, (\frac{1}{3})^2)su(2)$ fractional supersymmetry, extending equations (7.4-5) by adjoining the analytic part, read in the harmonic formulation [58], as:

$$S[\Phi] = \int d^2 z d^2 \theta^+_{\pm 1/3} d^2 \theta^+_{\mp 2/3} du L^{+4}(\Phi) \tag{7.11}$$

where L^{+4} is the functional of the superfield Φ.

7.2 q-Deformed $su(2)$ Liouville field theory

The aim of this section is to set up some crucial ingredients towards building the q-deformed analogue of the 2D $su(2)$ Liouville field theory, using the previous analysis studied in chapter 4, 5 and 6. Note also that there exist a correspondence between the second Hamiltonian structure of integrable systems and the Virasoro conformal algebra representing the symmetry of 2D Liouville field theory.

7.2.1 q-Deformed spin-2 current u_2

Let consider the integrable q-KdV equation, discussed in chapter 5, which take the following expression:

$$\dot{u}_2 = \frac{1+\bar{q}+\bar{q}^4}{1+\bar{q}^2} u_2 u_2^{'} - \frac{1+\bar{q}+\bar{q}^2}{(1+\bar{q})^2} u_2^{'''}. \tag{7.12}$$

Applying the Miura transformation connecting the dynamical current u_2 with the scalar field $\phi \equiv \phi(z, \bar{z})$ to the q-deformed KdV Lax operator as follows:

$$L_2 = (\partial^2 + u_2) = (\partial + A)(\partial + B), \tag{7.13}$$

where A and B are spin-1 fields. This two fields satisfy the relation:

$$\begin{aligned} A &= -\bar{q}B \\ AB + B^{'} &= u_2 \end{aligned} \tag{7.14}$$

with $B^{'} = (\partial B)$.
This equations have the solution

$$\begin{aligned} A &= -\partial\varphi \\ B &= q\partial\varphi, \end{aligned} \tag{7.15}$$

from which we conclude that

$$u_2 = q(\partial^2\varphi - (\partial\varphi)^2). \tag{7.16}$$

This relation shows that u_2 is a q-deformed spin two current, which behaves like energy momentum tensor of 2D Liouville conformal field theory.

7.2.2 q-Deformed Euler-Lagrange equations

Consider the q-deformed Liouville action which we write as [65], [69]:

$$S[\varphi] = \int d^2z\{\partial\varphi\bar{\partial}\varphi + \frac{2}{b}exp(b\varphi)\} \tag{7.17}$$

where b is q-dependent coefficients which can be determined using dimensional arguments and conservation of the induced conserved current.
The variational principle applied to the q-Liouville action S reads as:

$$\delta S[\varphi] = 0 \longleftrightarrow \int d^2z\{\frac{\partial L}{\partial\varphi}\delta\varphi + \frac{\partial L}{\partial(\partial\varphi)}(\delta\varphi)\} = 0, \tag{7.18}$$

where the lagrangian is given by

$$L = \partial\varphi\bar{\partial}\varphi + \frac{2}{b}exp(b\varphi); \quad \partial = \partial_q$$

and where the variation δ is required to satisfy $[\delta, \partial]_q = 0$, which means that $\partial\delta = \delta\partial$. Using these remarks and the equation (4.3), we obtain:

$$\begin{aligned} \partial(\frac{\partial L}{\partial(\partial\varphi)}\delta\varphi) &\equiv (\frac{\partial L}{\partial(\partial\varphi)}\delta\varphi)' \\ &= (\frac{\partial L}{\partial(\partial\varphi)})'\delta\varphi + \bar{q}^x\frac{\partial L}{\partial(\partial\varphi)}\partial(\delta\varphi) \end{aligned} \tag{7.19}$$

where x is the conformal dimension of $(\frac{\partial L}{\partial(\partial\varphi)})$, we obtain the q-deformed Euler Lagrange equation given by:

$$\frac{\partial L}{\partial\varphi} - q^x\partial\frac{\partial L}{\partial(\partial\varphi)} = 0 \tag{7.20}$$

for the q-Liouville lagrangian density $L = \partial\varphi\bar{\partial}\varphi + \frac{2}{b}exp(b\varphi)$.
An algebraic computations give (see appendix A):

$$\begin{aligned} \frac{\partial L}{\partial\varphi} &= \frac{2}{b}\frac{\partial exp(b\varphi)}{\partial\varphi} = 2exp(b\varphi) \\ \partial\frac{\partial L}{\partial(\partial\varphi)} &= \partial\bar{\partial}\varphi. \end{aligned} \tag{7.21}$$

Using this relation, we derive the following q-deformed Liouville equation of motion:

$$2exp(b\varphi) - q^x\partial\bar{\partial}\varphi = 0. \tag{7.22}$$

The dimensional arguments show that $x = 1$ as the conformal dimension of the lagrangian is $\widetilde{L} = 2$.
To determine the coefficient constant b, we use the conservation of the q-deformed current eq(7.26), then we have

$$0 = \bar{\partial}T(\varphi) = q\partial(\partial\bar{\partial}\varphi) - q\bar{\partial}(\partial\varphi)^2. \tag{7.23}$$

This equation fixe the coefficient $b = (1 + \bar{q})$ with $\widetilde{\varphi} = 0$ and then:

$$\bar{\partial}(\partial\varphi)^2 = (1 + \bar{q})\partial\varphi\partial\bar{\partial}\varphi \tag{7.24}$$

Finally, we have:

$$\partial\bar{\partial}\varphi - 2\bar{q}e^{(1+\bar{q})\varphi} = 0, \quad \bar{q} = q^{-1} \tag{7.25}$$

Setting $q = 1$, we recover the well known Liouville equation $\partial\bar{\partial}\varphi = 2e^{2\varphi}$ associated to the Liouville Lagrangian $L = \partial\varphi\bar{\partial}\varphi + exp(2\varphi)$.

7.2.3 q-Deformed $su(2)$ Liouville field theory

The q-deformed form of the conserved current can be written as:

$$T(\phi) \equiv q\partial^2\varphi - q(\partial\varphi)^2 \tag{7.26}$$

whose conservation is assured by the equation of motion eq(7.18)

$$\bar{\partial}T(\varphi) = 0. \tag{7.27}$$

Note that this conservation law combined with eq(7.25) fixes the q-coefficient number $b = (1 + \bar{q})$ in the exponential eq(7.17). Some remarks are in order:
1. The action eq(7.17) is invariant and generalize the $su(2)$ standard Liouville theory.
2. As for the standard studies, the coefficient number in the exponential Liouville potential is connected to the Cartan matrix of some simple Lie algebra. An important task is to look for the interpretation of the coefficient b appearing in our exponential, from the Lie algebraic point of view.
3. The coefficient $b = (1 + \bar{q})$ coincide in the classical limit case with the number 2 which is the Cartan matrix of the $su(2)$ Lie algebra.
4. The q-KdV Lax operator defined in eq(7.13), shows the existence of an $su(2)$

symmetry; which can be recovered also from the Liouville action.
If we redefine the scalar field φ to be as:

$$\Phi = \frac{\bar{q}+1}{2}\varphi, \tag{7.28}$$

We can easily read the $su(2)$ symmetry from the Liouville action which becomes as:

$$S[\Phi[= \int d^2z\{\lambda\partial\Phi\bar{\partial}\Phi + \frac{2}{\bar{q}+1}exp(2\Phi)\} \tag{7.29}$$

with λ is a parameter which take the value

$$\lambda = (\frac{\bar{q}+1}{2})^2.$$

Then the q-deformed Liouville equation of motion take the form:

$$\partial\bar{\partial}\Phi - \bar{q}(\bar{q}+1)exp(2\Phi) = 0. \tag{7.30}$$

We can also think to generalize the above q-deformed $su(2)$ Liouville field theory to the $su(n)$ conformal Toda field theory. We set:

$$S_{su(n)Toda} = \int d^2z\{\partial\phi\bar{\partial}\phi + \eta(q)\sum_{i=1}^{n-1} exp(\alpha_i\phi)\} \tag{7.31}$$

where $\phi = \sum_{j=1}^{n-1}\alpha_j\phi_j$ and α_j are the simple root of the $su(n)$ Lie algebra whose Cartan matrix is defined as:

$$K_{ij} = \alpha_i\alpha_j \tag{7.32}$$

and where $\eta(q)$ is a function of the parameter q, which can easily be fixed given the corresponding model in the generalised KdV hierarchy.

Chapter 8

Quantum groups and noncommutative space

Noncommutative geometry is based on noncommutative coordinates $\hat{x}^i$ and $\hat{x}^j$ satisfying the relation $[\hat{x}^i, \hat{x}^j] \neq 0$, and which means that this two coordinates are noncommutative operators. More general the theory of noncommutative geometry is based on the idea of replacing ordinary coordinates with noncommuting operators.
In this chapter, we will formulate some aspects of noncommutative geometry mathematically and we will be mainly concerned with quantum algebra and quantum spaces. We start by studying quantum planes and quantum groups and their differentials calculus on the noncommutative space [76]. Here, we will give some differential relations and as an example we will discuss the Manin plane. In the second step, we will treat the star product of functions and as examples we will take the three type of noncommutative structure namely canonical structure, Lie algebra structure and quantum space structure. We end this chapter by formulating gauge theory on noncommutative space. We will see that this gauge theorie is based on the idea that multiplication of a field by a coordinate or a function is not covariant only if that function does not commute with gauge transformation. This can be resolved by adding an appropriate noncommutative gauge potentials and thus introducing covariant coordinate in analogy to the covariant derivative of ordinary gauge theory [77], [78].

8.1 Quantum planes

In this section we study the quantum planes, which have the following algebraic structure:

$$x^1 x^2 = q x^2 x^1, \tag{8.1}$$

where q is the deformation parameter. In the general case the relation (8.1) is written as:

$$(1-\hat{R})^{ab}_{cd}x^c x^d = 0, \tag{8.2}$$

where $\hat{R}$ is the 4 by 4 matrix which satisfy the Yang Baxter equation

$$\hat{R}_{12}\hat{R}_{13}\hat{R}_{23} = \hat{R}_{23}\hat{R}_{13}\hat{R}_{12}$$

with $(R_{13})^{ijk}_{lmn} = \delta^j_m R^{ik}_{ln}$, $\hat{R}_{12}$ and $\hat{R}_{23}$ have the same relation.
$\hat{R}$ is defined by:

$$\hat{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q^{-1}\lambda & q^{-1} & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{8.3}$$

with $\lambda = q - q^{-1}$ and the rows and columns of $\hat{R}$ are labelled by 11,12,21 and 22. It is easy to verify that this matrix satisfies the characteristic equation:

$$(\hat{R}-1)(\hat{R}+q^{-2}) = 0 \tag{8.4}$$

This means that R̂ has two eigenvalues

$$\begin{aligned} \lambda &= 1 \qquad \mathit{Multiplicity} \quad 3 \\ \lambda &= -q^{-2} \quad \mathit{Multiplicity} \quad 1 \end{aligned} \tag{8.5}$$

The projectors that project on the respective eigenspaces follow from the characteristic equation, and are given by:

$$\begin{aligned} P_S &= \frac{1}{1+q^{-2}}(\hat{R}+q^{-2}) \\ P_A &= \frac{1}{1+q^{-2}}(1-\hat{R}) \\ \hat{R} &= P_S - q^{-2}P_A, \end{aligned} \tag{8.6}$$

where P_S is the q-deformed generalisation of the symmetriser and P_A of the antisymmetriser. Note that for $q=1$, we find the classical symetriser and antisymetriser.
A natural definition of the quantum plan algebra is [79]:

$$(P_A)^{ab}_{cd}x^c x^d = 0. \tag{8.7}$$

Substituting the expression of P_A in equation (8.7), we find:

$$0 = (P_A)^{ab}_{cd}x^c x^d = \frac{1}{1+q^{-2}}(1-\hat{R})^{ab}_{cd}x^c x^d. \tag{8.8}$$

In two dimensions this relation reduce to the equation (8.1).
The symmetry algebra $SU_q(2)$ is generated by T^+, T^- and T^3, which satisfy the following defining relations [80]:

$$\begin{aligned} \frac{1}{q}T^+T^- - qT^-T^+ &= T^3, \\ q^2T^3T^+ - \frac{1}{q^2}T^+T^3 &= (q+\frac{1}{q})T^+, \\ q^2T^-T^3 - \frac{1}{q^2}T^3T^- &= (q+\frac{1}{q})T^-, \end{aligned} \tag{8.9}$$

this algebra contains the following (2j+1) dimensional representations caracterised by [79]:

$$\begin{aligned} T^3|j,m> &= \frac{1}{q}[2m]_{q^{-2}}|j,m> \\ T^+|j,m> &= \frac{1}{q}\sqrt{[j+m+1]_{q^{-2}}[j-m]_{q^2}}|j,m+1> \\ T^-|j,m> &= q\sqrt{[j+m]_{q^{-2}}[j-m+1]_{q^2}}|j,m-1> \\ \vec{T}^2|j,m> &= [j]_{q^{-2}}[j+1]_{q^2}|j,m> \end{aligned} \tag{8.10}$$

where the q-number and the Casimir operator $\vec{T}^2$ are defined by [79]:

$$\begin{aligned} \vec{T}^2 &= q^2(T^-T^+ + \frac{1}{\lambda^2})\tau^{-1/2} + \frac{1}{\lambda^2}(\tau^{1/2} - 1 - q^2) \\ [x]_{q^n} &= \frac{1-q^{nx}}{1-q^n}, \end{aligned} \tag{8.11}$$

with $\tau = 1 - \lambda T^3$. The Casimir operator $\vec{T}^2$ commutes with T^+, T^- and T^3.
For a two dimensional representation $j = \frac{1}{2}$ the relations (8.10) become:

$$\begin{aligned} T^+|\frac{1}{2},-\frac{1}{2}> &= q^{-1}|\frac{1}{2},\frac{1}{2}> \\ T^+|\frac{1}{2},\frac{1}{2}> &= 0 \end{aligned}$$

$$\begin{aligned} T^-|\frac{1}{2},-\frac{1}{2}> &= 0 \\ T^-|\frac{1}{2},\frac{1}{2}> &= q|\frac{1}{2},-\frac{1}{2}> \end{aligned} \tag{8.12}$$

$$
\begin{aligned}
T^3|\frac{1}{2},-\frac{1}{2}> &= -q|\frac{1}{2},-\frac{1}{2}> \\
T^3|\frac{1}{2},\frac{1}{2}> &= q^{-1}|\frac{1}{2},\frac{1}{2}> .
\end{aligned}
$$

We can identify the vectors of a representation with the elements of a quantum plane. The elements of a quantum plane can be multiplied. This product, we can identify with the tensor product of the representations to obtain its transformation properties. Like in the case of the Hopf algebra, The $SU_q(2)$ algebra allows the following coproduct:

$$
\begin{aligned}
\Delta(T^3) &= T^3 \otimes 1 + \tau \otimes T^3 \\
\Delta(T^+) &= T^+ \otimes 1 + \tau^{\frac{1}{2}} \otimes T^+ \\
\Delta(T^-) &= T^- \otimes 1 + \tau^{\frac{1}{2}} \otimes T^- .
\end{aligned}
\tag{8.13}
$$

By identifying $x^1 \sim |\frac{1}{2},-\frac{1}{2}>$ and $x^2 \sim |\frac{1}{2},\frac{1}{2}>$ in eqs(8.12-13), we find the action of the generators T^+, T^- and T^3 on the coordinates x^1 and x^2 as:

$$
\begin{aligned}
T^+x^1 &= qx^1T^+ + \frac{1}{q}x^2 \\
T^+x^2 &= \frac{1}{q}x^2T^+ \\
T^-x^1 &= qx^1T^- \\
T^-x^2 &= \frac{1}{q}x^2T^- + qx^1 \\
T^3x^1 &= q^2x^1T^3 - qx^1 \\
T^3x^2 &= \frac{1}{q^2}x^2T^3 + \frac{1}{q}x^2
\end{aligned}
\tag{8.14}
$$

where

$$
\begin{aligned}
\Delta(T^3)x^i &= \tau(x^i)T^3 + T^3(x^i) \quad i = 1,2 \\
\Delta(T^+)x^i &= \tau(x^i)T^+ + T^+(x^i) \\
\Delta(T^-)x^i &= \tau(x^i)T^+ + T^-(x^i
\end{aligned}
\tag{8.15}
$$

In the classical limit, the symmetry algebra fulfills the usual relations:

$$
\begin{aligned}
[T^+, T^-] &= T^3 \\
[T^3, T^+] &= 2T^+ \\
[T^-, T^3] &= 2T^- .
\end{aligned}
\tag{8.16}
$$

8.2 Quantum groups

Quantum groups are a generalisation of the concept of groups. More precisely, a quantum group is a deformation of a group that, for particular values of the deformation parameter, coincides with the group.
To start let x^i and y^i with $i = 1, 2$, two quantum planes. We define the following variables A,B,C and D as:

$$\begin{aligned} A &= x^1 y^1 \\ B &= x^2 y^1 \\ C &= x^1 y^2 \\ D &= x^2 y^2. \end{aligned} \tag{8.17}$$

Using equations (8.12), the $SU_q(2)$ generators act on this coordinates as:

$$\begin{aligned} T^3 A &= q^4 A T^3 - q(q^2+1)A \\ T^3 B &= B T^3 \\ T^3 C &= C T^3 \\ T^3 D &= \frac{1}{q^4} D T^3 - \frac{1}{q}(\frac{1}{q^2}+1)D \\ \\ T^+ A &= q^2 A T^+ + B + \frac{1}{q} C \\ T^+ B &= B T^+ + \frac{1}{q} D \\ T^+ C &= C T^+ + \frac{1}{q^2} D \\ T^+ D &= \frac{1}{q^2} D T^+ \\ \\ T^- A &= q^2 A T^- \\ T^- B &= B T^- + q^2 A \\ T^- C &= C T^- + qA \\ T^- D &= \frac{1}{q^2} D T^- + C + qB. \end{aligned} \tag{8.18}$$

From the equation (8.10), and for $j = 0$ and $j = 1$, we find the following representations:

$$\begin{aligned} T^3|1,1> &= \frac{1}{q}(1+\frac{1}{q^2})|1,1> \\ T^3|1,0> &= 0 \\ T^3|1,-1> &= -q(1+q^2)|1,-1> \end{aligned}$$

$$\begin{aligned} T^+|1,1> &= 0 \\ T^+|1,0> &= \frac{1}{q^2}\sqrt{1+q^2}|1,1> \\ T^+|1,-1> &= \frac{1}{q}\sqrt{1+q^2}|1,0> \end{aligned} \tag{8.19}$$

$$\begin{aligned} T^-|1,1> &= \sqrt{1+q^2}|1,0> \\ T^-|1,0> &= q\sqrt{1+q^2}|1,-1> \\ T^-|1,-1> &= 0 \end{aligned}$$

Identifying the two equations (8.18) and (8.19) we obtain the relations [80]:

$$\begin{aligned} |1,-1> &\sim A =: X^- \\ |1,0> &\sim \frac{1}{\sqrt{1+q^2}}(B+qC) =: X^0 \\ |1,0> &\sim D =: X^+ \\ |0,0> &\sim C - qB =: Y \end{aligned} \tag{8.20}$$

Let now consider the two copies of the quantum plane (X^i, Y) and $(\widetilde{X}^i, \widetilde{Y})$ with $i = +, -, 0$ which are written as the product of the quantum planes (x, y) and (u, v). We choose the following expressions:

$$\begin{aligned} ux &= -q\hat{R}xu \\ uv &= -\hat{R}vu \\ yv &= -\hat{R}vy \\ yx &= -q\hat{R}xy \end{aligned} \tag{8.21}$$

Using equations (8.20) and (8.21), we find the following different relations [81]:

$$\begin{aligned} X^+\widetilde{X}^+ &= q^2\widetilde{X}^+X^+ \\ X^+\widetilde{X}^0 &= \widetilde{X}^0X^+ \\ X^+\widetilde{X}^- &= q^{-2}\widetilde{X}^-X^+ \\ X^-\widetilde{X}^+ &= q^{-2}\widetilde{X}^+X^- + \lambda^2(1+q^{-2})\widetilde{X}^-X^+ + \lambda q^{-2}\widetilde{X}^0X^0 \\ X^-\widetilde{X}^- &= q^2\widetilde{X}^-X^- \\ X^-\widetilde{X}^0 &= \widetilde{X}^0X^- + \lambda q^{-1}(1+q)\widetilde{X}^-X^0 \\ X^0\widetilde{X}^+ &= \widetilde{X}^+X^0 + \lambda(q+q^{-1})\widetilde{X}^0X^+ \\ X^0\widetilde{X}^- &= \widetilde{X}^-X^0 \\ X^0\widetilde{X}^0 &= \lambda(1+q^{-2})\widetilde{X}^-X^+ + \widetilde{X}^0X^0 \end{aligned} \tag{8.22}$$

$$
\begin{aligned}
X^+\widetilde{Y} &= \widetilde{Y}X^+ \\
X^0\widetilde{Y} &= \widetilde{Y}X^0 \\
X^-\widetilde{Y} &= \widetilde{Y}X^- \\
Y\widetilde{Y} &= \widetilde{Y}Y \\
Y\widetilde{X}^+ &= \widetilde{X}^+Y \\
Y\widetilde{X}^0 &= \widetilde{X}^0Y \\
Y\widetilde{X}^- &= \widetilde{X}^-Y,
\end{aligned}
$$

where

$$X \sim uy, \qquad \widetilde{X} \sim xv \tag{8.23}$$

and where

$$
\begin{aligned}
Y &= u^1y^2 - qu^2y^1, \\
\widetilde{Y} &= x^1v^2 - qx^2v^1
\end{aligned}
\tag{8.24}
$$

In this relations the Y and $\widetilde{Y}$ commute with all X^i and $\widetilde{X}^i$, for $i = +,-,0$. The two copies planes X^i and $\widetilde{X}^i$, $i = +,-,0$ construct the three dimensional plan which is caracterised by the $\hat{R}$ matrix and satisfy the following relation:

$$X^A\widetilde{X}^B = \hat{R}^{AB}_{CD}\widetilde{X}^C X^D \tag{8.25}$$

with $\hat{R}^{AB}_{CD}$ is the 9×9 matrix given by [81]:

$$
\hat{R} = \begin{pmatrix}
q^2 & & & & & & & & \\
& q^2 & & & & & & & \\
& & 0 & 1 & & & & & \\
& & 1 & \lambda(q+q^{-1}) & & & & & \\
& & & & 0 & 1 & & & \\
& & & & 1 & \lambda q^{-1}(1+q) & & & \\
& & & & & & & 0 & q^{-2} \\
& & & & & & 0 & 1 & \lambda(1+q^{-2}) \\
& & & & & & q^{-2} & \lambda q^{-2} & \lambda^2(1+q^{-2})
\end{pmatrix}
\tag{8.26}
$$

The rows and columns of $\hat{R}$ are labelled by $++, --, +0, 0+, 0-, -0, +-, 00$ and $-+$.

8.3 Differentials

It is known through the work of Wess [79], [82] that one can define a consistant differential calculus on the noncommutative space of a quantum group. In this

section we will give some differential relations and as an example we will study the Manin plane [83].
The commutation relation of the differential ∂_A are the same as the coordinate and fulfill [76]:

$$P^{ij}_{Akl}\partial_i\partial_j = 0. \tag{8.27}$$

this equation is comming form the assumptions on the exterior derivative d such that

$$d = \xi^A\partial_A,$$

where ξ^A is the coordinate differentials and d is the exterior derivative, which satisfy the same properties as in the classical case,

$$\begin{aligned} d^2 &= 0 \\ dx^A &= \xi^A + x^A d, \end{aligned} \tag{8.28}$$

ξ^A are suposed to anticommute, this means that:

$$P^{AB}_{SCD}\xi^C\xi^D = 0, \tag{8.29}$$

where P_S is the q-deformed symmetriser which obey the relation

$$\hat{R} = \lambda_1 P_S + \lambda_2 P_A,$$

with λ_1 and λ_2 are the eigenvalues of the matrix $\hat{R}$ and P_A the q-deformed anti-symmetriser. Consequently, the differentials satisfy a modified Leibniz rule

$$\partial_A(fg) = (\partial_A f)g + O^B_A f(\partial_B g), \tag{8.30}$$

where the operator O^B_A is a homomorphism, such that [76]:

$$O^B_A(fg) = O^C_A(f)O^B_C(g).$$

As example for this section we take the known two dimensional Manin plane.
The Manin plane is generated by two coordinate $\hat{x}$ and $\hat{y}$, which satisfy the relation:

$$\hat{x}\hat{y} = q\hat{y}\hat{x}. \tag{8.31}$$

The differentials obey the same relation, except for some scaling factor,

$$\partial_x\partial_y = \frac{1}{q}\partial_y\partial_x. \tag{8.32}$$

Then the relations between differentials and the coordinate $\hat{x}$ and $\hat{y}$ are given by:

$$\begin{aligned} \partial_x\hat{x} &= 1 + q^2\hat{x}\partial_x + q\lambda\hat{y}\partial_y \\ \partial_x\hat{y} &= q\hat{y}\partial_x, \end{aligned} \tag{8.33}$$

similarly, we have for ∂_y

$$\begin{aligned}\partial_y \hat{y} &= 1 + q^2 \hat{y} \partial_y \\ \partial_y \hat{x} &= q \hat{x} \partial_y,\end{aligned} \tag{8.34}$$

with $\lambda = q - \frac{1}{q}$.

8.4 Star product

The framework of deformation quantization allows to map the associative algebra of functions on noncommutative space to an algebra of functions on a commutative space by means of star product which we denote by $(\star)$. There are several ways to construct a star product. A particulary efficient way of computing this $\star$-products is Weyl quantization [84].

In this section, we will treat the star product of functions and as examples we will take three types of noncommutative spaces namely canonical structure, Lie algebra structure and quantum space structure.

8.4.1 Definition of star product:

1: Definition of Poisson Bracket:

A Poisson Bracket is defined by a bi-linear map $\{\}$ such that: $C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$ where M is a smooth manifold. the functions $f, g, h \in C^\infty(M)$ satisfy:

- $\{f, g\} = -\{g, f\}$, antisymmetry,
- $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$, Jacobi identity
- $\{f, gh\} = \{f, g\}h + \{f, h\}g$, Leibniz rule.

The Poisson Bracket can be written in function of an antisymmetric tensor as:

$$\{f, g\} = \theta^{ij} \partial_i f \partial_j g \tag{8.35}$$

where $\theta^{ij} = -\theta^{ji}$. From the antisymmetry and the Jacoby identity, we conclude that θ^{ij} satisfy:

$$\theta^{ij} \partial_j \theta^{kl} + \theta^{kj} \partial_j \theta^{li} + \theta^{lj} \partial_j \theta^{ik} = 0. \tag{8.36}$$

2. Definition of star product:

A star product is a deformation of a Poisson structure on a manifold.

Let now f and $g \in C^\infty(M)$, with $C^\infty(M)[[h]]$ is the algebra of formal power series in the formal parameter h with coefficients in $C^\infty(M)$. We define the star product $\star$: $C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)[[h]]$ by:

$$f \star g = \sum_{n=0}^{\infty} h^n C_n(f,g), \tag{8.37}$$

with $C_n : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$ are local bi-differential operators. This definition eq(8.37) satisfy the following axioms:

(i) $\star$ is associative product

(ii)$C_0(f,g) = fg$, classical limit.

(iii)$\frac{1}{h}[f \overset{\star}{,} g] = -i\{f,g\}$ in the limit $h \longrightarrow 0$, semiclassical limit.

with

$$[f \overset{\star}{,} g] = f \star g - g \star f.$$

The bi-differential operators obey the expression:

$$C_k(f,g) = (-1)^k C_k(g,f), \tag{8.38}$$

which means that the commutator $[f \overset{\star}{,} g]$ can be written in function of the parameter h as:

$$[f \overset{\star}{,} g] = 2 \sum_{n=0}^{\infty} h^{2n+1} C_{2n+1}(f,g). \tag{8.39}$$

For the first order of h of this equation, we obtain:

$$f \star g = fg + \frac{h}{2} \theta^{ij} \partial_i f \partial_j g + O(h^2). \tag{8.40}$$

8.4.2 Star product of functions

Let $\hat{A}$ be the noncommutative algebra of functions defined on the noncommutative space such that:

$$\hat{A} = \frac{C << \hat{x}^1, \ldots, \hat{x}^n >>}{R} \tag{8.41}$$

where R is the ideal generated by the commutation relations of the coordinate functions, and the commutative algebra of functions:

$$A = \frac{C << x^1, \ldots, x^n >>}{[x^i, x^j]} \equiv C[[x^1, \ldots, x^n]], \tag{8.42}$$

where $[x^i, x^j] = 0$.
We want now to construct a vector space isomorphism noted by W by choosing a basis ordering in $\hat{A}$ which must satisfy the Poincaré-Birkhoff-Witt property. The elements of $\hat{A}$ are maped as:

$$\begin{aligned} W : A &\longrightarrow \hat{A}, \\ x^i &\longrightarrow \hat{x}^i \qquad (8.43) \\ x^i x^j &\longrightarrow \frac{1}{2}(\hat{x}^i\hat{x}^j + \hat{x}^j\hat{x}^i) \equiv: \hat{x}^i\hat{x}^j : \qquad (8.44) \end{aligned}$$

which transform the coordinate x to $\hat{x}$. The ordering is indicated by (: :). So, for a function f we have:

$$W(f(x^i, x^j)) =: f(\hat{x}^i, \hat{x}^j) : \qquad (8.45)$$

The noncommutative multiplication $\star$ in A for two functions f and g is defined then by:

$$\begin{aligned} W(f \star g) &= W(f).W(g) = \hat{f}.\hat{g} \\ (A, \star) &= (\hat{A}, .), \end{aligned} \qquad (8.46)$$

where $f, g \in A$ and $\hat{f}, \hat{g} \in \hat{A}$. The commutation relation between $\hat{x}^i$ and $\hat{x}^j$ is given by:

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}(\hat{x}) \qquad (8.47)$$

The Weyl quantization procedure is given by using the Fourier transformation [77], as :

$$W(f) = \frac{1}{(2\pi)^{n/2}} \int d^n k e^{ik_j\hat{x}^j} \widetilde{f}(k), \qquad (8.48)$$

with

$$\widetilde{f}(k) = \frac{1}{(2\pi)^{n/2}} \int d^n x e^{-ik_j x^j} f(x), \qquad (8.49)$$

here we have replaced the commutative coordinates x by noncommutative ones $\hat{x}$ in the inverse Fourier transformation eq(8.48). The exponential is ordred according to the prescription choosen for the symmetric ordering or normal ordering. Using the equation (8.46), we can write the deformed product of the functions $(f \star g)$ as:

$$W(f).W(g) = \frac{1}{(2\pi)^n} \int d^n k d^n p e^{ik_i\hat{x}^i} e^{ip_j\hat{x}^j} \widetilde{f}(k)\widetilde{f}(p) = W(f \star g). \qquad (8.50)$$

The product of the two exponentials have to be rearranged into one exponential by help of the Campbell-Baker-Hausdorff formula defined by:

$$e^A e^B = e^{\{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[[A,B],B]-\frac{1}{12}[[A,B],A]+\ldots\}}. \qquad (8.51)$$

Now we need to specify $\theta^{ij}(\hat{x})$ eq(8.47) in order to evaluate the Campbell-Baker-Hausdorff formula. For this aim let us consider some examples.

8.4.3 Examples

8.4.3.1 Canonical structure

The noncommutative canonical structure is defined by:

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}, \quad \theta^{ij} \in C \tag{8.52}$$

Using the Campbell-Baker-Hausdorff formula we find:

$$e^{ik_i\hat{x}^i} e^{ip_j\hat{x}^j} = e^{i(k_j+p_j)\hat{x}^j - \frac{i}{2}k_i p_j \theta^{ij}}, \tag{8.53}$$

where terms with more than one commutator will vanish, due to the constant commutation relations (8.52).
Substituting this expression in eq(8.50), we obtain:

$$W(f \star g) = \frac{1}{(2\pi)^n} \int d^n k d^n p e^{i(k_j+p_j)\hat{x}^j - \frac{i}{2}k_i \theta^{ij} p_j} \widetilde{f}(k)\widetilde{g}(p) \tag{8.54}$$

which can be reads as:

$$(f \star g)(x) = \frac{1}{(2\pi)^n} \int d^n k d^n p e^{i(k_j+p_j)x^j - \frac{i}{2}k_i \theta^{ij} p_j} \widetilde{f}(k)\widetilde{g}(p). \tag{8.55}$$

Then we get the Moyal-Weyl star product [77]:

$$f \star g = e^{\frac{i}{2}\frac{\partial}{\partial x^i}\theta^{ij}\frac{\partial}{\partial y^j}} f(x)g(y)|_{y\to x} \tag{8.56}$$

8.4.3.2 Lie algebra structure

The second example that we will study is the Lie algebra case given by:

$$[\hat{x}^i, \hat{x}^j] = i\lambda^{ij}_k \hat{x}^k, \quad \lambda^{ij}_k \in C \tag{8.57}$$

where λ^{ij}_k is structure constants. In this case the product of the two exponential is:

$$e^{ik_i\hat{x}^i} e^{ip_j\hat{x}^j} = e^{iP_i(k,p)\hat{x}^i}, \tag{8.58}$$

where $P_i(k,p)$ are the parameters of a group element obtained by multiplying two group elements, one parametrized by k and the other by p. From the Campbell-Baker-Hausdorff formula we get:

$$P_i(k,p) = k_i + p_i + \frac{1}{2}g_i(k,p), \tag{8.59}$$

where all the terms containing more than one commutator are collected in $g_i(k,p)$. Using this results, eq(8.50) becomes:

$$f \star g = \frac{1}{(2\pi)^n} \int d^n k d^n p e^{iP_i(k,p)x^i} \tilde{f}(k)\tilde{g}(p) \tag{8.60}$$

Then the star product take the form:

$$f \star g = e^{\frac{i}{2}x^i g_i(i\frac{\partial}{\partial y}, i\frac{\partial}{\partial z})} f(y)g(z)|_{\substack{y\to x \\ z\to x}}. \tag{8.61}$$

The first terms of $g_i(i\frac{\partial}{\partial y}, i\frac{\partial}{\partial z})$ are given by [85]:

$$\begin{aligned} g_i(i\frac{\partial}{\partial y}, i\frac{\partial}{\partial z}) &= \lambda_i^{jk} i \frac{\partial}{\partial y^j}\frac{\partial}{\partial z^k} - \frac{i}{6}\lambda_l^{jk}\lambda_i^{ml}(\frac{\partial}{\partial y^j}\frac{\partial}{\partial y^m}\frac{\partial}{\partial z^k} + \frac{\partial}{\partial z^j}\frac{\partial}{\partial z^m}\frac{\partial}{\partial y^k}) \\ &+ \frac{1}{24}\lambda_m^{jk}\lambda_n^{rm}\lambda_i^{sn}(\frac{\partial}{\partial y^j}\frac{\partial}{\partial y^r}\frac{\partial}{\partial z^k}\frac{\partial}{\partial z^s} + \frac{\partial}{\partial z^j}\frac{\partial}{\partial z^r}\frac{\partial}{\partial y^k}\frac{\partial}{\partial y^s}) \\ &+ \ldots \end{aligned} \tag{8.62}$$

8.4.3.3 Quantum space structure

This structure take the form:

$$\hat{x}^i\hat{x}^j = q^{-1}\hat{R}^{ij}_{kl}\hat{x}^k\hat{x}^l, \tag{8.63}$$

where R^{ij}_{kl} is the dimensionless braid- or simply R-matrix of the quantum space with q its deformation parameter.
As example here we take the Manin plane [83], and the quantum plane $GL_q(n)$.

• **Manin plane**:

The Manin plane fulfill the equation:

$$\hat{x}\hat{y} = q\hat{y}\hat{x}. \tag{8.64}$$

To compute the star product of this plane we will not use the Weyl quantisation but the normal ordering which allow us to placed all the $\hat{x}$ operators befor $\hat{y}$. For this two operators we have:

$$\begin{aligned} :\hat{x}\hat{x}: &= \hat{x}\hat{x} \\ :\hat{y}\hat{y}: &= \hat{y}\hat{y} \\ :\hat{x}\hat{y}: &= =:\hat{y}\hat{x}:= \hat{x}\hat{y}. \end{aligned} \tag{8.65}$$

Then, we obtain for monomials:

$$\begin{aligned} \hat{x}^{n_1}\hat{y}^{m_1}\hat{x}^{n_2}\hat{y}^{m_2} &= q^{-m_1 n_2}\hat{x}^{n_1+n_2}\hat{y}^{m_1+m_2} \\ :\hat{x}^{n_1}\hat{y}^{m_1}::\hat{x}^{n_2}\hat{y}^{m_2}: &= q^{-m_1 n_2}:\hat{x}^{n_1+n_2}\hat{y}^{m_1+m_2}: \\ &= W(q^{-x'\frac{\partial}{\partial x'}y\frac{\partial}{\partial y}}:x^{n_1}y^{m_1}x'^{n_2}y'^{m_2}:|_{\substack{x'\to x\\ y'\to y}}) \end{aligned} \tag{8.66}$$

From this equation we have the star product for the Manin plane given by:

$$f\star g = q^{-x'\frac{\partial}{\partial x'}y\frac{\partial}{\partial y}}f(x,y)g(x',y')|_{\substack{x'\to x\\ y'\to y}}. \tag{8.67}$$

- **Quantum plane $\mathbf{GL_q(N)}$:**

The covariant quantum plane $GL_q(N)$ contain the coordinates $\hat{x}^1,\ldots,\hat{x}^N$ which satisfy the relation:

$$\hat{x}^i\hat{x}^j = q\hat{x}^j\hat{x}^i, \quad i<j \quad For \quad q\in C. \tag{8.68}$$

Using the weyl quantisation, the star product of this algebra has the form [85]:

$$f\star g = e^{\frac{-h}{2}(\sum_{i<j} y^i\frac{\partial}{\partial y^i}x^j\frac{\partial}{\partial x^j}-x^i\frac{\partial}{\partial x^i}y^j\frac{\partial}{\partial y^j})}f(x)g(y)|_{y\to x} \tag{8.69}$$

where $h = lnq$ is a deformed parameter. Taking in this relation $f = x^i$ and $g = x^j$, we find the star product of eq(8.68) as $\hat{x}^i\star\hat{x}^j = q\hat{x}^j\star\hat{x}^i, \quad i<j$.

8.5 Gauge theory on noncommutative space

In this section we will concentrate on noncommutative geometry formulated in the star product formalism to formulate gauge theories on noncommutative space. In this study we will observe that this gauge theorie is based on the idea that multiplication of a field by a coordinate or a function is not covariant only if that function does not commute with gauge transformation [77]. This can be resolved by adding an appropriate noncommutative gauge potentials and thus introducing covariant coordinate in analogy to the covariant derivative of ordinary gauge theory.

8.5.1 Gauge transformations

Let $\psi(\hat{x})$ be field of the algebra $\hat{A}$ such that:

$$\psi(\hat{x}) = \psi(\hat{x}^1,\hat{x}^2,\ldots,\hat{x}^n)\in\hat{A}, \tag{8.70}$$

where $\hat{x}$ is the noncommutative coordinate and $\hat{A}$ is the algebra given by eq(8.41). We define the parameter of transformation which represent an infinitesimal gauge transformation by $\hat{\alpha}(\hat{x})$. The gauge transformation of the field ψ can be written in the form:

$$\hat{\delta}\hat{\psi}(\hat{x}) = i\hat{\alpha}\hat{\psi}(\hat{x}), \quad \hat{\alpha}, \hat{\psi} \in \hat{A} \tag{8.71}$$

This transformation is called a covariant transformation law of the field $\hat{\psi}$. Since $\hat{\alpha}$ is an element of the algebra $\hat{A}$ it is the equivalent of an abelian gauge transformation but if this parameter of transformation belonged to an algebra of matrices then it is the equivalent of a non abelian gauge transformation [86].
The non abelian gauge theory is based on a Lie group with Lie algebra

$$[T^a, T^b] = i f_c^{ab} T^c, \tag{8.72}$$

where T^a are generators of the Lie algebra and f_c^{ab} are its structure constants. The elements of the algebra are represented by $n \times n$ matrices and the fields are given by n-dimensional vectors carrying an irreducible representation of the gauge group.
In the usual formulation of a gauge theory, fields noted by ψ are considered to transform under gauge transformation with Lie algebra valued infinitisimal parameters:

$$\delta\psi(x) = i\alpha(x)\psi(x), \tag{8.73}$$

where $\alpha(x)$ is Lie algebra valued, defined by:

$$\alpha(x) = \alpha_a(x) T^a. \tag{8.74}$$

Using this equations we observe that the derivative of the field ψ does not transform covariantly, this means that:

$$\delta\partial_\mu\psi(x) \neq i\alpha(x)\partial_\mu\psi(x). \tag{8.75}$$

If we replace the usual derivative ∂_μ by covariant derivative noted by D_μ which we can write in function of the gauge petential $A_\mu(x)$ as:

$$D_\mu = \partial_\mu - igA_\mu(x), \tag{8.76}$$

then D_μ can be transform covariantly, where A_μ is written in the form:

$$\begin{aligned} A_\mu(x) &= A_{\mu a}(x) T^a \\ \delta A_\mu(x) &= \frac{1}{g}\partial_\mu\alpha(x) + [\alpha(x), A_\mu(x)]. \end{aligned} \tag{8.77}$$

8.5.2 Covariant coordinates

The coordinates $\hat{x}^i$ are invariant under the gauge transformation, this means that:

$$\delta \hat{x}^i = 0. \tag{8.78}$$

The product of a field and a coordinate does not transform covariantly, since $\hat{x}^i$ and $\hat{\alpha}(\hat{x})$ does not commute,

$$\begin{aligned} \delta(\hat{x}^i \hat{\psi}) &= i\hat{x}^i \hat{\alpha}(\hat{x})\hat{\psi} \\ &\neq i\hat{\alpha}(\hat{x})\hat{x}^i \hat{\psi}. \end{aligned} \tag{8.79}$$

The arguments are the same as before for the classical gauge theory, we introduce here also covariant coordinates given by

$$\hat{X}^i = \hat{x}^i + \hat{A}^i(\hat{x}) \tag{8.80}$$

and satisfying the relation:

$$\delta(\hat{X}^i \hat{\psi}) = i\hat{\alpha}\hat{X}^i \hat{\psi}, \tag{8.81}$$

where

$$\delta(\hat{X}^i \hat{\psi}) = \delta(\hat{X}^i)\hat{\psi} + \hat{X}^i \delta(\hat{\psi}). \tag{8.82}$$

Using this relation and the equation (8.71), we find:

$$\delta(\hat{X}^i) = i[\hat{\alpha}, \hat{X}^i]. \tag{8.83}$$

Then the gauge potential transform under a noncommutative gauge theory transformation in the following way:

$$\delta \hat{A}^i = i[\hat{\alpha}, \hat{A}^i] - i[\hat{x}^i, \hat{\alpha}]. \tag{8.84}$$

We can define also the tensor $\hat{T}^{ij}$ which are covariant objects. For our three case of structure we write this tensors as [77]:

- Canonical structure:

$$T^{ij} = [\hat{X}^i, \hat{X}^j] - i\theta^{ij}. \tag{8.85}$$

- Lie structure:

$$T^{ij} = [\hat{X}^i, \hat{X}^j] - i f_k^{ij} \hat{X}^k. \tag{8.86}$$

- Quantum space:

$$T^{ij} = \hat{X}^i \hat{X}^j - q^{-1} \hat{R}_{kl}^{ij} \hat{X}^k \hat{X}^l. \tag{8.87}$$

Using the general expression given by:

$$\begin{aligned} \delta\hat{A} &= i[\hat{\alpha},\hat{A}] \\ \delta\hat{B} &= i[\hat{\alpha},\hat{B}] \\ \delta(\hat{A}\hat{B}) &= i[\hat{\alpha},\hat{A}\hat{B}] \end{aligned} \tag{8.88}$$

we find for our three case structures the following relation:

$$\delta\hat{T}^{ij} = i[\hat{\alpha},\hat{T}^{ij}]. \tag{8.89}$$

Appendix A

q-Deformed exponential

The exponential function $exp(z)$ is also shown to take a q-deformed form. Indeed, from the equation (4.7) we can extract the following prime derivative:

$$(z^n)' \equiv (\partial z^n) = (\sum_{i=0}^{n-1} q^i) z^{n-1}, \tag{A.1}$$

and write the exponential $exp(z)$ as follows:

$$exp(z) = \sum_{n=0}^{n-1} \frac{z^n}{[n]_q!} \tag{A.2}$$

where we define the q-deformed factorial numbers as follows

$$[n]_q! = 1.(q+1).(q^2+q+1).(q^{n-1}+q^{n-2}+\ldots+q+1). \tag{A.3}$$

With this definition, we clearly see from eq(A.1) and (A.2), that

$$exp(z) \equiv \partial exp(z), \tag{A.4}$$

where

$$\sum_{i=0}^{n-1} q^i = \frac{[n]_q!}{[(n-1)]_q!} \tag{A.5}$$

Note that one can generalize these definitions of the exponential for an arbitrary function $f(z)$ of conformal spin $\widetilde{f}$ by exploiting just the results established before.

Appendix B

q-Deformed commutator and compatibility condition

The use of the q-deformed commutator (5.3) instead of the usual one, namely $[L, B] = LB - BL$, which is nothing but the $q = 1$ limit of eq(5.3), implies a nontrivial consideration of the Lax evolution equation (5.2) in terms of the two compatibility equations. To be more precise, let us recall how these equations give rise to the standard evolution Lax equation for $q = 1$ for arbitrary Lax pair L, B.
The compatibility equations are given by the following system of linear equations:

$$\begin{aligned} L\Psi &= \lambda\Psi \\ B\Psi &= \frac{\partial\Psi}{\partial t} \end{aligned} \tag{B.1}$$

We have

$$BL\Psi = B\lambda\Psi = \lambda B\Psi = \lambda\frac{\partial\Psi}{\partial t} = \frac{\partial\lambda\Psi}{\partial t} = \frac{\partial L\Psi}{\partial t} \tag{B.2}$$

which also gives

$$BL\Psi = \frac{\partial L\Psi}{\partial t} = \frac{\partial L}{\partial t}\Psi + L\frac{\partial\Psi}{\partial t} = \frac{\partial L}{\partial t}\Psi + LB\Psi. \tag{B.3}$$

We then have

$$[B, L]\Psi = (BL - LB)\Psi = \frac{\partial L}{\partial t}\Psi \tag{B.4}$$

which implies that

$$[B, L] = \frac{\partial L}{\partial t}. \tag{B.5}$$

In the q-deformed case, the situation is not trivial, since the commutator is indisponsable to ensure this compatibility is q-deformed. In fact, let us consider for simplicity the q-differential Lax pairs L_2 and $H_{2n+1} = (L_2^{\frac{2n+1}{2}})_+$ required to satisfy by analogy the Lax evolution equation (5.2)

$$\frac{\partial L_2}{\partial t_{2n+1}} = [H_{2n+1}, L_2]_q, \tag{B.6}$$

where the q-deformed commutator is defined in eq(5.3). As suspected, by simply performing algebraic computations, we obtain:

$$\begin{aligned}
[H_1, L_2]_q &= H_1 L_2 - \bar{q}^2 L_2 H_1 + (\bar{q}^2 - 1)\partial^3 + \dots \\
[H_3, L_2]_q &= H_3 L_2 - \bar{q}^6 L_2 H_3 + (\bar{q}^6 - 1)\partial^5 + \dots \\
[H_5, L_2]_q &= H_5 L_2 - \bar{q}^{10} L_2 H_5 + (\bar{q}^{10} - 1)\partial^7 + \dots \\
[H_7, L_2]_q &= H_7 L_2 - \bar{q}^{14} L_2 H_7 + (\bar{q}^{14} - 1)\partial^9 + \dots \\
&\vdots
\end{aligned} \tag{B.7}$$

This equation can be generalised for arbitrary order n of the q-KdV hierarchy as follows:

$$[H_{2n+1}, L_2]_q = H_{2n+1} L_2 - \bar{q}^{2(2n+1)} L_2 H_{2n+1} + (\bar{q}^{2(2n+1)} - 1)\partial^{2n+3} + \dots \tag{B.8}$$

The terms $(\bar{q}^{2(2n+1)} - 1)\partial^{2n+3} + \dots)$ in (B.8) are extra nonlinear q-differential operators and which are proportional to $(\bar{q} - 1)$. These extra terms vanish in the standard limit $(\bar{q} = 1)$ to give rise to the standard commutator (B.4) and (B.5)

$$[H_{2n+1}, L_2]_{q=1} = H_{2n+1} L_2 - L_2 H_{2n+1}. \tag{B.9}$$

The important remark here is that if the compatibility equations exist, they must be highly nonlinear with a dependance in $\bar{q}$ as they should take into account the presence of the nonlinear extra terms in the q-deformed commutators (B.7). The possibility to write the two compatibility linear equations can naturally be emerged as $\bar{q} = 1$ limit of the previous equations.

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