Noncommutative Gauge Theory
beyond the
Canonical Case

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Abstract

Canonically deformed spacetime, where the commutator of two coordinates is a constant, is the most commonly studied noncommutative space. Noncommutative gauge theories that have ordinary gauge theory as their commutative limit have been constructed there. But these theories have their drawbacks: First of all, constant noncommutativity can only be an approximation of a realistic theory, and therefore it is necessary to study more complicated space-dependent structures as well. Secondly, in the canonical case, the noncommutativity didn't fulfill the initial hope of curing the divergencies of quantum field theory. Therefore it is very desirable to understand noncommutative spaces that really admit finite QFTs.

These two aspects of going beyond the canonical case will be the main focus of this thesis. They will be addressed within two different formalisms, each of which is especially suited for the purpose.

In the first part noncommutative spaces created by $\star$-products are studied. In the case of nonconstant noncommutativity, the ordinary derivatives possess a deformed Leibniz rule, i.e. $\partial_i(f \star g) \neq \partial_i f \star g + f \star \partial_i g$. Therefore we construct new objects that still have an undeformed Leibniz rule. These derivations of the $\star$-product algebra can be gauged much in the same way as in the canonical case and lead to function-valued gauge fields. By linking the derivations to frames (vielbeins) of a curved manifold, it is possible to formulate noncommutative gauge theories that admit nonconstant noncommutativity and go to gauge theory on curved spacetime in the commutative limit. We are also able to express the dependence of the noncommutative quantities on their corresponding commutative counterparts by using Seiberg-Witten maps.

In the second part we will study noncommutative gauge theory in the matrix theory approach. There, the noncommutative space is the ground state of a matrix action, the fluctuations around this ground state creating the gauge theory. In the canonical case the matrices used are infinite-dimensional (they are the Fock-space representation of the Heisenberg algebra), leading to a number of problems, especially with divergencies. Therefore we construct gauge theory using finite dimensional matrices (fuzzy spaces). This gauge theory is finite, goes to gauge theory on a 4-dimensional manifold in the commutative limit and can also be used to regularize the noncommutative gauge theory of the canonical case. In particular, we are able to match parts of the known instanton sector of the canonical case with the instantons of the finite theory.

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Contents

Chapter 1

Introduction

There is a simple Gedankenexperiment showing that any quantum theory including gravity will make it impossible to measure distances smaller than the Planck length: Trying to measure smaller and smaller distances, we are forced to use test particles with more and more energy. But this energy will affect the geometry of space itself, creating black holes which finally become bigger than the distances we wanted to measure (see e.g. [36]). Below the Planck length, distance looses its meaning.

In the absence of a consistent formulation of quantum gravity, we do not know the exact nature of quantized spacetime, but it is clear that the usual notion of a differentiable manifold should be replaced by something reflecting the quantum nature of spacetime at very small distances. Following the well known ideas of quantum mechanics, the uncertainty in the measurement of the coordinates leads directly to the notion of noncommutative spaces.

There is another motivation for the introduction of noncommutative spacetime, this time coming from quantum field theory. There, the divergencies appearing in the quantization are UV-effects, and therefore related to small distances. The introduction of noncommutativity could work as a ultraviolet cut-off, making QFT finite. Even though the UV-divergencies are now well under control through the renormalization programme, they nevertheless suggest that spacetime should change its nature at very small distances.

To make spacetime noncommutative, the commutative algebra of functions is usually replaced by a noncommutative algebra generated by coordinates $\hat{x}^i$ with commutation relations

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}. \tag{1.1}$$

In the canonical case, this commutator is a constant, i.e. $\theta^{ij} \in \mathbb{R}$. Gauge theory on this space was studied in great detail in the last few years, mainly due to its appearance in string theory. But if we think that noncommutativity is an effect of quantum gravity, the canonical case can only be the simplest example.

Other, more complicated structures should be studied, especially structures that are related to curved backgrounds. But also in view of our second motivation, the canonical case proved to be disappointing: it doesn't cure the infinities of QFT, it rather adds new ones.

The aim of this thesis will be to extend noncommutative gauge theory beyond the canonical case, in the two directions mentioned above: towards noncommutative gauge theory on curved backgrounds in part I, and towards gauge theory models which are actually finite in part II. For these two goals, we will use two different approaches, two different ways in which noncommutative gauge theory can be formulated already in the canonical case: one using $\star$-products, the other one matrices.

The notion of a $\star$-product came first up when Groenewold [47] and Moyal [87] used Weyl's quantization prescription [100] to pull back the noncommutativity of the quantum mechanical position and momentum operators onto the classical phase space. Later on, it was generalized in the framework of deformation quantization [11, 12] to arbitrary symplectic and Poisson manifolds. A $\star$-product is an associative noncommutative product acting on functions on a manifold, the noncommutativity being controlled by a deformation parameter. Expanded in this parameter, one can write

$$f \star g = f \cdot g + \theta^{ij} \partial_i f \partial_j g + \mathcal{O}(\theta^2) \tag{1.2}$$

To zeroth order, the $\star$-product reproduces ordinary pointwise multiplication, higher orders are bidifferential operators acting on the functions. The first order term corresponds to a Poisson structure. While every $\star$-product corresponds to a Poisson structure, the opposite is also true: For on every Poisson manifold there is a $\star$-product quantizing the Poisson structure [71].

Interest in noncommutative gauge theory formulated with the help of $\star$-products triggered when it became clear that it appears in string theory as the low energy limit of open strings with a background B-field [26]. In this picture, the endpoints of the open strings on the D-brane cease to commute, and depending on the regularization used, their behavior can be described either by noncommutative Yang-Mills theory or by commutative Yang-Mills with background B-field. These two descriptions can be linked by a map from the noncommutative quantities to the commutative ones, the Seiberg-Witten map [95].

The approach to noncommutative gauge theory most important to this thesis was developed in Munich in a series of papers [78, 64, 63], noticing that multiplication with a noncommutative coordinate is no longer a covariant operation. Then, coordinates have to be gauged much in the same way that derivatives have to be gauged in commutative gauge theory, leading to covariant coordinates. As the $\star$-product in the canonical case behaves very much like an ordinary product with respect to differentiation and integration, the covariant coordinates can be used to formulate noncommutative gauge theory in close analogy to the commutative case. The noncommutative theory is finally linked to the commutative one by using Seiberg-Witten maps, allowing to deal with nonabelian gauge theory as well. This way, it was possible to construct a noncommutative version of the standard model [18, 81] and study its phenomenological implications [13, 89, 80, 81, 82]. The extension to supersymmetry is somewhat more complicated as the SW-maps in general become nonlocal [84], but for the case of a reduced $N = \frac{1}{2}$ supersymmetry it is still possible [85]. Lately, it was even possible to formulate noncommutative gravity [8] for the canonical case.

As the noncommutative gauge transformations contain translations in space, there can't exist local observables in noncommutative gauge theory. But it was realized that in momentum space, certain Wilson loops with fixed momentum are actually gauge invariant [59, 48, 31]. These Wilson loops do not quite close, but contain a gap corresponding to the noncommutativity and the momentum, which is why they are also referred to as open Wilson lines. These open Wilson lines were used to construct the inverse SW-map for the field strength to all orders [90]. Other approaches to calculating SW-maps include a solution for abelian gauge theory to all orders using the Kontsevich formality map [71, 65, 66], a cohomological procedure within the BRST formalism [17] and a refined analysis of its internal structure [25].

In the quantization of noncommutative gauge theory, the legs of diagrams can no longer be exchanged, leading to a distinction between planar and nonplanar diagrams [39]. The planar diagrams have the same high energy behavior as their commutative counterparts, but the nonplanar diagrams lead to what is called IR/UV-mixing [86]. The diagrams are made finite in the UV by oscillatory factors, but only for finite momentum. For vanishing momenta, the divergencies reappear, therefore mixing the UV and the IR behavior of the theory. There are many studies on the renormalization properties of such theories (see [37, 98] for references), but so far the only consistently renormalizable theory is ϕ^4-theory with a special potential term added [54].

There are several lines of research going beyond the canonical case [19]. Covariant coordinates and SW-maps can be constructed for arbitrary Poisson manifolds [65, 67, 66], but the limit to commutative gauge theory no longer is clear. On κ-deformed spacetime, it was possible to establish noncommutative gauge theory, the nonconstant commutator of the coordinates leading to derivative valued gauge fields [33, 34, 35]. Somewhat closer to our approach, gauge theory on the $E_q(2)$-covariant plane was studied using frames [83]. More recently, there have been attempts using coordinate transformations from the canonical case to more complicated algebras [30, 40, 93].

The first part of this thesis will be devoted to expanding noncommutative gauge theory to more general $\star$-products and relating it to gauge theory on curved spacetime.

In chapter 2, we will first discuss the canonical case. We introduce the $\star$-product usually used in this case, the Moyal-Weyl $\star$-product. With this, noncommutative gauge theory is formulated in the standard way. As this approach can only deal with $U(N)$ gauge groups, we introduce Seiberg-Witten maps to accommodate for general gauge groups. We end this chapter with discussing noncommutative observables.

In chapter 3, we start with the general definition of $\star$-products, and show how they arise out of ordering prescriptions of algebras. For a special ordering, the Weyl- (or symmetric) ordering, we then calculate the corresponding $\star$-product to second order for general algebras, a result already published in [15] together with Andreas Sykora. Two other $\star$-products are presented as well, the Jambor-Sykora $\star$-product [62] and Kontsevich's formality $\star$-product [71]. After discussing integration on such $\star$-product algebras, we end with concrete examples.

In chapter 4, we discuss derivatives and derivations on $\star$-product algebras. For the canonical case, the usual derivatives still had the undeformed Leibniz rule. For general $\star$-products, this is no longer the case. The derivatives acquire a nontrivial coproduct, which means that their Leibniz rule is deformed. But for our construction of gauge theory we will need objects that still have the usual Leibniz rule, i.e. derivations of the $\star$-product algebra. We are able to identify such objects by linking them to vector fields commuting with the Poisson structure corresponding to the $\star$-products. We explicitly construct these derivations for the three $\star$-products introduced in chapter 4, and end with the continuation of the example from chapter 3.

In chapter 5, we use the derivations to construct gauge theory. As the derivations have the usual Leibniz rule, they can be gauged in full analogy to the canonical case, leading to function-valued gauge fields and field strength. As we want the noncommutative gauge theory to have a meaningful commutative limit, we link it to gauge theory on curved spacetime by introducing frames. On the commutative side, frames can be introduced to diagonalize the metric. If they fulfill a compatibility condition with the Poisson structure of the noncommutative space, we can lift them to derivations of the $\star$-product algebra. Then we use these derivations to build a noncommutative gauge theory that in the commutative limit reduces to gauge theory on curved spacetime. We give an example where the spacetime of the commutative limit is a manifold with constant curvature. To deal with general gauge groups, we again introduce SW-maps from the noncommutative to the commutative quantities. For the Weyl-ordered $\star$-product, we calculate the SW-maps for all relevant quantities up to second order. For the formality $\star$-product we are able to construct the SW-maps to all orders for abelian gauge theory. The results of this chapter (and parts of the preceding chapter) have already been published in [15] together with Andreas Sykora.

In chapter 6, the last chapter on the $\star$-product approach, we start with noticing that covariant coordinates can be defined for any $\star$-product, and use them to construct noncommutative analogs of Wilson lines. These can then be used to build noncommutative observables and to extend the construction of the inverse SW-map of [90] to general $\star$-products with nondegenerate Poisson structure. This has been published in [16], again together with Andreas Sykora.

But $\star$-products aren't the only way to express the noncommutativity (1.1). In the canonical case, the algebra of the coordinates is nothing but the well known Heisenberg algebra, and we can use the creator and annihilator formalism to represent it. The coordinates then become infinite-dimensional matrices acting on a Fock space, the derivatives commutators with the coordinates and integration the trace over the Fock space. Gauge transformations are now unitary transformations, and we again have to gauge the coordinates x^i to get covariant coordinates $X^i = x^i + A^i$. The gauge theory action

$$S = c\, \mathbf{tr}\, ([X^i, X^j] - i\theta^{ij})^2 \tag{1.3}$$

can be expressed entirely in terms of the dynamical matrix variables X^i, reproducing the noncommutative space as the ground state, with the fluctuations forming the gauge theory. In the canonical case, this description is equivalent to the $\star$-product approach, but it is the better framework to address nonperturbative questions such as topological solutions.

The instanton sector of noncommutative gauge theory is very rich, and many classical constructions can be reformulated on the noncommutative side. In two dimensions, all instantons have been classified [50], but in four dimensions the picture is far more complicated. There are the generalizations of the two-dimensional instantons (which will become important in this thesis), but there are many other instantons as well, which can be found by using a noncommutative ADHM-construction or Nahm's equations (see [37] for references).

The quantization of the model of course is troubled by the same divergencies as the one constructed via $\star$-products, but the exact definition is quite nontrivial for another reason as well: the theory contains sectors with any rank of the gauge group $U(n)$ [50]. To have a well-defined theory and quantization prescription, a regularization of gauge theory on $\mathbb{R}^d_\theta$ is therefore very desirable.

Luckily, there is a number of cases (in particular certain quantized compact spaces such as fuzzy spheres and tori), which have finite dimensional matrix representations of size N. In the limit $N \to \infty$, they nevertheless approach a commutative space. Gauge theory on these spaces can be introduced much in the same way as in (1.3), but now the covariant coordinates X_i are finite-dimensional Hermitian matrices of size N. The conventional gauge theory is then correctly reproduced in the limit $N \to \infty$. This leads to a natural quantization prescription by simply integrating over these matrices, making everything finite and well defined.

In the 2-dimensional case, this matrix-model approach to gauge theory has been studied in considerable detail for the fuzzy sphere S^2_N [74, 21, 96, 57, 22] and the noncommutative torus $\mathbb{T}^2_\theta$ [3, 91, 92, 45], both on the classical and quantized level. It is well-known that $\mathbb{R}^2_\theta$ can be obtained as the scaling limit of these spaces S^2_N and $\mathbb{T}^2_N$ at least locally, which suggests a correspondence also for the gauge theories. This correspondence of gauge theories has been studied in great detail for the case of $\mathbb{T}^2_\theta \to \mathbb{R}^2_\theta$ [91, 44, 46] on the quantized level, exhibiting the role of certain instanton contributions.

In 4 dimensions, the quantization of gauge theory is more difficult, and a regularization using finite-dimensional matrix models is particularly important. The most obvious 4-dimensional spaces suitable for this purpose are $\mathbb{T}^4$, $S^2 \times S^2$ and $\mathbb{C}P^2$. On fuzzy $\mathbb{C}P^2_N$ [52, 2, 20], such a formulation of gauge theory was given in [53]. This can indeed be used to obtain $\mathbb{R}^4_\theta$ for the case of $U(2)$ -invariant θ^{ij}. The case of $\mathbb{R}^2 \times S^2_N$ as regularization of $\mathbb{R}^4_\theta$ with degenerate θ^{ij} was considered in [102, 103], exhibiting a relation with a conventional non-linear sigma model. A formulation of lattice gauge theory for even-dimensional tori has been discussed in [5, 4, 45]. Related "fuzzy" solutions of the string-theoretical matrix models [58] were studied e.g. in [60, 70], see also [69].

The second part of this thesis will be devoted to the construction of gauge theory on such a 4-dimensional fuzzy space, the product of two fuzzy spheres $S^2_N \times S^2_N$. Besides introducing fermions as well, we will use this model to regularize gauge theory in the canonical case, i.e. on $\mathbb{R}^4_\theta$, with a special interest in the behavior of the instanton sector.

For this, we will again study the canonical case in chapter 7, this time using the matrix-model approach. The coordinates become annihilation and creation operators on a Fock space, and gauge theory can be formulated as an infinite-dimensional matrix model having the space as its ground state. We explain why this theory contains sectors for every rank n of the gauge group $U(n)$, and construct the 4-dimensional generalization of the instantons found in [50].

In chapter 8, we first present the fuzzy sphere S^2_N introduced by John Madore in [73]. To go to 4 dimensions, we use the product of two such spheres to get to $S^2_N \times S^2_N$, and show how to get to the canonical case of $\mathbb{R}^4_\theta$ in a double scaling limit.

In chapter 9, we give a definition of $U(n)$ gauge theory on fuzzy $S^2_N \times S^2_N$. The action is a generalization of the approach of [96] for fuzzy S^2_N. It differs from similar string-theoretical matrix models [58] by adding a constraint-term, which ensures that the vacuum solution is stable and describes the product of 2 spheres. The fluctuations of the covariant coordinates then correspond as usual to the gauge fields, and the action reduces to ordinary Yang-Mills theory on $S^2 \times S^2$ in the limit $N \to \infty$.

We then discuss some features of the model, in particular a hidden $SO(6)$ invariance of the action which is broken explicitly by the constraint. This suggests some alternative formulations in terms of collective matrices, which are assembled from the individual covariant coordinates. This turns out to be very useful to construct a Dirac operator, and may help to eventually study the quantization of the model explicitly. The stability of the model without constraint is also discussed, and we show that the only flat directions of the $SO(6)$ -invariant action are fluctuations of the constant radial modes of the 2 spheres. The quantization of the model is defined by a finite integral over the matrix degrees of freedom, which is shown to be convergent due to the constraint term. We also give a gauge-fixed action with BRST symmetry.

We also include charged fermions in the fundamental representation of the gauge group, by giving a Dirac operator $\widehat{D}$ which in the large N limit reduces to the ordinary gauged Dirac operator on $S^2 \times S^2$. This Dirac operator inherits the $SO(6)$ symmetry of the embedding space $S^2 \times S^2 \subset \mathbb{R}^6$, and exactly anti-commutes with a chirality operator. The 4-dimensional physical Dirac spinors are obtained by suitable projections from 8-dimensional $SO(6)$ spinors. This projection however commutes with $\widehat{D}$ only in the large N limit, and is achieved by giving one of the 2 spinors a large mass. Weyl spinors can then be defined using the exact chirality operator. An alternative version of chirality is given by defining a Ginsparg-Wilson system.

As a further test of the proposed gauge theory, we study topologically non-trivial solutions (instantons) on $S^2_N \times S^2_N$. We find in particular a simple class of solutions which can be interpreted as $U(1)$ instantons with quantized flux, combined with a singular, localized flux tube. They are related to the “fluxon” solutions of $U(1)$ gauge theory on $\mathbb{R}^4_\theta$ [50] discussed in chapter 7. Solutions which can be interpreted as 2-dimensional spherical branes wrapping one of the two spheres are also found.

In chapter 10, we then study the relation of the model on $S^2_N \times S^2_N$ with Yang-Mills theory on $\mathbb{R}^4_\theta$, and demonstrate that the usual Yang-Mills action on $\mathbb{R}^4_\theta$ is recovered in the appropriate scaling limit. We show in detail how the $U(1)$ instantons (fluxons) on $\mathbb{R}^4_\theta$ of chapter 7 arise as limits of the above non-trivial solutions on $S^2_N \times S^2_N$. In particular, we are able to match the moduli space of n fluxons, corresponding to their location on $\mathbb{R}^4_\theta$ resp. $S^2_N \times S^2_N$. We find in particular that even though the field strength in the bulk vanishes in the limit of $\mathbb{R}^4_\theta$, it does contribute to the action on $S^2_N \times S^2_N$ with equal weight as the localized flux tube. This can be interpreted on $\mathbb{R}^4_\theta$ as a topological or surface term at infinity. Another unexpected feature on $S^2_N \times S^2_N$ is the appearance of certain superselection rules, restricting the possible instanton numbers. In other words, not all instanton numbers on $\mathbb{R}^4_\theta$ are reproduced for a given matrix size $\mathcal{N}$, however they can be found by considering matrices of different size. This depends on the precise form of the constraint term in the action, which is hence seen to imply also certain topological constraints.

Most of the results of the second part of this thesis have already been published in [14], together with Frank Meyer and Harold Steinacker.

Part I

The $\star$-product approach

The use of $\star$-products made noncommutativity more accessible to physicists, as they can be applied very intuitively without reference to any strong (and complicated) mathematical background. We can still work with ordinary functions on ordinary commutative space-time, introducing the noncommutativity through the $\star$-product. The $\star$-product reproduces the ordinary pointwise product to zeroth order in some deformation parameter, the higher orders are differential operators acting on the functions and produce the noncommutativity. Therefore, $\star$-products are a very convenient tool for deforming commutative theories. The naive prescription for constructing noncommutative theories would then be to take the commutative theory and replace ordinary multiplication by $\star$-multiplication. As the deformation depends on a parameter, we can get back the commutative theory by letting it go to zero. Corrections to the commutative theory can be calculated order by order.

As we will see, this simple prescription works surprisingly well in the canonical case where the commutator of two coordinates is a constant. This is mainly due to the fact that in this case the $\star$-product still behaves very much like the commutative product with respect to differentiation and integration. But if we go to more complicated structures, this is no longer the case. Derivatives acquire a deformed Leibniz rule and ordinary integration no longer has the trace property. Therefore, the recipe of just replacing ordinary multiplication with $\star$-multiplication no longer works. In order to nevertheless construct noncommutative gauge theory on these more complicated spaces, it will be necessary to first have a closer look especially at the behavior of the derivatives. We will be able to identify objects that still have an undeformed Leibniz rule (we will call them derivations of the $\star$-product algebra), using them as building blocks for gauge theory. By linking them to frames on a curved spacetime, we can also make sense of the measure function we have to introduce in order to make integration cyclic again.

Chapter 2

The canonical case

Noncommutative gauge theory in the canonical case, where the commutator of two coordinates is a constant, has been studied extensively in the last few years (see e.g. [37, 98] for reviews), mainly due to its appearance in string theory [95]. It would be beyond the scope of this thesis to review all the aspects of this fascinating field, so we will have to concentrate on what will be important for going beyond the canonical case in the chapters to follow. We will start with the most commonly used $\star$-product for the canonical case, the Moyal-Weyl $\star$-product. Only the most important features of this $\star$-product will be presented here, but we will come back to it at the beginning of chapter 3 with a more detailed analysis. After a quick look at commutative gauge theory, an introduction into how noncommutative gauge theory can be formulated with the help of this $\star$-product is given. This introduction will mainly follow the approach developed here in Munich [78, 64, 63, 18] using Seiberg-Witten maps. Finally we will present the noncommutative observables found in [31, 48, 59], as we will be able to generalize them later on in chapter 6.

2.1 The Moyal-Weyl $\star$-product

In the canonical case, the noncommutative coordinates fulfill commutation relations

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij} \tag{2.1}$$

with the constant noncommutativity parameter $\theta \in \mathbb{R}$. The noncommutative algebra generated by the noncommutative coordinates can be represented on the space of functions on $\mathbb{R}^n$ by introducing a noncommutative product, the Moyal-

Weyl[1] $\star$-product [47, 87]

$$f \star g = m \cdot e^{\frac{i}{2}\theta^{ij}\partial_i \otimes \partial_j} f \otimes g = fg + \frac{i}{2}\theta^{ij}\partial_i f \partial_j g + O(2), \tag{2.2}$$

with $m \cdot (f \otimes g) = fg$ and $\partial_i = \frac{\partial}{\partial x_i}$. The product is associative, as

$$\begin{aligned}
(f \star g) \star h &= m \cdot e^{\frac{i}{2}\theta^{kl}\partial_k \otimes \partial_l}(m \cdot e^{\frac{i}{2}\theta^{ij}\partial_i \otimes \partial_j} f \otimes g) \otimes h \\
&= m \cdot m \cdot e^{\frac{i}{2}\theta^{kl}(\partial_k \otimes 1 \otimes \partial_l + 1 \otimes \partial_k \otimes \partial_l)} e^{\frac{i}{2}\theta^{ij}\partial_i \otimes \partial_j \otimes 1} f \otimes g \otimes h \\
&= m \cdot m \cdot e^{\frac{i}{2}\theta^{ij}(\partial_i \otimes 1 \otimes \partial_j + \partial_i \otimes \partial_j \otimes 1)} e^{\frac{i}{2}\theta^{kl} 1 \otimes \partial_k \otimes \partial_l} f \otimes g \otimes h \\
&= m \cdot e^{\frac{i}{2}\theta^{ij}\partial_i \otimes \partial_j}(f \otimes (m \cdot e^{\frac{i}{2}\theta^{kl}\partial_k \otimes \partial_l} g \otimes h)) \\
&= f \star (g \star h)
\end{aligned} \tag{2.3}$$

and obviously reproduces (2.1). Furthermore, as θ is antisymmetric, usual complex conjugation is still an involution

$$\overline{f \star g} = m \cdot e^{-\frac{i}{2}\theta^{ij}\partial_i \otimes \partial_j} \overline{f} \otimes \overline{g} = \overline{g} \star \overline{f} \tag{2.4}$$

and integration has the trace property

$$\int d^n x \; f \star g = \int d^n x \; g \star f, \tag{2.5}$$

if the functions f and g vanish sufficiently fast at infinity (of course $f \star g$ has to be integrable in the first place).

Differentiation on this space is an inner operation, i.e. we have

$$i\theta^{\mu\nu}\partial_\nu = [x^\mu, \cdot\,], \tag{2.6}$$

which can easily be calculated from (2.2). This also means that the derivatives still have the usual Leibniz rule, i.e. we have

$$\partial_i(f \star g) = \partial_i f \star g + f \star \partial_i g. \tag{2.7}$$

2.2 Commutative gauge theory

Let us now recall some properties of a general commutative gauge theory. A non-abelian gauge theory is based on a Lie group with Lie algebra

$$[T^a, T^b] = i\, f^{ab}{}_c T^c. \tag{2.8}$$

[1] actually, Groenewold-Moyal $\star$-product would be the more appropriate name, as Groenewold was the first to introduce the $\star$-product in [47], but to avoid misunderstandings, we will nevertheless stick to the term usually used in the literature.

Matter fields transform under a Lie algebra valued infinitesimal parameter

$$\lambda = \lambda_a T^a \tag{2.9}$$

in the fundamental representation as

$$\delta_\lambda \psi = i\lambda\psi.$$

It follows that

$$(\delta_\lambda \delta_\xi - \delta_\xi \delta_\lambda)\psi = \delta_{i[\xi,\lambda]}\psi. \tag{2.10}$$

The commutator of two consecutive infinitesimal gauge transformation closes into an infinitesimal gauge transformation. As differentiation isn't a covariant operation, a Lie algebra valued gauge potential $a_i = a_{ia}T^a$ is introduced with the transformation property

$$\delta_\lambda a_i = \partial_i \lambda + i[\lambda, a_i]. \tag{2.11}$$

With this the covariant derivative of a field is

$$D_i \psi = \partial_i \psi - i a_i \psi. \tag{2.12}$$

The field strength of the gauge potential is defined to be the commutator of two covariant derivatives

$$f_{ij} = i[D_i, D_j] = \partial_i a_j - \partial_j a_i - i[a_i, a_j]. \tag{2.13}$$

For nonabelian gauge theory, the field strength is not invariant under gauge transformations, but rather transforms covariantly, i.e.

$$\delta_\lambda f = i[\lambda, f]. \tag{2.14}$$

The same is true for the Lagrangian density $f_{ij}f^{ij}$. In order to get a gauge invariant action, we have to use the trace over the representation of the gauge fields. As the trace is cyclic, the commutator with the gauge parameter vanishes and the action

$$S = \int dx^n \mathbf{tr} f_{ij} f^{ij} \tag{2.15}$$

becomes invariant.

2.3 Noncommutative gauge theory

To do noncommutative gauge theory in the $\star$-product approach, we can simply mimic the commutative construction, replacing the ordinary pointwise product with the $\star$-product.

Fields should now transform as

$$\delta_\Lambda \Psi = i\Lambda \star \Psi. \tag{2.16}$$

The commutator of two such gauge transformations should again be a gauge transformation, i.e we want

$$(\delta_\Lambda \delta_\Xi - \delta_\Xi \delta_\Lambda)\Psi = \delta_{i[\Xi \stackrel{\star}{,} \Lambda]}\Psi, \tag{2.17}$$

which is only possible for gauge groups $U(N)$, as for $\Lambda = \Lambda_a T^a$ and $\Xi = \Xi_a T^a$ the commutator

$$[\Xi \stackrel{\star}{,} \Lambda] = \frac{1}{2}[\Xi_a \stackrel{\star}{,} \Lambda_b]\{T^a, T^b\} + \frac{1}{2}\{\Xi_a \stackrel{\star}{,} \Lambda_b\}[T^a, T^b] \tag{2.18}$$

will only close into the Lie algebra for $u(N)$ in the fundamental representation. But general gauge groups can be implemented by using Seiberg-Witten maps (see chapter 2.4).

As coordinates do not transform under gauge transformations, multiplication from the left with coordinates no longer is a covariant operation, i.e.

$$\delta_\Lambda(x_i \star \Psi) = x_i \star \Lambda \star \Psi \neq \Lambda \star x_i \star \Psi. \tag{2.19}$$

This is very much like the situation in commutative gauge theory, where acting with a derivative from the left isn't a covariant operation. Following the procedure there, we introduce covariant coordinates X^i by adding a gauge field A_i as

$$X^i = x^i + \theta^{ij} A_j. \tag{2.20}$$

To make the X^i covariant, i.e. $\delta_\Lambda X^i = i[\Lambda \stackrel{\star}{,} X^i]$, the gauge field has to transform as

$$\delta_\Lambda(\theta^{ij} A_j) = -i[x^i \stackrel{\star}{,} \Lambda] + i[\Lambda \stackrel{\star}{,} \theta^{ij} A_j] \tag{2.21}$$

and therefore

$$\delta_\Lambda A_i = \partial_i \Lambda + i[\Lambda \stackrel{\star}{,} A_i], \tag{2.22}$$

in exact analogy to the commutative case. The commutator with the coordinate produces the derivative on the gauge parameter, as $[x^i \stackrel{\star}{,} f] = i\theta^{ij}\partial_j f$. More generally we can introduce a covariantizer D that applied to a function f renders it covariant [65]

$$\delta_\Lambda(D(f)) = i[\Lambda \stackrel{\star}{,} D(f)]. \tag{2.23}$$

We can now go on to formulate noncommutative gauge theory much in the same way as we formulated commutative gauge theory.

The covariant derivative D_i can be introduced as

$$D_i\Psi = \partial_i\Psi - iA_i \star \Psi, \tag{2.24}$$

the field strength F_{ij} as

$$F_{ij} = i[D_i \stackrel{\star}{,} D_j] = \partial_i A_j - \partial_j A_i - i[A_i \stackrel{\star}{,} A_j]. \tag{2.25}$$

The relation to the covariant coordinates subsists at this level with

$$-i([X^i \stackrel{\star}{,} X^j] - i\theta^{ij}) = \theta^{ik}\theta^{jl}F_{kl}. \tag{2.26}$$

For nondegenerate θ, the two descriptions - either at the level of covariant coordinates or covariant derivatives - are clearly equivalent.

In noncommutative gauge theory, the field strength F is not gauge invariant, even for gauge group $U(1)$. It rather transforms covariantly under gauge transformations, i.e.

$$\delta_\Lambda(F_{\mu\nu} \star F^{\mu\nu}) = i[\Lambda \stackrel{\star}{,} F_{\mu\nu} \star F^{\mu\nu}]. \tag{2.27}$$

Therefore even Abelian noncommutative gauge theory looks more like nonabelian gauge theory. But just inserting a trace over the representation of the gauge group no longer guarantees gauge invariance. To get gauge invariant expressions, we have to use the trace property of the integral. If we set the action for noncommutative gauge theory as

$$S = \int d^n x \, \mathbf{tr} \, F_{\mu\nu} \star F^{\mu\nu}, \tag{2.28}$$

this expression will transform as

$$\delta_\Lambda S = i \int d^n x \, \mathbf{tr} \, [\Lambda \stackrel{\star}{,} F_{\mu\nu} \star F^{\mu\nu}] = 0, \tag{2.29}$$

because the cyclicity of the integral annihilates the $\star$-part of the commutator, and the cyclicity of the trace annihilates the nonabelian part. This means that we cannot separate the trace over the representation of the gauge group and the integration as in the commutative case, we need both to get a gauge invariant action.

2.4 The Seiberg-Witten map

Up to now, we could only do noncommutative gauge theory for gauge groups$U(n)$ because of (2.18). We will now show how to implement general gauge groups by using Seiberg-Witten maps [95, 64].

As we have seen, the commutator of two noncommutative gauge transformations no longer closes into the Lie algebra for general gauge groups. The noncommutative gauge parameter and the noncommutative gauge potential will therefore have to be enveloping algebra valued. In principle, this should mean that we are left with infinitely many degrees of freedom. But the enveloping algebra valued parameters will only depend on their commutative counterparts, therefore preserving the right number of degrees of freedom. These Seiberg-Witten maps Λ, Ψ and A are now functionals of their classical counterparts and additionally of the gauge potential a.

They will transform as

$$\delta_\lambda \Psi_\psi[a] = i\Lambda_\lambda[a] \star \Psi_\psi[a] \tag{2.30}$$

and

$$\delta_\lambda A_i[a] = \partial_i \Lambda_\lambda[a] + i[\Lambda_\lambda[a] \stackrel{\star}{,} A_i[a]]. \tag{2.31}$$

The covariantizer $D[a]$ will now transform as

$$\delta_\lambda(D[a](f)) = i[\Lambda_\lambda[a] \stackrel{\star}{,} D[a](f)]. \tag{2.32}$$

Their dependence on the commutative fields is given by the requirement that their noncommutative transformation properties should be induced by the commutative ones (2.9) and (2.11) like

$$\begin{aligned} \Psi_\psi[a] + \delta_\lambda \Psi_\psi[a] &= \Psi_{\psi+\delta_\lambda\psi}[a + \delta_\lambda a], \\ A_i[a] + \delta_\lambda A_i[a] &= A_i[a + \delta_\lambda a], \\ \Lambda_\lambda[a] + \delta_\xi \Lambda_\lambda[a] &= \Lambda_\lambda[a + \delta_\xi a]. \end{aligned} \tag{2.33}$$

This means that it doesn't matter if we transform the noncommutative fields under the noncommutative gauge transformations or if we transform the commutative fields they depend on under commutative gauge transformations. This is why we do not differentiate in our notation between commutative and noncommutative gauge transformations, using $\delta_\lambda = \delta_{\Lambda_\lambda[a]}$. Additionally, to zeroth order in the deformation parameter, the noncommutative fields should be equal to their commutative counterparts, i.e.

$$\begin{aligned} \Psi_\psi[a] &= \psi + \mathcal{O}(\theta), \\ A_i[a] &= a_i + \mathcal{O}(\theta), \\ \Lambda_\lambda[a] &= \lambda + \mathcal{O}(\theta). \end{aligned} \tag{2.34}$$

The SW-maps (Seiberg-Witten maps) can be found order by order in the deformation parameter. Alternatively they can be calculated via a consistency condition. Although the gauge transformations do not have to close into the Lie algebra, there is still the requirement that the commutator of two Seiberg-Witten gauge transformations (2.30) should again be a Seiberg-Witten gauge transformation (2.30), i.e.

$$(\delta_\lambda \delta_\xi - \delta_\xi \delta_\lambda)\Psi = \delta_{i[\xi,\lambda]}\Psi \tag{2.35}$$

Written out this means that

$$-i\delta_\xi \Lambda_\lambda[a] + i\delta_\lambda \Lambda_\xi[a] + [\Lambda_\lambda[a] \stackrel{\star}{,} \Lambda_\xi[a]] = i\Lambda_{i[\xi,\lambda]}[a]. \tag{2.36}$$

This consistency condition for the the SW-map of the gauge parameter can be solved order by order. Then the solutions can be used to calculate the other SW-maps by inserting them into (2.33) and using (2.34).

There are also methods for constructing the SW-maps to all orders [65, 66, 90], which we will discuss later in chapters 5.3 and 6.2, where we extend them to more complicated $\star$-products.

2.5 Observables

One characteristic property of noncommutative gauge theory is the fact that there are no local observables. As the gauge group of noncommutative gauge theory also comprises translations in space, gauge invariant quantities (such as observables) cannot be local fields. But nevertheless observables can be constructed by integrating over special Wilson lines [31, 48, 59], which can be interpreted as the Fourier transform of a Wilson line with fixed momentum. Unlike the commutative case, where closed Wilson lines are gauge invariant, these noncommutative Wilson lines do not quite close. The gap between the endpoints is related to the momentum via the parameter of the noncommutativity. To see this, we will first present finite expressions for noncommutative gauge theory.

2.5.1 Finite gauge transformations

In a finite version of a noncommutative gauge theory, a scalar field should transform like

$$\phi' = g \star \phi, \tag{2.37}$$

where g is a function that is invertible with respect to the $\star$-product

$$g \star g^{-1} = g^{-1} \star g = 1. \tag{2.38}$$

Again, multiplication with a coordinate function is not covariant any more

$$(x^i \star \phi)' \neq x^i \star \phi'. \tag{2.39}$$

Just as in the infinitesimal formulation, covariant coordinates

$$X^i(x) = x^i + \theta^{ij} A_j(x) \tag{2.40}$$

can be introduced, transforming in the adjoint representation

$$X^{i\prime} = g \star X^i \star g^{-1}. \tag{2.41}$$

Now the product of a covariant coordinate with a field is again a field. In perfect analogy to the commutative case, the gauge field A^i transforms as

$$A^{i\prime} = ig \star \partial_i g^{-1} + g \star A^i \star g^{-1}. \tag{2.42}$$

This finite formulation of noncommutative gauge theory is equivalent to the infinitesimal formulation presented before. For details on this equivalence, see [63].

2.5.2 Wilson lines

Just think of a field ϕ transforming covariantly under a gauge transformation with gauge parameter $\lambda = l_i x^i$. The corresponding finite expression is

$$\phi(x) \to e_\star^{il_i x^i} \star \phi(x^k) \star e_\star^{-il_i x^i} = \phi(x^k - l_j \theta^{jk}), \tag{2.43}$$

i.e. a translation by $-l_j \theta^{jk}$. This means that noncommutative gauge transformations in fact contain translations in space! The $\star$ subscript on the exponential means that all the multiplications are done using the $\star$-product. But it is a special property of the Moyal-Weyl $\star$-product that the $\star$-exponential actually is the same as the ordinary one, i. e. we have $e_\star^{ix^i} = e^{ix^i}$, which is why we will drop the $\star$-subscript in the following.

The fact that translations are gauge transformations can be used to construct noncommutative analogs of Wilson lines. Such a Wilson line

$$W_l = e^{il_i X^i} \star e^{-il_i x^i} \tag{2.44}$$

has indeed the same transformation properties under a gauge transformation

$$W_l'(x) = g(x) \star W_l(x) \star g^{-1}(x - l_i \theta^{ij}). \tag{2.45}$$

as a Wilson line starting at x and ending at $x - l\theta$. Here we only treat straight Wilson lines, but for the canonical case they can also be generalized to noncommutative Wilson lines with arbitrary paths [59, 31, 48].

2.5.3 Observables

As space translations are included in the noncommutative gauge transformations, no local observables can be constructed. One has to integrate over the whole space to get gauge invariant objects. For this it is useful to look at the Fourier transform of the Wilson lines (2.44)

$$W_l(k) = \int d^n x\, W_l(x) \star e^{ik_i x^i} \tag{2.46}$$

Under a gauge transformation, it transforms as

$$\begin{aligned} W_l(k)' &= \int d^n x\, g(x) \star W_l(x) \star g^{-1}(x - l_i\theta^{ij}) \star e^{ik_i x^i} \\ &= \int d^n x\, g(x) \star W_l(x) \star e^{ik_i x^i} \star e^{-ik_i x^i} \star g^{-1}(x - l_i\theta^{ij}) \star e^{ik_i x^i} \\ &= \int d^n x\, g(x) \star W_l(x) \star e^{ik_i x^i} \star g^{-1}(x - l_i\theta^{ij} + k_i\theta^{ij}). \end{aligned} \tag{2.47}$$

This means that the so called open Wilson lines [59, 31, 48] defined as

$$U_l = W_l(l) = \int d^n x\, W_l(x) \star e^{il_i x^i}, \tag{2.48}$$

are gauge invariant. Here, the momentum k of the Wilson line corresponds to its length $r^j = l_i\theta^{ij}$ via the parameter of the noncommutativity, i.e. $r^j = k_i\theta^{ij}$. Using (2.44), this is of course even more obvious

$$U_l = \int d^n x\, W_l(x) \star e^{il_i x^i} = \int d^n x\, e^{il_i X^i} \star e^{-il_i x^i} \star e^{il_i x^i} = \int d^n x\, e^{il_i X^i}. \tag{2.49}$$

These open Wilson lines can even be generalized by inserting an arbitrary function f of the covariant coordinates as

$$\int d^{2n} x\, f(X) \star e^{il_i X^i} \tag{2.50}$$

without spoiling the gauge invariance.

Chapter 3

General $\star$-products

We had introduced the Moyal-Weyl $\star$-product for the canonical case without explaining how it can be derived. In order to introduce the notion of $\star$-products in general, we will first have a closer look at the canonical case again. Suppose we have a two-dimensional canonical algebra generated by the noncommutative coordinates $\widehat{x}$ and $\widehat{y}$ with relations

$$[\widehat{x}, \widehat{y}] = -i\theta. \tag{3.1}$$

To represent this algebra on the space $C_\infty(\mathbb{R}^2)$, we will define an ordering prescription ρ by mapping monomials in the commutative variables x and y to the monomials in the noncommutative variables $\widehat{x}$ and $\widehat{y}$ with all the $\widehat{x}$ on the left hand side and all the $\widehat{y}$ on the right hand side

$$\rho(x^n y^m) := \widehat{x}^n \widehat{y}^m. \tag{3.2}$$

This is called normal ordering. If we normal order a monomial, we get

$$\widehat{y}^m \widehat{x}^k = \sum_{i=0}^{min(m,k)} \frac{(i\theta)^i}{i!} \frac{m!}{(m-i)!} \frac{k!}{(k-i)!} \widehat{x}^{k-i} \widehat{y}^{m-i}. \tag{3.3}$$

If we multiply two such monomials and normal order the result, we therefore get

$$\begin{aligned} \rho(x^n y^m)\rho(x^k y^l) &= \sum_{i=0}^{min(m,k)} \frac{(i\theta)^i}{i!} \frac{m!}{(m-i)!} \frac{k!}{(k-i)!} \widehat{x}^{n+k-i} \widehat{y}^{l+m-i} \\ &= \sum_{i=0}^{min(m,k)} \frac{(i\theta)^i}{i!} \frac{m!}{(m-i)!} \frac{k!}{(k-i)!} \rho(x^{n+k-i} y^{l+m-i}) \end{aligned} \tag{3.4}$$

As the vector space of the noncommutative polynomials of a certain degree has the same dimension as the vector space of the commutative polynomials of the same degree, the ordering ρ can be inverted, giving

$$\begin{aligned}
\rho^{-1}(\rho(x^n y^m)\rho(x^k y^l)) &= \sum_{i=0}^{min(m,k)} \frac{(i\theta)^i}{i!} \frac{m!}{(m-i)!} \frac{k!}{(k-i)!} x^{n+k-i} y^{l+m-i} \quad (3.5)\\
&= \sum_{i=0}^{\infty} \frac{(i\theta)^i}{i!} \partial_y^i (x^n y^m) \partial_x^i (x^k y^l)\\
&= m \cdot e^{i\theta \partial_y \otimes \partial_x} (x^n y^m) \otimes (x^k y^l)\\
&= (x^n y^m) \star_n (x^k y^l).
\end{aligned}$$

Therefore get a new $\star$-product

$$f \star_n g = m \cdot e^{i\theta \partial_y \otimes \partial_x} f \otimes g = \rho^{-1}(\rho(f)\rho(g)) \quad (3.6)$$

for the algebra (3.1) by applying the ordering prescription ρ on polynomial functions! Let's compare it to the Moyal-Weyl $\star$-product (2.2). For the algebra (3.1) this read

$$f \star_w g = m \cdot e^{\frac{i}{2}\theta(\partial_y \otimes \partial_x - \partial_x \otimes \partial_y)} f \otimes g. \quad (3.7)$$

If we define a differential operator

$$T = e^{-\frac{i}{2}\theta \partial_x \partial_y}, \quad (3.8)$$

we can calculate

$$\begin{aligned}
f \star_w g &= m \cdot e^{\frac{i}{2}\theta(\partial_y \otimes \partial_x - \partial_x \otimes \partial_y)} f \otimes g \quad (3.9)\\
&= m \cdot e^{\frac{i}{2}\theta(\partial_y \otimes 1 + 1 \otimes \partial_y)(\partial_x \otimes 1 + 1 \otimes \partial_x)} \, e^{i\theta \partial_y \otimes \partial_x} \, e^{-\frac{i}{2}\theta(\partial_x \partial_y \otimes \partial_x \partial_y)} f \otimes g\\
&= e^{\frac{i}{2}\theta \partial_x \partial_y}((e^{-\frac{i}{2}\theta \partial_x \partial_y} f) \star_n (e^{-\frac{i}{2}\theta \partial_x \partial_y} g))\\
&= T^{-1}((Tf) \star_n (Tg))\\
&= T^{-1}\rho^{-1}(\rho T(f) \rho T(g)).
\end{aligned}$$

The two $\star$-products are related by the differential operator T, and the Moyal-Weyl $\star$-product can be expressed by an ordering prescription ρT. The ordering actually corresponds to symmetric ordering, e.g. we have

$$\begin{aligned}
\rho T(xy) &= \rho(e^{-\frac{i}{2}\theta \partial_x \partial_y} xy) = \rho(xy - \frac{i}{2}\theta) = \widehat{x}\widehat{y} - \frac{i}{2}\theta \quad (3.10)\\
&= \widehat{x}\widehat{y} - \frac{1}{2}[\widehat{x}, \widehat{y}] = \frac{1}{2}(\widehat{x}\widehat{y} + \widehat{y}\widehat{x}).
\end{aligned}$$

This method of constructing $\star$-products by applying an ordering prescription is not limited to the canonical case.

3.1 Definition

But before we have a look at more complicated $\star$-products, we will fist give an abstract definition. For this we will introduce a parameter h (which we think of as small) measuring the deformation, and express everything as formal power series in this parameter.

A $\star$-product on a manifold M is an associative $\mathbb{C}$-linear product

$$f \star g = fg + \sum_{i=1}^{\infty} h^i B_i(f, g), \tag{3.11}$$

where the B_i are bidifferential operators acting on $f, g \,\epsilon\, C_\infty(M)[[h]]$. From the associativity of the $\star$-product follows that the $\star$-commutator fulfills the Jacobi identity, i.e.

$$[f \stackrel{\star}{,} [g \stackrel{\star}{,} h]] + [g \stackrel{\star}{,} [h \stackrel{\star}{,} f]] + [h \stackrel{\star}{,} [f \stackrel{\star}{,} g]] = 0. \tag{3.12}$$

For the antisymmetric part π of the first order term B_1, i.e.

$$[f \stackrel{\star}{,} g] = h\,\pi(f, g) + \mathcal{O}(2), \tag{3.13}$$

this means that it has to be a Poisson structure. Expressed in some local coordinates as $\pi = \frac{1}{2}\pi^{ij}\partial_i \wedge \partial_j$, this means that it has to fulfill

$$\pi^{ij}\partial_j\pi^{kl} + \pi^{kj}\partial_j\pi^{li} + \pi^{lj}\partial_j\pi^{ik} = 0. \tag{3.14}$$

Therefore for every $\star$-product, there is a Poisson structure related to it. On the other hand, if we start with some Poisson structure, we can always construct a corresponding $\star$-product [71], the formality $\star$-product we will present in chapter 3.4.

3.2 $\star$-products by operator ordering

We will now show how to construct $\star$-products for associative algebras that are defined by commutator relations

$$\mathcal{R}\ : \quad [\hat{x}^i, \hat{x}^j] = ih\hat{c}^{ij} \tag{3.15}$$

in the same way as we constructed the normal ordered $\star$-product in (3.6). More abstractly, such an algebra can be defined as

$$\mathcal{A} = \mathbb{C}\langle \hat{x}^1, ..., \hat{x}^n \rangle[[h]]/\mathcal{R}, \tag{3.16}$$

where we allow formal power series in the deformation parameter h. As we treat h as a formal parameter, such an algebra always has the Poincare-Birkoff-Witt property, i.e. the vector space of the noncommutative polynomials of a certain degree has the same dimension as if the coordinates were commutative. Especially this means that we can map the basis of the commutative algebra $\mathbb{C}[x^1, ..., x^n][[h]]$ onto a basis of the noncommutative algebra $\mathcal{A}$. Such an ordering prescription

$$\rho : \mathbb{C}[x^1, ..., x^n][[h]] \to \mathbb{C}\langle \hat{x}^1, ..., \hat{x}^n \rangle[[h]]/\mathcal{R} \tag{3.17}$$

is then an isomorphism of vector spaces. With it, we can introduce a $\star$-product as

$$f \star g = \rho^{-1}(\rho(f)\rho(g)), \tag{3.18}$$

making the commutative algebra equipped with the $\star$-product isomorphic to the noncommutative algebra $\mathcal{A}$. Two ordering prescriptions ρ and ρ' of the same algebra are always related by a similarity transformation T as $\rho = \rho' T$ with

$$T = id + \sum_{i=1}^{\infty} h^i T_i, \tag{3.19}$$

where the T_i are differential operators. The corresponding $\star$-products $\star$ and $\star'$ are then related by

$$f \star g = \rho^{-1}(\rho(f)\rho(g)) = T^{-1}\rho'^{-1}(\rho'(Tf)T\rho'(Tg)) = T^{-1}((Tf) \star' (Tg)). \tag{3.20}$$

3.3 The Weyl-ordered $\star$-product

In this chapter we will construct the Weyl-ordered $\star$-product of a general noncommutative algebra up to second order. Weyl-ordering means that we use totally symmetric ordering for the generators. We start with an algebra generated by N elements $\hat{x}^i$ and relations

$$[\hat{x}^i, \hat{x}^j] = \hat{c}^{ij}(\hat{x}), \tag{3.21}$$

where we have suppressed the explicit dependence of $\widehat{c}$ on a formal deformation parameter, but we will always assume that it is at least of order 1. For such an algebra we will calculate a $\star$-product up to second order. Let

$$f(p) = \int d^n x\, f(x) e^{ip_i x^i} \tag{3.22}$$

be the Fourier transform of f. Then the Weyl ordered operator associated to f is defined by

$$W(f) = \int \frac{d^n p}{(2\pi)^n} f(p) e^{-ip_i \hat{x}^i} \tag{3.23}$$

(see e. g. [78]) . Every monomial of coordinate functions is mapped to the corresponding Weyl ordered monomial of the algebra. We note that

$$W(e^{iq_i x^i}) = e^{iq_i \hat{x}^i}. \tag{3.24}$$

The Weyl ordered ⋆-product is defined by the equation

$$W(f \star g) = W(f)W(g). \tag{3.25}$$

If we insert the Fourier transforms of f and g we get

$$f \star g = \int \frac{d^n k}{(2\pi)^n} \int \frac{d^n p}{(2\pi)^n} f(k) g(p)\, W^{-1}(e^{-ik_i \hat{x}^i} e^{-ip_i \hat{x}^i}). \tag{3.26}$$

We are therefore able to write down the ⋆-product of the two functions if we know the form of the last expression. For this we expand it in terms of commutators. We use

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B}} R(\hat{A}, \hat{B}) \tag{3.27}$$

with

$$\begin{aligned} R(\hat{A}, \hat{B}) &= 1 + \frac{1}{2}[\hat{A}, \hat{B}] \\ &- \frac{1}{6}[\hat{A} + 2\hat{B}, [\hat{A}, \hat{B}]] + \frac{1}{8}[\hat{A}, \hat{B}][\hat{A}, \hat{B}] + \mathcal{O}(3). \end{aligned} \tag{3.28}$$

If we set $\hat{A} = -ik_i \hat{x}^i$ and $\hat{B} = -ip_i \hat{x}^i$, the above-mentioned expression becomes

$$\begin{aligned} & W^{-1}(e^{-ik_i \hat{x}^i} e^{-ip_i \hat{x}^i}) = \\ & e^{-i(k_i+p_i)x^i} + \frac{1}{2}(-ik_i)(-ip_j) W^{-1}(e^{-i(k_i+p_i)\hat{x}^i}[\hat{x}^i, \hat{x}^j]) \\ & -\frac{1}{6}(-i)(k_m + 2p_m)(-ik_i)(-ip_j) W^{-1}(e^{-i(k_i+p_i)\hat{x}^i}[\hat{x}^m, [\hat{x}^i, \hat{x}^j]]) \\ & +\frac{1}{8}(-ik_m)(-ip_n)(-ik_i)(-ip_j) W^{-1}(e^{-i(k_i+p_i)\hat{x}^i}[\hat{x}^m, \hat{x}^n][\hat{x}^i, \hat{x}^j]) \\ & +\mathcal{O}(3). \end{aligned} \tag{3.29}$$

If we assume that the commutators of the generators are written in Weyl ordered form

$$\hat{c}^{ij} = W(c^{ij}), \tag{3.30}$$

we see that

$$[\hat{x}^m, [\hat{x}^i, \hat{x}^j]] = W(c^{ml} \partial_l c^{ij}) + \mathcal{O}(3), \tag{3.31}$$

$$[\hat{x}^m, \hat{x}^n][\hat{x}^i, \hat{x}^j] = W(c^{mn} c^{ij}) + \mathcal{O}(3). \tag{3.32}$$

Further we can derive

$$
\begin{aligned}
W^{-1}(e^{-iq_i\hat{x}^i}W(f(x))) &= W^{-1}\Big(\int \frac{d^np}{(2\pi)^n} f(p) e^{-i(q_i+p_i)\hat{x}^i} R(-iq_i\hat{x}^i, -ip_i\hat{x}^i)\Big) \\
&= W^{-1}\bigg(W\Big(\int\int \frac{d^np}{(2\pi)^n} f(p) e^{-i(q_i+p_i)x^i} \times \\
&\qquad (1+\frac{1}{2}(-ip_i)(-iq_j)[x_i,x_j])\Big)\bigg) + \mathcal{O}(2) \\
&= e^{-iq_ix^i} f(x) + \frac{1}{2} e^{-iq_ix^i}(-iq_i)c^{ij}\partial_j f(x) + \mathcal{O}(2),
\end{aligned}
\tag{3.33}
$$

using

$$
\partial_j f(x) = \int \frac{d^np}{(2\pi)^n} f(p)(-ip_j)e^{-ip_ix^i}.
$$

Putting all this together yields

$$
\begin{aligned}
W^{-1}(e^{-ik_i\hat{x}^i}e^{-ip_i\hat{x}^i}) &= e^{-i(k_i+p_i)x^i}\bigg(1+\frac{1}{2}c^{ij}(-ik_i)(-ip_j) \\
&+ \frac{1}{8}c^{mn}c^{ij}(-ik_m)(-ip_n)(-ik_i)(-ip_j) \\
&+ \frac{1}{12}c^{ml}\partial_l c^{ij}(-i)(k_m-p_m)(-ik_i)(-ip_j)\bigg) \\
&+ \mathcal{O}(3),
\end{aligned}
\tag{3.34}
$$

and we can write down the Weyl ordered ⋆-product up to second order for an arbitrary algebra

$$
\begin{aligned}
f\star g &= fg + \frac{1}{2}c^{ij}\partial_i f\partial_j g \\
&+ \frac{1}{8}c^{mn}c^{ij}\partial_m\partial_i f\partial_n\partial_j g \\
&+ \frac{1}{12}c^{ml}\partial_l c^{ij}(\partial_m\partial_i f\partial_j g - \partial_i f\partial_m\partial_j g) + \mathcal{O}(3).
\end{aligned}
\tag{3.35}
$$

Let us collect some properties of the just calculated ⋆-product. First

$$
[x^i \overset{\star}{,} x^j] = c^{ij} \tag{3.36}
$$

is the Weyl ordered commutator of the algebra. Further, if there is a conjugation on the algebra and if we assume that the noncommutative coordinates are real $\overline{\hat{x}^i} = \hat{x}^i$, then the Weyl ordered monomials are real, too. This is also true for the monomials of the commutative coordinate functions. Therefore this $\star$-product respects the ordinary complex conjugation

$$\overline{f \star g} = \overline{g} \star \overline{f}. \tag{3.37}$$

On the level of the Poisson tensor this means

$$\overline{c^{ij}} = -c^{ij}. \tag{3.38}$$

3.4 The formality $\star$-product

The Weyl-ordered $\star$-product of chapter (3.3) is very useful for explicit calculations, but these can only be done in a perturbative way order by order. Also, it is only known in general up to the second order we calculated here. For closed expressions and questions of existence, Kontsevich's formality $\star$-product [71] is the better choice. It is known to all orders and comes with a strong mathematical framework that can be used for further constructions.

This mathematical framework, known as Kontsevich's formality map [71], is a very useful tool for studying the relations between Poisson tensors and $\star$-products. To make use of the formality map we first want to recall some definitions. A polyvector field is a skew-symmetric tensor in the sense of differential geometry. Every n-polyvector field α may locally be written as

$$\alpha = \alpha^{i_1 \ldots i_n} \, \partial_{i_1} \wedge \ldots \wedge \partial_{i_n}. \tag{3.39}$$

We see that the space of polyvector fields can be endowed with a grading n. For polyvector fields there is a grading respecting bracket that in a natural way generalizes the Lie bracket $[\,\cdot\,,\,\cdot\,]_L$ of two vector fields, the Schouten-Nijenhuis bracket. For an exact definition see A.1.1. If π is a Poisson tensor, the Hamiltonian vector field H_f for a function f is

$$H_f = [\pi, f]_S = -\pi^{ij} \partial_i f \partial_j. \tag{3.40}$$

Note that $[\pi, \pi]_S = 0$ is the Jacobi identity of a Poisson tensor.

On the other hand a n-polydifferential operator is a multilinear map that maps n functions to a function. For example, we may write a 1-polydifferential operator D as

$$D(f) = D_0 f + D_1^i \partial_i f + D_2^{ij} \partial_i \partial_j f + \ldots. \tag{3.41}$$

The ordinary multiplication $\cdot$ is a 2-differential operator. It maps two functions to one function. Again the number n is a grading on the space of polydifferential operators. Now the Gerstenhaber bracket $[\,\cdot\,,\,\cdot\,]_G$ is natural and respects the grading. For an exact definition see A.1.2.

The formality map is a collection of skew-symmetric multilinear maps U_n, $n = 0, 1, \ldots$, that maps n polyvector fields to a m-differential operator. To be more specific let $\alpha_1, \ldots, \alpha_n$ be polyvector fields of grade $k_1, \ldots, k_n$. Then $U_n(\alpha_1, \ldots, \alpha_n)$ is a polydifferential operator of grade

$$m = 2 - 2n + \sum_i k_i. \tag{3.42}$$

In particular the map U_1 is a map from a k-vectorfield to a k-differential operator. It is defined by

$$U_1(\alpha^{i_1\ldots i_n}\partial_{i_1} \wedge \ldots \wedge \partial_{i_n})(f_1, \ldots, f_n) = \alpha^{i_1\ldots i_n}\partial_{i_1} f_1 \cdot \ldots \cdot \partial_{i_n} f_n. \tag{3.43}$$

The formality maps U_n fulfill the formality condition [71, 7]

$$Q_1' U_n(\alpha_1, \ldots, \alpha_n) + \frac{1}{2} \sum_{\substack{I \sqcup J = \{1,\ldots,n\} \\ I,J \neq \emptyset}} \epsilon(I, J) Q_2'(U_{|I|}(\alpha_I), U_{|J|}(\alpha_J)) \tag{3.44}$$

$$= \frac{1}{2} \sum_{i \neq j} \epsilon(i, j, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, n) U_{n-1}(Q_2(\alpha_i, \alpha_j), \alpha_1, \ldots, \widehat{\alpha}_i, \ldots, \widehat{\alpha}_j, \ldots, \alpha_n).$$

The hats stand for omitted symbols, $Q_1'(\Upsilon) = [\Upsilon, \mu]$ with μ being ordinary multiplication and $Q_2'(\Upsilon_1, \Upsilon_2) = (-1)^{(|\Upsilon_1|-1)|\Upsilon_2|}[\Upsilon_1, \Upsilon_2]_G$ with $|\Upsilon_s|$ being the degree of the polydifferential operator Υ_s, i.e. the number of functions it is acting on. For polyvectorfields $\alpha_s^{i_1\ldots i_{k_s}}\partial_{i_1} \wedge \ldots \wedge \partial_{i_{k_s}}$ of degree k_s we have $Q_2(\alpha_1, \alpha_2) = -(-1)^{(k_1-1)k_2}[\alpha_2, \alpha_1]_S$.

For a bivectorfield π we can now define a bidifferential operator

$$\star = \sum_{n=0}^{\infty} \frac{1}{n!} U_n(\pi, \ldots, \pi) \tag{3.45}$$

i.e.

$$f \star g = \sum_{n=0}^{\infty} \frac{1}{n!} U_n(\pi, \ldots, \pi)(f, g). \tag{3.46}$$

To see that the formality $\star$-product is associative, we first define the special map

$$\Phi(\alpha) = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} U_n(\alpha, \pi, \ldots, \pi). \tag{3.47}$$

Using the formality condition (3.44) we calculate that

$$[\star, \star]_G = \Phi([\pi, \pi]_S), \tag{3.48}$$

where $[\star, \star]_G = 0$ means that the $\star$-product is associative. This follows from the fact that π is a Poisson tensor, i.e $[\pi\,\pi]_S = 0$. Note that the definition (3.46) would be equally valid for general bivector fields π', but the resulting product would cease to be associative. Nevertheless the non-associativity would be controlled by (3.48).

3.5 The Jambor-Sykora $\star$-product

The formality $\star$-product of the last chapter is very useful for abstract proofs (and in fact we will use it for constructing a SW-map to all orders in chapter 5.3, but it is too complicated for explicit calculations. But for special cases where the Poisson structure can be expressed in terms of commuting vector fields, there is a $\star$-product that is both known to all orders and easy to handle in calculations, the Jambor-Sykora $\star$-product [62]. For commuting vectorfields $X_a = X_a^i \partial_i$ (i.e. $[X_a, X_b] = 0$) and a constant matrix σ the Jambor-Sykora $\star$-product reads

$$f \star_\sigma g = m \cdot e^{\sigma^{ab} X_a \otimes X_b} f \otimes g, \tag{3.49}$$

where $m \cdot (f \otimes g) = fg$. The constant matrix σ can be written as $\sigma = \sigma_{as} + \sigma_s$ with σ_{as} antisymmetric and σ_s symmetric. There is an equivalence transformation

$$\rho = e^{\frac{1}{2}\sigma_s^{ab} X_a X_b} \tag{3.50}$$

from the antisymmetric $\star$-product

$$f \star_{as} g = m \cdot e^{\sigma_{as}^{ab} X_a \otimes X_b} f \otimes g \tag{3.51}$$

to the full one (3.49)

$$f \star_{as} g = \rho^{-1}(\rho(f) \star_\sigma \rho(g)). \tag{3.52}$$

Note that for real vectorfields X_a and σ_{as} imaginary, ordinary complex conjugation is an involution of the antisymmetric $\star$-product, i.e.

$$\overline{f \star_{as} g} = \overline{g} \star_{as} \overline{f}. \tag{3.53}$$

For the full $\star$-product we can pull back this property by ρ and the involution now is $\rho\overline{\rho^{-1}}$. On a function f this reads $\rho(\overline{\rho^{-1}(f)})$, and we have

$$\begin{aligned} \rho(\overline{\rho^{-1}(f \star_\sigma g)}) &= \rho(\overline{\rho^{-1} f \star_{as} \rho^{-1} g}) \\ &= \rho(\overline{\rho^{-1} g} \star_{as} \overline{\rho^{-1} f}) \\ &= \rho(\overline{\rho^{-1} g}) \star_\sigma \rho(\overline{\rho^{-1} f}). \end{aligned} \tag{3.54}$$

3.6 Traces

For the Moyal-Weyl $\star$-product, ordinary integration still had the trace property, i.e. it was invariant under cyclic permutations of the elements in the integrand. Unluckily, this is in general no longer the case for the more complicated $\star$-products of this chapter. But the cyclicity of integration is crucial for turning gauge-covariant objects into gauge invariant ones. Therefore, we have to guarantee the trace property of the integral by introducing a measure function Ω. For many $\star$-products the trace may then be written as

$$tr\, f = \int d^{2n}x\, \Omega(x)\, f(x). \tag{3.55}$$

Due to the cyclicity of the trace the measure function Ω has to fulfill

$$\partial_i(\Omega\theta^{ij}) = 0 \tag{3.56}$$

which can easily be seen by using partial integration. If we take the Poisson structure θ^{ij} to be invertible, the inverse of the Pfaffian

$$\frac{1}{\Omega} = Pf(\theta) = \sqrt{det(\theta)} = \frac{1}{2^n n!}\epsilon_{i_1 i_2 \cdots i_{2n}}\theta^{i_1 i_2}\cdots\theta^{i_{2n-1}i_{2n}} \tag{3.57}$$

is a solution to this equation. Unluckily, there is no such formula for $\star$-products whose Poisson structures are not invertible.

If equation (3.56) is fulfilled, cyclicity is only guaranteed to first order. In principle we have to calculate higher orders of Ω according to the $\star$-product chosen. Nevertheless there can always be found a $\star$-product so that a measure function fulfilling (3.56) guarantees cyclicity to all orders [38].

3.7 Example: $\star$-products for the κ-deformed plane

We will exemplify the ideas of the last chapter by applying them to the algebra generated by x and y with commutation relations

$$[x, y] = -iax. \tag{3.58}$$

This is the 2-dimensional version of what is known as κ-deformed spacetime. The generalization to higher dimensions is straightforward.

3.7.1 The Weyl-ordered $\star$-product

The Poisson structure for this algebra quite obviously is

$$\{f, g\}_p = -iax\partial_x f\partial_y g + ia\partial_y f x\partial_x g, \tag{3.59}$$

and the Poisson tensor therefore

$$c^{ij} = -iax\delta_x^i\delta_y^j + iax\delta_y^i\delta_x^j. \tag{3.60}$$

As c^{ij} is linear in the coordinates, the Weyl-ordering of the expression doesn't play a role. Inserting (3.60) into (3.35) produces the Weyl-ordered $\star$-product up to second order for (3.58)

$$\begin{aligned} f\star g &= fg - \frac{ia}{2}x(\partial_x f\partial_y g - \partial_y f\partial_x g) \\ &-\frac{a^2}{8}x^2(\partial_x^2 f\partial_y^2 g - 2\partial_x\partial_y f\partial_x\partial_y g + \partial_y^2 f\partial_x^2 g) \\ &+\frac{a^2}{12}x(\partial_x\partial_y f\partial_y g + \partial_y f\partial_x\partial_y g - \partial_x f\partial_y^2 g - \partial_y^2 f\partial_x g) + \mathcal{O}(3). \end{aligned} \tag{3.61}$$

3.7.2 The Jambor-Sykora $\star$-product

If we choose vectorfields

$$X_1 = x\partial_x \quad \text{and} \quad X_2 = -a\partial_y \tag{3.62}$$

and $\sigma = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}$ (see also [62]), the Jambor-Sykora $\star$-product (3.49) will reproduce the algebra (3.58). This $\star$-product corresponds to normal ordering. It reads

$$f\star_\sigma g = m\cdot e^{-ia\,x\partial_x\otimes\partial_y} f\otimes g, \tag{3.63}$$

while the antisymmetric $\star$-product (3.51) reads

$$f\star_{as} g = m\cdot e^{-\frac{ia}{2}x\partial_x\otimes\partial_y + \frac{ia}{2}\partial_y\otimes x\partial_x} f\otimes g. \tag{3.64}$$

Notice that the antisymmetric $\star$-product differs (3.64) from the Weyl-ordered $\star$-product (3.61) at second order and therefore does not correspond to symmetric ordering. The equivalence transformation (3.52) between (3.63) and (3.64) is

$$\rho = e^{-\frac{i}{2}ax\partial_x\partial_y}. \tag{3.65}$$

Chapter 4

Derivatives and Derivations

We are now able to represent more complicated algebras on ordinary functions by using the $\star$-products of the last chapter. But there is an important element still missing: derivatives. In the canonical case, we could just use the ordinary derivatives to construct noncommutative actions. This was unproblematic as the usual derivatives had an undeformed Leibniz rule, i.e.

$$\partial_i(f \star g) = \partial_i f \star g + f \star \partial_i g. \tag{4.1}$$

But with more complicated $\star$-products, this is in general no longer the case. The derivatives do not only act on the functions, but also on the $\star$-product, which now depends on the coordinates. Symbolically we can write

$$\partial_i(f \star g) = \partial_i f \star g + f \star \partial_i g + f(\partial_i \star) g, \tag{4.2}$$

where $\partial_i \star$ means that the derivative is acting on the bidifferential operator $\star$ represents. This additional term can already be seen at the level of the Poisson structure. Take e.g. the Poisson structure (3.59) of the κ-deformed plane. The derivative ∂_y in the y-direction does not act on it, so that we still have

$$\partial_y\{f, g\}_p = \{\partial_y f, g\}_p + \{f, \partial_y g\}_p, \tag{4.3}$$

but in the x-direction, things are different:

$$\partial_x\{f, g\}_p = \partial_x(-iax\partial_x f \partial_y g + ia\partial_y f x \partial_x g) = \{\partial_x f, g\}_p + \{f, \partial_x g\}_p + \frac{\{f, g\}_p}{x} \tag{4.4}$$

for $x \neq 0$. The same is true for derivatives acting on $\star$-products: the usual Leibniz rule is deformed. For the antisymmetric $\star$-product on the κ-deformed plane, this deformed Leibniz rule reads

$$\partial_x(f \star_{as} g) = (\partial_x f) \star_{as} (e^{-\frac{i}{2}a\partial_y} g) + (e^{\frac{i}{2}a\partial_y} f) \star_{as} (\partial_x g), \tag{4.5}$$

see (4.36). Such derivatives with a deformed Leibniz rule can nevertheless be used to construct gauge theory [34, 33, 35], but it is far more involved than in the canonical case. Especially, the gauge fields associated to these derivatives become derivative valued (see also chapter 5.1.1).

Here, we will pursue a different approach. As we saw in (4.3), the derivative in the y-direction did act on the Poisson structure as in the canonical case. And on the antisymmetric $\star$-product, its Leibniz rule is indeed undeformed, i.e

$$\partial_y(f \star_{as} g) = (\partial_y f) \star_{as} g + f \star_{as} (\partial_y g), \tag{4.6}$$

see (4.35). Such a derivative can be gauged much in the same way as in the canonical case, leading to function valued gauge fields. But before we actually construct gauge theory in chapter (5), we will first have a closer look at objects that behave like ∂_y did in our example, i.e we will be looking for vector fields that commute with the Poisson structure and how we can use them to get differential operators that have an undeformed Leibniz rule with the corresponding $\star$-product. These differential operators with an undeformed Leibniz rule we will call derivations of the $\star$-product algebra.

4.1 Derivations

We will be able to identify derivations of $\star$-product algebras with what we call Poisson vector fields of the Poisson structure associated with the $\star$-product, i.e. vector fields X with

$$X\{f, g\}_p = \{Xf, g\}_p + \{f, Xg\}_p. \tag{4.7}$$

If we locally write

$$\{f, g\}_p = \pi(f, g) = \pi^{ij}\partial_i f \partial_j g \quad \text{and} \quad X = X^i\partial_i, \tag{4.8}$$

this is equivalent to saying that the Schouten-Nijenhuis bracket (see A.1.1) of the vector field X with the Poisson structure π vanishes

$$[X, \pi]_S = 0 \quad \Leftrightarrow \quad X^k\partial_k\pi^{ij} - \pi^{ik}\partial_k X^j + \pi^{jk}\partial_k X^i, \tag{4.9}$$

or that the vector field X commutes with the Poisson structure π. If we have such a Poisson vector field X, we are looking for a differential operator δ_X with the following property

$$\delta_X(f \star g) = \delta_X f \star g + f \star \delta_X g. \tag{4.10}$$

Such a map δ from the vectorfields to the differential operators, which maps the derivations of the Poisson manifold $T_\pi M = \{X \in TM | [X, \pi]_S = 0\}$ to the derivations of the $\star$-product $D_\star M = \{\delta \in D_{poly} | [\delta, \star]_G = 0\}$, can be constructed both for the Weyl ordered $\star$-product (see 4.2), for the formality $\star$-product (see 4.3) and the Jambor-Sykora $\star$-product (see 4.4). Here we want to investigate the general properties of such a map δ. For this we expand it on a local patch in terms of partial derivatives

$$\delta_X = \delta_X^i \partial_i + \delta_X^{ij} \partial_i \partial_j + \cdots . \tag{4.11}$$

Due to its property to be a derivation, δ_X is completely determined by the first term $\delta_X^i \partial_i$. This means that if the first term is zero, the other terms have to vanish, too. If further e is an arbitrary derivation of the $\star$-product, there must exist a vector field X_e such that

$$\delta_{X_e} = e. \tag{4.12}$$

If $X, Y \in T_\pi M$, then $[\delta_X, \delta_Y]$ is again a derivation of the $\star$-product and we can conclude that

$$[\delta_X, \delta_Y] = \delta_{[X,Y]_\star}, \tag{4.13}$$

where $[X, Y]_\star$ is a deformation of the ordinary Lie bracket of vector fields. Obviously it is linear, skew-symmetric and fulfills the Jacobi identity.

With the help of the map δ and the deformed bracket $[\,\cdot\,,\,\cdot\,]_\star$ it is also possible to construct noncommutative forms over the derivations of the$\star$-product algebra, a formulation we will present in appendix A.2.

4.2 Derivations for the Weyl-ordered $\star$-product

We now want to calculate the derivationsδ_X of the Weyl-ordered $\star$-product (3.35) from the derivations X of the Poisson structure c^{ij} up to second order. We assume that δ_X can be expanded in the following way

$$\delta_X = X^i \partial_i + \delta_X^{ij} \partial_i \partial_j + \delta_X^{ijk} \partial_i \partial_j \partial_k + \cdots . \tag{4.14}$$

Expanding the equation

$$\delta_X(f \star g) = \delta_X(f) \star g + f \star \delta_X(g) \tag{4.15}$$

order by order and using $[X, c]_S = 0$ we find that

$$\begin{aligned} \delta_X \quad = \quad & X^i \partial_i - \frac{1}{12} c^{lk} \partial_k c^{im} \partial_l \partial_m X^j \partial_i \partial_j \\ & + \frac{1}{24} c^{lk} c^{im} \partial_l \partial_i X^j \partial_k \partial_m \partial_j + \mathcal{O}(3). \end{aligned} \tag{4.16}$$

For $[\,\cdot\,,\,\cdot\,]_\star$ we simply calculate $[\delta_X, \delta_Y]$ and get

$$\begin{aligned}
[X,Y]_\star &= [X,Y]_L \qquad (4.17)\\
&-\frac{1}{12}(c^{lk}\partial_k c^{im}\partial_l\partial_m X^j \partial_i\partial_j Y^n - c^{lk}\partial_k c^{im}\partial_l\partial_m Y^j \partial_i\partial_j X^n)\partial_n\\
&+\frac{1}{24}(c^{lk}c^{im}\partial_l\partial_i X^j \partial_k\partial_m\partial_j Y^n - c^{lk}c^{im}\partial_l\partial_i Y^j \partial_k\partial_m\partial_j X^n)\partial_n\\
&+\mathcal{O}(3).
\end{aligned}$$

4.3 Derivations for the formality $\star$-product

We saw in chapter 3.4 that the formality $\star$-product can be constructed from the maps U_n from the polyvectorfields to the polydifferential operators as

$$f \star g = \sum_{n=0}^{\infty} \frac{1}{n!} U_n(\pi, \ldots, \pi)(f, g). \qquad (4.18)$$

With these maps, we can further define the special polydifferential operators

$$\begin{aligned}
\Phi(\alpha) &= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} U_n(\alpha, \pi, \ldots, \pi), \qquad (4.19)\\
\Psi(\alpha_1, \alpha_2) &= \sum_{n=2}^{\infty} \frac{1}{(n-2)!} U_n(\alpha_1, \alpha_2, \pi, \ldots, \pi). \qquad (4.20)
\end{aligned}$$

For X a vectorfield, we define

$$\delta_X = \Phi(X). \qquad (4.21)$$

Using formula (3.42) we see that it is indeed a 1-differential operator. We will now use the formality condition (3.44) to have a closer look at its properties.

For g a function and X and Y vectorfields, we see that $\Psi(X, Y)$ is a function and we go on to calculate

$$\begin{aligned}
[\delta_X, \star]_G &= \Phi([X, \pi]_S), \qquad (4.22)\\
[\delta_X, \delta_Y]_G + [\Psi(X,Y), \star]_G &= \delta_{[X,Y]_S} \qquad (4.23)\\
&\quad +\Psi([\pi, Y]_S, X) - \Psi([\pi, X]_S, Y).
\end{aligned}$$

If π is a Poisson tensor, i. e. $[\pi, \pi]_S = 0$ and if X and Y are Poisson vector fields, i. e. $[X, \pi]_S = [Y, \pi]_S = 0$, the relations (4.22) and (4.23) become

$$\begin{aligned}
\delta_X(f \star g) &= \delta_X(f) \star g + f \star \delta_X(g), \qquad (4.24)\\
([\delta_X, \delta_Y] - \delta_{[X,Y]_L})(g) &= [\Psi(X,Y) \stackrel{\star}{,} g]. \qquad (4.25)
\end{aligned}$$

when evaluated on functions. We see that δ really is the map we were looking for, i.e. it maps derivations of the Poisson structure to derivations of the associated formality $\star$-product.

Additionally the map δ preserves the bracket up to an inner derivation. This can be cast into the following form:

$$[\delta_X, \delta_Y] = \delta_{[X,Y]_\star} \tag{4.26}$$

with

$$[X,Y]_\star = [X,Y]_L + H_{\Phi^{-1}\Psi(X,Y)}. \tag{4.27}$$

4.4 Derivations for the Jambor-Sykora $\star$-product

While looking for derivatives for the Jambor-Sykora $\star$-products, we can confine ourselves to the antisymmetric case (3.51), as all the properties can be pulled back to the full one (3.49) via the equivalence transformation (3.52).

In the framework of Kontsevich's formality $\star$-product, we saw that we could construct a map from the vectorfields to the differential operators that maps derivations of the Poisson structure to derivations of the formality $\star$-product to all orders.

We will now look for such a map that maps derivations of the Poisson structure

$$\pi = \frac{1}{2}\sigma_{as}^{ab} X_a^i \partial_i \wedge X_b^j \partial_j \tag{4.28}$$

associated with the Jambor-Sykora $\star$-product to derivations of the antisymmetric Jambor-Sykora $\star$-product (3.51). As the vectorfields X_a commute with each other and σ_{as} is antisymmetric, (3.51) can be rewritten in terms of the Poisson structure as

$$f \star_{as} g = m(e^{\sigma_{as}^{ab} X_a^i \partial_i \otimes X_b^j \partial_j}(f \otimes g)) = m(e^{\pi}(f \otimes g)). \tag{4.29}$$

In this notation it is obvious that vectorfields commuting with the Poisson structure (4.28) will be derivations of the $\star$-product (4.29) as well. This means that for vectorfields Y with

$$[Y,\pi]_S = 0 \quad \Leftrightarrow \quad [Y \otimes 1 + 1 \otimes Y, \pi] = 0 \quad \Leftrightarrow \quad Y\{f,g\}_P = \{Yf,g\}_P + \{f,Yg\}_P$$

we also have

$$\begin{aligned} Y(f \star_{as} g) &= m((Y \otimes 1 + 1 \otimes Y)e^{\pi}(f \otimes g)) \\ &= m(e^{\pi}(Y \otimes 1 + 1 \otimes Y)(f \otimes g)) \\ &= Yf \star_{as} g + f \star_{as} Yg. \end{aligned} \tag{4.30}$$

We therefore do not get higher order terms, the map from the vectorfields to the differential operators is the inclusion. This also implies that the algebra of the vectorfields remains undeformed under quantization.

For the coproduct of general vectorfields acting on the $\star$-product (3.51), we get a deformed Leibniz rule

$$\begin{aligned} X(f \star_{as} g) = & \\ & \sum_{n=0}^{\infty} \frac{1}{n!} \sigma_{as}^{a_1 a_2} \ldots \sigma_{as}^{a_n b_n} ([\ldots[X, X_{a_1}], X_{a_2}], \ldots], X_{a_n}] f) \star_{as} (X_{b_1} \ldots X_{b_n} g) \\ & + \sum_{n=0}^{\infty} \frac{1}{n!} \sigma_{as}^{a_1 a_2} \ldots \sigma_{as}^{a_n b_n} (X_{a_1} \ldots X_{a_n} f) \star_{as} ([\ldots[X, X_{b_1}], X_{b_2}], \ldots], X_{b_n}] g). \end{aligned} \tag{4.31}$$

We can use the equivalence transformation (3.50) to pull back these structures to the full $\star$-product (3.49). Note that the Poisson structure of the full $\star$-product (3.49) only depends on the antisymmetric part of σ and therefore is (4.28), the same as for the antisymmetric $\star$-product (3.51).

A vectorfield Y commuting with the Poisson structure of the full $\star$-product (3.49) will be a derivation of the antisymmetric $\star$-product (3.51). We use (3.52) to get

$$\rho Y \rho^{-1} (f \star_{\sigma} g) = (\rho Y \rho^{-1} f) \star_{\sigma} g + f \star_{\sigma} (\rho Y \rho^{-1} g) \tag{4.32}$$

from

$$Y(f \star_{as} g) = Y f \star_{as} g + f \star_{as} Y g. \tag{4.33}$$

The map δ_X from the vectorfields to the differential operators for the full $\star$-product (3.49) is therefore given by

$$\delta_X = \rho X \rho^{-1}. \tag{4.34}$$

The algebra of these deformed vectorfields is isomorphic to the algebra of the undeformed vectorfields. Also the coalgebra of the deformed vectorfields with respect to the full $\star$-product (3.49) is isomorphic to the coalgebra of the undeformed vectorfields with respect to the antisymmetric $\star$-product (3.51).

4.5 Example: Derivatives and Derivations for the κ-deformed plane

We will now exemplify the ideas of this chapter by applying them to the κ-deformed plane we already studied in chapter 3.7. We will now concentrate on the Jambor-Sykora $\star$-products, where the basic mechanisms at work can be best seen.

4.5.1 The antisymmetric case

For the antisymmetric $\star$-product (3.64), the vectorfields are undeformed, but they may acquire nontrivial coproducts (4.31). The derivative in the y-direction commutes with the Poisson structure and is therefore a derivation of the $\star$-product:

$$\partial_y(f \star_{as} g) = (\partial_y f) \star_{as} g + f \star_{as} (\partial_y g). \tag{4.35}$$

But the derivative in the x-direction does not commute with the Poisson structure and has a deformed Leibniz rule:

$$\partial_x(f \star_{as} g) = (\partial_x f) \star_{as} (e^{-\frac{i}{2}a\partial_y} g) + (e^{\frac{i}{2}a\partial_y} f) \star_{as} (\partial_x g). \tag{4.36}$$

Multiplication from the left with a function (without $\star$-multiplication) also acquires a derivative quality:

$$\begin{aligned} y(f \star_{as} g) &= (yf) \star_{as} g - \frac{i}{2}af \star_{as} (x\partial_x g) \\ &= f \star_{as} (yg) + \frac{i}{2}a(x\partial_x f) \star_{as} g \end{aligned} \tag{4.37}$$

and

$$\begin{aligned} x(f \star_{as} g) &= (xf) \star_{as} (e^{\frac{i}{2}a\partial_y} g) \\ &= (e^{-\frac{i}{2}a\partial_y} f) \star_{as} (xg). \end{aligned} \tag{4.38}$$

This also implies the following relations

$$(xf) \star_{as} g = (e^{-\frac{i}{2}a\partial_y} f) \star_{as} (xe^{-\frac{i}{2}a\partial_y} g) \tag{4.39}$$

and

$$(yf) \star_{as} g = f \star_{as} (yg) + \frac{i}{2}ax\partial_x(f \star_{as} g). \tag{4.40}$$

If we combine (4.36) and (4.38) to calculate the coproduct of $x\partial_x$, we see that it is indeed a derivation of the $\star$-product, as expected.

$$x\partial_x(f \star_{as} g) = (x\partial_x f) \star_{as} g + f \star_{as} (x\partial_x g) \tag{4.41}$$

For the other vectorfields linear in the coordinates we get

$$x\partial_y(f \star_{as} g) = (x\partial_y f) \star_{as} (e^{\frac{i}{2}a\partial_y} g) + (e^{-\frac{i}{2}a\partial_y} f) \star_{as} (x\partial_y g), \tag{4.42}$$

$$\begin{aligned} y\partial_y(f \star_{as} g) &= (y\partial_y f) \star_{as} g + f \star_{as} (y\partial_y g) \\ &\quad -\frac{i}{2}a(\partial_y f) \star_{as} (x\partial_x g) + \frac{i}{2}a(x\partial_x f) \star_{as} (\partial_y g) \end{aligned} \tag{4.43}$$

and

$$\begin{aligned} y\partial_x(f \star_{as} g) &= (y\partial_x f) \star_{as} (e^{-\frac{i}{2}a\partial_y} g) + (e^{\frac{i}{2}a\partial_y} f) \star_{as} (y\partial_x g) \\ &\quad -\frac{i}{2}a(\partial_x f) \star_{as} (x\partial_x e^{-\frac{i}{2}a\partial_y} g) + \frac{i}{2}a(x\partial_x e^{\frac{i}{2}a\partial_y} f) \star_{as} (\partial_x g). \end{aligned} \tag{4.44}$$

4.5.2 The normal ordered case

Of course we can switch to the $\star$-product (3.63) corresponding to normal ordering by using the transformation (3.65)

$$\rho = e^{-\frac{i}{2}ax\partial_x\partial_y}. \tag{4.45}$$

The vectorfields are then mapped to the differential operators by applying (4.34). For the coordinates we get

$$\delta_y = \rho y \rho^{-1} = y - \frac{i}{2}ax\partial_x \quad \text{and} \quad \delta_x = \rho x \rho^{-1} = xe^{-\frac{i}{2}a\partial_y}, \tag{4.46}$$

revealing the derivative nature of multiplication of coordinates from the left. The derivative in the y-direction stays undeformed

$$\delta_{\partial_y} = \rho\partial_y\rho^{-1} = \partial_y, \tag{4.47}$$

the derivative in the x-direction becomes

$$\delta_{\partial_x} = \rho\partial_x\rho^{-1} = e^{\frac{i}{2}a\partial_y}\partial_x. \tag{4.48}$$

We can combine (4.46) and (4.48) to give

$$\delta_{x\partial_x} = \rho x\partial_x\rho^{-1} = \rho x\rho^{-1}\rho\partial_x\rho^{-1} = x\partial_x. \tag{4.49}$$

Note that the deformation δ acts trivially on $x\partial_x$, as it does commute with ρ. For the other vectorfields linear in the coordinates we get from (4.46,4.47,4.48)

$$\begin{aligned} \delta_{x\partial_y} &= x\partial_y e^{-\frac{i}{2}a\partial_y}, \\ \delta_{y\partial_y} &= y\partial_y - \frac{i}{2}ax\partial_x\partial_y, \\ \delta_{y\partial_x} &= y\partial_x e^{\frac{i}{2}a\partial_y} - \frac{i}{2}ax\partial_x^2 e^{\frac{i}{2}a\partial_y}. \end{aligned} \tag{4.50}$$

Chapter 5

Gauge theory on curved NC spaces

One hope associated with the application of noncommutative geometry in physics is a better description of quantized gravity. At least it should be possible to construct effective actions where traces of this unknown theory remain. If one believes that quantum gravity is in a sense a quantum field theory, then its observables are operators on a Hilbert space and therefore elements of an algebra. Some properties of this algebra should be reflected in the noncommutative geometry the effective actions are constructed on. As the noncommutativity should be induced by background gravitational fields, the classical limit of the effective actions should reduce to actions on curved spacetimes [75, 29].

In the canonical case, the gauge theory reduces in the commutative limit to a theory on flat spacetime. Therefore it is necessary to develop concepts working with more general algebras, since one would expect that curved backgrounds are related to algebras with nonconstant commutation relations. We will use the derivations of $\star$-product algebras we studied in chapter 4 to build covariant derivatives for noncommutative gauge theory. We will be able to write down a noncommutative action by linking these derivations to a frame field induced by a nonconstant metric. In the commutative limit, this action reduces to gauge theory on a curved manifold. As an example we will again study κ-deformed spacetime, where the action reduces in the commutative limit to scalar electrodynamics on a manifold with constant curvature.

We will also introduce Seiberg-Witten maps to do noncommutative gauge theory with arbitrary gauge groups. A proof of the existence of the Seiberg-Witten-map for an Abelian gauge potential will be given for the formality $\star$-product. We will also give explicit formulas for the Weyl ordered $\star$-product up to second order.

5.1 The general formalism

5.1.1 Noncommutative gauge theory

To do gauge theory on the noncommutative spaces equipped with the more complicated $\star$-products of chapter 3, we will try to follow the formalism of the canonical case as much as possible.

Fields in the fundamental representation will again transform as

$$\delta_\Lambda \Psi = i\Lambda \star \Psi. \tag{5.1}$$

The commutator of two such gauge transformations should again be a gauge transformation, i.e we again want

$$(\delta_\Lambda \delta_\Xi - \delta_\Xi \delta_\Lambda)\Psi = \delta_{i[\Xi \stackrel{\star}{,} \Lambda]}\Psi. \tag{5.2}$$

As in the canonical case, this is only possible for gauge groups $U(N)$. The first difference to the canonical case occurs when we look at the transformation properties of a derivative

$$\delta_\Lambda(\partial_i \Psi) = \partial_i(i\Lambda \star \Psi) = i(\partial_i \Lambda) \star \Psi + i\Lambda \star (\partial_i \Psi) + i\Lambda(\partial_i \star)\Psi. \tag{5.3}$$

The additional term $i\Lambda(\partial_i\star)\Psi$ is in general no longer zero, corresponding to a nontrivial coproduct of the derivative. If we now want to add a gauge field A_i to the derivative to make it gauge invariant, i.e.

$$D_i\Psi = \partial_i \Psi - iA_i \star \Psi, \tag{5.4}$$

the transformation properties of A_i also have to offset this new term to get

$$\delta_\Lambda(D_i\Psi) = i\Lambda \star D_i\Psi. \tag{5.5}$$

From this we get

$$\delta_\Lambda(A_i) \star \Psi = \partial_i \Lambda \star \Psi + i[\Lambda \stackrel{\star}{,} A_i] \star \Psi + \Lambda(\partial_i \star)\Psi, \tag{5.6}$$

which means that the gauge potential can no longer be a function, it has to be derivative valued. To see this better, we take as an example the $\star$-product (3.64) for the κ-deformed plane. The above formula then reads

$$\begin{aligned} \delta_\Lambda(A_x) \star_{as} \Psi \ &= \ (\partial_x \Lambda) \star_{as} (e^{-\frac{i}{2}a\partial_y}\Psi) \\ &\quad +((e^{\frac{i}{2}a\partial_y} - 1)\Lambda) \star_{as} (\partial_x \Psi) + i[\Lambda \stackrel{\star_{as}}{,} A_x] \star_{as} \Psi. \end{aligned} \tag{5.7}$$

To offset the terms coming from the deformed Leibniz rule for ∂_x (where additional derivatives act on the right hand side), the gauge field A_x has to become derivative valued. Gauge theory using such derivative valued gauge fields was constructed in [34, 33, 35], but we will try a different approach here.

We saw in chapter 2.3 that there is a different formulation for noncommutative gauge theory in terms of covariant coordinates. So let us see what happens if we try to gauge the coordinates with a more complicated $\star$-product. We want to have

$$\delta_\Lambda(X_i \star \Psi) = \delta_\Lambda((x_i + \widetilde{A}_i) \star \Psi) = i\Lambda \star X_i \star \Psi. \tag{5.8}$$

Therefore the gauge field $\widetilde{A}_i$ has to transform as

$$\delta_\Lambda \widetilde{A}_i = i[\Lambda \stackrel{\star}{,} x_i] + i[\Lambda \stackrel{\star}{,} \widetilde{A}_i]. \tag{5.9}$$

This means that $\widetilde{A}_i$ is still a function, because the commutator with a coordinate of course has an undeformed Leibniz rule. But there is a problem with this Ansatz: the gauge field $\widetilde{A}_i$ vanishes in the commutative limit. In the canonical case, this could be solved by defining a new field $(\theta^{-1})^{ij}\widetilde{A}_j$, but this is no longer possible as the now coordinate dependent θ^{-1} would spoil the transformation properties of the new object.

This is why we introduced derivations δ_X in chapter 4. They do have both an undeformed Leibniz rule and a nonvanishing commutative limit. So we introduce covariant derivations as

$$D_X = \delta_X - iA_X, \tag{5.10}$$

where X is a Poisson vector field. The gauge field A_X will transform as

$$\delta_\Lambda A_X = \delta_X \Lambda + i[\Lambda \stackrel{\star}{,} A_X]. \tag{5.11}$$

Then, a field strength $F_{X,Y}$ can be defined as

$$-iF_{X,Y} = [D_X \stackrel{\star}{,} D_Y] - D_{[X,Y]_\star}, \tag{5.12}$$

the properties of D and $[\cdot\,,\,\cdot]_\star$ making sure that the field strength is function-valued and transforms covariantly[1].

[1] This can also be expressed in the language of the noncommutative forms introduced in appendix A.2. A_X is the connection one form evaluated on the vector field X. It transforms like

$$\delta_\Lambda A = \delta\Lambda + i\Lambda \wedge A - iA \wedge \Lambda. \tag{5.13}$$

The covariant derivative of a field is now

$$D\Psi = \delta\Psi - iA \wedge \Psi, \tag{5.14}$$

and the field strength becomes

$$F = DA = \delta A - iA \wedge A. \tag{5.15}$$

5.1.2 Seiberg-Witten gauge theory

Up to now, we could only do noncommutative gauge theory for gauge groups$U(n)$, just as in the canonical case. We will now show how to implement the concept of Seiberg-Witten maps [95, 64] into our new setting of covariant derivations to be able to do gauge theory for general gauge groups.

Just as in the canonical case, the Seiberg-Witten maps for the fields will have to be enveloping algebra valued, but they will only depend on their commutative counterparts, therefore preserving the right number of degrees of freedom. Again we demand that their noncommutative transformation properties are determined by the transformation properties of the commutative fields they depend on.

Therefore the fields again transform as [63]

$$\delta_\alpha \Psi_\psi[a] = i\Lambda_\alpha[a] \star \Psi_\psi[a], \tag{5.17}$$

leading to the same consistency condition for the gauge parameter

$$i\delta_\alpha \Lambda_\beta - i\delta_\beta \Lambda_\alpha + [\Lambda_\alpha \stackrel{\star}{,} \Lambda_\beta] = i\Lambda_{-i[\alpha,\beta]}. \tag{5.18}$$

The transformation law for the covariantizer is now

$$\delta_\alpha(D[a](f)) = i[\Lambda_\alpha[a] \stackrel{\star}{,} D[a](f)]. \tag{5.19}$$

The Seiberg-Witten-map can be easily extended to the derivationsδ_X of the $\star$-product. The noncommutative covariant derivation $D_X[a]$ can be written with the help of a noncommutative gauge potential$A_X[a]$ now depending both on the commutative gauge potential$a$ and the Poisson vectorfield X

$$D_X[a]\Psi_\psi[a] = \delta_X \Psi_\psi[a] - iA_X[a] \star \Psi_\psi[a]. \tag{5.20}$$

It follows that the gauge potential has to transform like

$$\delta_\alpha A_X[a] = \delta_X \Lambda_\alpha[a] + i[\Lambda_\alpha[a] \stackrel{\star}{,} A_X[a]]. \tag{5.21}$$

We will give explicit formulas for the Seiberg-Witten maps in chapters 5.2 and 5.3.

One easily can show that the field strength is a covariant constant

$$DF = \delta F - iA \wedge F = 0. \tag{5.16}$$

5.1.3 Commutative actions with the frame formalism

To link the noncommutative constructions of the last chapters with commutative gauge theory, we first want to recall some aspects of classical differential geometry. Suppose we are working on a n-dimensional manifold M with metric $g_{\mu\nu}$. Then there are locally n derivatives ∂_μ which form a basis of the tangent space TM of the manifold. We can always make a local basis transformation to a frame (or "non-coordinate basis")

$$e_a = e_a{}^\mu(x)\partial_\mu, \tag{5.22}$$

(with $e_a{}^\mu(x)$ invertible, i.e. $e_a{}^\mu e^a{}_\nu = \delta^\mu_\nu$) where the metric is constant

$$\eta_{ab} = e_a{}^\mu e_b{}^\nu g_{\mu\nu}. \tag{5.23}$$

Since forms are dual to vector fields, they may be evaluated on the frame. For the gauge field we get

$$a_a = a(e_a), \tag{5.24}$$

leading to the covariant derivate

$$D_a\psi = (D\psi)(e_a) = e_a\psi - ia_a\psi. \tag{5.25}$$

The field strength becomes

$$f_{ab} = i[D_a, D_b] - iD([e_a, e_b]) = e_a a_b - e_b a_a - a([e_a, e_b]) - i[a_a, a_b]. \tag{5.26}$$

Locally this means that

$$a_a = e_a{}^\mu a_\mu\,, \qquad D_a\psi = e_a{}^\mu D_\mu \quad \text{ and } \quad f_{ab} = e_a{}^\mu e_b{}^\nu f_{\mu\nu}. \tag{5.27}$$

Using these definitions, the action for gauge theory on a curved manifold can be written in the two different bases as

$$\mathcal{S} = -\frac{1}{4}\int d^n x\,\sqrt{g}\,\eta^{ab}\eta^{cd} f_{ac} f_{bd} = -\frac{1}{4}\int d^n x\,\sqrt{g}\,g^{\mu\nu} g^{\rho\sigma} f_{\mu\rho} f_{\nu\sigma}, \tag{5.28}$$

where

$$\sqrt{g} = \sqrt{\det\left(g_{\mu\nu}\right)} = \sqrt{\det(e^a{}_\mu e^b{}_\nu \eta_{ab})} = \det\, e^a{}_\mu \tag{5.29}$$

is the measure function induced by the metric.

5.1.4 Gauge theory on curved noncommutative spacetime

In order to formulate gauge theory on a curved noncommutative spacetime, we need a frame e_a and a Poisson structure $\{\cdot\,,\,\cdot\}_p = \pi^{\mu\nu}\partial_\mu \wedge \partial_\nu$ that are compatible with each other. Compatibility means that the frame e_a commutes with the Poisson structure $\{\cdot\,,\,\cdot\}_p$, i.e.

$$e_a\{f,g\}_p = \{e_a f, g\}_p + \{f, e_a g\}_p, \tag{5.30}$$

and that the measure function $\sqrt{g}$ induced by the metric $g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab}$ is also a measure function for the Poisson manifold, i.e. that we have

$$\partial_\mu(\sqrt{g}\pi^{\mu\nu}) = 0.$$

We will call the $\star$-product algebra generated by quantizing such a Poisson structure a curved noncommutative space, as the gauge theory we will define on it in this chapter will reduce to gauge theory on a curved manifold in the commutative limit. In appendix A.3 we will propose a method how to find frames commuting with the Poisson structure in the context of quantum spaces. How to find Poisson structures compatible with a given frame by a construction based on differential equations can be found in [97].

For the gauge theory, we saw in chapter 5.1.1 that we can define a covariant derivative of a field by using a derivation δ_X

$$D_X\Psi_\psi = \delta_X\Psi_\psi - iA_X \star \Psi_\psi. \tag{5.31}$$

With this, a field strength could be defined as

$$-iF_{X,Y} = [D_X \overset{\star}{,} D_Y] - D_{[X,Y]_\star}. \tag{5.32}$$

The properties of $\delta.$ and $[\cdot\,,\,\cdot]_\star$ ensured that this really is a function and not a polydifferential operator.

On a curved noncommutative space, we can quantize the frame e_a with the map δ to get derivations of the $\star$-product. These we can use to define our covariant derivatives. The noncommutative covariant derivative (5.31) and field strength (5.32) evaluated on the frame e_a then read

$$D_a\Phi = D_{e_a}\Phi = \delta_{e_a}\Phi - iA_{e_a} \star \Phi, \tag{5.33}$$

$$-iF_{ab} = -iF_{e_a,e_b} = [D_{e_a} \overset{\star}{,} D_{e_b}] - D_{[e_a,e_b]_\star}. \tag{5.34}$$

The field strength will transform covariantly under gauge transformations, i.e. we have

$$\delta_\Lambda(F) = i[\Lambda \overset{\star}{,} F]. \tag{5.35}$$

To make the action gauge invariant, the integral has to have the trace property, i.e. it has to be invariant under cyclic permutations. For this we need a measure function Ω, which in our case will be the measure function induced by the metric plus possible higher orders in the noncommutativity (see also chapter 3.6), i.e. we will have

$$\Omega = \sqrt{g} + \mathcal{O}(1). \tag{5.36}$$

With this, we have a noncommuative gauge action

$$\mathcal{S} = -\frac{1}{4}\int d^n x\, \Omega\, \eta^{ab}\eta^{cd} F_{ac} \star F_{bd}. \tag{5.37}$$

that goes in the commutative limit

$$\mathcal{S} \to -\frac{1}{4}\int d^n x\, \sqrt{g}\, g^{\mu\nu} g^{\rho\sigma} f_{\mu\rho} f_{\nu\sigma} \tag{5.38}$$

to gauge theory on a curved manifold.

5.1.4.1 Scalars

For the noncommutative version of a scalar Lagrangian

$$\eta^{ab} D_a\overline{\phi} D_b\phi + m^2\overline{\phi}\phi, \tag{5.39}$$

we also need an involution $\bar{\cdot}$ of the $\star$-product, i.e.

$$\overline{(f \star g)} = \overline{g} \star \overline{f}. \tag{5.40}$$

To make the NC Lagrangian invariant under NC gauge transformations, the NC gauge parameter Λ and the NC gauge field A_X have to be invariant under this involution to get

$$\delta_\Lambda \overline{\phi} = \overline{(\Lambda \star \phi)} = \overline{\phi} \star \overline{\Lambda} = \overline{\Phi} \star \Lambda \tag{5.41}$$

and

$$\overline{(A_X \star \phi)} = \overline{\phi} \star \overline{A_X} = \overline{\phi} \star A_X. \tag{5.42}$$

For the Weyl-ordered $\star$-product, ordinary complex conjugation still is an involution, and the hermiticity of the NC gauge parameter Λ and the NC gauge field A_X can be checked explicitly on the formulas of the Seiberg-Witten map in chapter 5.2.

Putting everything together, we therefore end up with an action

$$\mathcal{S} = \int d^n x\, \Omega\, (-\frac{1}{4}\eta^{ab}\eta^{cd} F_{ac} \star F_{bd} + \eta^{ab} D_a\overline{\Phi} \star D_b\Phi - m^2\overline{\Phi} \star \Phi). \tag{5.43}$$

that is invariant under noncommutative gauge transformations

$$\delta_\Lambda S = 0 \tag{5.44}$$

and reduces in the commutative limit

$$\mathcal{S} \to \int d^n x \, \sqrt{g} \, (-\frac{1}{4} g^{\mu\nu} g^{\rho\sigma} f_{\mu\rho} f_{\nu\sigma} + g^{\mu\nu} D_\mu \bar{\phi} D_\nu \phi - m^2 \bar{\phi}\phi), \tag{5.45}$$

to scalar electrodynamics on a curved manifold.

5.1.4.2 Spinors

Even though it isn't clear how to define NC spinors on general curved spacetimes due to the nontrivial spin-connection, it should still be possible in two dimensions. There, the spin connection vanishes and the commutative spinor action can be written as

$$S_{spinor} = \frac{1}{2} \int d^2x \sqrt{g} \overline{\Psi} i \gamma^a e^\mu_a (\partial_\mu - iA_\mu + m)\Psi. \tag{5.46}$$

Note that with the usual gamma-matrices $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ and $\gamma^\mu = \gamma^a e^\mu_a$, we get $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. The noncommutative version of (5.46) is easily constructed, and we get

$$S_{spinor} = \frac{1}{2} \int d^2x \Omega \overline{\Psi} i \gamma^a (\delta_{e_a} - iA_{e_a} + m) \star \Psi \tag{5.47}$$

with $e_a = e^\mu_a \partial_\mu$, which is invariant under NC gauge transformations and reduces in the commutative limit to (5.46).

5.1.5 Example: A frame for κ-deformed spacetime

In this chapter we will construct a frame for the n-dimensional generalization of the κ-deformed plane studied in chapters 3.7 and 4.5. The relations of this quantum space[2] are

$$[\widehat{x}^0, \widehat{x}^i] = ia\widehat{x}^i \quad \text{for} \quad i \neq 0, \tag{5.48}$$

with a a real number. The Poisson structure for this space is

$$c^{\mu\nu} = iax^i \delta^\mu_0 \delta^\nu_i - iax^i \delta^\mu_i \delta^\nu_0. \tag{5.49}$$

[2]Compared to the two-dimensional example in (3.7) and (4.5), the coordinatex^0 corresponds to x and the coordinates x^i correspond to y. A Jambor-Sykora $\star$-product for the n-dimensional κ-deformed space reads e.g.

$$f \star g = m \cdot e^{-ia\, x^i \partial_i \otimes \partial_0} f \otimes g.$$

The derivative in the x^0-direction obviously commutes with this Poisson structure, and we can use it for the frame, setting $e_0{}^\mu = \delta_0^\mu$. For the other directions, we see that $\rho\partial_i$ with $\rho = \sqrt{\sum_{i=1}^{n-1}(x^i)^2}$ commutes with the Poisson structure, as we have

$$\rho\partial_i c^{\mu\nu} = ia\rho\delta_0^\mu\delta_i^\nu - ia\rho\delta_i^\mu\delta_0^\nu \quad \text{and} \quad c^{\mu\sigma}\partial_\sigma(\rho\delta_i^\nu) = ia\rho\delta_0^\mu\delta_i^\nu, \tag{5.50}$$

giving

$$\rho\partial_i\gamma^{\mu\nu} - c^{\mu\sigma}\partial_\sigma\rho\delta_i^\nu + c^{\nu\sigma}\partial_\sigma\rho\delta_i^\mu = 0. \tag{5.51}$$

For the frame, we can therefore take

$$\begin{aligned} e_o &= \partial_o, \\ e_i &= \rho\partial_i, \end{aligned} \tag{5.52}$$

leading to a commutative metric

$$g = (dx^0)^2 + \rho^{-2}((dx^1)^2 + \cdots + (dx^{n-1})^2). \tag{5.53}$$

We know that we can write

$$(dx^1)^2 + \cdots + (dx^{n-1})^2 = d\rho^2 + \rho^2 d\Omega_{n-2}^2, \tag{5.54}$$

where $d\Omega_{n-2}^2$ is the metric of the $n-2$ dimensional sphere. Therefore in this new coordinate system

$$g = (dx^0)^2 + (d\ln\rho)^2 + d\Omega_{n-2}^2 \tag{5.55}$$

and we see that the commutative space is a cross product of a two dimensional Euclidean space and a $n-2$-sphere. Therefore it is a space of constant non-vanishing curvature. Further

$$\sqrt{\det g} = \rho^{-(n-1)} \tag{5.56}$$

is both the measure function on this curved space and it fulfills

$$\partial_\mu(\sqrt{\det g}c^{\mu\nu}) = 0, \tag{5.57}$$

i.e. it also guarantees the cyclicity of the integral, see chapter 3.6.

We have found a frame compatible with the Poisson structure of κ-deformed spacetime and can therefore construct noncommutative gauge theory on this space. We will continue this example in chapter 5.2.6, where we will also have explicit formulas for the SW-maps, writing down an explicit action for the gauge theory.

5.2 Explicit formulas for the Seiberg-Witten map

We will now present a consistent solution for the Seiberg-Witten-maps up to second order for the Weyl ordered $\star$-product and non-abelian gauge transformations. The solutions have been chosen in such a way that they reproduce the ones obtained in [63] for the canonical case.

For calculating the Seiberg-Witten maps, we will write the Weyl ordered $\star$-product (3.35) expanded to second order as

$$f \star g = fg + f \star_1 g + f \star_2 g + \mathcal{O}(3) \tag{5.58}$$

with

$$f \star_1 g = \frac{1}{2} c^{ij} \partial_i f \, \partial_j g \tag{5.59}$$

and

$$f \star_2 g = \frac{1}{8} c^{mn} c^{ij} \partial_m \partial_i f \partial_n \partial_j g + \frac{1}{12} c^{ml} \partial_l c^{ij} (\partial_m \partial_i f \partial_j g - \partial_i f \partial_m \partial_j g). \tag{5.60}$$

5.2.1 The gauge parameter

The gauge parameter is equally expanded as

$$\Lambda_\alpha[a] = \Lambda^0_\alpha[a] + \Lambda^1_\alpha[a] + \Lambda^2_\alpha[a] + \mathcal{O}(3) \tag{5.61}$$

The solution for the gauge transformations is obtained by solving the consistency condition (5.18) order by order. To zeroth order, we clearly have $\Lambda^0_\alpha[a] = \alpha$.

To first order, the consistency condition reads

$$\begin{aligned} i\delta_\alpha \Lambda^1_\beta - i\delta_\beta \Lambda^1_\alpha + [\alpha, \Lambda^1_\beta] + [\Lambda^1_\alpha, \beta] - i\Lambda^1_{-i[\alpha,\beta]} &= -[\alpha \stackrel{\star_1}{,} \beta] \\ &= -\frac{1}{2} c^{ij} [\partial_i \alpha, \partial_j \beta] \end{aligned} \tag{5.62}$$

A solution to this equation is

$$\Lambda^1_\alpha[a] = -\frac{i}{4} c^{ij} \{\partial_i \alpha, a_j\}. \tag{5.63}$$

Note that this solution is not unique. Especially, we could always add terms solving the homogeneous part of (5.62).

To second order, the consistency condition reads

$$\begin{aligned}
& i\delta_\alpha\Lambda^2_\beta - i\delta_\beta\Lambda^2_\alpha + [\alpha, \Lambda^2_\beta] + [\Lambda^2_\alpha, \beta] - i\Lambda^2_{-i[\alpha,\beta]} \\
= \; & -[\alpha \stackrel{\star_1}{,} \Lambda^1_\beta] - [\Lambda^1_\alpha \stackrel{\star_1}{,} \beta] - [\Lambda^1_\alpha, \Lambda^1_\beta] - [\alpha \stackrel{\star_2}{,} \beta] \\
= \; & -\frac{1}{2}c^{ij}[\partial_i\alpha, \partial_j\Lambda^1_\beta] - \frac{1}{2}c^{ij}[\partial_i\Lambda^1_\alpha, \partial_j\beta] - [\Lambda^1_\alpha, \Lambda^1_\beta] \\
& -\frac{1}{8}c^{mn}c^{ij}[\partial_m\partial_i\alpha, \partial_n\partial_j\beta] - \frac{1}{12}c^{ml}\partial_l c^{ij}([\partial_m\partial_i\alpha, \partial_j\beta] - [\partial_i\alpha, \partial_m\partial_j\beta]).
\end{aligned} \tag{5.64}$$

Using the first order term (5.63), we calculate the second order term

$$\begin{aligned}
\Lambda^2_\alpha[a] \;=\; & +\frac{1}{32}c^{ij}c^{kl}\Big(4\{\partial_i\alpha, \{a_k, \partial_l a_j\}\} - 2i[\partial_i\partial_k\alpha, \partial_j a_l] \\
& +2[\partial_j a_l, [\partial_i\alpha, a_k]] - 2i[[a_j, a_l], [\partial_i\alpha, a_k]] \\
& +i\{\partial_i\alpha, \{a_k, [a_j, a_l]\}\} + \{a_j, \{a_l, [\partial_i\alpha, a_k]\}\}\Big) \\
& +\frac{1}{24}c^{kl}\partial_l c^{ij}\Big(\{\partial_i\alpha, \{a_k, a_j\}\} - 2i[\partial_i\partial_k\alpha, a_j]\Big).
\end{aligned} \tag{5.65}$$

5.2.2 Fields in the fundamental representation

In the same way a solution for the field Ψ in the fundamental representation is obtained by solving equation (5.17). We expand it to second order as

$$\Psi_\psi[a] = \Psi^0_\psi[a] + \Psi^1_\psi[a] + \Psi^2_\psi[a] + \mathcal{O}(3). \tag{5.66}$$

The zeroth order is the commutative field, i. e. $\Psi^0_\psi[a] = \psi$. To first order, the equation (5.17) reads

$$\delta_\alpha\Psi^1_\psi - i\alpha\Psi^1_\psi = i\alpha \star_1 \psi + i\Lambda^1_\alpha\psi = \frac{i}{2}c^{ij}\partial_i\alpha\partial_j\psi + i\Lambda^1_\alpha\psi, \tag{5.67}$$

which is solved using (5.63) to give

$$\Psi^1_\psi[a] = \frac{1}{4}c^{ij}\Big(2ia_i\partial_j\psi + a_i a_j\psi\Big). \tag{5.68}$$

To second order the equation (5.17) reads

$$\begin{aligned}
\delta_\alpha\Psi^2_\psi - i\alpha\Psi^2_\psi \;=\; & i\alpha \star_2 \psi + i\alpha \star_1 \Psi^1_\psi + i\Lambda^1_\alpha \star_1 \psi + i\Lambda^1_\alpha\Psi^1_\psi + i\Lambda^2_\alpha\psi \\
=\; & \frac{i}{8}c^{mn}c^{ij}\partial_m\partial_i\alpha, \partial_n\partial_j\psi + \frac{i}{12}c^{ml}\partial_l c^{ij}(\partial_m\partial_i\alpha\partial_j\psi - \partial_i\alpha\partial_m\partial_j\psi) \\
& +\frac{i}{2}c^{ij}\partial_i\alpha\partial_j\Psi^1_\psi + \frac{i}{2}c^{ij}\partial_i\Lambda^1_\alpha\partial_j\psi + i\Lambda^1_\alpha\Psi^1_\psi + i\Lambda^2_\alpha\psi.
\end{aligned} \tag{5.69}$$

Using the solutions to first order (5.63) and (5.68), a solution

$$\begin{aligned}\Psi^2_\psi[a] &= +\frac{1}{32}c^{ij}c^{kl}\Big(4i\partial_i a_k\partial_j\partial_l\psi - 4a_i a_k\partial_j\partial_l\psi - 8a_i\partial_j a_k\partial_l\psi \\ &\quad +4a_i\partial_k a_j\partial_l\psi + 4ia_i a_j a_k\partial_l\psi - 4ia_k a_j a_i\partial_l\psi \\ &\quad +4ia_j a_k a_i\partial_l\psi - 4\partial_j a_k a_i\partial_l\psi + 2\partial_i a_k\partial_j a_l\psi \\ &\quad -4ia_i a_l\partial_k a_j\psi - 4ia_i\partial_k a_j a_l\psi + 4ia_i\partial_j a_k a_l\psi \\ &\quad -3a_i a_j a_l a_k\psi - 4a_i a_k a_j a_l\psi - 2a_i a_l a_k a_j\psi\Big) \\ &\quad +\frac{1}{24}c^{kl}\partial_l c^{ij}\Big(2ia_j\partial_k\partial_i\psi + 2i\partial_k a_i\partial_j\psi + 2\partial_k a_i a_j\psi \\ &\quad -a_k a_i\partial_j\psi - 3a_i a_k\partial_j\psi - 2ia_j a_k a_i\psi\Big)\end{aligned} \tag{5.70}$$

can be calculated for the second order term.

5.2.3 The covariantizer

The covariantizer is expanded as well as

$$D[a](f) = D^0[a](f) + D^1[a](f) + D^2[a](f) + \mathcal{O}(3). \tag{5.71}$$

We will now solve (5.19) order to order. To zeroth order, we take D to be the identity, i.e. $D^0[a](f) = f$. To first order (5.19) reads

$$\delta_\alpha D^1(f) - i[\alpha, D^1(f)] = i[\alpha \overset{\star_1}{,} f] = \frac{i}{2}c^{ij}[\partial_i\alpha, \partial_j f], \tag{5.72}$$

having a solution

$$D^1[a](f) = ic^{ij}a_i\partial_j f. \tag{5.73}$$

To second order we get for (5.19)

$$\begin{aligned}\delta_\alpha D^2(f) - i[\alpha, D^2(f)] &= i[\alpha \overset{\star_2}{,} f] + i[\alpha \overset{\star_1}{,} D^1(f)] \\ &\quad +i[\Lambda^1_\alpha \overset{\star_1}{,} f] + i[\Lambda^1_\alpha, D^1(f)] \\ &= \frac{i}{8}c^{mn}c^{ij}[\partial_m\partial_i\alpha, \partial_n\partial_j f] \\ &\quad +\frac{i}{12}c^{ml}\partial_l c^{ij}([\partial_m\partial_i\alpha, \partial_j f] - [\partial_i\alpha, \partial_m\partial_j f]) \\ &\quad +\frac{i}{2}c^{ij}[\partial_i\alpha, \partial_j D^1(f)] + \frac{i}{2}c^{ij}[\partial_i\Lambda^1_\alpha, \partial_j f] + i[\Lambda^1_\alpha, D^1(f)],\end{aligned} \tag{5.74}$$

with a solution

$$D^2[a](f) = +\frac{1}{4}c^{ij}c^{kl}\Big(- 2\{a_i, \partial_j a_k\}\partial_l f + \{a_i, \partial_k a_j\}\partial_l f \tag{5.75}$$

$$+i\{a_i,[a_j,a_k]\}\partial_l f - \{a_i,a_k\}\partial_j\partial_l f\Big)$$
$$+\frac{1}{4}c^{il}\partial_l c^{jk}\{a_i,a_k\}\partial_j f.$$

5.2.4 The gauge field

Finally we want to calculate a SW-map for the gauge potential A_X evaluated on a Poisson vector field X. Again we expand it as

$$A_X[a] = A_X^0[a] + A_X^1[a] + A_X^2[a] + \mathcal{O}(3). \tag{5.76}$$

Expanding the equation (5.21) as well, we see that to zeroth order it is solved by $A_X^0[a] = X^n a_n$. To first order it reads

$$\begin{aligned}\delta_\alpha A_X^1 - i[\alpha, A_X^1] &= X^i\partial_i\Lambda_\alpha^1 + \delta_X^1\alpha + i[\alpha \stackrel{\star}{,}_1 X^n a_n] + i[\Lambda_\alpha^1, X^n a_n] \qquad (5.77)\\ &= X^i\partial_i\Lambda_\alpha^1 + \frac{i}{2}c^{ij}[\partial_i\alpha, \partial_j(X^n a_n)] + i[\Lambda_\alpha^1, X^n a_n].\end{aligned}$$

Using $[X,c]_S = 0$, we can calculate a solution

$$A_X^1[a] = \frac{i}{4}c^{kl}X^n\{a_k, \partial_l a_n + f_{ln}\} + \frac{i}{4}c^{kl}\partial_l X^n\{a_k, a_n\}, \tag{5.78}$$

where $f_{ij} = \partial_i a_j - \partial_j a_i - i[a_i, a_j]$ is the commutative field strength.

The equation to second order is

$$\begin{aligned}\delta_\alpha A_X^2 - i[\alpha, A_X^2] &= X^i\partial_i\Lambda_\alpha^2 + \delta_X^1\Lambda_\alpha^1 + \delta_X^2\alpha + i[\Lambda_\alpha^1, A_X^1] + i[\Lambda_\alpha^2, X^n a_n] \qquad (5.79)\\ &\quad +i[\alpha \stackrel{\star}{,}_2 X^n a_n] + i[\alpha \stackrel{\star}{,}_1 A_X^1] + i[\Lambda_\alpha^1 \stackrel{\star}{,}_1 X^n a_n]\\ &= X^i\partial_i\Lambda_\alpha^2 + \delta_X^1\Lambda_\alpha^1 + i[\Lambda_\alpha^1, A_X^1] + i[\Lambda_\alpha^2, X^n a_n].\\ &\quad +\frac{i}{8}c^{mn}c^{ij}[\partial_m\partial_i\alpha, \partial_n\partial_j(X^n a_n)]\\ &\quad +\frac{i}{12}c^{ml}\partial_l c^{ij}([\partial_m\partial_i\alpha, \partial_j(X^n a_n)] - [\partial_i\alpha, \partial_m\partial_j(X^n a_n)])\\ &\quad +\frac{i}{2}c^{ij}[\partial_i\alpha, \partial_j A_X^1] + \frac{i}{2}c^{ij}[\partial_i\Lambda_\alpha^1, \partial_j(X^n a_n)]\\ &\quad -\frac{1}{12}c^{lk}\partial_k c^{im}\partial_l\partial_m X^j\partial_i\partial_j\alpha + \frac{1}{24}c^{lk}c^{im}\partial_l\partial_i X^j\partial_k\partial_m\partial_j\alpha\end{aligned}$$

The second order solution for the NC gauge potential is

$$\begin{aligned}A_X^2[a] &= +\frac{1}{32}c^{kl}c^{ij}X^n\Big(-4i[\partial_k\partial_i a_n, \partial_l a_j] + 2i[\partial_k\partial_n a_i, \partial_l a_j] \qquad (5.80)\\ &\quad -4\{a_k,\{a_i,\partial_j f_{ln}\}\} - 2[[\partial_k a_i, a_n], \partial_l a_j] + 4\{\partial_l a_n, \{\partial_i a_k, a_j\}\}\end{aligned}$$

$$
\begin{aligned}
&-4\{a_k,\{f_{li},f_{jn}\}\}+i\{\partial_n a_j,\{a_l,[a_i,a_k]\}\}\\
&+i\{a_i,\{a_k,[\partial_n a_j,a_l]\}\}-4i[[a_i,a_l],[a_k,\partial_j a_n]]\\
&+2i[[a_i,a_l],[a_k,\partial_n a_j]]+\{a_i,\{a_k,[a_l,[a_j,a_n]]\}\}\\
&-\{a_k,\{[a_l,a_i],[a_j,a_n]\}\}-[[a_i,a_l],[a_k,[a_j,a_n]]]\Big)\\
+\frac{1}{32}c^{kl}c^{ij}\partial_j X^n\Big(&2i[\partial_k a_i,\partial_l a_n]+2i[\partial_i a_k,\partial_l a_n]\\
&+2i[\partial_i a_k,\partial_l a_n-\partial_n a_l]+4\{a_n,\{a_l,\partial_k a_i\}\}\\
&+4\{a_k,\{a_i,\partial_n a_l-\partial_l a_n\}\}-2i\{a_k,\{a_i,[a_n,a_l]\}\}\\
&+i\{a_i,\{a_l,[a_n,a_k]\}\}+i\{a_n,\{a_l,[a_i,a_k]\}\}\Big)\\
+\frac{1}{24}c^{kl}c^{ij}\partial_l\partial_j X^n\Big(&\partial_i\partial_k a_n-2i[a_i,\partial_k a_n]-\{a_n,\{a_k,a_i\}\}\Big)\\
+\frac{1}{24}c^{kl}\partial_l c^{ij}X^n\Big(&2i[a_j,\partial_k\partial_i a_n]+2i[\partial_k a_i,f_{jn}]\\
&-\{\partial_j a_n,\{a_k,a_i\}\}+2\{a_i,\{a_k,f_{nj}\}\}\Big)\\
+\frac{1}{24}c^{kl}\partial_l c^{ij}\partial_j X^n\Big(&-4i[a_i,\partial_k a_n]+2i[a_k,\partial_i a_n]-\{a_n,\{a_k,a_i\}\}\Big)\\
-\frac{1}{12}c^{kl}\partial_l c^{ij}\partial_j\partial_k X^n\partial_i a_n&+\mathcal{O}(3).
\end{aligned}
$$

5.2.5 Field strength, covariant derivative and action

We will now use the Seiberg-Witten maps of the preceding chapters to calculate actions for noncommutative gauge theory to first order. We start with calculating the field strength (5.32). It is

$$
\begin{aligned}
F_{ab}=F(X_a,X_b) &= [D_{X_a} \overset{\star}{,} D_{X_b}]-D_{[X_a,X_b]_\star} \\
&= X_a^k X_b^l f_{kl}+\frac{i}{2}c^{ij}\{a_i,\partial_j(X_a^k X_b^l f_{kl})\}\\
&\quad+\frac{i}{2}c^{ij}X_a^k X_b^l\{f_{jl},f_{ki}\}+\frac{1}{4}c^{ij}X_a^k X_b^l\{a_i,[a_j,f_{kl}]\}+\mathcal{O}(2).
\end{aligned}
\tag{5.81}
$$

The covariant derivative is

$$
\begin{aligned}
D_a\Phi=D_{X_a}\Phi &= \delta_{X_a}\Phi-iA_{X_a}\star\Phi \\
&= X_a^k D_k\phi+\frac{i}{2}X_a^k f_{ki}c^{ij}D_j\phi\\
&\quad+\frac{i}{2}c^{ij}a_i\partial_j(X_a^k D_k\phi)+\frac{1}{4}c^{ij}a_i a_j X_a^k D_k\phi+\mathcal{O}(2).
\end{aligned}
\tag{5.82}
$$

Using partial integration and the trace property of the integral, i.e. $\partial_\mu(\Omega c^{\mu\nu})=0$, we can calculate

$$\begin{aligned}
\widetilde{S}_{gauge} &= \int d^n x\, \Omega\, \eta^{ab} \eta^{cd} F_{ac} \star F_{bd} \qquad (5.83)\\
&= \int d^n x\, \Omega\, \eta^{ab} \eta^{cd} X_a^\mu X_c^\nu X_b^\rho X_d^\sigma f_{\mu\nu} f_{\rho\sigma} \\
&+ \int d^4 x\, \Omega\, \eta^{ab} \eta^{cd} \Big(\frac{i}{4} c^{ij} [a_i, [\partial_j (X_a^\mu X_c^\nu f_{\mu\nu}), X_b^\rho X_d^\sigma f_{\rho\sigma}]] \\
&+ \frac{i}{8} c^{ij} X_a^\mu X_c^\nu X_b^\rho X_d^\sigma \{f_{\mu\nu}, \{f_{ij}, f_{\rho\sigma}\}\} - \frac{1}{4} c^{ij} X_a^\mu X_c^\nu X_b^\rho X_d^\sigma [f_{\mu\nu} a_i, f_{\rho\sigma} a_j] \\
&- \frac{1}{4} c^{ij} X_a^\mu X_c^\nu X_b^\rho X_d^\sigma [a_i f_{\mu\nu}, a_j f_{\rho\sigma}] + \frac{i}{2} c^{ij} X_a^\mu X_c^\nu X_b^\rho X_d^\sigma \{f_{\mu\nu}, \{f_{j\sigma}, f_{\rho i}\}\} \Big) \\
&+ \mathcal{O}(2),
\end{aligned}$$

where we haven't done the trace over the gauge representation jet. Doing this now, the action for the gauge particles is

$$\begin{aligned}
S_{gauge} &= -\frac{1}{4} tr(\widetilde{S}_{gauge}) \qquad (5.84)\\
&= \int d^n x\, \Omega\, \eta^{ab} \eta^{cd} X_a^\mu X_c^\nu X_b^\rho X_d^\sigma \Big(-\frac{1}{4} tr(f_{\mu\nu} f_{\rho\sigma}) \\
&- \frac{i}{8} c^{ij} tr(f_{ij} f_{\mu\nu} f_{\rho\sigma}) - \frac{i}{2} c^{ij} tr(f_{\mu\nu} f_{j\sigma} f_{\rho i}) \Big) + \mathcal{O}(2).
\end{aligned}$$

With $\overline{c}^{ij} = -c^{ij}$ we get

$$\begin{aligned}
S_{scalar} &= \int d^n x\, \Omega\, \eta^{ab} \overline{D_a \Phi} \star D_b \Phi \qquad (5.85)\\
&= \int d^n x\, \Omega\, \eta^{ab} \Big(\overline{X}_a^\mu X_b^\nu \overline{D_\mu \phi} D_\nu \phi \\
&+ \frac{i}{2} c^{ij} \overline{X}_a^\mu X_b^\nu \overline{D_\mu \phi} f_{\nu i} D_j \phi + \frac{i}{2} c^{ij} \overline{X}_a^\mu X_b^\nu \overline{D_j \phi} f_{\nu i} D_\mu \phi \\
&+ \frac{i}{2} c^{ij} \overline{X}_a^\mu X_b^\nu \overline{D_\mu \phi} f_{ij} D_\nu \phi \Big) + \mathcal{O}(2)
\end{aligned}$$

for the scalar fields.

5.2.6 Example: A NC action on κ-deformed spacetime

Now we continue our example from chapter 5.1.5. There, we had already constructed a frame

$$e_0^\mu = \delta_0^\mu, \qquad (5.86)$$
$$e_i^\mu = \rho \delta_i^\mu \qquad (5.87)$$

with $\rho = \sqrt{x_i x^i}$ compatible with the Poisson structure

$$c^{\mu\nu} = ia\delta_0^\mu \delta_i^\nu x^i - ia\delta_0^\nu \delta_i^\mu x^i. \tag{5.88}$$

These we can plug into our solution of the Seiberg-Witten map and get

$$\begin{aligned}
\Lambda_\lambda[a] &= \lambda + \frac{a}{4}x^i\{\partial_0\lambda, a_i\} - \frac{a}{4}x^i\{\partial_i\lambda, a_0\} + \mathcal{O}(a^2), \\
\Phi_\phi[a] &= \phi - \frac{a}{2}x^i a_0 \partial_i\phi + \frac{a}{2}x^i a_i\partial_0\phi + \frac{ia}{4}x^i[a_0, a_i]\phi + \mathcal{O}(a^2), \\
A_{X_0} &= a_0 - \frac{a}{4}x^i\{a_0, \partial_i a_0 + f_{i0}\} + \frac{a}{4}x^i\{a_i, \partial_0 a_0\} + \mathcal{O}(a^2), \qquad (5.89)\\
A_{X_j} &= \rho a_j - \frac{a}{4}\rho\{a_j, a_0\} - \frac{a}{4}\rho x^i\{a_0, \partial_i a_j + f_{ij}\} \\
&\quad + \frac{a}{4}\rho x^i\{a_i, \partial_0 a_j + f_{0j}\} + \mathcal{O}(a^2), \\
\delta_{X_\mu} &= X_\mu^\nu \partial_\nu + \mathcal{O}(a^2).
\end{aligned}$$

The measure function induced by the frame (5.86,5.87) was

$$\Omega = \rho^{-(n-1)} = \sqrt{g}, \tag{5.90}$$

also guaranteeing the cyclicity of the integral. With this measure function the actions become

$$\begin{aligned}
S_{gauge} &= -\frac{1}{2}\int d^n x\, \rho^{3-n}\eta^{00}\eta^{ij}Tr(f_{0i}f_{0j}) \qquad (5.91)\\
&\quad -\frac{1}{4}\int d^4 x\, \rho^{5-n}\eta^{kl}\eta^{ij}Tr(f_{ki}f_{lj}) \\
&\quad -\frac{a}{2}\int d^n x\, \rho^{3-n}\eta^{00}\eta^{ij}x^p Tr(f_{0p}f_{0i}f_{0j}) \\
&\quad +\frac{a}{4}\int d^4 x\, \rho^{5-n}\eta^{kl}\eta^{ij}x^p Tr(f_{0p}f_{ki}f_{lj}) \\
&\quad -\frac{a}{2}\int d^n x\, \rho^{5-n}\eta^{kl}\eta^{ij}x^p Tr(f_{jp}\{f_{ki}, f_{l0}\}) + \mathcal{O}(a^2)
\end{aligned}$$

and

$$\begin{aligned}
S_{scalar} &= \int d^n x\, \rho^{1-n}\eta^{00}\overline{D_0\phi}D_0\phi + \int d^n x\, \rho^{3-n}\eta^{kl}\overline{D_k\phi}D_l\phi \qquad (5.92)\\
&\quad -\frac{a}{2}\int d^n x\, \rho^{3-n}\eta^{kl}x^i\overline{D_k\phi}f_{l0}D_i\phi + \frac{a}{2}\int d^n x\, \rho^{3-n}\eta^{kl}x^i\overline{D_k\phi}f_{li}D_0\phi \\
&\quad -\frac{a}{2}\int d^n x\, \rho^{3-n}\eta^{kl}x^i\overline{D_i\phi}f_{l0}D_k\phi + \frac{a}{2}\int d^n x\, \rho^{3-n}\eta^{kl}x^i\overline{D_0\phi}f_{li}D_k\phi \\
&\quad -a\int d^n x\, \rho^{3-n}\eta^{kl}x^i\overline{D_k\phi}f_{0i}D_l\phi + \mathcal{O}(a^2).
\end{aligned}$$

In the commutative limit $a \to 0$ the action reduces to scalar electrodynamics on a manifold with constant curvature.

5.3 Construction of the Seiberg-Witten maps to all orders

For explicit calculations, the Weyl ordered $\star$-product is the best choice, but it is only known to second order. For calculations to all orders, we can use the formality $\star$-product, which also comes with strong mathematical tools we can use for the construction of the Seiberg-Witten maps. We already saw how to construct derivations for the formality $\star$-product in chapter (4.3). We can use them to formulate NC gauge theory on any Poisson manifold. To relate the NC theory to commutative gauge theory, we need the Seiberg-Witten maps for the formality $\star$-product. In [65] and [66] the SW maps for the NC gauge parameter and the covariantizer were already constructed to all orders in θ. We will extend the method developed there to the SW map for covariant derivations.

5.3.1 Formality

We saw in chapter 3.4 that the formality $\star$-product can be constructed using the maps U_n from the polyvectorfields to the polydifferential operators as

$$f \star g = \sum_{n=0}^{\infty} \frac{1}{n!} U_n(\pi, \ldots, \pi)(f, g). \tag{5.93}$$

With these maps, we already introduced the special polydifferential operators

$$\begin{aligned} \Phi(\alpha) &= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} U_n(\alpha, \pi, \ldots, \pi), & (5.94) \\ \Psi(\alpha_1, \alpha_2) &= \sum_{n=2}^{\infty} \frac{1}{(n-2)!} U_n(\alpha_1, \alpha_2, \pi, \ldots, \pi) & (5.95) \end{aligned}$$

in chapter 4.3. For the construction of the Seiberg-Witten maps, we will need some additional relations, which we calculate using the formality condition (3.44).

For g a function, X and Y vectorfields and π and σ bivectorfields we see that both $\Phi(g)$ and $\Psi(X, Y)$ are functions and we go on to calculate

$$\begin{aligned} [\star, \star]_G &= \Phi([\pi, \pi]_S), & (5.96) \\ [\Phi(f), \star]_G &= -\Phi([f, \pi]_S), & (5.97) \end{aligned}$$

$$\begin{aligned}
[\delta_X, \star]_G &= \Phi([X,\pi]_S), && (5.98)\\
[\delta_X, \delta_Y]_G + [\Psi(X,Y), \star]_G &= \delta_{[X,Y]_S} && (5.99)\\
&\quad +\Psi([\pi,Y]_S, X) - \Psi([\pi,X]_S, Y), \\
[\Phi(\sigma), \Phi(g)]_G + [\Psi(\sigma,g), \star]_G &= -\delta_{[\sigma,g]_S} && (5.100)\\
&\quad -\Psi([\pi,g]_S, \sigma) - \Psi([\pi,\sigma]_S, g), \\
[\delta_X, \Phi(g)]_G &= \phi([X,g]_S) && (5.101)\\
&\quad -\Psi([\pi,g]_S, X) - \Psi([\pi,X]_S, g).
\end{aligned}$$

If π is a Poisson tensor, i. e. $[\pi,\pi]_S = 0$ and if X and Y are Poisson vector fields, i. e. $[X,\pi]_S = [Y,\pi]_S = 0$, the relations (5.96) to (5.99) become

$$\begin{aligned}
f \star (g \star h) &= (f \star g) \star h, && (5.102)\\
\delta_{H_f}(g) &= -[\Phi(f) \stackrel{\star}{,} g], && (5.103)\\
\delta_X(f \star g) &= \delta_X(f) \star g + f \star \delta_X(g), && (5.104)\\
([\delta_X, \delta_Y] - \delta_{[X,Y]_L})(g) &= [\Psi(X,Y) \stackrel{\star}{,} g] && (5.105)
\end{aligned}$$

when evaluated on functions. $[\,\cdot\,,\,\cdot\,]$ are now ordinary commutator brackets. $\star$ defines an associative product, the Hamiltonian vector fields are mapped to inner derivations and Poisson vector fields are mapped to outer derivations of the $\star$-product.

5.3.2 Semi-classical construction

We will first do the construction in the semi-classical limit, where the star commutator is replaced by the Poisson bracket. As in [65] and [66], we define, with the help of the Poisson tensor $\theta = \frac{1}{2}\theta^{kl}\partial_k \wedge \partial_l$

$$d_\theta = -[\cdot, \theta] \tag{5.106}$$

and (locally)

$$a_\theta = \theta^{ij} a_j \partial_i. \tag{5.107}$$

Note that the bracket used in the definition of d_θ is not the Schouten-Nijenhuis bracket (A.1.1). For polyvectorfields π_1 and π_2 it is

$$[\pi_1, \pi_2] = -[\pi_2, \pi_1]_S, \tag{5.108}$$

giving an extra minus sign for π_1 and π_2 both even (see B.2.2). Especially, we get for d_θ acting on a function g

$$d_\theta g = -[g,\theta] = [g,\theta]_S = \theta^{kl}\partial_l g \partial_k. \tag{5.109}$$

Now a parameter t and t-dependent $\theta_t = \frac{1}{2}\theta_t^{kl}\partial_k \wedge \partial_l$ and $X_t = X_t^k\partial_k$ are introduced, fulfilling

$$\partial_t\theta_t = f_\theta = -\theta_t f \theta_t \quad \text{and} \quad \partial_t X_t = -X_t f \theta_t, \tag{5.110}$$

where the multiplication is understood as ordinary matrix multiplication, e.g. $(\theta_t f \theta_t)^{ij} = \theta_t^{ik} f_{kl} \theta_t^{kj}$. Given the Poisson tensor θ and the Poisson vectorfield X, the formal solutions are

$$\theta_t = \theta \sum_{n=0}^{\infty} (-t\ f\theta)^n = \frac{1}{2}(\theta^{kl} - t\theta^{ki} f_{ij} \theta^{jl} + \ldots)\partial_k \wedge \partial_l \tag{5.111}$$

and

$$X_t = X \sum_{n=0}^{\infty} (-t\ f\theta)^n = X^k\partial_k - tX^i f_{ij}\theta^{jk}\partial_k + \ldots. \tag{5.112}$$

θ_t is still a Poisson tensor and X_t is still a Poisson vectorfield, i.e.

$$[\theta_t, \theta_t] = 0 \qquad \text{and} \qquad [X_t, \theta_t] = 0. \tag{5.113}$$

For the proof see B.1.

With this we calculate

$$f_\theta = \partial_t\theta_t = -\theta_t f \theta_t = -[a_\theta, \theta] = d_\theta a_\theta. \tag{5.114}$$

We now get the following commutation relations

$$[a_{\theta_t} + \partial_t, d_{\theta_t}(g)] \;=\; d_{\theta_t}((a_{\theta_t} + \partial_t)(g)), \tag{5.115}$$
$$[a_{\theta_t} + \partial_t, X_t] \;=\; -d_{\theta_t}(X_t^k a_k), \tag{5.116}$$

where g is some function which might also depend on t (see B.2.1).

To construct the Seiberg-Witten map for the gauge potential A_X, we first define

$$K_t = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}(a_{\theta_t} + \partial_t)^n. \tag{5.117}$$

With this, the semi-classical gauge parameter reads [65, 66]

$$\Lambda_\lambda[a] = K_t(\lambda)\Big|_{t=0}. \tag{5.118}$$

To see that this has indeed the right transformation properties under gauge transformations, we first note that the transformation properties of a_{θ_t} and $X_t^k a_k$ are

$$\delta_\lambda a_{\theta_t} = \theta_t^{kl} \partial_l \lambda \partial_k = d_{\theta_t} \lambda \tag{5.119}$$

and

$$\delta_\lambda (X_t^k a_k) = X_t^k \partial_k \lambda = [X_t, \lambda]. \tag{5.120}$$

Using (5.119,5.120) and the commutation relations (5.115,5.116), a rather tedious calculation (see B.3) shows that

$$\delta_\lambda K_t(X_t^k a_k) = X_t^k \partial_k K_t(\lambda) + d_{\theta_t}(K_t(\lambda)) K_t(X_t^k a_k). \tag{5.121}$$

Therefore, the semi-classical gauge potential is

$$A_X[a] = K_t(X_t^k a_k)\Big|_{t=0}. \tag{5.122}$$

5.3.3 Quantum construction

We can now use the Kontsevich formality map to quantize the semi-classical construction. All the semi-classical expressions can be mapped to their counterparts in the $\star$-product formalism without loosing the properties necessary for the construction. One higher order term will appear, fixing the transformation properties for the quantum objects.

The $\star$-product we will use is

$$\star = \sum_{n=0}^{\infty} \frac{1}{n!} U_n(\theta_t, \ldots, \theta_t). \tag{5.123}$$

We define

$$d_\star = -[\cdot, \star]_G \ , \tag{5.124}$$

which for functions f and g reads

$$d_\star(g)\, f = [f \stackrel{\star}{,} g]. \tag{5.125}$$

The bracket used in the definition of $d_\star$ is the Gerstenhaber bracket (A.1.2). We now calculate the commutators (5.115) and (5.116) in the new setting (see B.2.2). We get

$$\begin{aligned} [\Phi(a_{\theta_t}) + \partial_t, d_\star(\Phi(g))] &= d_\star((\Phi(a_{\theta_t}) + \partial_t)\Phi(g)), & (5.126)\\ [\Phi(a_{\theta_t}) + \partial_t, \Phi(X_t)] &= -d_\star(\Phi(X_t^k a_k) - \Psi(a_{\theta_t}, X_t)). & (5.127) \end{aligned}$$

The higher order term $\Psi(a_{\theta_t}, X_t)$ has appeared, but looking at the gauge transformation properties of the quantum objects we see that it is actually necessary. We get

$$\delta_\lambda \Phi(a_{\theta_t}) = \Phi(d_{\theta_t}\lambda) = d_\star \Phi(\lambda) \tag{5.128}$$

with (5.104) and (5.119) and

$$\begin{aligned} \delta_\lambda(\Phi(X_t^k a_k) - \Psi(a_\theta, X_t)) &= \Phi([X_t, \lambda]) - \Psi(d_\theta \lambda, X_t) \\ &= [\Phi(X_t), \Phi(\lambda)] - \Psi([\theta_t, \lambda], X_t) \\ &\quad + \Psi([\theta_t, X_t], \lambda) - \Psi(d_\theta \lambda, X_t) \\ &= [\Phi(X_t), \Phi(\lambda)] \\ &= \delta_{X_t} \Phi(\lambda), \end{aligned} \tag{5.129}$$

where the addition of the new term preserves the correct transformation property. With

$$K_t^\star = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\Phi(a_{\theta_t}) + \partial_t)^n, \tag{5.130}$$

a calculation analogous to the semi-classical case gives

$$\begin{aligned} \delta_\lambda(K_t^\star(\Phi(X_t^k a_k) - \Psi(a_{\theta_t}, X_t))) &= \delta_{X_t} K_t^\star(\Phi(\lambda)) \\ &\quad + d_\star(K_t^\star(\Phi(\lambda))) K_t^\star(\Phi(X_t^k a_k) - \Psi(a_{\theta_t}, X_t)). \end{aligned} \tag{5.131}$$

As in [65, 66], the NC gauge parameter is

$$\Lambda_\lambda[a] = K_t^\star(\Phi(\lambda))\Big|_{t=0}, \tag{5.132}$$

and we therefore get for the NC gauge potential

$$A_X[a] = K_t^\star(\Phi(X_t^k a_k) - \Psi(a_{\theta_t}, X_t))\Big|_{t=0}, \tag{5.133}$$

transforming with

$$\delta_\lambda A_X = \delta_X \Lambda_\lambda - [\Lambda_\lambda \stackrel{\star}{,} A_X]. \tag{5.134}$$

Chapter 6

Covariant coordinates

While we can only construct actions for noncommutative gauge theory if we have a frame commuting with the Poisson structure, covariant coordinates can always be defined. Therefore we can still extract information from the noncommutative gauge theory, even if we do not have the complete picture. We will use these covariant coordinates to generalize the open Wilson lines of chapter 2.5. In [90] these were used to give an exact formula for the inverse Seiberg-Witten map. We will generalize this construction for general $\star$-products with invertible Poisson structure θ^{ij}.

6.1 Wilson lines and observables

As we saw in chapter 5.1.1, multiplication with a coordinate from the left is not a covariant operation. For this, we can define covariant coordinates

$$X^i = x^i + A^i \tag{6.1}$$

for which we want

$$\delta_\Lambda(X^i \star \Psi) = \delta_\Lambda((x^i + A^i) \star \Psi) = i\Lambda \star X^i \star \Psi. \tag{6.2}$$

Therefore the gauge field A^i has to transform as

$$\delta_\Lambda A^i = i[\Lambda \stackrel{\star}{,} x^i] + i[\Lambda \stackrel{\star}{,} A^i]. \tag{6.3}$$

Even though the gauge field A^i vanishes in the commutative limit, its Seiberg-Witten map can nevertheless be calculated [65]. It starts with

$$A^i = \theta^{ij} a_j + \mathcal{O}(2). \tag{6.4}$$

We can use the covariant coordinates to construct noncommutative Wilson lines. As in the canonical case we can start with

$$W_l = e_\star^{il_iX^i} \star e_\star^{-il_ix^i}, \tag{6.5}$$

where $\star$ is now an arbitrary $\star$-product. The transformation property of W_l is now

$$W_l'(x) = g(x) \star W_l(x) \star g^{-1}(T_l x), \tag{6.6}$$

where

$$T_l x^j = e_\star^{il_ix^i} \star x^j \star e_\star^{-il_ix^i} \tag{6.7}$$

is an inner automorphism of the algebra, which can be interpreted as a quantized coordinate transformation. Note that the $e_\star^{-l_iX^i}$ do not close to a group for $\theta^{ij}(x)$ at least quadratic in the x's. Therefore it is not clear how to generalize NC Wilson lines for arbitrary curves as in [59]. If we replace commutators by Poisson brackets, the semi-classical limit of these coordinate transformations may be calculated

$$T_l x^k = e_\star^{il_i[x^i \stackrel{\star}{,} \cdot]} x^k \approx e^{-l_i\{x^i, \cdot\}} x^j = e^{-l_i\theta^{ij}\partial_j} x^k, \tag{6.8}$$

the formula becoming exact for θ^{ij} constant or linear in x. We see that the semi-classical coordinate transformation is the flow induced by the Hamiltonian vector field $-l_i\theta^{ij}\partial_j$. At the end we may expand W_l in terms of θ and get

$$W_l = e^{-il_i\theta^{ij}a_j} + \mathcal{O}(\theta^2), \tag{6.9}$$

where we have replaced A^i by its Seiberg-Witten expansion. We see that for l small this really is a Wilson line starting at x and ending at $x - l\theta$. For a given $\star$-product, the higher order corrections to this expression can in principle be calculated. Note that this expression would also depend on the specific choice of the Seiberg-Witten-map of the covariant coordinates.

If we have a measure function $\Omega(x)$ for our $\star$-product with $\partial_i(\Omega\theta^{ij}) = 0$, we can use the trace property of the integral (see chapter 3.6) to generalize the open Wilson lines of chapter 2.5. They read

$$U_l = \int d^{2n}x\, \Omega(x)\, W_l(x) \star e_\star^{il_ix^i} = \int d^{2n}x\, \Omega(x)\, e_\star^{il_iX^i(x)} \tag{6.10}$$

and are again gauge invariant objects. Of course, we can again insert a function f depending only on the covariant coordinates

$$f_l = \int d^{2n}x\, \Omega(x)\, f(X^i) \star e_\star^{il_iX^i(x)} \tag{6.11}$$

without spoiling the gauge invariance.

6.2 Inverse Seiberg-Witten-map

As an application of the above constructed observables we generalize [90] to arbitrary $\star$-products, i. e. we give a formula for the inverse Seiberg-Witten map for $\star$-products with invertible Poisson structure. In order to map noncommutative gauge theory to its commutative counterpart, we need a functional $f_{ij}[X]$ fulfilling

$$f_{ij}[g \star X \star g^{-1}] = f_{ij}[X], \tag{6.12}$$

$$df = 0 \tag{6.13}$$

and

$$f_{ij} = \partial_i a_j - \partial_j a_i + \mathcal{O}(\theta). \tag{6.14}$$

f is a gauge covariant field strength and reduces in the limit $\theta \to 0$ to the correct expression.

To prove the first and the second property we will only use the algebra properties of the $\star$-product and the cyclicity of the trace. All quantities with a hat will be elements of an algebra. With this let $\hat{X}^i$ be covariant coordinates in an algebra, transforming under gauge transformations like

$$\hat{X}^{i\prime} = \hat{g}\hat{X}^i\hat{g}^{-1} \tag{6.15}$$

with $\hat{g}$ an invertible element of the algebra. Now define

$$\hat{F}^{ij} = -i[\hat{X}^i, \hat{X}^j] \tag{6.16}$$

and

$$(\hat{F}^{n-1})_{ij} = \frac{1}{2^{n-1}(n-1)!}\epsilon_{iji_1i_2\cdots i_{2n-2}}\hat{F}^{i_1i_2}\cdots\hat{F}^{i_{2n-3}i_{2n-2}}. \tag{6.17}$$

Note that the space is $2n$ dimensional. Using the the symmetrized trace str, i.e.

$$str_{\hat{F},\hat{X}}\left(\hat{F}^q\hat{X}^r\right) = \frac{q!r!}{(q+r)!}tr\Big(\hat{F}^q\hat{X}^r + \tag{6.18}$$

$$\text{all other possible permutations of } q\ \hat{F}'\text{s and } r\ \hat{X}'s\Big)$$

see also [90], the expression

$$\mathcal{F}_{ij}(k) = str_{\hat{F},\hat{X}}\left((\hat{F}^{n-1})_{ij}e^{ik_j\hat{X}^j}\right) \tag{6.19}$$

clearly fulfills the first property due to the properties of the trace. Note that symmetrization is only necessary for space dimension bigger than 4 due to the cyclicity of the trace. In dimensions 2 and 4 we may replace str by the ordinary trace tr. $\mathcal{F}_{ij}(k)$ is the Fourier transform of a closed form if

$$k_{[i}\mathcal{F}_{jk]} = 0 \tag{6.20}$$

or if the current

$$J^{i_1 \cdots i_{2n-2}} = str_{\hat{F},X}\left(\hat{F}^{[i_1 i_2} \cdots \hat{F}^{i_{2n-3} i_{2n-2}]} e^{ik_j \hat{F}^j}\right) \tag{6.21}$$

is conserved, respectively

$$k_i J^{i\cdots} = 0. \tag{6.22}$$

This is easy to show, if one uses

$$\begin{aligned} &str_{\hat{F},\hat{X}}\left([k_i\hat{X}^i, \hat{X}^l] e^{ik_j\hat{X}^j} \cdots\right) \\ &\quad = str_{\hat{F},\hat{X}}\left([\hat{X}^l, e^{ik_j\hat{X}^j}] \cdots\right) = str_{\hat{F},\hat{X}}\left(e^{ik_j\hat{X}^j}[\hat{X}^l, \cdots]\right), \end{aligned} \tag{6.23}$$

which can be calculated by simple algebra.

To prove that $\mathcal{F}$ has the right commutative limit, we have to switch to the $\star$-product formalism and expand the formula in θ^{ij}. The expression (6.19) now becomes

$$\mathcal{F}[X]_{ij}(k) = \int \frac{d^{2n}x}{Pf(\theta)} \left((F_\star^{n-1})_{ij} \star e_\star^{ik_jX^j}\right)_{sym\,F,X}. \tag{6.24}$$

The expression in brackets has to be symmetrized in F^{ij} and X^i for $n > 2$. Up to second order in θ^{ij}, the commutator F^{ij} of two covariant coordinates is

$$F^{ij} = -i[X^i \stackrel{\star}{,} X^j] = \theta^{ij} - \theta^{ik} f_{kl} \theta^{lj} - \theta^{kl}\partial_l\theta^{ij} a_k + \mathcal{O}(3) \tag{6.25}$$

with $f_{ij} = \partial_i a_j - \partial_j a_i$ the ordinary field strength. Furthermore we have

$$e_\star^{ik_iX^i} = e^{ik_ix^i}(1 + ik_i\theta^{ij}a_j) + \mathcal{O}(2). \tag{6.26}$$

If we choose an antisymmetric $\star$-product, the symmetrization will annihilate all the first order terms of the $\star$-products between the F^{ij} and X^i, and therefore we get

$$\begin{aligned} &-\mathcal{F}[X]_{ij}(k) \\ &\quad = -2n \int \frac{d^{2n}x}{\epsilon\theta^n} \Big(\epsilon_{ij}\theta^{n-1} - (n-1)\epsilon_{ij}\theta^{n-2}\theta f \theta \end{aligned} \tag{6.27}$$

$$
\begin{aligned}
&\qquad -\theta^{kl}\partial_l(\epsilon_{ij}\theta^{n-1})a_k\Big)e^{ik_ix^i}+\mathcal{O}(1)\\
&= -2n\int\frac{d^{2n}x}{\epsilon\theta^n}\Big(\epsilon_{ij}\theta^{n-1}-(n-1)\epsilon_{ij}\theta^{n-2}\theta f\theta\\
&\qquad -\frac{1}{2}\epsilon_{ij}\theta^{n-1}f_{kl}\theta^{kl}\Big)e^{ik_ix^i}+\mathcal{O}(1)\\
&= d^{2n}x\left(\theta^{-1}_{ij}+2n(n-1)\frac{\epsilon_{ij}\theta^{n-2}\theta f\theta}{\epsilon\theta^n}\right.\\
&\qquad \left.-\frac{1}{2}\theta^{-1}_{ij}f_{kl}\theta^{kl}\right)e^{ik_ix^i}+\mathcal{O}(1),
\end{aligned}
$$

using partial integration and $\partial_i(\epsilon\theta^n\theta^{ij})=0$. To simplify notation we introduced

$$\epsilon_{ij}\theta^{n-1}=\epsilon_{iji_1j_1\cdots i_{n-1}j_{n-1}}\theta^{i_1j_1}\cdots\theta^{i_{n-1}j_{n-1}}\text{etc.} \tag{6.28}$$

In the last line we have used

$$\theta^{-1}_{ij}=-\frac{(\theta^{n-1})_{ij}}{Pf(\theta)}=-2n\frac{\epsilon_{ij}\theta^{n-1}}{\epsilon\theta^n}. \tag{6.29}$$

We will now have a closer look at the second term, noting that

$$\theta^{ij}\frac{\epsilon_{ij}\theta^{n-2}\theta f\theta}{\epsilon\theta^n}=-\frac{1}{2n}\theta^{-1}_{kl}\theta^{kr}f_{rs}\theta^{sl}=-\frac{1}{2n}f_{rs}\theta^{rs} \tag{6.30}$$

and therefore

$$\frac{\epsilon_{ij}\theta^{n-2}\theta f\theta}{\epsilon\theta^n}=a\frac{\epsilon_{ij}\theta^{n-1}}{\epsilon\theta^n}f_{rs}\theta^{rs}+bf_{ij} \tag{6.31}$$

with $a+b=-\frac{1}{2n}$. Taking e. g. $i=1, j=2$ we see that

$$
\begin{aligned}
\epsilon_{12\cdots kl}\theta^{n-2}\theta^{kr}f_{rs}\theta^{sl} &= \epsilon_{12\cdots kl}\theta^{n-2}(\theta^{k1}\theta^{2l}-\theta^{k2}\theta^{1l})f_{12}\\
&\quad +\text{terms without } f_{12}.
\end{aligned} \tag{6.32}
$$

Especially there are no terms involving $f_{12}\theta^{12}$ and we get for the two terms on the right hand side of (6.31)

$$2a\epsilon_{12}\theta^{n-1}f_{12}\theta^{12}=-2nb\epsilon_{12}\theta^{12}\theta^{n-1}f_{12} \tag{6.33}$$

and therefore $b=-\frac{a}{n}$. This has the solution

$$a=-\frac{1}{2(n-1)}\quad\text{and}\quad b=\frac{1}{2n(n-1)}. \tag{6.34}$$

With the resulting

$$2n(n-1)\frac{\epsilon_{ij}\theta^{n-2}\theta f\theta}{\epsilon\theta^n}=\frac{1}{2}\theta^{-1}_{ij}f_{kl}\theta^{kl}+f_{ij} \tag{6.35}$$

we finally get

$$-\mathcal{F}[X]_{ij}(k) = \int d^{2n}x \, \left(\theta^{-1}_{ij} + f_{ij}\right) e^{ik_i x^i} + \mathcal{O}(1). \tag{6.36}$$

Therefore

$$f[X]_{ij} = \mathcal{F}[X]_{ij}(k) - \mathcal{F}[x]_{ij}(k) \tag{6.37}$$

is a closed form that reduces in the commutative limit to the commutative Abelian field strength. We have found an expression for the inverse Seiberg-Witten map.

Part II

Matrix model approach

Though $\star$-products are a convenient tool for studying noncommutativity, their strength lies mainly in the perturbative regime. For other purposes, especially nonperturbative ones, a different approach using a different representation of the algebra of functions on noncommutative space is better suited.

If we take the simple example of a noncommutative plane with canonical noncommutativity

$$[x, y] = i\theta, \tag{6.38}$$

we see immediately that this is nothing but the Heisenberg algebra, for which we can use the well known Fock-space representation. In $2n$ dimensions, we can use n such pairs of coordinates which upon complexification become creation and annihilation operators on the Fock-space. Using this approach, it was possible to study many nonperturbative features of noncommutative field theory such as solitons and instantons (see e.g. [37] for references).

We will call this approach matrix model approach, as the gauge theory can be described as a matrix model having the noncommutative space as its ground state, the fluctuations creating the gauge theory. But noncommutative spacetime with canonical commutation relations has to be represented on an infinite-dimensional vectorspace, leading to a number of problems. First of all, there are the well known divergencies of noncommutative gauge theory. Then, the rank of the gauge group can't be fixed in this model [37]. Therefore we are looking for spaces that can be represented as finite-dimensional matrix algebras, where everything is well defined. The space on which we will base our constructions will be the fuzzy sphere [73], an N-dimensional matrix algebra corresponding to a truncation of the spherical harmonics on the sphere at angular momentum $N - 1$. To go to 4 dimensions, we will use the product of two such fuzzy spheres $S^2_N \times S^2_N$, generated by N^2-dimensional matrices. In one limit, this fuzzy space goes over to the product of two commutative spheres, but in a different limit, it also goes to noncommutative $\mathbb{R}^4$ with canonical commutation relations. Our interest will therefore be twofold: On one hand we will study gauge theory on this fuzzy space as the deformation of commutative gauge theory, on the other hand as a regularization of gauge theory on $\mathbb{R}^4_\theta$.

Chapter 7

The canonical case

Before we study gauge theory on a finite-dimensional fuzzy space, we first want to present the usual matrix model approach to noncommutative gauge theory on $\mathbb{R}^4_\theta$. After a quick look at the infinite-dimensional Fock-space representation of $\mathbb{R}^4_\theta$, we will show how gauge theory can be formulated as a matrix model with ground state $\mathbb{R}^4_\theta$. The fluctuations around this ground state will create the gauge theory. Finally we will have a look at a certain class of instantons, the so called fluxon solutions.

7.1 The Heisenberg algebra

In two dimensions, the coordinate algebra with canonical deformation

$$[x, y] = i\theta \tag{7.1}$$

is nothing but the well known Heisenberg algebra. But now the noncommutativity isn't between the coordinates and momenta, but between the coordinates themselves. Of course we can use the usual Fock space representation for this algebra by first defining

$$x_\pm := x \pm iy \tag{7.2}$$

with

$$[x_+, x_-] = 2\theta. \tag{7.3}$$

The Fock space is given by

$$\mathcal{H} = \{|n\rangle,\ n\epsilon\mathbb{N}_0\}, \tag{7.4}$$

where the creator and annihilator operators act as

$$x_-|n\rangle = \sqrt{2\theta}\sqrt{n+1}|n+1\rangle, \qquad x_+|n\rangle = \sqrt{2\theta}\sqrt{n}|n-1\rangle. \tag{7.5}$$

This can be generalized to higher dimensions. Any $2n$-dimensional algebra with canonical commutation relations can by suitable rotations be brought into a form where it consists of n pairs of noncommuting variables (7.1). As we will mostly be concerned with the 4-dimensional case in the following, we will present it here in more detail.

The most general noncommutative $\mathbb{R}^4_\theta$ is generated by coordinates subject to the commutation relations

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}\,, \tag{7.6}$$

where $\mu, \nu \in \{1, \ldots, 4\}$. Using suitable rotations, $\theta_{\mu\nu}$ can always be cast into the form

$$\theta_{\mu\nu} = \begin{pmatrix} 0 & \theta_{12} & 0 & 0 \\ -\theta_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_{34} \\ 0 & 0 & -\theta_{34} & 0 \end{pmatrix}\,. \tag{7.7}$$

To simplify the following formulas, we restrict our discussion from now on to the selfdual case

$$\theta_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\theta_{\rho\sigma} \tag{7.8}$$

and denote

$$\theta := \theta_{12} = \theta_{34}; \tag{7.9}$$

the generalizations to the antiselfdual and the general case are obvious. In terms of the complex coordinates

$$x_{\pm L} := x_1 \pm i x_2 \quad , \quad x_{\pm R} := x_3 \pm i x_4, \tag{7.10}$$

the commutation relations (7.6) take the form

$$[x_{+a}, x_{-b}] = 2\theta\delta_{ab}, \;\; [x_{+a}, x_{+b}] = [x_{-a}, x_{-b}] = 0, \tag{7.11}$$

where $a, b \in \{L, R\}$. The Fock-space representation $\mathcal{H}$ of (7.11) has the standard basis

$$\mathcal{H} = \{|n_1, n_2\rangle, \quad n_1, n_2 \in \mathbb{N}_0\}, \tag{7.12}$$

with

$$\begin{aligned} x_{-L}|n_1, n_2\rangle &= \sqrt{2\theta}\sqrt{n_1{+}1}|n_1{+}1, n_2\rangle, \quad x_{+L}|n_1, n_2\rangle = \sqrt{2\theta}\sqrt{n_1}|n_1{-}1, n_2\rangle \\ x_{-R}|n_1, n_2\rangle &= \sqrt{2\theta}\sqrt{n_2{+}1}|n_1, n_2{+}1\rangle, \quad x_{+R}|n_1, n_2\rangle = \sqrt{2\theta}\sqrt{n_2}|n_1, n_2{-}1\rangle\,. \end{aligned} \tag{7.13}$$

The derivatives on this space are inner, i.e. they are produced by the commutator with a coordinate

$$-i\theta^{\mu\nu}\partial_\nu \;\widehat{=}\; [\,\cdot\,, x^\mu], \tag{7.14}$$

just as in the $\star$-product formalism.

7.2 Noncommutative gauge theory

We can introduce gauge theory by using a matrix action

$$S = -\frac{(2\pi)^2}{2g^2\theta^2}\,\mathrm{tr}\,([X_\mu, X_\nu] - i\theta_{\mu\nu})^2, \tag{7.15}$$

where the X_μ are infinite-dimensional matrices, and the trace is over the Fock space (7.12). The action is obviously constructed in such a way as to have the Fock-space representation of $\mathbb{R}^4_\theta$ as its ground state. As we want the action to be invariant under unitary transformations

$$X_\mu \to U^\dagger X_\mu U, \tag{7.16}$$

we get fluctuations A_μ around the ground state x_μ as

$$X_\mu = x_\mu + A_\mu. \tag{7.17}$$

The fluctuations A_μ are understood as infinite-dimensional matrices acting on the Fock space (7.12) as well. They have to transform as

$$A_\mu \to U^\dagger[x_\mu, U] + U^\dagger A_\mu U \tag{7.18}$$

to make the X_μ gauge covariant. Remembering that the commutator with a coordinate produces a derivative, we recognize the correct transformation behavior for the gauge field. The gauge covariant field strength then reads

$$iF_{\mu\nu} = ([X_\mu, X_\nu] - i\theta_{\mu\nu}) = [x_\mu, A_\nu] - [x_\nu, A_\mu] + [A_\mu, A_\nu] \tag{7.19}$$

and the action (7.15) reads

$$S = \frac{(2\pi)^2}{2g^2\theta^2}\,\mathrm{tr}(F_{\mu\nu}F_{\mu\nu}). \tag{7.20}$$

To bring the action into a form where it resembles more the creator and annihilator representation, we can also use the complex covariant coordinates $X_{\pm a}$

$$X_{\pm L} = X_1 \pm iX_2 \quad , \quad X_{\pm R} = X_3 \pm iX_4 \tag{7.21}$$

and the corresponding field strength

$$F_{\alpha a,\beta b} = [X_{\alpha a}, X_{\beta b}] - 2\theta\varepsilon_{\alpha\beta}\delta_{ab} \tag{7.22}$$

with $a, b \in \{L, R\}$ and $\alpha, \beta \in \{+, -\}$. The action (7.15) can now be written in the form

$$S = \frac{\pi^2}{g^2\theta^2}\,\mathrm{tr}(\sum_a F_{+a,-a}F_{+a,-a} - \sum_{a,b} F_{+a,+b}F_{-a,-b}), \tag{7.23}$$

and the equations of motion are given by

$$\sum_{a,\alpha}[X_{\alpha a}, (F_{\alpha a,\beta b})^\dagger] = 0\,. \tag{7.24}$$

We now want to discuss a peculiar feature of this formulation of noncommutative gauge theory. Even though we did construct our action for $U(1)$, it actually contains sectors for every rank of the gauge group $U(n)$! This is related to the fact that in noncommutative gauge theories, the gauge group also contains transformations acting on spacetime itself. As the size of the matrices X_μ isn't fixed (they are infinite-dimensional operators), we can't seperate the gauge part of the unitary transformations from the spacetime part. This can be seen as follows: If we have

$$X_\mu = x_\mu \tag{7.25}$$

as a ground state of the theory, then of course

$$X'_\mu = \begin{pmatrix} x_\mu & 0 \\ 0 & x_\mu \end{pmatrix} \tag{7.26}$$

is equally a ground state. In fact, the direct sum of n solutions x_μ of the equations of motion will again be a solution. As the covariant coordinates $X_\mu = x_\mu + A_\mu$ corresponding to the ground state (7.25) produce a $U(1)$ theory, any such ground state $X'_\mu = x_\mu \otimes 1_{n\times n}$ can be viewed as the ground state of a $U(n)$ gauge theory, where the gauge degrees of freedom act on the right hand side of the tensor product. The corresponding covariant coordinate can then be written as

$$X'_\mu = x_\mu \otimes 1_{n\times n} + A_{\mu,a}T^a, \tag{7.27}$$

with the T^a are generalized Gellman matrices for $U(n)$, producing a $U(n)$ gauge theory. So the matrix action (7.15) cannot be restricted to one gauge group, it contains sectors with all $U(n)$. As we will see in chapter 10, this problem can be fixed in a regularized theory.

7.3 $U(1)$ instantons on $\mathbb{R}^4_\theta$

We will for the moment stick to the $U(1)$-sector of the theory and look for solutions of the equations of motion (7.24) which can be understood as instantons of the gauge theory.

On noncommutative $\mathbb{R}^2_\theta$, all $U(1)$-instantons were constructed and classified in [50]. They can be interpreted as localized flux solutions, sometimes called fluxons.

The situation on $\mathbb{R}^4_\theta$ is more complicated, and there are different types of nontrivial $U(1)$ instanton solutions on $\mathbb{R}^4_\theta$. Assuming that $\theta_{\mu\nu}$ is self-dual, there are two types of instantons: first, there exist straightforward generalizations of the two-dimensional localized fluxon solutions with self-dual field strength. As in the two-dimensional case, we will refer to these 4-dimensional solutions as fluxons.

There are other types of $U(1)$ instantons on $\mathbb{R}^4_\theta$, which were found through a noncommutative version of the ADHM equations [88, 41, 27, 55, 61], in particular anti-selfdual instantons which are much less localized than the fluxon solutions. However, we will concentrate on the generalizations of [50], as they will become important for us in chapter 10.

For the construction of the fluxons, let us consider a finite dimensional subvectorspace V_n of the Fock-space H of dimension n spanned by a finite set of vectors $|n_1, n_2\rangle \in \mathcal{H}$,

$$V_n = \langle \{ |i_k, j_k\rangle ; \;\; k = 1, ..., n\} \rangle \, . \tag{7.28}$$

We introduce a partial isometry[1] S mapping $\mathcal{H}$ to $\mathcal{H} \setminus V_n$, which has

$$S^\dagger S = \mathbb{1}, \tag{7.29}$$

$$SS^\dagger = \mathbb{1} - P_{V_n} \tag{7.30}$$

with the projection operator onto the subspace V_n

$$P_{V_n} := \sum_{k=1}^{n} |i_k, j_k\rangle\langle i_k, j_k|. \tag{7.31}$$

Following [50] one then finds solutions to the equations of motion given by[2]

$$X^{(n)}_{+L} := Sx_{+L}S^\dagger + \sum_{k=1}^{n} \gamma^L_k |i_k, j_k\rangle\langle i_k, j_k| \tag{7.32}$$

$$X^{(n)}_{+R} := Sx_{+R}S^\dagger + \sum_{k=1}^{n} \gamma^R_k |i_k, j_k\rangle\langle i_k, j_k| \, , \tag{7.33}$$

and $X^{(n)}_{-a} = (X^{(n)}_{+a})^\dagger$. Here $\gamma^{L,R}_k \in \mathbb{C}$ determine the position of the fluxons. The field strength $F_{\mu\nu}$ for this solution is

$$F_{\mu\nu} = P_{V_n} \theta_{\mu\nu}. \tag{7.34}$$

In particular, the action corresponding to the instanton solution (7.32,7.33) is proportional to the dimension of the subspace V_n

$$S[X^{(n)}_{\pm a}] = \frac{8\pi^2}{g^2} \operatorname{tr}(P_{V_n}) = \frac{8\pi^2}{g^2}\, n. \tag{7.35}$$

[1]If we index the basis of $\mathcal{H}$ as $|i_k, j_k\rangle$ with $k \in \mathbb{N}$ and assume that V is spanned by the first n vectors (which we can always get by using a suitable unitary transformation),S can be given by $S : |i_k, j_k\rangle \rightarrow |i_{k+n}, j_{k+n}\rangle$.

[2]Note that $[X^{(n)}_{+L}, X^{(n)}_{+R}] = [X^{(n)}_{+L}, X^{(n)}_{-R}] = [X^{(n)}_{-L}, X^{(n)}_{+R}] = [X^{(n)}_{-L}, X^{(n)}_{-R}] = 0$.

Since they can be interpreted as localized flux, these $U(1)$-instanton solutions for $\mathbb{R}^4_\theta$ are called fluxons. The localization can be seen as follows: recall [42] that the above projection operators can be represented on the space of commutative functions (using a normal-ordering prescription) as

$$|k^1, k^2\rangle\langle k^1, k^2| \cong \frac{1}{k^1!k^2!}(\frac{x^{-L}}{\sqrt{2\theta}})^{k^1}(\frac{x^{+L}}{\sqrt{2\theta}})^{k^1}(\frac{x^{-R}}{\sqrt{2\theta}})^{k^2}(\frac{x^{+R}}{\sqrt{2\theta}})^{k^2}e^{-\frac{x^{+L}x^{-L}}{2\theta}-\frac{x^{+R}x^{-R}}{2\theta}} \; . \tag{7.36}$$

Hence the above field strengths $F_{\mu\nu} = P_{V_n}\theta_{\mu\nu}$ are superpositions of Gauss-functions which are localized in a region in space of size $\sqrt{\theta}$.

Chapter 8

Fuzzy spaces

In this chapter we will present a 4-dimensional noncommutative space that has the advantage of having finite dimensional representations. Therefore, the gauge theory we will construct on it in chapter 9 will be well defined and all calculations will become finite. Using this space we will be able to regularize both $\mathbb{R}^4_\theta$ itself in chapter 8.3 and gauge theory on $\mathbb{R}^4_\theta$ in chapter 10.

8.1 The fuzzy sphere S^2_N

We start by recalling the definition of a 2-dimensional space, the fuzzy sphere introduced in [73]. The algebra of functions on the fuzzy sphere is the finite algebra S^2_N generated by Hermitian operators $x_i = (x_1, x_2, x_3)$ satisfying the defining relations

$$[x_i, x_j] = i\Lambda_N \epsilon_{ijk} x_k, \tag{8.1}$$
$$x_1^2 + x_2^2 + x_3^2 = R^2. \tag{8.2}$$

They are obtained from the N-dimensional representation of $su(2)$ with generators λ_i $(i = 1, 2, 3)$ and commutation relations

$$[\lambda_i, \lambda_j] = i\epsilon_{ijk}\lambda_k, \qquad \sum_{i=1}^{3} \lambda_i\lambda_i = \frac{N^2-1}{4} \tag{8.3}$$

(see Appendix C.1) by identifying

$$x_i = \Lambda_N\ \lambda_i, \qquad \Lambda_N = \frac{2R}{\sqrt{N^2-1}}. \tag{8.4}$$

The noncommutativity parameter Λ_N is of dimension length. The algebra of functions S^2_N therefore coincides with the simple matrix algebra $Mat(N, \mathbb{C})$. The normalized integral of a function $f \in S_N^2$ is given by the trace

$$\int\limits_{S_N^2} f = \frac{4\pi R^2}{N} \mathrm{tr}(f). \tag{8.5}$$

The functions on the fuzzy sphere can be mapped to functions on the commutative sphere S^2 using the decomposition into harmonics under the action

$$J_i f = [\lambda_i, f] \tag{8.6}$$

of the rotation group $SU(2)$. One obtains analogs of the spherical harmonics up to a maximal angular momentum $N-1$. Therefore S_N^2 is a regularization of S^2 with a UV cutoff, and the commutative sphere S^2 is recovered in the limit $N \to \infty$. Note also that for the standard representation (C.2), entries in the upper-left block of the matrices correspond to functions localized at $x_3 = R$. In particular, the fuzzy delta-function at the "north pole" is given by a suitably normalized projector of rank 1,

$$\delta^{(2)}{}_{NP}(x) = \frac{N}{4\pi R^2} \, |\frac{N-1}{2}\rangle\langle\frac{N-1}{2}| \tag{8.7}$$

where $|\frac{N-1}{2}\rangle$ is the highest weight state with maximal eigenvalue of λ_3. Delta-functions with arbitrary localization are obtained by rotating (8.7).

8.2 $S^2_{N_L} \times S^2_{N_R}$

The simplest 4-dimensional generalization of the above is the product $S^2_{N_L} \times S^2_{N_R}$ of 2 such fuzzy spheres, with generally independent parameters $N_{L,R}$. It is generated by a double set of representations of $su(2)$ commuting with each other, i. e. by λ_i^L, λ_i^R satisfying

$$\begin{aligned} [\lambda_i^L, \lambda_j^L] &= i\epsilon_{ijk}\lambda_k^L, \qquad [\lambda_i^R, \lambda_j^R] = i\epsilon_{ijk}\lambda_k^R, \\ [\lambda_i^L, \lambda_j^R] &= 0 \end{aligned} \tag{8.8}$$

for $i, j = 1, 2, 3$, and Casimirs

$$\sum_{i=1}^{3} \lambda_i^L \lambda_i^L = \frac{N_L^2 - 1}{4}, \qquad \sum_{i=1}^{3} \lambda_i^R \lambda_i^R = \frac{N_R^2 - 1}{4}. \tag{8.9}$$

This can be realized as a tensor product of 2 fuzzy sphere algebras

$$\lambda_i^L = \lambda_i \otimes 1_{N_R \times N_R}, \tag{8.10}$$

$$\lambda_i^R = 1_{N_L \times N_L} \otimes \lambda_i, \tag{8.11}$$

hence as algebra we have $S^2_{N_L} \times S^2_{N_R} \cong \mathrm{Mat}(\mathcal{N}, \mathbb{C})$ where

$$\mathcal{N} = N_L N_R. \tag{8.12}$$

The normalized coordinate functions are given by

$$x_i^{L,R} = \frac{2R}{\sqrt{(N^{L,R})^2 - 1}}\, \lambda_i^{L,R}, \qquad \sum (x_i^L)^2 = R^2 = \sum (x_i^R)^2. \tag{8.13}$$

This space[1] can be viewed as regularization of $S^2 \times S^2 \subset \mathbb{R}^6$, and admits the symmetry group $SU(2)_L \times SU(2)_R \subset SO(6)$. The generators $x_i^{L,R}$ should be viewed as coordinates in an embedding space $\mathbb{R}^6$. The normalized integral of a function $f \in S^2_{N_L} \times S^2_{N_R}$ is now given by

$$\int\limits_{S^2_{N_L} \times S^2_{N_R}} f = \frac{16\pi^2 R^4}{\mathcal{N}} \mathrm{tr}(f) = \frac{V}{\mathcal{N}} \mathrm{tr}(f), \tag{8.14}$$

where we define the volume $V := 16\pi^2 R^4$. We will mainly consider $N_L = N_R$ in the following.

8.3 The limit to the canonical case $\mathbb{R}^4_\theta$

It is well-known [28] that if a fuzzy sphere is blown up near a given point, it can be used to obtain a (compactified) noncommutative plane with canonical commutation relations: Consider the tangential coordinates $x_{1,2}$ near the north pole $x_3 = R$. Setting

$$R^2 = N\theta/2, \tag{8.15}$$

they satisfy the commutation relations

$$[x_1, x_2] \;=\; i\frac{2R}{N} x_3 \;=\; i\frac{2R}{N}\sqrt{R^2 - x_1^2 - x_2^2} \;=\; i\theta + O(1/N). \tag{8.16}$$

Therefore in the double scaling limit with $N,\ R \to \infty$ keeping θ fixed, we recover[2] the commutation relation of the canonical case,

$$[x_1, x_2] = i\theta \tag{8.17}$$

[1] In principle one could also introduce different radii $R^{L,R}$ for the 2 spheres, but for simplicity we will keep only one scale parameter R (and sometimes we will set $R = 1$).

[2] One could be more sophisticated and use the stereographic projections as in [28], which leads essentially to the same results.

up to corrections of order $\frac{1}{N}$. Similarly, starting with $S^2_{N_L} \times S^2_{N_R}$ and setting

$$R^2 = N_{L,R}\theta_{L,R}/2, \tag{8.18}$$

we obtain in the large N_L, N_R limit

$$\begin{aligned} [x_i^L, x_j^L] &= i\epsilon_{ij}\theta^L, \qquad [x_i^R, x_j^R] = i\epsilon_{ij}\theta^R, \\ [x_i^L, x_j^R] &= 0. \end{aligned} \tag{8.19}$$

This is the most general form of $\mathbb{R}^4_\theta$ with coordinates $(x_1, ..., x_4) \equiv (x_1^L, x_2^L, x_1^R, x_2^R)$ (after a suitable orthogonal transformation). The integral of a function $f(x)$ then becomes

$$\int\limits_{S^2_{N_L} \times S^2_{N_R}} f(x) \to 4\pi^2\theta_L\theta_R \mathrm{tr}(f(x)) =: \int\limits_{\mathbb{R}^4_\theta} f(x), \tag{8.20}$$

which has indeed the standard normalization, giving each "Planck cell" the appropriate volume.

Chapter 9

Gauge theory on fuzzy $S^2 \times S^2$

Now that we have the fuzzy space $S^2_{N_L} \times S^2_{N_R}$ corresponding to $\mathcal{N}^2$-dimensional matrices, we want to construct a matrix model having $S^2_{N_L} \times S^2_{N_R}$ as its ground state. As in the canonical case, the fluctuations around this ground state will produce a gauge theory. But as the matrices are now finite-dimensional, the model will be well defined and finite.

We will start with the most obvious formulation, gauging every coordinate seperately. But there is also a more elegant formulation using collective matrices. This will be especially usefull to introduce fermions, which can be embedded very naturally in this framework.

We will also study non-trivial solutions of the EOMs, identifying some of them as the monopoles on the commutative $S^2 \times S^2$, while others will become important in the limit to $\mathbb{R}^4_\theta$ in the following chapter.

9.1 Gauge theory

In the fuzzy case, it is natural to construct $S^2_L \times S^2_R$ as a submanifold of $\mathbb{R}^6$. We therefore consider a multi-matrix model with 6 dynamical fields (covariant coordinates) B^L_i and B^R_i ($i = 1, 2, 3$), which are $\mathcal{N} \times \mathcal{N}$ Hermitian matrices. As action we choose the following generalization of the action in [96],

$$S = \frac{1}{g^2} \int \frac{1}{2} F_{ia\,jb} F_{ia\,jb} + \varphi_L^2 + \varphi_R^2 \tag{9.1}$$

with $a, b = L, R$ and $i, j = 1, 2, 3$; summation over repeated indices is implied. Here $\varphi_{L,R}$ are defined as

$$\varphi_L := \frac{1}{R^2}(B^L_i B^L_i - \frac{N_L^2 - 1}{4}), \qquad \varphi_R := \frac{1}{R^2}(B^R_i B^R_i - \frac{N_R^2 - 1}{4}), \tag{9.2}$$

and R denotes the radius of the two spheres, which we keep explicitly to have the correct dimensions. The field strength is defined by

$$\begin{aligned} F_{iL\,jL} &= \frac{1}{R^2}(i[B_i^L, B_j^L] + \epsilon_{ijk} B_k^L), \\ F_{iR\,jR} &= \frac{1}{R^2}(i[B_i^R, B_j^R] + \epsilon_{ijk} B_k^R), \\ F_{iL\,jR} &= \frac{1}{R^2}(i[B_i^L, B_j^R]). \end{aligned} \tag{9.3}$$

This model (9.1) is manifestly invariant under $SU(2)_L \times SU(2)_R$ rotations acting in the obvious way, and $U(\mathcal{N})$ gauge transformations acting as $B_i^{L,R} \to UB_i^{L,R}U^{-1}$. We will see below that this reduces indeed to the $U(1)$ Yang-Mills action on $S^2 \times S^2$ in the commutative limit. Note that if the action (9.1) is considered as a matrix model, the radius drops out using (8.14). The equations of motion for B_i^L are

$$\begin{aligned} &\{B_i^L, B_j^L B_j^L - \frac{N_L^2 - 1}{4}\} + (B_i^L + i\epsilon_{ijk} B_j^L B_k^L) \\ &+ i\epsilon_{ijk}[B_j^L, (B_k^L + i\epsilon_{krs} B_r^L B_s^L)] + [B_j^R, [B_j^R, B_i^L]] = 0, \end{aligned} \tag{9.4}$$

and those for B_i^R are obtained by exchanging $L \leftrightarrow R$. By construction, the minimum or ground state of the action is given by $F = \varphi = 0$, hence $B_i^{L,R} = \lambda_i^{L,R}$ as in (8.10,8.11) up to gauge transformations; cp. [53] for a similar approach on $\mathbb{C}P^2$. We can therefore expand the covariant coordinates B_i^L and B_i^R around the ground state

$$B_i^a = \lambda_i^a + RA_i^a, \tag{9.5}$$

where $a \in \{L, R\}$ and A_i^a is small. Then $A_i^{L,R}$ transforms under gauge transformations as

$$A_i^{L,R} \to A_i'^{L,R} = UA_i^{L,R}U^{-1} + U[\lambda_i^{L,R}, U^{-1}], \tag{9.6}$$

and the field strength takes a more familiar form[1],

$$\begin{aligned} F_{iL\,jL} &= i([\frac{\lambda_i^L}{R}, A_j^L] - [\frac{\lambda_j^L}{R}, A_i^L] + [A_i^L, A_j^L]), \\ F_{iR\,jR} &= i([\frac{\lambda_i^R}{R}, A_j^R] - [\frac{\lambda_j^R}{R}, A_i^R] + [A_i^R, A_j^R]), \\ F_{iL\,jR} &= i([\frac{\lambda_i^L}{R}, A_j^R] - [\frac{\lambda_j^R}{R}, A_i^L] + [A_i^L, A_j^R]). \end{aligned} \tag{9.7}$$

So far, the spheres are described in terms of 3 Cartesian covariant coordinates each. In the commutative limit, we can separate the radial and tangential degrees

[1] We do not distinguish between upper and lower indices L, R.

of freedom. There are many ways to do this; perhaps the most elegant for the present purpose is to note that the terms $\int \varphi_L^2 + \varphi_R^2$ in the action imply that $\varphi_{L,R}$ is bounded for configurations with finite action. Using

$$\varphi_L = \frac{\lambda_i^L}{R} A_i^L + A_i^L \frac{\lambda_i^L}{R} + A_i^L A_i^L, \tag{9.8}$$

and similarly for φ_R it follows that

$$x_i A_i^a + A_i^a x_i = O(\frac{\varphi}{N}) \tag{9.9}$$

for finite A_i^a. This means that A_i^a is tangential in the (commutative) large N limit. Alternatively, one could consider $\phi_L = N\varphi_L$, which would acquire a mass of order N and decouple from the other fields[2]. The commutative limit of (9.1) therefore gives the standard action for electrodynamics on $S^2 \times S^2$,

$$S = \frac{1}{2g^2} \int\limits_{S^2 \times S^2} F_{ia\,jb}^t F_{ia\,jb}^t \tag{9.10}$$

with $a, b = L, R$ and $i, j = 1, 2, 3$. Here $F_{iL\,jR}^t$ denotes the usual tangential field strength. This can be seen most easily by noting that e.g. at the north pole $x_3^{L,R} = R$, one can replace

$$i[\frac{\lambda_i^{L,R}}{R}, \cdot] \quad \rightarrow \quad -\varepsilon_{ij} \frac{\partial}{\partial x_j^{L,R}} \tag{9.11}$$

in the commutative limit, so that upon identifying the commutative gauge fields $A_i^{(cl)}$ via

$$A_i^{(cl)L,R} = -\varepsilon_{ij} A_i^{L,R} \tag{9.12}$$

the field strength is given by the standard expression $F_{iL\,jR}^t = \partial_i^L A_j^{(cl)R} - \partial_j^R A_i^{(cl)L}$ etc.

$U(k)$ gauge theory

The above action generalizes immediately to the nonabelian case, keeping precisely the same action (9.1), (9.2), but replacing the matrices $B_i^{L,R}$ by $k\mathcal{N} \times k\mathcal{N}$ matrices, cp. [96]. The constraint term will then impose as ground state $\lambda_i^{L/R} \otimes 1_{k\times k}$. Expanding the covariant coordinates $B_i^{L,R} = \lambda_i^{L/R} \otimes 1_{k\times k} + A_{i,a}^{L/R} T^a$ in terms of the Gellman matrices T^a, the action (9.1) is the fuzzy version of nonabelian $U(k)$ Yang-Mills on $S^2 \times S^2$.

[2]The constraints $\varphi_L = 0 = \varphi_R$ could also be imposed by hand; however the suppression through the above terms in the action is more flexible, as we will see in chapter 9.5.

9.2 A formulation based on $SO(6)$

The above action can be cast into a nicer form by assembling the matrices $B_i^{L,R}$ into bigger collective matrices, following [96]. Since it is natural from the fuzzy point of view to embed $S^2 \times S^2 \subset \mathbb{R}^6$ with corresponding embedding of the symmetry group $SO(3)_L \times SO(3)_R \subset SO(6)$, we consider

$$B_\mu = (B_i^L, B_i^R) \tag{9.13}$$

to be the 6 -dimensional irrep of $so(6) \cong su(4)$. Since $(4) \otimes (4) = (6) \oplus (10)$, it is natural to introduce the intertwiners

$$\gamma_\mu = (\gamma_i^L, \gamma_i^R) = (\gamma_\mu)^{\alpha,\beta} \tag{9.14}$$

where α, β denote indices of (4). We could then assemble our dynamical fields into a single $4\mathcal{N} \times 4\mathcal{N}$ matrix

$$B = B_\mu \gamma_\mu \quad + \text{const} \cdot 1\!\!1. \tag{9.15}$$

Of course the most general such $4\mathcal{N} \times 4\mathcal{N}$ matrix contains far too many degrees of freedom, and we have to constrain these B further. Since $SU(4)$ acts on B as $B \to U^T B U$, the γ_μ can be chosen as totally anti-symmetric matrices, which precisely singles out the $(6) \subset (4) \otimes (4)$. One can moreover impose

$$(\gamma_i^L)^\dagger = \gamma_i^L, \qquad (\gamma_i^R)^\dagger = -\gamma_i^R, \tag{9.16}$$

and

$$\begin{aligned} \gamma_i^L \gamma_j^L &= \delta_{ij} + i\epsilon_{ijk}\gamma_k^L, && (9.17)\\ \gamma_i^R \gamma_j^R &= -\delta_{ij} - \epsilon_{ijk}\gamma_k^R, && (9.18)\\ [\gamma_i^L, \gamma_j^R] &= 0, && (9.19) \end{aligned}$$

which will be assumed from now on; we will give two explicit such representations in (D.5), (C.5). This would suggest to constrain B to be antisymmetric. However, the component fields B_μ are naturally considered as Hermitian rather than symmetric matrices. Furthermore, since the $\gamma_\mu = (\gamma_\mu)^{\alpha,\beta}$ have two upper indices, they do not form an algebra. There are two ways to proceed. We can either separate them again by introducing two $4\mathcal{N} \times 4\mathcal{N}$ matrices,

$$B^L = \frac{1}{2} + B_i^L \gamma_i^L, \qquad B^R = \frac{i}{2} + B_i^R \gamma_i^R, \tag{9.20}$$

breaking $SO(6) \to SO(3) \times SO(3)$. This will be pursued in Appendix D.1. Alternatively, we can use the γ_μ with the above properties to construct the 8×8 Gamma-matrices

$$\Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^{\mu\dagger} & 0 \end{pmatrix}, \tag{9.21}$$

which generate the $SO(6)$-Clifford algebra

$$\{\Gamma^\mu, \Gamma^\nu\} = \begin{pmatrix} \gamma^\mu\gamma^{\nu\dagger} + \gamma^\nu\gamma^{\mu\dagger} & 0 \\ 0 & \gamma^{\mu\dagger}\gamma^\nu + \gamma^{\nu\dagger}\gamma^\mu \end{pmatrix} = 2\delta^{\mu\nu}. \tag{9.22}$$

This suggests to consider the single Hermitian $8\mathcal{N} \times 8\mathcal{N}$ matrix

$$C = \Gamma^\mu B_\mu + C_0 = \begin{pmatrix} 0 & B^L \\ B^L & 0 \end{pmatrix} + \begin{pmatrix} 0 & B^R \\ -B^R & 0 \end{pmatrix} =: C^L + C^R, \tag{9.23}$$

where $C_0 = C_0^L + C_0^R$ denote the constant 8×8-matrices

$$C_0^L = -\frac{i}{2}\Gamma_1^L\Gamma_2^L\Gamma_3^L = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{9.24}$$

$$C_0^R = -\frac{i}{2}\Gamma_1^R\Gamma_2^R\Gamma_3^R = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{9.25}$$

in the above basis. Using the Clifford algebra and the above definitions one then finds

$$C^2 = B_\mu B_\mu + \frac{1}{2} + \Sigma_8^{\mu\nu} F_{\mu\nu}. \tag{9.26}$$

Here $\Sigma_8^{\mu\nu} = -\frac{i}{4}[\Gamma_\mu, \Gamma_\nu]$, and the field strength $F_{\mu\nu}$ coincides with the definition in (9.3) if written in the $L-R$ notation,

$$F_{ia\,jb} = i[B_{ia}, B_{jb}] + \delta_{ab}\epsilon_{ijk}B_{ka}. \tag{9.27}$$

Therefore the action

$$S_6 = \mathrm{Tr}((C^2 - \frac{N^2}{2})^2) = 8\mathrm{tr}(B_\mu B_\mu - \frac{N^2-1}{2})^2 + 4\mathrm{tr}F_{\mu\nu}F_{\mu\nu} \tag{9.28}$$

is quite close to what we want. The only difference is the term $(B_\mu B_\mu - \frac{N^2-1}{2})^2$ instead of $(B_{iL}B_{iL} - \frac{N_L^2-1}{4})^2 + (B_{iR}B_{iR} - \frac{N_R^2-1}{4})^2$, for $2N^2 = N_L^2 + N_R^2$. This difference is easy to understand: since (9.28) is $SO(6)$-invariant, the ground state should be some S^5. We therefore have to break this $SO(6)$- invariance explicitly, which will be done in the next chapter. However before doing that, let us try to understand action (9.28) better and see whether it leads to a meaningful 4-dimensional field theory. We show in Appendix D.2 by carefully integrating out the scalar components of $B_i^{L,R}$ that the $SO(6)$- invariant constraint term in (9.28) induces the second term in the following effective action

$$S_6^{\mathrm{eff}} \sim 4\mathrm{tr}\left(F_{\mu\nu}F_{\mu\nu} - (F_{iL}x_{iL} - F_{iR}x_{iR})\frac{1}{4(\frac{1}{2} - \partial_\mu\partial_\mu)}(F_{iL}x_{iL} - F_{iR}x_{iR})\right) \tag{9.29}$$

in the commutative limit, where $F_{iL} = \frac{1}{2}\epsilon_{ijk}F_{jL\,kL}$ etc. Comparing the second term with $F_{\mu\nu}F_{\mu\nu}$, we see that the zero mode of the Laplace operator $\partial_\mu\partial_\mu$ can produce a contribution that cancels the corresponding contribution from $F_{\mu\nu}F_{\mu\nu}$, but that all higher modes are smaller by at least a factor of $2(\frac{1}{2} - \partial_\mu\partial_\mu)$. Therefore, the action (9.28) is positive definite except for the obvious zero mode $\delta B_i^L = \epsilon, \ \ \delta B_i^R = -\epsilon$. This means that the geometry of $S_L^2 \times S_R^2$ is locally stable even with the $SO(6)$-symmetry unbroken, except for opposite fluctuations of the radii.

9.2.1 Breaking $SO(6) \to SO(3) \times SO(3)$

To obtain the original action (9.1) for $S^2 \times S^2$, we have to break the $SO(6)$-symmetry down to $SO(3) \times SO(3)$. We can do this by using the left and right gauge fields C^L and C^R introduced in (9.23) separately. Their squares are

$$\begin{aligned} C_L^2 &= B_{iL}B_{iL} + \frac{1}{4} + \begin{pmatrix} \gamma_L^i & 0 \\ 0 & \gamma_L^i \end{pmatrix} (B_{iL} + i\epsilon_{ijk}B_{jL}B_{kL}), \\ C_R^2 &= B_{iR}B_{iR} + \frac{1}{4} - i \begin{pmatrix} \gamma_R^i & 0 \\ 0 & \gamma_R^i \end{pmatrix} (B_{iR} + i\epsilon_{ijk}B_{jR}B_{kR}). \end{aligned} \tag{9.30}$$

As both γ_L^i, γ_R^i and $\gamma_L^i\gamma_R^j$ are traceless, we have

$$\begin{aligned} S_{\text{break}} &:= 2\mathrm{Tr}((C_L^2 - \frac{N_L^2}{4})(C_R^2 - \frac{N_R^2}{4})) \\ &= 16\mathrm{Tr}((B_{iL}B_{iL} - \frac{N_L^2 - 1}{4})(B_{iR}B_{iR} - \frac{N_R^2-1}{4})). \end{aligned} \tag{9.31}$$

With these terms we can recover our action as

$$\begin{aligned} S &= S_6 - S_{\text{break}} = \mathrm{Tr}\big((C^2 - \frac{N^2}{2})^2 - 2(C_L^2 - \frac{N_L^2}{4})(C_R^2 - \frac{N_R^2}{4})\big) \\ &= 8\,\mathrm{tr}\,\big((B_{iL}B_{iL} - \frac{N_L^2-1}{4})^2 + (B_{iR}B_{iR} - \frac{N_R^2-1}{4})^2 + \frac{1}{2}F_{\mu\nu}F_{\mu\nu}\big), \end{aligned} \tag{9.32}$$

which is precisely the action (9.1) for gauge theory on $S_{N_L}^2 \times S_{N_R}^2$ omitting the overall constants. Hence the action is formulated as a 2-matrix model, however with highly constrained matrices C_L, C_R. This formulation using the Gamma-matrices is very natural and useful if one wants to couple the gauge fields to fermions, as discussed in chapter 9.4.

For simplicity, we will only consider $N_L = N_R = N$ from now on.

9.3 Quantization

The quantization of the gauge theory defined by (9.1) or its reformulation (9.32) is straightforward in principle, by a path integral over the Hermitian matrices

$$Z[J] = \int dB_\mu e^{-S[B_\mu] + \mathrm{tr} B_\mu J_\mu}. \tag{9.33}$$

Note that there is no need to fix the gauge since the gauge group $U(\mathcal{N})$ is compact. The above path integral is well-defined and finite for any fixed $\mathcal{N}$. To see this, it is enough to show that the integral

$$\int dB_\mu e^{-(B_i^L B_i^L - \frac{N^2-1}{4})^2 - (B_i^R B_i^R - \frac{N^2-1}{4})^2} \tag{9.34}$$

converges, since the contributions from the field strength further suppress the integrand. This integral is obviously convergent for any fixed N.

For perturbative computations it is necessary to fix the gauge, and to substitute gauge invariance by BRST-invariance. Such a gauge-fixed action will be presented next.

9.3.1 BRST Symmetry

To construct a gauge-fixed BRST-invariant action, we have to introduce ghost fields c and anti-ghost fields $\bar{c}$. These are fermionic fields, more precisely $\mathcal{N} \times \mathcal{N}-$ matrices with entries which are Grassman variables.

The full gauge-fixed action reads:

$$S_{\mathrm{BRST}} = S + \frac{1}{\mathcal{N}} \mathrm{tr}\big(\bar{c}[\lambda_\mu, [B_\mu, c]] - (\frac{\alpha}{2} b - [\lambda_\mu, B_\mu])b\big)\,, \tag{9.35}$$

where b is an auxiliary (Nakanishi-Lautrup) field. This action is invariant with respect to the following BRST-transformations:

$$\begin{aligned} sB_\mu &= [B_\mu, c] \qquad & sc &= cc \\ s\bar{c} &= b & sb &= 0 \end{aligned} \tag{9.36}$$

(matrix product is understood), where the BRST-differential s acts on a product of fields as follows:

$$s(XY) = X(sY) + (-1)^{\varepsilon_Y}(sX)Y\,. \tag{9.37}$$

Here ε_Y denotes the Grassman-parity of Y

$$\varepsilon_Y = \begin{cases} 0 & Y \text{ bosonic} \\ 1 & Y \text{ fermionic}\,. \end{cases} \tag{9.38}$$

It is not difficult to check that these BRST-transformations are indeed nilpotent, i.e.

$$s^2 = 0\,. \tag{9.39}$$

Integrating out the auxiliary field b leads to the following action

$$S'_{\mathrm{BRST}} = S + \frac{1}{\mathcal{N}}\mathrm{tr}\big(\bar{c}[\lambda_\mu,[B_\mu,c]] - \frac{1}{2\alpha}[\lambda_\mu,B_\mu][\lambda_\nu,B_\nu]\big)\,. \tag{9.40}$$

Setting $\alpha = 1$ corresponds to the Feynman gauge. This is indeed what one would obtain by the Faddeev-Popov procedure. The action S' is invariant with respect to the following operations:

$$\begin{aligned} s'B_\mu &= [B_\mu,c] \\ s'c &= cc \\ s'\bar{c} &= [\lambda_\mu,B_\mu]\,. \end{aligned} \tag{9.41}$$

Since we have used the equations of motion of b, the BRST-differential s' is *not* nilpotent off-shell anymore, but we still have

$$s'^2|_{\mathrm{on-shell}} = 0\,. \tag{9.42}$$

9.4 Fermions

To introduce spinors on fuzzy $S^2 \times S^2$, we will first have to have a look at the commutative case. There, we will calculate the Dirac operator and bring it into a form which is more suitable for the fuzzy case. The formulation of fuzzy gauge theory using the $SO(6)$-Clifford algebra will proove very usefull, and the fuzzy Dirac operator will be a simple generalization of the commutative one. But this Dirac operator (because it is based on $SO(6)$ instead of $SO(3) \times SO(3)$) will be reducible, which is why we will have to introduce projectors onto the physical Dirac fermions. Chirality can be introduced either using the chirality operator inherited from $SO(6)$ or using a Ginsparg-Wilson system.

9.4.1 The commutative Dirac operator on $S^2 \times S^2$

To find a form of the commutative Dirac operator on $S^2 \times S^2$ which is suitable for the fuzzy case, one can generalize the approach of [51] for S^2, which is carried out in detail in Appendix D.3.3: One can write the flat $SO(6)$ Dirac operator D_6 in 2 different forms, using the usual flat Euclidean coordinates and also using the spherical coordinates of the spheres. Then one can relate D_6 with the curved four-dimensional Dirac operator D_4 on $S^2 \times S^2$ in the same spherical coordinates.

This leads to an explicit expression for D_4 involving only the angular momentum generators, which is easy to generalize to the fuzzy case. In terms of these tangential derivatives J_μ, the result becomes the simple expression

$$D_4 = \Gamma^\mu J_\mu + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \Gamma^\mu J_\mu + 2C_0, \tag{9.43}$$

which is clearly a $SO(3) \times SO(3)$-covariant Hermitian first-oder differential operator. Here Γ^μ generate the $SO(6)$ Clifford algebra (9.22), C_0 is defined in (9.25), and we put $R = 1$ for simplicity here. However this Dirac operator is reducible, acting on 8-dimensional spinors Ψ_8 corresponding to the $SO(6)$ Clifford algebra. Hence Ψ_8 should be a combination of two independent 4-component Dirac spinors on the 4-dimensional space $S^2 \times S^2$. To see this, we will construct explicit projectors projecting onto these 4-dimensional spinors, and identify the appropriate 4-dimensional chirality operators. This will provide us with the desired physical Dirac or Weyl fermions.

9.4.2 Chirality and projections for the spinors

There are 3 obvious operators which anti-commute with D_4. One is the usual 6-dimensional chirality operator

$$\Gamma := i\Gamma_1^L\Gamma_2^L\Gamma_3^L\Gamma_1^R\Gamma_2^R\Gamma_3^R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{9.44}$$

which satisfies

$$\{D_4, \Gamma\} = 0, \qquad \Gamma^\dagger = \Gamma, \qquad \Gamma^2 = 1. \tag{9.45}$$

The 8-component spinors Ψ_8 split accordingly into two 4-component spinors $\Psi_8 = \begin{pmatrix} \psi_\alpha \\ \overline{\psi}_{\overline{\beta}} \end{pmatrix}$, which transform as 4 resp. $\overline{4}$ under $so(6) \cong su(4)$; recall the related discussion in chapter 9.2. The other operators of interest are

$$\chi_L = \Gamma^{iL} x_{iL} \quad \text{and} \quad \chi_R = \Gamma^{iR} x_{iR}. \tag{9.46}$$

They preserve $SO(3) \times SO(3) \subset SO(6)$, and satisfy

$$\{D_4, \chi_{L,R}\} = 0 = \{\chi_L, \chi_R\} \tag{9.47}$$

as well as

$$\chi_{L,R}^2 = 1. \tag{9.48}$$

We will also use

$$\chi = \frac{1}{\sqrt{2}}\, \Gamma^\mu x_\mu = \frac{1}{\sqrt{2}}\, (\chi_L + \chi_R) \tag{9.49}$$

which satisfies similar relations. This means that

$$P_\pm = \frac{1}{2}(1 \pm i\chi_L\chi_R) \tag{9.50}$$

with

$$P_\pm^2 = P_\pm, \quad P_+ + P_- = 1 \quad \text{and} \quad P_+P_- = 0 \tag{9.51}$$

are Hermitian projectors commuting with the Dirac operator on $S^2 \times S^2$ as well as with Γ,

$$P_\pm^\dagger = P_\pm \quad \text{and} \quad [P_\pm, D_4] = [P_\pm, \Gamma] = 0. \tag{9.52}$$

Therefore they project onto subspaces which are preserved by D_4 and Γ. Hence the spinor Lagrangian can be written as

$$\Psi_8^\dagger D_4 \Psi_8 = \Psi_+^\dagger D_4 \Psi_+ + \Psi_-^\dagger D_4 \Psi_- \tag{9.53}$$

involving two Dirac spinors $\Psi_\pm = P_\pm \Psi_8$. In order to get one 4-component Dirac spinor, we can e.g. impose the constraint

$$P_+ \Psi_8 = \Psi_8, \tag{9.54}$$

or equivalently give one of the two components a large mass, by adding a term

$$M\, \Psi_8^\dagger P_- \Psi_8 \tag{9.55}$$

to the action with $M \to \infty$. The physical chirality operator is now identified using (9.52) and (9.45) as Γ acting on Ψ_+. It can be used to define 2-component Weyl spinors on $S^2 \times S^2$.

To make the above more explicit, consider the a pole of the spheres, i.e.

$$x_L = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad x_R = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{9.56}$$

In the basis (9.21) for the Clifford algebra we then get explicitly

$$P_\pm = \frac{1}{2}(1 \pm i \begin{pmatrix} -\gamma_L^1\gamma_R^1 & 0 \\ 0 & \gamma_L^1\gamma_R^1 \end{pmatrix}) = \frac{1}{2}(1 \pm \sigma_3 \otimes \sigma_3 \otimes \sigma_3). \tag{9.57}$$

This means that

$$P_+ = \mathrm{diag}(1,0,0,1,0,1,1,0) \tag{9.58}$$

projects onto a 4-dimensional subspace exactly as expected.

9.4.3 Gauged fuzzy Dirac and chirality operators

To find a fuzzy analogue of the Dirac operator (9.43) coupled to the gauge fields, we recall the connection between the gauge theory on $S^2 \times S^2$ and the $SO(6)$ Gamma matrices established in chapter 9.2. In the spirit of that chapter a natural fuzzy spinor action would involve

$$\Psi^\dagger C \Psi, \tag{9.59}$$

where Ψ is now a $8\mathcal{N} \times \mathcal{N}$-matrix (with Grassman entries). Of course, (9.59) does not have the appropriate commutative limit, but we can split C into a fuzzy Dirac operator $\widehat{D}$ and the operator $\widehat{\chi}$ defined by

$$\widehat{\chi}\Psi = \frac{\sqrt{2}}{N}\,(\Gamma^\mu \Psi \lambda_\mu - C_0 \Psi), \tag{9.60}$$

which generalizes (9.49); here we used the definition (9.24,9.25) of C_0. This operator satisfies

$$\widehat{\chi}^2 = 1, \tag{9.61}$$

and reduces to (9.49) in the commutative limit. Note also that $\widehat{\chi}$ commutes with gauge transformations, since the coordinates λ_μ are acting from the right in (9.60). Setting

$$\widehat{J}_\mu \Psi = [\lambda_\mu, \Psi], \tag{9.62}$$

we get for the fuzzy Dirac operator

$$\widehat{D} = C - \frac{N}{\sqrt{2}}\,\widehat{\chi} = \Gamma^\mu(\widehat{J}_\mu + A_\mu) + 2C_0 = \Gamma^\mu \mathcal{D}_\mu + 2C_0. \tag{9.63}$$

Here[3]

$$\widehat{\mathcal{D}}_\mu := \widehat{J}_\mu + A_\mu \tag{9.64}$$

is a covariant derivative operator, i.e. $\widehat{\mathcal{D}}_\mu \psi \to U \widehat{\mathcal{D}}_\mu \psi$ which is easily verified using (9.6). This $\widehat{D}$ clearly has the correct commutative limit (9.43) for vanishing A, and the gauge fields are coupled correctly. In particular, this definition of $\widehat{D}$ applies also to the topologically non-trivial solutions of chapter 9.5 without any modifications. Moreover, the chirality operator Γ as defined in (9.44) anti-commutes with $\widehat{D}$ also in the fuzzy case,

$$\{\widehat{D}, \Gamma\} = 0. \tag{9.65}$$

[3]We set $R = 1$ in this chapter for simplicity.

Furthermore, using some identities given at the beginning of chapter 9.2 we obtain for $\widehat{D}^2\psi$:

$$\begin{aligned}\widehat{D}^2\psi &= (\Sigma^{\mu\nu}F_{\mu\nu} + \widehat{\mathcal{D}}_\mu\widehat{\mathcal{D}}_\mu + \{\Gamma^\mu, C_0\}\widehat{\mathcal{D}}_\mu + 2)\psi \\ &=: (\Sigma^{\mu\nu}F_{\mu\nu} + \widehat{\Box} + 2)\psi,\end{aligned} \tag{9.66}$$

defining the covariant 4-dimensional Laplacian $\widehat{\Box}$ acting on the spinors. This corresponds to the usual expression for $\widehat{D}^2$ on curved spaces, and the constant 2 is due to the curvature scalar. Since $\widehat{D}^2$ and $\Sigma^{\mu\nu}F_{\mu\nu}$ are both Hermitian and commute with Γ and $\widehat{P}_\pm$ as defined in (9.69) in the large N limit, it follows that $\widehat{\Box}$ satisfies these properties as well.

9.4.4 Projections for the fuzzy spinors

For the fuzzy case, we can again consider the following operators

$$\begin{aligned}\widehat{\chi}_L\Psi &= \frac{2}{N}(\Gamma^{iL}\Psi\lambda_{iL} + C_0^L\Psi), \\ \widehat{\chi}_R\Psi &= \frac{2}{N}(\Gamma^{iR}\Psi\lambda_{iR} + C_0^R\Psi)\end{aligned} \tag{9.67}$$

which satisfy

$$\widehat{\chi}_{L,R}^2 = 1, \qquad \{\widehat{\chi}_L, \widehat{\chi}_R\} = 0\,. \tag{9.68}$$

This implies $(\widehat{\chi}_L\widehat{\chi}_R)^2 = -1$, and we can write down the projection operators

$$\widehat{P}_\pm = \frac{1}{2}(1 \pm i\widehat{\chi}_L\widehat{\chi}_R) \tag{9.69}$$

which have the commutative limit (9.50) and the properties (9.51). However, the projector no longer commutes with the fuzzy Dirac operator (9.63):

$$\begin{aligned}[\widehat{D}, \widehat{\chi}_L\widehat{\chi}_R] &= \{\widehat{D}, \widehat{\chi}_L\}\widehat{\chi}_R - \widehat{\chi}_L\{\widehat{D}, \widehat{\chi}_R\} \\ &= -\frac{2}{N}\Big(\big(2(\lambda_{iL} + A_{iL})\widehat{J}_{iL} - 2A_{iL}\lambda_{iL} + 2C_0^L\,\Gamma^{iL}\widehat{\mathcal{D}}_{iL} + 1\big)\widehat{\chi}_R \\ &\qquad -\widehat{\chi}_L\big(2(\lambda_{iR} + A_{iR})\widehat{J}_{iR} - 2A_{iR}\lambda_{iR} + 2C_0^R\,\Gamma^{iR}\widehat{\mathcal{D}}_{iR} + 1\big)\Big)\,,\end{aligned} \tag{9.70}$$

which only vanishes for $N \to \infty$ and tangential $\mathcal{A}_\mu$ (9.9). To reduce the degrees of freedom to one Dirac 4-spinor, we should therefore add a mass term

$$M\,\Psi_8^\dagger\widehat{P}_-\Psi_8 \tag{9.71}$$

which for $M \to \infty$ suppresses one of the spinors, rather than impose an exact constraint as in (9.54). This is gauge invariant since $\widehat{P}_\pm$ commutes with gauge transformations,

$$\widehat{P}_\pm\psi \to U\widehat{P}_\pm\psi\,. \tag{9.72}$$

The complete action for a Dirac fermion on fuzzy $S_N^2 \times S_N^2$ is therefore given by

$$S_{\text{Dirac}} = \int \Psi_8^\dagger(\widehat{D} + m)\Psi_8 + M\,\Psi_8^\dagger \widehat{P}_- \Psi_8 \tag{9.73}$$

with $M \to \infty$. The physical chirality operator is given by Γ (9.44), which allows to consider Weyl spinors as well.

9.4.5 The Ginsparg-Wilson relations

There is an alternative approach to introduce chirality on fuzzy spaces, using the Ginsparg-Wilson relations. These were initially designed to study chiral fermions on the lattice [43], but they proved to be applicable to fuzzy fermions as well [9, 10]. On the fuzzy sphere, the Dirac and the chirality operator can be cast into a form in which they fulfill these relations. This makes it possible to study issues such as topological properties and index theory [6, 101]. We will see that the same relations can be formulated for our model, too.

A Ginsparg-Wilson system consists of two involutions Γ and Γ', i.e.

$$\Gamma^2 = 1\ ;\ \Gamma^\dagger = \Gamma \quad \text{and} \quad \Gamma'^2 = 1\ ;\ \Gamma'^\dagger = \Gamma'. \tag{9.74}$$

In our case, these two involutions are defined as two different noncommutative versions of chirality, one acting from the left, the other one acting from the right

$$\Gamma\Psi = \frac{\sqrt{2}}{N}(\Gamma^\mu \lambda_\mu + C_0)\Psi, \tag{9.75}$$

$$\Gamma'\Psi = \frac{\sqrt{2}}{N}(\Gamma^\mu \Psi \lambda_\mu - C_0\Psi). \tag{9.76}$$

We recognize Γ' as the fuzzy operator (9.60). But also Γ has the commutative operator (9.49) as its limit.

In the Ginsparg-Wilson system, the Dirac operator was initially defined to be

$$d = \frac{1}{a}\Gamma(\Gamma - \Gamma'), \tag{9.77}$$

where a is the lattice spacing, but here we will choose

$$D = \frac{N}{2}\sqrt{2}(\Gamma - \Gamma'), \tag{9.78}$$

as this reproduces our fuzzy Dirac operator (9.63) (with gauge fields switched off). We can now define an alternative chirality operator

$$\chi = \frac{1}{2}(\Gamma + \Gamma'). \tag{9.79}$$

It fulfills

$$\begin{aligned} \{D, \chi\} &= 0, \\ 2N^2\chi^2 + D^2 &= 2N^2. \end{aligned} \tag{9.80}$$

Therefore χ exactly anticommutes with D, but it vanishes on the top modes of D, i.e. for $|D| = \sqrt{2}N$. But at least for every eigenstate Ψ_E with positive eigenvalue $E < \sqrt{2}N$

$$D\Psi_E = E\Psi, \tag{9.81}$$

the ungauged fuzzy Dirac operator has also an eigenstate $\Psi_{-E} = \chi\Psi_E$ with the negative eigenvalue $-E$ because of

$$D\Psi_{-E} = D\chi\Psi = -\chi D\Psi = -\chi E\Psi = -E\Psi_{-E}. \tag{9.82}$$

This can be used [6] to derive the following index theorem for D

$$Ind(D) = n_+ - n_- = Tr(\chi). \tag{9.83}$$

To include gauge fields, we can write

$$\Gamma_A = \frac{\sqrt{2}}{N}(\Gamma^\mu(\lambda_\mu + A_\mu) + C_0) = \frac{\sqrt{2}}{N}C. \tag{9.84}$$

With

$$D_A = \frac{N}{2}\sqrt{2}(\Gamma_A - \Gamma'), \tag{9.85}$$

$$\chi_A = \frac{1}{2}(\Gamma_A + \Gamma') \tag{9.86}$$

we now get

$$\begin{aligned} \{D, \chi\} &= \frac{N}{2}\sqrt{2}(\Gamma_{\mathcal{A}}^2 - 1) \\ &= \frac{N}{2}\sqrt{2}(\frac{2}{N^2}(B_\mu B_\mu + \frac{1}{2} + \Sigma_8^{\mu\nu}F_{\mu\nu}) - 1) \\ &= \frac{\sqrt{2}}{N}(B_\mu B_\mu + \frac{N^2-1}{2} + \Sigma_8^{\mu\nu}F_{\mu\nu}), \end{aligned} \tag{9.87}$$

which corresponds exactly to the result of [101] for the fuzzy sphere. Other results of [101] are therefore expected to hold in our case, too.

Alternatively, the gauge fields could also be introduced in a way that is closer to the Ginsparg-Wilson setting by normalizing $\Gamma_{\mathcal{A}}$.

9.5 Topologically non-trivial solutions on $S^2_N \times S^2_N$

We will now go back to pure gauge theory on $S^2_N \times S^2_N$, looking for non-trivial solutions of the equations of motion (9.4). We will find that the theory is rich in topological solutions, some corresponding to monopoles on the commutative limit $S^2 \times S^2$, others corresponding to the fluxon solutions found on the second limit $\mathbb{R}^4_\theta$.

In order to understand better the non-trivial solutions found below, we first note that the classical space $S^2 \times S^2$ is symplectic with symplectic form

$$\omega = \omega^L + \omega^R, \tag{9.88}$$

where

$$\omega^L = \frac{1}{4\pi R^3}\epsilon_{ijk}x_i^L dx_j^L dx_k^L \tag{9.89}$$

and similarly ω^R. The normalization is chosen such that

$$\int_{S^2_{L,R}} \omega^{L,R} = 1 = \int_{S^2\times S^2} \omega^L \wedge \omega^R \tag{9.90}$$

so that ω^L, ω^R generate the integer cohomology $H^*(S^2 \times S^2, \mathbb{Z})$. Noting that ω is self-dual while $\tilde{\omega} := \omega^L - \omega^R$ is anti-selfdual, it follows immediately that both $F = 2\pi\omega$ and $F = 2\pi\tilde{\omega}$ are solutions of the Abelian field equations. More generally, any

$$F^{(m_L,m_R)} = 2\pi m_L \omega^L + 2\pi m_R \omega^R \tag{9.91}$$

for any integers m_L, m_R is a solution. In bundle language, they correspond to products of 2 monopole bundles with connections and monopole number $m_{L,R}$ over $S^2_{L,R}$. Following the literature we will denote any such non-trivial solution as instanton.

9.5.1 Instantons and fluxons

We are interested in similar non-trivial solutions of the EOMs (9.4) in the fuzzy case. The monopole solutions on the fuzzy sphere S^2_N are given by representations λ_i^{N-m} of $su(2)$ of size $N-m$ [68], which lead to the classical monopole gauge fields in the commutative limit as shown in [96]. It is hence easy to guess that we will obtain solutions on $S^2_N \times S^2_N$ by taking products of these:

$$\begin{aligned} B_i^L &= \alpha^L \, \lambda_i^{N-m_L} \otimes 1\!\!1_{N-m_R}, && (9.92)\\ B_i^R &= \alpha^R \, 1\!\!1_{N-m_L} \otimes \lambda_i^{N-m_R} && (9.93)\end{aligned}$$

where $\lambda_i^{N-m_{L,R}}$ are the $N-m_{L,R}$ dimensional generators of $su(2)$. It is not difficult to verify that these are solutions of (9.4) with $\alpha^{L,R} = 1 + \frac{m_{L,R}}{N}$ for $m_{L,R} \ll N$, with field strength

$$F_{iLjL} = -\frac{m^L}{2R^3}\epsilon_{ijk}x_k^L, \quad F_{iRjR} = -\frac{m^R}{2R^3}\epsilon_{ijk}x_k^R, \quad F_{iLjR} = 0, \tag{9.94}$$

while $B \cdot B - \frac{N^2-1}{4} \to 0$ as $N \to \infty$. This means that $F = -2\pi m^L\omega^L - 2\pi m^R\omega^R$ in the commutative limit, so that indeed

$$\int\limits_{S_2^{L,R}} \frac{F}{2\pi} = -m^{L,R}. \tag{9.95}$$

Notice that the Ansatz (9.93) implies that all matrices have size $\mathcal{N} = (N - m_L)(N - m_R)$, which is inconsistent if we require that $\mathcal{N} = N^2$ in order to have the original $S_N^2 \times S_N^2$ vacuum. Therefore it appears that these solutions live in a different configuration space, similar as the commutative monopoles which live on different bundles. However, the situation is in fact more interesting: the above solutions can be embedded in the *same* configuration spaces of $N^2 \times N^2$ matrices as the vacuum solution if we combine them with other solutions, which have finite action in four dimensions[4]. They are in fact crucial to recover some of the known $U(1)$ instantons in the limit $S_N^2 \to \mathbb{R}_\theta^2$ resp. $S_N^2 \times S_N^2 \to \mathbb{R}_\theta^4$, as we will see. Consider the following Ansatz

$$B_i^{L,R} = \mathrm{diag}(d_{i,1}^{L,R}, ..., d_{i,n}^{L,R}) \tag{9.96}$$

in terms of diagonal matrices (ignoring the size of the matrices for the moment). These are solutions of (9.4) in two cases,

$$\sum_i d_{i,k}^{L,R} d_{i,k}^{L,R} = \begin{cases} \frac{N^2-3}{4}, & \text{typeA} \\ 0, & \text{typeB} \end{cases} \tag{9.97}$$

(i.e. $d_{i,k}^{L,R} = 0$ in type B). The associated field strength is

$$F_{iLjL} = \frac{\epsilon_{ijk}}{R^2}\,\mathrm{diag}(d_{k,1}^L, ..., d_{k,n}^L), \quad F_{LR} = 0, \tag{9.98}$$

and a similar formula for F_{iRjR}. The constraint term is then $(B \cdot B - \frac{N^2-1}{4}) \to -\frac{1}{2}$ for type A, and $(B \cdot B - \frac{N^2-1}{4}) \to -\frac{N^2-1}{4}$ for type B in the large N limit. In particular, only the type A solutions will have a finite contribution

$$S_{\text{fluxon}} = \frac{V}{g^2\mathcal{N}}\left(\frac{n}{4R^4} + \frac{2n}{R^4}\frac{N^2-3}{4}\right) \to \frac{8\pi^2}{g^2}\, n \tag{9.99}$$

[4] as opposed to 2 dimensions, where their action goes to infinity for $N \to \infty$.

to the action[5], which for $N \to \infty$ is only due to the field strength. We will see below that these type A solutions can be interpreted as a localized flux or vortex, and we will call them fluxons since they will lead in the scaling limit to solutions on $\mathbb{R}^4_\theta$ which we denoted as such [94, 49, 56].

One can now combine these fluxon solutions with the monopole solutions (9.93) in the form

$$\begin{aligned} B_i^L &= \begin{pmatrix} \alpha^L\, \lambda_i^{N-m_L} \otimes \mathbb{1}_{N-m_R} & 0 \\ 0 & \mathrm{diag}(d_{i,1}^L, ..., d_{i,n}^L) \end{pmatrix}, \\ B_i^R &= \begin{pmatrix} \alpha^R\, \mathbb{1}_{N-m_L} \otimes \lambda_i^{N-m_R} & 0 \\ 0 & \mathrm{diag}(d_{i,1}^R, ..., d_{i,n}^R) \end{pmatrix}. \end{aligned} \tag{9.100}$$

These are now matrices of size $\mathcal{N} = (N-m_L)(N-m_R)+n$, which must agree with $\mathcal{N} = N^2$. This is clearly possible for

$$m_L = -m_R = m, \quad n = m^2, \tag{9.101}$$

while for $m_L \neq -m_R$ the contribution from the fluxons would be infinite since n would be of order N. To understand these solutions, we can compute the gauge field from (9.5),

$$A_i^L = \frac{1}{R}\left(B_i^L - \lambda_i^N \otimes \mathbb{1}_N\right) = A_i^L(x^L, x^R). \tag{9.102}$$

To evaluate this, we first have to choose a gauge, i.e. a unitary transformation U for (9.100) which allows to express e.g. $\lambda_i^{N-m_L} \otimes \mathbb{1}_{N-m_R}$ in terms of $x_i^L \propto \lambda_i^N \otimes \mathbb{1}_N$ and $x_i^R \propto \mathbb{1}_N \otimes \lambda_i^N$. For example, in the case $m_L = -m_R = m$ this can be done using a unitary map

$$U: \quad \mathbb{C}^{N-m} \otimes \mathbb{C}^{N+m} \oplus \mathbb{C}^{m^2} \to \mathbb{C}^N \otimes \mathbb{C}^N, \tag{9.103}$$

mapping a $(N-m) \times (N+m)$ matrix into a $N \times N$ matrix by trivially matching the upper-left corner in the obvious way, and fitting $\mathbb{C}^{m^2}$ into the remaining lower-right corner. With this being understood, one can write

$$\begin{aligned} RA_i^L(x^L, x^R) &= (\alpha^L \lambda_i^{N-m} - \lambda_i^N) \otimes \mathbb{1}_{N+m} \\ &\quad + \lambda_i^N \otimes (\mathbb{1}_{N+m} - \mathbb{1}_N) + (\text{d-terms}) \\ &= A_i^{(m_L)}(x^L) \;+\; \mathrm{sing}(x_3^L = -R, x_3^R = -R) \end{aligned} \tag{9.104}$$

where $A_i^{(m_L)}(x^L)$ is indeed the gauge field of a monopole with charge m on S_L^2 in the large N limit, as was checked explicitly in [96]. Here $\mathrm{sing}(x_3^L = -R, x_3^R = -R)$

[5] A finite action can also be obtained for the type B solution using a slightly modified action (9.107), as discussed below.

indicates a field which is singular for large N and localized at the south pole of S_L^2 and S_R^2. It originates both from cutting and pasting the bottom and right border of the above matrices using U (leading to singular gauge fields but regular field strength at the south poles), as well as the d-block (leading to a singular field strength). To see this recall that in general for the standard representation (C.2) of fuzzy spheres, entries in the lower-right block of the matrices correspond to functions localized at $x_3 = -R$, cp. (8.7). The gauge field near this singularity will be studied in more detail in chapter 10.2. The field strength is

$$F_{iLjL} = -\frac{m^L}{2R^3}\epsilon_{ijk}x_k^L + \epsilon_{ijk}\frac{1}{R^2}\sum_{i=1}^{n} d_{k,i}^L P_i \tag{9.105}$$

in the commutative limit, where P_i are projectors in the algebra of functions on $S_N^2 \times S_N^2$ of rank 1; recalling (8.7), they should be interpreted as delta-functions $P_i = \frac{V}{N^2}\,\delta^{(4)}(x_3 = -R)$. Similar formulae hold for $A_i^R(x^L, x^R)$ and F_{iRjR}, while $F_{LR} = 0$.

We assumed above that these delta-functions are localized at the south poles $x_3^L = x_3^R = -R$. However, the location of these delta-functions can be chosen freely using gauge transformations. This can be seen by applying suitable successive gauge transformations using $N-k$-dimensional irreps of $SU(2)$ for $k = 0, 1, ..., m-1$, which from the classical point of view all correspond to global rotations, successively moving the individual delta-peaks. Therefore the solution (9.100) should in general be interpreted as a monopole on $S^2 \times S^2$ with monopole number $m_L = -m_R = m$, combined with a localized singular field strength characterized by its position and a vector $d_{k,i}^L$. We will see in chapter 10 that it becomes the fluxon solution in the planar limit $\mathbb{R}_\theta^4$.

The total action of these solutions (9.100) is the sum of the contributions from the monopole field plus the contribution from the fluxons (9.99), which both give the same contribution

$$S_{(m)} = \frac{4\pi^2}{g^2}\left(2m^2 + 2m^2\right) \tag{9.106}$$

in the large N limit, using (9.101). The first term is due to the global monopole field (9.94), and the second term is the contribution of the fluxons through the localized field strength.

The interpretation of these solutions depends on the scaling limit $N \to \infty$ which we want to consider. We have seen that in the commutative limit keeping $R = \text{const}$, these solutions become commutative monopoles on $S^2 \times S^2$ with magnetic charges $m_L = -m_R$, plus additional localized fluxon degrees of freedom. For large R, the field strength of the monopoles vanishes, leaving only the localized fluxons. In particular, we will see in the following chapter that in the scaling limit

$S^2_N \times S^2_N \to \mathbb{R}^4_\theta$ only the fluxons survive and become well-known solutions for gauge theory on $\mathbb{R}^4_\theta$.

A final remark is in order: if we fix the size $\mathcal{N}$ of the matrices, only certain fluxon and monopole numbers are allowed, given by (9.101). Otherwise the number n of fluxons and hence the action would diverge with N. This can be seen as an interesting feature of our model: viewed as a regularization of gauge theory on $\mathbb{R}^4_\theta$, this points to possible subtleties of defining the admissible field configurations in infinite-dimensional Hilbert spaces and relations with topological terms in the action. On the other hand, we could accommodate the most general solutions including also type B solutions (9.97) by modifying the action similar as in [96]. For example,

$$S = \frac{1}{g^2}\int \Big(\frac{4B_i^L B_i^L}{N^2 R^4}(B_i^L B_i^L - \frac{N_L^2-1}{4})^2 + \frac{4B_i^R B_i^R}{R^4}(B_i^R B_i^R - \frac{N_R^2-1}{4})^2 + \frac{1}{2}F_{ia,jb}F_{ia,jb}\Big) \tag{9.107}$$

leads to the same commutative action, but with a vanishing action for the Dirac string in the type B solutions.

9.5.2 Spherical branes

Consider the following solutions

$$\begin{aligned} B_i^L &= \begin{pmatrix} \alpha^L\,\lambda_i^{N-m} & 0 \\ 0 & \mathrm{diag}(d_{i,1},\dots,d_{i,m}) \end{pmatrix} \otimes 1\!\!1_N, \\ B_i^R &= 1\!\!1_N \otimes \lambda_i^N \end{aligned} \tag{9.108}$$

which are matrices of size $\mathcal{N} = N^2$. The corresponding field strength is

$$\begin{aligned} F_{iLjL} &= -\frac{m}{2R^3}\epsilon_{ijk}x_k^L + \epsilon_{ijk}\frac{1}{R^2}\sum_{i=1}^m d_{k,i}P_i \\ F_{RR} &= F_{LR} = 0 \end{aligned} \tag{9.109}$$

where P_i are projectors in the algebra of functions on S^2_L of rank 1 which should be interpreted as delta-functions $P_i = \frac{4\pi R^2}{N}\;\delta^{(2)}(x_3 = -R)$. In particular the gauge field $\mathcal{A}$ vanishes on S^2_R, while on S^2_L there is a monopole field together with a singularity at a point. This is similar to the fluxons of the previous chapter, but now only on S^2_L. This leads to the interpretation as a 2-dimensional brane located at a point on S^2_L. The action for these solutions is infinite. In the limit $S^2_N \times S^2_N \to \mathbb{R}^4_\theta$, the flux will be located at a 2-dimensional hyperplane. Such solutions for gauge theory on $\mathbb{R}^4_\theta$ were found in [1, 50], which would be recovered in the scaling limit $S^2_N \times S^2_N \to \mathbb{R}^4_\theta$.

Chapter 10

Gauge theory on $\mathbb{R}^4_\theta$ from $S^2_N \times S^2_N$

We saw in chapter (8.3) that $\mathbb{R}^4_\theta$ can be obtained as a scaling limit of fuzzy $S^2_{N_L} \times S^2_{N_R}$. Here we will extend this scaling also to the covariant coordinates B_μ, thereby relating the gauge theory on $S^2_{N_L} \times S^2_{N_R}$ to that on $\mathbb{R}^4_\theta$ and hence providing a regularization for the latter. We will in particular relate the instanton solutions on these two spaces.

On noncommutative $\mathbb{R}^2_\theta$, all $U(1)$-instantons were constructed and classified in [50]. One can indeed recover these instantons from corresponding solutions on S^2_N, as we will show below. However, since we are mainly interested in the 4-dimensional case here, we will only present the corresponding constructions on $S^2_{N_L} \times S^2_{N_R}$ resp. $\mathbb{R}^4_\theta$ here, without discussing the 2-dimensional case separately. It can be recovered in an obvious way from the considerations below.

The situation on $\mathbb{R}^4_\theta$ is more complicated, and there are different types of non-trivial $U(1)$ instanton solutions on $\mathbb{R}^4_\theta$. The instantons found by solving the noncommutative version of the ADHM equations [88, 41, 27, 55, 61] are hard to find in the fuzzy case, as this construction relies heavily on selfduality, a notion which isn't naturally available in our formulation of $S^2 \times S^2$ embedded in $\mathbb{R}^6$. But the four-dimensional fluxon solutions discussed in detail in chapter 7.3 can be recovered as scaling limits of the solutions (9.100) on $S^2_{N_L} \times S^2_{N_R}$. In particular, the moduli of the fluxon solutions on $\mathbb{R}^4_\theta$ will be related to the free parameters $d_i^{L,R}$ in (9.100). This supports our suggestion to use gauge theory on $S^2_{N_L} \times S^2_{N_R}$ as a regularization for gauge theory on $\mathbb{R}^4_\theta$.

10.1 The action

We saw in chapter 8.3 that the fuzzy space $S^2_{N_L} \times S^2_{N_R}$ has a scaling limit to $\mathbb{R}^4_\theta$, with θ cast in the following form:

$$\theta_{\mu\nu} = \begin{pmatrix} 0 & \theta_{12} & 0 & 0 \\ -\theta_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_{34} \\ 0 & 0 & -\theta_{34} & 0 \end{pmatrix} . \tag{10.1}$$

This scaling can also be applied to the covariant coordinates B_μ, connecting the gauge theory on $S^2_{N_L} \times S^2_{N_R}$ to that on $\mathbb{R}^4_\theta$ and therefore providing it with a regularisation. For the selfdual case (i.e. $\theta_{12} > 0$ and $\theta_{34} > 0$) we can define

$$X_{1,2} := \sqrt{\frac{2\theta_{12}}{N_L}}\, B^L_{1,2} , \tag{10.2}$$

$$X_{3,4} := \sqrt{\frac{2\theta_{34}}{N_R}}\, B^R_{1,2} , \tag{10.3}$$

$$\phi^{L,R} := B_3^{L,R} - \frac{N_{L,R}}{2} + \frac{1}{N_{L,R}}((B_1^{L,R})^2 + (B_2^{L,R})^2) , \tag{10.4}$$

The antiselfdual case ($\theta_{34} < 0$) can easily be reached by setting e.g.

$$X_{4/3} := \sqrt{\frac{2\theta_{34}}{N_R}} B^R_{1/2}, \tag{10.5}$$

but for simplicity we will limit us to the selfdual case in the following. The X will become the covariant coordinates on $\mathbb{R}^4_\theta$ in the limit $N_{L/R} \to \infty$, and the ϕ an auxiliary field. To see this we now blow up the spheres by setting

$$R^2 = \frac{1}{2} N_L \theta_{34} = \frac{1}{2} N_R \theta_{12}. \tag{10.6}$$

With this double scaling limit R, $N \to \infty$ keeping θ fixed we calculate for the field strength

$$\begin{aligned}
\frac{1}{R^2}([B_1^L, B_1^R]) &= \frac{1}{\theta_{12}\theta_{34}}[X_1, X_3], \quad \text{etc.,} && (10.7)\\
\frac{1}{R^2}(B_1^L + i[B_2^L, B_3^L]) &= \sqrt{\frac{1}{\theta_{12}\theta_{34}R^2}}\left(X_1 + i[X_2, \phi^L] - \frac{i}{2\theta_{12}}[X_2, (X_1)^2]\right) \\
\frac{1}{R^2}(B_2^L + i[B_3^L, B_1^L]) &= \sqrt{\frac{1}{\theta_{12}\theta_{34}R^2}}\left(X_2 + i[X_1, \phi^L] - \frac{i}{2\theta_{12}}[X_1, (X_2)^2]\right) \\
\frac{1}{R^2}(B_3^L + i[B_1^L, B_2^L]) &= \frac{1}{\theta_{12}\theta_{34}}\Big(\theta_{12} + i[X_1, X_2] \\
&\quad + \frac{\theta_{12}\theta_{34}}{R^2}\phi_L - \frac{\theta_{12}\theta_{34}^2}{2R^4}((X_1)^2 + (X_2)^2)\Big) .
\end{aligned}$$

Analogous expressions hold for B_i^R. For the potential term we get

$$\begin{aligned}\frac{1}{R^2}(B_i^L B_i^L - \frac{N_L^2-1}{4}) &= \frac{1}{\theta_{34}}\phi^L + \frac{2}{R^2}((\phi^L)^2 + \frac{1}{4}) \\ &\quad -\frac{1}{\theta_{12}R^2}\{\phi^L, (X_1)^2 + (X_2)^2\} \\ &\quad +\frac{1}{\theta_{12}^2 R^2}((X_1)^2 + (X_2)^2)^2.\end{aligned} \tag{10.8}$$

We immediately see that the only terms from action (9.1) involving $\phi^{L,R}$ are

$$\frac{1}{\theta_{34}^2}(\phi^L)^2 + \frac{1}{\theta_{12}^2}(\phi^R)^2 + O(\frac{1}{R}), \tag{10.9}$$

and therefore we can integrate them out in the limit $R \to \infty$. In the leading order in R the remaining terms give the standard action

$$S = -\frac{1}{2g^2\theta_{12}^2\theta_{34}^2}\int ([X_\mu, X_\nu] - i\theta_{\mu\nu})^2 \tag{10.10}$$

for a gauge theory on $\mathbb{R}^4_\theta$ for general $\theta_{\mu\nu}$. The X_μ are interpreted as covariant coordinates, which can be written as[1]

$$X_\mu = x_\mu + A_\mu. \tag{10.11}$$

Hence the gauge fields A_μ describe the fluctuations around the vacuum. In particular, note that our regularization procedure clearly fixes the rank of the gauge group, unlike in the naive definition on $\mathbb{R}^d_\theta$ as discussed in chapter 7.2. The generalization to the $U(n)$ case is obvious.

10.2 Instantons on $\mathbb{R}^4_\theta$ from $S^2_N \times S^2_N$

With the scaling limit of chapter 10.1, the gauge theory on $S^2_N \times S^2_N$ provides us with a regularization for the gauge theory on $\mathbb{R}^4_\theta$. Of course, such a regularization might affect the topological features of the theory, an effect we want to investigate in this chapter. For this, we will map the topologically nontrivial solutions found in chapter 9.5 on $S^2_N \times S^2_N$ to $\mathbb{R}^4_\theta$.

Consider again the solutions (9.100) that combine the fluxon solutions with the monopoles, with the fluxons at the north pole instead of the south pole because

[1] we do not distinguish between upper and lower indices

we want to study their structure. Their scaling limit as in (10.2) gives

$$X_i = \sqrt{\frac{2\theta}{N}} \begin{pmatrix} \mathrm{diag}(d^L_{i,1},...,d^L_{i,n}) & 0 \\ 0 & \alpha^L \ \lambda_i^{N-m} \otimes \mathbb{1} \end{pmatrix}, \tag{10.12}$$

$$X_{i+2} = \sqrt{\frac{2\theta}{N}} \begin{pmatrix} \mathrm{diag}(d^R_{i,1},...,d^R_{i,n}) & 0 \\ 0 & \alpha^R \ \mathbb{1} \otimes \lambda_i^{N+m} \end{pmatrix} \tag{10.13}$$

for $i = 1, 2$. Recalling that the rescaled $\lambda_{1,2}$ on $S^2_{N_L} \times S^2_{N_R}$ become the $x_\pm$'s on $\mathbb{R}^4_\theta$ in the scaling limit

$$\sqrt{\frac{2\theta}{N}}(\lambda_1^{L,R} \pm i\lambda_2^{L,R}) \to x_{\pm L,R},$$

we see that (10.12) and (10.13) become the instantons (7.32, 7.33) on $\mathbb{R}^4_\theta$,

$$X_1 + iX_2 \quad \to \quad X^{(n)}_{+L} = Sx_{+L}S^\dagger + \sum_{k=1}^{n} \gamma^L_k |i_k, j_k\rangle\langle i_k, j_k|, \tag{10.14}$$

$$X_3 + iX_4 \quad \to \quad X^{(n)}_{+R} = Sx_{+R}S^\dagger + \sum_{k=1}^{n} \gamma^R_k |i_k, j_k\rangle\langle i_k, j_k|. \tag{10.15}$$

Here the (d_i)-block acting on a basis $|i_k, j_k\rangle$ of $V_n \subset \mathcal{H} \cong \mathbb{C}^\mathcal{N}$ becomes the projector part of (10.14, 10.15) with

$$\sqrt{\frac{2\theta}{N}}\, d^{L,R}_{1,k} \quad \to \quad \mathrm{Re}\gamma^{L,R}_k, \tag{10.16}$$

$$\sqrt{\frac{2\theta}{N}}\, d^{L,R}_{2,k} \quad \to \quad \mathrm{Im}\gamma^{L,R}_k,$$

and the monopole block becomes $Sx_+S^\dagger$ where S is a partial isometry from $\mathcal{H}$ to $\mathcal{H}\backslash V_n$. Note that we can recover any value for the γ's in this scaling, solving the constraint $d_i d_i = \frac{N^2-3}{4}$ by $d_3 \approx \frac{N}{2}$. Therefore the full moduli space of the fluxon solutions (7.32, 7.33) on $\mathbb{R}^4_\theta$ can be recovered in this way. Furthermore, the meaning of the parameters $\gamma^{L,R}$ is easy to understand in our approach: Note first that using a rotation (which acts also on the indices) followed by a gauge transformation, the d_i can be fixed to be radial at the north pole, $d^{L,R}_i \sim (0, 0, N/2)$. This is a fluxon localized at the north pole. Now apply a translation at the north pole, which corresponds to a suitable rotation on the sphere. As the $\gamma^{1,2}_k$, according to (10.16), are the projections of the vectors $d^{L,R}_i$ onto the surface of the spheres, rotating the vector $d^{L,R}_i$ in the scaling limit amounts to a translation of the $\gamma^{1,2}_k$, which therefore parametrize the position of the fluxons.

It has been noted [37] that the $Sx_+S^\dagger$ correspond to a pure (but topologically nontrivial) gauge, which can qualitatively be seen already in two dimensions.

There, the partial isometry $S : |k\rangle \to |k+n\rangle$ is basically $(\frac{x_-}{\sqrt{x_-x_+}})^n \sim (\frac{x-iy}{r})^n \sim e^{in\varphi}$ and therefore the gauge field $\mathcal{A}_i = S\partial_i S^\dagger$ has a winding number n. The topological nature of the $Sx_+S^\dagger$ is even more evident in our setting, as they are the limit of the monopole solutions (9.92,9.93) on $S^2_N \times S^2_N$. Moreover, note that their contribution to the action (9.106) survives the scaling: even though the field strength vanishes as $R \to \infty$, the integral gives a finite contribution equal to the contribution of the fluxon part. This topological "surface term" is usually omitted in the literature on $\mathbb{R}^4_\theta$, but becomes apparent in the regularized theory.

So it seems that we recovered all the instantons of chapter 7.3, but in fact there is an important detail that we haven't discussed jet. It is the embedding of the n-dimensional fluxons and the $(N-m)(N+m)$-dimensional monopole solutions into the N^2-dimensional matrices of the ground state. Such an embedding is clearly only possible for $n = m^2$. This means that the regularized theory has a superselection rule for the dimension of the allowed instantons, a rule that did not exist in the unregularized theory[2].

One way to allow arbitrary instanton numbers is to allow the size $\mathcal{N}$ of the matrices to vary. However, this is less satisfactory as it destroys the unification of topological sectors, which is a beautiful feature of noncommutative gauge theory. On the other hand, the type B solutions (9.97) together with the changed action (9.107) might allow the construction of the missing instantons. The idea is to fill up the unnecessary $m^2 - n$ places with $d_i = 0$. The changed action would not suppress such solutions any more, and in fact they would not even contribute to the action. This amounts to adding a discrete sector to the theory which accommodates these type B solutions, but decouples from the rest of the model. Whether or not one wants to do this appears to be a matter of choice. This emphasizes again the importance of a careful regularization of the theory. It would be very interesting to see what happens in other regularizations e.g. using gauge theory on noncommutative tori or fuzzy $\mathbb{C}P^2$.

[2]Note that this is different in two dimensions. There, a rank n fluxon can be combined with a $(N-n)$-dimensional monopole block and all the instantons on $\mathbb{R}^2_\theta$ can be recovered. Furthermore, the actions for the fluxons and the monopoles scale differently with N. Therefore, in two dimensions, the action for the monopoles vanishes in the scaling limit that produces a gauge theory on $\mathbb{R}^2_\theta$ with rescaled coupling constant.

Appendix

Appendix A

Brackets, forms and frames

A.1 Definitions of the brackets

A.1.1 The Schouten-Nijenhuis bracket

The Schouten-Nijenhuis bracket for multivectorfields $\pi_s = \pi_s^{i_1 \dots i_{k_s}} \partial_{i_1} \wedge \dots \wedge \partial_{i_{k_s}}$ can be written as ([7],IV.2.1):

$$[\pi_1, \pi_2]_S = (-1)^{k_1 - 1} \pi_1 \bullet \pi_2 - (-1)^{k_1(k_2-1)} \pi_2 \bullet \pi_1, \tag{A.1}$$

$$\pi_1 \bullet \pi_2 = \sum_{l=1}^{k_1} (-1)^{l-1} \pi_1^{i_1 \dots i_{k_1}} \partial_l \pi_2^{j_1 \dots j_{k_2}} \partial_{i_1} \wedge \dots \wedge \widehat{\partial_{i_l}} \wedge \dots \wedge \partial_{i_{k_1}} \wedge \partial_{j_1} \wedge \dots \wedge \partial_{j_{k_2}}, \tag{A.2}$$

where the hat marks an omitted derivative.

For a function g, vectorfields $X = X^k \partial_k$ and $Y = Y^k \partial_k$ and a bivectorfield $\pi = \frac{1}{2} \pi^{kl} \partial_k \wedge \partial_l$ we get:

$$\begin{aligned} [X, g]_S &= X^k \partial_k g, \qquad \text{(A.3)} \\ [\pi, g]_S &= -\pi^{kl} \partial_k g \partial_l, \\ [X, \pi]_S &= \frac{1}{2} (X^k \partial_k \pi^{ij} - \pi^{ik} \partial_k X^j + \pi^{jk} \partial_k X^i) \partial_i \wedge \partial_j, \\ [\pi, \pi]_S &= \frac{1}{3} (\pi^{kl} \partial_l \pi^{ij} + \pi^{il} \partial_l \pi^{jk} + \pi^{jl} \partial_l \pi^{ki}) \partial_k \wedge \partial_i \wedge \partial_j. \end{aligned}$$

A.1.2 The Gerstenhaber bracket

The Gerstenhaber bracket for polydifferential operators A_s can be written as ([7],IV.3):

$$[A_1, A_2]_G = A_1 \circ A_2 - (-1)^{(|A_1|-1)(|A_2|-1)} A_2 \circ A_1, \tag{A.4}$$

$$\begin{aligned}&(A_1 \circ A_2)(f_1, \dots f_{m_1+m_2-1}) \\ &= \sum_{j=1}^{m_1} (-1)^{(m_2-1)(j-1)} A_1(f_1, \dots f_{j-1}, A_2(f_j, \dots, f_{j+m_2-1}), f_{j+m_2}, \dots, f_{m_1+m_2-1}),\end{aligned} \tag{A.5}$$

where $|A_s|$ is the degree of the polydifferential operator A_s, i.e. the number of functions it is acting on.

For functions g and f, differential operators $D_1$and D_2 of degree one and P of degree two we get

$$\begin{aligned}[D, g]_G &= D(g), \\ [P, g]_G(f) &= P(g, f) - P(f, g), \\ [D_1, D_2]_G(g) &= D_1(D_2(g)) - D_2(D_1(g)), \\ [P, D]_G(f, g) &= P(D(f), g) + P(f, D(g)) - D(P(f, g)).\end{aligned} \tag{A.6}$$

A.2 Noncommutative forms

We are now able to introduce noncommutative forms as well. If we have a map δ from the Poisson vector fields to the derivations of the $\star$-product algebra, we have seen that there is a natural Lie-algebra structure

$$[\delta_X, \delta_Y] = \delta_{[X,Y]_\star}, \tag{A.7}$$

over the space of these derivations. On this we can easily construct the Chevalley cohomology. Further, again with the map δ, we can lift derivations of the Poisson structure to derivations of the $\star$-product. Therefore it should be possible to pull back the Chevalley cohomology from the space of derivations to the Poisson vector fields. This will be done in the following.

A deformed k-form is defined to map k Poisson vector fields to a function and has to be skew-symmetric and linear over $\mathbb{C}$. This is a generalization of the undeformed case, where a form has to be linear over the algebra of functions. Functions are defined to be 0-forms. The space of forms $\Omega_\star M$ is now a $\star$-bimodule via

$$(f \star \omega \star g)(X_1, \dots, X_k) = f \star \omega(X_1, \dots, X_k) \star g. \tag{A.8}$$

As expected, the exterior differential is defined with the help of the map δ.

$$\delta\omega(X_0, \dots, X_k) = \tag{A.9}$$

$$\begin{aligned} & \sum_{i=0}^{k}(-1)^i\, \delta_{X_i}\omega(X_0,\ldots,\hat{X}_i,\ldots,X_k) \\ + & \sum_{0\le i<j\le k}(-1)^{i+j}\omega([X_i,X_j]_\star,X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k). \end{aligned}$$

With the properties of δ and $[\cdot,\cdot]_\star$ it follows that

$$\delta^2\omega = 0. \tag{A.10}$$

To be more explicit we give formulas for a function f, a one form A and a two form F

$$\begin{aligned} \delta f(X) &= \delta_X f, \\ \delta A(X,Y) &= \delta_X A_Y - \delta_Y A_X - A_{[X,Y]_\star}, \\ \delta F(X,Y,Z) &= \delta_X F_{Y,Z} - \delta_Y F_{X,Z} + \delta_Z F_{X,Y}, \\ & \quad -F_{[X,Y]_\star,Z} + F_{[X,Z]_\star,Y} - F_{[Y,Z]_\star,X}. \end{aligned} \tag{A.11}$$

A wedge product may be defined

$$\begin{aligned} & \omega_1\wedge\omega_2(X_1,\ldots,X_{p+q}) = \\ & \quad \frac{1}{p!q!}\sum_{I,J}\varepsilon(I,J)\,\omega_1(X_{i_1},\ldots,X_{i_p})\star\omega_2(X_{j_1},\ldots,X_{j_q}) \end{aligned} \tag{A.12}$$

where (I,J) is a partition of $(1,\ldots,p+q)$ and $\varepsilon(I,J)$ is the sign of the corresponding permutation. The wedge product is linear and associative and generalizes the bimodule structure (A.8). We note that it is no more graded commutative. We again give some formulas.

$$\begin{aligned} (f\wedge A)_X &= f\star A_X, \\ (A\wedge f)_X &= A_X\star f, \\ (A\wedge B)_{X,Y} &= A_X\star B_Y - A_Y\star B_X. \end{aligned} \tag{A.13}$$

The differential (A.9) fulfills the graded Leibniz rule

$$\delta(\omega_1\wedge\omega_2) = \delta\omega_1\wedge\omega_2 + (-1)^{k_2}\,\omega_1\wedge\delta\omega_2. \tag{A.14}$$

A.3 Frames

We will now propose a method how to find frames and Poisson structures of quantum groups that are compatible. On several quantum spaces deformed derivations have been constructed [99, 72, 24]. In most cases the deformed Leibniz rule may be written in the following form

$$\hat{\partial}_\mu(\hat{f}\hat{g}) = \hat{\partial}_\mu\hat{f}\hat{g} + \hat{T}_\mu{}^\nu(\hat{f})\hat{\partial}_\nu\hat{g}, \tag{A.15}$$

where $\hat{T}$ is an algebra morphism from the quantum space to its matrix ring

$$\hat{T}_\mu{}^\nu(\hat{f}\hat{g}) = \hat{T}_\mu{}^\alpha(\hat{f})\hat{T}_\alpha{}^\nu(\hat{g}). \tag{A.16}$$

Again in some cases it is possible to implement this morphism with some kind of inner morphism

$$\hat{T}_\mu{}^\nu(\hat{f}) = \hat{e}_\mu{}^a\hat{f}\hat{e}_a{}^\nu, \tag{A.17}$$

where $\hat{e}_a{}^\mu$ is an invertible matrix with entries from the quantum space. If we define

$$\hat{e}_a = \hat{e}_a{}^\mu\hat{\partial}_\mu, \tag{A.18}$$

the $\hat{e}_a$ are derivations

$$\hat{e}_a(\hat{f}\hat{g}) = \hat{e}_a(\hat{f})\hat{g} + \hat{f}\hat{e}_a(\hat{g}). \tag{A.19}$$

The dual formulation of this with covariant differential calculi on quantum spaces is the formalism with commuting frames investigated for example in [32, 76, 23, 77]. There one can additionally find how our formalism fits into the language of Connes' spectral triples.

We can now represent the quantum space with the help of a $\star$-product. For example, we can use the Weyl ordered $\star$-product constructed in chapter 3.3. Further we can calculate the action of the operators $\hat{e}_a$ on functions. Since the $\hat{e}_a$ are now derivations of a $\star$-product, there necessarily exist Poisson vector fields e_a with

$$\delta_{e_a} = \hat{e}_a. \tag{A.20}$$

Appendix B

Calculation of the SW-map to all orders

B.1 Calculation of $[\theta_t, \theta_t]$ and $[\theta_t, X_t]$

We want to show that θ_t is still a Poisson tensor and that X_t still commutes with θ_t. For this we first define $\theta(n)^k_l = (\theta f)^n = \theta^{ki} f_{ij} \ldots \theta^{rs} f_{sl} = f_{li}\theta^{ij} \ldots f_{rs}\theta^{sk} = (f\theta)^n$ and $\theta(n)^{kl} = \theta(f\theta)^n = \theta^{ki} f_{ij} \ldots f_{rs}\theta^{sl}$. In the calculations to follow we will sometimes drop the derivatives of the polyvectorfields and associate $\pi^{k_1 \ldots k_n}$ with $\pi^{k_1 \ldots k_n} \frac{1}{n}\partial_{k_1} \wedge \ldots \wedge \partial_{k_n}$ for simplicity. All the calculations are done locally.

We evaluate

$$
\begin{aligned}
[\theta_t, \theta_t]_S &= \theta_t^{kl}\partial_l\theta_t^{ij} + \text{c.p. in } (kij) \\
&= \sum_{n,m=0}^{\infty}\sum_{o=0}^{m}(-t)^{n+m}\theta(n)^k_r\theta(o)^i_s\theta(m-o)^j_p\theta^{rl}\partial_l\theta^{sp} + \text{c.p. in } (kij) \\
&\quad + \sum_{n,m=0}^{\infty}\sum_{o=0}^{m}(-t)^{n+m+1}\theta(n)^{kl}\theta(o)^{is}\theta(m-o)^{pj}\partial_l f_{sp} + \text{c.p. in } (kij) \\
&= \sum_{n,m,o=0}^{\infty}(-t)^{n+m+o}\theta(n)^k_r\theta(o)^i_s\theta(m)^j_p\theta^{rl}\partial_l\theta^{sp} + \text{c.p. in } (kij) \\
&\quad - \sum_{n,m,o=0}^{\infty}(-t)^{n+m+o+1}\theta(n)^{kl}\theta(o)^{is}\theta(m)^{jp}\partial_l f_{sp} + \text{c.p. in } (kij).
\end{aligned} \tag{B.1}
$$

The first part vanishes because θ_t is a Poisson tensor, i.e.

$$
[\theta, \theta]_S = \theta^{kl}\partial_l\theta^{ij} + \text{c.p. in } (kij) = 0, \tag{B.2}
$$

the second part because of

$$\partial_k f_{ij} + \text{c.p. in } (kij) = 0. \tag{B.3}$$

To prove that X_t still commutes with θ_t, we first note that

$$X_t = X \sum_{n=0}^{\infty} (-tf\theta) = X(1 - tf\theta_t). \tag{B.4}$$

With this we can write

$$\begin{aligned}
[X_t, \theta_t] &= [X, \theta_t] - t[Xf\theta_t, \theta_t] \\
&= X^n \partial_n \theta_t^{kl} - \theta_t^{kn} \partial_n X^l + \theta_t^{ln} \partial_n X^k \\
&\quad -tX^m f_{mi} \theta_t^{in} \partial_n \theta_t^{kl} + t\theta_t^{kn} \partial_n (X^m f_{mi} \theta_t^{il}) - t\theta_t^{ln} \partial_n (X^m f_{mi} \theta_t^{ik}) \\
&= X^n \partial_n \theta_t^{kl} - \theta_t^{kn} \partial_n X^l + \theta_t^{ln} \partial_n X^k \\
&\quad +t\theta_t^{kn} \partial_n X^m f_{mi} \theta_t^{il} - t\theta_t^{ln} \partial_n X^m f_{mi} \theta_t^{ik} \\
&\quad +t\theta_t^{kn} X^m \partial_n f_{mi} \theta_t^{il} - t\theta_t^{ln} X^m \partial_n f_{mi} \theta_t^{ik}.
\end{aligned} \tag{B.5}$$

In the last step we used (B.2). To go on we note that

$$t\theta_t^{kn} X^m \partial_n f_{mi} \theta_t^{il} - t\theta_t^{ln} X^m \partial_n f_{mi} \theta_t^{ik} = tX^n \theta_t^{km} \partial_n f_{mi} \theta_t^{il}, \tag{B.6}$$

where we used (B.3). Making use of the power series expansion and the fact that X commutes with θ, i.e.

$$[X, \theta] = X^n \partial_n \theta^{kl} - \theta^{kn} \partial_n X^l + \theta^{ln} \partial_n X^k = 0, \tag{B.7}$$

we further get

$$\begin{aligned}
X^n \partial_n \theta_t^{kl} + tX^n \theta_t^{km} \partial_n f_{mi} \theta_t^{il} &= \sum_{r,s=0}^{\infty} (-t)^{r+s} \theta(r)_i^k X^n \partial_n \theta^{ij} \theta(s)_j^l \\
&= \sum_{r,s=0}^{\infty} (-t)^{r+s} \theta(r)_i^k \theta^{in} \partial_n X^j \theta(s)_j^l \\
&\quad - \sum_{r,s=0}^{\infty} (-t)^{r+s} \theta(r)_i^k \theta^{jn} \partial_n X^i \theta(s)_j^l.
\end{aligned} \tag{B.8}$$

Therefore (B.5) reads

$$
\begin{aligned}
[X_t, \theta_t] &= \sum_{r,s=0}^{\infty} (-t)^{r+s} \theta(r)_i^k \theta(s)_j^l \theta^{in} \partial_n X^j \\
&\quad - \sum_{r,s=0}^{\infty} (-t)^{r+s} \theta(r)_i^k \theta(s)_j^l \theta^{jn} \partial_n X^i \\
&\quad - \theta_t^{kn} \partial_n X^l + \theta_t^{ln} \partial_n X^k + t\theta_t^{kn} \partial_n X^m f_{mi} \theta_t^{il} - t\theta_t^{ln} \partial_n X^m f_{mi} \theta_t^{ik} \\
&= 0.
\end{aligned}
\tag{B.9}
$$

B.2 Calculation of the commutators

B.2.1 Semi-classical construction

We calculate the commutator (5.115) (see also [66]), dropping the t-subscripts on θ_t for simplicity and using local expressions.

$$
\begin{aligned}
[a_\theta, d_\theta(g)] &= -\theta^{ij} a_j \partial_i \theta^{kl} \partial_k g \partial_l - \theta^{ij} a_j \theta^{kl} \partial_i \partial_k g \partial_l \\
&\quad + \theta^{kl} \partial_k g \partial_l \theta^{ij} a_j \partial_i + \theta^{kl} \partial_k g \theta^{ij} \partial_l a_j \partial_i \\
&= -\theta^{kl} \partial_k \theta^{ij} a_j \partial_i g \partial_l - \theta^{kl} \theta^{ij} a_j \partial_k \partial_i g \partial_l - \theta^{kl} \theta^{ij} \partial_j a_k \partial_i g \partial_l \\
&= +\theta^{ij} f_{jk} \theta^{kl} \partial_i g \partial_l - \theta^{kl} \partial_k (\theta^{ij} a_j \partial_i g) \partial_l \\
&= -d_{\theta f \theta} g + d_\theta(a_\theta(g)) \\
&= -\partial_t(d_\theta) g + d_\theta(a_\theta(g)).
\end{aligned}
\tag{B.10}
$$

For (5.116) we get

$$
\begin{aligned}
[a_\theta, X_t] &= \theta^{ij} a_j \partial_i X^k \partial_k - X^k \partial_k \theta^{ij} a_j \partial_i - X^k \theta^{ij} \partial_k a_j \partial_i \\
&= -\theta^{ij} X^k \partial_k a_j \partial_i - \theta^{ik} \partial_k X^j a_j \partial_i \\
&= X^k f_{ki} \theta^{ij} \partial_j + \theta^{ij} \partial_i (X^k a_k) \partial_j \\
&= -\partial_t X - d_\theta(X^k a_k).
\end{aligned}
\tag{B.11}
$$

B.2.2 Quantum construction

In [79], (5.97,5.98,5.101) have already been calculated, unluckily (and implicitly) using a different sign convention for the brackets of polyvectorfields. In [66], again a different sign convention is used, coinciding with the one in [79] in the relevant cases. In order to keep our formulas consistent with the ones used in [79, 66], we define our bracket on polyvectorfields π_1 and π_2 as in [79] to be

$$[\pi_1, \pi_2] = -[\pi_2, \pi_1]_S, \tag{B.12}$$

giving an extra minus sign for π_1 and π_2 both even. The bracket on polydifferential operators is always the Gerstenhaber bracket.

With these conventions and

$$d_\star = -[\cdot, \star] \tag{B.13}$$

we rewrite the formulas (5.101,5.99,5.97,5.98) so we can use them in the following

$$\begin{aligned}
[\Phi(X), \Phi(g)]_G &= \Phi([X, g]) + \Psi([\theta, g], X) - \Psi([\theta, X], g), && \text{(B.14)}\\
[\Phi(X), \Phi(Y)]_G &= d_\star \Psi(X, Y) && \text{(B.15)}\\
&\quad + \Phi([X, Y]) + \Psi([\theta, Y], X) - \Psi([\theta, X], Y), &&\\
d_\star \Phi(g) &= \Phi(d_\theta(g)), && \text{(B.16)}\\
d_\star \Phi(X) &= \Phi(d_\theta(X)). && \text{(B.17)}
\end{aligned}$$

For the calculation of the commutators of the quantum objects we first define

$$a_\star = \Phi(a_{\theta_t}) \tag{B.18}$$

and

$$f_\star = \Phi(f_{\theta_t}). \tag{B.19}$$

With (B.17) we get the quantum version of (5.114)

$$f_\star = d_\star a_\star. \tag{B.20}$$

For functions f and g we get

$$\begin{aligned}
\partial_t(f \star g) &= \sum_{n=0}^{\infty} \frac{1}{n!} \partial_t U_n(\theta_t, \ldots, \theta_t)(f, g) && \text{(B.21)}\\
&= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} U_n(f_\theta, \ldots, \theta_t)(f, g) &&\\
&= f_\star(f, g)
\end{aligned}$$

With these two formulas we can now calculate the quantum version of (5.115) as in [66]. On two functions f and g we have

$$
\begin{aligned}
\partial_t(f \star g) &= f_\star(f, g) \\
&= d_\star a_\star(f, g) \\
&= -[a_\star, \star](f, g) \\
&= -a_\star(f \star g) + a_\star(f) \star g + f \star a_\star(g),
\end{aligned}
\tag{B.22}
$$

where we used (A.6) in the last step. Therefore

$$
\begin{aligned}
[a_\star, d_\star(g)](f) &= a_\star(d_\star(g)(f)) - d_\star(g)(a_\star(f)) \\
&= a_\star([f \overset{\star}{,} g]) - [a_\star(f) \overset{\star}{,} g] \\
&= -\partial_t[f \overset{\star}{,} g] - [a_\star(g) \overset{\star}{,} f] \\
&= -\partial_t d_\star(g)(f) + d_\star(a_\star(g))(f).
\end{aligned}
\tag{B.23}
$$

For a function g which might also depend on t the quantum version of (5.115) now reads

$$
[a_\star + \partial_t, d_\star(g)] = d_\star(a_\star(g)). \tag{B.24}
$$

We go on to calculate the quantum version of (5.116). We first note that

$$
\partial_t \Phi(X_t) = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \partial_t U_n(X_t, \theta_t, \ldots, \theta_t) = \Phi(\partial_t X_t) + \Psi(f_\theta, X_t). \tag{B.25}
$$

With this we get

$$
\begin{aligned}
[\Phi(a_\theta), \Phi(X_t)] &= d_\star \Psi(a_\theta, X_t) + \Phi([a_\theta, X_t]) \\
&\quad -\Psi([\theta_t\, a_\theta]) + \Psi([\theta_t, X_t], a_\theta) \\
&= d_\star \Psi(a_\theta, X_t) + \Phi(-d_\theta(X_t^k a_k)) + \Phi(-\partial_t X_t) - \Psi(f_\theta, X_t) \\
&= -d_\star(\Phi(X_t^k a_k) - \Psi(a_\theta, X_t)) - \partial_t \Phi(X_t),
\end{aligned}
\tag{B.26}
$$

where we have used (B.15).

B.3 The transformation properties of K_t

To calculate the transformation properties of $K_t(X_t^k a_k)$, we first evaluate

$$
\begin{aligned}
\delta_\lambda((a_\theta + \partial_t)^n) X^k a_k &= \sum_{i=0}^{n-1} (a_\theta + \partial_t)^i d_\theta(\lambda)(a_\theta + \partial_t)^{n-1-i} X^k a_k \\
&= \sum_{i=0}^{n-1} \sum_{l=0}^{i} \binom{i}{l} d_\theta((a_\theta + \partial_t)^l(\lambda))(a_\theta + \partial_t)^{n-1-l} X^k a_k
\end{aligned}
\tag{B.27}
$$

and

$$
\begin{aligned}
&(a_\theta+\partial_t)^n\delta_\lambda(X^k a_k) \\
&= (a_\theta+\partial_t)^n X^k\partial_k\lambda \\
&= X^k\partial_k(a_\theta+\partial_t)^n - \sum_{i=0}^{n-1}(a_\theta+\partial_t)^i d_\theta(X^k a_k)(a_\theta+\partial_t)^{n-1-i}\lambda \\
&= X^k\partial_k(a_\theta+\partial_t)^n - \sum_{i=0}^{n-1}\sum_{j=0}^{n-1-i}\binom{n-1-i}{j}(-1)^{n-1-i-j}(a_\theta+\partial_t)^{i+j}\times \\
&\qquad\qquad d_\theta((a_\theta+\partial_t)^{n-1-i-j}(X^k a_k))(\lambda) \\
&= X^k\partial_k(a_\theta+\partial_t)^n + \sum_{i=0}^{n-1}\sum_{j=0}^{n-1-i}\binom{n-1-i}{j}(-1)^{n-1-i-j}(a_\theta+\partial_t)^{i+j}\times \\
&\qquad\qquad d_\theta(\lambda)((a_\theta+\partial_t)^{n-1-i-j}(X^k a_k)) \\
&= X^k\partial_k(a_\theta+\partial_t)^n + \sum_{i=0}^{n-1}\sum_{j=0}^{n-1-i}\sum_{l=0}^{i+j}\binom{n-1-i}{j}\binom{i+j}{l}(-1)^{n-1-i-j}\times \\
&\qquad\qquad d_\theta((a_\theta+\partial_t)^l(\lambda))((a_\theta+\partial_t)^{n-1-l}(X^k a_k)).
\end{aligned}
\tag{B.28}
$$

We go on by simplifying these expressions. Using

$$
\binom{i}{l} = \binom{i-1}{l} + \binom{i-1}{l-1} \quad \text{for} \quad i > l \tag{B.29}
$$

we get

$$
\sum_{m=l}^{n-1}\sum_{i=0}^{m}\binom{n-1-i}{m-i}\binom{m}{l}(-1)^{n-1-m} = \sum_{m=l}^{n-1}\binom{n}{m}\binom{m}{l}(-1)^{n-1-m}. \tag{B.30}
$$

Using (B.29) again two times and then using induction we go on to

$$
\sum_{m=l}^{n-1}\binom{n}{m}\binom{m}{l}(-1)^{n-1-m} = \sum_{i=0}^{l}\binom{n-1-i}{n-1-l}, \tag{B.31}
$$

giving, after using (B.29) again

$$
\sum_{i=0}^{l}\binom{n-1-i}{n-1-l} = \binom{n}{l}. \tag{B.32}
$$

Together with

$$\sum_{i=l}^{n-1} \binom{i}{l} = \binom{n}{l+1} \tag{B.33}$$

these formulas add up to give

$$\sum_{m=l}^{n-1} \sum_{i=0}^{m} \binom{n-1-i}{m-i} \binom{m}{l} (-1)^{n-1-m} + \sum_{i=l}^{n-1} \binom{i}{l} = \binom{n+1}{l+1} \tag{B.34}$$

and therefore

$$\delta_\lambda(K_t(X^k a_k)) = X^k \partial_k(K_t(\lambda)) + d_\theta(K_t(\lambda))K_t(X^k a_k). \tag{B.35}$$

Appendix C

Representations

C.1 The standard representation of the fuzzy sphere

The irreducible N-dimensional representation of the $su(2)$ algebra λ_i (8.3) is given by

$$(\lambda_3)_{kl} = \delta_{kl} \frac{N+1-2k}{2}, \tag{C.1}$$

$$(\lambda_+)_{kl} = \delta_{k+1,l}\sqrt{(N-k)k}, \tag{C.2}$$

where $k, l = 1, ..., N$ and $\lambda_\pm = \lambda_1 \pm i\lambda_2$.

C.2 Representation of the $SO(6)$- intertwiners and Clifford algebra

Latin indices i, j will run from 1 to 3, whereas Greek indices μ, ν, ... denote all the six dimensions, i.e. both the three left and the three right indices. We will use the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{C.3}$$

which satisfy

$$\sigma^i \sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k. \tag{C.4}$$

With these we define the 4-dimensional antisymmetric matrices

$$\begin{array}{lll} \gamma_L^1 = \sigma^1 \otimes \sigma^2, & \gamma_L^2 = \sigma^2 \otimes 1, & \gamma_L^3 = \sigma^3 \otimes \sigma^2, \\ \gamma_R^1 = i\,\sigma^2 \otimes \sigma^1, & \gamma_R^2 = i\,1 \otimes \sigma^2, & \gamma_R^3 = i\,\sigma^2 \otimes \sigma^3. \end{array} \tag{C.5}$$

They are the intertwiners between $SU(4) \otimes SU(4)$ and $SO(6)$ and fulfill the following relations:

$$\begin{aligned} (\gamma_L^i)^\dagger &= \gamma_L^i, \\ (\gamma_R^i)^\dagger &= -\gamma_R^i \end{aligned} \tag{C.6}$$

and

$$\begin{aligned} \gamma_L^i \gamma_L^j &= \delta^{ij} + i\epsilon_k^{ij}\gamma_L^k, \\ \gamma_R^i \gamma_R^j &= -\delta^{ij} - \epsilon_k^{ij}\gamma_R^k, \\ [\gamma_L^i, \gamma_R^j] &= 0. \end{aligned} \tag{C.7}$$

We can now define the 8-dimensional representation of the $SO(6)$-Clifford algebra as

$$\Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^{\mu\dagger} & 0 \end{pmatrix}, \tag{C.8}$$

with the desired anticommutation relations

$$\{\Gamma^\mu, \Gamma^\nu\} = \begin{pmatrix} \gamma^\mu\gamma^{\nu\dagger} + \gamma^\nu\gamma^{\mu\dagger} & 0 \\ 0 & \gamma^{\mu\dagger}\gamma^\nu + \gamma^{\nu\dagger}\gamma^\mu \end{pmatrix} = 2\delta^{\mu\nu}. \tag{C.9}$$

The chirality operator in this basis is

$$\Gamma = i\Gamma_L^1\Gamma_L^2\Gamma_L^3\Gamma_R^1\Gamma_R^2\Gamma_R^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{C.10}$$

The 8-dimensional $SO(6)$-rotations are generated by

$$\Sigma_8^{\mu\nu} = -\frac{i}{4}[\Gamma^\mu, \Gamma^\nu] = -\frac{i}{4}\begin{pmatrix} \gamma^\mu\gamma^{\nu\dagger} - \gamma^\nu\gamma^{\mu\dagger} & 0 \\ 0 & \gamma^{\mu\dagger}\gamma^\nu - \gamma^{\nu\dagger}\gamma^\mu \end{pmatrix}. \tag{C.11}$$

If we define

$$\Sigma^{\mu\nu} = -\frac{i}{4}(\gamma^\mu\gamma^{\nu\dagger} - \gamma^\nu\gamma^{\mu\dagger}) \quad \text{and} \quad \overline{\Sigma}^{\mu\nu} = -\frac{i}{4}(\gamma^{\mu\dagger}\gamma^\nu - \gamma^{\nu\dagger}\gamma^\mu), \tag{C.12}$$

the Clifford algebra transforms as

$$[\Sigma_8^{\mu\nu}, \Gamma^\sigma] = \begin{pmatrix} 0 & \Sigma^{\mu\nu}\gamma^\sigma - \gamma^\sigma\overline{\Sigma}^{\mu\nu} \\ \overline{\Sigma}^{\mu\nu}\gamma^{\sigma\dagger} - \gamma^{\sigma\dagger}\Sigma^{\mu\nu} & 0 \end{pmatrix}. \tag{C.13}$$

Explicitly we have

$$\Sigma^{iL\,jL} = -\tfrac{i}{4}[\gamma_L^i, \gamma_L^j] = \overline{\Sigma}^{iL\,jL}, \tag{C.14}$$

$$\Sigma^{iR\,jR} = \tfrac{i}{4}[\gamma_R^i, \gamma_R^j] = \overline{\Sigma}^{iR\,jR}, \tag{C.15}$$

$$\Sigma^{iR\,jL} = -\tfrac{i}{4}\{\gamma_R^i, \gamma_L^j\} = -\overline{\Sigma}^{iR\,jL} \tag{C.16}$$

and therefore

$$[\Sigma_8^{iL\,jL}, \Gamma^\sigma] = \begin{pmatrix} 0 & [\Sigma^{iL\,jL}, \gamma^\sigma] \\ [\Sigma^{iL\,jL}, \gamma^{\sigma\dagger}] & 0 \end{pmatrix}, \tag{C.17}$$

$$[\Sigma_8^{iR\,jR}, \Gamma^\sigma] = \begin{pmatrix} 0 & [\Sigma^{iR\,jR}, \gamma^\sigma] \\ [\Sigma^{iR\,jR}, \gamma^{\sigma\dagger}] & 0 \end{pmatrix}, \tag{C.18}$$

$$[\Sigma_8^{iR\,jL}, \Gamma^\sigma] = \begin{pmatrix} 0 & \{\Sigma^{iR\,jL}, \gamma^\sigma\} \\ -\{\Sigma^{iR\,jL}, \gamma^{\sigma\dagger}\} & 0 \end{pmatrix}. \tag{C.19}$$

Appendix D

Calculations for the matrix model approach

D.1 Alternative formulation using $4\mathcal{N} \times 4\mathcal{N}$ matrices

Let us rewrite the action (9.32) in terms of the $4\mathcal{N} \times 4\mathcal{N}$ matrices B_L, B_R (9.20). Noting that

$$C_L C_R + C_R C_L = \begin{pmatrix} -[B_L, B_R] & 0 \\ 0 & [B_L, B_R] \end{pmatrix} \tag{D.1}$$

we can rewrite S_6 (9.28) as

$$S_6 = 2\mathrm{Tr}\left(B_L^2 - B_R^2 - \frac{N^2}{2}\right)^2 + 2\mathrm{Tr}\left([B_L, B_R]^2\right), \tag{D.2}$$

where the trace is now over $4\mathcal{N} \times 4\mathcal{N}$ matrices. Similarly

$$S_{\text{break}} = -4\mathrm{Tr}\left(B_L^2 - \frac{N^2}{4}\right)\left(-B_R^2 - \frac{N^2}{4}\right) \tag{D.3}$$

and combined we recover (9.1) as

$$S = S_6 - S_{\text{break}} = 2\mathrm{Tr}\left((B_L^2 - \frac{N^2}{4})^2 + (-B_R^2 - \frac{N^2}{4})^2 + [B_L, B_R]^2\right). \tag{D.4}$$

This looks like a 2-matrix model, however the degrees of freedom B_L, B_R are still very much constrained and span only a small subspace of the $4\mathcal{N} \times 4\mathcal{N}$ matrices. We would like to find an intrinsic characterization without using the γ_μ explicitly. One possibility is to choose the γ_μ to be completely anti-symmetric matrices, see Appendix C.2. However this does not extend to B, since the B_μ should be Hermitian and not necessarily symmetric, and moreover the γ_μ are not Hermitian (the conjugate being the intertwiner $(6) \subset (\overline{4}) \otimes (\overline{4})$). Another possibility is provided by the following representation of the γ-matrices:

$$\gamma_L^i = \sigma^i \otimes 1\!\!1_{2\times 2}, \qquad \gamma_R^i = 1\!\!1_{2\times 2} \otimes i\sigma^i. \tag{D.5}$$

They satisfy the relations (9.16) – (9.19), but are not antisymmetric. Now note that

$$\gamma_R^i = iP\gamma_L^i P \tag{D.6}$$

where

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}(1 + \sigma^i \otimes \sigma^i) \tag{D.7}$$

permutes the two tensor factors and satisfies

$$P^2 = 1. \tag{D.8}$$

Therefore we can characterize the degrees of freedom in terms of 2 Hermitian $2\mathcal{N} \times 2\mathcal{N}$ matrices

$$X_L = B_L^i \sigma_i + \frac{1}{2}, \qquad X_R = B_R^i \sigma_i + \frac{1}{2} \tag{D.9}$$

which are arbitrary up to the constraint that $X^0_{L,R} = \frac{1}{2}$. Then

$$B_L = X_L \otimes 1\!\!1_{2\times 2}, \quad B_R = iP(X_R \otimes 1\!\!1_{2\times 2})P; \tag{D.10}$$

they could be extracted from a single complex matrix $\tilde{B} = (X_L + iX_R) \otimes 1\!\!1_{2\times 2}$. Furthermore, matrices of the form $X \otimes 1\!\!1_{2\times 2}$ are characterized through their spectrum, which is doubly degenerate; indeed any such Hermitian matrix can be cast into the above form using suitable unitary $SU(4\mathcal{N})$ transformations. Similarly, P can also be characterized intrinsically: any matrix P written as

$$P = P_0 \otimes 1\!\!1_{2\times 2} + P_i \otimes \sigma^i \tag{D.11}$$

which satisfies the constraints

$$P_0 = \frac{1}{2}, \quad P^2 = 1\!\!1 \tag{D.12}$$

is given by (D.7) up to an irrelevant unitary transformation $U \otimes 1\!\!1$. We could therefore write down the action (D.4) in terms of three matrices $B_L, -iPB_RP$ and P, all of which are characterized by their spectrum and constraints of the form $(..)_0 = \frac{1}{2}$. The hope is that such a reformulation may allow to apply some of the powerful methods from random matrix theory, in the spirit of [96].

D.2 Stability analysis of the $SO(6)$ - invariant action (9.28)

Consider the action (9.28). We will split off the radial degrees of freedom for large N by setting[1]

$$B_{iL} = \lambda_{iL} + A_{iL} = \lambda_{iL} + \mathcal{A}_{iL} + x_{iL}\Phi_L \tag{D.13}$$

requiring that $\lambda_{iL}\mathcal{A}_{iL} = 0$, and similarly for B_{iR}, The stability of our geometry will depend on the behavior of Φ^L and Φ^R. We calculate that

$$B_\mu B_\mu - \frac{N^2-1}{2} = N(\Phi_L + \Phi_R) + \Phi_L\Phi_L + \Phi_R\Phi_R + \mathcal{A}_\mu\mathcal{A}_\mu - [\lambda_\mu, \mathcal{A}_\mu] + \mathcal{O}(\frac{1}{N}), \tag{D.14}$$

where we used that $\lambda_{ia}\mathcal{A}_{ia} = 0$ and therefore both $\mathcal{A}_{ia}x_{ia} = \mathcal{O}(\frac{1}{N})$ and $\mathcal{A}_{ia}[\lambda_{ia}, \cdot] = \mathcal{O}(\frac{1}{N})$ for $a = L, R$. Setting

$$\begin{aligned} \Phi_L + \Phi_R &= \Phi_1, \\ \Phi_L - \Phi_R &= \Phi_2 \end{aligned} \tag{D.15}$$

we get

$$B_\mu B_\mu - \frac{N^2-1}{2} = N\Phi_1 + \Phi_1\Phi_1 + \Phi_2\Phi_2 + \mathcal{A}_\mu\mathcal{A}_\mu - [\lambda_\mu, \mathcal{A}_\mu] + \mathcal{O}(\frac{1}{N}). \tag{D.16}$$

In the limit $N \to \infty$ we can integrate out Φ_1, as it acquires an infinite mass. Alternatively we can rescale Φ_1 by setting $\phi_1 = \frac{1}{N}\Phi_1$. Then, all the terms involving ϕ_1 but the first one in (D.16) will be of order $\frac{1}{N}$ and we can equally integrate out ϕ_1.

The terms from

$$F_{iL}F_{iL} + F_{iR}F_{iR} - [B_{iL}, B_{iR}]^2 \tag{D.17}$$

with $F_{iL} = \frac{1}{2}\epsilon_{ijk}F_{jL\,kL}$ etc. involving the remaining Φ_2 will be (in the limit $N \to \infty$)

$$\frac{1}{2}\Phi_2\Phi_2 - J_\mu(\Phi_2)J_\mu(\Phi_2) - F_{iL}x_{iL}\Phi_2 + F_{iR}x_{iR}\Phi_2 \tag{D.18}$$

with the tangential derivatives $J_{ia} = -i\epsilon_{ijk}x_{ja}\partial_{ka}$. Calculating that

$$J_\mu\Phi_2 J_\mu\Phi_2 = -\partial_\mu\Phi_2\partial_\mu\Phi_2 - x_{iL}\partial_{iL}\Phi_2 x_{jL}\partial_{jL}\Phi_2 - x_{iR}\partial_{iR}\Phi_2 x_{jR}\partial_{jR}\Phi_2 \tag{D.19}$$

and using partial integration under the integral this gives

$$\frac{1}{2}\Phi_2\Phi_2 - \Phi_2\partial_\mu\partial_\mu\Phi_2 - x_{iL}\partial_{iL}\Phi_2 x_{jL}\partial_{jL}\Phi_2 - x_{iR}\partial_{iR}\Phi_2 x_{jR}\partial_{jR}\Phi_2 - F_{iL}x_{iL}\Phi_2 + F_{iR}x_{iR}\Phi_2 \tag{D.20}$$

[1] the fact that this leads to non-hermitian fields for finite N is not essential here

Expanding both Φ_2 and F in left and right spherical harmonics as

$$\Phi_2 = \sum_{klmn} c_{klmn} Y^L_{km} Y^R_{ln} \quad \text{and} \quad F_{ia} x_{ia} = \sum_{klmn} f^a_{klmn} Y^L_{km} Y^R_{ln} \tag{D.21}$$

we get for fixed $klmn$, setting $c = c_{klmn}$, $f^a = f^a_{klmn}$ and $p = \frac{1}{2} + l(l+1) + k(k+1)$ the following expression

$$pc^2 - cf^L + cf^R = p(c - \frac{1}{2p} f^L + \frac{1}{2p} f^R)^2 - \frac{1}{4p}(f^L - f^R)^2. \tag{D.22}$$

Integrating out the c's and putting everything back this leaves us with the additional term

$$-(F_{iL} x_{iL} - F_{iR} x_{iR}) \frac{1}{4(\frac{1}{2} - \partial_\mu \partial_\mu)} (F_{iL} x_{iL} - F_{iR} x_{iR}) \tag{D.23}$$

in the action (9.28).

D.3 The Dirac operator in spherical coordinates

For a general Riemannian manifold with metric

$$g = g_{\mu\nu} dx^\mu dx^\nu \tag{D.24}$$

the Christoffel symbols are given by

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda} (\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}). \tag{D.25}$$

We can change to a non-coordinate basis (labeled by Latin indices in contrast to the Greek indices for the coordinates) by introducing the vielbeins e^μ_a with

$$\begin{aligned} e^a_\mu e^\mu_b &= \delta^a_b, \\ g_{\mu\nu} &= e^a_\mu e^b_\nu \delta_{ab}, \qquad g^{\mu\nu} = e^\mu_a e^\nu_b \delta^{ab}. \end{aligned} \tag{D.26}$$

With these, the Dirac operator is given by

$$D = -i\gamma^a e^\mu_a (\partial_\mu + \frac{1}{4} \omega_{\mu ab} [\gamma^a, \gamma^b]), \tag{D.27}$$

where the γ^a form a flat Clifford algebra, i. e.

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab} \quad , \quad \gamma^{a\dagger} = \gamma^a \tag{D.28}$$

and the spin connection ω fulfills

$$\partial_\mu e^a_\nu - \Gamma^\lambda_{\mu\nu} e^a_\lambda + \omega_\mu{}^a{}_b \, e^b_\nu = 0. \tag{D.29}$$

D.3.1 The Dirac operator on $\mathbb{R}^6$ in spherical coordinates

We will now write down the flat $SO(6)$ Dirac operator D_6 by splitting $\mathbb{R}^6$ into $\mathbb{R}^3_L \times \mathbb{R}^3_R$ and introducing spherical coordinates on both the left and right hand side. The flat metric becomes

$$\begin{aligned} g_6 &= r_L^2\, d\theta_L \otimes d\theta_L + r_L^2 \sin^2\theta_L\, d\phi_L \otimes d\phi_L + dr_L \otimes dr_L \\ &\quad + r_R^2\, d\theta_R \otimes d\theta_R + r_R^2 \sin^2\theta_R\, d\phi_R \otimes d\phi_R + dr_R \otimes dr_R. \end{aligned} \tag{D.30}$$

Looking at the formula for the Christoffel symbols (D.25), we see that all the symbols with both right and left indices vanish. For the symbols with only right or only left indices we get

$$\begin{aligned} \Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta, \\ \Gamma^\phi_{\theta\phi} &= \frac{\cos\theta}{\sin\theta} = \Gamma^\phi_{\phi\theta}, \\ \Gamma^r_{\theta\theta} &= -r, \\ \Gamma^r_{\phi\phi} &= -r\sin^2\theta, \\ \Gamma^\theta_{r\theta} &= \frac{1}{r} = \Gamma^\theta_{\theta r}, \\ \Gamma^\phi_{r\phi} &= \frac{1}{r} = \Gamma^\phi_{\phi r}, \end{aligned} \tag{D.31}$$

where we have dropped the left or right subscript for simplicity. All other symbols vanish. We want to go to a non-coordinate basis by introducing the vielbeins

$$e^{1_L}_{\theta_L} = r_L; \quad e^{2_L}_{\phi_L} = r_L \sin\theta_L; \quad e^{3_L}_{r_L} = 1; \tag{D.32}$$

$$e^{1_R}_{\theta_R} = r_L; \quad e^{2_R}_{\phi_R} = r_R \sin\theta_R; \quad e^{3_R}_{r_R} = 1. \tag{D.33}$$

Calculating the spinor connection by (D.29), we again see that all the terms with both left and right indices vanish. The terms with only left or only right indices are

$$\begin{aligned} {\omega_\phi}^1{}_2 &= -\cos\theta &= -{\omega_\phi}^2{}_1, \\ {\omega_\phi}^2{}_3 &= \sin\theta &= -{\omega_\phi}^3{}_2, \\ {\omega_\theta}^1{}_3 &= 1 &= -{\omega_\theta}^3{}_1, \end{aligned} \tag{D.34}$$

where we again dropped the left or right subscripts. Putting all this together we see that D_6 splits up into a left part D_{3L} and a right part D_{3R} as

$$D_6 = D_{3L} + D_{3R} \tag{D.35}$$

with

$$D_{3L} = -i\overline{\Gamma}^1_L \frac{1}{r_L}(\partial_{\theta_L} + \frac{\cos\theta_L}{\sin\theta_L}) - i\overline{\Gamma}^2_L \frac{1}{r_L \sin\theta_L}\partial_{\phi_L} - i\overline{\Gamma}^3_L(\partial_{r_L} + \frac{1}{r_L}), \quad \text{(D.36)}$$

$$D_{3R} = -i\overline{\Gamma}^1_R \frac{1}{r_R}(\partial_{\theta_R} + \frac{\cos\theta_R}{\sin\theta_R}) - i\overline{\Gamma}^2_R \frac{1}{r_R \sin\theta_R}\partial_{\phi_R} - i\overline{\Gamma}^3_R(\partial_{r_R} + \frac{1}{r_R}) \quad \text{(D.37)}$$

where the $\overline{\Gamma}$ have to form a $SO(6)$ Clifford algebra.

D.3.2 The Dirac operator on $S^2 \times S^2$

We now want to calculate the curved Dirac operator D_4 on $S^2 \times S^2$ in the spherical coordinates of the spheres (they are the same spherical coordinates we used before, now restricted to the spheres). The metric on $S^2 \times S^2$ with radii r_L and r_R is

$$\begin{aligned} g_4 &= r_L^2\, d\theta_L \otimes d\theta_L + r_L^2 \sin^2\theta_L\, d\phi_L \otimes d\phi_L \\ &\quad + r_R^2\, d\theta_R \otimes d\theta_R + r_R^2 \sin^2\theta_R\, d\phi_R \otimes d\phi_R. \end{aligned} \quad \text{(D.38)}$$

The metric is the same as (D.30) restricted to the spheres, so the Christoffel symbols are the same as (D.31). Again introducing the vielbeins

$$e^{1_L}_{\theta_L} = r_L; \quad e^{2_L}_{\phi_L} = r_L \sin\theta_L; \quad \text{(D.39)}$$

$$e^{1_R}_{\theta_R} = r_L; \quad e^{2_R}_{\phi_R} = r_R \sin\theta_R, \quad \text{(D.40)}$$

we see that also the spin connection is the same as (D.34), and therefore we can again split D_4 into a right part D_{2R} and a left part D_{2L} as $D_4 = D_{2L} + D_{2R}$ with

$$D_{2L} = -i\widetilde{\Gamma}^1_L \frac{1}{r_L}(\partial_{\theta_L} + \frac{\cos\theta_L}{\sin\theta_L}) - i\widetilde{\Gamma}^2_L \frac{1}{r_L \sin\theta_L}\partial_{\phi_L}, \quad \text{(D.41)}$$

$$D_{2R} = -i\widetilde{\Gamma}^1_R \frac{1}{r_R}(\partial_{\theta_R} + \frac{\cos\theta_R}{\sin\theta_R}) - i\widetilde{\Gamma}^2_R \frac{1}{r_R \sin\theta_R}\partial_{\phi_R}, \quad \text{(D.42)}$$

where the $\widetilde{\Gamma}$ form a flat $SO(4)$ Clifford algebra.

D.3.3 $SO(3) \times SO(3)$-covariant form of the Dirac operator on $S^2 \times S^2$

The flat $SO(6)$ Dirac operator D_6 can be split into a left part D_{3L} and a right part D_{3R} using spherical coordinates in D.35. Of course, D_6 can also be written in the usual Euclidian coordinates as

$$D_6 = -i\Gamma^\mu \partial_\mu, \quad \text{(D.43)}$$

where again we can split it into a left and a right part as

$$D_6 = D_{3L} + D_{3R} \tag{D.44}$$

with

$$\begin{gathered} D_{3L} = -i\Gamma^i_L\partial_i, \qquad D_{3R} = -i\Gamma^i_R\partial_i, \\ \{D_{3L}, D_{3R}\} = 0. \end{gathered} \tag{D.45}$$

We have left open which representation of the $SO(6)$ Clifford algebra we want to use for the $\overline{\Gamma}$ in (D.36,D.37), but Γ in (D.45) is really the representation given by (9.21). We will now relate the two expressions for the Clifford algebra and the Dirac operator by first defining

$$J_{iL} = -i\epsilon_{ijk}x_{jL}\partial_{kL} \quad \text{and} \quad J_{iR} = -i\epsilon_{ijk}x_{jR}\partial_{kR} \tag{D.46}$$

and noting that

$$\left(\frac{\Gamma^i_L x_{iL}}{r_L}\right)^2 = \left(\frac{\Gamma^i_R x_{iR}}{r_R}\right)^2 = 1. \tag{D.47}$$

We calculate that

$$\left(\frac{\Gamma^j_L x_{jL}}{r_L}\right)^2 \Gamma^i_L\partial_{iL} = \left(\frac{\Gamma^j_L x_{jL}}{r_L}\right)\left(\frac{x_{iL}\partial_{iL}}{r_L} - \frac{1}{r_L}\begin{pmatrix} \gamma^i_L & 0 \\ 0 & \gamma^i_L \end{pmatrix} J_{iL}\right), \tag{D.48}$$

$$\left(\frac{\Gamma^j_R x_{jR}}{r_R}\right)^2 \Gamma^i_R\partial_{iR} = \left(\frac{\Gamma^j_R x_{jR}}{r_R}\right)\left(\frac{x_{iR}\partial_{iR}}{r_R} + \frac{i}{r_R}\begin{pmatrix} \gamma^i_R & 0 \\ 0 & \gamma^i_R \end{pmatrix} J_{iR}\right), \tag{D.49}$$

and therefore

$$D_{3L} = -i\left(\frac{\Gamma^j_L x_{jL}}{r_L}\right)\left(\partial_{r_L} - \frac{1}{r_L}\begin{pmatrix} \gamma^i_L & 0 \\ 0 & \gamma^i_L \end{pmatrix} J_{iL}\right), \tag{D.50}$$

$$D_{3R} = -i\left(\frac{\Gamma^j_R x_{jR}}{r_R}\right)\left(\partial_{r_R} + \frac{i}{r_R}\begin{pmatrix} \gamma^i_R & 0 \\ 0 & \gamma^i_R \end{pmatrix} J_{iR}\right). \tag{D.51}$$

Comparing this with (D.36,D.37) we see that

$$\overline{\Gamma}^3_L = \left(\frac{\Gamma^i_L x_{iL}}{r_L}\right) \quad \text{and} \quad \overline{\Gamma}^3_R = \left(\frac{\Gamma^i_R x_{iR}}{r_R}\right), \tag{D.52}$$

as the J_L and J_R have no radial components. From (D.50,D.51) we can also deduce that

$$[\overline{\Gamma}^i_L, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}] = 0 = [\overline{\Gamma}^i_R, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}] \tag{D.53}$$

and

$$\{\overline{\Gamma}^i_R, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\} = 0 = \{\overline{\Gamma}^i_L, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}. \tag{D.54}$$

The curved Dirac operator D_4 on $S^2 \times S^2$ expressed in the spherical coordinates of the spheres also splits up as $D_4 = D_{2L} + D_{2R}$ with right part D_{2R} and left part D_{2L} given in (D.41,D.42). Comparing this with (D.36,D.37), we see that the dependence on the tangential coordinates is the same in both expressions. With (D.53,D.54) we see that the matrices $-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\overline{\Gamma}^3_L\overline{\Gamma}^i_L$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\overline{\Gamma}^3_R\overline{\Gamma}^j_R$ for $i, j = 1, 2$ form a $SO(4)$ Clifford algebra and can therefore be used as the $\widetilde{\Gamma}$. Note that this representation is still reducible, a problem we deal with in chapter 9.4.2. Now we can get a simple relation between the D_3 restricted on the spheres and the D_2

$$-\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(i\overline{\Gamma}^3_L D_{3L}|_{res.} - \frac{1}{r_L}) \;=\; D_{2L}, \tag{D.55}$$

$$-i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(i\overline{\Gamma}^3_R D_{3R}|_{res.} - \frac{1}{r_R}) \;=\; D_{2R}. \tag{D.56}$$

Inserting (D.50,D.51) and using (D.52) together with (D.47) we find that

$$D_{2L} \;=\; \frac{1}{r_L}(\Gamma^i_L J_{iL} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}), \tag{D.57}$$

$$D_{2R} \;=\; \frac{1}{r_R}(\Gamma^i_R J_{iR} + i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}). \tag{D.58}$$

Setting $r_L = r_R = 1$ for simplicity, the Dirac operator D_4 on $S^2 \times S^2$ takes the form (9.43).

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